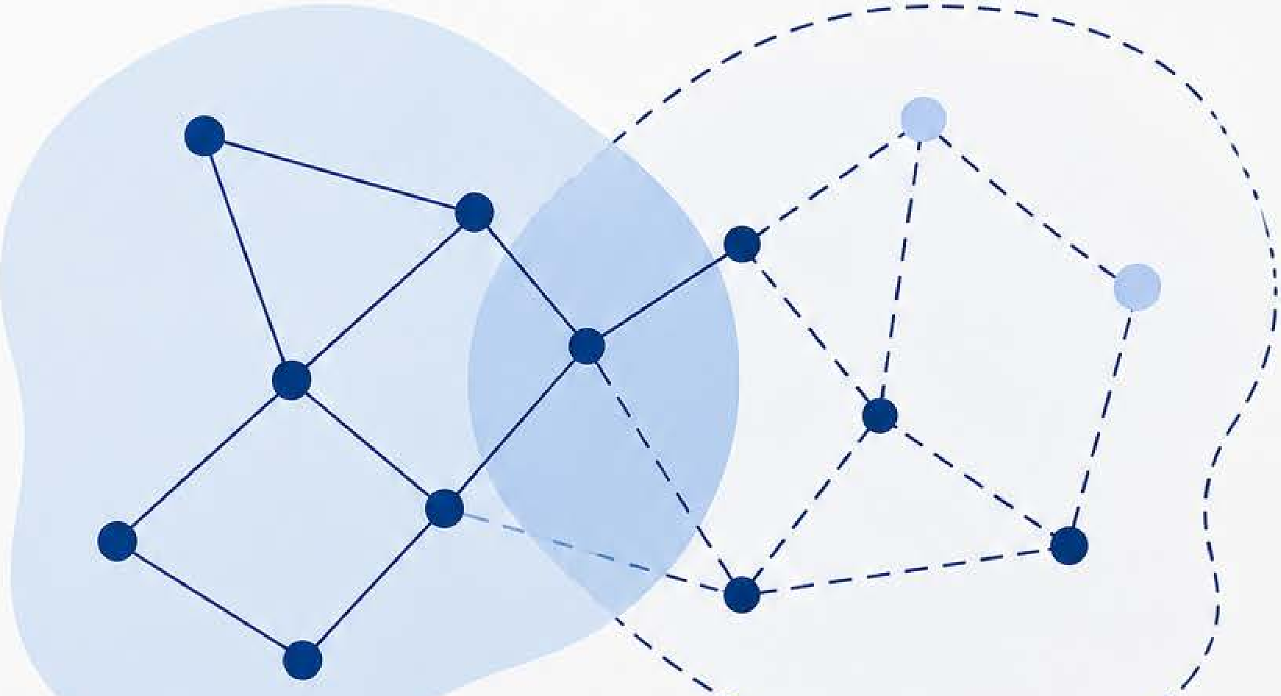
# A Comprehensive Survey of Soft Graphs and Rough Graphs



Takaaki Fujita
Florentin Smarandache

**Takaaki Fujita, Florentin Smarandache**

# A Comprehensive Survey of Soft Graphs and Rough Graphs

Neutrosophic Science International Association (NSIA)
Publishing House

Gallup - Guayaquil
United States of America – Ecuador
2026

*Editor:*

Neutrosophic Science International Association (NSIA)
Publishing House
https://fs.unm.edu/NSIA/

Division of Mathematics and Sciences
University of New Mexico
705 Gurley Ave., Gallup Campus
NM 87301, United States of America

University of Guayaquil
Av. Kennedy and Av. Delta
"Dr. Salvador Allende" University Campus
Guayaquil 090514, Ecuador

***Peer-Reviewers:***

**John Frederick D. Tapia**
Chemical Engineering Department, De La Salle University - Manila, 2401 Taft Avenue, Malate, Manila, Philippines
Email: *john.frederick.tapia@dlsu.edu.ph*

**Darren Chong**
Independent researcher, Singapore
Email: *darrenchong2001@yahoo.com.sg*

**Umit Cali**
Norwegian University of Science and Technology, NO-7491 Trondheim, Norway
Email: *umit.cali@ntnu.no*

ISBN 978-197250248-8

# Table of Contents

# Chapter 1

# Introduction

## 1.1 Graph Theory and Its Applications

Graph theory is a fundamental branch of mathematics concerned with networks composed of vertices, or nodes, and edges, or connections. It provides a rigorous framework for analyzing structural properties, connectivity patterns, paths, and interactions within such networks [1]. Owing to its broad applicability, graph theory has been extensively studied in numerous theoretical and practical domains, including real-world network analysis [2], graph neural networks [3, 4], Bayesian networks [5], bioinformatics [6, 7], protein-structure analysis [8, 9], circuit design [10], chemical graph theory [11], and graph databases [12, 13]. Major research directions also include the classification and structural analysis of graph families [14] and the development of efficient graph algorithms [15]. Extensions of classical graphs are also known, including HyperGraphs [16, 17], SuperHyperGraphs [18, 19], MultiGraphs [20, 21], and Digraphs.

## 1.2 Soft Set Theory

Classical set theory is based on exact membership and therefore does not always provide a sufficiently flexible framework for situations involving ambiguity, incomplete information, uncertainty, or vagueness. To address these limitations, numerous generalized set-theoretic frameworks have been developed, including fuzzy sets [22, 23], hyperfuzzy sets [24–26], neutrosophic sets [27, 28], double-valued neutrosophic sets [29–31], plithogenic sets [18,32,33], uncertain sets [34–36], rough sets [37,38], and soft sets [39].

Let $U$ be a universe and let $E$ be a set of parameters. A soft set over $U$ is a pair $(F, E)$, where

$$F : E \longrightarrow \mathcal{P}(U).$$

Thus, each parameter $e \in E$ is associated with a subset $F(e) \subseteq U$. Rather than assigning a single fixed classification to the elements of $U$, a soft set describes them through parameter-dependent subsets. This flexible parameterization has been applied in areas such as decision analysis, data processing, and knowledge representation; see, for example, [39,40]. The adaptability of soft-set theory has led to the introduction of numerous extensions. An *IndetermSoft set*, for example, permits indeterminacy in the parameters, the associated approximating subsets, the correspondence between them, or several of these components simultaneously [41, 42]. Other developments include *HyperSoft sets* [43, 44], *TreeSoft sets* [45–48], *bipolar soft sets* [49, 50], and *SuperHyperSoft sets* [51–53]. Surveys of soft-set theory are also available in the literature, including the work cited in [54].

Soft-set concepts have further been incorporated into graph-based frameworks, including Soft Graphs [55, 56] and Soft HyperGraphs [57]. Soft Graphs assign each parameter a subgraph of an ambient graph, enabling parameter-dependent representation of uncertain vertices, edges, relationships, and structural configurations within complex networks.

## 1.3 Rough Sets and Rough Graphs

A rough set represents an imprecisely specified concept through lower and upper approximations induced by an indiscernibility relation [58, 59]. The lower approximation contains elements that certainly belong to the target set, whereas the upper approximation contains elements that possibly belong to it. Their difference forms the boundary region, which captures the uncertainty inherent in the available information. Several extensions of Rough Sets are known, including Game-Theoretic Rough Sets [60], Probabilistic Rough Sets [61, 62], and Variable Precision Rough Sets [63–65].

A rough graph extends this approximation-based framework to graph structures [66–68]. Rough approximations may be applied to vertices, edges, or subgraphs, thereby distinguishing structural components that are certainly present, possibly present, or contained in a boundary region under a prescribed indiscernibility relation.

## 1.4 Our Contributions

Soft Graphs, Rough Graphs, and their numerous extensions have been studied extensively. However, relatively few works provide a unified survey of these graph-theoretic frameworks and their relationships. This book addresses this gap by presenting an organized introduction to Soft Graphs, Rough Graphs, and several of their principal extensions. The aim is to clarify their underlying mathematical structures, compare their modeling capabilities, and provide a useful reference for further theoretical and applied research.

# Chapter 2

# Preliminaries

In this chapter, we briefly review the basic definitions, terminology, and notation used throughout this book.

## 2.1 Soft Set

A Soft Set provides a simplified approach to parameterized decision modeling by mapping attributes, or parameters, to subsets of a universal set, thereby addressing uncertainty in a straightforward manner [39, 40, 69].

**Definition 2.1.1** (Soft Set)**.** [39, 40] Let $U$ be a universal set and $A$ be a set of attributes. A soft set over $U$ is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of $U$. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in $U$ associated with $\alpha$.

**Example 2.1.2** (Soft Set for Laptop Selection)**.** Let

$$U = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$$

be a set of laptops, and let

$$A = \{a_1, a_2, a_3, a_4\}$$

be an attribute set, where

$$a_1 = \text{affordable}, \qquad a_2 = \text{lightweight}, \qquad a_3 = \text{long battery life}, \qquad a_4 = \text{high performance}.$$

Choose

$$S = \{a_1, a_2, a_3\} \subseteq A$$

and define

$$\mathcal{F} : S \longrightarrow \mathcal{P}(U)$$

by

$$\mathcal{F}(a_1) = \{\ell_1, \ell_2, \ell_5\},$$
$$\mathcal{F}(a_2) = \{\ell_2, \ell_3, \ell_4\},$$

and

$$\mathcal{F}(a_3) = \{\ell_1, \ell_3, \ell_4\}.$$

Therefore,

$$(\mathcal{F}, S) = \left\{\begin{matrix}(a_1, \{\ell_1, \ell_2, \ell_5\}), \\ (a_2, \{\ell_2, \ell_3, \ell_4\}), \\ (a_3, \{\ell_1, \ell_3, \ell_4\})\end{matrix}\right\}$$

is a soft set over $U$.

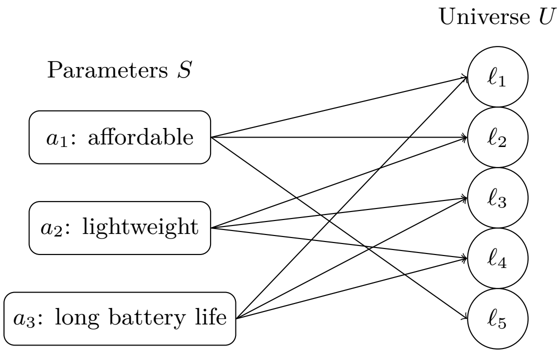



Figure 2.1: A soft set in which each parameter is associated with a subset of laptops.

## 2.2 Rough Set

A rough set approximates a subset using lower and upper bounds determined by equivalence classes, thereby capturing both certainty and uncertainty in membership [58, 70, 71].

**Definition 2.2.1** (Rough Set Approximation)**.** [59] Let $X$ be a nonempty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (also called an indiscernibility relation) on $X$. The relation $R$ partitions $X$ into disjoint equivalence classes, denoted by $[x]_R$ for each $x \in X$, where

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ are defined by:

1. *Lower Approximation*:
$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$
This set contains all elements whose entire equivalence class is contained within $U$; these elements *definitely* belong to $U$.

2. *Upper Approximation*:
$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$
This set contains all elements whose equivalence class has a nonempty intersection with $U$; these elements *possibly* belong to $U$.

Thus, the pair $(\underline{U}, \overline{U})$ forms the rough set representation of $U$, satisfying

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

**Example 2.2.2** (Rough Approximation of a Student Group)**.** Let

$$X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$$

be a set of students. Suppose that $R$ classifies students according to indistinguishable performance levels, with equivalence classes

$$C_1 = \{x_1, x_2\}, \qquad C_2 = \{x_3, x_4, x_5\}, \qquad C_3 = \{x_6, x_7\}, \qquad C_4 = \{x_8\}.$$

Consider the target set

$$U = \{x_1, x_2, x_3, x_6, x_7\}.$$

Since $C_1$ and $C_3$ are entirely contained in $U$, while $C_2$ only intersects $U$, we obtain

$$\underline{U} = C_1 \cup C_3 = \{x_1, x_2, x_6, x_7\},$$

and

$$\overline{U} = C_1 \cup C_2 \cup C_3 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}.$$

Thus, the boundary region is

$$\overline{U} \setminus \underline{U} = \{x_3, x_4, x_5\}.$$

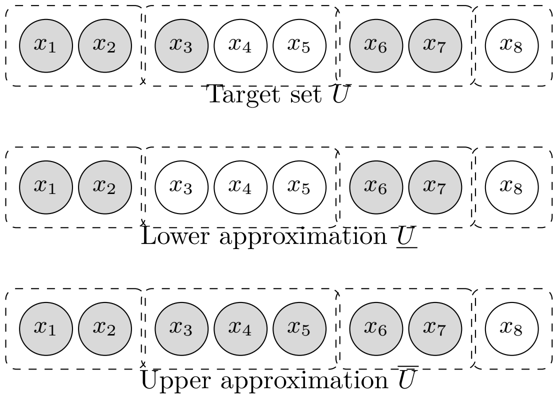



Figure 2.2: The target set and its lower and upper rough approximations. Shaded elements belong to the displayed set, and dashed boxes represent the equivalence classes of $R$.

## 2.3 Uncertain Set

An Uncertain Set assigns to each element a degree taken from a prescribed uncertainty model, thereby providing a common framework that includes fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic settings [72].

**Definition 2.3.1** (Uncertain Model)**.** [72] Let $U$ denote the class of all *uncertain models*. Each $M \in U$ is determined by the following data:

- a nonempty set $\mathrm{Dom}(M) \subseteq [0,1]^k$ of *admissible degree tuples*, for some fixed integer $k \geq 1$;
- a collection of model-dependent algebraic or geometric constraints imposed on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \leq 1$ in the intuitionistic fuzzy setting, or $0 \leq T + I + F \leq 3$ in the neutrosophic setting).

Typical examples are as follows:

- Fuzzy model: $\mathrm{Dom}(M) = [0,1]$;
- Intuitionistic fuzzy model:
$$\mathrm{Dom}(M) = \{(\mu, \nu) \in [0,1]^2 : \mu + \nu \leq 1\};$$
- Neutrosophic model:
$$\mathrm{Dom}(M) = \{(T, I, F) \in [0,1]^3 : 0 \leq T + I + F \leq 3\};$$
- Plithogenic model, together with many further extensions.

**Definition 2.3.2** (Uncertain Set (U-Set))**.** [72] Let $X$ be a nonempty universe, and let $M$ be a fixed uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *Uncertain Set of type $M$* (briefly, a *U-Set*) on $X$ is a pair
$$\mathcal{U} = (X, \mu_M),$$
where
$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$
is called the *uncertainty-degree function* (or membership map) of $\mathcal{U}$.

For each $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ records the degree, or collection of degrees, with which $x$ belongs to the uncertain set under the model $M$.

**Remark 2.3.3.** Several familiar structures appear as special cases:

- If $M$ is the fuzzy model with $\mathrm{Dom}(M) = [0,1]$, then $\mu_M : X \to [0,1]$ is an ordinary fuzzy membership function, and $\mathcal{U}$ is a fuzzy set.
- If $M$ is a neutrosophic model, then $\mu_M(x) = (T(x), I(x), F(x))$, and $\mathcal{U}$ becomes a neutrosophic set.
- Other choices of $M$ recover intuitionistic fuzzy sets, picture fuzzy sets, plithogenic sets, and many related models.

As indicated above, this framework admits many different specializations. For reference, Table 2.1 lists representative families of uncertainty sets (U-Sets), organized according to the dimension $k$ of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (cf. [36]).

Table 2.1: A catalogue of uncertainty-set families (U-Sets) organized by the dimension $k$ of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [36].

| $k$ | note | Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$ |
|---|---|---|
| 1 | | Fuzzy Set [22,73]; N-Fuzzy Set [74–76]; Shadowed Set [77–79] |
| 2 | | Intuitionistic Fuzzy Set [80,81]; Interval-Valued Fuzzy Set [82,83]; Pythagorean Fuzzy Set [84,85]; Fermatean Fuzzy Set [86,87]; q-Rung Orthopair Fuzzy Set [88,89], Vague Set [90,91]; Bipolar Fuzzy Set (two-component description) [92,93]; Variable Fuzzy Set [94–96]; Paraconsistent Fuzzy Set [97,98]; Bifuzzy Set [99,100] |
| 3 | | Single-Valued Neutrosophic Set [28,101]; Picture Fuzzy Set [102,103]; Cubic Set [104,105]; Spherical Fuzzy Set [106,107]; T-Spherical Fuzzy Set [108,109]; Tripolar Fuzzy Set (three-component formalisms) [110–113]; Neutrosophic Vague Set [114, 115] |
| 4 | | Quadripartitioned Neutrosophic Set [116,117]; Interval-Valued Intuitionistic Fuzzy Set [118,119]; Interval-Valued Pythagorean Fuzzy Set [120,121]; Linear Diophantine Fuzzy Set [122, 123]; Double-Valued Neutrosophic Set [29, 31, 124]; Dual Hesitant Fuzzy Set [125, 126]; Ambiguous Set [127–129]; Turiyam Neutrosophic Set [130–133] |
| 5 | | Pentapartitioned Neutrosophic Set [134–136]; Triple-Valued Neutrosophic Set [30, 137,138] |
| 6 | | Hexapartitioned Neutrosophic Set; Quadruple-Valued Neutrosophic Set [30, 139] Interval-Valued Neutrosophic Set [140,141]; |
| 7 | | Heptapartitioned Neutrosophic Set [142,143]; Quintuple-Valued Neutrosophic Set [30,144,145] |
| 8 | | Octapartitioned Neutrosophic Set [146,147] |
| 9 | | Nonapartitioned Neutrosophic Set [146,147] ; Neutrosophic Cubic Set [148,149] |
| $n$ | $(n \geq 1)$ | Multi-valued (Fuzzy) Sets [150]; MultiFuzzy Set [151]; $n$-Refined Fuzzy Set [152, 153] |
| $2n$ | $(n \geq 1)$ | $n$-Refined Intuitionistic Fuzzy Set [153]; Multi-Intuitionistic Fuzzy Set [151] |
| $3n$ | $(n \geq 1)$ | $n$-Refined Neutrosophic Set [153,154]; Multi-Neutrosophic Set [151,155] |

**Reading guide.** Within the U-Set framework [72], each model $M$ is determined by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ together with a membership map $\mu_M : X \to \mathrm{Dom}(M)$. Accordingly, the table groups representative families by the ambient dimension $k$, that is, by the number of numerical components assigned to each element.
(a) A common viewpoint in the literature is that neutrosophic sets form a broad umbrella encompassing several earlier multi-component fuzzy models and their generalizations; see [156].
(b) Ambiguous sets are often described as subclasses of certain four-component neutrosophic families; see [116,117,129].
(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [157].

We now introduce the corresponding notion of an Uncertain Graph.

**Definition 2.3.4** (Uncertain Graph)**.** Let $G = (V, E)$ be a finite undirected loopless graph, and let $M$ be an uncertain model with degree-domain $\mathrm{Dom}(M)$. An *Uncertain Graph of type $M$* is a triple

$$\mathcal{G}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns to every vertex $v \in V$ and every edge $e \in E$ an uncertainty degree $\mu_M(v)$ or $\mu_M(e)$ belonging to $\mathrm{Dom}(M)$.

If desired, one may additionally impose model-dependent consistency conditions relating vertex and edge degrees. For example, in fuzzy or intuitionistic fuzzy settings, the degree of an edge $e = \{u, v\}$ is often bounded in terms of the degrees of its incident vertices. However, such conditions depend on the particular model $M$ and are therefore not built into this general definition.

**Remark 2.3.5.** Well-known graph models are recovered by choosing $M$ appropriately:

- fuzzy graphs arise when $M$ is the fuzzy model and $\mu_M : V \cup E \to [0, 1]$;
- intuitionistic fuzzy graphs, neutrosophic graphs, plithogenic graphs, and related models are obtained from the corresponding choices of $M$.

For reference, Table 2.2 presents a catalogue of uncertainty-graph families (Uncertain Graphs), again classified by the dimension $k$ of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$.

Table 2.2: A catalogue of uncertainty-graph families (Uncertain Graphs) organized by the dimension $k$ of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$.

| $k$ | Representative uncertainty-graph type(s) $\mathcal{G}_M = (V, E, \mu_M)$ with $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0, 1]^k$ |
|---|---|
| 1 | Fuzzy graph [23, 73]; $N$-graph; shadowed-graph variants |
| 2 | Intuitionistic fuzzy graph [158]; vague graph [159]; bipolar fuzzy graph [160]; intuitionistic evidence graph; variable fuzzy graph; paraconsistent fuzzy graph; bifuzzy graph [161, 162] |
| 3 | Neutrosophic graph [163–165][(a)]; hesitant fuzzy graph [166]; tripolar fuzzy graph [167,168]; three-way fuzzy graph; picture fuzzy graph [169, 170]; spherical fuzzy graph [106, 171]; inconsistent intuitionistic fuzzy graph; ternary fuzzy / neutrosophic-fuzzy graph; neutrosophic vague graph |
| 4 | Quadripartitioned neutrosophic graph [172, 173]; double-valued neutrosophic graph [31]; dual hesitant fuzzy graph [174–176]; ambiguous graph [(b)]; local-neutrosophic graph; support-neutrosophic graph; turiyam neutrosophic graph [177][(c)] |
| 5 | Pentapartitioned neutrosophic graph [178]; triple-valued neutrosophic graph [179] |
| 6 | Hexapartitioned neutrosophic graph; quadruple-valued neutrosophic graph |
| 7 | Heptapartitioned neutrosophic graph [180]; quintuple-valued neutrosophic graph |
| 8 | Octapartitioned neutrosophic graph |
| 9 | Nonapartitioned neutrosophic graph |
| $n$ | $n$-refined fuzzy graph; multi-valued (fuzzy) graphs; multi-fuzzy graphs [181] |
| $2n$ | $n$-refined intuitionistic fuzzy graph; multi-intuitionistic fuzzy graphs |
| $3n$ | $n$-refined neutrosophic graph; multi-neutrosophic graphs |

(a) Neutrosophic graph models are often regarded as broad frameworks that can specialize to many degree-based graph formalisms under suitable additional constraints.
(b) Ambiguous graph models are commonly described as subclasses of certain quadripartitioned, and in some cases also double-valued, neutrosophic graph models.
(c) Turiyam neutrosophic graphs are reported as subclasses of certain quadripartitioned neutrosophic graph models.

## 2.4 Graph, HyperGraph, and SuperHyperGraph

We begin with the set-theoretic operations underlying SuperHyperGraphs and then recall the relevant graph-theoretic definitions. A hypergraph is a natural generalization of an ordinary graph in which an edge, usually called a *hyperedge*, may join any number of vertices rather than only two. We begin with the standard definition [182–186].

**Definition 2.4.1** (Hypergraph)**.** [184] A *hypergraph* is a pair

$$H = (V(H), E(H)),$$

where $V(H)$ is a nonempty set of vertices and $E(H)$ is a family of subsets of $V(H)$, called the *hyperedges* of $H$. In this book, all hypergraphs are assumed to be finite.

**Example 2.4.2** (A Collaboration Hypergraph)**.** Consider five researchers represented by

$$V(H) = \{v_1, v_2, v_3, v_4, v_5\}.$$

Suppose that three research teams are formed:

$$e_1 = \{v_1, v_2, v_3\}, \qquad e_2 = \{v_3, v_4\}, \qquad e_3 = \{v_2, v_4, v_5\}.$$

Then

$$H = (V(H), E(H)), \qquad E(H) = \{e_1, e_2, e_3\},$$

is a hypergraph. In particular, $e_1$ and $e_3$ represent multiway relationships involving three researchers.

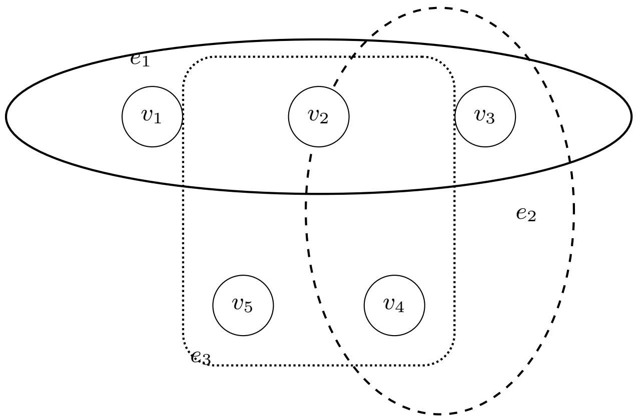



Figure 2.3: A hypergraph representing three overlapping research teams.

Hypergraphs are useful for modeling higher-order relationships in many areas, including computer science and biology [187, 188]. They are also actively studied in database theory [189, 190] and in hypergraph neural networks [191, 192].

A SuperHyperGraph is a higher-order structure whose supervertices belong to an iterated powerset of a base set, and whose superhyperedges are nonempty subsets thereof [18, 193, 194]. Moreover, related concepts such as Recursive SuperHyperGraphs [195–199], Meta-SuperHyperGraphs [182, 200], and Hierarchical SuperHyperGraphs [196, 200, 201] are also known. Their definitions and related notions are given below.

**Definition 2.4.3** (Base set)**.** A *base set* $S$ is the underlying universe of admissible objects in the setting under consideration, i.e.,

$$S = \{\, x \mid x \text{ is an admissible object in the given context} \,\}.$$

Accordingly, every element of $\mathcal{P}(S)$—and, more generally, of any iterated powerset—is ultimately formed from elements of $S$.

**Definition 2.4.4** (Powerset)**.** (see [202]) For a set $S$, its *powerset* is the family of all subsets of $S$:

$$\mathcal{P}(S) = \{\, A \mid A \subseteq S \,\}.$$

In particular, $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$.

**Definition 2.4.5** (Hypergraph)**.** [184, 203] A *hypergraph* is an ordered pair $H = (V, E)$ where

- $V$ is a finite set of *vertices*, and
- $E$ is a finite family of nonempty subsets of $V$, called *hyperedges*.

Thus a hyperedge may contain more than two vertices, allowing one to model genuinely multiway relations.

**Definition 2.4.6** ($n$-fold iterated powerset)**.** [204,205] Let $X$ be a set. Define $\mathcal{P}^1(X) := \mathcal{P}(X)$ and, for $n \geq 1$, set recursively

$$\mathcal{P}^{n+1}(X) := \mathcal{P}\big(\mathcal{P}^n(X)\big).$$

When excluding the empty set, we write

$$\mathcal{P}^n_*(X) := \mathcal{P}^n(X) \setminus \{\emptyset\}.$$

**Definition 2.4.7** ($n$-SuperHyperGraph)**.** (see [18,206]) Let $V_0$ be a finite, nonempty base set, and define the iterated powersets by

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \in \mathbb{N} \cup \{0\}).$$

For $n \geq 0$, an *$n$-SuperHyperGraph on $V_0$* is a pair

$$\mathrm{SHG}^{(n)} = (V, E)$$

satisfying

$$V \subseteq \mathcal{P}^n(V_0) \qquad \text{and} \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Elements of $V$ are called *$n$-supervertices*, and elements of $E$ are called *superhyperedges*; equivalently, each superhyperedge is a nonempty subset of the supervertex set $V$.

**Example 2.4.8** (A 1-SuperHyperGraph: Interdisciplinary Research Teams)**.** Let

$$V_0 = \{a, b, c, d, e, f\}$$

be a set of six researchers. Define three research teams by

$$S_1 := \{a, b\}, \qquad S_2 := \{c, d\}, \qquad S_3 := \{e, f\}.$$

Since $S_1, S_2, S_3 \in \mathcal{P}(V_0)$, the set

$$V := \{S_1, S_2, S_3\}$$

satisfies

$$V \subseteq \mathcal{P}(V_0) = \mathcal{P}^1(V_0).$$

Define the superhyperedges

$$e_1 := \{S_1, S_2\}, \qquad e_2 := \{S_2, S_3\}, \qquad e_3 := \{S_1, S_2, S_3\},$$

and put

$$E := \{e_1, e_2, e_3\}.$$

Then

$$\mathrm{SHG}^{(1)} = (V, E)$$

is a 1-SuperHyperGraph on $V_0$.

Here, $e_1$ represents collaboration between the first two research teams, $e_2$ represents collaboration between the second and third teams, and $e_3$ represents a joint project involving all three teams.

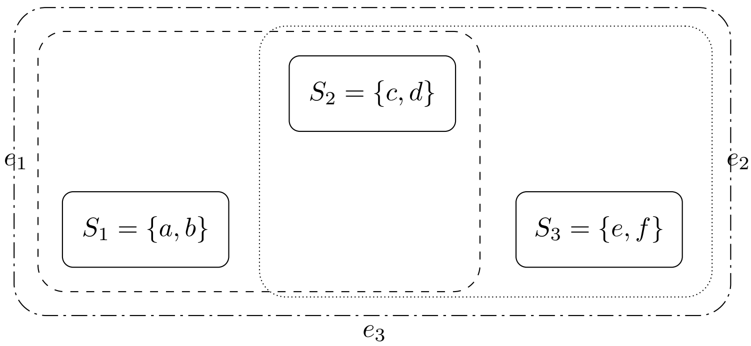



Figure 2.4: A 1-SuperHyperGraph whose supervertices are research teams.

**Example 2.4.9** (A 2-SuperHyperGraph: Coordinated Organizational Divisions)**.** Let

$$V_0 = \{a, b, c, d, e, f\}$$

be a set of employees. Consider the following first-level groups:

$$A_1 := \{a, b\}, \qquad A_2 := \{c\},$$

$$B_1 := \{d, e\}, \qquad B_2 := \{f\},$$

and

$$C_1 := \{b, e\}, \qquad C_2 := \{a, f\}.$$

Define three second-level organizational divisions by

$$T_1 := \{A_1, A_2\} = \big\{\{a, b\}, \{c\}\big\},$$

$$T_2 := \{B_1, B_2\} = \big\{\{d, e\}, \{f\}\big\},$$

and

$$T_3 := \{C_1, C_2\} = \big\{\{b, e\}, \{a, f\}\big\}.$$

Each $T_i$ is a set of subsets of $V_0$; hence,

$$T_i \in \mathcal{P}\big(\mathcal{P}(V_0)\big) = \mathcal{P}^2(V_0).$$

Therefore,

$$V := \{T_1, T_2, T_3\} \subseteq \mathcal{P}^2(V_0).$$

Define

$$f_1 := \{T_1, T_2\}, \qquad f_2 := \{T_2, T_3\}, \qquad f_3 := \{T_1, T_2, T_3\},$$

and let

$$E := \{f_1, f_2, f_3\}.$$

Then

$$\mathrm{SHG}^{(2)} = (V, E)$$

is a 2-SuperHyperGraph on $V_0$.

The superhyperedge $f_1$ represents coordination between the first and second divisions, $f_2$ represents coordination between the second and third divisions, and $f_3$ represents an organization-wide initiative involving all three divisions.

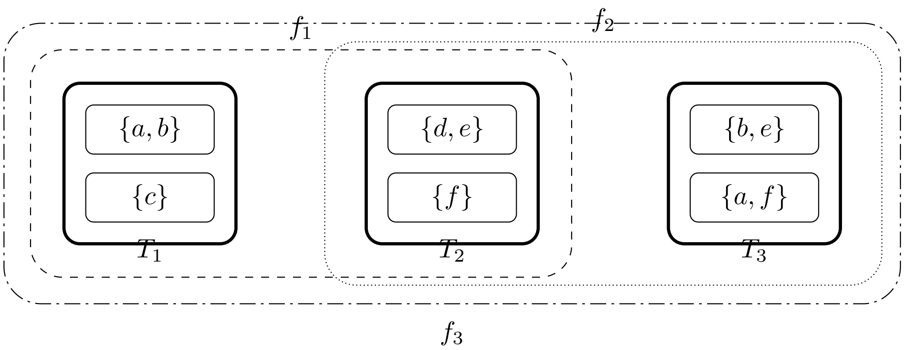



Figure 2.5: A 2-SuperHyperGraph whose supervertices are collections of employee groups.

# Chapter 3

# Soft Graph Series

This chapter examines Soft Graphs and their various extensions.

$$\text{Soft Graph extensions} = \underbrace{\text{uncertainty extensions}}_{\substack{\text{fuzzy, intuitionistic fuzzy,}\\ \text{neutrosophic, bipolar, rough, picture fuzzy}}} \cup \underbrace{\text{structural extensions}}_{\substack{\text{digraph, hypergraph,}\\ \text{directed hypergraph, superhypergraph}}} \cup \underbrace{\text{parameter extensions}}_{\substack{\text{hypersoft, multi-attribute,}\\ \text{expert and multilevel models}}}.$$

## 3.1 Soft Graph

A soft graph assigns each parameter a subgraph, enabling parameter-dependent representation of uncertain vertices, edges, and relationships within complex networks and decision-making systems effectively and systematically [207, 208].

**Definition 3.1.1** (Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, and define

$$\mathrm{Sub}(G^*) = \left\{ (W, F) : W \subseteq V,\ F \subseteq E \cap \binom{W}{2} \right\}.$$

Let $A \neq \emptyset$ be a parameter set. A mapping

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

is called a *soft graph over* $G^*$.

Equivalently, if

$$\Phi(a) = \big(V_a, E_a\big) \qquad (a \in A),$$

then

$$V_a \subseteq V, \qquad E_a \subseteq E \cap \binom{V_a}{2},$$

so that $\Phi(a)$ is a subgraph of $G^*$ for every $a \in A$.

**Example 3.1.2** (Soft Graph for Research-Team Selection)**.** Let

$$G^* = (V, E), \qquad V = \{v_1, v_2, v_3, v_4, v_5\},$$

where the vertices represent researchers, and let

$$E = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\big\}.$$

Consider the parameter set

$$A = \{a_{\mathrm{th}}, a_{\mathrm{comp}}\},$$

where $a_{\mathrm{th}}$ denotes theoretical expertise and $a_{\mathrm{comp}}$ denotes computational expertise. Define

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

by

$$\Phi(a_{\mathrm{th}}) = \big(\{v_1, v_2, v_3\}, \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\big\}\big)$$

and

$$\Phi(a_{\mathrm{comp}}) = \big(\{v_2, v_3, v_4, v_5\}, \big\{\{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\big\}\big)\,.$$

Each image of $\Phi$ is a subgraph of $G^*$; therefore, $\Phi$ defines a soft graph over $G^*$.

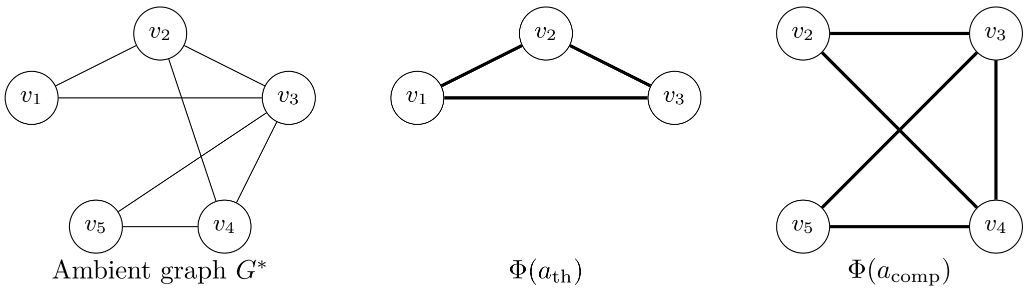



Figure 3.1: An ambient research-collaboration graph and its two parameter-dependent soft-graph slices.

## 3.2 HyperSoft Graph

A HyperSoft Graph assigns each tuple of attribute values a subgraph, enabling parameterized representation of uncertain relationships governed by multiple disjoint attribute sets simultaneously and systematically [209–212]. Related concepts, such as Hypersoft Hypergraphs, are also known [213].

**Definition 3.2.1** (HyperSoft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, and let

$$A_1, A_2, \ldots, A_m$$

be nonempty attribute-value domains. Define the hypersoft parameter space by

$$C_{\mathrm{H}} = A_1 \times A_2 \times \cdots \times A_m.$$

A mapping

$$\Phi_{\mathrm{H}} : C_{\mathrm{H}} \longrightarrow \mathrm{Sub}(G^*)$$

is called a *HyperSoft graph over $G^*$*.

Thus, for each attribute-value tuple

$$\gamma = (a_1, a_2, \ldots, a_m) \in C_{\mathrm{H}},$$

the value

$$\Phi_{\mathrm{H}}(\gamma) = \big(V_\gamma, E_\gamma\big)$$

is a subgraph of $G^*$, where

$$V_\gamma \subseteq V, \qquad E_\gamma \subseteq E \cap \binom{V_\gamma}{2}.$$

**Example 3.2.2** (HyperSoft Graph for Supplier Selection)**.** Consider a company that evaluates five candidate suppliers,

$$V = \{s_1, s_2, s_3, s_4, s_5\}.$$

Two suppliers are joined whenever they can participate in the same logistical collaboration. Let the underlying graph be

$$G^* = (V, E),$$

where

$$E = \big\{\{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_2, s_4\}, \{s_3, s_4\}, \{s_3, s_5\}, \{s_4, s_5\}\big\}.$$

Suppose that suppliers are assessed according to two attributes: cost level and reliability level. Define

$$A_1 = \{\text{low}, \text{medium}\}, \qquad A_2 = \{\text{high}, \text{moderate}\}.$$

Hence, the hypersoft parameter space is

$$C_{\mathrm{H}} = A_1 \times A_2 = \left\{ \begin{array}{l} (\text{low}, \text{high}), (\text{low}, \text{moderate}), \\ (\text{medium}, \text{high}), (\text{medium}, \text{moderate}) \end{array} \right\}.$$

Define

$$\Phi_{\mathrm{H}} : C_{\mathrm{H}} \longrightarrow \mathrm{Sub}(G^*)$$

by

$$\Phi_{\mathrm{H}}(\text{low}, \text{high}) = \big(\{s_1, s_2, s_3\}, \{\{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}\}\big),$$

$$\Phi_{\mathrm{H}}(\text{low}, \text{moderate}) = \big(\{s_1, s_3, s_5\}, \{\{s_1, s_3\}, \{s_3, s_5\}\}\big),$$

$$\Phi_{\mathrm{H}}(\text{medium}, \text{high}) = \big(\{s_2, s_3, s_4\}, \{\{s_2, s_3\}, \{s_2, s_4\}, \{s_3, s_4\}\}\big),$$

and

$$\Phi_{\mathrm{H}}(\text{medium}, \text{moderate}) = \big(\{s_3, s_4, s_5\}, \{\{s_3, s_4\}, \{s_3, s_5\}, \{s_4, s_5\}\}\big).$$

Each value of $\Phi_{\mathrm{H}}$ is a subgraph of $G^*$. Therefore, $\Phi_{\mathrm{H}}$ defines a HyperSoft Graph over $G^*$. For example,

$$\Phi_{\mathrm{H}}(\text{low}, \text{high})$$

represents the compatible supplier network consisting of suppliers having low cost and high reliability.

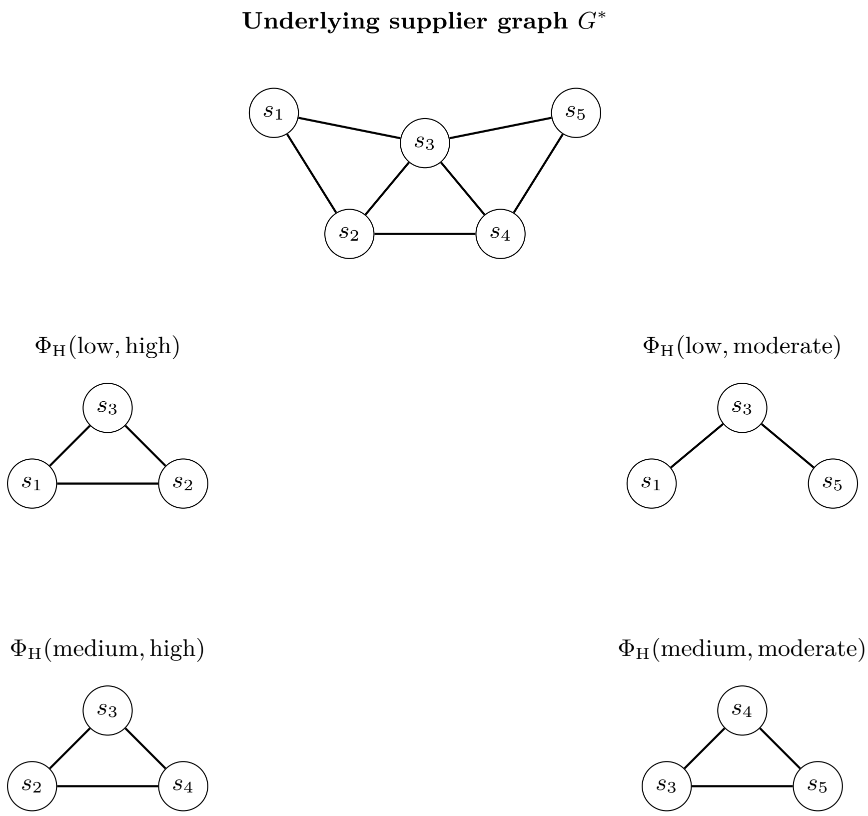



Figure 3.2: A HyperSoft Graph for supplier selection under cost and reliability attributes.

## 3.3 SuperHyperSoft Graph

A SuperHyperSoft Graph assigns each tuple of attribute-value subsets a subgraph, enabling flexible parameterized representation of uncertain relational structures under grouped multiattribute information and constraints [214].

**Definition 3.3.1** (SuperHyperSoft Graph)**.** [214] Let

$$G^* = (V, E)$$

be a finite simple graph, and let

$$A_1, A_2, \ldots, A_m$$

be nonempty pairwise-disjoint attribute-value domains. Define the SuperHyperSoft parameter space by

$$C_{\mathrm{SH}} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_m).$$

A mapping

$$\Phi_{\mathrm{SH}} : C_{\mathrm{SH}} \longrightarrow \mathrm{Sub}(G^*)$$

is called a *SuperHyperSoft graph over* $G^*$.

Thus, for every tuple

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in C_{\mathrm{SH}}, \qquad \alpha_i \subseteq A_i,$$

the value

$$\Phi_{\mathrm{SH}}(\boldsymbol{\alpha}) = \big(V_{\boldsymbol{\alpha}}, E_{\boldsymbol{\alpha}}\big)$$

is a subgraph of $G^*$, where

$$V_{\boldsymbol{\alpha}} \subseteq V, \qquad E_{\boldsymbol{\alpha}} \subseteq E \cap \binom{V_{\boldsymbol{\alpha}}}{2}.$$

## 3.4 Soft HyperGraph

A soft HyperGraph assigns each parameter a subhypergraph, enabling parameterized representation of uncertain multiway relationships among vertices through parameter-dependent vertex and hyperedge families in systems [215].

