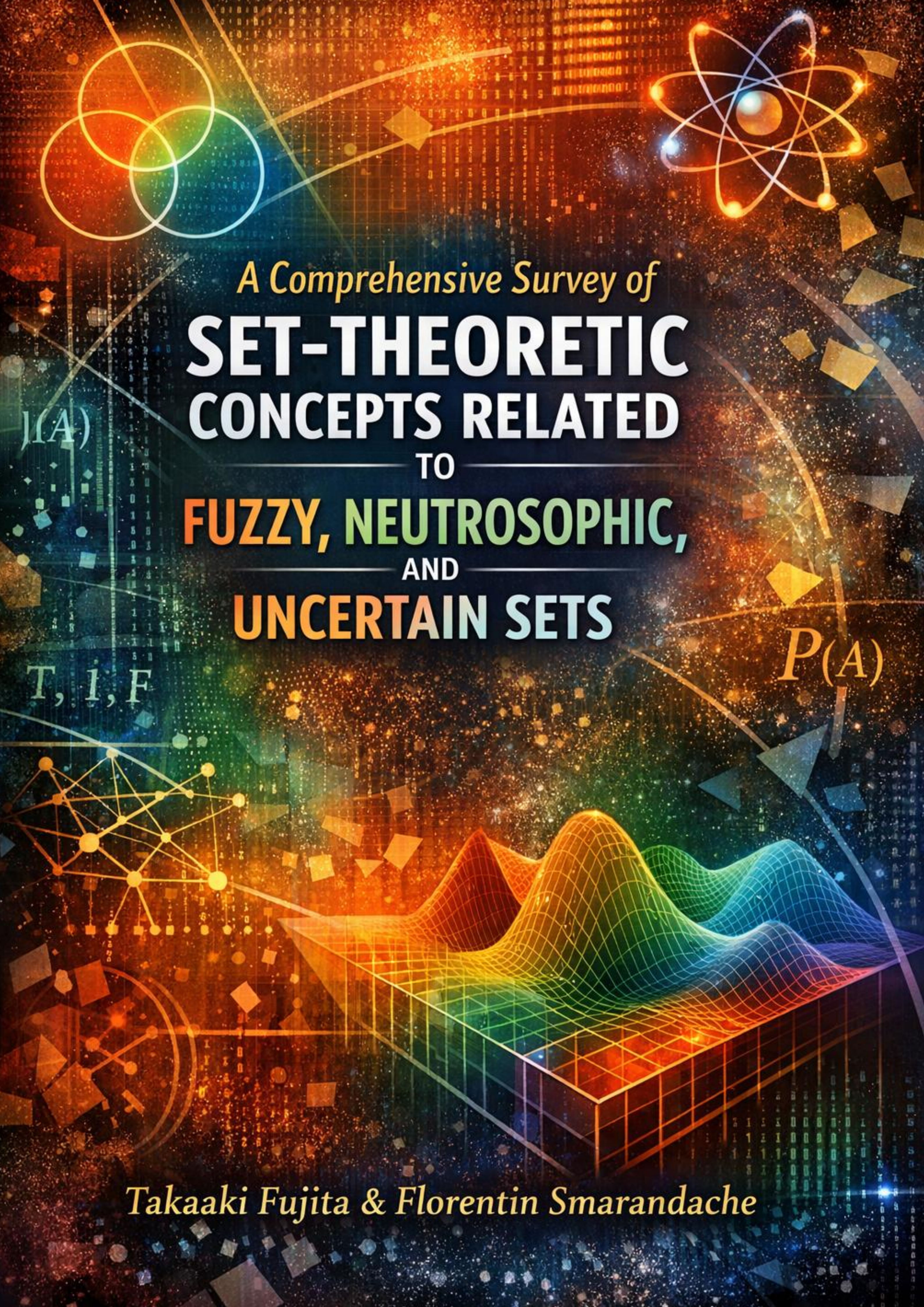

A Comprehensive Survey of

SET-THEORETIC CONCEPTS RELATED TO FUZZY, NEUTROSOPHIC, AND UNCERTAIN SETS

Takaaki Fujita & Florentin Smarandache

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Chapter 1

Introduction

1.1 Uncertain Sets

Real-world phenomena are often characterized by vagueness, partial truth, indeterminacy, incompleteness, and other forms of uncertainty. In order to describe such phenomena in a mathematically rigorous manner, many generalized set-theoretic frameworks have been developed over the past several decades. Representative examples include Fuzzy Sets [1, 2], Intuitionistic Fuzzy Sets [3,4], Hesitant Fuzzy Sets [5,6], Picture Fuzzy Sets [7,8], Quadripartitioned Neutrosophic Sets [9, 10], Pentapartitioned Neutrosophic Sets [11, 12], Plithogenic Sets [13, 14], HyperFuzzy Sets [15, 16], and HyperNeutrosophic Sets [17, 18].

These frameworks have found applications in a wide variety of fields, including decision science, chemistry, control systems, and machine learning [19]. Since different applications require different modes and levels of uncertainty representation, the choice of an appropriate generalized set model depends on the structure of the problem and on the number and type of uncertainty parameters needed to describe the underlying phenomenon.

1.2 Our Contributions

In this survey book, we provide a broad and systematic introduction to set concepts that are closely related to fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, plithogenic sets, and other uncertainty-oriented frameworks. Our aim is not only to review classical and modern developments, but also to organize a wide range of set-theoretic notions within a unified perspective that highlights their structural similarities, conceptual differences, and possible extensions.

More specifically, the book offers a comprehensive overview of these concepts from multiple perspectives, as summarized in Table 1.1.

Table 1.1: Chapter-Level Overview of Set Concepts

Ch.	Set Concept	Overview
2	Foundational Uncertainty-Related Sets	Reviews the principal baseline models used throughout the book, including fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, rough, soft, and uncertain sets.
3	Near Set	Studies sets characterized by descriptive nearness or perceptual similarity, together with major uncertainty-oriented extensions such as fuzzy, neutrosophic, soft, and rough variants.
4	Complex Set	Examines set models whose membership-related information is expressed in the complex domain, thereby allowing phase-sensitive or oscillatory interpretations and associated uncertainty-based extensions.
5	Interval Set	Develops interval-based representations in which membership or set-related information is described by intervals, thereby capturing imprecision through lower–upper bound structures.
6	Granular Set	Focuses on granularity-based set models in which information is organized through granules, approximations, or multiple levels of informational resolution.
7	Multiset	Considers collections that allow repeated occurrences of elements and extends this multiplicity-based viewpoint to fuzzy, neutrosophic, uncertain, soft, and rough settings.
8	Powerset	Investigates powerset-type constructions and their extensions, emphasizing how higher-order collections of subsets interact with uncertainty-based set representations.
9	Shadowed Set	Treats three-region models combining acceptance, rejection, and shadow zones, thereby capturing partial commitment and boundary uncertainty in a simplified form.
10	Grey Set	Studies grey-style set models designed to represent incomplete or partially known information, together with extensions adapted to uncertain and neutrosophic environments.
11	Flou Set	Examines flou-type set structures based on certain and maximal zones, providing an intermediate framework between crisp inclusion and boundary vagueness.
12	Controlled Set	Introduces set models equipped with control-oriented mechanisms or parameters, highlighting regulated forms of uncertainty handling and their soft and rough counterparts.
13	Weighted Set	Develops weighted versions of set-theoretic models in which elements, subsets, or membership-related data carry explicit importance coefficients or intensities.
14	Circular Set	Considers circular-type set models whose structure is associated with circular or radius-based representations, together with uncertainty-oriented generalizations.

Ch.	Set Concept	Overview
15	Cubic Set	Surveys cubic-style frameworks that combine interval-valued and point-valued information, along with neutrosophic, plithogenic, uncertain, rough, soft, and complex extensions.
16	Convex Set	Examines convexity-based set structures, including classical convex sets and convex uncertainty-oriented variants defined through suitable closure under convex combinations.
17	N-Set	Introduces negative-valued set models, including N-fuzzy, N-neutrosophic, and N-uncertain forms, in which membership-type information is expressed on negative scales.
18	Probabilistic Set	Studies probabilistic formulations of sets and related uncertainty models in which randomness or probability distributions are incorporated into the set description.
19	Graphic Set	Presents graphically structured set models in which set-related information is encoded or interpreted through graph-based or network-aware representations.
20	Ordered Set	Discusses order-theoretic set structures, including partial orders, total orders, preorders, and their fuzzy, neutrosophic, and uncertain extensions.
21	Lattice-Valued Set	Develops set models whose values lie in lattices rather than in the unit interval, thereby enabling algebraically richer representations of graded or uncertain membership.
22	Triangular Set	Treats triangular-type set representations, especially those inspired by triangular membership structures, and extends them to neutrosophic and uncertain contexts.
23	Dense Set	Investigates density-related set notions, including dense, somewhere dense, dense-in-itself, and nowhere dense forms, together with uncertainty-based analogues.
24	Baire Set	Studies Baire-type sets and corresponding fuzzy, neutrosophic, and uncertain variants, thereby linking set-theoretic uncertainty with topological category concepts.
25	Nonstandard Set	Introduces nonstandard extensions of set models, incorporating ideas compatible with infinitesimal or nonstandard-analysis-inspired viewpoints under uncertainty.
26	Dynamic Set	Examines time-dependent or state-dependent set models whose membership or structure evolves dynamically, including fuzzy, neutrosophic, uncertain, soft, rough, and related variants.
27	Linguistic Set	Studies set models described through linguistic terms rather than only numerical values, including neutrosophic and uncertain linguistic extensions.
28	Random Set	Studies set-theoretic models under probabilistic uncertainty, including random sets and their fuzzy, neutrosophic, uncertain, soft, and rough extensions.

Ch.	Set Concept	Overview
29	Admissible Set	Examines admissibility-based set models and their fuzzy, neutrosophic, and uncertain variants, emphasizing constraint-consistent membership and structurally valid set descriptions.
30	Named Set	Introduces named sets and their fuzzy, neutrosophic, and uncertain extensions, highlighting frameworks in which sets are explicitly associated with names or labels.
31	Naive Set	Discusses naive set-theoretic viewpoints and their fuzzy, neutrosophic, and uncertain extensions, focusing on direct membership-based formulations prior to axiomatic refinements.

As a further organizational guide, the chapter-level set concepts may be classified from several complementary viewpoints. The following classification tables 1.2, 1.3, 1.4 are not mutually exclusive; rather, they highlight different mathematical aspects of the same families of set concepts.

Table 1.2: Classification of chapter-level set concepts by primary mathematical viewpoint

Category	Chapters	Representative concepts
Foundational uncertainty models	2	Foundational uncertainty-related sets
Representation-oriented models	4, 5, 9, 10, 11, 14, 15, 17, 22, 27	Complex, Interval, Shadowed, Grey, Flou, Circular, Cubic, N-set, Triangular, Linguistic
Structure-expansion models	7, 8, 13, 19, 30	Multiset, Powerset, Weighted Set, Graphic Set, Named Set
Relation- and approximation-oriented models	3, 6, 23, 24, 29, 31	Near Set, Granular Set, Dense Set, Baire Set, Admissible Set, Naive Set
Algebraic and order-theoretic models	16, 20, 21	Convex Set, Ordered Set, Lattice-Valued Set
Time-, probability-, and state-sensitive models	18, 25, 26, 28	Probabilistic Set, Nonstandard Set, Dynamic Set, Random Set
Control-oriented models	12	Controlled Set

Table 1.3: Classification of chapter-level set concepts by principal mode of uncertainty representation

Mode of uncertainty representation	**Chapters**	**Representative concepts**
Approximation- or granule-based	3, 6, 23, 24	Near Set, Granular Set, Dense Set, Baire Set
Interval-, boundary-, or zone-based	5, 9, 10, 11, 14, 15, 22	Interval Set, Shadowed Set, Grey Set, Flou Set, Circular Set, Cubic Set, Triangular Set
Weighted, repeated, or higher-order collection based	7, 8, 13	Multiset, Powerset, Weighted Set
Order-, algebra-, or closure-based	16, 20, 21, 29	Convex Set, Ordered Set, Lattice-Valued Set, Admissible Set
Graph-, label-, or naming-based	19, 30	Graphic Set, Named Set
Probability-, randomness-, or dynamics-based	18, 26, 28	Probabilistic Set, Dynamic Set, Random Set
Nonclassical scale or domain based	4, 17, 25, 27	Complex Set, N-Set, Nonstandard Set, Linguistic Set
Direct baseline uncertainty frameworks	2	Fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, rough, soft, and uncertain sets

Table 1.4: Classification of chapter-level set concepts by supporting mathematical background

Supporting background	Chapters	Representative concepts
Topology and category	23, 24	Dense Set, Baire Set
Order and lattice theory	20, 21	Ordered Set, Lattice-Valued Set
Convexity and geometric structure	14, 16, 22	Circular Set, Convex Set, Triangular Set
Graph and network structure	19	Graphic Set
Granules, proximity, and approximation	3, 6	Near Set, Granular Set
Probability and stochastic structure	18, 28	Probabilistic Set, Random Set
Time or evolution	26	Dynamic Set
Nonstandard or infinitesimal viewpoint	25	Nonstandard Set
Labeling, naming, and parameter assignment	12, 27, 30	Controlled Set, Linguistic Set, Named Set
Membership-value enrichment	4, 5, 9, 10, 11, 15, 17	Complex, Interval, Shadowed, Grey, Flou, Cubic, N-set

Chapter 2

Preliminaries

This chapter introduces the notation and fundamental concepts used in the sequel.

2.1 Fuzzy Set

Fuzzy set theory generalizes the ordinary notion of a subset by allowing each element to belong to a set with a degree in the unit interval $[0, 1]$ [1, 20, 21]. We first recall the standard definition.

Definition 2.1.1 (Fuzzy set)**.** [1] Let X be a nonempty set. A *fuzzy set* A on X is determined by a function

$$\mu_A : X \to [0, 1],$$

called the *membership function* of A. Equivalently, one may represent A as

$$A = \{(x, \mu_A(x)) \mid x \in X\},$$

where $\mu_A(x)$ expresses the degree to which x belongs to A.

2.2 Intuitionistic Fuzzy Set

Intuitionistic fuzzy sets refine fuzzy sets by assigning to each element both a membership degree and a non-membership degree, thereby leaving room for an explicit hesitation part [4, 22]. The usual definition is given below.

Definition 2.2.1 (Intuitionistic fuzzy set)**.** [23] Let E be a nonempty set. An *intuitionistic fuzzy set* (IFS) A on E is of the form

$$A = \big\{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in E\big\},$$

where

$$\mu_A, \nu_A : E \to [0, 1]$$

denote the membership and non-membership functions, respectively, and satisfy

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \qquad \text{for all } x \in E.$$

The quantity

$$\pi_A(x) := 1 - \mu_A(x) - \nu_A(x)$$

is called the *hesitation degree* of x.

The classical fuzzy-set case is recovered when

$$\nu_A(x) = 1 - \mu_A(x) \qquad \text{for all } x \in E,$$

or equivalently when $\pi_A(x) = 0$ for every $x \in E$.

2.3 Neutrosophic Set

Neutrosophic sets describe uncertainty by assigning to each element three quantities: truth, indeterminacy, and falsity, usually taken in the interval $[0, 1]$ [24–27]. Because the indeterminacy component is handled explicitly, this framework extends both fuzzy sets and intuitionistic fuzzy sets in a flexible way [28].

Definition 2.3.1 (Neutrosophic set)**.** [29, 30] Let X be a nonempty set. A *neutrosophic set* (NS) A on X is specified by three mappings

$$T_A : X \to [0, 1], \qquad I_A : X \to [0, 1], \qquad F_A : X \to [0, 1],$$

where, for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively, of the statement "$x \in A$". These values satisfy

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3 \qquad \text{for all } x \in X.$$

2.4 Plithogenic Set

Plithogenic set theory extends uncertainty modeling by incorporating attribute-based appurtenance together with contradiction degrees between attribute values [31–34]. A standard formulation is as follows.

Definition 2.4.1 (Plithogenic Set)**.** [31, 32] Let P be a nonempty universe of discourse, and let v be a fixed attribute whose possible values form a nonempty set Pv. Let $s, t \in \mathbb{N}$.

A *plithogenic set* on (P, v, Pv) is a quintuple

$$PS = (P,\, v,\, Pv,\, pdf,\, pCF),$$

where

- $$pdf : P \times Pv \to [0,1]^s$$

  is the *degree of appurtenance function* (DAF); for $x \in P$ and $a \in Pv$, the value $pdf(x,a)$ gives the possibly vector-valued degree to which x belongs relative to the attribute value a;

- $$pCF : Pv \times Pv \to [0,1]^t$$

  is the *degree of contradiction function* (DCF), satisfying

  $$pCF(a,a) = 0, \qquad pCF(a,b) = pCF(b,a) \quad \text{for all } a,b \in Pv.$$

In plithogenic theory, one usually selects a *dominant attribute value* $a^* \in Pv$. Set-theoretic operations such as union and intersection are then constructed by combining appurtenance degrees with contradiction degrees relative to a^*, thereby capturing interaction and opposition among different attribute values.

Example 2.4.2 (A concrete real-life example of a plithogenic set)**.** Let

$$P = \{h_1, h_2, h_3\}$$

be a set of three houses for sale. Suppose that a buyer evaluates these houses with respect to the attribute

$$v = \text{“location quality”}.$$

Assume that the set of possible attribute values is

$$Pv = \{\text{urban}, \text{suburban}, \text{rural}\}.$$

We consider the scalar case

$$s = t = 1.$$

Thus, the degree of appurtenance function is

$$pdf : P \times Pv \to [0,1],$$

and the degree of contradiction function is

$$pCF : Pv \times Pv \to [0,1].$$

Assume that the buyer's appurtenance evaluations are given by

$$\begin{aligned}
&pdf(h_1,\text{urban}) = 0.90, \quad pdf(h_1,\text{suburban}) = 0.40, \quad pdf(h_1,\text{rural}) = 0.10,\\
&pdf(h_2,\text{urban}) = 0.55, \quad pdf(h_2,\text{suburban}) = 0.85, \quad pdf(h_2,\text{rural}) = 0.30,\\
&pdf(h_3,\text{urban}) = 0.15, \quad pdf(h_3,\text{suburban}) = 0.50, \quad pdf(h_3,\text{rural}) = 0.95.
\end{aligned}$$

These values mean, for example, that:

- h_1 strongly belongs to the class of houses with an urban location;
- h_2 is judged most compatible with a suburban location;
- h_3 is judged most compatible with a rural location.

Next, define the contradiction function by

$$pCF(\text{urban}, \text{urban}) = 0, \qquad pCF(\text{suburban}, \text{suburban}) = 0, \qquad pCF(\text{rural}, \text{rural}) = 0,$$

and

$$pCF(\text{urban}, \text{suburban}) = 0.3, \qquad pCF(\text{urban}, \text{rural}) = 0.8, \qquad pCF(\text{suburban}, \text{rural}) = 0.4.$$

By symmetry,

$$pCF(a, b) = pCF(b, a) \qquad \text{for all } a, b \in Pv.$$

This reflects the intuition that:

- urban and suburban are only mildly contradictory;
- urban and rural are strongly contradictory;
- suburban and rural are moderately contradictory.

Suppose now that the buyer chooses

$$a^* = \text{urban}$$

as the dominant attribute value. Then the plithogenic set

$$PS = (P, v, Pv, pdf, pCF)$$

represents the collection of houses together with:

- their appurtenance degrees relative to each possible location type, and
- the contradiction degrees among those location types, measured relative to the dominant preference urban.

Hence, PS is a concrete real-life example of a plithogenic set: it models a housing decision problem in which each object is evaluated not only by belongingness degrees with respect to several attribute values, but also by the contradictions among those values.

2.5 Rough Set

Rough set theory treats imprecision by replacing a target set with two approximations: a lower approximation, representing certainty, and an upper approximation, representing possibility. These are derived from an indiscernibility relation [35–38]. Related concepts such as game-theoretic rough sets [39] and dominance-based rough sets [40–43] are also known. The classical Pawlak construction is recalled below.

Definition 2.5.1 (Rough set approximations)**.** [44] Let X be a nonempty universe, and let $R \subseteq X \times X$ be an equivalence relation. For each $x \in X$, define the equivalence class of x by

$$[x]_R := \{\, y \in X \mid (x, y) \in R \,\}.$$

For any subset $U \subseteq X$, define:

1. *Lower approximation*:
$$\underline{U} := \{\, x \in X \mid [x]_R \subseteq U \,\}.$$
Thus, $\underline{U}$ consists of those elements whose entire equivalence classes lie inside U.
2. *Upper approximation*:
$$\overline{U} := \{\, x \in X \mid [x]_R \cap U \neq \varnothing \,\}.$$
Hence, $\overline{U}$ consists of those elements whose equivalence classes intersect U.

The pair $(\underline{U}, \overline{U})$ is called the *rough approximation* of U, and one always has

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

2.6 Soft Set

Soft sets model uncertainty by means of parameters: each parameter determines a subset of the universe, and the entire family of such subsets forms the soft description. This framework was introduced by Molodtsov and has since been used widely in uncertainty analysis and decision-making [45, 46]. As related concepts, HyperSoft Sets [47–49], TreeSoft Set [50–54], and SuperHyperSoft Sets [55–57] are also known.

Definition 2.6.1 (Soft set)**.** [46] Let U be a universe, let E be a set of parameters, and let $A \subseteq E$. Denote by $\mathcal{P}(U)$ the power set of U. A pair (F, A) is called a *soft set* over U if

$$F : A \to \mathcal{P}(U).$$

For each parameter $\epsilon \in A$, the subset $F(\epsilon) \subseteq U$ is called the *ϵ-approximation* of (F, A). Thus, a soft set is simply a parameterized family of subsets of the universe U.

2.7 Uncertain set

An *uncertain set* associates with each element a degree taken from a chosen uncertainty model, thereby providing a unifying umbrella for fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related frameworks [58, 59].

Definition 2.7.1 (Uncertain model)**.** [58] Let U denote the class of all *uncertain models*. Each $M \in U$ is determined by:

- a nonempty set $\mathrm{Dom}(M) \subseteq [0,1]^k$ of *admissible degree tuples* for some fixed integer $k \geq 1$; and
- model-specific algebraic or geometric constraints imposed on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \leq 1$ in the intuitionistic fuzzy setting, or $0 \leq T + I + F \leq 3$ in the neutrosophic setting).

Typical instances include:

- **Fuzzy model:** $\mathrm{Dom}(M) = [0,1]$;
- **Intuitionistic fuzzy model:** $\mathrm{Dom}(M) = \{(\mu,\nu) \in [0,1]^2 : \ \mu + \nu \leq 1\}$;
- **Neutrosophic model:** $\mathrm{Dom}(M) = \{(T,I,F) \in [0,1]^3 : \ 0 \leq T + I + F \leq 3\}$;
- **Plithogenic model,** and many further extensions.

Definition 2.7.2 (Uncertain set (U-set))**.** [58] Let X be a nonempty universe, and fix an uncertain model M with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *uncertain set of type M* (briefly, a *U-set*) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is the *uncertainty-degree function* (membership map) of $\mathcal{U}$. For $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ encodes the degree(s) to which x belongs to $\mathcal{U}$, as prescribed by the model M.

As noted in the remark, various generalizations are possible. For reference, Table 2.1 presents a catalogue of uncertainty-set families (U-Sets) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (cf. [59]).

Also, as a reference, a brief comparison of rough sets, soft sets, and uncertain sets is presented in Table 2.2. Rough sets, soft sets, and uncertain sets address different aspects of imperfect information. Rough sets are approximation-based, soft sets are parameter-based, and uncertain sets provide direct uncertainty-based descriptions at the element level.

Table 2.1: A catalogue of uncertainty-set families (U-Sets) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [59].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [1,2]; N-Fuzzy Set [60–62] Shadowed Set [63–65]
2		Intuitionistic Fuzzy Set [4,22]; Vague Set [66,67]; Bipolar Fuzzy Set (two-component description) [68–70]; Pythagorean Fuzzy Set [71,72]; Fermatean fuzzy Set [73,74]; Variable Fuzzy Set [75–77]; Paraconsistent Fuzzy Set [78, 79]; Bifuzzy Set [80,81]
3		Single-Valued Neutrosophic Set [26,30]; Picture Fuzzy Set [8,82]; Ternary Fuzzy Set [83]; Hesitant Fuzzy Set [6,84]; Spherical Fuzzy Set [85,86]; Tripolar Fuzzy Set (three-component formalisms) [87–89]; Neutrosophic Vague Set [90,91]
4		Quadripartitioned Neutrosophic Set [10,92]; Double-Valued Neutrosophic Set [93, 94]; Dual Hesitant Fuzzy Set [95, 96]; Ambiguous Set [97–99]; Turiyam Neutrosophic Set [100–103]
5		Pentapartitioned Neutrosophic Set [104–106]; Triple-Valued Neutrosophic Set [107–110]
6		Hexapartitioned Neutrosophic Set [111]; Bipolar Neutrosophic Set [112,113]; Bipolar Picture Fuzzy Sets [114,115]; Quadruple-Valued Neutrosophic Set [109,116]
7		Heptapartitioned Neutrosophic Set [117–119]; Quintuple-Valued Neutrosophic Set [109,120,121]
8		Octapartitioned Neutrosophic Set [111]; Bipolar Quadripartitioned Neutrosophic Set [122,123]; Bipolar Double-valued Neutrosophic Set
9		Nonapartitioned Neutrosophic Set [111]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [124]; MultiFuzzy Set [125]; n-Refined Fuzzy Set [126,127]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [127]; Multi-Intuitionistic Fuzzy Set [125]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [127,128]; Multi-Neutrosophic Set [125,129,130]

Reading guide. In the U-Set scheme [58], each model M is specified by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ and a membership map $\mu_M : X \to \mathrm{Dom}(M)$. The table groups representative families by the ambient dimension k (i.e., how many numerical components are stored per element).

(a) A widely cited viewpoint is that neutrosophic sets provide a unifying umbrella covering several earlier multi-component fuzzy models (and their generalizations); see [28].

(b) Ambiguous sets are commonly presented as subclasses of certain four-component neutrosophic families; see [10,92,99].

(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [131].

Table 2.2: A brief comparison of Rough, Soft, and Uncertain sets

Framework	**Basic representation**	**Main source of uncertainty**	**Main structural feature**	**Typical interpretation**
Rough set	A subset together with lower and upper approximations	Indiscernibility, granularity, or limited discernibility of objects	Boundary region generated by approximation	Used when objects cannot be sharply distinguished, so membership is described through approximation rather than direct assignment
Soft set	A parameterized family $F : A \to \mathcal{P}(U)$	Dependence on parameters, attributes, or viewpoints	Parameter-driven description of a subset family	Used when membership depends on selected parameters such as "cheap," "large," or "reliable"
Uncertain set	An element-wise uncertainty-bearing assignment (e.g. interval-, set-, or tuple-valued description, depending on the model)	Direct uncertainty in the membership information itself	Explicit modeling of imprecision or indeterminacy at the membership level	Used when the status of each element is itself uncertain and should be represented directly rather than only by approximation or parameterization

Chapter 3

Near Set and Its Extensions

In this chapter, we introduce near sets and their extensions.

3.1 Near set

A near set is a pair of subsets related by spatial or descriptive proximity, capturing resemblance through overlap, nearness relations, or shared feature-description pattern [132–136].

Definition 3.1.1 (Near sets)**.** [135, 136] Let X be a nonempty set, and let δ be a proximity relation on $\mathcal{P}(X)$. For subsets $A, B \subseteq X$, the pair (A, B) is called a *pair of near sets* if

$$A \,\delta\, B.$$

Equivalently, A is said to be *near* to B.

In the particular case where δ is the discrete proximity

$$A \,\delta\, B \iff A \cap B \neq \emptyset,$$

the sets A and B are called *spatially near sets*.

Definition 3.1.2 (Descriptive near sets)**.** (cf. [137, 138]) Let X be a nonempty set, and let

$$\Phi = \{\varphi_1, \ldots, \varphi_n\}$$

be a finite family of probe functions, where each

$$\varphi_i : X \to V_i$$

maps points of X to a feature space V_i. Define the description map

$$\Phi_X : X \to \prod_{i=1}^{n} V_i, \qquad \Phi_X(x) := (\varphi_1(x), \ldots, \varphi_n(x)).$$

For each subset $A \subseteq X$, define its description set by

$$Q_\Phi(A) := \{\Phi_X(a) : a \in A\}.$$

Two subsets $A, B \subseteq X$ are said to be *descriptively near* if

$$A \,\delta_\Phi\, B \iff Q_\Phi(A) \cap Q_\Phi(B) \neq \emptyset.$$

In this case, (A, B) is called a *pair of descriptive near sets.*

If, in addition,

$$A \cap B = \emptyset \quad \text{and} \quad A \,\delta_\Phi\, B,$$

then A and B are called *disjoint descriptive near sets.*

3.2 Near fuzzy set

Near fuzzy sets are fuzzy subsets whose alpha-cuts are proximate for some threshold, expressing closeness between vague concepts through level-set proximity rather than exact overlap [139–142].

Definition 3.2.1 (Near fuzzy set)**.** Let X be a nonempty set, let δ be a proximity relation on $\mathcal{P}(X)$, and let A, B be fuzzy sets on X with membership functions

$$\mu_A, \mu_B : X \to [0, 1].$$

For each $\alpha \in (0, 1]$, define the α-cuts

$$A_\alpha := \{x \in X \mid \mu_A(x) \geq \alpha\}, \qquad B_\alpha := \{x \in X \mid \mu_B(x) \geq \alpha\}.$$

Then A and B are called *near fuzzy sets at level* α if

$$A_\alpha \,\delta\, B_\alpha.$$

They are called *near fuzzy sets* if there exists some $\alpha \in (0, 1]$ such that

$$A_\alpha \,\delta\, B_\alpha.$$

Remark 3.2.2. If δ is the discrete proximity

$$C \,\delta\, D \iff C \cap D \neq \emptyset,$$

then A and B are near fuzzy sets if and only if

$$A_\alpha \cap B_\alpha \neq \emptyset$$

for some $\alpha \in (0, 1]$.

3.3 Near intuitionistic fuzzy set

Near intuitionistic fuzzy sets are intuitionistic fuzzy sets whose (α, β)-cuts are proximate for some threshold pair, expressing closeness through membership and non-membership conditions [139, 143].

Definition 3.3.1 (Near intuitionistic fuzzy set)**.** Let X be a nonempty set, let δ be a proximity relation on $\mathcal{P}(X)$, and let

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}, \qquad B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$$

be intuitionistic fuzzy sets on X.

For $\alpha, \beta \in [0, 1]$, define the (α, β)-cuts

$$A_{(\alpha,\beta)} := \{\, x \in X \mid \mu_A(x) \geq \alpha,\ \nu_A(x) \leq \beta \,\},$$

$$B_{(\alpha,\beta)} := \{\, x \in X \mid \mu_B(x) \geq \alpha,\ \nu_B(x) \leq \beta \,\}.$$

Then A and B are called *near intuitionistic fuzzy sets at level* (α, β) if

$$A_{(\alpha,\beta)} \,\delta\, B_{(\alpha,\beta)}.$$

They are called *near intuitionistic fuzzy sets* if there exists $(\alpha, \beta) \in [0, 1]^2$ such that

$$A_{(\alpha,\beta)} \,\delta\, B_{(\alpha,\beta)}.$$

Remark 3.3.2. If δ is the discrete proximity

$$C \,\delta\, D \iff C \cap D \neq \emptyset,$$

then A and B are near intuitionistic fuzzy sets exactly when

$$A_{(\alpha,\beta)} \cap B_{(\alpha,\beta)} \neq \emptyset$$

for some $(\alpha, \beta) \in [0, 1]^2$.

3.4 Near Neutrosophic Set

Near neutrosophic sets are neutrosophic sets whose thresholded truth-indeterminacy-falsity cuts are proximate at some level, expressing closeness between uncertain, indeterminate, and false information structures meaningfully [144].

Definition 3.4.1 (Near neutrosophic set)**.** Let X be a nonempty set, let δ be a proximity relation on $\mathcal{P}(X)$, and let

$$A = (T_A, I_A, F_A), \qquad B = (T_B, I_B, F_B)$$

be single-valued neutrosophic sets on X, where

$$T_A, I_A, F_A, T_B, I_B, F_B : X \to [0, 1].$$

For $\alpha, \beta, \gamma \in [0,1]$, define

$$A_{(\alpha,\beta,\gamma)} := \{\, x \in X \mid T_A(x) \geq \alpha,\ I_A(x) \leq \beta,\ F_A(x) \leq \gamma \,\},$$

$$B_{(\alpha,\beta,\gamma)} := \{\, x \in X \mid T_B(x) \geq \alpha,\ I_B(x) \leq \beta,\ F_B(x) \leq \gamma \,\}.$$

Then A and B are called *near neutrosophic sets at level* (α, β, γ) if

$$A_{(\alpha,\beta,\gamma)} \,\delta\, B_{(\alpha,\beta,\gamma)}.$$

They are called *near neutrosophic sets* if there exists

$$(\alpha, \beta, \gamma) \in [0,1]^3$$

such that

$$A_{(\alpha,\beta,\gamma)} \,\delta\, B_{(\alpha,\beta,\gamma)}.$$

Example 3.4.2 (A real-life example of near neutrosophic sets)**.** Let

$$X = \{h_1, h_2, h_3, h_4\}$$

be a set of four apartments under consideration by a person who wants to rent a home.

Define two single-valued neutrosophic sets on X:

- $A = (T_A, I_A, F_A)$: the set of apartments that are *affordable*;
- $B = (T_B, I_B, F_B)$: the set of apartments that are *convenient for commuting*.

Here, for each apartment $h \in X$,

- $T_A(h)$ and $T_B(h)$ represent the degree to which the apartment is considered affordable or convenient;
- $I_A(h)$ and $I_B(h)$ represent the degree of indeterminacy, for example due to incomplete information about costs, transportation, or hidden conditions;
- $F_A(h)$ and $F_B(h)$ represent the degree to which the apartment is considered not affordable or not convenient.

Assume the evaluations are given as follows:

$$A(h_1) = (0.82, 0.10, 0.12), \quad A(h_2) = (0.76, 0.18, 0.15),$$
$$A(h_3) = (0.61, 0.25, 0.28), \quad A(h_4) = (0.45, 0.30, 0.40),$$

and

$$B(h_1) = (0.65, 0.22, 0.21), \quad B(h_2) = (0.79, 0.14, 0.13),$$
$$B(h_3) = (0.72, 0.19, 0.18), \quad B(h_4) = (0.50, 0.27, 0.35).$$

Let the proximity relation δ on $\mathcal{P}(X)$ be defined by

$$C\,\delta\,D \iff C \cap D \neq \emptyset$$

for all $C, D \subseteq X$. Thus, two subsets are regarded as near whenever they share at least one apartment.

Now choose the level

$$(\alpha, \beta, \gamma) = (0.7, 0.2, 0.2).$$

Then

$$A_{(0.7,0.2,0.2)} = \{\, h \in X \mid T_A(h) \geq 0.7,\ I_A(h) \leq 0.2,\ F_A(h) \leq 0.2 \,\} = \{h_1, h_2\},$$

because h_1 and h_2 satisfy the required affordability conditions, while h_3 and h_4 do not.

Similarly,

$$B_{(0.7,0.2,0.2)} = \{\, h \in X \mid T_B(h) \geq 0.7,\ I_B(h) \leq 0.2,\ F_B(h) \leq 0.2 \,\} = \{h_2, h_3\}.$$

Since

$$A_{(0.7,0.2,0.2)} \cap B_{(0.7,0.2,0.2)} = \{h_2\} \neq \emptyset,$$

we obtain

$$A_{(0.7,0.2,0.2)}\,\delta\,B_{(0.7,0.2,0.2)}.$$

Hence, A and B are near neutrosophic sets at the level $(0.7, 0.2, 0.2)$.

In practical terms, this means that there exists at least one apartment, namely h_2, that is sufficiently affordable and sufficiently convenient for commuting, with low indeterminacy and low falsity in both evaluations. Therefore, the two criteria "affordable apartments" and "commuting-convenient apartments" are near to each other in this decision-making situation.

Theorem 3.4.3 (Well-definedness of near neutrosophic sets)**.** *Let A and B be single-valued neutrosophic sets on X, and let*

$$\alpha, \beta, \gamma \in [0, 1].$$

Then the sets

$$A_{(\alpha,\beta,\gamma)}, \qquad B_{(\alpha,\beta,\gamma)}$$

are well-defined subsets of X. Consequently, the notion of a near neutrosophic set is well-defined.

Proof. Since

$$T_A, I_A, F_A : X \to [0, 1],$$

for every $x \in X$ the inequalities

$$T_A(x) \geq \alpha, \qquad I_A(x) \leq \beta, \qquad F_A(x) \leq \gamma$$

are meaningful. Hence

$$A_{(\alpha,\beta,\gamma)} \subseteq X.$$

Similarly,

$$B_{(\alpha,\beta,\gamma)} \subseteq X.$$

Because δ is a proximity relation on $\mathcal{P}(X)$, the statement

$$A_{(\alpha,\beta,\gamma)}\,\delta\,B_{(\alpha,\beta,\gamma)}$$

is meaningful. Therefore the above definition is well-defined. □

3.5 Near Plithogenic Set

Near plithogenic sets are plithogenic sets whose evaluated attribute-based level cuts are proximate for some attribute and threshold, capturing closeness under contradiction-aware appurtenance information.

Definition 3.5.1 (Near plithogenic set)**.** Let

$$\mathfrak{P} = (P, v, Pv, pCF, s, t)$$

be a fixed plithogenic frame, let δ be a proximity relation on $\mathcal{P}(P)$, and let

$$A = (P, v, Pv, pdf_A, pCF), \qquad B = (P, v, Pv, pdf_B, pCF)$$

be plithogenic sets on the same frame. Let

$$\sigma : [0,1]^s \to [0,1]$$

be a fixed evaluation map. For $a \in Pv$ and $\lambda \in [0,1]$, define

$$A^{\sigma}_{(a,\lambda)} := \{\, x \in P \mid \sigma(pdf_A(x,a)) \geq \lambda \,\},$$

$$B^{\sigma}_{(a,\lambda)} := \{\, x \in P \mid \sigma(pdf_B(x,a)) \geq \lambda \,\}.$$

Then A and B are called *near plithogenic sets at level* (a, λ) if

$$A^{\sigma}_{(a,\lambda)} \,\delta\, B^{\sigma}_{(a,\lambda)}.$$

They are called *near plithogenic sets* if there exist

$$a \in Pv, \qquad \lambda \in [0,1]$$

such that

$$A^{\sigma}_{(a,\lambda)} \,\delta\, B^{\sigma}_{(a,\lambda)}.$$

Theorem 3.5.2 (Well-definedness of near plithogenic sets)**.** *Let A and B be plithogenic sets on the fixed frame $\mathfrak{P}$, let*

$$\sigma : [0,1]^s \to [0,1]$$

be an evaluation map, and let

$$a \in Pv, \qquad \lambda \in [0,1].$$

Then

$$A^{\sigma}_{(a,\lambda)}, \qquad B^{\sigma}_{(a,\lambda)}$$

are well-defined subsets of P. Consequently, the notion of a near plithogenic set is well-defined.

Proof. Since

$$pdf_A, pdf_B : P \times Pv \to [0,1]^s,$$

for every

$$x \in P, \qquad a \in Pv,$$

the values

$$pdf_A(x,a), \qquad pdf_B(x,a)$$

belong to $[0,1]^s$. Hence

$$\sigma(pdf_A(x,a)), \qquad \sigma(pdf_B(x,a))$$

are well-defined elements of $[0,1]$. Therefore the threshold conditions

$$\sigma(pdf_A(x,a)) \geq \lambda, \qquad \sigma(pdf_B(x,a)) \geq \lambda$$

are meaningful, and thus

$$A^{\sigma}_{(a,\lambda)} \subseteq P, \qquad B^{\sigma}_{(a,\lambda)} \subseteq P.$$

Because δ is a proximity relation on $\mathcal{P}(P)$, the relation

$$A^{\sigma}_{(a,\lambda)} \,\delta\, B^{\sigma}_{(a,\lambda)}$$

is meaningful. Hence the definition is well-defined. □

3.6 Near Uncertain Set

Near uncertain sets are uncertain sets whose evaluated level cuts are proximate at some threshold, expressing closeness between generalized uncertainty structures under a fixed assessment map.

Definition 3.6.1 (Near uncertain set)**.** Let X be a nonempty set, let δ be a proximity relation on $\mathcal{P}(X)$, let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$\eta : \mathrm{Dom}(M) \to [0,1]$$

be a fixed evaluation map. Let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M on X, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

For $\lambda \in [0,1]$, define

$$\mathcal{U}^{\eta}_{\lambda} := \{\, x \in X \mid \eta(\mu_{\mathcal{U}}(x)) \geq \lambda \,\},$$
$$\mathcal{V}^{\eta}_{\lambda} := \{\, x \in X \mid \eta(\mu_{\mathcal{V}}(x)) \geq \lambda \,\}.$$

Then $\mathcal{U}$ and $\mathcal{V}$ are called *near uncertain sets at level* λ if

$$\mathcal{U}^{\eta}_{\lambda} \,\delta\, \mathcal{V}^{\eta}_{\lambda}.$$

They are called *near uncertain sets* if there exists

$$\lambda \in [0,1]$$

such that

$$\mathcal{U}^{\eta}_{\lambda} \,\delta\, \mathcal{V}^{\eta}_{\lambda}.$$

Theorem 3.6.2 (Well-definedness of near uncertain sets)**.** *Let $\mathcal{U}$ and $\mathcal{V}$ be uncertain sets of type M on X, let*

$$\eta : \mathrm{Dom}(M) \to [0,1]$$

be an evaluation map, and let

$$\lambda \in [0,1].$$

Then

$$\mathcal{U}^{\eta}_{\lambda}, \qquad \mathcal{V}^{\eta}_{\lambda}$$

are well-defined subsets of X. Consequently, the notion of a near uncertain set is well-defined.

Proof. Since

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M),$$

for every $x \in X$ one has

$$\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M).$$

Because

$$\eta : \mathrm{Dom}(M) \to [0,1],$$

the values

$$\eta(\mu_{\mathcal{U}}(x)), \qquad \eta(\mu_{\mathcal{V}}(x))$$

are well-defined elements of $[0,1]$. Hence the conditions

$$\eta(\mu_{\mathcal{U}}(x)) \geq \lambda, \qquad \eta(\mu_{\mathcal{V}}(x)) \geq \lambda$$

are meaningful. Therefore

$$\mathcal{U}^{\eta}_{\lambda} \subseteq X, \qquad \mathcal{V}^{\eta}_{\lambda} \subseteq X.$$

Since δ is a proximity relation on $\mathcal{P}(X)$, the relation

$$\mathcal{U}^{\eta}_{\lambda} \, \delta \, \mathcal{V}^{\eta}_{\lambda}$$

is meaningful. Thus the definition is well-defined. □

3.7 Near Soft Set

Near soft sets are soft sets having proximate parameter-images for some parameter, expressing closeness between parameterized families of subsets through a proximity relation [145–147].

Definition 3.7.1 (Near soft set)**.** Let X be a nonempty set, let E be a nonempty set of parameters, and let

$$\delta$$

be a proximity relation on

$$\mathcal{P}(X).$$

Let

$$(F, E), \qquad (G, E)$$

be soft sets over X, where

$$F, G : E \to \mathcal{P}(X).$$

For a parameter $e \in E$, the soft sets (F, E) and (G, E) are called *near soft sets at the parameter* e if

$$F(e)\,\delta\,G(e).$$

They are called *near soft sets* if there exists some

$$e \in E$$

such that

$$F(e)\,\delta\,G(e).$$

Remark 3.7.2. If δ is the discrete proximity

$$A\,\delta\,B \iff A \cap B \neq \emptyset,$$

then (F, E) and (G, E) are near soft sets if and only if there exists

$$e \in E$$

such that

$$F(e) \cap G(e) \neq \emptyset.$$

3.8 Near Rough Set

Near rough sets are rough sets whose upper approximations are proximate, representing closeness between imprecise concepts through nearness of possible regions rather than exact equality [148, 149].

Definition 3.8.1 (Near rough set)**.** Let X be a nonempty set, let

$$R \subseteq X \times X$$

be an equivalence relation, and let δ be a proximity relation on

$$\mathcal{P}(X).$$

For subsets $A, B \subseteq X$, write

$$\mathrm{RS}_R(A) := (\underline{A}_R, \overline{A}_R), \qquad \mathrm{RS}_R(B) := (\underline{B}_R, \overline{B}_R),$$

where

$$\underline{A}_R := \{\, x \in X \mid [x]_R \subseteq A \,\}, \qquad \overline{A}_R := \{\, x \in X \mid [x]_R \cap A \neq \emptyset \,\},$$

and similarly for B.

Then $\mathrm{RS}_R(A)$ and $\mathrm{RS}_R(B)$ are called *near rough sets* if

$$\overline{A}_R\,\delta\,\overline{B}_R.$$

3.9 Tolerance Near Set

A tolerance near set is a pair of object sets regarded as near when some feature descriptions differ by at most a fixed tolerance threshold [150–153].

Definition 3.9.1 (Tolerance near set)**.** Let

$$(O, F)$$

be a perceptual system, where O is a nonempty set of perceptual objects and F is a nonempty set of real-valued probe functions on O. Let

$$B = \{\varphi_1, \dots, \varphi_\ell\} \subseteq F$$

be a finite family of probe functions, and let

$$\Phi_B : O \to \mathbb{R}^\ell$$

be the corresponding object-description mapping defined by

$$\Phi_B(x) = \big(\varphi_1(x), \dots, \varphi_\ell(x)\big) \qquad (x \in O).$$

For a fixed tolerance threshold

$$\varepsilon \geq 0,$$

define the perceptual tolerance relation

$$\cong_{B,\varepsilon} \subseteq O \times O$$

by

$$x \cong_{B,\varepsilon} y \quad \Longleftrightarrow \quad \big\|\Phi_B(x) - \Phi_B(y)\big\|_2 \leq \varepsilon.$$

Let

$$X, Y \subseteq O.$$

Then the pair

$$(X, Y)$$

is called a *tolerance near set* (or *tolerance near pair*) with respect to B and ε if

$$\exists\, x \in X,\ \exists\, y \in Y \quad \text{such that} \quad x \cong_{B,\varepsilon} y.$$

Equivalently,

$$(X, Y)$$

is a tolerance near set if there exist

$$x \in X, \qquad y \in Y$$

such that

$$\big\|\Phi_B(x) - \Phi_B(y)\big\|_2 \leq \varepsilon.$$

3.10 Perceptual Near Set

A perceptual near set is a set whose objects are regarded as near another set through shared perceptual descriptions, features, or indistinguishable probe-function values [154–157].

Definition 3.10.1 (Perceptual near set)**.** Let

$$(O, F)$$

be a perceptual system, where O is a nonempty set of perceptual objects and F is a nonempty set of real-valued probe functions

$$\varphi : O \to \mathbb{R}.$$

For each

$$B \subseteq F,$$

define the indiscernibility relation

$$\sim_B \subseteq O \times O$$

by

$$x \sim_B y \iff \varphi(x) = \varphi(y) \text{ for all } \varphi \in B.$$

Let

$$O/{\sim_B}$$

denote the quotient set of equivalence classes determined by $\sim_B$.

For subsets

$$X, Y \subseteq O,$$

we say that X is *near* to Y in the perceptual system (O, F), written

$$X \, \delta_F \, Y,$$

if there exist

$$B_1, B_2 \subseteq F, \qquad \varphi \in F,$$

and equivalence classes

$$A \in O/{\sim_{B_1}}, \qquad B \in O/{\sim_{B_2}}, \qquad C \in O/{\sim_{\{\varphi\}}},$$

such that

$$A \subseteq X, \qquad B \subseteq Y, \qquad A \subseteq C, \qquad B \subseteq C.$$

A subset

$$X \subseteq O$$

is called a *perceptual near set* if there exists a subset

$$Y \subseteq O$$

such that

$$X \, \delta_F \, Y.$$

As a reference, a brief comparison among near sets, tolerance near sets, and perceptual near sets is presented in Table 3.1. Near sets provide a general framework of descriptive nearness, tolerance near sets emphasize approximate similarity under admissible deviation, and perceptual near sets focus on similarity derived from perceptual or feature-based descriptions.

Table 3.1: A brief comparison among near sets, tolerance near sets, and perceptual near sets

Aspect	**Near set**	**Tolerance near set**	**Perceptual near set**
Basic idea	Objects are considered near when they are descriptively similar	Objects are considered near under a tolerance-based similarity relation	Objects are considered near according to perceptual or feature-based resemblance
Main source of nearness	General descriptive overlap	Similarity up to an allowed tolerance	Similarity in perceptual descriptions or observable features
Underlying relation	Usually a nearness or descriptive proximity relation	A tolerance relation, typically reflexive and symmetric	A perceptual nearness relation induced by feature descriptions
Strictness of comparison	General and flexible	More relaxed than exact equivalence, since small differences are allowed	Depends on perceptual criteria and chosen feature representation
Typical interpretation	Two object collections are near if they share descriptive characteristics	Two object collections are near if their descriptions are sufficiently close within tolerance bounds	Two object collections are near if they are perceived as similar from available sensory or descriptive information
Mathematical emphasis	Descriptive nearness and approximation	Tolerance-based grouping and approximate similarity	Perception-oriented description and feature comparison
Typical application viewpoint	Pattern classification and descriptive analysis	Approximate matching under allowable deviation	Image analysis, object recognition, and perceptual classification
Common feature	All three model nearness without requiring exact equality	All three model nearness without requiring exact equality	All three model nearness without requiring exact equality

Chapter 4

Complex Set and Its Extensions

In this chapter, we introduce complex sets and their extensions.

4.1 Complex set

Complex sets assign each element either zero or a unit-modulus complex number, encoding crisp membership together with phase information, as a special complex fuzzy case.

Definition 4.1.1 (Complex set)**.** Let U be a nonempty universe, and let

$$\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}.$$

A *complex set* on U is a set represented by a characteristic function

$$\chi_A : U \to \{0\} \cup \mathbb{S}^1,$$

that is,

$$\chi_A(x) = \begin{cases} e^{i\theta_A(x)}, & x \in A, \\ 0, & x \notin A, \end{cases}$$

for some phase function $\theta_A : A \to \mathbb{R}$.

Equivalently, a complex set is a crisp special case of a complex fuzzy set in which the amplitude takes only the values 0 and 1.

4.2 Complex Fuzzy set

A complex fuzzy set assigns each element a complex membership in the unit disk, combining amplitude and phase to represent graded belonging with periodic information [158–163].

Definition 4.2.1 (Complex fuzzy set)**.** [158–160] Let U be a nonempty universe, and let

$$\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}.$$

A *complex fuzzy set* on U is a set determined by a membership function

$$\mu_A : U \to \mathbb{D}.$$

Equivalently, for each $x \in U$,

$$\mu_A(x) = r_A(x)e^{i\omega_A(x)},$$

where

$$r_A(x) \in [0, 1], \qquad \omega_A(x) \in \mathbb{R}.$$

Here $r_A(x)$ is the amplitude of membership and $\omega_A(x)$ is its phase.

Example 4.2.2 (A real-life example of a complex fuzzy set)**.** Let

$$U = \{m_1, m_2, m_3, m_4\}$$

be a set of four machines in a factory.

Suppose we want to describe the set of machines that are *highly active in a periodic production cycle.* Because the activity of each machine has both

- an intensity aspect (how strongly the machine is used), and
- a timing aspect (at which stage of the daily production cycle the activity is concentrated),

it is natural to model this situation by a complex fuzzy set.

Define a complex fuzzy set

$$A = \{(m, \mu_A(m)) \mid m \in U\}$$

on U, where

$$\mu_A(m) = r_A(m)e^{i\omega_A(m)} \in \mathbb{D}.$$

Here,

- $r_A(m) \in [0, 1]$ represents the degree to which machine m is actively used;
- $\omega_A(m) \in \mathbb{R}$ represents the phase of usage in the production cycle.

For example, let

$$\mu_A(m_1) = 0.90e^{i\pi/6}, \qquad \mu_A(m_2) = 0.75e^{i\pi/2}, \qquad \mu_A(m_3) = 0.40e^{i5\pi/6}, \qquad \mu_A(m_4) = 0.60e^{i4\pi/3}.$$

Then:

- m_1 has a very high activity level, and its main activity occurs near the early stage of the cycle;
- m_2 is also strongly active, but its peak usage occurs later in the cycle;
- m_3 has relatively low activity;
- m_4 has moderate activity, with its usage concentrated in another phase of the cycle.

Thus, the complex fuzzy set A simultaneously represents both the strength of membership and the cyclic timing of membership. This is useful in practice when membership is not only vague in magnitude but also depends on periodic or oscillatory behavior.

4.3 Pure Complex Fuzzy Set

A pure complex fuzzy set assigns each element a complex value whose real and imaginary parts both lie in $[0, 1]$, giving two membership grades.

