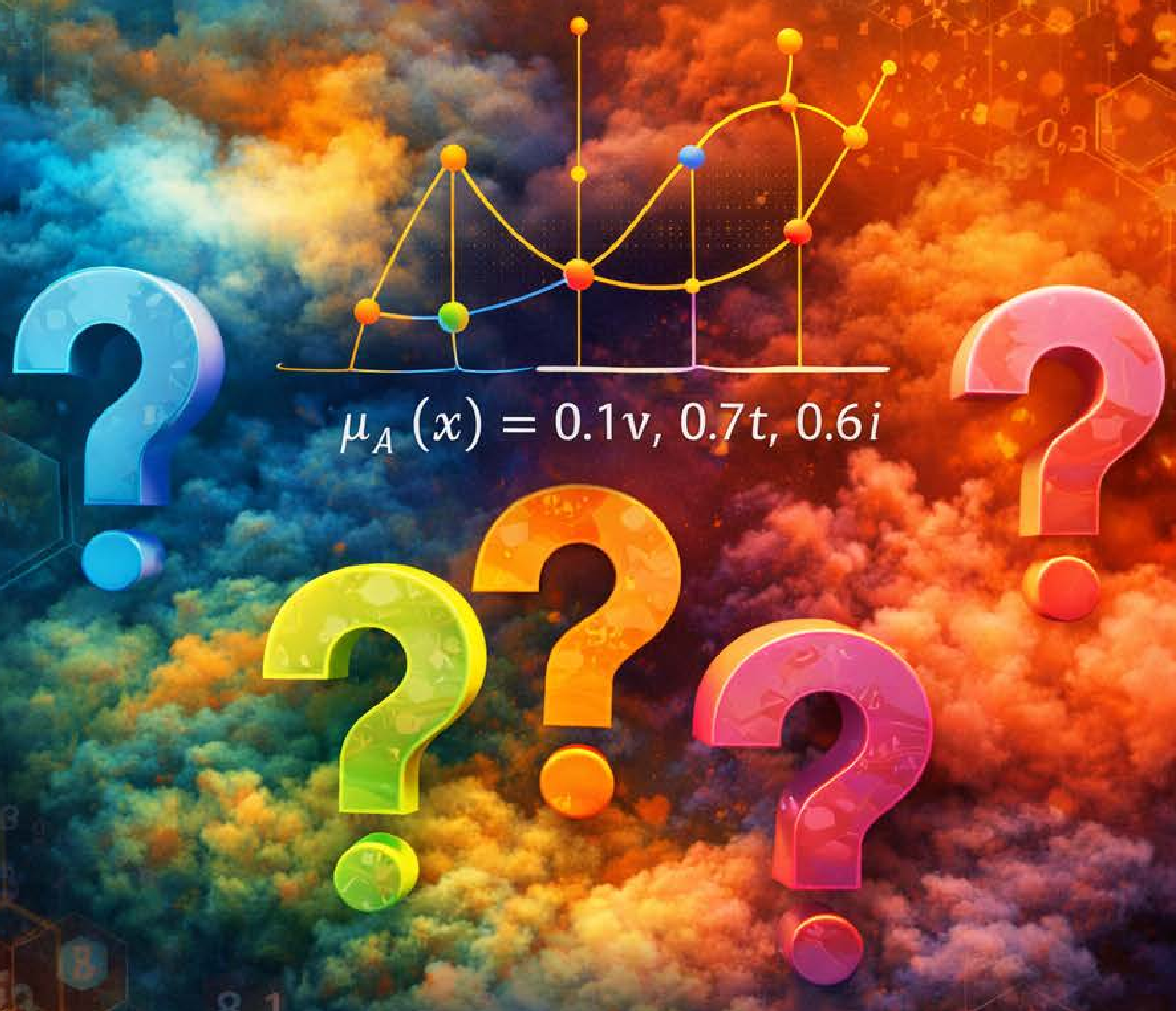
A Survey of FUZZY and UNCERTAIN Concepts in Applied Mathematics



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Chapter 1

Introduction

1.1 Uncertain Set

Real-world phenomena frequently involve vagueness, partial truth, incomplete information, and sometimes even inconsistency. To model such uncertainty in a mathematically rigorous manner, many generalized set-theoretic frameworks have been developed, including Fuzzy Sets [1], Intuitionistic Fuzzy Sets [2], Neutrosophic Sets [3, 4], Vague Sets [5], Hesitant Fuzzy Sets [6], Picture Fuzzy Sets [7], Quadripartitioned Neutrosophic Sets [8], PentaPartitioned Neutrosophic Sets [9], Plithogenic Sets [10], HyperFuzzy Sets [11], and HyperNeutrosophic Sets [12]. These models have been applied in a wide range of areas, including decision science, chemistry, control theory, and machine learning [13]. In practice, the choice of a suitable framework depends on the type of uncertainty to be represented and on the number and nature of the parameters needed to describe the underlying phenomenon.

In the classical fuzzy setting, each element x of a universe X is assigned a single membership degree

$$\mu(x) \in [0, 1],$$

which expresses the extent to which x belongs to the set under consideration [1]. An intuitionistic fuzzy set enriches this description by assigning to each element x a pair

$$(\mu(x), \nu(x)),$$

where $\mu, \nu : X \to [0, 1]$ denote the membership and nonmembership functions, respectively, and satisfy

$$0 \le \mu(x) + \nu(x) \le 1$$

for all $x \in X$ [2, 14]. The remaining quantity

$$1 - \mu(x) - \nu(x)$$

is interpreted as the hesitation margin.

A neutrosophic set further generalizes this representation by assigning to each element x a triple

$$(T(x), I(x), F(x)),$$

where $T(x)$, $I(x)$, and $F(x)$ denote, respectively, the degrees of truth, indeterminacy, and falsity, typically taking values in $[0,1]$. Unlike the intuitionistic fuzzy setting, these three components are not required to sum to 1, which makes it possible to represent incomplete, inconsistent, or redundant information in a more flexible way [14,15]. [1] More broadly, neutrosophy emphasizes the role of neutrality and indeterminacy, leading to the development of neutrosophic logic, probability, statistics, measure, integral, and many related mathematical frameworks [13,17].

Plithogenic sets extend these ideas even further by describing each element through attribute values together with corresponding appurtenance degrees, while also incorporating a contradiction (or dissimilarity) function between distinct attribute values [10,18,19]. This additional structure allows context-sensitive aggregation of heterogeneous and possibly conflicting information, thereby refining classical fuzzy, intuitionistic fuzzy, and neutrosophic models (e.g. [17,20]).

For convenience, Table 1.1 summarizes the canonical information associated with each element in several representative extensions of set theory used throughout this book.

Table 1.1: Representative set extensions and the canonical information associated with each element.

Set Type	**Canonical data attached to each element**
Fuzzy Set	Membership mapping $\mu : X \to [0,1]$.
Intuitionistic Fuzzy Set	Membership μ and nonmembership ν satisfying $\mu(x)+\nu(x) \le 1$; the remainder $1-\mu(x)-\nu(x)$ represents hesitation.
Neutrosophic Set	Triple (T, I, F) with $T, I, F \in [0,1]$, representing truth, indeterminacy, and falsity, treated as mutually independent components.
Plithogenic Set	Attribute values together with appurtenance degrees and a contradiction function between attribute values; equivalently, one may use a tuple such as $(P, v, Pv, \mathrm{pdf}, \mathrm{pCF})$, where $\mathrm{pdf} : P \times Pv \to [0,1]^s$ encodes s-dimensional appurtenance and $\mathrm{pCF} : Pv \times Pv \to [0,1]^t$ is a symmetric contradiction map.

A brief comparison of the fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic frameworks is given in Table 1.2. Fuzzy sets use one membership degree, intuitionistic fuzzy sets add non-membership, neutrosophic sets separate truth, indeterminacy, and falsity, while plithogenic sets further incorporate multi-attribute contradiction-aware aggregation.

[1] In an intuitionistic fuzzy set, indeterminacy is represented only implicitly through the hesitation margin $1-\mu(x)-\nu(x)$, whereas a neutrosophic set introduces indeterminacy explicitly as an independent component $I(x)$. It is also often noted in the literature that neutrosophic sets generalize several important extensions, including intuitionistic fuzzy sets, picture fuzzy sets, Pythagorean fuzzy sets, spherical fuzzy sets, and q-rung orthopair fuzzy sets [16].

Table 1.2: Comparison of fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic frameworks

Framework	**Degree representation**	**Basic constraint**	**Main feature**	**Typical limitation**
Fuzzy set	One membership degree $\mu(x) \in [0, 1]$	No additional degree is required	Models gradual belongingness by a single value	Does not explicitly separate support, opposition, and indeterminacy
Intuitionistic fuzzy set	Membership $\mu(x)$ and non-membership $\nu(x)$	$\mu(x), \nu(x) \in [0, 1]$, $\mu(x) + \nu(x) \leq 1$	Explicitly represents acceptance, rejection, and hesitation	Indeterminacy is implicit, given by $1 - \mu(x) - \nu(x)$
Neutrosophic set	Truth $T(x)$, indeterminacy $I(x)$, and falsity $F(x)$	Typically $T(x), I(x), F(x) \in [0, 1]$	Separates truth, indeterminacy, and falsity independently	More flexible but harder to interpret and aggregate consistently
Plithogenic set	Appurtenance degrees together with attribute values and contradiction degrees	Degree structure depends on the chosen underlying model	Incorporates multi-attribute evaluation and contradiction-aware aggregation	More general and expressive, but mathematically and computationally more complex

1.2 Our Contributions

Fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, plithogenic sets, and many related frameworks have been studied extensively in a broad spectrum of theoretical and applied disciplines. Nevertheless, the relevant concepts are scattered across many branches of applied mathematics, and a unified survey covering these extensions in a systematic way remains highly desirable.

In this book, we present a broad and organized survey of extension concepts in applied mathematics that are formulated by means of fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, plithogenic sets, and related uncertainty-oriented frameworks. Our aim is not only to collect these concepts in a single reference, but also to clarify their mathematical relationships, highlight their structural similarities and differences, and provide a coherent overview of their roles in applied mathematical modeling. For the reader's convenience, a concise classification of the fuzzy topics covered in this book is presented in Table 1.3.

Table 1.3: A more detailed concise classification of the fuzzy topics covered in this book

Category	Subcategory	Topics	Main focus
Discrete and structural mathematics	Graph, network, and combinatorial structures	Fuzzy Graph, Fuzzy HyperGraph, Fuzzy SuperHyperGraph, Fuzzy Matroid	Relations, dependence, higher-order structure
Discrete and structural mathematics	Formal and computational structures	Fuzzy Language, Fuzzy Automata, Fuzzy Petri Net, Fuzzy Decision Tree	Computation, transition, rule-based systems
Algebraic and linear structures	Algebraic systems	Fuzzy Algebra, Fuzzy Group, Fuzzy Semigroup, Fuzzy Ring, Fuzzy Semiring	Uncertain algebraic operations
Algebraic and linear structures	Order and lattice-based structures	Fuzzy Lattice, Fuzzy Boolean Algebra, Fuzzy Poset	Ordering, logic, hierarchical structure
Algebraic and linear structures	Linear and representation structures	Fuzzy Vector Space, Fuzzy Matrix, Fuzzy Module	Linear representation and computation
Analytical and spatial mathematics	Probability and dynamical models	Fuzzy Probability, Fuzzy Differential Equations, Fuzzy Fixed Point	Dynamics, evolution, uncertainty in systems
Analytical and spatial mathematics	Metric, topological, and geometric structures	Fuzzy Metric Space, Fuzzy Topology, Fuzzy Geometry, Fuzzy Distance Measure	Distance, continuity, spatial reasoning
Quantitative and measurement tools	Numerical and interval-type representations	Fuzzy Number, Fuzzy Fibonacci Number, Fuzzy Sequence	Quantitative uncertain values
Quantitative and measurement tools	Aggregation and information measures	Fuzzy Weighted Average, Fuzzy Integral, Fuzzy Entropy, Fuzzy Measure	Aggregation, evaluation, uncertainty quantification

Chapter 2

Preliminaries

This chapter collects the basic notation and background used throughout the book.

2.1 Fuzzy set

Fuzzy set theory extends the classical notion of a subset by allowing graded membership, quantified by a value in $[0,1]$ [1, 21, 22]. The core definition is recalled next.

Definition 2.1.1 (Fuzzy set)**.** [1] Let X be a nonempty set. A *fuzzy set* A on X is specified by a membership function

$$\mu_A : X \to [0,1].$$

Equivalently, one may write

$$A = \{(x, \mu_A(x)) \mid x \in X\},$$

where $\mu_A(x)$ indicates the degree to which x belongs to A.

A particularly common numeric object in fuzzy decision analysis is the *triangular fuzzy number* (TFN) [23, 24]. It models an imprecise quantity through a simple piecewise-linear membership profile.

Definition 2.1.2 (Triangular fuzzy number (TFN))**.** [25, 26] Let X be a universe. A *triangular fuzzy number* (TFN) $\tilde{x}$ is a fuzzy set on $\mathbb{R}$

whose membership function $\mu_{\tilde{x}} : \mathbb{R} \to [0,1]$ is determined by three real parameters

$$\tilde{x} = (l_x, m_x, u_x) \in \mathbb{R}^3, \qquad l_x \le m_x \le u_x,$$

via

$$\mu_{\tilde{x}}(t) = \begin{cases} 0, & t < l_x, \\ \dfrac{t - l_x}{m_x - l_x}, & l_x \le t \le m_x, \quad (m_x > l_x), \\ \dfrac{u_x - t}{u_x - m_x}, & m_x \le t \le u_x, \quad (u_x > m_x), \\ 0, & t > u_x. \end{cases}$$

If $l_x = m_x$ (resp. $m_x = u_x$), the corresponding middle expression is interpreted in the limiting sense, so that $\mu_{\tilde{x}}(m_x) = 1$ and the membership curve remains triangular. If, in addition, $0 < l_x \leq m_x \leq u_x$, then $\tilde{x}$ is called a *positive* TFN.

Here m_x represents the modal (most plausible) value, while l_x and u_x serve as lower and upper bounds.

2.2 Intuitionistic fuzzy set

Intuitionistic fuzzy sets enrich fuzzy sets by recording both membership and non-membership information, leaving room for an explicit hesitation component [2]. A standard formulation is as follows.

Definition 2.2.1 (Intuitionistic fuzzy set)**.** [27] Let E be a nonempty set. An *intuitionistic fuzzy set* (IFS) A on E is given by

$$A = \big\{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in E \big\},$$

where

$$\mu_A, \nu_A : E \longrightarrow [0,1]$$

are, respectively, the membership and non-membership functions, and for each $x \in E$,

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

The remaining part,

$$\pi_A(x) := 1 - \mu_A(x) - \nu_A(x),$$

is called the *hesitation degree* at x.

The classical fuzzy-set situation is recovered when $\nu_A(x) = 1 - \mu_A(x)$ for all $x \in E$, equivalently $\pi_A(x) = 0$ for every x.

2.3 Neutrosophic Set

Neutrosophic sets represent uncertainty by assigning three (in general independent) degrees to each element: truth T, indeterminacy I, and falsity F, typically taken in $[0,1]$ [4, 14, 15, 28]. This explicit indeterminacy component provides a flexible generalization of both fuzzy sets and intuitionistic fuzzy sets.

Definition 2.3.1 (Neutrosophic set)**.** [29, 30] Let X be a nonempty set. A *neutrosophic set* (NS) A on X is described by three functions

$$T_A : X \to [0,1], \qquad I_A : X \to [0,1], \qquad F_A : X \to [0,1],$$

where, for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ quantify the degrees of truth, indeterminacy, and falsity of the statement "$x \in A$", respectively. They satisfy

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 2.3.2 (A neutrosophic set on a finite universe)**.** Let

$$X = \{x_1, x_2, x_3\}.$$

Define a neutrosophic set A on X by assigning to each element $x \in X$ the triple

$$\big(T_A(x), I_A(x), F_A(x)\big)$$

as follows:

$$\big(T_A(x_1), I_A(x_1), F_A(x_1)\big) = (0.8, 0.1, 0.2),$$

$$\big(T_A(x_2), I_A(x_2), F_A(x_2)\big) = (0.5, 0.3, 0.4),$$

$$\big(T_A(x_3), I_A(x_3), F_A(x_3)\big) = (0.2, 0.6, 0.7).$$

Then A is a neutrosophic set on X, since for each $x \in X$ one has

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Indeed,

$$0.8 + 0.1 + 0.2 = 1.1, \qquad 0.5 + 0.3 + 0.4 = 1.2, \qquad 0.2 + 0.6 + 0.7 = 1.5,$$

and all of these values belong to the interval $[0, 3]$.

Hence, one may write

$$A = \{\langle x_1, 0.8, 0.1, 0.2\rangle,\ \langle x_2, 0.5, 0.3, 0.4\rangle,\ \langle x_3, 0.2, 0.6, 0.7\rangle\}.$$

For reference, a comparison between fuzzy sets and neutrosophic sets is presented in Table 2.1. Neutrosophic sets are particularly effective for situations in which truth, falsity, and indeterminacy coexist independently, and they are therefore regarded as capable of representing real-world uncertainty in a more refined manner.

Table 2.1: Comparison between fuzzy sets and neutrosophic sets

Aspect	**Fuzzy Set**	**Neutrosophic Set**
Basic representation	Each element is described by a single membership degree $\mu(x) \in [0, 1]$.	Each element is described by three independent degrees: truth $T(x)$, indeterminacy $I(x)$, and falsity $F(x)$.
Information structure	Only the degree of membership is explicitly represented.	Membership-related, indeterminate, and non-membership-related information are represented separately.
Indeterminacy	Indeterminacy is not explicitly modeled as an independent component.	Indeterminacy is explicitly modeled by $I(x)$.
Contradictory information	Cannot directly express simultaneous strong support and strong opposition within the standard model.	Can express conflicting evidence by allowing both $T(x)$ and $F(x)$ to be high at the same time.
Incomplete information	Missing information is not explicitly separated from low membership.	Can distinguish incomplete information from falsity by means of the indeterminacy degree.
Expressiveness	Simpler and easier to interpret.	More expressive and suitable for inconsistent, incomplete, and indeterminate environments.
Typical use	Appropriate when uncertainty is adequately captured by gradual membership alone.	Appropriate when truth, falsity, and indeterminacy must be treated independently.

2.4 Plithogenic Set

A plithogenic set assigns to each element one or more appurtenance degrees associated with relevant attribute values, together with contradiction degrees that quantify the dissimilarity among those values [18, 31–33]. In this way, plithogenic sets provide a flexible framework for multi-attribute uncertainty modeling, where aggregation is performed in a contradiction-aware manner. As a result, they can represent nuanced, heterogeneous, and even conflicting information more faithfully than classical fuzzy, intuitionistic fuzzy, or neutrosophic frameworks.

Definition 2.4.1 (Plithogenic Set)**.** [18, 31] Let P be a nonempty universe of discourse, and let v be a (fixed) attribute whose possible values form a nonempty set Pv. Fix dimensions $s, t \in \mathbb{N}$.

A *plithogenic set* on (P, v, Pv) is a quintuple

$$PS = (P, v, Pv, pdf, pCF),$$

where

- $pdf : P \times Pv \longrightarrow [0,1]^s$ is the *degree of appurtenance function* (DAF); for $x \in P$ and $a \in Pv$, $pdf(x,a)$ is the (possibly vector–valued) membership degree of x corresponding to the attribute value a;
- $pCF : Pv \times Pv \longrightarrow [0,1]^t$ is the *degree of contradiction function* (DCF), satisfying
$$pCF(a,a) = 0, \qquad pCF(a,b) = pCF(b,a) \quad \text{for all } a,b \in Pv.$$

In plithogenic theory, a (typically fixed) *dominant attribute value* $a^* \in Pv$ is chosen, and set–theoretic operations (such as union and intersection) are defined by combining the appurtenance degrees *pdf* with the contradiction degrees $pCF(\,\cdot\,, a^*)$ in order to model interaction and opposition between different attribute values.

Example 2.4.2 (A plithogenic set on a finite universe)**.** Let
$$P = \{p_1, p_2\}$$
be a universe of discourse consisting of two objects, and let v be the attribute "color preference." Assume that the set of possible attribute values is
$$Pv = \{\text{red}, \text{blue}, \text{green}\}.$$
Take
$$s = t = 1.$$

Define the degree of appurtenance function
$$pdf : P \times Pv \to [0,1]$$
by
$$pdf(p_1, \text{red}) = 0.9, \qquad pdf(p_1, \text{blue}) = 0.4, \qquad pdf(p_1, \text{green}) = 0.2,$$
$$pdf(p_2, \text{red}) = 0.3, \qquad pdf(p_2, \text{blue}) = 0.8, \qquad pdf(p_2, \text{green}) = 0.5.$$

Define the degree of contradiction function
$$pCF : Pv \times Pv \to [0,1]$$
by
$$pCF(a,a) = 0 \qquad \text{for all } a \in Pv,$$
and
$$pCF(\text{red}, \text{blue}) = pCF(\text{blue}, \text{red}) = 0.6,$$
$$pCF(\text{red}, \text{green}) = pCF(\text{green}, \text{red}) = 0.3,$$
$$pCF(\text{blue}, \text{green}) = pCF(\text{green}, \text{blue}) = 0.5.$$

Then
$$PS = (P, v, Pv, pdf, pCF)$$
is a plithogenic set.

If the dominant attribute value is chosen as
$$a^* = \text{red},$$
then the contradiction degrees with respect to the dominant value are
$$pCF(\text{red}, \text{red}) = 0, \qquad pCF(\text{blue}, \text{red}) = 0.6, \qquad pCF(\text{green}, \text{red}) = 0.3.$$
Hence, the appurtenance degrees corresponding to "blue" and "green" are interpreted relative to their levels of contradiction with the dominant value "red" when defining plithogenic set operations.

2.5 Rough Set

Rough set theory models imprecision by approximating a target subset through a *certain* part (lower approximation) and a *possible* part (upper approximation), constructed from an indiscernibility relation [34–38]. The classical Pawlak approximations are recalled below.

Definition 2.5.1 (Rough set approximations)**.** [39] Let X be a nonempty universe, and let $R \subseteq X \times X$ be an equivalence (indiscernibility) relation. For $x \in X$, write the equivalence class of x as

$$[x]_R := \{ y \in X \mid (x, y) \in R \}.$$

Given any subset $U \subseteq X$, define:

1. *Lower approximation*:
$$\underline{U} := \{ x \in X \mid [x]_R \subseteq U \}.$$
Elements of $\underline{U}$ are those that belong to U with certainty (their entire class is contained in U).

2. *Upper approximation*:
$$\overline{U} := \{ x \in X \mid [x]_R \cap U \neq \varnothing \}.$$
Elements of $\overline{U}$ are those that may belong to U (their class meets U).

The pair $(\underline{U}, \overline{U})$ is the rough-set representation of U, and it always satisfies

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

2.6 Soft set

Soft sets provide a parameter-based description of uncertainty: each parameter selects a subset of the universe, and the family of all such selections forms the model. This framework was introduced by Molodtsov (1999) and has been widely used in decision problems [40–42].

Definition 2.6.1 (Soft set)**.** [42] Let U be a universe set and let E be a set of parameters. Take $A \subseteq E$ and denote by $\mathcal{P}(U)$ the power set of U. A pair (F, A) is called a *soft set* over U if

$$F : A \to \mathcal{P}(U).$$

For each parameter $\epsilon \in A$, the subset $F(\epsilon) \subseteq U$ is called the *ϵ-approximation* of (F, A). Thus, a soft set is a parameterized family of subsets of U.

As extension concepts of soft sets, notions such as HyperSoft Set [43, 44], IndetermSoft Set [45, 46], SuperHyperSoft Set [47, 48], TreeSoft Set [49, 50], and ForestSoft Set [51, 52] are also known.

2.7 Uncertain set

An *uncertain set* associates with each element a degree taken from a chosen uncertainty model, thereby providing a unifying umbrella for fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related frameworks [53–56].

Definition 2.7.1 (Uncertain model)**.** [54] Let U denote the class of all *uncertain models*. Each $M \in U$ is determined by:

- a nonempty set $\mathrm{Dom}(M) \subseteq [0,1]^k$ of *admissible degree tuples* for some fixed integer $k \geq 1$; and
- model-specific algebraic or geometric constraints imposed on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \leq 1$ in the intuitionistic fuzzy setting, or $0 \leq T + I + F \leq 3$ in the neutrosophic setting).

Typical instances include:

- **Fuzzy model:** $\mathrm{Dom}(M) = [0,1]$;
- **Intuitionistic fuzzy model:** $\mathrm{Dom}(M) = \{(\mu,\nu) \in [0,1]^2 : \ \mu + \nu \leq 1\}$;
- **Neutrosophic model:** $\mathrm{Dom}(M) = \{(T,I,F) \in [0,1]^3 : \ 0 \leq T + I + F \leq 3\}$;
- **Plithogenic model,** and many further extensions.

Definition 2.7.2 (Uncertain set (U-set))**.** [54] Let X be a nonempty universe, and fix an uncertain model M with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *uncertain set of type M* (briefly, a *U-set*) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is the *uncertainty-degree function* (membership map) of $\mathcal{U}$. For $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ encodes the degree(s) to which x belongs to $\mathcal{U}$, as prescribed by the model M.

Example 2.7.3 (A concrete uncertain set)**.** Let

$$U = \{u_1, u_2, u_3, u_4\}$$

be a universe of discourse, where

$$u_1 = \text{“low risk”}, \quad u_2 = \text{“moderate risk”}, \quad u_3 = \text{“high risk”}, \quad u_4 = \text{“critical risk”}.$$

Suppose that an uncertain set A on U is described by an uncertainty-valued function

$$\lambda_A : U \to [0,1],$$

where $\lambda_A(u)$ represents the degree to which the element u belongs to the set under uncertainty.

Define

$$\lambda_A(u_1) = 0.20, \qquad \lambda_A(u_2) = 0.55, \qquad \lambda_A(u_3) = 0.80, \qquad \lambda_A(u_4) = 0.35.$$

Then the uncertain set A can be written as

$$A = \{(u_1, 0.20),\ (u_2, 0.55),\ (u_3, 0.80),\ (u_4, 0.35)\}.$$

In this example, the element u_3 has the highest degree in A, so it is regarded as the most strongly associated with the uncertain set, whereas u_1 has the lowest degree.

As noted in the remark, various generalizations are possible. For reference, Table 2.2 presents a catalogue of uncertainty-set families (U-Sets) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (cf. [55]).

Table 2.2: A catalogue of uncertainty-set families (U-Sets) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [55].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [1, 57]; N-Fuzzy Set [58–60] Shadowed Set [61–63]
2		Intuitionistic Fuzzy Set [2, 64]; Vague Set [5, 65]; Bipolar Fuzzy Set (two-component description) [66]; Variable Fuzzy Set [67–69]; Paraconsistent Fuzzy Set [70, 71]; Bifuzzy Set [72, 73]
3		Single-Valued Neutrosophic Set [28,30]; Picture Fuzzy Set [7,74]; Spherical Fuzzy Set [75,76]; Tripolar Fuzzy Set (three-component formalisms) [77–79]; Neutrosophic Vague Set [80, 81]
4		Quadripartitioned Neutrosophic Set [8, 82]; Double-Valued Neutrosophic Set [83, 84]; Dual Hesitant Fuzzy Set [85, 86]; Ambiguous Set [87–89]; Turiyam Neutrosophic Set [90–93]
5		Pentapartitioned Neutrosophic Set [94–96]; Triple-Valued Neutrosophic Set [97–99]
6		Hexapartitioned Neutrosophic Set [100]; Bipolar Neutrosophic Set [101, 102]; Quadruple-Valued Neutrosophic Set [98, 103]
7		Heptapartitioned Neutrosophic Set [100, 104, 105]; Quintuple-Valued Neutrosophic Set [98, 106, 107]
8		Octapartitioned Neutrosophic Set [100, 108]
9		Nonapartitioned Neutrosophic Set [100, 108]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [109]; MultiFuzzy Set [110]; n-Refined Fuzzy Set [111, 112]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [112]; Multi-Intuitionistic Fuzzy Set [110]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [112–115]; Multi-Neutrosophic Set [110, 116]

Reading guide. In the U-Set scheme [54], each model M is specified by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ and a membership map $\mu_M : X \to \mathrm{Dom}(M)$. The table groups representative families by the ambient dimension k (i.e., how many numerical components are stored per element).

(a) A widely cited viewpoint is that neutrosophic sets provide a unifying umbrella covering several earlier multi-component fuzzy models (and their generalizations); see [16].

(b) Ambiguous sets are commonly presented as subclasses of certain four-component neutrosophic families; see [8, 82, 89].

(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [117].

Chapter 3

Discrete and Structural Applied Mathematics

In this chapter, we discuss fuzzy discrete and structural applied mathematics and its extensions. The following concise comparison of the uncertain concepts covered in Chapter 3 is presented in Table 3.1.

Table 3.1: A concise comparison of the uncertain concepts covered in Chapter 3.

Concept	**Base structure**	**Main focus**
Uncertain Graph	Graph	Pairwise uncertain relations, connectivity, and network structure.
Uncertain HyperGraph	Hypergraph	Uncertain multi-element relations represented by hyperedges.
Uncertain SuperHyperGraph	SuperHyperGraph	Uncertain higher-order and hierarchical relational structure.
Uncertain Directed Graph	Directed graph	Uncertain oriented connections, reachability, and flow.
Uncertain Matroid	Matroid	Uncertain independence, dependence, and combinatorial structure.
Uncertain Decision Tree	Decision tree	Uncertain branching rules, tests, and decision outcomes.
Uncertain Petri Net	Petri net	Uncertain transitions, concurrency, and process dynamics.
Uncertain Language	Formal language	Uncertain membership and representation of strings or words.
Uncertain Automata	Automaton	Uncertain state transitions and acceptance behavior.
Uncertain HyperStructure and SuperHyperStructure	Hyperstructure / SuperHyperStructure	Uncertain multi-valued operations and generalized higher-order algebraic structure.
Uncertain Fibonacci Numbers	Fibonacci-type sequence	Uncertain recursive numerical behavior and sequence structure.

3.1 Uncertain Graph

A fuzzy graph assigns each vertex and edge a membership degree in $[0, 1]$, representing uncertain connectivity and relationship strength as graded, rather than binary, links [22, 118, 119].

Definition 3.1.1 (Fuzzy graph)**.** [22] A *fuzzy graph* on a vertex set V is a pair $G = (\sigma, \mu)$ consisting of:

- A vertex membership function $\sigma : V \to [0, 1]$, where $\sigma(x)$ gives the degree to which $x \in V$ belongs to the graph.
- An edge membership function $\mu : V \times V \to [0, 1]$, which is a fuzzy relation on σ, satisfying

$$\mu(x, y) \;\leq\; \sigma(x) \wedge \sigma(y), \quad \forall\, x, y \in V,$$

where $\wedge$ denotes the minimum operator.

The associated *crisp graph* $G^* = (\sigma^*, \mu^*)$ is determined by

$$\sigma^* = \{\, x \in V \mid \sigma(x) > 0 \,\}, \qquad \mu^* = \{\, (x, y) \in V \times V \mid \mu(x, y) > 0 \,\}.$$

A *fuzzy subgraph* $H = (\sigma', \mu')$ of G is obtained by choosing a subset $X \subseteq V$ and defining

- a restricted vertex membership $\sigma' : X \to [0, 1]$,
- an edge membership $\mu' : X \times X \to [0, 1]$ such that

$$\mu'(x, y) \;\leq\; \sigma'(x) \wedge \sigma'(y), \quad \forall\, x, y \in X.$$

We now state the uncertain graph-theoretic notions.

Definition 3.1.2 (Uncertain graph)**.** Let $G = (V, E)$ be a finite, undirected, loopless graph, and let M be an uncertain model with degree-domain $\mathrm{Dom}(M)$. An *uncertain graph of type M* is a triple

$$\mathcal{G}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns an uncertainty degree in $\mathrm{Dom}(M)$ to each vertex $v \in V$ and each edge $e \in E$. Optionally, one may impose model-dependent consistency relations between vertex- and edge-degrees (e.g., bounding $\mu_M(e)$ in terms of $\mu_M(u)$ and $\mu_M(v)$ for $e = \{u, v\}$ in fuzzy or intuitionistic fuzzy settings), but such constraints are dictated by the chosen model M and are not fixed at the level of this general definition.

Example 3.1.3 (A concrete uncertain graph)**.** Let

$$G = (V, E)$$

be the finite undirected loopless graph with

$$V = \{v_1, v_2, v_3, v_4\}$$

and

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}\big\}.$$

Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) = [0, 1].$$

Define

$$\mathcal{G}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \to [0, 1]$$

is given by the following assignments:

$$\mu_M(v_1) = 0.90, \qquad \mu_M(v_2) = 0.70, \qquad \mu_M(v_3) = 0.80, \qquad \mu_M(v_4) = 0.60,$$

and

$$\mu_M(\{v_1, v_2\}) = 0.65, \qquad \mu_M(\{v_2, v_3\}) = 0.70, \qquad \mu_M(\{v_3, v_4\}) = 0.55, \qquad \mu_M(\{v_1, v_4\}) = 0.50.$$

Then $\mathcal{G}_M$ is an uncertain graph of type M, since each vertex and each edge of G is assigned an uncertainty degree in $[0, 1]$.

Moreover, if one wishes to impose the fuzzy-style consistency condition

$$\mu_M(\{u, v\}) \le \min\{\mu_M(u), \mu_M(v)\} \qquad (\{u, v\} \in E),$$

then this example also satisfies it, because

$$0.65 \le \min\{0.90, 0.70\} = 0.70,$$

$$0.70 \le \min\{0.70, 0.80\} = 0.70,$$

$$0.55 \le \min\{0.80, 0.60\} = 0.60,$$

and

$$0.50 \le \min\{0.90, 0.60\} = 0.60.$$

Hence, $\mathcal{G}_M$ provides a concrete example of an uncertain graph.

For convenience, Table 3.2 lists representative uncertainty-graph families, organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$.

Table 3.2: A catalogue of uncertainty-graph families (uncertain graphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-graph type(s) $\mathcal{G}_M = (V, E, \mu_M)$ with $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$
1	Fuzzy graph; N-graph; shadowed-graph variants
2	Intuitionistic fuzzy graph [120]; vague graph [121]; bipolar fuzzy graph [122]; intuitionistic evidence graph; variable fuzzy graph; paraconsistent fuzzy graph; bifuzzy graph [123, 124]
3	Neutrosophic graph [4][(a)]; hesitant fuzzy graph [125]; tripolar fuzzy graph; three-way fuzzy graph; picture fuzzy graph [126, 127]; spherical fuzzy graph [75]; inconsistent intuitionistic fuzzy graph; ternary fuzzy / neutrosophic-fuzzy graph; neutrosophic vague graph
4	Quadripartitioned neutrosophic graph [128, 129]; double-valued neutrosophic graph [83]; dual hesitant fuzzy graph [130]; ambiguous graph[(b)]; local-neutrosophic graph; support-neutrosophic graph; turiyam neutrosophic graph [131][(c)]
5	Pentapartitioned neutrosophic graph [132]; triple-valued neutrosophic graph [133]
6	Hexapartitioned neutrosophic graph; quadruple-valued neutrosophic graph [133]
7	Heptapartitioned neutrosophic graph [134]; quintuple-valued neutrosophic graph [133]
8	Octapartitioned neutrosophic graph
9	Nonapartitioned neutrosophic graph
n	n-refined fuzzy graph; multi-valued (fuzzy) graphs; multi-fuzzy graphs [135]
$2n$	n-refined intuitionistic fuzzy graph; multi-intuitionistic fuzzy graphs
$3n$	n-refined neutrosophic graph; multi-neutrosophic graphs

(a) Neutrosophic graph models are often treated as broad frameworks that can specialize to many degree-based graph formalisms under suitable constraints.
(b) Ambiguous-graph models are commonly presented as subclasses of certain quadripartitioned and also double-valued neutrosophic graph models.
(c) Turiyam neutrosophic graphs are reported as subclasses of certain quadripartitioned neutrosophic graph models.

For example, as related concepts of graphs, regular graphs [136, 137], planar graphs [138–140], complete graphs [141, 142], directed graphs [143, 144], bidirected graphs [145, 146], and multidirected graphs [147] are also known and have been studied extensively.

3.2 Uncertain HyperGraph

A HyperGraph is a pair of vertices and hyperedges where each hyperedge may join any number of vertices naturally generalizing ordinary graphs beyond pairwise relations [148–152]. A fuzzy HyperGraph assigns membership degrees to vertices and hyperedges representing uncertain participation and higher-order relations by values in the interval $[0, 1]$ [57, 153–156]. An uncertain HyperGraph is a HyperGraph equipped with uncertainty degrees on vertices and hyperedges, modeling ambiguous participation and higher-order relations under a specified uncertainty framework (cf. [56, 157]).

Definition 3.2.1 (Uncertain HyperGraph). [56] Let $H = (V, E)$ be a HyperGraph and let M be an uncertain model with degree–domain $\mathrm{Dom}(M)$. An *Uncertain HyperGraph of type M* is a triple

$$\mathcal{H}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns an uncertainty degree to each vertex $v \in V$ and each hyperedge $e \in E$.

As in the graph case, possible relations between vertex and hyperedge degrees (for instance, bounds of $\mu_M(e)$ in terms of $\mu_M(v)$ for $v \in e$) are governed by the chosen model M and its constraints.

For suitable choices of M, this framework yields fuzzy HyperGraphs, intuitionistic fuzzy HyperGraphs, neutrosophic HyperGraphs, plithogenic HyperGraphs, and many further extensions. We present the catalogue of uncertainty-HyperGraph families (Uncertain HyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ in Table 3.3.

Table 3.3: A catalogue of uncertainty-HyperGraph families (Uncertain HyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-HyperGraph family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy HyperGraph [158–160]:* $\mu_M : V \cup E \to [0,1]$.
2	*Intuitionistic-fuzzy HyperGraph [161–163]:* $\mu_M : V \cup E \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic HyperGraph [164–167]:* $\mu_M : V \cup E \to [0,1]^3$ (e.g., (T, I, F)).
4	*Quadripartitioned Neutrosophic / four-component uncertainty HyperGraph:* $\mu_M : V \cup E \to [0,1]^4$.
5	*Pentapartitioned Neutrosophic / five-component uncertainty HyperGraph:* $\mu_M : V \cup E \to [0,1]^5$.
k	*k-component uncertainty HyperGraph:* $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

3.3 Uncertain SuperHyperGraph

A superHyperGraph is a higher-order HyperGraph whose supervertices may themselves be set-based objects, while superedges connect collections of such supervertices hierarchically [56]. A fuzzy superHyperGraph assigns membership degrees to supervertices and superedges, representing uncertain hierarchical participation and higher-order connectivity within the interval $[0,1]$ [168, 169]. An uncertain superHyperGraph equips supervertices and superedges with model-dependent uncertainty degrees, capturing ambiguous hierarchical relations under generalized frameworks such as fuzzy or neutrosophic settings.

Definition 3.3.1 (Uncertain n-SuperHyperGraph)**.** [56] Let V_0 be a finite base set and let $n \in \mathbb{N}_0$. Assume that an n-SuperHyperGraph on V_0 is given by

$$\mathrm{SHG}^{(n)} = (V_n, E),$$

where

$$\emptyset \neq V_n \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad \emptyset \neq E \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\},$$

so that each n-superedge $e \in E$ is a nonempty subset of the n-supervertex set V_n.

Let M be a fixed uncertain model with degree–domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *Uncertain n-SuperHyperGraph of type M* is a triple

$$\mathcal{S}_M^{(n)} = (V_n, E, \mu_M),$$

where

$$\mu_M : V_n \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns to each n-supervertex $v \in V_n$ and each n-superedge $e \in E$ an uncertainty degree $\mu_M(v)$ or $\mu_M(e)$ in $\mathrm{Dom}(M)$.

Any additional relations between the degrees of n-superedges and the degrees of the n-supervertices they contain (for example, model- specific bounds or aggregations) are imposed by the chosen uncertain model M and are not fixed at the level of this general definition.

For $n = 0$ and $V_0 = V_n$, the above notion reduces to an Uncertain HyperGraph of type M.

Particular choices of the model M recover well–known uncertain SuperHyperGraph types:

- Fuzzy n-SuperHyperGraphs (when M is fuzzy);
- Intuitionistic fuzzy, neutrosophic, and plithogenic n-SuperHyperGraphs for the corresponding models M;
- More exotic variants (e.g. q-rung orthopair, picture fuzzy, refined neutrosophic) are obtained by choosing the appropriate degree–domain $\mathrm{Dom}(M)$.

Regarding the catalogue of uncertainty-superHyperGraph families (Uncertain n-SuperHyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, we list them in Table 3.4.

Table 3.4: A catalogue of uncertainty-superHyperGraph families (Uncertain n-SuperHyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-superHyperGraph family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy n-SuperHyperGraph [170]:* $\mu_M : V_n \cup E \to [0,1]$.
2	*Intuitionistic-fuzzy n-SuperHyperGraph [170]:* $\mu_M : V_n \cup E \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic n-SuperHyperGraph [171–173]:* $\mu_M : V_n \cup E \to [0,1]^3$ (e.g., (T, I, F)).
4	*Quadripartitioned / four-component uncertainty n-SuperHyperGraph:* $\mu_M : V_n \cup E \to [0,1]^4$.
k	*k-component uncertainty n-SuperHyperGraph (Plithogenic n-SuperHyperGraph [174, 175]):* $\mu_M : V_n \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics) .

3.4 Uncertain Directed Graph

A Fuzzy directed graph assigns membership degrees to vertices and directed edges, modeling uncertain asymmetric relationships where arc strengths depend on ordered pairs quantitatively [176, 177].

Definition 3.4.1 (Fuzzy Directed Graph / Fuzzy Digraph)**.** Let

$$G^* = (V, A)$$

be a finite directed graph, where V is a nonempty set of vertices and

$$A \subseteq V \times V$$

is the set of directed edges (arcs). A *fuzzy directed graph* (or *fuzzy digraph*) on G^* is a pair

$$G = (\sigma, \mu),$$

where

$$\sigma : V \to [0,1]$$

is a fuzzy set of vertices and

$$\mu : A \to [0,1]$$

is a fuzzy set of directed edges such that

$$\mu(x,y) \le \min\{\sigma(x), \sigma(y)\} \qquad \text{for all } (x,y) \in A.$$

Here $\sigma(x)$ denotes the membership degree of the vertex x, and $\mu(x,y)$ denotes the membership degree of the directed edge from x to y.

In general, $\mu(x,y)$ and $\mu(y,x)$ are independent whenever both (x,y) and (y,x) belong to A; hence μ need not be symmetric.

An uncertain directed graph assigns uncertainty degrees to vertices and directed edges under a general uncertainty model, while preserving arc-to-vertex compatibility through a score-based constraint.

Definition 3.4.2 (Uncertain Directed Graph / Uncertain Digraph)**.** Let

$$G^* = (V, A)$$

be a finite directed graph, where V is a nonempty set of vertices and

$$A \subseteq V \times V$$

is the set of directed edges (arcs).

Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Let

$$\sigma_M : V \to \mathrm{Dom}(M), \qquad \mu_M : A \to \mathrm{Dom}(M)$$

be uncertainty-degree functions, so that

$$\mathcal{U}_V = (V, \sigma_M) \qquad \text{and} \qquad \mathcal{U}_A = (A, \mu_M)$$

are uncertain sets of type M on V and A, respectively.

Assume that the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0,1];$$
- a binary arc-compatibility operator
$$\Gamma_A : [0,1]^2 \to [0,1].$$

Then the pair

$$G_M = (\sigma_M, \mu_M)$$

is called an *uncertain directed graph of type M* on G^* if, for every

$$(x, y) \in A,$$

one has

$$S_M\big(\mu_M(x,y)\big) \;\le\; \Gamma_A\big(S_M(\sigma_M(x)),\, S_M(\sigma_M(y))\big).$$

Here $\sigma_M(x)$ is the uncertainty degree of the vertex x, and $\mu_M(x, y)$ is the uncertainty degree of the directed edge from x to y.

In general, whenever both $(x, y) \in A$ and $(y, x) \in A$, the values

$$\mu_M(x,y) \qquad \text{and} \qquad \mu_M(y,x)$$

are independent; hence uncertain directed graphs are not required to be symmetric.

Remark 3.4.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_A(a,b) = \min\{a,b\},$$

then Definition 3.4.2 reduces to the usual fuzzy directed graph condition

$$\mu(x,y) \le \min\{\sigma(x), \sigma(y)\} \qquad \text{for all } (x,y) \in A.$$

Thus uncertain directed graphs recover fuzzy directed graphs as a special case.

Theorem 3.4.4 (Well-definedness of uncertain directed graphs)**.** *Let*

$$G^* = (V, A)$$

be a finite directed graph, let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$\sigma_M : V \to \mathrm{Dom}(M), \qquad \mu_M : A \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_A : [0,1]^2 \to [0,1]$$

be given.

Then the notion of uncertain directed graph in Definition 3.4.2 is mathematically well-defined.