**Definition 3.4.1** (Soft HyperGraph)**.** Let

$$\mathcal{H} = (V, E)$$

be a hypergraph, where

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

and let $C \neq \emptyset$ be a set of parameters.

A *soft HyperGraph* over $\mathcal{H}$ is a quadruple

$$\mathcal{H}_{\text{soft}} = (\mathcal{H}, C, A, B),$$

where

$$A : C \longrightarrow \mathcal{P}(V), \qquad B : C \longrightarrow \mathcal{P}(E),$$

such that, for every $c \in C$,

$$B(c) \subseteq \{e \in E : e \subseteq A(c)\}.$$

For each $c \in C$, the hypergraph

$$\mathcal{H}[c] = \big(A(c), B(c)\big)$$

is called the *parameter slice* or *soft subhypergraph* at $c$.

**Example 3.4.2** (Soft HyperGraph for Hospital Emergency Coordination)**.** Consider the set of hospital departments

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\},$$

where

$$\begin{array}{ll} v_1 = \text{Emergency Department}, & v_2 = \text{Intensive Care Unit}, \\ v_3 = \text{Laboratory}, & v_4 = \text{Pharmacy}, \\ v_5 = \text{Information Technology}, & v_6 = \text{Security}. \end{array}$$

Let

$$E = \{e_1, e_2, e_3, e_4\},$$

with

$$e_1 = \{v_1, v_2, v_3\}, \qquad e_2 = \{v_1, v_4\},$$
$$e_3 = \{v_5, v_6\}, \qquad e_4 = \{v_2, v_4, v_5\}.$$

Thus,

$$\mathcal{H} = (V, E)$$

is a hypergraph representing coordination groups among hospital departments.

Let

$$C = \{c_{\text{med}}, c_{\text{cyb}}\},$$

where $c_{\text{med}}$ denotes a medical emergency and $c_{\text{cyb}}$ denotes a cybersecurity emergency. Define

$$A(c_{\text{med}}) = \{v_1, v_2, v_3, v_4\}, \qquad B(c_{\text{med}}) = \{e_1, e_2\},$$

and
$$A(c_{\text{cyb}}) = \{v_2, v_4, v_5, v_6\}, \qquad B(c_{\text{cyb}}) = \{e_3, e_4\}.$$
Since
$$e_1, e_2 \subseteq A(c_{\text{med}})$$
and
$$e_3, e_4 \subseteq A(c_{\text{cyb}}),$$
we have
$$B(c) \subseteq \{e \in E : e \subseteq A(c)\} \qquad \text{for every } c \in C.$$
Therefore,
$$\mathcal{H}_{\text{soft}} = (\mathcal{H}, C, A, B)$$
is a soft HyperGraph.

Its two parameter slices are
$$\mathcal{H}[c_{\text{med}}] = (\{v_1, v_2, v_3, v_4\}, \{e_1, e_2\})$$
and
$$\mathcal{H}[c_{\text{cyb}}] = (\{v_2, v_4, v_5, v_6\}, \{e_3, e_4\}).$$

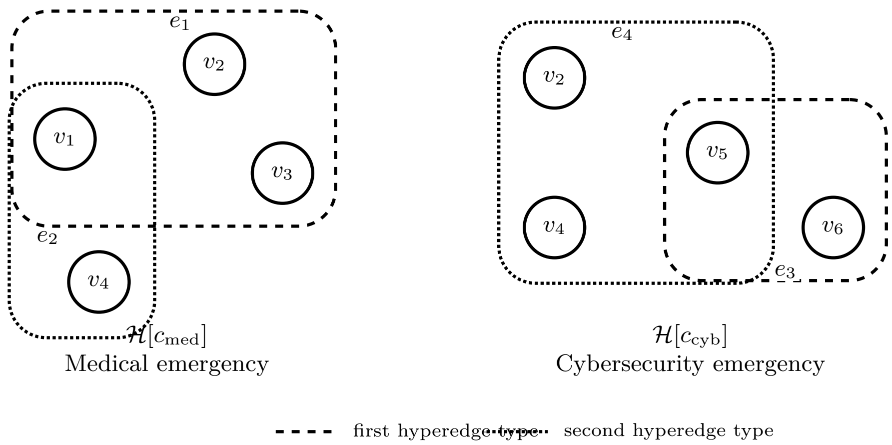



Figure 3.3: Two parameter slices of a soft HyperGraph for hospital emergency coordination. The medical-emergency slice activates $e_1$ and $e_2$, whereas the cybersecurity-emergency slice activates $e_3$ and $e_4$.

The example illustrates how the same ambient hospital hypergraph can produce different subhypergraphs according to the emergency parameter.

## 3.5 Soft SuperHyperGraph

A soft SuperHyperGraph assigns each parameter a sub-SuperHyperGraph, enabling parameterized modeling of uncertain higher-order relations among supervertices and superedges within complex hierarchical networks and systems [216].

**Definition 3.5.1** (Soft $n$-SuperHyperGraph)**.** Let $V_0$ be a nonempty finite base set, let $n \geq 1$, and let
$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$
be an $n$-SuperHyperGraph satisfying
$$\mathcal{V} \subseteq \mathcal{P}^{\langle n \rangle}(V_0), \qquad \mathcal{E} \subseteq \mathcal{P}(\mathcal{V}) \setminus \{\emptyset\}.$$

Let $C \neq \emptyset$ be a set of parameters.

A *soft n-SuperHyperGraph* over $\mathcal{G}$ is a quadruple

$$\mathcal{G}_{\mathrm{soft}} = (\mathcal{G}, C, A, B),$$

where

$$A : C \longrightarrow \mathcal{P}(\mathcal{V}), \qquad B : C \longrightarrow \mathcal{P}(\mathcal{E}),$$

such that, for every $c \in C$,

$$B(c) \subseteq \{e \in \mathcal{E} : e \subseteq A(c)\}.$$

For each $c \in C$, the $n$-SuperHyperGraph

$$\mathcal{G}[c] = \big(A(c), B(c)\big)$$

is called the *parameter slice* or *soft sub-n-SuperHyperGraph* at $c$. A soft $n$-SuperHyperGraph is also called a *soft SuperHyperGraph* when the level $n$ is understood.

**Example 3.5.2** (Soft 2-SuperHyperGraph for Project Organization)**.** Let $n = 2$ and let

$$V_0 = \{a, b, c, d, e, f\}$$

be a set of project members. Define the first-level supervertices

$$S_1 = \{a, b\}, \qquad S_2 = \{c, d\}, \qquad S_3 = \{e, f\}, \qquad S_4 = \{b, c\},$$

and the second-level supervertices

$$T_1 = \{S_1, S_2\}, \qquad T_2 = \{S_2, S_3\}, \qquad T_3 = \{S_1, S_4\}.$$

Thus,

$$\mathcal{V} = \{S_1, S_2, S_3, S_4, T_1, T_2, T_3\} \subseteq \mathcal{P}^{\langle 2 \rangle}(V_0).$$

Let

$$\mathcal{E} = \{e_1, e_2, e_3, e_4\},$$

where

$$\begin{aligned} e_1 &= \{S_1, S_2, T_1\}, \\ e_2 &= \{S_2, S_3, T_2\}, \\ e_3 &= \{S_1, S_4, T_3\}, \\ e_4 &= \{S_1, S_2, S_4\}. \end{aligned}$$

Then

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

is a 2-SuperHyperGraph.

Consider the parameter set

$$C = \{c_{\mathrm{dev}}, c_{\mathrm{ops}}\},$$

where $c_{\mathrm{dev}}$ denotes development activities and $c_{\mathrm{ops}}$ denotes operational activities. Define

$$A(c_{\mathrm{dev}}) = \{S_1, S_2, S_4, T_1, T_3\},$$

$$B(c_{\mathrm{dev}}) = \{e_1, e_3, e_4\},$$

and

$$A(c_{\mathrm{ops}}) = \{S_2, S_3, T_2\}, \qquad B(c_{\mathrm{ops}}) = \{e_2\}.$$

Since

$$e \subseteq A(c) \qquad \text{for every } e \in B(c)$$

and every $c \in C$, the structure

$$\mathcal{G}_{\text{soft}} = (\mathcal{G}, C, A, B)$$

is a soft 2-SuperHyperGraph. Its parameter slices are

$$\mathcal{G}[c_{\text{dev}}] = \big(\{S_1, S_2, S_4, T_1, T_3\}, \{e_1, e_3, e_4\}\big)$$

and

$$\mathcal{G}[c_{\text{ops}}] = \big(\{S_2, S_3, T_2\}, \{e_2\}\big).$$

## 3.6 Recursive Soft SuperHyperGraph

A Recursive SuperHyperGraph organizes nested supervertices and superedges across multiple levels, preserving recursive containment and representing hierarchical higher-order relational structures [195, 217]. A Recursive Soft SuperHyperGraph assigns each parameter a recursively closed sub-SuperHyperGraph, preserving nested supervertices, superedges, and levelwise hierarchical structure consistently.

**Definition 3.6.1** (Recursive Soft SuperHyperGraph)**.** Let $r \geq 0$, let $V_0$ be a finite nonempty set, and recursively define nonempty level sets

$$V_k \subseteq \mathcal{P}(V_{k-1}) \setminus \{\emptyset\}, \qquad k = 1, \dots, r.$$

Put

$$V^* := \bigcup_{k=0}^{r} V_k,$$

and let

$$\mathcal{G}^* = (\{V_k\}_{k=0}^{r}, E), \qquad E \subseteq \mathcal{P}(V^*) \setminus \{\emptyset\},$$

be an ambient recursive SuperHyperGraph.

Let $C \neq \emptyset$ be a parameter set. A *recursive soft SuperHyperGraph* over $\mathcal{G}^*$ is a triple

$$\mathcal{G}_{\text{RSSH}} = (\mathcal{G}^*, C, \Phi),$$

where, for every $c \in C$,

$$\Phi(c) = (\{V_k(c)\}_{k=0}^{r}, E(c))$$

satisfies

$$V_0(c) \subseteq V_0,$$

$$V_k(c) \subseteq V_k \cap \big(\mathcal{P}(V_{k-1}(c)) \setminus \{\emptyset\}\big), \qquad k = 1, \dots, r,$$

and

$$E(c) \subseteq \left\{ e \in E : e \subseteq \bigcup_{k=0}^{r} V_k(c) \right\}.$$

Thus, each parameter determines a recursively closed sub-SuperHyperGraph of $\mathcal{G}^*$.

**Example 3.6.2** (Recursive Soft SuperHyperGraph for a Software Project)**.** Let $r = 2$ and consider the base-level task set

$$V_0 = \{x_1, x_2, x_3, x_4, x_5, x_6\},$$

where

$x_1$ : requirements analysis, $x_2$ : system design,
$x_3$ : implementation, $x_4$ : software testing,
$x_5$ : deployment, $x_6$ : system monitoring.

Define the first-level supervertices

$$S_1 = \{x_1, x_2\}, \qquad S_2 = \{x_3, x_4\}, \qquad S_3 = \{x_5, x_6\},$$

and put

$$V_1 = \{S_1, S_2, S_3\}.$$

Define the second-level supervertices

$$T_1 = \{S_1, S_2\}, \qquad T_2 = \{S_2, S_3\},$$

so that

$$V_2 = \{T_1, T_2\}.$$

Let

$$E = \{e_1, e_2, e_3, e_4, e_5\},$$

where

$$\begin{aligned} &e_1 = \{x_1, x_2, S_1\}, \quad e_2 = \{x_3, x_4, S_2\}, \\ &e_3 = \{x_5, x_6, S_3\}, \quad e_4 = \{S_1, S_2, T_1\}, \\ &e_5 = \{S_2, S_3, T_2\}. \end{aligned}$$

Then

$$\mathcal{G}^* = (\{V_0, V_1, V_2\}, E)$$

is an ambient recursive SuperHyperGraph.

Consider the parameter set

$$C = \{c_{\text{dev}}, c_{\text{ops}}\},$$

where $c_{\text{dev}}$ denotes the development phase and $c_{\text{ops}}$ denotes the deployment-and-operation phase.

Define

$$\Phi(c_{\text{dev}}) = (\{V_0(c_{\text{dev}}), V_1(c_{\text{dev}}), V_2(c_{\text{dev}})\}, E(c_{\text{dev}})),$$

where

$$V_0(c_{\text{dev}}) = \{x_1, x_2, x_3, x_4\},$$
$$V_1(c_{\text{dev}}) = \{S_1, S_2\}, \qquad V_2(c_{\text{dev}}) = \{T_1\},$$

and

$$E(c_{\text{dev}}) = \{e_1, e_2, e_4\}.$$

Similarly, define

$$\Phi(c_{\text{ops}}) = (\{V_0(c_{\text{ops}}), V_1(c_{\text{ops}}), V_2(c_{\text{ops}})\}, E(c_{\text{ops}})),$$

where

$$V_0(c_{\text{ops}}) = \{x_3, x_4, x_5, x_6\},$$
$$V_1(c_{\text{ops}}) = \{S_2, S_3\}, \qquad V_2(c_{\text{ops}}) = \{T_2\},$$

and

$$E(c_{\text{ops}}) = \{e_2, e_3, e_5\}.$$

Indeed,

$$T_1 \subseteq V_1(c_{\text{dev}}) \qquad \text{and} \qquad T_2 \subseteq V_1(c_{\text{ops}}),$$

and every selected first-level supervertex consists entirely of selected base vertices. Moreover, every selected superhyperedge is contained in the union of the corresponding selected levels. Therefore,

$$\mathcal{G}_{\text{RSSH}} = (\mathcal{G}^*, C, \Phi)$$

is a Recursive Soft SuperHyperGraph.

## 3.7 TreeSoft Graph

A TreeSoft Set maps subsets of a hierarchical attribute tree to subsets of a universe, representing structured multilevel parameter-dependent uncertainty [48, 218–222]. A TreeSoft Graph maps hierarchical attribute-tree subsets to parameter-dependent vertex and edge sets, representing structured multilevel uncertainty within graph networks [46].

**Definition 3.7.1** (TreeSoft Graph)**.** Let $H$ be a finite nonempty vertex set, let

$$\mathcal{T} = \mathrm{Tree}(A)$$

be a finite rooted attribute tree, and define

$$E_H := \binom{H}{2}.$$

A *TreeSoft Graph* on $(H, \mathcal{T})$ is a pair of mappings

$$F_V : \mathcal{P}(\mathcal{T}) \longrightarrow \mathcal{P}(H), \qquad F_E : \mathcal{P}(\mathcal{T}) \longrightarrow \mathcal{P}(E_H),$$

such that, for every $X \subseteq \mathcal{T}$,

$$F_E(X) \subseteq \binom{F_V(X)}{2}.$$

The graph

$$G[X] := \big(F_V(X), F_E(X)\big)$$

is called the *TreeSoft graph slice* associated with $X$.

**Example 3.7.2** (TreeSoft Graph for Research-Team Selection)**.** Let

$$H = \{p_1, p_2, p_3, p_4, p_5\}$$

be a set of researchers. Consider the rooted attribute tree

$$\mathcal{T} = \{A, S, M, C, D, I, L\},$$

where $A$ is the root and

$$A \longrightarrow \{S, D\}, \qquad S \longrightarrow \{M, C\}, \qquad D \longrightarrow \{I, L\}.$$

The attributes are interpreted as

$$\begin{array}{rlrl} S: & \text{suitable expertise,} & M: & \text{mathematical expertise,} \\ C: & \text{computational expertise,} & D: & \text{availability,} \\ I: & \text{immediately available,} & L: & \text{available later.} \end{array}$$

Assign to each tree node $t \in \mathcal{T}$ a subset $Q_t \subseteq H$:

$$\begin{array}{ll} Q_A = H, & \\ Q_S = \{p_1, p_2, p_3, p_4\}, & Q_M = \{p_1, p_2, p_4\}, \\ Q_C = \{p_2, p_3, p_4\}, & Q_D = \{p_1, p_3, p_4, p_5\}, \\ Q_I = \{p_1, p_3, p_4\}, & Q_L = \{p_5\}. \end{array}$$

Let the admissible collaboration edges be

$$E^* = \big\{\{p_1, p_2\}, \{p_1, p_3\}, \{p_2, p_3\}, \{p_2, p_4\}, \{p_3, p_4\}, \{p_4, p_5\}\big\}.$$

For every $X \subseteq \mathcal{T}$, define

$$F_V(X) = \begin{cases} H, & X = \emptyset, \\ \bigcap_{t \in X} Q_t, & X \neq \emptyset, \end{cases}$$

and

$$F_E(X) = E^* \cap \binom{F_V(X)}{2}.$$

Then

$$F_E(X) \subseteq \binom{F_V(X)}{2}$$

for every $X \subseteq \mathcal{T}$, so $(F_V, F_E)$ is a TreeSoft Graph.

For example, take

$$X = \{S, I\}.$$

Then

$$F_V(X) = Q_S \cap Q_I = \{p_1, p_3, p_4\},$$

and

$$F_E(X) = \big\{\{p_1, p_3\}, \{p_3, p_4\}\big\}.$$

Hence the corresponding TreeSoft graph slice is

$$G[X] = \big(\{p_1, p_3, p_4\}, \big\{\{p_1, p_3\}, \{p_3, p_4\}\big\}\big).$$

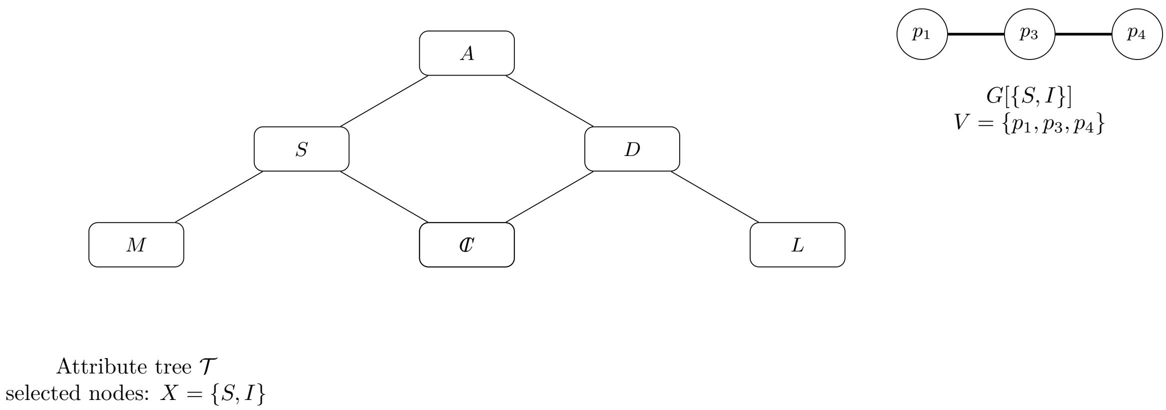



Figure 3.4: An attribute tree and the TreeSoft graph slice determined by the attributes "suitable expertise" and "immediately available."

## 3.8 ForestSoft Graph

A ForestSoft Set integrates multiple disjoint TreeSoft structures, representing parameter-dependent uncertainty across several hierarchical attribute trees within one unified framework [223–226]. A ForestSoft Graph unifies TreeSoft Graphs over multiple disjoint attribute trees, jointly representing parameter-dependent vertices and edges under multi-hierarchical uncertainty [46].

**Definition 3.8.1** (ForestSoft Graph)**.** Let

$$\{\mathcal{T}_t\}_{t \in I}$$

be a finite family of pairwise disjoint attribute trees, and let

$$\mathcal{F} := \bigsqcup_{t \in I} \mathcal{T}_t$$

be their forest. Suppose that, for each $t \in I$,

$$G_t = \big(F_V^{(t)}, F_E^{(t)}\big)$$

is a TreeSoft Graph on $(H, \mathcal{T}_t)$.

The corresponding *ForestSoft Graph* is the pair

$$G_{\mathcal{F}} = (F_V, F_E),$$

where, for every $X \subseteq \mathcal{F}$,

$$F_V(X) := \bigcup_{t \in I} F_V^{(t)}(X \cap \mathcal{T}_t)\,,$$

and

$$F_E(X) := \bigcup_{t \in I} F_E^{(t)}(X \cap \mathcal{T}_t)\,.$$

Then

$$F_E(X) \subseteq \binom{F_V(X)}{2},$$

and

$$G_{\mathcal{F}}[X] := \big(F_V(X), F_E(X)\big)$$

is called the *ForestSoft graph slice* associated with $X$.

**Example 3.8.2** (ForestSoft Graph for Research-Team Formation)**.** Let

$$H = \{p_1, p_2, p_3, p_4, p_5\}$$

be a set of researchers. Consider two pairwise disjoint attribute trees:

$$\mathcal{T}_E = \{E, M, C\}, \qquad \mathcal{T}_A = \{A, I, L\},$$

where

$$E \longrightarrow \{M, C\}, \qquad A \longrightarrow \{I, L\}.$$

Here, $M$ and $C$ represent mathematical and computational expertise, while $I$ and $L$ represent immediate and later availability. The corresponding forest is

$$\mathcal{F} = \mathcal{T}_E \sqcup \mathcal{T}_A.$$

Assign the following vertex subsets:

$$\begin{aligned} Q_E &= H, & Q_M &= \{p_1, p_2, p_4\}, & Q_C &= \{p_2, p_3, p_5\}, \\ Q_A &= H, & Q_I &= \{p_2, p_3, p_4\}, & Q_L &= \{p_1, p_5\}. \end{aligned}$$

For $t \in \{E, A\}$ and $Y \subseteq \mathcal{T}_t$, define

$$F_V^{(t)}(Y) = \begin{cases} \emptyset, & Y = \emptyset, \\ \bigcap_{a \in Y} Q_a, & Y \neq \emptyset. \end{cases}$$

Let the admissible edge sets for the two trees be

$$K_E = \big\{\{p_1, p_2\}, \{p_2, p_4\}, \{p_2, p_3\}, \{p_3, p_5\}\big\},$$

and

$$K_A = \big\{\{p_2, p_3\}, \{p_3, p_4\}, \{p_1, p_5\}\big\}.$$

Define

$$F_E^{(t)}(Y) = K_t \cap \binom{F_V^{(t)}(Y)}{2}.$$

Thus,

$$G_E = (F_V^{(E)}, F_E^{(E)}) \quad \text{and} \quad G_A = (F_V^{(A)}, F_E^{(A)})$$

are TreeSoft Graphs.

Consider the forest-node set

$$X = \{M, I\} \subseteq \mathcal{F}.$$

Since

$$X \cap \mathcal{T}_E = \{M\}, \qquad X \cap \mathcal{T}_A = \{I\},$$

we obtain

$$F_V^{(E)}(\{M\}) = \{p_1, p_2, p_4\}, \qquad F_E^{(E)}(\{M\}) = \big\{\{p_1, p_2\}, \{p_2, p_4\}\big\},$$

and

$$F_V^{(A)}(\{I\}) = \{p_2, p_3, p_4\}, \qquad F_E^{(A)}(\{I\}) = \big\{\{p_2, p_3\}, \{p_3, p_4\}\big\}.$$

Therefore,

$$F_V(X) = \{p_1, p_2, p_3, p_4\},$$

and

$$F_E(X) = \big\{\{p_1, p_2\}, \{p_2, p_3\}, \{p_2, p_4\}, \{p_3, p_4\}\big\}.$$

Hence the corresponding ForestSoft graph slice is

$$G_{\mathcal{F}}[X] = \big(\{p_1, p_2, p_3, p_4\}, \big\{\{p_1, p_2\}, \{p_2, p_3\}, \{p_2, p_4\}, \{p_3, p_4\}\big\}\big).$$

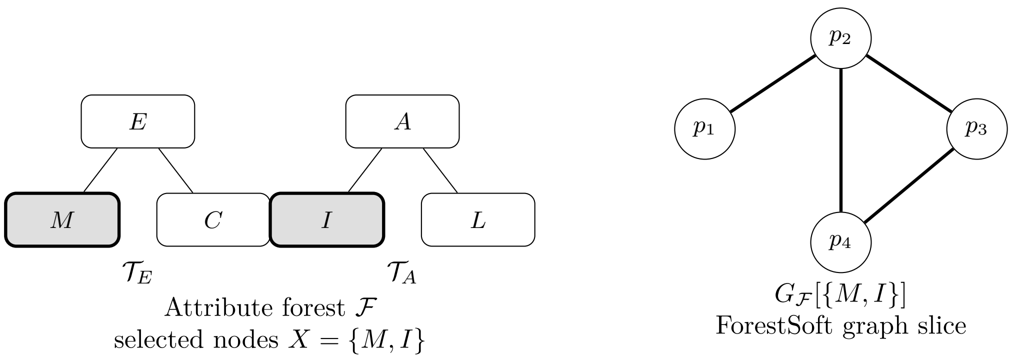



Figure 3.5: A ForestSoft Graph obtained by combining the expertise and availability TreeSoft graph slices.

## 3.9 Indetermsoft Graph

An IndetermSoft Set permits indeterminacy in parameters, universes, mappings, or approximate subsets, representing unclear, uncertain, conflicting, or non-unique information systematically [227–230]. An IndetermSoft Graph maps parameters to vertex and edge subsets while allowing indeterminacy in attributes, mappings, vertices, edges, or relationships [214].

**Definition 3.9.1** (IndetermSoft Graph). [214] Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$ be a parameter set, and let

$$\mathcal{P}^+\big(\mathrm{Sub}(G^*)\big) := \mathcal{P}\big(\mathrm{Sub}(G^*)\big) \setminus \{\emptyset\}.$$

An *IndetermSoft Graph* over $G^*$ is a mapping

$$\Phi_{\mathrm{I}} : A \longrightarrow \mathcal{P}^{+}\big(\mathrm{Sub}(G^*)\big).$$

Thus, for each parameter $a \in A$, the set

$$\Phi_{\mathrm{I}}(a)$$

is a nonempty family of admissible subgraphs of $G^*$, representing the possible vertex and edge configurations associated with $a$.

The structure is genuinely indeterminate if

$$\big|\Phi_{\mathrm{I}}(a)\big| > 1$$

for at least one $a \in A$. If every $\Phi_{\mathrm{I}}(a)$ is a singleton, the structure reduces to an ordinary Soft Graph.

**Example 3.9.2** (IndetermSoft Graph for Research-Team Selection)**.** Consider the finite simple graph

$$G^* = (V, E),$$

where

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

represents five researchers and

$$E = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\big\}.$$

An edge indicates that the corresponding researchers have an established collaboration relationship.

Let

$$A = \{a_{\mathrm{exp}}, a_{\mathrm{av}}\},$$

where

$$a_{\mathrm{exp}} = \text{“suitable expertise”}$$

and

$$a_{\mathrm{av}} = \text{“sufficient availability”}.$$

Suppose that the information concerning expertise is indeterminate. In particular, the available assessments lead to two admissible candidate subgraphs:

$$H_{\mathrm{exp}}^{(1)} = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\})$$

and

$$H_{\mathrm{exp}}^{(2)} = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}).$$

The first assessment selects $v_1, v_2, v_3$, whereas the second assessment additionally regards $v_4$ as having suitable expertise.

For the availability parameter, suppose that the information is unambiguous and yields the single subgraph

$$H_{\mathrm{av}} = (\{v_2, v_3, v_4\}, \{\{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}).$$

Define

$$\Phi_{\mathrm{I}} : A \longrightarrow \mathcal{P}^{+}\big(\mathrm{Sub}(G^*)\big)$$

by

$$\Phi_{\mathrm{I}}(a_{\mathrm{exp}}) = \Big\{H_{\mathrm{exp}}^{(1)}, H_{\mathrm{exp}}^{(2)}\Big\}$$

and

$$\Phi_{\mathrm{I}}(a_{\mathrm{av}}) = \{H_{\mathrm{av}}\}.$$

Every member of these families is a subgraph of $G^*$. Moreover,

$$|\Phi_{\mathrm{I}}(a_{\exp})| = 2 > 1,$$

while

$$|\Phi_{\mathrm{I}}(a_{\mathrm{av}})| = 1.$$

Hence,

$$(G^*, A, \Phi_{\mathrm{I}})$$

defines a genuinely indeterminate IndetermSoft Graph.

## 3.10 Multisoft Graph

A MultiSoft Set assigns subsets of a universe to multilevel parameter combinations, representing information governed by several parameter groups simultaneously (cf. [231, 232]). A MultiSoft Graph assigns induced subgraphs to multilevel parameter subsets, representing parameter-dependent network structures governed by several disjoint parameter blocks [233].

**Definition 3.10.1** (MultiSoft Graph)**.** [233] Let

$$G = (V, E)$$

be a finite simple graph, and let

$$P_1, P_2, \ldots, P_n$$

be pairwise disjoint nonempty parameter sets. Put

$$P := \bigcup_{i=1}^{n} P_i,$$

and let

$$\emptyset \neq A \subseteq \mathcal{P}(P).$$

A mapping

$$F : A \longrightarrow \mathcal{P}(V)$$

defines a *MultiSoft Graph* over $G$. For each $a \in A$, set

$$V_a := F(a), \qquad E_a := \big\{\{u, v\} \in E : u, v \in V_a\big\}.$$

The corresponding slice is the induced subgraph

$$G[a] := (V_a, E_a) = G[V_a].$$

Thus, a MultiSoft Graph is a family

$$\{G[a] : a \in A\}$$

of induced subgraphs indexed by subsets of a multilevel parameter system $\{P_i\}_{i=1}^{n}$.

## 3.11 Fuzzy Soft Graph

A fuzzy soft set assigns fuzzy subsets to parameters, representing graded membership and parameter-dependent uncertainty within a unified mathematical framework [234–236]. A fuzzy soft graph combines fuzzy membership with soft-set parameterization, assigning each parameter a fuzzy subgraph to represent uncertain vertices and edges through graded relationships [237, 238]. Related concepts, such as Fuzzy HyperSoft Graphs [209, 211, 213] and Fuzzy Soft Expert Graphs [239, 240], have also been studied in the literature.

**Definition 3.11.1** (Fuzzy Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, and let $A \neq \emptyset$ be a set of parameters. A *fuzzy soft graph* over $G^*$ is a quadruple

$$\mathcal{G} = \big(G^*, A, \mu_V, \mu_E\big),$$

where

$$\mu_V : A \times V \longrightarrow [0, 1], \qquad \mu_E : A \times E \longrightarrow [0, 1],$$

such that, for every $a \in A$ and every edge $e = \{u, v\} \in E$,

$$\mu_E(a, e) \leq \min\big\{\mu_V(a, u), \mu_V(a, v)\big\}.$$

For each $a \in A$, the pair

$$\mathcal{G}_a = \big(\mu_V^a, \mu_E^a\big),$$

where

$$\mu_V^a(v) = \mu_V(a, v), \qquad \mu_E^a(e) = \mu_E(a, e),$$

is a fuzzy subgraph of $G^*$. Thus, a fuzzy soft graph is a parameterized family

$$\mathcal{G} = \{\mathcal{G}_a : a \in A\}$$

of fuzzy subgraphs of $G^*$.

**Example 3.11.2** (Fuzzy Soft Communication Network)**.** Let

$$G^* = (V, E),$$

where

$$V = \{v_1, v_2, v_3, v_4\}$$

represents four communication stations and

$$E = \{e_{12}, e_{23}, e_{34}, e_{14}, e_{13}\}, \qquad e_{ij} = \{v_i, v_j\}.$$

Consider the parameter set

$$A = \{a_{\mathrm{r}}, a_{\mathrm{s}}\},$$

where $a_{\mathrm{r}}$ denotes reliability and $a_{\mathrm{s}}$ denotes signal strength.

For $a_{\mathrm{r}}$, define

| $v$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
|---|---|---|---|---|
| $\mu_V(a_{\mathrm{r}}, v)$ | 0.9 | 0.8 | 0.7 | 0.6 |

and

| $e$ | $e_{12}$ | $e_{23}$ | $e_{34}$ | $e_{14}$ | $e_{13}$ |
|---|---|---|---|---|---|
| $\mu_E(a_{\mathrm{r}}, e)$ | 0.75 | 0.65 | 0.55 | 0.50 | 0.60 |

.

For $a_{\mathrm{s}}$, define

| $v$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
|---|---|---|---|---|
| $\mu_V(a_{\mathrm{s}}, v)$ | 0.5 | 0.7 | 0.9 | 0.8 |

and

| $e$ | $e_{12}$ | $e_{23}$ | $e_{34}$ | $e_{14}$ | $e_{13}$ |
|---|---|---|---|---|---|
| $\mu_E(a_{\mathrm{s}}, e)$ | 0.45 | 0.65 | 0.75 | 0.40 | 0.50 |

.

For every parameter and every edge $e_{ij}$,

$$\mu_E(a, e_{ij}) \le \min\{\mu_V(a, v_i), \mu_V(a, v_j)\}.$$

For example,

$$\mu_E(a_{\mathrm{r}}, e_{12}) = 0.75 \le \min\{0.9, 0.8\} = 0.8.$$

Therefore, these assignments define a fuzzy soft graph over $G^*$.

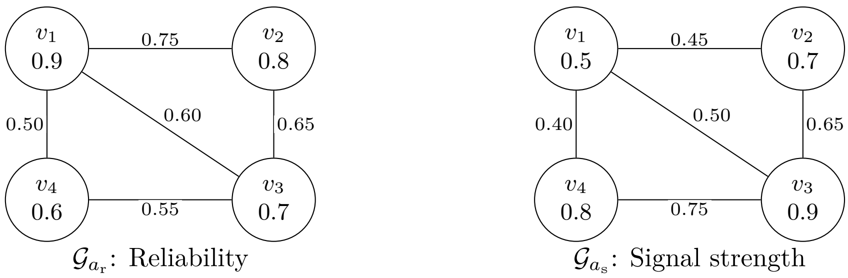



Figure 3.6: Two parameter-dependent fuzzy subgraphs of the communication network. Vertex and edge membership degrees are displayed beside the corresponding elements.

## 3.12 Intuitionistic Fuzzy Soft Graph

An intuitionistic fuzzy soft set assigns parameter-dependent membership and non-membership degrees, explicitly representing hesitation and uncertainty within soft-set models systematically [241–244]. An Intuitionistic Fuzzy Graph assigns membership and non-membership degrees to vertices and edges, representing uncertain relationships with explicit hesitation information [245–247]. An intuitionistic fuzzy soft graph assigns each parameter an intuitionistic fuzzy subgraph, representing membership, non-membership, and hesitation degrees of vertices and edges under parameterized uncertainty [248–251].

**Definition 3.12.1** (Intuitionistic Fuzzy Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, and let $A \neq \emptyset$ be a set of parameters. An *intuitionistic fuzzy soft graph* over $G^*$ is a sextuple

$$\mathcal{G} = \big(G^*, A, \mu_V, \nu_V, \mu_E, \nu_E\big),$$

where

$$\mu_V, \nu_V : A \times V \longrightarrow [0, 1]$$

and

$$\mu_E, \nu_E : A \times E \longrightarrow [0, 1].$$

For every $a \in A$ and $v \in V$, assume that

$$0 \le \mu_V(a, v) + \nu_V(a, v) \le 1,$$

and, for every $a \in A$ and $e \in E$,

$$0 \le \mu_E(a,e) + \nu_E(a,e) \le 1.$$

Moreover, for every $a \in A$ and every edge $e = \{u,v\} \in E$, suppose that

$$\mu_E(a,e) \le \min\bigl\{\mu_V(a,u), \mu_V(a,v)\bigr\},$$

and

$$\nu_E(a,e) \ge \max\bigl\{\nu_V(a,u), \nu_V(a,v)\bigr\}.$$

For each $a \in A$, define

$$\mathcal{G}_a = \Bigl((\mu_V^a, \nu_V^a), (\mu_E^a, \nu_E^a)\Bigr),$$

where

$$\mu_V^a(v) = \mu_V(a,v), \qquad \nu_V^a(v) = \nu_V(a,v),$$

and

$$\mu_E^a(e) = \mu_E(a,e), \qquad \nu_E^a(e) = \nu_E(a,e).$$

Then each $\mathcal{G}_a$ is an intuitionistic fuzzy subgraph of $G^*$, and

$$\mathcal{G} = \{\mathcal{G}_a : a \in A\}$$

is called an *intuitionistic fuzzy soft graph*.

## 3.13 Neutrosophic Soft Graph

A Neutrosophic Graph assigns truth, indeterminacy, and falsity degrees to vertices and edges, representing uncertain and inconsistent network relationships flexibly [165, 252, 253]. A Neutrosophic Soft Graph assigns parameter-dependent truth, indeterminacy, and falsity degrees to vertices and edges while preserving endpoint-consistency constraints systematically [254–256].

**Definition 3.13.1** (Neutrosophic Soft Graph)**.** Let

$$G = (V,E)$$

be a finite simple graph, and let $C \neq \emptyset$ be a parameter set. A *Neutrosophic Soft Graph* over $G$ is the structure

$$\mathcal{G}_{\mathrm{NS}} = \bigl(G, C, T_V, I_V, F_V, T_E, I_E, F_E\bigr),$$

where

$$T_V, I_V, F_V : C \times V \longrightarrow [0,1]$$

and

$$T_E, I_E, F_E : C \times E \longrightarrow [0,1]$$

assign parameter-dependent truth, indeterminacy, and falsity degrees to vertices and edges, respectively.

For every $c \in C$, $v \in V$, and $e \in E$,

$$0 \le T_V(c,v) + I_V(c,v) + F_V(c,v) \le 3,$$

$$0 \le T_E(c,e) + I_E(c,e) + F_E(c,e) \le 3.$$

Moreover, for every edge $e = \{u,v\} \in E$,

$$T_E(c,e) \le \min\{T_V(c,u), T_V(c,v)\},$$

$$I_E(c,e) \le \min\{I_V(c,u), I_V(c,v)\},$$

and

$$F_E(c,e) \ge \max\{F_V(c,u), F_V(c,v)\}.$$

## 3.14 Plithogenic Soft Graph

A Plithogenic Graph assigns multi-attribute appurtenance and contradiction degrees to vertices and edges, representing heterogeneous uncertainty within complex network structures [257–259]. A Plithogenic Soft Graph assigns parameter-dependent appurtenance and contradiction degrees to vertices and edges, modeling multi-attribute uncertainty across graph slices.

**Definition 3.14.1** (Plithogenic Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $C \neq \emptyset$ be a parameter set, and let $A_V$ and $A_E$ be attribute-value sets for vertices and edges, respectively. Fix $r \geq 1$.

A *Plithogenic Soft Graph* over $G^*$ is a structure

$$\mathcal{G}_{\mathrm{PS}} = \big(G^*, C, \mu_V, \mu_E, \delta_V, \delta_E\big),$$

where

$$\mu_V : C \times V \longrightarrow [0,1]^r, \qquad \mu_E : C \times E \longrightarrow [0,1]^r$$

are parameter-dependent degree-of-appurtenance functions, and

$$\delta_V : C \times A_V \times A_V \longrightarrow [0,1], \qquad \delta_E : C \times A_E \times A_E \longrightarrow [0,1]$$

are parameter-dependent degree-of-contradiction functions satisfying

$$\delta_X(c, a, a) = 0, \qquad \delta_X(c, a, b) = \delta_X(c, b, a),$$

for $X \in \{V, E\}$, $c \in C$, and admissible $a, b$.

Moreover, for every $c \in C$ and every edge $e = \{u, v\} \in E$,

$$\mu_E(c, e) \leq_{\mathrm{cw}} \mu_V(c, u) \wedge_{\mathrm{cw}} \mu_V(c, v),$$

where $\leq_{\mathrm{cw}}$ and $\wedge_{\mathrm{cw}}$ denote componentwise order and componentwise minimum, respectively. Thus, every parameter $c$ determines a plithogenic graph slice of the underlying graph $G^*$.

## 3.15 Uncertain Soft Graph

An Uncertain Soft Graph assigns parameter-dependent uncertain subgraphs to each parameter, modeling graded vertex and edge information under generalized uncertainty.

**Definition 3.15.1** (Graph-Compatible Uncertainty Model)**.** Let $M$ be an uncertainty model with a nonempty degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. Let

$$\mathcal{C}_M \subseteq \mathrm{Dom}(M) \times \mathrm{Dom}(M) \times \mathrm{Dom}(M)$$

be a model-dependent admissibility relation.

For degrees

$$\alpha, \beta, \gamma \in \mathrm{Dom}(M),$$

the condition

$$(\alpha, \beta, \gamma) \in \mathcal{C}_M$$

means that $\gamma$ is an admissible uncertainty degree for an edge whose endpoint degrees are $\alpha$ and $\beta$.

The pair

$$\mathfrak{M} = \big(\mathrm{Dom}(M), \mathcal{C}_M\big)$$

is called a *graph-compatible uncertainty model.*

**Remark 3.15.2.** The admissibility relation $\mathcal{C}_M$ depends on the selected uncertainty model. For example, in the fuzzy setting,

$$\mathrm{Dom}(M) = [0, 1]$$

and one may take

$$\mathcal{C}_M = \left\{(\alpha, \beta, \gamma) \in [0, 1]^3 : \gamma \leq \min\{\alpha, \beta\}\right\}.$$

Other choices of $M$ yield intuitionistic fuzzy, neutrosophic, picture fuzzy, spherical fuzzy, plithogenic, and related graph models.

**Definition 3.15.3** (Uncertain Soft Graph)**.** Let

$$G^* = (V, E), \qquad E \subseteq \binom{V}{2},$$

be a finite simple graph, let $A \neq \emptyset$ be a set of parameters, and let

$$\mathfrak{M} = \bigl(\mathrm{Dom}(M), \mathcal{C}_M\bigr)$$

be a graph-compatible uncertainty model.

An *Uncertain Soft Graph of type $M$ over $G^*$* is a mapping

$$\Phi_M : A \longrightarrow \mathrm{USub}_M(G^*),$$

where $\mathrm{USub}_M(G^*)$ denotes the family of all $M$-uncertain subgraphs of $G^*$.

More explicitly, for every parameter $a \in A$,

$$\Phi_M(a) = \mathcal{G}_{M,a} = \bigl(V_a, E_a, \mu_{V,a}, \mu_{E,a}\bigr),$$

where

$$V_a \subseteq V, \qquad E_a \subseteq E \cap \binom{V_a}{2},$$

and

$$\mu_{V,a} : V_a \longrightarrow \mathrm{Dom}(M), \qquad \mu_{E,a} : E_a \longrightarrow \mathrm{Dom}(M)$$

are uncertainty-degree functions satisfying

$$\bigl(\mu_{V,a}(u), \mu_{V,a}(v), \mu_{E,a}(\{u, v\})\bigr) \in \mathcal{C}_M$$

for every edge

$$\{u, v\} \in E_a.$$

Thus, every parameter $a \in A$ determines an $M$-uncertain subgraph

$$\mathcal{G}_{M,a} = \bigl(V_a, E_a, \mu_{V,a}, \mu_{E,a}\bigr)$$

of the ambient graph $G^*$. The complete Uncertain Soft Graph may equivalently be written as

$$\mathcal{G}_M^{\mathrm{soft}} = (G^*, A, \{\mathcal{G}_{M,a}\}_{a \in A}).$$

**Remark 3.15.4** (Catalogue of Uncertain Soft Graph Families)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$ be a parameter set, and let $M$ be an uncertainty model whose degree-domain satisfies

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \geq 1$.