Definition 4.3.1 (Pure complex fuzzy set)**.** Let X be a nonempty set, and define

$$\mathbb{C}_{[0,1]} := \{\, a + \mathrm{i}b \in \mathbb{C} \mid a, b \in [0, 1] \,\} = [0, 1] + \mathrm{i}[0, 1].$$

A *pure complex fuzzy set* A on X is determined by two functions

$$\mu_A^{(r)}, \mu_A^{(i)} : X \to [0, 1].$$

Its *pure complex membership function* is the mapping

$$\mu_A : X \to \mathbb{C}_{[0,1]}$$

defined by

$$\mu_A(x) := \mu_A^{(r)}(x) + \mathrm{i}\,\mu_A^{(i)}(x) \qquad \text{for all } x \in X.$$

Here $\mu_A^{(r)}(x)$ and $\mu_A^{(i)}(x)$ are called the real and imaginary membership grades of x, respectively.

Equivalently, one may represent A as

$$A = \{(x, \mu_A(x)) \mid x \in X\}.$$

4.4 Complex Intuitionistic Fuzzy Set

A complex intuitionistic fuzzy set assigns complex membership and non-membership degrees, combining amplitudes and phases while preserving intuitionistic constraints that limit simultaneous acceptance and rejection [164–167].

Definition 4.4.1 (Complex intuitionistic fuzzy set)**.** Let X be a nonempty set, and let

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| \le 1\}$$

be the closed unit disk in the complex plane. A *complex intuitionistic fuzzy set* (CIFS) A on X is of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where

$$\mu_A, \nu_A : X \to \mathbb{D}$$

are the *complex membership function* and the *complex non-membership function*, respectively.

Equivalently, for each $x \in X$, one may write

$$\mu_A(x) = r_A(x)e^{\mathrm{i}2\pi\omega_A(x)}, \qquad \nu_A(x) = s_A(x)e^{\mathrm{i}2\pi\psi_A(x)},$$

where

$$r_A(x),\ s_A(x),\ \omega_A(x),\ \psi_A(x) \in [0,1]$$

and

$$r_A(x) + s_A(x) \le 1, \qquad \omega_A(x) + \psi_A(x) \le 1 \qquad \text{for all } x \in X.$$

Here $r_A(x)$ and $s_A(x)$ denote the amplitudes of membership and non-membership, while $\omega_A(x)$ and $\psi_A(x)$ denote their normalized phase components.

4.5 Complex Neutrosophic Set

A complex neutrosophic set assigns each element complex truth, indeterminacy, and falsity degrees in the unit disk, combining amplitudes and phases within neutrosophic constraints [168–171].

Definition 4.5.1 (Complex neutrosophic set)**.** [168,169] Let X be a nonempty set, and let

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| \le 1\}$$

be the closed unit disk in the complex plane. A *complex neutrosophic set* A on X is of the form

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\},$$

where

$$T_A, I_A, F_A : X \to \mathbb{D}$$

are called the *complex truth-membership function*, the *complex indeterminacy-membership function*, and the *complex falsity-membership function*, respectively.

Equivalently, for each $x \in X$, one may write

$$T_A(x) = p_A(x)e^{2\pi\mathrm{i}\,\theta_A(x)}, \qquad I_A(x) = q_A(x)e^{2\pi\mathrm{i}\,\varphi_A(x)}, \qquad F_A(x) = r_A(x)e^{2\pi\mathrm{i}\,\psi_A(x)},$$

where

$$p_A(x), q_A(x), r_A(x) \in [0,1], \qquad \theta_A(x), \varphi_A(x), \psi_A(x) \in [0,1].$$

The quantities $p_A(x)$, $q_A(x)$, and $r_A(x)$ are called the *truth amplitude*, *indeterminacy amplitude*, and *falsity amplitude*, respectively, and they satisfy

$$0 \le p_A(x) + q_A(x) + r_A(x) \le 3 \qquad \text{for all } x \in X.$$

4.6 Complex Plithogenic Set

A complex plithogenic set assigns complex appurtenance degrees relative to attribute values, while contradiction degrees between values model interaction, opposition, and context-dependent membership behavior [172].

Definition 4.6.1 (Complex plithogenic set)**.** [172] Let X be a nonempty universe, let a be a fixed attribute, and let V be a nonempty set of possible values of a. Let

$$c : V \times V \to [0, 1]$$

be a contradiction degree function satisfying

$$c(v, v) = 0, \qquad c(v_1, v_2) = c(v_2, v_1) \quad \text{for all } v, v_1, v_2 \in V.$$

Let

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$$

be the closed unit disk in the complex plane.

A *complex plithogenic set* on the frame (X, a, V, c) is a quintuple

$$P = (X, a, V, d, c),$$

where

$$d : X \times V \to \mathbb{D}$$

is called the *complex degree of appurtenance function*.

Equivalently, for each $x \in X$ and $v \in V$, one may write

$$d(x, v) = r(x, v)e^{2\pi \mathrm{i}\, \omega(x,v)},$$

where

$$r(x, v) \in [0, 1], \qquad \omega(x, v) \in [0, 1].$$

Here $r(x, v)$ represents the amplitude of appurtenance, while $\omega(x, v)$ represents its phase component.

4.7 Complex Uncertain Set

A complex uncertain set assigns admissible complex degree-tuples whose moduli satisfy a fixed uncertain model, extending generalized uncertainty representation with phase-sensitive information.

Definition 4.7.1 (Complex degree-domain induced by an uncertain model)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \geq 1$. Let

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$$

be the closed unit disk in the complex plane.

The *complex degree-domain induced by M* is defined by

$$\mathrm{Dom}_{\mathbb{C}}(M) := \left\{(z_1, \ldots, z_k) \in \mathbb{D}^k \;\middle|\; \big(|z_1|, \ldots, |z_k|\big) \in \mathrm{Dom}(M)\right\}.$$

Definition 4.7.2 (Complex uncertain set)**.** Let X be a nonempty set, and let M be an uncertain model with complex degree-domain

$$\mathrm{Dom}_{\mathbb{C}}(M) \subseteq \mathbb{D}^k.$$

A *complex uncertain set of type M* on X is a pair

$$\mathcal{U}^{\mathbb{C}} = (X, \mu_M^{\mathbb{C}}),$$

where

$$\mu_M^{\mathbb{C}} : X \to \mathrm{Dom}_{\mathbb{C}}(M)$$

is called the *complex uncertainty-degree function.*

Equivalently, for each $x \in X$,

$$\mu_M^{\mathbb{C}}(x) = (z_1(x), \dots, z_k(x))$$

with

$$z_j(x) \in \mathbb{D} \qquad (j = 1, \dots, k),$$

and

$$\big(|z_1(x)|, \dots, |z_k(x)|\big) \in \mathrm{Dom}(M).$$

Thus, the modulus tuple of each complex degree is admissible for the underlying uncertain model M.

Theorem 4.7.3 (Well-definedness of complex uncertain sets)**.** *Let X be a nonempty set, and let M be an uncertain model with*

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Then the induced complex degree-domain

$$\mathrm{Dom}_{\mathbb{C}}(M)$$

is a well-defined nonempty subset of $\mathbb{D}^k$. Consequently, every mapping

$$\mu_M^{\mathbb{C}} : X \to \mathrm{Dom}_{\mathbb{C}}(M)$$

determines a well-defined complex uncertain set of type M on X.

Moreover, for every $x \in X$, if

$$\mu_M^{\mathbb{C}}(x) = (z_1, \dots, z_k),$$

then

$$(|z_1|, \dots, |z_k|) \in \mathrm{Dom}(M).$$

Conversely, every complex uncertain set of type M arises in this way.

Proof. Since M is an uncertain model, the set

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

is nonempty. Take any

$$(r_1, \dots, r_k) \in \mathrm{Dom}(M).$$

Each r_j belongs to $[0, 1]$, hence also to $\mathbb{D}$ when regarded as a complex number. Therefore

$$(r_1, \dots, r_k) \in \mathbb{D}^k$$

and

$$\big(|r_1|, \dots, |r_k|\big) = (r_1, \dots, r_k) \in \mathrm{Dom}(M).$$

Thus

$$(r_1, \dots, r_k) \in \mathrm{Dom}_{\mathbb{C}}(M),$$

so $\mathrm{Dom}_{\mathbb{C}}(M)$ is nonempty.

Next, by definition,

$$\mathrm{Dom}_{\mathbb{C}}(M) \subseteq \mathbb{D}^k.$$

Hence $\mathrm{Dom}_{\mathbb{C}}(M)$ is a well-defined subset of $\mathbb{D}^k$.

Now let

$$\mu_M^{\mathbb{C}} : X \to \mathrm{Dom}_{\mathbb{C}}(M)$$

be any mapping. Then for each $x \in X$,

$$\mu_M^{\mathbb{C}}(x) \in \mathrm{Dom}_{\mathbb{C}}(M).$$

Therefore, if

$$\mu_M^{\mathbb{C}}(x) = (z_1, \dots, z_k),$$

one has

$$(z_1, \dots, z_k) \in \mathbb{D}^k$$

and

$$(|z_1|, \dots, |z_k|) \in \mathrm{Dom}(M).$$

So every value of $\mu_M^{\mathbb{C}}$ is an admissible complex degree of type M, and hence

$$\mathcal{U}^{\mathbb{C}} = (X, \mu_M^{\mathbb{C}})$$

is a well-defined complex uncertain set.

Conversely, by the definition of a complex uncertain set, any such object is precisely given by a mapping

$$\mu_M^{\mathbb{C}} : X \to \mathrm{Dom}_{\mathbb{C}}(M).$$

Therefore every complex uncertain set of type M arises in this way. □

4.8 Complex Rough Set

A complex rough set encodes lower and upper rough approximations by complex-valued membership, using distinct real-imaginary combinations to represent certainty, possibility, and exclusion.

Definition 4.8.1 (Complex Rough Set)**.** Let U be a nonempty universe, and let

$$R \subseteq U \times U$$

be an equivalence relation. For any subset $A \subseteq U$, define the Pawlak lower and upper approximations of A by

$$\underline{A}_R := \{\, x \in U \mid [x]_R \subseteq A \,\}, \qquad \overline{A}_R := \{\, x \in U \mid [x]_R \cap A \neq \emptyset \,\},$$

where

$$[x]_R := \{\, y \in U \mid (x, y) \in R \,\}.$$

Define the *complex rough alphabet* by

$$D_{\mathrm{CR}} := \{0, \mathrm{i}, 1 + \mathrm{i}\} \subset \mathbb{C}.$$

The *complex rough membership function* of A is the mapping

$$\mu_A^{\mathrm{CR}} : U \to D_{\mathrm{CR}}$$

defined by

$$\mu_A^{\mathrm{CR}}(x) := \mathbf{1}_{\underline{A}_R}(x) + \mathrm{i}\, \mathbf{1}_{\overline{A}_R}(x), \qquad x \in U,$$

where $\mathbf{1}_S$ denotes the indicator function of a subset $S \subseteq U$.

Then the pair

$$\mathrm{CRS}_R(A) := (U, \mu_A^{\mathrm{CR}})$$

is called the *complex rough set* of A with respect to R.

4.9 Complex Soft Set

A complex soft set assigns each parameter a complex fuzzy set, combining parameterization with amplitude-phase membership values to model complex-valued uncertain descriptions [173].

Definition 4.9.1 (Complex soft set)**.** Let U be a nonempty universe, let E be a nonempty set of parameters, and let

$$A \subseteq E$$

be nonempty. Define the closed unit disk by

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

Let

$$\mathrm{CFS}(U) := \{\mu : U \to \mathbb{D}\}$$

denote the family of all complex fuzzy sets on U.

A *complex soft set* over U with parameter set A is a pair

$$(F, A),$$

where

$$F : A \to \mathrm{CFS}(U).$$

Thus, for each parameter $e \in A$, the value $F(e)$ is a complex fuzzy set on U.

Equivalently, a complex soft set may be represented by a mapping

$$\mu : A \times U \to \mathbb{D}, \qquad \mu(e, u) := F(e)(u),$$

such that, for every fixed $e \in A$, the section

$$\mu_e : U \to \mathbb{D}, \qquad \mu_e(u) := \mu(e, u),$$

is a complex fuzzy membership function on U.

Moreover, for each $(e, u) \in A \times U$, one may write

$$\mu(e, u) = r(e, u)e^{\,\mathrm{i}\omega(e,u)},$$

where

$$r(e, u) \in [0, 1], \qquad \omega(e, u) \in [0, 2\pi].$$

Here $r(e, u)$ is the amplitude of membership and $\omega(e, u)$ is its phase.

4.10 Complex multisets

A complex multiset is a multiset whose elements are complex numbers, each assigned a non-negative integer multiplicity, usually with finite or countably supported occurrence structure [174, 175].

Definition 4.10.1 (Complex multiset)**.** Let

$$\mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

A *complex multiset* is a function

$$m_M : \mathbb{C} \to \mathbb{N}_0$$

such that its support

$$\mathrm{supp}(m_M) := \{z \in \mathbb{C} \mid m_M(z) \neq 0\}$$

is finite.

For each

$$z \in \mathbb{C},$$

the value

$$m_M(z)$$

is called the *multiplicity* of z in M.

Equivalently, a complex multiset may be written in the form

$$M = \{\, z^{(m_M(z))} \mid z \in \mathrm{supp}(m_M) \,\},$$

where $z^{(m_M(z))}$ indicates that the complex number z occurs with multiplicity $m_M(z)$.

Chapter 5

Interval Set and Its Extensions

In this chapter, we introduce interval sets and their extensions.

5.1 Interval set

An interval set is the family of all subsets lying between fixed lower and upper bounds, representing bounded set-valued uncertainty on a universe [176–179].

Definition 5.1.1 (Interval set)**.** [176] Let U be a nonempty universe, and let $\mathcal{P}(U)$ be its power set. For subsets $A_\ell, A_u \subseteq U$ with $A_\ell \subseteq A_u$, the family

$$A = [A_\ell, A_u] := \{\, X \in \mathcal{P}(U) \mid A_\ell \subseteq X \subseteq A_u \,\}$$

is called an *interval set* on U.

5.2 Multi-Interval sets

Multi-interval sets are finite families of interval sets on one universe, representing several lower-upper bounded subset ranges simultaneously within a single structured framework.

Definition 5.2.1 (Multi-interval set)**.** Let U be a nonempty universe, and let $n \geq 1$ be an integer. A *multi-interval set* on U is an n-tuple

$$\mathcal{A} = \big([A_{1,\ell}, A_{1,u}], \ldots, [A_{n,\ell}, A_{n,u}]\big),$$

where, for each $i = 1, \ldots, n$,

$$A_{i,\ell}, A_{i,u} \subseteq U \qquad \text{and} \qquad A_{i,\ell} \subseteq A_{i,u},$$

and

$$[A_{i,\ell}, A_{i,u}] := \{\, X \in \mathcal{P}(U) \mid A_{i,\ell} \subseteq X \subseteq A_{i,u} \,\}$$

is an interval set on U.

Equivalently, a multi-interval set is a finite family of interval sets on the same universe U.

Theorem 5.2.2 (Well-definedness of multi-interval sets)**.** *Let*

$$\mathcal{A} = \big([A_{1,\ell}, A_{1,u}], \ldots, [A_{n,\ell}, A_{n,u}]\big)$$

be as in the above definition. Then $\mathcal{A}$ is well-defined. More precisely, for each

$$i = 1, \ldots, n,$$

the family

$$[A_{i,\ell}, A_{i,u}] = \{ X \in \mathcal{P}(U) \mid A_{i,\ell} \subseteq X \subseteq A_{i,u} \}$$

is a well-defined nonempty subset of $\mathcal{P}(U)$. Hence the tuple $\mathcal{A}$ is a well-defined finite family of interval sets on U.

Proof. Fix $i \in \{1, \ldots, n\}$. Since

$$A_{i,\ell} \subseteq A_{i,u} \subseteq U,$$

both $A_{i,\ell}$ and $A_{i,u}$ belong to $\mathcal{P}(U)$. Therefore the condition

$$A_{i,\ell} \subseteq X \subseteq A_{i,u}$$

is meaningful for subsets $X \subseteq U$, and thus

$$[A_{i,\ell}, A_{i,u}] \subseteq \mathcal{P}(U)$$

is well-defined.

Moreover,

$$A_{i,\ell} \in [A_{i,\ell}, A_{i,u}],$$

because

$$A_{i,\ell} \subseteq A_{i,\ell} \subseteq A_{i,u}.$$

Hence

$$[A_{i,\ell}, A_{i,u}] \neq \emptyset.$$

Since i was arbitrary, each component

$$[A_{i,\ell}, A_{i,u}]$$

is a well-defined interval set on U. Consequently,

$$\mathcal{A} = \big([A_{1,\ell}, A_{1,u}], \ldots, [A_{n,\ell}, A_{n,u}]\big)$$

is a well-defined multi-interval set. □

5.3 Interval Rough sets

Interval rough sets approximate an interval set by applying lower and upper rough approximations to both endpoints, producing interval-valued lower and upper approximation bounds (cf. [180]).

Definition 5.3.1 (Interval rough set)**.** Let U be a nonempty universe, and let

$$R \subseteq U \times U$$

be an equivalence relation. For each subset $X \subseteq U$, define the Pawlak lower and upper approximations by

$$\underline{X}_R := \{ x \in U \mid [x]_R \subseteq X \}, \qquad \overline{X}_R := \{ x \in U \mid [x]_R \cap X \neq \emptyset \},$$

where

$$[x]_R := \{ y \in U \mid (x, y) \in R \}.$$

Let

$$A = [A_\ell, A_u]$$

be an interval set on U, where

$$A_\ell \subseteq A_u \subseteq U.$$

The *lower interval approximation* and *upper interval approximation* of A are defined by

$$\underline{A}_R^I := \big[\underline{A_\ell}_R, \underline{A_u}_R\big], \qquad \overline{A}_R^I := \big[\overline{A_\ell}_R, \overline{A_u}_R\big].$$

Then the pair

$$\mathrm{IRS}_R(A) := \big(\underline{A}_R^I, \overline{A}_R^I\big)$$

is called the *interval rough set* of A with respect to the approximation space (U, R).

5.4 Iwinski rough sets

Iwinski rough sets are rough sets determined by definable lower and upper bounds under an equivalence relation, forming an interval of admissible subsets (cf. [181, 182]).

Definition 5.4.1 (Iwinski rough set)**.** Let U be a nonempty universe, and let

$$E \subseteq U \times U$$

be an equivalence relation. Denote by

$$U/E$$

the partition of U induced by E, and let

$$\sigma(U/E)$$

be the σ-algebra of all definable subsets of U generated by the partition U/E.

A pair

$$(A_\ell, A_u)$$

is called an *Iwinski rough set* on the approximation space (U, E) if

$$A_\ell, A_u \in \sigma(U/E) \qquad \text{and} \qquad A_\ell \subseteq A_u.$$

Equivalently, an Iwinski rough set may be written in interval form as

$$[A_\ell, A_u] = \{ A \subseteq U \mid A_\ell \subseteq A \subseteq A_u \},$$

where both endpoints A_ℓ and A_u are definable with respect to E.

5.5 Lattice-valued interval sets

A lattice-valued interval set is an interval in a lattice, consisting of all elements between fixed lower and upper bounds under the lattice order [183, 184].

Definition 5.5.1 (Lattice-valued interval set). [183] Let $(L, \vee, \wedge)$ be a lattice. A *lattice-valued interval set* is an interval

$$[x_\ell, x_u] = \{x \in L \mid x_\ell \le x \le x_u\}$$

with $x_\ell, x_u \in L$ and $x_\ell \le x_u$.

5.6 Interval Soft set

An interval soft set assigns each parameter an interval-valued set, representing parameterized uncertainty through lower and upper membership bounds instead of single exact values [185, 186].

Definition 5.6.1 (Interval soft set). [185, 186] Let U be a nonempty universe, let E be a set of parameters, and let

$$A \subseteq E.$$

Denote by

$$I(\mathcal{P}(U))$$

the family of all interval sets on U, that is,

$$I(\mathcal{P}(U)) := \{[X_\ell, X_u] \mid X_\ell, X_u \subseteq U,\ X_\ell \subseteq X_u\},$$

where

$$[X_\ell, X_u] := \{\, X \in \mathcal{P}(U) \mid X_\ell \subseteq X \subseteq X_u \,\}.$$

A pair

$$(F, A)$$

is called an *interval soft set* over U if

$$F : A \to I(\mathcal{P}(U)).$$

Equivalently, for each parameter $e \in A$, there exist subsets

$$F_\ell(e), F_u(e) \subseteq U \qquad \text{with} \qquad F_\ell(e) \subseteq F_u(e)$$

such that

$$F(e) = [F_\ell(e), F_u(e)].$$

5.7 Interval-valued fuzzy set

An interval-valued fuzzy set assigns to each element a membership interval in $[0, 1]$, thereby representing uncertainty by means of lower and upper bounds [187–189]. As related concepts to interval-valued fuzzy sets, notions such as hyperfuzzy sets are also known [15,16,190–192].

Definition 5.7.1 (Interval-valued fuzzy set)**.** [187,188] Let U be a nonempty universe, and let

$$\mathbb{I}([0,1]) := \{\, [a,b] \subseteq [0,1] \mid 0 \le a \le b \le 1 \,\}.$$

An *interval-valued fuzzy set* on U is a set determined by a membership mapping

$$\mu_A : U \to \mathbb{I}([0,1]).$$

Equivalently, for each $x \in U$,

$$\mu_A(x) = [\mu_A^-(x), \mu_A^+(x)], \qquad 0 \le \mu_A^-(x) \le \mu_A^+(x) \le 1.$$

Example 5.7.2 (A real-life example of an interval-valued fuzzy set)**.** Let

$$U = \{a_1, a_2, a_3, a_4\}$$

be a set of four job applicants for a company position.

Suppose that the hiring committee wants to describe the set of applicants who are *suitable for the job*. In practice, the suitability of an applicant may not be expressed by a single precise number, because different interviewers may give slightly different evaluations, and some information may still be incomplete. For this reason, it is natural to use an interval-valued fuzzy set.

Define an interval-valued fuzzy set

$$A = \{(a, \mu_A(a)) \mid a \in U\}$$

on U, where

$$\mu_A(a) \in \mathbb{I}([0,1])$$

represents an interval of possible suitability degrees for applicant a.

For example, let

$$\mu_A(a_1) = [0.80, 0.90], \qquad \mu_A(a_2) = [0.60, 0.75], \qquad \mu_A(a_3) = [0.45, 0.65], \qquad \mu_A(a_4) = [0.20, 0.40].$$

These intervals can be interpreted as follows:

- applicant a_1 is regarded as highly suitable, with evaluation lying between 0.80 and 0.90;
- applicant a_2 is fairly suitable, but the exact assessment is somewhat uncertain;
- applicant a_3 has a moderate and more uncertain suitability level;
- applicant a_4 is considered to have relatively low suitability.

Thus, A is an interval-valued fuzzy set representing the vague and partially uncertain assessment of job suitability. The use of intervals is appropriate here because the committee does not commit to a single exact membership degree, but rather allows a range of plausible evaluations for each applicant.

5.8 Interval-valued intuitionistic fuzzy set

An interval-valued intuitionistic fuzzy set assigns each element membership and non-membership intervals, preserving hesitation through bounded uncertainty and intuitionistic constraints [193–195].

Definition 5.8.1 (Interval-valued intuitionistic fuzzy set)**.** [193–195] Let X be a nonempty set. An *interval-valued intuitionistic fuzzy set* (IVIFS) A on X is of the form

$$A = \{\langle x,\, [\mu_A^-(x), \mu_A^+(x)],\, [\nu_A^-(x), \nu_A^+(x)]\rangle \mid x \in X\},$$

where

$$\mu_A^-, \mu_A^+, \nu_A^-, \nu_A^+ : X \to [0,1]$$

satisfy

$$0 \le \mu_A^-(x) \le \mu_A^+(x) \le 1, \qquad 0 \le \nu_A^-(x) \le \nu_A^+(x) \le 1,$$

and

$$\mu_A^+(x) + \nu_A^+(x) \le 1 \qquad \text{for all } x \in X.$$

Here $[\mu_A^-(x), \mu_A^+(x)]$ and $[\nu_A^-(x), \nu_A^+(x)]$ are called the *membership interval* and *non-membership interval* of x, respectively.

5.9 Interval-valued neutrosophic set

An interval-valued neutrosophic set assigns each element truth, indeterminacy, and falsity intervals, enabling flexible representation of imprecise and incomplete information [196–199]. As a related concept, the HyperNeutrosophic Set is also known [18, 200–202].

Definition 5.9.1 (Interval-valued neutrosophic set)**.** [196–199] Let X be a nonempty set. An *interval-valued neutrosophic set* (IVNS) A on X is of the form

$$A = \{\langle x,\, [T_A^-(x), T_A^+(x)],\, [I_A^-(x), I_A^+(x)],\, [F_A^-(x), F_A^+(x)]\rangle \mid x \in X\},$$

where

$$T_A^-, T_A^+, I_A^-, I_A^+, F_A^-, F_A^+ : X \to [0,1]$$

satisfy

$$0 \le T_A^-(x) \le T_A^+(x) \le 1,$$
$$0 \le I_A^-(x) \le I_A^+(x) \le 1,$$
$$0 \le F_A^-(x) \le F_A^+(x) \le 1$$

for all $x \in X$.

Here

$$[T_A^-(x), T_A^+(x)], \qquad [I_A^-(x), I_A^+(x)], \qquad [F_A^-(x), F_A^+(x)]$$

are called the *truth-membership interval*, *indeterminacy-membership interval*, and *falsity-membership interval* of x, respectively.

5.10 Interval-valued plithogenic set

An interval-valued plithogenic set assigns each element and attribute value an interval appurtenance degree, combining bounded uncertainty with contradiction-aware, attribute-based plithogenic membership modeling in contexts.

Definition 5.10.1 (Interval-valued plithogenic set)**.** [59] Let P be a nonempty universe, let v be a fixed attribute, let Pv be a nonempty set of possible values of v, and let $s, t \in \mathbb{N}$. Assume that

$$pCF : Pv \times Pv \to [0,1]^t$$

satisfies

$$pCF(a,a) = 0, \qquad pCF(a,b) = pCF(b,a) \quad \text{for all } a, b \in Pv.$$

Let

$$\mathbb{I}([0,1]) := \{\, [\alpha,\beta] \subseteq [0,1] \mid 0 \le \alpha \le \beta \le 1 \,\}.$$

An *interval-valued plithogenic set* on the frame (P, v, Pv, pCF) is a quintuple

$$IPS = (P, v, Pv, ipdf, pCF),$$

where

$$ipdf : P \times Pv \to \mathbb{I}([0,1])^s$$

is called the *interval-valued degree of appurtenance function.*

Equivalently, for each $x \in P$ and $a \in Pv$,

$$ipdf(x,a) = \big([\,\ell_1(x,a), u_1(x,a)\,], \dots, [\,\ell_s(x,a), u_s(x,a)\,]\big),$$

where

$$\ell_j(x,a), u_j(x,a) \in [0,1], \qquad \ell_j(x,a) \le u_j(x,a) \quad (j = 1, \dots, s).$$

Theorem 5.10.2 (Well-definedness of interval-valued plithogenic sets)**.** *Let*

$$IPS = (P, v, Pv, ipdf, pCF)$$

be as above. Then IPS is well-defined. More precisely, for every

$$x \in P, \qquad a \in Pv,$$

the value

$$ipdf(x,a) \in \mathbb{I}([0,1])^s$$

is a well-defined s-tuple of closed intervals in $[0,1]$, and the fixed contradiction map pCF satisfies the plithogenic conditions.

Proof. Let

$$x \in P, \qquad a \in Pv.$$

Since

$$ipdf : P \times Pv \to \mathbb{I}([0,1])^s,$$

the value $ipdf(x,a)$ is by definition an element of $\mathbb{I}([0,1])^s$. Hence it has the form

$$ipdf(x,a) = \big([\,\ell_1(x,a), u_1(x,a)\,], \dots, [\,\ell_s(x,a), u_s(x,a)\,]\big)$$

with

$$0 \leq \ell_j(x, a) \leq u_j(x, a) \leq 1 \qquad (j = 1, \dots, s).$$

Therefore each

$$[\, \ell_j(x, a), u_j(x, a) \,]$$

is a well-defined closed interval contained in $[0, 1]$, and so $ipdf(x, a)$ is a well-defined interval-valued appurtenance degree.

Moreover, the contradiction map pCF is fixed and satisfies

$$pCF(a, a) = 0, \qquad pCF(a, b) = pCF(b, a) \quad \text{for all } a, b \in Pv.$$

Thus all components required in the definition of an interval-valued plithogenic set are well-defined. □

5.11 Interval-valued uncertain set

An interval-valued uncertain set assigns each element a nonempty interval of admissible uncertainty degrees between lower and upper bounds within a fixed uncertain model framework [59].

Definition 5.11.1 (Componentwise order on Dom(M))**.** Let M be an uncertain model with

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \geq 1$. For

$$a = (a_1, \dots, a_k), \qquad b = (b_1, \dots, b_k) \in \mathrm{Dom}(M),$$

define

$$a \preceq b \iff a_j \leq b_j \quad \text{for all } j = 1, \dots, k.$$

Definition 5.11.2 (Interval-valued uncertain set)**.** Let X be a nonempty set, and let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

An *interval-valued uncertain set of type M on X* is a pair

$$\mathcal{U}_I = (X, \Gamma_M),$$

where

$$\Gamma_M : X \to \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}$$

is induced by two functions

$$\underline{\mu}_M, \overline{\mu}_M : X \to \mathrm{Dom}(M)$$

such that

$$\underline{\mu}_M(x) \preceq \overline{\mu}_M(x) \qquad \text{for all } x \in X,$$

and

$$\Gamma_M(x) := [\, \underline{\mu}_M(x), \overline{\mu}_M(x) \,]_{\mathrm{Dom}(M)} := \{\, d \in \mathrm{Dom}(M) \mid \underline{\mu}_M(x) \preceq d \preceq \overline{\mu}_M(x) \,\}.$$

The set $\Gamma_M(x)$ is called the *interval-valued uncertainty degree* of x.

Theorem 5.11.3 (Well-definedness of interval-valued uncertain sets). *Let X be a nonempty set, let M be an uncertain model, and let*

$$\underline{\mu}_M, \overline{\mu}_M : X \to \mathrm{Dom}(M)$$

satisfy

$$\underline{\mu}_M(x) \preceq \overline{\mu}_M(x) \qquad \textit{for all } x \in X.$$

Then the mapping

$$\Gamma_M : X \to \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}$$

defined by

$$\Gamma_M(x) = [\, \underline{\mu}_M(x), \overline{\mu}_M(x) \,]_{\mathrm{Dom}(M)}$$

is well-defined. In particular, for every $x \in X$,

$$\Gamma_M(x) \neq \emptyset \qquad \textit{and} \qquad \Gamma_M(x) \subseteq \mathrm{Dom}(M).$$

Proof. Fix $x \in X$. Since

$$\underline{\mu}_M(x), \overline{\mu}_M(x) \in \mathrm{Dom}(M)$$

and

$$\underline{\mu}_M(x) \preceq \overline{\mu}_M(x),$$

it follows immediately that

$$\underline{\mu}_M(x) \in [\, \underline{\mu}_M(x), \overline{\mu}_M(x) \,]_{\mathrm{Dom}(M)}.$$

Hence

$$\Gamma_M(x) \neq \emptyset.$$

By definition,

$$\Gamma_M(x) = \{\, d \in \mathrm{Dom}(M) \mid \underline{\mu}_M(x) \preceq d \preceq \overline{\mu}_M(x) \,\},$$

so every element of $\Gamma_M(x)$ belongs to $\mathrm{Dom}(M)$. Therefore

$$\Gamma_M(x) \subseteq \mathrm{Dom}(M).$$

Thus

$$\Gamma_M(x) \in \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}.$$

Since $x \in X$ was arbitrary, the mapping Γ_M is well-defined. □

Table 5.1 presents a catalogue of interval-valued uncertainty-set families, organized by the number k of interval-valued components.

Table 5.1: A catalogue of interval-valued uncertainty-set families, organized by the number k of interval-valued components (equivalently, by endpoint encodings in $[0,1]^{2k}$).

k	note	Representative interval-valued U-Set model(s)
2		Interval-valued Intuitionistic Fuzzy Sets; Interval-valued vague sets [203–205]; Interval-valued Pythagorean Fuzzy Set [206, 207]; Interval-valued Fermatean Fuzzy Set [74, 208]; Interval-valued Bipolar Fuzzy Set [209, 210].
3		Interval-valued Picture Fuzzy Set [211, 212]; Interval-valued Hesitant Fuzzy Set [213, 214]; Interval-valued Spherical Fuzzy Set [215, 216]; Interval-valued Neutrosophic Set.
4		Quadripartitioned neutrosophic sets [217–219].
k	$(k \geq 1)$	Interval-valued Plithogenic Set.
$3k$	$(k \geq 1)$	Interval-valued refined neutrosophic Set [220, 221].

Chapter 6

Granular Set and Its Extensions

In this chapter, we introduce granular sets and their extensions.

6.1 Granular Set

A granular set is a subset exactly representable as a union of granules from a chosen granulation covering the universe [222–225].

Definition 6.1.1 (Granulation)**.** Let X be a nonempty set. A family

$$\mathcal{G} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$$

is called a *granulation* of X if

$$\bigcup_{G \in \mathcal{G}} G = X.$$

Each element $G \in \mathcal{G}$ is called a *granule.*

Definition 6.1.2 (Granular set)**.** [222, 223] Let X be a nonempty set, and let $\mathcal{G}$ be a granulation of X. A subset $A \subseteq X$ is called a *granular set* with respect to $\mathcal{G}$ if there exists a subfamily

$$\mathcal{A} \subseteq \mathcal{G}$$

such that

$$A = \bigcup_{G \in \mathcal{A}} G.$$

Thus, a granular set is a set exactly representable as a union of granules.

6.2 Multi-Granular Set

A multi-granular set is a subset representable as a union of granules in each of several granulations, remaining simultaneously granular under multiple knowledge structures.

Definition 6.2.1 (Multi-granular set)**.** Let X be a nonempty set, and let

$$\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m$$

be granulations of X, where $m \geq 2$. A subset

$$A \subseteq X$$

is called a *multi-granular set* with respect to

$$\mathfrak{G} := (\mathcal{G}_1, \dots, \mathcal{G}_m)$$

if, for each

$$i \in \{1, \dots, m\},$$

there exists a subfamily

$$\mathcal{A}_i \subseteq \mathcal{G}_i$$

such that

$$A = \bigcup_{G \in \mathcal{A}_i} G.$$

Equivalently, A is a set that is granular with respect to each granulation

$$\mathcal{G}_1, \dots, \mathcal{G}_m.$$

Theorem 6.2.2 (Well-definedness of multi-granular sets)**.** *Let X be a nonempty set, and let*

$$\mathcal{G}_1, \dots, \mathcal{G}_m$$

be granulations of X, where $m \geq 2$. Assume that, for each

$$i \in \{1, \dots, m\},$$

there exists a subfamily

$$\mathcal{A}_i \subseteq \mathcal{G}_i$$

such that

$$A = \bigcup_{G \in \mathcal{A}_i} G.$$

Then A is a well-defined subset of X. Hence the notion of a multi-granular set is well-defined.

Proof. Fix

$$i \in \{1, \dots, m\}.$$

Since $\mathcal{G}_i$ is a granulation of X, every granule

$$G \in \mathcal{G}_i$$

satisfies

$$G \subseteq X.$$

Therefore, for every subfamily

$$\mathcal{A}_i \subseteq \mathcal{G}_i,$$

one has

$$\bigcup_{G \in \mathcal{A}_i} G \subseteq X.$$

By assumption,

$$A = \bigcup_{G \in \mathcal{A}_i} G,$$

and hence

$$A \subseteq X.$$

Thus A is a well-defined subset of X.

Since this holds for each

$$i = 1, \dots, m,$$

the equalities

$$A = \bigcup_{G \in \mathcal{A}_i} G$$

are meaningful for all given granulations. Therefore A is granular with respect to every

$$\mathcal{G}_i,$$

and the concept of a multi-granular set is well-defined. □

6.3 Granular Rough Set

A granular rough set approximates a target subset by unions of granules fully contained in it or merely intersecting it [226–228].

Definition 6.3.1 (Granular rough set)**.** Let X be a nonempty set, let $\mathcal{G}$ be a granulation of X, and let $A \subseteq X$. Define the *granular lower approximation* and *granular upper approximation* of A by

$$\underline{A}_{\mathcal{G}} := \bigcup \{\, G \in \mathcal{G} \mid G \subseteq A \,\},$$

and

$$\overline{A}_{\mathcal{G}} := \bigcup \{\, G \in \mathcal{G} \mid G \cap A \neq \emptyset \,\}.$$

The pair

$$\mathrm{GRS}_{\mathcal{G}}(A) := \big(\underline{A}_{\mathcal{G}}, \overline{A}_{\mathcal{G}}\big)$$

is called the *granular rough set* determined by A with respect to $\mathcal{G}$.

If

$$\underline{A}_{\mathcal{G}} = \overline{A}_{\mathcal{G}},$$

then A is called *granularly exact*; otherwise, it is called *granularly rough.*

Table 6.1: A brief comparison between rough sets and granular rough sets

Aspect	Rough set	Granular rough set
Basic idea	Describes a set by lower and upper approximations based on indiscernibility	Describes a set by lower and upper approximations based on a given granulation
Underlying structure	Usually built from an equivalence relation or approximation space	Built from an explicit family of granules $\mathcal{G}$ on the universe
Approximation unit	Equivalence classes or indiscernibility classes	Granules $G \in \mathcal{G}$
Lower approximation	Union of classes entirely contained in the target set	Union of granules entirely contained in the target set
Upper approximation	Union of classes having nonempty intersection with the target set	Union of granules having nonempty intersection with the target set
Main emphasis	Vagueness caused by limited discernibility among objects	Vagueness described through granule-based information organization
Flexibility	Often tied to a specific indiscernibility relation	More flexible, since different granulations may be chosen directly
Exactness	Exact when lower and upper approximations coincide	Granularly exact when $\underline{A}_{\mathcal{G}} = \overline{A}_{\mathcal{G}}$
Common feature	Both represent incomplete knowledge through approximation boundaries	Both represent incomplete knowledge through approximation boundaries

Remark 6.3.2. For every $A \subseteq X$, one has

$$\underline{A}_{\mathcal{G}} \subseteq A \subseteq \overline{A}_{\mathcal{G}}.$$

If $\mathcal{G}$ is a partition induced by an equivalence relation R on X, then this definition reduces to the classical Pawlak rough-set approximations.

As a reference, a comparison between rough sets and granular rough sets is presented in Table 6.1.

Example 6.3.3 (A real-life example of a granular rough set)**.** Let

$$X = \{s_1, s_2, s_3, s_4, s_5, s_6\}$$

be a set of six students in a school.

Suppose that the school counselor wants to study the set

$$A = \{s_1, s_2, s_3\},$$

where A is interpreted as the set of students who are considered to be *at risk of poor academic performance* according to detailed individual records.

Assume, however, that the counselor does not always work with individual students directly, but instead uses groups formed from similar study habits. Let the granulation of X be

$$\mathcal{G} = \big\{\{s_1, s_2\}, \{s_3, s_4\}, \{s_5, s_6\}\big\}.$$

Thus, each granule represents a small group of students who are treated as similar for practical evaluation.

We now compute the granular approximations of A.

First, the granular lower approximation is

$$\underline{A}_{\mathcal{G}} = \bigcup \{ G \in \mathcal{G} \mid G \subseteq A \}.$$

Among the granules of $\mathcal{G}$, only

$$\{s_1, s_2\} \subseteq A$$

holds. Therefore,

$$\underline{A}_{\mathcal{G}} = \{s_1, s_2\}.$$

Next, the granular upper approximation is

$$\overline{A}_{\mathcal{G}} = \bigcup \{ G \in \mathcal{G} \mid G \cap A \neq \emptyset \}.$$

Here,

$$\{s_1, s_2\} \cap A \neq \emptyset, \qquad \{s_3, s_4\} \cap A \neq \emptyset,$$

but

$$\{s_5, s_6\} \cap A = \emptyset.$$

Hence,

$$\overline{A}_{\mathcal{G}} = \{s_1, s_2, s_3, s_4\}.$$

Thus, the granular rough set determined by A with respect to $\mathcal{G}$ is

$$\mathrm{GRS}_{\mathcal{G}}(A) = \big(\{s_1, s_2\}, \{s_1, s_2, s_3, s_4\}\big).$$

Since

$$\underline{A}_{\mathcal{G}} \neq \overline{A}_{\mathcal{G}},$$

the set A is granularly rough.

In practical terms, this means that students s_1 and s_2 can be confidently classified as at risk based on the available grouping, while students s_3 and s_4 belong to a group that partially overlaps with the at-risk students and therefore cannot be classified with full certainty. This illustrates how a granular rough set models decision-making under limited group-based information.

6.4 Multigranulation Rough Set

A multigranulation rough set approximates a target set using multiple equivalence relations, combining several granulations through joint lower and upper approximations [229].

Definition 6.4.1 (Multigranulation rough set)**.** Let U be a nonempty universe, and let

$$R_1, R_2, \ldots, R_m$$

be equivalence relations on U, where $m \geq 1$. For each $i = 1, \ldots, m$ and each $x \in U$, write

$$[x]_{R_i} := \{\, y \in U \mid (x, y) \in R_i \,\}.$$

Let

$$X \subseteq U.$$

The *multigranulation lower approximation* and *multigranulation upper approximation* of X with respect to

$$\mathcal{R} := \{R_1, \ldots, R_m\}$$

are defined by

$$\underline{X}^{M}_{\mathcal{R}} := \{\, x \in U \mid \exists i \in \{1, \ldots, m\} \text{ such that } [x]_{R_i} \subseteq X \,\},$$

and

$$\overline{X}^{M}_{\mathcal{R}} := \{\, x \in U \mid \forall i \in \{1, \ldots, m\},\ [x]_{R_i} \cap X \neq \emptyset \,\}.$$

Equivalently,

$$\underline{X}^{M}_{\mathcal{R}} = \bigcup_{i=1}^{m} \underline{X}_{R_i}, \qquad \overline{X}^{M}_{\mathcal{R}} = \bigcap_{i=1}^{m} \overline{X}_{R_i},$$

where

$$\underline{X}_{R_i} := \{\, x \in U \mid [x]_{R_i} \subseteq X \,\}, \qquad \overline{X}_{R_i} := \{\, x \in U \mid [x]_{R_i} \cap X \neq \emptyset \,\}.$$

The pair

$$\mathrm{MGRS}_{\mathcal{R}}(X) := \big(\underline{X}^{M}_{\mathcal{R}}, \overline{X}^{M}_{\mathcal{R}}\big)$$

is called the *multigranulation rough set* of X with respect to the family $\mathcal{R}$.

Chapter 7

Multisets and Its Extensions

In this chapter, we discuss multisets and their extensions.

7.1 Multisets

A multiset assigns each element a nonnegative integer multiplicity, allowing repeated occurrences while preserving the underlying base set structure.

Definition 7.1.1 (Multiset)**.** A *multiset* on a set X is a mapping

$$m : X \to \mathbb{N}_0$$

assigning to each element $x \in X$ its number of occurrences $m(x)$.

7.2 Iterated Multisets

Iterated multisets are multisets whose elements are themselves multisets, recursively repeated finitely many times to represent hierarchically layered multiplicity structures [230].

Definition 7.2.1 (n-fold iterated multiset)**.** Let X be a set, and let

$$\mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

Write

$$\mathcal{M}(Y) := \{m : Y \to \mathbb{N}_0 \mid \operatorname{supp}(m) := \{y \in Y \mid m(y) > 0\} \text{ is finite}\}$$

for the set of all finite multisets on a set Y.

For each integer $n \geq 1$, define recursively

$$\mathcal{M}^1(X) := \mathcal{M}(X), \qquad \mathcal{M}^{n+1}(X) := \mathcal{M}\big(\mathcal{M}^n(X)\big).$$

An element of

$$\mathcal{M}^n(X)$$

is called an *n-fold iterated multiset* over X.

Equivalently, an iterated multiset is a multiset whose elements are themselves multisets, recursively repeated a finite number of times.

7.3 Fuzzy Multisets

A fuzzy multiset assigns each element multiplicities across membership degrees in $[0, 1]$, allowing repeated occurrences of the same element with possibly different fuzzy grades simultaneously [231, 232].

Definition 7.3.1 (Fuzzy multiset)**.** [231, 232] Let X be a nonempty set, and let

$$\mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

A *fuzzy multiset* A on X is a mapping

$$A : X \to \mathbb{N}_0^{[0,1]}.$$

Equivalently, A may be identified with a mapping

$$\widetilde{A} : X \times [0, 1] \to \mathbb{N}_0$$

defined by

$$\widetilde{A}(x, \lambda) := A(x)(\lambda) \qquad \text{for all } (x, \lambda) \in X \times [0, 1].$$

For each $x \in X$ and $\lambda \in [0, 1]$, the value

$$\widetilde{A}(x, \lambda)$$

represents the multiplicity of occurrences of x having membership degree λ in A.

7.4 Intuitionistic Fuzzy Multisets

An intuitionistic fuzzy multiset assigns each element sequences of membership and non-membership degrees, with every paired component satisfying the intuitionistic constraint $\mu + \nu \leq 1$ at all points [233, 234].

Definition 7.4.1 (Intuitionistic fuzzy multiset)**.** Let X be a nonempty set. An *intuitionistic fuzzy multiset* on X is given by two sequences of functions

$$(\mu_A^j)_{j \in J}, \qquad (\nu_A^j)_{j \in J},$$

where each

$$\mu_A^j, \nu_A^j : X \to [0, 1]$$

and

$$0 \leq \mu_A^j(x) + \nu_A^j(x) \leq 1 \qquad \text{for all } x \in X,\ j \in J.$$

If $J = \{1, \dots, n\}$ for some $n \in \mathbb{N}$, then n is called the *dimension* of the intuitionistic fuzzy multiset.

7.5 Neutrosophic Multisets

A neutrosophic multiset assigns each element sequences of truth, indeterminacy, and falsity degrees, allowing multiple neutrosophic evaluations of the same element within one structure [235–239].

Definition 7.5.1 (Neutrosophic multiset)**.** [235–237] Let X be a nonempty set, and let $p \in \mathbb{N}$. A *neutrosophic multiset* A on X is of the form

$$A = \left\{ \langle x, \left(T_A^1(x), \ldots, T_A^p(x)\right), \left(I_A^1(x), \ldots, I_A^p(x)\right), \left(F_A^1(x), \ldots, F_A^p(x)\right) \rangle \mid x \in X \right\},$$

where, for each $j = 1, \ldots, p$,

$$T_A^j, I_A^j, F_A^j : X \to [0,1]$$

satisfy

$$0 \le T_A^j(x) + I_A^j(x) + F_A^j(x) \le 3 \qquad \text{for all } x \in X.$$

Here

$$\left(T_A^1(x), \ldots, T_A^p(x)\right), \quad \left(I_A^1(x), \ldots, I_A^p(x)\right), \quad \left(F_A^1(x), \ldots, F_A^p(x)\right)$$

are called the truth-membership, indeterminacy-membership, and falsity-membership sequences of x, respectively, and p is called the *dimension* of A.

Example 7.5.2 (A real-life example of a neutrosophic multiset)**.** Let

$$X = \{c_1, c_2, c_3\}$$

be a set of three job candidates for a company position, and let

$$p = 3.$$

Assume that the three dimensions correspond to the evaluations of three interviewers.

We define a neutrosophic multiset

$$A$$

on X, where A represents the collection of candidates who are *suitable for the job.* For each candidate $c \in X$:

- $T_A^j(c)$ denotes the degree to which the jth interviewer believes that c is suitable;
- $I_A^j(c)$ denotes the degree of indeterminacy in the jth interviewer's judgment;
- $F_A^j(c)$ denotes the degree to which the jth interviewer believes that c is not suitable.

Consider

$$\begin{aligned} A = \{ &\langle c_1, (0.85, 0.80, 0.78), (0.10, 0.12, 0.15), (0.05, 0.08, 0.07) \rangle, \\ &\langle c_2, (0.60, 0.55, 0.50), (0.20, 0.25, 0.30), (0.20, 0.20, 0.20) \rangle, \\ &\langle c_3, (0.40, 0.35, 0.45), (0.25, 0.30, 0.20), (0.35, 0.35, 0.35) \rangle \}. \end{aligned}$$

For example, for candidate c_1,

$$\left(T_A^1(c_1), T_A^2(c_1), T_A^3(c_1)\right) = (0.85, 0.80, 0.78),$$

$$\big(I_A^1(c_1), I_A^2(c_1), I_A^3(c_1)\big) = (0.10, 0.12, 0.15),$$
$$\big(F_A^1(c_1), F_A^2(c_1), F_A^3(c_1)\big) = (0.05, 0.08, 0.07).$$

Thus, all three interviewers judge c_1 to be highly suitable, with relatively low indeterminacy and falsity.

Similarly, candidate c_2 receives moderate support, while candidate c_3 receives lower truth-membership and higher falsity-membership values.

Hence, A is a neutrosophic multiset of dimension 3, since each candidate is described not by a single triple (T, I, F), but by three parallel truth, indeterminacy, and falsity sequences corresponding to three different evaluations.

7.6 Plithogenic Multisets

A plithogenic multiset records multiplicities of appurtenance degrees for each element and attribute value, combining repeated graded information with contradiction-aware plithogenic structure in one framework.

Definition 7.6.1 (Plithogenic multiset)**.** Let

$$\mathfrak{P} := (P, v, Pv, pCF, s, t)$$

be a plithogenic frame, where P is a nonempty universe, v is a fixed attribute, Pv is the set of possible values of v, and

$$pCF : Pv \times Pv \to [0,1]^t$$

satisfies

$$pCF(a,a) = 0, \qquad pCF(a,b) = pCF(b,a) \quad \text{for all } a, b \in Pv.$$

Let

$$\mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

A *plithogenic multiset* on the frame $\mathfrak{P}$ is a quintuple

$$PM = (P, v, Pv, mpdf, pCF),$$

where

$$mpdf : P \times Pv \to \mathbb{N}_0^{[0,1]^s}.$$

Equivalently, PM may be represented by a mapping

$$\widetilde{mpdf} : P \times Pv \times [0,1]^s \to \mathbb{N}_0$$

defined by

$$\widetilde{mpdf}(x, a, \lambda) := mpdf(x, a)(\lambda) \qquad \text{for all } (x, a, \lambda) \in P \times Pv \times [0,1]^s.$$

For each $x \in P$, $a \in Pv$, and $\lambda \in [0,1]^s$, the value

$$\widetilde{mpdf}(x, a, \lambda)$$

is called the *multiplicity of the appurtenance degree* λ *of* x *relative to the attribute value* a.

Theorem 7.6.2 (Well-definedness of plithogenic multisets)**.** *Let*

$$\mathfrak{P} = (P, v, Pv, pCF, s, t)$$

be a plithogenic frame. Then every mapping

$$mpdf : P \times Pv \to \mathbb{N}_0^{[0,1]^s}$$

determines a well-defined plithogenic multiset

$$PM = (P, v, Pv, mpdf, pCF).$$

Conversely, every plithogenic multiset on the frame $\mathfrak{P}$ arises in this way.

Proof. Since P and Pv are sets, the Cartesian product

$$P \times Pv$$

is a set. Since $s \in \mathbb{N}$, the Cartesian power

$$[0, 1]^s$$

is also a set. Hence

$$\mathbb{N}_0^{[0,1]^s}$$

is the set of all functions from $[0, 1]^s$ to $\mathbb{N}_0$, and therefore the mapping

$$mpdf : P \times Pv \to \mathbb{N}_0^{[0,1]^s}$$

is meaningful.

For each $(x, a) \in P \times Pv$, the value

$$mpdf(x, a)$$

is a function

$$mpdf(x, a) : [0, 1]^s \to \mathbb{N}_0.$$

Thus, for every $\lambda \in [0, 1]^s$, the quantity

$$mpdf(x, a)(\lambda) \in \mathbb{N}_0$$

is well-defined, so the equivalent map

$$\widetilde{mpdf} : P \times Pv \times [0, 1]^s \to \mathbb{N}_0$$

is also well-defined.

Moreover, the contradiction map pCF is fixed as part of the frame and already satisfies

$$pCF(a, a) = 0, \qquad pCF(a, b) = pCF(b, a) \quad \text{for all } a, b \in Pv.$$

Hence all components of

$$(P, v, Pv, mpdf, pCF)$$

are well-defined, and this object is a plithogenic multiset on $\mathfrak{P}$.