Proof. First, since V and A are sets and

$$\sigma_M : V \to \mathrm{Dom}(M), \qquad \mu_M : A \to \mathrm{Dom}(M),$$

the pairs

$$\mathcal{U}_V = (V, \sigma_M) \qquad \text{and} \qquad \mathcal{U}_A = (A, \mu_M)$$

are well-defined uncertain sets of type M.

Next, fix any arc

$$(x,y) \in A.$$

Because σ_M and μ_M take values in $\mathrm{Dom}(M)$, one has

$$\sigma_M(x) \in \mathrm{Dom}(M), \qquad \sigma_M(y) \in \mathrm{Dom}(M), \qquad \mu_M(x,y) \in \mathrm{Dom}(M).$$

Applying the score map S_M, we obtain

$$S_M(\sigma_M(x)) \in [0,1], \qquad S_M(\sigma_M(y)) \in [0,1], \qquad S_M(\mu_M(x,y)) \in [0,1].$$

Hence

$$\big(S_M(\sigma_M(x)),\, S_M(\sigma_M(y))\big) \in [0,1]^2.$$

Since

$$\Gamma_A : [0,1]^2 \to [0,1],$$

it follows that

$$\Gamma_A\big(S_M(\sigma_M(x)),\, S_M(\sigma_M(y))\big) \in [0,1].$$

Therefore both sides of the compatibility condition

$$S_M\big(\mu_M(x,y)\big) \;\le\; \Gamma_A\big(S_M(\sigma_M(x)),\, S_M(\sigma_M(y))\big)$$

are well-defined real numbers in $[0,1]$. Consequently, the inequality is meaningful for every arc $(x,y) \in A$.

Hence all objects and conditions appearing in Definition 3.4.2 are mathematically meaningful, and the notion of uncertain directed graph is well-defined. □

For reference, a catalogue of representative uncertainty-directed-graph families is presented in Table 3.5.

Table 3.5: A catalogue of representative uncertainty-directed-graph families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-directed-graph family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy Directed Graph [176, 178, 179]:* $\mu_M : V \cup A \to [0,1]$.
2	*Intuitionistic Fuzzy Directed Graph [180, 181]:* $\mu_M : V \cup A \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic Directed Graph [91, 144, 182–184]:* $\mu_M : V \cup A \to [0,1]^3$ (e.g., (T, I, F)).
4	*Quadripartitioned / four-component uncertainty directed graph:* $\mu_M : V \cup A \to [0,1]^4$.
k	*k-component uncertainty directed graph [185]:* $\mu_M : V \cup A \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

3.5 Uncertain Matroid

Fuzzy matroids generalize independence by degrees: subsets have membership/weight measuring "independent-like" status, extending rank, closure, circuits, and greedy optimization to settings with ambiguous dependence information [186–188].

Definition 3.5.1 (Goetschel–Voxman fuzzy matroid)**.** Let E be a finite ground set and let

$$\mathcal{F}(E) := [0,1]^E = \{\mu : E \to [0,1]\}$$

be the family of all fuzzy subsets of E. For $\mu, \nu \in \mathcal{F}(E)$ write $\mu \le \nu$ if $\mu(e) \le \nu(e)$ for all $e \in E$, and $\mu < \nu$ if $\mu \le \nu$ and $\mu \ne \nu$. Define

$$\mathrm{supp}(\mu) := \{e \in E : \mu(e) > 0\}, \qquad m(\mu) := \min\{\mu(e) : e \in \mathrm{supp}(\mu)\} \quad (\text{when } \mathrm{supp}(\mu) \ne \varnothing),$$

and the pointwise join $(\mu \vee \nu)(e) := \max\{\mu(e), \nu(e)\}$.

A pair $M = (E, \Psi)$ is called a *(Goetschel–Voxman) fuzzy matroid* if $\Psi \subseteq \mathcal{F}(E)$ is nonempty and satisfies the following axioms:

(FI1) **(Hereditary property)** If $\mu \in \Psi$ and $\nu \in \mathcal{F}(E)$ with $\nu \le \mu$, then $\nu \in \Psi$.

(FI2) **(Exchange property)** If $\mu, \nu \in \Psi$ and

$$0 < |\operatorname{supp}(\mu)| < |\operatorname{supp}(\nu)|,$$

then there exists $\omega \in \Psi$ such that

$$\mu < \omega \leq \mu \vee \nu \quad \text{and} \quad m(\omega) \geq \min\{m(\mu), m(\nu)\}.$$

The members of Ψ are called the *independent fuzzy sets* of M.

The notion of an uncertain matroid should reduce the multidimensional uncertainty data in $\mathrm{Dom}(M) \subseteq [0,1]^k$ to a comparable scalar level when one speaks about support, hereditary reduction, and augmentation. For this purpose, we fix a score map.

Definition 3.5.2 (Scored uncertain subsets on a finite ground set)**.** Let E be a finite set, let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, and let

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

be a fixed *score map*. Assume that a distinguished *null degree*

$$0_M \in \mathrm{Dom}(M) \qquad \text{satisfies} \qquad S_M(0_M) = 0.$$

Write

$$\mathcal{U}_M(E) := \{\mu : E \to \mathrm{Dom}(M)\}$$

for the family of all uncertain sets of type M on E. For $\mu, \nu \in \mathcal{U}_M(E)$, define:

(i) the *score-order*

$$\mu \preceq_M \nu \quad \Longleftrightarrow \quad S_M\big(\mu(e)\big) \leq S_M\big(\nu(e)\big) \text{ for all } e \in E;$$

(ii) the *scored support*

$$\operatorname{supp}_M(\mu) := \{\, e \in E \mid S_M(\mu(e)) > 0 \,\};$$

(iii) for $e \in E$, the *one-element augmentation of μ by ν at e*

$$\mu \oplus_e \nu \in \mathcal{U}_M(E)$$

defined by

$$(\mu \oplus_e \nu)(f) := \begin{cases} \nu(e), & f = e, \\ \mu(f), & f \neq e. \end{cases}$$

Also define the *zero uncertain set*

$$0_E^M : E \to \mathrm{Dom}(M), \qquad 0_E^M(e) := 0_M \quad (e \in E).$$

Definition 3.5.3 (Uncertain Matroid)**.** Let E be a finite ground set, let M be an uncertain model, and let S_M and 0_M be as in Definition 3.5.2.

An *uncertain matroid of type M* is a pair

$$\mathcal{M}_M = (E, \mathcal{I}_M),$$

where

$$\mathcal{I}_M \subseteq \mathcal{U}_M(E)$$

is a family of uncertain sets, called the *independent uncertain sets*, satisfying:

(UM1) **(Nonempty / zero independent)**

$$0_E^M \in \mathcal{I}_M.$$

(UM2) **(Hereditary property)** If $\mu \in \mathcal{I}_M$ and $\nu \in \mathcal{U}_M(E)$ satisfy

$$\nu \preceq_M \mu,$$

then

$$\nu \in \mathcal{I}_M.$$

(UM3) **(Exchange property)** If $\mu, \nu \in \mathcal{I}_M$ and

$$|\operatorname{supp}_M(\mu)| < |\operatorname{supp}_M(\nu)|,$$

then there exists an element

$$e \in \operatorname{supp}_M(\nu) \setminus \operatorname{supp}_M(\mu)$$

such that

$$\mu \oplus_e \nu \in \mathcal{I}_M.$$

The elements of $\mathcal{I}_M$ are called *independent uncertain sets.*

Remark 3.5.4. Definition 3.5.3 depends on the chosen score map S_M. For the fuzzy model M with $\mathrm{Dom}(M) = [0,1]$, taking $S_M = \mathrm{id}_{[0,1]}$ yields the natural one-dimensional uncertain/fuzzy version of independence on graded subsets.

Theorem 3.5.5 (Well-definedness of uncertain matroids)**.** *Let E be a finite set, let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, let $S_M : \mathrm{Dom}(M) \to [0,1]$ be a score map, and let $0_M \in \mathrm{Dom}(M)$ satisfy $S_M(0_M) = 0$. Then the notion of uncertain matroid in Definition 3.5.3 is well-defined. More precisely:*

(a) *$\mathcal{U}_M(E)$ is well-defined, and each $\mu \in \mathcal{U}_M(E)$ is an uncertain set of type M on E.*

(b) *The relation $\preceq_M$ on $\mathcal{U}_M(E)$ is a preorder.*

(c) *For every $\mu \in \mathcal{U}_M(E)$, the set $\operatorname{supp}_M(\mu)$ is a well-defined finite subset of E.*

(d) *For every $\mu, \nu \in \mathcal{U}_M(E)$ and $e \in E$, the augmentation $\mu \oplus_e \nu$ is a well-defined element of $\mathcal{U}_M(E)$.*

(e) *If* $e \in \mathrm{supp}_M(\nu) \setminus \mathrm{supp}_M(\mu)$, *then*

$$\mathrm{supp}_M(\mu \oplus_e \nu) = \mathrm{supp}_M(\mu) \cup \{e\}.$$

Hence the exchange axiom (UM3) *is meaningful.*

Consequently, the axioms (UM1)–(UM3) *define a mathematically valid class of structures.*

Proof. **(a)** By definition,

$$\mathcal{U}_M(E) = \{\mu : E \to \mathrm{Dom}(M)\}.$$

Since E is a set and $\mathrm{Dom}(M) \neq \varnothing$ is a set of admissible degree-tuples, the collection of all such maps is well-defined. Each $\mu \in \mathcal{U}_M(E)$ is exactly an uncertain set of type M on E.

(b) We show that $\preceq_M$ is reflexive and transitive.

Reflexivity follows because for every $\mu \in \mathcal{U}_M(E)$ and every $e \in E$,

$$S_M(\mu(e)) \le S_M(\mu(e)).$$

Hence $\mu \preceq_M \mu$.

For transitivity, assume $\lambda \preceq_M \mu$ and $\mu \preceq_M \nu$. Then for every $e \in E$,

$$S_M(\lambda(e)) \le S_M(\mu(e)) \le S_M(\nu(e)),$$

so $\lambda \preceq_M \nu$. Therefore $\preceq_M$ is a preorder.

(c) For $\mu \in \mathcal{U}_M(E)$,

$$\mathrm{supp}_M(\mu) = \{e \in E : S_M(\mu(e)) > 0\}$$

is a subset of E, hence is well-defined. Since E is finite, $\mathrm{supp}_M(\mu)$ is finite as well.

(d) Fix $\mu, \nu \in \mathcal{U}_M(E)$ and $e \in E$. For each $f \in E$, the value $(\mu \oplus_e \nu)(f)$ is either $\mu(f)$ or $\nu(e)$. Both belong to $\mathrm{Dom}(M)$, because μ, ν are maps into $\mathrm{Dom}(M)$. Thus

$$\mu \oplus_e \nu : E \to \mathrm{Dom}(M),$$

so $\mu \oplus_e \nu \in \mathcal{U}_M(E)$.

(e) Assume $e \in \mathrm{supp}_M(\nu) \setminus \mathrm{supp}_M(\mu)$. Then

$$S_M(\nu(e)) > 0 \qquad \text{and} \qquad S_M(\mu(e)) = 0.$$

For $f \neq e$, one has $(\mu \oplus_e \nu)(f) = \mu(f)$, so

$$f \in \mathrm{supp}_M(\mu \oplus_e \nu) \iff f \in \mathrm{supp}_M(\mu).$$

At $f = e$,

$$(\mu \oplus_e \nu)(e) = \nu(e),$$

and since $S_M(\nu(e)) > 0$, we have $e \in \mathrm{supp}_M(\mu \oplus_e \nu)$. Therefore

$$\mathrm{supp}_M(\mu \oplus_e \nu) = \mathrm{supp}_M(\mu) \cup \{e\}.$$

This shows in particular that if

$$|\,\mathrm{supp}_M(\mu)| < |\,\mathrm{supp}_M(\nu)|,$$

then $\mathrm{supp}_M(\nu) \setminus \mathrm{supp}_M(\mu) \neq \varnothing$, so the element required in (UM3) is sought in a nonempty set, and the augmented uncertain set is a legitimate member of $\mathcal{U}_M(E)$.

Hence all objects and axioms appearing in Definition 3.5.3 are well-defined. □

A catalogue of fuzzy-matroid families classified by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ is presented in Table 3.6.

Table 3.6: A catalogue of fuzzy-matroid families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (conceptual overview).

k	Representative uncertainty-matroid type(s) on a ground set E (degree assignment to elements/independent sets, depending on the model)
1	Fuzzy matroids (e.g. Goetschel–Voxman style: independence family of fuzzy subsets $\Psi \subseteq [0,1]^E$) [189].
2	Intuitionistic fuzzy matroids (membership/nonmembership pair) [190, 191]; Pythagorean fuzzy matroids (orthopair pair under a Pythagorean constraint)[(a)] [192, 193].
m	m-polar fuzzy matroids (an m-tuple of membership degrees, i.e. $\mu : E \to [0,1]^m$)[(b)] [194].
3	Neutrosophic matroids (truth/indeterminacy/falsity degrees) [195, 196].

[(a)] Pythagorean fuzzy models are commonly formulated as an orthopair $(\mu, \nu) \in [0,1]^2$ satisfying $\mu^2 + \nu^2 \leq 1$; hence it is natural to catalogue them as $k = 2$ (two coordinates with an additional constraint).
[(b)] In m-polar settings, each element carries m membership coordinates (e.g. along m "poles"); cataloguing as $k = m$ records the ambient degree-domain dimension.

Apart from Uncertain Matroid, related concepts such as Antimatroid [197, 198], Polymatroid [199, 200], and Delta-matroid [201, 202] are also known.

3.6 Uncertain Decision Tree

A decision tree is a rule-based model splitting data by feature tests, forming branches to predict classes or values [203, 204]. A fuzzy decision tree replaces crisp splits with fuzzy membership functions, enabling soft rules, smoother boundaries, and robustness to noise [205–207].

Definition 3.6.1 (Fuzzy decision tree (attribute-value / fuzzy-partition form))**.** [205–207] Let $X \neq \varnothing$ be a finite set of training objects, and let

$$\mathcal{F}(X) := \{\, A : X \to [0,1] \,\}$$

denote the family of all fuzzy subsets of X. For $A \in \mathcal{F}(X)$, define its (sigma-count) fuzzy cardinality by

$$M(A) := \sum_{x \in X} A(x).$$

Fix $m \geq 1$ attributes. For each attribute $i \in \{1, \ldots, m\}$, let $X_i \subseteq \mathcal{F}(X)$ be the (finite) set of fuzzy attribute-values (linguistic values), and assume $|X_i| \geq 1$. Fix a t-norm T (e.g. $T = \min$) and define the fuzzy intersection

$$(A \wedge_T B)(x) := T\big(A(x), B(x)\big) \qquad (x \in X).$$

A *fuzzy decision tree* is a rooted directed tree $\mathcal{T}$ whose nodes are fuzzy subsets of X (i.e. elements of $\mathcal{F}(X)$) such that:

(a) Every node N of $\mathcal{T}$ satisfies $N \in \mathcal{F}(X)$.

(b) For every non-leaf node N, there exists an attribute index $i \in \{1, \ldots, m\}$ for which the set of children of N is exactly

$$\mathrm{ch}(N) = \{\, A \wedge_T N \;:\; A \in X_i \,\} \subseteq \mathcal{F}(X).$$

(c) Each leaf is assigned one or more class labels (classification decisions).

Example 3.6.2 (A concrete fuzzy decision tree)**.** Let

$$X = \{x_1, x_2, x_3\}$$

be a finite set of training objects, where

$$x_1 = \text{object } 1, \qquad x_2 = \text{object } 2, \qquad x_3 = \text{object } 3.$$

Take two attributes:

$$\text{Temperature} \quad \text{and} \quad \text{Humidity}.$$

Thus, $m = 2$.

For the attribute *Temperature*, define the set of fuzzy attribute-values

$$X_1 = \{A_{\text{Low}}, A_{\text{High}}\} \subseteq \mathcal{F}(X)$$

by

$$A_{\text{Low}}(x_1) = 0.8, \qquad A_{\text{Low}}(x_2) = 0.3, \qquad A_{\text{Low}}(x_3) = 0.1,$$
$$A_{\text{High}}(x_1) = 0.2, \qquad A_{\text{High}}(x_2) = 0.7, \qquad A_{\text{High}}(x_3) = 0.9.$$

For the attribute *Humidity*, define

$$X_2 = \{B_{\text{Dry}}, B_{\text{Wet}}\} \subseteq \mathcal{F}(X)$$

by

$$B_{\text{Dry}}(x_1) = 0.9, \qquad B_{\text{Dry}}(x_2) = 0.4, \qquad B_{\text{Dry}}(x_3) = 0.2,$$
$$B_{\text{Wet}}(x_1) = 0.1, \qquad B_{\text{Wet}}(x_2) = 0.6, \qquad B_{\text{Wet}}(x_3) = 0.8.$$

Let the t-norm be

$$T(a, b) = \min\{a, b\}.$$

Define the root node $N_0 \in \mathcal{F}(X)$ by

$$N_0(x) = 1 \qquad (\forall x \in X).$$

Hence,

$$M(N_0) = 1 + 1 + 1 = 3.$$

We first split the root by the attribute *Temperature*. By Definition 3.6.1, the children of N_0 are exactly

$$\mathrm{ch}(N_0) = \{A \wedge_T N_0 : A \in X_1\}.$$

Since $N_0(x) = 1$ for all $x \in X$, we obtain

$$N_1 := A_{\mathrm{Low}} \wedge_T N_0 = A_{\mathrm{Low}}, \qquad N_2 := A_{\mathrm{High}} \wedge_T N_0 = A_{\mathrm{High}}.$$

Therefore,

$$N_1 = \{(x_1, 0.8), (x_2, 0.3), (x_3, 0.1)\},$$
$$N_2 = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0.9)\}.$$

Their fuzzy cardinalities are

$$M(N_1) = 0.8 + 0.3 + 0.1 = 1.2, \qquad M(N_2) = 0.2 + 0.7 + 0.9 = 1.8.$$

Next, split the node N_2 by the attribute *Humidity*. Its children are

$$\mathrm{ch}(N_2) = \{B \wedge_T N_2 : B \in X_2\}.$$

Using the minimum t-norm, we obtain

$$N_{21} := B_{\mathrm{Dry}} \wedge_T N_2, \qquad N_{22} := B_{\mathrm{Wet}} \wedge_T N_2,$$

where

$$N_{21}(x_1) = \min\{0.9, 0.2\} = 0.2, \qquad N_{21}(x_2) = \min\{0.4, 0.7\} = 0.4, \qquad N_{21}(x_3) = \min\{0.2, 0.9\} = 0.2,$$

and

$$N_{22}(x_1) = \min\{0.1, 0.2\} = 0.1, \qquad N_{22}(x_2) = \min\{0.6, 0.7\} = 0.6, \qquad N_{22}(x_3) = \min\{0.8, 0.9\} = 0.8.$$

Thus,

$$N_{21} = \{(x_1, 0.2), (x_2, 0.4), (x_3, 0.2)\},$$
$$N_{22} = \{(x_1, 0.1), (x_2, 0.6), (x_3, 0.8)\},$$

with

$$M(N_{21}) = 0.2 + 0.4 + 0.2 = 0.8, \qquad M(N_{22}) = 0.1 + 0.6 + 0.8 = 1.5.$$

Now assign class labels to the leaves as follows:

$$N_1 \longmapsto \text{Class } C_1, \qquad N_{21} \longmapsto \text{Class } C_2, \qquad N_{22} \longmapsto \text{Class } C_3.$$

Hence, the resulting rooted directed tree is a fuzzy decision tree:

- the root is N_0;
- N_0 has children N_1 and N_2, obtained from the fuzzy attribute-values in X_1;
- N_2 has children N_{21} and N_{22}, obtained from the fuzzy attribute-values in X_2;
- the leaves N_1, N_{21}, and N_{22} are assigned classification decisions.

Its structure may be summarized as

$$N_0 \xrightarrow{\text{Temperature}} \{N_1, N_2\}, \qquad N_2 \xrightarrow{\text{Humidity}} \{N_{21}, N_{22}\}.$$

Therefore, this is a concrete example of a fuzzy decision tree in the sense of Definition 3.6.1.

Definition 3.6.3 (Soft/complete fuzzy decision tree (membership-propagation semantics)). Let U be a universe of objects (instances). A *soft (fuzzy) decision tree* for a (crisp or fuzzy) target class C consists of a rooted directed tree whose nodes v correspond to fuzzy subsets $S_v \subseteq U$ with membership $\mu_v : U \to [0,1]$, and:

- each internal node v selects an attribute a and a *discriminator* (soft split) $\sigma_v : U \to [0,1]$ (e.g. a piecewise-linear function governed by a cut-point α and width β), and for a binary split defines the children v_L, v_R by

$$\mu_{v_L}(o) := \mu_v(o)\,\sigma_v(o), \qquad \mu_{v_R}(o) := \mu_v(o)\,\bigl(1 - \sigma_v(o)\bigr) \qquad (o \in U).$$

- each leaf ℓ carries a numeric label $L_\ell \in \mathbb{R}$ (often $L_\ell \in [0,1]$ for class-membership prediction).

For an input object $o \in U$, the tree output (estimated class-membership) is the leaf-aggregation

$$\widehat{\mu}_C(o) := \frac{\displaystyle\sum_{\ell \in \text{Leaves}} \mu_\ell(o)\, L_\ell}{\displaystyle\sum_{\ell \in \text{Leaves}} \mu_\ell(o)},$$

with the usual convention that the denominator is positive (e.g. $\mu_{\text{root}}(o) = 1$).

Using an Uncertain Set, we define an Uncertain decision tree (U-tree) as follows.

Definition 3.6.4 (Uncertain decision tree (U-tree) as a measurable partition rule). Let $(\Omega, \mathcal{F})$ be a measurable space of instances and let Y be a label space (e.g. $Y = \{1, \dots, K\}$ for classification or $Y = \mathbb{R}$ for regression). Fix an uncertain model M with $\mathrm{Dom}(M) \neq \emptyset$ and an admissible score S_M.

Assume d uncertain-valued features

$$X_j : \Omega \to \mathrm{Dom}(M) \qquad (j = 1, \dots, d),$$

and let $\mathcal{S} = \{S_v\}_{v \in V}$ be the node set of a finite rooted directed tree $\mathcal{T}$ with root r. An *uncertain decision tree* is specified by:

- a finite rooted tree $\mathcal{T} = (V, E)$;
- for each internal node $v \in V$ a *split rule* determined by an index $j(v) \in \{1, \dots, d\}$ and a measurable set $B_v \subseteq \mathbb{R}$, producing two children v_L, v_R and the measurable tests

$$o \in S_{v_L} \iff S_M(X_{j(v)}(o)) \in B_v, \qquad o \in S_{v_R} \iff S_M(X_{j(v)}(o)) \notin B_v;$$

- for each leaf ℓ a prediction value $h(\ell) \in Y$.

The induced predictor $H : \Omega \to Y$ is

$$H(o) := h(\ell(o)),$$

where $\ell(o)$ is the unique leaf reached by starting at the root and applying the tests along the unique root-to-leaf path.

Definition 3.6.5 (Soft uncertain decision tree (membership-propagation form))**.** Let $(\Omega, \mathcal{F})$ be a measurable space. Fix an uncertain model M and an admissible score S_M. A *soft uncertain decision tree* is a finite rooted tree $\mathcal{T} = (V, E)$ with:

- a root membership $\mu_r : \Omega \to [0, 1]$ (typically $\mu_r \equiv 1$);
- for each internal node v, a feature index $j(v)$ and a measurable *soft split* function

$$\sigma_v : \Omega \to [0, 1], \qquad \sigma_v(o) = \Sigma_v\big(S_M(X_{j(v)}(o))\big)$$

  for some measurable $\Sigma_v : \mathbb{R} \to [0, 1]$ (e.g. a sigmoid or triangular membership around a cut-point); with children v_L, v_R defined by the membership propagation

$$\mu_{v_L}(o) := \mu_v(o)\, \sigma_v(o), \qquad \mu_{v_R}(o) := \mu_v(o)\, \big(1 - \sigma_v(o)\big);$$

- for each leaf ℓ, a numeric label $L_\ell \in \mathbb{R}$ (or a class-probability vector in Δ^{K-1}).

The tree output is the leaf aggregation

$$\widehat{H}(o) := \begin{cases} \dfrac{\sum_{\ell \in \text{Leaves}} \mu_\ell(o)\, L_\ell}{\sum_{\ell \in \text{Leaves}} \mu_\ell(o)}, & \sum_\ell \mu_\ell(o) > 0, \\ 0, & \sum_\ell \mu_\ell(o) = 0, \end{cases}$$

(with 0 replaced by any fixed default output when the denominator is 0).

Theorem 3.6.6 (Well-definedness of uncertain decision trees)**.** *In Definitions 3.6.4 and 3.6.5, assume:*

(i) S_M *is admissible (finite everywhere);*

(ii) all sets $B_v \subseteq \mathbb{R}$ *and all functions* $\Sigma_v : \mathbb{R} \to [0, 1]$ *are measurable;*

(iii) $\mathcal{T}$ *is finite.*

Then:

(a) *The hard uncertain decision tree predictor* $H : \Omega \to Y$ *is well-defined.*

(b) *In the soft uncertain decision tree, each node membership* μ_v *is well-defined and satisfies* $0 \leq \mu_v \leq 1$.

(c) *For every* $o \in \Omega$*, the soft output* $\widehat{H}(o)$ *is well-defined as a real number under the stated denominator convention.*

Proof. (a) Since $\mathcal{T}$ is a rooted tree, each instance o follows a unique path from the root to a leaf by repeatedly applying the binary test "$S_M(X_{j(v)}(o)) \in B_v$". Because $S_M(X_{j(v)}(o))$ is a finite real and B_v is a fixed set, the test outcome is defined at each internal node. Finiteness of $\mathcal{T}$ implies the path terminates at a leaf $\ell(o)$ in finitely many steps. Thus $H(o) = h(\ell(o))$ is defined for all o.

(b) By induction on depth. At the root, $\mu_r(o) \in [0,1]$. Suppose $\mu_v(o) \in [0,1]$ for some node v. Since $\sigma_v(o) = \Sigma_v(S_M(X_{j(v)}(o))) \in [0,1]$, we have $\mu_{v_L}(o) = \mu_v(o)\sigma_v(o) \in [0,1]$ and $\mu_{v_R}(o) = \mu_v(o)(1-\sigma_v(o)) \in [0,1]$. Thus all node memberships are well-defined and bounded in $[0,1]$.

(c) For each o, the set of leaves is finite and each $\mu_\ell(o) \in [0,1]$, so the sums $\sum_\ell \mu_\ell(o)$ and $\sum_\ell \mu_\ell(o)L_\ell$ are finite real numbers. If the denominator is positive, the ratio is well-defined; if it is 0, the definition assigns a fixed default value, so $\widehat{H}(o)$ is well-defined for all o. □

Related concepts of decision trees under uncertainty-aware models are listed in Table 3.7.

Table 3.7: Related concepts of decision trees under uncertainty-aware models.

k	**Related decision tree concept(s)**
2	Intuitionistic Fuzzy Decision Trees [208]
3	Hesitant Fuzzy Decision Trees [209, 210]
3	Neutrosophic Decision Trees [211, 212]

Besides Uncertain Decision Trees, several other related decision-tree concepts are also known, including Boosted Decision Trees [213, 214], Decision Hypertrees [215], Rough Decision Trees [216, 217], Directed Decision Trees [218, 219], Complete Decision Trees [220, 221], Phonetic Decision Trees [222, 223], and Multi-Criteria Decision Trees [224, 225].

3.7 Uncertain Petri Net

A Petri net is a directed bipartite graph of places and transitions, modeling concurrency, synchronization, resource flow, and reachability dynamically [226–228].

Definition 3.7.1 (Petri net)**.** [226–228] A *Petri net* is a tuple

$$\mathcal{N} = (P, T, F, W),$$

where

1. P is a finite set of *places*;
2. T is a finite set of *transitions*;
3. $P \cap T = \varnothing$;
4. $$F \subseteq (P \times T) \cup (T \times P)$$
   is the *flow relation* (or set of directed arcs);
5. $$W : F \to \mathbb{N}_{>0}$$
   is the *weight function*.

Thus (P, T, F) is a directed bipartite graph: arcs are allowed only from places to transitions or from transitions to places.

Definition 3.7.2 (Fuzzy Petri Net). [226, 229] A *fuzzy Petri net* is a tuple

$$\mathcal{N}_F = (P, T, F, \mu_0, \omega, \theta, \gamma),$$

where

1. P is a finite set of *places*;
2. T is a finite set of *transitions*;
3. $P \cap T = \varnothing$;
4. $$F \subseteq (P \times T) \cup (T \times P)$$
   is the *flow relation*;
5. $$\mu_0 : P \to [0, 1]$$
   is the *initial marking*, where $\mu_0(p)$ denotes the initial truth degree (or token value) of the place $p \in P$;
6. $$\omega : F \cap (P \times T) \to (0, 1]$$
   is the *input weight function*;
7. $$\theta : T \to (0, 1]$$
   is the *threshold function*;
8. $$\gamma : F \cap (T \times P) \to (0, 1]$$
   is the *certainty factor function* (or *output strength function*).

For each transition $t \in T$, define its input and output place sets by

$$^{\bullet}t := \{\, p \in P \mid (p,t) \in F \,\}, \qquad t^{\bullet} := \{\, p \in P \mid (t,p) \in F \,\}.$$

A *marking* of $\mathcal{N}_F$ is any map

$$\mu : P \to [0,1].$$

Given a marking μ, a transition $t \in T$ is said to be *enabled* at μ if

$$\min_{p \in {}^{\bullet}t} \big(\mu(p)\, \omega(p,t)\big) \geq \theta(t).$$

If t is enabled at μ, then the firing of t produces a new marking

$$\mu'_t : P \to [0,1]$$

defined by

$$\mu'_t(p) = \begin{cases} 0, & p \in {}^{\bullet}t \setminus t^{\bullet}, \\ \max\Big\{ \mu(p),\ \gamma(t,p) \min_{q \in {}^{\bullet}t} \big(\mu(q)\omega(q,t)\big) \Big\}, & p \in t^{\bullet}, \\ \mu(p), & p \notin {}^{\bullet}t \cup t^{\bullet}. \end{cases}$$

An uncertain Petri net assigns uncertainty degrees, under a general uncertainty model, to markings, input-arc weights, transition thresholds, and output certainties of a Petri-net structure.

Definition 3.7.3 (Uncertain Petri Net)**.** Let

$$\mathcal{N}^* = (P, T, F)$$

be a finite Petri-net structure, where

1. P is a finite set of places;
2. T is a finite set of transitions;
3. $P \cap T = \varnothing$;
4. $$F \subseteq (P \times T) \cup (T \times P)$$
   is the flow relation.

Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Assume that the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0,1];$$
- a compatible lift map
$$L_M : [0,1] \to \mathrm{Dom}(M)$$
satisfying
$$S_M\big(L_M(r)\big) = r \qquad \text{for all } r \in [0,1].$$

An *uncertain Petri net of type M* is a tuple

$$\mathcal{N}_M = (P, T, F, \mu_{0,M}, \omega_M, \theta_M, \gamma_M; S_M, L_M),$$

where

1. $$\mu_{0,M} : P \to \mathrm{Dom}(M)$$
   is the *initial uncertain marking*;
2. $$\omega_M : F \cap (P \times T) \to \mathrm{Dom}(M)$$
   is the *uncertain input-weight function*;
3. $$\theta_M : T \to \mathrm{Dom}(M)$$
   is the *uncertain threshold function*;
4. $$\gamma_M : F \cap (T \times P) \to \mathrm{Dom}(M)$$
   is the *uncertain output-certainty function.*

For each transition $t \in T$, define

$$^{\bullet}t := \{\, p \in P \mid (p,t) \in F \,\}, \qquad t^{\bullet} := \{\, p \in P \mid (t,p) \in F \,\}.$$

A *marking* of $\mathcal{N}_M$ is any map
$$\mu_M : P \to \mathrm{Dom}(M).$$
Its associated score-realization is the map

$$\overline{\mu} : P \to [0,1], \qquad \overline{\mu}(p) := S_M\big(\mu_M(p)\big).$$

Similarly, define

$$\overline{\omega}(p,t) := S_M\big(\omega_M(p,t)\big), \qquad \overline{\theta}(t) := S_M\big(\theta_M(t)\big), \qquad \overline{\gamma}(t,p) := S_M\big(\gamma_M(t,p)\big).$$

For a marking μ_M and a transition $t \in T$, define the *enabling strength* of t at μ_M by

$$E_{\mu_M}(t) := \min_{p \in {}^\bullet t}\big(\overline{\mu}(p)\,\overline{\omega}(p,t)\big),$$

with the convention

$$\min \varnothing = 1.$$

Then t is said to be *enabled* at μ_M if

$$E_{\mu_M}(t) \geq \overline{\theta}(t).$$

If t is enabled at μ_M, then the *score-updated marking*

$$\overline{\mu}'_t : P \to [0,1]$$

is defined by

$$\overline{\mu}'_t(p) = \begin{cases} 0, & p \in {}^\bullet t \setminus t^\bullet, \\ \max\Big\{\overline{\mu}(p),\, \overline{\gamma}(t,p)\, E_{\mu_M}(t)\Big\}, & p \in t^\bullet, \\ \overline{\mu}(p), & p \notin {}^\bullet t \cup t^\bullet. \end{cases}$$

The corresponding *updated uncertain marking*

$$\mu'_{M,t} : P \to \mathrm{Dom}(M)$$

is then given by

$$\mu'_{M,t}(p) := L_M\big(\overline{\mu}'_t(p)\big) \qquad \text{for all } p \in P.$$

Remark 3.7.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1], \qquad S_M = \mathrm{id}_{[0,1]}, \qquad L_M = \mathrm{id}_{[0,1]},$$

then Definition 3.7.3 reduces to the usual fuzzy Petri net:

$$E_\mu(t) = \min_{p \in {}^\bullet t}\big(\mu(p)\omega(p,t)\big),$$

t is enabled exactly when

$$E_\mu(t) \geq \theta(t),$$

and the firing rule becomes

$$\mu'_t(p) = \begin{cases} 0, & p \in {}^\bullet t \setminus t^\bullet, \\ \max\Big\{\mu(p),\, \gamma(t,p)E_\mu(t)\Big\}, & p \in t^\bullet, \\ \mu(p), & p \notin {}^\bullet t \cup t^\bullet. \end{cases}$$

Thus uncertain Petri nets recover fuzzy Petri nets as a special case.

Theorem 3.7.5 (Well-definedness of uncertain Petri nets)**.** *Let*

$$\mathcal{N}_M = (P, T, F, \mu_{0,M}, \omega_M, \theta_M, \gamma_M; S_M, L_M)$$

be as in Definition 3.7.3. Then:

1. *the enabling strength*
$$E_{\mu_M}(t)$$
is well-defined for every marking $\mu_M : P \to \mathrm{Dom}(M)$ *and every* $t \in T$*;*

2. *the enabledness condition*
$$E_{\mu_M}(t) \geq \overline{\theta}(t)$$
is meaningful;

3. *whenever* t *is enabled at* μ_M*, the score-updated marking*
$$\overline{\mu}'_t : P \to [0,1]$$
is well-defined;

4. *the updated map*
$$\mu'_{M,t} : P \to \mathrm{Dom}(M)$$
is a well-defined uncertain marking on P*.*

Hence the notion of uncertain Petri net, together with its firing rule, is mathematically well-defined.

Proof. Let $\mu_M : P \to \mathrm{Dom}(M)$ be any marking and let $t \in T$.

Since
$$\mu_M(p) \in \mathrm{Dom}(M) \qquad \text{for all } p \in P,$$
and
$$\omega_M(p,t) \in \mathrm{Dom}(M) \qquad \text{for all } (p,t) \in F \cap (P \times T),$$
the score map $S_M : \mathrm{Dom}(M) \to [0,1]$ yields
$$\overline{\mu}(p) = S_M(\mu_M(p)) \in [0,1] \qquad \text{and} \qquad \overline{\omega}(p,t) = S_M(\omega_M(p,t)) \in [0,1].$$
Therefore, for each $p \in {}^\bullet t$,
$$\overline{\mu}(p)\,\overline{\omega}(p,t) \in [0,1].$$
Hence
$$\min_{p \in {}^\bullet t}\big(\overline{\mu}(p)\,\overline{\omega}(p,t)\big)$$
is well-defined when ${}^\bullet t \neq \varnothing$, and if ${}^\bullet t = \varnothing$, then by the convention $\min \varnothing = 1$, the value $E_{\mu_M}(t)$ is also well-defined. This proves (1).

Next, since
$$\theta_M(t) \in \mathrm{Dom}(M),$$
one has
$$\overline{\theta}(t) = S_M(\theta_M(t)) \in [0,1].$$
Thus both sides of
$$E_{\mu_M}(t) \geq \overline{\theta}(t)$$
are real numbers in $[0,1]$, so the enabledness condition is meaningful. This proves (2).

Assume now that t is enabled at μ_M. For each $p \in P$, consider the three cases in the definition of $\overline{\mu}'_t(p)$.

If

$$p \in {}^\bullet t \setminus t^\bullet,$$

then

$$\overline{\mu}'_t(p) = 0 \in [0, 1].$$

If

$$p \in t^\bullet,$$

then

$$\overline{\mu}(p) \in [0, 1], \qquad \overline{\gamma}(t, p) = S_M(\gamma_M(t, p)) \in [0, 1], \qquad E_{\mu_M}(t) \in [0, 1].$$

Hence

$$\overline{\gamma}(t, p)\, E_{\mu_M}(t) \in [0, 1],$$

and therefore

$$\max\Big\{\overline{\mu}(p),\, \overline{\gamma}(t, p)\, E_{\mu_M}(t)\Big\} \in [0, 1].$$

If

$$p \notin {}^\bullet t \cup t^\bullet,$$

then

$$\overline{\mu}'_t(p) = \overline{\mu}(p) \in [0, 1].$$

Thus in every case,

$$\overline{\mu}'_t(p) \in [0, 1].$$

Hence

$$\overline{\mu}'_t : P \to [0, 1]$$

is well-defined. This proves (3).

Finally, because

$$L_M : [0, 1] \to \mathrm{Dom}(M),$$

the assignment

$$\mu'_{M,t}(p) := L_M\big(\overline{\mu}'_t(p)\big)$$

belongs to $\mathrm{Dom}(M)$ for every $p \in P$. Therefore

$$\mu'_{M,t} : P \to \mathrm{Dom}(M)$$

is a well-defined map, that is, a well-defined uncertain marking on P. This proves (4).

Consequently, every object and every condition appearing in Definition 3.7.3 is mathematically meaningful, and the uncertain firing rule indeed produces a valid uncertain marking. Therefore the notion of uncertain Petri net is well-defined. $\square$

Related Petri-net concepts under fuzzy and uncertainty-aware frameworks are listed in Table 3.8.

Table 3.8: Related Petri-net concepts under fuzzy and uncertainty-aware frameworks

k	**Concept**	**Reference(s)**
1	Fuzzy Petri Nets	[230–232]
2	Intuitionistic Fuzzy Petri Nets	[226, 233, 234]
3	Picture Fuzzy Petri Nets	[235–237]
3	Spherical Fuzzy Petri Nets	[238, 239]
3	Neutrosophic Petri Nets	[240, 241]

3.8 Uncertain Language

Fuzzy language is a formal language assigning each word a membership degree in $[0, 1]$, generalizing crisp languages by allowing partial acceptance and graded linguistic uncertainty [242–244].

Definition 3.8.1 (Fuzzy Language)**.** [245, 246] Let Σ be an alphabet and let Σ^* be the set of all finite words over Σ. A *fuzzy language* over Σ is a function

$$L : \Sigma^* \to [0, 1].$$

That is, a fuzzy language is a fuzzy subset of Σ^*.

An *uncertain language* extends the notion of a fuzzy language by assigning to each word an admissible uncertainty degree taken from a prescribed uncertain model. Thus it may encode fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, or other uncertainty patterns on formal languages.

Definition 3.8.2 (Uncertain Language)**.** Let Σ be an alphabet, and let

$$\Sigma^* := \bigcup_{n \in \mathbb{N}_0} \Sigma^n$$

denote the set of all finite words over Σ, where $\Sigma^0 := \{\varepsilon\}$ and ε is the empty word.

Fix an uncertain model M with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

An *uncertain language of type M over* Σ is an uncertain set of type M on Σ^*; that is, it is a pair

$$\mathcal{L}_M = (\Sigma^*, \lambda_M),$$

where

$$\lambda_M : \Sigma^* \longrightarrow \mathrm{Dom}(M)$$

is a map assigning to each word $w \in \Sigma^*$ an admissible uncertainty degree $\lambda_M(w) \in \mathrm{Dom}(M)$.

The value $\lambda_M(w)$ is called the *uncertainty degree* (or *acceptance degree*) of the word w in $\mathcal{L}_M$.

Remark 3.8.3. Definition 3.8.2 unifies several familiar language concepts:

- If M is the fuzzy model, so that $\mathrm{Dom}(M) = [0, 1]$, then
$$\lambda_M : \Sigma^* \to [0, 1],$$
and $\mathcal{L}_M$ is exactly a *fuzzy language*.
- If M is an intuitionistic fuzzy model, then each word is assigned a pair $(\mu(w), \nu(w))$ with $\mu(w)+\nu(w) \leq 1$.
- If M is a neutrosophic model, then each word is assigned a triple $(T(w), I(w), F(w))$.
- Other uncertainty-aware language formalisms are obtained similarly by changing the model M.

Theorem 3.8.4 (Well-definedness of uncertain languages)**.** *Let Σ be an alphabet, and let M be an uncertain model with nonempty degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$. Then the notion of uncertain language in Definition 3.8.2 is well-defined.*

More precisely:

(a) *The set Σ^* of all finite words over Σ is well-defined.*

(b) *The collection*
$$\mathrm{ULang}_M(\Sigma) := \{\lambda : \Sigma^* \to \mathrm{Dom}(M)\}$$
of all M-valued word maps is well-defined.

(c) *For every $\lambda \in \mathrm{ULang}_M(\Sigma)$, the pair*
$$(\Sigma^*, \lambda)$$
is an uncertain set of type M on Σ^*, hence an uncertain language of type M.

(d) *In the special case $\mathrm{Dom}(M) = [0, 1]$, Definition 3.8.2 reduces exactly to the classical notion of a fuzzy language.*

Proof. **(a)** For each $n \in \mathbb{N}_0$, the Cartesian power Σ^n is a well-defined set. Hence
$$\Sigma^* = \bigcup_{n\in\mathbb{N}_0} \Sigma^n$$
is a well-defined set, namely the free monoid generated by Σ. Its elements are precisely the finite words over Σ.

(b) Since Σ^* is a set by part (a) and $\mathrm{Dom}(M)$ is a nonempty set by the definition of uncertain model, the family of all functions from Σ^* to $\mathrm{Dom}(M)$ is well-defined. Therefore
$$\mathrm{ULang}_M(\Sigma) = \{\lambda : \Sigma^* \to \mathrm{Dom}(M)\}$$
is well-defined.

(c) Let $\lambda \in \mathrm{ULang}_M(\Sigma)$. Then, by definition, λ is a map

$$\lambda : \Sigma^* \to \mathrm{Dom}(M).$$

Hence the pair

$$(\Sigma^*, \lambda)$$

is exactly an uncertain set of type M on the universe Σ^*. Therefore it is an uncertain language of type M.

(d) If $\mathrm{Dom}(M) = [0,1]$, then an uncertain language of type M is a map

$$\lambda : \Sigma^* \to [0,1],$$

which is precisely the standard definition of a fuzzy language. Thus Definition 3.8.2 recovers fuzzy languages as a special case.

Therefore all objects appearing in Definition 3.8.2 are mathematically meaningful, and the concept of uncertain language is well-defined. □

A catalogue of representative uncertain-language families is presented in Table 3.9.

Table 3.9: A catalogue of representative uncertain-language families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertain-language family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy Language:* $\mu_M : \Sigma^* \to [0,1]$.
2	*Intuitionistic Fuzzy Language [247]:* $\mu_M : \Sigma^* \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic Language (cf. [248, 249]):* $\mu_M : \Sigma^* \to [0,1]^3$ (e.g., (T, I, F)).

In addition, related concepts include natural language [250, 251] and large language models [252, 253], both of which are well known and play major roles in fields such as artificial intelligence.

3.9 Uncertain Automata

Automata are abstract machines with states and transitions processing input symbols, accepting or rejecting strings, modeling computation [254, 255]. Fuzzy automata assign degrees to transitions and acceptance, so each input string is recognized with a graded value [242, 256].

Definition 3.9.1 (Fuzzy automaton over a complete residuated lattice)**.** [242, 256] Let

$$L = (L, \wedge, \vee, \otimes, \to, 0, 1)$$

be a *complete residuated lattice* (so $\vee, \wedge$ are complete joins/meets and $\otimes$ is a commutative monoid operation adjoint to $\to$). Let X be a finite alphabet.