An Uncertain Soft Graph of type $M$ over $G^*$ may be represented by

$$\mathcal{G}_M^{\text{soft}} = \big(G^*, A, \mu_{V,M}, \mu_{E,M}\big),$$

where

$$\mu_{V,M} : A \times V \longrightarrow \mathrm{Dom}(M)$$

and

$$\mu_{E,M} : A \times E \longrightarrow \mathrm{Dom}(M)$$

assign parameter-dependent uncertainty degrees to the vertices and edges, respectively.

For each $a \in A$, define

$$\mu_{V,M}^a(v) := \mu_{V,M}(a, v), \qquad \mu_{E,M}^a(e) := \mu_{E,M}(a, e).$$

Then

$$\mathcal{G}_{M,a} = \big(V, E, \mu_{V,M}^a, \mu_{E,M}^a\big)$$

is the uncertainty-graph slice associated with the parameter $a$. Whenever required by the chosen uncertainty model $M$, suitable model-dependent endpoint-consistency conditions are imposed on $\mu_{V,M}$ and $\mu_{E,M}$.

For reference, Table 3.1 presents a catalogue of uncertainty-soft-graph families, called Uncertain Soft Graphs, classified according to the dimension $k$ of the underlying degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

## 3.16 Soft Expert Graph

A Soft Expert Set assigns subsets of a universe to parameter–expert–opinion triples, representing expert-specific agreement and disagreement under parameterized uncertainty [270–274]. Related concepts, such as HyperSoft Expert Sets, are also known in the literature [275–277]. A Soft Expert Graph assigns each parameter–expert–opinion triple an induced subgraph, representing expert-specific agreement or disagreement about vertices, edges, and structural relationships within a network [239, 278].

**Definition 3.16.1** (Soft Expert Graph)**.** [239, 278] Let $G = (V, E)$ be a finite simple graph, $P$ a nonempty set of parameters, $X$ a nonempty set of experts, and

$$O = \{0, 1\},$$

where 1 and 0 represent agreement and disagreement, respectively. Let

$$A \subseteq P \times X \times O$$

be nonempty, and let

$$F : A \longrightarrow \mathcal{P}(V)$$

be a mapping.

For each $\alpha = (p, x, o) \in A$, define

$$V_\alpha := F(\alpha)$$

and

$$E_\alpha := \big\{\{u, v\} \in E : u, v \in V_\alpha\big\}.$$

Then

$$G_\alpha := (V_\alpha, E_\alpha) = G[V_\alpha]$$

Table 3.1: A catalogue of Uncertain Soft Graph families organized by the dimension $k$ of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

| $k$ | Representative Uncertain Soft Graph type(s) $\mathcal{G}_M^{\mathrm{soft}} = (G^*, A, \mu_{V,M}, \mu_{E,M})$, where $\mu_{V,M} : A \times V \to \mathrm{Dom}(M)$ and $\mu_{E,M} : A \times E \to \mathrm{Dom}(M)$ |
|---|---|
| 1 | Fuzzy soft graph [260, 261]; $N$-fuzzy soft graph; shadowed soft graph variants |
| 2 | Intuitionistic fuzzy soft graph [248,249,262]; vague soft graph [263]; bipolar fuzzy soft graph [264]; intuitionistic evidence soft graph; variable fuzzy soft graph; paraconsistent fuzzy soft graph; bifuzzy soft graph |
| 3 | Neutrosophic soft graph [178, 250, 254, 255][a]; hesitant fuzzy soft graph[d] [265]; tripolar fuzzy soft graph; three-way fuzzy soft graph; picture fuzzy soft graph [266–268]; spherical fuzzy soft graph [269]; inconsistent intuitionistic fuzzy soft graph; ternary fuzzy soft graph; neutrosophic-fuzzy soft graph; neutrosophic vague soft graph |
| 4 | Quadripartitioned neutrosophic soft graph [117]; double-valued neutrosophic soft graph; dual hesitant fuzzy soft graph[d]; ambiguous soft graph[b]; local-neutrosophic soft graph; support-neutrosophic soft graph; Turiyam neutrosophic soft graph[c] |
| 5 | Pentapartitioned neutrosophic soft graph; triple-valued neutrosophic soft graph |
| 6 | Hexapartitioned neutrosophic soft graph; quadruple-valued neutrosophic soft graph |
| 7 | Heptapartitioned neutrosophic soft graph; quintuple-valued neutrosophic soft graph |
| 8 | Octapartitioned neutrosophic soft graph |
| 9 | Nonapartitioned neutrosophic soft graph |
| $n$ | $n$-refined fuzzy soft graph; multi-valued fuzzy soft graph; multi-fuzzy soft graph |
| $2n$ | $n$-refined intuitionistic fuzzy soft graph; multi-intuitionistic fuzzy soft graph |
| $3n$ | $n$-refined neutrosophic soft graph; multi-neutrosophic soft graph |

[a] Neutrosophic soft graph models may serve as broad parameterized frameworks capable of specializing to several degree-based soft graph models when suitable model-dependent constraints are imposed.
[b] An ambiguous soft graph is obtained by using an ambiguous-set degree model within the Uncertain Soft Graph framework. Ambiguous-set models are commonly related to certain four-component neutrosophic models.
[c] A Turiyam neutrosophic soft graph is obtained from a four-component Turiyam neutrosophic degree model. Such models are commonly regarded as specializations of quadripartitioned neutrosophic frameworks.
[d] Hesitant and dual hesitant fuzzy degrees are naturally set-valued. Their placement in a fixed-dimensional row presupposes a selected finite-coordinate representation. Without such a representation, their natural degree-domains need not be subsets of one fixed finite-dimensional cube $[0,1]^k$.
*Citation note.* The references in this table identify representative sources for the underlying uncertainty models or their graph-theoretic realizations. The corresponding soft graph type is obtained by introducing the parameter-dependent maps $\mu_{V,M}$ and $\mu_{E,M}$ specified above.

is the subgraph of $G$ induced by $V_\alpha$.

The structure

$$\mathrm{SEG} := (G, (F, A))$$

is called a *Soft Expert Graph* over $G$. Thus, it is a family of induced subgraphs indexed by parameter–expert–opinion triples.

**Example 3.16.2** (Soft Expert Graph for Supplier Evaluation)**.** Let

$$V = \{s_1, s_2, s_3, s_4, s_5\}$$

be a set of candidate suppliers, and let

$$E = \big\{\{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_2, s_4\}, \{s_3, s_4\}, \{s_3, s_5\}, \{s_4, s_5\}\big\}.$$

Thus,

$$G = (V, E)$$

is the ambient supplier-network graph, where an edge indicates an established compatibility or cooperation relationship.

Consider the parameter set

$$P = \{p_q, p_d\},$$

where

$$p_q = \text{high quality}, \qquad p_d = \text{reliable delivery},$$

and the expert set

$$X = \{x_1, x_2\}.$$

Let

$$O = \{0, 1\},$$

where 1 denotes agreement and 0 denotes disagreement.

For simplicity, consider

$$A = \{(p_q, x_1, 1), (p_q, x_2, 1), (p_d, x_1, 1), (p_d, x_2, 0)\}.$$

Define

$$F : A \longrightarrow \mathcal{P}(V)$$

by

$$F(p_q, x_1, 1) = \{s_1, s_2, s_3\},$$
$$F(p_q, x_2, 1) = \{s_2, s_3, s_4\},$$
$$F(p_d, x_1, 1) = \{s_3, s_4, s_5\},$$

and

$$F(p_d, x_2, 0) = \{s_1, s_3, s_5\}.$$

For example, for

$$\alpha = (p_q, x_1, 1),$$

we have

$$V_\alpha = \{s_1, s_2, s_3\}.$$

Hence

$$E_\alpha = \big\{\{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}\big\},$$

and therefore

$$G_\alpha = G[\{s_1, s_2, s_3\}]$$

is the triangle induced by $s_1, s_2, s_3$.

Similarly, for

$$\beta = (p_d, x_1, 1),$$

we obtain

$$V_\beta = \{s_3, s_4, s_5\},$$

and

$$E_\beta = \big\{\{s_3, s_4\}, \{s_3, s_5\}, \{s_4, s_5\}\big\}.$$

Thus

$$G_\beta = G[\{s_3, s_4, s_5\}]$$

is another induced triangle.

Consequently,

$$\text{SEG} = (G, (F, A))$$

is a Soft Expert Graph whose slices describe supplier subnetworks selected according to parameters, experts, and their opinions.

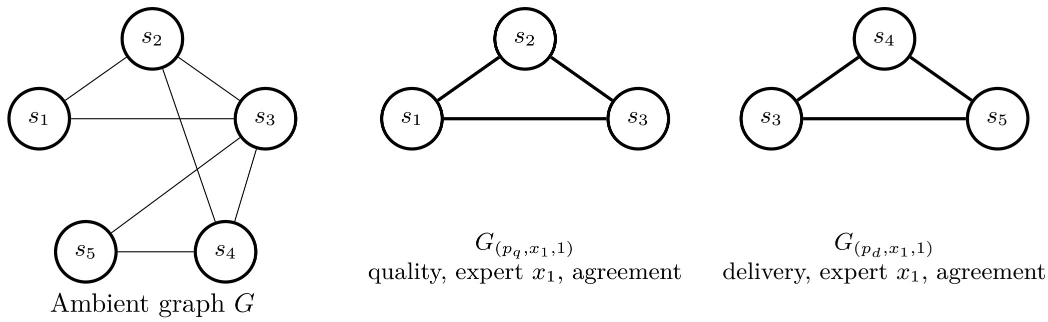



Figure 3.7: An ambient supplier graph and two representative slices of a Soft Expert Graph. Each slice is the induced subgraph determined by a parameter–expert–opinion triple.

## 3.17 Multisoft Multiexpert Graph

A MultiSoft MultiExpert Graph assigns induced subgraphs to multilevel parameter–expert–opinion configurations, allowing multiplicities and jointly representing interacting attributes, expert assessments, repeated scenarios, and network structure [233].

**Definition 3.17.1** (MultiSoft MultiExpert Graph)**.** Let $G = (V, E)$ be a finite simple graph, and let

$$P_1, P_2, \ldots, P_n$$

be pairwise disjoint nonempty parameter sets. Put

$$P := \bigcup_{i=1}^{n} P_i.$$

Let $X$ be a nonempty set of experts, let $O = \{0, 1\}$, and define

$$Z := P \times X \times O.$$

For a set $Y$, let

$$\mathcal{M}(Y) := \{m : Y \longrightarrow \mathbb{N}_0 : \mathrm{supp}(m) \text{ is finite}\}$$

denote the collection of finite multisets on $Y$. Let

$$A \in \mathcal{M}\big(\mathcal{P}(Z)\big)$$

and let

$$F : \mathcal{P}(Z) \longrightarrow \mathcal{P}(V)$$

be a mapping.

For every $\alpha \subseteq Z$ satisfying $A(\alpha) > 0$, define

$$V_\alpha := F(\alpha)$$

and

$$E_\alpha := \big\{\{u, v\} \in E : u, v \in V_\alpha\big\}.$$

Then

$$G_\alpha := (V_\alpha, E_\alpha) = G[V_\alpha]$$

is the subgraph of $G$ induced by $V_\alpha$.

The structure

$$\mathrm{MSMEG} := (G, (F, A), \{P_i\}_{i=1}^{n}, X, O)$$

is called a *MultiSoft MultiExpert Graph* over $G$. It represents a multiset-indexed family of induced subgraphs determined by multilevel parameters, experts, and their opinions.

## 3.18 Soft Digraph

A soft digraph represents a soft set as a directed graph whose vertices and arcs encode parameter-dependent memberships, using a universal vertex for unmatched elements [279–282].

**Definition 3.18.1** (Soft Digraph)**.** Let

$$U = \{h_j : j \in \Delta\}, \qquad P = \{p_1, p_2, \ldots, p_n\},$$

and let $(F, P)$ be a soft set over $U$, where

$$F : P \longrightarrow \mathcal{P}(U).$$

Introduce an additional parameter $p_a \notin P$ such that

$$F(p_a) = \emptyset.$$

The *Soft Digraph* corresponding to $(F, P)$ is the digraph

$$D_F = (V_F, A_F),$$

where

$$V_F = P \cup \{p_a\}$$

and

$$A_F = \big\{(p_i, p_j) : h_j \in F(p_i),\ 1 \le j \le n\big\} \cup \big\{(p_i, p_a) : h_j \in F(p_i),\ j > n\big\}.$$

The additional vertex $p_a$ is called the *universal vertex*.

**Example 3.18.2** (Soft Digraph for a Soft Set)**.** Let

$$U = \{h_1, h_2, h_3, h_4, h_5\}$$

and

$$P = \{p_1, p_2, p_3, p_4\}.$$

Define a soft set $(F, P)$ over $U$ by

$$F(p_1) = \{h_2, h_5\}, \qquad F(p_2) = \{h_1, h_3\},$$
$$F(p_3) = \{h_2, h_4\}, \qquad F(p_4) = \{h_3\}.$$

Introduce the additional parameter $p_a \notin P$ such that

$$F(p_a) = \emptyset.$$

By Definition 3.18.1, the corresponding Soft Digraph is

$$D_F = (V_F, A_F), \qquad V_F = \{p_1, p_2, p_3, p_4, p_a\}.$$

Since $h_2 \in F(p_1)$ and $h_5 \in F(p_1)$, we obtain

$$(p_1, p_2) \in A_F, \qquad (p_1, p_a) \in A_F.$$

Similarly,

$$(p_2, p_1), (p_2, p_3) \in A_F,$$
$$(p_3, p_2), (p_3, p_4) \in A_F,$$

and

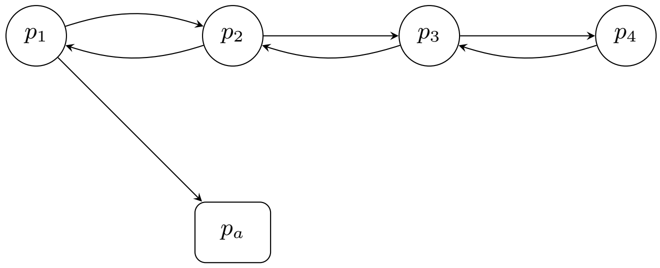
$$(p_4, p_3) \in A_F.$$

Hence

$$A_F = \{(p_1, p_2), (p_1, p_a), (p_2, p_1), (p_2, p_3), (p_3, p_2), (p_3, p_4), (p_4, p_3)\}.$$

Therefore, the Soft Digraph of $(F, P)$ is

$$D_F = \big(\{p_1, p_2, p_3, p_4, p_a\}, \{(p_1, p_2), (p_1, p_a), (p_2, p_1), (p_2, p_3), (p_3, p_2), (p_3, p_4), (p_4, p_3)\}\big).$$




Soft Digraph $D_F$

Figure 3.8: A Soft Digraph associated with the soft set $(F, P)$. The vertex $p_a$ is the universal vertex.

## 3.19 Bipolar Soft Graph

A Bipolar Soft Set assigns disjoint positive and negative approximations to opposite parameters, representing supportive and opposing information simultaneously and systematically [283–286]. A Bipolar Soft Graph assigns disjoint positive and negative parameterized subgraphs, representing supportive and opposing relational information within one network.

**Definition 3.19.1** (Bipolar Soft Set)**.** [283] Let $U \neq \emptyset$ be a universe, let $A \neq \emptyset$ be a parameter set, and let

$$\neg A := \{\neg a : a \in A\}$$

be the corresponding set of opposite parameters, with

$$A \cap \neg A = \emptyset.$$

A *Bipolar Soft Set* over $U$ is a triple

$$(F, G, A),$$

where

$$F : A \longrightarrow \mathcal{P}(U), \qquad G : \neg A \longrightarrow \mathcal{P}(U),$$

such that

$$F(a) \cap G(\neg a) = \emptyset \qquad \text{for every } a \in A.$$

Here, $F(a)$ and $G(\neg a)$ represent the positive and negative approximations associated with $a$, respectively.

**Definition 3.19.2** (Bipolar Soft Graph)**.** Let

$$G^* = (V, E), \qquad E \subseteq \binom{V}{2},$$

be a finite simple graph. Let $A \neq \emptyset$ be a parameter set, and let

$$\neg A := \{\neg a : a \in A\}$$

be its set of opposite parameters.

A *Bipolar Soft Graph* over $G^*$ is a structure

$$\mathcal{G}_{\mathrm{BS}} = \big(G^*, A, F_V^+, F_E^+, F_V^-, F_E^-\big),$$

where

$$F_V^+ : A \longrightarrow \mathcal{P}(V), \qquad F_E^+ : A \longrightarrow \mathcal{P}(E),$$

and

$$F_V^- : \neg A \longrightarrow \mathcal{P}(V), \qquad F_E^- : \neg A \longrightarrow \mathcal{P}(E),$$

satisfy, for every $a \in A$,

$$F_E^+(a) \subseteq E \cap \binom{F_V^+(a)}{2},$$

$$F_E^-(\neg a) \subseteq E \cap \binom{F_V^-(\neg a)}{2},$$

and

$$F_V^+(a) \cap F_V^-(\neg a) = \emptyset.$$

The corresponding positive and negative graph slices are

$$G_a^+ = \big(F_V^+(a), F_E^+(a)\big)$$

and

$$G_{\neg a}^- = \big(F_V^-(\neg a), F_E^-(\neg a)\big),$$

respectively.

**Example 3.19.3** (Bipolar Soft Graph for Project-Team Selection)**.** Consider the finite simple graph

$$G^* = (V, E),$$

where

$$V = \{v_1, v_2, v_3, v_4, v_5\},$$

and

$$E = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_4, v_5\}\big\}.$$

Interpret the vertices as five researchers in a collaboration network.

Let

$$A = \{a\},$$

where $a$ denotes the parameter

$$a = \text{“suitable for the core project team”}.$$

Then

$$\neg A = \{\neg a\},$$

where $\neg a$ denotes the opposite parameter

$$\neg a = \text{“not suitable for the core project team”}.$$

Define the positive vertex and edge maps by

$$F_V^+(a) = \{v_1, v_2, v_3\},$$

and

$$F_E^+(a) = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\big\}.$$

Thus,

$$F_E^+(a) \subseteq E \cap \binom{F_V^+(a)}{2}.$$

Define the negative vertex and edge maps by

$$F_V^-(\neg a) = \{v_4, v_5\},$$

and

$$F_E^-(\neg a) = \big\{\{v_4, v_5\}\big\}.$$

Hence,

$$F_E^-(\neg a) \subseteq E \cap \binom{F_V^-(\neg a)}{2}.$$

Moreover,

$$F_V^+(a) \cap F_V^-(\neg a) = \{v_1, v_2, v_3\} \cap \{v_4, v_5\} = \emptyset.$$

Therefore, all conditions in Definition 3.19.2 are satisfied, and

$$\mathcal{G}_{\mathrm{BS}} = \big(G^*, A, F_V^+, F_E^+, F_V^-, F_E^-\big)$$

is a Bipolar Soft Graph over $G^*$.

The corresponding positive and negative graph slices are

$$G_a^+ = \big(\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}\big),$$

and

$$G_{\neg a}^- = \big(\{v_4, v_5\}, \{\{v_4, v_5\}\}\big),$$

respectively.

Here, $G_a^+$ represents the positively selected core team, whereas $G_{\neg a}^-$ represents the negatively classified researchers under the opposite parameter. An illustrative diagram is shown in Figure 3.9.

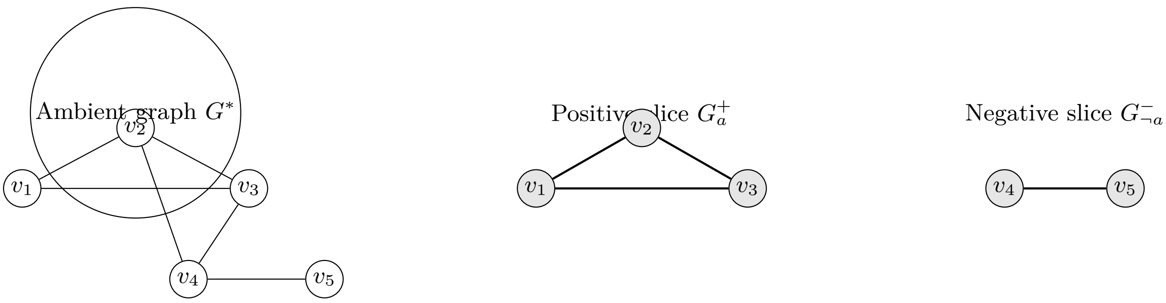



Figure 3.9: A Bipolar Soft Graph consisting of an ambient graph, a positive graph slice, and a negative graph slice.

**Theorem 3.19.4** (Well-Definedness of a Bipolar Soft Graph)**.** *For every $a \in A$, the structures $G_a^+$ and $G_{\neg a}^-$ are vertex-disjoint subgraphs of $G^*$. Consequently, the Bipolar Soft Graph is well-defined.*

*Proof.* By definition,

$$F_E^+(a) \subseteq E \cap \binom{F_V^+(a)}{2},$$

so every endpoint of every edge in $F_E^+(a)$ belongs to $F_V^+(a)$. Hence $G_a^+$ is a subgraph of $G^*$. The same argument proves that $G_{\neg a}^-$ is a subgraph.

Moreover,

$$F_V^+(a) \cap F_V^-(\neg a) = \emptyset.$$

Therefore the two slices are vertex-disjoint. In particular, they cannot share an edge, because any common edge would have both endpoints in the empty intersection of their vertex sets. □

## 3.20 Multipolar Soft Graph

An m-polar Soft Set assigns each parameter an m-tuple of subsets, representing multiple perspectives or polar evaluations over one universe [36]. An m-polar Soft Graph assigns each parameter multiple graph slices, representing polar perspectives on vertices, edges, and network relationships simultaneously.

**Definition 3.20.1** ($m$-Polar Soft Set)**.** Let $U \neq \emptyset$ be a universe, let $E \neq \emptyset$ be a parameter set, and let $m \geq 2$. An *$m$-Polar Soft Set* over $U$ is a pair

$$(F, E),$$

where

$$F : E \longrightarrow \big(\mathcal{P}(U)\big)^m.$$

For each $e \in E$, write

$$F(e) = \big(F_1(e), F_2(e), \ldots, F_m(e)\big), \qquad F_i(e) \subseteq U.$$

The subset $F_i(e)$ is called the $i$-th polar approximation of $U$ under the parameter $e$.

**Definition 3.20.2** ($m$-Polar Soft Graph)**.** Let

$$G^* = (V, E), \qquad E \subseteq \binom{V}{2},$$

be a finite simple graph, let $P \neq \emptyset$ be a parameter set, and let $m \geq 2$.

An *$m$-Polar Soft Graph* over $G^*$ is a structure

$$\mathcal{G}_{m\mathrm{PS}} = \big(G^*, P, m, F_V, F_E\big),$$

where

$$F_V : P \longrightarrow \big(\mathcal{P}(V)\big)^m$$

and

$$F_E : P \longrightarrow \big(\mathcal{P}(E)\big)^m.$$

For every $p \in P$, write

$$F_V(p) = \big(V_1(p), V_2(p), \ldots, V_m(p)\big)$$

and

$$F_E(p) = \big(E_1(p), E_2(p), \ldots, E_m(p)\big),$$

where

$$E_i(p) \subseteq E \cap \binom{V_i(p)}{2}, \qquad i = 1, \ldots, m.$$

The $i$-th polar graph slice under $p$ is

$$G_i[p] = \big(V_i(p), E_i(p)\big).$$

**Theorem 3.20.3** (Well-Definedness of an $m$-Polar Soft Graph)**.** *For every $p \in P$ and $i \in \{1, \ldots, m\}$, the slice $G_i[p]$ is a subgraph of $G^*$. Hence*

$$p \longmapsto \big(G_1[p], G_2[p], \ldots, G_m[p]\big)$$

*is a well-defined mapping into the ordered $m$-tuples of subgraphs of $G^*$.*

*Proof.* For every $p \in P$ and $i \in \{1, \ldots, m\}$,

$$V_i(p) \subseteq V$$

and

$$E_i(p) \subseteq E \cap \binom{V_i(p)}{2}.$$

Thus every edge of $E_i(p)$ belongs to $E$ and has both endpoints in $V_i(p)$. Therefore $G_i[p]$ is a subgraph of $G^*$, which proves the assertion. $\square$

## 3.21 ClusterSoft Graph

The ClusterSoft Set construction uses the union-based aggregation presented in the literature; the ClusterSoft Graph is its corresponding graph-theoretic extension (cf. [287]).

**Definition 3.21.1** (ClusterSoft Set)**.** Let $U \neq \emptyset$ be a universe, let $I \neq \emptyset$ be a finite index set, and let

$$\{(F_i, A_i) : i \in I\}$$

be a finite family of soft sets over $U$, where

$$A_i \neq \emptyset, \qquad F_i : A_i \longrightarrow \mathcal{P}(U).$$

For each $i \in I$, define the aggregated image

$$F_i^{\cup} := \bigcup_{a \in A_i} F_i(a).$$

Let

$$\mathcal{C} = \{C_j : j \in J\}$$

be a partition of $I$; that is,

$$C_j \neq \emptyset, \qquad \bigcup_{j \in J} C_j = I, \qquad C_j \cap C_k = \emptyset \quad (j \neq k).$$

The mapping

$$\Gamma : \mathcal{C} \longrightarrow \mathcal{P}(U)$$

defined by

$$\Gamma(C_j) = \bigcup_{i \in C_j} F_i^{\cup} = \bigcup_{i \in C_j} \bigcup_{a \in A_i} F_i(a)$$

is called the *ClusterSoft Set* induced by $\{(F_i, A_i) : i \in I\}$ and $\mathcal{C}$.

Thus, each cluster $C_j$ aggregates the objects associated with all soft sets whose indices belong to $C_j$.

**Definition 3.21.2** (ClusterSoft Graph)**.** Let

$$G^* = (V, E), \qquad E \subseteq \binom{V}{2},$$

be a finite simple graph, and define

$$\mathrm{Sub}(G^*) = \left\{ (W, F) : W \subseteq V,\ F \subseteq E \cap \binom{W}{2} \right\}.$$

Let $I \neq \emptyset$ be a finite index set. For every $i \in I$, let

$$\Phi_i : A_i \longrightarrow \mathrm{Sub}(G^*)$$

be a soft graph over $G^*$, where $A_i \neq \emptyset$. Write

$$\Phi_i(a) = \big(V_{i,a}, E_{i,a}\big), \qquad a \in A_i.$$

For each $i \in I$, define its aggregated graph by

$$V_i^{\cup} = \bigcup_{a \in A_i} V_{i,a}, \qquad E_i^{\cup} = \bigcup_{a \in A_i} E_{i,a},$$

and put

$$H_i^{\cup} = \left(V_i^{\cup}, E_i^{\cup}\right).$$

Let

$$\mathcal{C} = \{C_j : j \in J\}$$

be a partition of $I$. Define

$$\Psi : \mathcal{C} \longrightarrow \mathrm{Sub}(G^*)$$

by

$$\Psi(C_j) = \bigcup_{i \in C_j} H_i^{\cup} = \left( \bigcup_{i \in C_j} V_i^{\cup}, \bigcup_{i \in C_j} E_i^{\cup} \right).$$

Equivalently,

$$\Psi(C_j) = \left( \bigcup_{i \in C_j} \bigcup_{a \in A_i} V_{i,a}, \bigcup_{i \in C_j} \bigcup_{a \in A_i} E_{i,a} \right).$$

The structure

$$\mathcal{G}_{\mathrm{CS}} = (G^*, \{\Phi_i : i \in I\}, \mathcal{C}, \Psi)$$

is called a *ClusterSoft Graph* over $G^*$.

Indeed, every edge contained in the edge set of $\Psi(C_j)$ belongs to $E$ and has both endpoints in the vertex set of $\Psi(C_j)$; hence $\Psi(C_j)$ is a well-defined subgraph of $G^*$.

## 3.22 Double-Framed Soft Graph

A Double-Framed Soft Set uses positive and negative mappings satisfying algebraic inclusion conditions to model dual parameter-dependent information over a universe [288–291]. Related concepts, such as Double-Framed HyperSoft Sets, are also known in the literature [292–294]. A Double-Framed Soft Graph assigns positive and negative subgraph frames satisfying algebraic inclusion conditions, representing parameter-dependent network structures within ambient graphs.

**Definition 3.22.1** (Double-Framed Soft Set)**.** [288–290] Let $U \neq \emptyset$ be a universe, and let

$$(A, *)$$

be a nonempty set equipped with a binary operation

$$* : A \times A \longrightarrow A.$$

A *Double-Framed Soft Set* over $U$ is a triple

$$\langle \alpha, \beta; A \rangle,$$

where

$$\alpha, \beta : A \longrightarrow \mathcal{P}(U)$$

satisfy, for all $a, b \in A$,

$$\alpha(a * b) \supseteq \alpha(a) \cap \alpha(b),$$

and

$$\beta(a * b) \subseteq \beta(a) \cup \beta(b).$$

Here, $\alpha(a)$ and $\beta(a)$ are called the *positive frame* and the *negative frame* associated with $a$, respectively.

**Definition 3.22.2** (Double-Framed Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, and define

$$\mathrm{Sub}(G^*) = \left\{ (W, F) : W \subseteq V,\ F \subseteq E \cap \binom{W}{2} \right\}.$$

For

$$H_i = (W_i, F_i) \in \mathrm{Sub}(G^*), \qquad i = 1, 2,$$

define

$$H_1 \cap H_2 = (W_1 \cap W_2, F_1 \cap F_2),$$
$$H_1 \cup H_2 = (W_1 \cup W_2, F_1 \cup F_2),$$

and write

$$H_1 \preceq H_2$$

whenever $W_1 \subseteq W_2$ and $F_1 \subseteq F_2$.

Let $(A, *)$ be a nonempty set equipped with a binary operation. A *Double-Framed Soft Graph* over $G^*$ is a triple

$$\langle \Phi^+, \Phi^-; A \rangle,$$

where

$$\Phi^+, \Phi^- : A \longrightarrow \mathrm{Sub}(G^*)$$

satisfy, for all $a, b \in A$,

$$\Phi^+(a) \cap \Phi^+(b) \preceq \Phi^+(a * b),$$

and

$$\Phi^-(a * b) \preceq \Phi^-(a) \cup \Phi^-(b).$$

The subgraphs $\Phi^+(a)$ and $\Phi^-(a)$ are called the *positive graph frame* and the *negative graph frame* associated with $a$, respectively.

## 3.23 Ranked Soft Graph

A Ranked Soft Set assigns each parameter an ordered partition of the universe, representing objects at distinct parameter-dependent satisfaction levels systematically [295]. A Ranked Soft Graph assigns parameter-dependent ranks to vertices and edges, producing nested threshold subgraphs that preserve endpoint compatibility across levels.

**Definition 3.23.1** (Ranked Soft Set)**.** [295] Let $U \neq \emptyset$ be a finite universe, let $A \neq \emptyset$ be a parameter set, and fix an integer $k \geq 0$. Define

$$\mathrm{RPart}_k(U) = \left\{ (U^0, U^1, \ldots, U^k) : U = \bigsqcup_{i=0}^{k} U^i \right\},$$

where $\bigsqcup$ denotes disjoint union and a larger index represents a higher rank.

A *Ranked Soft Set* over $U$ is a mapping

$$\mathcal{R} : A \longrightarrow \mathrm{RPart}_k(U).$$

For each $a \in A$, write

$$\mathcal{R}(a) = \bigl(U_a^0, U_a^1, \ldots, U_a^k\bigr).$$

Thus,

$$U = \bigsqcup_{i=0}^{k} U_a^i,$$

and the elements of $U_a^j$ satisfy the parameter $a$ at a higher rank than those of $U_a^i$ whenever $i < j$.

**Definition 3.23.2** (Ranked Soft Graph)**.** Let

$$G^* = (V, E), \qquad E \subseteq \binom{V}{2},$$

be a finite simple graph, let $A \neq \emptyset$ be a parameter set, and fix an integer $k \geq 0$.

A *Ranked Soft Graph* over $G^*$ is a structure

$$\mathcal{G}_{\mathrm{RS}} = \big(G^*, A, \rho_V, \rho_E\big),$$

where

$$\rho_V : A \times V \longrightarrow \{0, 1, \ldots, k\},$$
$$\rho_E : A \times E \longrightarrow \{0, 1, \ldots, k\},$$

and, for every $a \in A$ and every edge $e = \{u, v\} \in E$,

$$\rho_E(a, e) \leq \min\{\rho_V(a, u), \rho_V(a, v)\}.$$

For $a \in A$ and $i \in \{0, \ldots, k\}$, define

$$V_a^i = \{v \in V : \rho_V(a, v) = i\}, \qquad E_a^i = \{e \in E : \rho_E(a, e) = i\}.$$

Then

$$V = \bigsqcup_{i=0}^{k} V_a^i, \qquad E = \bigsqcup_{i=0}^{k} E_a^i,$$

are the parameter-dependent ranked partitions of the vertices and edges.

Moreover, the threshold sets

$$V_a^{\geq i} = \{v \in V : \rho_V(a, v) \geq i\}, \qquad E_a^{\geq i} = \{e \in E : \rho_E(a, e) \geq i\}$$

determine a subgraph

$$G_a^{\geq i} = \big(V_a^{\geq i}, E_a^{\geq i}\big)$$

of $G^*$ for every $a \in A$ and $i \in \{0, \ldots, k\}$.

## 3.24 Type-2 Soft Graph

A Type-2 Soft Set extends the ordinary Soft Set framework by assigning to each primary parameter a Type-1 Soft Set, thereby providing a two-level hierarchical representation of parameter-dependent information [296–299]. Similarly, a Type-3 Soft Set assigns a Type-2 Soft Set to each primary parameter and thus introduces a three-level nested parameter structure.

Related higher-order uncertainty models include Type-3 Fuzzy Sets [300–302], Type-3 Intuitionistic Fuzzy Sets [303, 304], and Type-3 Neutrosophic Sets [305, 306].

In graph-theoretic settings, a Type-2 Soft Graph introduces two levels of parameterization by associating each primary parameter with a Soft Graph indexed by a corresponding set of secondary parameters [307, 308].

**Definition 3.24.1** (Type-2 Soft Graph)**.** Let

$$G = (V, E)$$

be a finite simple graph, and denote by

$$\mathrm{Sub}(G)$$

the family of all subgraphs of $G$.

Define

$$\mathfrak{S}(G) = \{(\Psi, B) : \emptyset \neq B \subseteq V,\ \Psi : B \longrightarrow \mathrm{Sub}(G)\}.$$

Thus, each element of $\mathfrak{S}(G)$ is a Type-1 Soft Graph over $G$.

Let

$$A \subseteq \{x \in V : N_G(x) \neq \emptyset\}$$

be a nonempty set of primary parameters, where

$$N_G(x) = \{y \in V : \{x, y\} \in E\}$$

denotes the neighborhood of $x$.

A *Type-2 Soft Graph* over $G$ is a mapping

$$\Phi^{(2)} : A \longrightarrow \mathfrak{S}(G)$$

such that, for every $x \in A$,

$$\Phi^{(2)}(x) = (\Phi_x, N_G(x)),$$

where

$$\Phi_x : N_G(x) \longrightarrow \mathrm{Sub}(G)$$

is a Type-1 Soft Graph over $G$.

Equivalently, a Type-2 Soft Graph may be represented as the two-level parameterized family

$$\{G_{x,z} = \Phi_x(z) : x \in A,\ z \in N_G(x)\},$$

where every

$$G_{x,z} \in \mathrm{Sub}(G).$$

Hence, the first parameter $x$ selects a neighborhood-based Type-1 Soft Graph, while the secondary parameter $z \in N_G(x)$ selects a particular subgraph of $G$.

**Remark 3.24.2.** The two parameter levels of a Type-2 Soft Graph can be represented schematically as

$$x \longmapsto (z \longmapsto G_{x,z}), \qquad x \in A, \quad z \in N_G(x).$$

Thus, the construction is genuinely hierarchical: primary parameters determine Soft Graphs, whereas secondary parameters determine the individual subgraphs contained in those Soft Graphs.

**Example 3.24.3** (Type-2 Soft Graph)**.** Let

$$G = (V, E)$$

be the finite simple graph with

$$V = \{v_1, v_2, v_3, v_4\}$$

and

$$E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}.$$

Choose the set of primary parameters
$$A = \{v_1, v_2\}.$$
The corresponding neighborhoods are
$$N_G(v_1) = \{v_2, v_3\}$$
and
$$N_G(v_2) = \{v_1, v_3, v_4\}.$$

For the primary parameter $v_1$, define
$$\Phi_{v_1} : N_G(v_1) \longrightarrow \mathrm{Sub}(G)$$
by
$$\Phi_{v_1}(v_2) = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\}),$$
and
$$\Phi_{v_1}(v_3) = (\{v_1, v_3, v_4\}, \{\{v_1, v_3\}, \{v_3, v_4\}\}).$$

Thus,
$$\Phi^{(2)}(v_1) = (\Phi_{v_1}, N_G(v_1)) \in \mathfrak{S}(G).$$

For the primary parameter $v_2$, define
$$\Phi_{v_2} : N_G(v_2) \longrightarrow \mathrm{Sub}(G)$$
by
$$\Phi_{v_2}(v_1) = (\{v_1, v_2\}, \{\{v_1, v_2\}\}),$$
$$\Phi_{v_2}(v_3) = (\{v_2, v_3, v_4\}, \{\{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}),$$
and
$$\Phi_{v_2}(v_4) = (\{v_2, v_4\}, \{\{v_2, v_4\}\}).$$

Hence,
$$\Phi^{(2)}(v_2) = (\Phi_{v_2}, N_G(v_2)) \in \mathfrak{S}(G).$$

Therefore, the mapping
$$\Phi^{(2)} : A \longrightarrow \mathfrak{S}(G),$$
defined by
$$\Phi^{(2)}(v_1) = (\Phi_{v_1}, N_G(v_1))$$
and
$$\Phi^{(2)}(v_2) = (\Phi_{v_2}, N_G(v_2)),$$
is a Type-2 Soft Graph over $G$.

Equivalently, the associated two-level parameterized family is
$$\{G_{v_1,v_2}, G_{v_1,v_3}, G_{v_2,v_1}, G_{v_2,v_3}, G_{v_2,v_4}\},$$
where
$$G_{x,z} = \Phi_x(z).$$

This example illustrates the hierarchical nature of a Type-2 Soft Graph: the primary parameter $x$ selects a Type-1 Soft Graph, while the secondary parameter $z \in N_G(x)$ selects a particular subgraph of $G$.

## 3.25 Type-$n$ Soft Graph

A Type-$n$ Soft Graph recursively parameterizes soft graphs through $n$ levels, providing a hierarchical representation of graph structures under nested parameter systems. Related concepts include Type-$n$ Fuzzy Sets [309–311], Type-$n$ Intuitionistic Fuzzy Sets [312, 313], and Type-$n$ Neutrosophic Sets [314–317].

**Definition 3.25.1** (Type-$n$ Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $P \neq \emptyset$ be a set of parameters, and let $n \geq 1$ be an integer.

Denote by

$$\mathrm{Sub}(G^*)$$

the family of all subgraphs of $G^*$.

Define recursively the classes

$$\mathfrak{SG}^{(n)}(G^*, P)$$

as follows.

For $n = 1$, set

$$\mathfrak{SG}^{(1)}(G^*, P) = \{(F, A) : \emptyset \neq A \subseteq P,\ F : A \longrightarrow \mathrm{Sub}(G^*)\}.$$

For every $n \geq 2$, define

$$\mathfrak{SG}^{(n)}(G^*, P) = \Big\{(F, A) : \emptyset \neq A \subseteq P,\ F : A \longrightarrow \mathfrak{SG}^{(n-1)}(G^*, P)\Big\}.$$

An element

$$\widetilde{G}^{(n)} = (G^*, F, A), \qquad (F, A) \in \mathfrak{SG}^{(n)}(G^*, P),$$

is called a *Type-n Soft Graph* over $G^*$.

More explicitly, for each $a_1 \in A$,

$$F(a_1) = (F_{a_1}, A_{a_1}) \in \mathfrak{SG}^{(n-1)}(G^*, P).$$

Iterating this decomposition, every admissible parameter chain

$$a_1 \in A, \quad a_2 \in A_{a_1}, \quad \ldots, \quad a_n \in A_{a_1 \cdots a_{n-1}}$$

terminates in a subgraph

$$F_{a_1 \cdots a_{n-1}}(a_n) \in \mathrm{Sub}(G^*).$$

**Theorem 3.25.2** (Well-Definedness of Type-$n$ Soft Graphs)**.** *For every integer $n \geq 1$, the recursive construction in Definition 3.25.1 is well-defined. In particular, every admissible parameter chain of length $n$ terminates in a uniquely specified subgraph of $G^*$.*

*Proof.* We proceed by induction on $n$.

For $n = 1$, let
$$(F, A) \in \mathfrak{SG}^{(1)}(G^*, P).$$
By definition,
$$F : A \longrightarrow \mathrm{Sub}(G^*),$$
and therefore, for every $a_1 \in A$,
$$F(a_1) \in \mathrm{Sub}(G^*).$$
Hence, every Type-1 Soft Graph assigns a well-defined subgraph of $G^*$ to each parameter.

Assume now that the statement holds for some $n - 1 \geq 1$. Let
$$(F, A) \in \mathfrak{SG}^{(n)}(G^*, P).$$
Then, by definition,
$$F : A \longrightarrow \mathfrak{SG}^{(n-1)}(G^*, P).$$
Thus, for every $a_1 \in A$,
$$F(a_1) = (F_{a_1}, A_{a_1}) \in \mathfrak{SG}^{(n-1)}(G^*, P).$$

By the induction hypothesis, every admissible parameter chain
$$a_2, \ldots, a_n$$
within $F(a_1)$ terminates in a well-defined subgraph
$$F_{a_1 \cdots a_{n-1}}(a_n) \in \mathrm{Sub}(G^*).$$

Since $a_1$ was arbitrary, every admissible parameter chain
$$a_1, a_2, \ldots, a_n$$
of the Type-$n$ Soft Graph terminates in a subgraph of $G^*$. Moreover, since every mapping appearing in the recursive construction is a function, the terminal subgraph is uniquely determined by the chosen parameter chain.