Conversely, by definition, any plithogenic multiset on the frame $\mathfrak{P}$ is precisely given by such a mapping

$$mpdf : P \times Pv \to \mathbb{N}_0^{[0,1]^s}.$$

Therefore every plithogenic multiset on $\mathfrak{P}$ arises in this way. □

7.7 Uncertain Multisets

An uncertain multiset assigns each element multiplicities indexed by admissible uncertainty degrees from a fixed model, representing repeated occurrences together with generalized uncertainty information.

Definition 7.7.1 (Uncertain multiset)**.** Let X be a nonempty set, let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$, and let

$$\mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

An *uncertain multiset of type M* on X is a pair

$$\mathcal{U}_M = (X, m_M),$$

where

$$m_M : X \to \mathbb{N}_0^{\mathrm{Dom}(M)}$$

is called the *uncertainty-multiplicity function.*

Equivalently, $\mathcal{U}_M$ may be represented by a mapping

$$\widetilde{m}_M : X \times \mathrm{Dom}(M) \to \mathbb{N}_0$$

defined by

$$\widetilde{m}_M(x, d) := m_M(x)(d) \qquad \text{for all } (x, d) \in X \times \mathrm{Dom}(M).$$

For each $x \in X$ and $d \in \mathrm{Dom}(M)$, the value

$$\widetilde{m}_M(x, d)$$

is called the *multiplicity of occurrences of x having uncertainty degree d.*

Theorem 7.7.2 (Well-definedness of uncertain multisets)**.** *Let X be a nonempty set, and let M be an uncertain model with admissible degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Then every mapping

$$m_M : X \to \mathbb{N}_0^{\mathrm{Dom}(M)}$$

determines a well-defined uncertain multiset

$$\mathcal{U}_M = (X, m_M)$$

of type M on X. Conversely, every uncertain multiset of type M on X arises in this way.

Proof. Since $\mathrm{Dom}(M) \subseteq [0,1]^k$, the set

$$\mathrm{Dom}(M)$$

is well-defined. Therefore

$$\mathbb{N}_0^{\mathrm{Dom}(M)}$$

is the set of all functions from $\mathrm{Dom}(M)$ to $\mathbb{N}_0$.

Now let

$$m_M : X \to \mathbb{N}_0^{\mathrm{Dom}(M)}.$$

Then, for each $x \in X$, the value

$$m_M(x)$$

is a function

$$m_M(x) : \mathrm{Dom}(M) \to \mathbb{N}_0.$$

Hence, for every

$$d \in \mathrm{Dom}(M),$$

the integer

$$m_M(x)(d) \in \mathbb{N}_0$$

is well-defined. Thus the associated map

$$\widetilde{m}_M : X \times \mathrm{Dom}(M) \to \mathbb{N}_0, \qquad \widetilde{m}_M(x,d) := m_M(x)(d),$$

is well-defined, and therefore

$$\mathcal{U}_M = (X, m_M)$$

is a well-defined uncertain multiset of type M on X.

Conversely, by definition, any uncertain multiset of type M on X is precisely a pair

$$(X, m_M)$$

with

$$m_M : X \to \mathbb{N}_0^{\mathrm{Dom}(M)}.$$

Hence every uncertain multiset of type M on X arises in this way. □

As a reference, extensions of fuzzy multiset models organized by the number k of membership components are presented in Table 7.1.

Table 7.1: Extensions of fuzzy multiset models organized by the number k of membership components

k	**Concept and References**
1	Fuzzy Multisets
2	Intuitionistic Fuzzy Multisets; Pythagorean Fuzzy Multisets [240–242]
3	Neutrosophic Multisets; Picture Fuzzy Multisets [243]; Hesitant Fuzzy Multisets [244, 245]

7.8 Soft Multisets

A soft multiset is a parameterized family of whole submultisets of a universal multiset, assigning multiplicity-preserving selections to elements under different parameters consistently (cf. [246–248]).

Definition 7.8.1 (Soft multiset)**.** [249] Let X be a nonempty set, and let

$$\mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

A *multiset* on X is a mapping

$$m : X \to \mathbb{N}_0.$$

Fix a multiset

$$U : X \to \mathbb{N}_0,$$

called the *universal multiset*, and let E be a set of parameters with

$$A \subseteq E.$$

A multiset

$$M : X \to \mathbb{N}_0$$

is called a *whole submultiset* of U if

$$M(x) \in \{0, U(x)\} \qquad \text{for all } x \in X.$$

Denote by

$$PW(U) := \{\, M : X \to \mathbb{N}_0 \mid M(x) \in \{0, U(x)\} \text{ for all } x \in X \,\}$$

the family of all whole submultisets of U.

Then a pair

$$(F, A)$$

is called a *soft multiset* over U if

$$F : A \to PW(U).$$

Equivalently, a soft multiset is a parameterized family of whole submultisets of the universal multiset U.

7.9 Rough Multisets

Rough multisets approximate a submultiset by lower and upper multisets using equivalence classes, preserving multiplicities while representing certainty and possibility (cf. [250–253]).

Definition 7.9.1 (Rough multiset)**.** [250] Let X be a nonempty set, and let

$$C_M : X \to \mathbb{N}_0$$

be a multiset on X, where

$$\mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

Let

$$R \subseteq X \times X$$

be an equivalence relation, and let A be a submultiset of M, that is,

$$C_A(x) \leq C_M(x) \qquad \text{for all } x \in X.$$

Define the support of A by

$$A^* := \{\, x \in X \mid C_A(x) > 0 \,\}.$$

For each $x \in X$, let

$$[x]_R := \{\, y \in X \mid (x, y) \in R \,\}.$$

The *lower multiset approximation* and *upper multiset approximation* of A with respect to R are the multisets

$$\underline{A}_R, \qquad \overline{A}_R$$

whose count functions are defined by

$$C_{\underline{A}_R}(x) := \begin{cases} C_M(x), & [x]_R \subseteq A^*, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$C_{\overline{A}_R}(x) := \begin{cases} C_M(x), & [x]_R \cap A^* \neq \emptyset, \\ 0, & \text{otherwise}. \end{cases}$$

Then the pair

$$\mathrm{RMS}_R(A) := (\underline{A}_R, \overline{A}_R)$$

is called the *rough multiset* of A with respect to the approximation space (X, R).

7.10 Group-Valued Multisets

A group-valued multiset assigns each element a multiplicity from a group, usually with only finitely many elements having non-identity values [254, 255].

Definition 7.10.1 (Group-valued multiset)**.** [254] Let X be a nonempty set, and let

$$(G, \cdot, e)$$

be a group with identity element e. A *group-valued multiset* on X is a function

$$m_M : X \to G$$

such that its algebraic support

$$\mathrm{algsupp}(m_M) := \{x \in X \mid m_M(x) \neq e\}$$

is finite.

For each

$$x \in X,$$

the value

$$m_M(x) \in G$$

is called the *group-valued multiplicity* of x in M.

Equivalently, a group-valued multiset may be written as

$$M = \{(x, m_M(x)) \mid x \in X,\ m_M(x) \neq e\},$$

where only finitely many elements of X have multiplicity different from the identity element of G.

Example 7.10.2 (A concrete real-life example of a group-valued multiset)**.** Let

$$X = \{b_1, b_2, b_3, b_4\}$$

be a set of four book titles in a library, and let

$$(G, \cdot, e) = (\mathbb{Z}, +, 0)$$

be the additive group of integers.

We interpret the group-valued multiplicity of each book as its *net circulation balance* during one day:

- a positive integer means that more copies were returned than borrowed;
- a negative integer means that more copies were borrowed than returned;
- 0 means that there was no net change.

Define

$$m_M : X \to \mathbb{Z}$$

by

$$m_M(b_1) = 3, \qquad m_M(b_2) = -2, \qquad m_M(b_3) = 1, \qquad m_M(b_4) = 0.$$

Then:

- $m_M(b_1) = 3$ means that title b_1 had a net increase of three returned copies;
- $m_M(b_2) = -2$ means that title b_2 had two more borrowings than returns;
- $m_M(b_3) = 1$ means that title b_3 had one more return than borrowing;

- $m_M(b_4) = 0$ means that title b_4 had no net circulation change.

The algebraic support of m_M is

$$\operatorname{algsupp}(m_M) = \{x \in X \mid m_M(x) \neq 0\} = \{b_1, b_2, b_3\},$$

which is finite. Hence, m_M defines a group-valued multiset on X.

Equivalently, the group-valued multiset may be written as

$$M = \{(b_1, 3), (b_2, -2), (b_3, 1)\},$$

where the title b_4 is omitted because its multiplicity is the identity element 0.

Thus, M is a concrete real-life example of a group-valued multiset: it records net gains and losses of library circulation using integer-valued multiplicities from the group $(\mathbb{Z}, +, 0)$.

7.11 Triangular multiset

A triangular multiset assigns each element a multiplicity from the nonnegative integers or infinity, representing repeated occurrences by a generalized count-based multiset over a set framework [256].

Definition 7.11.1 (Triangular multiset)**.** Let D be a nonempty set, and let

$$\mathbb{N}_{+\infty} := \mathbb{N}_0 \cup \{+\infty\}, \qquad \mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

A *triangular multiset* (or *t-multiset*) on D is a function

$$A : D \to \mathbb{N}_{+\infty}.$$

For each

$$x \in D,$$

the value

$$A(x) \in \mathbb{N}_{+\infty}$$

is called the *multiplicity* of x in A.

Equivalently, $A(x) = m$ means that the element x occurs m times in the triangular multiset A.

The support of A is defined by

$$\operatorname{supp}(A) := \{x \in D \mid A(x) \neq 0\}.$$

Example 7.11.2 (A concrete real-life example of a triangular multiset)**.** Let

$$D = \{\text{apple}, \text{banana}, \text{orange}, \text{milk}\}$$

be a set of products in a grocery store.

Suppose that a store manager wants to record the available stock of these products on a shelf. Define a triangular multiset

$$A : D \to \mathbb{N}_{+\infty}$$

by

$$A(\text{apple}) = 5, \qquad A(\text{banana}) = 3, \qquad A(\text{orange}) = 0, \qquad A(\text{milk}) = 2.$$

This means that:

- apple occurs 5 times in the multiset;
- banana occurs 3 times;
- orange does not occur, since its multiplicity is 0;
- milk occurs 2 times.

Equivalently, the triangular multiset A may be represented informally as

$$A = \{\text{apple}, \text{apple}, \text{apple}, \text{apple}, \text{apple}, \text{banana}, \text{banana}, \text{banana}, \text{milk}, \text{milk}\}.$$

The support of A is

$$\text{supp}(A) = \{x \in D \mid A(x) \neq 0\} = \{\text{apple}, \text{banana}, \text{milk}\}.$$

Thus, A is a concrete real-life example of a triangular multiset, where each product is assigned a multiplicity indicating how many copies of it are present.

Chapter 8

Power Set and Its Extensions

In this chapter, we examine the power set and its extensions.

8.1 Powersets

A powerset is the set of all subsets of a given set, collecting every possible selection of its elements.

Definition 8.1.1 (Power set)**.** Let X be a set. The *power set* of X is the set of all subsets of X, and is denoted by

$$\mathcal{P}(X) = \{A \subseteq X\}.$$

8.2 n-th PowerSet

The n-th power set is obtained by repeatedly applying the power-set operation, forming sets of subsets through n recursive stages.

Definition 8.2.1 (n-th power set)**.** [257, 258] Let X be a set. The *n-th power set* of X, denoted by

$$\mathcal{P}^n(X),$$

is defined recursively by

$$\mathcal{P}^1(X) := \mathcal{P}(X), \qquad \mathcal{P}^{n+1}(X) := \mathcal{P}\big(\mathcal{P}^n(X)\big) \quad \text{for } n \geq 1.$$

As extension concepts based on the n-th power set, notions such as SuperHyperGraph [259–262], SuperHyperStructure [258,263,264], and SuperHyperAlgebra [265–267] are also known.

8.3 Multi Powerset

A multi powerset consists of all ordered m-tuples of subsets of a set, combining several subset selections in one structure.

Definition 8.3.1 (Multi powerset)**.** Let X be a nonempty set, and let $m \in \mathbb{N}$. The *multi powerset of order* m of X, denoted by

$$\mathrm{MPow}_m(X),$$

is defined by

$$\mathrm{MPow}_m(X) := \big(\mathcal{P}(X)\big)^m.$$

Equivalently,

$$\mathrm{MPow}_m(X) = \{\, (A_1, \ldots, A_m) \mid A_i \subseteq X \text{ for all } i = 1, \ldots, m \,\}.$$

Thus, an element of $\mathrm{MPow}_m(X)$ is an m-tuple of subsets of X.

Theorem 8.3.2 (Well-definedness of the multi powerset)**.** *Let X be a nonempty set, and let $m \in \mathbb{N}$. Then $\mathrm{MPow}_m(X)$ is well-defined. More precisely:*

1. *every element*
$$(A_1, \ldots, A_m) \in \mathrm{MPow}_m(X)$$
is an m-tuple of subsets of X;
2. *conversely, every m-tuple*
$$(A_1, \ldots, A_m)$$
with
$$A_i \subseteq X \qquad \text{for all } i = 1, \ldots, m$$
belongs to $\mathrm{MPow}_m(X)$.

Proof. Since X is a set, its power set

$$\mathcal{P}(X)$$

is well-defined. Because $m \in \mathbb{N}$, the Cartesian power

$$\big(\mathcal{P}(X)\big)^m$$

is also well-defined.

Now let

$$(A_1, \ldots, A_m) \in \mathrm{MPow}_m(X) = \big(\mathcal{P}(X)\big)^m.$$

By definition of Cartesian power,

$$A_i \in \mathcal{P}(X) \qquad \text{for all } i = 1, \ldots, m.$$

Hence

$$A_i \subseteq X \qquad \text{for all } i = 1, \ldots, m.$$

This proves (1).

Conversely, let

$$(A_1, \dots, A_m)$$

be an m-tuple such that

$$A_i \subseteq X \qquad \text{for all } i = 1, \dots, m.$$

Then

$$A_i \in \mathcal{P}(X) \qquad \text{for all } i = 1, \dots, m.$$

Therefore

$$(A_1, \dots, A_m) \in \big(\mathcal{P}(X)\big)^m = \mathrm{MPow}_m(X).$$

This proves (2). Hence $\mathrm{MPow}_m(X)$ is well-defined. □

8.4 Fractional Powersets

A fractional powerset is a set whose iterated powerset becomes bijective to a given set after a specified finite number of steps [268].

Definition 8.4.1 (Fractional powerset)**.** [268] For a set X, define the iterated powersets recursively by

$$\mathcal{P}^0(X) := X, \qquad \mathcal{P}^{n+1}(X) := \mathcal{P}\big(\mathcal{P}^n(X)\big) \quad (n \geq 0).$$

Let $m \geq 1$ be an integer, and let U be a finite nonempty set. A set V is called an *m-root powerset* of U if there exists a bijection

$$\vartheta : \mathcal{P}^{m-1}(V) \to U.$$

In that case, we write

$$\mathcal{P}^{\langle 1/m \rangle}(U) := V$$

up to bijection.

Any such m-root powerset is called a *fractional powerset* of U.

8.5 Fuzzy Powerset

A fuzzy powerset is the collection of all fuzzy subsets of a set, represented by membership functions from the set into the unit interval precisely [269–272].

Definition 8.5.1 (Fuzzy powerset)**.** [269,270] Let X be a nonempty set. The *fuzzy powerset* of X, denoted by

$$\mathcal{F}(X),$$

is the set of all fuzzy subsets of X, that is,

$$\mathcal{F}(X) := [0,1]^X = \{\mu \mid \mu : X \to [0,1]\}.$$

Each element $\mu \in \mathcal{F}(X)$ is a membership function representing a fuzzy subset of X.

8.6 Intuitionistic Fuzzy Powerset

An intuitionistic fuzzy powerset contains all intuitionistic fuzzy subsets of a set, represented by membership and non-membership functions satisfying $\mu(x) + \nu(x) \leq 1$ for every element of X (cf. [273]).

Definition 8.6.1 (Intuitionistic fuzzy powerset)**.** Let X be a nonempty set. The *intuitionistic fuzzy powerset* of X, denoted by

$$\mathrm{IFPow}(X),$$

is defined by

$$\mathrm{IFPow}(X) := \Big\{ (\mu, \nu) \in [0,1]^X \times [0,1]^X \;\Big|\; \mu(x) + \nu(x) \leq 1 \text{ for all } x \in X \Big\}.$$

Equivalently, $\mathrm{IFPow}(X)$ is the set of all intuitionistic fuzzy sets on X.

Theorem 8.6.2 (Well-definedness of the intuitionistic fuzzy powerset)**.** *Let X be a nonempty set. Then* $\mathrm{IFPow}(X)$ *is well-defined. More precisely, every*

$$(\mu, \nu) \in \mathrm{IFPow}(X)$$

determines an intuitionistic fuzzy set on X, and conversely every intuitionistic fuzzy set on X belongs to $\mathrm{IFPow}(X)$*.*

Proof. Let $(\mu, \nu) \in \mathrm{IFPow}(X)$. By definition,

$$\mu, \nu : X \to [0,1]$$

and

$$\mu(x) + \nu(x) \leq 1 \qquad \text{for all } x \in X.$$

Hence the pair (μ, ν) satisfies exactly the defining conditions of an intuitionistic fuzzy set on X. Therefore each element of $\mathrm{IFPow}(X)$ determines an intuitionistic fuzzy set.

Conversely, let A be an intuitionistic fuzzy set on X. Then A is given by two functions

$$\mu_A, \nu_A : X \to [0,1]$$

such that

$$\mu_A(x) + \nu_A(x) \leq 1 \qquad \text{for all } x \in X.$$

Thus

$$(\mu_A, \nu_A) \in \mathrm{IFPow}(X).$$

Therefore $\mathrm{IFPow}(X)$ is well-defined. □

8.7 Neutrosophic Powerset

A neutrosophic powerset contains all single-valued neutrosophic subsets of a set, represented by truth, indeterminacy, and falsity functions whose sums remain between zero and three (cf. [274]).

Definition 8.7.1 (Neutrosophic powerset)**.** Let X be a nonempty set. The *neutrosophic powerset* of X, denoted by

$$\mathrm{NPow}(X),$$

is defined by

$$\mathrm{NPow}(X) := \left\{ (T, I, F) \in [0,1]^X \times [0,1]^X \times [0,1]^X \;\middle|\; 0 \le T(x) + I(x) + F(x) \le 3 \text{ for all } x \in X \right\}.$$

Equivalently, $\mathrm{NPow}(X)$ is the set of all single-valued neutrosophic sets on X.

Example 8.7.2 (A concrete example of a neutrosophic powerset)**.** Let

$$X = \{p_1, p_2, p_3\}$$

be a set of three restaurants in a city.

Consider the decision problem of choosing a restaurant for dinner. For each restaurant, we assign:

- a truth-membership degree, representing how strongly the restaurant is considered suitable;
- an indeterminacy-membership degree, representing uncertainty due to incomplete or conflicting information;
- a falsity-membership degree, representing how strongly the restaurant is considered unsuitable.

Define

$$T, I, F : X \to [0,1]$$

by

$$\begin{aligned} T(p_1) &= 0.8, & T(p_2) &= 0.5, & T(p_3) &= 0.3, \\ I(p_1) &= 0.1, & I(p_2) &= 0.3, & I(p_3) &= 0.4, \\ F(p_1) &= 0.2, & F(p_2) &= 0.4, & F(p_3) &= 0.6. \end{aligned}$$

Then, for each element of X, we have

$$T(p_1) + I(p_1) + F(p_1) = 0.8 + 0.1 + 0.2 = 1.1 \le 3,$$

$$T(p_2) + I(p_2) + F(p_2) = 0.5 + 0.3 + 0.4 = 1.2 \le 3,$$

$$T(p_3) + I(p_3) + F(p_3) = 0.3 + 0.4 + 0.6 = 1.3 \le 3.$$

Hence,

$$(T, I, F) \in \mathrm{NPow}(X).$$

Thus, (T, I, F) is an element of the neutrosophic powerset of X. In practical terms, it represents one possible neutrosophic evaluation of all restaurants in X: restaurant p_1 is judged to be highly suitable, p_2 is moderately suitable with more uncertainty, and p_3 is relatively unsuitable.

Therefore, $\mathrm{NPow}(X)$ is the collection of all such neutrosophic assessments on the set X.

Theorem 8.7.3 (Well-definedness of the neutrosophic powerset)**.** *Let X be a nonempty set. Then* $\mathrm{NPow}(X)$ *is well-defined. More precisely, every*

$$(T, I, F) \in \mathrm{NPow}(X)$$

determines a single-valued neutrosophic set on X, and conversely every single-valued neutrosophic set on X belongs to $\mathrm{NPow}(X)$*.*

Proof. Let $(T, I, F) \in \mathrm{NPow}(X)$. By definition,

$$T, I, F : X \to [0, 1]$$

and

$$0 \leq T(x) + I(x) + F(x) \leq 3 \qquad \text{for all } x \in X.$$

Hence (T, I, F) satisfies the defining conditions of a single-valued neutrosophic set on X. Therefore each element of $\mathrm{NPow}(X)$ determines a single-valued neutrosophic set.

Conversely, let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic set on X. Then

$$T_A, I_A, F_A : X \to [0, 1]$$

and

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3 \qquad \text{for all } x \in X.$$

Thus

$$(T_A, I_A, F_A) \in \mathrm{NPow}(X).$$

Hence $\mathrm{NPow}(X)$ is well-defined. □

8.8 Plithogenic Powerset

A plithogenic powerset collects all admissible appurtenance functions on a fixed plithogenic frame, preserving attribute values and contradiction structure while varying membership information across elements.

Definition 8.8.1 (Plithogenic frame)**.** Let P be a nonempty set, let v be a fixed attribute, let Pv be a nonempty set of possible values of v, and let $s, t \in \mathbb{N}$. Suppose

$$pCF : Pv \times Pv \to [0,1]^t$$

satisfies

$$pCF(a,a) = 0 \qquad \text{and} \qquad pCF(a,b) = pCF(b,a) \quad \text{for all } a, b \in Pv.$$

Then

$$\mathfrak{P} := (P, v, Pv, pCF, s, t)$$

is called a *plithogenic frame*.

Definition 8.8.2 (Plithogenic powerset)**.** Let

$$\mathfrak{P} := (P, v, Pv, pCF, s, t)$$

be a plithogenic frame. The *plithogenic powerset* of $\mathfrak{P}$, denoted by

$$\mathrm{PlPow}(\mathfrak{P}),$$

is defined by

$$\mathrm{PlPow}(\mathfrak{P}) := \{\, pdf : P \times Pv \to [0,1]^s \,\}.$$

Each element

$$pdf \in \mathrm{PlPow}(\mathfrak{P})$$

determines a plithogenic set

$$(P, v, Pv, pdf, pCF).$$

Theorem 8.8.3 (Well-definedness of the plithogenic powerset)**.** *Let*

$$\mathfrak{P} := (P, v, Pv, pCF, s, t)$$

be a plithogenic frame. Then $\mathrm{PlPow}(\mathfrak{P})$ *is well-defined. More precisely, every*

$$pdf \in \mathrm{PlPow}(\mathfrak{P})$$

determines a plithogenic set on the fixed frame $\mathfrak{P}$*, and conversely every plithogenic set on the frame* $\mathfrak{P}$ *arises in this way.*

Proof. Since P and Pv are nonempty sets, the Cartesian product

$$P \times Pv$$

is a set. Since $s \in \mathbb{N}$, the codomain

$$[0,1]^s$$

is also a set. Therefore the collection

$$\{\, pdf : P \times Pv \to [0,1]^s \,\}$$

is well-defined.

Now let

$$pdf \in \mathrm{PlPow}(\mathfrak{P}).$$

By definition,

$$pdf : P \times Pv \to [0,1]^s.$$

Because pCF is fixed in the frame and already satisfies

$$pCF(a,a) = 0, \qquad pCF(a,b) = pCF(b,a) \quad \text{for all } a,b \in Pv,$$

the quintuple

$$(P, v, Pv, pdf, pCF)$$

satisfies the defining conditions of a plithogenic set.

Conversely, any plithogenic set on the fixed frame $\mathfrak{P}$ has exactly the form

$$(P, v, Pv, pdf, pCF)$$

for some map

$$pdf : P \times Pv \to [0,1]^s.$$

Hence

$$pdf \in \mathrm{PlPow}(\mathfrak{P}).$$

Therefore $\mathrm{PlPow}(\mathfrak{P})$ is well-defined. □

8.9 Uncertain Powerset

An uncertain powerset consists of all uncertainty-degree functions from a set into a fixed admissible degree-domain, thereby collecting every uncertain set of that model.

Definition 8.9.1 (Uncertain powerset)**.** Let X be a nonempty set, and let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. The *uncertain powerset of type* M on X, denoted by

$$\mathrm{UPow}_M(X),$$

is defined by

$$\mathrm{UPow}_M(X) := \mathrm{Dom}(M)^X = \{\, \mu : X \to \mathrm{Dom}(M) \,\}.$$

Equivalently, $\mathrm{UPow}_M(X)$ is the set of all uncertain sets of type M on X.

Theorem 8.9.2 (Well-definedness of the uncertain powerset)**.** *Let X be a nonempty set, and let M be an uncertain model. Then* $\mathrm{UPow}_M(X)$ *is well-defined. More precisely, every*

$$\mu \in \mathrm{UPow}_M(X)$$

determines an uncertain set of type M on X, and conversely every uncertain set of type M on X belongs to $\mathrm{UPow}_M(X)$*.*

Proof. Let
$$\mu \in \mathrm{UPow}_M(X).$$
Then, by definition,
$$\mu : X \to \mathrm{Dom}(M).$$
Since $\mathrm{Dom}(M)$ is the admissible degree-domain of the uncertain model M, every value
$$\mu(x) \in \mathrm{Dom}(M)$$
automatically satisfies all model-specific constraints encoded in M. Hence μ is exactly an uncertainty-degree function of a U-set of type M on X. Therefore each element of $\mathrm{UPow}_M(X)$ determines an uncertain set of type M.

Conversely, let
$$\mathcal{U} = (X, \mu_M)$$
be an uncertain set of type M on X. By definition,
$$\mu_M : X \to \mathrm{Dom}(M).$$
Thus
$$\mu_M \in \mathrm{UPow}_M(X).$$
Therefore $\mathrm{UPow}_M(X)$ is well-defined. □

Table 8.1: A catalogue of powerset-based uncertainty families, organized by the number k of membership components

k	**Representative powerset model(s)**
1	Fuzzy Powerset
2	Intuitionistic Fuzzy Powerset
3	Neutrosophic Powerset

8.10 Soft Powerset

A soft powerset is the collection of all soft sets over a universe, allowing arbitrary parameter subsets and parameterized subset assignments from the universe (cf. [275, 276]).

Definition 8.10.1 (Soft powerset)**.** Let U be a nonempty universe and let E be a set of parameters. The *soft powerset* of U relative to E, denoted by
$$\mathrm{SPow}(U, E),$$
is defined by
$$\mathrm{SPow}(U, E) := \{\, (F, A) \mid A \subseteq E,\ F : A \to \mathcal{P}(U) \,\}.$$
Equivalently, $\mathrm{SPow}(U, E)$ is the set of all soft sets over U whose parameter domains are subsets of E.

Example 8.10.2 (A concrete example of a soft powerset)**.** Let

$$U = \{h_1, h_2, h_3, h_4\}$$

be a set of four houses for sale, and let

$$E = \{\text{cheap}, \text{spacious}, \text{near school}\}$$

be a set of parameters.

Consider a buyer who wants to describe houses according to some selected attributes. Choose

$$A = \{\text{cheap}, \text{near school}\} \subseteq E.$$

Define a mapping

$$F : A \to \mathcal{P}(U)$$

by

$$F(\text{cheap}) = \{h_1, h_3\}, \qquad F(\text{near school}) = \{h_2, h_3, h_4\}.$$

Then the pair

$$(F, A)$$

is a soft set over U relative to the parameter set E. Hence,

$$(F, A) \in \mathrm{SPow}(U, E).$$

This means that:

- the houses h_1 and h_3 are regarded as cheap;
- the houses h_2, h_3, and h_4 are regarded as near a school.

Therefore, (F, A) is a concrete element of the soft powerset $\mathrm{SPow}(U, E)$. In practical terms, the soft powerset consists of all possible parameterized descriptions of houses over U using parameters taken from E.

Theorem 8.10.3 (Well-definedness of the soft powerset)**.** *Let U be a nonempty universe and let E be a set of parameters. Then* $\mathrm{SPow}(U, E)$ *is well-defined. More precisely, every*

$$(F, A) \in \mathrm{SPow}(U, E)$$

is a soft set over U, and conversely every soft set over U with parameter domain contained in E belongs to $\mathrm{SPow}(U, E)$*.*

Proof. Let
$$(F, A) \in \mathrm{SPow}(U, E).$$
Then, by definition,
$$A \subseteq E \qquad \text{and} \qquad F : A \to \mathcal{P}(U).$$
Hence, for each parameter $\epsilon \in A$, the value
$$F(\epsilon) \subseteq U$$
is a subset of the universe U. Therefore (F, A) satisfies exactly the defining conditions of a soft set over U.

Conversely, let (F, A) be a soft set over U with
$$A \subseteq E.$$
By definition,
$$F : A \to \mathcal{P}(U).$$
Thus
$$(F, A) \in \mathrm{SPow}(U, E).$$
Hence $\mathrm{SPow}(U, E)$ is well-defined. □

8.11 Rough Powerset

A rough powerset is the collection of all rough approximations induced by an equivalence relation, gathering every lower-upper approximation pair of subsets (cf. [277]).

Definition 8.11.1 (Rough powerset)**.** Let X be a nonempty set, and let $R \subseteq X \times X$ be an equivalence relation. For each subset $A \subseteq X$, define its lower and upper approximations by
$$\underline{A}_R := \{\, x \in X \mid [x]_R \subseteq A \,\},$$
$$\overline{A}_R := \{\, x \in X \mid [x]_R \cap A \neq \emptyset \,\},$$
where
$$[x]_R := \{\, y \in X \mid (x, y) \in R \,\}.$$
The *rough powerset* of the approximation space (X, R), denoted by
$$\mathrm{RPow}(X, R),$$
is defined by
$$\mathrm{RPow}(X, R) := \big\{ (\underline{A}_R, \overline{A}_R) \mid A \subseteq X \big\}.$$
Equivalently, $\mathrm{RPow}(X, R)$ is the set of all rough sets induced by the approximation space (X, R).

Theorem 8.11.2 (Well-definedness of the rough powerset)**.** *Let X be a nonempty set, and let R be an equivalence relation on X. Then* $\mathrm{RPow}(X, R)$ *is well-defined. More precisely, for every subset $A \subseteq X$:*

1. *both*
$$\underline{A}_R \subseteq X \qquad \text{and} \qquad \overline{A}_R \subseteq X;$$

2. *one has*
$$\underline{A}_R \subseteq \overline{A}_R;$$

3. *hence*
$$(\underline{A}_R, \overline{A}_R) \in \mathcal{P}(X) \times \mathcal{P}(X)$$
is a well-defined rough set.

Proof. Let $A \subseteq X$.

By definition,
$$\underline{A}_R = \{ x \in X \mid [x]_R \subseteq A \},$$
so every element of $\underline{A}_R$ belongs to X. Hence
$$\underline{A}_R \subseteq X.$$
Similarly,
$$\overline{A}_R = \{ x \in X \mid [x]_R \cap A \neq \emptyset \},$$
so every element of $\overline{A}_R$ also belongs to X. Hence
$$\overline{A}_R \subseteq X.$$
This proves (1).

Now let
$$x \in \underline{A}_R.$$
Then
$$[x]_R \subseteq A.$$
Because $x \in [x]_R$, it follows that
$$[x]_R \cap A \neq \emptyset.$$
Therefore
$$x \in \overline{A}_R.$$
Hence
$$\underline{A}_R \subseteq \overline{A}_R.$$
This proves (2).

From (1) and (2), the pair
$$(\underline{A}_R, \overline{A}_R)$$
is a pair of subsets of X with lower part contained in upper part. Therefore it is a well-defined rough set. Since $A \subseteq X$ was arbitrary, $\mathrm{RPow}(X, R)$ is well-defined. □

Chapter 9

Shadowed Set and its Extensions

In this chapter, we introduce shadowed sets and their extensions.

9.1 Shadowed Set

A shadowed set approximates a fuzzy set by assigning definite exclusion, definite inclusion, or an unresolved interval through two thresholds, simplifying uncertainty representation for analysis [226, 278–280].

Definition 9.1.1 (Shadowed set)**.** [63] Let U be a nonempty universe, and let A be a fuzzy set on U with membership function

$$\mu_A : U \to [0,1].$$

Fix two thresholds $\alpha, \beta \in [0,1]$ such that

$$\alpha \leq \beta.$$

The *shadowed set* induced by A with respect to (α, β) is the set-valued mapping

$$\mu_{\mathrm{Sh}_{\alpha,\beta}(A)} : U \to \mathcal{P}([0,1])$$

defined by

$$\mu_{\mathrm{Sh}_{\alpha,\beta}(A)}(x) = \begin{cases} \{0\}, & \mu_A(x) \leq \alpha, \\ \{1\}, & \mu_A(x) \geq \beta, \\ [0,1], & \alpha < \mu_A(x) < \beta, \end{cases} \qquad x \in U.$$

Here $\{0\}$ means definite exclusion, $\{1\}$ means definite inclusion, and $[0,1]$ denotes the shadowed region of unresolved uncertainty.

A comparison between interval sets and shadowed sets is presented in Table 9.1, and a comparison between grey sets and shadowed sets is presented in Table 9.2.

Table 9.1: A brief comparison between interval sets and shadowed sets

Aspect	**Interval set**	**Shadowed set**
Basic idea	Assigns to each element an interval of possible membership values	Converts a fuzzy membership value into one of three regions: definite exclusion, definite inclusion, or shadowed uncertainty
Membership form	Each element is described by an interval $[\mu^-(x), \mu^+(x)] \subseteq [0,1]$	Each element is described by $\{0\}, \{1\}$, or $[0,1]$ according to fixed thresholds
Main source of uncertainty	Uncertainty is represented directly as interval-valued membership	Uncertainty is represented through threshold-based discretization of fuzzy membership
Structural feature	Lower and upper bounds are assigned explicitly	Two thresholds α, β determine the certain and shadowed regions
Interpretation	The exact membership degree is unknown but lies within a specified interval	The element is classified as definitely out, definitely in, or unresolved
Level of granularity	Usually preserves more detailed interval information	Provides a coarser three-region representation
Typical viewpoint	Interval-valued extension of fuzzy-set modeling	Simplified uncertainty model emphasizing approximation and decision regions
Common feature	Both allow non-crisp membership representation	Both allow non-crisp membership representation

9.2 Shadowed OffSet

A shadowed offset is a set-valued extension of a fuzzy offset that preserves exact offset-membership outside a threshold band and assigns the whole offset interval inside it [281].

Definition 9.2.1 (Shadowed Offset)**.** [281] Let X be a nonempty universe, and let $\widetilde{A}$ be a fuzzy offset on X with membership function

$$\mu_{\widetilde{A}} : X \to [\Psi, \Omega], \qquad \Psi < 0 < 1 < \Omega.$$

Fix thresholds $\alpha, \beta \in [\Psi, \Omega]$ such that

$$\Psi \leq \alpha < \beta \leq \Omega.$$

The *shadowed offset* of $\widetilde{A}$ with respect to (α, β), denoted by

$$SO_{\alpha,\beta}(\widetilde{A}),$$

is the set-valued mapping

$$\mu_{SO_{\alpha,\beta}(\widetilde{A})} : X \to \mathcal{P}([\Psi, \Omega])$$

Table 9.2: A brief comparison between grey sets and shadowed sets

Aspect	Grey set	Shadowed set
Basic idea	Assigns to each element a grey interval expressing incomplete or partially known membership	Converts a fuzzy membership value into definite exclusion, definite inclusion, or a shadowed region
Membership form	Each element is described by a grey interval $[\underline{\mu}(x), \overline{\mu}(x)] \subseteq [0,1]$	Each element is described by $\{0\}, \{1\}$, or $[0,1]$
Main source of uncertainty	Incomplete or insufficient information about the actual membership degree	Threshold-based simplification of fuzzy membership into three decision regions
Structural feature	Uses lower and upper grey bounds	Uses two thresholds α, β to induce certain and uncertain zones
Interpretation	The actual membership is only known to lie within a grey range	The actual status is reduced to definitely low, definitely high, or unresolved
Information retention	Retains interval-type information about partial knowledge	Retains only a coarse three-valued approximation
Typical viewpoint	Often associated with grey-system theory and incomplete-information analysis	Often associated with simplified fuzzy representation and decision-oriented approximation
Common feature	Both describe uncertainty by non-crisp membership values	Both describe uncertainty by non-crisp membership values

defined by

$$\mu_{SO_{\alpha,\beta}(\widetilde{A})}(x) = \begin{cases} \{\mu_{\widetilde{A}}(x)\}, & \mu_{\widetilde{A}}(x) \leq \alpha \text{ or } \mu_{\widetilde{A}}(x) \geq \beta, \\ [\Psi, \Omega], & \alpha < \mu_{\widetilde{A}}(x) < \beta, \end{cases} \qquad x \in X.$$

Here the singleton $\{\mu_{\widetilde{A}}(x)\}$ preserves the exact offset-membership value, while $[\Psi, \Omega]$ represents the shadowed offset region of unresolved uncertainty.

9.3 Intuitionistic Shadowed set

An intuitionistic shadowed set approximates intuitionistic fuzzy information by assigning each element to accepted, rejected, or shadowed regions using membership and non-membership thresholds for classification (cf. [282]).

Definition 9.3.1 (Intuitionistic shadowed set)**.** Let X be a nonempty set, and let

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

be an intuitionistic fuzzy set on X, where

$$\mu_A, \nu_A : X \to [0,1]$$

satisfy

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \qquad \text{for all } x \in X.$$

Fix thresholds

$$w, p \in [0, 1].$$

Define the *intuitionistic admissible region* by

$$\Delta_{\text{IFS}} := \{(u, v) \in [0, 1]^2 \mid u + v \le 1\}.$$

The *intuitionistic shadowed set* induced by A with respect to (w, p) is the set-valued mapping

$$\text{ISh}_{w,p}(A) : X \to \mathcal{P}(\Delta_{\text{IFS}}) \setminus \{\emptyset\}$$

defined by

$$\text{ISh}_{w,p}(A)(x) = \begin{cases} \{(1, 0)\}, & \mu_A(x) \ge w \text{ and } \nu_A(x) < p, \\ \{(0, 1)\}, & \mu_A(x) < w \text{ and } \nu_A(x) \ge p, \\ \Delta_{\text{IFS}}, & \text{otherwise}, \end{cases} \qquad x \in X.$$

Equivalently, each element is assigned to one of three regions:

- the *accepted region*, represented by the singleton $\{(1, 0)\}$;
- the *rejected region*, represented by the singleton $\{(0, 1)\}$;
- the *shadowed region*, represented by the full admissible intuitionistic triangle Δ_{IFS}.

Theorem 9.3.2 (Well-definedness of intuitionistic shadowed sets)**.** *Let A be an intuitionistic fuzzy set on a nonempty set X, and let $w, p \in [0, 1]$. Then the mapping*

$$\text{ISh}_{w,p}(A) : X \to \mathcal{P}(\Delta_{\text{IFS}}) \setminus \{\emptyset\}$$

given in Definition 9.3.1 is well-defined. More precisely:

1. *for every $x \in X$, the value $\text{ISh}_{w,p}(A)(x)$ is a nonempty subset of Δ_{IFS};*
2. *for every $x \in X$ and every*
$$(u, v) \in \text{ISh}_{w,p}(A)(x),$$
one has
$$u, v \in [0, 1] \qquad \textit{and} \qquad u + v \le 1.$$

Proof. Fix $x \in X$. Since A is an intuitionistic fuzzy set, we have

$$\mu_A(x) \in [0, 1], \qquad \nu_A(x) \in [0, 1], \qquad \text{and} \qquad \mu_A(x) + \nu_A(x) \le 1.$$

Now exactly one of the two alternatives

$$\mu_A(x) \geq w \qquad \text{or} \qquad \mu_A(x) < w$$

holds, and exactly one of the two alternatives

$$\nu_A(x) < p \qquad \text{or} \qquad \nu_A(x) \geq p$$

holds. Hence exactly one of the following four mutually exclusive situations occurs:

$$(\mu_A(x) \geq w,\ \nu_A(x) < p),$$

$$(\mu_A(x) < w,\ \nu_A(x) \geq p),$$
$$(\mu_A(x) \geq w,\ \nu_A(x) \geq p),$$
$$(\mu_A(x) < w,\ \nu_A(x) < p).$$

According to Definition 9.3.1, in the first case

$$\mathrm{ISh}_{w,p}(A)(x) = \{(1,0)\},$$

in the second case

$$\mathrm{ISh}_{w,p}(A)(x) = \{(0,1)\},$$

and in the remaining two cases

$$\mathrm{ISh}_{w,p}(A)(x) = \Delta_{\mathrm{IFS}}.$$

Therefore a unique value is assigned to each $x \in X$.

It remains to verify that each possible value belongs to

$$\mathcal{P}(\Delta_{\mathrm{IFS}}) \setminus \{\emptyset\}.$$

First,

$$(1,0) \in \Delta_{\mathrm{IFS}}$$

because $1, 0 \in [0,1]$ and $1 + 0 = 1 \leq 1$. Hence

$$\{(1,0)\} \subseteq \Delta_{\mathrm{IFS}}$$

and it is nonempty. Similarly,

$$(0,1) \in \Delta_{\mathrm{IFS}},$$

so

$$\{(0,1)\} \subseteq \Delta_{\mathrm{IFS}}$$

and it is nonempty. Finally, Δ_{IFS} is clearly a nonempty subset of itself, since for example

$$(0,0) \in \Delta_{\mathrm{IFS}}.$$

Thus, for every $x \in X$,

$$\mathrm{ISh}_{w,p}(A)(x) \in \mathcal{P}(\Delta_{\mathrm{IFS}}) \setminus \{\emptyset\}.$$

This proves part (1).

For part (2), let

$$(u, v) \in \mathrm{ISh}_{w,p}(A)(x).$$

If

$$\mathrm{ISh}_{w,p}(A)(x) = \{(1, 0)\},$$

then $(u, v) = (1, 0)$, and hence

$$u, v \in [0, 1], \qquad u + v = 1 \leq 1.$$

If

$$\mathrm{ISh}_{w,p}(A)(x) = \{(0, 1)\},$$

then $(u, v) = (0, 1)$, and again

$$u, v \in [0, 1], \qquad u + v = 1 \leq 1.$$

If

$$\mathrm{ISh}_{w,p}(A)(x) = \Delta_{\mathrm{IFS}},$$

then by the definition of Δ_{IFS} one directly has

$$u, v \in [0, 1] \qquad \text{and} \qquad u + v \leq 1.$$

Hence every representative pair selected from the intuitionistic shadowed value is an admissible intuitionistic pair.

Therefore, $\mathrm{ISh}_{w,p}(A)$ is well-defined. □

9.4 Neutrosophic Shadowed Set

A neutrosophic shadowed set approximates truth, indeterminacy, and falsity functions by assigning each component the values 0, 1, or $[0, 1]$ through thresholds for simplified uncertainty.

Definition 9.4.1 (Scalar shadowing operator)**.** Let

$$\Sigma := \big\{\{0\},\ [0, 1],\ \{1\}\big\} \subseteq \mathcal{P}([0, 1]).$$

For real numbers $\alpha, \beta \in [0, 1]$ with

$$\alpha < \beta,$$

define the *scalar shadowing operator*

$$s_{\alpha,\beta} : [0, 1] \to \Sigma$$

by

$$s_{\alpha,\beta}(u) = \begin{cases} \{0\}, & u \leq \alpha, \\ [0, 1], & \alpha < u < \beta, \\ \{1\}, & u \geq \beta. \end{cases}$$

Definition 9.4.2 (Neutrosophic shadowed set)**.** Let X be a nonempty set, and let

$$A = (T_A, I_A, F_A)$$

be a single-valued neutrosophic set on X, where

$$T_A, I_A, F_A : X \to [0, 1]$$

satisfy

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3 \qquad \text{for all } x \in X.$$

Fix threshold pairs

$$(\alpha_T, \beta_T), \quad (\alpha_I, \beta_I), \quad (\alpha_F, \beta_F)$$

with

$$0 \le \alpha_T < \beta_T \le 1, \qquad 0 \le \alpha_I < \beta_I \le 1, \qquad 0 \le \alpha_F < \beta_F \le 1.$$

The *neutrosophic shadowed set* induced by A is the quadruple

$$\mathrm{NSh}(A) = \big(X, \widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A\big),$$

where

$$\widetilde{T}_A : X \to \Sigma, \qquad \widetilde{I}_A : X \to \Sigma, \qquad \widetilde{F}_A : X \to \Sigma$$

are defined componentwise by

$$\widetilde{T}_A(x) := s_{\alpha_T, \beta_T}\big(T_A(x)\big),$$
$$\widetilde{I}_A(x) := s_{\alpha_I, \beta_I}\big(I_A(x)\big),$$
$$\widetilde{F}_A(x) := s_{\alpha_F, \beta_F}\big(F_A(x)\big)$$

for all $x \in X$.

Equivalently,

$$\widetilde{T}_A(x) = \begin{cases} \{0\}, & T_A(x) \le \alpha_T, \\ [0, 1], & \alpha_T < T_A(x) < \beta_T, \\ \{1\}, & T_A(x) \ge \beta_T, \end{cases}$$

$$\widetilde{I}_A(x) = \begin{cases} \{0\}, & I_A(x) \le \alpha_I, \\ [0, 1], & \alpha_I < I_A(x) < \beta_I, \\ \{1\}, & I_A(x) \ge \beta_I, \end{cases}$$

and

$$\widetilde{F}_A(x) = \begin{cases} \{0\}, & F_A(x) \le \alpha_F, \\ [0, 1], & \alpha_F < F_A(x) < \beta_F, \\ \{1\}, & F_A(x) \ge \beta_F. \end{cases}$$

Thus, each neutrosophic component is converted into a three-valued shadowed form: definitely low ($\{0\}$), shadowed/undetermined ($[0, 1]$), or definitely high ($\{1\}$).

Example 9.4.3 (A concrete example of a neutrosophic shadowed set)**.** Let

$$X = \{d_1, d_2, d_3, d_4\}$$

be a set of four medical cases under preliminary hospital evaluation.

Suppose that

$$A = (T_A, I_A, F_A)$$

is a single-valued neutrosophic set on X, where:

- $T_A(d)$ represents the degree to which case d is assessed as *high priority*;
- $I_A(d)$ represents the degree of *indeterminacy* in that assessment, caused for example by incomplete test results or conflicting symptoms;
- $F_A(d)$ represents the degree to which case d is assessed as *not high priority.*

Assume the evaluations are:

$$\begin{aligned} A(d_1) &= (0.85, 0.20, 0.10), \\ A(d_2) &= (0.55, 0.45, 0.30), \\ A(d_3) &= (0.25, 0.60, 0.70), \\ A(d_4) &= (0.65, 0.15, 0.20). \end{aligned}$$

Choose the threshold pairs

$$(\alpha_T, \beta_T) = (0.4, 0.7), \qquad (\alpha_I, \beta_I) = (0.3, 0.5), \qquad (\alpha_F, \beta_F) = (0.2, 0.6).$$

Then the shadowed truth-membership component is obtained as follows:

$$\widetilde{T}_A(d_1) = \{1\}, \qquad \widetilde{T}_A(d_2) = [0, 1], \qquad \widetilde{T}_A(d_3) = \{0\}, \qquad \widetilde{T}_A(d_4) = [0, 1],$$

because

$$T_A(d_1) = 0.85 \geq 0.7, \quad 0.4 < T_A(d_2) = 0.55 < 0.7, \quad T_A(d_3) = 0.25 \leq 0.4, \quad 0.4 < T_A(d_4) = 0.65 < 0.7.$$

Similarly, the shadowed indeterminacy-membership component is

$$\widetilde{I}_A(d_1) = \{0\}, \qquad \widetilde{I}_A(d_2) = [0, 1], \qquad \widetilde{I}_A(d_3) = \{1\}, \qquad \widetilde{I}_A(d_4) = \{0\},$$

since

$$I_A(d_1) = 0.20 \leq 0.3, \quad 0.3 < I_A(d_2) = 0.45 < 0.5, \quad I_A(d_3) = 0.60 \geq 0.5, \quad I_A(d_4) = 0.15 \leq 0.3.$$

The shadowed falsity-membership component is

$$\widetilde{F}_A(d_1) = \{0\}, \qquad \widetilde{F}_A(d_2) = [0, 1], \qquad \widetilde{F}_A(d_3) = \{1\}, \qquad \widetilde{F}_A(d_4) = \{0\},$$

because

$$F_A(d_1) = 0.10 \le 0.2, \quad 0.2 < F_A(d_2) = 0.30 < 0.6, \quad F_A(d_3) = 0.70 \ge 0.6, \quad F_A(d_4) = 0.20 \le 0.2.$$

Hence, the neutrosophic shadowed set induced by A is

$$\mathrm{NSh}(A) = \big(X, \widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A\big),$$

where

$$\begin{aligned} d_1 &\mapsto (\{1\}, \{0\}, \{0\}), \\ d_2 &\mapsto ([0,1], [0,1], [0,1]), \\ d_3 &\mapsto (\{0\}, \{1\}, \{1\}), \\ d_4 &\mapsto ([0,1], \{0\}, \{0\}). \end{aligned}$$

In practical terms:

- d_1 is clearly judged to be a high-priority case, with low indeterminacy and low rejection;
- d_2 lies in the shadowed region for all three components, meaning that its status is still unclear;
- d_3 is clearly not high priority and is accompanied by high indeterminacy and high falsity;
- d_4 has a moderately high priority assessment, but not enough to be classified as definitely high.

Thus, this neutrosophic shadowed set provides a simplified three-level classification of medical priority by transforming each neutrosophic component into the values $\{0\}$, $[0,1]$, or $\{1\}$.

Theorem 9.4.4 (Well-definedness of neutrosophic shadowed sets)**.** *Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic set on a nonempty set X, and let*

$$0 \le \alpha_T < \beta_T \le 1, \qquad 0 \le \alpha_I < \beta_I \le 1, \qquad 0 \le \alpha_F < \beta_F \le 1.$$

Then the neutrosophic shadowed set

$$\mathrm{NSh}(A) = \big(X, \widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A\big)$$

defined in Definition 9.4.2 is well-defined. More precisely:

1. *for every $x \in X$,*
$$\widetilde{T}_A(x),\ \widetilde{I}_A(x),\ \widetilde{F}_A(x) \in \Sigma;$$
2. *hence*
$$\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A : X \to \Sigma$$
are well-defined functions;

3. *for every* $x \in X$ *and every choice*

$$t \in \widetilde{T}_A(x), \qquad i \in \widetilde{I}_A(x), \qquad f \in \widetilde{F}_A(x),$$

one has

$$t, i, f \in [0,1] \quad \textit{and} \quad 0 \le t + i + f \le 3.$$

Proof. Fix $x \in X$.

We first consider the truth component. Since A is a single-valued neutrosophic set, we have

$$T_A(x) \in [0,1].$$

Because

$$\alpha_T < \beta_T,$$

exactly one of the following three mutually exclusive cases holds:

$$T_A(x) \le \alpha_T, \qquad \alpha_T < T_A(x) < \beta_T, \qquad T_A(x) \ge \beta_T.$$

Therefore, by the definition of s_{α_T,β_T}, exactly one of the values

$$\{0\},\ [0,1],\ \{1\}$$

is assigned to $\widetilde{T}_A(x)$. Hence

$$\widetilde{T}_A(x) \in \Sigma.$$

The same argument applies to $I_A(x) \in [0,1]$ and $F_A(x) \in [0,1]$, yielding

$$\widetilde{I}_A(x) \in \Sigma \qquad \text{and} \qquad \widetilde{F}_A(x) \in \Sigma.$$

Since $x \in X$ was arbitrary, it follows that

$$\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A : X \to \Sigma$$

are well-defined functions. This proves (1) and (2).

For (3), let

$$t \in \widetilde{T}_A(x), \qquad i \in \widetilde{I}_A(x), \qquad f \in \widetilde{F}_A(x).$$

Each of the sets

$$\widetilde{T}_A(x),\ \widetilde{I}_A(x),\ \widetilde{F}_A(x)$$

is either $\{0\}$, $[0,1]$, or $\{1\}$. Hence

$$t, i, f \in [0,1].$$

Therefore,

$$0 \le t + i + f \le 1 + 1 + 1 = 3.$$

So every representative triple selected from the shadowed neutrosophic degrees remains an admissible neutrosophic triple in the interval sense.

Consequently, $\mathrm{NSh}(A)$ is well-defined. □

9.5 Uncertain Shadowed Set

An uncertain shadowed set classifies model-admissible uncertainty degrees into lower exact, shadowed, or upper exact regions through an evaluation map and thresholds, simplifying representation considerably.