A (nondeterministic) *L-fuzzy automaton* over X is a quadruple

$$\mathcal{A} = (A, \delta, \sigma, \tau),$$

where A is a nonempty finite set (states),

$$\delta : A \times X \times A \to L$$

is the fuzzy transition function, and

$$\sigma, \tau : A \to L$$

are the fuzzy sets of initial and terminal states, respectively (often written as $\sigma(a)$ and $\tau(a)$).

Define the extended transition degree $\delta^* : A \times X^* \times A \to L$ by

$$\delta^*(a, \varepsilon, b) = \begin{cases} 1, & a = b, \\ 0, & a \neq b, \end{cases} \qquad \delta^*(a, ux, b) = \bigvee_{c \in A} \Big(\delta^*(a, u, c) \otimes \delta(c, x, b) \Big),$$

equivalently, for $u = x_1 x_2 \cdots x_n \in X^+$,

$$\delta^*(a, u, b) = \bigvee_{(c_1, \ldots, c_{n-1}) \in A^{n-1}} \delta(a, x_1, c_1) \otimes \delta(c_1, x_2, c_2) \otimes \cdots \otimes \delta(c_{n-1}, x_n, b).$$

The *fuzzy language recognized by* $\mathcal{A}$ is the map $L(\mathcal{A}) : X^* \to L$ defined by

$$L(\mathcal{A})(u) = \bigvee_{a, b \in A} \sigma(a) \otimes \delta^*(a, u, b) \otimes \tau(b), \qquad u \in X^*,$$

so $L(\mathcal{A})(u)$ is the acceptance degree of the word u.

Since a general uncertainty model M has degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$, there is, in general, no canonical notion of multiplication, supremum, or order on $\mathrm{Dom}(M)$. Therefore, in order to define an automaton of type M, we explicitly fix the operators used for sequential composition and nondeterministic aggregation.

Definition 3.9.2 (Uncertain Automaton)**.** Let Σ be a finite alphabet, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Assume that the following data are fixed:

- distinguished degrees
$$0_M, 1_M \in \mathrm{Dom}(M);$$
- a binary *sequential-composition operator*
$$\odot_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M);$$
- for each integer $n \geq 1$, a symmetric n-ary *choice-aggregation operator*
$$\boxplus_M^{(n)} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M).$$

An *uncertain automaton of type M* over Σ is a quadruple

$$\mathcal{A}_M = (A, \delta, \sigma, \tau),$$

where

1. A is a nonempty finite set of states;
2. $$\delta : A \times \Sigma \times A \to \mathrm{Dom}(M)$$
   is the *uncertain transition function*;
3. $$\sigma, \tau : A \to \mathrm{Dom}(M)$$
   are the uncertain sets of initial and terminal states, respectively.

Let $A = \{a_1, \ldots, a_n\}$. The *extended uncertain transition function*

$$\delta_M^* : A \times \Sigma^* \times A \to \mathrm{Dom}(M)$$

is defined recursively by

$$\delta_M^*(a, \varepsilon, b) = \begin{cases} 1_M, & a = b, \\ 0_M, & a \neq b, \end{cases}$$

and, for $u \in \Sigma^*$, $x \in \Sigma$, and $a, b \in A$,

$$\delta_M^*(a, ux, b) = \boxplus_M^{(n)} \Big(\delta_M^*(a, u, a_1) \odot_M \delta(a_1, x, b), \ \ldots, \ \delta_M^*(a, u, a_n) \odot_M \delta(a_n, x, b) \Big).$$

The *uncertain language recognized by* $\mathcal{A}_M$ is the mapping

$$L_M(\mathcal{A}_M) : \Sigma^* \to \mathrm{Dom}(M)$$

defined, for $u \in \Sigma^*$, by

$$L_M(\mathcal{A}_M)(u) = \boxplus_M^{(n^2)} \Big((\sigma(a_i) \odot_M \delta_M^*(a_i, u, a_j)) \odot_M \tau(a_j) \Big)_{1 \leq i, j \leq n}.$$

Thus $L_M(\mathcal{A}_M)$ is an uncertain set of type M on Σ^*.

Remark 3.9.3. If M is the fuzzy model, i.e. $\mathrm{Dom}(M) = [0, 1]$, and one takes

$$0_M = 0, \qquad 1_M = 1,$$

together with a suitable fuzzy composition operator $\odot_M$ and a join-type aggregation $\boxplus_M$, then Definition 3.9.2 reduces to the usual fuzzy-automaton scheme.

Theorem 3.9.4 (Well-definedness of uncertain automata)**.** *Let*

$$\mathcal{A}_M = (A, \delta, \sigma, \tau)$$

be an uncertain automaton of type M as in Definition 3.9.2. Then:

(a) *the extended transition function*

$$\delta_M^* : A \times \Sigma^* \times A \to \mathrm{Dom}(M)$$

is well-defined;

(b) *the recognized language*

$$L_M(\mathcal{A}_M) : \Sigma^* \to \mathrm{Dom}(M)$$

is well-defined;

(c) *consequently, $L_M(\mathcal{A}_M)$ is an uncertain set of type M on Σ^*.*

Proof. Let $A = \{a_1, \ldots, a_n\}$, where $n = |A| \geq 1$.

(a) Well-definedness of δ_M^*. We prove by induction on the length of the word $u \in \Sigma^*$ that

$$\delta_M^*(a, u, b) \in \mathrm{Dom}(M) \qquad \text{for all } a, b \in A.$$

Base case: $u = \varepsilon$. By definition,

$$\delta_M^*(a, \varepsilon, b) = \begin{cases} 1_M, & a = b, \\ 0_M, & a \neq b. \end{cases}$$

Since $0_M, 1_M \in \mathrm{Dom}(M)$, it follows that

$$\delta_M^*(a, \varepsilon, b) \in \mathrm{Dom}(M) \qquad \text{for all } a, b \in A.$$

Inductive step: Assume that for some $u \in \Sigma^*$,

$$\delta_M^*(a, u, c) \in \mathrm{Dom}(M) \qquad \text{for all } a, c \in A.$$

Let $x \in \Sigma$ and $b \in A$. For each $r \in \{1, \ldots, n\}$,

$$\delta_M^*(a, u, a_r) \in \mathrm{Dom}(M) \qquad \text{and} \qquad \delta(a_r, x, b) \in \mathrm{Dom}(M),$$

hence, by closure of $\odot_M$,

$$\delta_M^*(a, u, a_r) \odot_M \delta(a_r, x, b) \in \mathrm{Dom}(M).$$

Therefore the n-tuple

$$\Big(\delta_M^*(a, u, a_1) \odot_M \delta(a_1, x, b), \ \ldots, \ \delta_M^*(a, u, a_n) \odot_M \delta(a_n, x, b)\Big)$$

belongs to $\mathrm{Dom}(M)^n$. Since

$$\boxplus_M^{(n)} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M),$$

we obtain

$$\delta_M^*(a, ux, b) \in \mathrm{Dom}(M).$$

Hence δ_M^* is well-defined on $A \times \Sigma^* \times A$.

Moreover, because each $\boxplus_M^{(n)}$ is assumed symmetric, the value of

$$\delta_M^*(a, ux, b)$$

does not depend on the particular enumeration $A = \{a_1, \ldots, a_n\}$. Thus the recursive definition is unambiguous.

(b) Well-definedness of $L_M(\mathcal{A}_M)$. Fix $u \in \Sigma^*$. By part (a),

$$\delta_M^*(a_i, u, a_j) \in \mathrm{Dom}(M) \qquad \text{for all } 1 \le i, j \le n.$$

Also,

$$\sigma(a_i), \tau(a_j) \in \mathrm{Dom}(M).$$

Hence, by closure of $\odot_M$,

$$(\sigma(a_i) \odot_M \delta_M^*(a_i, u, a_j)) \odot_M \tau(a_j) \in \mathrm{Dom}(M) \qquad \text{for all } 1 \le i, j \le n.$$

Thus the n^2-tuple

$$\Big((\sigma(a_i) \odot_M \delta_M^*(a_i, u, a_j)) \odot_M \tau(a_j)\Big)_{1 \le i,j \le n}$$

belongs to $\mathrm{Dom}(M)^{n^2}$. Since

$$\boxplus_M^{(n^2)} : \mathrm{Dom}(M)^{n^2} \to \mathrm{Dom}(M),$$

it follows that

$$L_M(\mathcal{A}_M)(u) \in \mathrm{Dom}(M).$$

Therefore

$$L_M(\mathcal{A}_M) : \Sigma^* \to \mathrm{Dom}(M)$$

is a well-defined mapping.

Again, symmetry of $\boxplus_M^{(n^2)}$ guarantees that this value is independent of the enumeration of A.

(c) Uncertain-set interpretation. By definition of an uncertain set of type M, any mapping

$$\mu : \Sigma^* \to \mathrm{Dom}(M)$$

is an uncertain set of type M on Σ^*. Since part (b) shows that

$$L_M(\mathcal{A}_M) : \Sigma^* \to \mathrm{Dom}(M),$$

the recognized language is an uncertain set of type M on Σ^*. □

A catalogue of uncertainty-automata families classified by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ is presented in Table 3.10.

Table 3.10: A catalogue of uncertainty-automata families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (conceptual overview).

k	Representative uncertainty-automaton type(s) $\mathcal{A}_M = (Q, \Sigma, \delta_M, I_M, F_M)$ with degree data in $\mathrm{Dom}(M) \subseteq [0,1]^k$
1	Fuzzy automata (transition/initial/final degrees in $[0,1]$).
2	Intuitionistic fuzzy automata [257, 258]; bipolar fuzzy automata [259, 260].
3	Hesitant fuzzy automata [261][(a)]; neutrosophic automata [262–264].

[(a)] Hesitant fuzzy automata are naturally set-valued (a transition may carry a finite set of grades in $[0,1]$). Placing them at $k = 3$ corresponds to restricting to hesitant sets of size 3 or encoding each hesitant set by three canonical summary grades (e.g. min/mean/max) for catalogue purposes.

Besides Uncertain Automata, many other extensions of automata are also known. For example, these include cellular automata [265, 266], quantum automata [267], deterministic automata [268, 269], probabilistic automata [270, 271], and pushdown automata [272–274].

3.10 Uncertain HyperStructure and SuperHyperStructure

Hyperstructures map m-tuples of elements to nonempty subsets [275–278]; SuperHyperStructures lift operations to iterated powersets, enabling higher-order composition [279–281]. Fuzzy hyperstructures attach membership degrees to elements or results [277, 282, 283]; fuzzy SuperHyperStructures assign grades to iterated-powerset inputs/outputs, enforcing graded closure axioms.

Definition 3.10.1 (Hyperstructure (m-ary hyperoperation))**.** [278,284] Let $H \neq \varnothing$ be a set and let $m \geq 1$ be an integer. Write $\mathcal{P}^*(H) := \mathcal{P}(H) \setminus \{\varnothing\}$. A (single-operation) *hyperstructure of arity* m is a pair

$$(H, \circ),$$

where the *hyperoperation* is a mapping

$$\circ : H^m \longrightarrow \mathcal{P}^*(H).$$

(With several hyperoperations, one considers a family $\{\circ_i : H^{m_i} \to \mathcal{P}^*(H)\}_{i \in I}$.)

Definition 3.10.2 (Fuzzy subhypergroup (Davvaz-type) / fuzzy hyperstructure)**.** [277,282,283] Let $(H, \cdot)$ be a (binary) hypergroup (or Hv-group) and let $\mu : H \to [0, 1]$ be a fuzzy subset of H. Then μ is called a *fuzzy subhypergroup* of $(H, \cdot)$ if the following axioms hold:

(i) For all $x, y \in H$,

$$\min\{\mu(x), \mu(y)\} \ \leq \ \inf_{\alpha \in x \cdot y} \mu(\alpha).$$

(ii) For all $x, a \in H$ there exists $y \in H$ such that $x \in a \cdot y$ and

$$\min\{\mu(a), \mu(x)\} \ \leq \ \mu(y).$$

(iii) For all $x, a \in H$ there exists $z \in H$ such that $x \in z \cdot a$ and

$$\min\{\mu(a), \mu(x)\} \ \leq \ \mu(z).$$

In this situation, the triple $(H, \cdot, \mu)$ may be regarded as a *fuzzy hyperstructure* (i.e., a hyperstructure equipped with a membership profile compatible with the hyperoperation).

Definition 3.10.3 (Iterated nonempty powersets)**.** Let $H \neq \varnothing$ be a set and define recursively

$$\mathcal{P}^{*0}(H) := H, \qquad \mathcal{P}^{*(k+1)}(H) := \mathcal{P}^*\big(\mathcal{P}^{*k}(H)\big) \quad (k \geq 0),$$

where $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\varnothing\}$ for any set X.

Definition 3.10.4 (SuperHyperStructure)**.** [280, 281] Fix integers $r \geq 1$, $n \geq 1$, and $m \geq 1$. A *SuperHyperStructure of type* $(r, n; m)$ over H is a pair

$$\mathbb{S} = \big(\mathcal{P}^{*n}(H), \star\big),$$

where the *SuperHyperOperator* is a mapping

$$\star : \big(\mathcal{P}^{*r}(H)\big)^m \longrightarrow \mathcal{P}^{*n}(H),$$

together with chosen *SuperHyperAxioms* imposed on $\star$ (depending on the intended category: SuperHyperGroupoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, etc.).

Definition 3.10.5 (Fuzzy SuperHyperStructure (min-convention))**.** Let $\mathbb{S} = \big(\mathcal{P}^{*n}(H), \star\big)$ be a SuperHyperStructure of type $(r, n; m)$. Assume fuzzy membership functions

$$\mu_r : \mathcal{P}^{*r}(H) \to [0,1], \qquad \mu_n : \mathcal{P}^{*n}(H) \to [0,1].$$

The triple

$$\widetilde{\mathbb{S}} = \big(\mathcal{P}^{*n}(H), \star; \mu_r, \mu_n\big)$$

is called a *fuzzy SuperHyperStructure* (of type $(r, n; m)$, in the min-convention) if for all $A_1, \ldots, A_m \in \mathcal{P}^{*r}(H)$,

$$\mu_n\big(\star(A_1, \ldots, A_m)\big) \;\geq\; \min\{\mu_r(A_1), \ldots, \mu_r(A_m)\}.$$

More generally, one may replace min by a chosen t-norm T_m and require $\mu_n(\star(\mathbf{A})) \geq T_m(\mu_r(A_1), \ldots, \mu_r(A_m))$.

Definition 3.10.6 (Uncertain SuperHyperStructure)**.** Let $H \neq \varnothing$, and let

$$\mathbb{S} = \big(\mathcal{P}^{*n}(H), \star\big)$$

be a SuperHyperStructure of type $(r, n; m)$, where

$$\star : \big(\mathcal{P}^{*r}(H)\big)^m \longrightarrow \mathcal{P}^{*n}(H).$$

Let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. Fix:

- an *uncertainty-degree function* on the input layer

$$\mu_r : \mathcal{P}^{*r}(H) \longrightarrow \mathrm{Dom}(M),$$

  so that

$$\mathcal{U}_r := \big(\mathcal{P}^{*r}(H), \mu_r\big)$$

  is an uncertain set of type M on $\mathcal{P}^{*r}(H)$;

- an *uncertainty-degree function* on the output layer

$$\mu_n : \mathcal{P}^{*n}(H) \longrightarrow \mathrm{Dom}(M),$$

  so that

$$\mathcal{U}_n := \big(\mathcal{P}^{*n}(H), \mu_n\big)$$

  is an uncertain set of type M on $\mathcal{P}^{*n}(H)$;

- an *admissible score map*

$$S_M : \mathrm{Dom}(M) \longrightarrow [0,1];$$

- an m-ary *uncertainty-aggregation operator*

$$\Gamma_m : [0,1]^m \longrightarrow [0,1].$$

Then the tuple

$$\widetilde{\mathbb{S}}_M = \big(\mathcal{P}^{*n}(H), \star; \mu_r, \mu_n, S_M, \Gamma_m\big)$$

is called an *uncertain SuperHyperStructure of type* $(r, n; m)$ *and model* M if, for all

$$A_1, \ldots, A_m \in \mathcal{P}^{*r}(H),$$

the following compatibility condition holds:

$$S_M\big(\mu_n(\star(A_1, \ldots, A_m))\big) \;\geq\; \Gamma_m\big(S_M(\mu_r(A_1)), \ldots, S_M(\mu_r(A_m))\big).$$

Remark 3.10.7. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1], \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_m(x_1, \ldots, x_m) = \min\{x_1, \ldots, x_m\},$$

then Definition 3.10.6 reduces to the usual min-type fuzzy SuperHyperStructure condition

$$\mu_n(\star(A_1, \ldots, A_m)) \geq \min\{\mu_r(A_1), \ldots, \mu_r(A_m)\}.$$

Theorem 3.10.8 (Well-definedness of uncertain SuperHyperStructures)**.** *Let $H \neq \varnothing$, let*

$$\mathbb{S} = \big(\mathcal{P}^{*n}(H), \star\big)$$

be a SuperHyperStructure of type $(r, n; m)$, let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, and let

$$\mu_r : \mathcal{P}^{*r}(H) \to \mathrm{Dom}(M), \qquad \mu_n : \mathcal{P}^{*n}(H) \to \mathrm{Dom}(M),$$
$$S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_m : [0,1]^m \to [0,1]$$

be given. Then:

(a) $\mathcal{U}_r = \big(\mathcal{P}^{*r}(H), \mu_r\big)$ *and* $\mathcal{U}_n = \big(\mathcal{P}^{*n}(H), \mu_n\big)$ *are well-defined uncertain sets of type M.*

(b) *For every*

$$A_1, \ldots, A_m \in \mathcal{P}^{*r}(H),$$

the value

$$\star(A_1, \ldots, A_m)$$

is a well-defined element of $\mathcal{P}^{*n}(H)$.

(c) *For every*

$$A_1, \ldots, A_m \in \mathcal{P}^{*r}(H),$$

both sides of

$$S_M\big(\mu_n(\star(A_1, \ldots, A_m))\big) \ \geq \ \Gamma_m\big(S_M(\mu_r(A_1)), \ldots, S_M(\mu_r(A_m))\big)$$

are well-defined real numbers in $[0,1]$.

(d) *Consequently, Definition 3.10.6 determines a mathematically well-defined class of structures.*

Proof. (a) By the recursive definition of iterated nonempty powersets,

$$\mathcal{P}^{*0}(H) = H, \qquad \mathcal{P}^{*(s+1)}(H) = \mathcal{P}^{*}\big(\mathcal{P}^{*s}(H)\big) \quad (s \geq 0).$$

Since $H \neq \varnothing$, it follows inductively that $\mathcal{P}^{*r}(H) \neq \varnothing$ and $\mathcal{P}^{*n}(H) \neq \varnothing$. Because

$$\mu_r : \mathcal{P}^{*r}(H) \to \mathrm{Dom}(M) \qquad \text{and} \qquad \mu_n : \mathcal{P}^{*n}(H) \to \mathrm{Dom}(M),$$

the pairs

$$\mathcal{U}_r = \big(\mathcal{P}^{*r}(H), \mu_r\big) \qquad \text{and} \qquad \mathcal{U}_n = \big(\mathcal{P}^{*n}(H), \mu_n\big)$$

are, by the definition of an uncertain set, uncertain sets of type M.

(b) Since $\mathbb{S}$ is a SuperHyperStructure of type $(r, n; m)$, its SuperHyperOperator is, by definition, a map

$$\star : \big(\mathcal{P}^{*r}(H)\big)^m \to \mathcal{P}^{*n}(H).$$

Hence for every

$$A_1, \ldots, A_m \in \mathcal{P}^{*r}(H),$$

the value

$$\star(A_1, \ldots, A_m)$$

exists and belongs to $\mathcal{P}^{*n}(H)$.

(c) Fix

$$A_1, \ldots, A_m \in \mathcal{P}^{*r}(H).$$

By part (b),

$$\star(A_1, \ldots, A_m) \in \mathcal{P}^{*n}(H).$$

Applying μ_n, we obtain

$$\mu_n(\star(A_1, \ldots, A_m)) \in \mathrm{Dom}(M).$$

Since

$$S_M : \mathrm{Dom}(M) \to [0, 1],$$

it follows that

$$S_M\big(\mu_n(\star(A_1, \ldots, A_m))\big) \in [0, 1].$$

Also, for each $i = 1, \ldots, m$,

$$\mu_r(A_i) \in \mathrm{Dom}(M),$$

hence

$$S_M(\mu_r(A_i)) \in [0, 1].$$

Therefore

$$\big(S_M(\mu_r(A_1)), \ldots, S_M(\mu_r(A_m))\big) \in [0, 1]^m.$$

Since

$$\Gamma_m : [0, 1]^m \to [0, 1],$$

we obtain

$$\Gamma_m\big(S_M(\mu_r(A_1)), \ldots, S_M(\mu_r(A_m))\big) \in [0, 1].$$

Thus both sides of the compatibility inequality are well-defined elements of $[0, 1]$, and the comparison

$$S_M\big(\mu_n(\star(A_1, \ldots, A_m))\big) \;\geq\; \Gamma_m\big(S_M(\mu_r(A_1)), \ldots, S_M(\mu_r(A_m))\big)$$

is meaningful.

(d) By parts (a)–(c), every object appearing in Definition 3.10.6 is well-defined: the input and output uncertain sets exist, the SuperHyperOperator produces a valid output in the target layer, and the compatibility condition is a legitimate scalar inequality in $[0, 1]$. Hence the notion of uncertain SuperHyperStructure is mathematically well-defined. □

3.11 Uncertain Fibonacci Numbers

Fuzzy Fibonacci numbers are fuzzy-number extensions of Fibonacci numbers, satisfying Fibonacci-type recurrence while representing each term by graded intervals or membership functions capturing uncertainty explicitly [285–287]. Related concepts such as Neutrosophic Fibonacci Numbers [288, 289] are also known.

Definition 3.11.1 (Fuzzy Fibonacci Number)**.** [285–287] Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence. For each $n \geq 1$ and $\alpha \in [0,1]$, the n-th *fuzzy Fibonacci number* $\widetilde{F}_n$ is the fuzzy number whose α-cut is

$$[\widetilde{F}_n]_\alpha = \Big[F_{n-1} + \alpha\big(F_n - F_{n-1}\big),\ F_{n+1} - \alpha\big(F_{n+1} - F_n\big)\Big].$$

Together with

$$[\widetilde{F}_0]_\alpha = [\,1-\alpha,\ 1+\alpha\,],$$

these numbers form the fuzzy Fibonacci sequence.

Example 3.11.2 (A fuzzy Fibonacci number)**.** Using the Fibonacci sequence

$$F_0 = 0, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8,$$

consider the fuzzy Fibonacci number $\widetilde{F}_5$. By Definition, for each $\alpha \in [0,1]$,

$$[\widetilde{F}_5]_\alpha = \Big[F_4 + \alpha(F_5 - F_4),\ F_6 - \alpha(F_6 - F_5)\Big].$$

Substituting $F_4 = 3$, $F_5 = 5$, and $F_6 = 8$, we obtain

$$[\widetilde{F}_5]_\alpha = \big[3 + 2\alpha,\ 8 - 3\alpha\big].$$

In particular,

$$[\widetilde{F}_5]_0 = [3, 8], \qquad [\widetilde{F}_5]_{1/2} = [4, 6.5], \qquad [\widetilde{F}_5]_1 = [5, 5].$$

Hence, $\widetilde{F}_5$ is a triangular fuzzy number centered at the classical Fibonacci number $F_5 = 5$, with support interval $[3, 8]$.

Uncertain Fibonacci numbers are uncertainty-model-based extensions of Fibonacci numbers, represented as uncertain sets on $\mathbb{R}$ whose score-induced profiles are triangular around the classical Fibonacci values.

Definition 3.11.3 (Uncertain Fibonacci Number)**.** Let

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0)$$

be the Fibonacci sequence.

Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Assume that the following data are fixed:

- a score map

$$S_M : \mathrm{Dom}(M) \to [0,1];$$

- a lift map

$$L_M : [0, 1] \to \mathrm{Dom}(M)$$

  such that

$$S_M(L_M(r)) = r \qquad \text{for all } r \in [0, 1].$$

For each $n \geq 1$, define a function

$$\lambda_n : \mathbb{R} \to [0, 1]$$

by

$$\lambda_n(x) = \begin{cases} 0, & x < F_{n-1}, \\ \dfrac{x - F_{n-1}}{F_n - F_{n-1}}, & F_{n-1} \leq x \leq F_n, \\ \dfrac{F_{n+1} - x}{F_{n+1} - F_n}, & F_n \leq x \leq F_{n+1}, \\ 0, & x > F_{n+1}. \end{cases}$$

Then define

$$\mu_n^M : \mathbb{R} \to \mathrm{Dom}(M)$$

by

$$\mu_n^M(x) := L_M\big(\lambda_n(x)\big) \qquad (x \in \mathbb{R}).$$

The uncertain set

$$\widetilde{F}_{n,M} := (\mathbb{R}, \mu_n^M)$$

is called the n-th *uncertain Fibonacci number of type M*.

Its score-induced α-cut is defined by

$$[\widetilde{F}_{n,M}]_\alpha^{S_M} := \{\, x \in \mathbb{R} \mid S_M(\mu_n^M(x)) \geq \alpha \,\}, \qquad \alpha \in (0, 1].$$

Equivalently, for every $n \geq 1$ and every $\alpha \in (0, 1]$,

$$[\widetilde{F}_{n,M}]_\alpha^{S_M} = \Big[F_{n-1} + \alpha(F_n - F_{n-1}),\ F_{n+1} - \alpha(F_{n+1} - F_n)\Big].$$

If one wishes to include the initial term, one may define

$$\widetilde{F}_{0,M} := (\mathbb{R}, \mu_0^M),$$

where

$$\mu_0^M(x) = \begin{cases} L_M(1), & x = 0, \\ L_M(0), & x \neq 0. \end{cases}$$

Remark 3.11.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0, 1], \qquad S_M = \mathrm{id}_{[0,1]}, \qquad L_M = \mathrm{id}_{[0,1]},$$

then $\widetilde{F}_{n,M}$ becomes the triangular fuzzy number with peak F_n and support $[F_{n-1}, F_{n+1}]$. Hence Definition 3.11.3 recovers the usual triangular-type fuzzy Fibonacci number for $n \geq 1$.

Theorem 3.11.5 (Well-definedness of uncertain Fibonacci numbers). *Let M, S_M, and L_M be as in Definition 3.11.3. Then:*

1. *for every $n \geq 1$, the map*
$$\mu_n^M : \mathbb{R} \to \mathrm{Dom}(M)$$
is well-defined, so
$$\widetilde{F}_{n,M} = (\mathbb{R}, \mu_n^M)$$
is a well-defined uncertain set of type M on $\mathbb{R}$;
2. *for every $n \geq 1$ and every $\alpha \in (0,1]$, the score-induced α-cut*
$$[\widetilde{F}_{n,M}]_\alpha^{S_M}$$
is well-defined and satisfies
$$[\widetilde{F}_{n,M}]_\alpha^{S_M} = \Big[F_{n-1} + \alpha(F_n - F_{n-1}),\ F_{n+1} - \alpha(F_{n+1} - F_n)\Big];$$
3. *the optional initial term $\widetilde{F}_{0,M}$ is also well-defined.*

Consequently, the family
$$\big(\widetilde{F}_{n,M}\big)_{n\geq 1} \quad (\textit{and also } (\widetilde{F}_{n,M})_{n\geq 0} \textit{ if } \widetilde{F}_{0,M} \textit{ is included})$$
forms a mathematically well-defined sequence of uncertain Fibonacci numbers.

Proof. Fix $n \geq 1$.

First, since
$$F_{n-1} < F_n < F_{n+1},$$
the denominators
$$F_n - F_{n-1} \qquad \text{and} \qquad F_{n+1} - F_n$$
are strictly positive. Hence the piecewise formula defining λ_n is meaningful.

Next, we show that
$$\lambda_n(x) \in [0,1] \qquad \text{for all } x \in \mathbb{R}.$$
Indeed:

- if $x < F_{n-1}$ or $x > F_{n+1}$, then $\lambda_n(x) = 0$;
- if $F_{n-1} \leq x \leq F_n$, then
$$0 \leq x - F_{n-1} \leq F_n - F_{n-1},$$
so
$$0 \leq \frac{x - F_{n-1}}{F_n - F_{n-1}} \leq 1;$$

- if $F_n \le x \le F_{n+1}$, then
$$0 \le F_{n+1} - x \le F_{n+1} - F_n,$$
so
$$0 \le \frac{F_{n+1} - x}{F_{n+1} - F_n} \le 1.$$

Therefore $\lambda_n : \mathbb{R} \to [0,1]$ is well-defined.

Since
$$L_M : [0,1] \to \mathrm{Dom}(M),$$
it follows that
$$\mu_n^M(x) = L_M(\lambda_n(x)) \in \mathrm{Dom}(M) \qquad \text{for all } x \in \mathbb{R}.$$
Hence
$$\mu_n^M : \mathbb{R} \to \mathrm{Dom}(M)$$
is a well-defined map, and thus
$$\widetilde{F}_{n,M} = (\mathbb{R}, \mu_n^M)$$
is a well-defined uncertain set of type M. This proves (1).

For $\alpha \in (0,1]$, we have
$$S_M(\mu_n^M(x)) = S_M(L_M(\lambda_n(x))) = \lambda_n(x) \qquad (x \in \mathbb{R}).$$
Therefore
$$[\widetilde{F}_{n,M}]_\alpha^{S_M} = \{\, x \in \mathbb{R} \mid \lambda_n(x) \ge \alpha \,\}.$$

Now solve the inequality $\lambda_n(x) \ge \alpha$. On the left branch $F_{n-1} \le x \le F_n$,
$$\frac{x - F_{n-1}}{F_n - F_{n-1}} \ge \alpha \quad \Longleftrightarrow \quad x \ge F_{n-1} + \alpha(F_n - F_{n-1}).$$
On the right branch $F_n \le x \le F_{n+1}$,
$$\frac{F_{n+1} - x}{F_{n+1} - F_n} \ge \alpha \quad \Longleftrightarrow \quad x \le F_{n+1} - \alpha(F_{n+1} - F_n).$$
Hence
$$[\widetilde{F}_{n,M}]_\alpha^{S_M} = \Big[F_{n-1} + \alpha(F_n - F_{n-1}),\ F_{n+1} - \alpha(F_{n+1} - F_n)\Big].$$
This proves (2).

Finally, for the optional initial term $\widetilde{F}_{0,M}$, the values $L_M(1)$ and $L_M(0)$ both belong to $\mathrm{Dom}(M)$, so
$$\mu_0^M : \mathbb{R} \to \mathrm{Dom}(M)$$
is also well-defined. Thus $\widetilde{F}_{0,M}$ is a well-defined uncertain set. This proves (3).

Therefore all uncertain Fibonacci numbers defined above are mathematically well-defined. □

Chapter 4

Algebraic and Linear Applied Structures

In this chapter, we present algebraic and linear applied structures. A concise comparison of the algebraic and linear uncertain concepts covered in Chapter 4 is presented in Table 4.1.

Table 4.1: A concise comparison of the algebraic and linear uncertain concepts covered in Chapter 4.

Concept	**Base structure**	**Main focus**
Uncertain Algebra	Algebra	Uncertain algebraic operations and general algebraic structure.
Uncertain Group	Group	Uncertain binary operation, identity, and inverse structure.
Uncertain Topology	Topological space	Uncertain open sets, continuity, and topological structure.
Uncertain Lattice	Lattice	Uncertain order, meet, and join operations.
Uncertain Vector	Vector space	Uncertain linear combination and vector structure.
Uncertain Matrices	Matrix space	Uncertain entries and matrix-based representation.
Uncertain Semigroup	Semigroup	Uncertain associative operation and algebraic composition.
Uncertain Ring	Ring	Uncertain addition, multiplication, and distributive structure.
Uncertain Semiring	Semiring	Uncertain additive and multiplicative operations without full ring requirements.
Uncertain Boolean Algebra	Boolean algebra	Uncertain logical operations, complement, and order structure.
Uncertain Module	Module	Uncertain scalar action and additive structure over a ring.
Uncertain Poset	Partially ordered set	Uncertain partial order and hierarchical comparison.
Uncertain Manifold	Manifold	Uncertain local Euclidean structure and geometric continuity.

Continued on the next page

Concept	Base structure	Main focus
Uncertain HyperAlgebra	Hyperalgebra	Uncertain multi-valued algebraic operations.
Uncertain MultiAlgebra	Multialgebra	Uncertain generalized operations with multiple outputs or rules.
Uncertain Ideal	Ideal	Uncertain absorption and substructure within algebraic systems.
Uncertain Filter	Filter	Uncertain upward closure and intersection structure in ordered settings.

4.1 Uncertain Algebra

Fuzzy algebra extends algebraic operations to graded truth: elements, relations, or laws hold with degrees. Uses t-norms, t-conorms, and residuation to generalize rings, lattices, and logic algebras [290–292]. In fuzzy algebra, the underlying carrier may be equipped with graded membership, and algebraic closure is required only *to a degree* (typically via a chosen t-norm), rather than in a crisp yes/no sense. This viewpoint unifies many "fuzzy substructure" notions (fuzzy subgroups, fuzzy subrings, etc.).

Definition 4.1.1 (T-fuzzy algebra / fuzzy subalgebra)**.** Let τ be a (single-sorted) algebraic signature, i.e., a family $\tau = \{ f_i \mid i \in I \}$ of operation symbols, where each f_i has arity $n_i \in \mathbb{N}$. Let

$$\mathbf{A} = \big(A, (f_i^{\mathbf{A}})_{i\in I}\big)$$

be a τ-algebra, where $f_i^{\mathbf{A}} : A^{n_i} \to A$.

Let $\mu : A \to [0,1]$ be a fuzzy subset of A (a membership function). Fix an n-ary triangular norm (t-norm) $T_n : [0,1]^n \to [0,1]$ for each arity n (e.g. induced from a binary t-norm by associativity), so that each T_n is monotone in every argument and $T_n(1,\dots,1) = 1$.

We call μ a *T-fuzzy subalgebra* of $\mathbf{A}$ if, for every $i \in I$ and all $a_1,\dots,a_{n_i} \in A$,

$$\mu\big(f_i^{\mathbf{A}}(a_1,\dots,a_{n_i})\big) \;\ge\; T_{n_i}\big(\mu(a_1),\dots,\mu(a_{n_i})\big).$$

In this case, the triple $(\mathbf{A},\mu,T)$ (or simply (A,μ) when T is understood) is called a *T-fuzzy algebra.*

Special case (min-fuzzy algebra). If $T_n(x_1,\dots,x_n) = \min\{x_1,\dots,x_n\}$ for all n, then μ is called a *(min-)fuzzy subalgebra*, and $(\mathbf{A},\mu)$ is called a *fuzzy algebra.* This is the same "min" convention often used in foundational fuzzy algebraic structures.

Remark 4.1.2 (α-cut characterization for the min case)**.** Assume $T = \min$ and define, for $\alpha \in (0,1]$,

$$A_\alpha := \{a \in A : \mu(a) \ge \alpha\}.$$

If μ is a min-fuzzy subalgebra, then each nonempty A_α is a (crisp) subalgebra of $\mathbf{A}$: indeed, if $a_j \in A_\alpha$ for all j, then

$$\mu\big(f_i^{\mathbf{A}}(a_1,\dots,a_{n_i})\big) \ge \min_j \mu(a_j) \ge \alpha,$$

hence $f_i^{\mathbf{A}}(a_1,\dots,a_{n_i}) \in A_\alpha$. Conversely, if every A_α is a subalgebra, then μ satisfies the min-closure inequality in Definition 4.1.1.

Example 4.1.3 (Fuzzy group as a fuzzy algebra)**.** Let $\tau = \{\cdot, {}^{-1}, e\}$ be the group signature, and let $\mathbf{G} = (G, \cdot, {}^{-1}, e)$ be a group. A fuzzy subset $\mu : G \to [0,1]$ is a min-fuzzy subalgebra of $\mathbf{G}$ precisely when

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\} \quad \text{and} \quad \mu(x^{-1}) = \mu(x) \qquad (x, y \in G),$$

which matches the standard definition of a fuzzy subgroup (in the "min" convention).

Definition 4.1.4 (Uncertain Algebra)**.** Let $\tau = \{f_i\}_{i\in I}$ be a single-sorted algebraic signature such that each operation symbol f_i has arity $n_i \geq 1$. Let

$$\mathbf{A} = \big(A, (f_i^{\mathbf{A}})_{i\in I}\big)$$

be a τ-algebra, where

$$f_i^{\mathbf{A}} : A^{n_i} \to A \qquad (i \in I).$$

Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Let

$$\mu_M : A \to \mathrm{Dom}(M)$$

be an uncertainty-degree function, so that

$$\mathcal{U}_M = (A, \mu_M)$$

is an uncertain set of type M on A.

Assume moreover that, for each integer $n \geq 1$, an n-ary aggregation map

$$\Gamma_n : [0,1]^n \to [0,1]$$

is fixed, and that an admissible score map

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

is given.

Then $\mathcal{U}_M = (A, \mu_M)$ is called an *uncertain algebra of type* (τ, M) (or an *uncertain* τ*-algebra*) if, for every $i \in I$ and all $a_1, \ldots, a_{n_i} \in A$,

$$S_M\Big(\mu_M\big(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})\big)\Big) \;\geq\; \Gamma_{n_i}\big(S_M(\mu_M(a_1)), \ldots, S_M(\mu_M(a_{n_i}))\big).$$

Remark 4.1.5. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_n(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\},$$

then Definition 4.1.4 reduces to the usual min-type fuzzy algebra condition

$$\mu\big(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})\big) \geq \min\{\mu(a_1), \ldots, \mu(a_{n_i})\}.$$

Theorem 4.1.6 (Well-definedness of uncertain algebras)**.** *Let $\tau = \{f_i\}_{i\in I}$ be a single-sorted algebraic signature with arities $n_i \geq 1$, let*

$$\mathbf{A} = \big(A, (f_i^{\mathbf{A}})_{i\in I}\big)$$

be a τ-algebra, let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, and let

$$\mu_M : A \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_n : [0,1]^n \to [0,1] \quad (n \geq 1)$$

be given.

Then:

(a) (A, μ_M) *is a well-defined uncertain set of type M on A.*

(b) *For every $i \in I$ and every $a_1, \ldots, a_{n_i} \in A$, the value*

$$S_M\Big(\mu_M\big(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})\big)\Big)$$

is a well-defined element of $[0,1]$.

(c) *For every $i \in I$ and every $a_1, \ldots, a_{n_i} \in A$, the value*

$$\Gamma_{n_i}\big(S_M(\mu_M(a_1)), \ldots, S_M(\mu_M(a_{n_i}))\big)$$

is a well-defined element of $[0,1]$.

(d) *Consequently, the compatibility inequality in Definition 4.1.4 is meaningful for every basic operation and every input tuple, so Definition 4.1.4 determines a mathematically well-defined class of structures.*

Proof. (a) By the definition of an uncertain set of type M, any pair

$$(A, \mu_M) \qquad \text{with} \qquad \mu_M : A \to \mathrm{Dom}(M)$$

is an uncertain set of type M on A. Since μ_M is such a map, (A, μ_M) is well-defined.

(b) Fix $i \in I$ and $a_1, \ldots, a_{n_i} \in A$. Because

$$f_i^{\mathbf{A}} : A^{n_i} \to A,$$

the element

$$f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})$$

belongs to A. Applying $\mu_M : A \to \mathrm{Dom}(M)$, we obtain

$$\mu_M\big(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})\big) \in \mathrm{Dom}(M).$$

Since $S_M : \mathrm{Dom}(M) \to [0,1]$, it follows that

$$S_M\Big(\mu_M\big(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})\big)\Big) \in [0,1].$$

Hence the left-hand side is well-defined.

(c) For each $j = 1, \ldots, n_i$, since $a_j \in A$ and $\mu_M : A \to \mathrm{Dom}(M)$, we have

$$\mu_M(a_j) \in \mathrm{Dom}(M).$$

Applying the score map S_M, we obtain

$$S_M(\mu_M(a_j)) \in [0,1] \qquad (j = 1, \ldots, n_i).$$

Therefore

$$\big(S_M(\mu_M(a_1)), \ldots, S_M(\mu_M(a_{n_i}))\big) \in [0,1]^{n_i}.$$

Because

$$\Gamma_{n_i} : [0,1]^{n_i} \to [0,1],$$

it follows that

$$\Gamma_{n_i}\big(S_M(\mu_M(a_1)), \ldots, S_M(\mu_M(a_{n_i}))\big) \in [0,1].$$

Hence the right-hand side is well-defined.

(d) By parts (b) and (c), for every basic operation $f_i^{\mathbf{A}}$ and every tuple $(a_1, \ldots, a_{n_i}) \in A^{n_i}$, both sides of

$$S_M\Big(\mu_M\big(f_i^{\mathbf{A}}(a_1, \ldots, a_{n_i})\big)\Big) \;\geq\; \Gamma_{n_i}\big(S_M(\mu_M(a_1)), \ldots, S_M(\mu_M(a_{n_i}))\big)$$

are well-defined real numbers in $[0,1]$. Therefore the inequality is meaningful, and the notion of uncertain algebra is mathematically well-defined. □

Representative extensions of fuzzy algebra are listed in Table 4.2.

Table 4.2: Representative extensions of fuzzy algebra (conceptual overview).

k	Representative uncertainty-algebra type(s) $\mathbf{A}_M = (A, \Omega, \mu_M)$ with degree map $\mu_M : A \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (or set-valued grades embedded into $[0,1]^k$)
1	Fuzzy algebras / fuzzy subalgebras (graded closure under operations; typically via a t-norm or min) [290–292].
2	Intuitionistic fuzzy algebras: membership–nonmembership pair $\big(\mu(a), \nu(a)\big) \in [0,1]^2$ with $\mu(a) + \nu(a) \leq 1$ [293, 294].
3	Hesitant fuzzy algebras (3-valued hesitant profile, e.g. three representative hesitant grades per element)[(a)] [295]; neutrosophic algebras: truth/indeterminacy/falsity degrees $\big(T(a), I(a), F(a)\big) \in [0,1]^3$ (typically treated as independent coordinates) [296–298].

Note. The above k-catalogue is a bookkeeping device: in algebraic settings, axioms are imposed on how the degrees propagate through operations (e.g. $\mu(f(a_1, \ldots, a_m))$ bounded below by a t-norm of the input degrees).
[(a)] Hesitant grades are naturally set-valued; placing them at $k = 3$ means adopting a fixed 3-tuple representation (e.g. restricting to hesitant sets of size 3, or extracting three canonical/summary grades).

4.2 Uncertain Group

A fuzzy group assigns each group element a membership degree, requiring compatibility with multiplication and inversion, capturing subgroup-like graded structure [299, 300].

Definition 4.2.1 (T-fuzzy subgroup (fuzzy group))**.** [301, 302] Let $(G, \circ, e, (\cdot)^{-1})$ be a (crisp) group. Let $T : [0,1]^2 \to [0,1]$ be a (binary) t-norm. A mapping $\mu : G \to [0,1]$ (a fuzzy subset of G) is called a *T-fuzzy subgroup* of G (or a *fuzzy group* on G) if

$$T\big(\mu(a), \mu(b)\big) \;\le\; \mu\big(a \circ b^{-1}\big) \qquad \text{for all } a, b \in G.$$

To extend fuzzy groups to a general uncertainty model M, one cannot assume that the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ carries a canonical order or a canonical t-norm. Therefore, we compare uncertainty degrees through a score map and aggregate input scores through a fixed binary operator.

Definition 4.2.2 (Uncertain Group)**.** Let $(G, \circ, e, (\cdot)^{-1})$ be a (crisp) group, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Let

$$\mu_M : G \to \mathrm{Dom}(M)$$

be an uncertainty-degree function, so that

$$\mathcal{U}_M = (G, \mu_M)$$

is an uncertain set of type M on G.

Assume that the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0,1];$$
- a binary aggregation operator
$$\Gamma_2 : [0,1]^2 \to [0,1].$$

Then $\mathcal{U}_M = (G, \mu_M)$ is called an *uncertain group of type M* if, for all $a, b \in G$,

$$S_M\big(\mu_M(a \circ b^{-1})\big) \;\ge\; \Gamma_2\big(S_M(\mu_M(a)),\, S_M(\mu_M(b))\big).$$

Remark 4.2.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes $\Gamma_2 = T$, where $T : [0,1]^2 \to [0,1]$ is a binary t-norm, then Definition 4.2.2 reduces to the usual T-fuzzy subgroup condition

$$T\big(\mu(a), \mu(b)\big) \le \mu(a \circ b^{-1}) \qquad (a, b \in G).$$

Hence uncertain groups generalize fuzzy groups through the uncertain-set framework.

Theorem 4.2.4 (Well-definedness of uncertain groups)**.** *Let $(G, \circ, e, (\cdot)^{-1})$ be a group, let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, and let*

$$\mu_M : G \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_2 : [0,1]^2 \to [0,1]$$

be given.

Then:

(a) *(G, μ_M) is a well-defined uncertain set of type M on G.*

(b) *For every $a, b \in G$, the value*

$$S_M\big(\mu_M(a \circ b^{-1})\big)$$

is a well-defined element of $[0, 1]$.