Therefore,
$$\mathfrak{SG}^{(n)}(G^*, P)$$
is well-defined for every $n \geq 1$. □

**Remark 3.25.3.** The Type-$n$ Soft Graph framework provides a hierarchy of increasingly deep parameterizations rather than a literal chain of set-theoretic inclusions.

For $n = 1$, the recursive construction reduces to an ordinary Soft Graph. For $n = 2$, each primary parameter determines a Type-1 Soft Graph, yielding a general two-level parameterized Soft Graph. In particular, it reduces to the Type-2 Soft Graph introduced above when the secondary parameter set associated with each primary parameter $x$ is chosen to be the neighborhood $N_G(x)$.

More generally, increasing $n$ introduces additional nested parameter levels. Thus, the progression
$$\text{Type-1} \longrightarrow \text{Type-2} \longrightarrow \cdots \longrightarrow \text{Type-}n$$
should be understood as an increase in parameterization depth, rather than as ordinary set inclusion.

## 3.26 Interval-Valued Fuzzy Soft Graph

An Interval-Valued Fuzzy Soft Graph assigns interval-valued fuzzy memberships to vertices and edges under parameters, representing uncertainty through bounded membership ranges [318, 319].

**Definition 3.26.1** (Interval-Valued Fuzzy Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, and let $A \neq \emptyset$ be a set of parameters.

An *Interval-Valued Fuzzy Soft Graph* over $G^*$ is a quadruple

$$\widetilde{G} = (G^*, \mathcal{F}, \mathcal{K}, A),$$

where, for every $a \in A$,

$$\mathcal{F}(a) : V \longrightarrow \mathrm{Int}([0, 1])$$

is an interval-valued fuzzy set on $V$, and

$$\mathcal{K}(a) : E \longrightarrow \mathrm{Int}([0, 1])$$

is an interval-valued fuzzy set on $E$.

Write

$$\mathcal{F}(a)(v) = \left[\mu^-_{\mathcal{F}(a)}(v), \mu^+_{\mathcal{F}(a)}(v)\right]$$

for $v \in V$, and

$$\mathcal{K}(a)(\{u, v\}) = \left[\mu^-_{\mathcal{K}(a)}(\{u, v\}), \mu^+_{\mathcal{K}(a)}(\{u, v\})\right]$$

for $\{u, v\} \in E$.

For every $a \in A$ and every $\{u, v\} \in E$, the following compatibility conditions are required:

$$\mu^-_{\mathcal{K}(a)}(\{u, v\}) \leq \min\left\{\mu^-_{\mathcal{F}(a)}(u), \mu^-_{\mathcal{F}(a)}(v)\right\},$$

and

$$\mu^+_{\mathcal{K}(a)}(\{u, v\}) \leq \min\left\{\mu^+_{\mathcal{F}(a)}(u), \mu^+_{\mathcal{F}(a)}(v)\right\}.$$

Thus, for each parameter $a \in A$,

$$\big(\mathcal{F}(a), \mathcal{K}(a)\big)$$

is an interval-valued fuzzy graph of $G^*$, and $\widetilde{G}$ is a parameterized family of interval-valued fuzzy graphs.

## 3.27 Interval-Valued Uncertain Soft Graph

An Interval-Valued Uncertain Soft Graph extends an Uncertain Soft Graph by assigning an interval of admissible uncertainty values to each vertex and edge under every parameter.

**Definition 3.27.1** (Interval-Valued Uncertain Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$ be a set of parameters, and let

$$(L, \preceq, \wedge)$$

be a meet-semilattice of uncertainty values.

For $x^-, x^+ \in L$ with $x^- \preceq x^+$, write

$$[x^-, x^+]_L = \{x \in L : x^- \preceq x \preceq x^+\}.$$

For every $a \in A$, let

$$\underline{\omega}_{V,a}, \overline{\omega}_{V,a} : V \longrightarrow L$$

and

$$\underline{\omega}_{E,a}, \overline{\omega}_{E,a} : E \longrightarrow L$$

satisfy

$$\underline{\omega}_{V,a}(v) \preceq \overline{\omega}_{V,a}(v) \qquad (v \in V),$$

and

$$\underline{\omega}_{E,a}(e) \preceq \overline{\omega}_{E,a}(e) \qquad (e \in E).$$

Define the interval-valued uncertainty degrees by

$$\Omega_{V,a}(v) = \left[\underline{\omega}_{V,a}(v), \overline{\omega}_{V,a}(v)\right]_L,$$

and

$$\Omega_{E,a}(e) = \left[\underline{\omega}_{E,a}(e), \overline{\omega}_{E,a}(e)\right]_L.$$

Moreover, for every edge $e = \{u, v\} \in E$, assume that

$$\underline{\omega}_{E,a}(e) \preceq \underline{\omega}_{V,a}(u) \wedge \underline{\omega}_{V,a}(v),$$

and

$$\overline{\omega}_{E,a}(e) \preceq \overline{\omega}_{V,a}(u) \wedge \overline{\omega}_{V,a}(v).$$

Then

$$\widetilde{\mathcal{G}}_{IU} = (G^*, \{\Omega_{V,a}\}_{a \in A}, \{\Omega_{E,a}\}_{a \in A}, A)$$

is called an *Interval-Valued Uncertain Soft Graph*.

**Theorem 3.27.2** (Well-Definedness)**.** *The Interval-Valued Uncertain Soft Graph in Definition 3.27.1 is well-defined. In particular, every vertex and edge is assigned a nonempty interval of uncertainty values, and the graph compatibility conditions are preserved at both interval endpoints.*

*Proof.* Fix $a \in A$.

For every $v \in V$, we have
$$\underline{\omega}_{V,a}(v) \preceq \overline{\omega}_{V,a}(v).$$
Hence,
$$\Omega_{V,a}(v) = \big[\underline{\omega}_{V,a}(v), \overline{\omega}_{V,a}(v)\big]_L \neq \emptyset,$$
because both endpoints belong to the corresponding order interval.

Similarly, for every $e \in E$,
$$\underline{\omega}_{E,a}(e) \preceq \overline{\omega}_{E,a}(e),$$
and therefore
$$\Omega_{E,a}(e) = \big[\underline{\omega}_{E,a}(e), \overline{\omega}_{E,a}(e)\big]_L \neq \emptyset.$$

Now let $e = \{u, v\} \in E$. By assumption,
$$\underline{\omega}_{E,a}(e) \preceq \underline{\omega}_{V,a}(u) \wedge \underline{\omega}_{V,a}(v),$$
and
$$\overline{\omega}_{E,a}(e) \preceq \overline{\omega}_{V,a}(u) \wedge \overline{\omega}_{V,a}(v).$$
Thus, both the lower and upper edge uncertainty degrees satisfy the required incidence compatibility conditions with their endpoint vertices.

Consequently, for every parameter $a \in A$, the mappings $\Omega_{V,a}$ and $\Omega_{E,a}$ define valid interval-valued uncertainty assignments on $V$ and $E$, respectively. Therefore,
$$\widetilde{\mathcal{G}}_{IU}$$
is a well-defined Interval-Valued Uncertain Soft Graph. □

**Remark 3.27.3.** If
$$\underline{\omega}_{V,a}(v) = \overline{\omega}_{V,a}(v)$$
for every $v \in V$, and
$$\underline{\omega}_{E,a}(e) = \overline{\omega}_{E,a}(e)$$
for every $e \in E$, then all uncertainty intervals are degenerate. Hence, an ordinary Uncertain Soft Graph is recovered as a special case of an Interval-Valued Uncertain Soft Graph.

Related concepts, such as Interval-Valued Intuitionistic Fuzzy Soft Graphs [320, 321] and Interval-Valued Neutrosophic Soft Graphs [178, 322, 323], are also known.

## 3.28 Pythagorean Fuzzy Soft Graph

A Pythagorean Fuzzy Soft Graph parameterizes Pythagorean fuzzy graphs, modeling membership and non-membership degrees satisfying squared-sum constraints under uncertainty flexibly [260, 324–327].

**Definition 3.28.1** (Pythagorean Fuzzy Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, and let $A \neq \emptyset$ be a set of parameters.

A *Pythagorean Fuzzy Soft Graph* over $G^*$ is a quadruple

$$\widetilde{G}_P = (G^*, \mathcal{F}, \mathcal{K}, A),$$

where, for every $a \in A$,

$$\mathcal{F}(a) = \big(\mu_{\mathcal{F}(a)}, \nu_{\mathcal{F}(a)}\big)$$

is a Pythagorean fuzzy set on $V$, and

$$\mathcal{K}(a) = \big(\mu_{\mathcal{K}(a)}, \nu_{\mathcal{K}(a)}\big)$$

is a Pythagorean fuzzy set on $E$.

Thus, for every $v \in V$,

$$0 \leq \mu_{\mathcal{F}(a)}(v)^2 + \nu_{\mathcal{F}(a)}(v)^2 \leq 1,$$

and, for every $e \in E$,

$$0 \leq \mu_{\mathcal{K}(a)}(e)^2 + \nu_{\mathcal{K}(a)}(e)^2 \leq 1.$$

Moreover, for every edge $\{u, v\} \in E$,

$$\mu_{\mathcal{K}(a)}(\{u, v\}) \leq \min\left\{\mu_{\mathcal{F}(a)}(u), \mu_{\mathcal{F}(a)}(v)\right\},$$

and

$$\nu_{\mathcal{K}(a)}(\{u, v\}) \geq \max\left\{\nu_{\mathcal{F}(a)}(u), \nu_{\mathcal{F}(a)}(v)\right\}.$$

Hence, for each $a \in A$,

$$\big(\mathcal{F}(a), \mathcal{K}(a)\big)$$

is a Pythagorean fuzzy graph of $G^*$, and $\widetilde{G}_P$ is a parameterized family of Pythagorean fuzzy graphs.

**Example 3.28.2** (Pythagorean Fuzzy Soft Graph)**.** Let

$$G^* = (V, E),$$

where

$$V = \{v_1, v_2, v_3\}$$

and

$$E = \{e_{12}, e_{23}\}, \qquad e_{12} = \{v_1, v_2\}, \qquad e_{23} = \{v_2, v_3\}.$$

Let the parameter set be

$$A = \{a_1, a_2\}.$$

For the parameter $a_1$, define the Pythagorean fuzzy vertex degrees by

| $v$ | $v_1$ | $v_2$ | $v_3$ |
|---|---|---|---|
| $\mu_{\mathcal{F}(a_1)}(v)$ | 0.8 | 0.7 | 0.6 |
| $\nu_{\mathcal{F}(a_1)}(v)$ | 0.3 | 0.4 | 0.5 |

and the edge degrees by

| $e$ | $e_{12}$ | $e_{23}$ |
|---|---|---|
| $\mu_{\mathcal{K}(a_1)}(e)$ | 0.6 | 0.5 |
| $\nu_{\mathcal{K}(a_1)}(e)$ | 0.5 | 0.6 |

.

For the parameter $a_2$, define

| $v$ | $v_1$ | $v_2$ | $v_3$ |
|---|---|---|---|
| $\mu_{\mathcal{F}(a_2)}(v)$ | 0.6 | 0.9 | 0.5 |
| $\nu_{\mathcal{F}(a_2)}(v)$ | 0.6 | 0.2 | 0.7 |

and

| $e$ | $e_{12}$ | $e_{23}$ |
|---|---|---|
| $\mu_{\mathcal{K}(a_2)}(e)$ | 0.5 | 0.4 |
| $\nu_{\mathcal{K}(a_2)}(e)$ | 0.7 | 0.8 |

.

All vertex and edge degrees satisfy the Pythagorean condition. For example,

$$0.8^2 + 0.3^2 = 0.73 \leq 1,$$

and

$$0.6^2 + 0.5^2 = 0.61 \leq 1.$$

Moreover, for $a_1$ and $e_{12} = \{v_1, v_2\}$,

$$0.6 \leq \min\{0.8, 0.7\} = 0.7,$$

and

$$0.5 \geq \max\{0.3, 0.4\} = 0.4.$$

Similarly, for $e_{23} = \{v_2, v_3\}$,

$$0.5 \leq \min\{0.7, 0.6\} = 0.6,$$

and

$$0.6 \geq \max\{0.4, 0.5\} = 0.5.$$

The same conditions are satisfied for the parameter $a_2$. Therefore,

$$\widetilde{G}_P = (G^*, \mathcal{F}, \mathcal{K}, A)$$

is a Pythagorean Fuzzy Soft Graph.

## 3.29 *q*-Rung Orthopair Fuzzy Soft Graph

A q-Rung Orthopair Fuzzy Set assigns membership and non-membership degrees whose qth powers sum to at most one for element [328–331]. A q-Rung Orthopair Fuzzy Graph assigns q-rung membership and non-membership degrees to vertices and edges while preserving incidence compatibility conditions [332–334]. A $q$-Rung Orthopair Fuzzy Soft Graph parameterizes $q$-rung orthopair fuzzy graphs, representing membership and non-membership uncertainty under flexible $q$-rung constraints.

**Definition 3.29.1** ($q$-Rung Orthopair Fuzzy Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$ be a set of parameters, and let $q \geq 1$.

A *$q$-Rung Orthopair Fuzzy Soft Graph* over $G^*$ is a quadruple

$$\widetilde{G}_q = (G^*, \mathcal{F}, \mathcal{K}, A),$$

where, for every $a \in A$,

$$\mathcal{F}(a) = \left(\mu_{\mathcal{F}(a)}, \nu_{\mathcal{F}(a)}\right)$$

is a $q$-rung orthopair fuzzy set on $V$, and

$$\mathcal{K}(a) = \left(\mu_{\mathcal{K}(a)}, \nu_{\mathcal{K}(a)}\right)$$

is a $q$-rung orthopair fuzzy set on $E$.

Thus, for every $v \in V$,

$$0 \leq \mu_{\mathcal{F}(a)}(v)^q + \nu_{\mathcal{F}(a)}(v)^q \leq 1,$$

and, for every $e \in E$,

$$0 \leq \mu_{\mathcal{K}(a)}(e)^q + \nu_{\mathcal{K}(a)}(e)^q \leq 1.$$

Moreover, for every edge $\{u, v\} \in E$,

$$\mu_{\mathcal{K}(a)}(\{u, v\}) \leq \min\left\{\mu_{\mathcal{F}(a)}(u), \mu_{\mathcal{F}(a)}(v)\right\},$$

and

$$\nu_{\mathcal{K}(a)}(\{u, v\}) \geq \max\left\{\nu_{\mathcal{F}(a)}(u), \nu_{\mathcal{F}(a)}(v)\right\}.$$

Hence, for every $a \in A$,

$$(\mathcal{F}(a), \mathcal{K}(a))$$

is a $q$-rung orthopair fuzzy graph of $G^*$, and $\widetilde{G}_q$ is called a *$q$-Rung Orthopair Fuzzy Soft Graph.*

**Example 3.29.2** (3-Rung Orthopair Fuzzy Soft Graph)**.** Let

$$G^* = (V, E),$$

where

$$V = \{v_1, v_2, v_3\}$$

and

$$E = \{e_{12}, e_{23}\}, \qquad e_{12} = \{v_1, v_2\}, \qquad e_{23} = \{v_2, v_3\}.$$

Let

$$A = \{a_1, a_2\}$$

and take

$$q = 3.$$

For the parameter $a_1$, define the vertex degrees by

| $v$ | $v_1$ | $v_2$ | $v_3$ |
|---|---|---|---|
| $\mu_{\mathcal{F}(a_1)}(v)$ | 0.80 | 0.75 | 0.65 |
| $\nu_{\mathcal{F}(a_1)}(v)$ | 0.70 | 0.70 | 0.75 |

and the edge degrees by

| $e$ | $e_{12}$ | $e_{23}$ |
|---|---|---|
| $\mu_{\mathcal{K}(a_1)}(e)$ | 0.70 | 0.60 |
| $\nu_{\mathcal{K}(a_1)}(e)$ | 0.72 | 0.78 |

.

For the parameter $a_2$, define

| $v$ | $v_1$ | $v_2$ | $v_3$ |
|---|---|---|---|
| $\mu_{\mathcal{F}(a_2)}(v)$ | 0.65 | 0.85 | 0.70 |
| $\nu_{\mathcal{F}(a_2)}(v)$ | 0.70 | 0.50 | 0.65 |

and

| $e$ | $e_{12}$ | $e_{23}$ |
|---|---|---|
| $\mu_{\mathcal{K}(a_2)}(e)$ | 0.60 | 0.65 |
| $\nu_{\mathcal{K}(a_2)}(e)$ | 0.72 | 0.70 |

.

We verify the 3-rung orthopair condition. For example, for $a_1$ and $v_1$,

$$0.80^3 + 0.70^3 = 0.512 + 0.343 = 0.855 \leq 1.$$

For $a_1$ and $e_{12}$,

$$0.70^3 + 0.72^3 = 0.343 + 0.373248 = 0.716248 \leq 1.$$

The vertex–edge compatibility conditions are also satisfied. For $a_1$ and $e_{12} = \{v_1, v_2\}$,

$$0.70 \leq \min\{0.80, 0.75\} = 0.75,$$

and

$$0.72 \geq \max\{0.70, 0.70\} = 0.70.$$

For $e_{23} = \{v_2, v_3\}$,

$$0.60 \leq \min\{0.75, 0.65\} = 0.65,$$

and

$$0.78 \geq \max\{0.70, 0.75\} = 0.75.$$

Similarly, all vertex and edge degrees associated with $a_2$ satisfy the 3-rung orthopair condition and the corresponding vertex–edge compatibility conditions.

Therefore,

$$\widetilde{G}_3 = (G^*, \mathcal{F}, \mathcal{K}, A)$$

is a 3-Rung Orthopair Fuzzy Soft Graph, and hence an instance of a $q$-Rung Orthopair Fuzzy Soft Graph.

Notice that

$$0.80^2 + 0.70^2 = 1.13 > 1.$$

Thus, the degree pair $(0.80, 0.70)$ is not admissible in a Pythagorean fuzzy model, although

$$0.80^3 + 0.70^3 = 0.855 \leq 1.$$

This example therefore also illustrates the greater flexibility of the $q$-rung orthopair framework for $q > 2$.

**Theorem 3.29.3** (Generalization of Pythagorean Fuzzy Soft Graphs)**.** *Every Pythagorean Fuzzy Soft Graph is a q-Rung Orthopair Fuzzy Soft Graph for every $q \geq 2$. In particular, when $q = 2$, the notion of a q-Rung Orthopair Fuzzy Soft Graph reduces exactly to that of a Pythagorean Fuzzy Soft Graph. Furthermore, for $q > 2$, the q-rung framework is strictly more general.*

*Proof.* Let

$$\widetilde{G}_P = (G^*, \mathcal{F}, \mathcal{K}, A)$$

be a Pythagorean Fuzzy Soft Graph. Then, for every $a \in A$ and $v \in V$,

$$\mu_{\mathcal{F}(a)}(v)^2 + \nu_{\mathcal{F}(a)}(v)^2 \leq 1,$$

and, for every $e \in E$,

$$\mu_{\mathcal{K}(a)}(e)^2 + \nu_{\mathcal{K}(a)}(e)^2 \leq 1.$$

Since $0 \leq x \leq 1$ implies

$$x^q \leq x^2 \qquad \text{for every } q \geq 2,$$

we obtain

$$\mu_{\mathcal{F}(a)}(v)^q + \nu_{\mathcal{F}(a)}(v)^q \leq \mu_{\mathcal{F}(a)}(v)^2 + \nu_{\mathcal{F}(a)}(v)^2 \leq 1.$$

Similarly,

$$\mu_{\mathcal{K}(a)}(e)^q + \nu_{\mathcal{K}(a)}(e)^q \leq 1.$$

The vertex–edge compatibility conditions are unchanged. Therefore, every Pythagorean Fuzzy Soft Graph satisfies the defining conditions of a $q$-Rung Orthopair Fuzzy Soft Graph for every $q \geq 2$.

When $q = 2$, the defining condition

$$\mu^q + \nu^q \leq 1$$

becomes

$$\mu^2 + \nu^2 \leq 1,$$

which is precisely the Pythagorean fuzzy condition. Hence, Pythagorean Fuzzy Soft Graphs are exactly the $q = 2$ case.

To see that the generalization is strict for $q > 2$, choose $c \in [0, 1]$ such that

$$2^{-1/2} < c \leq 2^{-1/q}.$$

Then

$$2c^q \leq 1,$$

whereas

$$2c^2 > 1.$$

Thus, an orthopair $(c, c)$ may satisfy the $q$-rung condition while failing the Pythagorean condition. Consequently, for $q > 2$, the $q$-Rung Orthopair Fuzzy Soft Graph framework admits configurations that are not Pythagorean Fuzzy Soft Graphs. □

**Remark 3.29.4.** The relationship among the corresponding soft graph models may be summarized as

Intuitionistic Fuzzy Soft Graph $\subseteq$ Pythagorean Fuzzy Soft Graph $\subseteq$ $q$-Rung Orthopair Fuzzy Soft Graph,

where the first model is recovered at $q = 1$, the second at $q = 2$, and larger values of $q$ provide a wider admissible region for the membership and non-membership degrees.

## 3.30 $q$-Rung Orthopair Uncertain Soft Graph

A $q$-Rung Orthopair Uncertain Soft Graph represents parameterized graph uncertainty through membership, non-membership, and hesitation degrees whose $q$th powers satisfy a normalized uncertainty constraint.

**Definition 3.30.1** ($q$-Rung Orthopair Uncertain Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$ be a set of parameters, and let $q \geq 1$.

For each $a \in A$, assign to every vertex $v \in V$ three values

$$\mu_{V,a}(v), \quad \nu_{V,a}(v), \quad \pi_{V,a}(v) \in [0,1],$$

satisfying

$$\mu_{V,a}(v)^q + \nu_{V,a}(v)^q + \pi_{V,a}(v)^q = 1.$$

Similarly, assign to every edge $e \in E$ three values

$$\mu_{E,a}(e), \quad \nu_{E,a}(e), \quad \pi_{E,a}(e) \in [0,1],$$

satisfying

$$\mu_{E,a}(e)^q + \nu_{E,a}(e)^q + \pi_{E,a}(e)^q = 1.$$

Here,

$$\mu, \qquad \nu, \qquad \pi$$

represent membership, non-membership, and hesitation degrees, respectively.

For every edge $e = \{u, v\} \in E$, assume that

$$\mu_{E,a}(e)^q \leq \min\{\mu_{V,a}(u)^q, \mu_{V,a}(v)^q\},$$

and

$$\nu_{E,a}(e)^q \geq \max\{\nu_{V,a}(u)^q, \nu_{V,a}(v)^q\}.$$

Equivalently, the hesitation degrees are determined by

$$\pi_{V,a}(v)^q = 1 - \mu_{V,a}(v)^q - \nu_{V,a}(v)^q,$$

and

$$\pi_{E,a}(e)^q = 1 - \mu_{E,a}(e)^q - \nu_{E,a}(e)^q.$$

The parameterized structure

$$\widetilde{\mathcal{G}}_{qU} = (G^*, \{\mu_{V,a}, \nu_{V,a}, \pi_{V,a}\}_{a \in A}, \{\mu_{E,a}, \nu_{E,a}, \pi_{E,a}\}_{a \in A}, A)$$

is called a *$q$-Rung Orthopair Uncertain Soft Graph.*

**Theorem 3.30.2** (Well-Definedness)**.** *The q-Rung Orthopair Uncertain Soft Graph given in Definition 3.30.1 is well-defined.*

*Proof.* Fix $a \in A$.

For every $v \in V$,
$$\mu_{V,a}(v)^q + \nu_{V,a}(v)^q \leq 1.$$
Hence,
$$1 - \mu_{V,a}(v)^q - \nu_{V,a}(v)^q \geq 0,$$
and therefore
$$\pi_{V,a}(v) = (1 - \mu_{V,a}(v)^q - \nu_{V,a}(v)^q)^{1/q}$$
is a well-defined element of $[0, 1]$. Consequently,
$$\mu_{V,a}(v)^q + \nu_{V,a}(v)^q + \pi_{V,a}(v)^q = 1.$$

Likewise, for every $e \in E$,
$$\mu_{E,a}(e)^q + \nu_{E,a}(e)^q \leq 1$$
implies that
$$\pi_{E,a}(e) = (1 - \mu_{E,a}(e)^q - \nu_{E,a}(e)^q)^{1/q} \in [0, 1],$$
and hence
$$\mu_{E,a}(e)^q + \nu_{E,a}(e)^q + \pi_{E,a}(e)^q = 1.$$

Moreover, for every $e = \{u, v\} \in E$,
$$\mu_{E,a}(e)^q \leq \min \{\mu_{V,a}(u)^q, \mu_{V,a}(v)^q\},$$
and
$$\nu_{E,a}(e)^q \geq \max \{\nu_{V,a}(u)^q, \nu_{V,a}(v)^q\}.$$

Since $x \mapsto x^q$ is monotonically increasing on $[0, 1]$, these conditions are equivalent to
$$\mu_{E,a}(e) \leq \min \{\mu_{V,a}(u), \mu_{V,a}(v)\},$$
and
$$\nu_{E,a}(e) \geq \max \{\nu_{V,a}(u), \nu_{V,a}(v)\}.$$

Thus, all vertex and edge uncertainty degrees satisfy both the $q$-rung orthopair constraints and the graph incidence compatibility conditions. Therefore,
$$\widetilde{\mathcal{G}}_{qU}$$
is well-defined. □

**Remark 3.30.3.** The defining uncertainty relations are expressed entirely through $q$th powers:
$$\mu^q + \nu^q + \pi^q = 1.$$
In particular, $q = 1$ yields the intuitionistic-type constraint, whereas $q = 2$ yields the Pythagorean-type constraint. Increasing $q$ enlarges the admissible region of membership and non-membership degrees.

## Chapter 4

# Rough Graph Series

This chapter introduces rough graphs and their extensions.

$$\begin{aligned}\text{Rough Graph extensions} = \ & \underbrace{\text{approximation-based extensions}}_{\substack{\text{edge rough, vertex rough, vertex–edge rough,}\\ \text{weighted rough, and directed rough graphs}}} \\ & \cup \underbrace{\text{hybrid uncertainty extensions}}_{\substack{\text{fuzzy rough, intuitionistic fuzzy rough,}\\ \text{neutrosophic rough, soft rough,}\\ \text{and hypersoft rough graphs}}} \\ & \cup \underbrace{\text{higher-order structural extensions}}_{\substack{\text{rough hypergraphs, rough superhypergraphs,}\\ \text{rough } n\text{-superhypergraphs, and}\\ \text{recursive rough superhypergraphs}}}.\end{aligned}$$

## 4.1 Rough Graph

The term *Rough Graph* is used in the literature for several closely related graph models based on rough-set theory. Two useful formulations are distinguished in this book. The first represents a rough graph through lower and upper approximations of graph elements, whereas the second represents rough information through a rough-membership function attached to the vertices of a graph. These formulations emphasize different aspects of roughness and should not be regarded as mathematically identical.

#### Formulation I: Approximation-Based Rough Graph

An *approximation-based Rough Graph* models uncertainty in the edge set of a graph by partitioning the edges through an equivalence relation and approximating a target edge set by its lower and upper approximations [68, 335–337].

**Definition 4.1.1** (Approximation-Based Rough Graph)**.** Let

$$G = (V, E)$$

be a graph, where $V$ is the vertex set and $E$ is the edge set. Let $R_E$ be an equivalence relation on $E$, possibly induced by a collection of edge attributes. For each $e \in E$, denote its equivalence class by

$$[e]_{R_E} = \{f \in E : eR_E f\}.$$

For any target edge set $X \subseteq E$, define the lower approximation by

$$\underline{R_E}(X) = \{e \in E : [e]_{R_E} \subseteq X\},$$

and the upper approximation by

$$\overline{R_E}(X) = \{e \in E : [e]_{R_E} \cap X \neq \emptyset\}.$$

The corresponding lower- and upper-approximation graphs are

$$G_*^{R_E}(X) = \big(V, \underline{R_E}(X)\big)$$

and

$$G^*_{R_E}(X) = \big(V, \overline{R_E}(X)\big),$$

respectively.

The target edge set $X$ is called $R_E$-*exact* if

$$\underline{R_E}(X) = X = \overline{R_E}(X).$$

Otherwise, $X$ is called $R_E$-*rough*.

The ordered pair

$$\mathcal{G}_{R_E}(X) = \big(G_*^{R_E}(X), G^*_{R_E}(X)\big)$$

is called an *Approximation-Based Rough Graph determined by* $X$.

**Example 4.1.2** (A Rough Graph Determined by Edge Types)**.** Let

$$G = (V, E), \qquad V = \{v_1, v_2, v_3, v_4, v_5\},$$

with

$$E = \{e_1, e_2, e_3, e_4, e_5\},$$

where

$$e_1 = \{v_1, v_2\}, \quad e_2 = \{v_2, v_3\}, \quad e_3 = \{v_3, v_4\}, \quad e_4 = \{v_4, v_5\}, \quad e_5 = \{v_5, v_1\}.$$

Suppose that $R_E$ has the equivalence classes

$$[e_1]_{R_E} = [e_2]_{R_E} = \{e_1, e_2\}, \qquad [e_3]_{R_E} = \{e_3\}, \qquad [e_4]_{R_E} = [e_5]_{R_E} = \{e_4, e_5\}.$$

Consider

$$X = \{e_1, e_2, e_4\}.$$

Since

$$\{e_1, e_2\} \subseteq X, \qquad \{e_4, e_5\} \cap X \neq \emptyset,$$

but

$$\{e_4, e_5\} \not\subseteq X,$$

we obtain

$$\underline{R_E}(X) = \{e_1, e_2\}$$

and

$$\overline{R_E}(X) = \{e_1, e_2, e_4, e_5\}.$$

Therefore,

$$\mathcal{G}_{R_E}(X) = \left( (V, \{e_1, e_2\}), (V, \{e_1, e_2, e_4, e_5\}) \right).$$

Since

$$\underline{R_E}(X) \neq \overline{R_E}(X),$$

the target edge set $X$ is $R_E$-rough.

### Formulation II: Rough-Membership-Function-Based Rough Graph

A second formulation represents roughness quantitatively through a rough-membership function. In a finite approximation space, the rough membership of a vertex measures the proportion of its equivalence class that belongs to a prescribed target set.

**Definition 4.1.3** (Rough-Membership-Function-Based Rough Graph)**.** Let

$$G = (V, E)$$

be a finite simple graph, let $R_V$ be an equivalence relation on $V$, and let

$$X \subseteq V$$

be a target vertex set.

For every $v \in V$, let

$$[v]_{R_V} = \{u \in V : vR_V u\}$$

denote the equivalence class of $v$. Define the rough-membership function of $X$ by

$$\omega_X(v) = \frac{|[v]_{R_V} \cap X|}{|[v]_{R_V}|}, \qquad v \in V.$$

Then

$$\omega_X : V \longrightarrow [0, 1].$$

The structure

$$\mathcal{R}_X = (G, \omega_X) = (V, E, \omega_X)$$

is called a *Rough-Membership-Function-Based Rough Graph.*

In particular,

$$\omega_X(v) = 1$$

means that the entire equivalence class of $v$ belongs to $X$, whereas

$$\omega_X(v) = 0$$

means that its equivalence class does not intersect $X$. Values satisfying

$$0 < \omega_X(v) < 1$$

represent boundary or partially supported vertices.

**Example 4.1.4** (A Rough-Membership-Function-Based Rough Graph)**.** Let

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and let

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}\}.$$

Suppose that the equivalence relation $R_V$ induces the classes

$$C_1 = \{v_1, v_2\}, \qquad C_2 = \{v_3, v_4, v_5\}, \qquad C_3 = \{v_6\}.$$

Let

$$X = \{v_1, v_2, v_3, v_5\}.$$

Then

$$\omega_X(v_1) = \omega_X(v_2) = \frac{2}{2} = 1,$$

$$\omega_X(v_3) = \omega_X(v_4) = \omega_X(v_5) = \frac{2}{3},$$

and

$$\omega_X(v_6) = 0.$$

Hence,

$$\mathcal{R}_X = (V, E, \omega_X)$$

is a Rough-Membership-Function-Based Rough Graph. The vertices $v_1$ and $v_2$ belong completely to the rough target concept, $v_3, v_4, v_5$ belong to its boundary region, and $v_6$ lies outside the corresponding upper approximation.

**Remark 4.1.5** (Relationship Between the Two Formulations)**.** The two formulations describe different aspects of rough information. The approximation-based formulation represents roughness explicitly through a pair of lower and upper graphs,

$$\left(G_*^{R_E}(X), G^*_{R_E}(X)\right),$$

whereas the rough-membership formulation represents roughness through a degree function

$$\omega_X : V \to [0, 1].$$

Thus, schematically,

$$\text{Approximation-Based Rough Graph} \quad \Longrightarrow \quad \text{lower/upper graph pair,}$$

whereas

$$\text{Rough-Membership-Function-Based Rough Graph} \quad \Longrightarrow \quad \text{graph with rough membership degrees.}$$

These notions are related through rough-set theory, but they are not identical constructions. Therefore, the intended formulation should be stated explicitly whenever the term "Rough Graph" is used.

## 4.2 Rough HyperGraph

A rough hypergraph employs lower and upper approximations to model uncertain vertices and hyperedges, distinguishing certain, possible, and boundary multiway relationships induced by indiscernibility relations [338–341].

**Definition 4.2.1** (Rough Hypergraph)**.** Let $Q$ be a nonempty finite set, and let $\varphi$ be an equivalence relation on $Q$. For $A \subseteq Q$, define the lower and upper approximations of $A$ by

$$\underline{\varphi}(A) = \{x \in Q : [x]_\varphi \subseteq A\}, \qquad \overline{\varphi}(A) = \{x \in Q : [x]_\varphi \cap A \neq \emptyset\}.$$

Let

$$\mathcal{M} \subseteq \mathcal{P}(Q) \setminus \{\emptyset\}$$

be a family of nonempty subsets of $Q$, and let $\psi$ be an equivalence relation on $\mathcal{M}$. For a target hyperedge family

$$\mathcal{D} \subseteq \mathcal{M},$$

define

$$\underline{\psi}(\mathcal{D}) = \{F \in \mathcal{M} : [F]_\psi \subseteq \mathcal{D}\},$$

and

$$\overline{\psi}(\mathcal{D}) = \{F \in \mathcal{M} : [F]_\psi \cap \mathcal{D} \neq \emptyset\}.$$

Assume that

$$\underline{\psi}(\mathcal{D}) \subseteq \mathcal{P}\big(\underline{\varphi}(A)\big) \setminus \{\emptyset\}$$

and

$$\overline{\psi}(\mathcal{D}) \subseteq \mathcal{P}\big(\overline{\varphi}(A)\big) \setminus \{\emptyset\}.$$

Then the ordered pair

$$\mathcal{R} = \left(\underline{\mathcal{R}}, \overline{\mathcal{R}}\right),$$

where

$$\underline{\mathcal{R}} = \left(\underline{\varphi}(A), \underline{\psi}(\mathcal{D})\right)$$

and

$$\overline{\mathcal{R}} = \left(\overline{\varphi}(A), \overline{\psi}(\mathcal{D})\right),$$

is called a *rough hypergraph*. The hypergraphs $\underline{\mathcal{R}}$ and $\overline{\mathcal{R}}$ are called the *lower-approximation hypergraph* and the *upper-approximation hypergraph*, respectively.

## 4.3 Rough SuperHyperGraph

A rough SuperHyperGraph applies lower and upper approximations to supervertices and superedges, distinguishing definite, possible, and boundary higher-order structures under indiscernibility relations within complex networks [342].

**Definition 4.3.1** (Rough SuperHyperGraph)**.** [342] Let $S$ be a nonempty finite set, and let

$$\mathcal{V} \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$$

be a family of supervertices. Let

$$\mathcal{M} \subseteq \mathcal{P}(\mathcal{V}) \setminus \{\emptyset\}$$

be a family of candidate superedges.

Let $\rho$ and $\sigma$ be equivalence relations on $\mathcal{V}$ and $\mathcal{M}$, respectively. For $\mathcal{A} \subseteq \mathcal{V}$ and $\mathcal{B} \subseteq \mathcal{M}$, define

$$\underline{\rho}(\mathcal{A}) = \{X \in \mathcal{V} : [X]_\rho \subseteq \mathcal{A}\}, \qquad \overline{\rho}(\mathcal{A}) = \{X \in \mathcal{V} : [X]_\rho \cap \mathcal{A} \neq \emptyset\},$$

and

$$\underline{\sigma}(\mathcal{B}) = \{F \in \mathcal{M} : [F]_\sigma \subseteq \mathcal{B}\}, \qquad \overline{\sigma}(\mathcal{B}) = \{F \in \mathcal{M} : [F]_\sigma \cap \mathcal{B} \neq \emptyset\}.$$

Assume that

$$\underline{\sigma}(\mathcal{B}) \subseteq \mathcal{P}\big(\underline{\rho}(\mathcal{A})\big) \setminus \{\emptyset\}$$

and

$$\overline{\sigma}(\mathcal{B}) \subseteq \mathcal{P}\big(\overline{\rho}(\mathcal{A})\big) \setminus \{\emptyset\}.$$

Then

$$\mathfrak{R} = \big(\underline{\mathfrak{R}}, \overline{\mathfrak{R}}\big),$$

where

$$\underline{\mathfrak{R}} = \big(\underline{\rho}(\mathcal{A}), \underline{\sigma}(\mathcal{B})\big)$$

and

$$\overline{\mathfrak{R}} = \left(\overline{\rho}(\mathcal{A}), \overline{\sigma}(\mathcal{B})\right),$$

is called a *rough SuperHyperGraph*. The structures $\underline{\mathfrak{R}}$ and $\overline{\mathfrak{R}}$ are called its *lower-approximation SuperHyperGraph* and *upper-approximation SuperHyperGraph*, respectively.

## 4.4 Recursive Rough SuperHyperGraph

A Recursive Rough SuperHyperGraph applies lower and upper approximations to recursively nested supervertices and superedges, representing certain and possible structures.

**Definition 4.4.1** (Recursive Rough SuperHyperGraph)**.** Let $r \geq 0$, let $V_0$ be a finite nonempty set, and recursively define

$$V_k \subseteq \mathcal{P}(V_{k-1}) \setminus \{\emptyset\}, \qquad k = 1, \ldots, r.$$

Put

$$V^* := \bigcup_{k=0}^{r} V_k, \qquad \mathcal{G} = (V^*, E), \qquad E \subseteq \mathcal{P}(V^*) \setminus \{\emptyset\}.$$

A subset $X \subseteq V^*$ is called *recursively closed* if

$$x \in X \cap V_k \implies x \subseteq X \cap V_{k-1}, \qquad k = 1, \ldots, r.$$

For $Y \subseteq V^*$, let

$$\operatorname{rker}(Y)$$

denote the largest recursively closed subset of $Y$, and let

$$\operatorname{rcl}(Y)$$

denote the smallest recursively closed subset containing $Y$.

Let $\sim_V$ and $\sim_E$ be equivalence relations on $V^*$ and $E$, respectively, where $\sim_V$ preserves levels. For a recursive sub-SuperHyperGraph

$$\mathcal{H} = (X, F), \qquad F \subseteq \{e \in E : e \subseteq X\},$$

define

$$\underline{X} = \operatorname{rker}\big(\{x \in V^* : [x]_{\sim_V} \subseteq X\}\big),$$

$$\overline{X} = \mathrm{rcl}\big(\{x \in V^* : [x]_{\sim_V} \cap X \neq \emptyset\}\big),$$

and

$$\underline{F} = \{e \in E : [e]_{\sim_E} \subseteq F \text{ and } e \subseteq \underline{X}\},$$
$$\overline{F} = \big\{e \in E : [e]_{\sim_E} \cap F \neq \emptyset \text{ and } e \subseteq \overline{X}\big\}.$$

The pair

$$\mathcal{R}(\mathcal{H}) = \big(\underline{\mathcal{H}}, \overline{\mathcal{H}}\big),$$

where

$$\underline{\mathcal{H}} = (\underline{X}, \underline{F}), \qquad \overline{\mathcal{H}} = (\overline{X}, \overline{F}),$$

is called a *Recursive Rough SuperHyperGraph.*

## 4.5 Fuzzy Rough Graph

A fuzzy rough set combines fuzzy membership with rough approximations, representing uncertain concepts through graded lower and upper approximation boundaries [343–345]. A fuzzy rough graph combines fuzzy membership with rough lower and upper approximations, representing uncertain vertices and edges through graded inclusion and indiscernibility relations simultaneously [333, 346–348]. Related concepts, such as Fuzzy Rough Digraphs, are also known [349–351].

**Definition 4.5.1** (Fuzzy Rough Graph)**.** Let $V$ be a nonempty finite set and let

$$E^* = \binom{V}{2}.$$

Let $R : V \times V \to [0,1]$ and $S : E^* \times E^* \to [0,1]$ be fuzzy tolerance relations. Let

$$\mu : V \to [0,1] \qquad \text{and} \qquad \nu : E^* \to [0,1]$$

be fuzzy sets of vertices and edges, respectively.

The lower and upper approximations of $\mu$ with respect to $R$ are defined by

$$\underline{R}(\mu)(x) = \bigwedge_{y \in V} \left((1 - R(x,y)) \vee \mu(y)\right),$$

and

$$\overline{R}(\mu)(x) = \bigvee_{y \in V} \left(R(x,y) \wedge \mu(y)\right), \qquad x \in V.$$

Similarly, the lower and upper approximations of $\nu$ with respect to $S$ are defined by

$$\underline{S}(\nu)(e) = \bigwedge_{f \in E^*} \left((1 - S(e,f)) \vee \nu(f)\right),$$

and

$$\overline{S}(\nu)(e) = \bigvee_{f \in E^*} \left(S(e,f) \wedge \nu(f)\right), \qquad e \in E^*.$$

Assume that, for every $e = \{x, y\} \in E^*$,

$$\underline{S}(\nu)(e) \leq \underline{R}(\mu)(x) \wedge \underline{R}(\mu)(y),$$

and

$$\overline{S}(\nu)(e) \leq \overline{R}(\mu)(x) \wedge \overline{R}(\mu)(y).$$

Then

$$\mathcal{G} = \left(\underline{\mathcal{G}}, \overline{\mathcal{G}}\right),$$

where

$$\underline{\mathcal{G}} = (\underline{R}(\mu), \underline{S}(\nu))$$

and

$$\overline{\mathcal{G}} = \left(\overline{R}(\mu), \overline{S}(\nu)\right),$$

is called a *fuzzy rough graph*. The fuzzy graphs $\underline{\mathcal{G}}$ and $\overline{\mathcal{G}}$ are called the *lower-approximation graph* and the *upper-approximation graph*, respectively.