Definition 9.5.1 (Uncertain shadowing frame)**.** Let M be an uncertain model with admissible degree-domain

$$D := \mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. Assume that:

1. $D \neq \emptyset$;
2. $\sigma : D \to [0,1]$ is a fixed evaluation map;
3. $d^-, d^+ \in D$ are two distinguished admissible points.

Then the quadruple

$$\mathfrak{S} = (M, \sigma, d^-, d^+)$$

is called an *uncertain shadowing frame*.

Definition 9.5.2 (Uncertain shadowed set)**.** Let X be a nonempty set, let

$$\mathcal{U} = (X, \mu_M)$$

be an uncertain set of type M, where

$$\mu_M : X \to D = \mathrm{Dom}(M),$$

and let

$$\mathfrak{S} = (M, \sigma, d^-, d^+)$$

be an uncertain shadowing frame. Fix thresholds $\alpha, \beta \in [0,1]$ such that

$$\alpha < \beta.$$

The *uncertain shadowed set* induced by $\mathcal{U}$ with respect to $(\mathfrak{S}, \alpha, \beta)$ is the set-valued mapping

$$\mathrm{USh}_{\mathfrak{S};\alpha,\beta}(\mathcal{U}) : X \to \mathcal{P}(D) \setminus \{\emptyset\}$$

defined by

$$\mathrm{USh}_{\mathfrak{S};\alpha,\beta}(\mathcal{U})(x) = \begin{cases} \{d^-\}, & \sigma(\mu_M(x)) \leq \alpha, \\ D, & \alpha < \sigma(\mu_M(x)) < \beta, \\ \{d^+\}, & \sigma(\mu_M(x)) \geq \beta, \end{cases} \qquad x \in X.$$

Here $\{d^-\}$ is the *lower exact region*, $\{d^+\}$ is the *upper exact region*, and D is the *shadowed region*.

Theorem 9.5.3 (Well-definedness of uncertain shadowed sets)**.** *Let X be a nonempty set, let*

$$\mathcal{U} = (X, \mu_M)$$

be an uncertain set of type M, and let

$$\mathfrak{S} = (M, \sigma, d^-, d^+)$$

be an uncertain shadowing frame. Fix thresholds

$$\alpha, \beta \in [0,1], \qquad \alpha < \beta.$$

Then

$$\mathrm{USh}_{\mathfrak{S};\alpha,\beta}(\mathcal{U}) : X \to \mathcal{P}(D) \setminus \{\emptyset\}$$

is well-defined. Moreover, for every $x \in X$ and every

$$z \in \mathrm{USh}_{\mathfrak{S};\alpha,\beta}(\mathcal{U})(x),$$

one has

$$z \in D = \mathrm{Dom}(M).$$

Hence every representative value selected from the uncertain shadowed set is admissible for the model M.

Proof. Fix $x \in X$. Since $\mu_M : X \to D$, we have

$$\mu_M(x) \in D.$$

Because $\sigma : D \to [0,1]$, it follows that

$$\sigma(\mu_M(x)) \in [0,1].$$

Since $\alpha < \beta$, exactly one of the following three mutually exclusive cases holds:

$$\sigma(\mu_M(x)) \le \alpha, \qquad \alpha < \sigma(\mu_M(x)) < \beta, \qquad \sigma(\mu_M(x)) \ge \beta.$$

Therefore exactly one of the values

$$\{d^-\}, \qquad D, \qquad \{d^+\}$$

is assigned to

$$\mathrm{USh}_{\mathfrak{S};\alpha,\beta}(\mathcal{U})(x).$$

Since $d^-, d^+ \in D$ and $D \neq \emptyset$, each of these three values is a nonempty subset of D. Hence

$$\mathrm{USh}_{\mathfrak{S};\alpha,\beta}(\mathcal{U})(x) \in \mathcal{P}(D) \setminus \{\emptyset\}.$$

As $x \in X$ was arbitrary, the mapping is well-defined.

Now let

$$z \in \mathrm{USh}_{\mathfrak{S};\alpha,\beta}(\mathcal{U})(x).$$

If

$$\mathrm{USh}_{\mathfrak{S};\alpha,\beta}(\mathcal{U})(x) = \{d^-\},$$

then $z = d^- \in D$. If

$$\mathrm{USh}_{\mathfrak{S};\alpha,\beta}(\mathcal{U})(x) = \{d^+\},$$

then $z = d^+ \in D$. If

$$\mathrm{USh}_{\mathfrak{S};\alpha,\beta}(\mathcal{U})(x) = D,$$

then trivially $z \in D$. Thus in all cases $z \in D = \mathrm{Dom}(M)$. □

9.6 Shadowed Soft Set

A shadowed soft set parameterizes shadowed memberships, assigning each element under each parameter definite exclusion, definite inclusion, or unresolved uncertainty via parameter-dependent thresholds for analysis [283].

Definition 9.6.1 (Fuzzy soft set)**.** [284, 285] Let U be a nonempty universe and let E be a nonempty set of parameters. A *fuzzy soft set* on (U, E) is a mapping

$$F : E \to [0,1]^U.$$

Equivalently, for each $e \in E$, there exists a membership function

$$\mu_e : U \to [0,1]$$

such that

$$F(e) = \mu_e.$$

Definition 9.6.2 (Shadowed soft set)**.** [283] Let $F : E \to [0,1]^U$ be a fuzzy soft set on (U, E). Let

$$\alpha, \beta : E \to [0,1]$$

be threshold functions such that

$$\alpha(e) < \beta(e) \qquad \text{for all } e \in E.$$

Define

$$\Sigma_{\mathrm{Sh}} := \big\{\{0\}, [0,1], \{1\}\big\} \subseteq \mathcal{P}([0,1]).$$

The *shadowed soft set* induced by F with respect to (α, β) is the mapping

$$\mathrm{ShSoft}_{\alpha,\beta}(F) : E \to \Sigma_{\mathrm{Sh}}^U$$

defined by

$$\big(\mathrm{ShSoft}_{\alpha,\beta}(F)(e)\big)(x) = \begin{cases} \{0\}, & F(e)(x) \le \alpha(e), \\ [0,1], & \alpha(e) < F(e)(x) < \beta(e), \\ \{1\}, & F(e)(x) \ge \beta(e), \end{cases} \qquad e \in E,\ x \in U.$$

Theorem 9.6.3 (Well-definedness of shadowed soft sets)**.** *Let $F : E \to [0,1]^U$ be a fuzzy soft set, and let*

$$\alpha, \beta : E \to [0,1]$$

satisfy

$$\alpha(e) < \beta(e) \qquad \textit{for all } e \in E.$$

Then

$$\mathrm{ShSoft}_{\alpha,\beta}(F) : E \to \Sigma_{\mathrm{Sh}}^U$$

is well-defined. Equivalently, for every $e \in E$, the mapping

$$\mathrm{ShSoft}_{\alpha,\beta}(F)(e) : U \to \Sigma_{\mathrm{Sh}}$$

is a well-defined shadowed-membership function.

Proof. Fix $e \in E$ and $x \in U$. Since $F : E \to [0,1]^U$, one has

$$F(e)(x) \in [0,1].$$

Also,

$$\alpha(e), \beta(e) \in [0,1] \qquad \text{and} \qquad \alpha(e) < \beta(e).$$

Hence exactly one of the following three mutually exclusive cases holds:

$$F(e)(x) \leq \alpha(e), \qquad \alpha(e) < F(e)(x) < \beta(e), \qquad F(e)(x) \geq \beta(e).$$

Therefore exactly one of the values

$$\{0\}, \qquad [0,1], \qquad \{1\}$$

is assigned to

$$\big(\mathrm{ShSoft}_{\alpha,\beta}(F)(e)\big)(x).$$

Since

$$\{0\}, [0,1], \{1\} \in \Sigma_{\mathrm{Sh}},$$

it follows that

$$\big(\mathrm{ShSoft}_{\alpha,\beta}(F)(e)\big)(x) \in \Sigma_{\mathrm{Sh}}.$$

As $x \in U$ was arbitrary,

$$\mathrm{ShSoft}_{\alpha,\beta}(F)(e) : U \to \Sigma_{\mathrm{Sh}}$$

is well-defined. Since $e \in E$ was arbitrary, the mapping

$$\mathrm{ShSoft}_{\alpha,\beta}(F) : E \to \Sigma_{\mathrm{Sh}}^U$$

is well-defined. □

9.7 Shadowed Rough Set

A shadowed rough set approximates rough membership values by assigning each element definite exclusion, definite inclusion, or unresolved uncertainty through two thresholds in approximation spaces (cf. [286]).

Definition 9.7.1 (Rough membership function)**.** Let X be a finite nonempty set, let R be an equivalence relation on X, and let $A \subseteq X$. For each $x \in X$, define

$$[x]_R := \{y \in X \mid (x,y) \in R\}.$$

The *rough membership function* of A with respect to R is the mapping

$$\rho_A^R : X \to [0,1]$$

given by

$$\rho_A^R(x) := \frac{|[x]_R \cap A|}{|[x]_R|}.$$

Definition 9.7.2 (Shadowed rough set)**.** Let X be a finite nonempty set, let R be an equivalence relation on X, let $A \subseteq X$, and fix thresholds

$$\alpha, \beta \in [0,1], \qquad \alpha < \beta.$$

Define

$$\Sigma_{\mathrm{Sh}} := \big\{\{0\}, [0,1], \{1\}\big\} \subseteq \mathcal{P}([0,1]).$$

The *shadowed rough set* induced by A in the approximation space (X, R) is the set-valued mapping

$$\mathrm{ShR}^{R}_{\alpha,\beta}(A) : X \to \Sigma_{\mathrm{Sh}}$$

defined by

$$\mathrm{ShR}^{R}_{\alpha,\beta}(A)(x) = \begin{cases} \{0\}, & \rho^R_A(x) \le \alpha, \\ [0,1], & \alpha < \rho^R_A(x) < \beta, \\ \{1\}, & \rho^R_A(x) \ge \beta, \end{cases} \qquad x \in X.$$

Theorem 9.7.3 (Well-definedness of shadowed rough sets)**.** *Let X be a finite nonempty set, let R be an equivalence relation on X, let $A \subseteq X$, and let*

$$\alpha, \beta \in [0,1], \qquad \alpha < \beta.$$

Then the rough membership function

$$\rho^R_A : X \to [0,1]$$

is well-defined, and consequently

$$\mathrm{ShR}^{R}_{\alpha,\beta}(A) : X \to \Sigma_{\mathrm{Sh}}$$

is well-defined.

Proof. Fix $x \in X$. Since R is an equivalence relation, the equivalence class $[x]_R$ contains x itself. Hence

$$[x]_R \neq \emptyset.$$

Because X is finite, the set $[x]_R$ is finite, and therefore

$$|[x]_R| \in \mathbb{N}.$$

In particular,

$$|[x]_R| > 0.$$

Also,

$$[x]_R \cap A \subseteq [x]_R,$$

so

$$0 \le |[x]_R \cap A| \le |[x]_R|.$$

Dividing by the positive integer $|[x]_R|$, we obtain

$$0 \le \rho^R_A(x) = \frac{|[x]_R \cap A|}{|[x]_R|} \le 1.$$

Thus

$$\rho^R_A(x) \in [0,1].$$

Since $x \in X$ was arbitrary, the function

$$\rho_A^R : X \to [0,1]$$

is well-defined.

Now fix $x \in X$. Since $\rho_A^R(x) \in [0,1]$ and $\alpha < \beta$, exactly one of the three mutually exclusive cases

$$\rho_A^R(x) \le \alpha, \qquad \alpha < \rho_A^R(x) < \beta, \qquad \rho_A^R(x) \ge \beta$$

holds. Therefore exactly one of the three values

$$\{0\}, \qquad [0,1], \qquad \{1\}$$

is assigned to

$$\mathrm{ShR}^R_{\alpha,\beta}(A)(x).$$

Since each of these values belongs to Σ_{Sh}, it follows that

$$\mathrm{ShR}^R_{\alpha,\beta}(A)(x) \in \Sigma_{\mathrm{Sh}}.$$

As $x \in X$ was arbitrary, the mapping

$$\mathrm{ShR}^R_{\alpha,\beta}(A) : X \to \Sigma_{\mathrm{Sh}}$$

is well-defined. □

Chapter 10

Grey Set and its Extensions

In this chapter, we introduce grey sets and their extensions.

10.1 Grey set

A grey set assigns each element an interval of possible membership values, bounded by lower and upper functions, thereby modeling incomplete information and uncertainty quantitatively [287–291].

Definition 10.1.1 (Grey set)**.** [287, 288] Let U be a nonempty universe. A *grey set* G on U is determined by two functions

$$\underline{\mu}_G, \overline{\mu}_G : U \to [0, 1]$$

such that

$$\underline{\mu}_G(x) \leq \overline{\mu}_G(x) \qquad \text{for all } x \in U.$$

Equivalently, each element $x \in U$ is assigned the interval-valued membership

$$\mu_G(x) := [\, \underline{\mu}_G(x), \overline{\mu}_G(x) \,] \subseteq [0, 1].$$

The interval $\mu_G(x)$ represents the grey membership degree of x in G.

Remark 10.1.2. If

$$\underline{\mu}_G(x) = \overline{\mu}_G(x) \qquad \text{for all } x \in U,$$

then the grey set reduces to an ordinary fuzzy set.

As a reference, a brief comparison between interval sets and grey sets is presented in Table 10.1.

Table 10.1: A brief comparison between interval sets and grey sets

Aspect	**Interval set**	**Grey set**
Basic idea	Represents membership by an interval of possible values	Represents membership by a grey interval expressing incomplete or partially known information
Membership form	Each element is assigned an interval $[\mu^-(x), \mu^+(x)] \subseteq [0,1]$	Each element is assigned a grey number or grey interval $[\underline{\mu}(x), \overline{\mu}(x)] \subseteq [0,1]$
Main emphasis	Interval-valued uncertainty in membership	Uncertainty caused by incomplete, insufficient, or partially known information
Interpretation	The exact membership degree is not fixed, but lies somewhere in a known interval	The exact membership degree is not fully known and is described only within a grey range
Typical viewpoint	Often treated as an extension of fuzzy-set modeling	Often treated in the framework of grey-system theory and incomplete-information analysis
Structural similarity	Interval endpoints give lower and upper membership bounds	Lower and upper grey bounds also determine a membership range
Main difference in usage	Usually emphasizes interval-based representation itself	Usually emphasizes the epistemic meaning of partial knowledge or information deficiency
Common feature	Both use lower and upper bounds to describe non-crisp membership	Both use lower and upper bounds to describe non-crisp membership

10.2 MultiGrey Set

A MultiGrey set assigns each element several interval-valued membership degrees, representing multiple bounded uncertainty layers simultaneously within one unified grey framework.

Definition 10.2.1 (MultiGrey set)**.** Let U be a nonempty universe, and let $n \geq 1$ be an integer. A *MultiGrey set* G of dimension n on U is determined by two families of functions

$$\underline{\mu}_G^1, \dots, \underline{\mu}_G^n : U \to [0,1], \qquad \overline{\mu}_G^1, \dots, \overline{\mu}_G^n : U \to [0,1],$$

such that

$$\underline{\mu}_G^i(x) \leq \overline{\mu}_G^i(x) \qquad \text{for all } x \in U,\ i = 1, \dots, n.$$

Equivalently, for each $x \in U$, one assigns the n-tuple of interval-valued memberships

$$\mu_G(x) := \big([\underline{\mu}_G^1(x), \overline{\mu}_G^1(x)], \dots, [\underline{\mu}_G^n(x), \overline{\mu}_G^n(x)]\big).$$

Each interval

$$[\underline{\mu}_G^i(x), \overline{\mu}_G^i(x)]$$

is called the *i-th grey membership degree* of x.

Example 10.2.2 (A concrete example of a MultiGrey set)**.** Let

$$U = \{p_1, p_2, p_3\}$$

be a set of three products sold by a company, and let

$$n = 2.$$

Assume that the two dimensions correspond to the following two evaluation aspects:

- the first grey membership degree represents the interval-valued assessment of *customer satisfaction*;
- the second grey membership degree represents the interval-valued assessment of *market potential*.

Define a MultiGrey set

$$G$$

of dimension 2 on U by assigning to each product $p \in U$ a pair of grey membership intervals as follows:

$$\mu_G(p_1) = ([0.70, 0.85], [0.60, 0.80]),$$
$$\mu_G(p_2) = ([0.45, 0.65], [0.75, 0.90]),$$
$$\mu_G(p_3) = ([0.20, 0.40], [0.30, 0.50]).$$

Equivalently, the lower and upper membership functions are given by

$$\underline{\mu}_G^1(p_1) = 0.70, \quad \overline{\mu}_G^1(p_1) = 0.85, \qquad \underline{\mu}_G^2(p_1) = 0.60, \quad \overline{\mu}_G^2(p_1) = 0.80,$$
$$\underline{\mu}_G^1(p_2) = 0.45, \quad \overline{\mu}_G^1(p_2) = 0.65, \qquad \underline{\mu}_G^2(p_2) = 0.75, \quad \overline{\mu}_G^2(p_2) = 0.90,$$
$$\underline{\mu}_G^1(p_3) = 0.20, \quad \overline{\mu}_G^1(p_3) = 0.40, \qquad \underline{\mu}_G^2(p_3) = 0.30, \quad \overline{\mu}_G^2(p_3) = 0.50.$$

For each product and for each dimension, the lower bound does not exceed the upper bound. Hence,

$$\underline{\mu}_G^i(p_j) \le \overline{\mu}_G^i(p_j) \qquad \text{for all } i = 1, 2 \text{ and } j = 1, 2, 3.$$

Therefore, G is a MultiGrey set of dimension 2 on U.

In practical terms:

- p_1 has relatively high customer satisfaction and moderately high market potential;
- p_2 has medium customer satisfaction but very strong market potential;
- p_3 has low evaluations in both aspects.

Thus, this MultiGrey set models a real-life decision situation in which each product is evaluated by several interval-valued criteria rather than by only one single grey membership interval.

Theorem 10.2.3 (Well-definedness of MultiGrey sets). *Let U be a nonempty universe, let $n \geq 1$, and let*

$$\underline{\mu}_G^i, \overline{\mu}_G^i : U \to [0,1] \qquad (i = 1, \ldots, n)$$

satisfy

$$\underline{\mu}_G^i(x) \leq \overline{\mu}_G^i(x) \qquad \text{for all } x \in U,\ i = 1, \ldots, n.$$

Then, for every $x \in U$ and every $i = 1, \ldots, n$, the interval

$$[\underline{\mu}_G^i(x), \overline{\mu}_G^i(x)]$$

is a well-defined closed subinterval of $[0,1]$. Consequently,

$$\mu_G : U \to \big(\mathbb{I}([0,1])\big)^n, \qquad \mu_G(x) = \big([\underline{\mu}_G^1(x), \overline{\mu}_G^1(x)], \ldots, [\underline{\mu}_G^n(x), \overline{\mu}_G^n(x)]\big),$$

where

$$\mathbb{I}([0,1]) := \{[a,b] \subseteq [0,1] \mid 0 \leq a \leq b \leq 1\},$$

is a well-defined mapping. Hence G is a well-defined MultiGrey set on U.

Proof. Fix $x \in U$ and $i \in \{1, \ldots, n\}$. Since

$$\underline{\mu}_G^i, \overline{\mu}_G^i : U \to [0,1],$$

one has

$$\underline{\mu}_G^i(x) \in [0,1], \qquad \overline{\mu}_G^i(x) \in [0,1].$$

By assumption,

$$\underline{\mu}_G^i(x) \leq \overline{\mu}_G^i(x).$$

Therefore

$$[\underline{\mu}_G^i(x), \overline{\mu}_G^i(x)]$$

is a well-defined closed interval contained in $[0,1]$, that is,

$$[\underline{\mu}_G^i(x), \overline{\mu}_G^i(x)] \in \mathbb{I}([0,1]).$$

Since this holds for every $i = 1, \ldots, n$, the tuple

$$\mu_G(x) = \big([\underline{\mu}_G^1(x), \overline{\mu}_G^1(x)], \ldots, [\underline{\mu}_G^n(x), \overline{\mu}_G^n(x)]\big)$$

belongs to

$$\big(\mathbb{I}([0,1])\big)^n.$$

As $x \in U$ was arbitrary, the map

$$\mu_G : U \to \big(\mathbb{I}([0,1])\big)^n$$

is well-defined. Hence G is a well-defined MultiGrey set on U. □

10.3 Grey OffSet

Grey offset assigns each element an interval-valued membership within an extended range beyond $[0, 1]$, modeling uncertainty when degrees may fall below zero or exceed one.

Definition 10.3.1 (Grey offset)**.** Let X be a nonempty set, and let $\Psi, \Omega \in \mathbb{R}$ satisfy

$$\Psi < 0 < 1 < \Omega.$$

A *grey offset* on X is determined by two functions

$$\underline{\mu}_G, \overline{\mu}_G : X \to [\Psi, \Omega]$$

such that

$$\underline{\mu}_G(x) \leq \overline{\mu}_G(x) \qquad \text{for all } x \in X.$$

Equivalently, each element $x \in X$ is assigned the interval-valued offset-membership

$$\mu_G(x) := [\,\underline{\mu}_G(x), \overline{\mu}_G(x)\,] \subseteq [\Psi, \Omega].$$

The interval $\mu_G(x)$ is called the *grey offset degree* of x in G.

Theorem 10.3.2 (Well-definedness of grey offsets)**.** *Let X be a nonempty set, and let*

$$\underline{\mu}_G, \overline{\mu}_G : X \to [\Psi, \Omega]$$

satisfy

$$\underline{\mu}_G(x) \leq \overline{\mu}_G(x) \qquad \textit{for all } x \in X,$$

where $\Psi < 0 < 1 < \Omega$. Then the mapping

$$\mu_G : X \to \big\{[a, b] \subseteq [\Psi, \Omega] \mid \Psi \leq a \leq b \leq \Omega\big\}$$

defined by

$$\mu_G(x) := [\,\underline{\mu}_G(x), \overline{\mu}_G(x)\,]$$

is well-defined.

Proof. Fix $x \in X$. Since

$$\underline{\mu}_G(x), \overline{\mu}_G(x) \in [\Psi, \Omega]$$

and

$$\underline{\mu}_G(x) \leq \overline{\mu}_G(x),$$

the set

$$[\,\underline{\mu}_G(x), \overline{\mu}_G(x)\,]$$

is a nonempty closed interval contained in $[\Psi, \Omega]$. Hence

$$\mu_G(x) \in \big\{[a, b] \subseteq [\Psi, \Omega] \mid \Psi \leq a \leq b \leq \Omega\big\}.$$

Since $x \in X$ was arbitrary, the mapping μ_G is well-defined. □

10.4 Neutrosophic Grey Set

A neutrosophic grey set assigns each element interval-valued truth, indeterminacy, and falsity degrees, capturing bounded uncertainty in all three neutrosophic components simultaneously (cf. [292, 293]).

Definition 10.4.1 (Neutrosophic grey set)**.** Let X be a nonempty set. A *neutrosophic grey set* A on X is specified by six functions

$$\underline{T}_A, \overline{T}_A, \underline{I}_A, \overline{I}_A, \underline{F}_A, \overline{F}_A : X \to [0, 1]$$

such that, for all $x \in X$,

$$\underline{T}_A(x) \le \overline{T}_A(x), \qquad \underline{I}_A(x) \le \overline{I}_A(x), \qquad \underline{F}_A(x) \le \overline{F}_A(x).$$

Equivalently, each element $x \in X$ is assigned three interval-valued neutrosophic components

$$T_A(x) := [\,\underline{T}_A(x), \overline{T}_A(x)\,], \qquad I_A(x) := [\,\underline{I}_A(x), \overline{I}_A(x)\,], \qquad F_A(x) := [\,\underline{F}_A(x), \overline{F}_A(x)\,].$$

Here $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the grey truth, grey indeterminacy, and grey falsity intervals of x, respectively.

Theorem 10.4.2 (Well-definedness of neutrosophic grey sets)**.** *Let A be as in the above definition. Then:*

1. *for every $x \in X$, the intervals*

$$T_A(x), \quad I_A(x), \quad F_A(x)$$

are well-defined elements of

$$\big\{[a, b] \subseteq [0, 1] \mid 0 \le a \le b \le 1\big\};$$

2. *for every $x \in X$ and every choice*

$$t \in T_A(x), \qquad i \in I_A(x), \qquad f \in F_A(x),$$

one has

$$t, i, f \in [0, 1] \qquad \textit{and} \qquad 0 \le t + i + f \le 3.$$

Hence every representative triple selected from a neutrosophic grey set is an admissible neutrosophic triple.

Proof. Fix $x \in X$. Since

$$\underline{T}_A(x), \overline{T}_A(x) \in [0, 1] \quad \text{and} \quad \underline{T}_A(x) \le \overline{T}_A(x),$$

the interval

$$T_A(x) = [\,\underline{T}_A(x), \overline{T}_A(x)\,]$$

is a well-defined nonempty closed interval contained in $[0, 1]$. The same argument shows that

$$I_A(x) = [\,\underline{I}_A(x), \overline{I}_A(x)\,] \quad \text{and} \quad F_A(x) = [\,\underline{F}_A(x), \overline{F}_A(x)\,]$$

are also well-defined intervals in $[0,1]$. This proves (1).

Now let

$$t \in T_A(x), \qquad i \in I_A(x), \qquad f \in F_A(x).$$

Since each of the intervals $T_A(x), I_A(x), F_A(x)$ is contained in $[0,1]$, we have

$$t, i, f \in [0,1].$$

Therefore,

$$0 \le t + i + f \le 1 + 1 + 1 = 3.$$

Thus every representative triple is an admissible neutrosophic triple. This proves (2). □

10.5 Uncertain Grey Set

An uncertain grey set assigns each element a nonempty interval of admissible uncertainty degrees between lower and upper model-consistent bounds within a fixed uncertain framework.

Definition 10.5.1 (Componentwise order on $\mathrm{Dom}(M)$)**.** Let M be an uncertain model with

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \ge 1$. For

$$a = (a_1, \dots, a_k), \quad b = (b_1, \dots, b_k) \in \mathrm{Dom}(M),$$

define

$$a \preceq b \iff a_j \le b_j \quad \text{for all } j = 1, \dots, k.$$

Definition 10.5.2 (Uncertain grey set)**.** Let X be a nonempty set, and let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

An *uncertain grey set of type M on X* is a pair

$$\mathcal{G}_M = (X, \Gamma_M),$$

where

$$\Gamma_M : X \to \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}$$

is induced by two functions

$$\underline{\mu}_M, \overline{\mu}_M : X \to \mathrm{Dom}(M)$$

satisfying

$$\underline{\mu}_M(x) \preceq \overline{\mu}_M(x) \qquad \text{for all } x \in X,$$

and

$$\Gamma_M(x) := [\,\underline{\mu}_M(x), \overline{\mu}_M(x)\,]_{\mathrm{Dom}(M)} := \{\, d \in \mathrm{Dom}(M) \mid \underline{\mu}_M(x) \preceq d \preceq \overline{\mu}_M(x)\,\}.$$

The set $\Gamma_M(x)$ is called the *uncertain grey degree set* of x.

Theorem 10.5.3 (Well-definedness of uncertain grey sets)**.** *Let X be a nonempty set, let M be an uncertain model, and let*

$$\underline{\mu}_M, \overline{\mu}_M : X \to \mathrm{Dom}(M)$$

satisfy

$$\underline{\mu}_M(x) \preceq \overline{\mu}_M(x) \qquad \text{for all } x \in X.$$

Then the mapping

$$\Gamma_M : X \to \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}$$

defined by

$$\Gamma_M(x) = [\, \underline{\mu}_M(x), \overline{\mu}_M(x) \,]_{\mathrm{Dom}(M)}$$

is well-defined. Moreover, for every $x \in X$ and every

$$d \in \Gamma_M(x),$$

one has

$$d \in \mathrm{Dom}(M).$$

Hence every representative degree selected from an uncertain grey set is admissible for the uncertain model M.

Proof. Fix $x \in X$. Since

$$\underline{\mu}_M(x), \overline{\mu}_M(x) \in \mathrm{Dom}(M)$$

and

$$\underline{\mu}_M(x) \preceq \overline{\mu}_M(x),$$

it follows that

$$\underline{\mu}_M(x) \in [\, \underline{\mu}_M(x), \overline{\mu}_M(x) \,]_{\mathrm{Dom}(M)}.$$

Therefore

$$[\, \underline{\mu}_M(x), \overline{\mu}_M(x) \,]_{\mathrm{Dom}(M)} \neq \emptyset.$$

By definition, every element of this set belongs to $\mathrm{Dom}(M)$. Hence

$$\Gamma_M(x) \in \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}.$$

Since $x \in X$ was arbitrary, the mapping Γ_M is well-defined.

Now let

$$d \in \Gamma_M(x).$$

Again by definition of $\Gamma_M(x)$, one has

$$d \in \mathrm{Dom}(M).$$

Thus every representative degree selected from $\Gamma_M(x)$ is admissible for the model M. □

10.6 Grey Soft Set

A grey soft set assigns each parameter a grey set, using lower and upper membership functions to represent parameterized uncertainty over a universe [294].

Definition 10.6.1 (Grey powerset)**.** Let U be a nonempty universe. Define

$$\mathrm{GPow}(U) := \Big\{ (\underline{\mu}, \overline{\mu}) \in [0,1]^U \times [0,1]^U \;\Big|\; \underline{\mu}(x) \le \overline{\mu}(x) \text{ for all } x \in U \Big\}.$$

Each element of $\mathrm{GPow}(U)$ is a grey set on U.

Definition 10.6.2 (Grey soft set)**.** Let U be a nonempty universe, let E be a set of parameters, and let $A \subseteq E$. A pair

$$(F, A)$$

is called a *grey soft set* over U if

$$F : A \to \mathrm{GPow}(U).$$

Equivalently, for each parameter $\epsilon \in A$, there exist functions

$$\underline{\mu}_{\epsilon}, \overline{\mu}_{\epsilon} : U \to [0,1]$$

such that

$$F(\epsilon) = (\underline{\mu}_{\epsilon}, \overline{\mu}_{\epsilon})$$

and

$$\underline{\mu}_{\epsilon}(x) \le \overline{\mu}_{\epsilon}(x) \qquad \text{for all } x \in U.$$

Theorem 10.6.3 (Well-definedness of grey soft sets)**.** *Let U be a nonempty universe, let E be a set of parameters, let $A \subseteq E$, and let*

$$F : A \to \mathrm{GPow}(U).$$

Then (F, A) is a well-defined grey soft set over U. More precisely, for every $\epsilon \in A$, the value $F(\epsilon)$ is a well-defined grey set on U.

Proof. Fix $\epsilon \in A$. Since

$$F(\epsilon) \in \mathrm{GPow}(U),$$

there exist functions

$$\underline{\mu}_{\epsilon}, \overline{\mu}_{\epsilon} : U \to [0,1]$$

such that

$$F(\epsilon) = (\underline{\mu}_{\epsilon}, \overline{\mu}_{\epsilon})$$

and

$$\underline{\mu}_{\epsilon}(x) \le \overline{\mu}_{\epsilon}(x) \qquad \text{for all } x \in U.$$

Hence, by the definition of a grey set, $F(\epsilon)$ is a grey set on U.

Since $\epsilon \in A$ was arbitrary, every parameter is assigned a grey set on U. Therefore

$$F : A \to \mathrm{GPow}(U)$$

defines a well-defined grey soft set (F, A) over U. □

10.7 Grey Rough Set

A grey rough set represents rough approximations by interval-valued membership, assigning exact inclusion, boundary uncertainty, or exclusion through lower and upper approximation induced intervals [295–298].

Definition 10.7.1 (Grey rough set)**.** Let X be a nonempty set, let $R \subseteq X \times X$ be an equivalence relation, and let $A \subseteq X$. Define the lower and upper approximations of A by

$$\underline{A}_R := \{\, x \in X \mid [x]_R \subseteq A \,\}, \qquad \overline{A}_R := \{\, x \in X \mid [x]_R \cap A \neq \emptyset \,\},$$

where

$$[x]_R := \{\, y \in X \mid (x, y) \in R \,\}.$$

The *grey rough set* induced by A in the approximation space (X, R) is the grey set

$$\mathrm{GRSet}_R(A) = (X, \underline{\mu}_{A,R}, \overline{\mu}_{A,R}),$$

where

$$\underline{\mu}_{A,R} : X \to [0, 1], \qquad \overline{\mu}_{A,R} : X \to [0, 1]$$

are defined by

$$\underline{\mu}_{A,R}(x) := \begin{cases} 1, & x \in \underline{A}_R, \\ 0, & x \notin \underline{A}_R, \end{cases} \qquad \overline{\mu}_{A,R}(x) := \begin{cases} 1, & x \in \overline{A}_R, \\ 0, & x \notin \overline{A}_R. \end{cases}$$

Equivalently, the interval-valued membership of x is

$$\mu_{A,R}(x) = [\, \underline{\mu}_{A,R}(x), \overline{\mu}_{A,R}(x) \,].$$

Thus,

$$\mu_{A,R}(x) = \begin{cases} [1, 1], & x \in \underline{A}_R, \\ [0, 1], & x \in \overline{A}_R \setminus \underline{A}_R, \\ [0, 0], & x \in X \setminus \overline{A}_R. \end{cases}$$

Theorem 10.7.2 (Well-definedness of grey rough sets)**.** *Let X be a nonempty set, let R be an equivalence relation on X, and let $A \subseteq X$. Then the grey rough set*

$$\mathrm{GRSet}_R(A) = (X, \underline{\mu}_{A,R}, \overline{\mu}_{A,R})$$

is well-defined. More precisely:

1. *the functions*

   $$\underline{\mu}_{A,R}, \overline{\mu}_{A,R} : X \to [0, 1]$$

   are well-defined;

2. *for every $x \in X$,*

   $$\underline{\mu}_{A,R}(x) \le \overline{\mu}_{A,R}(x).$$

   Hence

   $$[\, \underline{\mu}_{A,R}(x), \overline{\mu}_{A,R}(x) \,]$$

   is a well-defined grey membership interval.

Proof. For every $x \in X$, each of the values

$$\underline{\mu}_{A,R}(x), \quad \overline{\mu}_{A,R}(x)$$

is either 0 or 1. Hence both functions are well-defined maps from X to $[0,1]$. This proves (1).

It remains to show that

$$\underline{\mu}_{A,R}(x) \le \overline{\mu}_{A,R}(x) \qquad \text{for all } x \in X.$$

Equivalently, it is enough to prove

$$\underline{A}_R \subseteq \overline{A}_R.$$

Let

$$x \in \underline{A}_R.$$

Then

$$[x]_R \subseteq A.$$

Since $x \in [x]_R$, it follows that

$$[x]_R \cap A \neq \emptyset.$$

Thus

$$x \in \overline{A}_R.$$

Therefore

$$\underline{A}_R \subseteq \overline{A}_R.$$

Consequently, if $\underline{\mu}_{A,R}(x) = 1$, then $\overline{\mu}_{A,R}(x) = 1$ as well, and hence

$$\underline{\mu}_{A,R}(x) \le \overline{\mu}_{A,R}(x)$$

for all $x \in X$.

Thus each interval

$$[\,\underline{\mu}_{A,R}(x), \overline{\mu}_{A,R}(x)\,]$$

is a well-defined grey membership interval, and the grey rough set is well-defined. □

Chapter 11

Flou Set Theory

In this chapter, we discuss Flou set theory.

11.1 Flou set

A flou set $A = (E, F)$ distinguishes three regions: elements certainly in A (those in E), elements possibly in A (those in $F \setminus E$), and elements not in A (those in $U \setminus F$) [299–302]. The flou set is a concept that was defined long ago, and it can be regarded as a framework capable of generalizing concepts such as fuzzy sets.

Definition 11.1.1 (Flou set)**.** [300] Let U be a nonempty universe. A *flou set* on U is a pair

$$A = (E, F)$$

of subsets of U such that

$$E \subseteq F \subseteq U.$$

The set E is called the *certain zone* of A, the set F is called the *maximal zone* of A, and the difference

$$F \setminus E$$

is called the *flou zone*.

11.2 MultiFlou set

A MultiFlou set is a nested sequence of subsets whose successive differences form disjoint flou layers, representing multiple levels of certainty and ambiguity.

Definition 11.2.1 (n-flou set)**.** Let U be a nonempty universe, and let $n \geq 2$ be an integer. An *n-flou set* on U is an n-tuple

$$A = (F_1, F_2, \ldots, F_n)$$

of subsets of U such that

$$F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq U.$$

The set F_1 is called the *certain core* of A, and F_n is called the *maximal zone* of A.

For $j = 2, \ldots, n$, the set

$$L_j := F_j \setminus F_{j-1}$$

is called the *j-th flou layer* of A, and we also put

$$L_1 := F_1.$$

A *MultiFlou set* is an n-flou set for some integer $n \geq 2$.

Example 11.2.2 (A concrete example of an n-flou set)**.** Let

$$U = \{r_1, r_2, r_3, r_4, r_5\}$$

be a set of five applicants for a research scholarship, and let

$$n = 3.$$

Suppose that a committee classifies applicants according to the degree to which they are regarded as suitable for receiving the scholarship. Because the assessment is not completely sharp, the committee uses three nested levels:

- applicants who are definitely recommended;
- applicants who are reasonably acceptable;
- applicants who are still considered possible candidates.

Define

$$F_1 = \{r_1, r_2\}, \qquad F_2 = \{r_1, r_2, r_3\}, \qquad F_3 = \{r_1, r_2, r_3, r_4\}.$$

Then

$$F_1 \subseteq F_2 \subseteq F_3 \subseteq U.$$

Hence,

$$A = (F_1, F_2, F_3)$$

is a 3-flou set on U.

Here:

- $F_1 = \{r_1, r_2\}$ is the certain core, consisting of applicants who are definitely recommended;
- $F_3 = \{r_1, r_2, r_3, r_4\}$ is the maximal zone, consisting of all applicants who are still considered possible scholarship recipients.

The flou layers are:

$$L_1 = F_1 = \{r_1, r_2\},$$

$$L_2 = F_2 \setminus F_1 = \{r_3\},$$

$$L_3 = F_3 \setminus F_2 = \{r_4\}.$$

Thus:

- r_1 and r_2 belong to the most certain level of acceptance;
- r_3 belongs to the second flou layer, meaning that the committee finds this applicant acceptable but not as strongly as those in the core;
- r_4 belongs to the third flou layer, meaning that this applicant is only marginally retained as a possible candidate;
- r_5 does not belong to the maximal zone, so this applicant is excluded from consideration.

Therefore, $A = (F_1, F_2, F_3)$ is a concrete real-life example of an n-flou set, where nested subsets represent successive levels of certainty in a scholarship selection process.

Theorem 11.2.3 (Well-definedness of MultiFlou sets)**.** *Let*

$$A = (F_1, F_2, \dots, F_n)$$

be an n-flou set on a nonempty universe U. Then the family of layers

$$L_1 := F_1, \qquad L_j := F_j \setminus F_{j-1} \quad (j = 2, \dots, n)$$

is well-defined and satisfies:

1.

$$L_j \subseteq U \qquad \text{for all } j = 1, \dots, n;$$

2. the sets $L_1, \dots, L_n$ are pairwise disjoint;

3.

$$F_j = \bigcup_{i=1}^{j} L_i \qquad \text{for all } j = 1, \dots, n.$$

In particular,

$$F_n = \bigcup_{i=1}^{n} L_i.$$

Hence every MultiFlou set admits a unique layered decomposition of its maximal zone.

Proof. For each $j = 1, \dots, n$, since $F_j \subseteq U$, one has $L_j \subseteq U$. Thus each layer is a well-defined subset of U.

Next, let $1 \leq i < j \leq n$. If $x \in L_i$, then $x \in F_i \subseteq F_{j-1}$ because

$$F_i \subseteq F_{j-1}.$$

On the other hand, if $x \in L_j$, then

$$x \in F_j \setminus F_{j-1},$$

so $x \notin F_{j-1}$. Hence no element can belong to both L_i and L_j, and therefore

$$L_i \cap L_j = \emptyset.$$

Thus the layers are pairwise disjoint.

Finally, for $j = 1$,

$$F_1 = L_1.$$

Assume now that $j \geq 2$. Since

$$L_j = F_j \setminus F_{j-1},$$

we have the disjoint union

$$F_j = F_{j-1} \cup L_j.$$

Applying this identity recursively yields

$$F_j = L_1 \cup L_2 \cup \cdots \cup L_j = \bigcup_{i=1}^{j} L_i.$$

This proves all assertions. □

11.3 Flou Powerset

A flou powerset is the collection of all ordered pairs (E,F) of subsets with $E \subseteq F$, representing every possible flou set.

Definition 11.3.1 (Flou powerset)**.** Let U be a nonempty universe. The *flou powerset* of U, denoted by

$$\mathrm{FlPow}(U),$$

is defined by

$$\mathrm{FlPow}(U) := \big\{(E, F) \in \mathcal{P}(U) \times \mathcal{P}(U) \mid E \subseteq F\big\}.$$

Equivalently, $\mathrm{FlPow}(U)$ is the set of all flou sets on U.

Theorem 11.3.2 (Well-definedness of the flou powerset)**.** *Let U be a nonempty universe. Then* $\mathrm{FlPow}(U)$ *is well-defined. More precisely:*

1. *every element*

$$(E, F) \in \mathrm{FlPow}(U)$$

   determines a flou set on U;

2. *every flou set on U belongs to* $\mathrm{FlPow}(U)$.

Proof. Let

$$(E, F) \in \mathrm{FlPow}(U).$$

By definition,

$$E, F \subseteq U \qquad \text{and} \qquad E \subseteq F.$$

Hence (E, F) satisfies exactly the defining conditions of a flou set on U.

Conversely, let (E, F) be a flou set on U. Then, by definition,

$$E \subseteq F \subseteq U.$$

Therefore

$$(E, F) \in \mathrm{FlPow}(U).$$

Thus $\mathrm{FlPow}(U)$ is well-defined. □

11.4 Flou Soft Set

A flou soft set assigns to each parameter a flou set, enabling parameterized representation of certain zones, maximal zones, and intermediate ambiguity.

Definition 11.4.1 (Flou soft set)**.** Let U be a nonempty universe, let E be a set of parameters, and let

$$A \subseteq E.$$

A pair

$$(F, A)$$

is called a *flou soft set* over U if

$$F : A \to \mathrm{FlPow}(U).$$

Equivalently, for each parameter $\varepsilon \in A$, there exist subsets

$$C_\varepsilon, M_\varepsilon \subseteq U$$

such that

$$C_\varepsilon \subseteq M_\varepsilon \qquad \text{and} \qquad F(\varepsilon) = (C_\varepsilon, M_\varepsilon).$$

Here C_ε is the certain zone and M_ε is the maximal zone associated with the parameter ε.

Theorem 11.4.2 (Well-definedness of flou soft sets)**.** *Let U be a nonempty universe, let E be a set of parameters, let $A \subseteq E$, and let*

$$F : A \to \mathrm{FlPow}(U).$$

Then (F, A) is a well-defined flou soft set over U. More precisely, for every $\varepsilon \in A$, the value $F(\varepsilon)$ is a flou set on U.

Proof. Fix $\varepsilon \in A$. Since
$$F(\varepsilon) \in \mathrm{FlPow}(U),$$
there exist subsets
$$C_\varepsilon, M_\varepsilon \subseteq U$$
such that
$$F(\varepsilon) = (C_\varepsilon, M_\varepsilon) \qquad \text{and} \qquad C_\varepsilon \subseteq M_\varepsilon.$$
Hence $F(\varepsilon)$ satisfies the defining conditions of a flou set on U.

Since $\varepsilon \in A$ was arbitrary, every parameter in A is assigned a flou set on U. Therefore the pair (F, A) is a well-defined flou soft set over U. □

11.5 Flou Rough Set

A flou rough set is the pair of Pawlak lower and upper approximations, viewing the lower region as certain and the boundary as flou.

Definition 11.5.1 (Flou rough set)**.** Let U be a nonempty universe, let
$$R \subseteq U \times U$$
be an equivalence relation, and let $X \subseteq U$. Define the Pawlak lower and upper approximations of X by
$$\underline{X}_R := \{\, x \in U \mid [x]_R \subseteq X \,\}, \qquad \overline{X}_R := \{\, x \in U \mid [x]_R \cap X \neq \emptyset \,\},$$
where
$$[x]_R := \{\, y \in U \mid (x, y) \in R \,\}.$$
The pair
$$\mathrm{FRS}_R(X) := (\underline{X}_R, \overline{X}_R)$$
is called the *flou rough set* of X with respect to the approximation space (U, R).

Thus, the certain zone of $\mathrm{FRS}_R(X)$ is $\underline{X}_R$, the maximal zone is $\overline{X}_R$, and the flou zone is
$$\overline{X}_R \setminus \underline{X}_R.$$

Theorem 11.5.2 (Well-definedness of flou rough sets)**.** *Let U be a nonempty universe, let R be an equivalence relation on U, and let $X \subseteq U$. Then*
$$\mathrm{FRS}_R(X) = (\underline{X}_R, \overline{X}_R)$$
is a well-defined flou set on U. Equivalently,
$$\underline{X}_R \subseteq \overline{X}_R \subseteq U.$$

Proof. By definition of lower and upper approximation, both

$$\underline{X}_R = \{\, x \in U \mid [x]_R \subseteq X \,\} \quad \text{and} \quad \overline{X}_R = \{\, x \in U \mid [x]_R \cap X \neq \emptyset \,\}$$

are subsets of U. Hence

$$\underline{X}_R \subseteq U \qquad \text{and} \qquad \overline{X}_R \subseteq U.$$

It remains to prove that

$$\underline{X}_R \subseteq \overline{X}_R.$$

Let

$$x \in \underline{X}_R.$$

Then

$$[x]_R \subseteq X.$$

Since R is an equivalence relation, one has

$$x \in [x]_R.$$

Therefore

$$[x]_R \cap X \neq \emptyset,$$

and hence

$$x \in \overline{X}_R.$$

Thus

$$\underline{X}_R \subseteq \overline{X}_R.$$

Consequently,

$$\underline{X}_R \subseteq \overline{X}_R \subseteq U,$$

so the pair

$$(\underline{X}_R, \overline{X}_R)$$

is a well-defined flou set on U. □

Chapter 12

Controlled Set Theory

In this chapter, we introduce controlled set theory.

12.1 Controlled Set

A controlled set is a graded set in which every membership value has a complementary counterpart, ensuring balance through the simultaneous presence of a and 1—a [303–306]. A controlled set, arising in recent algebraic set-theoretic research, requires every membership degree to be balanced by another element realizing its complementary degree within itself.

Definition 12.1.1 (α-set)**.** Let E be a nonempty set, and let

$$\alpha : E \to [0, 1]$$

be a mapping. The pair (E, α) is called an *α-set.*

Definition 12.1.2 (Controlled set)**.** [303,304] Let (E, α) be an α-set. Then (E, α) is called a *controlled set* (or an *α-controlled set*) if

$$\forall x \in E,\ \exists y \in E \text{ such that } \alpha(y) = 1 - \alpha(x).$$

Equivalently,

$$1 - \alpha(E) \subseteq \alpha(E),$$

where

$$\alpha(E) := \{\alpha(x) \mid x \in E\}.$$

Thus, every membership degree occurring in E has its complementary degree also realized by some element of E.

Definition 12.1.3 (Control set of an element)**.** Let (E, α) be a controlled set and let $a \in E$. The *control set* of a is

$$\mathrm{Ctrl}(a) := \{\, b \in E \mid \alpha(b) = 1 - \alpha(a) \,\}.$$

Example 12.1.4 (A concrete example of a controlled set)**.** Let

$$E = \{s_1, s_2, s_3, s_4\}$$

be a set of four devices in a laboratory, and let

$$\alpha : E \to [0, 1]$$

represent the degree to which each device is currently regarded as *reliable for immediate use*.

Assume that

$$\alpha(s_1) = 0.2, \qquad \alpha(s_2) = 0.8, \qquad \alpha(s_3) = 0.4, \qquad \alpha(s_4) = 0.6.$$

Then

$$\alpha(E) = \{0.2, 0.8, 0.4, 0.6\}.$$

Now observe that

$$1 - 0.2 = 0.8, \qquad 1 - 0.8 = 0.2, \qquad 1 - 0.4 = 0.6, \qquad 1 - 0.6 = 0.4.$$

Hence, for every membership degree occurring in E, its complementary degree also occurs in E. Therefore,

$$1 - \alpha(E) = \{0.8, 0.2, 0.6, 0.4\} \subseteq \alpha(E),$$

so (E, α) is a controlled set.

In practical terms, this means that each reliability level appearing in the system is balanced by another device whose reliability degree is exactly complementary to it. For example, a device with reliability degree 0.2 is matched by another device with reliability degree 0.8.

Moreover, the control sets of the elements are:

$$\mathrm{Ctrl}(s_1) = \{s_2\}, \qquad \mathrm{Ctrl}(s_2) = \{s_1\},$$

$$\mathrm{Ctrl}(s_3) = \{s_4\}, \qquad \mathrm{Ctrl}(s_4) = \{s_3\}.$$

Thus, (E, α) is a concrete real-life example of a controlled set, and each control set identifies the devices whose reliability degrees are complementary to one another.

12.2 Neutrosophic Controlled Set

A neutrosophic controlled set assigns every element a control partner whose truth, indeterminacy, and falsity degrees are exactly the componentwise complements of its degrees.

Definition 12.2.1 (Neutrosophic control complement)**.** Let

$$\mathcal{D}_N := \{(t,i,f) \in [0,1]^3 \mid 0 \le t+i+f \le 3\}.$$

Define the mapping

$$c_N : \mathcal{D}_N \to [0,1]^3$$

by

$$c_N(t,i,f) := (1-t,\ 1-i,\ 1-f).$$

The mapping c_N is called the *neutrosophic control complement*.

Definition 12.2.2 (Neutrosophic controlled set)**.** Let X be a nonempty set, and let

$$A = (T_A, I_A, F_A)$$

be a single-valued neutrosophic set on X, where

$$T_A, I_A, F_A : X \to [0,1]$$

satisfy

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3 \qquad \text{for all } x \in X.$$

Then A is called a *neutrosophic controlled set* if

$$\forall x \in X,\ \exists y \in X \quad \text{such that} \quad (T_A(y), I_A(y), F_A(y)) = c_N\big(T_A(x), I_A(x), F_A(x)\big).$$

Equivalently,

$$\forall x \in X,\ \exists y \in X \quad \text{such that} \quad \begin{cases} T_A(y) = 1 - T_A(x), \\ I_A(y) = 1 - I_A(x), \\ F_A(y) = 1 - F_A(x). \end{cases}$$

Definition 12.2.3 (Control set of an element in a neutrosophic controlled set)**.** Let $A = (T_A, I_A, F_A)$ be a neutrosophic controlled set on X, and let $x \in X$. The *control set* of x is defined by

$$\mathrm{Ctrl}_A(x) := \{y \in X \mid (T_A(y), I_A(y), F_A(y)) = (1 - T_A(x),\ 1 - I_A(x),\ 1 - F_A(x))\}.$$

Theorem 12.2.4 (Well-definedness of neutrosophic controlled sets)**.** *Let X be a nonempty set, and let*

$$A = (T_A, I_A, F_A)$$

be a single-valued neutrosophic set on X. Then:

1. *the neutrosophic control complement*

   $$c_N : \mathcal{D}_N \to \mathcal{D}_N$$

   is well-defined;

2. *if A satisfies the defining condition of a neutrosophic controlled set, then for every $x \in X$,*

   $$\mathrm{Ctrl}_A(x) \neq \emptyset.$$

   Hence the notion of a neutrosophic controlled set is well-defined.

Proof. Let

$$(t, i, f) \in \mathcal{D}_N.$$

Then

$$t, i, f \in [0, 1] \qquad \text{and} \qquad 0 \le t + i + f \le 3.$$

Hence

$$1 - t,\ 1 - i,\ 1 - f \in [0, 1].$$

Also,

$$(1 - t) + (1 - i) + (1 - f) = 3 - (t + i + f),$$

so

$$0 \le 3 - (t + i + f) \le 3.$$

Therefore

$$c_N(t, i, f) = (1 - t, 1 - i, 1 - f) \in \mathcal{D}_N.$$

Thus $c_N : \mathcal{D}_N \to \mathcal{D}_N$ is well-defined. This proves (1).

Now assume that A is a neutrosophic controlled set, and let $x \in X$. By definition, there exists $y \in X$ such that

$$(T_A(y), I_A(y), F_A(y)) = (1 - T_A(x),\ 1 - I_A(x),\ 1 - F_A(x)).$$

Hence

$$y \in \mathrm{Ctrl}_A(x),$$

so

$$\mathrm{Ctrl}_A(x) \neq \emptyset.$$

This proves (2). Therefore the notion is well-defined. □

12.3 Uncertain Controlled Set

An uncertain controlled set assigns each element a partner whose uncertainty degree is obtained through a fixed involutive control map on the model domain.