(c) *For every $a, b \in G$, the value*

$$\Gamma_2\big(S_M(\mu_M(a)),\, S_M(\mu_M(b))\big)$$

is a well-defined element of $[0, 1]$.

(d) *Consequently, the compatibility inequality in Definition 4.2.2 is meaningful for every pair $a, b \in G$, so Definition 4.2.2 determines a mathematically well-defined class of structures.*

Proof. (a) By the definition of an uncertain set of type M, any pair

$$(G, \mu_M) \qquad \text{with} \qquad \mu_M : G \to \mathrm{Dom}(M)$$

is an uncertain set of type M on G. Hence (G, μ_M) is well-defined.

(b) Fix $a, b \in G$. Since G is a group, the inverse $b^{-1} \in G$ exists, and therefore

$$a \circ b^{-1} \in G.$$

Because $\mu_M : G \to \mathrm{Dom}(M)$, it follows that

$$\mu_M(a \circ b^{-1}) \in \mathrm{Dom}(M).$$

Applying the score map $S_M : \mathrm{Dom}(M) \to [0, 1]$, one obtains

$$S_M\big(\mu_M(a \circ b^{-1})\big) \in [0, 1].$$

Thus the left-hand side of the defining inequality is well-defined.

(c) Since $a, b \in G$ and $\mu_M : G \to \mathrm{Dom}(M)$, one has

$$\mu_M(a), \mu_M(b) \in \mathrm{Dom}(M).$$

Applying S_M, we obtain

$$S_M(\mu_M(a)),\, S_M(\mu_M(b)) \in [0, 1].$$

Hence

$$\big(S_M(\mu_M(a)),\, S_M(\mu_M(b))\big) \in [0, 1]^2.$$

Because $\Gamma_2 : [0, 1]^2 \to [0, 1]$, it follows that

$$\Gamma_2\big(S_M(\mu_M(a)),\, S_M(\mu_M(b))\big) \in [0, 1].$$

Thus the right-hand side of the defining inequality is well-defined.

(d) By parts (b) and (c), for every $a, b \in G$, both sides of

$$S_M\big(\mu_M(a \circ b^{-1})\big) \;\geq\; \Gamma_2\big(S_M(\mu_M(a)),\, S_M(\mu_M(b))\big)$$

are well-defined real numbers in $[0, 1]$. Therefore the inequality is meaningful for every pair $(a, b) \in G \times G$. Hence the notion of uncertain group in Definition 4.2.2 is mathematically well-defined. □

For reference, a catalogue of fuzzy-group families classified by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ is presented in Table 4.3.

Table 4.3: A catalogue of fuzzy-group families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (conceptual overview).

k	Representative uncertainty-group type(s) on a group G (degree assignment for each $g \in G$)
1	Fuzzy group (Rosenfeld-type) with $\mu : G \to [0,1]$; complex fuzzy group (one complex-valued grade per element)[(a)] [303].
2	Intuitionistic fuzzy group (membership/nonmembership pair) [304,305]; vague group (often equivalent to an intuitionistic-type pair under different semantics) [306,307]; Pythagorean fuzzy subgroups [308–310]; bipolar fuzzy group (positive/negative membership pair) [311,312].
3	Hesitant fuzzy group (triple-valued hesitant profile, e.g. three representative hesitant grades per element)[(b)] [313, 314]; neutrosophic group $(T(g), I(g), F(g)) \in [0,1]^3$ [276, 315]; spherical fuzzy group (typically $(\mu, \nu, \pi) \in [0,1]^3$ under a spherical constraint) [316]; Picture Fuzzy group [317–319].

[(a)] Complex grades are values in $\mathbb{C}$ (often restricted to the unit disk). Cataloguing as $k = 1$ means "one complex coordinate"; encoding as $(\Re, \mathrm{Im})$ or (r, θ) would correspond to $k = 2$.
[(b)] Hesitant membership is naturally set-valued. Placing it at $k = 3$ means adopting a fixed 3-tuple representation (e.g. selecting three canonical/summary grades from the hesitant set, or restricting to hesitant sets of size 3).

In addition to Uncertain Groups, related concepts such as monoids [320], quasigroups [321–323], polygroups [324,325], transformation groups [326,327], hypergroups [328–331], superhypergroups [332–334], topological groups [335,336], local groups [337,338], soft groups [339], and rough groups [340,341] are also known.

4.3 Uncertain Topology

Topology studies open sets and continuity, defining neighborhoods and convergence without distances, via axioms closed under unions and intersections. Fuzzy topology replaces crisp opens with membership functions, closed under suprema and finite infima, enabling graded continuity [342,343].

Definition 4.3.1 (Fuzzy topology)**.** [344, 345] Let $X \neq \varnothing$ be a set and let $I := [0,1]$. Write $I^X := \{\mu : X \to I\}$ for the set of all fuzzy subsets of X.

A family $\tau \subseteq I^X$ is called a *fuzzy topology* on X if it satisfies:

(FT1) **(Constants)** For every $a \in I$, the constant map $a_X \in I^X$ defined by $a_X(x) := a$ for all $x \in X$ belongs to τ.

(FT2) **(Arbitrary joins)** For every family $\{\mu_i\}_{i \in J} \subseteq \tau$ (with $J \neq \varnothing$), the pointwise supremum $\bigvee_{i \in J} \mu_i \in I^X$ defined by

$$\Big(\bigvee_{i \in J} \mu_i\Big)(x) := \sup_{i \in J} \mu_i(x) \qquad (x \in X)$$

belongs to τ.

(FT3) **(Finite meets)** For all $\mu, \nu \in \tau$, the pointwise infimum $\mu \wedge \nu \in I^X$ defined by

$$(\mu \wedge \nu)(x) := \min\{\mu(x), \nu(x)\} \qquad (x \in X)$$

belongs to τ.

The pair (X, τ) is called a *fuzzy topological space*, and members of τ are called *fuzzy open sets*.

Definition 4.3.2 (Fuzzy continuity)**.** Let (X, τ_X) and (Y, τ_Y) be fuzzy topological spaces. A function $f : X \to Y$ is called *(fuzzy) continuous* if for every $\nu \in \tau_Y$, the inverse image (pullback) fuzzy set $f^{\leftarrow}(\nu) \in I^X$ given by

$$f^{\leftarrow}(\nu) := \nu \circ f \quad (\text{i.e., } (f^{\leftarrow}(\nu))(x) = \nu(f(x)))$$

belongs to τ_X.

To extend fuzzy topology to the general uncertain-set setting, one must assume that the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ carries enough order structure to support arbitrary joins and finite meets. Accordingly, we work with an uncertain model whose degree-domain is equipped with a complete lattice order.

Definition 4.3.3 (Uncertain Topology)**.** Let $X \neq \varnothing$ be a set, and let M be an uncertain model with degree-domain

$$D := \mathrm{Dom}(M) \subseteq [0,1]^k.$$

Assume that D is equipped with a partial order $\leq_M$ such that

$$(D, \leq_M)$$

is a complete lattice. Denote its least and greatest elements by

$$0_M := \bigwedge D, \qquad 1_M := \bigvee D.$$

Let

$$\mathcal{U}_M(X) := \{\mu : X \to D\}$$

be the family of all uncertain sets of type M on X.

For a nonempty family $\{\mu_i\}_{i \in I} \subseteq \mathcal{U}_M(X)$, define the *pointwise join* $\bigvee_{i \in I} \mu_i \in \mathcal{U}_M(X)$ by

$$\Big(\bigvee_{i \in I} \mu_i\Big)(x) := \bigvee_{i \in I} \mu_i(x) \qquad (x \in X),$$

where the join on the right-hand side is taken in the complete lattice D.

For $\mu, \nu \in \mathcal{U}_M(X)$, define the *pointwise meet* $\mu \wedge_M \nu \in \mathcal{U}_M(X)$ by

$$(\mu \wedge_M \nu)(x) := \mu(x) \wedge \nu(x) \qquad (x \in X),$$

where $\wedge$ denotes the lattice meet in D.

For each $d \in D$, define the *constant uncertain set*

$$d_X : X \to D, \qquad d_X(x) := d \quad (x \in X).$$

A family

$$\tau_M \subseteq \mathcal{U}_M(X)$$

is called an *uncertain topology of type M* on X if it satisfies:

(UT1) **(Constants)** For every $d \in D$, the constant uncertain set d_X belongs to τ_M.

(UT2) **(Arbitrary joins)** For every nonempty family $\{\mu_i\}_{i\in I} \subseteq \tau_M$,

$$\bigvee_{i\in I} \mu_i \in \tau_M.$$

(UT3) **(Finite meets)** For all $\mu, \nu \in \tau_M$,

$$\mu \wedge_M \nu \in \tau_M.$$

The pair

$$(X, \tau_M)$$

is called an *uncertain topological space*, and each $\mu \in \tau_M$ is called an *uncertain open set* (of type M).

Definition 4.3.4 (Uncertain Continuity)**.** Let (X, τ_X) and (Y, τ_Y) be uncertain topological spaces of the same model M. A map

$$f : X \to Y$$

is called *uncertainly continuous* if, for every $\nu \in \tau_Y$, the pullback

$$f^{\leftarrow}(\nu) := \nu \circ f : X \to D$$

belongs to τ_X; that is,

$$\nu \in \tau_Y \quad \Longrightarrow \quad \nu \circ f \in \tau_X.$$

Remark 4.3.5. If M is the fuzzy model, then

$$D = \mathrm{Dom}(M) = [0, 1]$$

with the usual order. In that case,

$$\bigvee_{i\in I} \mu_i$$

is the ordinary pointwise supremum and

$$\mu \wedge_M \nu$$

is the ordinary pointwise minimum. Hence Definition 4.3.3 reduces to the standard notion of fuzzy topology.

Theorem 4.3.6 (Well-definedness of uncertain topology)**.** *Let $X \neq \varnothing$, let M be an uncertain model with degree-domain*

$$D = \mathrm{Dom}(M) \subseteq [0, 1]^k,$$

and assume that $(D, \leq_M)$ is a complete lattice. Then the notion of uncertain topology in Definition 4.3.3 is well-defined. More precisely:

(a) *Every element of $\mathcal{U}_M(X)$ is a well-defined uncertain set of type M on X.*

(b) *For every $d \in D$, the constant map $d_X : X \to D$ is a well-defined element of $\mathcal{U}_M(X)$.*

(c) *For every nonempty family $\{\mu_i\}_{i\in I} \subseteq \mathcal{U}_M(X)$, the pointwise join*

$$\bigvee_{i\in I} \mu_i$$

is a well-defined element of $\mathcal{U}_M(X)$.

(d) *For all* $\mu, \nu \in \mathcal{U}_M(X)$*, the pointwise meet*

$$\mu \wedge_M \nu$$

is a well-defined element of $\mathcal{U}_M(X)$*.*

(e) *Consequently, the axioms* (UT1)–(UT3) *are meaningful, and Definition 4.3.3 determines a mathematically valid class of structures.*

Proof. (a) By definition,

$$\mathcal{U}_M(X) = \{\mu : X \to D\}.$$

Since X is a set and $D = \mathrm{Dom}(M)$ is the degree-domain of the uncertain model M, every $\mu \in \mathcal{U}_M(X)$ is precisely an uncertain set of type M on X. Thus $\mathcal{U}_M(X)$ is well-defined.

(b) Fix $d \in D$. Define

$$d_X : X \to D, \qquad d_X(x) := d \quad (x \in X).$$

Because $d \in D$, one has $d_X(x) \in D$ for every $x \in X$. Hence

$$d_X \in \mathcal{U}_M(X).$$

Therefore every constant uncertain set is well-defined.

(c) Let $\{\mu_i\}_{i \in I} \subseteq \mathcal{U}_M(X)$ be a nonempty family. Fix $x \in X$. Since each $\mu_i : X \to D$, the set

$$\{\mu_i(x) : i \in I\}$$

is a nonempty subset of D. Because D is a complete lattice, the join

$$\bigvee_{i \in I} \mu_i(x)$$

exists in D. Therefore the assignment

$$x \longmapsto \bigvee_{i \in I} \mu_i(x)$$

defines a map

$$\bigvee_{i \in I} \mu_i : X \to D.$$

Hence

$$\bigvee_{i \in I} \mu_i \in \mathcal{U}_M(X).$$

So arbitrary pointwise joins are well-defined.

(d) Let $\mu, \nu \in \mathcal{U}_M(X)$. For each $x \in X$, one has

$$\mu(x) \in D \qquad \text{and} \qquad \nu(x) \in D.$$

Since D is a lattice, the meet

$$\mu(x) \wedge \nu(x)$$

exists in D. Therefore the assignment

$$x \longmapsto \mu(x) \wedge \nu(x)$$

defines a map

$$\mu \wedge_M \nu : X \to D.$$

Thus

$$\mu \wedge_M \nu \in \mathcal{U}_M(X).$$

So finite pointwise meets are well-defined.

(e) By parts (b)–(d), each clause in Definition 4.3.3 refers to well-defined members of $\mathcal{U}_M(X)$. Hence the conditions (UT1)–(UT3) are meaningful, and the concept of uncertain topology is mathematically well-defined. ☐

Theorem 4.3.7 (Well-definedness of uncertain continuity). *Let (X, τ_X) and (Y, τ_Y) be uncertain topological spaces of the same model M, and let*

$$f : X \to Y$$

be a function. Then, for every $\nu \in \tau_Y$, the pullback

$$f^{\leftarrow}(\nu) = \nu \circ f$$

is a well-defined uncertain set of type M on X. Consequently, Definition 4.3.4 is mathematically well-defined.

Proof. Let $\nu \in \tau_Y$. Since $\tau_Y \subseteq \mathcal{U}_M(Y)$, one has

$$\nu : Y \to D.$$

Because $f : X \to Y$, the composition

$$\nu \circ f : X \to D$$

is well-defined. Hence

$$\nu \circ f \in \mathcal{U}_M(X),$$

that is, $f^{\leftarrow}(\nu)$ is an uncertain set of type M on X. Therefore the condition

$$\nu \in \tau_Y \Longrightarrow \nu \circ f \in \tau_X$$

is meaningful, and uncertain continuity is well-defined. ☐

The catalogue of representative uncertainty-topology families by the dimension k of the degree-domain is presented in Table 4.4.

Table 4.4: A catalogue of representative uncertainty-topology families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-topology family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy Topology:* $\mu_M : \tau_M \to [0,1]$, where τ_M denotes a family of fuzzy open sets on X.
2	*Intuitionistic Fuzzy Topology [346–348]:* $\mu_M : \tau_M \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic Topology [349, 350]:* $\mu_M : \tau_M \to [0,1]^3$ (e.g., (T, I, F)).
k	*k-component uncertainty topology:* $\mu_M : \tau_M \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

Besides the above, related concepts such as soft topology [351, 352], rough topology [353, 354], Alexandroff Topology [355, 356], Bitopological Space [357, 358], Proximity Space [359, 360], and hypertopology [361, 362] are also known.

4.4 Uncertain Lattice

A lattice is a partially ordered set where any two elements have a join and meet, modeling combination and intersection [363]. A fuzzy lattice equips elements with graded order or membership, defining join/meet compatibility via degrees, often using min or t-norm [364–366].

Definition 4.4.1 (Fuzzy lattice via a fuzzy order relation)**.** Let $L = (L, \wedge, \vee, \leq)$ be a (crisp) lattice. A *fuzzy lattice* is a pair $\langle L, \lambda \rangle$ such that $\lambda : L \times L \to [0, 1]$ is a fuzzy relation satisfying

$$\lambda(x, y) = 1 \quad \Longleftrightarrow \quad x \leq y \qquad (x, y \in L).$$

Thus $\lambda(x, y)$ may be interpreted as the degree to which x is below y, while it coincides with the crisp order at degree 1.

Definition 4.4.2 (Inclusion measure producing a fuzzy lattice)**.** Let L be a (complete) lattice with least element O. A function $r : L \times L \to [0, 1]$ is called an *inclusion measure* if for all $u, w, x, y \in L$:

(C1) $r(x, O) = 0$ whenever $x \neq O$;

(C2) $r(x, x) = 1$ for every $x \in L$;

(C3) $u \leq w \;\Rightarrow\; r(x, u) \leq r(x, w)$ (*consistency*);

(C4) $x \wedge y < x \;\Rightarrow\; r(x, y) < 1$.

(For non-complete lattices, the condition (C0) is dropped.)

To extend fuzzy lattices to a general uncertain-set framework, we regard a lattice as an algebra with two binary operations $\wedge$ and $\vee$, and we compare uncertainty degrees through a score map. Since a general degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$ need not carry a canonical order or a canonical binary conjunction/disjunction, we explicitly fix aggregation operators for meet and join.

Definition 4.4.3 (Uncertain Lattice)**.** Let

$$(L, \wedge, \vee)$$

be a (crisp) lattice, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Let

$$\mu_M : L \to \mathrm{Dom}(M)$$

be an uncertainty-degree function, so that

$$\mathcal{U}_M = (L, \mu_M)$$

is an uncertain set of type M on L.

Assume that the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0,1];$$
- a binary aggregation operator for meet
$$\Gamma_\wedge : [0,1]^2 \to [0,1];$$
- a binary aggregation operator for join
$$\Gamma_\vee : [0,1]^2 \to [0,1].$$

Then $\mathcal{U}_M = (L, \mu_M)$ is called an *uncertain lattice of type M* if, for all $x, y \in L$,

$$S_M\big(\mu_M(x \wedge y)\big) \;\geq\; \Gamma_\wedge\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big),$$

and

$$S_M\big(\mu_M(x \vee y)\big) \;\geq\; \Gamma_\vee\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big).$$

Remark 4.4.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_\wedge(a,b) = \min\{a,b\}, \qquad \Gamma_\vee(a,b) = \min\{a,b\},$$

then Definition 4.4.3 becomes the usual min-type closure condition

$$\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}, \qquad \mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}.$$

Thus uncertain lattices recover the standard fuzzy-lattice-style membership propagation as a special case.

Theorem 4.4.5 (Well-definedness of uncertain lattices)**.** *Let*

$$(L, \wedge, \vee)$$

be a lattice, let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, and let

$$\mu_M : L \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_\wedge, \Gamma_\vee : [0,1]^2 \to [0,1]$$

be given.

Then:

(a) (L, μ_M) *is a well-defined uncertain set of type M on L.*

(b) *For every $x, y \in L$, the values*

$$S_M\big(\mu_M(x \wedge y)\big) \qquad \text{and} \qquad S_M\big(\mu_M(x \vee y)\big)$$

are well-defined elements of $[0,1]$.

(c) *For every $x, y \in L$, the values*

$$\Gamma_\wedge\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \qquad \text{and} \qquad \Gamma_\vee\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

are well-defined elements of $[0,1]$.

(d) *Consequently, the defining inequalities in Definition 4.4.3 are meaningful for every pair $x, y \in L$, so Definition 4.4.3 determines a mathematically well-defined class of structures.*

Proof. (a) By the definition of an uncertain set of type M, any pair

$$(L, \mu_M) \qquad \text{with} \qquad \mu_M : L \to \mathrm{Dom}(M)$$

is an uncertain set of type M on L. Hence (L, μ_M) is well-defined.

(b) Fix $x, y \in L$. Since $(L, \wedge, \vee)$ is a lattice, both

$$x \wedge y \in L \qquad \text{and} \qquad x \vee y \in L$$

exist and belong to L. Because $\mu_M : L \to \mathrm{Dom}(M)$, it follows that

$$\mu_M(x \wedge y) \in \mathrm{Dom}(M) \qquad \text{and} \qquad \mu_M(x \vee y) \in \mathrm{Dom}(M).$$

Applying the score map $S_M : \mathrm{Dom}(M) \to [0, 1]$, we obtain

$$S_M\big(\mu_M(x \wedge y)\big) \in [0, 1] \qquad \text{and} \qquad S_M\big(\mu_M(x \vee y)\big) \in [0, 1].$$

Thus the left-hand sides of the defining inequalities are well-defined.

(c) Since $x, y \in L$ and $\mu_M : L \to \mathrm{Dom}(M)$, one has

$$\mu_M(x), \mu_M(y) \in \mathrm{Dom}(M).$$

Applying S_M, we get

$$S_M(\mu_M(x)),\ S_M(\mu_M(y)) \in [0, 1].$$

Hence

$$\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big) \in [0, 1]^2.$$

Because

$$\Gamma_\wedge : [0, 1]^2 \to [0, 1] \qquad \text{and} \qquad \Gamma_\vee : [0, 1]^2 \to [0, 1],$$

it follows that

$$\Gamma_\wedge\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big) \in [0, 1]$$

and

$$\Gamma_\vee\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big) \in [0, 1].$$

Thus the right-hand sides are well-defined.

(d) By parts (b) and (c), for every $x, y \in L$, both sides of

$$S_M\big(\mu_M(x \wedge y)\big) \ \geq\ \Gamma_\wedge\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big)$$

and

$$S_M\big(\mu_M(x \vee y)\big) \ \geq\ \Gamma_\vee\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big)$$

are well-defined real numbers in $[0, 1]$. Therefore both inequalities are meaningful for every pair $x, y \in L$. Hence the notion of uncertain lattice in Definition 4.4.3 is mathematically well-defined. □

For reference, a catalogue of representative uncertainty-lattice families is presented in Table 4.5.

Table 4.5: A catalogue of representative uncertainty-lattice families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-lattice family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy Lattice:* $\mu_M : L \to [0,1]$, where L is a lattice.
2	*Intuitionistic Fuzzy Lattice [367–369]:* $\mu_M : L \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic Lattice [370, 371]:* $\mu_M : L \to [0,1]^3$ (e.g., (T, I, F)).
k	*k-component uncertainty lattice:* $\mu_M : L \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

In addition to uncertain lattices, related concepts such as semilattices [372], complete lattices [373, 374], bounded lattices [375, 376], distributive lattices [377, 378], residuated lattices [379, 380], rough lattices [381, 382], and soft lattices [383, 384] are also known.

4.5 Uncertain Vector

Fuzzy vector is a vector-space element modeled as a fuzzy set or singleton, assigning membership degrees to vectors and representing imprecise linear entities quantitatively formally [385–387].

Definition 4.5.1 (Fuzzy vector (fuzzy point / fuzzy singleton))**.** [385–387] Let E be a (crisp) vector space over a field $\mathbb{F}$. A *fuzzy vector* on E is a fuzzy set $\tilde{v}$ on E, i.e., a map $\mu_{\tilde{v}} : E \to [0,1]$.

A fundamental and widely used special case is a *fuzzy point* (or *fuzzy singleton*): given $v \in E$ and $\alpha \in (0,1]$, define $\tilde{v}_\alpha$ by

$$\mu_{\tilde{v}_\alpha}(x) := \begin{cases} \alpha, & x = v, \\ 0, & x \neq v. \end{cases}$$

The value α is the *grade* (height) of the fuzzy vector, and $\mathrm{supp}(\tilde{v}_\alpha) = \{v\}$.

Definition 4.5.2 (Fuzzy addition and scalar multiplication (extension principle))**.** Let $\tilde{u}, \tilde{v}$ be fuzzy vectors on E with membership functions $\mu_{\tilde{u}}, \mu_{\tilde{v}} : E \to [0,1]$. Define their *fuzzy sum* $\tilde{u} \oplus \tilde{v}$ by

$$\mu_{\tilde{u}\oplus\tilde{v}}(z) := \sup_{\substack{x,y\in E \\ x+y=z}} \min\{\mu_{\tilde{u}}(x),\, \mu_{\tilde{v}}(y)\}, \qquad z \in E.$$

For $c \in \mathbb{F}$, define *fuzzy scalar multiplication* $c \odot \tilde{u}$ by

$$\mu_{c\odot\tilde{u}}(z) := \sup_{\substack{x\in E \\ cx=z}} \mu_{\tilde{u}}(x), \qquad z \in E,$$

(with the convention $\sup \varnothing = 0$).

Let E be a vector space over a field $\mathbb{F}$, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Since $\mathrm{Dom}(M)$ need not carry a canonical arithmetic structure, we define uncertain vectors through uncertain sets and compare their degrees via a score map.

Definition 4.5.3 (Uncertain Vector)**.** An *uncertain vector of type M* on E is an uncertain set

$$\widetilde{v}_M = (E, \mu_M),$$

where

$$\mu_M : E \to \mathrm{Dom}(M)$$

is an uncertainty-degree function. Equivalently, an uncertain vector on E is an uncertain set of type M whose universe is the vector space E.

Definition 4.5.4 (Uncertain Point / Uncertain Singleton)**.** Fix $v \in E$ and a degree $d \in \mathrm{Dom}(M)$. Assume that a distinguished null degree

$$0_M \in \mathrm{Dom}(M)$$

is fixed.

The *uncertain point* (or *uncertain singleton*) at v with degree d is the uncertain vector

$$\widetilde{v}_{M,d} = (E, \mu^M_{v,d}),$$

where

$$\mu^M_{v,d}(x) := \begin{cases} d, & x = v, \\ 0_M, & x \neq v. \end{cases} \qquad (x \in E).$$

Definition 4.5.5 (Addition and Scalar Multiplication of Uncertain Vectors)**.** Let $\widetilde{u}_M = (E, \mu_M)$ and $\widetilde{v}_M = (E, \nu_M)$ be uncertain vectors of type M on E. Assume that the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0,1];$$
- a binary aggregation operator
$$\Gamma_2 : [0,1]^2 \to [0,1].$$

The *score-induced uncertain sum* of $\widetilde{u}_M$ and $\widetilde{v}_M$ is the mapping

$$\mu_M \boxplus \nu_M : E \to [0,1]$$

defined by

$$(\mu_M \boxplus \nu_M)(z) := \sup_{\substack{x,y \in E \\ x+y=z}} \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\nu_M(y))\big), \qquad z \in E.$$

For $c \in \mathbb{F}$, the *score-induced scalar multiple* of $\widetilde{u}_M$ by c is the mapping

$$(c \odot_M \mu_M) : E \to [0,1]$$

defined by

$$(c \odot_M \mu_M)(z) := \sup_{\substack{x \in E \\ cx=z}} S_M(\mu_M(x)), \qquad z \in E,$$

with the convention $\sup \varnothing = 0$.

Remark 4.5.6. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1], \qquad S_M = \mathrm{id}_{[0,1]}, \qquad \Gamma_2(a,b) = \min\{a,b\},$$

then Definition 4.5.5 reduces to the usual fuzzy-vector operations

$$\mu_{\tilde{u}\oplus\tilde{v}}(z) = \sup_{\substack{x,y\in E \\ x+y=z}} \min\{\mu_{\tilde{u}}(x), \mu_{\tilde{v}}(y)\},$$

and

$$\mu_{c\odot\tilde{u}}(z) = \sup_{\substack{x\in E \\ cx=z}} \mu_{\tilde{u}}(x).$$

Theorem 4.5.7 (Well-definedness of uncertain vectors)**.** *Let E be a vector space over $\mathbb{F}$, let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, and let*

$$\mu_M : E \to \mathrm{Dom}(M)$$

be a mapping. Then:

(a) *(E, μ_M) is a well-defined uncertain vector of type M.*

(b) *For every $v \in E$ and every $d \in \mathrm{Dom}(M)$, the uncertain point $\widetilde{v}_{M,d}$ in Definition 4.5.4 is well-defined.*

(c) *If $S_M : \mathrm{Dom}(M) \to [0,1]$ and $\Gamma_2 : [0,1]^2 \to [0,1]$ are fixed, then the score-induced sum*

$$\mu_M \boxplus \nu_M : E \to [0,1]$$

and the score-induced scalar multiple

$$c \odot_M \mu_M : E \to [0,1]$$

in Definition 4.5.5 are well-defined for all uncertain vectors (E, μ_M), (E, ν_M) and all $c \in \mathbb{F}$.

Proof. (a) By the definition of an uncertain set of type M, any pair

$$(E, \mu_M) \qquad \text{with} \qquad \mu_M : E \to \mathrm{Dom}(M)$$

is an uncertain set of type M on E. Since an uncertain vector is defined to be exactly such an uncertain set on the vector space E, the pair (E, μ_M) is a well-defined uncertain vector.

(b) Fix $v \in E$ and $d \in \mathrm{Dom}(M)$. The mapping

$$\mu^M_{v,d} : E \to \mathrm{Dom}(M)$$

is defined by

$$\mu^M_{v,d}(x) = d \quad \text{if } x = v, \qquad \mu^M_{v,d}(x) = 0_M \quad \text{if } x \neq v.$$

Since both d and 0_M belong to $\mathrm{Dom}(M)$, it follows that

$$\mu^M_{v,d}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in E.$$

Hence $\mu^M_{v,d}$ is a well-defined map from E to $\mathrm{Dom}(M)$, so $\widetilde{v}_{M,d}$ is a well-defined uncertain vector.

(c) Let (E, μ_M) and (E, ν_M) be uncertain vectors of type M, and let $z \in E$.

For every pair $(x, y) \in E \times E$ satisfying $x + y = z$, one has

$$\mu_M(x), \nu_M(y) \in \mathrm{Dom}(M),$$

hence

$$S_M(\mu_M(x)),\ S_M(\nu_M(y)) \in [0,1].$$

Therefore

$$\Gamma_2\big(S_M(\mu_M(x)),\ S_M(\nu_M(y))\big) \in [0,1].$$

So the set

$$\big\{\Gamma_2\big(S_M(\mu_M(x)),\ S_M(\nu_M(y))\big) :\ x, y \in E,\ x + y = z\big\}$$

is a subset of $[0,1]$. Since $[0,1]$ is complete with respect to suprema, the value

$$(\mu_M \boxplus \nu_M)(z) = \sup_{\substack{x,y \in E \\ x+y=z}} \Gamma_2\big(S_M(\mu_M(x)),\ S_M(\nu_M(y))\big)$$

exists and belongs to $[0,1]$. Thus $\mu_M \boxplus \nu_M : E \to [0,1]$ is well-defined.

Now let $c \in \mathbb{F}$ and $z \in E$. For every $x \in E$ such that $cx = z$, one has

$$\mu_M(x) \in \mathrm{Dom}(M), \qquad S_M(\mu_M(x)) \in [0,1].$$

Hence

$$\{S_M(\mu_M(x)) :\ x \in E,\ cx = z\} \subseteq [0,1].$$

Again, since $[0,1]$ is complete, the supremum

$$(c \odot_M \mu_M)(z) = \sup_{\substack{x \in E \\ cx=z}} S_M(\mu_M(x))$$

exists in $[0,1]$, with the convention $\sup \varnothing = 0$. Therefore $c \odot_M \mu_M : E \to [0,1]$ is well-defined.

This proves that both the score-induced addition and scalar multiplication are well-defined. □

For reference, representative uncertainty-vector-space families and their degree representations are listed in Table 4.6.

Table 4.6: Representative uncertainty-vector-space families and their degree representations.

Uncertainty-vector-space family	**Degree form**	**Typical degree-domain / representation**
Fuzzy Vector Spaces	scalar	$\mu_M : V \to [0,1]$.
Intuitionistic Fuzzy Vector Spaces [388–390]	2-component	$\mu_M : V \to [0,1]^2$ (typically (membership, non-membership)).
Hesitant Fuzzy Vector Spaces [391]	set-valued (3-component)	$\mu_M : V \to \mathcal{P}_{\mathrm{fin}}([0,1]) \setminus \{\varnothing\}$, where each vector is assigned a finite set of possible membership degrees.
Neutrosophic Vector Spaces [392–394]	3-component	$\mu_M : V \to [0,1]^3$ (typically (T, I, F)).

In addition, as related concepts of vector spaces, notions such as semantic vector spaces [395–397], Affine Spaces [398, 399], Normed Vector Space [400, 401], Inner Product Space [402, 403], Banach Space [404, 405], Topological Vector Space [406, 407], and hypervectors [408, 409] have also been studied.

4.6 Uncertain Matrices

A fuzzy matrix is a matrix with entries in $[0, 1]$, representing membership grades of a fuzzy relation between two finite sets [176, 410, 411].

Definition 4.6.1 (Fuzzy matrix)**.** [176, 410, 411] Let $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$ be finite index sets. A matrix

$$A = (a_{ij}) \in \mathbb{R}^{n\times m}$$

is called a *fuzzy matrix* if

$$a_{ij} \in [0, 1] \qquad (i \in I,\ j \in J).$$

Equivalently, A can be viewed as the *membership matrix* of a fuzzy relation $R : X \times Y \to [0, 1]$ on finite sets $X = \{x_i : i \in I\}$ and $Y = \{y_j : j \in J\}$ via

$$a_{ij} = R(x_i, y_j) \in [0, 1].$$

An uncertain matrix is an uncertainty-model-based matrix whose entries belong to a general degree-domain, equivalently representing an uncertain set on a finite Cartesian product.

Definition 4.6.2 (Uncertain Matrix)**.** Let

$$I = \{1, \ldots, n\} \qquad \text{and} \qquad J = \{1, \ldots, m\}$$

be finite index sets, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

An *uncertain matrix of type M* is a mapping

$$A_M : I \times J \to \mathrm{Dom}(M).$$

Writing

$$A_M(i, j) = a_{ij}^{(M)} \in \mathrm{Dom}(M) \qquad (i \in I,\ j \in J),$$

we denote the corresponding uncertain matrix by

$$A_M = \big(a_{ij}^{(M)}\big)_{n\times m}.$$

Equivalently, A_M is an uncertain set of type M on the finite set

$$I \times J.$$

If $X = \{x_i : i \in I\}$ and $Y = \{y_j : j \in J\}$ are finite sets, then A_M may also be viewed as the membership matrix of an uncertain relation

$$R_M : X \times Y \to \mathrm{Dom}(M)$$

via

$$a_{ij}^{(M)} = R_M(x_i, y_j).$$

Remark 4.6.3. If an admissible score map

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

is fixed, then every uncertain matrix

$$A_M = \big(a_{ij}^{(M)}\big)_{n\times m}$$

induces a score matrix

$$\overline{A}_M = \big(\overline{a}_{ij}\big)_{n\times m} \in [0,1]^{n\times m}, \qquad \overline{a}_{ij} := S_M\big(a_{ij}^{(M)}\big).$$

This score matrix gives a scalar realization of the uncertain matrix.

Remark 4.6.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1],$$

then Definition 4.6.2 reduces to the usual definition of a fuzzy matrix:

$$A = (a_{ij}) \in [0,1]^{n\times m}.$$

Thus fuzzy matrices are special cases of uncertain matrices.

Theorem 4.6.5 (Well-definedness of uncertain matrices)**.** *Let $I = \{1,\ldots,n\}$, $J = \{1,\ldots,m\}$, and let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Then every mapping

$$A_M : I \times J \to \mathrm{Dom}(M)$$

determines a unique uncertain matrix

$$A_M = \big(a_{ij}^{(M)}\big)_{n\times m}, \qquad a_{ij}^{(M)} := A_M(i,j),$$

which is equivalently a well-defined uncertain set of type M on $I \times J$.

Moreover, if a score map

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

is fixed, then the induced score matrix

$$\overline{A}_M = \big(S_M(a_{ij}^{(M)})\big)_{n\times m}$$

is a well-defined matrix in $[0,1]^{n\times m}$.

Proof. Since I and J are finite sets, their Cartesian product

$$I \times J$$

is also a well-defined finite set.

Now let

$$A_M : I \times J \to \mathrm{Dom}(M)$$

be any mapping. For each pair

$$(i, j) \in I \times J,$$

the value

$$A_M(i, j)$$

belongs to $\mathrm{Dom}(M)$ by definition of the codomain. Hence, if we set

$$a_{ij}^{(M)} := A_M(i, j),$$

then every entry $a_{ij}^{(M)}$ is a well-defined element of $\mathrm{Dom}(M)$. Therefore

$$A_M = \big(a_{ij}^{(M)}\big)_{n\times m}$$

is a well-defined $n \times m$ matrix with entries in $\mathrm{Dom}(M)$.

On the other hand, an uncertain set of type M on a base set U is precisely a mapping

$$\mu : U \to \mathrm{Dom}(M).$$

Taking

$$U = I \times J$$

and

$$\mu = A_M,$$

it follows immediately that A_M is also a well-defined uncertain set of type M on $I \times J$. This establishes the equivalence between uncertain matrices and uncertain sets on $I \times J$.

Finally, suppose that a score map

$$S_M : \mathrm{Dom}(M) \to [0, 1]$$

is fixed. Since each entry

$$a_{ij}^{(M)} \in \mathrm{Dom}(M),$$

the value

$$S_M(a_{ij}^{(M)})$$

is a well-defined element of $[0, 1]$. Hence

$$\overline{A}_M = \big(S_M(a_{ij}^{(M)})\big)_{n\times m}$$

is a well-defined matrix in $[0, 1]^{n\times m}$.

Therefore the notion of uncertain matrix is mathematically well-defined, and its score realization is also well-defined whenever a score map is given. □

A catalogue of representative uncertainty-matrix families classified by the dimension k of the degree-domain is presented in Table 4.7.

Table 4.7: A catalogue of representative uncertainty-matrix families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-matrix family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy Matrix:* $\mu_M : I \times J \to [0,1]$.
2	*Intuitionistic Fuzzy Matrix [412, 413]:* $\mu_M : I \times J \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic Matrix [414–417]:* $\mu_M : I \times J \to [0,1]^3$ (e.g., (T, I, F)). (cf. Hesitant Fuzzy Matrices [418–420], Picture Fuzzy Matrices [421, 422])
k	*k-component uncertainty matrix:* $\mu_M : I \times J \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

Other than uncertain matrices, several related concepts are also known, such as block matrices [423, 424], sparse matrices [425, 426], band matrices [427], Toeplitz matrices [428], circulant matrices [429], and triangular matrices [430].

4.7 Uncertain Semigroup

Fuzzy semigroup is a semigroup equipped with a membership function preserving multiplication through a minimum condition, modeling algebraic closure and graded participation simultaneously under uncertainty [431, 432].

Definition 4.7.1 (Fuzzy Semigroup)**.** [431, 432] Let $(S, \cdot)$ be a semigroup. A *fuzzy semigroup* on S is a fuzzy set

$$\mu : S \to [0,1]$$

satisfying

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\} \qquad \text{for all } x, y \in S.$$

Equivalently, μ is a fuzzy subsemigroup of S. The pair (S, μ) is also called a fuzzy semigroup.

An uncertain semigroup is a semigroup equipped with an uncertainty-degree function from the underlying set into a general degree-domain, such that the semigroup product is compatible with the uncertainty degrees through a score-based aggregation rule.

Definition 4.7.2 (Uncertain Semigroup)**.** Let $(S, \cdot)$ be a semigroup, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

An *uncertain semigroup of type M* on S is an uncertain set

$$\mathcal{S}_M = (S, \mu_M),$$

where

$$\mu_M : S \to \mathrm{Dom}(M),$$

for which the following data are fixed:

- an admissible score map

$$S_M : \mathrm{Dom}(M) \to [0,1];$$

- a binary aggregation operator
$$\Gamma_2 : [0,1]^2 \to [0,1].$$

These data are required to satisfy

$$S_M\big(\mu_M(xy)\big) \ge \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \qquad \text{for all } x, y \in S.$$

Equivalently, the uncertainty degree of the product xy is at least the aggregated score of the uncertainty degrees of x and y.

The pair (S, μ_M) is also called an *uncertain subsemigroup* of type M.

Remark 4.7.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes
$$\Gamma_2(a,b) = \min\{a,b\},$$
then Definition 4.7.2 reduces to the usual fuzzy semigroup condition

$$\mu(xy) \ge \min\{\mu(x), \mu(y)\} \qquad \text{for all } x, y \in S.$$

Thus fuzzy semigroups are special cases of uncertain semigroups.

Theorem 4.7.4 (Well-definedness of uncertain semigroups)**.** *Let $(S,\cdot)$ be a semigroup, let M be an uncertain model with degree-domain*
$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$
and let
$$\mu_M : S \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_2 : [0,1]^2 \to [0,1]$$
be given. Then:

(a) (S, μ_M) *is a well-defined uncertain set of type M on S.*

(b) *For every $x, y \in S$, the value*
$$S_M\big(\mu_M(xy)\big)$$
is a well-defined element of $[0,1]$.

(c) *For every $x, y \in S$, the value*
$$\Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$
is a well-defined element of $[0,1]$.

(d) *Consequently, the compatibility inequality in Definition 4.7.2*
$$S_M\big(\mu_M(xy)\big) \ge \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$
is meaningful for all $x, y \in S$.

Hence Definition 4.7.2 determines a mathematically well-defined class of structures.

Proof. (a) By the definition of an uncertain set of type M, any pair

$$(S, \mu_M) \qquad \text{with} \qquad \mu_M : S \to \mathrm{Dom}(M)$$

is an uncertain set of type M on S. Since μ_M is such a map, (S, μ_M) is well-defined.

(b) Let $x, y \in S$. Because $(S, \cdot)$ is a semigroup, the product

$$xy \in S$$

is well-defined. Since

$$\mu_M : S \to \mathrm{Dom}(M),$$

it follows that

$$\mu_M(xy) \in \mathrm{Dom}(M).$$

Applying the score map

$$S_M : \mathrm{Dom}(M) \to [0, 1],$$

we obtain

$$S_M\big(\mu_M(xy)\big) \in [0, 1].$$

Hence this quantity is well-defined.

(c) Again let $x, y \in S$. Since

$$\mu_M(x), \mu_M(y) \in \mathrm{Dom}(M),$$

we have

$$S_M(\mu_M(x)) \in [0, 1] \qquad \text{and} \qquad S_M(\mu_M(y)) \in [0, 1].$$

Therefore

$$\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \in [0, 1]^2.$$

Because

$$\Gamma_2 : [0, 1]^2 \to [0, 1],$$

it follows that

$$\Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \in [0, 1].$$

Hence this quantity is also well-defined.

(d) By (b) and (c), both sides of

$$S_M\big(\mu_M(xy)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

are well-defined real numbers in $[0, 1]$. Therefore the inequality is meaningful for every pair $x, y \in S$.

Hence all objects and conditions appearing in Definition 4.7.2 are mathematically meaningful, and the notion of uncertain semigroup is well-defined. □

For reference, representative uncertainty-semigroup families and their degree representations are presented in Table 4.8.

Table 4.8: Representative uncertainty-semigroup families and their degree representations.

Uncertainty-semigroup family	**Degree form**	**Typical degree-domain / representation**
Fuzzy Semigroup	scalar	$\mu_M : S \to [0,1]$, where S is a semigroup.
Intuitionistic Fuzzy Semigroup [304, 433]	2-component	$\mu_M : S \to [0,1]^2$, typically (membership, non-membership).
Hesitant Fuzzy Semigroup [434, 435]	set-valued	$\mu_M : S \to \mathcal{P}_{\text{fin}}([0,1]) \setminus \{\varnothing\}$, where each element is assigned a finite set of possible membership degrees.
Picture Fuzzy Semigroup	3-component	$\mu_M : S \to [0,1]^3$, typically (positive, neutral, negative), with a model-specific constraint such as $$p(x) + n(x) + q(x) \le 1.$$
Neutrosophic Semigroup [436–438]	3-component	$\mu_M : S \to [0,1]^3$, typically (T, I, F).

As concepts related to uncertain semigroups, several other extensions are also known, including soft semigroups [439, 440], rough semigroups [441], inverse semigroups [442], regular semigroups [443], completely regular semigroups [444], commutative semigroups [445], and ordered semigroups [446].

4.8 Uncertain Ring

Fuzzy ring is a ring equipped with a membership function compatible with subtraction and multiplication, modeling graded algebraic belonging while preserving subring-type structure under uncertainty [447–450].

Definition 4.8.1 (Fuzzy Ring)**.** [447, 448] Let $(R, +, \cdot)$ be a ring. A *fuzzy ring* on R is a fuzzy set

$$\mu : R \to [0,1]$$

such that, for all $x, y \in R$,

$$\mu(x - y) \ge \min\{\mu(x), \mu(y)\} \qquad \text{and} \qquad \mu(xy) \ge \min\{\mu(x), \mu(y)\}.$$

Equivalently, μ is a fuzzy subring of R. The pair (R, μ) is also called a fuzzy ring.

An uncertain ring is a ring equipped with an uncertainty-degree function into a general degree-domain, such that subtraction and multiplication are compatible with the uncertainty structure through score-based aggregation.

Definition 4.8.2 (Uncertain Ring)**.** Let $(R, +, \cdot)$ be a ring, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

An *uncertain ring of type M* on R is an uncertain set

$$\mathcal{R}_M = (R, \mu_M),$$

where

$$\mu_M : R \to \mathrm{Dom}(M),$$

for which the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0,1];$$
- a binary aggregation operator
$$\Gamma_2 : [0,1]^2 \to [0,1].$$

These data are required to satisfy

$$S_M\big(\mu_M(x-y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \qquad \text{for all } x, y \in R,$$

and

$$S_M\big(\mu_M(xy)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \qquad \text{for all } x, y \in R.$$

Equivalently, the uncertainty degree of both the ring difference $x - y$ and the product xy is at least the aggregated score of the uncertainty degrees of x and y.

Remark 4.8.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_2(a,b) = \min\{a,b\},$$

then Definition 4.8.2 reduces to the usual fuzzy ring condition

$$\mu(x-y) \geq \min\{\mu(x), \mu(y)\} \qquad \text{and} \qquad \mu(xy) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in R$. Thus fuzzy rings are special cases of uncertain rings.