**Example 4.5.2** (A Fuzzy Rough Triangle)**.** Let

$$V = \{v_1, v_2, v_3\}, \qquad E^* = \{e_{12}, e_{23}, e_{13}\},$$

where $e_{ij} = \{v_i, v_j\}$. Define universal fuzzy tolerance relations by

$$R(v_i, v_j) = 1 \qquad \text{and} \qquad S(e, f) = 1$$

for all $v_i, v_j \in V$ and $e, f \in E^*$.

Let the vertex-membership function be

$$\mu(v_1) = 0.8, \qquad \mu(v_2) = 0.6, \qquad \mu(v_3) = 0.9,$$

and let the edge-membership function be

$$\nu(e_{12}) = 0.5, \qquad \nu(e_{23}) = 0.4, \qquad \nu(e_{13}) = 0.6.$$

Since $R$ is universal, for every $x \in V$,

$$\underline{R}(\mu)(x) = \min_{y \in V} \mu(y) = 0.6,$$

whereas

$$\overline{R}(\mu)(x) = \max_{y \in V} \mu(y) = 0.9.$$

Similarly, for every $e \in E^*$,

$$\underline{S}(\nu)(e) = \min_{f \in E^*} \nu(f) = 0.4,$$

and

$$\overline{S}(\nu)(e) = \max_{f \in E^*} \nu(f) = 0.6.$$

Consequently, for every edge $e = \{x, y\} \in E^*$,

$$0.4 \leq \min\{0.6, 0.6\}$$

and

$$0.6 \leq \min\{0.9, 0.9\}.$$

Hence

$$\underline{\mathcal{G}} = \left(\underline{R}(\mu), \underline{S}(\nu)\right)$$

and

$$\overline{\mathcal{G}} = \left(\overline{R}(\mu), \overline{S}(\nu)\right)$$

are well-defined fuzzy graphs. Since their membership values differ,

$$\mathcal{G} = \left(\underline{\mathcal{G}}, \overline{\mathcal{G}}\right)$$

is a non-exact fuzzy rough graph.

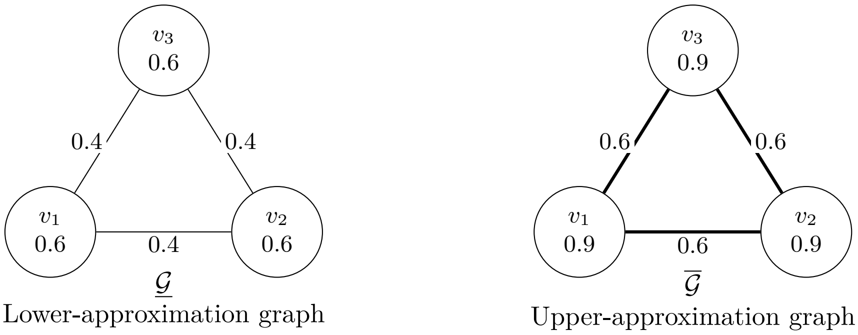



Figure 4.1: The lower- and upper-approximation fuzzy graphs. The values inside the vertices and beside the edges are their membership degrees.

## 4.6 Intuitionistic Fuzzy Rough Graph

An intuitionistic fuzzy rough set uses membership and non-membership degrees with lower and upper approximations to represent uncertain, hesitant concepts [352–354]. An intuitionistic fuzzy rough graph combines membership, non-membership, and hesitation degrees with lower and upper approximations to model uncertain vertices and edges under indiscernibility relations [355–357].

**Definition 4.6.1** (Intuitionistic Fuzzy Rough Graph)**.** Let $V$ be a nonempty finite set and let

$$E^* = \big\{\{x, y\} \subseteq V : x \neq y\big\}.$$

Let

$$\underline{A} = \big(\underline{\mu}_V, \underline{\nu}_V\big), \qquad \overline{A} = \big(\overline{\mu}_V, \overline{\nu}_V\big)$$

be the lower and upper intuitionistic fuzzy approximations of a vertex set on $V$, and let

$$\underline{B} = \big(\underline{\mu}_E, \underline{\nu}_E\big), \qquad \overline{B} = \big(\overline{\mu}_E, \overline{\nu}_E\big)$$

be the corresponding lower and upper intuitionistic fuzzy approximations of an edge set on $E^*$.

Assume that, for every $x \in V$,

$$0 \leq \underline{\mu}_V(x) + \underline{\nu}_V(x) \leq 1, \qquad 0 \leq \overline{\mu}_V(x) + \overline{\nu}_V(x) \leq 1,$$

and, for every $e \in E^*$,

$$0 \leq \underline{\mu}_E(e) + \underline{\nu}_E(e) \leq 1, \qquad 0 \leq \overline{\mu}_E(e) + \overline{\nu}_E(e) \leq 1.$$

Moreover, for every edge $e = \{x, y\} \in E^*$, suppose that

$$\underline{\mu}_E(e) \leq \min\big\{\underline{\mu}_V(x), \underline{\mu}_V(y)\big\},$$
$$\underline{\nu}_E(e) \geq \max\big\{\underline{\nu}_V(x), \underline{\nu}_V(y)\big\},$$

and

$$\overline{\mu}_E(e) \leq \min\big\{\overline{\mu}_V(x), \overline{\mu}_V(y)\big\},$$
$$\overline{\nu}_E(e) \geq \max\big\{\overline{\nu}_V(x), \overline{\nu}_V(y)\big\}.$$

Then

$$\mathcal{G} = \big(\underline{\mathcal{G}}, \overline{\mathcal{G}}\big),$$

where

$$\underline{\mathcal{G}} = (\underline{A}, \underline{B}) \qquad \text{and} \qquad \overline{\mathcal{G}} = \big(\overline{A}, \overline{B}\big),$$

is called an *intuitionistic fuzzy rough graph.* The graphs $\underline{\mathcal{G}}$ and $\overline{\mathcal{G}}$ are called its *lower-approximation intuitionistic fuzzy graph* and *upper-approximation intuitionistic fuzzy graph*, respectively.

## 4.7 Neutrosophic Rough Graph

A neutrosophic rough set combines truth, indeterminacy, and falsity degrees with lower and upper approximations to model inconsistent uncertain information [358–360]. A neutrosophic rough graph combines truth, indeterminacy, and falsity degrees with lower and upper approximations, representing uncertain vertices and edges under indiscernibility relations more flexibly [361].

**Definition 4.7.1** (Neutrosophic Rough Graph)**.** Let $V$ be a nonempty finite set, and let

$$E^* = \big\{\{x, y\} \subseteq V : x \neq y\big\}.$$

Let

$$\underline{A} = \big(\underline{T}_V, \underline{I}_V, \underline{F}_V\big), \qquad \overline{A} = \big(\overline{T}_V, \overline{I}_V, \overline{F}_V\big)$$

be the lower and upper neutrosophic approximations on $V$, and let

$$\underline{B} = \big(\underline{T}_E, \underline{I}_E, \underline{F}_E\big), \qquad \overline{B} = \big(\overline{T}_E, \overline{I}_E, \overline{F}_E\big)$$

be the corresponding lower and upper neutrosophic approximations on $E^*$, where all component functions take values in $[0, 1]$.

Assume that, for every edge $e = \{x, y\} \in E^*$,

$$\underline{T}_E(e) \leq \min\{\underline{T}_V(x), \underline{T}_V(y)\},$$

$$\underline{I}_E(e) \leq \max\{\underline{I}_V(x), \underline{I}_V(y)\}, \qquad \underline{F}_E(e) \leq \max\{\underline{F}_V(x), \underline{F}_V(y)\},$$

and

$$\overline{T}_E(e) \leq \min\{\overline{T}_V(x), \overline{T}_V(y)\},$$

$$\overline{I}_E(e) \leq \max\{\overline{I}_V(x), \overline{I}_V(y)\}, \qquad \overline{F}_E(e) \leq \max\{\overline{F}_V(x), \overline{F}_V(y)\}.$$

Then

$$\mathcal{G} = \big(\underline{\mathcal{G}}, \overline{\mathcal{G}}\big),$$

where

$$\underline{\mathcal{G}} = (\underline{A}, \underline{B}) \qquad \text{and} \qquad \overline{\mathcal{G}} = \big(\overline{A}, \overline{B}\big),$$

is called a *neutrosophic rough graph.* The neutrosophic graphs $\underline{\mathcal{G}}$ and $\overline{\mathcal{G}}$ are called the *lower-approximation graph* and the *upper-approximation graph*, respectively.

## 4.8 Plithogenic Rough Graph

A plithogenic rough set combines appurtenance and contradiction degrees with lower and upper approximations to model heterogeneous uncertain information systematically [362]. A Plithogenic Rough Graph combines plithogenic appurtenance and contradiction degrees with lower and upper approximations of uncertain vertices and edges.

**Definition 4.8.1** (Plithogenic Rough Graph)**.** Let

$$G = (V, E)$$

be a finite simple graph, and let $\sim_V$ and $\sim_E$ be equivalence relations on $V$ and $E$, respectively. Let $A \neq \emptyset$ be a finite attribute-value set, and let

$$\mu_V : V \times A \longrightarrow [0, 1], \qquad \mu_E : E \times A \longrightarrow [0, 1]$$

be degree-of-appurtenance functions. Let

$$\delta : A \times A \longrightarrow [0, 1]$$

be a degree-of-contradiction function satisfying

$$\delta(a, a) = 0, \qquad \delta(a, b) = \delta(b, a).$$

For every $e = \{u, v\} \in E$ and $a \in A$, require

$$\mu_E(e, a) \leq \min\{\mu_V(u, a), \mu_V(v, a)\}.$$

Let

$$H = (X, F), \qquad X \subseteq V, \qquad F \subseteq E \cap \binom{X}{2}.$$

Define

$$\underline{X} = \{v \in V : [v]_{\sim_V} \subseteq X\}, \qquad \overline{X} = \{v \in V : [v]_{\sim_V} \cap X \neq \emptyset\},$$

and

$$\underline{F} = \{e \in E : [e]_{\sim_E} \subseteq F,\ e \subseteq \underline{X}\},$$
$$\overline{F} = \{e \in E : [e]_{\sim_E} \cap F \neq \emptyset,\ e \subseteq \overline{X}\}.$$

The pair

$$\mathcal{R}_P(H) = \big((\underline{X}, \underline{F}, \mu_V|_{\underline{X} \times A}, \mu_E|_{\underline{F} \times A}, \delta), (\overline{X}, \overline{F}, \mu_V|_{\overline{X} \times A}, \mu_E|_{\overline{F} \times A}, \delta)\big)$$

is called a *Plithogenic Rough Graph*.

## 4.9 Uncertain Rough Graph

An Uncertain Rough Graph applies lower and upper approximations to uncertainty-weighted vertices and edges, distinguishing certain, possible, and boundary structures.

**Definition 4.9.1** (Uncertain Rough Graph)**.** Let

$$G = (V, E)$$

be a finite simple graph, and let $(L, \preceq, \wedge)$ be a meet-semilattice of uncertainty values. Let

$$\omega_V : V \longrightarrow L, \qquad \omega_E : E \longrightarrow L$$

satisfy

$$\omega_E(\{u, v\}) \preceq \omega_V(u) \wedge \omega_V(v)$$

for every $\{u, v\} \in E$.

Let $\sim_V$ and $\sim_E$ be equivalence relations on $V$ and $E$, respectively, and let

$$H = (X, F), \qquad X \subseteq V, \qquad F \subseteq E \cap \binom{X}{2}.$$

Define

$$\underline{X} = \{v \in V : [v]_{\sim_V} \subseteq X\}, \qquad \overline{X} = \{v \in V : [v]_{\sim_V} \cap X \neq \emptyset\},$$
$$\underline{F} = \{e \in E : [e]_{\sim_E} \subseteq F,\ e \subseteq \underline{X}\},$$

and

$$\overline{F} = \{e \in E : [e]_{\sim_E} \cap F \neq \emptyset,\ e \subseteq \overline{X}\}.$$

The pair

$$\mathcal{R}_U(H) = \big((\underline{X}, \underline{F}, \omega_V|_{\underline{X}}, \omega_E|_{\underline{F}}), (\overline{X}, \overline{F}, \omega_V|_{\overline{X}}, \omega_E|_{\overline{F}})\big)$$

is called an *Uncertain Rough Graph.*

**Remark 4.9.2** (Catalogue of Uncertain Rough Graph Families)**.** For reference, Table 4.1 presents a catalogue of uncertainty-rough-graph families, called Uncertain Rough Graphs, classified according to the dimension $k$ of the underlying uncertainty-degree domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Table 4.1: A catalogue of Uncertain Rough Graph families organized by the dimension $k$ of the underlying degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

| $k$ | Representative Uncertain Rough Graph type(s) $\mathcal{R}_M(H) = (\underline{\mathcal{G}}_M(H), \overline{\mathcal{G}}_M(H))$ induced by uncertainty-degree maps $\mu_{V,M} : V \to \mathrm{Dom}(M)$ and $\mu_{E,M} : E \to \mathrm{Dom}(M)$ |
|---|---|
| 1 | Fuzzy rough graph [347,363]; $N$-fuzzy rough graph; shadowed rough graph variants |
| 2 | Intuitionistic fuzzy rough graph [364–366]; vague rough graph; bipolar fuzzy rough graph [367]; intuitionistic evidence rough graph; variable fuzzy rough graph; paraconsistent fuzzy rough graph; bifuzzy rough graph |
| 3 | Neutrosophic rough graph [361][(a)]; hesitant fuzzy rough graph[(d)]; tripolar fuzzy rough graph; three-way fuzzy rough graph; picture fuzzy rough graph; spherical fuzzy rough graph [333]; inconsistent intuitionistic fuzzy rough graph; ternary fuzzy rough graph; neutrosophic-fuzzy rough graph; neutrosophic vague rough graph |
| 4 | Quadripartitioned neutrosophic rough graph; double-valued neutrosophic rough graph; dual hesitant fuzzy rough graph[(d)]; ambiguous rough graph[(b)]; local-neutrosophic rough graph; support-neutrosophic rough graph; Turiyam neutrosophic rough graph[(c)] |
| 5 | Pentapartitioned neutrosophic rough graph; triple-valued neutrosophic rough graph |
| 6 | Hexapartitioned neutrosophic rough graph; quadruple-valued neutrosophic rough graph |
| 7 | Heptapartitioned neutrosophic rough graph; quintuple-valued neutrosophic rough graph |
| 8 | Octapartitioned neutrosophic rough graph |
| 9 | Nonapartitioned neutrosophic rough graph |
| $n$ | $n$-refined fuzzy rough graph; multi-valued fuzzy rough graph; multi-fuzzy rough graph |
| $2n$ | $n$-refined intuitionistic fuzzy rough graph; multi-intuitionistic fuzzy rough graph |
| $3n$ | $n$-refined neutrosophic rough graph; multi-neutrosophic rough graph |

(a) Neutrosophic rough graph models may be viewed as broad approximation-based frameworks capable of specializing to several degree-based rough graph models under suitable constraints.
(b) An ambiguous rough graph is obtained by applying the lower and upper graph-approximation operators to an uncertainty graph whose degrees belong to an ambiguous-set model. Ambiguous-set models are commonly related to certain four-component neutrosophic models.
(c) A Turiyam neutrosophic rough graph is induced by a four-component Turiyam neutrosophic degree model and suitable lower and upper approximations. Such models are commonly regarded as specializations of quadripartitioned neutrosophic frameworks.
(d) Hesitant and dual hesitant fuzzy degrees are naturally set-valued. Their placement in a fixed-dimensional row presupposes a selected finite-coordinate representation. Otherwise, their degree-domains need not be subsets of a single finite-dimensional cube $[0,1]^k$.
*Dimension note.* The integer $k$ records the dimension of the underlying uncertainty degree assigned to each vertex or edge. It does not count the lower and upper approximations as additional degree components. Thus, the rough pair

$$(\underline{\mathcal{G}}_M(H), \overline{\mathcal{G}}_M(H))$$

remains classified according to the dimension of $\mathrm{Dom}(M)$, rather than according to $2k$.

## 4.10 Soft Rough Graph

A soft rough set combines soft-set parameterization with rough lower and upper approximations to represent parameter-dependent uncertainty in information systems [368–370]. Related concepts, such as hypersoft rough sets [371–376], are also known in the literature. A soft rough graph combines soft-set parameterization with lower and upper approximations to represent uncertain vertices, edges, parameter-dependent subgraphs, and boundary information systematically within networks [377–379].

**Definition 4.10.1** (Soft Rough Graph). [377–379] Let

$$G = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$ be a set of parameters, and let

$$F : A \longrightarrow \mathcal{P}(V), \qquad K : A \longrightarrow \mathcal{P}(E)$$

be set-valued mappings such that

$$K(a) \subseteq E \cap \binom{F(a)}{2} \qquad \text{for every } a \in A.$$

Thus, each

$$G_a = (F(a), K(a))$$

is a subgraph of $G$.

For a nonempty target set $X \subseteq V$, define

$$A_*(X) = \{a \in A : F(a) \subseteq X\}$$

and

$$A^*(X) = \{a \in A : F(a) \cap X \neq \emptyset\}.$$

The lower approximations of the vertex and edge sets are

$$F_*(X) = \bigcup_{a \in A_*(X)} F(a), \qquad K_*(X) = \bigcup_{a \in A_*(X)} K(a),$$

whereas their upper approximations are

$$F^*(X) = \bigcup_{a \in A^*(X)} F(a), \qquad K^*(X) = \bigcup_{a \in A^*(X)} K(a).$$

Then

$$\widetilde{G}_X = (G_*(X), G^*(X)),$$

where

$$G_*(X) = \big(F_*(X), K_*(X)\big)$$

and

$$G^*(X) = \big(F^*(X), K^*(X)\big),$$

is called a *soft rough graph* of $G$. The graphs $G_*(X)$ and $G^*(X)$ are called the *lower-approximation graph* and the *upper-approximation graph*, respectively.

## 4.11 Edge Rough Graph

An edge rough graph approximates an uncertain target edge set through equivalence classes, producing lower and upper subgraphs that distinguish certain, possible, and boundary edges [380].

**Definition 4.11.1** (Edge Rough Graph)**.** Let

$$G = (V, E)$$

be a finite graph, and let $R_E$ be an equivalence relation on the edge set $E$. For each $e \in E$, denote its equivalence class under $R_E$ by

$$[e]_{R_E} = \{f \in E : e\, R_E\, f\}.$$

For a target edge set $X \subseteq E$, define its lower and upper approximations by

$$\underline{R_E}(X) = \{e \in E : [e]_{R_E} \subseteq X\},$$

and

$$\overline{R_E}(X) = \{e \in E : [e]_{R_E} \cap X \neq \emptyset\}.$$

For any $F \subseteq E$, let

$$V(F) = \{v \in V : v \text{ is incident with some edge in } F\}.$$

The corresponding lower- and upper-approximation graphs are

$$G_*^E(X) = \big(V(\underline{R_E}(X)), \underline{R_E}(X)\big)$$

and

$$G_E^*(X) = \big(V(\overline{R_E}(X)), \overline{R_E}(X)\big),$$

respectively.

The ordered pair

$$\mathcal{G}_{R_E}(X) = \big(G_*^E(X), G_E^*(X)\big)$$

is called the *edge rough graph determined by* $X$. The graphs $G_*^E(X)$ and $G_E^*(X)$ are called the *lower edge-approximation graph* and the *upper edge-approximation graph*, respectively.

## 4.12 Vertex Rough Graph

A vertex rough graph approximates an uncertain target vertex set through equivalence classes, producing lower and upper subgraphs that identify certain, possible, and boundary vertices [381, 382].

**Definition 4.12.1** (Vertex Rough Graph)**.** Let

$$G = (V, E)$$

be a finite simple graph, let $R$ be an equivalence relation on $V$, and let

$$H = (W, F)$$

be a subgraph of $G$, where $W \subseteq V$ and $F \subseteq E$.

For each $v \in V$, let

$$[v]_R = \{u \in V : uRv\}$$

denote the equivalence class of $v$. The lower and upper approximations of $W$ are defined by

$$\underline{R}(W) = \{v \in V : [v]_R \subseteq W\},$$

and

$$\overline{R}(W) = \{v \in V : [v]_R \cap W \neq \emptyset\}.$$

For any $U \subseteq V$, write

$$E_G[U] = \big\{\{x, y\} \in E : x, y \in U\big\}.$$

Define the lower- and upper-approximation graphs of $H$ by

$$\underline{R}_V(H) = (\underline{R}(W), F \cap E_G[\underline{R}(W)])$$

and

$$\overline{R}_V(H) = \left(\overline{R}(W), E_G[\overline{R}(W)]\right),$$

respectively.

The ordered pair

$$\mathcal{R}_V(H) = \left(\underline{R}_V(H), \overline{R}_V(H)\right)$$

is called the *vertex rough graph of $H$ with respect to $R$*. The graphs $\underline{R}_V(H)$ and $\overline{R}_V(H)$ are called the *lower vertex-approximation graph* and the *upper vertex-approximation graph*, respectively.

**Example 4.12.2** (A Vertex Rough Graph)**.** Let

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and let

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_1\}, \{v_1, v_3\}, \{v_3, v_5\}\big\}.$$

Define an equivalence relation $R$ on $V$ by the classes

$$\{v_1, v_2\}, \qquad \{v_3, v_4\}, \qquad \{v_5, v_6\}.$$

Consider the subgraph

$$H = (W, F),$$

where

$$W = \{v_1, v_2, v_3, v_5\}$$

and

$$F = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_3, v_5\}\big\}.$$

Only the class $\{v_1, v_2\}$ is entirely contained in $W$, while every equivalence class intersects $W$. Hence

$$\underline{R}(W) = \{v_1, v_2\}, \qquad \overline{R}(W) = V.$$

Therefore,

$$\underline{R}_V(H) = \left(\{v_1, v_2\}, \big\{\{v_1, v_2\}\big\}\right)$$

and

$$\overline{R}_V(H) = (V, E).$$

Thus,

$$\mathcal{R}_V(H) = \left(\underline{R}_V(H), \overline{R}_V(H)\right)$$

is a non-exact vertex rough graph.

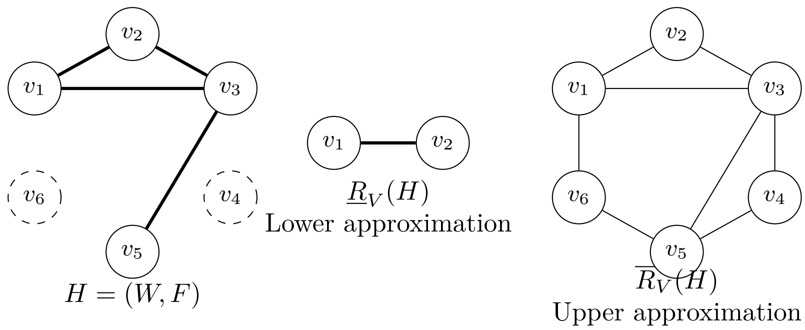



Figure 4.2: A target subgraph and its lower and upper vertex-approximation graphs. Dashed vertices in the first panel belong to the ambient graph but not to $W$.

## 4.13 Weighted Rough Graph

A weighted rough graph integrates edge or vertex weights with lower and upper approximations, preserving quantitative network information while distinguishing certain, possible, and boundary structures [67, 341, 383].

**Definition 4.13.1** (Weighted Rough Graph)**.** Let

$$G = (V, E, w)$$

be a finite weighted graph, where

$$w : E \longrightarrow \mathbb{R}_{\geq 0}$$

is an edge-weight function, and let $R$ be an equivalence relation on $E$.

For a target edge set $X \subseteq E$, define its lower and upper approximations by

$$\underline{R}(X) = \{e \in E : [e]_R \subseteq X\}, \qquad \overline{R}(X) = \{e \in E : [e]_R \cap X \neq \emptyset\}.$$

For $F \subseteq E$, let

$$V(F) = \{v \in V : v \text{ is incident with an edge of } F\}.$$

The lower- and upper-approximation graphs are

$$G_*^R(X) = \big(V(\underline{R}(X)), \underline{R}(X)\big)$$

and

$$G_R^*(X) = \big(V(\overline{R}(X)), \overline{R}(X)\big),$$

respectively.

Let

$$\Phi : \{\text{finite multisets in } \mathbb{R}_{\geq 0}\} \longrightarrow \mathbb{R}_{\geq 0}$$

be a fixed aggregation function. For each equivalence class $C \in E/R$, define its *class weight* by

$$\omega_R(C) = \Phi\big(\{w(e) : e \in C\}\big).$$

Then

$$\mathcal{G}_{R,w}(X) = \big(G_*^R(X), G_R^*(X), \omega_R\big)$$

is called a *weighted rough graph*.

**Example 4.13.2** (A Weighted Rough Transportation Network)**.** Let

$$G = (V, E, w), \qquad V = \{v_1, v_2, v_3, v_4, v_5\},$$

where

$$\begin{aligned} e_1 &= \{v_1, v_2\}, \quad w(e_1) = 2, \\ e_2 &= \{v_2, v_3\}, \quad w(e_2) = 4, \\ e_3 &= \{v_3, v_4\}, \quad w(e_3) = 3, \\ e_4 &= \{v_4, v_5\}, \quad w(e_4) = 7, \\ e_5 &= \{v_5, v_1\}, \quad w(e_5) = 5. \end{aligned}$$

Assume that $R$ has the equivalence classes

$$C_1 = \{e_1, e_2\}, \qquad C_2 = \{e_3\}, \qquad C_3 = \{e_4, e_5\}.$$

For the target edge set

$$X = \{e_1, e_2, e_4\},$$

we obtain

$$\underline{R}(X) = \{e_1, e_2\}, \qquad \overline{R}(X) = \{e_1, e_2, e_4, e_5\}.$$

Hence

$$G_*^R(X) = \big(\{v_1, v_2, v_3\}, \{e_1, e_2\}\big)$$

and

$$G_R^*(X) = \big(V, \{e_1, e_2, e_4, e_5\}\big).$$

Choose the arithmetic-mean aggregator

$$\Phi(\{x_1, \ldots, x_k\}) = \frac{1}{k}\sum_{i=1}^{k} x_i.$$

The corresponding class weights are

$$\omega_R(C_1) = 3, \qquad \omega_R(C_2) = 3, \qquad \omega_R(C_3) = 6.$$

Therefore,

$$\mathcal{G}_{R,w}(X) = \big(G_*^R(X), G_R^*(X), \omega_R\big)$$

is a weighted rough graph.

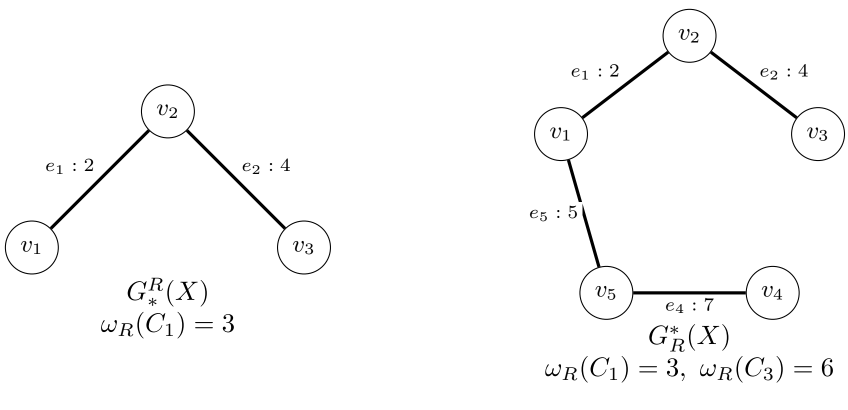



Figure 4.3: The lower and upper approximation graphs. Edge labels display the original edge and its weight.

## 4.14 Rough Digraph

A rough digraph applies lower and upper approximations to directed vertices or arcs, representing definite, possible, and boundary relationships while preserving orientation within uncertain networks [367, 384–386].

**Definition 4.14.1** (Rough Digraph)**.** Let

$$D = (V, A)$$

be a finite digraph, where

$$A \subseteq V \times V,$$

and let

$$H = (W, B)$$

be a subdigraph of $D$, where

$$W \subseteq V, \qquad B \subseteq A \cap (W \times W).$$

Let $R_V$ and $R_A$ be equivalence relations on $V$ and $A$, respectively. For $x \in V$ and $a \in A$, denote their equivalence classes by

$$[x]_{R_V} = \{y \in V : xR_V y\}$$

and

$$[a]_{R_A} = \{b \in A : aR_A b\}.$$

Define the lower and upper approximations of $W$ by

$$\underline{R_V}(W) = \{x \in V : [x]_{R_V} \subseteq W\},$$

$$\overline{R_V}(W) = \{x \in V : [x]_{R_V} \cap W \neq \emptyset\},$$

and those of $B$ by

$$\underline{R_A}(B) = \{a \in A : [a]_{R_A} \subseteq B\},$$

$$\overline{R_A}(B) = \{a \in A : [a]_{R_A} \cap B \neq \emptyset\}.$$

For $U \subseteq V$, let

$$A_D[U] = A \cap (U \times U).$$

The lower- and upper-approximation digraphs of $H$ are defined by

$$\underline{\mathcal{R}}(H) = \left(\underline{R_V}(W), \underline{R_A}(B) \cap A_D[\underline{R_V}(W)]\right)$$

and

$$\overline{\mathcal{R}}(H) = \left(\overline{R_V}(W), \overline{R_A}(B) \cap A_D[\overline{R_V}(W)]\right),$$

respectively.

The ordered pair

$$\mathcal{R}(H) = \left(\underline{\mathcal{R}}(H), \overline{\mathcal{R}}(H)\right)$$

is called the *rough digraph of* $H$ *with respect to* $(R_V, R_A)$.

## 4.15 Soft-Covering-Based Rough Graph

A soft-covering-based rough graph derives lower and upper graph approximations from parameterized coverings, identifying definite, possible, and boundary vertices or edges without requiring equivalence relations [387, 388].

**Definition 4.15.1** (Soft-Covering-Based Rough Graph)**.** Let

$$G = (V, E)$$

be a finite simple graph, and let $A \neq \emptyset$ be a finite parameter set. Suppose that

$$\lambda : A \longrightarrow \mathcal{P}(V) \setminus \{\emptyset\}, \qquad \mu : A \longrightarrow \mathcal{P}(E) \setminus \{\emptyset\}$$

satisfy

$$\bigcup_{a \in A} \lambda(a) = V, \qquad \bigcup_{a \in A} \mu(a) = E,$$

and

$$\mu(a) \subseteq E_G[\lambda(a)] \qquad \text{for every } a \in A,$$

where, for $U \subseteq V$,

$$E_G[U] = \big\{\{x, y\} \in E : x, y \in U\big\}.$$

Thus,

$$\mathcal{C}_V = \{\lambda(a) : a \in A\} \qquad \text{and} \qquad \mathcal{C}_E = \{\mu(a) : a \in A\}$$

are coverings of $V$ and $E$, respectively.

For $v \in V$, define its minimal vertex descriptions by

$$\mathrm{Md}_\lambda(v) = \left\{ \lambda(a) : \begin{array}{l} v \in \lambda(a), \text{ and there is no } b \in A \\ \text{such that } v \in \lambda(b) \subsetneq \lambda(a) \end{array} \right\}.$$

Similarly, for $e \in E$, define

$$\mathrm{Md}_\mu(e) = \left\{ \mu(a) : \begin{array}{l} e \in \mu(a), \text{ and there is no } b \in A \\ \text{such that } e \in \mu(b) \subsetneq \mu(a) \end{array} \right\}.$$

Let

$$H = (X, Y)$$

be a subgraph of $G$, where

$$X \subseteq V, \qquad Y \subseteq E_G[X].$$

Define the lower and upper soft-covering approximations of $X$ by

$$\underline{\lambda}(X) = \bigcup_{\substack{a \in A \\ \lambda(a) \subseteq X}} \lambda(a),$$

and

$$\overline{\lambda}(X) = \bigcup_{v \in X} \bigcup \mathrm{Md}_\lambda(v).$$

Similarly, define the lower and upper soft-covering approximations of $Y$ by

$$\underline{\mu}(Y) = \bigcup_{\substack{a \in A \\ \mu(a) \subseteq Y}} \mu(a),$$

and

$$\overline{\mu}(Y) = \bigcup_{e \in Y} \bigcup \mathrm{Md}_\mu(e).$$

The corresponding lower- and upper-approximation graphs are

$$\underline{\mathcal{R}}(H) = \left(\underline{\lambda}(X), \underline{\mu}(Y) \cap E_G[\underline{\lambda}(X)]\right)$$

and

$$\overline{\mathcal{R}}(H) = \left(\overline{\lambda}(X), \overline{\mu}(Y) \cap E_G[\overline{\lambda}(X)]\right),$$

respectively.

If

$$\underline{\lambda}(X) \neq \overline{\lambda}(X) \qquad \text{and} \qquad \underline{\mu}(Y) \neq \overline{\mu}(Y),$$

then the ordered pair

$$\mathcal{R}_{\lambda,\mu}(H) = \left(\underline{\mathcal{R}}(H), \overline{\mathcal{R}}(H)\right)$$

is called the *soft-covering-based rough graph of $H$*.

# Chapter 5

# Special Classes of Soft Graphs

This chapter introduces several special classes of Soft Graphs.

## 5.1 Soft Tree

A Soft Tree is a special class of Soft Graph in which every parameter-dependent graph is a tree. This concept provides a parameterized representation of connected and acyclic graph structures [207, 307, 389].

**Definition 5.1.1** (Soft Tree)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$ be a parameter set, and let

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

be a Soft Graph over $G^*$. For each $a \in A$, write

$$\Phi(a) = G_a = (V_a, E_a),$$

where

$$V_a \subseteq V, \qquad E_a \subseteq E \cap \binom{V_a}{2}.$$

The Soft Graph $\Phi$ is called a *Soft Tree* if, for every $a \in A$, the graph $G_a$ is a tree.

Equivalently, for every $a \in A$,

$$G_a \text{ is connected}$$

and

$$G_a \text{ contains no cycle.}$$

Thus, each parameter $a$ determines a tree contained in the ambient graph $G^*$.

**Example 5.1.2** (A Soft Tree)**.** Let the ambient graph be

$$G^* = (V, E),$$

where

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\big\}.$$

Let

$$A = \{a_1, a_2, a_3\}$$

be the parameter set. Define the Soft Graph

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

as follows:

$$\Phi(a_1) = G_{a_1} = (V_{a_1}, E_{a_1}),$$

where

$$V_{a_1} = \{v_1, v_2, v_3\}, \qquad E_{a_1} = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\};$$

$$\Phi(a_2) = G_{a_2} = (V_{a_2}, E_{a_2}),$$

where

$$V_{a_2} = \{v_2, v_3, v_4\}, \qquad E_{a_2} = \big\{\{v_2, v_3\}, \{v_3, v_4\}\big\};$$

and

$$\Phi(a_3) = G_{a_3} = (V_{a_3}, E_{a_3}),$$

where

$$V_{a_3} = \{v_3, v_4, v_5\}, \qquad E_{a_3} = \big\{\{v_3, v_4\}, \{v_3, v_5\}\big\}.$$

Each parameter-dependent graph is connected and contains no cycle. Indeed,

$$G_{a_1} \cong P_3, \qquad G_{a_2} \cong P_3, \qquad G_{a_3} \cong P_3,$$

where $P_3$ denotes the path graph on three vertices. In particular, each $G_{a_i}$ has three vertices and two edges, and is connected. Hence every $G_{a_i}$ is a tree.

Therefore,

$$\Phi = \{G_{a_1}, G_{a_2}, G_{a_3}\}$$

is a Soft Tree over $G^*$.

## 5.2 Complete Soft Graph

A complete graph is a graph in which every pair of distinct vertices is connected by exactly one edge to each other (cf. [390–392]). A Complete Soft Graph is a Soft Graph whose parameter-dependent graph is complete for every parameter [393–395].

**Definition 5.2.1** (Complete Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$, and let

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

be a Soft Graph over $G^*$. For each $a \in A$, write

$$\Phi(a) = G_a = (V_a, E_a).$$

The Soft Graph $\Phi$ is called a *Complete Soft Graph* if $G_a$ is a complete graph for every $a \in A$.

Equivalently,

$$E_a = \binom{V_a}{2} \qquad \text{for every } a \in A.$$

Hence, for each parameter $a$, every pair of distinct vertices $u, v \in V_a$ satisfies

$$\{u, v\} \in E_a.$$

In particular,

$$G_a \cong K_{|V_a|} \qquad \text{for every } a \in A.$$

**Example 5.2.2** (Complete Soft Graph)**.** Let

$$G^* = (V, E)$$

be the complete graph $K_4$, where

$$V = \{v_1, v_2, v_3, v_4\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\big\}.$$

Let the parameter set be

$$A = \{a_1, a_2, a_3\}.$$

Define the Soft Graph

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

by

$$\Phi(a_1) = G_{a_1} = (V_{a_1}, E_{a_1}),$$

where

$$V_{a_1} = \{v_1, v_2, v_3\}$$

and

$$E_{a_1} = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\big\},$$

and by

$$\Phi(a_2) = G_{a_2} = (V_{a_2}, E_{a_2}),$$

where

$$V_{a_2} = \{v_2, v_3, v_4\}$$

and

$$E_{a_2} = \big\{\{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\big\}.$$

Finally, let

$$\Phi(a_3) = G_{a_3} = (V_{a_3}, E_{a_3}),$$

where

$$V_{a_3} = \{v_1, v_4\}$$

and

$$E_{a_3} = \big\{\{v_1, v_4\}\big\}.$$

Then

$$E_{a_1} = \binom{V_{a_1}}{2}, \qquad E_{a_2} = \binom{V_{a_2}}{2}, \qquad E_{a_3} = \binom{V_{a_3}}{2}.$$

Hence,

$$G_{a_1} \cong K_3, \qquad G_{a_2} \cong K_3, \qquad G_{a_3} \cong K_2.$$

Therefore, every parameter-dependent graph $G_a$ is complete, and thus $\Phi$ is a *Complete Soft Graph.*

## 5.3 Regular Soft Graph

A regular graph is one where every vertex has identical degree, so all vertices have the same number of neighbors (cf. [396–398]). A Regular Soft Graph is a Soft Graph for which each parameter-dependent graph is regular [207, 399].

**Definition 5.3.1** (Regular Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$, and let

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

be a Soft Graph over $G^*$. For each $a \in A$, write

$$\Phi(a) = G_a = (V_a, E_a).$$

For $v \in V_a$, let

$$\deg_{G_a}(v) = \big|\{u \in V_a : \{u, v\} \in E_a\}\big|$$

denote the degree of $v$ in the parameter graph $G_a$.

The Soft Graph $\Phi$ is called a *Regular Soft Graph* if, for every $a \in A$, there exists a nonnegative integer $r_a$ such that

$$\deg_{G_a}(v) = r_a \qquad \text{for every } v \in V_a.$$

Equivalently, every parameter graph $G_a$ is an $r_a$-regular graph.

If there exists a single integer $r \geq 0$ such that

$$\deg_{G_a}(v) = r \qquad \text{for every } a \in A \text{ and every } v \in V_a,$$

then $\Phi$ may more specifically be called a *uniformly $r$-regular Soft Graph.*

**Example 5.3.2** (Regular Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph with

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_4\}\big\}.$$

Let the parameter set be

$$A = \{a_1, a_2\}.$$

Define the Soft Graph

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

by

$$\Phi(a_1) = G_{a_1} = (V_{a_1}, E_{a_1}),$$

where

$$V_{a_1} = \{v_1, v_2, v_3\}$$

and

$$E_{a_1} = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}\big\},$$

and by

$$\Phi(a_2) = G_{a_2} = (V_{a_2}, E_{a_2}),$$

where

$$V_{a_2} = \{v_4, v_5, v_6\}$$

and

$$E_{a_2} = \big\{\{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_4\}\big\}.$$

For the first parameter graph $G_{a_1}$, we have

$$\deg_{G_{a_1}}(v_1) = \deg_{G_{a_1}}(v_2) = \deg_{G_{a_1}}(v_3) = 2.$$

Hence, $G_{a_1}$ is a 2-regular graph.

Similarly, for the second parameter graph $G_{a_2}$,

$$\deg_{G_{a_2}}(v_4) = \deg_{G_{a_2}}(v_5) = \deg_{G_{a_2}}(v_6) = 2.$$

Thus, $G_{a_2}$ is also a 2-regular graph.

Therefore, every parameter-dependent graph $G_a$ is regular, and hence $\Phi$ is a *Regular Soft Graph*. Moreover, since the same degree $r = 2$ occurs for every parameter and every corresponding vertex, $\Phi$ is, more specifically, a *uniformly* 2*-regular Soft Graph*.

## 5.4 Irregular Soft Graph

An irregular graph is one whose vertices do not all have the same degree, so vertex degrees differ among vertices [400–402]. An Irregular Soft Graph is a Soft Graph whose parameter-dependent graphs fail to be regular [207].

**Definition 5.4.1** (Irregular Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$, and let

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

be a Soft Graph over $G^*$. For every $a \in A$, write

$$\Phi(a) = G_a = (V_a, E_a).$$

The Soft Graph $\Phi$ is called an *Irregular Soft Graph* if $G_a$ is an irregular graph for every $a \in A$; equivalently, for every $a \in A$, the graph $G_a$ is not regular.

Thus, for every $a \in A$, there exist vertices $u, v \in V_a$ such that

$$\deg_{G_a}(u) \neq \deg_{G_a}(v).$$

Hence, no parameter graph $G_a$ has a constant vertex degree.

**Example 5.4.2** (Irregular Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph with

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_4, v_7\}\big\}.$$

Let the parameter set be

$$A = \{a_1, a_2\}.$$

Define the Soft Graph

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

as follows.

For the parameter $a_1$, let

$$\Phi(a_1) = G_{a_1} = (V_{a_1}, E_{a_1}),$$

where

$$V_{a_1} = \{v_1, v_2, v_3\}$$

and

$$E_{a_1} = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\}.$$

Then

$$\deg_{G_{a_1}}(v_1) = 1, \qquad \deg_{G_{a_1}}(v_2) = 2, \qquad \deg_{G_{a_1}}(v_3) = 1.$$

Hence, $G_{a_1}$ is not regular.