Definition 12.3.1 (Controlled uncertain model)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \ge 1$. A mapping

$$C_M : \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

is called a *control map* on M if

$$C_M(C_M(d)) = d \qquad \text{for all } d \in \mathrm{Dom}(M).$$

The pair (M, C_M) is called a *controlled uncertain model.*

Definition 12.3.2 (Uncertain controlled set)**.** Let X be a nonempty set, and let (M, C_M) be a controlled uncertain model. An uncertain set

$$\mathcal{U} = (X, \mu_M), \qquad \mu_M : X \to \mathrm{Dom}(M),$$

is called an *uncertain controlled set of type* (M, C_M) if

$$\forall x \in X,\ \exists y \in X \quad \text{such that} \quad \mu_M(y) = C_M(\mu_M(x)).$$

Definition 12.3.3 (Control set of an element in an uncertain controlled set)**.** Let $\mathcal{U} = (X, \mu_M)$ be an uncertain controlled set of type (M, C_M), and let $x \in X$. The *control set* of x is

$$\mathrm{Ctrl}_{\mathcal{U}}(x) := \{\, y \in X \mid \mu_M(y) = C_M(\mu_M(x)) \,\}.$$

Theorem 12.3.4 (Well-definedness of uncertain controlled sets)**.** *Let X be a nonempty set, let (M, C_M) be a controlled uncertain model, and let*

$$\mathcal{U} = (X, \mu_M)$$

be an uncertain set of type M. Then:

1. *for every $x \in X$,*
$$C_M(\mu_M(x)) \in \mathrm{Dom}(M);$$
2. *if $\mathcal{U}$ satisfies the defining condition of an uncertain controlled set, then for every $x \in X$,*
$$\mathrm{Ctrl}_{\mathcal{U}}(x) \neq \emptyset.$$

   Hence the notion of an uncertain controlled set is well-defined.

Proof. Let $x \in X$. Since

$$\mu_M : X \to \mathrm{Dom}(M),$$

we have

$$\mu_M(x) \in \mathrm{Dom}(M).$$

Because

$$C_M : \mathrm{Dom}(M) \to \mathrm{Dom}(M),$$

it follows that

$$C_M(\mu_M(x)) \in \mathrm{Dom}(M).$$

This proves (1).

Now assume that $\mathcal{U}$ is an uncertain controlled set. By definition, for each $x \in X$ there exists $y \in X$ such that

$$\mu_M(y) = C_M(\mu_M(x)).$$

Hence

$$y \in \mathrm{Ctrl}_{\mathcal{U}}(x),$$

and therefore

$$\mathrm{Ctrl}_{\mathcal{U}}(x) \neq \emptyset.$$

This proves (2). Thus the notion is well-defined. □

Remark 12.3.5. If

$$\mathrm{Dom}(M) = [0, 1] \quad \text{and} \quad C_M(a) = 1 - a,$$

then an uncertain controlled set reduces to an ordinary controlled set.

12.4 Controlled Soft Set

A controlled soft set assigns each parameter a fuzzy-valued set where every membership degree has a complementary counterpart, ensuring parameterwise balance across the universe.

Definition 12.4.1 (α-soft set)**.** Let U be a nonempty universe, let E be a set of parameters, and let $A \subseteq E$. A pair

$$(F, A)$$

is called an *α-soft set* over U if

$$F : A \to [0, 1]^U.$$

Equivalently, for each parameter $\varepsilon \in A$, the value $F(\varepsilon)$ is a mapping

$$F(\varepsilon) : U \to [0, 1].$$

Definition 12.4.2 (Controlled soft set)**.** Let (F, A) be an α-soft set over U. Then (F, A) is called a *controlled soft set* if, for every parameter $\varepsilon \in A$, the α-set

$$(U, F(\varepsilon))$$

is a controlled set; that is,

$$\forall \varepsilon \in A,\ \forall x \in U,\ \exists y \in U \quad \text{such that} \quad F(\varepsilon)(y) = 1 - F(\varepsilon)(x).$$

Definition 12.4.3 (Parameterized control set)**.** Let (F, A) be a controlled soft set, let $\varepsilon \in A$, and let $x \in U$. The *control set of x with respect to the parameter ε* is

$$\mathrm{Ctrl}_{F,\varepsilon}(x) := \{\, y \in U \mid F(\varepsilon)(y) = 1 - F(\varepsilon)(x) \,\}.$$

Theorem 12.4.4 (Well-definedness of controlled soft sets)**.** *Let U be a nonempty universe, let E be a set of parameters, let $A \subseteq E$, and let*

$$(F, A)$$

be an α-soft set over U. Then:

1. *for every $\varepsilon \in A$, the pair*

   $$(U, F(\varepsilon))$$

   is an α-set;

2. *if (F, A) satisfies the defining condition of a controlled soft set, then for every $\varepsilon \in A$ and every $x \in U$,*

   $$\mathrm{Ctrl}_{F,\varepsilon}(x) \neq \emptyset.$$

   Hence the notion of a controlled soft set is well-defined.

Proof. Let $\varepsilon \in A$. Since

$$F : A \to [0, 1]^U,$$

the value $F(\varepsilon)$ is a function

$$F(\varepsilon) : U \to [0, 1].$$

Therefore

$$(U, F(\varepsilon))$$

is an α-set. This proves (1).

Now assume that (F, A) is a controlled soft set. Let $\varepsilon \in A$ and $x \in U$. By definition, there exists $y \in U$ such that

$$F(\varepsilon)(y) = 1 - F(\varepsilon)(x).$$

Hence

$$y \in \mathrm{Ctrl}_{F,\varepsilon}(x),$$

so

$$\mathrm{Ctrl}_{F,\varepsilon}(x) \neq \emptyset.$$

This proves (2). Therefore the notion is well-defined. □

12.5 Controlled Rough Set

A controlled rough set uses rough membership values such that each element has another whose rough membership degree is exactly the complement of its own.

Definition 12.5.1 (Rough membership function)**.** Let X be a finite nonempty set, let $R \subseteq X \times X$ be an equivalence relation, and let $A \subseteq X$. For each $x \in X$, define

$$[x]_R := \{\, y \in X \mid (x, y) \in R \,\}.$$

The *rough membership function* of A with respect to R is the mapping

$$\rho_A^R : X \to [0, 1]$$

given by

$$\rho_A^R(x) := \frac{|[x]_R \cap A|}{|[x]_R|}.$$

Definition 12.5.2 (Controlled rough set)**.** Let X be a finite nonempty set, let R be an equivalence relation on X, and let $A \subseteq X$. The pair

$$(X, \rho_A^R)$$

is called a *controlled rough set* if

$$\forall x \in X,\ \exists y \in X \quad \text{such that} \quad \rho_A^R(y) = 1 - \rho_A^R(x).$$

Definition 12.5.3 (Control set in a controlled rough set)**.** Let (X, ρ_A^R) be a controlled rough set, and let $x \in X$. The *control set* of x is

$$\mathrm{Ctrl}_{A,R}(x) := \{\, y \in X \mid \rho_A^R(y) = 1 - \rho_A^R(x) \,\}.$$

Theorem 12.5.4 (Well-definedness of controlled rough sets)**.** *Let X be a finite nonempty set, let R be an equivalence relation on X, and let $A \subseteq X$. Then:*

1. *the rough membership function*
$$\rho_A^R : X \to [0,1]$$
is well-defined;

2. *if* (X, ρ_A^R) *satisfies the defining condition of a controlled rough set, then for every* $x \in X$*,*
$$\mathrm{Ctrl}_{A,R}(x) \neq \emptyset.$$
Hence the notion of a controlled rough set is well-defined.

Proof. Let $x \in X$. Since R is an equivalence relation, one has
$$x \in [x]_R,$$
and hence
$$[x]_R \neq \emptyset.$$
Because X is finite, the set $[x]_R$ is finite, so
$$|[x]_R| \in \mathbb{N} \quad \text{and} \quad |[x]_R| > 0.$$
Also,
$$[x]_R \cap A \subseteq [x]_R,$$
so
$$0 \leq |[x]_R \cap A| \leq |[x]_R|.$$
Dividing by the positive number $|[x]_R|$, we obtain
$$0 \leq \rho_A^R(x) = \frac{|[x]_R \cap A|}{|[x]_R|} \leq 1.$$
Thus
$$\rho_A^R(x) \in [0,1].$$
Since $x \in X$ was arbitrary, the map
$$\rho_A^R : X \to [0,1]$$
is well-defined. This proves (1).

Now assume that (X, ρ_A^R) is a controlled rough set. Let $x \in X$. By definition, there exists $y \in X$ such that
$$\rho_A^R(y) = 1 - \rho_A^R(x).$$
Hence
$$y \in \mathrm{Ctrl}_{A,R}(x),$$
and therefore
$$\mathrm{Ctrl}_{A,R}(x) \neq \emptyset.$$
This proves (2). Therefore the notion is well-defined. □

Chapter 13

Weighted Set Theory

In this chapter, we introduce weighted set theory.

13.1 Weighted Set

A weighted set assigns each element of a set a real-valued weight, representing quantitative importance, cost, strength, or relevance together with ordinary set membership.

Definition 13.1.1 (Weighted set)**.** Let S be a nonempty set. A *weighted set* is an ordered pair

$$(S, w),$$

where

$$w : S \to \mathbb{R}$$

is a function assigning to each element $x \in S$ a real number $w(x)$, called the *weight* of x.

Equivalently, a weighted set may be represented as

$$\{(x, w(x)) \mid x \in S\}.$$

Concepts in which the idea of a weighted set is frequently applied include weighted graphs [307, 308] and weighted hypergraphs [309, 310].

13.2 Weighted Multiset

A weighted multiset records both multiplicity and weight for each element, representing repeated occurrences together with quantitative importance or cost.

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Definition 13.2.1 (Weighted multiset)**.** Let X be a nonempty set, and let

$$\mathbb{N}_0 := \{0, 1, 2, \dots\}.$$

A *weighted multiset* on X is a triple

$$A = (X, m_A, w_A),$$

where

$$m_A : X \to \mathbb{N}_0$$

is the *multiplicity function* and

$$w_A : X \to \mathbb{R}$$

is the *weight function*.

For each $x \in X$, the value

$$m_A(x)$$

represents the number of occurrences of x in A, while

$$w_A(x)$$

represents the weight assigned to x.

The *support* of A is defined by

$$\operatorname{supp}(A) := \{\, x \in X \mid m_A(x) > 0 \,\}.$$

Equivalently, the weighted multiset A may be represented as

$$A = \{(x, m_A(x), w_A(x)) \mid x \in X,\ m_A(x) > 0\}.$$

13.3 Weighted Powerset

A weighted powerset is the collection of all subsets of a set together with real-valued weight functions assigned to their elements.

Definition 13.3.1 (Weighted powerset)**.** Let X be a set. The *weighted powerset* of X, denoted by

$$\mathrm{WPow}(X),$$

is defined by

$$\mathrm{WPow}(X) := \{\, (A, w) \mid A \subseteq X,\ w : A \to \mathbb{R} \,\}.$$

Equivalently, $\mathrm{WPow}(X)$ is the set of all weighted subsets of X, where each subset

$$A \subseteq X$$

is equipped with a real-valued weight function on A.

Theorem 13.3.2 (Well-definedness of the weighted powerset)**.** *Let X be a set. Then $\mathrm{WPow}(X)$ is well-defined. More precisely:*

1. *every element*
$$(A, w) \in \mathrm{WPow}(X)$$
consists of a subset
$$A \subseteq X$$
and a well-defined function
$$w : A \to \mathbb{R};$$

2. *conversely, every pair*
$$(A, w)$$
with
$$A \subseteq X \qquad \textit{and} \qquad w : A \to \mathbb{R}$$
belongs to WPow(X).

Hence WPow(X) *is exactly the collection of all weighted subsets of* X*.*

Proof. Since X is a set, its powerset
$$\mathcal{P}(X)$$
is well-defined. For each
$$A \in \mathcal{P}(X),$$
the collection
$$\mathbb{R}^A$$
of all functions from A to $\mathbb{R}$ is well-defined. Therefore the class
$$\{\, (A, w) \mid A \subseteq X,\ w : A \to \mathbb{R} \,\}$$
is well-defined.

Now let
$$(A, w) \in \mathrm{WPow}(X).$$
By definition,
$$A \subseteq X \qquad \text{and} \qquad w : A \to \mathbb{R}.$$
Thus (A, w) is a weighted subset of X. This proves (1).

Conversely, let
$$A \subseteq X$$
and let
$$w : A \to \mathbb{R}$$
be a function. Then, by the definition of WPow(X),
$$(A, w) \in \mathrm{WPow}(X).$$
This proves (2).

Therefore WPow(X) is well-defined and consists exactly of all weighted subsets of X. □

13.4 Complex Weighted Set

A complex weighted set assigns each element of a set a complex number, representing quantitative magnitude together with phase-sensitive or oscillatory information.

Definition 13.4.1 (Complex weighted set)**.** Let S be a nonempty set. A *complex weighted set* on S is an ordered pair

$$(S, w),$$

where

$$w : S \to \mathbb{C}$$

is a function assigning to each element $x \in S$ a complex number $w(x)$, called the *complex weight* of x.

Equivalently, a complex weighted set may be represented as

$$\{(x, w(x)) \mid x \in S\}.$$

Theorem 13.4.2 (Well-definedness of complex weighted sets)**.** *Let S be a nonempty set, and let*

$$w : S \to \mathbb{C}$$

be a function. Then the pair

$$(S, w)$$

is a well-defined complex weighted set on S.

Conversely, every complex weighted set on S arises from such a function

$$w : S \to \mathbb{C}.$$

Proof. Since S is a set and

$$w : S \to \mathbb{C}$$

is a function, for every

$$x \in S$$

the value

$$w(x) \in \mathbb{C}$$

is uniquely determined. Hence the assignment of a complex weight to each element of S is well-defined, and therefore

$$(S, w)$$

is a well-defined complex weighted set.

Conversely, by definition, a complex weighted set on S is precisely an ordered pair

$$(S, w)$$

with

$$w : S \to \mathbb{C}.$$

Thus every complex weighted set on S arises in this way. □

Remark 13.4.3. If

$$w(S) \subseteq \mathbb{R},$$

then a complex weighted set reduces to an ordinary weighted set.

13.5 Weighted Fuzzy Set

A weighted fuzzy set assigns each element both a membership degree and a weight, combining graded belonging with relative importance in one representation [311, 312].

Definition 13.5.1 (Weighted fuzzy set)**.** [311, 312] Let X be a nonempty set. A *weighted fuzzy set* A on X is an ordered pair

$$A = (\mu_A, w_A),$$

where

$$\mu_A : X \to [0, 1]$$

is the *membership function* of A, and

$$w_A : X \to [0, 1]$$

is the *weight function* of A.

Equivalently, A may be represented as

$$A = \{\langle x, \mu_A(x), w_A(x)\rangle \mid x \in X\}.$$

Here $\mu_A(x)$ expresses the degree to which x belongs to A, while $w_A(x)$ expresses the relative importance or impact of x in the weighted fuzzy set.

Example 13.5.2 (A concrete example of a weighted fuzzy set)**.** Let

$$X = \{e_1, e_2, e_3, e_4\}$$

be a set of four employees in a company.

Suppose that a manager wants to describe the fuzzy set of employees who are *suitable for assignment to an important project.* For each employee $e \in X$:

- $\mu_A(e)$ represents the degree to which e is considered suitable for the project;
- $w_A(e)$ represents the importance or impact of e in the project, for example based on experience, responsibility, or expected contribution.

Define a weighted fuzzy set

$$A = (\mu_A, w_A)$$

on X by

$$\begin{aligned}
\mu_A(e_1) &= 0.90, & w_A(e_1) &= 0.95,\\
\mu_A(e_2) &= 0.75, & w_A(e_2) &= 0.80,\\
\mu_A(e_3) &= 0.60, & w_A(e_3) &= 0.50,\\
\mu_A(e_4) &= 0.40, & w_A(e_4) &= 0.30.
\end{aligned}$$

Equivalently,

$$A = \{\langle e_1, 0.90, 0.95\rangle, \langle e_2, 0.75, 0.80\rangle, \langle e_3, 0.60, 0.50\rangle, \langle e_4, 0.40, 0.30\rangle\}.$$

In this example:

- e_1 has a very high suitability degree and also a very high weight, so this employee is both highly qualified and highly important for the project;
- e_2 is also a strong candidate with substantial importance;
- e_3 has moderate suitability and medium impact;
- e_4 has relatively low suitability and low importance.

Thus, A is a weighted fuzzy set on X, since each employee is described not only by a fuzzy membership degree but also by an additional weight expressing practical significance in the decision-making process.

13.6 Weighted Intuitionistic Fuzzy Set

A weighted intuitionistic fuzzy set assigns membership, non-membership, and weight to each element, representing graded evaluation, hesitation, and relative importance simultaneously (cf. [313–315]).

Definition 13.6.1 (Weighted intuitionistic fuzzy set)**.** Let X be a nonempty set. A *weighted intuitionistic fuzzy set* A on X is an ordered triple

$$A = (\mu_A, \nu_A, w_A),$$

where

$$\mu_A, \nu_A, w_A : X \to [0, 1]$$

satisfy

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \qquad \text{for all } x \in X.$$

Here

$$\mu_A(x)$$

is called the *membership degree* of x,

$$\nu_A(x)$$

is called the *non-membership degree* of x, and

$$w_A(x)$$

is called the *weight* of x.

Equivalently, A may be represented as

$$A = \{\langle x, \mu_A(x), \nu_A(x), w_A(x) \rangle \mid x \in X\}.$$

The hesitation degree of x is defined by

$$\pi_A(x) := 1 - \mu_A(x) - \nu_A(x).$$

13.7 Weighted Neutrosophic Set

A weighted neutrosophic set assigns truth, indeterminacy, falsity, and weight to each element, combining neutrosophic information with relative importance in one structure (cf. [209, 316–318]).

Definition 13.7.1 (Weighted neutrosophic set)**.** Let X be a nonempty set. A *weighted neutrosophic set* A on X is an ordered quadruple

$$A = (T_A, I_A, F_A, w_A),$$

where

$$T_A, I_A, F_A, w_A : X \to [0,1]$$

satisfy

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3 \qquad \text{for all } x \in X.$$

Here

$$T_A(x), \quad I_A(x), \quad F_A(x)$$

denote the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree of x, respectively, and

$$w_A(x)$$

denotes the weight of x.

Equivalently, A may be represented as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x), w_A(x)\rangle \mid x \in X\}.$$

13.8 Weighted Uncertain Set

A weighted uncertain set assigns each element both a model-admissible uncertainty degree and a weight, combining generalized uncertainty representation with relative importance in one framework.

Definition 13.8.1 (Weighted uncertain set)**.** Let X be a nonempty set, and let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \ge 1$. A *weighted uncertain set of type M on X* is a triple

$$\mathcal{U}_w = (X, \mu_M, w),$$

where

$$\mu_M : X \to \mathrm{Dom}(M)$$

is the *uncertainty-degree function* and

$$w : X \to [0,1]$$

is the *weight function*.

Equivalently, for each $x \in X$,

$$\mathcal{U}_w(x) = \big(\mu_M(x), w(x)\big),$$

where $\mu_M(x)$ is the uncertain degree of x and $w(x)$ is the weight of x.

Theorem 13.8.2 (Well-definedness of weighted uncertain sets)**.** *Let X be a nonempty set, let M be an uncertain model with*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$\mu_M : X \to \mathrm{Dom}(M), \qquad w : X \to [0,1].$$

Then the triple

$$\mathcal{U}_w = (X, \mu_M, w)$$

is a well-defined weighted uncertain set of type M on X.

Conversely, every weighted uncertain set of type M on X arises in this way.

Proof. Since

$$\mu_M : X \to \mathrm{Dom}(M),$$

for every $x \in X$ one has

$$\mu_M(x) \in \mathrm{Dom}(M).$$

Hence the uncertain degree of each element is admissible for the model M.

Also, since

$$w : X \to [0,1],$$

for every $x \in X$ one has

$$w(x) \in [0,1].$$

Therefore each element $x \in X$ is assigned both a well-defined uncertain degree

$$\mu_M(x)$$

and a well-defined weight

$$w(x).$$

Thus

$$\mathcal{U}_w = (X, \mu_M, w)$$

is a well-defined weighted uncertain set of type M on X.

Conversely, by definition, any weighted uncertain set of type M on X is precisely a triple

$$(X, \mu_M, w)$$

with

$$\mu_M : X \to \mathrm{Dom}(M) \qquad \text{and} \qquad w : X \to [0,1].$$

Hence every weighted uncertain set of type M on X arises in this way. □

Table 13.1: A catalogue of weighted uncertainty-set families, organized by the number k of membership components

k	**Representative weighted-set model(s)**
1	Weighted Fuzzy Set.
2	Weighted Intuitionistic Fuzzy Set.
3	Weighted Neutrosophic Set [316–318].
3	Weighted Hesitant Fuzzy Set [319, 320].

13.9 Weighted Soft Set

A weighted soft set is a soft set equipped with parameter weights, allowing each parameter to contribute according to its relative importance in decision processes [173, 321].

Definition 13.9.1 (Weighted soft set)**.** Let U be a nonempty universe, let E be a set of parameters, and let

$$A \subseteq E.$$

A *weighted soft set* over U with parameter set A is a triple

$$(F, A, w),$$

where

$$F : A \to \mathcal{P}(U)$$

is a soft set over U, and

$$w : A \to [0, \infty)$$

is a *weight function* assigning to each parameter $a \in A$ a nonnegative real number $w(a)$.

Equivalently, a weighted soft set may be represented as

$$(F, A, w) = \{\, (a, F(a), w(a)) \mid a \in A \,\}.$$

Here $F(a) \subseteq U$ is the approximation of the parameter a, and $w(a)$ expresses the relative importance of a.

13.10 Weighted Rough Set

A weighted rough set approximates a target subset using lower and upper regions derived from weighted attributes, incorporating relative importance into rough set uncertainty analysis [322–326].

Definition 13.10.1 (Weighted information system)**.** Let

$$S = (U, C, \{V_a\}_{a\in C}, \{f_a\}_{a\in C})$$

be an information system, where:

- U is a finite nonempty set of objects;

- C is a finite nonempty set of condition attributes;
- for each $a \in C$, V_a is the domain of a;
- for each $a \in C$, $f_a : U \to V_a$ is the value map of a.

A *weight function* on C is a mapping

$$w : C \to [0, 1]$$

such that

$$\sum_{a \in C} w(a) = 1.$$

The pair (S, w) is called a *weighted information system*.

Definition 13.10.2 (Weighted rough set)**.** Let (S, w) be a weighted information system, let

$$B \subseteq C$$

be nonempty, and let

$$\lambda \in [0, 1].$$

For each $x \in U$, define the *weighted neighborhood* of x with respect to (B, w, λ) by

$$[x]_{B,w,\lambda} := \left\{ y \in U \;\middle|\; \sum_{\substack{a \in B \\ f_a(x) = f_a(y)}} w(a) \geq \lambda \right\}.$$

For any subset $X \subseteq U$, define the *weighted lower approximation* and *weighted upper approximation* of X by

$$\underline{X}^{w}_{B,\lambda} := \{\, x \in U \mid [x]_{B,w,\lambda} \subseteq X \,\},$$

and

$$\overline{X}^{w}_{B,\lambda} := \{\, x \in U \mid [x]_{B,w,\lambda} \cap X \neq \emptyset \,\}.$$

Then the pair

$$\mathrm{WRS}_{B,\lambda}(X) := \left(\underline{X}^{w}_{B,\lambda}, \overline{X}^{w}_{B,\lambda} \right)$$

is called the *weighted rough set* of X with respect to (B, w, λ).

Chapter 14

Circular Set Theory

In this chapter, we discuss circular set theory.

14.1 Circular Set

A circular set assigns each element a crisp membership center and a radius, producing a disk-shaped membership region in the complex plane.

Definition 14.1.1 (Circular set)**.** Let X be a nonempty set. A *circular set* on X is a pair

$$A = (\chi_A, r_A),$$

where

$$\chi_A : X \to \{0, 1\}$$

is the characteristic function of a crisp set, and

$$r_A : X \to [0, \infty)$$

is a radius function.

For each $x \in X$, define the associated *circular membership region* by

$$O_A(x) := \{\, z \in \mathbb{C} \mid |z - \chi_A(x)| \leq r_A(x) \,\}.$$

Equivalently, one may represent A as

$$A = \{\langle x, \chi_A(x), r_A(x)\rangle \mid x \in X\}.$$

Example 14.1.2 (A concrete real-life example of a circular set)**.** Let

$$X = \{m_1, m_2, m_3\}$$

be a set of three medical test items for a patient, where

$$m_1 = \text{blood pressure}, \qquad m_2 = \text{blood sugar}, \qquad m_3 = \text{cholesterol}.$$

Suppose that a doctor wants to describe the set of test items that are judged to be *abnormal*, while also recording the uncertainty or tolerance around that binary judgment.

Define a circular set

$$A = (\chi_A, r_A)$$

on X by

$$\chi_A(m_1) = 1, \qquad \chi_A(m_2) = 0, \qquad \chi_A(m_3) = 1,$$

and

$$r_A(m_1) = 0.20, \qquad r_A(m_2) = 0.35, \qquad r_A(m_3) = 0.10.$$

Equivalently,

$$A = \{\langle m_1, 1, 0.20\rangle, \langle m_2, 0, 0.35\rangle, \langle m_3, 1, 0.10\rangle\}.$$

Thus:

- m_1 and m_3 belong to the crisp abnormality set, since $\chi_A(m_1) = \chi_A(m_3) = 1$;
- m_2 does not belong to that crisp set, since $\chi_A(m_2) = 0$;
- the radius $r_A(x)$ describes the size of the circular uncertainty region around the crisp value $\chi_A(x)$ in the complex plane.

The associated circular membership regions are

$$O_A(m_1) = \{z \in \mathbb{C} \mid |z - 1| \leq 0.20\},$$

$$O_A(m_2) = \{z \in \mathbb{C} \mid |z - 0| \leq 0.35\},$$

$$O_A(m_3) = \{z \in \mathbb{C} \mid |z - 1| \leq 0.10\}.$$

In practical terms, this means that:

- blood pressure is classified as abnormal, with a moderate uncertainty radius;
- blood sugar is classified as normal, but with a larger uncertainty radius;
- cholesterol is classified as abnormal, with a relatively small uncertainty radius.

Hence, A is a concrete real-life example of a circular set, where each medical test item is described by a crisp classification together with a circular tolerance region in the complex plane. As a reference, an illustrative figure is provided in Fig. 14.1.

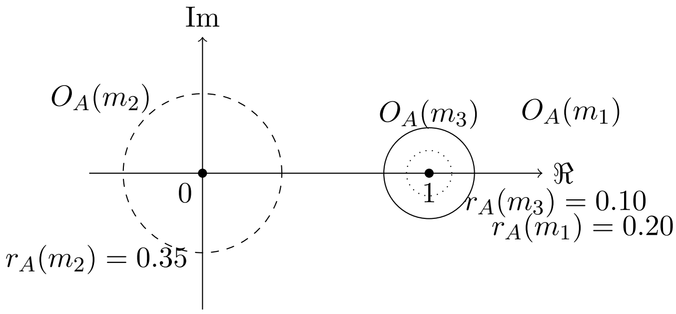

Figure 14.1: Circular membership regions for the circular set in Example. The test items m_1 and m_3 are centered at 1, while m_2 is centered at 0.

14.2 Circular Intuitionistic Fuzzy Set

A circular intuitionistic fuzzy set assigns membership, non-membership, and radius to each element, yielding circular regions inside the intuitionistic admissible domain [22, 327–329].

Definition 14.2.1 (Circular intuitionistic fuzzy set)**.** [22] Let X be a nonempty set, and define

$$L^* := \{(a, b) \in [0, 1]^2 \mid a + b \leq 1\}.$$

A *circular intuitionistic fuzzy set* on X is a triple

$$A = (\mu_A, \nu_A, r_A),$$

where

$$\mu_A, \nu_A : X \to [0, 1], \qquad r_A : X \to [0, \infty),$$

such that

$$\mu_A(x) + \nu_A(x) \leq 1 \qquad \text{for all } x \in X.$$

For each $x \in X$, define the associated *circular intuitionistic membership region* by

$$O_A(x) := \{\, (a, b) \in L^* \mid (a - \mu_A(x))^2 + (b - \nu_A(x))^2 \leq r_A(x)^2 \,\}.$$

Equivalently, one may represent A as

$$A = \{\langle x, \mu_A(x), \nu_A(x), r_A(x)\rangle \mid x \in X\}.$$

14.3 Circular Neutrosophic Set

A circular neutrosophic set assigns truth, indeterminacy, falsity, and radius to each element, producing Euclidean ball regions around neutrosophic membership triples.

Definition 14.3.1 (Circular neutrosophic set)**.** Let X be a nonempty set, and define

$$D_N := \{(t, i, f) \in [0, 1]^3 \mid 0 \leq t + i + f \leq 3\}.$$

A *circular neutrosophic set* on X is a quadruple

$$A = (T_A, I_A, F_A, r_A),$$

where

$$T_A, I_A, F_A : X \to [0,1], \qquad r_A : X \to [0,\infty),$$

such that

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3 \qquad \text{for all } x \in X.$$

For each $x \in X$, define the center

$$c_A(x) := \big(T_A(x), I_A(x), F_A(x)\big) \in D_N$$

and the associated *circular neutrosophic region*

$$O_A(x) := \{u \in D_N \mid \|u - c_A(x)\| \le r_A(x)\},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^3$.

Equivalently, one may represent A as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x), r_A(x)\rangle \mid x \in X\}.$$

Theorem 14.3.2 (Well-definedness of circular neutrosophic sets)**.** *Let*

$$A = (T_A, I_A, F_A, r_A)$$

be a circular neutrosophic set on a nonempty set X. Then the mapping

$$O_A : X \to \mathcal{P}(D_N) \setminus \{\emptyset\}$$

given by

$$O_A(x) = \{u \in D_N \mid \|u - c_A(x)\| \le r_A(x)\}$$

is well-defined. In particular, for every $x \in X$,

$$O_A(x) \ne \emptyset \qquad \textit{and} \qquad O_A(x) \subseteq D_N.$$

Proof. Fix $x \in X$. Since

$$T_A(x), I_A(x), F_A(x) \in [0,1]$$

and

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3,$$

it follows that

$$c_A(x) = \big(T_A(x), I_A(x), F_A(x)\big) \in D_N.$$

Also,

$$r_A(x) \in [0,\infty),$$

so

$$\|c_A(x) - c_A(x)\| = 0 \le r_A(x).$$

Hence

$$c_A(x) \in O_A(x).$$

Therefore

$$O_A(x) \ne \emptyset.$$

By definition, every element of $O_A(x)$ belongs to D_N, so

$$O_A(x) \subseteq D_N.$$

Thus

$$O_A(x) \in \mathcal{P}(D_N) \setminus \{\emptyset\}.$$

Since $x \in X$ was arbitrary, the map

$$O_A : X \to \mathcal{P}(D_N) \setminus \{\emptyset\}$$

is well-defined. □

14.4 Circular Uncertain Set

A circular uncertain set assigns each element an admissible uncertainty center and radius, representing uncertainty degrees by neighborhoods around model-valid points.

Definition 14.4.1 (Circular uncertain set)**.** Let X be a nonempty set, and let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. A *circular uncertain set of type* M on X is a triple

$$\mathcal{U}^\circ = (X, \mu_M, r),$$

where

$$\mu_M : X \to \mathrm{Dom}(M)$$

is the uncertainty-degree function and

$$r : X \to [0, \infty)$$

is the radius function.

For each $x \in X$, define the *circular uncertainty region* of x by

$$O_{\mathcal{U}}(x) := \{d \in \mathrm{Dom}(M) \mid \|d - \mu_M(x)\| \leq r(x)\},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^k$.

Equivalently, one may represent $\mathcal{U}^\circ$ as

$$\mathcal{U}^\circ = \{\langle x, \mu_M(x), r(x)\rangle \mid x \in X\}.$$

Theorem 14.4.2 (Well-definedness of circular uncertain sets)**.** *Let*

$$\mathcal{U}^\circ = (X, \mu_M, r)$$

be a circular uncertain set of type M on a nonempty set X. Then the mapping

$$O_{\mathcal{U}} : X \to \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}$$

given by

$$O_{\mathcal{U}}(x) = \{d \in \mathrm{Dom}(M) \mid \|d - \mu_M(x)\| \leq r(x)\}$$

is well-defined. In particular, for every $x \in X$,

$$O_{\mathcal{U}}(x) \neq \emptyset \qquad \textit{and} \qquad O_{\mathcal{U}}(x) \subseteq \mathrm{Dom}(M).$$

Proof. Fix $x \in X$. Since

$$\mu_M : X \to \mathrm{Dom}(M),$$

one has

$$\mu_M(x) \in \mathrm{Dom}(M).$$

Also, since

$$r(x) \in [0, \infty),$$

we have

$$\|\mu_M(x) - \mu_M(x)\| = 0 \leq r(x).$$

Hence

$$\mu_M(x) \in O_{\mathcal{U}}(x),$$

so

$$O_{\mathcal{U}}(x) \neq \emptyset.$$

By definition, every element of

$$O_{\mathcal{U}}(x)$$

belongs to

$$\mathrm{Dom}(M),$$

and therefore

$$O_{\mathcal{U}}(x) \subseteq \mathrm{Dom}(M).$$

Thus

$$O_{\mathcal{U}}(x) \in \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}.$$

Since $x \in X$ was arbitrary, the mapping

$$O_{\mathcal{U}} : X \to \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}$$

is well-defined. □

Remark 14.4.3. If

$$r(x) = 0 \qquad \text{for all } x \in X,$$

then a circular uncertain set reduces to an ordinary uncertain set of type M.

Chapter 15

Cubic Set Theory

Cubic set theory is presented below.

15.1 Cubic Set

A cubic set assigns each element both an interval-valued fuzzy membership and an ordinary fuzzy membership, combining bounded uncertainty with precise grade information simultaneously [330–332].

Definition 15.1.1 (Cubic set)**.** [330–332] Let X be a nonempty set. A *cubic set* A on X is an ordered pair

$$A = (\widetilde{\mu}_A, \lambda_A),$$

where

$$\widetilde{\mu}_A : X \to \mathbb{I}([0,1]), \qquad \lambda_A : X \to [0,1],$$

and

$$\mathbb{I}([0,1]) := \{[a,b] \subseteq [0,1] \mid 0 \leq a \leq b \leq 1\}.$$

Equivalently, for each $x \in X$,

$$A(x) = \big([\mu_A^-(x), \mu_A^+(x)], \lambda_A(x)\big),$$

where

$$0 \leq \mu_A^-(x) \leq \mu_A^+(x) \leq 1, \qquad \lambda_A(x) \in [0,1].$$

Thus, a cubic set is a combination of an interval-valued fuzzy membership and an ordinary fuzzy membership on the same universe.

Example 15.1.2 (A concrete real-life example of a cubic set)**.** Let

$$X = \{a_1, a_2, a_3\}$$

be a set of three apartments for rent.

Suppose that a tenant wants to describe the set of apartments that are *suitable for living.* For each apartment $a \in X$,

- the interval-valued membership

$$\widetilde{\mu}_A(a) = [\mu_A^-(a), \mu_A^+(a)]$$

  represents a plausible range of suitability under incomplete information;

- the single-valued membership

$$\lambda_A(a)$$

  represents a current representative evaluation given by the tenant.

Define a cubic set

$$A = (\widetilde{\mu}_A, \lambda_A)$$

on X by

$$A(a_1) = ([0.70, 0.90], 0.80), \qquad A(a_2) = ([0.40, 0.65], 0.55), \qquad A(a_3) = ([0.20, 0.45], 0.30).$$

Equivalently,

$$\widetilde{\mu}_A(a_1) = [0.70, 0.90], \quad \lambda_A(a_1) = 0.80,$$
$$\widetilde{\mu}_A(a_2) = [0.40, 0.65], \quad \lambda_A(a_2) = 0.55,$$
$$\widetilde{\mu}_A(a_3) = [0.20, 0.45], \quad \lambda_A(a_3) = 0.30.$$

Thus:

- apartment a_1 is evaluated as highly suitable;
- apartment a_2 is evaluated as moderately suitable;
- apartment a_3 is evaluated as rather weakly suitable.

Hence, A is a cubic set on X, since each apartment is described by both an interval-valued fuzzy membership and an ordinary fuzzy membership. An illustrative diagram is shown in Fig. 15.1.

15.2 Neutrosophic Cubic Set

A neutrosophic cubic set combines interval-valued and single-valued neutrosophic information, assigning each element bounded and exact truth, indeterminacy, and falsity degrees simultaneously [333–337].

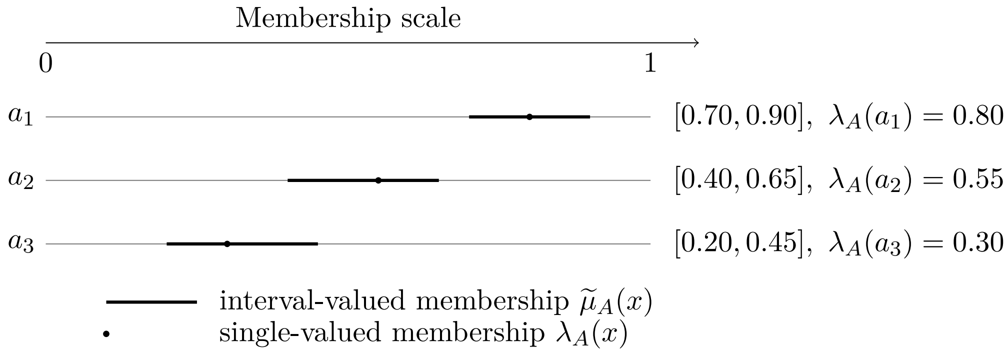

Figure 15.1: A graphical illustration of the cubic set in the example. For each apartment, the segment represents the interval-valued membership and the dot represents the ordinary fuzzy membership.

Definition 15.2.1 (Neutrosophic cubic set)**.** Let X be a nonempty set, and let

$$\mathbb{I}([0,1]) := \{[a,b] \subseteq [0,1] \mid 0 \leq a \leq b \leq 1\}.$$

A *neutrosophic cubic set* A on X is an ordered pair

$$A = (\widetilde{N}_A, N_A),$$

where

$$\widetilde{N}_A = \{\langle x, \widetilde{T}_A(x), \widetilde{I}_A(x), \widetilde{F}_A(x)\rangle \mid x \in X\}$$

is an interval-valued neutrosophic set on X, with

$$\widetilde{T}_A, \widetilde{I}_A, \widetilde{F}_A : X \to \mathbb{I}([0,1]),$$

and

$$N_A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$$

is a single-valued neutrosophic set on X, with

$$T_A, I_A, F_A : X \to [0,1].$$

Equivalently, for each $x \in X$,

$$A(x) = \big([\underline{T}_A(x), \overline{T}_A(x)], [\underline{I}_A(x), \overline{I}_A(x)], [\underline{F}_A(x), \overline{F}_A(x)]; T_A(x), I_A(x), F_A(x)\big),$$

where

$$0 \leq \underline{T}_A(x) \leq \overline{T}_A(x) \leq 1,$$

$$0 \leq \underline{I}_A(x) \leq \overline{I}_A(x) \leq 1,$$

$$0 \leq \underline{F}_A(x) \leq \overline{F}_A(x) \leq 1,$$

and

$$T_A(x), I_A(x), F_A(x) \in [0,1].$$

15.3 Plithogenic Cubic Set

A plithogenic cubic set combines interval-valued and single-valued appurtenance degrees for each element and attribute value, together with contradiction information on a fixed plithogenic frame [338–341].

Definition 15.3.1 (Plithogenic cubic set)**.** [340, 341] Let

$$\mathfrak{P} := (P, v, Pv, pCF, s, t)$$

be a plithogenic frame, where P is a nonempty universe, v is a fixed attribute, Pv is the set of possible values of v, and

$$pCF : Pv \times Pv \to [0, 1]^t$$

satisfies

$$pCF(a, a) = 0, \qquad pCF(a, b) = pCF(b, a) \quad \text{for all } a, b \in Pv.$$

Let

$$\mathbb{I}([0, 1]) := \{[\,\alpha, \beta\,] \subseteq [0, 1] \mid 0 \le \alpha \le \beta \le 1\}.$$

A *plithogenic cubic set* on the frame $\mathfrak{P}$ is a sextuple

$$PCS = (P, v, Pv, ipdf, pdf, pCF),$$

where

$$ipdf : P \times Pv \to \big(\mathbb{I}([0, 1])\big)^s$$

is an interval-valued degree of appurtenance function, and

$$pdf : P \times Pv \to [0, 1]^s$$

is a single-valued degree of appurtenance function.

Equivalently, for each $x \in P$ and $a \in Pv$,

$$PCS(x, a) = \big(ipdf(x, a), pdf(x, a)\big),$$

where

$$ipdf(x, a) = \big([\,\ell_1(x, a), u_1(x, a)\,], \dots, [\,\ell_s(x, a), u_s(x, a)\,]\big)$$

with

$$0 \le \ell_j(x, a) \le u_j(x, a) \le 1 \qquad (j = 1, \dots, s),$$

and

$$pdf(x, a) = (p_1(x, a), \dots, p_s(x, a)) \in [0, 1]^s.$$

Theorem 15.3.2 (Well-definedness of plithogenic cubic sets)**.** *Let*

$$PCS = (P, v, Pv, ipdf, pdf, pCF)$$

be as above. Then PCS is well-defined. More precisely, for every

$$x \in P, \qquad a \in Pv,$$

the values

$$ipdf(x, a) \in \big(\mathbb{I}([0, 1])\big)^s \qquad \text{and} \qquad pdf(x, a) \in [0, 1]^s$$

are well-defined, and the fixed contradiction map pCF satisfies the plithogenic conditions. Conversely, every such choice of ipdf and pdf on the same frame $\mathfrak{P}$ determines a plithogenic cubic set.

Proof. Since

$$ipdf : P \times Pv \to \big(\mathbb{I}([0,1])\big)^s,$$

for each

$$x \in P, \qquad a \in Pv,$$

the value $ipdf(x,a)$ is an s-tuple of closed intervals in $[0,1]$. Hence each component

$$[\,\ell_j(x,a), u_j(x,a)\,]$$

is well-defined with

$$0 \le \ell_j(x,a) \le u_j(x,a) \le 1.$$

Also, since

$$pdf : P \times Pv \to [0,1]^s,$$

the value

$$pdf(x,a)$$

is a well-defined s-tuple of real numbers in $[0,1]$.

Moreover, pCF is fixed as part of the frame and satisfies

$$pCF(a,a) = 0, \qquad pCF(a,b) = pCF(b,a) \quad \text{for all } a,b \in Pv.$$

Therefore all components of

$$(P, v, Pv, ipdf, pdf, pCF)$$

are well-defined, so PCS is a well-defined plithogenic cubic set.

Conversely, any pair of maps

$$ipdf : P \times Pv \to \big(\mathbb{I}([0,1])\big)^s, \qquad pdf : P \times Pv \to [0,1]^s$$

together with the fixed contradiction map pCF on the same frame $\mathfrak{P}$ gives exactly such a sextuple. Hence every such choice determines a plithogenic cubic set. □

15.4 Uncertain Cubic Set

An uncertain cubic set assigns each element both an interval-valued uncertain degree and a single uncertain degree within the same admissible uncertainty model.

Definition 15.4.1 (Componentwise order on Dom(M))**.** Let M be an uncertain model with

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \ge 1$. For

$$a = (a_1, \ldots, a_k), \qquad b = (b_1, \ldots, b_k) \in \mathrm{Dom}(M),$$

define

$$a \preceq b \iff a_j \le b_j \quad \text{for all } j = 1, \ldots, k.$$

Definition 15.4.2 (Uncertain cubic set)**.** Let X be a nonempty set, and let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

An *uncertain cubic set of type M on X* is a triple

$$\mathcal{C}_M = (X, \Gamma_M, \mu_M),$$

where

$$\mu_M : X \to \mathrm{Dom}(M)$$

is an uncertain degree function, and

$$\Gamma_M : X \to \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}$$

is induced by two functions

$$\underline{\mu}_M, \overline{\mu}_M : X \to \mathrm{Dom}(M)$$

satisfying

$$\underline{\mu}_M(x) \preceq \overline{\mu}_M(x) \qquad \text{for all } x \in X,$$

and

$$\Gamma_M(x) := [\,\underline{\mu}_M(x), \overline{\mu}_M(x)\,]_{\mathrm{Dom}(M)} := \{\, d \in \mathrm{Dom}(M) \mid \underline{\mu}_M(x) \preceq d \preceq \overline{\mu}_M(x)\,\}.$$

Equivalently, for each $x \in X$, the value

$$\mathcal{C}_M(x) = \big(\Gamma_M(x), \mu_M(x)\big)$$

consists of an interval-valued uncertain degree and a single-valued uncertain degree.

Theorem 15.4.3 (Well-definedness of uncertain cubic sets)**.** *Let*

$$\mathcal{C}_M = (X, \Gamma_M, \mu_M)$$

be as above. Then $\mathcal{C}_M$ is well-defined. More precisely:

1. *for every $x \in X$,*
$$\Gamma_M(x) \in \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\};$$
2. *for every $x \in X$,*
$$\mu_M(x) \in \mathrm{Dom}(M);$$
3. *hence*
$$\mathcal{C}_M(x) = \big(\Gamma_M(x), \mu_M(x)\big)$$
is a well-defined cubic uncertain value.

Conversely, every such choice of Γ_M and μ_M determines an uncertain cubic set of type M.

Proof. Fix $x \in X$. Since

$$\underline{\mu}_M(x), \overline{\mu}_M(x) \in \mathrm{Dom}(M)$$

and

$$\underline{\mu}_M(x) \preceq \overline{\mu}_M(x),$$

it follows that

$$\underline{\mu}_M(x) \in [\,\underline{\mu}_M(x), \overline{\mu}_M(x)\,]_{\mathrm{Dom}(M)} = \Gamma_M(x).$$

Hence

$$\Gamma_M(x) \neq \emptyset.$$

By definition, every element of $\Gamma_M(x)$ belongs to $\mathrm{Dom}(M)$, so

$$\Gamma_M(x) \subseteq \mathrm{Dom}(M).$$

Therefore

$$\Gamma_M(x) \in \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}.$$

This proves (1).

Next, since

$$\mu_M : X \to \mathrm{Dom}(M),$$

one has

$$\mu_M(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

This proves (2).

By (1) and (2), for every $x \in X$ the pair

$$\big(\Gamma_M(x), \mu_M(x)\big)$$

is well-defined, so

$$\mathcal{C}_M = (X, \Gamma_M, \mu_M)$$

is a well-defined uncertain cubic set. This proves (3).

Conversely, any pair of mappings

$$\Gamma_M : X \to \mathcal{P}(\mathrm{Dom}(M)) \setminus \{\emptyset\}, \qquad \mu_M : X \to \mathrm{Dom}(M),$$

with Γ_M induced by some

$$\underline{\mu}_M, \overline{\mu}_M : X \to \mathrm{Dom}(M)$$

satisfying

$$\underline{\mu}_M(x) \preceq \overline{\mu}_M(x) \qquad \text{for all } x \in X,$$

gives exactly an uncertain cubic set of type M. □

15.5 Rough Cubic Set

A rough cubic set approximates a cubic set through lower and upper cubic operators induced by a relation, capturing boundary uncertainty in cubic form [342–344].

Definition 15.5.1 (Rough cubic set)**.** [342, 343] Let X be a nonempty universe, let

$$R = \langle \widetilde{R}, r \rangle$$

be a cubic relation on X, and let

$$A = \langle \widetilde{A}, \lambda \rangle$$

be a cubic set on X. Define two rough operators

$$N(A)(x) := \bigwedge_{y \in X} \big(A(y) \, t \, R^c(y, x) \big), \qquad H(A)(x) := \bigvee_{y \in X} \big(A(y) \, u \, R(x, y) \big),$$

where t and u denote the P-union and P-intersection of cubic values, respectively. Then the pair

$$\big(N(A), H(A) \big)$$

is called the *rough cubic set* of A induced by the cubic relation R.

15.6 Stable Cubic set

Stable cubic sets are cubic sets whose precise membership lies inside the associated interval membership for every element, ensuring internal consistency between both descriptions always [345, 346].

Definition 15.6.1 (Stable cubic set)**.** Let X be a nonempty set, and let

$$A = \langle \widetilde{A}, \lambda \rangle$$

be a cubic set on X, where

$$\widetilde{A}(x) = [A^-(x), A^+(x)] \qquad (x \in X).$$

For each $x \in X$, define the evaluative interval

$$E_A(x) := \big[\lambda(x) - A^-(x), \; A^+(x) - \lambda(x) \big].$$

Then A is called a *stable cubic set* if, for every $x \in X$,

$$\lambda(x) - A^-(x) \geq 0 \qquad \text{and} \qquad A^+(x) - \lambda(x) \geq 0.$$

Equivalently,

$$A^-(x) \leq \lambda(x) \leq A^+(x) \qquad \text{for all } x \in X.$$

Example 15.6.2 (A concrete real-life example of a stable cubic set)**.** Let

$$X = \{s_1, s_2, s_3\}$$

be a set of three students applying for a scholarship.

Suppose that a selection committee wants to describe the set of students who are *academically strong.* For each student $s \in X$,

- $\widetilde{A}(s) = [A^-(s), A^+(s)]$ represents an interval-valued assessment of academic strength, reflecting variation among different reviewers;
- $\lambda(s)$ represents a single representative score, such as the committee's final summarized evaluation.

Define a cubic set

$$A = \langle \widetilde{A}, \lambda \rangle$$

on X by

$$\begin{aligned} \widetilde{A}(s_1) &= [0.70, 0.90], & \lambda(s_1) &= 0.80, \\ \widetilde{A}(s_2) &= [0.50, 0.75], & \lambda(s_2) &= 0.60, \\ \widetilde{A}(s_3) &= [0.30, 0.55], & \lambda(s_3) &= 0.40. \end{aligned}$$

Thus,

$$A = \{\langle s_1, [0.70, 0.90], 0.80\rangle,\ \langle s_2, [0.50, 0.75], 0.60\rangle,\ \langle s_3, [0.30, 0.55], 0.40\rangle\}.$$

Now we check the stability condition.

For s_1,

$$0.70 \le 0.80 \le 0.90,$$

so

$$E_A(s_1) = [0.80 - 0.70,\ 0.90 - 0.80] = [0.10, 0.10].$$

For s_2,

$$0.50 \le 0.60 \le 0.75,$$

so

$$E_A(s_2) = [0.60 - 0.50,\ 0.75 - 0.60] = [0.10, 0.15].$$

For s_3,

$$0.30 \le 0.40 \le 0.55,$$

so

$$E_A(s_3) = [0.40 - 0.30,\ 0.55 - 0.40] = [0.10, 0.15].$$

Hence, for every $s \in X$,

$$A^-(s) \le \lambda(s) \le A^+(s).$$

Therefore, A is a stable cubic set.

In practical terms, this means that for each student, the committee's representative evaluation lies inside the corresponding interval of plausible academic assessments. Thus, the single-valued score is consistent with the interval-valued assessment for every student.

15.7 Soft Cubic set

A soft cubic set is a parameterized family of cubic sets, assigning each parameter an interval-valued and single-valued membership description on the same universe [347–349].

Definition 15.7.1 (Soft cubic set)**.** Let U be a nonempty universe, let E be a set of parameters, and let

$$A \subseteq E.$$

Write

$$\mathbb{I}([0,1]) := \{[a,b] \subseteq [0,1] \mid 0 \leq a \leq b \leq 1\},$$

and let

$$\mathrm{Cub}(U) := \Big\{(\widetilde{\mu}, \lambda) \;\Big|\; \widetilde{\mu} : U \to \mathbb{I}([0,1]),\ \lambda : U \to [0,1]\Big\}.$$

A *soft cubic set* over U with parameter set A is a pair

$$(F, A),$$

where

$$F : A \to \mathrm{Cub}(U).$$

Equivalently, for each

$$e \in A$$

and each

$$x \in U,$$

one has

$$F(e)(x) = \big([\mu_e^-(x), \mu_e^+(x)], \lambda_e(x)\big),$$

where

$$\mu_e^-, \mu_e^+, \lambda_e : U \to [0,1]$$

satisfy

$$0 \leq \mu_e^-(x) \leq \mu_e^+(x) \leq 1, \qquad \lambda_e(x) \in [0,1].$$

Thus, a soft cubic set is a parameterized family of cubic sets on U.

15.8 Complex Cubic set

Complex cubic sets assign each element a complex interval-valued membership and a single complex membership within the closed unit disk [330].