Theorem 4.8.4 (Well-definedness of uncertain rings)**.** *Let $(R, +, \cdot)$ be a ring, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$\mu_M : R \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_2 : [0,1]^2 \to [0,1]$$

be given. Then:

(a) *(R, μ_M) is a well-defined uncertain set of type M on R.*

(b) *For every $x, y \in R$, the quantities*

$$S_M\big(\mu_M(x-y)\big) \qquad \text{and} \qquad S_M\big(\mu_M(xy)\big)$$

are well-defined elements of $[0,1]$.

(c) *For every $x, y \in R$, the quantity*

$$\Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

is a well-defined element of $[0,1]$.

(d) *Consequently, both defining inequalities in Definition 4.8.2,*

$$S_M\big(\mu_M(x-y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

and

$$S_M\big(\mu_M(xy)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big),$$

are meaningful for all $x, y \in R$.

Hence Definition 4.8.2 determines a mathematically well-defined class of structures.

Proof. (a) By the definition of an uncertain set of type M, any pair

$$(R, \mu_M) \qquad \text{with} \qquad \mu_M : R \to \mathrm{Dom}(M)$$

is an uncertain set of type M on R. Hence (R, μ_M) is well-defined.

(b) Let $x, y \in R$. Since $(R, +, \cdot)$ is a ring, the difference

$$x - y \in R$$

and the product

$$xy \in R$$

are both well-defined. Because

$$\mu_M : R \to \mathrm{Dom}(M),$$

it follows that

$$\mu_M(x-y) \in \mathrm{Dom}(M) \qquad \text{and} \qquad \mu_M(xy) \in \mathrm{Dom}(M).$$

Applying the score map

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

we obtain

$$S_M\big(\mu_M(x-y)\big) \in [0,1] \qquad \text{and} \qquad S_M\big(\mu_M(xy)\big) \in [0,1].$$

Thus both quantities are well-defined.

(c) Again let $x, y \in R$. Since

$$\mu_M(x), \mu_M(y) \in \mathrm{Dom}(M),$$

we have

$$S_M(\mu_M(x)),\, S_M(\mu_M(y)) \in [0,1].$$

Hence

$$\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \in [0,1]^2.$$

Because

$$\Gamma_2 : [0,1]^2 \to [0,1],$$

it follows that

$$\Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \in [0,1].$$

So this quantity is well-defined.

(d) By parts (b) and (c), all terms appearing in

$$S_M\big(\mu_M(x-y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

and

$$S_M\big(\mu_M(xy)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

are well-defined real numbers in $[0,1]$. Therefore both inequalities are meaningful for every $x, y \in R$.

Hence all objects and conditions appearing in Definition 4.8.2 are mathematically meaningful, and the notion of uncertain ring is well-defined. □

For reference, representative uncertainty-ring families and their degree representations are summarized in Table 4.9.

Table 4.9: Representative uncertainty-ring families and their degree representations.

Uncertainty-ring family	**Degree form**	**Typical degree-domain / representation**
Intuitionistic Fuzzy Ring [451, 452]	2-component	$\mu_M : R \to [0,1]^2$, typically (membership, non-membership).
Bipolar Fuzzy Ring [453, 454]	bipolar	Usually represented by two functions $$\mu_M^+ : R \to [0,1], \qquad \mu_M^- : R \to [-1,0],$$ or equivalently by $\mu_M : R \to [0,1] \times [-1,0]$, encoding positive and negative membership degrees.
Neutrosophic Ring [455–457]	3-component	$\mu_M : R \to [0,1]^3$, typically (T, I, F).
Plithogenic Ring [458–460]	attribute-based / multi-component	Typically described through appurtenance degrees together with contradiction information relative to attribute values; e.g., $$\mu_M : R \times Pv \to [0,1]^s$$ with an associated contradiction map on the attribute-value set.

Related concepts such as Near-Ring [461], Soft Ring [462, 463], Rough Ring [464, 465], HyperRing [283, 466, 467], and SuperHyperRing [334] are also known.

4.9 Uncertain Semiring

Fuzzy semiring is a semiring equipped with a membership function compatible with addition and multiplication, expressing graded subsemiring structure and algebraic closure under uncertainty [468–470].

Definition 4.9.1 (Fuzzy Semiring)**.** [468–470] Let $(S, +, \cdot)$ be a semiring. A *fuzzy semiring* on S is a fuzzy set

$$\mu : S \to [0,1]$$

such that, for all $x, y \in S$,

$$\mu(x+y) \geq \min\{\mu(x), \mu(y)\} \qquad \text{and} \qquad \mu(xy) \geq \min\{\mu(x), \mu(y)\}.$$

Equivalently, μ is a fuzzy subsemiring of S. The pair (S, μ) is also called a fuzzy semiring.

An uncertain semiring is a semiring equipped with an uncertainty-degree function into a general degree-domain, such that both addition and multiplication are compatible with the uncertainty structure through score-based aggregation.

Definition 4.9.2 (Uncertain Semiring)**.** Let $(S, +, \cdot)$ be a semiring, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

An *uncertain semiring of type M* on S is an uncertain set

$$\mathcal{S}_M = (S, \mu_M),$$

where

$$\mu_M : S \to \mathrm{Dom}(M),$$

for which the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0, 1];$$
- a binary aggregation operator
$$\Gamma_2 : [0, 1]^2 \to [0, 1].$$

These data are required to satisfy

$$S_M\big(\mu_M(x + y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \qquad \text{for all } x, y \in S,$$

and

$$S_M\big(\mu_M(xy)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \qquad \text{for all } x, y \in S.$$

Equivalently, the uncertainty degree of both the sum $x + y$ and the product xy is at least the aggregated score of the uncertainty degrees of x and y.

Remark 4.9.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0, 1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_2(a, b) = \min\{a, b\},$$

then Definition 4.9.2 reduces to the usual fuzzy semiring condition

$$\mu(x + y) \geq \min\{\mu(x), \mu(y)\} \qquad \text{and} \qquad \mu(xy) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in S$. Thus fuzzy semirings are special cases of uncertain semirings.

Theorem 4.9.4 (Well-definedness of uncertain semirings)**.** *Let $(S, +, \cdot)$ be a semiring, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

and let

$$\mu_M : S \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0, 1], \qquad \Gamma_2 : [0, 1]^2 \to [0, 1]$$

be given. Then:

(a) (S, μ_M) *is a well-defined uncertain set of type* M *on* S*.*

(b) *For every* $x, y \in S$*, the quantities*

$$S_M\big(\mu_M(x+y)\big) \qquad \text{and} \qquad S_M\big(\mu_M(xy)\big)$$

are well-defined elements of $[0,1]$*.*

(c) *For every* $x, y \in S$*, the quantity*

$$\Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

is a well-defined element of $[0,1]$*.*

(d) *Consequently, both defining inequalities in Definition 4.9.2,*

$$S_M\big(\mu_M(x+y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

and

$$S_M\big(\mu_M(xy)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big),$$

are meaningful for all $x, y \in S$*.*

Hence Definition 4.9.2 determines a mathematically well-defined class of structures.

Proof. (a) By the definition of an uncertain set of type M, any pair

$$(S, \mu_M) \qquad \text{with} \qquad \mu_M : S \to \mathrm{Dom}(M)$$

is an uncertain set of type M on S. Hence (S, μ_M) is well-defined.

(b) Let $x, y \in S$. Since $(S, +, \cdot)$ is a semiring, both

$$x + y \in S \qquad \text{and} \qquad xy \in S$$

are well-defined. Because

$$\mu_M : S \to \mathrm{Dom}(M),$$

it follows that

$$\mu_M(x+y) \in \mathrm{Dom}(M) \qquad \text{and} \qquad \mu_M(xy) \in \mathrm{Dom}(M).$$

Applying the score map

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

we obtain

$$S_M\big(\mu_M(x+y)\big) \in [0,1] \qquad \text{and} \qquad S_M\big(\mu_M(xy)\big) \in [0,1].$$

Thus both quantities are well-defined.

(c) Again let $x, y \in S$. Since

$$\mu_M(x), \mu_M(y) \in \mathrm{Dom}(M),$$

we have

$$S_M(\mu_M(x)),\, S_M(\mu_M(y)) \in [0,1].$$

Hence

$$\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \in [0,1]^2.$$

Because

$$\Gamma_2 : [0,1]^2 \to [0,1],$$

it follows that

$$\Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \in [0,1].$$

So this quantity is well-defined.

(d) By parts (b) and (c), all terms appearing in

$$S_M\big(\mu_M(x+y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

and

$$S_M\big(\mu_M(xy)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

are well-defined real numbers in $[0,1]$. Therefore both inequalities are meaningful for every $x, y \in S$.

Hence all objects and conditions appearing in Definition 4.9.2 are mathematically meaningful, and the notion of uncertain semiring is well-defined. □

For reference, representative uncertainty-semiring families and their degree representations are presented in Table 4.10.

Table 4.10: Representative uncertainty-semiring families and their degree representations.

Uncertainty-semiring family	**Degree form**	**Typical degree-domain / representation**
Fuzzy Semiring [469]	scalar	$\mu_M : S \to [0,1]$, where S is a semiring.
Intuitionistic Fuzzy Semiring [471, 472]	2-component	$\mu_M : S \to [0,1]^2$, typically (membership, non-membership).
Hesitant Fuzzy Semiring	set-valued	$\mu_M : S \to \mathcal{P}_{\text{fin}}([0,1]) \setminus \{\varnothing\}$, where each element is assigned a finite set of possible membership degrees.
Neutrosophic Semiring [473–476]	3-component	$\mu_M : S \to [0,1]^3$, typically (T, I, F).

As concepts other than uncertain semirings, several related notions are also known, including soft semirings [477], rough semirings [478, 479], hemirings [480, 481], commutative semirings [482], idempotent semirings [483], selective semirings [484], and complete semirings [485].

4.10 Uncertain Boolean Algebra

Boolean algebra is an algebraic structure with binary operations, unary complement, and distinguished bounds, modeling logical reasoning, set operations, and distributive complemented order relations abstractly [486–490]. Fuzzy Boolean algebra is a complemented distributive fuzzy lattice, extending classical Boolean structure through graded order or membership while retaining algebraic negation and lattice behavior [491, 492].

Definition 4.10.1 (Fuzzy Boolean Algebra)**.** [491, 493] Let (L, R) be a bounded fuzzy lattice with least element 0 and greatest element 1. Then (L, R) is called a *fuzzy Boolean algebra* if

1. (L, R) is distributive; and
2. for every $a \in L$, there exists an element $a' \in L$ such that

$$a \wedge a' = 0, \qquad a \vee a' = 1.$$

Equivalently, a fuzzy Boolean algebra is a complemented distributive fuzzy lattice.

An uncertain Boolean algebra is a Boolean algebra equipped with an uncertainty-degree function into a general degree-domain, such that meet, join, and complement are compatible with the uncertainty structure through score-based conditions.

Definition 4.10.2 (Uncertain Boolean Algebra)**.** Let

$$\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$$

be a Boolean algebra, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

An *uncertain Boolean algebra of type M* on $\mathcal{B}$ is an uncertain set

$$\mathcal{B}_M = (B, \mu_M),$$

where

$$\mu_M : B \to \mathrm{Dom}(M),$$

for which the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0, 1];$$
- a binary aggregation operator
$$\Gamma_2 : [0, 1]^2 \to [0, 1].$$

These data are required to satisfy, for all $x, y \in B$,

$$S_M\big(\mu_M(x \wedge y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big),$$

$$S_M\big(\mu_M(x \vee y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big),$$

and

$$S_M\big(\mu_M(x')\big) \geq S_M\big(\mu_M(x)\big).$$

Equivalently, the uncertainty degree is compatible with the Boolean operations $\wedge$, $\vee$, and complement.

Remark 4.10.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_2(a,b) = \min\{a,b\},$$

then Definition 4.10.2 reduces to the membership-based fuzzy Boolean algebra conditions

$$\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\},$$

$$\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\},$$

and

$$\mu(x') \geq \mu(x) \qquad (x, y \in B).$$

Thus membership-type fuzzy Boolean algebras are special cases of uncertain Boolean algebras.

Theorem 4.10.4 (Well-definedness of uncertain Boolean algebras)**.** *Let*

$$\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$$

be a Boolean algebra, let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$\mu_M : B \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_2 : [0,1]^2 \to [0,1]$$

be given. Then:

1. *(B, μ_M) is a well-defined uncertain set of type M on B;*
2. *for all $x, y \in B$, the quantities*

   $$S_M\big(\mu_M(x \wedge y)\big), \qquad S_M\big(\mu_M(x \vee y)\big), \qquad S_M\big(\mu_M(x')\big)$$

   are well-defined elements of $[0,1]$;
3. *for all $x, y \in B$, the quantity*

   $$\Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

   is a well-defined element of $[0,1]$;
4. *consequently, all defining inequalities in Definition 4.10.2 are meaningful for every $x, y \in B$.*

Hence Definition 4.10.2 determines a mathematically well-defined class of structures.

Proof. Since

$$\mu_M : B \to \mathrm{Dom}(M),$$

the pair

$$(B, \mu_M)$$

is, by definition, an uncertain set of type M on B. Hence (B, μ_M) is well-defined.

Now let $x, y \in B$. Because

$$\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$$

is a Boolean algebra, the elements

$$x \wedge y, \qquad x \vee y, \qquad x'$$

all belong to B. Since

$$\mu_M : B \to \mathrm{Dom}(M),$$

it follows that

$$\mu_M(x \wedge y), \qquad \mu_M(x \vee y), \qquad \mu_M(x')$$

all belong to $\mathrm{Dom}(M)$. Applying the score map

$$S_M : \mathrm{Dom}(M) \to [0, 1],$$

we obtain

$$S_M\big(\mu_M(x \wedge y)\big) \in [0, 1], \qquad S_M\big(\mu_M(x \vee y)\big) \in [0, 1], \qquad S_M\big(\mu_M(x')\big) \in [0, 1].$$

Thus the left-hand sides of the defining inequalities are well-defined.

Also, since

$$\mu_M(x), \mu_M(y) \in \mathrm{Dom}(M),$$

one has

$$S_M(\mu_M(x)),\ S_M(\mu_M(y)) \in [0, 1].$$

Hence

$$\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big) \in [0, 1]^2.$$

Because

$$\Gamma_2 : [0, 1]^2 \to [0, 1],$$

it follows that

$$\Gamma_2\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big) \in [0, 1].$$

So the right-hand side appearing in the meet and join conditions is also well-defined.

Therefore every term in

$$S_M\big(\mu_M(x \wedge y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big),$$

$$S_M\big(\mu_M(x \vee y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big),$$

and

$$S_M\big(\mu_M(x')\big) \geq S_M\big(\mu_M(x)\big)$$

is a well-defined real number in $[0, 1]$. Hence all three inequalities are meaningful for every $x, y \in B$.

Thus the notion of uncertain Boolean algebra is mathematically well-defined. □

Proposition 4.10.5 (Complement-score invariance)**.** *Let (B, μ_M) be an uncertain Boolean algebra of type M. Then, for every $x \in B$,*

$$S_M\big(\mu_M(x')\big) = S_M\big(\mu_M(x)\big).$$

Proof. By Definition 4.10.2,

$$S_M\big(\mu_M(x')\big) \geq S_M\big(\mu_M(x)\big) \qquad \text{for all } x \in B.$$

Apply the same condition to x'. Since Boolean complementation is involutive, one has

$$(x')' = x.$$

Therefore

$$S_M\big(\mu_M((x')')\big) \geq S_M\big(\mu_M(x')\big),$$

that is,

$$S_M\big(\mu_M(x)\big) \geq S_M\big(\mu_M(x')\big).$$

Combining the two inequalities yields

$$S_M\big(\mu_M(x')\big) = S_M\big(\mu_M(x)\big).$$

□

4.11 Uncertain Module

Fuzzy module is a module equipped with a membership function compatible with addition and scalar action, expressing graded submodule structure and uncertainty-sensitive linear behavior formally [494–496].

Definition 4.11.1 (Fuzzy Module)**.** [494, 495] Let X be a ring, let $\mu_R : X \to [0,1]$ be a fuzzy ring on X, and let Y be a left X-module. A fuzzy set

$$\mu_M : Y \to [0,1]$$

is called a *fuzzy module* on Y over the fuzzy ring μ_R if, for all $x, y \in Y$ and all $\lambda \in X$,

$$\mu_M(x+y) \geq \min\{\mu_M(x), \mu_M(y)\},$$

$$\mu_M(\lambda x) \geq \min\{\mu_R(\lambda), \mu_M(x)\},$$

and

$$\mu_M(0) = 1.$$

In this case, the pair (Y, μ_M) is called a fuzzy module over the fuzzy ring (X, μ_R).

Remark 4.11.2. If X is regarded as an ordinary ring, one usually replaces the scalar condition by

$$\mu_M(\lambda x) \geq \mu_M(x) \qquad (\lambda \in X,\ x \in Y).$$

An uncertain module is a module equipped with an uncertainty-degree function into a general degree-domain, such that addition and scalar multiplication are compatible with the uncertainty structure through score-based aggregation.

Definition 4.11.3 (Uncertain Module)**.** Let $(R, +, \cdot)$ be a ring, let

$$\mathcal{R}_{M_R} = (R, \mu_R)$$

be an uncertain ring of type M_R, and let E be a left R-module.

Let M_E be an uncertain model with degree-domain

$$\mathrm{Dom}(M_E) \subseteq [0, 1]^{k_E}.$$

An *uncertain module of type* M_E on E over the uncertain ring $\mathcal{R}_{M_R}$ is an uncertain set

$$\mathcal{E}_{M_E} = (E, \mu_E),$$

where

$$\mu_E : E \to \mathrm{Dom}(M_E),$$

for which the following data are fixed:

- a score map for the ring degrees
$$S_R : \mathrm{Dom}(M_R) \to [0, 1];$$
- a score map for the module degrees
$$S_E : \mathrm{Dom}(M_E) \to [0, 1];$$
- a binary aggregation operator for addition
$$\Gamma_+ : [0, 1]^2 \to [0, 1];$$
- a binary aggregation operator for scalar multiplication
$$\Gamma_\odot : [0, 1]^2 \to [0, 1].$$

These data are required to satisfy, for all $u, v \in E$ and all $r \in R$,

$$S_E\big(\mu_E(u + v)\big) \geq \Gamma_+\big(S_E(\mu_E(u)),\ S_E(\mu_E(v))\big),$$
$$S_E\big(\mu_E(ru)\big) \geq \Gamma_\odot\big(S_R(\mu_R(r)),\ S_E(\mu_E(u))\big),$$

and

$$S_E\big(\mu_E(0_E)\big) = 1.$$

Equivalently, the uncertainty degree of the module sum $u + v$ and the scalar multiple ru is controlled by the aggregated scores of the corresponding arguments, and the zero vector has maximal score.

Remark 4.11.4. If both M_R and M_E are the fuzzy model, so that

$$\mathrm{Dom}(M_R) = \mathrm{Dom}(M_E) = [0, 1], \qquad S_R = S_E = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_+(a, b) = \Gamma_\odot(a, b) = \min\{a, b\},$$

then Definition 4.11.3 reduces to the usual fuzzy-module conditions

$$\mu_E(u + v) \geq \min\{\mu_E(u), \mu_E(v)\},$$
$$\mu_E(ru) \geq \min\{\mu_R(r), \mu_E(u)\},$$

and

$$\mu_E(0_E) = 1.$$

Thus fuzzy modules are special cases of uncertain modules.

Theorem 4.11.5 (Well-definedness of uncertain modules)**.** *Let* $(R, +, \cdot)$ *be a ring, let*

$$\mathcal{R}_{M_R} = (R, \mu_R)$$

be an uncertain ring of type M_R*, let* E *be a left* R*-module, and let*

$$\mu_E : E \to \mathrm{Dom}(M_E), \qquad S_R : \mathrm{Dom}(M_R) \to [0,1], \qquad S_E : \mathrm{Dom}(M_E) \to [0,1],$$

$$\Gamma_+ : [0,1]^2 \to [0,1], \qquad \Gamma_\odot : [0,1]^2 \to [0,1]$$

be given. Then:

(a) (E, μ_E) *is a well-defined uncertain set of type* M_E *on* E*.*

(b) *For every* $u, v \in E$ *and every* $r \in R$*, the quantities*

$$S_E\big(\mu_E(u+v)\big), \qquad S_E\big(\mu_E(ru)\big), \qquad S_E\big(\mu_E(0_E)\big)$$

are well-defined elements of $[0,1]$*.*

(c) *For every* $u, v \in E$ *and every* $r \in R$*, the quantities*

$$\Gamma_+\big(S_E(\mu_E(u)),\, S_E(\mu_E(v))\big)$$

and

$$\Gamma_\odot\big(S_R(\mu_R(r)),\, S_E(\mu_E(u))\big)$$

are well-defined elements of $[0,1]$*.*

(d) *Consequently, all defining conditions in Definition 4.11.3 are meaningful.*

Hence Definition 4.11.3 determines a mathematically well-defined class of structures.

Proof. (a) By the definition of an uncertain set of type M_E, any pair

$$(E, \mu_E) \qquad \text{with} \qquad \mu_E : E \to \mathrm{Dom}(M_E)$$

is an uncertain set of type M_E on E. Hence (E, μ_E) is well-defined.

(b) Let $u, v \in E$ and $r \in R$. Since E is a left R-module, the sum

$$u + v \in E,$$

the scalar multiple

$$ru \in E,$$

and the zero element

$$0_E \in E$$

are all well-defined.

Because

$$\mu_E : E \to \mathrm{Dom}(M_E),$$

it follows that

$$\mu_E(u+v) \in \mathrm{Dom}(M_E), \qquad \mu_E(ru) \in \mathrm{Dom}(M_E), \qquad \mu_E(0_E) \in \mathrm{Dom}(M_E).$$

Applying

$$S_E : \mathrm{Dom}(M_E) \to [0,1],$$

we obtain

$$S_E\big(\mu_E(u+v)\big) \in [0,1], \qquad S_E\big(\mu_E(ru)\big) \in [0,1], \qquad S_E\big(\mu_E(0_E)\big) \in [0,1].$$

Thus these quantities are well-defined.

(c) Since

$$\mu_E(u), \mu_E(v) \in \mathrm{Dom}(M_E),$$

we have

$$S_E(\mu_E(u)),\ S_E(\mu_E(v)) \in [0,1].$$

Hence

$$\big(S_E(\mu_E(u)),\ S_E(\mu_E(v))\big) \in [0,1]^2.$$

Because

$$\Gamma_+ : [0,1]^2 \to [0,1],$$

it follows that

$$\Gamma_+\big(S_E(\mu_E(u)),\ S_E(\mu_E(v))\big) \in [0,1].$$

Also, since

$$\mu_R(r) \in \mathrm{Dom}(M_R),$$

we have

$$S_R(\mu_R(r)) \in [0,1].$$

Together with

$$S_E(\mu_E(u)) \in [0,1],$$

this gives

$$\big(S_R(\mu_R(r)),\ S_E(\mu_E(u))\big) \in [0,1]^2.$$

Because

$$\Gamma_\odot : [0,1]^2 \to [0,1],$$

it follows that

$$\Gamma_\odot\big(S_R(\mu_R(r)),\ S_E(\mu_E(u))\big) \in [0,1].$$

Thus both right-hand sides are well-defined.

(d) By parts (b) and (c), every term appearing in

$$S_E\big(\mu_E(u+v)\big) \geq \Gamma_+\big(S_E(\mu_E(u)),\ S_E(\mu_E(v))\big),$$

$$S_E\big(\mu_E(ru)\big) \geq \Gamma_\odot\big(S_R(\mu_R(r)),\ S_E(\mu_E(u))\big),$$

and

$$S_E\big(\mu_E(0_E)\big) = 1$$

is a well-defined real number. Therefore all defining conditions in Definition 4.11.3 are meaningful.

Hence the notion of uncertain module is mathematically well-defined. □

For reference, representative uncertainty-module families and their degree representations are presented in Table 4.11.

Table 4.11: Representative uncertainty-module families and their degree representations.

Uncertainty-module family	**Degree form**	**Typical degree-domain / representation**
Fuzzy Modules	scalar	Typically represented by a fuzzy subset $$\mu_M : E \to [0,1],$$ where E is a module over a ring R, together with compatibility conditions for addition and scalar multiplication.
Intuitionistic Fuzzy Modules [496–498]	2-component	Typically represented by $$\mu_M : E \to [0,1]^2,$$ with components such as $(\text{membership}, \text{non-membership})$, together with module-compatibility conditions.
Hesitant Fuzzy Modules [499, 500]	set-valued	Typically represented by $$\mu_M : E \to \mathcal{P}_{\text{fin}}([0,1]) \setminus \{\varnothing\},$$ where each module element is assigned a finite set of possible membership degrees.
Neutrosophic Modules [501–504]	3-component	Typically represented by $$\mu_M : E \to [0,1]^3,$$ with components such as (T, I, F), together with module-compatibility conditions.

4.12 Uncertain Poset

Poset is a set equipped with a reflexive, antisymmetric, and transitive order relation, expressing partial comparability among elements without requiring every pair be comparable therein [505–507]. Fuzzy poset is a set endowed with an L-valued partial order, where graded comparability satisfies fuzzy reflexivity, transitivity, and antisymmetry in uncertainty-aware ordering structures naturally [508–512].

Definition 4.12.1 (Fuzzy Poset)**.** [508, 509, 513] Let $L = (L, \wedge, \vee, \otimes, \to, 0, 1)$ be a complete residuated lattice, and let X be a nonempty set. A mapping

$$e : X \times X \to L$$

is called a *fuzzy partial order* on X if, for all $x, y, z \in X$,

1. $$e(x, x) = 1;$$
2. $$e(x, y) \otimes e(y, z) \le e(x, z);$$
3. $$e(x, y) = 1 \text{ and } e(y, x) = 1 \implies x = y.$$

The pair (X, e) is called a *fuzzy partially ordered set*, or briefly, a *fuzzy poset*.

An uncertain poset is a set equipped with an uncertainty-valued binary relation, regarded as an uncertain set on the Cartesian product, whose reflexivity, transitivity, and antisymmetry are expressed through score-based conditions.

Definition 4.12.2 (Uncertain Poset)**.** Let X be a nonempty set, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

An *uncertain relation of type M* on X is an uncertain set

$$\mathcal{E}_M = (X \times X, e_M),$$

where

$$e_M : X \times X \to \mathrm{Dom}(M).$$

Fix:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0,1];$$
- a binary transitivity operator
$$\Gamma_{\leq} : [0,1]^2 \to [0,1].$$

Then (X, e_M) is called an *uncertain partially ordered set of type M*, or briefly an *uncertain poset*, if, for all $x, y, z \in X$, the following conditions hold:

1. $$S_M\big(e_M(x,x)\big) = 1;$$
2. $$\Gamma_{\leq}\big(S_M(e_M(x,y)),\, S_M(e_M(y,z))\big) \leq S_M\big(e_M(x,z)\big);$$
3. $$S_M\big(e_M(x,y)\big) = 1 \quad \text{and} \quad S_M\big(e_M(y,x)\big) = 1 \quad \Longrightarrow \quad x = y.$$

Here $e_M(x, y)$ represents the uncertainty degree to which x is below y in the uncertain order.

Remark 4.12.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_{\leq}(a,b) = a \otimes b$$

for a fixed t-norm $\otimes$ on $[0, 1]$, then Definition 4.12.2 reduces to the usual notion of a fuzzy poset:

$$e(x,x) = 1,$$
$$e(x,y) \otimes e(y,z) \leq e(x,z),$$
$$e(x,y) = 1 \text{ and } e(y,x) = 1 \implies x = y.$$

Thus fuzzy posets are special cases of uncertain posets.

Theorem 4.12.4 (Well-definedness of uncertain posets)**.** *Let X be a nonempty set, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$e_M : X \times X \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_{\leq} : [0,1]^2 \to [0,1]$$

be given. Then:

1. *$(X \times X, e_M)$ is a well-defined uncertain set of type M on $X \times X$;*
2. *for all $x, y, z \in X$, the quantities*
$$S_M\big(e_M(x,x)\big), \qquad S_M\big(e_M(x,y)\big), \qquad S_M\big(e_M(y,z)\big), \qquad S_M\big(e_M(x,z)\big), \qquad S_M\big(e_M(y,x)\big)$$
are well-defined elements of $[0,1]$;
3. *for all $x, y, z \in X$, the quantity*
$$\Gamma_{\leq}\big(S_M(e_M(x,y)),\, S_M(e_M(y,z))\big)$$
is a well-defined element of $[0,1]$;
4. *consequently, all defining conditions in Definition 4.12.2 are meaningful.*

Hence Definition 4.12.2 determines a mathematically well-defined class of structures.

Proof. Since X is a nonempty set, the Cartesian product

$$X \times X$$

is also a well-defined set. Because

$$e_M : X \times X \to \mathrm{Dom}(M),$$

the pair

$$(X \times X, e_M)$$

is, by definition, an uncertain set of type M on $X \times X$. This proves the first assertion.

Now let $x, y, z \in X$. Since

$$(x,x),\ (x,y),\ (y,z),\ (x,z),\ (y,x) \in X \times X,$$

and since

$$e_M : X \times X \to \mathrm{Dom}(M),$$

it follows that

$$e_M(x,x),\ e_M(x,y),\ e_M(y,z),\ e_M(x,z),\ e_M(y,x) \in \mathrm{Dom}(M).$$

Applying the score map

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

we obtain

$$S_M\big(e_M(x,x)\big), \quad S_M\big(e_M(x,y)\big), \quad S_M\big(e_M(y,z)\big), \quad S_M\big(e_M(x,z)\big), \quad S_M\big(e_M(y,x)\big) \in [0,1].$$

Hence all score values appearing in Definition 4.12.2 are well-defined.

Moreover, since
$$S_M\big(e_M(x,y)\big) \in [0,1] \qquad \text{and} \qquad S_M\big(e_M(y,z)\big) \in [0,1],$$
we have
$$\big(S_M(e_M(x,y)),\, S_M(e_M(y,z))\big) \in [0,1]^2.$$
Because
$$\Gamma_{\leq} : [0,1]^2 \to [0,1],$$
it follows that
$$\Gamma_{\leq}\big(S_M(e_M(x,y)),\, S_M(e_M(y,z))\big) \in [0,1].$$
Therefore the transitivity inequality
$$\Gamma_{\leq}\big(S_M(e_M(x,y)),\, S_M(e_M(y,z))\big) \leq S_M\big(e_M(x,z)\big)$$
is meaningful.

Likewise, the reflexivity condition
$$S_M\big(e_M(x,x)\big) = 1$$
and the antisymmetry condition
$$S_M\big(e_M(x,y)\big) = 1 \quad \text{and} \quad S_M\big(e_M(y,x)\big) = 1 \quad \Longrightarrow \quad x = y$$
are meaningful, because all score values involved are well-defined real numbers in $[0,1]$, while $x = y$ is an ordinary equality in the base set X.

Hence all objects and all conditions in Definition 4.12.2 are mathematically meaningful. Therefore the notion of uncertain poset is well-defined. □

4.13 Uncertain Manifold

A neutrosophic manifold is a topological space locally modeled on neutrosophic coordinates, with transition maps preserving componentwise smoothness and algebra (cf. [514–517]). A neutrosophic manifold generalizes both classical manifolds [518, 519] and fuzzy manifolds [520–522].

Definition 4.13.1 (Neutrosophic Manifold)**.** (cf. [514–517]) Let $\mathbb{N}n$ denote the set of neutrosophic numbers (e.g. triples or literals based on T, I, F components) equipped with a topology making $\mathbb{N}n^d$ a Hausdorff, second-countable space. A *d-dimensional neutrosophic C^k manifold* is a Hausdorff, second-countable topological space X together with an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ such that $U_i \subseteq X$ are open, $\varphi_i : U_i \to \Omega_i \subseteq \mathbb{N}n^d$ are homeomorphisms onto open sets, and for all i, j with $U_i \cap U_j \neq \varnothing$ the transition maps $\varphi_j \circ \varphi_i^{-1}$: $\varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are *neutrosophically C^k*: each coordinate function decomposes into (T, I, F)-components that are C^k (as real maps on representatives) and respect the neutrosophic algebra on $\mathbb{N}n$. Charts and changes of charts are required to be compatible with the neutrosophic product topology and with the (T, I, F)-componentwise operations.

An uncertain manifold is a classical manifold equipped with local uncertain-set structures on its coordinate domains, compatible across overlaps through the chart transition maps.

Definition 4.13.2 (Uncertain C^r Manifold)**.** Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^q$$

for some integer $q \geq 1$, and let $d \geq 0$ and $r \in \mathbb{N} \cup \{0, \infty\}$.

A *d-dimensional uncertain C^r manifold of type M* is a pair

$$\mathfrak{X}_M = (X, \mathcal{A}_M),$$

where X is a Hausdorff, second-countable topological space and

$$\mathcal{A}_M = \{(U_i, \varphi_i, \nu_i^M)\}_{i \in I}$$

is a family satisfying the following conditions:

1. each $U_i \subseteq X$ is open, and
$$X = \bigcup_{i \in I} U_i;$$
2. for each $i \in I$,
$$\varphi_i : U_i \to \Omega_i \subseteq \mathbb{R}^d$$
is a homeomorphism from U_i onto an open set Ω_i;
3. for each $i \in I$,
$$\nu_i^M : \Omega_i \to \mathrm{Dom}(M)$$
is an uncertainty-degree function, so that
$$(\Omega_i, \nu_i^M)$$
is an uncertain set of type M on Ω_i;
4. whenever
$$U_i \cap U_j \neq \varnothing,$$
the transition map
$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$
is of class C^r, and the local uncertainty-degree functions are compatible in the sense that
$$\nu_j^M \circ \left(\varphi_j \circ \varphi_i^{-1}\right) = \nu_i^M \qquad \text{on } \varphi_i(U_i \cap U_j).$$

The family $\mathcal{A}_M$ is called an *uncertain C^r atlas* of type M on X.

Remark 4.13.3. Under the compatibility condition in Definition 4.13.2, one obtains an induced global uncertainty-degree function

$$\mu_X^M : X \to \mathrm{Dom}(M)$$

by setting

$$\mu_X^M(x) := \nu_i^M\left(\varphi_i(x)\right) \qquad \text{for any } i \in I \text{ such that } x \in U_i.$$

Thus every uncertain manifold carries a canonical underlying uncertain set

$$(X, \mu_X^M).$$

Remark 4.13.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0, 1],$$

then an uncertain manifold becomes a manifold equipped with chartwise fuzzy membership functions

$$\nu_i : \Omega_i \to [0, 1]$$

that agree on overlaps.

If M is the neutrosophic model, so that

$$\mathrm{Dom}(M) = \{(T, I, F) \in [0, 1]^3 \mid 0 \leq T + I + F \leq 3\},$$

then each chart carries a neutrosophic degree function

$$\nu_i^M = (T_i, I_i, F_i) : \Omega_i \to \mathrm{Dom}(M),$$

and one obtains an uncertain manifold of neutrosophic type.

Theorem 4.13.5 (Well-definedness of uncertain manifolds)**.** *Let*

$$\mathfrak{X}_M = (X, \mathcal{A}_M)$$

be as in Definition 4.13.2. Then:

1. *for each $i \in I$, the pair*

$$(\Omega_i, \nu_i^M)$$

is a well-defined uncertain set of type M;

2. *for each $i, j \in I$ with $U_i \cap U_j \neq \varnothing$, the transition map*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is a well-defined C^r map between open subsets of $\mathbb{R}^d$;

3. *the induced global map*

$$\mu_X^M : X \to \mathrm{Dom}(M), \qquad \mu_X^M(x) := \nu_i^M\big(\varphi_i(x)\big) \quad (x \in U_i),$$

is well-defined, that is, independent of the chosen chart containing x;

4. *consequently,*

$$(X, \mu_X^M)$$

is a well-defined uncertain set of type M on X.

Hence the notion of uncertain C^r manifold is mathematically well-defined.

Proof. (1) Fix $i \in I$. By Definition 4.13.2, the set $\Omega_i \subseteq \mathbb{R}^d$ is open, hence nonempty or empty as an ordinary set, and

$$\nu_i^M : \Omega_i \to \mathrm{Dom}(M)$$

is a map into the degree-domain of the uncertain model M. Therefore

$$(\Omega_i, \nu_i^M)$$

is, by definition, an uncertain set of type M on Ω_i.

(2) Let $i, j \in I$ and suppose that

$$U_i \cap U_j \neq \varnothing.$$

Since

$$\varphi_i : U_i \to \Omega_i \qquad \text{and} \qquad \varphi_j : U_j \to \Omega_j$$

are homeomorphisms onto open subsets of $\mathbb{R}^d$, their restrictions yield a bijection

$$\varphi_i : U_i \cap U_j \to \varphi_i(U_i \cap U_j)$$

with inverse

$$\varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to U_i \cap U_j.$$

Hence the composition

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is well-defined. By assumption in Definition 4.13.2, it is of class C^r.

(3) Let $x \in X$. Because the family $\{U_i\}_{i\in I}$ covers X, there exists at least one index $i \in I$ such that

$$x \in U_i.$$

Define

$$\mu_X^M(x) := \nu_i^M\big(\varphi_i(x)\big).$$

We must show that this value does not depend on the chosen chart.

Assume that

$$x \in U_i \cap U_j.$$

Then

$$\varphi_i(x) \in \varphi_i(U_i \cap U_j).$$

Using the compatibility condition,

$$\nu_j^M \circ (\varphi_j \circ \varphi_i^{-1}) = \nu_i^M \qquad \text{on } \varphi_i(U_i \cap U_j),$$

we obtain

$$\nu_i^M\big(\varphi_i(x)\big) = \nu_j^M\big((\varphi_j \circ \varphi_i^{-1})(\varphi_i(x))\big) = \nu_j^M\big(\varphi_j(x)\big).$$

Hence the value assigned to x is independent of the chosen chart, so

$$\mu_X^M : X \to \mathrm{Dom}(M)$$

is well-defined.

(4) For every $x \in X$, choose $i \in I$ with $x \in U_i$. Since

$$\varphi_i(x) \in \Omega_i \qquad \text{and} \qquad \nu_i^M : \Omega_i \to \mathrm{Dom}(M),$$

it follows that

$$\mu_X^M(x) = \nu_i^M\big(\varphi_i(x)\big) \in \mathrm{Dom}(M).$$

Therefore

$$\mu_X^M : X \to \mathrm{Dom}(M)$$

is a well-defined mapping. By the definition of a U-set, the pair

$$(X, \mu_X^M)$$

is an uncertain set of type M on X.

Thus all constituents of Definition 4.13.2 are mathematically meaningful, and the notion of uncertain manifold is well-defined. □

4.14 Uncertain HyperAlgebra

Fuzzy hyperalgebra is a hyperalgebra whose operations assign fuzzy subsets to input tuples, generalizing algebraic hyperstructures by allowing graded multivalued outcomes under uncertainty [523–526].

Definition 4.14.1 (Fuzzy hyperalgebra)**.** [524] Let H be a nonempty set, and let

$$\mathcal{F}(H) := [0,1]^H$$

denote the family of all fuzzy subsets of H.

For each $i \in I$, let $n_i \geq 1$ be an integer. A mapping

$$\beta_i : H^{n_i} \to \mathcal{F}(H)$$

is called an *n_i-ary fuzzy hyperoperation* on H.

A *fuzzy hyperalgebra* is a structure

$$\mathcal{H} = \big(H, (\beta_i)_{i \in I}\big),$$

where H is a nonempty set and each β_i is an n_i-ary fuzzy hyperoperation on H.

Equivalently, for every

$$(x_1, \ldots, x_{n_i}) \in H^{n_i},$$

the value

$$\beta_i(x_1, \ldots, x_{n_i})$$

is a fuzzy subset of H, that is, a membership function

$$\beta_i(x_1, \ldots, x_{n_i}) : H \to [0,1].$$

Thus a fuzzy hyperalgebra is an algebraic hyperstructure in which each operation assigns to every tuple of inputs a fuzzy subset of H rather than a single element or a crisp subset.

Using Uncertain Sets, we define Uncertain HyperAlgebra as follows.

Definition 4.14.2 (Uncertain HyperAlgebra)**.** Let H be a nonempty set, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Denote by

$$\mathcal{U}_M(H) := \{\mu : H \to \mathrm{Dom}(M)\}$$

the family of all uncertain sets of type M on H.

For each $i \in I$, let $n_i \geq 1$ be an integer. A mapping

$$\beta_i^M : H^{n_i} \to \mathcal{U}_M(H)$$

is called an *n_i-ary uncertain hyperoperation of type M* on H.

An *uncertain hyperalgebra of type M* is a structure

$$\mathcal{H}_M = \big(H, (\beta_i^M)_{i\in I}\big),$$

where H is a nonempty set and each β_i^M is an n_i-ary uncertain hyperoperation of type M on H.

Equivalently, for every $i \in I$ and every

$$(x_1, \ldots, x_{n_i}) \in H^{n_i},$$

the value

$$\beta_i^M(x_1, \ldots, x_{n_i})$$

is an uncertain set of type M on H, that is, a map

$$\beta_i^M(x_1, \ldots, x_{n_i}) : H \to \mathrm{Dom}(M).$$

Thus an uncertain hyperalgebra is a hyperalgebra in which each operation assigns to every tuple of inputs an uncertain subset of H rather than a single element or a crisp subset.

Remark 4.14.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1],$$

then an uncertain hyperalgebra reduces to a fuzzy hyperalgebra.

Theorem 4.14.4 (Well-definedness of uncertain hyperalgebras)**.** *Let H be a nonempty set, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and for each $i \in I$, let

$$\beta_i^M : H^{n_i} \to \mathcal{U}_M(H)$$

be a mapping. Then:

1. *for every $i \in I$ and every*

   $$(x_1, \ldots, x_{n_i}) \in H^{n_i},$$

   the value

   $$\beta_i^M(x_1, \ldots, x_{n_i})$$

   is a well-defined uncertain set of type M on H;

2. *equivalently, for every $i \in I$, the associated evaluation map*

$$\widetilde{\beta}_i^M : H^{n_i} \times H \to \mathrm{Dom}(M), \qquad \widetilde{\beta}_i^M(x_1, \ldots, x_{n_i}; y) := \beta_i^M(x_1, \ldots, x_{n_i})(y),$$

is well-defined;

3. *consequently,*

$$\mathcal{H}_M = \big(H, (\beta_i^M)_{i \in I}\big)$$

is a mathematically well-defined uncertain hyperalgebra of type M.

Proof. Fix $i \in I$. Since

$$\beta_i^M : H^{n_i} \to \mathcal{U}_M(H),$$

for every tuple

$$(x_1, \ldots, x_{n_i}) \in H^{n_i},$$

the value

$$\beta_i^M(x_1, \ldots, x_{n_i})$$

belongs to $\mathcal{U}_M(H)$. By definition of $\mathcal{U}_M(H)$, this means that

$$\beta_i^M(x_1, \ldots, x_{n_i}) : H \to \mathrm{Dom}(M)$$

is a well-defined mapping. Hence $\beta_i^M(x_1, \ldots, x_{n_i})$ is a well-defined uncertain set of type M on H. This proves (1).

Now define

$$\widetilde{\beta}_i^M : H^{n_i} \times H \to \mathrm{Dom}(M)$$

by

$$\widetilde{\beta}_i^M(x_1, \ldots, x_{n_i}; y) := \beta_i^M(x_1, \ldots, x_{n_i})(y).$$

Because $\beta_i^M(x_1, \ldots, x_{n_i})$ is a map from H to $\mathrm{Dom}(M)$, the value

$$\beta_i^M(x_1, \ldots, x_{n_i})(y)$$

is defined for every $y \in H$, and it belongs to $\mathrm{Dom}(M)$. Therefore

$$\widetilde{\beta}_i^M$$

is a well-defined mapping. This proves (2).

Since the above argument applies to every $i \in I$, each β_i^M is a well-defined n_i-ary uncertain hyperoperation on H. Therefore the structure

$$\mathcal{H}_M = \big(H, (\beta_i^M)_{i \in I}\big)$$

is a well-defined uncertain hyperalgebra of type M. This proves (3). □

As a related concept, structures such as SuperHyperAlgebra [527, 528] are also known.

4.15 Uncertain MultiAlgebra

Multialgebra is an algebraic structure whose operations assign nonempty sets of outputs to input tuples, generalizing ordinary algebras by allowing multivalued results systematically across operations (cf. [529–531]). Fuzzy multialgebra is a multialgebra whose operations return nonzero fuzzy subsets, generalizing multivalued algebraic structures by incorporating graded uncertainty into each operation output in practice (cf. [532]).

Definition 4.15.1 (Fuzzy multialgebra)**.** Let A be a nonempty set, and let

$$\tau = (n_\gamma)_{\gamma < o(\tau)}$$

be a type, where each $n_\gamma \in \mathbb{N}$.

Define

$$\mathcal{F}^*(A) := \{\mu : A \to [0,1] \mid \mu \neq 0\},$$

the family of all nonzero fuzzy subsets of A.

For each $\gamma < o(\tau)$, a mapping

$$f_\gamma : A^{n_\gamma} \to \mathcal{F}^*(A)$$

is called a *fuzzy n_γ-ary multioperation* on A.