For the parameter $a_2$, let

$$\Phi(a_2) = G_{a_2} = (V_{a_2}, E_{a_2}),$$

where

$$V_{a_2} = \{v_4, v_5, v_6, v_7\}$$

and

$$E_{a_2} = \big\{\{v_4, v_5\}, \{v_4, v_6\}, \{v_4, v_7\}\big\}.$$

In this case,

$$\deg_{G_{a_2}}(v_4) = 3,$$

whereas

$$\deg_{G_{a_2}}(v_5) = \deg_{G_{a_2}}(v_6) = \deg_{G_{a_2}}(v_7) = 1.$$

Therefore, $G_{a_2}$ is also not regular.

Since every parameter-dependent graph $G_a$ contains vertices of different degrees, each $G_a$ is irregular. Consequently,

$$\Phi$$

is an *Irregular Soft Graph*.

## 5.5 Connected Soft Graph

Connectedness of a Soft Graph can be formulated in terms of paths appearing within its parameter-dependent graph parts [403–405].

**Definition 5.5.1** (Connected Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$, and let

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

be a Soft Graph over $G^*$. For every $a \in A$, write

$$\Phi(a) = G_a = (V_a, E_a),$$

and define the vertex support of the Soft Graph by

$$V_\Phi := \bigcup_{a \in A} V_a.$$

The Soft Graph $\Phi$ is called a *Connected Soft Graph* if, for every pair of vertices $u, v \in V_\Phi$, there exists a parameter $a \in A$ such that $u, v \in V_a$ and there exists a $u$-$v$ path entirely contained in $G_a$.

Equivalently,

$$\forall\, u, v \in V_\Phi, \quad \exists\, a \in A \quad \text{such that} \quad u, v \in V_a \text{ and } u \overset{G_a}{\rightsquigarrow} v,$$

where

$$u \overset{G_a}{\rightsquigarrow} v$$

means that $u$ and $v$ are joined by a path in $G_a$.

**Example 5.5.2** (Connected Soft Graph)**.** Let
$$G^* = (V, E)$$
be the finite simple graph with
$$V = \{v_1, v_2, v_3, v_4\}$$
and
$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\big\}.$$

Let the parameter set be
$$A = \{a_1, a_2\}.$$
Define the Soft Graph
$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$
by
$$\Phi(a_1) = G_{a_1} = (V_{a_1}, E_{a_1}),$$
where
$$V_{a_1} = \{v_1, v_2, v_3, v_4\}$$
and
$$E_{a_1} = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\big\}.$$
Thus, $G_{a_1}$ is the path
$$v_1 - v_2 - v_3 - v_4.$$

For the second parameter, let
$$\Phi(a_2) = G_{a_2} = (V_{a_2}, E_{a_2}),$$
where
$$V_{a_2} = \{v_1, v_3\}$$
and
$$E_{a_2} = \emptyset.$$

The vertex support of $\Phi$ is
$$V_\Phi = V_{a_1} \cup V_{a_2} = \{v_1, v_2, v_3, v_4\}.$$

Now consider arbitrary vertices $u, v \in V_\Phi$. Since
$$u, v \in V_{a_1}$$
and $G_{a_1}$ is connected, there exists a $u$-$v$ path entirely contained in $G_{a_1}$. For example,
$$v_1 \overset{G_{a_1}}{\rightsquigarrow} v_4$$
via the path
$$v_1 - v_2 - v_3 - v_4,$$
while
$$v_2 \overset{G_{a_1}}{\rightsquigarrow} v_4$$
via
$$v_2 - v_3 - v_4.$$

Therefore,
$$\forall\, u, v \in V_\Phi, \quad \exists\, a \in A \quad \text{such that} \quad u, v \in V_a \text{ and } u \overset{G_a}{\rightsquigarrow} v.$$
Hence, $\Phi$ is a *Connected Soft Graph.*

Notice that $G_{a_2}$ itself is disconnected. This illustrates that, under the present definition, it is not necessary for every parameter-dependent graph $G_a$ to be connected; it is sufficient that each pair of vertices in $V_\Phi$ be connected within at least one parameter graph.

## 5.6 Complement Soft Graph

The complement operation can be applied parameterwise to the graph parts of a Soft Graph (cf. [406]).

**Definition 5.6.1** (Complement Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$, and let

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

be a Soft Graph over $G^*$. For every $a \in A$, write

$$\Phi(a) = G_a = (V_a, E_a).$$

For each $a \in A$, define the complement of the parameter graph $G_a$ by

$$\overline{G_a} = \big(V_a, \overline{E}_a\big),$$

where

$$\overline{E}_a = \binom{V_a}{2} \setminus E_a.$$

The *Complement Soft Graph* of $\Phi$, denoted by

$$\overline{\Phi},$$

is the parameterized family

$$\overline{\Phi} = \{\overline{G_a} : a \in A\},$$

or equivalently the mapping

$$\overline{\Phi} : A \longrightarrow \mathrm{Sub}(K_V), \qquad a \longmapsto \overline{G_a},$$

where $K_V$ denotes the complete graph on $V$.

Thus, for distinct $u, v \in V_a$,

$$\{u, v\} \in \overline{E}_a \quad \Longleftrightarrow \quad \{u, v\} \notin E_a.$$

In particular,

$$\overline{\overline{\Phi}} = \Phi.$$

**Example 5.6.2** (Complement Soft Graph)**.** Let

$$G^* = (V, E)$$

be the finite simple graph with

$$V = \{v_1, v_2, v_3, v_4\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\big\}.$$

Let the parameter set be

$$A = \{a_1, a_2\}.$$

Define the Soft Graph

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

as follows.

For the parameter $a_1$, let

$$\Phi(a_1) = G_{a_1} = (V_{a_1}, E_{a_1}),$$

where

$$V_{a_1} = \{v_1, v_2, v_3\}$$

and

$$E_{a_1} = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\}.$$

Since

$$\binom{V_{a_1}}{2} = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\big\},$$

the complement edge set is

$$\overline{E}_{a_1} = \binom{V_{a_1}}{2} \setminus E_{a_1} = \big\{\{v_1, v_3\}\big\}.$$

Therefore,

$$\overline{G_{a_1}} = \big(\{v_1, v_2, v_3\}, \big\{\{v_1, v_3\}\big\}\big).$$

For the parameter $a_2$, let

$$\Phi(a_2) = G_{a_2} = (V_{a_2}, E_{a_2}),$$

where

$$V_{a_2} = \{v_2, v_3, v_4\}$$

and

$$E_{a_2} = \big\{\{v_2, v_3\}, \{v_3, v_4\}\big\}.$$

In this case,

$$\binom{V_{a_2}}{2} = \big\{\{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\big\},$$

and hence

$$\overline{E}_{a_2} = \binom{V_{a_2}}{2} \setminus E_{a_2} = \big\{\{v_2, v_4\}\big\}.$$

Thus,

$$\overline{G_{a_2}} = \big(\{v_2, v_3, v_4\}, \big\{\{v_2, v_4\}\big\}\big).$$

Accordingly, the Complement Soft Graph of $\Phi$ is

$$\overline{\Phi} = \big\{\overline{G_{a_1}}, \overline{G_{a_2}}\big\},$$

or equivalently,

$$\overline{\Phi}(a_1) = \overline{G_{a_1}}, \qquad \overline{\Phi}(a_2) = \overline{G_{a_2}}.$$

Moreover, taking the complement once again gives

$$\overline{\overline{E}}_{a_1} = \binom{V_{a_1}}{2} \setminus \overline{E}_{a_1} = E_{a_1}$$

and

$$\overline{\overline{E}}_{a_2} = \binom{V_{a_2}}{2} \setminus \overline{E}_{a_2} = E_{a_2}.$$

Hence,

$$\overline{\overline{\Phi}} = \Phi.$$

Therefore, this construction provides a concrete example of a *Complement Soft Graph*.

## 5.7 Hamiltonian Soft Graph

A Hamiltonian graph contains a cycle that visits every vertex exactly once before returning to the starting vertex, called a Hamiltonian cycle [407–410]. A Hamiltonian Soft Graph is a Soft Graph in which every parameter-dependent graph contains a Hamiltonian cycle [411].

**Definition 5.7.1** (Hamiltonian Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph, let $A \neq \emptyset$, and let

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

be a Soft Graph over $G^*$. For each $a \in A$, write

$$\Phi(a) = G_a = (V_a, E_a).$$

A parameter graph $G_a$ is called a *Hamiltonian part* of $\Phi$ if it contains a cycle passing through every vertex of $V_a$ exactly once, except for the initial vertex, which is repeated at the end of the cycle.

The Soft Graph $\Phi$ is called a *Hamiltonian Soft Graph* if every parameter graph $G_a$ is a Hamiltonian part; that is,

$$G_a \text{ is Hamiltonian} \qquad \text{for every } a \in A.$$

Equivalently, for every $a \in A$, there exists an ordering

$$v_1, v_2, \ldots, v_{n_a}$$

of all vertices of $V_a$, where

$$n_a = |V_a| \geq 3,$$

such that

$$\{v_i, v_{i+1}\} \in E_a \qquad (i = 1, \ldots, n_a - 1)$$

and

$$\{v_{n_a}, v_1\} \in E_a.$$

**Example 5.7.2** (Hamiltonian Soft Graph)**.** Let

$$G^* = (V, E)$$

be the complete graph $K_4$, where

$$V = \{v_1, v_2, v_3, v_4\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\big\}.$$

Let the parameter set be

$$A = \{a_1, a_2\}.$$

Define the Soft Graph

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

as follows.

For the parameter $a_1$, let
$$\Phi(a_1) = G_{a_1} = (V_{a_1}, E_{a_1}),$$
where
$$V_{a_1} = \{v_1, v_2, v_3, v_4\}$$
and
$$E_{a_1} = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\big\}.$$
Thus, $G_{a_1}$ is the cycle graph $C_4$. In particular,
$$v_1 - v_2 - v_3 - v_4 - v_1$$
is a Hamiltonian cycle of $G_{a_1}$, since it visits every vertex of $V_{a_1}$ exactly once before returning to $v_1$.

For the parameter $a_2$, let
$$\Phi(a_2) = G_{a_2} = (V_{a_2}, E_{a_2}),$$
where
$$V_{a_2} = \{v_1, v_2, v_3\}$$
and
$$E_{a_2} = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}\big\}.$$
Hence,
$$G_{a_2} \cong K_3 \cong C_3,$$
and
$$v_1 - v_2 - v_3 - v_1$$
is a Hamiltonian cycle of $G_{a_2}$.

Therefore,
$$G_{a_1} \text{ is Hamiltonian} \qquad \text{and} \qquad G_{a_2} \text{ is Hamiltonian.}$$
Since every parameter-dependent graph $G_a$ contains a Hamiltonian cycle, the Soft Graph
$$\Phi$$
is a *Hamiltonian Soft Graph*.

## 5.8 Line Graph of a Soft Graph

A line graph transforms each edge of a graph into a vertex, connecting vertices whenever their corresponding edges share endpoints (cf. [412,413]). The line graph construction extends naturally to Soft Graphs by applying the classical line graph operation to each parameter-dependent graph [414].

**Definition 5.8.1** (Line Graph of a Soft Graph)**.** Let
$$G^* = (V, E)$$
be a finite simple graph, let $A \neq \emptyset$, and let
$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$
be a Soft Graph over $G^*$. For each $a \in A$, write
$$\Phi(a) = G_a = (V_a, E_a).$$

The line graph of the parameter graph $G_a$, denoted by

$$L(G_a),$$

is the graph

$$L(G_a) = \big(E_a, E_a^L\big),$$

where the vertices of $L(G_a)$ are the edges of $G_a$, and

$$E_a^L = \left\{ \{e, f\} \in \binom{E_a}{2} : e \cap f \neq \emptyset \right\}.$$

Thus, two vertices $e, f \in E_a$ of $L(G_a)$ are adjacent if and only if the corresponding edges $e$ and $f$ of $G_a$ share a common endpoint.

The *Line Graph of the Soft Graph* $\Phi$, denoted by

$$L(\Phi),$$

is defined parameterwise by

$$L(\Phi) = \{L(G_a) : a \in A\}.$$

Equivalently, since

$$L(G_a) \subseteq L(G^*) \qquad \text{for every } a \in A,$$

the line graph of $\Phi$ may be regarded as the Soft Graph

$$L(\Phi) : A \longrightarrow \mathrm{Sub}\big(L(G^*)\big), \qquad a \longmapsto L(G_a).$$

**Example 5.8.2** (Line Graph of a Soft Graph)**.** Let

$$G^* = (V, E)$$

be a finite simple graph with

$$V = \{v_1, v_2, v_3, v_4\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_4\}\big\}.$$

Let the parameter set be

$$A = \{a_1, a_2\}.$$

Define the Soft Graph

$$\Phi : A \longrightarrow \mathrm{Sub}(G^*)$$

as follows.

For the parameter $a_1$, let

$$\Phi(a_1) = G_{a_1} = (V_{a_1}, E_{a_1}),$$

where

$$V_{a_1} = \{v_1, v_2, v_3\}$$

and

$$E_{a_1} = \big\{e_1, e_2\big\},$$

with

$$e_1 = \{v_1, v_2\}, \qquad e_2 = \{v_2, v_3\}.$$

Since
$$e_1 \cap e_2 = \{v_2\} \neq \emptyset,$$
the two vertices $e_1$ and $e_2$ are adjacent in the line graph. Therefore,
$$L(G_{a_1}) = \big(\{e_1, e_2\}, \{\{e_1, e_2\}\}\big).$$
Hence,
$$L(G_{a_1}) \cong K_2.$$

For the parameter $a_2$, let
$$\Phi(a_2) = G_{a_2} = (V_{a_2}, E_{a_2}),$$
where
$$V_{a_2} = \{v_2, v_3, v_4\}$$
and
$$E_{a_2} = \big\{e_3, e_4, e_5\big\},$$
with
$$e_3 = \{v_2, v_3\}, \qquad e_4 = \{v_3, v_4\}, \qquad e_5 = \{v_2, v_4\}.$$

We have
$$e_3 \cap e_4 = \{v_3\}, \qquad e_4 \cap e_5 = \{v_4\}, \qquad e_5 \cap e_3 = \{v_2\}.$$
Thus, every pair of distinct edges in $E_{a_2}$ shares a common endpoint. Consequently,
$$E^L_{a_2} = \big\{\{e_3, e_4\}, \{e_4, e_5\}, \{e_5, e_3\}\big\},$$
and hence
$$L(G_{a_2}) = \big(\{e_3, e_4, e_5\}, E^L_{a_2}\big) \cong K_3.$$

Therefore, the line graph of the Soft Graph $\Phi$ is
$$L(\Phi) = \big\{L(G_{a_1}), L(G_{a_2})\big\}.$$

Equivalently,
$$L(\Phi)(a_1) = L(G_{a_1}), \qquad L(\Phi)(a_2) = L(G_{a_2}).$$

Thus, this example illustrates how the classical line graph operation is applied independently to each parameter-dependent graph of a Soft Graph.

## 5.9 Rough Fuzzy Graph

A Rough Fuzzy Graph applies rough lower and upper approximations to fuzzy vertex and edge memberships under indiscernibility relations systematically.

**Definition 5.9.1** (Rough Fuzzy Graph)**.** Let $V$ be a finite nonempty set and let

$$E^* = \binom{V}{2}.$$

Let $R_V$ and $R_E$ be equivalence relations on $V$ and $E^*$, respectively. Let

$$\mu : V \longrightarrow [0,1], \qquad \nu : E^* \longrightarrow [0,1]$$

define a fuzzy graph, so that

$$\nu(\{x,y\}) \leq \min\{\mu(x), \mu(y)\} \qquad \text{for all } \{x,y\} \in E^*.$$

For $x \in V$ and $e \in E^*$, define

$$\underline{\mu}(x) = \inf_{z \in [x]_{R_V}} \mu(z), \qquad \overline{\mu}(x) = \sup_{z \in [x]_{R_V}} \mu(z),$$

and

$$\underline{\nu}(e) = \inf_{f \in [e]_{R_E}} \nu(f), \qquad \overline{\nu}(e) = \sup_{f \in [e]_{R_E}} \nu(f).$$

Suppose that

$$\underline{\nu}(\{x,y\}) \leq \min\{\underline{\mu}(x), \underline{\mu}(y)\}$$

and

$$\overline{\nu}(\{x,y\}) \leq \min\{\overline{\mu}(x), \overline{\mu}(y)\}$$

for every $\{x,y\} \in E^*$.

Then

$$\underline{G} = (\underline{\mu}, \underline{\nu}) \qquad \text{and} \qquad \overline{G} = (\overline{\mu}, \overline{\nu})$$

are fuzzy graphs, and the ordered pair

$$\mathcal{G}_{RF} = (\underline{G}, \overline{G})$$

is called a *Rough Fuzzy Graph.*

For reference, a comparison between Rough Fuzzy Graphs and Fuzzy Rough Graphs is presented in Table 5.1.

## 5.10 Rough Neutrosophic Graph

A Rough Neutrosophic Graph applies rough approximations to truth, indeterminacy, and falsity degrees assigned to vertices and edges under indiscernibility (cf. [415]).

**Definition 5.10.1** (Rough Neutrosophic Graph)**.** Let $V$ be a finite nonempty set and let

$$E^* = \binom{V}{2}.$$

Let $R_V$ and $R_E$ be equivalence relations on $V$ and $E^*$, respectively.

Let

$$T_V, I_V, F_V : V \longrightarrow [0,1]$$

Table 5.1: Comparison between Rough Fuzzy Graph and Fuzzy Rough Graph

| **Aspect** | **Rough Fuzzy Graph** | **Fuzzy Rough Graph** |
|---|---|---|
| Basic idea | Applies rough approximations to an underlying fuzzy graph. | Combines fuzzy approximation mechanisms with rough-set-based graph modeling. |
| Construction order | Fuzzy structure $\to$ Rough approximation. | Rough approximation framework $\to$ Fuzzy representation. |
| Main information | Fuzzy membership degrees of vertices and edges are approximated by lower and upper rough approximations. | Lower and upper approximations are determined using fuzzy relations, fuzzy neighborhoods, or graded indiscernibility. |
| Typical representation | A pair of lower and upper approximated fuzzy graphs. | A graph equipped with fuzzy lower and upper approximation structures. |
| Interpretation | Represents uncertainty in an already fuzzy graph through rough approximations. | Represents rough uncertainty through fuzzy or graded approximation mechanisms. |

and

$$T_E, I_E, F_E : E^* \longrightarrow [0,1]$$

define a single-valued neutrosophic graph.

For $x \in V$, define the lower approximations by

$$\underline{T}_V(x) = \inf_{z\in[x]_{R_V}} T_V(z), \qquad \underline{I}_V(x) = \inf_{z\in[x]_{R_V}} I_V(z),$$

$$\underline{F}_V(x) = \sup_{z\in[x]_{R_V}} F_V(z),$$

and the upper approximations by

$$\overline{T}_V(x) = \sup_{z\in[x]_{R_V}} T_V(z), \qquad \overline{I}_V(x) = \sup_{z\in[x]_{R_V}} I_V(z),$$

$$\overline{F}_V(x) = \inf_{z\in[x]_{R_V}} F_V(z).$$

Similarly, for $e \in E^*$, define

$$\underline{T}_E(e) = \inf_{f\in[e]_{R_E}} T_E(f), \qquad \underline{I}_E(e) = \inf_{f\in[e]_{R_E}} I_E(f),$$

$$\underline{F}_E(e) = \sup_{f\in[e]_{R_E}} F_E(f),$$

and

$$\overline{T}_E(e) = \sup_{f\in[e]_{R_E}} T_E(f), \qquad \overline{I}_E(e) = \sup_{f\in[e]_{R_E}} I_E(f),$$

$$\overline{F}_E(e) = \inf_{f\in[e]_{R_E}} F_E(f).$$

If the resulting lower and upper degree functions satisfy the corresponding neutrosophic graph compatibility conditions, then

$$\underline{G}_N = (\underline{T}_V, \underline{I}_V, \underline{F}_V, \underline{T}_E, \underline{I}_E, \underline{F}_E)$$

and

$$\overline{G}_N = (\overline{T}_V, \overline{I}_V, \overline{F}_V, \overline{T}_E, \overline{I}_E, \overline{F}_E)$$

are neutrosophic graphs.

The ordered pair

$$\mathcal{G}_{RN} = (\underline{G}_N, \overline{G}_N)$$

is called a *Rough Neutrosophic Graph*.

For reference, a comparison between Neutrosophic Rough Graphs and Rough Neutrosophic Graphs is presented in Table 5.2. Since the terminology for these two concepts is not always used consistently across the literature, readers should be aware of this distinction.

Table 5.2: Comparison between Neutrosophic Rough Graph and Rough Neutrosophic Graph

| **Aspect** | **Neutrosophic Rough Graph** | **Rough Neutrosophic Graph** |
|---|---|---|
| Basic idea | Extends a rough graph framework by incorporating neutrosophic truth, indeterminacy, and falsity information. | Applies rough lower and upper approximations to an underlying neutrosophic graph. |
| Construction order | Rough structure → Neutrosophic extension. | Neutrosophic structure → Rough approximation. |
| Main information | Rough approximations are enriched with truth, indeterminacy, and falsity degrees. | Neutrosophic vertex and edge degrees are represented through lower and upper rough approximations. |
| Typical representation | A rough graph equipped with neutrosophic membership information. | A pair of lower and upper approximated neutrosophic graphs. |
| Interpretation | Represents rough structural uncertainty using neutrosophic information. | Represents uncertainty in a neutrosophic graph through rough approximations. |

## 5.11 Rough Uncertain Graph

A Rough Uncertain Graph first constructs lower and upper rough approximations of a graph structure and then equips the resulting approximation graphs with uncertainty degrees on their vertices and edges. Thus, it represents rough structural uncertainty together with generalized uncertainty information.

**Definition 5.11.1** (Rough Uncertain Graph)**.** Let

$$G = (V, E)$$

be a finite simple graph, and let $\sim_V$ and $\sim_E$ be equivalence relations on $V$ and $E$, respectively.

Let

$$H = (X, F), \qquad X \subseteq V, \qquad F \subseteq E \cap \binom{X}{2}$$

be a subgraph of $G$. Define the lower and upper vertex approximations by

$$\underline{X} = \{v \in V : [v]_{\sim_V} \subseteq X\}, \qquad \overline{X} = \{v \in V : [v]_{\sim_V} \cap X \neq \emptyset\},$$

and the corresponding edge approximations by

$$\underline{F} = \{e \in E : [e]_{\sim_E} \subseteq F,\ e \subseteq \underline{X}\},$$

and

$$\overline{F} = \{e \in E : [e]_{\sim_E} \cap F \neq \emptyset,\ e \subseteq \overline{X}\}.$$

Let $(L, \preceq, \wedge)$ be a meet-semilattice of uncertainty values. Assign uncertainty-degree maps

$$\underline{\omega}_V : \underline{X} \longrightarrow L, \qquad \underline{\omega}_E : \underline{F} \longrightarrow L,$$

and

$$\overline{\omega}_V : \overline{X} \longrightarrow L, \qquad \overline{\omega}_E : \overline{F} \longrightarrow L,$$

such that

$$\underline{\omega}_E(\{u, v\}) \preceq \underline{\omega}_V(u) \wedge \underline{\omega}_V(v)$$

for every $\{u, v\} \in \underline{F}$, and

$$\overline{\omega}_E(\{u, v\}) \preceq \overline{\omega}_V(u) \wedge \overline{\omega}_V(v)$$

for every $\{u, v\} \in \overline{F}$.

Define

$$\underline{\mathcal{G}}_U(H) = (\underline{X}, \underline{F}, \underline{\omega}_V, \underline{\omega}_E)$$

and

$$\overline{\mathcal{G}}_U(H) = \left(\overline{X}, \overline{F}, \overline{\omega}_V, \overline{\omega}_E\right).$$

The ordered pair

$$\mathcal{RU}(H) = \left(\underline{\mathcal{G}}_U(H), \overline{\mathcal{G}}_U(H)\right)$$

is called a *Rough Uncertain Graph.*

**Remark 5.11.2.** The construction order is important. In a Rough Uncertain Graph, the rough lower and upper graph approximations are constructed first, and uncertainty degrees are subsequently assigned to the resulting approximation graphs. Thus, schematically,

$$\text{Graph} \longrightarrow \text{Rough approximations} \longrightarrow \text{Uncertain graphs.}$$

This construction should be distinguished from an Uncertain Rough Graph, in which an uncertain graph is specified first and rough approximations are then applied to that uncertain structure.

**Remark 5.11.3** (Catalogue of Rough Uncertain Graph Families)**.** For reference, Table 5.3 presents representative specializations of Rough Uncertain Graphs, classified according to the dimension $k$ of the underlying uncertainty-degree domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

## 5.12 Covering-Based Rough Graph

A Covering-Based Rough Graph uses overlapping coverings instead of partitions to construct lower and upper approximations of structures under uncertainty [416].

Table 5.3: A catalogue of Rough Uncertain Graph families organized by the dimension $k$ of the underlying degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

| $k$ | Representative Rough Uncertain Graph specialization(s) obtained by equipping the lower and upper rough graph approximations with uncertainty degrees in $\mathrm{Dom}(M)$ |
|---|---|
| 1 | Rough fuzzy graph; rough $N$-fuzzy graph; rough shadowed graph variants |
| 2 | Rough intuitionistic fuzzy graph; rough vague graph; rough bipolar fuzzy graph; rough intuitionistic evidence graph; rough variable fuzzy graph; rough paraconsistent fuzzy graph; rough bifuzzy graph |
| 3 | Rough neutrosophic graph[(a)]; rough hesitant fuzzy graph[(d)]; rough tripolar fuzzy graph; rough three-way fuzzy graph; rough picture fuzzy graph; rough spherical fuzzy graph; rough inconsistent intuitionistic fuzzy graph; rough ternary fuzzy graph; rough neutrosophic-fuzzy graph; rough neutrosophic vague graph |
| 4 | Rough quadripartitioned neutrosophic graph; rough double-valued neutrosophic graph; rough dual hesitant fuzzy graph[(d)]; rough ambiguous graph[(b)]; rough local-neutrosophic graph; rough support-neutrosophic graph; rough Turiyam neutrosophic graph[(c)] |
| 5 | Rough pentapartitioned neutrosophic graph; rough triple-valued neutrosophic graph |
| 6 | Rough hexapartitioned neutrosophic graph; rough quadruple-valued neutrosophic graph |
| 7 | Rough heptapartitioned neutrosophic graph; rough quintuple-valued neutrosophic graph |
| 8 | Rough octapartitioned neutrosophic graph |
| 9 | Rough nonapartitioned neutrosophic graph |
| $n$ | Rough $n$-refined fuzzy graph; rough multi-valued fuzzy graph; rough multi-fuzzy graph |
| $2n$ | Rough $n$-refined intuitionistic fuzzy graph; rough multi-intuitionistic fuzzy graph |
| $3n$ | Rough $n$-refined neutrosophic graph; rough multi-neutrosophic graph |

[(a)] A Rough Neutrosophic Graph is obtained by first constructing lower and upper rough graph approximations and then assigning neutrosophic truth, indeterminacy, and falsity degrees to their vertices and edges.
[(b)] A rough ambiguous graph is obtained by equipping the lower and upper rough graph approximations with degrees belonging to an ambiguous-set model. Ambiguous-set models are commonly related to certain four-component neutrosophic models.
[(c)] A rough Turiyam neutrosophic graph is obtained by assigning a four-component Turiyam neutrosophic degree model to the lower and upper rough graph approximations. Such degree models are commonly regarded as specializations of quadripartitioned neutrosophic frameworks.
[(d)] Hesitant and dual hesitant fuzzy degrees are naturally set-valued. Their placement in a fixed-dimensional row presupposes a selected finite-coordinate representation. Otherwise, their degree-domains need not be subsets of a single finite-dimensional cube $[0,1]^k$.
*Dimension note.* The integer $k$ records the dimension of the uncertainty degree assigned to each vertex or edge of each rough approximation graph. It does not count the lower and upper approximations as additional degree components. Hence, the pair

$$\left(\underline{\mathcal{G}}_M(H), \overline{\mathcal{G}}_M(H)\right)$$

is classified according to the dimension of $\mathrm{Dom}(M)$, rather than according to $2k$.
*Terminology note.* The terminology for constructions such as Rough Fuzzy Graph versus Fuzzy Rough Graph, and Rough Neutrosophic Graph versus Neutrosophic Rough Graph, is not completely uniform throughout the literature. In this book, the prefix "Rough" indicates that the rough graph structure is constructed first and is subsequently equipped with the corresponding uncertainty model.

**Definition 5.12.1** (Covering-Based Rough Graph)**.** Let

$$G = (V, E)$$

be a finite simple graph, and let

$$\mathcal{C}_V$$

be a covering of $V$; that is,

$$\emptyset \notin \mathcal{C}_V, \qquad \bigcup_{C \in \mathcal{C}_V} C = V.$$

For $X \subseteq V$, define the covering-based lower and upper approximations by

$$\underline{\mathcal{C}}_V(X) = \bigcup \{C \in \mathcal{C}_V : C \subseteq X\},$$

and

$$\overline{\mathcal{C}}_V(X) = \bigcup \{C \in \mathcal{C}_V : C \cap X \neq \emptyset\}.$$

The covering $\mathcal{C}_V$ induces an edge covering $\mathcal{C}_V^{(e)}$ of $E$ by the nonempty sets

$$E_{C,D} = \big\{\{u, v\} \in E : u \in C,\ v \in D\big\}, \qquad C, D \in \mathcal{C}_V.$$

Let

$$H = (X, F)$$

be a subgraph of $G$. Using $\mathcal{C}_V^{(e)}$, let

$$\underline{\mathcal{C}}_V^{(e)}(F) \qquad \text{and} \qquad \overline{\mathcal{C}}_V^{(e)}(F)$$

denote the lower and upper covering approximations of $F$.

Define

$$\underline{\mathcal{C}}_G(H) = \Big(\underline{\mathcal{C}}_V(X),\ \underline{\mathcal{C}}_V^{(e)}(F) \cap E_G[\underline{\mathcal{C}}_V(X)]\Big),$$

and

$$\overline{\mathcal{C}}_G(H) = \Big(\overline{\mathcal{C}}_V(X),\ \overline{\mathcal{C}}_V^{(e)}(F)\Big),$$

where

$$E_G[W] = \big\{\{u, v\} \in E : u, v \in W\big\}$$

for $W \subseteq V$.

The ordered pair

$$\mathcal{R}_{\mathcal{C}}(H) = \big(\underline{\mathcal{C}}_G(H), \overline{\mathcal{C}}_G(H)\big)$$

is called a *Covering-Based Rough Graph*.

# Chapter 6

# Special Classes of Rough Graphs

This chapter introduces several special classes of Rough Graphs. In this chapter, we use the rough-membership-function-based formulation of Rough Graphs in order to introduce several special graph classes.

## 6.1 Regular Rough Graph

A regular Rough Graph is a Rough Graph whose vertices all have equal degree, equivalently equal maximum and minimum degrees [417, 418].

**Definition 6.1.1** (Regular Rough Graph)**.** [418] Let

$$\mathcal{R} = (V, E, \omega)$$

be a Rough Graph, where

$$\omega : V \longrightarrow [0, 1].$$

For each $v \in V$, let

$$\deg_{\mathcal{R}}(v)$$

denote the degree of $v$. Define

$$\Delta^{+}(\mathcal{R}) = \max_{v \in V} \deg_{\mathcal{R}}(v)$$

and

$$\Delta^{-}(\mathcal{R}) = \min_{v \in V} \deg_{\mathcal{R}}(v).$$

The Rough Graph $\mathcal{R}$ is called a *Regular Rough Graph* if

$$\Delta^{+}(\mathcal{R}) = \Delta^{-}(\mathcal{R}).$$

Equivalently, there exists an integer $r \geq 0$ such that

$$\deg_{\mathcal{R}}(v) = r \qquad \text{for every } v \in V.$$

In this case, $\mathcal{R}$ is called an $r$-regular Rough Graph.

**Example 6.1.2** (Regular Rough Graph)**.** Let

$$\mathcal{R} = (V, E, \omega)$$

be a Rough Graph with vertex set

$$V = \{v_1, v_2, v_3, v_4\}$$

and edge set

$$E = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\big\}.$$

Thus, the underlying graph of $\mathcal{R}$ is the complete graph $K_4$.

Define the rough membership function

$$\omega : V \longrightarrow [0, 1]$$

by

$$\omega(v_1) = 0.6, \qquad \omega(v_2) = 0.7, \qquad \omega(v_3) = 0.8, \qquad \omega(v_4) = 0.9.$$

Since every vertex is adjacent to the other three vertices, we have

$$\deg_{\mathcal{R}}(v_1) = \deg_{\mathcal{R}}(v_2) = \deg_{\mathcal{R}}(v_3) = \deg_{\mathcal{R}}(v_4) = 3.$$

Therefore,

$$\Delta^+(\mathcal{R}) = 3$$

and

$$\Delta^-(\mathcal{R}) = 3.$$

Hence,

$$\Delta^+(\mathcal{R}) = \Delta^-(\mathcal{R}).$$

Consequently, $\mathcal{R}$ is a *Regular Rough Graph.* More specifically, it is a 3-regular Rough Graph.

## 6.2 Complete Rough Graph

A complete Rough Graph is a Rough Graph where every vertex is adjacent to every other vertex in the graph [418].

**Definition 6.2.1** (Complete Rough Graph)**.** [418] Let

$$\mathcal{R} = (V, E, \omega)$$

be a Rough Graph. The graph $\mathcal{R}$ is called a *Complete Rough Graph* if every pair of distinct vertices is adjacent; that is,

$$\{u, v\} \in E \qquad \text{for all distinct } u, v \in V.$$

Equivalently,

$$E = \binom{V}{2}.$$

If

$$|V| = n,$$

then the complete Rough Graph has

$$|E| = \binom{n}{2} = \frac{n(n-1)}{2}$$

edges and is $(n-1)$-regular.

**Example 6.2.2** (Complete Rough Graph)**.** Let

$$\mathcal{R} = (V, E, \omega)$$

be a Rough Graph with

$$V = \{v_1, v_2, v_3, v_4\}.$$

Define the edge set by

$$E = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\big\}.$$

Let the rough membership function

$$\omega : V \longrightarrow [0, 1]$$

be given by

$$\omega(v_1) = 0.6, \qquad \omega(v_2) = 0.7, \qquad \omega(v_3) = 0.8, \qquad \omega(v_4) = 0.9.$$

Since every pair of distinct vertices is adjacent, we have

$$E = \binom{V}{2}.$$

Hence, the underlying graph is isomorphic to the complete graph

$$K_4.$$

Since

$$|V| = 4,$$

the number of edges is

$$|E| = \binom{4}{2} = \frac{4(4-1)}{2} = 6.$$

Moreover, every vertex is adjacent to the other three vertices, so

$$\deg_{\mathcal{R}}(v_1) = \deg_{\mathcal{R}}(v_2) = \deg_{\mathcal{R}}(v_3) = \deg_{\mathcal{R}}(v_4) = 3.$$

Therefore, $\mathcal{R}$ is a *Complete Rough Graph.* Furthermore, it is a 3-regular Rough Graph.

## 6.3 Rough Path

A Rough Path is a rough walk with distinct vertices, where each vertex has rough membership value at least 0.5 [418].

**Definition 6.3.1** (Rough Path)**.** [418] Let

$$\mathcal{R} = (V, E, \omega)$$

be a Rough Graph. A sequence of distinct vertices

$$P = v_1, v_2, \ldots, v_k$$

with $k \geq 2$ is called a *Rough Path* if

$$\{v_i, v_{i+1}\} \in E \qquad \text{for } i = 1, \ldots, k-1,$$

and

$$\omega(v_i) \geq \frac{1}{2} \qquad \text{for } i = 1, \ldots, k.$$

Thus, a Rough Path is a rough walk in which all vertices are distinct.

**Example 6.3.2** (Rough Path)**.** Let

$$\mathcal{R} = (V, E, \omega)$$

be a Rough Graph with

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_1, v_3\}\big\}.$$

Define the rough membership function

$$\omega : V \longrightarrow [0, 1]$$

by

$$\omega(v_1) = 0.8, \qquad \omega(v_2) = 0.6, \qquad \omega(v_3) = 0.9, \qquad \omega(v_4) = 0.7, \qquad \omega(v_5) = 0.4.$$

Consider the vertex sequence

$$P = v_1, v_2, v_3, v_4.$$

The vertices $v_1, v_2, v_3, v_4$ are distinct, and the consecutive pairs satisfy

$$\{v_1, v_2\} \in E, \qquad \{v_2, v_3\} \in E, \qquad \{v_3, v_4\} \in E.$$

Moreover,

$$\omega(v_1) = 0.8 \geq \frac{1}{2}, \qquad \omega(v_2) = 0.6 \geq \frac{1}{2},$$
$$\omega(v_3) = 0.9 \geq \frac{1}{2}, \qquad \omega(v_4) = 0.7 \geq \frac{1}{2}.$$

Therefore,

$$P = v_1, v_2, v_3, v_4$$

is a *Rough Path* in $\mathcal{R}$.

On the other hand, although

$$\{v_4, v_5\} \in E,$$

the extended sequence

$$v_1, v_2, v_3, v_4, v_5$$

is not a Rough Path under the stated definition because

$$\omega(v_5) = 0.4 < \frac{1}{2}.$$

Thus, this example illustrates that both adjacency of consecutive vertices and the rough membership threshold are required for a sequence to form a Rough Path.

## 6.4 Rough Cycle

A Rough Cycle is a closed rough walk with distinct internal vertices and membership values of at least 0.5 throughout [418].

**Definition 6.4.1** (Rough Cycle). [418] Let

$$\mathcal{R} = (V, E, \omega)$$

be a Rough Graph. A sequence

$$C = v_1, v_2, \ldots, v_k, v_1, \qquad k \geq 3,$$

is called a *Rough Cycle* if

$$v_1, v_2, \ldots, v_k$$

are distinct,

$$\{v_i, v_{i+1}\} \in E \qquad \text{for } i = 1, \ldots, k-1,$$

$$\{v_k, v_1\} \in E,$$

and

$$\omega(v_i) \geq \frac{1}{2} \qquad \text{for } i = 1, \ldots, k.$$

Equivalently, a Rough Cycle is a closed Rough Path whose initial vertex is repeated only at the end.

**Example 6.4.2** (Rough Cycle). Let

$$\mathcal{R} = (V, E, \omega)$$

be a Rough Graph with

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}, \{v_4, v_5\}\big\}.$$

Define the rough membership function

$$\omega : V \longrightarrow [0, 1]$$

by

$$\omega(v_1) = 0.8, \qquad \omega(v_2) = 0.7, \qquad \omega(v_3) = 0.9, \qquad \omega(v_4) = 0.6, \qquad \omega(v_5) = 0.4.$$

Consider the sequence

$$C = v_1, v_2, v_3, v_4, v_1.$$

The vertices

$$v_1, v_2, v_3, v_4$$

are distinct, and the consecutive pairs satisfy

$$\{v_1, v_2\} \in E, \qquad \{v_2, v_3\} \in E, \qquad \{v_3, v_4\} \in E,$$

and

$$\{v_4, v_1\} \in E.$$

Moreover,

$$\omega(v_1) = 0.8 \geq \frac{1}{2}, \qquad \omega(v_2) = 0.7 \geq \frac{1}{2},$$

$$\omega(v_3) = 0.9 \geq \frac{1}{2}, \qquad \omega(v_4) = 0.6 \geq \frac{1}{2}.$$

Therefore,

$$C = v_1, v_2, v_3, v_4, v_1$$

is a *Rough Cycle* in $\mathcal{R}$.

Notice that although

$$\{v_4, v_5\} \in E,$$

the vertex $v_5$ cannot be included in a Rough Cycle satisfying the stated membership condition because

$$\omega(v_5) = 0.4 < \frac{1}{2}.$$

Hence, this example illustrates that a Rough Cycle must be a closed sequence of distinct vertices, except for the repeated initial vertex, with adjacent consecutive vertices and rough membership values at least 1/2.

## 6.5 Rough Tree

A Rough Tree extends the classical notion of a tree to the framework of Rough Graphs by requiring the underlying graph structure to be connected and acyclic.

**Definition 6.5.1** (Rough Tree)**.** Let

$$\mathcal{R} = (V, E, \omega)$$

be a Rough Graph, where

$$\omega : V \longrightarrow [0, 1]$$

is its rough membership function.

The Rough Graph $\mathcal{R}$ is called a *Rough Tree* if its underlying graph

$$G_{\mathcal{R}} = (V, E)$$

is a tree.

Equivalently, $\mathcal{R}$ is a Rough Tree if

$$G_{\mathcal{R}} \text{ is connected}$$

and

$$G_{\mathcal{R}} \text{ contains no cycle.}$$

For a finite Rough Graph, this is equivalent to

$$|E| = |V| - 1$$

together with the connectedness of $G_{\mathcal{R}}$.

Thus, a Rough Tree preserves the rough membership information associated with its vertices while its underlying graph has the classical tree structure.

**Example 6.5.2** (Rough Tree)**.** Let

$$\mathcal{R} = (V, E, \omega)$$

be a Rough Graph with

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_4, v_5\}\big\}.$$

Define the rough membership function

$$\omega : V \longrightarrow [0, 1]$$

by

$$\omega(v_1) = 0.8, \qquad \omega(v_2) = 0.7, \qquad \omega(v_3) = 0.6, \qquad \omega(v_4) = 0.9, \qquad \omega(v_5) = 0.5.$$

The underlying graph

$$G_{\mathcal{R}} = (V, E)$$

is connected. Indeed, every vertex can be reached from every other vertex through a sequence of adjacent vertices. For example,

$$v_1 - v_2 - v_4 - v_5$$

is a path connecting $v_1$ and $v_5$.

Moreover, $G_{\mathcal{R}}$ contains no cycle. Since

$$|V| = 5$$

and

$$|E| = 4,$$

we have

$$|E| = |V| - 1.$$

Thus, because $G_{\mathcal{R}}$ is connected and has exactly $|V| - 1$ edges, it is a tree.

Therefore,

$$\mathcal{R} = (V, E, \omega)$$

is a *Rough Tree*. The rough membership values

$$\omega(v_1), \omega(v_2), \omega(v_3), \omega(v_4), \omega(v_5)$$

provide the rough information associated with the vertices, while the underlying graph preserves the classical tree structure.