Definition 15.8.1 (Complex cubic set)**.** Let X be a nonempty set, and let

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$$

be the closed unit disk in the complex plane. Define

$$\mathcal{CI}(\mathbb{D}) := \Big\{\big(r^- e^{\mathrm{i}\theta^-},\, r^+ e^{\mathrm{i}\theta^+}\big) \in \mathbb{D} \times \mathbb{D} \;\Big|\; 0 \leq r^- \leq r^+ \leq 1,\ 0 \leq \theta^- \leq \theta^+ \leq 2\pi\Big\}.$$

A *complex cubic set* C on X is an ordered pair

$$C = (\widetilde{\eta}, \mu),$$

where

$$\widetilde{\eta} : X \to \mathcal{CI}(\mathbb{D})$$

is a complex interval-valued membership function, and

$$\mu : X \to \mathbb{D}$$

is a complex membership function.

Equivalently, for each

$$x \in X,$$

one may write

$$C(x) = \Big(r^-(x)e^{\mathrm{i}\theta^-(x)},\ r^+(x)e^{\mathrm{i}\theta^+(x)};\ \rho(x)e^{\mathrm{i}\varphi(x)}\Big),$$

where

$$0 \le r^-(x) \le r^+(x) \le 1, \qquad 0 \le \theta^-(x) \le \theta^+(x) \le 2\pi,$$

and

$$\rho(x) \in [0,1], \qquad \varphi(x) \in [0,2\pi].$$

Thus, a complex cubic set combines a complex interval-valued fuzzy description with a complex fuzzy description on the same universe.

15.9 HyperCubic Set

A hypercubic set assigns each element multiple interval-valued and single-valued memberships, extending cubic sets to m components and reducing to a cubic set when $m = 1$ [350, 351].

Definition 15.9.1 (m-hypercubic set)**.** Let X be a nonempty set, and let $m \ge 1$ be an integer. Define

$$\mathbb{I}([0,1]) := \{[a,b] \subseteq [0,1] \mid 0 \le a \le b \le 1\}.$$

An *m-hypercubic set H* on X is an ordered pair

$$H = (\widetilde{\mu}_H, \lambda_H),$$

where

$$\widetilde{\mu}_H : X \to \big(\mathbb{I}([0,1])\big)^m$$

is an m-tuple of interval-valued membership functions, and

$$\lambda_H : X \to [0,1]^m$$

is an m-tuple of single-valued membership functions.

Equivalently, for each $x \in X$,

$$H(x) = \big([\mu^-_{H,1}(x), \mu^+_{H,1}(x)], \ldots, [\mu^-_{H,m}(x), \mu^+_{H,m}(x)]; \lambda_{H,1}(x), \ldots, \lambda_{H,m}(x)\big),$$

where, for every $j = 1, \ldots, m$,

$$0 \le \mu^-_{H,j}(x) \le \mu^+_{H,j}(x) \le 1, \qquad \lambda_{H,j}(x) \in [0,1].$$

In particular, when $m = 1$, an m-hypercubic set reduces to an ordinary cubic set.

15.10 Linguistic Cubic Set

A linguistic cubic set assigns each element a linguistic interval and a linguistic label, combining interval-valued and single-valued qualitative memberships within one ordered framework.

Definition 15.10.1 (Linguistic term set)**.** Let

$$\mathcal{L} = \{\ell_0, \ell_1, \ldots, \ell_g\}$$

be a finite nonempty totally ordered set of linguistic labels, written as

$$\ell_0 \preceq \ell_1 \preceq \cdots \preceq \ell_g.$$

Define

$$\mathbb{I}(\mathcal{L}) := \{[\ell_i, \ell_j] \mid 0 \leq i \leq j \leq g\},$$

where

$$[\ell_i, \ell_j] := \{\ell \in \mathcal{L} \mid \ell_i \preceq \ell \preceq \ell_j\}.$$

The set

$$\mathbb{I}(\mathcal{L})$$

is called the *linguistic interval family* induced by $\mathcal{L}$.

Definition 15.10.2 (Linguistic cubic set)**.** Let X be a nonempty set, and let $\mathcal{L}$ be a linguistic term set. A *linguistic cubic set* A on X is an ordered pair

$$A = (\widetilde{\lambda}_A, \lambda_A),$$

where

$$\widetilde{\lambda}_A : X \to \mathbb{I}(\mathcal{L})$$

is a linguistic interval-valued membership function, and

$$\lambda_A : X \to \mathcal{L}$$

is a linguistic membership function.

Equivalently, there exist mappings

$$\lambda_A^-, \lambda_A^+ : X \to \mathcal{L}$$

such that

$$\lambda_A^-(x) \preceq \lambda_A^+(x) \qquad \text{for all } x \in X,$$

and

$$\widetilde{\lambda}_A(x) = [\lambda_A^-(x), \lambda_A^+(x)].$$

Thus, for each

$$x \in X,$$

one may write

$$A(x) = \big([\lambda_A^-(x), \lambda_A^+(x)],\ \lambda_A(x)\big),$$

where

$$\lambda_A^-(x), \lambda_A(x), \lambda_A^+(x) \in \mathcal{L}$$

and

$$\lambda_A^-(x) \preceq \lambda_A^+(x).$$

Theorem 15.10.3 (Well-definedness of linguistic cubic sets). *Let X be a nonempty set, let*

$$\mathcal{L} = \{\ell_0, \ell_1, \ldots, \ell_g\}$$

be a finite nonempty totally ordered linguistic term set, and let

$$\lambda_A^-, \lambda_A^+, \lambda_A : X \to \mathcal{L}$$

be mappings such that

$$\lambda_A^-(x) \preceq \lambda_A^+(x) \qquad \text{for all } x \in X.$$

Define

$$\widetilde{\lambda}_A : X \to \mathbb{I}(\mathcal{L})$$

by

$$\widetilde{\lambda}_A(x) := [\lambda_A^-(x), \lambda_A^+(x)].$$

Then

$$A = (\widetilde{\lambda}_A, \lambda_A)$$

is a well-defined linguistic cubic set on X.

Conversely, every linguistic cubic set on X arises in this way from suitable mappings

$$\lambda_A^-, \lambda_A^+, \lambda_A : X \to \mathcal{L}$$

with

$$\lambda_A^-(x) \preceq \lambda_A^+(x) \qquad \text{for all } x \in X.$$

Proof. Fix

$$x \in X.$$

Since

$$\lambda_A^-(x), \lambda_A^+(x) \in \mathcal{L}$$

and

$$\lambda_A^-(x) \preceq \lambda_A^+(x),$$

the set

$$[\lambda_A^-(x), \lambda_A^+(x)] = \{\ell \in \mathcal{L} \mid \lambda_A^-(x) \preceq \ell \preceq \lambda_A^+(x)\}$$

is a well-defined nonempty interval of linguistic labels. Hence

$$[\lambda_A^-(x), \lambda_A^+(x)] \in \mathbb{I}(\mathcal{L}).$$

Therefore

$$\widetilde{\lambda}_A(x)$$

is well-defined for every

$$x \in X.$$

Also, since

$$\lambda_A : X \to \mathcal{L},$$

the value

$$\lambda_A(x) \in \mathcal{L}$$

is well-defined for every

$$x \in X.$$

Thus the pair

$$A(x) = \big(\widetilde{\lambda}_A(x), \lambda_A(x)\big)$$

is well-defined for all

$$x \in X,$$

and consequently

$$A = (\widetilde{\lambda}_A, \lambda_A)$$

is a well-defined linguistic cubic set on X.

Conversely, let

$$A = (\widetilde{\lambda}_A, \lambda_A)$$

be a linguistic cubic set on X. By definition,

$$\widetilde{\lambda}_A(x) \in \mathbb{I}(\mathcal{L}) \qquad \text{and} \qquad \lambda_A(x) \in \mathcal{L}$$

for every

$$x \in X.$$

Since each element of

$$\mathbb{I}(\mathcal{L})$$

has the form

$$[\ell_i, \ell_j] \qquad \text{with } \ell_i \preceq \ell_j,$$

there exist linguistic labels

$$\lambda_A^-(x), \lambda_A^+(x) \in \mathcal{L}$$

such that

$$\widetilde{\lambda}_A(x) = [\lambda_A^-(x), \lambda_A^+(x)] \qquad \text{and} \qquad \lambda_A^-(x) \preceq \lambda_A^+(x).$$

Hence every linguistic cubic set arises in this way. □

Chapter 16

Convex Set Theory

In this chapter, we examine Convex Set Theory.

16.1 Convex Sets

A convex set is a subset of a real vector space that contains every convex combination of any two of its points, equivalently every line segment joining them [352, 352–354].

Definition 16.1.1 (Convex set). [352, 353] Let V be a real vector space, and let

$$C \subseteq V.$$

Then C is called a *convex set* if, for all

$$x, y \in C \qquad \text{and} \qquad \lambda \in [0, 1],$$

one has

$$\lambda x + (1 - \lambda)y \in C.$$

Equivalently, C is convex if it contains every line segment joining any two of its points.

16.2 Convex Fuzzy Sets

A convex fuzzy set is a fuzzy set on a convex domain whose membership value at each convex combination is at least the minimum membership of the endpoints [355, 356].

Definition 16.2.1 (Convex fuzzy set). [355, 356] Let X be a nonempty convex subset of a real vector space V, and let

$$\mu_A : X \to [0, 1]$$

be a fuzzy set on X. Then A is called a *convex fuzzy set* if, for all

$$x, y \in X \qquad \text{and} \qquad \lambda \in [0, 1],$$

one has

$$\mu_A\big(\lambda x + (1 - \lambda)y\big) \geq \min\{\mu_A(x), \mu_A(y)\}.$$

Equivalently, A is convex if every α-cut

$$A_\alpha := \{x \in X \mid \mu_A(x) \geq \alpha\}, \qquad \alpha \in (0, 1],$$

is a convex subset of X.

16.3 Convex Neutrosophic Sets

A convex neutrosophic set is a neutrosophic set whose truth-membership remains convex, while indeterminacy and falsity do not exceed the larger endpoint values (cf. [357]).

Definition 16.3.1 (Convex neutrosophic set)**.** Let V be a real vector space, let $X \subseteq V$ be a nonempty convex set, and let

$$A = (T_A, I_A, F_A)$$

be a single-valued neutrosophic set on X, where

$$T_A, I_A, F_A : X \to [0, 1].$$

Then A is called a *convex neutrosophic set* if, for all

$$x, y \in X \qquad \text{and} \qquad \lambda \in [0, 1],$$

one has

$$T_A\big(\lambda x + (1 - \lambda)y\big) \geq \min\{T_A(x), T_A(y)\},$$
$$I_A\big(\lambda x + (1 - \lambda)y\big) \leq \max\{I_A(x), I_A(y)\},$$

and

$$F_A\big(\lambda x + (1 - \lambda)y\big) \leq \max\{F_A(x), F_A(y)\}.$$

16.4 Convex Uncertain Sets

A convex uncertain set is an uncertain set whose evaluated uncertainty degree at any convex combination is at least the minimum of endpoint evaluations.

Definition 16.4.1 (η-level cut of an uncertain set)**.** Let V be a real vector space, let

$$X \subseteq V$$

be a nonempty set, let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \geq 1$, and let

$$\eta : \mathrm{Dom}(M) \to [0, 1]$$

be a fixed evaluation map. Let

$$\mathcal{U} = (X, \mu_M)$$

be an uncertain set of type M on X, where

$$\mu_M : X \to \mathrm{Dom}(M).$$

For each

$$\alpha \in [0, 1],$$

define the *η-level cut* of $\mathcal{U}$ by

$$\mathcal{U}^\eta_\alpha := \{\, x \in X \mid \eta(\mu_M(x)) \geq \alpha \,\}.$$

Definition 16.4.2 (Convex uncertain set)**.** Let V be a real vector space, let

$$X \subseteq V$$

be a nonempty convex set, let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$\eta : \mathrm{Dom}(M) \to [0,1]$$

be a fixed evaluation map. An uncertain set

$$\mathcal{U} = (X, \mu_M)$$

of type M on X is called a *convex uncertain set* (with respect to η) if, for all

$$x, y \in X \qquad \text{and} \qquad \lambda \in [0,1],$$

one has

$$\eta\big(\mu_M(\lambda x + (1-\lambda)y)\big) \geq \min\big\{\eta(\mu_M(x)), \eta(\mu_M(y))\big\}.$$

Theorem 16.4.3 (Well-definedness and cut characterization of convex uncertain sets)**.** *Let V be a real vector space, let*

$$X \subseteq V$$

be a nonempty convex set, let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

let

$$\eta : \mathrm{Dom}(M) \to [0,1]$$

be a fixed evaluation map, and let

$$\mathcal{U} = (X, \mu_M)$$

be an uncertain set of type M on X. Then:

1. *the composite map*

   $$\eta \circ \mu_M : X \to [0,1]$$

   is well-defined;

2. *for every*

   $$\alpha \in [0,1],$$

   the set

   $$\mathcal{U}_\alpha^\eta = \{\, x \in X \mid \eta(\mu_M(x)) \geq \alpha \,\}$$

   is a well-defined subset of X;

3. *the condition*

   $$\eta\big(\mu_M(\lambda x + (1-\lambda)y)\big) \geq \min\big\{\eta(\mu_M(x)), \eta(\mu_M(y))\big\}$$

   is well-formed for all

   $$x, y \in X, \qquad \lambda \in [0,1];$$

4. *moreover, $\mathcal{U}$ is a convex uncertain set if and only if every η-level cut*

$$\mathcal{U}_\alpha^\eta \qquad (\alpha \in [0,1])$$

is a convex subset of X.

Hence the notion of a convex uncertain set is well-defined.

Proof. Since

$$\mu_M : X \to \mathrm{Dom}(M)$$

and

$$\eta : \mathrm{Dom}(M) \to [0,1],$$

it follows that, for every

$$x \in X,$$

one has

$$\mu_M(x) \in \mathrm{Dom}(M) \qquad \text{and} \qquad \eta(\mu_M(x)) \in [0,1].$$

Therefore the composite map

$$\eta \circ \mu_M : X \to [0,1]$$

is well-defined. This proves (1).

Now fix

$$\alpha \in [0,1].$$

Because

$$\eta(\mu_M(x)) \in [0,1]$$

for every

$$x \in X,$$

the inequality

$$\eta(\mu_M(x)) \geq \alpha$$

is meaningful. Hence

$$\mathcal{U}_\alpha^\eta = \{\, x \in X \mid \eta(\mu_M(x)) \geq \alpha \,\}$$

is a well-defined subset of X. This proves (2).

Next, let

$$x, y \in X \qquad \text{and} \qquad \lambda \in [0,1].$$

Since X is convex,

$$\lambda x + (1-\lambda)y \in X.$$

Therefore

$$\mu_M(\lambda x + (1-\lambda)y)$$

is defined and belongs to

$$\mathrm{Dom}(M).$$

Applying η, one gets

$$\eta\big(\mu_M(\lambda x + (1-\lambda)y)\big) \in [0,1].$$

Also,

$$\eta(\mu_M(x)),\ \eta(\mu_M(y)) \in [0,1],$$

so

$$\min\big\{\eta(\mu_M(x)), \eta(\mu_M(y))\big\}$$

is well-defined. Hence the inequality in the definition is well-formed. This proves (3).

It remains to prove (4).

Assume first that $\mathcal{U}$ is a convex uncertain set. Let

$$\alpha \in [0,1], \qquad x, y \in \mathcal{U}_\alpha^\eta, \qquad \lambda \in [0,1].$$

Then

$$\eta(\mu_M(x)) \geq \alpha \qquad \text{and} \qquad \eta(\mu_M(y)) \geq \alpha.$$

By convexity of $\mathcal{U}$,

$$\eta\big(\mu_M(\lambda x + (1-\lambda)y)\big) \geq \min\big\{\eta(\mu_M(x)), \eta(\mu_M(y))\big\} \geq \alpha.$$

Thus

$$\lambda x + (1-\lambda)y \in \mathcal{U}_\alpha^\eta.$$

Hence $\mathcal{U}_\alpha^\eta$ is convex.

Conversely, assume that every set

$$\mathcal{U}_\alpha^\eta \qquad (\alpha \in [0,1])$$

is convex. Let

$$x, y \in X \qquad \text{and} \qquad \lambda \in [0,1].$$

Put

$$\alpha := \min\big\{\eta(\mu_M(x)), \eta(\mu_M(y))\big\} \in [0,1].$$

Then

$$x, y \in \mathcal{U}_\alpha^\eta.$$

By convexity of $\mathcal{U}_\alpha^\eta$,

$$\lambda x + (1-\lambda)y \in \mathcal{U}_\alpha^\eta.$$

Therefore

$$\eta\big(\mu_M(\lambda x + (1-\lambda)y)\big) \geq \alpha = \min\big\{\eta(\mu_M(x)), \eta(\mu_M(y))\big\}.$$

Hence $\mathcal{U}$ is a convex uncertain set.

Therefore all assertions hold, and the notion of a convex uncertain set is well-defined. □

16.5 Convex soft sets

A convex soft set is a soft set whose value at every convex combination of parameters contains the intersection of the values at those parameters [358, 359].

Definition 16.5.1 (Convex soft set)**.** Let U be a nonempty universe, and let E be a nonempty convex subset of a real vector space. A soft set over U with parameter set E is a mapping

$$F : E \to \mathcal{P}(U).$$

Then F is called a *convex soft set* if, for all

$$e_1, e_2 \in E \qquad \text{and} \qquad \lambda \in [0, 1],$$

one has

$$F\big(\lambda e_1 + (1 - \lambda)e_2\big) \supseteq F(e_1) \cap F(e_2).$$

Example 16.5.2 (A concrete real-life example of a convex soft set)**.** Let

$$U = \{h_1, h_2, h_3, h_4\}$$

be a set of four apartments for rent, and let

$$E = [0, 1] \subseteq \mathbb{R}.$$

We interpret each parameter

$$e \in E$$

as a required minimum level of overall living comfort.

Assume that the apartments have comfort scores given by

$$c(h_1) = 0.9, \qquad c(h_2) = 0.7, \qquad c(h_3) = 0.5, \qquad c(h_4) = 0.3.$$

Define a soft set

$$F : E \to \mathcal{P}(U)$$

by

$$F(e) := \{\, h \in U \mid c(h) \geq e \,\}.$$

Thus, for each threshold e, the set $F(e)$ consists of all apartments whose comfort score is at least e.

For example,

$$F(0.2) = \{h_1, h_2, h_3, h_4\},$$
$$F(0.6) = \{h_1, h_2\},$$
$$F(0.8) = \{h_1\}.$$

We show that F is a convex soft set. Let

$$e_1, e_2 \in E, \qquad \lambda \in [0, 1],$$

and take any apartment

$$h \in F(e_1) \cap F(e_2).$$

Then

$$c(h) \geq e_1 \qquad \text{and} \qquad c(h) \geq e_2.$$

Hence

$$c(h) \geq \max\{e_1, e_2\} \geq \lambda e_1 + (1-\lambda)e_2.$$

Therefore,

$$h \in F\big(\lambda e_1 + (1-\lambda)e_2\big).$$

Since h was arbitrary, it follows that

$$F\big(\lambda e_1 + (1-\lambda)e_2\big) \supseteq F(e_1) \cap F(e_2).$$

Thus, F is a convex soft set.

In practical terms, this means that if an apartment satisfies two comfort requirements e_1 and e_2, then it also satisfies any intermediate comfort requirement between them. Therefore, the parameterized family F forms a convex soft set.

16.6 Convex Hypersoft sets

A convex hypersoft set is a hypersoft set whose image at each componentwise convex combination contains the intersection of the corresponding multiattribute parameter-images [358].

Definition 16.6.1 (Convex hypersoft set)**.** Let U be a nonempty universe, and let

$$A_1, \dots, A_n$$

be nonempty convex subsets of real vector spaces

$$V_1, \dots, V_n,$$

respectively. Put

$$G := A_1 \times \cdots \times A_n \subseteq V_1 \times \cdots \times V_n.$$

A hypersoft set over U with multi-attribute parameter domain G is a mapping

$$H : G \to \mathcal{P}(U).$$

Then H is called a *convex hypersoft set* if, for all

$$\omega, \mu \in G \qquad \text{and} \qquad \lambda \in [0,1],$$

one has

$$H\big(\lambda\omega + (1-\lambda)\mu\big) \supseteq H(\omega) \cap H(\mu),$$

where the convex combination is taken componentwise in the product space

$$V_1 \times \cdots \times V_n.$$

Equivalently, if

$$\omega = (a_1, \dots, a_n), \qquad \mu = (b_1, \dots, b_n),$$

then

$$\lambda\omega + (1-\lambda)\mu = \big(\lambda a_1 + (1-\lambda)b_1, \dots, \lambda a_n + (1-\lambda)b_n\big) \in G,$$

and

$$H(\lambda\omega + (1-\lambda)\mu) \supseteq H(\omega) \cap H(\mu).$$

16.7 Convex rough sets

A convex rough set is a rough approximation pair whose lower and upper approximation sets are both convex in the underlying vector space [360, 361].

Definition 16.7.1 (Convex rough set)**.** Let V be a real vector space, let

$$X \subseteq V$$

be a nonempty set, and let

$$R \subseteq X \times X$$

be an equivalence relation. For any subset

$$A \subseteq X,$$

define the Pawlak lower and upper approximations of A by

$$\underline{A}_R := \{\, x \in X \mid [x]_R \subseteq A \,\}, \qquad \overline{A}_R := \{\, x \in X \mid [x]_R \cap A \neq \emptyset \,\},$$

where

$$[x]_R := \{\, y \in X \mid (x, y) \in R \,\}.$$

The pair

$$\mathrm{RS}_R(A) := (\underline{A}_R, \overline{A}_R)$$

is called a *convex rough set* if both

$$\underline{A}_R \quad \text{and} \quad \overline{A}_R$$

are convex subsets of V; that is, for all

$$x, y \in \underline{A}_R \quad \text{and} \quad \lambda \in [0, 1],$$

one has

$$\lambda x + (1 - \lambda) y \in \underline{A}_R,$$

and for all

$$x, y \in \overline{A}_R \quad \text{and} \quad \lambda \in [0, 1],$$

one has

$$\lambda x + (1 - \lambda) y \in \overline{A}_R.$$

16.8 Concave sets

A concave set is a subset of a vector space whose complement is convex, so convex combinations of outside points remain outside the set [362–364].

Definition 16.8.1 (Concave set)**.** Let V be a real vector space, and let

$$C \subseteq V.$$

Then C is called a *concave set* if its complement

$$V \setminus C$$

is a convex set; that is, for all

$$x, y \in V \setminus C \qquad \text{and} \qquad \lambda \in [0, 1],$$

one has

$$\lambda x + (1 - \lambda) y \in V \setminus C.$$

16.9 Closed convex sets

A closed convex set is a convex subset of a topological vector space that is also closed under the ambient topology [365–368].

Definition 16.9.1 (Closed convex set)**.** Let V be a real topological vector space, and let

$$C \subseteq V.$$

Then C is called a *closed convex set* if:

1. C is *convex*, that is, for all

   $$x, y \in C \qquad \text{and} \qquad \lambda \in [0, 1],$$

   one has

   $$\lambda x + (1 - \lambda)y \in C;$$

2. C is *closed* in the topology of V.

16.10 Plane convex sets

A plane convex set is a subset of the plane containing every line segment joining any two of its points [369–371].

Definition 16.10.1 (Plane convex set)**.** A subset

$$C \subseteq \mathbb{R}^2$$

is called a *plane convex set* if, for all

$$x, y \in C \qquad \text{and} \qquad \lambda \in [0, 1],$$

one has

$$\lambda x + (1 - \lambda)y \in C.$$

Equivalently, C is a plane convex set if it contains the entire line segment joining any two of its points.

Chapter 17

N-Sets

In this chapter, we introduce N-sets.

17.1 Negative Set (N-Set)

A negative set assigns each element a membership degree in the interval $[-1, 0]$, representing negatively graded belonging, opposition, deficiency, or reverse preference.

Definition 17.1.1 (Negative Set (N-Set))**.** Let X be a nonempty set. A *negative set* (briefly, an *N-set*) on X is a mapping

$$\mu_A : X \to [-1, 0].$$

The pair

$$A = (X, \mu_A)$$

is called a *negative set* on X, and $\mu_A(x)$ is called the *negative membership degree* of $x \in X$ in A.

17.2 N-Fuzzy Set

An N-fuzzy set assigns to each element of a nonempty set a negative membership degree in $[-1, 0]$, providing a well-defined negative-valued fuzzy representation [335, 372, 373].

Definition 17.2.1 (N-fuzzy set)**.** Let X be a nonempty set. An *N-fuzzy set* A on X is a mapping

$$\mu_A^N : X \to [-1, 0].$$

Equivalently, one may represent A as

$$A = \{(x, \mu_A^N(x)) \mid x \in X\}.$$

The value

$$\mu_A^N(x)$$

is called the *negative membership degree* of x in A.

Example 17.2.2 (A concrete real-life example of an N-fuzzy set)**.** Let

$$X = \{p_1, p_2, p_3, p_4\}$$

be a set of four products sold in an online store.

Suppose that a customer wants to describe the set of products that are *unsuitable for purchase.* In this situation, instead of using positive membership degrees in $[0, 1]$, we use negative membership degrees in $[-1, 0]$, where values closer to -1 indicate stronger non-preference or stronger unsuitability.

Define an N-fuzzy set

$$A$$

on X by

$$\mu_A^N(p_1) = -0.90, \qquad \mu_A^N(p_2) = -0.65, \qquad \mu_A^N(p_3) = -0.30, \qquad \mu_A^N(p_4) = -0.10.$$

Equivalently,

$$A = \{(p_1, -0.90), (p_2, -0.65), (p_3, -0.30), (p_4, -0.10)\}.$$

Here:

- p_1 has a strongly negative membership degree, so it is regarded as highly unsuitable for purchase;
- p_2 is also considered unsuitable, but less strongly than p_1;
- p_3 has only a mild degree of unsuitability;
- p_4 is close to 0, so it is only slightly unsuitable.

Since

$$\mu_A^N : X \to [-1, 0],$$

the mapping μ_A^N defines an N-fuzzy set on X. Thus, A is a concrete real-life example of an N-fuzzy set representing negative customer evaluation of products.

Theorem 17.2.3 (Well-definedness of N-fuzzy sets)**.** *Let X be a nonempty set, and let*

$$\mu_A^N : X \to [-1, 0]$$

be a function. Then

$$A = (X, \mu_A^N)$$

is a well-defined N-fuzzy set on X.

Conversely, every N-fuzzy set on X arises from such a function

$$\mu_A^N : X \to [-1, 0].$$

Proof. Since

$$\mu_A^N : X \to [-1, 0]$$

is a function, for every

$$x \in X$$

the value

$$\mu_A^N(x) \in [-1, 0]$$

is uniquely determined. Hence each element of X is assigned a well-defined negative membership degree, and therefore

$$A = (X, \mu_A^N)$$

is a well-defined N-fuzzy set on X.

Conversely, by definition, any N-fuzzy set on X is precisely determined by a function

$$\mu_A^N : X \to [-1, 0].$$

Thus every N-fuzzy set on X arises in this way. □

17.3 N-Neutrosophic Set

An N-neutrosophic set assigns each element negative truth, indeterminacy, and falsity degrees in $[-1, 0]$, with total negative membership constrained between -3 and 0 [335].

Definition 17.3.1 (N-neutrosophic set)**.** Let X be a nonempty set. An *N-neutrosophic set* A on X is of the form

$$A = \{\langle x, T_A^N(x), I_A^N(x), F_A^N(x)\rangle \mid x \in X\},$$

where

$$T_A^N, I_A^N, F_A^N : X \to [-1, 0]$$

satisfy

$$-3 \le T_A^N(x) + I_A^N(x) + F_A^N(x) \le 0 \qquad \text{for all } x \in X.$$

Here

$$T_A^N(x), \quad I_A^N(x), \quad F_A^N(x)$$

are called the *negative truth-membership degree*, *negative indeterminacy-membership degree*, and *negative falsity-membership degree* of x, respectively.

Theorem 17.3.2 (Well-definedness of N-neutrosophic sets)**.** *Let X be a nonempty set, and let*

$$T_A^N, I_A^N, F_A^N : X \to [-1, 0]$$

satisfy

$$-3 \le T_A^N(x) + I_A^N(x) + F_A^N(x) \le 0 \qquad \textit{for all } x \in X.$$

Then

$$A = \{\langle x, T_A^N(x), I_A^N(x), F_A^N(x)\rangle \mid x \in X\}$$

is a well-defined N-neutrosophic set on X.

Conversely, every N-neutrosophic set on X arises from such functions

$$T_A^N, I_A^N, F_A^N : X \to [-1, 0].$$

Proof. Since

$$T_A^N, I_A^N, F_A^N : X \to [-1, 0],$$

for every

$$x \in X$$

the three values

$$T_A^N(x), \quad I_A^N(x), \quad F_A^N(x)$$

are well-defined elements of $[-1, 0]$.

Moreover, by assumption,

$$-3 \le T_A^N(x) + I_A^N(x) + F_A^N(x) \le 0 \qquad \text{for all } x \in X.$$

Hence each triple

$$\big(T_A^N(x), I_A^N(x), F_A^N(x)\big)$$

is admissible. Therefore

$$A = \{\langle x, T_A^N(x), I_A^N(x), F_A^N(x)\rangle \mid x \in X\}$$

is a well-defined N-neutrosophic set on X.

Conversely, by definition, any N-neutrosophic set on X is precisely determined by three functions

$$T_A^N, I_A^N, F_A^N : X \to [-1, 0]$$

satisfying the above inequality. Thus every N-neutrosophic set on X arises in this way. □

17.4 N-Uncertain Set

An N-uncertain set assigns each element an admissible negative uncertainty degree from a fixed model, extending uncertain-set theory from nonnegative to negative-valued domains.

Definition 17.4.1 (N-neutrosophic set)**.** Let X be a nonempty set. An *N-neutrosophic set* A on X is of the form

$$A = \{\langle x, T_A^N(x), I_A^N(x), F_A^N(x)\rangle \mid x \in X\},$$

where

$$T_A^N, I_A^N, F_A^N : X \to [-1, 0]$$

satisfy

$$-3 \le T_A^N(x) + I_A^N(x) + F_A^N(x) \le 0 \qquad \text{for all } x \in X.$$

Here

$$T_A^N(x), \quad I_A^N(x), \quad F_A^N(x)$$

are called the *negative truth-membership degree*, *negative indeterminacy-membership degree*, and *negative falsity-membership degree* of x, respectively.

Theorem 17.4.2 (Well-definedness of N-neutrosophic sets)**.** *Let X be a nonempty set, and let*

$$T_A^N, I_A^N, F_A^N : X \to [-1, 0]$$

satisfy

$$-3 \leq T_A^N(x) + I_A^N(x) + F_A^N(x) \leq 0 \qquad \text{for all } x \in X.$$

Then

$$A = \{\langle x, T_A^N(x), I_A^N(x), F_A^N(x)\rangle \mid x \in X\}$$

is a well-defined N-neutrosophic set on X.

Conversely, every N-neutrosophic set on X arises from such functions

$$T_A^N, I_A^N, F_A^N : X \to [-1, 0].$$

Proof. Since

$$T_A^N, I_A^N, F_A^N : X \to [-1, 0],$$

for every

$$x \in X$$

the three values

$$T_A^N(x), \quad I_A^N(x), \quad F_A^N(x)$$

are well-defined elements of $[-1, 0]$.

Moreover, by assumption,

$$-3 \leq T_A^N(x) + I_A^N(x) + F_A^N(x) \leq 0 \qquad \text{for all } x \in X.$$

Hence each triple

$$\big(T_A^N(x), I_A^N(x), F_A^N(x)\big)$$

is admissible. Therefore

$$A = \{\langle x, T_A^N(x), I_A^N(x), F_A^N(x)\rangle \mid x \in X\}$$

is a well-defined N-neutrosophic set on X.

Conversely, by definition, any N-neutrosophic set on X is precisely determined by three functions

$$T_A^N, I_A^N, F_A^N : X \to [-1, 0]$$

satisfying the above inequality. Thus every N-neutrosophic set on X arises in this way. □

Chapter 18

Probabilistic Set Theory

In this chapter, we discuss probabilistic set theory.

18.1 Probabilistic Set

A probabilistic set is a measurable subset of a probability space, whose occurrence is described by membership through an indicator function and quantified by probability [81, 374].

Definition 18.1.1 (Probabilistic set)**.** Let

$$(X, \mathcal{A}, \mathbb{P})$$

be a probability space. A *probabilistic set* on X is a measurable subset

$$A \in \mathcal{A}.$$

Equivalently, it may be represented by its indicator function

$$\chi_A : X \to \{0, 1\}, \qquad \chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

The quantity

$$\mathbb{P}(A)$$

is called the *probability* of the set A.

18.2 Probabilistic Fuzzy Set

A probabilistic fuzzy set assigns each element a random membership degree in $[0, 1]$, so belonging is modeled by measurable uncertainty rather than a fixed value [375–378].

Definition 18.2.1 (Probabilistic fuzzy set)**.** (cf. [375–378]) Let X be a nonempty set, and let

$$(\Omega, \mathcal{F}, \mathbb{P})$$

be a probability space. A *probabilistic fuzzy set* A on X is a mapping

$$\mu_A : X \times \Omega \to [0, 1]$$

such that, for every fixed

$$x \in X,$$

the section

$$\mu_A(x, \cdot) : \Omega \to [0, 1]$$

is an $\mathcal{F}$-measurable random variable.

Equivalently, for each $x \in X$, the membership degree of x in A is not a single number but an $[0, 1]$-valued random variable.

If, for a fixed $x \in X$, the random variable

$$\mu_A(x, \cdot)$$

admits a probability density function on $[0, 1]$, that density is called the *secondary probability density function* of the membership degree of x.

18.3 Probabilistic Neutrosophic Set

A probabilistic neutrosophic set assigns each element random truth, indeterminacy, and falsity degrees, representing neutrosophic information through measurable uncertainty on a probability space [379].

Definition 18.3.1 (Probabilistic neutrosophic set)**.** Let X be a nonempty set, and let

$$(\Omega, \mathcal{F}, \mathbb{P})$$

be a probability space. A *probabilistic neutrosophic set* A on X is given by three mappings

$$T_A, I_A, F_A : X \times \Omega \to [0, 1]$$

such that, for every fixed

$$x \in X,$$

the sections

$$T_A(x, \cdot), \qquad I_A(x, \cdot), \qquad F_A(x, \cdot)$$

are $\mathcal{F}$-measurable random variables, and

$$0 \le T_A(x, \omega) + I_A(x, \omega) + F_A(x, \omega) \le 3$$

for all

$$(x, \omega) \in X \times \Omega.$$

Equivalently, for each $x \in X$, the truth-membership, indeterminacy-membership, and falsity-membership degrees of x are $[0, 1]$-valued random variables.

18.4 Probabilistic Uncertain Set

A probabilistic uncertain set assigns each element a random admissible uncertainty degree from a fixed model, extending uncertain-set theory into a probabilistic framework.

Definition 18.4.1 (Borel structure on $\mathrm{Dom}(M)$)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. Denote by

$$\mathcal{B}([0,1]^k)$$

the Borel σ-algebra on $[0,1]^k$, and equip $\mathrm{Dom}(M)$ with the subspace Borel σ-algebra

$$\mathcal{B}(\mathrm{Dom}(M)) := \{\, B \cap \mathrm{Dom}(M) \mid B \in \mathcal{B}([0,1]^k) \,\}.$$

Definition 18.4.2 (Probabilistic uncertain set)**.** Let X be a nonempty set, let

$$(\Omega, \mathcal{F}, \mathbb{P})$$

be a probability space, and let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

A *probabilistic uncertain set of type M* on X is a pair

$$\mathcal{U}_M^{\mathbb{P}} = (X, \mu_M),$$

where

$$\mu_M : X \times \Omega \to \mathrm{Dom}(M)$$

satisfies the following condition:

for every fixed

$$x \in X,$$

the section

$$\mu_M(x, \cdot) : \Omega \to \mathrm{Dom}(M)$$

is

$$(\mathcal{F}, \mathcal{B}(\mathrm{Dom}(M)))\text{-measurable.}$$

Equivalently, for each $x \in X$, the uncertainty degree of x is not a fixed element of $\mathrm{Dom}(M)$, but a $\mathrm{Dom}(M)$-valued random variable on

$$(\Omega, \mathcal{F}, \mathbb{P}).$$

Theorem 18.4.3 (Well-definedness of probabilistic uncertain sets)**.** *Let X be a nonempty set, let*

$$(\Omega, \mathcal{F}, \mathbb{P})$$

be a probability space, let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$\mu_M : X \times \Omega \to \mathrm{Dom}(M)$$

be a mapping such that, for every fixed

$$x \in X,$$

the section

$$\mu_M(x, \cdot) : \Omega \to \mathrm{Dom}(M)$$

is

$$(\mathcal{F}, \mathcal{B}(\mathrm{Dom}(M)))\text{-}\textit{measurable.}$$

Then

$$\mathcal{U}_M^{\mathbb{P}} = (X, \mu_M)$$

is a well-defined probabilistic uncertain set of type M on X.

Moreover:

1. *for every*

   $$x \in X, \qquad \omega \in \Omega,$$

   one has

   $$\mu_M(x, \omega) \in \mathrm{Dom}(M);$$

2. *for every fixed*

   $$x \in X,$$

   the section

   $$\mu_M(x, \cdot)$$

   is a well-defined $\mathrm{Dom}(M)$*-valued random variable;*

3. *conversely, every family*

   $$\{Z_x : \Omega \to \mathrm{Dom}(M)\}_{x \in X}$$

   of

   $$(\mathcal{F}, \mathcal{B}(\mathrm{Dom}(M)))\text{-}\textit{measurable}$$

   $\mathrm{Dom}(M)$*-valued random variables determines a unique probabilistic uncertain set of type M on X by*

   $$\mu_M(x, \omega) := Z_x(\omega).$$

Proof. Since

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

the set

$$\mathrm{Dom}(M)$$

is well-defined, and hence the subspace Borel σ-algebra

$$\mathcal{B}(\mathrm{Dom}(M)) = \{\, B \cap \mathrm{Dom}(M) \mid B \in \mathcal{B}([0, 1]^k) \,\}$$

is well-defined as well.

Now let

$$x \in X \qquad \text{and} \qquad \omega \in \Omega.$$

Because

$$\mu_M : X \times \Omega \to \mathrm{Dom}(M),$$

the value

$$\mu_M(x, \omega)$$

is a well-defined element of

$$\mathrm{Dom}(M).$$

Thus every realized uncertainty degree is admissible for the model M. This proves (1).

Next, fix

$$x \in X.$$

By assumption, the section

$$\mu_M(x, \cdot) : \Omega \to \mathrm{Dom}(M)$$

is

$$(\mathcal{F}, \mathcal{B}(\mathrm{Dom}(M)))\text{-measurable.}$$

Therefore

$$\mu_M(x, \cdot)$$

is a well-defined $\mathrm{Dom}(M)$-valued random variable on

$$(\Omega, \mathcal{F}, \mathbb{P}).$$

Hence each element

$$x \in X$$

is assigned a random admissible uncertainty degree, and so

$$\mathcal{U}_M^{\mathbb{P}} = (X, \mu_M)$$

is a well-defined probabilistic uncertain set of type M on X. This proves (2) and the main statement.

Conversely, let

$$\{Z_x : \Omega \to \mathrm{Dom}(M)\}_{x \in X}$$

be a family of

$$(\mathcal{F}, \mathcal{B}(\mathrm{Dom}(M)))\text{-measurable}$$

$\mathrm{Dom}(M)$-valued random variables. Define

$$\mu_M : X \times \Omega \to \mathrm{Dom}(M)$$

by

$$\mu_M(x, \omega) := Z_x(\omega).$$

Then, for each fixed

$$x \in X,$$

the section

$$\mu_M(x, \cdot) = Z_x$$

is measurable. Hence

$$(X, \mu_M)$$

satisfies the definition of a probabilistic uncertain set of type M.

Uniqueness is immediate, because the map

$$\mu_M$$

is completely determined by the values

$$\mu_M(x, \omega) = Z_x(\omega) \qquad (x \in X,\ \omega \in \Omega).$$

This proves (3). □

18.5 Probabilistic Rough Set

A probabilistic rough set approximates a target set using conditional probabilities over equivalence classes, with thresholds distinguishing positive, boundary, and negative decision regions [39, 380–382].

Definition 18.5.1 (Probabilistic rough set)**.** Let U be a nonempty universe, let

$$E \subseteq U \times U$$

be an equivalence relation, and let

$$C \subseteq U.$$

For each $x \in U$, denote by

$$[x]_E$$

the equivalence class of x, and let

$$P(C \mid [x]_E)$$

be the conditional probability that an object in $[x]_E$ belongs to C. Let

$$0 \leq \beta < \alpha \leq 1.$$

The *probabilistic rough set* of C with respect to the threshold pair

$$(\alpha, \beta)$$

is the approximation determined by

$$\underline{C}_{\alpha,\beta} := \{\, x \in U \mid P(C \mid [x]_E) \geq \alpha \,\},$$

and

$$\overline{C}_{\alpha,\beta} := \{\, x \in U \mid P(C \mid [x]_E) > \beta \,\}.$$

Equivalently, it classifies objects into positive, boundary, and negative regions according to whether the conditional probability is high, intermediate, or low.

Chapter 19

Graphic Set Theory

In this chapter, we explain graphic set theory.

19.1 Graphic Set

A graphic set assigns to each subgraph of a fixed graph a subset of a universe, linking graph structure with ordinary set-valued information.

Definition 19.1.1 (Graphic set)**.** Let U be a nonempty universe, and let

$$G = (V, E)$$

be a finite graph. Denote by

$$\mathcal{P}(G) := \{\, H = (V_H, E_H) \mid V_H \subseteq V,\ E_H \subseteq E \cap (V_H \times V_H) \,\}$$

the set of all subgraphs of G.

A *graphic set* on (U, G) is a mapping

$$S : \mathcal{P}(G) \to \mathcal{P}(U).$$

Equivalently, to each subgraph

$$H \in \mathcal{P}(G),$$

the mapping S assigns a subset

$$S(H) \subseteq U.$$

19.2 Graphic Neutrosophic Set

A graphic neutrosophic set assigns to each subgraph a neutrosophic set on the universe, giving truth, indeterminacy, and falsity degrees for every element [383].

Definition 19.2.1 (Graphic neutrosophic set)**.** [383] Let U be a nonempty universe, let

$$G = (V, E)$$

be a finite graph, and let $\mathcal{P}(G)$ be the set of all subgraphs of G. A *graphic neutrosophic set* on (U, G) is a mapping

$$F : \mathcal{P}(G) \to \mathcal{N}(U),$$

where

$$\mathcal{N}(U) := \{(T, I, F) \mid T, I, F : U \to [0,1],\ 0 \le T(x) + I(x) + F(x) \le 3 \text{ for all } x \in U\}.$$

Equivalently, for each

$$H \in \mathcal{P}(G)$$

and each

$$x \in U,$$

one has

$$F(H)(x) = \big(T_H(x), I_H(x), F_H(x)\big),$$

where

$$T_H(x), I_H(x), F_H(x) \in [0,1]$$

and

$$0 \le T_H(x) + I_H(x) + F_H(x) \le 3.$$

Example 19.2.2 (A concrete real-life example of a graphic neutrosophic set)**.** Let

$$U = \{r_1, r_2, r_3\}$$

be a set of three delivery regions in a city, where

$$r_1 = \text{North district}, \qquad r_2 = \text{Central district}, \qquad r_3 = \text{South district}.$$

Let

$$G = (V, E)$$

be a road network graph, where

$$V = \{a, b, c\}$$

represents three distribution centers, and

$$E = \{\{a, b\}, \{b, c\}\}$$

represents direct road connections between them.

Each subgraph

$$H \in \mathcal{P}(G)$$

is interpreted as a currently available transportation configuration of the road network. For each such subgraph, we assign a neutrosophic set on U describing the degree to which each delivery region is regarded as *well serviceable* under that road configuration.

Define
$$F : \mathcal{P}(G) \to \mathcal{N}(U)$$
as follows.

First, consider the full network
$$H_1 = (\{a, b, c\}, \{\{a, b\}, \{b, c\}\}).$$
Assume
$$F(H_1)(r_1) = (0.90, 0.05, 0.10), \qquad F(H_1)(r_2) = (0.95, 0.03, 0.05), \qquad F(H_1)(r_3) = (0.85, 0.08, 0.12).$$
This means that when all roads are available, all three regions are highly serviceable, with low indeterminacy and low falsity.

Next, consider the subgraph
$$H_2 = (\{a, b, c\}, \{\{a, b\}\}),$$
where the road between b and c is unavailable. Suppose that
$$F(H_2)(r_1) = (0.88, 0.06, 0.12), \qquad F(H_2)(r_2) = (0.70, 0.15, 0.25), \qquad F(H_2)(r_3) = (0.35, 0.30, 0.55).$$
Here the South district becomes much less serviceable, and the uncertainty also increases because one main road is missing.

Finally, consider the subgraph
$$H_3 = (\{b, c\}, \{\{b, c\}\}),$$
which represents a smaller active network involving only centers b and c. Let
$$F(H_3)(r_1) = (0.20, 0.25, 0.70), \qquad F(H_3)(r_2) = (0.65, 0.20, 0.25), \qquad F(H_3)(r_3) = (0.80, 0.10, 0.15).$$
Thus, under this subgraph, the North district is poorly served, while the South district remains relatively well served.

In each case, for every $r_i \in U$, the assigned values satisfy
$$0 \leq T_{H_j}(r_i) + I_{H_j}(r_i) + F_{H_j}(r_i) \leq 3,$$
so each
$$F(H_j)$$
is a single-valued neutrosophic set on U.

Therefore, F is a concrete example of a graphic neutrosophic set on (U, G): each subgraph of the road network determines a neutrosophic evaluation of how well the city regions can be served under that particular transportation structure.

19.3 Graphic Uncertain Set

A graphic uncertain set assigns to each subgraph an uncertain set of fixed model type, so every element receives an admissible uncertainty degree tuple.

Definition 19.3.1 (Graphic uncertain set)**.** Let U be a nonempty universe, let

$$G = (V, E)$$

be a finite graph, and let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \geq 1$. Let $\mathcal{P}(G)$ be the set of all subgraphs of G.

A *graphic uncertain set of type* M on (U, G) is a mapping

$$F : \mathcal{P}(G) \to \mathrm{Dom}(M)^U.$$

Equivalently, for each

$$H \in \mathcal{P}(G),$$

the value

$$F(H) : U \to \mathrm{Dom}(M)$$

is an uncertain set of type M on U. Thus, each subgraph of G determines an M-valued uncertainty-degree function on U.

19.4 Graphic Soft Set

A graphic soft set is a parameterized family of graphic sets, assigning to each parameter a subgraph-indexed set-valued mapping on the universe [384].

Definition 19.4.1 (Graphic soft set)**.** [384] Let U be a nonempty universe, let

$$G = (V, E)$$

be a finite graph, let $\mathcal{P}(G)$ be the set of all subgraphs of G, let E_0 be a set of parameters, and let

$$A \subseteq E_0.$$

A *graphic soft set* over (U, G) is a pair

$$(\Phi, A),$$

where

$$\Phi : A \to \mathcal{P}(U)^{\mathcal{P}(G)}.$$

Equivalently, for each parameter

$$a \in A,$$

the value

$$\Phi(a) : \mathcal{P}(G) \to \mathcal{P}(U)$$

is a graphic set on (U, G). Hence a graphic soft set is a parameterized family of graphic sets.

Example 19.4.2 (A concrete real-life example of a graphic soft set)**.** Let

$$U = \{z_1, z_2, z_3\}$$

be a set of three city zones, where

$$z_1 = \text{downtown}, \qquad z_2 = \text{residential area}, \qquad z_3 = \text{industrial area}.$$

Let

$$G = (V, E)$$

be a finite graph representing a transportation network, where

$$V = \{v_1, v_2, v_3\}$$

are three stations and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\}$$

are direct connections between them.

Let the parameter set be

$$E_0 = \{\text{high demand}, \text{emergency service}, \text{night service}\},$$

and choose

$$A = \{\text{high demand}, \text{emergency service}\} \subseteq E_0.$$

We define a mapping

$$\Phi : A \to \mathcal{P}(U)^{\mathcal{P}(G)}.$$

Thus, for each parameter $a \in A$, the value

$$\Phi(a) : \mathcal{P}(G) \to \mathcal{P}(U)$$

assigns to each subgraph $H \in \mathcal{P}(G)$ a subset of city zones in U.

For simplicity, consider the following two subgraphs of G:

$$H_1 = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\}),$$

the full transportation network, and

$$H_2 = (\{v_1, v_2\}, \{\{v_1, v_2\}\}),$$

a reduced network in which only one connection is available.

Define

$$\Phi(\text{high demand})(H_1) = \{z_1, z_2\}, \qquad \Phi(\text{high demand})(H_2) = \{z_1\},$$

and

$$\Phi(\text{emergency service})(H_1) = \{z_1, z_2, z_3\}, \qquad \Phi(\text{emergency service})(H_2) = \{z_1, z_2\}.$$

These assignments may be interpreted as follows:

- under the full network H_1, the zones z_1 and z_2 are classified as high-demand service areas;
- under the reduced network H_2, only z_1 remains in the high-demand category;
- under the parameter *emergency service*, all three zones are serviceable when the full network is available;
- under the reduced network, only z_1 and z_2 remain covered for emergency service.

Hence, each parameter in A determines a graphic set on (U, G), and therefore

$$(\Phi, A)$$

is a graphic soft set over (U, G).

In practical terms, this graphic soft set describes how different service-related parameters produce different zone selections depending on the currently available transportation subgraph.

Chapter 20

Partially Ordered Set Theory

In this chapter, we examine partially ordered set theory.

20.1 Partially Ordered Set

A partially ordered set is a set equipped with a binary relation that is reflexive, antisymmetric, and transitive, allowing some elements to remain incomparable [385–388].

Definition 20.1.1 (Partially ordered set)**.** [385, 386] A *partially ordered set* (or *poset*) is a pair

$$(P, \leq)$$

consisting of a set P and a binary relation

$$\leq \subseteq P \times P$$

such that, for all $x, y, z \in P$:

1. $$x \leq x \qquad \text{(reflexivity);}$$

2. if
$$x \leq y \quad \text{and} \quad y \leq x,$$
then
$$x = y \qquad \text{(antisymmetry);}$$

3. if
$$x \leq y \quad \text{and} \quad y \leq z,$$
then
$$x \leq z \qquad \text{(transitivity).}$$

Example 20.1.2 (A concrete real-life example of a partially ordered set)**.** Let

$$P = \{r, d, u, i, s\}$$

be a set of five tasks in a software development project, where

$$r = \text{requirements analysis}, \qquad d = \text{database design}, \qquad u = \text{user-interface design},$$

$$i = \text{implementation}, \qquad s = \text{system integration}.$$

Define a binary relation

$$\leq$$

on P by declaring

$$x \leq y$$

to mean that task x must be completed no later than task y, that is, x is a prerequisite for y (possibly $x = y$).

Assume the project dependencies are as follows:

$$r \leq d, \qquad r \leq u, \qquad d \leq i, \qquad u \leq i, \qquad i \leq s.$$

Together with reflexivity and the relations implied by transitivity, this gives the partial order

$$(P, \leq).$$

For example, by transitivity,

$$r \leq i \qquad \text{and} \qquad r \leq s,$$

because requirements analysis must be completed before implementation and system integration.

On the other hand, d and u are incomparable:

$$d \not\leq u \qquad \text{and} \qquad u \not\leq d,$$

since database design and user-interface design can proceed independently after the requirements analysis stage.

Thus, $(P, \leq)$ is a partially ordered set:

- reflexivity holds because every task is regarded as preceding itself;
- antisymmetry holds because if two tasks precede each other, then they are the same task;
- transitivity holds because prerequisite chains can be composed.

Hence, this poset models a real-life project schedule in which some tasks are ordered by dependency, while others remain incomparable. An illustrative Hasse diagram of this partially ordered set is shown in Fig. 20.1.

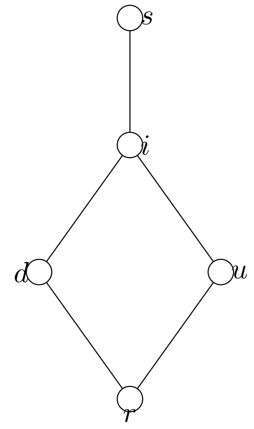

Figure 20.1: Hasse diagram of the project-task poset in the example. Here r denotes requirements analysis, d database design, u user-interface design, i implementation, and s system integration.

20.2 Totally ordered set

A totally ordered set is a poset in which every two elements are comparable, so for any x and y, either $x \leq y$ or $y \leq x$ holds [389–392].

Definition 20.2.1 (Totally ordered set)**.** [389, 390] A *totally ordered set* (or *linear order*) is a partially ordered set

$$(T, \leq)$$

such that, for all $x, y \in T$, one has

$$x \leq y \quad \text{or} \quad y \leq x.$$

This additional property is called *totality* (or *comparability*).