A *fuzzy multialgebra* of type τ is a structure

$$\mathcal{A} = \big(A, (f_\gamma)_{\gamma < o(\tau)}\big),$$

where A is a nonempty set and each f_γ is a fuzzy n_γ-ary multioperation on A.

Equivalently, for every $\gamma < o(\tau)$ and every

$$(a_0, \ldots, a_{n_\gamma - 1}) \in A^{n_\gamma},$$

the value

$$f_\gamma(a_0, \ldots, a_{n_\gamma - 1})$$

is a nonzero fuzzy subset of A, that is, a function

$$f_\gamma(a_0, \ldots, a_{n_\gamma - 1}) : A \to [0,1].$$

Using Uncertain Sets, we define Uncertain multialgebra as follows.

Definition 4.15.2 (Uncertain multialgebra)**.** Let A be a nonempty set, let

$$\tau = (n_\gamma)_{\gamma < o(\tau)}$$

be a type, where each $n_\gamma \in \mathbb{N}$, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Fix a score map
$$S_M : \mathrm{Dom}(M) \to [0,1].$$

Define
$$\mathcal{U}_M^*(A) := \{\mu : A \to \mathrm{Dom}(M) \mid \exists a \in A \text{ such that } S_M(\mu(a)) > 0\},$$
the family of all *nonzero uncertain subsets of type M* on A.

For each $\gamma < o(\tau)$, a mapping
$$f_\gamma^M : A^{n_\gamma} \to \mathcal{U}_M^*(A)$$
is called an *uncertain n_γ-ary multioperation of type M* on A.

An *uncertain multialgebra of type τ and M* is a structure
$$\mathcal{A}_M = \big(A, (f_\gamma^M)_{\gamma<o(\tau)}\big),$$
where A is a nonempty set and each f_γ^M is an uncertain n_γ-ary multioperation on A.

Equivalently, for every $\gamma < o(\tau)$ and every
$$(a_0, \dots, a_{n_\gamma - 1}) \in A^{n_\gamma},$$
the value
$$f_\gamma^M(a_0, \dots, a_{n_\gamma-1})$$
is a nonzero uncertain subset of type M on A, that is, a mapping
$$f_\gamma^M(a_0, \dots, a_{n_\gamma-1}) : A \to \mathrm{Dom}(M)$$
such that
$$\exists b \in A \quad S_M\big(f_\gamma^M(a_0, \dots, a_{n_\gamma-1})(b)\big) > 0.$$

Remark 4.15.3. If M is the fuzzy model, so that
$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$
then
$$\mathcal{U}_M^*(A) = \{\mu : A \to [0,1] \mid \mu \neq 0\} = \mathcal{F}^*(A),$$
and Definition 4.15.2 reduces to the usual definition of a fuzzy multialgebra.

Theorem 4.15.4 (Well-definedness of uncertain multialgebras)**.** *Let A, τ, M, and S_M be as in Definition 4.15.2. Suppose that, for each $\gamma < o(\tau)$,*
$$f_\gamma^M : A^{n_\gamma} \to \mathcal{U}_M^*(A)$$
is a mapping. Then:

1. *for every $\gamma < o(\tau)$ and every*
$$(a_0, \dots, a_{n_\gamma-1}) \in A^{n_\gamma},$$
the value
$$f_\gamma^M(a_0, \dots, a_{n_\gamma-1})$$
is a well-defined nonzero uncertain subset of type M on A;

2. *equivalently, for every $\gamma < o(\tau)$, the evaluation map*

$$\widetilde{f}_\gamma^M : A^{n_\gamma} \times A \to \mathrm{Dom}(M), \qquad \widetilde{f}_\gamma^M(a_0, \ldots, a_{n_\gamma - 1}; b) := f_\gamma^M(a_0, \ldots, a_{n_\gamma - 1})(b),$$

   is well-defined;

3. *consequently,*

$$\mathcal{A}_M = \big(A, (f_\gamma^M)_{\gamma < o(\tau)}\big)$$

   is a mathematically well-defined uncertain multialgebra.

Proof. Fix $\gamma < o(\tau)$. Since

$$f_\gamma^M : A^{n_\gamma} \to \mathcal{U}_M^*(A),$$

for every tuple

$$(a_0, \ldots, a_{n_\gamma - 1}) \in A^{n_\gamma},$$

the value

$$f_\gamma^M(a_0, \ldots, a_{n_\gamma - 1})$$

belongs to $\mathcal{U}_M^*(A)$. By definition of $\mathcal{U}_M^*(A)$, this means that

$$f_\gamma^M(a_0, \ldots, a_{n_\gamma - 1}) : A \to \mathrm{Dom}(M)$$

is a well-defined mapping and there exists some $b \in A$ such that

$$S_M\big(f_\gamma^M(a_0, \ldots, a_{n_\gamma - 1})(b)\big) > 0.$$

Hence

$$f_\gamma^M(a_0, \ldots, a_{n_\gamma - 1})$$

is a well-defined nonzero uncertain subset of type M on A. This proves (1).

Now define

$$\widetilde{f}_\gamma^M : A^{n_\gamma} \times A \to \mathrm{Dom}(M)$$

by

$$\widetilde{f}_\gamma^M(a_0, \ldots, a_{n_\gamma - 1}; b) := f_\gamma^M(a_0, \ldots, a_{n_\gamma - 1})(b).$$

Because

$$f_\gamma^M(a_0, \ldots, a_{n_\gamma - 1}) : A \to \mathrm{Dom}(M),$$

the value

$$f_\gamma^M(a_0, \ldots, a_{n_\gamma - 1})(b)$$

is defined for every $b \in A$, and it belongs to $\mathrm{Dom}(M)$. Therefore

$$\widetilde{f}_\gamma^M$$

is a well-defined mapping. This proves (2).

Since the above argument applies to every $\gamma < o(\tau)$, each

$$f_\gamma^M$$

is a well-defined uncertain multioperation on A. Therefore the structure

$$\mathcal{A}_M = \big(A, (f_\gamma^M)_{\gamma < o(\tau)}\big)$$

is a well-defined uncertain multialgebra. This proves (3). □

4.16 Uncertain Ideal

Fuzzy ideal is a ring-valued membership structure closed under subtraction and absorbent under multiplication, expressing graded ideal membership while preserving algebraic stability and inclusion behavior (cf. [533–535]).

Definition 4.16.1 (Fuzzy Ideal)**.** Let $(R, +, \cdot)$ be a ring. A fuzzy set

$$\mu : R \to [0, 1]$$

is called a *fuzzy ideal* of R if, for all $x, y \in R$,

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$$

and

$$\mu(xy) \geq \max\{\mu(x), \mu(y)\}.$$

Equivalently, μ is a fuzzy subgroup of $(R, +)$ and is absorbent with respect to multiplication.

An uncertain ideal is an uncertain subset of a ring whose uncertainty degree is stable under subtraction and absorbent under multiplication through score-based aggregation.

Definition 4.16.2 (Uncertain Ideal)**.** Let $(R, +, \cdot)$ be a ring, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

An *uncertain ideal of type* M in R is an uncertain set

$$\mathcal{I}_M = (R, \mu_M),$$

where

$$\mu_M : R \to \mathrm{Dom}(M),$$

for which the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0, 1];$$
- a subtraction-aggregation operator
$$\Gamma_- : [0, 1]^2 \to [0, 1];$$
- a multiplication-absorption operator
$$\Gamma_\cdot : [0, 1]^2 \to [0, 1].$$

These data are required to satisfy, for all $x, y \in R$,

$$S_M\big(\mu_M(x - y)\big) \geq \Gamma_-\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big),$$

and

$$S_M\big(\mu_M(xy)\big) \geq \Gamma_\cdot\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big).$$

Equivalently, the uncertainty degree of $x - y$ is bounded below by the aggregated scores of x and y, while the uncertainty degree of the product xy is bounded below by an absorption-type aggregation of their scores.

Remark 4.16.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_-(a,b) = \min\{a,b\}, \qquad \Gamma_\cdot(a,b) = \max\{a,b\},$$

then Definition 4.16.2 reduces to the usual fuzzy ideal condition

$$\mu(x-y) \geq \min\{\mu(x), \mu(y)\}, \qquad \mu(xy) \geq \max\{\mu(x), \mu(y)\}$$

for all $x, y \in R$. Thus fuzzy ideals are special cases of uncertain ideals.

Theorem 4.16.4 (Well-definedness of uncertain ideals)**.** *Let $(R, +, \cdot)$ be a ring, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$\mu_M : R \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_- : [0,1]^2 \to [0,1], \qquad \Gamma_\cdot : [0,1]^2 \to [0,1]$$

be given. Then:

1. *(R, μ_M) is a well-defined uncertain set of type M on R;*
2. *for every $x, y \in R$, the quantities*

$$S_M\big(\mu_M(x-y)\big) \qquad \textit{and} \qquad S_M\big(\mu_M(xy)\big)$$

   are well-defined elements of $[0,1]$;
3. *for every $x, y \in R$, the quantities*

$$\Gamma_-\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \qquad \textit{and} \qquad \Gamma_\cdot\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

   are well-defined elements of $[0,1]$;
4. *consequently, both defining inequalities in Definition 4.16.2 are meaningful for all $x, y \in R$.*

Hence Definition 4.16.2 determines a mathematically well-defined class of structures.

Proof. (1) By the definition of an uncertain set of type M, any pair

$$(R, \mu_M) \qquad \text{with} \qquad \mu_M : R \to \mathrm{Dom}(M)$$

is an uncertain set of type M on R. Hence (R, μ_M) is well-defined.

(2) Let $x, y \in R$. Since $(R, +, \cdot)$ is a ring, both

$$x - y \in R \qquad \text{and} \qquad xy \in R$$

are well-defined. Because

$$\mu_M : R \to \mathrm{Dom}(M),$$

it follows that

$$\mu_M(x-y) \in \mathrm{Dom}(M) \qquad \text{and} \qquad \mu_M(xy) \in \mathrm{Dom}(M).$$

Applying the score map

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

one obtains

$$S_M\big(\mu_M(x-y)\big) \in [0,1] \qquad \text{and} \qquad S_M\big(\mu_M(xy)\big) \in [0,1].$$

Thus both quantities are well-defined.

(3) Again let $x, y \in R$. Since

$$\mu_M(x), \mu_M(y) \in \mathrm{Dom}(M),$$

one has

$$S_M(\mu_M(x)),\ S_M(\mu_M(y)) \in [0,1].$$

Hence

$$\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big) \in [0,1]^2.$$

Because

$$\Gamma_- : [0,1]^2 \to [0,1] \qquad \text{and} \qquad \Gamma_\cdot : [0,1]^2 \to [0,1],$$

it follows that

$$\Gamma_-\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big) \in [0,1]$$

and

$$\Gamma_\cdot\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big) \in [0,1].$$

Thus both right-hand sides are well-defined.

(4) By (2) and (3), all terms appearing in

$$S_M\big(\mu_M(x-y)\big) \geq \Gamma_-\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big)$$

and

$$S_M\big(\mu_M(xy)\big) \geq \Gamma_\cdot\big(S_M(\mu_M(x)),\ S_M(\mu_M(y))\big)$$

are well-defined real numbers in $[0,1]$. Therefore both inequalities are meaningful for every $x, y \in R$.

Hence all objects and all conditions appearing in Definition 4.16.2 are mathematically meaningful, and the notion of uncertain ideal is well-defined. □

4.17 Uncertain Filter

Fuzzy filter is a graded multiplicative filter on a commutative ring, whose level cuts are filters, preserving unity, excluding zero, and factor closure conditions consistently (cf. [536, 537]).

Definition 4.17.1 (Fuzzy Filter in a Ring)**.** Since the notion of a *filter in a ring* is not completely standard, we adopt the multiplicative-filter convention.

Let R be a commutative ring with identity $1 \neq 0$. A fuzzy set

$$\mu : R \to [0,1]$$

is called a *fuzzy filter* on R if, for every $\alpha \in (0,1]$, the α-cut

$$F_\alpha := \{\, x \in R \mid \mu(x) \geq \alpha \,\}$$

is a filter of R, that is,

$$1 \in F_\alpha, \qquad 0 \notin F_\alpha,$$
$$x, y \in F_\alpha \implies xy \in F_\alpha,$$

and

$$xy \in F_\alpha \implies x \in F_\alpha \text{ and } y \in F_\alpha.$$

Equivalently, μ is a fuzzy filter on R if and only if

$$\mu(1) = 1, \qquad \mu(0) = 0,$$

and

$$\mu(xy) = \min\{\mu(x), \mu(y)\} \qquad \text{for all } x, y \in R.$$

An uncertain filter is an uncertain subset of a commutative ring with identity whose score-induced level cuts are multiplicative filters.

Definition 4.17.2 (Uncertain Filter in a Ring)**.** Let R be a commutative ring with identity $1 \neq 0$, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Let

$$\mu_M : R \to \mathrm{Dom}(M)$$

be an uncertain set of type M on R, and fix a score map

$$S_M : \mathrm{Dom}(M) \to [0,1].$$

For each $\alpha \in (0,1]$, define the score-induced α-cut by

$$F_\alpha^{S_M} := \{\, x \in R \mid S_M(\mu_M(x)) \geq \alpha \,\}.$$

Then μ_M is called an *uncertain filter of type M* on R if, for every $\alpha \in (0,1]$, the set $F_\alpha^{S_M}$ is a filter of R, that is,

$$1 \in F_\alpha^{S_M}, \qquad 0 \notin F_\alpha^{S_M},$$
$$x, y \in F_\alpha^{S_M} \implies xy \in F_\alpha^{S_M},$$

and

$$xy \in F_\alpha^{S_M} \implies x \in F_\alpha^{S_M} \text{ and } y \in F_\alpha^{S_M}.$$

Proposition 4.17.3 (Pointwise characterization)**.** *Let R, M, μ_M, and S_M be as in Definition 4.17.2. Then μ_M is an uncertain filter on R if and only if*

$$S_M(\mu_M(1)) = 1, \qquad S_M(\mu_M(0)) = 0,$$

and

$$S_M(\mu_M(xy)) = \min\big\{S_M(\mu_M(x)),\, S_M(\mu_M(y))\big\} \qquad \text{for all } x, y \in R.$$

Proof. Assume first that μ_M is an uncertain filter. We prove the three conditions.

For every $\alpha \in (0,1]$, since $1 \in F_\alpha^{S_M}$, we have

$$S_M(\mu_M(1)) \geq \alpha.$$

Because this holds for every $\alpha \in (0,1]$, it follows that

$$S_M(\mu_M(1)) = 1.$$

Likewise, for every $\alpha \in (0,1]$, since $0 \notin F_\alpha^{S_M}$, we have

$$S_M(\mu_M(0)) < \alpha.$$

Because this holds for every $\alpha \in (0,1]$, it follows that

$$S_M(\mu_M(0)) = 0.$$

Now let $x, y \in R$. Set

$$a := S_M(\mu_M(x)), \qquad b := S_M(\mu_M(y)).$$

We show that

$$S_M(\mu_M(xy)) = \min\{a, b\}.$$

First, let $\alpha \leq \min\{a,b\}$. Then

$$x, y \in F_\alpha^{S_M},$$

so by filter closure under multiplication,

$$xy \in F_\alpha^{S_M}.$$

Hence

$$S_M(\mu_M(xy)) \geq \alpha.$$

Since this is true for every $\alpha \leq \min\{a,b\}$, we obtain

$$S_M(\mu_M(xy)) \geq \min\{a, b\}.$$

Conversely, let

$$\alpha \leq S_M(\mu_M(xy)).$$

Then
$$xy \in F_\alpha^{S_M}.$$
By the factor property of a filter,
$$x \in F_\alpha^{S_M} \qquad \text{and} \qquad y \in F_\alpha^{S_M}.$$
Thus
$$a \geq \alpha, \qquad b \geq \alpha,$$
so
$$\min\{a, b\} \geq \alpha.$$
Since this holds for every $\alpha \leq S_M(\mu_M(xy))$, it follows that
$$\min\{a, b\} \geq S_M(\mu_M(xy)).$$
Therefore
$$S_M(\mu_M(xy)) = \min\{a, b\}.$$

This proves the necessity.

For the converse, assume that
$$S_M(\mu_M(1)) = 1, \qquad S_M(\mu_M(0)) = 0,$$
and
$$S_M(\mu_M(xy)) = \min\big\{S_M(\mu_M(x)),\, S_M(\mu_M(y))\big\}$$
for all $x, y \in R$. Fix $\alpha \in (0, 1]$.

Since
$$S_M(\mu_M(1)) = 1 \geq \alpha,$$
we have
$$1 \in F_\alpha^{S_M}.$$
Since
$$S_M(\mu_M(0)) = 0 < \alpha,$$
we have
$$0 \notin F_\alpha^{S_M}.$$

Now let $x, y \in F_\alpha^{S_M}$. Then
$$S_M(\mu_M(x)) \geq \alpha, \qquad S_M(\mu_M(y)) \geq \alpha.$$
Hence
$$S_M(\mu_M(xy)) = \min\big\{S_M(\mu_M(x)),\, S_M(\mu_M(y))\big\} \geq \alpha,$$
so
$$xy \in F_\alpha^{S_M}.$$

Finally, let $xy \in F_\alpha^{S_M}$. Then
$$S_M(\mu_M(xy)) \geq \alpha.$$

Using the assumed identity,

$$\min\{S_M(\mu_M(x)), S_M(\mu_M(y))\} \geq \alpha.$$

Therefore

$$S_M(\mu_M(x)) \geq \alpha \qquad \text{and} \qquad S_M(\mu_M(y)) \geq \alpha,$$

which means

$$x \in F_\alpha^{S_M} \qquad \text{and} \qquad y \in F_\alpha^{S_M}.$$

Thus $F_\alpha^{S_M}$ is a filter of R for every $\alpha \in (0, 1]$. Hence μ_M is an uncertain filter. □

Theorem 4.17.4 (Well-definedness of uncertain filters)**.** *Let R be a commutative ring with identity $1 \neq 0$, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

and let

$$\mu_M : R \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0, 1]$$

be given. Then:

1. *for every $\alpha \in (0, 1]$, the set*
$$F_\alpha^{S_M} = \{\, x \in R \mid S_M(\mu_M(x)) \geq \alpha \,\}$$
is well-defined;
2. *the statements*
$$1 \in F_\alpha^{S_M}, \qquad 0 \notin F_\alpha^{S_M},$$
$$x, y \in F_\alpha^{S_M} \Longrightarrow xy \in F_\alpha^{S_M},$$
and
$$xy \in F_\alpha^{S_M} \Longrightarrow x, y \in F_\alpha^{S_M}$$
are meaningful for every $\alpha \in (0, 1]$;
3. *consequently, Definition 4.17.2 determines a mathematically well-defined class of structures.*

Proof. Fix $\alpha \in (0, 1]$. Since

$$\mu_M : R \to \mathrm{Dom}(M)$$

and

$$S_M : \mathrm{Dom}(M) \to [0, 1],$$

the composition

$$S_M \circ \mu_M : R \to [0, 1]$$

is a well-defined map. Therefore, for each $x \in R$, the inequality

$$S_M(\mu_M(x)) \geq \alpha$$

is meaningful. Hence the subset

$$F_\alpha^{S_M} = \{\, x \in R \mid S_M(\mu_M(x)) \geq \alpha \,\}$$

is well-defined. This proves (1).

Next, since $1, 0 \in R$, the statements

$$1 \in F_\alpha^{S_M} \qquad \text{and} \qquad 0 \notin F_\alpha^{S_M}$$

are meaningful.

Also, if $x, y \in R$, then $xy \in R$ because R is a ring. Thus the implications

$$x, y \in F_\alpha^{S_M} \Longrightarrow xy \in F_\alpha^{S_M}$$

and

$$xy \in F_\alpha^{S_M} \Longrightarrow x, y \in F_\alpha^{S_M}$$

are meaningful statements about elements of the set R. This proves (2).

Therefore all conditions appearing in Definition 4.17.2 are mathematically meaningful, and the notion of uncertain filter is well-defined. This proves (3). □

As related concepts, notions such as Ultrafilter [538], Ultraproduct, Quasi-Filter [539], and Weak Filter [540] are also known.

Chapter 5

Analytical and Spatial Applied Mathematics

In this chapter, we examine analytical and spatial applied mathematics. A concise comparison of the analytical and spatial uncertain concepts covered in Chapter 5 is presented in Table 5.1.

Table 5.1: A concise comparison of the analytical and spatial uncertain concepts covered in Chapter 5.

Concept	**Base structure**	**Main focus**
Uncertain Probability	Probability space	Uncertain events, likelihood, and probabilistic reasoning.
Uncertain Metric Spaces	Metric space	Uncertain distance and generalized spatial structure.
Uncertain Distance Measure	Distance function	Uncertain dissimilarity and comparative measurement between objects.
Uncertain Geometry	Geometric space	Uncertain spatial form, shape, and geometric relations.
Uncertain Differential Equations	Differential equation	Uncertain dynamics, change, and evolution of systems.
Uncertain Normed Space	Normed space	Uncertain magnitude, length, and functional structure.
Uncertain Fixed Point Theory	Fixed point framework	Existence and behavior of fixed points under uncertainty.
Uncertain BCK/BCI–Algebras	BCK/BCI-algebra	Uncertain logical-algebraic operations and implication-like structure.

5.1 Uncertain Probability

Fuzzy probability models uncertainty when probabilities are imprecise: events have fuzzy likelihoods, represented by membership functions, intervals, or possibility distributions, enabling robust inference under vague information [541–543].

Definition 5.1.1 (Fuzzy probability (normalized weighted-cardinality model))**.** Let Ω be a finite set (universe of elementary outcomes).

(1) Fuzzy sample space. A *fuzzy sample space* is a fuzzy set $\widetilde{S}$ on Ω with membership function

$$\mu_{\widetilde{S}} : \Omega \to [0,1],$$

normalized by

$$\sum_{\omega \in \Omega} \mu_{\widetilde{S}}(\omega) = 1.$$

(2) Fuzzy event. A *fuzzy event* is a fuzzy set $\widetilde{E}$ on Ω with membership function

$$\mu_{\widetilde{E}} : \Omega \to [0,1]$$

such that $\mu_{\widetilde{E}}(\omega) \le \mu_{\widetilde{S}}(\omega)$ for all $\omega \in \Omega$ (i.e., $\widetilde{E}$ is a fuzzy subset of $\widetilde{S}$).

(3) Weighted size and fuzzy probability. For any fuzzy set $\widetilde{A}$ on Ω, define its *weighted size* by

$$|\widetilde{A}|_w := \sum_{\omega \in \Omega} \mu_{\widetilde{A}}(\omega).$$

The *fuzzy probability* of the fuzzy event $\widetilde{E}$ (relative to $\widetilde{S}$) is

$$P_f(\widetilde{E}) \; := \; \frac{|\widetilde{E}|_w}{|\widetilde{S}|_w} \; = \; \sum_{\omega \in \Omega} \mu_{\widetilde{E}}(\omega),$$

where the last equality uses $|\widetilde{S}|_w = 1$.

Remark 5.1.2 (Reduction to a classical probability mass function)**.** If $\mu_{\widetilde{S}}(\omega) = p(\omega)$ is a classical pmf on Ω and $\mu_{\widetilde{E}}(\omega) = \mathbb{1}_E(\omega)\, p(\omega)$ for a crisp event $E \subseteq \Omega$, then

$$P_f(\widetilde{E}) = \sum_{\omega \in E} p(\omega),$$

which coincides with the ordinary probability of E.

Uncertain probability is an uncertainty-model-based extension of fuzzy probability in which sample spaces and events are represented as uncertain sets and probabilities are computed through score-induced weighted cardinalities.

Definition 5.1.3 (Uncertain Probability)**.** Let Ω be a finite nonempty set, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Fix a score map

$$S_M : \mathrm{Dom}(M) \to [0,1].$$

(1) Uncertain sample space. An *uncertain sample space of type M* is an uncertain set

$$\widetilde{S}_M = (\Omega, \mu_{\widetilde{S}_M}), \qquad \mu_{\widetilde{S}_M} : \Omega \to \mathrm{Dom}(M),$$

satisfying the normalization condition

$$\sum_{\omega \in \Omega} S_M\big(\mu_{\widetilde{S}_M}(\omega)\big) = 1.$$

(2) Uncertain event. An *uncertain event of type M* relative to $\widetilde{S}_M$ is an uncertain set

$$\widetilde{E}_M = (\Omega, \mu_{\widetilde{E}_M}), \qquad \mu_{\widetilde{E}_M} : \Omega \to \mathrm{Dom}(M),$$

such that

$$S_M\big(\mu_{\widetilde{E}_M}(\omega)\big) \le S_M\big(\mu_{\widetilde{S}_M}(\omega)\big) \qquad \text{for all } \omega \in \Omega.$$

(3) Score-weighted size. For any uncertain set

$$\widetilde{A}_M = (\Omega, \mu_{\widetilde{A}_M})$$

of type M on Ω, define its *score-weighted size* by

$$|\widetilde{A}_M|_{S_M} := \sum_{\omega \in \Omega} S_M\big(\mu_{\widetilde{A}_M}(\omega)\big).$$

(4) Uncertain probability. The *uncertain probability* of the uncertain event $\widetilde{E}_M$ relative to the uncertain sample space $\widetilde{S}_M$ is defined by

$$P_M(\widetilde{E}_M) := \frac{|\widetilde{E}_M|_{S_M}}{|\widetilde{S}_M|_{S_M}}.$$

By the normalization of $\widetilde{S}_M$, this becomes

$$P_M(\widetilde{E}_M) = \sum_{\omega \in \Omega} S_M\big(\mu_{\widetilde{E}_M}(\omega)\big).$$

Remark 5.1.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

then Definition 5.1.3 reduces exactly to the normalized weighted-cardinality model of fuzzy probability:

$$P_f(\widetilde{E}) = \sum_{\omega \in \Omega} \mu_{\widetilde{E}}(\omega).$$

Thus fuzzy probability is a special case of uncertain probability.

Remark 5.1.5 (Reduction to a classical probability mass function)**.** If M is the fuzzy model,

$$\mu_{\widetilde{S}}(\omega) = p(\omega)$$

is a classical probability mass function on Ω, and

$$\mu_{\widetilde{E}}(\omega) = \mathbb{1}_E(\omega)\, p(\omega)$$

for a crisp event $E \subseteq \Omega$, then

$$P_f(\widetilde{E}) = \sum_{\omega \in E} p(\omega),$$

which is the ordinary probability of E.

Theorem 5.1.6 (Well-definedness of uncertain probability)**.** *Let Ω be a finite nonempty set, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

be a score map. Let

$$\widetilde{S}_M = (\Omega, \mu_{\widetilde{S}_M})$$

be an uncertain sample space of type M, and let

$$\widetilde{E}_M = (\Omega, \mu_{\widetilde{E}_M})$$

be an uncertain event relative to $\widetilde{S}_M$. Then:

1. *the score-weighted sizes*

   $$|\widetilde{S}_M|_{S_M} \qquad \text{and} \qquad |\widetilde{E}_M|_{S_M}$$

   are well-defined real numbers;
2. *one has*

   $$0 \leq |\widetilde{E}_M|_{S_M} \leq |\widetilde{S}_M|_{S_M} = 1;$$

3. *therefore*

   $$P_M(\widetilde{E}_M) = \frac{|\widetilde{E}_M|_{S_M}}{|\widetilde{S}_M|_{S_M}}$$

   is a well-defined real number in $[0,1]$.

Hence the notion of uncertain probability is mathematically well-defined.

Proof. Since Ω is finite and

$$\mu_{\widetilde{S}_M}, \mu_{\widetilde{E}_M} : \Omega \to \mathrm{Dom}(M),$$

for every $\omega \in \Omega$ the values

$$\mu_{\widetilde{S}_M}(\omega), \qquad \mu_{\widetilde{E}_M}(\omega)$$

belong to $\mathrm{Dom}(M)$. Applying the score map

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

we obtain

$$S_M\big(\mu_{\widetilde{S}_M}(\omega)\big) \in [0,1], \qquad S_M\big(\mu_{\widetilde{E}_M}(\omega)\big) \in [0,1]$$

for all $\omega \in \Omega$.

Because Ω is finite, the sums

$$|\widetilde{S}_M|_{S_M} = \sum_{\omega \in \Omega} S_M\big(\mu_{\widetilde{S}_M}(\omega)\big)$$

and

$$|\widetilde{E}_M|_{S_M} = \sum_{\omega \in \Omega} S_M\big(\mu_{\widetilde{E}_M}(\omega)\big)$$

are finite sums of real numbers in $[0,1]$, hence are well-defined real numbers. This proves (1).

Next, by the definition of an uncertain event,

$$S_M\big(\mu_{\widetilde{E}_M}(\omega)\big) \le S_M\big(\mu_{\widetilde{S}_M}(\omega)\big) \qquad \text{for all } \omega \in \Omega.$$

Summing over Ω, we obtain

$$|\widetilde{E}_M|_{S_M} \le |\widetilde{S}_M|_{S_M}.$$

Also, each term in the sum defining $|\widetilde{E}_M|_{S_M}$ is nonnegative, so

$$|\widetilde{E}_M|_{S_M} \ge 0.$$

By the normalization condition on $\widetilde{S}_M$,

$$|\widetilde{S}_M|_{S_M} = \sum_{\omega\in\Omega} S_M\big(\mu_{\widetilde{S}_M}(\omega)\big) = 1.$$

Hence

$$0 \le |\widetilde{E}_M|_{S_M} \le 1.$$

This proves (2).

Finally, since

$$|\widetilde{S}_M|_{S_M} = 1 \ne 0,$$

the quotient

$$P_M(\widetilde{E}_M) = \frac{|\widetilde{E}_M|_{S_M}}{|\widetilde{S}_M|_{S_M}}$$

is well-defined. Moreover, by (2),

$$0 \le P_M(\widetilde{E}_M) \le 1.$$

Thus $P_M(\widetilde{E}_M)$ is a well-defined real number in $[0,1]$. This proves (3).

Therefore the notion of uncertain probability is mathematically well-defined. □

A catalogue of fuzzy-probability families is presented in Table 5.2.

Table 5.2: A catalogue of fuzzy-probability families by the dimension k of the degree-domain $\mathrm{Dom}(P) \subseteq [0,1]^k$ (conceptual overview).

k	Representative fuzzy-probability type(s) with $P_M : \mathcal{E} \to \mathrm{Dom}(P) \subseteq [0,1]^k$
1	Fuzzy probability $P_F(E) \in [0,1]$ (graded likelihood).
2	Intuitionistic fuzzy probability $\big(P_T(E), P_F(E)\big) \in [0,1]^2$ with $P_T(E) + P_F(E) \le 1$ [541, 542]; orthopair-style variants (e.g. Pythagorean/q-rung) under alternative constraints on (P_T, P_F).
3	Hesitant fuzzy probability (3-valued hesitant profile, e.g. three representative plausible probabilities per event)[(a)] [6, 544]; neutrosophic probability $\big(P_T(E), P_I(E), P_F(E)\big) \in [0,1]^3$ (truth/indeterminacy/falsity likelihoods), typically with $0 \le P_T + P_I + P_F \le 3$ [545–551].
$s+t$	Plithogenic probability: an attribute-aware degree vector in $[0,1]^s$ together with contradiction coordinates in $[0,1]^t$ (concatenated as a $[0,1]^{s+t}$-vector for bookkeeping)[(b)] [552, 553].

[(a)] Hesitant fuzzy probability is naturally set-valued ($H(E) \subseteq [0,1]$). Placing it at $k = 3$ means adopting a fixed 3-tuple representation, e.g. restricting to hesitant sets of size 3, or extracting three canonical/summary values (such as min/mean/max) from $H(E)$.

[(b)] In plithogenic settings, contradiction information is often given by a separate function on attribute values; concatenating appurtenance and contradiction coordinates yields a single degree-domain $\subseteq [0,1]^{s+t}$ suitable for "dimension-k" cataloguing.

In addition, besides Uncertain Probability, related concepts such as Conditional Probability [554–556], Imprecise Probability [557, 558], HyperProbability [334, 559, 560], Geometric Probability [561, 562], and Joint Probability [563, 564] are also known.

5.2 Uncertain Metric spaces

A fuzzy metric space replaces distance by a function $M(x, y, t) \in [0, 1]$ measuring nearness over time, satisfying symmetry, triangle-type t-norm inequality, and convergence [565–567].

Definition 5.2.1 (Continuous triangular norm (t-norm)). [568] A mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a *(continuous) t-norm* if:

(T1) $a * b = b * a$ for all $a, b \in [0, 1]$ (commutativity),

(T2) $a * 1 = a$ for all $a \in [0, 1]$ (unit element),

(T3) $(a * b) * c = a * (b * c)$ for all $a, b, c \in [0, 1]$ (associativity),

(T4) if $a \le c$ and $b \le d$, then $a * b \le c * d$ (monotonicity),

(T5) $*$ is continuous on $[0, 1]^2$.

Definition 5.2.2 (Fuzzy metric space (Kramosil–Michálek)). [565–567] Let X be a nonempty set and let $*$ be a continuous t-norm. A mapping

$$M : X \times X \times [0, \infty) \longrightarrow [0, 1]$$

is called a *fuzzy metric* on X if for all $x, y, z \in X$ the following hold:

(FM1) $M(x, y, 0) = 0$;

(FM2) $\big[M(x, y, t) = 1 \text{ for all } t > 0\big] \iff x = y$;

(FM3) $M(x, y, t) = M(y, x, t)$ for all $t \ge 0$;

(FM4) $M(x, z, t + s) \ge M(x, y, t) * M(y, z, s)$ for all $t, s \ge 0$;

(FM5) for each $x, y \in X$, the function $t \mapsto M(x, y, t)$ is left-continuous on $[0, \infty)$ and

$$\lim_{t \to \infty} M(x, y, t) = 1.$$

The triple $(X, M, *)$ is called a *fuzzy metric space.*

An uncertain metric space is a set equipped with a time-dependent uncertainty-valued nearness function, viewed as a family of uncertain sets on the Cartesian product, whose score-realization satisfies fuzzy-metric-type axioms.

Definition 5.2.3 (Uncertain metric space)**.** Let X be a nonempty set, let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let $*$ be a continuous t-norm on $[0,1]$.

A mapping

$$\mathfrak{M}_M : X \times X \times [0,\infty) \to \mathrm{Dom}(M)$$

is called an *uncertain metric of type M* on X if the following conditions hold.

For each $t \geq 0$, define

$$\mathfrak{M}_M^t : X \times X \to \mathrm{Dom}(M), \qquad \mathfrak{M}_M^t(x,y) := \mathfrak{M}_M(x,y,t).$$

Then each

$$(X \times X, \mathfrak{M}_M^t)$$

is an uncertain set of type M on $X \times X$.

Fix an admissible score map

$$S_M : \mathrm{Dom}(M) \to [0,1].$$

Write

$$\overline{\mathfrak{M}}(x,y,t) := S_M\big(\mathfrak{M}_M(x,y,t)\big) \qquad (x,y \in X,\ t \geq 0).$$

We require that, for all $x,y,z \in X$, the following hold:

1. $$\overline{\mathfrak{M}}(x,y,0) = 0;$$
2. $$\big[\overline{\mathfrak{M}}(x,y,t) = 1 \text{ for all } t > 0\big] \quad \Longleftrightarrow \quad x = y;$$
3. $$\overline{\mathfrak{M}}(x,y,t) = \overline{\mathfrak{M}}(y,x,t) \qquad \text{for all } t \geq 0;$$
4. $$\overline{\mathfrak{M}}(x,z,t+s) \geq \overline{\mathfrak{M}}(x,y,t) * \overline{\mathfrak{M}}(y,z,s) \qquad \text{for all } t,s \geq 0;$$
5. for each fixed $x,y \in X$, the function
$$t \longmapsto \overline{\mathfrak{M}}(x,y,t)$$
is left-continuous on $[0,\infty)$, and
$$\lim_{t\to\infty} \overline{\mathfrak{M}}(x,y,t) = 1.$$

In this case, the triple

$$(X, \mathfrak{M}_M, *)$$

is called an *uncertain metric space of type M*.

Remark 5.2.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

then Definition 5.2.3 reduces exactly to the usual Kramosil–Michálek fuzzy metric space. Thus fuzzy metric spaces are special cases of uncertain metric spaces.

Theorem 5.2.5 (Well-definedness of uncertain metric spaces)**.** *Let X be a nonempty set, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

let

$$\mathfrak{M}_M : X \times X \times [0,\infty) \to \mathrm{Dom}(M)$$

be a mapping, let

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

be a score map, and let $*$ be a continuous t-norm on $[0,1]$. Then:

1. *for each $t \geq 0$, the map*
$$\mathfrak{M}_M^t : X \times X \to \mathrm{Dom}(M)$$
is a well-defined uncertain set of type M on $X \times X$;
2. *the score-realization*
$$\overline{\mathfrak{M}} : X \times X \times [0,\infty) \to [0,1], \qquad \overline{\mathfrak{M}}(x,y,t) = S_M\big(\mathfrak{M}_M(x,y,t)\big),$$
is a well-defined mapping;
3. *all expressions appearing in Definition 5.2.3 are meaningful real numbers in $[0,1]$;*
4. *consequently, the notion of uncertain metric space is mathematically well-defined.*

Moreover, if the conditions in Definition 5.2.3 hold, then

$$(X, \overline{\mathfrak{M}}, *)$$

is a well-defined fuzzy metric space.

Proof. For each fixed $t \geq 0$, define

$$\mathfrak{M}_M^t(x,y) := \mathfrak{M}_M(x,y,t) \qquad (x,y \in X).$$

Since

$$\mathfrak{M}_M : X \times X \times [0,\infty) \to \mathrm{Dom}(M),$$

it follows that for every $x, y \in X$,

$$\mathfrak{M}_M^t(x,y) \in \mathrm{Dom}(M).$$

Hence

$$\mathfrak{M}_M^t : X \times X \to \mathrm{Dom}(M)$$

is a well-defined map, that is, a well-defined uncertain set of type M on $X \times X$. This proves (1).

Next, because
$$S_M : \mathrm{Dom}(M) \to [0,1],$$
for every $(x, y, t) \in X \times X \times [0, \infty)$ one has
$$S_M\big(\mathfrak{M}_M(x, y, t)\big) \in [0, 1].$$
Therefore
$$\overline{\mathfrak{M}}(x, y, t) := S_M\big(\mathfrak{M}_M(x, y, t)\big)$$
defines a well-defined mapping
$$\overline{\mathfrak{M}} : X \times X \times [0, \infty) \to [0, 1].$$
This proves (2).

Now every quantity appearing in
$$\overline{\mathfrak{M}}(x, y, 0), \qquad \overline{\mathfrak{M}}(x, y, t), \qquad \overline{\mathfrak{M}}(y, x, t), \qquad \overline{\mathfrak{M}}(x, z, t+s)$$
is a real number in $[0, 1]$. Since
$$* : [0,1]^2 \to [0,1]$$
is a t-norm, the product
$$\overline{\mathfrak{M}}(x, y, t) * \overline{\mathfrak{M}}(y, z, s)$$
is also a well-defined element of $[0, 1]$. Thus the triangle-type inequality
$$\overline{\mathfrak{M}}(x, z, t+s) \geq \overline{\mathfrak{M}}(x, y, t) * \overline{\mathfrak{M}}(y, z, s)$$
is meaningful.

Likewise, the statements
$$\overline{\mathfrak{M}}(x, y, 0) = 0,$$
$$\overline{\mathfrak{M}}(x, y, t) = \overline{\mathfrak{M}}(y, x, t),$$
and
$$\big[\overline{\mathfrak{M}}(x, y, t) = 1 \text{ for all } t > 0\big] \iff x = y$$
are meaningful, since all occurring terms are well-defined real numbers. Also, for fixed $x, y \in X$, the map
$$t \mapsto \overline{\mathfrak{M}}(x, y, t)$$
is an ordinary real-valued function on $[0, \infty)$, so left-continuity and the limit
$$\lim_{t \to \infty} \overline{\mathfrak{M}}(x, y, t) = 1$$
are standard well-defined analytic notions. This proves (3).

Therefore all objects and all conditions in Definition 5.2.3 are mathematically meaningful, so the notion of uncertain metric space is well-defined. This proves (4).

Finally, if the axioms in Definition 5.2.3 hold, then the score-realization $\overline{\mathfrak{M}}$ satisfies precisely the Kramosil–Michálek fuzzy metric axioms with respect to the t-norm $*$. Hence
$$(X, \overline{\mathfrak{M}}, *)$$
is a well-defined fuzzy metric space. □

A catalogue of representative uncertainty-metric-space families by the dimension k of the degree-domain is presented in Table 5.3.

Table 5.3: A catalogue of representative uncertainty-metric-space families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-metric-space family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy Metric Spaces:* $\mathfrak{M}_M : X \times X \times [0,\infty) \to [0,1]$.
2	*Intuitionistic Fuzzy Metric Spaces [567, 569]:* $\mathfrak{M}_M : X \times X \times [0,\infty) \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic Metric Spaces [570–575]:* $\mathfrak{M}_M : X \times X \times [0,\infty) \to [0,1]^3$ (e.g., (T, I, F)).
k	*k-component uncertainty metric spaces:* $\mathfrak{M}_M : X \times X \times [0,\infty) \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

5.3 Uncertain distance measure

Fuzzy distance measure is a normalized metric on fuzzy sets, quantifying dissimilarity through bounded, symmetric, identity-preserving, and triangle-inequality-respecting comparisons under graded uncertainty mathematically [576–578].

Definition 5.3.1 (Normalized fuzzy distance measure between fuzzy sets)**.** [579–581] Let $U \neq \varnothing$ be a universe and let

$$\mathcal{F}(U) := [0,1]^U = \{\mu : U \to [0,1]\}$$

be the family of all fuzzy sets on U. A mapping

$$d : \mathcal{F}(U) \times \mathcal{F}(U) \longrightarrow [0,1]$$

is called a *(normalized) fuzzy distance measure* if for all $A, B, C \in \mathcal{F}(U)$:

(FD1) $0 \leq d(A,B) \leq 1$;

(FD2) $d(A,B) = 0 \iff A = B$;

(FD3) $d(A,B) = d(B,A)$;

(FD4) $d(A,C) \leq d(A,B) + d(B,C)$.

An uncertain distance measure compares uncertain sets by first transforming their uncertainty degrees into scalar scores in $[0,1]$, and then applying a normalized fuzzy distance measure.

Definition 5.3.2 (Uncertain distance measure)**.** Let $U \neq \varnothing$ be a universe, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Let

$$\mathcal{U}_M(U) := \{\mu : U \to \mathrm{Dom}(M)\}$$

denote the family of all uncertain sets of type M on U.

Assume that an injective score map

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

is fixed, and let

$$d : \mathcal{F}(U) \times \mathcal{F}(U) \to [0, 1]$$

be a normalized fuzzy distance measure, where

$$\mathcal{F}(U) := [0, 1]^U.$$

For uncertain sets

$$\widetilde{A}_M = (U, \mu_{\widetilde{A}_M}), \qquad \widetilde{B}_M = (U, \mu_{\widetilde{B}_M}),$$

define their score-induced fuzzy sets

$$\overline{A}_M, \overline{B}_M \in \mathcal{F}(U)$$

by

$$\mu_{\overline{A}_M}(x) := S_M\big(\mu_{\widetilde{A}_M}(x)\big), \qquad \mu_{\overline{B}_M}(x) := S_M\big(\mu_{\widetilde{B}_M}(x)\big) \qquad (x \in U).$$

Then the mapping

$$d_M : \mathcal{U}_M(U) \times \mathcal{U}_M(U) \to [0, 1]$$

defined by

$$d_M(\widetilde{A}_M, \widetilde{B}_M) := d(\overline{A}_M, \overline{B}_M)$$

is called an *uncertain distance measure of type M on U*.

Remark 5.3.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0, 1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

then

$$d_M(\widetilde{A}, \widetilde{B}) = d(\widetilde{A}, \widetilde{B}),$$

so the above definition reduces exactly to the usual normalized fuzzy distance measure.

Theorem 5.3.4 (Well-definedness of uncertain distance measures)**.** *Let $U \neq \varnothing$, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

let

$$S_M : \mathrm{Dom}(M) \to [0, 1]$$

be an injective score map, and let

$$d : \mathcal{F}(U) \times \mathcal{F}(U) \to [0, 1]$$

be a normalized fuzzy distance measure. Then the mapping

$$d_M : \mathcal{U}_M(U) \times \mathcal{U}_M(U) \to [0, 1], \qquad d_M(\widetilde{A}_M, \widetilde{B}_M) := d(\overline{A}_M, \overline{B}_M),$$

is a well-defined normalized distance measure on $\mathcal{U}_M(U)$. More precisely, for all

$$\widetilde{A}_M, \widetilde{B}_M, \widetilde{C}_M \in \mathcal{U}_M(U),$$

the following hold:

1.

$$0 \leq d_M(\widetilde{A}_M, \widetilde{B}_M) \leq 1;$$

2.
$$d_M(\widetilde{A}_M, \widetilde{B}_M) = 0 \quad \Longleftrightarrow \quad \widetilde{A}_M = \widetilde{B}_M;$$

3.
$$d_M(\widetilde{A}_M, \widetilde{B}_M) = d_M(\widetilde{B}_M, \widetilde{A}_M);$$

4.
$$d_M(\widetilde{A}_M, \widetilde{C}_M) \le d_M(\widetilde{A}_M, \widetilde{B}_M) + d_M(\widetilde{B}_M, \widetilde{C}_M).$$

Hence the notion of uncertain distance measure is mathematically well-defined.