# Chapter 7

# Conclusion

This book has presented a unified survey of Soft Graphs, Rough Graphs, and several of their principal extensions. Soft Graphs provide a parameterized framework in which each parameter determines a subgraph of an ambient graph, while Rough Graphs use lower and upper approximations to distinguish certain, possible, and boundary structures under indiscernibility. By organizing these models within a common graph-theoretic perspective, the discussion clarifies their definitions, structural properties, relationships, and roles in representing uncertain network information. The surveyed extensions demonstrate that parameterization, graded membership, approximation, bipolarity, neutrosophy, plithogenicity, and higher-order structures can be incorporated into graph models in systematic ways. These frameworks offer useful foundations for decision-making, information systems, social networks, engineering, and other applications involving incomplete or ambiguous relational data. Future research may further develop algorithms, computational complexity results, comparative studies, benchmark datasets, and real-world implementations, while also exploring deeper connections with hypergraphs, SuperHyperGraphs, machine learning, and dynamic network analysis.

# Disclaimer

## Funding

This study was conducted without any financial support from external organizations or grants.

## Acknowledgments

We would like to express our sincere gratitude to everyone who provided valuable insights, support, and encouragement throughout this research. We also extend our thanks to the readers for their interest and to the authors of the referenced works, whose scholarly contributions have greatly influenced this study. Lastly, we are deeply grateful to the publishers and reviewers who facilitated the dissemination of this work.

## Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

## Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

## Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this book.

## Code Availability

No code or software was developed for this study.

**Use of Generative AI and AI-Assisted Tools**

The authors used generative AI and AI-assisted tools only for limited language-editing tasks, such as checking English grammar and improving readability. The mathematical definitions, results, interpretations, and final verification remain the responsibility of the authors.

## Disclaimer (Others)

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

# Appendix (List of Tables)

# Appendix (List of Figures)

# Bibliography


[1] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[2] Biswajit Bera, Sk Amanathulla, and Sanat Kumar Mahato. A comprehensive study of picture fuzzy planar graphs with real-world applications. *Journal of uncertain Systems*, 16(04):2350009, 2023.

[3] Franco Scarselli, Marco Gori, Ah Chung Tsoi, Markus Hagenbuchner, and Gabriele Monfardini. The graph neural network model. *IEEE transactions on neural networks*, 20(1):61–80, 2008.

[4] Zonghan Wu, Shirui Pan, Fengwen Chen, Guodong Long, Chengqi Zhang, and S Yu Philip. A comprehensive survey on graph neural networks. *IEEE transactions on neural networks and learning systems*, 32(1):4–24, 2020.

[5] Alessandro Pagano, Raffaele Giordano, and Ivan Portoghese. A pipe ranking method for water distribution network resilience assessment based on graph-theory metrics aggregated through bayesian belief networks. *Water Resources Management*, 36(13):5091–5106, 2022.

[6] Tero Aittokallio and Benno Schwikowski. Graph-based methods for analysing networks in cell biology. *Briefings in bioinformatics*, 7(3):243–255, 2006.

[7] Yuanyuan Tian, Richard C Mceachin, Carlos Santos, David J States, and Jignesh M Patel. Saga: a subgraph matching tool for biological graphs. *Bioinformatics*, 23(2):232–239, 2007.

[8] Martin Charles Golumbic, Haim Kaplan, and Ron Shamir. On the complexity of dna physical mapping. *Advances in Applied Mathematics*, 15(3):251–261, 1994.

[9] Mirko Torrisi, Gianluca Pollastri, and Quan Le. Deep learning methods in protein structure prediction. *Computational and Structural Biotechnology Journal*, 18:1301–1310, 2020.

[10] Yuxiang Wei, Shiqi Wang, and Yun Li. Graph theory based machine learning for analog circuit design. In *2023 28th International Conference on Automation and Computing (ICAC)*, pages 1–6. IEEE, 2023.

[11] Nenad Trinajstic. *Chemical graph theory*. CRC press, 2018.

[12] Renzo Angles and Claudio Gutierrez. Survey of graph database models. *ACM Computing Surveys (CSUR)*, 40(1):1–39, 2008.

[13] Justin J Miller. Graph database applications and concepts with neo4j. In *Proceedings of the southern association for information systems conference, Atlanta, GA, USA*, pages 141–147, 2013.

[14] Andreas Brandstädt, Van Bang Le, and Jeremy P Spinrad. *Graph classes: a survey*. SIAM, 1999.

[15] Bonnie Berger, Rohit Singht, and Jinbo Xu. Graph algorithms for biological systems analysis. In *Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 142–151, 2008.

[16] George Karypis, Rajat Aggarwal, Vipin Kumar, and Shashi Shekhar. Multilevel hypergraph partitioning: Application in vlsi domain. In *Proceedings of the 34th annual Design Automation Conference*, pages 526–529, 1997.

[17] Yue Gao, Zizhao Zhang, Haojie Lin, Xibin Zhao, Shaoyi Du, and Changqing Zou. Hypergraph learning: Methods and practices. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 44(5):2548–2566, 2020.

[18] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[19] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[20] Nahlia Rakhmawati, Agus Widodo, Noor Hidayat, and Abdul Rouf Alghofari. Optimal path with interval value of intuitionistic fuzzy number in multigraph. *IAENG International Journal of Computer Science*, 51(1):39–44, 2024.

[21] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Subset Vertex Multigraphs and Neutrosophic Multigraphs for Social Multi Networks*. Infinite Study, 2019.

[22] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[23] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[24] Takaaki Fujita and Florentin Smarandache. *A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models*. Infinite Study, 2025.

[25] Takaaki Fujita. A short note on hyperfuzzy directed graphs. *International Journal of Emerging Trends in Engineering and Development (IJETED)*, 16(4):18/1–18/7, 2026.

[26] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[27] Madeleine Al Tahan, Saba Al-Kaseasbeh, and Bijan Davvaz. Neutrosophic quadruple hv-modules and their fundamental module. *Neutrosophic Sets and Systems*, 72:304–325, 2024.

[28] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[29] Qiuyan Zhao and Wentao Li. Incorporating intelligence in multiple-attribute decision-making using algorithmic framework and double-valued neutrosophic sets: Varied applications to employment quality evaluation for university graduates. *Neutrosophic Sets and Systems*, 76:59–78, 2025.

[30] Takaaki Fujita. Triple-valued neutrosophic set, quadruple-valued neutrosophic set, quintuple-valued neutrosophic set, and double-valued indetermsoft set. *Neutrosophic Systems with Applications*, 25(5):3, 2025.

[31] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[32] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, and Mohammed M Al-Shamiri. Fuzzy graph structure model based on plithogenic set theory with applications to the traveling salesman problem (tsp). *Applied Soft Computing*, page 115883, 2026.

[33] Hamiyet Merkepçi. On novel results about the algebraic properties of symbolic 3-plithogenic and 4-plithogenic real square matrices. *Symmetry*, 15(8):1494, 2023.

[34] Takaaki Fujita and Florentin Smarandache. *A Survey of Fuzzy and Uncertain Concepts in Applied Mathematics*. Infinite Study, 2025.

[35] Takaaki Fujita and Florentin Smarandache. *A comprehensive survey of set-theoretic concepts related to fuzzy, neutrosophic, and uncertain sets*. Infinite Study, 2026.

[36] Takaaki Fujita and Florentin Smarandache. *A Dynamic Survey of Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Extensional Sets*. Neutrosophic Science International Association (NSIA), 2025.

[37] Salvatore Greco, Benedetto Matarazzo, and Roman Slowinski. Rough sets theory for multicriteria decision analysis. *European journal of operational research*, 129(1):1–47, 2001.

[38] Zdzisław Pawlak and Andrzej Skowron. Rudiments of rough sets. *Information sciences*, 177(1):3–27, 2007.

[39] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[40] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[41] Florentin Smarandache. New types of soft sets: Hypersoft set, indetermsoft set, indetermhypersoft set, and treesoft set. *International Journal of Neutrosophic Science*, 2023.

[42] Bhargavi Krishnamurthy and Sajjan G Shiva. Indetermsoft-set-based d* extra lite framework for resource provisioning in cloud computing. *Algorithms*, 17(11):479, 2024.

[43] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Hypersoft expert set with application in decision making for recruitment process. *Neutrosophic Sets and Systems*, 2021.

[44] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[45] Takaaki Fujita, Arif Mehmood, Ajoy Kanti Das, Suman Das, Volkan Duran, Arkan A Ghaib, and Talal Al-Hawary. Multitree-soft, pseudotree-soft set, hypertree-soft, andtree-to-tree-soft set. *Neutrosophic Computing and Machine Learning*, 42:65–87, 2026.

[46] Takaaki Fujita, Ajoy Kanti Das, Arif Mehmood, Suman Das, and Volkan Duran. Decision analytics applications of the relationship between treesoft graphs and forestsoft graphs. *Applied Decision Analytics*, 2(1):73–92, 2026.

[47] Zhouyi Jin. Using the treesoft set and aroman approach for evaluating environmental design quality in urban parks: An integrated methodology. *Neutrosophic Sets and Systems*, 83(1):17, 2025.

[48] M Myvizhi, Ahmed A Metwaly, and Ahmed M Ali. Treesoft approach for refining air pollution analysis: A case study. *Neutrosophic Sets and Systems*, 68(1):17, 2024.

[49] Fazli Amin, Zia Ullah, Sidra Niaz, Sk A Shezan, Naveed Ahmad, Rayan Hamza Alsisi, Nowshad Amin, and Farooq Ahmad. A decision-making framework based on complex fuzzy bipolar soft sets for investment strategy optimization. *Scientific Reports*, 2026.

[50] Hind Y Saleh, Areen A Salih, Baravan A Asaad, and Ramadhan A Mohammed. Binary bipolar soft points and topology on binary bipolar soft sets with their symmetric properties. *Symmetry*, 16(1):23, 2023.

[51] Zigeng Wen, Jie Liu, and Jianchang Huang. Hybrid superhypersoft model for evaluation of comprehensive transportation efficiency: Analysis and results. *Neutrosophic Sets and Systems*, 81(1):46, 2025.

[52] Manal Elzain Mohamed Abdalla, Anns Uzair, AiIman Ishtiaq, Muhammad Tahir, and Muhammad Kamran. Algebraic structures and practical implications of interval-valued fermatean neutrosophic super hypersoft sets in healthcare. *Spectrum of Operational Research*, 2(1):240–259, 2025.

[53] Mona Mohamed, Ahmed M AbdelMouty, Khalid Mohamed, and Florentin Smarandache. Superhypersoft-driven evaluation of smart transportation in centroidous-moosra: Real-world insights for the uav era. *Neutrosophic Sets and Systems*, 78:149–163, 2025.

[54] Takaaki Fujita and Florentin Smarandache. A dynamic survey of soft set theory and its extensions. *arXiv preprint arXiv:2602.21268*, 2026.

[55] Bobin George, Jinta Jose, and Rajesh K Thumbakara. Soft graph products: A soft subgraph perspective. *New Mathematics and Natural Computation*, pages 1–26, 2026.

[56] Bobin George, Jinta Jose, and Rajesh K Thumbakara. Cartesian product and composition of soft semigraphs. *New Mathematics and Natural Computation*, 21(01):37–60, 2025.

[57] Takaaki Fujita and Florentin Smarandache. *Neutrosophic soft n-super-hypergraphs with real-world applications.* Infinite Study, 2025.

[58] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[59] Zdzislaw Pawlak. Rough set theory and its applications to data analysis. *Cybernetics & Systems*, 29(7):661–688, 1998.

[60] Nouman Azam and JingTao Yao. Analyzing uncertainties of probabilistic rough set regions with game-theoretic rough sets. *International journal of approximate reasoning*, 55(1):142–155, 2014.

[61] Wojciech Ziarko. Probabilistic rough sets. In *Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing: 10th International Conference, RSFDGrC 2005, Regina, Canada, August 31-September 3, 2005, Proceedings, Part I 10*, pages 283–293. Springer, 2005.

[62] Yiyu Yao. Two semantic issues in a probabilistic rough set model. *Fundamenta Informaticae*, 108(3-4):249–265, 2011.

[63] Suyun Zhao, Eric CC Tsang, and Degang Chen. The model of fuzzy variable precision rough sets. *IEEE transactions on Fuzzy Systems*, 17(2):451–467, 2009.

[64] Chao-Ton Su and Jyh-Hwa Hsu. Precision parameter in the variable precision rough sets model: an application. *Omega*, 34(2):149–157, 2006.

[65] Malcolm Beynon. Reducts within the variable precision rough sets model: a further investigation. *European journal of operational research*, 134(3):592–605, 2001.

[66] Noor Rehman, Nasir Shah, Muhammad Irfan Ali, and Choonkil Park. Uncertainty measurement for neighborhood based soft covering rough graphs with applications. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 113:2515–2535, 2019.

[67] S Robinson Chellathurai and L Jesmalar. Core in rough graph and weighted rough graph. *International Journal of Contemporary Mathematical Sciences*, 11(6):251–265, 2016.

[68] Tong He, Yong Chen, and Kaiquan Shi. Weighted rough graph and its application. In *Sixth International Conference on Intelligent Systems Design and Applications*, volume 1, pages 486–491. IEEE, 2006.

[69] Daniela Gifu. Soft sets extensions: Innovating healthcare claims analysis. *Applied Sciences*, 14(19):8799, 2024.

[70] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

[71] Zdzislaw Pawlak, Jerzy Grzymala-Busse, Roman Slowinski, and Wojciech Ziarko. Rough sets. *Communications of the ACM*, 38(11):88–95, 1995.

[72] Takaaki Fujita and Florentin Smarandache. A unified framework for $u$-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.

[73] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[74] P. R. Kavyasree and B. Surender Reddy. N-cubic sets applied to linear spaces. *Kragujevac Journal of Mathematics*, 2022.

[75] Gundeti Soujanya and B. Surender Reddy. N-cubic picture fuzzy linear spaces. *Indian Journal Of Science And Technology*, 2024.

[76] Young Bae Jun, Seok Zun Song, and Seon Jeong Kim. N-hyper sets. *Mathematics*, 2018.

[77] Witold Pedrycz. Shadowed sets: representing and processing fuzzy sets. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 28(1):103–109, 1998.

[78] Mingjie Cai, Qingguo Li, and Guangming Lang. Shadowed sets of dynamic fuzzy sets. *Granular Computing*, 2:85 – 94, 2016.

[79] Gianpiero Cattaneo and Davide Ciucci. Shadowed sets and related algebraic structures. *Fundamenta Informaticae*, 55(3-4):255–284, 2003.

[80] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[81] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[82] RA Borzooei and Hossein Rashmanlou. Cayley interval-valued fuzzy graphs. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, 78(3):83–94, 2016.

[83] Madhumangal Pal, Sovan Samanta, and Hossein Rashmanlou. Some results on interval-valued fuzzy graphs. *International Journal of Computer Science and Electronics Engineering*, 3(3):205–211, 2015.

[84] Xindong Peng. New operations for interval-valued pythagorean fuzzy set. *Scientia Iranica*, 26:1049–1076, 2018.

[85] Weijian Ren and Yuhong Du. Development and application of an interval-valued pythagorean fuzzy set-based grade evaluation method to foreign fiber content. *Journal of Natural Fibers*, 20, 2023.

[86] Manish Kumar Gunjan, Amal Kumar Adak, and Davood Darvishi Salookolaei. Fermatean fuzzy artinian and noetherian rings. *Asian Research Journal of Mathematics*, 21(5):179–190, 2025.

[87] Fei Gao, Meihong Han, Siyang Wang, and Jie Gao. A novel fermatean fuzzy bwm-vikor based multi-criteria decision-making approach for selecting health care waste treatment technology. *Eng. Appl. Artif. Intell.*, 127:107451, 2024.

[88] Ping Wang, Jie Wang, Guiwu Wei, and Cun Wei. Similarity measures of q-rung orthopair fuzzy sets based on cosine function and their applications. *Mathematics*, 2019.

[89] Muhmamet Deveci, Dragan Pamucar, Ilgin Gokasar, Mario Köppen, and Brij Bhooshan Gupta. Personal mobility in metaverse with autonomous vehicles using q-rung orthopair fuzzy sets based opa-rafsi model. *IEEE Transactions on Intelligent Transportation Systems*, 24:15642–15651, 2023.

[90] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[91] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[92] Juanjuan Lu, Linli Zhu, and Wei Gao. Structured representation of fuzzy data by bipolar fuzzy hypergraphs. In *International Conference on Machine Learning for Cyber Security*, pages 663–676. Springer, 2022.

[93] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pages 305–309, 1994.

[94] Chen Shouyu and Guo Yu. Variable fuzzy sets and its application in comprehensive risk evaluation for flood-control engineering system. *Fuzzy Optimization and Decision Making*, 5(2):153–162, 2006.

[95] Wen-chuan Wang, Dong-mei Xu, Kwok-wing Chau, and Guan-jun Lei. Assessment of river water quality based on theory of variable fuzzy sets and fuzzy binary comparison method. *Water resources management*, 28(12):4183–4200, 2014.

[96] Yunhai Fang, Xilai Zheng, Hui Peng, Huan Wang, and Jia Xin. A new method of the relative membership degree calculation in variable fuzzy sets for water quality assessment. *Ecological indicators*, 98:515–522, 2019.

[97] Vasile Patrascu. A new penta-valued logic based knowledge representation. *arXiv preprint arXiv:1502.05562*, 2015.

[98] Vasile Patrascu. Similarity, cardinality and entropy for bipolar fuzzy set in the framework of penta-valued representation. *arXiv preprint arXiv:1506.02060*, 2015.

[99] Yong-Uk Cho, Young-Bae Jun, and Seok-Zun Song. Bifuzzy ideals of pseudo mv-algebras. *JOURNAL OF APPLIED MATHEMATICS AND COMPUTING*, 22(1/2):475, 2006.

[100] Tadeusz Gerstenkorn and Jacek Mańko. Bifuzzy probabilistic sets. *Fuzzy sets and systems*, 71(2):207–214, 1995.

[101] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[102] Bui Cong Cuong. Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, 30:409, 2015.

[103] Sankar Das, Ganesh Ghorai, and Madhumangal Pal. Picture fuzzy tolerance graphs with application. *Complex & Intelligent Systems*, 8(1):541–554, 2022.

[104] G Muhiuddin, Sun Shin Ahn, Chang Su Kim, and Young Bae Jun. Stable cubic sets. *J. Comput. Anal. Appl*, 23(5):802–819, 2017.

[105] Madad Khan, Young Bae Jun, Muhammad Gulistan, and Naveed Yaqoob. The generalized version of jun's cubic sets in semigroups. *Journal of Intelligent & Fuzzy Systems*, 28(2):947–960, 2015.

[106] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[107] Fatma Kutlu Gündoğdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *Journal of intelligent & fuzzy systems*, 36(1):337–352, 2019.

[108] Khushbakhat Asif, Muhammad Kamran Jamil, Hanen Karamti, Muhammad Azeem, and Kifayat Ullah. Randić energies for t-spherical fuzzy hamacher graphs and their applications in decision making for business plans. *Computational and Applied Mathematics*, 42:1–32, 2023.

[109] Abhishek Guleria and Rakesh Kumar Bajaj. T-spherical fuzzy graphs: operations and applications in various selection processes. *Arabian Journal for Science and Engineering*, 45(3):2177–2193, 2020.

[110] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a $\gamma$-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[111] Takaaki Fujita. A mathematical study of tripolar intuitionistic fuzzy graphs. *International Journal of Computer Application (IJCA)*, 16(4):16/1–16/11, 2026.

[112] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, page 731743, 2019.

[113] M Murali Krishna Rao. Tripolar fuzzy interior ideals of $\gamma$-semigroup. *Ann. Fuzzy Math. Inform*, 15(2):199–206, 2018.

[114] Shawkat Alkhazaleh. Neutrosophic vague set theory. *Critical Review*, 10:29–39, 2015.

[115] Ashraf Al-Quran, Hazwani Hashim, and Lazim Abdullah. A hybrid approach of interval neutrosophic vague sets and dematel with new linguistic variable. *Symmetry*, 12(2):275, 2020.

[116] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.

[117] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[118] Mouhamed Bayane Bouraima, Ertugrul Ayyıldız, Melike Erdogan, and Dragan Pamucar. An interval-valued intuitionistic fuzzy group decision model for evaluation of cross-border railway development. *Cognitive Computation*, 18(1):22, 2026.

[119] Ravindar Raj Chinnappan and Dhanasekar Sundaram. Image enhancement framework combining interval-valued intuitionistic fuzzy sets and fractional sobel operator. *Artificial Intelligence Review*, 58(9):291, 2025.

[120] Yukun Hu, Hong Wang, Chunhui Li, Ang Xu, Li Yi, Chao Liang, Bingyang Xi, and Fang Dai. An integrated approach for design concept evaluation based on interval-valued pythagorean fuzzy set and consensus model. *Kybernetes*, 55(8):3611–3636, 2026.

[121] Yoojeong Oh, Hans Pasman, Safyan Akram Khan, and Sunhwa Park. Hydrogen storage selection for saudi arabia: A multi-criteria decision making under interval-valued pythagorean fuzzy environment. *International Journal of Hydrogen Energy*, 105:1281–1293, 2025.

[122] Abdul Wahab Mustafa, Zia Bashir, Jawad Ali, and Muhammed I Syam. A unified framework of information measures for complex linear diophantine fuzzy sets with application to chronic kidney disease. *Scientific Reports*, 2026.

[123] Masooma Raza Hashmi, Muhammad Riaz, Arshid Mahmood, Muhammad Ajmaeen, and Muhammad Aslam. Optimizing performance metrics with new distance and similarity measures using control parameters of linear diophantine fuzzy sets. *Engineering Applications of Artificial Intelligence*, 162:112557, 2025.

[124] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[125] Bin Zhu and Zeshui Xu. Some results for dual hesitant fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 26(4):1657–1668, 2014.

[126] José Carlos R Alcantud, Gustavo Santos-García, Xindong Peng, and Jianming Zhan. Dual extended hesitant fuzzy sets. *Symmetry*, 11(5):714, 2019.

[127] Florentin Smarandache. Ambiguous set is a subclass of the double refined indeterminacy neutrosophic set, and of the refined neutrosophic set in general. *Neutrosophic Sets & Systems*, 58, 2023.

[128] Pritpal Singh et al. Ambiguous set theory: A new approach to deal with unconsciousness and ambiguousness of human perception. *Full Length Article*, 5(1):52–2, 2023.

[129] Florentin Smarandache and F Smarandache. Ambiguous set brings nothing new. *nidus idearum*, page 24, 2024.

[130] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[131] Takaaki Fujita. Review of rough turiyam neutrosophic directed graphs and rough pentapartitioned neutrosophic directed graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:48–79, 2025.

[132] Takaaki Fujita and Florentin Smarandache. Pythagorean, fermatean, and complex turiyam neutrosophic graphs. *SciNexuses*, 2:39–63, 2025.

[133] Prem Kumar Singh, Naveen Surathu, Ghattamaneni Surya Prakash, et al. Turiyam based four way unknown profile characterization on social networks. *Full Length Article*, 10(2):27–7, 2024.

[134] Rama Mallick and Surapati Pramanik. Pentapartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 35:49, 2020.

[135] Radha R. and Stanis Arul Mary. Pentapartitioned neutrosophic pythagorean set. *International Research Journal on Advanced Science Hub*, 2021.

[136] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.

[137] Jiarong Jia. Triple-valued neutrosophic off for public digital cultural service level evaluation in craft and art museums. *Neutrosophic Sets and Systems*, 88(1):44, 2025.

[138] Hongxin Wang. Professional identity formation in traditional chinese medicine students: An educational perspective using triple-valued neutrosophic set. *Neutrosophic Sets and Systems*, 88:83–92, 2025.

[139] Fang Sun, Mifeng Ren, Yujing Shi, and Xuanbai Feng. Quadruple-valued neutrosophic offset for maintenance platforms design quality evaluation of digital monitoring and remote operation in new energy power generation systems. *Neutrosophic Sets and Systems*, 88:940–953, 2025.

[140] Maha Mohammed Saeed, Sami Ullah Khan, Fatima Suriyya, Arif Mehmood, and Jamil J Hamja. Interval-valued complex neutrosophic sets and complex neutrosophic soft topological spaces. *International Journal of Analysis and Applications*, 23:132–132, 2025.

[141] Samaneh Salari and Ali Karimi. Introducing an integrated approach for fire safety assessment in healthcare facilities by interval valued neutrosophic-ahp and fuzzy inference system. *Heliyon*, 11(2), 2025.

[142] Raed Hatamleh, Nasir Odat, Abdallah Al-Husban, Arif Mehmood Khattak, Alaa M Abd El-latif, Husham M Attaalfadeel, Walid Abdelfattah, and Ahmad M Abdel-Mageed. Mapping love: A heptapartitioned neutrosophic machine learning study of university students' romantic sensations. *European Journal of Pure and Applied Mathematics*, 18(3):6484–6484, 2025.

[143] Maha Mohammed Saeed, Raed Hatamleh, Hamza Ali Abujabal, Yahya Khan, Abdallah Al-Husban, Amy A Laja, Sulfaisa MM Pangilan, Cris L Armada, Jamil J Hamja, and Arif Mehmood Khattak. Heptapartitioned neutrosophic soft topologies and machine learning techniques for exploring romantic feelings. *European Journal of Pure and Applied Mathematics*, 18(3):6221–6221, 2025.

[144] Wenwen Meng. Quintuple-valued neutrosophic offset for quality evaluation of cross-border e-commerce talent training based on artificial intelligence. *Neutrosophic Sets and Systems*, 88(1):54, 2025.

[145] Yuan Sun. Assessing the performance of basic education informatization in the digital age using quintuple-valued neutrosophic set. *Neutrosophic Sets and Systems*, 88(1):18, 2025.

[146] Takaaki Fujita, Raed Hatamleh, and Ahmed Heilat. Advanced partitioned neutrosophic offsets, oversets, and undersets: Modeling "neither agree nor disagree" and beyond. *Statistics, Optimization & Information Computing*, 2026.

[147] Takaaki Fujita. Advanced partitioned neutrosophic sets: Formalization of hexa-, hepta-, octa-, nona-, and deca-partitioned structures. *Abhath Journal of Basic and Applied Sciences*, 4(2):40–60, 2025.

[148] CV Malavika, Kalaivani Chandran, Sofia Jennifer John, and Florentin Smarandache. Python based tool for neutrosophic cubic sets and their operations. *Journal of Fuzzy Extension and Applications*, 7(2):592–614, 2026.

[149] Adil Darvesh, Khalid Naeem, and Syed Bilal Hussain Shah. Scrutiny of neutrosophic cubic fuzzy data with different parametric values using bonferroni weighted mean: A robust mcdm approach. *Decision making advances*, 3(1):245–265, 2025.

[150] Giovanni Panti. Multi-valued logics. In *Quantified representation of uncertainty and imprecision*, pages 25–74. Springer, 1998.

[151] Fujita Takaaki and Arif Mehmood. Iterative multifuzzy set, iterative multineutrosophic set, iterative multisoft set, and multiplithogenic sets. *Neutrosophic Computing and Machine Learning*, 41:1–30, 2025.

[152] Muhammad Arshad, Muhammad Saeed, and Atiqe Ur Rahman. Convexity cum concavity on refined fuzzy set with some properties. *Uncertainty Discourse and Applications*, 1(1):140–150, 2024.

[153] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[154] Vakkas Uluçay. Some concepts on interval-valued refined neutrosophic sets and their applications. *Journal of Ambient Intelligence and Humanized Computing*, 12(7):7857–7872, 2021.

[155] Florentin Smarandache. *Introduction to the multineutrosophic set*. Infinite Study, 2023.

[156] Florentin Smarandache. Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set), pythagorean fuzzy set, spherical fuzzy set, and q-rung orthopair fuzzy set, while neutrosophication is a generalization of regret theory, grey system theory, and three-ways decision (revisited). *Journal of New Theory*, 29:1–31, 2019.

[157] Florentin Smarandache. *Nidus Idearum. Scilogs, XIII: Structure/NeutroStructure/AntiStructure*. Infinite Study, 2024.

[158] Hossein Rashmanlou, Sovan Samanta, Madhumangal Pal, and Rajab Ali Borzooei. Intuitionistic fuzzy graphs with categorical properties. *Fuzzy information and Engineering*, 7(3):317–334, 2015.

[159] Zehui Shao, Saeed Kosari, Muhammad Shoaib, and Hossein Rashmanlou. Certain concepts of vague graphs with applications to medical diagnosis. *Frontiers in physics*, 8:357, 2020.

[160] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[161] Sundas Shahzadi and Muhammad Akram. Coloring of bifuzzy graphs. *Ital. J. Pure Appl. Math*, 36:429–444, 2016.

[162] Muhammad Akram and Karamat H Dar. Interval-valued bifuzzy graphs. In *Proceedings Of The International Conference On Algebra 2010: Advances in Algebraic Structures*, pages 1–10. World Scientific, 2012.

[163] Suman Das, Rakhal Das, and Surapati Pramanik. Single valued pentapartitioned neutrosophic graphs. *Neutrosophic Sets and Systems*, 50(1):225–238, 2022.

[164] S krishna Prabha, Said Broumi, Souhail Dhouib, and Mohamed Talea. Implementation of circle-breaking algorithm on fermatean neutrosophic graph to discover shortest path. *Neutrosophic Sets and Systems*, 72:256–271, 2024.

[165] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, 10:86–101, 2016.

[166] Xinyu Guo, Bingjie Tian, and Xuedong Tian. Hfgnn-proto: Hesitant fuzzy graph neural network-based prototypical network for few-shot text classification. *Electronics*, 11(15):2423, 2022.

[167] Takaaki Fujita. On tripolar fuzzy offgraph. *INTERNATIONAL JOURNAL OF COMPUTER APPLICATION (IJCA)*, 16(4):13, 2026.

[168] Takaaki Fujita. A mathematical study of tripolar neutrosophic graphs. *INTERNATIONAL JOURNAL OF RESEARCH IN ENGINEERING & SCIENCE (IJRES)*, 16(4):13, 2026.

[169] Sk Amanathulla, Biswajit Bera, and Madhumangal Pal. Balanced picture fuzzy graph with application. *Artificial Intelligence Review*, 54(7):5255–5281, 2021.

[170] Waqar Arif, Waheed Ahmad Khan, Asghar Khan, and Hossein Rashmanlou. Some indices of picture fuzzy graphs and their applications. *Computational and Applied Mathematics*, 42:1–23, 2023.

[171] B. Mohamed Harif and A. Nazeera Begam. On pseudo regular spherical fuzzy graphs. *Ratio Mathematica*, 50, 2023.

[172] M Ramya, S Murali, R Radha, and R Princy. A note on different types of product of fermatean quadripartitioned neutrosophic graphs. *Neutrosophic Sets and Systems*, 97:237–264, 2026.

[173] V Divya and J Jesintha Rosline. Fermatean quadripartitioned neutrosophic fuzzy graph. *Reliability: Theory & Applications*, 20(3 (86)):816–828, 2025.

[174] Tianming Zhang, Dayong Wang, and Yejun Xu. Consensus stability among composite decision makers in the framework of hesitant fuzzy graph model with application to doctor–patient disputes. *Scientific Reports*, 15(1):24665, 2025.

[175] Nikita Prajapati and Gautam Patel. Extending hesitant fuzzy graphs to bipolar hesitant frameworks for capturing positive and negative uncertainty. *International Journal of Computer Information Systems and Industrial Management Applications*, 18(8s):678–688, 2026.

[176] Sumera Naz, Muhammad Akram, Samirah Alsulami, and Faiza Ziaa. Decision-making analysis under interval-valued q-rung orthopair dual hesitant fuzzy environment. *Int. J. Comput. Intell. Syst.*, 14(1):332–357, 2021.

[177] Takaaki Fujita and Florentin Smarandache. *A compact exploration of turiyam neutrosophic competition graphs.* Infinite Study, 2024.

[178] Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache, and Faruk Karaaslan. Interval valued neutrosophic soft graphs. *Collected Papers. Volume VI: On Neutrosophic Theory and Applications*, page 475, 2022.

[179] Takaaki Fujita, Arif Mehmood, and A Ghaib Arkan. Single-valued, double-valued, triple-valued, quadruple-valued, and quintuple-valued neutrosophic graph. *Neutrosophic Systems with Applications*, 26(2):2, 2026.

[180] M Myvizhi, Ahmed Abdelhafeez, Mohamed Refaat Abdellah, Mohamed Eassa, Mohamed S Sawah, Hussam Elbehiery, et al. Single-valued neutrosophic graph with heptapartitioend structure. *Neutrosophic Sets and Systems*, 80:728–748, 2025.

[181] R Muthuraj, K Krithika, and S Revathi. Various product on multi fuzzy graphs. *Ratio Mathematica*, 44:231, 2022.

[182] Takaaki Fujita. Metahypergraphs, metasuperhypergraphs, and iterated metagraphs: Modeling graphs of graphs, hypergraphs of hypergraphs, superhypergraphs of superhypergraphs, and beyond. *Current Research in Interdisciplinary Studies*, 4(5):1–23, 2025.

[183] Muhammad Akram and Anam Luqman. Intuitionistic single-valued neutrosophic hypergraphs. *Opsearch*, 54:799–815, 2017.

[184] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[185] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.

[186] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: A survey. In *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázne, Czech Republic, August 27–31, 2001 Proceedings 26*, pages 37–57. Springer, 2001.

[187] Song Feng, Emily Heath, Brett Jefferson, Cliff Joslyn, Henry Kvinge, Hugh D Mitchell, Brenda Praggastis, Amie J Eisfeld, Amy C Sims, Larissa B Thackray, et al. Hypergraph models of biological networks to identify genes critical to pathogenic viral response. *BMC bioinformatics*, 22(1):287, 2021.

[188] Nicole Pearcy, Jonathan J Crofts, and Nadia Chuzhanova. Hypergraph models of metabolism. *International Journal of Biological, Veterinary, Agricultural and Food Engineering*, 8(8):752–756, 2014.

[189] Md Tanvir Alam, Chowdhury Farhan Ahmed, Md Samiullah, and Carson K Leung. Mining frequent patterns from hypergraph databases. In *Pacific-Asia Conference on Knowledge Discovery and Data Mining*, pages 3–15. Springer, 2021.

[190] Borislav Iordanov. Hypergraphdb: a generalized graph database. In *Web-Age Information Management: WAIM 2010 International Workshops: IWGD 2010, XMLDM 2010, WCMT 2010, Jiuzhaigou Valley, China, July 15-17, 2010 Revised Selected Papers 11*, pages 25–36. Springer, 2010.

[191] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.

[192] Yihe Dong, Will Sawin, and Yoshua Bengio. Hnhn: Hypergraph networks with hyperedge neurons. *arXiv preprint arXiv:2006.12278*, 2020.

[193] Yan Cao. Integrating treesoft and hypersoft paradigms into urban elderly care evaluation: A comprehensive n-superhypergraph approach. *Neutrosophic Sets and Systems*, 85:852–873, 2025.

[194] Nelly Hodelín Amable, Elizabeth Esther Vergel De Salazar, Martha Gloria Martínez Isaac, Olivia Catalina Olavarría Sánchez, and Johanna Mariuxi Solís Palma. Representation of motivational dynamics in school environments through plithogenic n-superhypergraphs with family participation. *Neutrosophic Sets and Systems*, 92:570–583, 2025.

[195] Takaaki Fujita. Recursive hypergraphs and recursive superhypergraphs: Exploring more hierarchical and generalized graph concepts. *Neutrosophic Knowledge*, 10:76–90, 2026.

[196] Fujita Takaaki. Short introduction to superhypergraph theory with some applications. *International Journal of Open Problems in Computer Science and Mathematics*, 19(2):74–86, 2026.

[197] Takaaki Fujita. Recursive intuitionistic fuzzy superhypergraphs. *Journal of Analytical Uncertainty*, 1(2):52–64, 2026.

[198] Takaaki Fujita and Florentin Smarandache. *Hyperstructures and Superhyperstructures: Theory and Selected Applications*. Infinite Study, 2026.

[199] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (VII): About Directed SuperHyperGraph*. Infinite Study, 2026.

[200] Takaaki Fujita and Florentin Smarandache. *Representing Higher-Order Networks: A Survey of Graph-Based Frameworks*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[201] T. Fujita and Florentin Smarandache. *Hyperstructures and Superhyperstructures: Theory and Selected Applications*. Neutrosophic Science International Association (NSIA) Publishing House, March 2026.

[202] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[203] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[204] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[205] Huda E Khali, Gonca D GÜNGÖR, and Muslim A Noah Zaina. Neutrosophic superhyper bi-topological spaces: Original notions and new insights. *Neutrosophic Sets and Systems*, 51(1):3, 2022.

[206] Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today*. Infinite Study, 2022.

[207] Muhammad Akram and Saira Nawaz. Certain types of soft graphs. *UPB Sci. Bull., Ser. A*, 78(4):67–82, 2016.

[208] Jinta Jose, Bobin George, and Rajesh K Thumbakara. Soft graphs: A comprehensive survey. *New Mathematics and Natural Computation*, pages 1–52, 2024.

[209] Muhammad Saeed, Atiqe Ur Rahman, Muhammad Ahsan, and Florentin Smarandache. Theory of hypersoft sets: Axiomatic properties, aggregation operations, relations, functions. *Collected Papers, Volume XVII: A Multi-Disciplinary Approach to Uncertainty, Indeterminacy, and Complex Systems*, page 462, 2025.

[210] Atiqe Ur Rahman, Diaa S. Metwally, M. M. Abd El-Raouf, M. A. El-Qurashi, Hamiden Abd El-Wahed Khalifa, and Khadiga Wadi Nahar Tajer. Application in multi-attribute decision-making using possibility single-valued neutrosophic hypersoft graphs. *Journal of Statistics Applications & Probability*, 15(1):69–96, 2026.

[211] Muhammad Saeed, Muhammad Amad Sarwar, Atiqe Ur Rahman, and Sana Naz. Representation of fuzzy hypersoft set in graphs. *Palestine journal of mathematics*, 12(1):836–47, 2023.

[212] Muhammad Saeed, Muhammad Imran Harl, Muhammad Haris Saeed, and Ibrahim Mekawy. Theoretical framework for a decision support system for micro-enterprise supermarket investment risk assessment using novel picture fuzzy hypersoft graph. *Plos one*, 18(3):e0273642, 2023.

[213] R Keerthana and S Venkatesh. A systematic study of intercropping agricultural system using fuzzy and inverse fuzzy hypersoft hypergraph. *Journal of Applied Mathematics and Computing*, 71(3):4067–4094, 2025.

[214] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[215] Bobin George, Jinta Jose, and Rajesh K Thumbakara. An introduction to soft hypergraphs. *Journal of Prime Research in Mathematics*, 18(1):43–59, 2022.

[216] Takaaki Fujita and Florentin Smarandache. *Soft directed n-superhypergraphs with some real-world applications*. Infinite Study, 2025.

[217] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (VIII): Hierarchical SuperHyperGraph, Recursive SuperHyperGraph, and Related Graph Theory*. Infinite Study, 2026.

[218] Vakkas Uluçay, Zeynep Baser, and Seyyed Ahmad Edalatpanah. Euclidean distance on treesoft sets and application for electric vehicle selection. *Journal of fuzzy extension and applications*, 6(4):814–826, 2025.

[219] T Fujita and F Smarandache. Quantum-treesoft set and quantum-forestsoft set. *Journal of Analysis and Applications*, 23(2):87–116, 2025.

[220] Zigeng Wen, Jie Liu, and Jianchang Huang. Decision support model with treesoft set for assessment of coupled and coordinated development of digital transportation technology and industry. *Neutrosophic Sets & Systems*, 82:478, 2025.

[221] Yue Wang. Comprehensive analysis of application of multi-criteria decision-making methods with treesoft set for teaching quality improvement of college english learning process. *Neutrosophic Sets and Systems*, 82(1):37, 2025.

[222] Adebisi Sunday Adesina. Further operations (complement, intersection, union) for indetermsoft set, indetermhypersoft set, and treesoft set and their applications. *Journal of Basic & Applied Sciences*, 21:47–52, 2025.

[223] Hairong Luo. Forestsoft set approach for estimating innovation and entrepreneurship education in universities through a hierarchical and uncertainty-aware analytical framework. *Neutrosophic Sets and Systems*, 86:332–342, 2025.

[224] Takaaki Fujita and Florentin Smarandache. *Quantum-TreeSoft Set and Quantum-ForestSoft Set*. Infinite Study, 2025.

[225] Li Song, Jianyong Liu, Han Ding, and Wenhui Zhang. Forestsoft set for mechanical automation production control systems analysis based on an intelligent manufacturing environment. *Neutrosophic Sets and Systems*, 85:229–254, 2025.

[226] Jianguo Liu and Ruohan Liu. A relational forestsoft set approach to modeling and optimizing teaching quality in university english translation programs amid digital transformation. *Neutrosophic Sets and Systems*, 86:184–200, 2025.

[227] Savita Rathee, Diksha Goyal, Meenakshi Hooda, Ridam Girdhar, and Savita Ahlawat. On operations of indetermsoft sets. *Neutrosophic Sets & Systems*, 99:503, 2026.

[228] Takaaki Fujita, Ajoy Kanti Das, and Suman Das. Indetermsoft sets: A multi-valued framework for modeling indeterminacy in decision science. *Journal of Contemporary Decision Science*, 2(1):97–113, 2026.

[229] Takaaki Fujita, K Anitha, Arif Mehmood, Arkan A Ghaib, and Atiqe Ur Rahman. Modeling directional uncertainty for sustainability: Indetermsoft and indetermhypersoft multi-directed sets. *International Scientific Spectrum*, 1(1):43–79, 2025.

[230] Takaaki Fujita and Florentin Smarandache. Indetermsuperhypersoft sets, indetermforestsoft sets, and indetermtreesoft sets. *SuperHyperTopologies and SuperHyperStructures with their Applications*, page 23, 2025.

[231] Shawkat Alkhazaleh, Abdul Razak Salleh, Nasruddin Hassan, and Abd Ghafur Ahmad. Multisoft sets. In *Proc. 2nd International Conference on Mathematical Sciences*, pages 910–917, 2010.

[232] Florentin Smarandache. *Practical applications of IndetermSoft Set and IndetermHyperSoft Set and introduction to TreeSoft Set as an extension of the MultiSoft Set*. Infinite Study, 2022.

[233] Takaaki Fujita. Note for soft multiexpert graph and multisoft multiexpert graph. *Integra: Journal of Integrated Mathematics and Computer Science*, 3(1):6–15, 2026.

[234] Orhan Dalkılıç. A stability-oriented virtual fuzzy soft set framework inspired by topological perspectives for decision analysis. *International Journal of Systems Science*, pages 1–17, 2026.

[235] D Rajalakshmi, K Kannan, A Menaga, G Subashini, and S Punitha. Advancing sustainable agriculture through multi-parameter fuzzy soft set–based plant disease classification. *Scientific Reports*, 2026.

[236] Ajoy Kanti Das, Nandini Gupta, Suman Patra, Carlos Granados, and Suman Das. An innovative weighted fuzzy soft set-based model for multi-criteria groundwater quality assessment in urban areas: A case study of agartala, india. *Geofísica Internacional*, 65(3):2269–2296, 2026.