20.3 Fuzzy Partially Ordered Set

A fuzzy partially ordered set assigns each ordered pair a degree in $[0, 1]$, satisfying fuzzy reflexivity, antisymmetry at degree one, and transitivity via minimum condition [393–395].

Definition 20.3.1 (Fuzzy partially ordered set)**.** [393, 394] Let X be a nonempty set. A *fuzzy partial order* on X is a mapping

$$e : X \times X \to [0, 1]$$

such that, for all $x, y, z \in X$,

1. $$e(x, x) = 1 \qquad \text{(reflexivity);}$$
2. if
$$e(x, y) = 1 \quad \text{and} \quad e(y, x) = 1,$$
then
$$x = y \qquad \text{(antisymmetry);}$$

3.
$$e(x,y) \wedge e(y,z) \le e(x,z) \qquad \text{(transitivity).}$$

The pair
$$(X, e)$$
is called a *fuzzy partially ordered set* (or *fuzzy poset*).

20.4 Neutrosophic Partially Ordered Set

A neutrosophic partially ordered set assigns truth, indeterminacy, and falsity degrees to each ordered pair, satisfying reflexivity, antisymmetry, and transitivity by componentwise conditions on triples [396].

Definition 20.4.1 (Neutrosophic partially ordered set)**.** Let X be a nonempty set. A *neutrosophic partial order* on X is a mapping
$$R_N : X \times X \to [0,1]^3, \qquad R_N(x,y) = \big(T(x,y), I(x,y), F(x,y)\big),$$
such that, for all $x, y, z \in X$:

1.
$$T(x,x) = 1, \qquad I(x,x) = 0, \qquad F(x,x) = 0 \qquad \text{(reflexivity);}$$

2. if
$$T(x,y) = T(y,x) = 1, \qquad I(x,y) = I(y,x) = 0, \qquad F(x,y) = F(y,x) = 0,$$
then
$$x = y \qquad \text{(antisymmetry);}$$

3.
$$T(x,z) \ge \min\{T(x,y), T(y,z)\},$$
$$I(x,z) \le \max\{I(x,y), I(y,z)\},$$
$$F(x,z) \le \max\{F(x,y), F(y,z)\} \qquad \text{(transitivity).}$$

The pair
$$(X, R_N)$$
is called a *neutrosophic partially ordered set.*

20.5 Uncertain Partially Ordered Set

An uncertain partially ordered set is an uncertain relation on $X \times X$ valued in an order frame, satisfying uncertain reflexivity, antisymmetry, and transitivity of prescribed type.

Definition 20.5.1 (Componentwise order on $\mathrm{Dom}(M)$)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. For

$$a = (a_1, \ldots, a_k), \qquad b = (b_1, \ldots, b_k) \in \mathrm{Dom}(M),$$

define

$$a \preceq b \iff a_j \leq b_j \quad \text{for all } j = 1, \ldots, k.$$

Definition 20.5.2 (Uncertain order frame)**.** Let M be an uncertain model with

$$D := \mathrm{Dom}(M) \subseteq [0,1]^k.$$

Assume that:

1. a distinguished element
$$e_M \in D$$
is fixed;
2. a binary operation
$$\otimes : D \times D \to D$$
is fixed.

Then

$$\mathfrak{O}_M := (M, \preceq, e_M, \otimes)$$

is called an *uncertain order frame*.

Definition 20.5.3 (Uncertain partially ordered set)**.** Let X be a nonempty set, and let

$$\mathfrak{O}_M = (M, \preceq, e_M, \otimes)$$

be an uncertain order frame, where

$$D = \mathrm{Dom}(M).$$

An *uncertain partial order of type $\mathfrak{O}_M$ on X* is an uncertain set

$$\mathcal{R} = (X \times X, \mu_R)$$

of type M on $X \times X$, that is,

$$\mu_R : X \times X \to D,$$

satisfying, for all $x, y, z \in X$:

1.
$$\mu_R(x,x) = e_M \qquad \text{(uncertain reflexivity);}$$

2. if
$$\mu_R(x,y) = e_M \quad \text{and} \quad \mu_R(y,x) = e_M,$$
then
$$x = y \qquad \text{(uncertain antisymmetry);}$$

3.
$$\mu_R(x,y) \otimes \mu_R(y,z) \preceq \mu_R(x,z) \qquad \text{(uncertain transitivity).}$$

The pair
$$(X, \mathcal{R})$$
is called an *uncertain partially ordered set.*

Theorem 20.5.4 (Well-definedness of uncertain partially ordered sets)**.** *Let X be a nonempty set, and let*
$$\mathfrak{O}_M = (M, \preceq, e_M, \otimes)$$
be an uncertain order frame with
$$D = \mathrm{Dom}(M) \subseteq [0,1]^k.$$
Suppose that
$$\mu_R : X \times X \to D$$
satisfies the three axioms in the above definition. Then
$$\mathcal{R} = (X \times X, \mu_R)$$
is a well-defined uncertain set of type M on $X \times X$, and hence
$$(X, \mathcal{R})$$
is a well-defined uncertain partially ordered set.

Conversely, every uncertain partially ordered set on X arises from such a map
$$\mu_R : X \times X \to D$$
satisfying those three axioms.

Proof. Since X is a set, the Cartesian product
$$X \times X$$
is also a set. Since
$$\mu_R : X \times X \to D = \mathrm{Dom}(M),$$
the pair
$$\mathcal{R} = (X \times X, \mu_R)$$
is, by definition, an uncertain set of type M on $X \times X$.

Next, the componentwise relation

$$\preceq$$

is well-defined on D, because $D \subseteq [0,1]^k$ and coordinatewise comparison is meaningful for all elements of D. Also,

$$e_M \in D$$

by assumption, so the equality

$$\mu_R(x,x) = e_M$$

is meaningful for every $x \in X$. Furthermore, because

$$\otimes : D \times D \to D,$$

for any $x, y, z \in X$ one has

$$\mu_R(x,y) \otimes \mu_R(y,z) \in D,$$

and therefore the comparison

$$\mu_R(x,y) \otimes \mu_R(y,z) \preceq \mu_R(x,z)$$

is meaningful as well.

Hence all three axioms are well-formed statements on the uncertain set

$$\mathcal{R} = (X \times X, \mu_R).$$

Therefore

$$(X, \mathcal{R})$$

is a well-defined uncertain partially ordered set.

Conversely, by the definition of an uncertain partially ordered set, any such object is precisely an uncertain set

$$\mathcal{R} = (X \times X, \mu_R)$$

of type M whose degree map

$$\mu_R : X \times X \to D$$

satisfies the above three axioms. Thus every uncertain partially ordered set arises in this way. □

20.6 Preordered set

A preordered set is a set equipped with a binary relation that is reflexive and transitive, allowing comparable structure without necessarily requiring antisymmetry [397–400].

Definition 20.6.1 (Preordered set)**.** [397, 398] A *preordered set* (or *preorder*) is a pair

$$(P, \preceq)$$

consisting of a set P and a binary relation

$$\preceq \subseteq P \times P$$

such that, for all $x, y, z \in P$,

1.
$$x \preceq x \qquad \text{(reflexivity);}$$

2. if
$$x \preceq y \quad \text{and} \quad y \preceq z,$$
then
$$x \preceq z \qquad \text{(transitivity).}$$

Chapter 21

Lattice-valued Set Theory

In this chapter, we explain lattice-valued set theory.

21.1 Lattice-valued set

A lattice-valued set assigns to each element of a nonempty set a membership value from a lattice, replacing numeric grades with abstract ordered membership values [401–405].

Definition 21.1.1 (Lattice)**.** A *lattice* is a partially ordered set

$$(L, \leq)$$

such that, for every

$$x, y \in L,$$

both the greatest lower bound

$$x \wedge y$$

and the least upper bound

$$x \vee y$$

exist in L. Here

$$x \wedge y$$

is called the *meet* of x and y, and

$$x \vee y$$

is called the *join* of x and y.

As extended concepts of lattices, fuzzy lattices [406, 407], neutrosophic lattices [396], and hyperlattices [408, 409] are also known. Next, we present the definition of a lattice-valued set based on a lattice.

Definition 21.1.2 (Lattice-valued set)**.** Let X be a nonempty set, and let

$$(L, \vee, \wedge)$$

be a lattice. A *lattice-valued set* A on X is a mapping

$$\mu_A : X \to L.$$

For each $x \in X$, the value $\mu_A(x)$ is called the *membership value* of x in A.

Equivalently, A may be represented as

$$A = \{(x, \mu_A(x)) \mid x \in X\}.$$

21.2 Lattice-valued Fuzzy Set

A lattice-valued fuzzy set, or L-fuzzy set, generalizes fuzzy sets by taking membership degrees in a lattice rather than only in the unit interval [410–412].

Definition 21.2.1 (Lattice-valued fuzzy set)**.** Let X be a nonempty set, and let

$$(L, \leq)$$

be a lattice. A *lattice-valued fuzzy set* (or *L-fuzzy set*) on X is a mapping

$$\mu_A : X \to L.$$

For each $x \in X$, the value $\mu_A(x) \in L$ is called the *membership value* of x in A.

Equivalently, one may represent A as

$$A = \{(x, \mu_A(x)) \mid x \in X\}.$$

21.3 Lattice-valued Neutrosophic Set

A lattice-valued neutrosophic set assigns truth, indeterminacy, and falsity values from a complete distributive lattice to each element, extending neutrosophic modeling beyond numerical scales [413–415].

Definition 21.3.1 (Lattice-valued neutrosophic set)**.** [413] Let X be a nonempty set, and let

$$(L, \leq)$$

be a nontrivial complete distributive lattice. A *lattice-valued neutrosophic set* A on X is determined by three mappings

$$T_A, I_A, F_A : X \to L,$$

called the *truth-membership function*, *indeterminacy-membership function*, and *falsity-membership function*, respectively.

Equivalently, A may be represented as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}.$$

Example 21.3.2 (A concrete real-life example of a lattice-valued neutrosophic set)**.** Let

$$X = \{p_1, p_2, p_3\}$$

be a set of three patients in a hospital, and let

$$L = \left\{0, \frac{1}{2}, 1\right\}$$

be the three-element chain with the usual order

$$0 \le \frac{1}{2} \le 1.$$

Then $(L, \le)$ is a nontrivial complete distributive lattice.

Suppose that we want to describe the set of patients who *require urgent observation.* We interpret the lattice values as follows:

$$0 = \text{low}, \qquad \frac{1}{2} = \text{moderate}, \qquad 1 = \text{high}.$$

Define three mappings

$$T_A, I_A, F_A : X \to L$$

by

$$\begin{aligned} T_A(p_1) &= 1, & I_A(p_1) &= \frac{1}{2}, & F_A(p_1) &= 0, \\ T_A(p_2) &= \frac{1}{2}, & I_A(p_2) &= 1, & F_A(p_2) &= \frac{1}{2}, \\ T_A(p_3) &= 0, & I_A(p_3) &= \frac{1}{2}, & F_A(p_3) &= 1. \end{aligned}$$

Equivalently, the lattice-valued neutrosophic set A can be written as

$$A = \left\{ \left\langle p_1, 1, \frac{1}{2}, 0 \right\rangle, \left\langle p_2, \frac{1}{2}, 1, \frac{1}{2} \right\rangle, \left\langle p_3, 0, \frac{1}{2}, 1 \right\rangle \right\}.$$

This means that:

- patient p_1 has a high truth-membership degree for urgent observation, moderate indeterminacy, and low falsity;
- patient p_2 has moderate truth, high indeterminacy, and moderate falsity, so the decision is quite uncertain;
- patient p_3 has low truth-membership and high falsity-membership, so this patient is not regarded as requiring urgent observation.

Therefore, A is a concrete example of a lattice-valued neutrosophic set on X, since each patient is assigned truth, indeterminacy, and falsity values from the lattice L rather than from the real interval $[0, 1]$.

21.4 Lattice-valued Uncertain Set

A lattice-valued uncertain set assigns each element an admissible lattice-valued degree tuple whose image under an order-preserving map satisfies the constraints of a fixed uncertain model.

Definition 21.4.1 (Induced lattice-valued degree-domain)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. Let

$$(L, \vee, \wedge)$$

be a lattice, and let

$$\eta : L \to [0,1]$$

be an order-preserving map.

Define

$$\eta^{(k)} : L^k \to [0,1]^k$$

by

$$\eta^{(k)}(a_1, \ldots, a_k) := (\eta(a_1), \ldots, \eta(a_k)).$$

Assume that

$$\mathrm{Dom}_L(M) := (\eta^{(k)})^{-1}(\mathrm{Dom}(M)) = \{\, a \in L^k \mid \eta^{(k)}(a) \in \mathrm{Dom}(M) \,\} \neq \emptyset.$$

Then $\mathrm{Dom}_L(M)$ is called the *lattice-valued degree-domain induced by* M.

Definition 21.4.2 (Lattice-valued uncertain set)**.** Let X be a nonempty set, let M be an uncertain model, and let $\mathrm{Dom}_L(M)$ be the induced lattice-valued degree-domain as above. A *lattice-valued uncertain set of type* (M, L, η) on X is a pair

$$\mathcal{U}_L = (X, \mu_{M,L}),$$

where

$$\mu_{M,L} : X \to \mathrm{Dom}_L(M)$$

is called the *lattice-valued uncertainty-degree function.*

Equivalently, for each $x \in X$,

$$\mu_{M,L}(x) = (a_1(x), \ldots, a_k(x)) \in L^k$$

and

$$\big(\eta(a_1(x)), \ldots, \eta(a_k(x))\big) \in \mathrm{Dom}(M).$$

Theorem 21.4.3 (Well-definedness of lattice-valued uncertain sets)**.** *Let X be a nonempty set, let M be an uncertain model with*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

let $(L, \vee, \wedge)$ be a lattice, and let

$$\eta : L \to [0,1]$$

be an order-preserving map such that

$$\mathrm{Dom}_L(M) \neq \emptyset.$$

Then $\mathrm{Dom}_L(M)$ *is a well-defined nonempty subset of* L^k*. Consequently, every mapping*

$$\mu_{M,L} : X \to \mathrm{Dom}_L(M)$$

determines a well-defined lattice-valued uncertain set of type (M, L, η) *on* X*.*

Conversely, every lattice-valued uncertain set of type (M, L, η) *arises in this way.*

Proof. Since

$$\eta : L \to [0, 1]$$

is a map, the componentwise map

$$\eta^{(k)} : L^k \to [0, 1]^k$$

is well-defined. Because

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

the inverse image

$$(\eta^{(k)})^{-1}(\mathrm{Dom}(M))$$

is a well-defined subset of L^k. By assumption,

$$\mathrm{Dom}_L(M) = (\eta^{(k)})^{-1}(\mathrm{Dom}(M)) \neq \emptyset.$$

Hence $\mathrm{Dom}_L(M)$ is a well-defined nonempty subset of L^k.

Now let

$$\mu_{M,L} : X \to \mathrm{Dom}_L(M).$$

Then, for every $x \in X$,

$$\mu_{M,L}(x) \in \mathrm{Dom}_L(M) \subseteq L^k.$$

By the definition of $\mathrm{Dom}_L(M)$, this implies

$$\eta^{(k)}\big(\mu_{M,L}(x)\big) \in \mathrm{Dom}(M).$$

Therefore each value of $\mu_{M,L}$ is an admissible lattice-valued uncertainty degree of type (M, L, η), and so

$$\mathcal{U}_L = (X, \mu_{M,L})$$

is a well-defined lattice-valued uncertain set on X.

Conversely, by definition, every lattice-valued uncertain set of type (M, L, η) is precisely a pair

$$\mathcal{U}_L = (X, \mu_{M,L})$$

with

$$\mu_{M,L} : X \to \mathrm{Dom}_L(M).$$

Hence every such object arises in this way. □

21.5 L-valued up-sets and down-sets

A lattice-valued up-set assigns each element a lattice value that never decreases along the order, so larger elements receive greater or equal values [416]. A lattice-valued down-set assigns each element a lattice value that never increases along the order, so larger elements receive smaller or equal values [416, 417].

Definition 21.5.1 (Lattice-valued up-set)**.** Let

$$(X, \leq)$$

be a poset, and let

$$(L, \leq)$$

be a complete lattice with bottom element 0 and top element 1. A mapping

$$\alpha : X \to L$$

is called a *lattice-valued up-set* (or *L-valued up-set*) on X if, for all

$$x, y \in X,$$

one has

$$x \leq y \implies \alpha(x) \leq \alpha(y).$$

Equivalently, α is an isotone mapping from X to L.

Definition 21.5.2 (Lattice-valued down-set)**.** Let

$$(X, \leq)$$

be a poset, and let

$$(L, \leq)$$

be a complete lattice with bottom element 0 and top element 1. A mapping

$$\mu : X \to L$$

is called a *lattice-valued down-set* (or *L-valued down-set*) on X if, for all

$$x, y \in X,$$

one has

$$x \leq y \implies \mu(y) \leq \mu(x).$$

Equivalently, μ is an antitone mapping from X to L.

21.6 L-powersets

An L-powerset is the set of all L-valued subsets of a set, represented by membership functions from the set into a lattice [418–420].

Definition 21.6.1 (L-powerset)**.** Let X be a nonempty set, and let

$$(L, \leq)$$

be a complete lattice. The L-*powerset* of X, denoted by

$$\mathcal{P}_L(X),$$

is defined by

$$\mathcal{P}_L(X) := L^X = \{\mu \mid \mu : X \to L\}.$$

Each element

$$\mu \in \mathcal{P}_L(X)$$

is called an L-*subset* (or *lattice-valued subset*) of X.

Equivalently, the L-powerset of X is the set of all lattice-valued membership functions on X.

Chapter 22

Triangular Set Theory

In this chapter, we explain triangular set theory.

22.1 Triangular Fuzzy Sets

A triangular fuzzy set is a fuzzy number on $\mathbb{R}$ determined by three parameters (a, b, c), with linear rise, peak at b, and linear decline [421–423].

Definition 22.1.1 (Triangular fuzzy set)**.** [422, 423] Let $a, b, c \in \mathbb{R}$ satisfy

$$a \leq b \leq c, \qquad a < c.$$

A *triangular fuzzy set* A on $\mathbb{R}$ is the fuzzy set with membership function

$$\mu_A : \mathbb{R} \to [0, 1]$$

defined by

$$\mu_A(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \leq x \leq b,\ a < b, \\ \dfrac{c-x}{c-b}, & b \leq x \leq c,\ b < c, \\ 0, & x > c. \end{cases}$$

Equivalently, one writes

$$A = (a, b, c).$$

Here $[a, c]$ is the support of A, and b is its peak point.

22.2 Triangular Neutrosophic Sets

A triangular neutrosophic set assigns each element triangular truth, indeterminacy, and falsity values, using ordered triples in $[0,1]$ to represent neutrosophic uncertainty [424–428].

Definition 22.2.1 (Triangular neutrosophic set)**.** [424, 425] Let X be a nonempty set, and let

$$\mathbb{T}([0,1]) := \{(a,b,c) \in [0,1]^3 \mid a \le b \le c\}$$

denote the set of all triangular fuzzy numbers on $[0,1]$.

A *triangular neutrosophic set* A on X is of the form

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\},$$

where

$$T_A, I_A, F_A : X \to \mathbb{T}([0,1]).$$

For each $x \in X$, write

$$T_A(x) = (t_1(x), t_2(x), t_3(x)), \quad I_A(x) = (i_1(x), i_2(x), i_3(x)), \quad F_A(x) = (f_1(x), f_2(x), f_3(x)),$$

with

$$0 \le t_1(x) \le t_2(x) \le t_3(x) \le 1,$$

$$0 \le i_1(x) \le i_2(x) \le i_3(x) \le 1,$$

$$0 \le f_1(x) \le f_2(x) \le f_3(x) \le 1,$$

and

$$0 \le t_3(x) + i_3(x) + f_3(x) \le 3 \qquad \text{for all } x \in X.$$

Here $T_A(x)$, $I_A(x)$, and $F_A(x)$ are called the triangular truth-membership, triangular indeterminacy-membership, and triangular falsity-membership values of x, respectively.

Example 22.2.2 (A concrete example of a MultiGrey set)**.** Let

$$U = \{p_1, p_2, p_3\}$$

be a set of three products sold by a company, and let

$$n = 2.$$

Assume that the two dimensions correspond to the following two evaluation aspects:

- the first grey membership degree represents the interval-valued assessment of *customer satisfaction*;
- the second grey membership degree represents the interval-valued assessment of *market potential*.

Define a MultiGrey set

$$G$$

of dimension 2 on U by assigning to each product $p \in U$ a pair of grey membership intervals as follows:

$$\mu_G(p_1) = ([0.70, 0.85], [0.60, 0.80]),$$
$$\mu_G(p_2) = ([0.45, 0.65], [0.75, 0.90]),$$
$$\mu_G(p_3) = ([0.20, 0.40], [0.30, 0.50]).$$

Equivalently, the lower and upper membership functions are given by

$$\begin{aligned} &\underline{\mu}_G^1(p_1) = 0.70, \quad \overline{\mu}_G^1(p_1) = 0.85, \qquad \underline{\mu}_G^2(p_1) = 0.60, \quad \overline{\mu}_G^2(p_1) = 0.80, \\ &\underline{\mu}_G^1(p_2) = 0.45, \quad \overline{\mu}_G^1(p_2) = 0.65, \qquad \underline{\mu}_G^2(p_2) = 0.75, \quad \overline{\mu}_G^2(p_2) = 0.90, \\ &\underline{\mu}_G^1(p_3) = 0.20, \quad \overline{\mu}_G^1(p_3) = 0.40, \qquad \underline{\mu}_G^2(p_3) = 0.30, \quad \overline{\mu}_G^2(p_3) = 0.50. \end{aligned}$$

For each product and for each dimension, the lower bound does not exceed the upper bound. Hence,

$$\underline{\mu}_G^i(p_j) \le \overline{\mu}_G^i(p_j) \qquad \text{for all } i = 1, 2 \text{ and } j = 1, 2, 3.$$

Therefore, G is a MultiGrey set of dimension 2 on U.

In practical terms:

- p_1 has relatively high customer satisfaction and moderately high market potential;
- p_2 has medium customer satisfaction but very strong market potential;
- p_3 has low evaluations in both aspects.

Thus, this MultiGrey set models a real-life decision situation in which each product is evaluated by several interval-valued criteria rather than by only one single grey membership interval.

22.3 Triangular Uncertain Sets

A triangular uncertain set assigns each element lower, modal, and upper admissible uncertainty degrees, ordered componentwise, within a fixed uncertain model domain.

Definition 22.3.1 (Componentwise order on Dom(M))**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \ge 1$. For

$$a = (a_1, \dots, a_k), \qquad b = (b_1, \dots, b_k) \in \mathrm{Dom}(M),$$

define

$$a \preceq b \iff a_j \le b_j \quad \text{for all } j = 1, \dots, k.$$

Definition 22.3.2 (Triangular uncertain set)**.** Let X be a nonempty set, and let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

A *triangular uncertain set of type* M on X is a quadruple

$$\mathcal{T}_M = (X, \mu_M^-, \mu_M^0, \mu_M^+),$$

where

$$\mu_M^-, \mu_M^0, \mu_M^+ : X \to \mathrm{Dom}(M)$$

satisfy

$$\mu_M^-(x) \preceq \mu_M^0(x) \preceq \mu_M^+(x) \qquad \text{for all } x \in X.$$

The maps μ_M^-, μ_M^0, and μ_M^+ are called the *lower*, *modal*, and *upper uncertainty-degree functions*, respectively.

Equivalently, for each $x \in X$, the triple

$$\big(\mu_M^-(x), \mu_M^0(x), \mu_M^+(x)\big)$$

is called the *triangular uncertain degree* of x.

Theorem 22.3.3 (Well-definedness of triangular uncertain sets)**.** *Let*

$$\mathcal{T}_M = (X, \mu_M^-, \mu_M^0, \mu_M^+)$$

be a triangular uncertain set of type M *on* X*. Then:*

1. *each pair*

$$(X, \mu_M^-), \qquad (X, \mu_M^0), \qquad (X, \mu_M^+)$$

   is a well-defined uncertain set of type M *on* X*;*

2. *for every* $x \in X$*, the comparisons*

$$\mu_M^-(x) \preceq \mu_M^0(x) \preceq \mu_M^+(x)$$

   are well-defined in $\mathrm{Dom}(M)$*;*

3. *consequently,* $\mathcal{T}_M$ *is a well-defined triangular uncertain set.*

Conversely, any three uncertain sets of type M *on* X *whose degree functions satisfy*

$$\mu_M^-(x) \preceq \mu_M^0(x) \preceq \mu_M^+(x) \qquad \textit{for all } x \in X$$

determine a unique triangular uncertain set of type M*.*

Proof. Since
$$\mu_M^-, \mu_M^0, \mu_M^+ : X \to \mathrm{Dom}(M),$$
each of the pairs
$$(X, \mu_M^-), \qquad (X, \mu_M^0), \qquad (X, \mu_M^+)$$
satisfies exactly the definition of an uncertain set of type M on X. Hence (1) holds.

Next, because
$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$
every value
$$\mu_M^-(x),\ \mu_M^0(x),\ \mu_M^+(x)$$
is a k-tuple of real numbers in $[0,1]$. Therefore the componentwise relation $\preceq$ is meaningful on these values, and so the inequalities
$$\mu_M^-(x) \preceq \mu_M^0(x) \preceq \mu_M^+(x)$$
are well-defined for every $x \in X$. This proves (2).

By (1) and (2), the quadruple
$$\mathcal{T}_M = (X, \mu_M^-, \mu_M^0, \mu_M^+)$$
consists of three well-defined uncertain sets arranged in triangular order. Hence $\mathcal{T}_M$ is a well-defined triangular uncertain set. This proves (3).

Conversely, let
$$(X, \mu_M^-), \qquad (X, \mu_M^0), \qquad (X, \mu_M^+)$$
be uncertain sets of type M on X such that
$$\mu_M^-(x) \preceq \mu_M^0(x) \preceq \mu_M^+(x) \qquad \text{for all } x \in X.$$
Then, by definition, the quadruple
$$(X, \mu_M^-, \mu_M^0, \mu_M^+)$$
is a triangular uncertain set of type M. Uniqueness is immediate, since the object is completely determined by its three degree functions. □

As a reference, a catalogue of representative triangular set families organized by the number of numerical parameters used in their basic description is presented in Table 22.1.

Table 22.1: A catalogue of representative triangular set families organized by the number of numerical parameters used in their basic description.

k	note	Representative triangular set family(ies)
3		Triangular Fuzzy Set
6		Triangular Intuitionistic Fuzzy Set [429–431]; Triangular bipolar fuzzy Set [432, 433]; Triangular pythagorean fuzzy set [434–436]; Triangular Fermatean fuzzy set [437, 438]
9		Triangular Spherical Fuzzy Set [439]; Triangular Picture Fuzzy Set [440, 441]; Triangular Neutrosophic Set [424, 442, 443]; Triangular Hesitant Fuzzy Set [444–446]

Reading guide. Here k denotes the number of numerical parameters appearing in one basic elementwise triangular description. A triangular fuzzy value uses three parameters; triangular intuitionistic, spherical, and neutrosophic variants inherit this structure componentwise. Triangular hesitant fuzzy sets are listed as *variable*, since each element may contain a finite family of triangular fuzzy values rather than a single fixed-length tuple.

Chapter 23

Dense Set Theory

In this chapter, we present dense set theory.

23.1 Dense Set

A dense set in a topological space is a subset whose closure equals the whole space, meeting every nonempty open set nontrivially [447–450].

Definition 23.1.1 (Dense set). [447,448] Let (X, τ) be a topological space, and let $A \subseteq X$. Then A is called a *dense set* in X if

$$\overline{A} = X.$$

Equivalently, for every nonempty open set $U \in \tau$,

$$U \cap A \neq \emptyset.$$

Example 23.1.2 (A concrete real-life example of a dense set). Let

$$X = [0, 1] \subseteq \mathbb{R}$$

be the set of all possible normalized control settings of a machine, where each value in $[0, 1]$ represents a continuously adjustable operating level.

In practice, an operator usually enters only rational values, such as

$$0, \quad \frac{1}{2}, \quad 0.73, \quad \frac{19}{25},$$

rather than arbitrary real numbers. Hence consider the set

$$A = \mathbb{Q} \cap [0, 1].$$

This set represents all rational control settings available within the interval $[0, 1]$.

We claim that A is dense in X. Indeed, let

$$U \subseteq [0,1]$$

be any nonempty open set in the subspace topology of $[0,1]$. Then U contains an open interval of real numbers inside $[0,1]$. Since the rational numbers are dense in $\mathbb{R}$, every nonempty open interval contains at least one rational number. Therefore,

$$U \cap A \neq \emptyset.$$

Equivalently,

$$\overline{A} = [0,1] = X.$$

Thus, $A = \mathbb{Q} \cap [0,1]$ is a dense set in X.

In practical terms, this means that although the machine setting is theoretically continuous, rational settings can approximate every possible operating level arbitrarily well. Hence the set of rationally selectable control values is dense in the full control interval.

23.2 Dense Fuzzy sets

A dense fuzzy set in a fuzzy topological space is a fuzzy set whose closure is the total fuzzy set everywhere [451–453].

Definition 23.2.1 (Fuzzy dense set)**.** Let (X, T) be a fuzzy topological space, and let

$$\lambda : X \to [0,1]$$

be a fuzzy set on X. Then λ is called a *fuzzy dense set* in (X, T) if

$$\mathrm{cl}(\lambda) = 1_X,$$

where 1_X denotes the total fuzzy set on X.

Equivalently, there exists no fuzzy closed set μ on X such that

$$\lambda < \mu < 1_X.$$

23.3 Dense Neutrosophic sets

A dense neutrosophic set is a neutrosophic set whose neutrosophic closure equals the total neutrosophic set, intersecting every nonempty open set nontrivially [454].

Definition 23.3.1 (Dense neutrosophic set)**.** Let $(X, \mathcal{T})$ be a neutrosophic topological space, and let

$$A = (T_A, I_A, F_A)$$

be a neutrosophic set on X. Then A is called a *dense neutrosophic set* in X if

$$\mathrm{Cl}(A) = 1_X,$$

where $\mathrm{Cl}(A)$ denotes the neutrosophic closure of A, and

$$1_X = (1, 1, 1)$$

denotes the total neutrosophic set on X.

Equivalently, every nonempty neutrosophic open set of X intersects A nontrivially.

Example 23.3.2 (A concrete real-life example of a dense neutrosophic set)**.** Let

$$X = \mathbb{R}$$

represent all possible time instants during a continuous monitoring process in a smart factory, and let

$$\mathcal{T}$$

be the usual topology on $\mathbb{R}$, interpreted as a neutrosophic topological space in the standard way.

Suppose that a sensor network sends verification signals at all rational time instants. We define a neutrosophic set

$$A = (T_A, I_A, F_A)$$

on X by

$$T_A(t) = \begin{cases} 1, & t \in \mathbb{Q}, \\ 0, & t \notin \mathbb{Q}, \end{cases} \qquad I_A(t) = 0, \qquad F_A(t) = \begin{cases} 0, & t \in \mathbb{Q}, \\ 1, & t \notin \mathbb{Q}. \end{cases}$$

Thus, A represents the set of time instants at which the monitoring system definitely records a verification signal, namely the rational times.

Now let U be any nonempty neutrosophic open set in X. In particular, in the usual topology on $\mathbb{R}$, every nonempty open set contains a nonempty open interval. Since the rational numbers are dense in $\mathbb{R}$, every nonempty open interval contains at least one rational number. Hence,

$$U \cap A \neq 0_X$$

in the neutrosophic sense; that is, every nonempty neutrosophic open set intersects A nontrivially.

Therefore, the neutrosophic closure of A is the total neutrosophic set:

$$\mathrm{Cl}(A) = 1_X.$$

Hence, A is a dense neutrosophic set in X.

In practical terms, this means that no matter how small a time window one chooses during the monitoring process, that window always contains some verification signal times from the sensor system. Thus, the set of verification times is dense in the whole observation timeline.

23.4 Dense Uncertain sets

A dense uncertain set is an uncertain set whose every positive evaluation-based level cut is dense in the underlying topological space.

Definition 23.4.1 (Dense uncertain set)**.** Let (X, τ) be a topological space, let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$, and let

$$\eta : \mathrm{Dom}(M) \to [0,1]$$

be a fixed evaluation map. Let

$$\mathcal{U} = (X, \mu_M)$$

be an uncertain set of type M on X, where

$$\mu_M : X \to \mathrm{Dom}(M).$$

For each

$$\lambda \in [0,1],$$

define the η-level cut of $\mathcal{U}$ by

$$\mathcal{U}^{\eta}_{\lambda} := \{\, x \in X \mid \eta(\mu_M(x)) \geq \lambda \,\}.$$

Then $\mathcal{U}$ is called a *dense uncertain set* (with respect to η) if, for every

$$\lambda \in (0,1],$$

the set

$$\mathcal{U}^{\eta}_{\lambda}$$

is dense in X.

Theorem 23.4.2 (Well-definedness of dense uncertain sets)**.** *Let (X, τ) be a topological space, let M be an uncertain model, let*

$$\eta : \mathrm{Dom}(M) \to [0,1]$$

be an evaluation map, and let

$$\mathcal{U} = (X, \mu_M)$$

be an uncertain set of type M on X. Then, for every

$$\lambda \in [0,1],$$

the set

$$\mathcal{U}^{\eta}_{\lambda} = \{\, x \in X \mid \eta(\mu_M(x)) \geq \lambda \,\}$$

is a well-defined subset of X. Consequently, the notion of a dense uncertain set is well-defined.

Proof. Since
$$\mu_M : X \to \mathrm{Dom}(M),$$
for every
$$x \in X$$
one has
$$\mu_M(x) \in \mathrm{Dom}(M).$$
Because
$$\eta : \mathrm{Dom}(M) \to [0,1],$$
it follows that
$$\eta(\mu_M(x)) \in [0,1] \qquad \text{for all } x \in X.$$
Hence, for every
$$\lambda \in [0,1],$$
the inequality
$$\eta(\mu_M(x)) \geq \lambda$$
is meaningful. Therefore
$$\mathcal{U}_\lambda^\eta \subseteq X$$
is a well-defined subset of X.

Since density is a property of subsets of the topological space (X, τ), the statement
$$\mathcal{U}_\lambda^\eta \text{ is dense in } X$$
is meaningful for every
$$\lambda \in (0,1].$$
Thus the definition of a dense uncertain set is well-defined. □

23.5 Somewhere Dense Sets

A somewhere dense set is a subset whose closure has nonempty interior, so it becomes dense in at least one nonempty open region [449, 455, 456].

Definition 23.5.1 (Somewhere dense set)**.** [448, 450] Let (X, τ) be a topological space, and let
$$A \subseteq X.$$
Then A is called a *somewhere dense set* in X if
$$\mathrm{int}(\overline{A}) \neq \emptyset,$$
where
$$\overline{A}$$
denotes the closure of A and
$$\mathrm{int}(\overline{A})$$
denotes the interior of $\overline{A}$.

23.6 Dense-in-itself Sets

A dense-in-itself set is a subset with no isolated points, meaning every point is a limit point of the set [457–459].

Definition 23.6.1 (Dense-in-itself set)**.** Let (X, τ) be a topological space, and let

$$A \subseteq X.$$

Then A is called *dense in itself* if every point of A is a limit point of A; that is, for every

$$x \in A$$

and every open set

$$U \in \tau$$

with

$$x \in U,$$

one has

$$(U \setminus \{x\}) \cap A \neq \emptyset.$$

Equivalently, A has no isolated points.

23.7 Nowhere Dense Sets

A nowhere dense set is a subset whose closure has empty interior, so it is not dense in any nonempty open region [460–463].

Definition 23.7.1 (Nowhere dense set)**.** [464–466] Let (X, τ) be a topological space, and let

$$A \subseteq X.$$

Then A is called a *nowhere dense set* in X if

$$\text{int}(\overline{A}) = \emptyset,$$

where

$$\overline{A}$$

denotes the closure of A and

$$\text{int}(\overline{A})$$

denotes the interior of $\overline{A}$.

23.8 Meager set

A meager set is a subset of a topological space expressible as a countable union of nowhere dense sets, representing topologically small or negligible structure [467–469].

Definition 23.8.1 (Meager set)**.** Let X be a topological space, and let

$$A \subseteq X.$$

Then A is called a *meager set* (or a *set of first category*) in X if there exists a sequence

$$\{A_n\}_{n=1}^{\infty}$$

of nowhere dense subsets of X such that

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Equivalently, A is meager in X if it can be written as a countable union of subsets whose closures have empty interior, that is,

$$A = \bigcup_{n=1}^{\infty} A_n \qquad \text{with} \qquad \operatorname{int}\big(\overline{A_n}\big) = \emptyset \quad \text{for all } n \in \mathbb{N}.$$

23.9 Residual set

Residual sets are subsets whose complements are meager, equivalently containing a dense G_δ subset, and represent topologically large or generic portions of space [470–472].

Definition 23.9.1 (Residual set)**.** Let X be a topological space, and let

$$A \subseteq X.$$

Then A is called a *residual set* in X if its complement

$$X \setminus A$$

is a meager set in X.

Equivalently, A is residual in X if there exists a sequence

$$\{U_n\}_{n=1}^{\infty}$$

of open dense subsets of X such that

$$\bigcap_{n=1}^{\infty} U_n \subseteq A.$$

In other words, a residual set is a set containing a dense G_δ subset of X.

Chapter 24

Baire Set Theory

In this chapter, we present Baire set theory.

24.1 Baire Set

A Baire set is a subset of a topological space expressible as an open set minus a meager set, or equivalently, open intersect residual [473–475].

Definition 24.1.1 (Baire set)**.** [473–475] Let (X, τ) be a topological space. Recall that a subset

$$M \subseteq X$$

is called *meager* (or *of first category*) if it is a countable union of nowhere dense sets, and a subset

$$R \subseteq X$$

is called *residual* if

$$X \setminus R$$

is meager.

A subset

$$B \subseteq X$$

is called a *Baire set* if there exist an open set

$$U \in \tau$$

and a residual set

$$R \subseteq X$$

such that

$$B = U \cap R.$$

Equivalently, there exist an open set U and a meager set M such that

$$B = U \setminus M.$$

24.2 Fuzzy Baire sets

A fuzzy Baire set in a fuzzy topological space is the meet of a nonzero fuzzy open set and a fuzzy residual set [476–480].

Definition 24.2.1 (Fuzzy Baire set)**.** Let (X, T) be a fuzzy topological space, and let λ be a fuzzy set on X. Then λ is called a *fuzzy Baire set* if there exist a nonzero fuzzy open set μ in (X, T) and a fuzzy residual set δ in (X, T) such that

$$\lambda = \mu \wedge \delta.$$

24.3 Neutrosophic Baire sets

A neutrosophic Baire set is a neutrosophic set whose every threshold cut set is a Baire set in the underlying topological space.

Definition 24.3.1 (Neutrosophic Baire set)**.** Let (X, τ) be a topological space, and let

$$A = (T_A, I_A, F_A)$$

be a single-valued neutrosophic set on X, where

$$T_A, I_A, F_A : X \to [0, 1].$$

For

$$(\alpha, \beta, \gamma) \in [0, 1]^3,$$

define the cut set

$$A_{(\alpha,\beta,\gamma)} := \{\, x \in X \mid T_A(x) \geq \alpha,\ I_A(x) \leq \beta,\ F_A(x) \leq \gamma \,\}.$$

Then A is called a *neutrosophic Baire set* if, for every

$$(\alpha, \beta, \gamma) \in [0, 1]^3,$$

the set

$$A_{(\alpha,\beta,\gamma)}$$

is a Baire set in X.

Example 24.3.2 (A concrete real-life example of a neutrosophic Baire set)**.** Let

$$X = \mathbb{R}$$

represent the positions along a straight road, and let X carry the usual topology.

Suppose that a wireless transmitter is installed at position 0, and we want to describe the neutrosophic set of locations having *good signal quality*. Define a single-valued neutrosophic set

$$A = (T_A, I_A, F_A)$$

on X by

$$T_A(x) = \max\{0,\, 1 - |x|\}, \qquad I_A(x) = 0, \qquad F_A(x) = 1 - T_A(x)$$

for all $x \in X$.

Here:

- $T_A(x)$ measures the degree to which the signal at position x is regarded as good;
- $I_A(x) = 0$ means that we assume no additional indeterminacy in this simplified model;
- $F_A(x)$ measures the degree to which the signal is regarded as not good.

Now fix any

$$(\alpha, \beta, \gamma) \in [0, 1]^3.$$

Then the corresponding cut set is

$$A_{(\alpha,\beta,\gamma)} = \{\, x \in \mathbb{R} \mid T_A(x) \geq \alpha,\ I_A(x) \leq \beta,\ F_A(x) \leq \gamma \,\}.$$

Since $I_A(x) = 0$ for all x, the condition

$$I_A(x) \leq \beta$$

is always satisfied. Also, because

$$F_A(x) = 1 - T_A(x),$$

the condition

$$F_A(x) \leq \gamma$$

is equivalent to

$$T_A(x) \geq 1 - \gamma.$$

Hence,

$$A_{(\alpha,\beta,\gamma)} = \{\, x \in \mathbb{R} \mid T_A(x) \geq c \,\}, \qquad c := \max\{\alpha, 1 - \gamma\}.$$

If $c > 1$, then

$$A_{(\alpha,\beta,\gamma)} = \varnothing.$$

If $0 \leq c \leq 1$, then

$$T_A(x) \geq c \quad \Longleftrightarrow \quad 1 - |x| \geq c \quad \Longleftrightarrow \quad |x| \leq 1 - c,$$

so

$$A_{(\alpha,\beta,\gamma)} = [-(1 - c),\, 1 - c].$$

Therefore, for every $(\alpha, \beta, \gamma) \in [0, 1]^3$, the cut set

$$A_{(\alpha,\beta,\gamma)}$$

is either empty or a closed interval in $\mathbb{R}$. Since every closed set in a topological space is a Baire set, each

$$A_{(\alpha,\beta,\gamma)}$$

is a Baire set in X.

Hence, A is a neutrosophic Baire set.

In practical terms, this means that for any prescribed thresholds of truth, indeterminacy, and falsity, the set of road positions satisfying those signal-quality requirements is always a topologically regular set (in fact, a closed interval or the empty set). Thus, the neutrosophic model of signal quality forms a concrete example of a neutrosophic Baire set.

Theorem 24.3.3 (Well-definedness of neutrosophic Baire sets)**.** *Let (X, τ) be a topological space, and let*

$$A = (T_A, I_A, F_A)$$

be a single-valued neutrosophic set on X. Then, for every

$$(\alpha, \beta, \gamma) \in [0, 1]^3,$$

the set

$$A_{(\alpha,\beta,\gamma)} = \{ x \in X \mid T_A(x) \geq \alpha,\ I_A(x) \leq \beta,\ F_A(x) \leq \gamma \}$$

is a well-defined subset of X. Consequently, the notion of a neutrosophic Baire set is well-defined.

Proof. Since

$$T_A, I_A, F_A : X \to [0, 1],$$

for each

$$x \in X$$

the values

$$T_A(x), \qquad I_A(x), \qquad F_A(x)$$

belong to $[0, 1]$. Hence the inequalities

$$T_A(x) \geq \alpha, \qquad I_A(x) \leq \beta, \qquad F_A(x) \leq \gamma$$

are meaningful for every

$$(\alpha, \beta, \gamma) \in [0, 1]^3.$$

Therefore

$$A_{(\alpha,\beta,\gamma)} \subseteq X$$

is a well-defined subset.

Since Baire sets are subsets of X, the statement

$$A_{(\alpha,\beta,\gamma)} \text{ is a Baire set}$$

is meaningful for every

$$(\alpha, \beta, \gamma) \in [0, 1]^3.$$

Thus the definition of a neutrosophic Baire set is well-defined. □

24.4 Uncertain Baire sets

An uncertain Baire set is an uncertain set whose every evaluation-based level cut is a Baire set in the underlying topological space.

Definition 24.4.1 (Uncertain Baire set)**.** Let (X, τ) be a topological space, let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$, and let

$$\eta : \mathrm{Dom}(M) \to [0,1]$$

be a fixed evaluation map. Let

$$\mathcal{U} = (X, \mu_M)$$

be an uncertain set of type M on X, where

$$\mu_M : X \to \mathrm{Dom}(M).$$

For each

$$\lambda \in [0,1],$$

define the η-level cut

$$\mathcal{U}^{\eta}_{\lambda} := \{\, x \in X \mid \eta(\mu_M(x)) \geq \lambda \,\}.$$

Then $\mathcal{U}$ is called an *uncertain Baire set* if, for every

$$\lambda \in [0,1],$$

the set

$$\mathcal{U}^{\eta}_{\lambda}$$

is a Baire set in X.

Theorem 24.4.2 (Well-definedness of uncertain Baire sets)**.** *Let (X, τ) be a topological space, let M be an uncertain model, let*

$$\eta : \mathrm{Dom}(M) \to [0,1]$$

be an evaluation map, and let

$$\mathcal{U} = (X, \mu_M)$$

be an uncertain set of type M on X. Then, for every

$$\lambda \in [0,1],$$

the set

$$\mathcal{U}^{\eta}_{\lambda} = \{\, x \in X \mid \eta(\mu_M(x)) \geq \lambda \,\}$$

is a well-defined subset of X. Consequently, the notion of an uncertain Baire set is well-defined.

Proof. Since

$$\mu_M : X \to \mathrm{Dom}(M),$$

one has

$$\mu_M(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Because

$$\eta : \mathrm{Dom}(M) \to [0,1],$$

it follows that

$$\eta(\mu_M(x)) \in [0,1] \qquad \text{for all } x \in X.$$

Hence, for every

$$\lambda \in [0, 1],$$

the inequality

$$\eta(\mu_M(x)) \geq \lambda$$

is meaningful, and therefore

$$\mathcal{U}_\lambda^\eta \subseteq X$$

is a well-defined subset of X.

Since Baire sets are subsets of X, the statement

$$\mathcal{U}_\lambda^\eta \text{ is a Baire set}$$

is meaningful for every

$$\lambda \in [0, 1].$$

Thus the definition of an uncertain Baire set is well-defined. □

Chapter 25

Nonstandard Set Theory

In this chapter, we introduce nonstandard set theory.

25.1 Nonstandard Set

A nonstandard set is an internal subset of a nonstandard extension ${}^*X$, allowing ordinary set membership to be studied with infinitesimal and infinite elements.

Definition 25.1.1 (Nonstandard set)**.** Assume a fixed nonstandard enlargement

$$X \longmapsto {}^*X.$$

Let X be a nonempty set. A *nonstandard set* on X is an internal subset

$$A \subseteq {}^*X.$$

Equivalently, A may be represented by its internal characteristic function

$$\chi_A : {}^*X \to \{0,1\}, \qquad \chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

25.2 Nonstandard Fuzzy Set

A nonstandard fuzzy set assigns to each element of ${}^*X$ an internal membership degree in ${}^*[0,1]$, within a nonstandard analysis framework.

Definition 25.2.1 (Nonstandard fuzzy set)**.** Assume a fixed nonstandard enlargement

$$X \longmapsto {}^*X.$$

Let X be a nonempty set. A *nonstandard fuzzy set* A on X is determined by an internal mapping

$$\mu_A : {}^*X \to {}^*[0,1].$$

For each

$$x \in {}^*X,$$

the value

$$\mu_A(x) \in {}^*[0,1]$$

is called the *nonstandard membership degree* of x in A.

Equivalently, one may write

$$A = \{(x, \mu_A(x)) \mid x \in {}^*X\}.$$

25.3 Nonstandard Neutrosophic Set

A nonstandard neutrosophic set assigns each element of ${}^*X$ internal truth, indeterminacy, and falsity degrees in ${}^*[0,1]$, extending neutrosophic modeling nonstandardly [481].

Definition 25.3.1 (Nonstandard neutrosophic set)**.** Assume a fixed nonstandard enlargement

$$X \longmapsto {}^*X.$$

Let X be a nonempty set. A *nonstandard neutrosophic set* A on X is specified by three internal mappings

$$T_A, I_A, F_A : {}^*X \to {}^*[0,1]$$

such that, for every

$$x \in {}^*X,$$

one has

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$$

in

$${}^*\mathbb{R}.$$

Here

$$T_A(x), \qquad I_A(x), \qquad F_A(x)$$

represent the nonstandard truth-membership, indeterminacy-membership, and falsity-membership degrees of x, respectively.

Equivalently, one may write

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in {}^*X\}.$$

Example 25.3.2 (A concrete real-life example of a nonstandard neutrosophic set)**.** Let

$$X = \mathbb{R}$$

represent the set of all possible deviations from a target voltage in a precision manufacturing process. Thus, a value

$$x \in X$$

means that the measured voltage differs from the ideal target by x units.

Now pass to the nonstandard enlargement

$$^*X = {}^*\mathbb{R},$$

so that, in addition to ordinary real deviations, we may also consider infinitesimal deviations that are smaller than any positive real tolerance.

Let

$$\varepsilon \in {}^*\mathbb{R}$$

be a fixed positive infinitesimal. We define three internal mappings

$$T_A, I_A, F_A : {}^*\mathbb{R} \to {}^*[0,1]$$

by

$$T_A(x) = \begin{cases} 1, & |x| \le \varepsilon, \\ \frac{1}{2}, & \varepsilon < |x| \le 2\varepsilon, \\ 0, & |x| > 2\varepsilon, \end{cases}$$

$$I_A(x) = \begin{cases} 0, & |x| \le \varepsilon, \\ \frac{1}{2}, & \varepsilon < |x| \le 2\varepsilon, \\ 0, & |x| > 2\varepsilon, \end{cases}$$

and

$$F_A(x) = \begin{cases} 0, & |x| \le \varepsilon, \\ 0, & \varepsilon < |x| \le 2\varepsilon, \\ 1, & |x| > 2\varepsilon. \end{cases}$$

Then, for every

$$x \in {}^*\mathbb{R},$$

we have

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 1 \le 3.$$

Hence,

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$$

holds in

$$^*\mathbb{R}.$$

Therefore,

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in {}^*\mathbb{R}\}$$

is a nonstandard neutrosophic set on X.

Its practical meaning is as follows:

- if the voltage deviation x is infinitesimally small, that is,

  $$|x| \le \varepsilon,$$

  then the measurement is regarded as fully acceptable, so the truth-membership is 1;

- if the deviation is still extremely small but slightly larger, namely

$$\varepsilon < |x| \leq 2\varepsilon,$$

  then the status is partially acceptable and partially indeterminate;

- if the deviation exceeds this infinitesimal band, then the measurement is regarded as unacceptable, so the falsity-membership is 1.

Thus, this nonstandard neutrosophic set models a realistic engineering situation in which one wants to distinguish perfect agreement with a target, an infinitesimally uncertain transition zone, and clear deviation, using hyperreal-valued infinitesimal tolerances.

25.4 Nonstandard Uncertain Set

A nonstandard uncertain set assigns each element of ${}^*X$ an internal admissible degree from ${}^*!\,\mathrm{Dom}(M)$, extending uncertain-set theory to nonstandard domains.

Definition 25.4.1 (Nonstandard degree-domain induced by an uncertain model)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. Under a fixed nonstandard enlargement, define

$${}^*\mathrm{Dom}(M) \subseteq \left({}^*[0,1]\right)^k$$

to be the nonstandard extension of

$$\mathrm{Dom}(M).$$

The set

$${}^*\mathrm{Dom}(M)$$

is called the *nonstandard degree-domain induced by M*.

Definition 25.4.2 (Nonstandard uncertain set)**.** Assume a fixed nonstandard enlargement

$$X \longmapsto {}^*X.$$

Let X be a nonempty set, and let M be an uncertain model with induced nonstandard degree-domain

$${}^*\mathrm{Dom}(M) \subseteq \left({}^*[0,1]\right)^k.$$

A *nonstandard uncertain set of type M* on X is a pair

$$\mathcal{U}^*_M = ({}^*X, \mu^*_M),$$

where

$$\mu^*_M : {}^*X \to {}^*\mathrm{Dom}(M)$$

is an internal mapping, called the *nonstandard uncertainty-degree function.*

Equivalently, for each

$$x \in {}^*X,$$

the value

$$\mu^*_M(x) \in {}^*\mathrm{Dom}(M)$$

is a nonstandard admissible uncertainty degree of type M.

Chapter 26

Dynamic Set Theory

In this chapter, we discuss Dynamic Set Theory.

26.1 Dynamic Set

A dynamic set is a time-indexed family of subsets of a universe, allowing element membership to change over time according to the index parameter.

Definition 26.1.1 (Dynamic set)**.** Let X be a nonempty set, and let

$$\mathbb{T}$$

be a nonempty time domain. A *dynamic set* on X over $\mathbb{T}$ is a mapping

$$A : \mathbb{T} \to \mathcal{P}(X).$$

Equivalently, a dynamic set is a family

$$\{A_t\}_{t\in\mathbb{T}}$$

of subsets of X, where

$$A_t := A(t) \subseteq X \qquad \text{for each } t \in \mathbb{T}.$$

Thus, the membership status of an element may vary with time.