Proof. Let
$$\widetilde{A}_M = (U, \mu_{\widetilde{A}_M}), \qquad \widetilde{B}_M = (U, \mu_{\widetilde{B}_M})$$
be arbitrary uncertain sets of type M on U.

Since
$$\mu_{\widetilde{A}_M}, \mu_{\widetilde{B}_M} : U \to \mathrm{Dom}(M)$$
and
$$S_M : \mathrm{Dom}(M) \to [0,1],$$
the compositions
$$\mu_{\overline{A}_M} = S_M \circ \mu_{\widetilde{A}_M}, \qquad \mu_{\overline{B}_M} = S_M \circ \mu_{\widetilde{B}_M}$$
are well-defined mappings from U to $[0,1]$. Therefore
$$\overline{A}_M, \overline{B}_M \in \mathcal{F}(U)$$
are well-defined fuzzy sets. Since
$$d : \mathcal{F}(U) \times \mathcal{F}(U) \to [0,1],$$
it follows that
$$d_M(\widetilde{A}_M, \widetilde{B}_M) = d(\overline{A}_M, \overline{B}_M)$$
is a well-defined real number in $[0,1]$. This proves (1).

For (2), assume first that
$$d_M(\widetilde{A}_M, \widetilde{B}_M) = 0.$$
Then
$$d(\overline{A}_M, \overline{B}_M) = 0.$$
Because d is a normalized fuzzy distance measure, one obtains
$$\overline{A}_M = \overline{B}_M,$$
that is,
$$S_M\big(\mu_{\widetilde{A}_M}(x)\big) = S_M\big(\mu_{\widetilde{B}_M}(x)\big) \qquad \text{for all } x \in U.$$
Since S_M is injective, it follows that
$$\mu_{\widetilde{A}_M}(x) = \mu_{\widetilde{B}_M}(x) \qquad \text{for all } x \in U.$$

Hence
$$\widetilde{A}_M = \widetilde{B}_M.$$
Conversely, if
$$\widetilde{A}_M = \widetilde{B}_M,$$
then clearly
$$\overline{A}_M = \overline{B}_M,$$
and therefore
$$d_M(\widetilde{A}_M, \widetilde{B}_M) = d(\overline{A}_M, \overline{B}_M) = 0.$$

For (3),
$$d_M(\widetilde{A}_M, \widetilde{B}_M) = d(\overline{A}_M, \overline{B}_M) = d(\overline{B}_M, \overline{A}_M) = d_M(\widetilde{B}_M, \widetilde{A}_M),$$
because d is symmetric.

For (4), let
$$\widetilde{C}_M = (U, \mu_{\widetilde{C}_M}) \in \mathcal{U}_M(U).$$
Then
$$d_M(\widetilde{A}_M, \widetilde{C}_M) = d(\overline{A}_M, \overline{C}_M).$$
Since d satisfies the triangle inequality,
$$d(\overline{A}_M, \overline{C}_M) \le d(\overline{A}_M, \overline{B}_M) + d(\overline{B}_M, \overline{C}_M).$$
Hence
$$d_M(\widetilde{A}_M, \widetilde{C}_M) \le d_M(\widetilde{A}_M, \widetilde{B}_M) + d_M(\widetilde{B}_M, \widetilde{C}_M).$$

Thus all axioms of a normalized distance measure are satisfied, and the uncertain distance measure is well-defined. □

5.4 Uncertain geometry

Fuzzy geometry studies geometric objects and relations with graded membership or incidence, extending points, lines, regions, and spatial structure to uncertain and imprecise settings [582–585].

Definition 5.4.1 (Fuzzy geometry (incidence-style model on $\mathbb{R}^2$))**.** [582, 583] Let $X = \mathbb{R}^2$ and let
$$\mathcal{F}(X) := \{\, A : X \to [0,1] \,\}$$
denote the set of all fuzzy subsets of X.

(1) Underlying fuzzy topological space. A *(proper) fuzzy topology* on X is a family $\mathcal{T} \subseteq \mathcal{F}(X)$ satisfying:
$$k_0, k_1 \in \mathcal{T}, \qquad A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}, \qquad \{A_j\}_{j\in J} \subseteq \mathcal{T} \Rightarrow \bigcup_{j\in J} A_j \in \mathcal{T},$$
where $(A \cap B)(x) := \min\{A(x), B(x)\}$ and $\left(\bigcup_{j\in J} A_j\right)(x) := \sup_{j\in J} A_j(x)$, and $k_c(x) \equiv c$ for $c \in [0,1]$ ("proper" means $k_c \in \mathcal{T}$ for every $c \in [0,1]$).[1]

(2) Fuzzy points and fuzzy lines. Fix parameters $\sigma > 0$ and $d_0 > 0$.

[1]This is the standard fuzzy-topology closure pattern under min-intersection and sup-union.

- For each $p_0 \in X$, a *fuzzy point centered at* p_0 is the fuzzy set

$$P_{p_0}(x) := \exp\left(-\frac{\|x - p_0\|^2}{\sigma^2}\right), \qquad x \in X.$$

- For coefficients $(a, b, c) \in \mathbb{R}^3$ with $(a, b) \neq (0, 0)$, a *fuzzy line* is the fuzzy set

$$L_{a,b,c}(x) := \max\left\{0,\ 1 - \frac{|ax_1 + bx_2 + c|}{d_0}\right\}, \qquad x = (x_1, x_2) \in X.$$

(3) Fuzzy incidence. Define the *fuzzy incidence degree*

$$I : \mathcal{P} \times \mathcal{L} \to [0, 1], \qquad I\big(P_{p_0}, L_{a,b,c}\big) := L_{a,b,c}(p_0),$$

where $\mathcal{P} := \{P_{p_0} : p_0 \in X\}$ and $\mathcal{L} := \{L_{a,b,c} : (a, b) \neq (0, 0),\ c \in \mathbb{R}\}$.

The tuple

$$\mathsf{FG} := (X, \mathcal{T}, \mathcal{P}, \mathcal{L}, I)$$

is called a *fuzzy geometry* (of point–line incidence type) on X.

Uncertain geometry studies geometric objects and incidence relations under a general uncertainty model, by replacing fuzzy subsets with uncertain sets whose score-realizations recover ordinary fuzzy geometric profiles.

Definition 5.4.2 (Uncertain geometry (incidence-style model on $\mathbb{R}^2$))**.** Let

$$X = \mathbb{R}^2,$$

and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Denote by

$$\mathcal{U}_M(X) := \{\mu : X \to \mathrm{Dom}(M)\}$$

the family of all uncertain sets of type M on X.

Assume that the following data are fixed:

- a score map

$$S_M : \mathrm{Dom}(M) \to [0, 1];$$

- a lift map

$$L_M : [0, 1] \to \mathrm{Dom}(M)$$

  such that

$$S_M\big(L_M(r)\big) = r \qquad \text{for all } r \in [0, 1].$$

For every uncertain set

$$A_M = (X, \mu_{A_M}), \qquad \mu_{A_M} \in \mathcal{U}_M(X),$$

define its *score-image* $\overline{A}_M \in [0,1]^X$ by

$$\mu_{\overline{A}_M}(x) := S_M\big(\mu_{A_M}(x)\big) \qquad (x \in X).$$

(1) Underlying uncertain topological space. An *uncertain topology of type* M on X is a family

$$\mathcal{T}_M \subseteq \mathcal{U}_M(X)$$

such that its score-image family

$$\overline{\mathcal{T}}_M := \{\overline{A}_M \mid A_M \in \mathcal{T}_M\} \subseteq [0,1]^X$$

is a proper fuzzy topology on X; namely, if $k_c(x) \equiv c$ for $c \in [0,1]$, then

$$k_c \in \overline{\mathcal{T}}_M \qquad (c \in [0,1]),$$

$$A, B \in \overline{\mathcal{T}}_M \implies A \cap B \in \overline{\mathcal{T}}_M,$$

$$\{A_j\}_{j \in J} \subseteq \overline{\mathcal{T}}_M \implies \bigcup_{j \in J} A_j \in \overline{\mathcal{T}}_M,$$

where

$$(A \cap B)(x) := \min\{A(x), B(x)\}, \qquad \Big(\bigcup_{j \in J} A_j\Big)(x) := \sup_{j \in J} A_j(x).$$

(2) Uncertain points and uncertain lines. Fix parameters

$$\sigma > 0, \qquad d_0 > 0.$$

For each $p_0 \in X$, define the *uncertain point centered at* p_0 as the uncertain set

$$P^M_{p_0} = (X, \mu^M_{P_{p_0}})$$

with

$$\mu^M_{P_{p_0}}(x) := L_M\left(\exp\left(-\frac{\|x - p_0\|^2}{\sigma^2}\right)\right), \qquad x \in X.$$

For coefficients $(a, b, c) \in \mathbb{R}^3$ with $(a, b) \neq (0, 0)$, define the *uncertain line* as the uncertain set

$$L^M_{a,b,c} = (X, \mu^M_{L_{a,b,c}})$$

with

$$\mu^M_{L_{a,b,c}}(x) := L_M\left(\max\left\{0,\, 1 - \frac{|ax_1 + bx_2 + c|}{d_0}\right\}\right), \qquad x = (x_1, x_2) \in X.$$

Set

$$\mathcal{P}_M := \{P^M_{p_0} : p_0 \in X\}, \qquad \mathcal{L}_M := \{L^M_{a,b,c} : (a, b) \neq (0, 0),\ c \in \mathbb{R}\}.$$

(3) Uncertain incidence. Define the *uncertain incidence degree*

$$I_M : \mathcal{P}_M \times \mathcal{L}_M \to \mathrm{Dom}(M)$$

by

$$I_M(P^M_{p_0}, L^M_{a,b,c}) := \mu^M_{L_{a,b,c}}(p_0) = L_M\left(\max\left\{0,\ 1 - \frac{|ap_{0,1} + bp_{0,2} + c|}{d_0}\right\}\right),$$

where $p_0 = (p_{0,1}, p_{0,2})$.

The tuple

$$\mathsf{UG}_M := (X, \mathcal{T}_M, \mathcal{P}_M, \mathcal{L}_M, I_M)$$

is called an *uncertain geometry of point–line incidence type* on X.

Theorem 5.4.3 (Well-definedness of uncertain geometry)**.** *Let M, S_M, L_M, σ, d_0, and $\mathcal{T}_M$ be as in Definition 5.4.2. Then:*

1. *for every $p_0 \in X$, the uncertain point*
$$P^M_{p_0} = (X, \mu^M_{P_{p_0}})$$
is a well-defined uncertain set of type M on X;
2. *for every $(a, b, c) \in \mathbb{R}^3$ with $(a, b) \neq (0, 0)$, the uncertain line*
$$L^M_{a,b,c} = (X, \mu^M_{L_{a,b,c}})$$
is a well-defined uncertain set of type M on X;
3. *the incidence map*
$$I_M : \mathcal{P}_M \times \mathcal{L}_M \to \mathrm{Dom}(M)$$
is well-defined;
4. *the score-images of uncertain points and uncertain lines are exactly the corresponding fuzzy point and fuzzy line profiles:*
$$S_M(\mu^M_{P_{p_0}}(x)) = \exp\left(-\frac{\|x - p_0\|^2}{\sigma^2}\right),$$
$$S_M(\mu^M_{L_{a,b,c}}(x)) = \max\left\{0,\ 1 - \frac{|ax_1 + bx_2 + c|}{d_0}\right\};$$
5. *consequently, UG_M is a mathematically well-defined uncertain geometry, and its score-realization is an ordinary fuzzy geometry of the same point–line incidence type.*

Proof. (1) Fix $p_0 \in X$. For every $x \in X$,

$$\|x - p_0\|^2 \geq 0 \qquad \text{and} \qquad \sigma > 0,$$

so

$$\exp\left(-\frac{\|x - p_0\|^2}{\sigma^2}\right) \in (0, 1] \subseteq [0, 1].$$

Since

$$L_M : [0, 1] \to \mathrm{Dom}(M),$$

it follows that

$$\mu^M_{P_{p_0}}(x) = L_M\left(\exp\left(-\frac{\|x-p_0\|^2}{\sigma^2}\right)\right) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Hence

$$\mu^M_{P_{p_0}} : X \to \mathrm{Dom}(M)$$

is a well-defined map, so $P^M_{p_0}$ is a well-defined uncertain set of type M on X.

(2) Fix $(a,b,c) \in \mathbb{R}^3$ with $(a,b) \neq (0,0)$. For every $x = (x_1,x_2) \in X$, the quantity

$$|ax_1 + bx_2 + c|$$

is a well-defined nonnegative real number. Since $d_0 > 0$,

$$1 - \frac{|ax_1 + bx_2 + c|}{d_0}$$

is well-defined, and therefore

$$\max\left\{0,\, 1 - \frac{|ax_1 + bx_2 + c|}{d_0}\right\} \in [0,1].$$

Applying L_M, we obtain

$$\mu^M_{L_{a,b,c}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Thus

$$\mu^M_{L_{a,b,c}} : X \to \mathrm{Dom}(M)$$

is a well-defined map, so $L^M_{a,b,c}$ is a well-defined uncertain set of type M on X.

(3) Let

$$P^M_{p_0} \in \mathcal{P}_M, \qquad L^M_{a,b,c} \in \mathcal{L}_M.$$

Since $p_0 \in X$ and

$$\mu^M_{L_{a,b,c}} : X \to \mathrm{Dom}(M),$$

the value

$$\mu^M_{L_{a,b,c}}(p_0)$$

is a well-defined element of $\mathrm{Dom}(M)$. Therefore

$$I_M\big(P^M_{p_0}, L^M_{a,b,c}\big) = \mu^M_{L_{a,b,c}}(p_0)$$

is well-defined. Hence

$$I_M : \mathcal{P}_M \times \mathcal{L}_M \to \mathrm{Dom}(M)$$

is well-defined.

(4) By the defining property

$$S_M\big(L_M(r)\big) = r \qquad (r \in [0,1]),$$

we have, for every $x \in X$,

$$S_M\big(\mu^M_{P_{p_0}}(x)\big) = S_M\left(L_M\left(\exp\left(-\frac{\|x-p_0\|^2}{\sigma^2}\right)\right)\right) = \exp\left(-\frac{\|x-p_0\|^2}{\sigma^2}\right),$$

and

$$S_M(\mu^M_{L_{a,b,c}}(x)) = S_M\left(L_M\left(\max\left\{0,\, 1 - \frac{|ax_1 + bx_2 + c|}{d_0}\right\}\right)\right) = \max\left\{0,\, 1 - \frac{|ax_1 + bx_2 + c|}{d_0}\right\}.$$

Thus the score-images of the uncertain point and uncertain line are precisely the corresponding fuzzy point and fuzzy line profiles.

Similarly,

$$S_M(I_M(P^M_{p_0}, L^M_{a,b,c})) = \max\left\{0,\, 1 - \frac{|ap_{0,1} + bp_{0,2} + c|}{d_0}\right\},$$

which is exactly the fuzzy incidence degree of the corresponding fuzzy point and fuzzy line.

(5) By assumption, $\mathcal{T}_M \subseteq \mathcal{U}_M(X)$ is such that its score-image family

$$\overline{\mathcal{T}}_M = \{\overline{A}_M : A_M \in \mathcal{T}_M\}$$

is a proper fuzzy topology on X. By parts (1)–(4), all other constituents

$$\mathcal{P}_M, \qquad \mathcal{L}_M, \qquad I_M$$

are well-defined. Therefore the tuple

$$\mathsf{UG}_M = (X, \mathcal{T}_M, \mathcal{P}_M, \mathcal{L}_M, I_M)$$

is mathematically well-defined.

Moreover, its score-realization

$$(X, \overline{\mathcal{T}}_M, \overline{\mathcal{P}}_M, \overline{\mathcal{L}}_M, \overline{I}_M)$$

is exactly a fuzzy geometry of point–line incidence type on X. Hence the uncertain geometry is well-defined. □

Remark 5.4.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1], \qquad S_M = \mathrm{id}_{[0,1]}, \qquad L_M = \mathrm{id}_{[0,1]},$$

then Definition 5.4.2 reduces exactly to the original fuzzy geometry model.

Uncertain Geometry can generalize Intuitionistic Fuzzy Geometry, Neutrosophic Geometry [586, 587], and related frameworks.

5.5 Uncertain Differential Equations

A *fuzzy differential equation (FDE)* is a differential equation in which uncertainty is modeled by fuzzy sets (typically fuzzy numbers), e.g. via fuzzy coefficients/parameters and/or fuzzy initial/boundary data [588–590]. In many works, the class of fuzzy sets is taken to be the class of fuzzy numbers (normal, convex, upper semicontinuous, compactly supported fuzzy subsets of $\mathbb{R}$).

Definition 5.5.1 (Hukuhara difference and (lateral) fuzzy H-derivative)**.** Let F^n denote the space of compact, convex fuzzy subsets of $\mathbb{R}^n$, equipped with the usual level-set structure and the metric D induced from the Hausdorff metric on α-cuts.

(1) Hukuhara (H) difference. For $u, v \in F^n$, if there exists $w \in F^n$ such that $u = v + w$, then w is called the *Hukuhara difference* of u and v and is denoted by $u \ominus_H v := w$.

(2) Lateral H-derivative (two forms). Let $T = (a, b)$ and let $X : T \to F^n$ be a fuzzy mapping. We say that X is (lateral) H-differentiable at $t_0 \in T$ if there exists $X'(t_0) \in F^n$ such that either

(Form I) for all sufficiently small $h > 0$, the differences $X(t_0 + h) \ominus_H X(t_0)$ and $X(t_0) \ominus_H X(t_0 - h)$ exist and

$$\lim_{h \to 0^+} \frac{X(t_0 + h) \ominus_H X(t_0)}{h} = \lim_{h \to 0^+} \frac{X(t_0) \ominus_H X(t_0 - h)}{h} = X'(t_0),$$

or

(Form II) for all sufficiently small $h < 0$, the differences $X(t_0 + h) \ominus_H X(t_0)$ and $X(t_0) \ominus_H X(t_0 - h)$ exist and

$$\lim_{h \to 0^-} \frac{X(t_0 + h) \ominus_H X(t_0)}{h} = \lim_{h \to 0^-} \frac{X(t_0) \ominus_H X(t_0 - h)}{h} = X'(t_0),$$

where limits are taken in the metric space (F^n, D) .

Definition 5.5.2 (Fuzzy differential equation (Cauchy/initial value problem))**.** Let $[0, a] \subset \mathbb{R}$ be a time interval, let $x_0 \in F^n$ be a fuzzy initial value, and let

$$F : [0, a] \times F^n \longrightarrow F^n$$

be a continuous fuzzy mapping. The *fuzzy initial value problem* (or *fuzzy differential equation*) is

$$x'(t) = F\big(t, x(t)\big), \qquad x(0) = x_0, \tag{5.1}$$

where $x : [0, a] \to F^n$ is unknown and the derivative $x'(t)$ is understood in a specified fuzzy sense (e.g. Form I or Form II of the lateral H-derivative in Definition above) [2].

A mapping $x : [0, a] \to F^n$ is called a *solution* of (5.1) (in the chosen differentiability form) if x is H-differentiable at every $t \in (0, a)$ in that form and satisfies (5.1) in F^n.

Let F^n denote the space of compact, convex fuzzy subsets of $\mathbb{R}^n$, equipped with the usual level-set structure and the metric D induced by the Hausdorff metric on α-cuts.

Definition 5.5.3 (Uncertain state space and score-realization)**.** Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Assume that the following data are fixed:

- a score map

$$S_M : \mathrm{Dom}(M) \to [0, 1];$$

- a lift map
$$L_M : [0,1] \to \mathrm{Dom}(M)$$
such that
$$S_M\big(L_M(r)\big) = r \qquad \text{for all } r \in [0,1].$$

Define
$$\mathbb{U}_M^n := \left\{ A_M : \mathbb{R}^n \to \mathrm{Dom}(M) \;\middle|\; \overline{A}_M := S_M \circ A_M \in F^n \right\}.$$
An element of $\mathbb{U}_M^n$ is called an *uncertain n-vector of type M*.

For $A_M \in \mathbb{U}_M^n$, the fuzzy set
$$\overline{A}_M := S_M \circ A_M \in F^n$$
is called the *score-realization* of A_M.

Moreover, define the lifting operator
$$\Lambda_M : F^n \to \mathbb{U}_M^n$$
by
$$\big(\Lambda_M(u)\big)(\xi) := L_M(u(\xi)) \qquad (\xi \in \mathbb{R}^n,\ u \in F^n).$$

Definition 5.5.4 (Score-H-derivative)**.** Let $T = (a,b) \subseteq \mathbb{R}$, and let
$$X_M : T \to \mathbb{U}_M^n$$
be an uncertain-valued mapping. Define its score-realization
$$\overline{X} : T \to F^n, \qquad \overline{X}(t) := \overline{X_M(t)} = S_M \circ X_M(t).$$

We say that X_M is *score-H-differentiable* at $t_0 \in T$ if $\overline{X}$ is H-differentiable at t_0 in the chosen fuzzy sense (for example, Form I or Form II of the lateral H-derivative).

In that case, the *score-H-derivative* of X_M at t_0 is defined by
$$D_H^{[S]} X_M(t_0) := \Lambda_M\big(\overline{X}'(t_0)\big) \in \mathbb{U}_M^n.$$
Equivalently,
$$\overline{D_H^{[S]} X_M(t_0)} = \overline{X}'(t_0).$$

Definition 5.5.5 (Uncertain differential equation / uncertain initial value problem)**.** Let $[0,a] \subset \mathbb{R}$, let
$$X_{0,M} \in \mathbb{U}_M^n,$$
and let
$$F_M : [0,a] \times \mathbb{U}_M^n \to \mathbb{U}_M^n$$
be an uncertain-valued mapping.

For $(t, A_M) \in [0,a] \times \mathbb{U}_M^n$, define the score-realized right-hand side by
$$\overline{F}_M(t, A_M) := \overline{F_M(t, A_M)} = S_M \circ F_M(t, A_M) \in F^n.$$

The *uncertain initial value problem* (or *uncertain differential equation*) of type M is

$$D_H^{[S]} X_M(t) = F_M\big(t, X_M(t)\big), \qquad X_M(0) = X_{0,M}, \tag{5.2}$$

where $X_M : [0,a] \to \mathbb{U}_M^n$ is unknown.

A mapping

$$X_M : [0,a] \to \mathbb{U}_M^n$$

is called a *solution* of (5.2) if its score-realization

$$\overline{X}(t) := S_M \circ X_M(t)$$

is H-differentiable at every $t \in (0,a)$ in the chosen fuzzy sense and satisfies

$$\overline{X}'(t) = \overline{F}_M\big(t, X_M(t)\big) \qquad \text{in } F^n$$

for all $t \in (0,a)$, together with

$$\overline{X}(0) = \overline{X_{0,M}}.$$

Remark 5.5.6. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1], \qquad S_M = \mathrm{id}_{[0,1]}, \qquad L_M = \mathrm{id}_{[0,1]},$$

then

$$\mathbb{U}_M^n = F^n, \qquad D_H^{[S]} = D_H, \qquad \overline{F}_M = F_M,$$

and Definition 5.5.5 reduces exactly to the usual fuzzy differential equation.

Theorem 5.5.7 (Well-definedness of uncertain differential equations)**.** *Let M, S_M, L_M, $\mathbb{U}_M^n$, and F_M be as in Definitions 5.5.3–5.5.5. Then:*

1. *for every $A_M \in \mathbb{U}_M^n$, the score-realization*

   $$\overline{A}_M = S_M \circ A_M$$

   is a well-defined element of F^n;

2. *for every $u \in F^n$, the lifted object*

   $$\Lambda_M(u)$$

   is a well-defined element of $\mathbb{U}_M^n$, and

   $$\overline{\Lambda_M(u)} = u;$$

3. *if $X_M : T \to \mathbb{U}_M^n$ is such that its score-realization $\overline{X}$ is H-differentiable at $t_0 \in T$, then the score-H-derivative*

   $$D_H^{[S]} X_M(t_0) = \Lambda_M\big(\overline{X}'(t_0)\big)$$

   is well-defined in $\mathbb{U}_M^n$;

4. *for every $(t, A_M) \in [0,a] \times \mathbb{U}_M^n$, the score-realized right-hand side*

   $$\overline{F}_M(t, A_M) = S_M \circ F_M(t, A_M)$$

   is a well-defined element of F^n;

5. *consequently, the uncertain differential equation* (5.2) *is mathematically meaningful, and its solution concept is well-defined.*

Proof. (1) By definition of $\mathbb{U}_M^n$, an element

$$A_M : \mathbb{R}^n \to \mathrm{Dom}(M)$$

belongs to $\mathbb{U}_M^n$ precisely when

$$\overline{A}_M := S_M \circ A_M$$

belongs to F^n. Hence the score-realization is well-defined.

(2) Let $u \in F^n$. By definition,

$$\big(\Lambda_M(u)\big)(\xi) = L_M(u(\xi)) \qquad (\xi \in \mathbb{R}^n).$$

Since $u(\xi) \in [0,1]$ and $L_M : [0,1] \to \mathrm{Dom}(M)$, it follows that

$$\Lambda_M(u) : \mathbb{R}^n \to \mathrm{Dom}(M)$$

is a well-defined mapping. Moreover,

$$\overline{\Lambda_M(u)}(\xi) = S_M\big(L_M(u(\xi))\big) = u(\xi) \qquad (\xi \in \mathbb{R}^n),$$

because $S_M \circ L_M = \mathrm{id}_{[0,1]}$. Therefore

$$\overline{\Lambda_M(u)} = u \in F^n,$$

and hence $\Lambda_M(u) \in \mathbb{U}_M^n$. This proves (2).

(3) Assume that $\overline{X}$ is H-differentiable at t_0. Then

$$\overline{X}'(t_0) \in F^n$$

is well-defined in the fuzzy sense. By (2), the lifted object

$$\Lambda_M\big(\overline{X}'(t_0)\big)$$

is a well-defined element of $\mathbb{U}_M^n$. Hence

$$D_H^{[S]} X_M(t_0)$$

is well-defined.

(4) Let $(t, A_M) \in [0,a] \times \mathbb{U}_M^n$. Since

$$F_M : [0,a] \times \mathbb{U}_M^n \to \mathbb{U}_M^n,$$

the value

$$F_M(t, A_M)$$

belongs to $\mathbb{U}_M^n$. By (1), its score-realization

$$\overline{F}_M(t, A_M) = S_M \circ F_M(t, A_M)$$

is therefore a well-defined element of F^n.

(5) By (3), the left-hand side of

$$D_H^{[S]} X_M(t) = F_M\big(t, X_M(t)\big)$$

has a well-defined score-realization

$$\overline{D_H^{[S]} X_M(t)} = \overline{X}'(t) \in F^n.$$

By (4), the right-hand side has a well-defined score-realization

$$\overline{F}_M\big(t, X_M(t)\big) \in F^n.$$

Hence the defining equation

$$\overline{X}'(t) = \overline{F}_M\big(t, X_M(t)\big)$$

is a meaningful equality in F^n, and the initial condition

$$\overline{X}(0) = \overline{X_{0,M}}$$

is also meaningful in F^n.

Therefore the uncertain differential equation (5.2) and its solution concept are mathematically well-defined. □

A catalogue of uncertainty-differential-equation families classified by the dimension k is presented in Table 5.4.

Table 5.4: A catalogue of uncertainty-differential-equation families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (conceptual overview).

k	Representative uncertainty-differential-equation type(s) for an unknown state $x : [0,T] \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (or a fuzzy-number-valued state whose membership degrees are k-dimensional)
1	Fuzzy differential equation (FDE): $x(t)$ is a fuzzy number (single membership profile) and the derivative is defined in a suitable fuzzy sense (e.g. Hukuhara-type), with fuzzy initial/boundary data [591–594].
2	Intuitionistic fuzzy differential equation (IFDE): each state value carries a membership–nonmembership pair $(\mu(t), \nu(t)) \in [0,1]^2$ (typically with $\mu(t) + \nu(t) \le 1$) [595–598].
3	Picture fuzzy differential equation (PFDE): each state value carries a triple (positive/neutral/negative) $(\mu(t), \eta(t), \nu(t)) \in [0,1]^3$ (typically with $\mu(t) + \eta(t) + \nu(t) \le 1)^{(a)}$ [599]; Neutrosophic differential equation (NDE): each state value carries a truth/indeterminacy/falsity triple $(T(t), I(t), F(t)) \in [0,1]^3$ (often allowing $0 \le T(t) + I(t) + F(t) \le 3)^{(b)}$ [600–606].

(a) The precise constraint for picture-fuzzy states depends on the adopted picture-fuzzy set convention; the standard one uses $\mu + \eta + \nu \le 1$.
(b) Some neutrosophic differential-equation books use specialized variants (e.g. bipolar neutrosophic settings), but the ambient degree domain remains a 3-tuple of independent coordinates.

In addition to uncertain differential equations, related concepts such as fractional differential equations [607, 608], delay differential equations [609, 610], neutral differential equations [611, 612], stochastic differential equations [613, 614], differential inclusions [615, 616], and differential-algebraic equations [617, 618] are also well known.

5.6 Uncertain Normed Space

Fuzzy normed space is a vector space equipped with a fuzzy norm measuring degrees of smallness, generalizing classical normed spaces through uncertainty-sensitive distance behavior patterns [619–621].

Definition 5.6.1 (Fuzzy normed space)**.** (cf. [622]) Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, and let $*$ be a continuous t-norm on $[0, 1]$. A mapping

$$N : V \times \mathbb{R} \to [0, 1]$$

is called a *fuzzy norm* on V if, for all $x, y \in V$, all $s, t > 0$, and all scalars $c \neq 0$, the following conditions hold:

1. $$N(x, t) = 0 \qquad \text{for all } t \leq 0;$$
2. $$x = 0 \quad \Longleftrightarrow \quad N(x, t) = 1 \text{ for all } t > 0;$$
3. $$N(cx, t) = N\left(x, \frac{t}{|c|}\right);$$
4. $$N(x + y, s + t) \geq N(x, s) * N(y, t);$$
5. for each fixed $x \in V$, the function
$$t \mapsto N(x, t)$$
is nondecreasing on $\mathbb{R}$, and
$$\lim_{t\to\infty} N(x, t) = 1;$$
6. if $x \neq 0$, then
$$t \mapsto N(x, t)$$
is continuous on $(0, \infty)$.

In this case, the pair (V, N) is called a *fuzzy normed space.*

An uncertain normed space is a vector space equipped with an uncertainty-valued norm whose score-realization satisfies the axioms of a fuzzy norm.

Definition 5.6.2 (Uncertain normed space)**.** Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

and let $*$ be a continuous t-norm on $[0, 1]$.

An *uncertain norm of type M* on V is a mapping

$$N_M : V \times \mathbb{R} \to \mathrm{Dom}(M)$$

such that, for each fixed $t \in \mathbb{R}$, the map

$$N_M^t : V \to \mathrm{Dom}(M), \qquad N_M^t(x) := N_M(x,t),$$

is an uncertain set of type M on V.

Fix an admissible score map

$$S_M : \mathrm{Dom}(M) \to [0,1].$$

Define the score-realization

$$\overline{N}_M : V \times \mathbb{R} \to [0,1]$$

by

$$\overline{N}_M(x,t) := S_M\big(N_M(x,t)\big).$$

Then N_M is called an *uncertain norm* on V if, for all $x, y \in V$, all $s, t > 0$, and all scalars $c \neq 0$, the following conditions hold:

1. $$\overline{N}_M(x,t) = 0 \qquad \text{for all } t \leq 0;$$
2. $$x = 0 \quad \Longleftrightarrow \quad \overline{N}_M(x,t) = 1 \text{ for all } t > 0;$$
3. $$\overline{N}_M(cx,t) = \overline{N}_M\left(x, \frac{t}{|c|}\right);$$
4. $$\overline{N}_M(x+y, s+t) \geq \overline{N}_M(x,s) * \overline{N}_M(y,t);$$
5. for each fixed $x \in V$, the function
$$t \mapsto \overline{N}_M(x,t)$$
is nondecreasing on $\mathbb{R}$, and
$$\lim_{t\to\infty} \overline{N}_M(x,t) = 1;$$
6. if $x \neq 0$, then
$$t \mapsto \overline{N}_M(x,t)$$
is continuous on $(0,\infty)$.

In this case, the triple

$$(V, N_M, *)$$

is called an *uncertain normed space of type M*.

Remark 5.6.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

then

$$\overline{N}_M = N_M,$$

and Definition 5.6.2 reduces exactly to the usual definition of a fuzzy normed space. Thus fuzzy normed spaces are special cases of uncertain normed spaces.

Theorem 5.6.4 (Well-definedness of uncertain normed spaces)**.** *Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

let

$$N_M : V \times \mathbb{R} \to \mathrm{Dom}(M)$$

be a mapping, let

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

be a score map, and let $*$ be a continuous t-norm on $[0,1]$. Then:

1. *for each fixed $t \in \mathbb{R}$, the map*

$$N_M^t : V \to \mathrm{Dom}(M), \qquad N_M^t(x) := N_M(x,t),$$

   is a well-defined uncertain set of type M on V;

2. *the score-realization*

$$\overline{N}_M : V \times \mathbb{R} \to [0,1], \qquad \overline{N}_M(x,t) = S_M\big(N_M(x,t)\big),$$

   is a well-defined mapping;

3. *all expressions appearing in Definition 5.6.2 are meaningful real numbers in $[0,1]$;*

4. *consequently, the notion of uncertain normed space is mathematically well-defined.*

Moreover, if the axioms in Definition 5.6.2 hold, then

$$(V, \overline{N}_M, *)$$

is a well-defined fuzzy normed space.

Proof. For each fixed $t \in \mathbb{R}$, define

$$N_M^t(x) := N_M(x,t) \qquad (x \in V).$$

Since

$$N_M : V \times \mathbb{R} \to \mathrm{Dom}(M),$$

it follows that for every $x \in V$,

$$N_M^t(x) = N_M(x,t) \in \mathrm{Dom}(M).$$

Hence

$$N_M^t : V \to \mathrm{Dom}(M)$$

is a well-defined map, that is, a well-defined uncertain set of type M on V. This proves (1).

Next, because

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

for every $(x,t) \in V \times \mathbb{R}$ one has

$$S_M\big(N_M(x,t)\big) \in [0,1].$$

Therefore

$$\overline{N}_M(x,t) := S_M\big(N_M(x,t)\big)$$

defines a well-defined mapping

$$\overline{N}_M : V \times \mathbb{R} \to [0,1].$$

This proves (2).

Now every quantity appearing in

$$\overline{N}_M(x,t), \qquad \overline{N}_M(cx,t), \qquad \overline{N}_M\left(x, \frac{t}{|c|}\right), \qquad \overline{N}_M(x+y, s+t)$$

is a real number in $[0,1]$. Since

$$* : [0,1]^2 \to [0,1]$$

is a t-norm, the product

$$\overline{N}_M(x,s) * \overline{N}_M(y,t)$$

is also a well-defined element of $[0,1]$. Hence the inequality

$$\overline{N}_M(x+y, s+t) \geq \overline{N}_M(x,s) * \overline{N}_M(y,t)$$

is meaningful.

Likewise, the equalities

$$\overline{N}_M(x,t) = 0, \qquad \overline{N}_M(cx,t) = \overline{N}_M\left(x, \frac{t}{|c|}\right),$$

the biconditional

$$x = 0 \iff \overline{N}_M(x,t) = 1 \text{ for all } t > 0,$$

the monotonicity of

$$t \mapsto \overline{N}_M(x,t),$$

and the limit and continuity conditions are all meaningful, because $\overline{N}_M$ is an ordinary real-valued function on $V \times \mathbb{R}$. This proves (3).

Therefore all objects and all conditions in Definition 5.6.2 are mathematically meaningful, so the notion of uncertain normed space is well-defined. This proves (4).

Finally, if the axioms in Definition 5.6.2 hold, then the score-realization $\overline{N}_M$ satisfies exactly the usual axioms of a fuzzy norm with respect to the t-norm $*$. Hence

$$(V, \overline{N}_M, *)$$

is a well-defined fuzzy normed space. □

5.7 Uncertain Fixed Point Theory

Fixed point theory studies conditions under which mappings admit invariant points, and analyzes existence, uniqueness, approximation, and stability of such points across various mathematical structures [623, 624]. Fuzzy fixed point theory studies conditions ensuring mappings on fuzzy spaces or fuzzy-valued mappings admit points invariant under the mapping, extending classical fixed point analysis [625–627].

Definition 5.7.1 (Fuzzy fixed point)**.** [628] Let X be a nonempty set, and let

$$F : X \to \mathcal{F}(X)$$

be a fuzzy mapping, where $\mathcal{F}(X)$ denotes the family of all fuzzy subsets of X. For each $x \in X$, write

$$\mu_{F(x)} : X \to [0, 1]$$

for the membership function of the fuzzy set $F(x)$.

Fix $\alpha \in (0, 1]$. A point $x^* \in X$ is called an *α-fuzzy fixed point* of F if

$$x^* \in [F(x^*)]_\alpha,$$

where

$$[F(x^*)]_\alpha := \{y \in X \mid \mu_{F(x^*)}(y) \geq \alpha\}.$$

Equivalently,

$$\mu_{F(x^*)}(x^*) \geq \alpha.$$

If the level α is understood from the context, then x^* is simply called a *fuzzy fixed point* of F.

An uncertain fixed point is a point that belongs, at a prescribed score level, to the uncertain set assigned to itself by an uncertain set-valued mapping.

Definition 5.7.2 (Uncertain fixed point)**.** Let X be a nonempty set, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Define

$$\mathcal{U}_M(X) := \{\mu : X \to \mathrm{Dom}(M)\},$$

the family of all uncertain sets of type M on X.

Let

$$F_M : X \to \mathcal{U}_M(X)$$

be an uncertain mapping. For each $x \in X$, write

$$\mu_{F_M(x)} : X \to \mathrm{Dom}(M)$$

for the uncertainty-degree function of the uncertain set $F_M(x)$.

Fix a score map

$$S_M : \mathrm{Dom}(M) \to [0, 1],$$

and let $\alpha \in (0, 1]$.

The *score-induced α-cut* of $F_M(x)$ is defined by

$$[F_M(x)]_\alpha^{S_M} := \{\, y \in X \mid S_M(\mu_{F_M(x)}(y)) \geq \alpha \,\}.$$

A point $x^* \in X$ is called an *α-uncertain fixed point of type M* of F_M if

$$x^* \in [F_M(x^*)]_\alpha^{S_M}.$$

Equivalently,

$$S_M\big(\mu_{F_M(x^*)}(x^*)\big) \geq \alpha.$$

If the level α is understood from the context, then x^* is simply called an *uncertain fixed point*.

Remark 5.7.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0, 1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

then Definition 5.7.2 reduces exactly to the usual notion of an α-fuzzy fixed point:

$$x^* \in [F(x^*)]_\alpha \quad \Longleftrightarrow \quad \mu_{F(x^*)}(x^*) \geq \alpha.$$

Thus fuzzy fixed points are special cases of uncertain fixed points.

Theorem 5.7.4 (Well-definedness of uncertain fixed points)**.** *Let X be a nonempty set, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

let

$$F_M : X \to \mathcal{U}_M(X)$$

be an uncertain mapping, and let

$$S_M : \mathrm{Dom}(M) \to [0, 1]$$

be a score map. Then:

1. *for every $x \in X$, the score-induced α-cut*

$$[F_M(x)]_\alpha^{S_M} = \{\, y \in X \mid S_M(\mu_{F_M(x)}(y)) \geq \alpha \,\}$$

is a well-defined subset of X;

2. *for every $x^* \in X$, the membership statement*

$$x^* \in [F_M(x^*)]_\alpha^{S_M}$$

is meaningful;

3. *the condition*

$$x^* \in [F_M(x^*)]_\alpha^{S_M}$$

is equivalent to

$$S_M\big(\mu_{F_M(x^*)}(x^*)\big) \geq \alpha;$$

4. *consequently, the notion of α-uncertain fixed point is mathematically well-defined.*

Proof. Let $x \in X$. Since

$$F_M : X \to \mathcal{U}_M(X),$$

the value

$$F_M(x)$$

belongs to $\mathcal{U}_M(X)$. By definition of $\mathcal{U}_M(X)$, this means that

$$\mu_{F_M(x)} : X \to \mathrm{Dom}(M)$$

is a well-defined mapping.

Because

$$S_M : \mathrm{Dom}(M) \to [0, 1],$$

the composition

$$S_M \circ \mu_{F_M(x)} : X \to [0, 1]$$

is also well-defined. Therefore, for every $y \in X$, the inequality

$$S_M(\mu_{F_M(x)}(y)) \geq \alpha$$

is meaningful. Hence the set

$$[F_M(x)]_\alpha^{S_M} = \{\, y \in X \mid S_M(\mu_{F_M(x)}(y)) \geq \alpha \,\}$$

is a well-defined subset of X. This proves (1).

Now let $x^* \in X$. Since

$$[F_M(x^*)]_\alpha^{S_M} \subseteq X$$

is well-defined, the statement

$$x^* \in [F_M(x^*)]_\alpha^{S_M}$$

is meaningful. This proves (2).

By the definition of the score-induced α-cut,

$$x^* \in [F_M(x^*)]_\alpha^{S_M}$$

holds if and only if

$$S_M(\mu_{F_M(x^*)}(x^*)) \geq \alpha.$$

Thus (3) follows immediately.

Since all objects and conditions appearing in Definition 5.7.2 are mathematically meaningful, the notion of α-uncertain fixed point is well-defined. This proves (4). □

For reference, a catalogue of representative uncertainty-fixed-point-theory families is presented in Table 5.5.

Table 5.5: A catalogue of representative uncertainty-fixed-point-theory families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-fixed-point-theory family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy Fixed Point Theory [629, 630]:* $F_M : X \to [0,1]^X$, or equivalently $F_M : X \to \mathcal{F}(X)$.
2	*Intuitionistic Fuzzy Fixed Point Theory [631, 632]:* $F_M : X \to ([0,1]^2)^X$, or equivalently $F_M : X \to \mathcal{IF}(X)$ (e.g., (membership, non-membership)).
3	*Neutrosophic Fixed Point Theory [633–638]:* $F_M : X \to ([0,1]^3)^X$, or equivalently $F_M : X \to \mathcal{NS}(X)$ (e.g., (T, I, F)).
k	*k-component uncertainty fixed point theory:* $F_M : X \to \mathrm{Dom}(M)^X$, or equivalently $F_M : X \to \mathcal{U}_M(X)$ (model-specific semantics).

In addition to uncertain fixed points, related concepts such as soft fixed points [639, 640], asymptotic fixed points [641, 642], approximate fixed points [643, 644], best proximity points [645, 646], coupled fixed points [647, 648], and tripled fixed points [649, 650] are also known.

5.8 Uncertain BCK/BCI–Algebras

Fuzzy BCK/BCI-algebra is a BCK or BCI algebra equipped with a membership function preserving the operation fuzzily, with zero having maximal membership degree among elements [651–653].

Definition 5.8.1 (Fuzzy BCK/BCI-algebra)**.** [651, 652] Let $(X, *, 0)$ be a BCK-algebra (respectively, a BCI-algebra), and let

$$\mu : X \to [0,1]$$

be a fuzzy set on X. Then μ is called a *fuzzy BCK-algebra* (respectively, a *fuzzy BCI-algebra*) on X if, for all $x, y \in X$,

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

and

$$\mu(0) \geq \mu(x).$$

In this case, the pair (X, μ) is called a fuzzy BCK-algebra (respectively, fuzzy BCI-algebra).

An uncertain BCK/BCI-algebra is a BCK- or BCI-algebra equipped with an uncertainty-degree function into a general degree-domain, such that the algebraic operation is compatible with the uncertainty structure through score-based aggregation and the zero element has maximal score.

Definition 5.8.2 (Uncertain BCK/BCI-algebra)**.** Let $(X, *, 0)$ be a BCK-algebra (respectively, a BCI-algebra), and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

An *uncertain BCK-algebra* (respectively, *uncertain BCI-algebra*) of type M on X is an uncertain set

$$\mathcal{X}_M = (X, \mu_M),$$

where

$$\mu_M : X \to \mathrm{Dom}(M),$$

for which the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0,1];$$
- a binary aggregation operator
$$\Gamma_2 : [0,1]^2 \to [0,1].$$

These data are required to satisfy, for all $x, y \in X$,

$$S_M\big(\mu_M(x * y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big),$$

and

$$S_M\big(\mu_M(0)\big) \geq S_M\big(\mu_M(x)\big).$$

In this case, the pair (X, μ_M) is called an *uncertain BCK-algebra* (respectively, an *uncertain BCI-algebra*) of type M.