[237] P Chellamani, D Ajay, Said Broumi, and T Antony Alphonse Ligori. An approach to decision-making via picture fuzzy soft graphs. *Granular computing*, pages 1–22, 2021.

[238] Muhammad Akram and Saira Nawaz. Fuzzy soft graphs with applications. *Journal of Intelligent & Fuzzy Systems*, 30(6):3619–3632, 2016.

[239] Mehmet Şahīn and Vakkas Ulucay. Fuzzy soft expert graphs with application. *Asian Journal of Mathematics and Computer Research*, 26(4):216–229, 2019.

[240] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Collected Papers, Volume XVI: Neutrosophic Theories and Applications; Neutrosophic Statistics; Advances in Plithogenic Sets; Logic & Thought; Physics & Cosmology*, page 444, 2025.

[241] Sania Saleem, Maria Riaz, Fatima Razaq, and Muhammad Saeed. Dynamic intuitionistic fuzzy soft set-based waspas model for multi-criteria decision making in renewable energy selection. *Argumentation Based Systems Journal*, 2:208–235, 2026.

[242] Qonita Qurratu Aini, Imam Mukhlash, Fatia Fatimah, and Dian Winda Setyawati. Energy of intuitionistic fuzzy soft set for multi-criteria decision-making. In *International Conference on Intelligent and Fuzzy Systems*, pages 399–406. Springer, 2025.

[243] Thanh Nha Nguyen and Quang-Thinh Bui. A framework of near-intuitionistic fuzzy soft sets with applications to uncertainty-based diagnosis. *Journal of Intelligent & Fuzzy Systems*, page 18758967261435701, 2026.

[244] Gözde Yaylalı Umul. Decision-making method that prioritizes user ranking by using intuitionistic fuzzy soft set. *Karadeniz Fen Bilimleri Dergisi*, 15(2):764–786, 2025.

[245] A Al-Jarrah, MO Massa'deh, AEK Baareh, MA Aljarrah, RM Alogla, and N Abu-Darwish. A study on r-edge regular intuitionistic fuzzy graphs. *J Math Comput Sci*, 23(4):279–288, 2021.

[246] Naveed Yaqoob, Muhammad Gulistan, Seifedine Kadry, and Hafiz Abdul Wahab. Complex intuitionistic fuzzy graphs with application in cellular network provider companies. *Mathematics*, 7(1):35, 2019.

[247] Sankar Sahoo and Madhumangal Pal. Product of intuitionistic fuzzy graphs and degree. *Journal of Intelligent & Fuzzy Systems*, 32(1):1059–1067, 2017.

[248] MG Karunambigai and V Gnaneswaran. Matrix representation of intuitionistic fuzzy soft graph. *International Journal of Special Education*, 41(7s):1095–1103, 2026.

[249] Uttam Mondal, Madhumangal Pal, and Kalyani Das. A study on bipolar intuitionistic fuzzy soft graphs and its application. *TWMS Journal of Applied and Engineering Mathematics*, 2025.

[250] G Suriyakumar, VJ Sudhakar, Syed Tahir Hussainy, and Mana Donganont. Complex neutrosophic vague soft graphs. *Neutrosophic Sets and Systems*, 87:785–810, 2025.

[251] Sundas Shahzadi and Muhammad Akram. Edge regular intuitionistic fuzzy soft graphs. *Journal of Intelligent & Fuzzy Systems*, 31(3):1881–1895, 2016.

[252] Siti Nurul Fitriah Mohamad, Roslan Hasni, Florentin Smarandache, and Binyamin Yusoff. Novel concept of energy in bipolar single-valued neutrosophic graphs with applications. *Axioms*, 10(3):172, 2021.

[253] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[254] Yıldıray Çelik. Renewed structure of neutrosophic soft graphs and its application in decision-making problem. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 72(4):1055–1076, 2023.

[255] RV Jaikumar[1], R Sundareswaran, and Said Broumi. Integrity and domination integrity in neutrosophic soft graphs. *Neutrosophic Sets and Systems, vol. 53/2023: An International Journal in Information Science and Engineering*, page 165, 2023.

[256] Vakkas Uluçay. Q-neutrosophic soft graphs in operations management and communication network. *Soft Computing*, 25(13):8441–8459, 2021.

[257] Prem Kumar Singh. Intuitionistic plithogenic graph and it's-cut for knowledge processing tasks. *Neutrosophic Sets and Systems*, 49(1):5, 2022.

[258] Prem Kumar Singh et al. Single-valued plithogenic graph for handling multi-valued attribute data and its context. *Int. J. Neutrosophic Sci*, 15:98–112, 2021.

[259] Prem Kumar Singh et al. Dark data analysis using intuitionistic plithogenic graphs. *International Journal of Neutrosophic Sciences*, 16(2):80–100, 2021.

[260] R Sivasamy, M Mohammed Jabarulla, and Bijan Davvaz. Some operations on pythagorean fuzzy soft graphs. *Journal of Multiple-Valued Logic & Soft Computing*, 46(5/6):551, 2025.

[261] Waheed Ahmad Khan, Arsh E Mah Niaz, Trung Tuan Nguyen, Minh Hoan Pham, Thi Minh Ngoc Tong, and Hai Van Pham. A fuzzy soft planar graph with application in image segmentation. *Scientific Reports*, 15(1):21809, 2025.

[262] Ali Asghar Talebi, Hossein Rashmanlou, and Seyed Hossein Sadati. Interval-valued intuitionistic fuzzy soft graph. *TWMS Journal Of Applied And Engineering Mathematics*, 2023.

[263] S Satham Hussain, R Jahir Hussain, Ghulam Muhiuddin, and P Anitha. Operations on neutrosophic vague soft graphs. *Neutrosophic Sets and Systems, vol. 51/2022: An International Journal in Information Science and Engineering*, page 254, 2022.

[264] Umair Amin, Aliya Fahmi, Yaqoob Naveed, Aqsa Farid, and Muhammad Arshad Shehzad Hassan. Domination in bipolar fuzzy soft graphs. *J. Intell. Fuzzy Syst.*, 46:6369–6382, 2024.

[265] L Subha Lakshmi and NR Santhi Maheswari. A study on constant and regular hesitancy fuzzy soft graphs. *International J. Math. Combin.*, pages 33–43, 2023.

[266] RV Jaikumar, Raman Sundareswaran, Marayanagaraj Shanmugapriya, Said Broumi, and Talal Ali Al-Hawary. Vulnerability parameters in picture fuzzy soft graphs and their applications to locate a diagnosis center in cities. *Journal of Fuzzy Extension and Applications*, 5(1):86–99, 2024.

[267] M Vijaya and D Hema. Some products of picture fuzzy soft graph. *Arya Bhatta Journal of Mathematics and Informatics*, 13(1):75–82, 2021.

[268] Atiqe Ur Rahman, Muhammad Saeed, Ebenezer Bonyah, and Muhammad Arshad. Graphical exploration of generalized picture fuzzy hypersoft information with application in human resource management multiattribute decision-making. *Mathematical Problems in Engineering*, 2022(1):6435368, 2022.

[269] Sundas Shahzadi, Fariha Zafar, and Moeed Nawaz. Spherical fuzzy soft competition graphs with application. *Journal of Applied Mathematics and Computing*, 71(4):4615–4665, 2025.

[270] Noreen Mushtaq, Wajid Ali, Afshan Qayyum, Faiza Tufail, and Amal Kumar Adak. An algebraic framework for generalized interval-valued intuitionistic fuzzy soft expert sets. *Intelligent Systems Research and Applications Journal*, 2:399–427, 2026.

[271] Sumyyah Al-Hijjawi, Abd Ghafur Ahmad, and Shawkat Alkhazaleh. Effective neutrosophic soft expert set (enses). In *AIP Conference Proceedings*, volume 3150, page 020006. AIP Publishing LLC, 2024.

[272] Hazwani Hashim, Nur Syahida Mohd Noor, Noor Azzah Awang, Ashraf Al-Quran, and Lazim Abdullah. An integrated topsis model with neutrosophic vague soft expert sets for uncertain decision making. *International Journal of Analysis and Applications*, 24:218–218, 2026.

[273] Sagvan Y Musa, Zanyar A Ameen, Wafa Alagal, and Baravan A Asaad. A robust methodology for multi-criteria group decision-making: intuitionistic fuzzy n-bipolar soft expert sets in cybersecurity risk assessment for financial institutions. *Complex & Intelligent Systems*, 2026.

[274] Yanan Chen, Xiaoguang Zhou, and Xin He. Decision-making method based on n-soft fuzzy expert set. *Soft Computing*, 30(1):73–89, 2026.

[275] Muhammad IHSAN, Muhammad SAEED, and Muhammad Yasir HAYAT MALIK. An innovative group decision-making approach utilizing picture fuzzy hypersoft expert set. *Sigma*, 44(1):210–221, 2026.

[276] Muhammad Ihsan, Muhammad Saeed, Atiqe Ur Rahman, Hüseyin Kamacı, Nehad Ali Shah, and Wajaree Weera. An madm-based fuzzy parameterized framework for solar panels evaluation in a fuzzy hypersoft expert set environment. *AIMS Mathematics*, 8(2):3403–3427, 2023.

[277] Takaaki Fujita. Superhypersoft rough set, superhypersoft expert set, and bipolar superhypersoft set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 270, 2025.

[278] Vakkas Uluçay and Memet Şahin. Decision-making method based on neutrosophic soft expert graphs. In *Neutrosophic Graph Theory and Algorithms*, pages 33–76. IGI Global Scientific Publishing, 2020.

[279] Bobin George, Jinta Jose, and Rajesh K Thumbakara. Combined matrices associated with soft digraphs. *Mathematical Modeling for Computer Applications*, pages 231–239, 2024.

[280] Kalyan Sinha and Pinaki Majumdar. Soft digraphs, operations and applications. *Annals of Fuzzy Mathematics and Informatics*, 12(6):791–804, 2016.

[281] Kalyan Sinha and Pinaki Majumdar. Properties and applications of soft digraphs. *Malaya Journal of Matematik (MJM)*, 7(3, 2019):451–457, 2019.

[282] N Sarala and S Tharani. An introduction to fuzzy soft digraphs. *IOSR J Math*, 13:46–54, 2017.

[283] Maha M Saeed, Sagvan Y Musa, Baravan A Asaad, and Zanyar A Ameen. Pythagorean fuzzy n-bipolar soft sets-based multi-criteria decision-making framework for sustainability evaluation and risk assessment in manufacturing industries. *Scientific Reports*, 15(1):29648, 2025.

[284] Asghar Khan, Muhammad Izhar, and Mohammed M. Khalaf. Generalised multi-fuzzy bipolar soft sets and its application in decision making. *J. Intell. Fuzzy Syst.*, 37:2713–2725, 2019.

[285] Tareq M Al-Shami. Bipolar soft sets: relations between them and ordinary points and their applications. *Complexity*, 2021(1):6621854, 2021.

[286] Hanan Al-Saadi. A new generalization of bipolar soft structures: Theory and applications. *Mathematics*, 14(10):1692, 2026.

[287] Takaaki Fujita and Florentin Smarandache. An introduction to advanced soft set variants: Superhypersoft sets, indetermsuperhypersoft sets, indetermtreesoft sets, bihypersoft sets, graphicsoft sets, and beyond. *Neutrosophic Sets and Systems*, 82:817–843, 2025.

[288] Asghar Khan, Muhammad Izhar, and Mohammed M Khalaf. Double-framed soft la-semigroups. *Journal of Intelligent & Fuzzy Systems*, 33(6):3339–3353, 2017.

[289] Muhammad Saeed, Hafiz Inam ul Haq, and Mubashir Ali. Extension of double frame soft set to double frame hypersoft set (dfss to dfhss). *HyperSoft Set Methods in Engineering*, 2:18–27, 2024.

[290] Tauseef Asif, Asghar Khan, and Jian Tang. Fully prime double-framed soft ordered semigroups. *Open Journal of Science and Technology*, 2(4):17–25, 2019.

[291] Muhammad Izhar, Tariq Mahmood, Asghar Khan, Muhammad Farooq, and Kostaq Hila. Double-framed soft set theory applied to abel-grassmann's hypergroupoids. *New Mathematics and Natural Computation*, 18(03):819–841, 2022.

[292] Ajoy Kanti Das, Florentin Smarandache, Rakhal Das, and Suman Das. *A comprehensive study on decision-making algorithms in retail and project management using double framed hypersoft sets*. Infinite Study, 2024.

[293] Minyan Chen. Double framed hypersoft set for studies factors that influence and ways to improve vocational college instruction in innovation and entrepreneurship. *Neutrosophic Sets and Systems*, 85(1):5, 2025.

[294] Lingling Chen. Measuring teaching success in college foreign literature programs: An evaluation perspective using double framed superhypersoft set. *Neutrosophic Sets and Systems*, 85(1):7, 2025.

[295] Gustavo Santos-García and José Carlos R Alcantud. Ranked soft sets. *Expert Systems*, 40(6):e13231, 2023.

[296] Khizar Hayat, Muhammad Irfan Ali, Bing-Yuan Cao, and Xiao-Peng Yang. A new type-2 soft set: Type-2 soft graphs and their applications. *Advances in Fuzzy Systems*, 2017(1):6162753, 2017.

[297] K Bhargavi, B Sathish Babu, and Sajjan G Shiva. Type-2-soft-set based uncertainty aware task offloading framework for fog computing using apprenticeship learning. *Cybernetics and Information Technologies*, 23(1):38–58, 2023.

[298] Qikun Ye. Q-neutrosophic type-2 soft set (t2-qnss) for tourism development potential analysis of" intangible cultural heritage" resources from the perspective of cultural and tourism integrations. *Neutrosophic Sets and Systems*, 90(1):43, 2025.

[299] Sk Nazmul. On type-2 soft topologies. *New Trends in Mathematical Sciences*, 5(4):182–194, 2017.

[300] Manas Ghosh and Jamuna Kanta Sing. Interval type-3 fuzzy set for rank-level fusion using multiple feature vectors: Theory and application to face recognition. *Applied Soft Computing*, 175:113102, 2025.

[301] Dhan Jeet Singh, Teena Sharma, and Nishchal K Verma. Interval type-3 fuzzy systems. In *Fuzzy Systems with Applications to Modeling and Control of an Aircraft*, pages 77–96. Springer, 2026.

[302] Oscar Castillo. Type-3 fuzzy modeling of restaurant selection decision-making. In *Intelligent System Design Based on Soft Computing Models*, pages 51–60. Springer, 2025.

[303] Oscar Castillo and Patricia Melin. Towards interval type-3 intuitionistic fuzzy sets and systems. *Mathematics*, 10(21):4091, 2022.

[304] Patricia Melin and Oscar Castillo. Aggregation in ensemble neural models with type-3 and intuitionistic fuzzy logic. In *New Directions on Hybrid Intelligent Systems Based on Neural Networks, Fuzzy Logic, and Optimization Algorithms*, pages 73–83. Springer, 2024.

[305] Hui Liu. Neutrosophic type-iii sets for multi-layered uncertainty modeling in quality evaluation of university ideological and political education. *Neutrosophic Sets and Systems*, 86(1):13, 2025.

[306] Florentin Smarandache and Maikel Leyva. *Type-k Neutrosophic Sets*. Infinite Study, 2026.

[307] Khizar Hayat, Bing-Yuan Cao, Muhammad Irfan Ali, Faruk Karaaslan, and Zejian Qin. Characterizations of certain types of type 2 soft graphs. *Discrete Dynamics in Nature and Society*, 2018(1):8535703, 2018.

[308] Sundas Shahzadi, Musavarah Sarwar, and Muhammad Akram. Decision-making approach with fuzzy type-2 soft graphs. *Journal of Mathematics*, 2020(1):8872446, 2020.

[309] MH Fazel Zarandi, R Gamasaee, and Oscar Castillo. Type-1 to type-n fuzzy logic and systems. In *Fuzzy Logic in Its 50th Year: New Developments, Directions and Challenges*, pages 129–157. Springer, 2016.

[310] Saikat Maity, Sanjay Chakraborty, Saroj Kumar Pandey, Indrajit De, and Sourasish Nath. Type-n fuzzy logic-the next level of type-1 and type-2 fuzzy logic. *International Journal of Intelligent Engineering Informatics*, 11(4):353–389, 2023.

[311] Janet Aisbett, John T Rickard, and David Morgenthaler. Intersection and union of type-n fuzzy sets. In *International Conference on Fuzzy Systems*, pages 1–8. IEEE, 2010.

[312] Kanika Bhalla, Sonika Dahiya, and Anjana Gosain. Kernelized type 2 intuitionistic fuzzy c means: a robust approach for brain mri image segmentation. *Discover Computing*, 28(1):170, 2025.

[313] Tina Verma. Solving fuzzy matrix games with triangular type-2 intuitionistic fuzzy numbers for water management problem. *Water Resources Management*, 40(8):371, 2026.

[314] Muhammad Touqeer, Ehtisham Rasool, Ali Ahmadian, Mehdi Salimi, and Soheil Salahshour. Decision making based on interval type-2 neutrosophic numbers involving the optimal selection of a house. *Soft Computing*, 30(2):809–821, 2026.

[315] Hend Ahmed. Generative ai in soft computing: Methodological implications and a literature survey on mcdm and type-2 neutrosophic applications. *International Journal of Computers and Informatics (Zagazig University)*, 9:17–26, 2025.

[316] Ömer Faruk Görçün, Hande Küçükönder, and Ahmet Çalık. Prioritizing sustainable energy strategies using multi-criteria decision-making models in type-2 neutrosophic environment. *Optimization in Sustainable Energy: Methods and Applications*, pages 113–182, 2026.

[317] Saad M Darwish, Noha A Bagi, and Noha A El-Shoafy. Intelligent automated essay scoring under uncertainty using type 2 neutrosophic ontologies. *Scientific Reports*, 16(1):17186, 2026.

[318] Onur Zihni, Yıldıray Çelik, and Güven Kara. Interval-valued fuzzy soft graphs. *Topological Algebra and its Applications*, 5(1):19–27, 2017.

[319] Xiaoli Qiang, Qian-Ru Xiao, Aysha Khan, AA Talebi, Arun Kumar Sivaraman, and M Mojahedfar. A study on interval-valued fuzzy graph with application in energy industry management. *Discrete Dynamics in Nature and Society*, 2022(1):8499577, 2022.

[320] Ali Asghar Talebi, Hossein Rashmanlou, and Seyed Hossein Sadati. Interval-valued intuitionistic fuzzy soft graph. *TWMS Journal Of Applied And Engineering Mathematics*, 2023.

[321] Uttam Mondal, Madhumangal Pal, and Kalyani Das. Interval-valued intuitionistic fuzzy soft graphs with application. *TWMS Journal Of Applied And Engineering Mathematics*, 2024.

[322] Yildiray Celik and Guven Kara. *An application of interval-valued neutrosophic soft graphs in a decision making problem.* Infinite Study, 2018.

[323] Yildiray Celik and Guven Kara. *Combination of interval-valued neutrosophic soft sets and graph theory.* Infinite Study, 2018.

[324] Gulfam Shahzadi, Muhammad Akram, and Bijan Davvaz. Pythagorean fuzzy soft graphs with applications. *Journal of Intelligent & Fuzzy Systems*, 38(4):4977–4991, 2020.

[325] Mohammed Jabarulla Mohamed and Sivasamy Rajamanickam. Strong interval–valued pythagorean fuzzy soft graphs. *Ratio Mathematica*, 46, 2023.

[326] Adnan Khan, Muhammad Farman, and Ali Akgül. Decision making under pythagorean fuzzy soft environment. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 31(05):773–793, 2023.

[327] Muhammad Akram and Hafiza Saba Nawaz. Inter-specific competition among trees in pythagorean fuzzy soft environment. *Complex & Intelligent Systems*, 8(2):863–884, 2022.

[328] Jawad Ali. Norm-based distance measure of q-rung orthopair fuzzy sets and its application in decision-making. *Computational and applied mathematics*, 42(4):184, 2023.

[329] Palash Dutta, Bornali Saikia, and Abhilash Kangsha Banik. Novel distance measure for q-rung orthopair fuzzy sets with application to transportation problem. *International journal of information technology*, pages 1–18, 2024.

[330] Donglai Wang, Yige Yuan, Zhe Liu, Sijia Zhu, and Zhifang Sun. Novel distance measures of q-rung orthopair fuzzy sets and their applications. *Symmetry*, 16(5):574, 2024.

[331] B Yusoff, A Kilicman, D Pratama, and R Hasni. Circular q-rung orthopair fuzzy set and its algebraic properties. *Malaysian journal of mathematical sciences*, 17(3):363–378, 2023.

[332] Hussain AlSalman and Bader Fahad Alkhamees. Graphical analysis of q-rung orthopair fuzzy information with application. *Mathematical Problems in Engineering*, 2022(1):9650995, 2022.

[333] Wedad Khaled Alshaman and Kholood Mohammad Alsager. A q-rung orthopair spherical fuzzy rough graph approach for modeling uncertain agricultural networks. *AIMS Mathematics*, 11(7):21607–21653, 2026.

[334] Andleeb Kausar, Nabilah Abughazalah, and Naveed Yaqoob. Multi polar q-rung orthopair fuzzy graphs with some topological indices. *Symmetry*, 15(12):2131, 2023.

[335] Tong He. Representation form of rough graph. *Applied Mechanics and Materials*, 157:874–877, 2012.

[336] Saibal Majumder and Samarjit Kar. Multi-criteria shortest path for rough graph. *Journal of Ambient Intelligence and Humanized Computing*, 9(6):1835–1859, 2018.

[337] Tong He. Rough properties of rough graph. *Applied Mechanics and Materials*, 157:517–520, 2012.

[338] Musavarah Sarwar. A theoretical investigation based on the rough approximations of hypergraphs. *Journal of Mathematics*, 2022(1):1540004, 2022.

[339] John G Stell. Relational granularity for hypergraphs. In *International Conference on Rough Sets and Current Trends in Computing*, pages 267–276. Springer, 2010.

[340] Ye Li, Lei Yang, Binbin Sang, Guoyin Wang, Shuyin Xia, Weihua Xu, and Jianhang Yu. An efficient feature selection method using the granular-ball divergence-based fuzzy rough hypergraph. *IEEE Transactions on Knowledge and Data Engineering*, 2026.

[341] Ayman Elsaid Ammar, El Atik, and Abd El Fattah. Class shortest path algorithm in weighted rough hypergraphs with some medical applications. *Egyptian Journal of Artificial Intelligence*, 5(3):1–10, 2026.

[342] Takaaki Fujita. Review of some superhypergraph classes: Directed, bidirected, soft, and rough. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*, 2024.

[343] Yujie Qin, Guoping Lin, Jiye Liang, Jianwei Lin, and Yaojin Lin. A fuzzy rough set framework for interval-valued multilabel feature selection with label enhancement. *IEEE Transactions on Fuzzy Systems*, 2026.

[344] Yuqi Zhou, Ye Li, Lei Yang, Binbin Sang, Shaoguo Cui, Shuyin Xia, and Weihua Xu. Unsupervised feature selection via pseudo-label guided bi-level granular-ball hypergraph. *Expert Systems with Applications*, page 133729, 2026.

[345] Ambika Hazarika, Ansuman Kumar, and Anindya Halder. Semi-supervised ensemble learning with interval type-2 fuzzy-rough sets for parkinson's disease prediction from multi-omics. *Mammalian Genome*, 37(1):89, 2026.

[346] Noorjahan Shaik and Sharief Basha Shaik. Similarity measures and energy-based decision-making in intuitionistic fuzzy rough graphs. *International Journal of Computational Intelligence Systems*, 2026.

[347] Dali Shi, Salah Eldin Abbas, Hossam M Khiamy, and Ismail Ibedou. Fuzzy rough graphs via fuzzy graph ideals with applications. *AIMS Mathematics*, 11(1):2979–3007, 2026.

[348] Noorjahan Shaik and Sharief Basha Shaik. Wiener index application in intuitionistic fuzzy rough graphs for transport network flow. *Scientific Reports*, 15, 2025.

[349] Danyang Wang and Ping Zhu. Fuzzy rough digraph based on strength of connectedness with application. *Neural Computing and Applications*, 35(16):11847–11866, 2023.

[350] Aliya Fahmi, Aziz Khan, Kinza Gull, Aiman Mukheimer, Aliya Thabet Abdeliawad, Arshia Hashmi, and Rajermani Thinakaran. Domination in rough\(m\)-polar fuzzy digraphs based on trade networking. *European Journal of Pure and Applied Mathematics*, 18(3):6094–6094, 2025.

[351] Uzma Ahmad and Tahira Batool. Domination in rough fuzzy digraphs with application. *Soft Computing*, 27(5):2425–2442, 2023.

[352] Pankhuri Jain, Anoop Tiwari, and Tanmoy Som. Intuitionistic fuzzy rough set model based on k-means and its application to enhance prediction of aptamer–protein interacting pairs. *Journal of Ambient Intelligence and Humanized Computing*, 15(9):3575–3586, 2024.

[353] Muhammad Kamraz Khan, Kamran, Muhammad Sajjad Ali Khan, Ahmad Aloqaily, and Nabil Mlaiki. Covering-based intuitionistic hesitant fuzzy rough set models and their application to decision-making problems. *Symmetry*, 16(6):693, 2024.

[354] Neelam Kumari, Varsha Gautam, and Shivam Shreevastava. A review on rough, fuzzy rough and intuitionistic fuzzy rough sets based techniques for attribute reduction. *International Journal of System Assurance Engineering and Management*, pages 1–15, 2026.

[355] Noorjahan Shaik and Sharief Basha Shaik. Energy of an intuitionistic fuzzy rough graph and their application. *OPSEARCH*, pages 1–35, 2026.

[356] Shaik Noorjahan and Sharief Basha Shaik. Decision-making using the correlation coefficient measures of intuitionistic fuzzy rough graph. *Hacettepe Journal of Mathematics and Statistics*, 53(6):1774–1797, 2024.

[357] Ramesh Obbu, Tarakaramu Nainaru, G YuvaRoopa Lakshmi, Sarvar Iskandarov, Akbar Toyirov, Sardor Sabirov, KK Prashanth, et al. Planarity of intuitionistic pythagorean fuzzy graphs and its application in decision making. *European Journal of Pure and Applied Mathematics*, 18(3):6372–6372, 2025.

[358] Ajoy Kanti Das, Nandini Gupta, Rajat Das, Carlos Granados, and Suman Das. An innovative approach to neutrosophic rough soft set in intelligent multi-criteria decision-making. *OPSEARCH*, pages 1–32, 2025.

[359] Li Zeng. A neutrosophic-rough set support for ai-driven training quality evaluation of interdisciplinary communication talent in the converged media era. *Neutrosophic Sets and Systems*, 93(1):16, 2025.

[360] Ajoy Kanti Das, Nanini Gupta, Rajat Das, Rajib Mallik, and Suman Das. Intervalvalued neutrosophic rough soft set based intelligent multi-criteria decision-making framework. *Boletim da Sociedade Paranaense de Matemática (3rd Series)*, 43:1–26, 2025.

[361] Muhammad Akram. Neutrosophic soft rough graphs. In *Single-Valued Neutrosophic Graphs*, pages 329–359. Springer, 2018.

[362] Takaaki Fujita. Plithogenic rough sets. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 152, 2025.

[363] Uzma Ahmad, Iqra Nawaz, and Said Broumi. Connectivity index of directed rough fuzzy graphs and its application in traffic flow network. *Granular Computing*, 8(6):1561–1582, 2023.

[364] Hafsa Masood Malik and Muhammad Akram. A new approach based on intuitionistic fuzzy rough graphs for decision-making. *Journal of Intelligent & Fuzzy Systems*, 34(4):2325–2342, 2018.

[365] Noorjahan Shaik and Sharief Basha Shaik. Similarity measures and energy-based decision-making in intuitionistic fuzzy rough graphs. *International Journal of Computational Intelligence Systems*, 19(1):120, 2026.

[366] Shaik Noorjahan and Shaik Sharief Basha. The laplacian energy of an intuitionistic fuzzy rough graph and its utilisation in decision-making. *Operations Research and Decisions*, 35, 2025.

[367] Anessa Arif, Aliya Fahmi, Aziz Khan, Aiman Mukheimer, Thabet Abdeljawad, and Rajermani Thinakaran. Domination in bipolar fuzzy rough digraphs with applications to decision-making. *European Journal of Pure and Applied Mathematics*, 18(4):6216–6216, 2025.

[368] Esraa M Alsawaby, Heba I Mustafa, Abdelaziz E Radwan, Ragabia Hussein Sallam, and Ali TST. On lattice generalizations of bipolar soft rough approximations. *Filomat*, 40(8):2847–2869, 2026.

[369] Shanshan Lyu. A comprehensive evaluation of teaching standards in dance education at the university level: Neutrosophic soft rough set model. *Neutrosophic Sets and Systems*, 93(1):4, 2025.

[370] Liwen Ma and Meifeng Li. Covering rough set models, fuzzy rough set models and soft rough set models induced by covering similarity. *Information Sciences*, 689:121520, 2025.

[371] Muhammad Abdullah, Khuram Ali Khan, Jaroslav Frnda, and Atiqe Ur Rahman. A novel algorithmic multi-attribute decision-making framework for the evaluation of energy systems using rough approximations of hypersoft sets. *Heliyon*, 10(23), 2024.

[372] T FUJITA and F SMARANDACHE. A short note for hypersoft rough graphs. *HYPERSOFT SET METHODS IN ENGINEERING*, 3:1–25, 2024.

[373] Hüseyin Kamacı. On hybrid structures of hypersoft sets and rough sets. *International Journal of Modern Science and Technology*, 6(4):69–82, 2021.

[374] Takaaki Fujita. Superhypersoft rough set, superhypersoft expert set, and bipolar superhypersoft set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 270, 2025.

[375] VS Subha and R Selvakumar. A new approach to neutrosophic hypersoft rough sets. *Neutrosophic Sets and Systems*, 58:226–247, 2023.

[376] Muhammad Abdullah, Khuram Ali Khan, Atiqe Ur Rahman, and Michael Kikomba Kahungu. An intelligent decision-support system for climate change mitigation using similarity measures of hypersoft rough sets. *International Journal of Computational Intelligence Systems*, 19(1):108, 2026.

[377] R Noor, I Irshad, and I Javaid. Soft rough graphs. *arXiv preprint arXiv:1707.05837*, 2017.

[378] Nasir Shah, Noor Rehman, Muhammad Shabir, and Muhammad Irfan Ali. Another approach to roughness of soft graphs with applications in decision making. *Symmetry*, 10(5):145, 2018.

[379] Takaaki Fujita and Florentin Smarandache. *Handbook of Rough Set Extensions and Uncertainty Models*. Infinite Study, 2026.

[380] Meilian Liang, Binmei Liang, Linna Wei, and Xiaodong Xu. Edge rough graph and its application. In *2011 Eighth International Conference on Fuzzy Systems and Knowledge Discovery (FSKD)*, volume 1, pages 335–338. IEEE, 2011.

[381] Rajesh K Thumbakara and Viji George. Overlap–density vertex rough graph. *New Mathematics and Natural Computation*, pages 1–26, 2026.

[382] Mathew Bibin, John Sunil Jacob, and Garg Harish. Vertex rough graphs. *Complex & Intelligent Systems*, 6(2):347–353, 2020.

[383] S Dhanalakshmi and N Parvathi. Minimum spanning tree of a rough graph having weights as trapezoidal fuzzy numbers. *Global Journal of Pure and Applied Mathematics (GJPAM)*, 11(1):2015, 2015.

[384] Danyang Wang and Ping Zhu. A novel framework for trust network analysis: Connectivity-based intuitionistic fuzzy rough digraph. *International Journal of Approximate Reasoning*, page 109564, 2025.

[385] Takaaki Fujita. Extensions of multidirected graphs: Fuzzy, neutrosophic, plithogenic, rough, soft, hypergraph, and superhypergraph variants. *International journal of topology*, 2(3):11, 2025.

[386] Muhammad Akram, Nabeela Ishfaq, Sidra Sayed, and Florentin Smarandache. Decision-making approach based on neutrosophic rough information. *Algorithms*, 11(5):59, 2018.

[387] Choonkil Park, Nasir Shah, Noor Rehman, Abbas Ali, Muhammad Irfan Ali, and Muhammad Shabir. Soft covering based rough graphs and corresponding decision making. *Open Mathematics*, 17(1):423–438, 2019.

[388] S Pavithra and A Manimaran. Uncertainty measure for z-soft covering based rough graphs with application. *Journal of Intelligent & Fuzzy Systems*, 44(4):5789–5802, 2023.

[389] Jinta Jose, Rajesh K Thumbakara, JD Thenge Mashale, Bobin George, and Sijo P George. Homomorphic and restricted homomorphic products of soft graphs. *Journal of the Nigerian Society of Physical Sciences*, pages 2391–2391, 2025.

[390] MS Sunitha and A Vijayakumar. *Studies on fuzzy graphs*. PhD thesis, Department of Mathematics, 2001.

[391] J Uma Maheswari, K Dillibabu, and A Anbarasan. Fuzzy detour $\mu$-distance on direct sum of two complete fuzzy graphs. In *AIP Conference Proceedings*, volume 3345, page 020274. AIP Publishing LLC, 2026.

[392] Zengtai Gong and Jing Zhang. Chromatic number of fuzzy graphs: Operations, fuzzy graph coloring, and applications. *Axioms*, 11(12):697, 2022.

[393] Muhammad Akram and Fariha Zafar. On soft trees. *Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica*, (2):82–95, 2015.

[394] Bobin George, Jinta Jose, and Rajesh K Thumbakara. Spanning soft trees. In *Optimization Techniques and Associated Applications*, pages 42–48. CRC Press, 2024.

[395] Monika Rianti Helmi, Syafrizal Sy, Admi Nazra, Nurul Hanifa, Noverina Alfiany, et al. Soft graphs of the barbell star graph. *Jurnal Matematika UNAND*, 14(4):366–375, 2025.

[396] M Vijaya and S Anitha. Q fuzzy regular graphs. *Journal of Algebraic Statistics*, 13(2):2528, 2022.

[397] Wei Xiao, Arindam Dey, and Le Hoang Son. A study on regular picture fuzzy graph with applications in communication networks. *Journal of Intelligent & Fuzzy Systems*, 39(3):3633–3645, 2020.

[398] Renuka Kolandasamy, D Ramesh, Aiyared Iampan, Gadde Sambasiva Rao, Sravani Jayanti, and Sabenabanu Abdulkadhar. Regular fuzzy graphs with chromatic numbers. *International Journal of Analysis and Applications*, 23:173–173, 2025.

[399] M Baghernejad and Rajab Ali Borzooei. Results on soft graphs and soft multigraphs with application in controlling urban traffic flows. *Soft Computing*, 27(16):11155–11175, 2023.

[400] A Nagoor Gani and SR Latha. On irregular fuzzy graphs. *Applied mathematical sciences*, 6(11):517–523, 2012.

[401] Liangsong Huang, Yu Hu, Yuxia Li, PK Kishore Kumar, Dipak Koley, and Arindam Dey. A study of regular and irregular neutrosophic graphs with real life applications. *Mathematics*, 7(6):551, 2019.

[402] Sankar Sahoo and Madhumangal Pal. Certain types of edge irregular intuitionistic fuzzy graphs. *Journal of Intelligent & Fuzzy Systems*, 34(1):295–305, 2018.

[403] Jyoti D Thenge, B Surendranath Reddy, and Rupali S Jain. Connected soft graph. *New Mathematics and Natural Computation*, 16(02):305–318, 2020.

[404] Bobin George, Jinta Jose, and Rajesh K Thumbakara. Connectedness in soft semigraphs. *New Mathematics and Natural Computation*, 20(01):157–182, 2024.

[405] Jinta Jose, Bobin George, and Rajesh K Thumbakara. Connectedness and some measurements in soft directed graphs. *Southeast Asian Bulletin of Mathematics*, 49(4):539, 2025.

[406] K Kalaiarasi and L Mahalakshmi. An introduction to fuzzy strong graphs, fuzzy soft graphs, complement of fuzzy strong and soft graphs. *Global Journal of Pure and Applied Mathematics*, 13(6):2235–2254, 2017.

[407] Shanookha Ali, Sunil Mathew, and John N Mordeson. Hamiltonian fuzzy graphs with application to human trafficking. *Information Sciences*, 550:268–284, 2021.

[408] Muhammad Tanveer Hussain, Fahad Ur Rehman, and Tabasam Rashid. An algorithmic approach to minimize road accidents in the highway system using hamiltonian fuzzy influence graphs. *Neural Computing and Applications*, 37(6):5019–5038, 2025.

[409] J Greeda and V Vinoba. Agent-based fuzzy hamiltonian graph systems with artificial intelligence in the private region. *Procedia Computer Science*, 230:608–619, 2023.

[410] Abdul Muneera, T Nageswara Rao, and RVN Srinivasa Rao. Fuzzy eulerian and fuzzy hamiltonian graphs with their applications. *International Journal of Recent Technology and Engineering*, 8(4), 2019.

[411] Rajesh K Thumbakara, Jinta Jose, and Bobin George. Hamiltonian soft graphs. *Ganita*, 72(1):145–151, 2022.

[412] Sumera Naz and Muhammad Aslam Malik. Single-valued neutrosophic line graphs. *TWMS Journal of Applied and Engineering Mathematics*, 8(2):483–494, 2018.

[413] S Satham Hussain, R Jahir Hussain, and Ghulam Muhiuddin. *Neutrosophic vague line graphs*, volume 36. Infinite Study, 2020.

[414] R. K. Thumbakara, B. George, and J. Jose. Subdivision graph, power and line graph of a soft graph. *Communications in Mathematics and Applications*, 13(1):75–85, 2022.

[415] Muhammad Akram and Fariha Zafar. *Hybrid soft computing models applied to graph theory*, volume 380. Springer, 2020.

[416] Hamdy Hafez, Kamal A Dib, R El-Shanawany, and AA El Atik. A study on rough graphs. *Mathematics in Applied Sciences and Engineering*, pages 1–34, 2020.

[417] Aruna Devi Ramsankar and Anitha Krishnamoorthy. Exploring metric dimensions for dimensionality reduction and navigation in rough graphs. *Mathematical Modelling of Engineering Problems*, 11(4):1037, 2024.

[418] R Aruna Devi and K Anitha. Construction of rough graph to handle uncertain pattern from an information system. *arXiv preprint arXiv:2205.10127*, 2022.

# A Comprehensive Survey of Soft Graphs and Rough Graphs

**Takaaki Fujita** [1] * **and Florentin Smarandache**[2]
[1] Independent Researcher, Tokyo, Japan.
Email: Takaaki.fujita060@gmail.com
[2] University of New Mexico, Gallup Campus, NM 87301, USA.
Email: fsmarandache@gmail.com

## Abstract

Graph theory provides a fundamental mathematical language for analyzing networks whose entities and relationships are represented by vertices and edges. Soft Graphs extend this framework by assigning a subgraph of an ambient graph to each parameter, thereby enabling parameter-dependent representations of uncertain vertices, edges, relationships, and structural configurations. Rough Graphs incorporate approximation-based reasoning into graph structures by applying lower and upper approximations to vertices, edges, or subgraphs. This approach distinguishes structural components that are certainly present, possibly present, or located within a boundary region under a prescribed indiscernibility relation.

Soft Graphs, Rough Graphs, and their numerous extensions have been studied extensively. Nevertheless, comparatively few works provide a unified and systematic account of these frameworks and the relationships among them. This book addresses this gap by presenting an organized introduction to Soft Graphs, Rough Graphs, and several of their principal extensions, with particular emphasis on their mathematical foundations, structural features, interconnections, and roles in modeling uncertain networks.

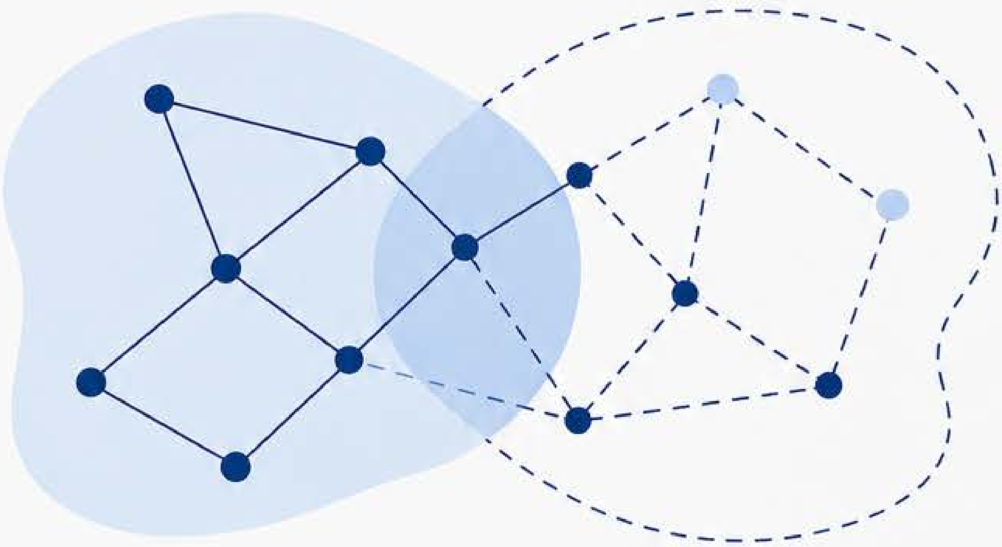
*Keywords:* Soft Graph, Rough Graph, Soft Set, Rough Set

*A Comprehensive Survey of Soft Graphs and Rough Graphs* presents a unified and systematic account of two powerful approaches to modeling uncertainty in complex networks: Soft Graphs and Rough Graphs.

Soft Graphs associate parameters with subgraphs of an ambient graph, enabling parameter-dependent representations of uncertain vertices, edges, relationships, and structural configurations. Rough Graphs incorporate approximation-based reasoning by applying lower and upper approximations to vertices, edges, or subgraphs, distinguishing components that are certainly present, possibly present, or located within a boundary region under an indiscernibility relation.

The book develops a wide spectrum of models and extensions, including HyperSoft, SuperHyperSoft, fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, bipolar, multipolar, and interval-valued formulations, as well as Soft and Rough HyperGraphs and SuperHyperGraphs. Special classes, structural properties, connectivity, paths, cycles, and covering-based models are also systematically explored.

Designed for researchers and advanced students, this volume serves as a comprehensive reference and a foundation for future developments in uncertain graph theory and its applications.



ISBN 978-197250248-8

9 781972 502488