26.2 Dynamic Fuzzy Set

A dynamic fuzzy set assigns each element a time-dependent membership degree in $[0, 1]$, forming a family of fuzzy sets evolving over time [64, 482].

Definition 26.2.1 (Dynamic fuzzy set)**.** Let X be a nonempty set, and let

$$\mathbb{T}$$

be a nonempty time domain. A *dynamic fuzzy set* A on X over $\mathbb{T}$ is determined by a mapping

$$\mu_A : X \times \mathbb{T} \to [0, 1].$$

For each

$$t \in \mathbb{T},$$

the section

$$\mu_{A,t} : X \to [0, 1], \qquad \mu_{A,t}(x) := \mu_A(x, t),$$

is a fuzzy set on X.

Equivalently, a dynamic fuzzy set is a family

$$\{A_t\}_{t\in\mathbb{T}}$$

of fuzzy sets on X, where the membership degree of $x \in X$ at time t is

$$\mu_A(x, t).$$

26.3 Dynamic Neutrosophic Set

A dynamic neutrosophic set assigns time-dependent truth, indeterminacy, and falsity degrees to each element, producing a time-indexed family of neutrosophic sets (cf. [483–486]).

Definition 26.3.1 (Dynamic neutrosophic set)**.** Let X be a nonempty set, and let

$$\mathbb{T}$$

be a nonempty time domain. A *dynamic neutrosophic set* A on X over $\mathbb{T}$ is specified by three mappings

$$T_A, I_A, F_A : X \times \mathbb{T} \to [0, 1]$$

such that, for all

$$x \in X, \qquad t \in \mathbb{T},$$

one has

$$0 \le T_A(x, t) + I_A(x, t) + F_A(x, t) \le 3.$$

For each fixed

$$t \in \mathbb{T},$$

the triple

$$\big(T_{A,t}, I_{A,t}, F_{A,t}\big), \qquad T_{A,t}(x) := T_A(x, t),\ I_{A,t}(x) := I_A(x, t),\ F_{A,t}(x) := F_A(x, t),$$

defines a neutrosophic set on X.

Equivalently, a dynamic neutrosophic set is a time-indexed family of neutrosophic sets on X.

26.4 Dynamic Uncertain Set

A dynamic uncertain set assigns each element a time-dependent admissible degree from a fixed uncertain model, yielding uncertain sets that evolve over time.

Definition 26.4.1 (Dynamic uncertain set)**.** Let X be a nonempty set, let

$$\mathbb{T}$$

be a nonempty time domain, and let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. A *dynamic uncertain set of type M on X over $\mathbb{T}$* is a pair

$$\mathcal{U}_M^{\mathrm{dyn}} = (X, \mu_M^{\mathrm{dyn}}),$$

where

$$\mu_M^{\mathrm{dyn}} : X \times \mathbb{T} \to \mathrm{Dom}(M)$$

is called the *dynamic uncertainty-degree function.*

For each fixed

$$t \in \mathbb{T},$$

the section

$$\mu_{M,t} : X \to \mathrm{Dom}(M), \qquad \mu_{M,t}(x) := \mu_M^{\mathrm{dyn}}(x,t),$$

defines an uncertain set of type M on X.

Equivalently, a dynamic uncertain set is a family

$$\{\mathcal{U}_{M,t}\}_{t\in\mathbb{T}}$$

of uncertain sets of type M on X, indexed by time.

26.5 Dynamic Soft Set

A dynamic soft set is a time-indexed family of soft sets, where parameter subsets and their associated approximations may vary over time [173, 487, 488].

Definition 26.5.1 (Dynamic soft set)**.** Let U be a nonempty universe, let E be a set of parameters, and let

$$T$$

be a nonempty time domain. A *dynamic soft set* over U with respect to E and T is a family

$$\mathcal{S} = \{(F_t, A_t)\}_{t\in T},$$

where, for each

$$t \in T,$$

one has

$$A_t \subseteq E$$

and

$$F_t : A_t \to \mathcal{P}(U).$$

Equivalently, a dynamic soft set is a time-indexed family of soft sets on U.

26.6 Dynamic Rough Set

A dynamic rough set is a time-indexed family of rough approximations, where target sets and equivalence relations may change with time [489–491].

Definition 26.6.1 (Dynamic rough set)**.** Let U be a nonempty universe, let

$$T$$

be a nonempty time domain, and let

$$\{R_t\}_{t\in T}$$

be a family of equivalence relations on U. Let

$$\{A_t\}_{t\in T}$$

be a family of subsets of U, where

$$A_t \subseteq U \qquad \text{for all } t \in T.$$

For each

$$t \in T,$$

define the lower and upper approximations of A_t with respect to R_t by

$$\underline{A_t}^{R_t} := \{\, x \in U \mid [x]_{R_t} \subseteq A_t \,\},$$

and

$$\overline{A_t}^{R_t} := \{\, x \in U \mid [x]_{R_t} \cap A_t \neq \emptyset \,\},$$

where

$$[x]_{R_t} := \{\, y \in U \mid (x, y) \in R_t \,\}.$$

Then the family

$$\mathcal{R}^{\text{dyn}} := \left\{ \left(\underline{A_t}^{R_t}, \overline{A_t}^{R_t}\right) \right\}_{t\in T}$$

is called a *dynamic rough set* on U. Equivalently, a dynamic rough set is a time-indexed family of rough sets whose target sets and/or approximation relations may vary with time.

26.7 Dynamic Grey Set

A dynamic grey set assigns each element time-dependent lower and upper membership bounds in $[0, 1]$, giving a time-indexed family of grey sets.

Definition 26.7.1 (Dynamic grey set)**.** Let X be a nonempty universe, and let

$$\mathbb{T}$$

be a nonempty time domain. A *dynamic grey set* G on X over $\mathbb{T}$ is determined by two mappings

$$\underline{\mu}_G, \overline{\mu}_G : X \times \mathbb{T} \to [0, 1]$$

such that

$$\underline{\mu}_G(x, t) \le \overline{\mu}_G(x, t) \qquad \text{for all } x \in X,\ t \in \mathbb{T}.$$

For each fixed

$$t \in \mathbb{T},$$

the time-section

$$G_t$$

is defined by the interval-valued membership

$$\mu_{G_t}(x) := \left[\underline{\mu}_G(x,t), \overline{\mu}_G(x,t)\right] \qquad (x \in X).$$

Equivalently, a dynamic grey set is a time-indexed family

$$\{G_t\}_{t\in\mathbb{T}}$$

of grey sets on X.

Theorem 26.7.2 (Well-definedness of dynamic grey sets)**.** *Let X be a nonempty universe, let*

$$\mathbb{T}$$

be a nonempty time domain, and let

$$\underline{\mu}_G, \overline{\mu}_G : X \times \mathbb{T} \to [0,1]$$

satisfy

$$\underline{\mu}_G(x,t) \le \overline{\mu}_G(x,t) \qquad \textit{for all } x \in X,\ t \in \mathbb{T}.$$

Then, for every

$$t \in \mathbb{T} \quad \textit{and} \quad x \in X,$$

the interval

$$\mu_{G_t}(x) = \left[\underline{\mu}_G(x,t), \overline{\mu}_G(x,t)\right]$$

is a well-defined closed subinterval of $[0,1]$*. Consequently, each*

$$G_t$$

is a well-defined grey set on X, and hence

$$\{G_t\}_{t\in\mathbb{T}}$$

is a well-defined dynamic grey set.

Proof. Fix

$$t \in \mathbb{T} \quad \text{and} \quad x \in X.$$

Since

$$\underline{\mu}_G, \overline{\mu}_G : X \times \mathbb{T} \to [0,1],$$

one has

$$\underline{\mu}_G(x,t) \in [0,1] \qquad \text{and} \qquad \overline{\mu}_G(x,t) \in [0,1].$$

By assumption,

$$\underline{\mu}_G(x,t) \le \overline{\mu}_G(x,t).$$

Therefore

$$\left[\underline{\mu}_G(x,t), \overline{\mu}_G(x,t)\right]$$

is a well-defined closed interval contained in $[0,1]$.

Since this holds for every

$$x \in X,$$

the mapping

$$\mu_{G_t} : X \to \{[a, b] \subseteq [0, 1] \mid 0 \leq a \leq b \leq 1\}$$

is well-defined, and thus

$$G_t$$

is a well-defined grey set on X.

Since

$$t \in \mathbb{T}$$

was arbitrary, all time-sections

$$G_t$$

are well-defined. Hence

$$\{G_t\}_{t \in \mathbb{T}}$$

is a well-defined dynamic grey set. □

26.8 Dynamic Near Set

A dynamic near set consists of two time-indexed subsets that remain near each other at every time under a fixed proximity relation.

Definition 26.8.1 (Dynamic near set)**.** Let X be a nonempty set, let

$$\mathbb{T}$$

be a nonempty time domain, and let

$$\delta$$

be a proximity relation on

$$\mathcal{P}(X).$$

Let

$$A, B : \mathbb{T} \to \mathcal{P}(X)$$

be dynamic sets on X. Then the pair

$$(A, B)$$

is called a *dynamic near set* on X (with respect to δ) if, for every

$$t \in \mathbb{T},$$

one has

$$A(t) \, \delta \, B(t).$$

Equivalently, writing

$$A_t := A(t), \qquad B_t := B(t),$$

the pair

$$(A_t, B_t)$$

is a pair of near sets for every

$$t \in \mathbb{T}.$$

Theorem 26.8.2 (Well-definedness of dynamic near sets)**.** *Let X be a nonempty set, let*

$$\mathbb{T}$$

be a nonempty time domain, let

$$\delta$$

be a proximity relation on

$$\mathcal{P}(X),$$

and let

$$A, B : \mathbb{T} \to \mathcal{P}(X)$$

be mappings. Then, for every

$$t \in \mathbb{T},$$

the sets

$$A(t), \qquad B(t)$$

are well-defined subsets of X, and the statement

$$A(t)\,\delta\,B(t)$$

is well-formed. Consequently, if

$$A(t)\,\delta\,B(t) \qquad \text{for all } t \in \mathbb{T},$$

then

$$(A, B)$$

is a well-defined dynamic near set on X.

Conversely, every family

$$\{(A_t, B_t)\}_{t\in\mathbb{T}}$$

of near-set pairs on X determines a unique dynamic near set by setting

$$A(t) := A_t, \qquad B(t) := B_t \qquad (t \in \mathbb{T}).$$

Proof. Since

$$A, B : \mathbb{T} \to \mathcal{P}(X),$$

for every

$$t \in \mathbb{T},$$

one has

$$A(t) \in \mathcal{P}(X) \qquad \text{and} \qquad B(t) \in \mathcal{P}(X).$$

Hence

$$A(t) \subseteq X \qquad \text{and} \qquad B(t) \subseteq X,$$

so both are well-defined subsets of X.

Because

$$\delta$$

is a proximity relation on

$$\mathcal{P}(X),$$

the relation

$$A(t)\,\delta\,B(t)$$

is meaningful for every

$$t \in \mathbb{T}.$$

Therefore, if

$$A(t)\,\delta\,B(t) \qquad \text{for all } t \in \mathbb{T},$$

then

$$(A, B)$$

satisfies the definition of a dynamic near set and is well-defined.

Conversely, let

$$\{(A_t, B_t)\}_{t\in\mathbb{T}}$$

be a family such that

$$A_t \subseteq X, \qquad B_t \subseteq X, \qquad A_t\,\delta\,B_t \qquad \text{for all } t \in \mathbb{T}.$$

Define

$$A : \mathbb{T} \to \mathcal{P}(X), \qquad A(t) := A_t,$$

and

$$B : \mathbb{T} \to \mathcal{P}(X), \qquad B(t) := B_t.$$

Then

$$A(t)\,\delta\,B(t) \qquad \text{for all } t \in \mathbb{T},$$

so

$$(A, B)$$

is a dynamic near set. Uniqueness is immediate from the definitions

$$A(t) = A_t, \qquad B(t) = B_t.$$

□

Chapter 27

Linguistic Set Theory

In this chapter, we discuss linguistic set theory.

27.1 Linguistic Set

A linguistic set assigns each element a label from an ordered linguistic term set, representing membership qualitatively rather than by numerical degrees.

Definition 27.1.1 (Linguistic term set)**.** Let

$$\mathcal{L} = \{\ell_0, \ell_1, \ldots, \ell_g\}$$

be a finite nonempty totally ordered set of linguistic labels, written as

$$\ell_0 \prec \ell_1 \prec \cdots \prec \ell_g.$$

The set $\mathcal{L}$ is called a *linguistic term set.*

Definition 27.1.2 (Linguistic set)**.** Let X be a nonempty set, and let $\mathcal{L}$ be a linguistic term set. A *linguistic set* A on X is a mapping

$$\lambda_A : X \to \mathcal{L}.$$

For each

$$x \in X,$$

the value

$$\lambda_A(x) \in \mathcal{L}$$

is called the *linguistic membership value* of x in A.

Equivalently, one may represent A as

$$A = \{(x, \lambda_A(x)) \mid x \in X\}.$$

27.2 Linguistic Neutrosophic Set

A linguistic neutrosophic set assigns each element linguistic truth, indeterminacy, and falsity labels, expressing neutrosophic evaluation through ordered qualitative terms [492].

Definition 27.2.1 (Linguistic neutrosophic set)**.** Let X be a nonempty set, and let $\mathcal{L}$ be a linguistic term set. A *linguistic neutrosophic set* A on X is determined by three mappings

$$T_A, I_A, F_A : X \to \mathcal{L},$$

called the *linguistic truth-membership function*, the *linguistic indeterminacy-membership function*, and the *linguistic falsity-membership function*, respectively.

Equivalently, one may write

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}.$$

Thus, for each

$$x \in X,$$

the triple

$$\big(T_A(x), I_A(x), F_A(x)\big) \in \mathcal{L}^3$$

represents the linguistic truth, indeterminacy, and falsity assessments of x.

Example 27.2.2 (A concrete real-life example of a linguistic neutrosophic set)**.** Let

$$X = \{a_1, a_2, a_3\}$$

be a set of three job applicants, and let

$$\mathcal{L} = \{\text{very low}, \text{low}, \text{medium}, \text{high}, \text{very high}\}$$

be a linguistic term set.

Suppose that a hiring committee wants to describe the set of applicants who are *suitable for a managerial position*. Because the evaluation is expressed in natural language rather than precise numerical values, the committee uses a linguistic neutrosophic set.

Define a linguistic neutrosophic set

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$$

on X by

$$A = \{\langle a_1, \text{high}, \text{low}, \text{low}\rangle,\ \langle a_2, \text{medium}, \text{medium}, \text{low}\rangle,\ \langle a_3, \text{low}, \text{high}, \text{high}\rangle\}.$$

Equivalently, the three linguistic mappings are given by

$$T_A(a_1) = \text{high}, \qquad I_A(a_1) = \text{low}, \qquad F_A(a_1) = \text{low},$$
$$T_A(a_2) = \text{medium}, \qquad I_A(a_2) = \text{medium}, \qquad F_A(a_2) = \text{low},$$
$$T_A(a_3) = \text{low}, \qquad I_A(a_3) = \text{high}, \qquad F_A(a_3) = \text{high}.$$

This means that:

- applicant a_1 is regarded as highly suitable, with low uncertainty and low opposition;
- applicant a_2 is considered moderately suitable, but the evaluation is also moderately uncertain;
- applicant a_3 is considered weakly suitable, with high uncertainty and high unsuitability.

Therefore, A is a linguistic neutrosophic set on X, since each applicant is described by linguistic truth, indeterminacy, and falsity assessments taken from the linguistic term set $\mathcal{L}$.

27.3 Linguistic Uncertain Set

A linguistic uncertain set assigns each element a tuple of linguistic labels whose numerical interpretation, under a scale, satisfies a fixed uncertain model.

Definition 27.3.1 (Linguistic scale)**.** Let $\mathcal{L} = \{\ell_0, \ell_1, \ldots, \ell_g\}$ be a linguistic term set. A mapping

$$\iota : \mathcal{L} \to [0, 1]$$

is called a *linguistic scale* if it is order-preserving; that is,

$$\ell_i \preceq \ell_j \quad \Longrightarrow \quad \iota(\ell_i) \le \iota(\ell_j).$$

Definition 27.3.2 (Linguistic degree-domain induced by an uncertain model)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \ge 1$, let $\mathcal{L}$ be a linguistic term set, and let

$$\iota : \mathcal{L} \to [0, 1]$$

be a linguistic scale. Define

$$\iota^{(k)} : \mathcal{L}^k \to [0, 1]^k$$

by

$$\iota^{(k)}(a_1, \ldots, a_k) := \big(\iota(a_1), \ldots, \iota(a_k)\big).$$

Then

$$\mathrm{Dom}_{\mathcal{L}}(M) := \big(\iota^{(k)}\big)^{-1}(\mathrm{Dom}(M)) = \{\, a \in \mathcal{L}^k \mid \iota^{(k)}(a) \in \mathrm{Dom}(M) \,\}$$

is called the *linguistic degree-domain induced by M*.

Definition 27.3.3 (Linguistic uncertain set)**.** Let X be a nonempty set, let M be an uncertain model, let $\mathcal{L}$ be a linguistic term set, and let

$$\mathrm{Dom}_{\mathcal{L}}(M)$$

be the linguistic degree-domain induced by M through a linguistic scale

$$\iota : \mathcal{L} \to [0, 1].$$

A *linguistic uncertain set of type $(M, \mathcal{L}, \iota)$* on X is a pair

$$\mathcal{U}_{\mathcal{L}} = (X, \mu_{M,\mathcal{L}}),$$

where

$$\mu_{M,\mathcal{L}} : X \to \mathrm{Dom}_{\mathcal{L}}(M)$$

is called the *linguistic uncertainty-degree function*.

Equivalently, for each

$$x \in X,$$

one has

$$\mu_{M,\mathcal{L}}(x) = (a_1(x), \ldots, a_k(x)) \in \mathcal{L}^k$$

and

$$\big(\iota(a_1(x)), \ldots, \iota(a_k(x))\big) \in \mathrm{Dom}(M).$$

Thus each element of X is assigned a k-tuple of linguistic labels whose numerical interpretation is admissible for the uncertain model M.

Chapter 28

Random Set Theory

In this chapter, we consider Random Set Theory.

28.1 Random Set

A random set is a measurable mapping from a probability space into nonempty compact subsets of a metric space, assigning a random compact set outcome.

Definition 28.1.1 (Random set)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let X be a metric space. Denote by

$$K(X)$$

the family of all nonempty compact subsets of X, equipped with the Borel σ-algebra induced by the Hausdorff metric. A *random set* on X is a measurable mapping

$$R : \Omega \to K(X).$$

Equivalently, for each $\omega \in \Omega$, the value $R(\omega)$ is a nonempty compact subset of X, and for every Borel set $B \subseteq K(X)$ one has

$$R^{-1}(B) \in \mathcal{A}.$$

As a reference, a brief comparison between probabilistic sets and random sets is presented in Table 28.1. Probabilistic sets usually describe uncertainty at the element level, whereas random sets describe uncertainty through probability-driven variation of whole subsets.

Table 28.1: A brief comparison between probabilistic sets and random sets

Aspect	Probabilistic set	Random set
Basic idea	Assigns to each element a probability-related degree	Assigns to each outcome a set-valued realization
Underlying object	Usually an elementwise mapping on a fixed universe	Usually a set-valued measurable mapping from a probability space
Main source of uncertainty	Uncertainty in the degree of membership or occurrence of each element	Uncertainty in which subset of the universe is realized
Representation style	Pointwise probability-type description	Outcome-dependent set-valued description
Typical interpretation	Expresses how likely or plausible each element is	Expresses a random selection of subsets generated by chance
Mathematical emphasis	Membership-oriented probabilistic evaluation	Measurable set-valued structure and distribution of sets
Common feature	Both model uncertainty using probabilistic ideas	Both model uncertainty using probabilistic ideas

28.2 Random Fuzzy Set

A random fuzzy set assigns each outcome a fuzzy set whose nonempty compact alpha-cuts vary measurably, combining fuzzy uncertainty with probabilistic randomness [493–496].

Definition 28.2.1 (Random fuzzy set)**.** [493,494] Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let X be a metric space. Let $\mathcal{F}_c(X)$ denote the class of all normalized upper semicontinuous fuzzy sets

$$A : X \to [0, 1]$$

such that, for every $\alpha \in (0, 1]$, the α-cut

$$A_\alpha := \{x \in X \mid A(x) \geq \alpha\}$$

is a nonempty compact subset of X. A *random fuzzy set* on X is a mapping

$$\widetilde{A} : \Omega \to \mathcal{F}_c(X)$$

such that, for every $\alpha \in (0, 1]$, the set-valued mapping

$$\omega \mapsto (\widetilde{A}(\omega))_\alpha$$

is a random set in $K(X)$.

28.3 Random Neutrosophic Set

A random neutrosophic set assigns each outcome a neutrosophic set, so truth, indeterminacy, and falsity degrees vary measurably under probabilistic uncertainty (cf. [497, 498]).

Definition 28.3.1 (Random neutrosophic set)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let X be a nonempty set. A *random neutrosophic set* on X is a triple of mappings

$$T_N, I_N, F_N : \Omega \times X \to [0, 1]$$

such that, for every fixed $\omega \in \Omega$, the section

$$N_\omega := \big(T_N(\omega, \cdot),\ I_N(\omega, \cdot),\ F_N(\omega, \cdot)\big)$$

is a single-valued neutrosophic set on X. Equivalently, one may write

$$N : \Omega \to \mathcal{N}(X), \qquad N(\omega) = N_\omega,$$

where $\mathcal{N}(X)$ denotes the class of all single-valued neutrosophic sets on X, and N is measurable.

28.4 Random Uncertain Set

A random uncertain set assigns each outcome an uncertain set, allowing model-based uncertainty degrees to vary measurably across probabilistic realizations and random environments.

Definition 28.4.1 (Random uncertain set)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let X be a nonempty set. Let $\mathcal{U}(X)$ denote the class of uncertain sets on X. A *random uncertain set* on X is a measurable mapping

$$\Xi : \Omega \to \mathcal{U}(X).$$

Equivalently, for each $\omega \in \Omega$, the value

$$\Xi(\omega)$$

is an uncertain set on X, and the assignment of uncertain sets varies measurably with respect to ω.

If uncertain sets on X are represented by membership mappings of the form

$$\mu : X \to \mathcal{V},$$

where $\mathcal{V}$ is the chosen uncertainty-value space, then a random uncertain set may be written as

$$\mu_\Xi : \Omega \times X \to \mathcal{V}$$

such that, for every $\omega \in \Omega$, the section

$$\mu_\Xi(\omega, \cdot)$$

is an uncertain set on X.

28.5 Random Soft Set

A random soft set assigns each outcome a soft set, so parameterized subset information changes measurably according to randomness in the underlying probability space.

Definition 28.5.1 (Random soft set)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let U be a nonempty universe, and let E be a nonempty set of parameters. Define

$$\mathcal{S}(U, E) := \{F \mid F : E \to \mathcal{P}(U)\},$$

the class of all soft sets on U with parameter set E.

A *random soft set* on U with parameter set E is a measurable mapping

$$\Xi : \Omega \to \mathcal{S}(U, E).$$

Equivalently, one may write

$$\Xi(\omega) = F_\omega, \qquad F_\omega : E \to \mathcal{P}(U),$$

so that, for each $\omega \in \Omega$, the section F_ω is a soft set on U.

Equivalently again, a random soft set may be represented by a mapping

$$\Phi : \Omega \times E \to \mathcal{P}(U)$$

such that, for every fixed $\omega \in \Omega$, the mapping

$$e \mapsto \Phi(\omega, e)$$

is a soft set on U.

28.6 Random Rough Set

A random rough set assigns each outcome a rough set, so lower and upper approximations vary measurably according to randomness in the underlying probability space [499–501].

Definition 28.6.1 (Random rough set)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let U be a nonempty set. Define

$$\mathcal{R}(U) := \{(L, M) \in \mathcal{P}(U) \times \mathcal{P}(U) \mid L \subseteq M\},$$

the class of all rough-set pairs on U.

A *random rough set* on U is a measurable mapping

$$\Gamma : \Omega \to \mathcal{R}(U).$$

Equivalently, for each $\omega \in \Omega$,

$$\Gamma(\omega) = \big(L_\omega, M_\omega\big),$$

where

$$L_\omega \subseteq M_\omega \subseteq U.$$

Here L_ω and M_ω are interpreted as the lower and upper approximations, respectively.

In particular, if for each $\omega \in \Omega$ there exist an approximation relation R_ω on U and a subset $X_\omega \subseteq U$ such that

$$L_\omega = \underline{R_\omega}(X_\omega), \qquad M_\omega = \overline{R_\omega}(X_\omega),$$

then Γ is a random rough set generated by random approximation data.

Chapter 29

Admissible Set Theory

In this chapter, we discuss Admissible set theory.

29.1 Admissible Set

An admissible set is the subset of candidate elements satisfying all prescribed inequality and equality constraints, representing exactly the feasible or allowable solutions under consideration [502–504].

Definition 29.1.1 (Admissible set)**.** [502, 503] Let X be a nonempty set of candidate elements, and let

$$g_i : X \to \mathbb{R} \qquad (i = 1, \dots, m)$$

and

$$h_j : X \to \mathbb{R} \qquad (j = 1, \dots, n)$$

be constraint functions. The *admissible set* determined by these constraints is the subset

$$C := \left\{ x \in X \;\middle|\; g_i(x) \le 0 \text{ for all } i = 1, \dots, m,\ h_j(x) = 0 \text{ for all } j = 1, \dots, n \right\}.$$

Each element of C is called an *admissible element* (or *admissible solution*).

29.2 Fuzzy Admissible Set

A fuzzy admissible set assigns each candidate element a degree of admissibility in $[0, 1]$, often obtained by aggregating fuzzy satisfaction degrees of constraints appropriately overall (cf. [505–507]).

Definition 29.2.1 (Fuzzy admissible set)**.** Let X be a nonempty set of candidate elements. A *fuzzy admissible set* on X is a fuzzy set

$$\widetilde{C} = \{(x, \mu_{\widetilde{C}}(x)) \mid x \in X\},$$

where

$$\mu_{\widetilde{C}} : X \to [0,1]$$

assigns to each $x \in X$ its degree of admissibility.

If the admissibility conditions are given by fuzzy constraint-satisfaction functions

$$\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n : X \to [0,1],$$

and if $T : [0,1]^{m+n} \to [0,1]$ is a chosen aggregation operator (for example, a t-norm or the minimum operator), then one may define

$$\mu_{\widetilde{C}}(x) = T\big(\mu_1(x), \dots, \mu_m(x), \nu_1(x), \dots, \nu_n(x)\big).$$

In particular, using the minimum operator,

$$\mu_{\widetilde{C}}(x) = \min\{\mu_1(x), \dots, \mu_m(x), \nu_1(x), \dots, \nu_n(x)\}.$$

29.3 Neutrosophic Admissible Set

A neutrosophic admissible set assigns each element truth, indeterminacy, and falsity degrees of admissibility, commonly combining constraints by minimum truth and maximum indeterminacy and falsity.

Definition 29.3.1 (Neutrosophic admissible set)**.** Let X be a nonempty set. A *neutrosophic admissible set* on X is a single-valued neutrosophic set

$$\mathcal{A}_N = \big(T_{\mathcal{A}_N}, I_{\mathcal{A}_N}, F_{\mathcal{A}_N}\big),$$

where

$$T_{\mathcal{A}_N}, I_{\mathcal{A}_N}, F_{\mathcal{A}_N} : X \to [0,1].$$

For each $x \in X$, the values

$$T_{\mathcal{A}_N}(x), \qquad I_{\mathcal{A}_N}(x), \qquad F_{\mathcal{A}_N}(x)$$

are interpreted, respectively, as the truth-degree, indeterminacy-degree, and falsity-degree of admissibility of x.

If

$$\mathcal{C}_k = (T_k, I_k, F_k) \qquad (k = 1, \dots, m)$$

is a finite family of neutrosophic constraint-satisfaction mappings on X, then one natural induced neutrosophic admissible set is given by

$$T_{\mathcal{A}_N}(x) = \min_{1 \le k \le m} T_k(x),$$

$$I_{\mathcal{A}_N}(x) = \max_{1 \le k \le m} I_k(x),$$

$$F_{\mathcal{A}_N}(x) = \max_{1 \le k \le m} F_k(x), \qquad x \in X.$$

Theorem 29.3.2 (Well-definedness of neutrosophic admissible sets)**.** *Let X be a nonempty set, and let*

$$T, I, F : X \to [0, 1]$$

be functions. Then

$$\mathcal{A}_N = (T, I, F)$$

is a well-defined neutrosophic admissible set on X.

In particular, if

$$\mathcal{C}_k = (T_k, I_k, F_k) \qquad (k = 1, \ldots, m)$$

is a finite family of neutrosophic constraint-satisfaction mappings on X, and if

$$T(x) = \min_{1 \le k \le m} T_k(x), \qquad I(x) = \max_{1 \le k \le m} I_k(x), \qquad F(x) = \max_{1 \le k \le m} F_k(x),$$

then

$$\mathcal{A}_N = (T, I, F)$$

is a well-defined neutrosophic admissible set on X.

Conversely, every neutrosophic admissible set on X is determined by such a triple of functions

$$T, I, F : X \to [0, 1].$$

Proof. Since

$$T, I, F : X \to [0, 1]$$

are functions, for each

$$x \in X$$

the values

$$T(x), \quad I(x), \quad F(x)$$

are uniquely determined elements of $[0, 1]$. Hence each element $x \in X$ is assigned a unique neutrosophic admissibility triple, and therefore

$$\mathcal{A}_N = (T, I, F)$$

is a well-defined neutrosophic admissible set on X.

For the induced construction, let

$$x \in X.$$

Since each

$$T_k(x),\ I_k(x),\ F_k(x) \in [0, 1],$$

the finite minimum

$$\min_{1 \le k \le m} T_k(x)$$

and the finite maxima

$$\max_{1 \le k \le m} I_k(x), \qquad \max_{1 \le k \le m} F_k(x)$$

also belong to $[0, 1]$. Thus the functions

$$T,\ I,\ F : X \to [0, 1]$$

are well defined, and so

$$\mathcal{A}_N = (T, I, F)$$

is a well-defined neutrosophic admissible set on X.

Conversely, by definition, any neutrosophic admissible set on X is precisely a single-valued neutrosophic set on X, and hence is determined by a triple of functions

$$T, I, F : X \to [0, 1].$$

This completes the proof. □

29.4 Uncertain Admissible Set

An uncertain admissible set assigns to each element a nonempty set of possible admissibility degrees, often formed from minima of compatible constraint values under uncertainty.

Definition 29.4.1 (Uncertain admissible set)**.** Let X be a nonempty set, and let

$$\mathcal{P}^*([0, 1]) := \mathcal{P}([0, 1]) \setminus \{\emptyset\}.$$

An *uncertain admissible set* on X is an uncertain set

$$\mathcal{A}_U = (X, \mu_{\mathcal{A}_U}),$$

where

$$\mu_{\mathcal{A}_U} : X \to \mathcal{P}^*([0, 1]).$$

For each $x \in X$, the value

$$\mu_{\mathcal{A}_U}(x)$$

is interpreted as the set of possible admissibility degrees of x.

If

$$\mu_1, \ldots, \mu_m : X \to \mathcal{P}^*([0, 1])$$

is a finite family of uncertain constraint-satisfaction mappings on X, then one natural induced uncertain admissible set is defined by

$$\mu_{\mathcal{A}_U}(x) := \Big\{ \min(\alpha_1, \ldots, \alpha_m) \ \Big|\ \alpha_k \in \mu_k(x) \text{ for all } k = 1, \ldots, m \Big\}, \qquad x \in X.$$

Theorem 29.4.2 (Well-definedness of uncertain admissible sets)**.** *Let X be a nonempty set, and let*

$$\mu : X \to \mathcal{P}^*([0, 1])$$

be a function. Then

$$\mathcal{A}_U = (X, \mu)$$

is a well-defined uncertain admissible set on X.

In particular, if

$$\mu_1, \ldots, \mu_m : X \to \mathcal{P}^*([0,1])$$

is a finite family of uncertain constraint-satisfaction mappings on X, and if

$$\mu(x) := \Big\{ \min(\alpha_1, \ldots, \alpha_m) \ \Big| \ \alpha_k \in \mu_k(x) \text{ for all } k = 1, \ldots, m \Big\},$$

then

$$\mathcal{A}_U = (X, \mu)$$

is a well-defined uncertain admissible set on X.

Conversely, every uncertain admissible set on X arises from such a function

$$\mu : X \to \mathcal{P}^*([0,1]).$$

Proof. Since

$$\mu : X \to \mathcal{P}^*([0,1])$$

is a function, for every

$$x \in X$$

the value

$$\mu(x)$$

is uniquely determined and satisfies

$$\mu(x) \in \mathcal{P}^*([0,1]).$$

Hence each element of X is assigned a unique nonempty subset of $[0,1]$ as its uncertain admissibility degree, and therefore

$$\mathcal{A}_U = (X, \mu)$$

is a well-defined uncertain admissible set on X.

For the induced construction, fix

$$x \in X.$$

Since each

$$\mu_k(x) \in \mathcal{P}^*([0,1]),$$

every $\mu_k(x)$ is nonempty. Choose

$$\alpha_k \in \mu_k(x) \qquad (k = 1, \ldots, m).$$

Then

$$\min(\alpha_1, \ldots, \alpha_m) \in [0,1],$$

so the set

$$\mu(x) = \Big\{ \min(\alpha_1, \ldots, \alpha_m) \ \Big| \ \alpha_k \in \mu_k(x) \text{ for all } k = 1, \ldots, m \Big\}$$

is nonempty and satisfies

$$\mu(x) \subseteq [0,1].$$

Hence

$$\mu(x) \in \mathcal{P}^*([0,1]).$$

Since this is true for every

$$x \in X,$$

the mapping

$$\mu : X \to \mathcal{P}^*([0, 1])$$

is well defined, and thus

$$\mathcal{A}_U = (X, \mu)$$

is a well-defined uncertain admissible set on X.

Conversely, by definition, every uncertain admissible set on X is precisely an uncertain set determined by a function

$$\mu : X \to \mathcal{P}^*([0, 1]).$$

This completes the proof. □

Chapter 30

Named Set

In this chapter, we discuss named set theory.

30.1 Named Set

A named set is a structure assigning to each element of a support set a name from a designated name set through a naming map [508–510].

Definition 30.1.1 (Named set)**.** Let X be a nonempty set, let N be a nonempty set, and let

$$\alpha : X \to N$$

be a mapping. Then the triple

$$\Gamma = (X, \alpha, N)$$

is called a *named set*, where X is called the *support*, N is called the *set of names*, and α is called the *naming map*.

Equivalently, a named set may be represented by

$$\Gamma = \{(x, \alpha(x)) \mid x \in X\}.$$

30.2 Fuzzy Named Set

A fuzzy named set assigns each element both a name and a membership degree in $[0, 1]$, combining naming structure with fuzzy graded belonging information formally.

Definition 30.2.1 (Fuzzy named set)**.** Let X be a nonempty set, let N be a nonempty set, let

$$\alpha : X \to N$$

be a naming map, and let

$$\mu : X \to [0,1]$$

be a fuzzy membership function. Then the quadruple

$$\Gamma_F = (X, \alpha, N, \mu)$$

is called a *fuzzy named set.*

For each

$$x \in X,$$

the value

$$\alpha(x) \in N$$

is the name assigned to x, and

$$\mu(x) \in [0,1]$$

is the degree of membership of x.

Equivalently, a fuzzy named set may be represented by

$$\Gamma_F = \{(x, \alpha(x), \mu(x)) \mid x \in X\}.$$

30.3 Neutrosophic Named Set

A neutrosophic named set assigns each element a name together with truth, indeterminacy, and falsity degrees, extending named sets by three-valued neutrosophic membership information formally.

Definition 30.3.1 (Neutrosophic named set)**.** Let X be a nonempty set, let N be a nonempty set, and let

$$\alpha : X \to N$$

be a naming map. Let

$$T, I, F : X \to [0,1]$$

be three functions such that, for every

$$x \in X,$$

one has

$$0 \le T(x) + I(x) + F(x) \le 3.$$

Then the sextuple

$$\Gamma_N = (X, \alpha, N, T, I, F)$$

is called a *neutrosophic named set.*

For each

$$x \in X,$$

the value

$$\alpha(x) \in N$$

is the name assigned to x, while

$$T(x), \qquad I(x), \qquad F(x)$$

denote, respectively, the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree associated with x.

Equivalently, a neutrosophic named set may be represented by

$$\Gamma_N = \{(x, \alpha(x), T(x), I(x), F(x)) \mid x \in X\}.$$

Example 30.3.2 (A concrete real-life example of a neutrosophic named set)**.** Let

$$X = \{p_1, p_2, p_3\}$$

be a set of three patients in a hospital, and let

$$N = \{\text{critical}, \text{stable}, \text{observation}\}$$

be a set of medical status labels.

Define a naming map

$$\alpha : X \to N$$

by

$$\alpha(p_1) = \text{critical}, \qquad \alpha(p_2) = \text{stable}, \qquad \alpha(p_3) = \text{observation}.$$

Suppose that the hospital also records, for each patient, neutrosophic degrees describing the extent to which the assigned status is regarded as appropriate. Define

$$T, I, F : X \to [0, 1]$$

by

$$T(p_1) = 0.90, \qquad I(p_1) = 0.05, \qquad F(p_1) = 0.10,$$
$$T(p_2) = 0.80, \qquad I(p_2) = 0.10, \qquad F(p_2) = 0.15,$$
$$T(p_3) = 0.60, \qquad I(p_3) = 0.25, \qquad F(p_3) = 0.20.$$

Then, for each patient, we have

$$0 \le T(x) + I(x) + F(x) \le 3.$$

Indeed,

$$T(p_1) + I(p_1) + F(p_1) = 0.90 + 0.05 + 0.10 = 1.05 \le 3,$$
$$T(p_2) + I(p_2) + F(p_2) = 0.80 + 0.10 + 0.15 = 1.05 \le 3,$$
$$T(p_3) + I(p_3) + F(p_3) = 0.60 + 0.25 + 0.20 = 1.05 \le 3.$$

Hence,

$$\Gamma_N = (X, \alpha, N, T, I, F)$$

is a neutrosophic named set.

Equivalently, it may be written as

$\Gamma_N = \{(p_1, \text{critical}, 0.90, 0.05, 0.10),\ (p_2, \text{stable}, 0.80, 0.10, 0.15),\ (p_3, \text{observation}, 0.60, 0.25, 0.20)\}$.

In practical terms:

- patient p_1 is named *critical*, and this label has high truth-membership with low indeterminacy and low falsity;
- patient p_2 is named *stable*, and this label is also strongly supported;
- patient p_3 is named *observation*, but this assignment involves more uncertainty than the others.

Thus, this example shows how a neutrosophic named set combines an ordinary naming system with neutrosophic evaluations of the assigned names.

30.4 Uncertain Named Set

An uncertain named set assigns each element a name and a nonempty set of possible membership degrees, representing naming with set-valued uncertainty in a framework.

Definition 30.4.1 (Uncertain named set)**.** Let X be a nonempty set, let N be a nonempty set, and let

$$\alpha : X \to N$$

be a naming map. Define

$$\mathcal{P}^*([0, 1]) := \mathcal{P}([0, 1]) \setminus \{\emptyset\}.$$

Let

$$\mu_U : X \to \mathcal{P}^*([0, 1])$$

be an uncertain membership mapping. Then the quadruple

$$\Gamma_U = (X, \alpha, N, \mu_U)$$

is called an *uncertain named set*.

For each

$$x \in X,$$

the value

$$\alpha(x) \in N$$

is the name assigned to x, and

$$\mu_U(x) \subseteq [0, 1], \qquad \mu_U(x) \neq \emptyset,$$

is the set of possible membership degrees of x.

Equivalently, an uncertain named set may be represented by

$$\Gamma_U = \{(x, \alpha(x), \mu_U(x)) \mid x \in X\}.$$

Chapter 31

Naive Set Theory

In this chapter, we discuss naive set theory.

31.1 Naive Set

A naive set is the collection of all elements in a universe satisfying a given property, formed directly by unrestricted comprehension in naive set theory [511, 512].

Definition 31.1.1 (Naive set)**.** Let U be a nonempty universe, and let $\varphi(x)$ be a formula in the underlying language with free variable x. A *naive set* determined by φ is the collection

$$A = \{x \in U \mid \varphi(x)\}.$$

Thus, in naive set theory, sets are formed by unrestricted comprehension: every well-formed property $\varphi(x)$ determines a set consisting of exactly those elements of U that satisfy φ.

31.2 Naive Fuzzy Set

A naive fuzzy set assigns to each universe element the truth degree of a fuzzy predicate, thereby forming a fuzzy set by unrestricted fuzzy comprehension (cf. [513–515]).

Definition 31.2.1 (Naive fuzzy set)**.** Let U be a nonempty universe, and let $\varphi(x)$ be a fuzzy predicate on U whose truth value

$$\|\varphi(x)\|$$

belongs to $[0, 1]$ for each $x \in U$. A *naive fuzzy set* determined by φ is a fuzzy set

$$A = (U, \mu_A), \qquad \mu_A : U \to [0, 1],$$

defined by

$$\mu_A(x) = \|\varphi(x)\| \qquad (x \in U).$$

Equivalently, one may write

$$A = \{(x, \mu_A(x)) \mid x \in U\} = \{(x, \|\varphi(x)\|) \mid x \in U\}.$$

Hence a naive fuzzy set is obtained by an unrestricted fuzzy comprehension principle: every fuzzy property determines a fuzzy set whose membership degree at each element is the truth degree of that property.

31.3 Naive Neutrosophic Set

A naive neutrosophic set assigns each element truth, indeterminacy, and falsity degrees from a neutrosophic predicate, thereby producing a neutrosophic set through unrestricted comprehension scheme.

Definition 31.3.1 (Naive neutrosophic set)**.** Let U be a nonempty universe, and let $\varphi(x)$ be a neutrosophic predicate on U such that, for each $x \in U$, the evaluation of $\varphi(x)$ is given by a triple

$$\|\varphi(x)\|_N = \big(T_\varphi(x), I_\varphi(x), F_\varphi(x)\big) \in [0,1]^3,$$

where

$$0 \le T_\varphi(x) + I_\varphi(x) + F_\varphi(x) \le 3.$$

A *naive neutrosophic set* determined by φ is the single-valued neutrosophic set

$$A_N = \big(U, T_{A_N}, I_{A_N}, F_{A_N}\big),$$

defined by

$$T_{A_N}(x) = T_\varphi(x), \qquad I_{A_N}(x) = I_\varphi(x), \qquad F_{A_N}(x) = F_\varphi(x) \qquad (x \in U).$$

Equivalently, one may write

$$A_N = \left\{ \big(x, T_\varphi(x), I_\varphi(x), F_\varphi(x)\big) \;\middle|\; x \in U \right\}.$$

Thus a naive neutrosophic set is obtained by an unrestricted neutrosophic comprehension principle: every neutrosophic property determines a neutrosophic set whose truth, indeterminacy, and falsity degrees at each element are exactly the corresponding evaluation degrees of that property.

Theorem 31.3.2 (Well-definedness of naive neutrosophic sets)**.** *Let U be a nonempty universe, and let $\varphi(x)$ be a neutrosophic predicate on U such that, for every $x \in U$,*

$$\|\varphi(x)\|_N = \big(T_\varphi(x), I_\varphi(x), F_\varphi(x)\big) \in [0,1]^3$$

with

$$0 \le T_\varphi(x) + I_\varphi(x) + F_\varphi(x) \le 3.$$

Then the mapping

$$A_N = \big(U, T_{A_N}, I_{A_N}, F_{A_N}\big),$$

defined by

$$T_{A_N}(x) = T_\varphi(x), \qquad I_{A_N}(x) = I_\varphi(x), \qquad F_{A_N}(x) = F_\varphi(x),$$

is a well-defined naive neutrosophic set on U.

Conversely, every naive neutrosophic set on U arises from such a neutrosophic predicate.

Proof. For each

$$x \in U,$$

the values

$$T_\varphi(x), \qquad I_\varphi(x), \qquad F_\varphi(x)$$

are uniquely determined by the evaluation of the predicate $\varphi(x)$. By assumption,

$$T_\varphi(x),\ I_\varphi(x),\ F_\varphi(x) \in [0,1]$$

and

$$0 \leq T_\varphi(x) + I_\varphi(x) + F_\varphi(x) \leq 3.$$

Hence the assignments

$$x \mapsto T_\varphi(x), \qquad x \mapsto I_\varphi(x), \qquad x \mapsto F_\varphi(x)$$

define three well-defined functions

$$T_{A_N}, I_{A_N}, F_{A_N} : U \to [0,1].$$

Therefore

$$A_N = \big(U, T_{A_N}, I_{A_N}, F_{A_N}\big)$$

is a well-defined single-valued neutrosophic set on U, and thus a well-defined naive neutrosophic set.

Conversely, let

$$A_N = \big(U, T, I, F\big)$$

be a naive neutrosophic set on U. Define a neutrosophic predicate $\varphi_A(x)$ by setting

$$\|\varphi_A(x)\|_N = (T(x), I(x), F(x)) \qquad (x \in U).$$

Then the naive neutrosophic set determined by φ_A is exactly A_N. Hence every naive neutrosophic set on U arises from such a neutrosophic predicate. □

31.4 Naive Uncertain Set

A naive uncertain set assigns each element a nonempty set of possible membership degrees, obtained from an uncertain predicate by unrestricted uncertain comprehension in theory.

Definition 31.4.1 (Naive uncertain set)**.** Let U be a nonempty universe, and define

$$\mathcal{P}^*([0,1]) := \mathcal{P}([0,1]) \setminus \{\emptyset\}.$$

Let $\psi(x)$ be an uncertain predicate on U such that, for each $x \in U$, the evaluation of $\psi(x)$ is a nonempty subset of $[0,1]$, written as

$$\|\psi(x)\|_U \in \mathcal{P}^*([0,1]).$$

A *naive uncertain set* determined by ψ is the uncertain set

$$A_U = (U, \mu_{A_U}),$$

where

$$\mu_{A_U} : U \to \mathcal{P}^*([0,1])$$

is defined by

$$\mu_{A_U}(x) = \|\psi(x)\|_U \qquad (x \in U).$$

Equivalently, one may write

$$A_U = \left\{ \big(x, \|\psi(x)\|_U\big) \;\middle|\; x \in U \right\}.$$

Thus a naive uncertain set is obtained by an unrestricted uncertain comprehension principle: every uncertain property determines an uncertain set whose membership value at each element is the corresponding set of possible truth degrees of that property.

Theorem 31.4.2 (Well-definedness of naive uncertain sets)**.** *Let U be a nonempty universe, and let $\psi(x)$ be an uncertain predicate on U such that, for every $x \in U$,*

$$\|\psi(x)\|_U \in \mathcal{P}^*([0,1]).$$

Then the mapping

$$A_U = (U, \mu_{A_U}), \qquad \mu_{A_U}(x) = \|\psi(x)\|_U,$$

is a well-defined naive uncertain set on U.

Conversely, every naive uncertain set on U arises from such an uncertain predicate.

Proof. For each

$$x \in U,$$

the value

$$\|\psi(x)\|_U$$

is uniquely determined by the evaluation of the uncertain predicate $\psi(x)$. By assumption,

$$\|\psi(x)\|_U \in \mathcal{P}^*([0,1]),$$

that is, $\|\psi(x)\|_U$ is a nonempty subset of $[0,1]$. Hence the assignment

$$x \mapsto \|\psi(x)\|_U$$

defines a well-defined function

$$\mu_{A_U} : U \to \mathcal{P}^*([0,1]).$$

Therefore

$$A_U = (U, \mu_{A_U})$$

is a well-defined uncertain set on U, and thus a well-defined naive uncertain set.

Conversely, let

$$A_U = (U, \mu)$$

be a naive uncertain set on U, where

$$\mu : U \to \mathcal{P}^*([0,1]).$$

Define an uncertain predicate $\psi_A(x)$ by setting

$$\|\psi_A(x)\|_U = \mu(x) \qquad (x \in U).$$

Then the naive uncertain set determined by ψ_A is exactly A_U. Hence every naive uncertain set on U arises from such an uncertain predicate. □

Chapter 32

Conclusion

In this survey book, we have provided a broad and systematic introduction to set concepts closely related to fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, plithogenic sets, and other uncertainty-oriented frameworks. In particular, we have organized a wide range of set-theoretic models and their extensions, clarified their basic definitions and structural viewpoints, and highlighted how these concepts can be understood within a unified framework for representing vagueness, indeterminacy, partial truth, and complex uncertainty.

We hope that future research will further develop both the theoretical foundations and practical applications of these concepts. Promising directions include extensions based on topological spaces and algebraic structures, deeper investigations of order-theoretic and lattice-theoretic properties, and the construction of new models for decision-making under uncertainty. In addition, further studies on hypergraphs and superhypergraphs may provide richer mathematical frameworks for describing higher-order relations, multi-agent systems, and complex interconnected data. We also expect that these developments will contribute to broader applications in artificial intelligence, data analysis, optimization, and related areas of decision support and intelligent systems.

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Data Availability

This work is purely theoretical and mathematical in nature; therefore, no empirical data or computational datasets were used. Future studies may build upon these results through data-driven, computational, or experimental approaches.

Ethical Statement

This study did not involve human participants, animals, or personal data. Accordingly, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the content or publication of this book.

Use of Generative AI and AI-Assisted Tools

The authors used generative AI and AI-assisted tools only for limited support tasks, such as English grammar and language refinement. These tools were not used in any manner that would compromise academic integrity or violate ethical standards.

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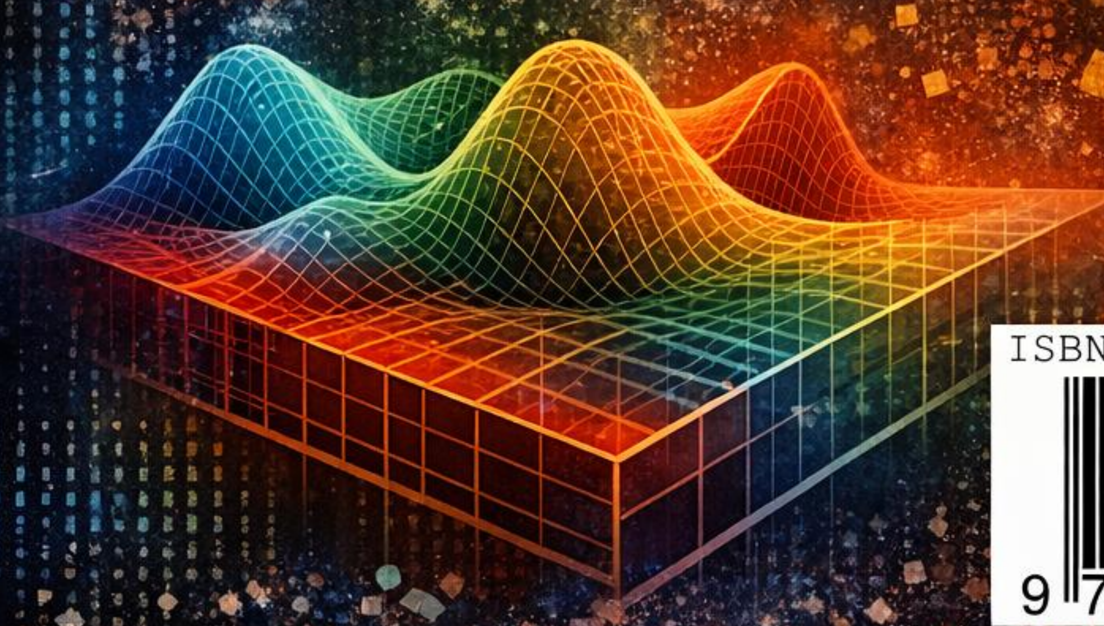
Abstract

This book provides a comprehensive and systematic survey of modern set-theoretic frameworks developed to model uncertainty, vagueness, indeterminacy, and partial truth in real-world phenomena. It synthesizes a wide spectrum of classical and advanced theories, including fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, **plithogenic** sets, and numerous related extensions.

The work organizes these concepts across multiple structural and conceptual dimensions—such as **interval, granular, probilistic,** and **complex extensions**—highlighting their mathematical foundations, interrelations, and distinguishing characteristics.

By unifying diverse approaches under a coherent analytical perspective, the book clarifies how different uncertainty models complement or generalize one another. It also **explores** emerging directions in set theory, including **hybrid and dynamic frameworks,** offering insights into their applicability **in fields such as decision-making, control systems,** artificial intelligence, and **data analysis.**



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