Remark 5.8.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

and if one takes

$$\Gamma_2(a,b) = \min\{a,b\},$$

then Definition 5.8.2 reduces to the usual fuzzy BCK/BCI-algebra condition

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \qquad \text{and} \qquad \mu(0) \geq \mu(x)$$

for all $x, y \in X$. Thus fuzzy BCK/BCI-algebras are special cases of uncertain BCK/BCI-algebras.

Theorem 5.8.4 (Well-definedness of uncertain BCK/BCI-algebras)**.** *Let $(X, *, 0)$ be a BCK-algebra (respectively, a BCI-algebra), let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$\mu_M : X \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1], \qquad \Gamma_2 : [0,1]^2 \to [0,1]$$

be given. Then:

(a) *(X, μ_M) is a well-defined uncertain set of type M on X.*

(b) *For every $x, y \in X$, the quantities*

$$S_M\big(\mu_M(x * y)\big), \qquad S_M\big(\mu_M(0)\big), \qquad S_M\big(\mu_M(x)\big), \qquad S_M\big(\mu_M(y)\big)$$

are well-defined elements of $[0,1]$.

(c) *For every $x, y \in X$, the quantity*

$$\Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

is a well-defined element of $[0,1]$.

(d) *Consequently, both defining inequalities in Definition 5.8.2,*

$$S_M\big(\mu_M(x * y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

and

$$S_M\big(\mu_M(0)\big) \geq S_M\big(\mu_M(x)\big),$$

are meaningful for all $x, y \in X$.

Hence Definition 5.8.2 determines a mathematically well-defined class of structures.

Proof. (a) By the definition of an uncertain set of type M, any pair

$$(X, \mu_M) \qquad \text{with} \qquad \mu_M : X \to \mathrm{Dom}(M)$$

is an uncertain set of type M on X. Hence (X, μ_M) is well-defined.

(b) Let $x, y \in X$. Since $(X, *, 0)$ is a BCK-algebra or a BCI-algebra, the binary operation

$$* : X \times X \to X$$

is well-defined, and the constant 0 is a distinguished element of X. Therefore

$$x * y \in X, \qquad 0 \in X, \qquad x \in X, \qquad y \in X.$$

Because

$$\mu_M : X \to \mathrm{Dom}(M),$$

it follows that

$$\mu_M(x * y), \qquad \mu_M(0), \qquad \mu_M(x), \qquad \mu_M(y) \in \mathrm{Dom}(M).$$

Applying the score map

$$S_M : \mathrm{Dom}(M) \to [0, 1],$$

we obtain

$$S_M\big(\mu_M(x * y)\big), \qquad S_M\big(\mu_M(0)\big), \qquad S_M\big(\mu_M(x)\big), \qquad S_M\big(\mu_M(y)\big) \in [0, 1].$$

Thus all these quantities are well-defined.

(c) Since

$$S_M(\mu_M(x)),\, S_M(\mu_M(y)) \in [0, 1],$$

we have

$$\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \in [0, 1]^2.$$

Because

$$\Gamma_2 : [0, 1]^2 \to [0, 1],$$

it follows that

$$\Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big) \in [0, 1].$$

Hence this quantity is well-defined.

(d) By parts (b) and (c), both sides of

$$S_M\big(\mu_M(x*y)\big) \geq \Gamma_2\big(S_M(\mu_M(x)),\, S_M(\mu_M(y))\big)$$

are well-defined real numbers in $[0,1]$, and both sides of

$$S_M\big(\mu_M(0)\big) \geq S_M\big(\mu_M(x)\big)$$

are also well-defined real numbers in $[0,1]$. Therefore both inequalities are meaningful for every $x, y \in X$.

Hence all objects and conditions appearing in Definition 5.8.2 are mathematically meaningful, and the notion of uncertain BCK/BCI-algebra is well-defined. □

Representative BCK/BCI-algebra frameworks under uncertainty are listed in Table **??**.

Table 5.6: A catalogue of representative uncertainty-BCK/BCI-algebra families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-BCK/BCI-algebra family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy BCK/BCI-Algebras:* $\mu_M : X \to [0,1]$.
2	*Intuitionistic Fuzzy BCK/BCI-Algebras [654, 655]:* $\mu_M : X \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic BCK/BCI-Algebras [656, 657]:* $\mu_M : X \to [0,1]^3$ (e.g., (T, I, F)).
k	*k-component uncertainty BCK/BCI-algebras:* $\mu_M : X \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

In addition to uncertain BCK/BCI-algebras, related concepts of BCK/BCI-algebras such as BE-algebras [658], BCH-algebras [659, 660], BF-algebras [661], KU-algebras [662, 663], QS-algebras [664, 665], and soft BCK/BCI-algebras [666, 667] are known.

Chapter 6

Quantitative and Measurement Tools

In this chapter, quantitative and measurement tools are presented. A concise comparison of the quantitative and measurement-oriented uncertain concepts covered in Chapter 6 is presented in Table 6.1.

Table 6.1: A concise comparison of the quantitative and measurement-oriented uncertain concepts covered in Chapter 6.

Concept	**Base structure**	**Main focus**
Uncertain Weighted Average	Weighted aggregation	Uncertain aggregation of values with weighted importance.
Uncertain Entropy	Entropy measure	Uncertain quantification of uncertainty, dispersion, or information content.
Uncertain Numbers	Numerical representation	Uncertain numerical values and their mathematical representation.

6.1 Uncertain Weighted Average

Fuzzy weighted average aggregates fuzzy numbers using fuzzy weights and the extension principle, producing an imprecise weighted mean that preserves graded uncertainty in inputs faithfully [668–672].

Definition 6.1.1 (Fuzzy weighted average (type-1, via the extension principle))**.** Let $n \geq 1$. Let $\tilde{w}_1, \dots, \tilde{w}_n$ and $\tilde{x}_1, \dots, \tilde{x}_n$ be (type-1) fuzzy numbers on $\mathbb{R}_{\geq 0}$ and $\mathbb{R}$, with membership functions $\mu_{\tilde{w}_i} : \mathbb{R}_{\geq 0} \to [0, 1]$ and $\mu_{\tilde{x}_i} : \mathbb{R} \to [0, 1]$. Assume the denominator is well-defined in the sense that, for every $\alpha \in (0, 1]$,

$$\inf\Big\{ \sum_{i=1}^{n} w_i \;\Big|\; w_i \in [\tilde{w}_i]_\alpha \Big\} \;>\; 0.$$

Define the crisp weighted-average mapping

$$f:\ (\mathbb{R}_{\geq 0})^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad f(w_1, \dots, w_n;\, x_1, \dots, x_n) := \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}.$$

The *fuzzy weighted average* (FWA) of $(\tilde{w}_i)_{i=1}^n$ and $(\tilde{x}_i)_{i=1}^n$ is the fuzzy number $\widetilde{y} = \mathrm{FWA}(\tilde{w}_1, \dots, \tilde{w}_n; \tilde{x}_1, \dots, \tilde{x}_n)$ on $\mathbb{R}$ defined as the fuzzy image of the inputs through f by Zadeh's extension principle:

$$\mu_{\widetilde{y}}(y) := \sup_{\substack{w_i \in \mathbb{R}_{\geq 0},\ x_i \in \mathbb{R} \\ y = f(w_1, \dots, w_n; x_1, \dots, x_n)}} \min\bigl\{\mu_{\tilde{w}_1}(w_1), \dots, \mu_{\tilde{w}_n}(w_n), \mu_{\tilde{x}_1}(x_1), \dots, \mu_{\tilde{x}_n}(x_n)\bigr\}.$$

(Equivalently, one may replace min by a chosen t-norm.)

Uncertain weighted average aggregates uncertain numbers by applying a fuzzy weighted average to their score-realizations and then lifting the resulting fuzzy number back to the uncertainty model.

Definition 6.1.2 (Uncertain numbers and score-realization)**.** Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Assume that the following data are fixed:

- a score map
$$S_M : \mathrm{Dom}(M) \to [0, 1];$$
- a lift map
$$L_M : [0, 1] \to \mathrm{Dom}(M)$$
such that
$$S_M\bigl(L_M(r)\bigr) = r \qquad \text{for all } r \in [0, 1].$$

An *uncertain number of type M* on $\mathbb{R}$ is an uncertain set

$$\widetilde{A}_M = (\mathbb{R}, \mu_{\widetilde{A}_M}), \qquad \mu_{\widetilde{A}_M} : \mathbb{R} \to \mathrm{Dom}(M),$$

whose score-realization

$$\overline{A}_M : \mathbb{R} \to [0, 1], \qquad \mu_{\overline{A}_M}(x) := S_M\bigl(\mu_{\widetilde{A}_M}(x)\bigr),$$

is a fuzzy number on $\mathbb{R}$.

Similarly, an *uncertain nonnegative number of type M* is an uncertain set

$$\widetilde{W}_M = (\mathbb{R}_{\geq 0}, \mu_{\widetilde{W}_M}), \qquad \mu_{\widetilde{W}_M} : \mathbb{R}_{\geq 0} \to \mathrm{Dom}(M),$$

whose score-realization

$$\overline{W}_M : \mathbb{R}_{\geq 0} \to [0, 1], \qquad \mu_{\overline{W}_M}(w) := S_M\bigl(\mu_{\widetilde{W}_M}(w)\bigr),$$

is a fuzzy number on $\mathbb{R}_{\geq 0}$.

Definition 6.1.3 (Uncertain weighted average)**.** Let $n \geq 1$. Let

$$\widetilde{W}_{1,M}, \dots, \widetilde{W}_{n,M}$$

be uncertain nonnegative numbers of type M, and let

$$\widetilde{X}_{1,M}, \dots, \widetilde{X}_{n,M}$$

be uncertain numbers of type M.

Write

$$\overline{W}_{i,M} := S_M \circ \mu_{\widetilde{W}_{i,M}} \qquad (i = 1, \dots, n),$$

and

$$\overline{X}_{i,M} := S_M \circ \mu_{\widetilde{X}_{i,M}} \qquad (i = 1, \dots, n)$$

for their score-realized fuzzy numbers.

Assume that for every $\alpha \in (0, 1]$,

$$\inf\Big\{ \sum_{i=1}^{n} w_i \;\Big|\; w_i \in [\overline{W}_{i,M}]_\alpha \Big\} > 0.$$

Define the crisp weighted-average mapping

$$f : (\mathbb{R}_{\geq 0})^n \times \mathbb{R}^n \to \mathbb{R}, \qquad f(w_1, \dots, w_n; x_1, \dots, x_n) := \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}.$$

Let

$$\overline{Y}_M = \mathrm{FWA}(\overline{W}_{1,M}, \dots, \overline{W}_{n,M}; \overline{X}_{1,M}, \dots, \overline{X}_{n,M})$$

denote the fuzzy weighted average of the score-realized fuzzy numbers, defined by Zadeh's extension principle:

$$\mu_{\overline{Y}_M}(y) := \sup_{\substack{w_i \in \mathbb{R}_{\geq 0},\ x_i \in \mathbb{R} \\ y = f(w_1, \dots, w_n; x_1, \dots, x_n)}} \min\Big\{ \mu_{\overline{W}_{1,M}}(w_1), \dots, \mu_{\overline{W}_{n,M}}(w_n), \mu_{\overline{X}_{1,M}}(x_1), \dots, \mu_{\overline{X}_{n,M}}(x_n) \Big\}.$$

The *uncertain weighted average of type M* is the uncertain number

$$\widetilde{Y}_M = \mathrm{UWA}_M(\widetilde{W}_{1,M}, \dots, \widetilde{W}_{n,M}; \widetilde{X}_{1,M}, \dots, \widetilde{X}_{n,M})$$

defined by

$$\mu_{\widetilde{Y}_M}(y) := L_M\big(\mu_{\overline{Y}_M}(y)\big) \qquad (y \in \mathbb{R}).$$

Equivalently,

$$\overline{Y}_M = S_M \circ \mu_{\widetilde{Y}_M}.$$

Remark 6.1.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0, 1], \qquad S_M = \mathrm{id}_{[0,1]}, \qquad L_M = \mathrm{id}_{[0,1]},$$

then Definition 6.1.3 reduces exactly to the usual type-1 fuzzy weighted average.

Theorem 6.1.5 (Well-definedness of uncertain weighted averages)**.** *Let*

$$\widetilde{W}_{1,M}, \dots, \widetilde{W}_{n,M}$$

be uncertain nonnegative numbers of type M, and let

$$\widetilde{X}_{1,M}, \dots, \widetilde{X}_{n,M}$$

be uncertain numbers of type M. Assume that for every $\alpha \in (0, 1]$,

$$\inf\Big\{ \sum_{i=1}^{n} w_i \;\Big|\; w_i \in [\overline{W}_{i,M}]_\alpha \Big\} > 0.$$

Then:

1. *the fuzzy weighted average*

$$\overline{Y}_M = \mathrm{FWA}(\overline{W}_{1,M}, \ldots, \overline{W}_{n,M}; \overline{X}_{1,M}, \ldots, \overline{X}_{n,M})$$

is well-defined as a fuzzy number on $\mathbb{R}$*;*

2. *the map*

$$\mu_{\widetilde{Y}_M} : \mathbb{R} \to \mathrm{Dom}(M), \qquad \mu_{\widetilde{Y}_M}(y) = L_M\big(\mu_{\overline{Y}_M}(y)\big),$$

is well-defined;

3. *the score-realization of* $\widetilde{Y}_M$ *is exactly* $\overline{Y}_M$*, namely*

$$S_M\big(\mu_{\widetilde{Y}_M}(y)\big) = \mu_{\overline{Y}_M}(y) \qquad \text{for all } y \in \mathbb{R};$$

4. *consequently,*

$$\widetilde{Y}_M = \mathrm{UWA}_M(\widetilde{W}_{1,M}, \ldots, \widetilde{W}_{n,M}; \widetilde{X}_{1,M}, \ldots, \widetilde{X}_{n,M})$$

is a well-defined uncertain number of type M*.*

Proof. Since each

$$\widetilde{W}_{i,M}$$

is an uncertain nonnegative number of type M, its score-realization

$$\overline{W}_{i,M}$$

is a fuzzy number on $\mathbb{R}_{\geq 0}$. Likewise, each

$$\widetilde{X}_{i,M}$$

has a score-realization

$$\overline{X}_{i,M}$$

that is a fuzzy number on $\mathbb{R}$.

By the hypothesis

$$\inf\Big\{\sum_{i=1}^{n} w_i \;\Big|\; w_i \in [\overline{W}_{i,M}]_\alpha\Big\} > 0 \qquad (\alpha \in (0,1]),$$

the denominator

$$\sum_{i=1}^{n} w_i$$

in the crisp weighted-average map

$$f(w_1, \ldots, w_n;\, x_1, \ldots, x_n) = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

is strictly positive on every α-cut combination of the score-realized fuzzy weights. Hence f is well-defined on the relevant domain determined by the extension principle.

Therefore the fuzzy weighted average

$$\overline{Y}_M = \mathrm{FWA}(\overline{W}_{1,M}, \ldots, \overline{W}_{n,M}; \overline{X}_{1,M}, \ldots, \overline{X}_{n,M})$$

is well-defined as a fuzzy number on $\mathbb{R}$. This proves (1).

Next, because

$$\mu_{\overline{Y}_M}(y) \in [0,1] \qquad \text{for all } y \in \mathbb{R},$$

and because

$$L_M : [0,1] \to \mathrm{Dom}(M),$$

the value

$$L_M\big(\mu_{\overline{Y}_M}(y)\big)$$

belongs to $\mathrm{Dom}(M)$ for every $y \in \mathbb{R}$. Hence

$$\mu_{\widetilde{Y}_M}(y) := L_M\big(\mu_{\overline{Y}_M}(y)\big)$$

defines a well-defined mapping

$$\mu_{\widetilde{Y}_M} : \mathbb{R} \to \mathrm{Dom}(M).$$

This proves (2).

For every $y \in \mathbb{R}$, we have

$$S_M\big(\mu_{\widetilde{Y}_M}(y)\big) = S_M\Big(L_M\big(\mu_{\overline{Y}_M}(y)\big)\Big) = \mu_{\overline{Y}_M}(y),$$

because

$$S_M \circ L_M = \mathrm{id}_{[0,1]}.$$

Thus the score-realization of $\widetilde{Y}_M$ is exactly $\overline{Y}_M$. This proves (3).

Since $\overline{Y}_M$ is a fuzzy number and $\widetilde{Y}_M$ has $\overline{Y}_M$ as its score-realization, it follows from Definition 6.1.2 that $\widetilde{Y}_M$ is an uncertain number of type M. This proves (4).

Hence the uncertain weighted average is mathematically well-defined. □

Representative uncertainty-weighted-average families and their degree representations are listed in Table 6.2.

Table 6.2: Representative uncertainty-weighted-average families and their degree representations.

Uncertainty-weighted-average family	Degree form	Typical degree-domain / representation
Fuzzy Weighted Average	scalar	Typically defined for fuzzy weights and fuzzy arguments with $$\widetilde{w}_i : \mathbb{R}_{\geq 0} \to [0,1], \qquad \widetilde{x}_i : \mathbb{R} \to [0,1],$$ and the aggregated result is a fuzzy number obtained by the extension principle or by α-cut-based computation.
Intuitionistic Fuzzy Weighted Average [673–675]	2-component	Typically defined for intuitionistic fuzzy weights and arguments with $$\widetilde{w}_i : \mathbb{R}_{\geq 0} \to [0,1]^2, \qquad \widetilde{x}_i : \mathbb{R} \to [0,1]^2,$$ where the two components usually represent (membership, non-membership).
Neutrosophic Weighted Average [676–679]	3-component	Typically defined for neutrosophic weights and arguments with $$\widetilde{w}_i : \mathbb{R}_{\geq 0} \to [0,1]^3, \qquad \widetilde{x}_i : \mathbb{R} \to [0,1]^3,$$ where the three components usually represent (T, I, F).
Spherical Fuzzy Weighted Average [680]	3-component	Typically defined for spherical fuzzy weights and arguments with $$\widetilde{w}_i : \mathbb{R}_{\geq 0} \to [0,1]^3, \qquad \widetilde{x}_i : \mathbb{R} \to [0,1]^3,$$ where the components usually represent (membership, non-membership, hesitancy), subject to the spherical constraint $$\mu^2 + \nu^2 + \pi^2 \leq 1.$$

6.2 Uncertain entropy

Fuzzy entropy quantifies the amount of vagueness in a fuzzy set, measuring ambiguity, maximal fuzziness, and deviation from crispness through entropy-based uncertainty indices [681–683].

Definition 6.2.1 (Fuzzy entropy measure (De Luca–Termini axioms))**.** Let $X = \{x_1, \ldots, x_n\}$ be a finite universe and let A be a fuzzy set on X with membership $\mu_A : X \to [0,1]$. Write A^c for the complement, $\mu_{A^c}(x) := 1 - \mu_A(x)$. A mapping

$$E : [0,1]^X \longrightarrow \mathbb{R}_{\geq 0}, \qquad A \longmapsto E(A),$$

is called a *fuzzy entropy measure* if it satisfies:

(FE1) **(Crispness)** $E(A) = 0$ iff $\mu_A(x_i) \in \{0,1\}$ for all i (i.e. A is crisp).

(FE2) **(Maximal fuzziness)** $E(A)$ is maximal iff $\mu_A(x_i) = \frac{1}{2}$ for all i.

(FE3) **(Monotonicity under sharpening)** If $A^\sharp$ is a *sharpening* of A, meaning

$$\mu_{A^\sharp}(x_i) \leq \mu_A(x_i) \text{ whenever } \mu_A(x_i) \leq \tfrac{1}{2}, \qquad \mu_{A^\sharp}(x_i) \geq \mu_A(x_i) \text{ whenever } \mu_A(x_i) \geq \tfrac{1}{2},$$

then $E(A^\sharp) \leq E(A)$.

(FE4) **(Complement invariance)** $E(A) = E(A^c)$.

Definition 6.2.2 (De Luca–Termini fuzzy entropy (log-form))**.** Let $X = \{x_1, \ldots, x_n\}$ and $A \in [0,1]^X$ as above. Fix a logarithm base $b > 1$ and a constant $K > 0$ (often $K = 1/n$ for normalization). The *De Luca–Termini fuzzy entropy* of A is

$$H_{\mathrm{DLT}}(A) := -K \sum_{i=1}^{n} \Big(\mu_A(x_i) \log_b \mu_A(x_i) + \big(1 - \mu_A(x_i)\big) \log_b \big(1 - \mu_A(x_i)\big) \Big),$$

with the convention $0 \log_b 0 := 0$.

Uncertain entropy quantifies the amount of uncertainty of an uncertain set by first transforming its uncertainty degrees into scalar scores in $[0,1]$, and then applying a fuzzy entropy measure.

Definition 6.2.3 (Uncertain set of type M)**.** Let

$$X = \{x_1, \ldots, x_n\}$$

be a finite nonempty set, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

An *uncertain set of type M* on X is a pair

$$A_M = (X, \mu_{A_M}),$$

where

$$\mu_{A_M} : X \to \mathrm{Dom}(M).$$

Definition 6.2.4 (Uncertain entropy measure induced by a score map)**.** Let $X = \{x_1, \ldots, x_n\}$ and let M be as in Definition 6.2.3. Fix a score map

$$S_M : \mathrm{Dom}(M) \to [0,1].$$

For an uncertain set

$$A_M = (X, \mu_{A_M}),$$

define its *score-realization* $\overline{A}_M \in [0,1]^X$ by

$$\mu_{\overline{A}_M}(x) := S_M\big(\mu_{A_M}(x)\big) \qquad (x \in X).$$

Let

$$E : [0,1]^X \to \mathbb{R}_{\geq 0}$$

be a fuzzy entropy measure. The *uncertain entropy measure of type M* induced by E and S_M is the mapping

$$E_M : \mathcal{U}_M(X) \to \mathbb{R}_{\geq 0}, \qquad E_M(A_M) := E(\overline{A}_M),$$

where

$$\mathcal{U}_M(X) := \{\mu : X \to \mathrm{Dom}(M)\}.$$

Definition 6.2.5 (De Luca–Termini uncertain entropy)**.** Let $X = \{x_1, \dots, x_n\}$, let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

be a score map.

For an uncertain set

$$A_M = (X, \mu_{A_M}),$$

write

$$s_i := S_M\big(\mu_{A_M}(x_i)\big) \qquad (i = 1, \dots, n).$$

Fix a logarithm base $b > 1$ and a constant $K > 0$. The *De Luca–Termini uncertain entropy* of A_M is defined by

$$H^M_{\mathrm{DLT}}(A_M) := -K \sum_{i=1}^{n} \Big(s_i \log_b s_i + (1 - s_i) \log_b (1 - s_i) \Big),$$

with the convention

$$0 \log_b 0 := 0.$$

Equivalently,

$$H^M_{\mathrm{DLT}}(A_M) = H_{\mathrm{DLT}}(\overline{A}_M),$$

where H_{DLT} denotes the ordinary De Luca–Termini fuzzy entropy of the score-realization $\overline{A}_M$.

Remark 6.2.6. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

then

$$H^M_{\mathrm{DLT}}(A_M) = -K \sum_{i=1}^{n} \Big(\mu_A(x_i) \log_b \mu_A(x_i) + (1 - \mu_A(x_i)) \log_b (1 - \mu_A(x_i)) \Big),$$

which is exactly the usual De Luca–Termini fuzzy entropy.

Theorem 6.2.7 (Well-definedness of uncertain entropy)**.** *Let $X = \{x_1, \dots, x_n\}$ be finite, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

be a score map. Then:

1. *for every uncertain set*

   $$A_M = (X, \mu_{A_M}),$$

   the score-realization

   $$\overline{A}_M \in [0,1]^X$$

   is well-defined;

2. *if*

$$E : [0,1]^X \to \mathbb{R}_{\geq 0}$$

is a fuzzy entropy measure, then the induced mapping

$$E_M(A_M) := E(\overline{A}_M)$$

is a well-defined uncertain entropy measure on $\mathcal{U}_M(X)$*;*

3. *for every uncertain set* A_M*, the quantity*

$$H^M_{\mathrm{DLT}}(A_M)$$

in Definition 6.2.5 is a well-defined real number in $\mathbb{R}_{\geq 0}$*.*

Proof. Let

$$A_M = (X, \mu_{A_M})$$

be an uncertain set of type M on X. Since

$$\mu_{A_M} : X \to \mathrm{Dom}(M)$$

and

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

for each $x_i \in X$ one has

$$S_M\big(\mu_{A_M}(x_i)\big) \in [0,1].$$

Hence the mapping

$$\mu_{\overline{A}_M} : X \to [0,1], \qquad \mu_{\overline{A}_M}(x_i) := S_M\big(\mu_{A_M}(x_i)\big),$$

is well-defined. Therefore

$$\overline{A}_M \in [0,1]^X$$

is a well-defined fuzzy set. This proves (1).

Now let

$$E : [0,1]^X \to \mathbb{R}_{\geq 0}$$

be a fuzzy entropy measure. Since $\overline{A}_M \in [0,1]^X$, the value

$$E(\overline{A}_M)$$

is well-defined and belongs to $\mathbb{R}_{\geq 0}$. Therefore

$$E_M(A_M) := E(\overline{A}_M)$$

is a well-defined mapping

$$E_M : \mathcal{U}_M(X) \to \mathbb{R}_{\geq 0}.$$

This proves (2).

For (3), set

$$s_i := S_M\big(\mu_{A_M}(x_i)\big) \in [0,1] \qquad (i = 1, \ldots, n).$$

Thus each term

$$s_i \log_b s_i \qquad \text{and} \qquad (1 - s_i)\log_b(1 - s_i)$$

is well-defined under the convention $0 \log_b 0 := 0$. Since X is finite, the sum

$$\sum_{i=1}^{n} \Big(s_i \log_b s_i + (1 - s_i) \log_b (1 - s_i) \Big)$$

is finite and therefore well-defined.

Moreover, because $0 \leq s_i \leq 1$, one has

$$\log_b s_i \leq 0 \qquad \text{whenever } s_i \in (0, 1],$$

and

$$\log_b (1 - s_i) \leq 0 \qquad \text{whenever } 1 - s_i \in (0, 1].$$

Hence

$$s_i \log_b s_i + (1 - s_i) \log_b (1 - s_i) \leq 0$$

for every i. Since $K > 0$, it follows that

$$H^M_{\mathrm{DLT}}(A_M) \geq 0.$$

Therefore

$$H^M_{\mathrm{DLT}}(A_M) \in \mathbb{R}_{\geq 0}$$

is well-defined. This proves (3). □

Representative uncertainty-entropy families and their degree representations are presented in Table 6.3.

Table 6.3: Representative uncertainty-entropy families and their degree representations.

Uncertainty-entropy family	**Degree form**	**Typical degree-domain / representation**
Fuzzy Entropy	scalar	$E_M : [0,1]^X \to \mathbb{R}_{\geq 0}$, based on a membership function $\mu_M : X \to [0, 1]$.
Intuitionistic Fuzzy Entropy [684, 685]	2-component	$E_M : \mathcal{IF}(X) \to \mathbb{R}_{\geq 0}$, where $\mu_M : X \to [0, 1]^2$, typically $(\text{membership}, \text{non-membership})$.
Hesitant Fuzzy Entropy [686, 687]	set-valued	$E_M : \mathcal{HF}(X) \to \mathbb{R}_{\geq 0}$, where each element is assigned a finite set of possible membership degrees, e.g., $\mu_M : X \to \mathcal{P}_{\text{fin}}([0, 1]) \setminus \{\varnothing\}$.
Picture Fuzzy Entropy [688, 689]	3-component	$E_M : \mathcal{PF}(X) \to \mathbb{R}_{\geq 0}$, where $\mu_M : X \to [0, 1]^3$, typically $(\text{positive}, \text{neutral}, \text{negative})$, with a model-specific constraint such as $$p(x) + n(x) + q(x) \leq 1.$$
Spherical Fuzzy Entropy [690–692]	3-component	$E_M : \mathcal{SF}(X) \to \mathbb{R}_{\geq 0}$, where $\mu_M : X \to [0, 1]^3$, typically $(\text{membership}, \text{non-membership}, \text{hesitancy})$, with the spherical constraint $$\mu(x)^2 + \nu(x)^2 + \pi(x)^2 \leq 1.$$
Neutrosophic Entropy [693–695]	3-component	$E_M : \mathcal{NS}(X) \to \mathbb{R}_{\geq 0}$, where $\mu_M : X \to [0, 1]^3$, typically (T, I, F).

As related concepts beyond the above uncertain entropy, multiscale fuzzy entropy [696, 697], Tsallis entropy [698–700], Cross entropy [701–704], Differential entropy [705–707], Conditional entropy [708, 708, 709], and hyperentropy [710] are also known.

6.3 Uncertain Numbers

Fuzzy numbers are normalized, convex fuzzy sets on $\mathbb{R}$ with bounded support; each -cut is a closed interval. Arithmetic uses α-cuts/extension principle, representing imprecise quantities and tolerances [711, 712].

Definition 6.3.1 (Fuzzy number (see [711, 712]))**.** A fuzzy set A on $\mathbb{R}$ with membership function

$$\mu_A : \mathbb{R} \longrightarrow [0, 1]$$

is called a *fuzzy number* if it satisfies:

(i) **Normality:** there exists $x_0 \in \mathbb{R}$ such that $\mu_A(x_0) = 1$.

(ii) **Fuzzy convexity:** for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$\mu_A\big(\lambda x + (1 - \lambda)y\big) \;\geq\; \min\{\mu_A(x),\, \mu_A(y)\}.$$

(iii) **Upper semi-continuity:** μ_A is upper semi-continuous on $\mathbb{R}$.

(iv) **Compact support:** the support $\mathrm{supp}(A) = \{\, x \in \mathbb{R} \mid \mu_A(x) > 0\}$ is a compact (hence bounded) subset of $\mathbb{R}$.

Uncertain numbers are uncertain sets on $\mathbb{R}$ whose score-realizations are fuzzy numbers, thus extending fuzzy numbers to general uncertainty-degree domains.

Definition 6.3.2 (Uncertain number)**.** Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

and fix a score map

$$S_M : \mathrm{Dom}(M) \to [0, 1].$$

An *uncertain set of type M* on $\mathbb{R}$ is a pair

$$A_M = (\mathbb{R}, \mu_{A_M}),$$

where

$$\mu_{A_M} : \mathbb{R} \to \mathrm{Dom}(M).$$

For such an uncertain set, define its *score-realization*

$$\overline{A}_M : \mathbb{R} \to [0, 1]$$

by

$$\mu_{\overline{A}_M}(x) := S_M\big(\mu_{A_M}(x)\big) \qquad (x \in \mathbb{R}).$$

Then A_M is called an *uncertain number of type M* if its score-realization $\overline{A}_M$ is a fuzzy number on $\mathbb{R}$; equivalently, if the following conditions hold:

(i) **Normality:** there exists $x_0 \in \mathbb{R}$ such that

$$S_M\big(\mu_{A_M}(x_0)\big) = 1;$$

(ii) **Score-convexity:** for all $x, y \in \mathbb{R}$ and all $\lambda \in [0,1]$,

$$S_M\big(\mu_{A_M}(\lambda x + (1-\lambda)y)\big) \geq \min\big\{S_M(\mu_{A_M}(x)), S_M(\mu_{A_M}(y))\big\};$$

(iii) **Upper semi-continuity:** the function

$$x \longmapsto S_M\big(\mu_{A_M}(x)\big)$$

is upper semi-continuous on $\mathbb{R}$;

(iv) **Compact score-support:** the set

$$\operatorname{supp}_{S_M}(A_M) := \{\, x \in \mathbb{R} \mid S_M(\mu_{A_M}(x)) > 0 \,\}$$

is compact.

Definition 6.3.3 (Score-induced α-cut)**.** Let A_M be an uncertain set of type M on $\mathbb{R}$, and let $\alpha \in (0,1]$. The *score-induced α-cut* of A_M is defined by

$$[A_M]_\alpha^{S_M} := \{\, x \in \mathbb{R} \mid S_M(\mu_{A_M}(x)) \geq \alpha \,\}.$$

Remark 6.3.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

then Definition 6.3.2 reduces exactly to the usual definition of a fuzzy number. Thus fuzzy numbers are special cases of uncertain numbers.

Theorem 6.3.5 (Well-definedness of uncertain numbers)**.** *Let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let

$$S_M : \mathrm{Dom}(M) \to [0,1]$$

be a score map. Let

$$A_M = (\mathbb{R}, \mu_{A_M})$$

be an uncertain set of type M on $\mathbb{R}$. Then:

1. *the score-realization*

$$\overline{A}_M : \mathbb{R} \to [0,1], \qquad \mu_{\overline{A}_M}(x) = S_M(\mu_{A_M}(x)),$$

is a well-defined fuzzy set on $\mathbb{R}$;

2. *if A_M satisfies the conditions in Definition 6.3.2, then $\overline{A}_M$ is a fuzzy number on $\mathbb{R}$;*

3. *for every $\alpha \in (0,1]$, the score-induced α-cut*

$$[A_M]_\alpha^{S_M}$$

is a nonempty compact interval in $\mathbb{R}$;

4. *consequently, the notion of uncertain number is mathematically well-defined.*

Proof. Since
$$\mu_{A_M} : \mathbb{R} \to \mathrm{Dom}(M)$$
and
$$S_M : \mathrm{Dom}(M) \to [0,1],$$
the composition
$$x \longmapsto S_M(\mu_{A_M}(x))$$
is a well-defined map from $\mathbb{R}$ to $[0,1]$. Hence
$$\overline{A}_M : \mathbb{R} \to [0,1]$$
is a well-defined fuzzy set on $\mathbb{R}$. This proves (1).

If A_M satisfies (i)–(iv) of Definition 6.3.2, then by construction its score-realization $\overline{A}_M$ is normal, fuzzy convex, upper semi-continuous, and has compact support. Therefore $\overline{A}_M$ is a fuzzy number on $\mathbb{R}$. This proves (2).

Fix $\alpha \in (0,1]$. By Definition 6.3.3,
$$[A_M]_\alpha^{S_M} = \{\, x \in \mathbb{R} \mid \mu_{\overline{A}_M}(x) \geq \alpha \,\},$$
so $[A_M]_\alpha^{S_M}$ is exactly the ordinary α-cut of the fuzzy number $\overline{A}_M$.

Because $\overline{A}_M$ is normal, there exists $x_0 \in \mathbb{R}$ such that
$$\mu_{\overline{A}_M}(x_0) = 1 \geq \alpha,$$
hence
$$x_0 \in [A_M]_\alpha^{S_M},$$
so the α-cut is nonempty.

Next, since $\mu_{\overline{A}_M}$ is upper semi-continuous, the set
$$\{\, x \in \mathbb{R} \mid \mu_{\overline{A}_M}(x) \geq \alpha \,\}$$
is closed. Moreover, if $x \in [A_M]_\alpha^{S_M}$, then necessarily
$$\mu_{\overline{A}_M}(x) \geq \alpha > 0,$$
so
$$x \in \mathrm{supp}_{S_M}(A_M).$$
Hence
$$[A_M]_\alpha^{S_M} \subseteq \mathrm{supp}_{S_M}(A_M).$$
Since $\mathrm{supp}_{S_M}(A_M)$ is compact, $[A_M]_\alpha^{S_M}$ is a closed subset of a compact set, and therefore compact.

It remains to prove that $[A_M]_\alpha^{S_M}$ is an interval. Let $x, y \in [A_M]_\alpha^{S_M}$ and let $\lambda \in [0,1]$. Then
$$S_M(\mu_{A_M}(x)) \geq \alpha, \qquad S_M(\mu_{A_M}(y)) \geq \alpha.$$

By score-convexity,

$$S_M\big(\mu_{A_M}(\lambda x + (1-\lambda)y)\big) \geq \min\big\{S_M(\mu_{A_M}(x)), S_M(\mu_{A_M}(y))\big\} \geq \alpha.$$

Therefore

$$\lambda x + (1-\lambda)y \in [A_M]_\alpha^{S_M}.$$

Thus $[A_M]_\alpha^{S_M}$ is convex. Since every nonempty compact convex subset of $\mathbb{R}$ is a compact interval, it follows that $[A_M]_\alpha^{S_M}$ is a nonempty compact interval.

This proves (3), and (4) follows immediately. □

A catalogue of uncertainty-number families is presented in Table 6.4.

Table 6.4: A catalogue of uncertainty-number families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (conceptual overview).

k	Representative uncertainty-number type(s) $\widetilde{x}_M$ with degree data in $\mathrm{Dom}(M) \subseteq [0,1]^k$
1	Fuzzy number (single membership profile on $\mathbb{R}$).
2	Intuitionistic fuzzy number [713, 714]; vague number [715, 716][(a)]; bipolar fuzzy number [717, 718].
3	Hesitant fuzzy number [719, 720][(b)]; neutrosophic number [721–726]; picture fuzzy number [727, 728]; spherical fuzzy number [729, 730].
4	Quadripartitioned neutrosophic number [731]; Double-Valued Neutrosophic Number [732]
$s+t$	Plithogenic number [733, 734][(c)].

[(a)] Vague numbers are commonly presented with "true" and "false" membership functions; this is typically equivalent to an intuitionistic-fuzzy (orthopair) representation, hence $k = 2$.
[(b)] Hesitant fuzzy numbers are naturally set-valued ($H(x) \subseteq [0,1]$). Placing them at $k = 3$ means adopting a fixed 3-tuple representation (e.g. restricting to hesitant sets of size 3, or extracting three canonical/summary values such as min /mean/ max).
[(c)] Plithogenic numbers are attribute-aware and use contradiction information; concatenating appurtenance and contradiction coordinates yields a single degree-domain $\subseteq [0,1]^{s+t}$ for the "dimension-k" catalogue format.

Chapter 7

Other Fuzzy Mathematics

In this chapter, we describe other topics in fuzzy mathematics. A concise comparison of the uncertain concepts covered in Chapter 7 is presented in Table 7.1.

Table 7.1: A concise comparison of the uncertain concepts covered in Chapter 7.

Concept	**Base structure**	**Main focus**
Uncertain Measure	Measure space	Uncertain set functions and generalized measurement of size.
Uncertain Integral	Integral framework	Uncertain aggregation and accumulation over functions or sets.
Uncertain Optimization	Optimization problem	Uncertain objective values, constraints, and solution behavior.
Uncertain Cardinality	Set cardinality	Uncertain assessment of the size or extent of a set.
Uncertain Quantifiers	Quantifier framework	Uncertain interpretation of universal and existential statements.
Uncertain Function	Function space	Uncertain mappings between domains and codomains.

7.1 Uncertain Measure

Fuzzy measure assigns monotone graded values to sets, generalizing measures without requiring additivity and modeling uncertainty, importance, or interaction among events or subsets in applications [735–738].

Definition 7.1.1 (Fuzzy measure)**.** [735, 736] Let X be a nonempty set, and let $\mathcal{A}$ be a σ-algebra on X. A set function

$$g : \mathcal{A} \to [0, 1]$$

is called a *fuzzy measure* on $(X, \mathcal{A})$ if it satisfies the following conditions:

1. $g(\varnothing) = 0$;

2. whenever $A, B \in \mathcal{A}$ and $A \subseteq B$, one has
$$g(A) \le g(B);$$
3. for every increasing sequence $(A_n)_{n\ge1} \subseteq \mathcal{A}$ with
$$A_1 \subseteq A_2 \subseteq \cdots,$$
one has
$$g\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} g(A_n);$$
4. for every decreasing sequence $(A_n)_{n\ge1} \subseteq \mathcal{A}$ with
$$A_1 \supseteq A_2 \supseteq \cdots,$$
one has
$$g\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} g(A_n).$$

If, in addition,
$$g(X) = 1,$$
then g is called a *normalized fuzzy measure.*

An uncertain measure is an uncertainty-valued set function on a σ-algebra whose score-realization satisfies the axioms of a fuzzy measure.

Definition 7.1.2 (Uncertain measure)**.** Let X be a nonempty set, let $\mathcal{A}$ be a σ-algebra on X, and let M be an uncertain model with degree-domain
$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$
A mapping
$$g_M : \mathcal{A} \to \mathrm{Dom}(M)$$
is called an *uncertain measure of type M* on $(X, \mathcal{A})$ if the following data are fixed:

- an admissible score map
$$S_M : \mathrm{Dom}(M) \to [0,1].$$

Define the score-realization
$$\overline{g}_M : \mathcal{A} \to [0,1]$$
by
$$\overline{g}_M(A) := S_M\big(g_M(A)\big) \qquad (A \in \mathcal{A}).$$
We require that $\overline{g}_M$ be a fuzzy measure on $(X, \mathcal{A})$; namely:

1. $$\overline{g}_M(\varnothing) = 0;$$

2. whenever $A, B \in \mathcal{A}$ and $A \subseteq B$, one has
$$\overline{g}_M(A) \leq \overline{g}_M(B);$$

3. for every increasing sequence $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ with
$$A_1 \subseteq A_2 \subseteq \cdots,$$
one has
$$\overline{g}_M\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \overline{g}_M(A_n);$$

4. for every decreasing sequence $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ with
$$A_1 \supseteq A_2 \supseteq \cdots,$$
one has
$$\overline{g}_M\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \overline{g}_M(A_n).$$

If, in addition,
$$\overline{g}_M(X) = 1,$$
equivalently,
$$S_M\big(g_M(X)\big) = 1,$$
then g_M is called a *normalized uncertain measure.*

Remark 7.1.3. If M is the fuzzy model, so that
$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$
then Definition 7.1.2 reduces exactly to the usual definition of a fuzzy measure. Thus fuzzy measures are special cases of uncertain measures.

Theorem 7.1.4 (Well-definedness of uncertain measures)**.** *Let X be a nonempty set, let $\mathcal{A}$ be a σ-algebra on X, let M be an uncertain model with degree-domain*
$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$
and let
$$g_M : \mathcal{A} \to \mathrm{Dom}(M), \qquad S_M : \mathrm{Dom}(M) \to [0,1]$$
be given. Then:

1. *the score-realization*
$$\overline{g}_M : \mathcal{A} \to [0,1], \qquad \overline{g}_M(A) = S_M\big(g_M(A)\big),$$
is a well-defined set function on $\mathcal{A}$;

2. *all quantities and limits appearing in Definition 7.1.2 are meaningful real numbers;*

3. *consequently, the notion of uncertain measure is mathematically well-defined.*

Moreover, if the axioms in Definition 7.1.2 hold, then $\overline{g}_M$ *is a well-defined fuzzy measure on* $(X, \mathcal{A})$.

Proof. Since

$$g_M : \mathcal{A} \to \mathrm{Dom}(M)$$

and

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

for every $A \in \mathcal{A}$ the value

$$S_M\big(g_M(A)\big)$$

belongs to $[0,1]$. Therefore the assignment

$$\overline{g}_M(A) := S_M\big(g_M(A)\big) \qquad (A \in \mathcal{A})$$

defines a well-defined mapping

$$\overline{g}_M : \mathcal{A} \to [0,1].$$

This proves (1).

Next, because $\mathcal{A}$ is a σ-algebra on X, one has

$$\varnothing \in \mathcal{A}, \qquad A, B \in \mathcal{A},\ A \subseteq B \implies A, B \in \mathcal{A},$$

and for every sequence $(A_n)_{n\geq 1} \subseteq \mathcal{A}$,

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}, \qquad \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}.$$

Hence the quantities

$$\overline{g}_M(\varnothing), \qquad \overline{g}_M(A), \qquad \overline{g}_M(B), \qquad \overline{g}_M\left(\bigcup_{n=1}^{\infty} A_n\right), \qquad \overline{g}_M\left(\bigcap_{n=1}^{\infty} A_n\right)$$

are all well-defined real numbers in $[0,1]$.

Moreover, for every $n \geq 1$,

$$\overline{g}_M(A_n) \in [0,1].$$

Thus the sequence

$$\big(\overline{g}_M(A_n)\big)_{n\geq 1}$$

is a bounded sequence of real numbers, so the expressions

$$\lim_{n\to\infty} \overline{g}_M(A_n)$$

appearing in the continuity-from-below and continuity-from-above axioms are standard real limits whenever they are required to exist by the axioms.

Therefore all inequalities and equalities in Definition 7.1.2,

$$\overline{g}_M(A) \leq \overline{g}_M(B),$$

$$\overline{g}_M\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \overline{g}_M(A_n),$$

and

$$\overline{g}_M\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \overline{g}_M(A_n),$$

are meaningful statements about real numbers. This proves (2).

Hence all objects and conditions appearing in Definition 7.1.2 are mathematically meaningful, and the notion of uncertain measure is well-defined. This proves (3).

Finally, if the axioms in Definition 7.1.2 hold, then by definition $\overline{g}_M$ satisfies the axioms of a fuzzy measure on $(X, \mathcal{A})$. Therefore $\overline{g}_M$ is a well-defined fuzzy measure. □

For reference, representative uncertainty-measure families and their degree representations are presented in Table 7.2.

Table 7.2: Representative uncertainty-measure families and their degree representations.

Uncertainty-measure family	**Degree form**	**Typical degree-domain / representation**
Fuzzy Measure	scalar	$g_M : \mathcal{A} \to [0,1]$, where $\mathcal{A}$ is a σ-algebra on X.
Intuitionistic Fuzzy Measure [739, 740]	2-component	$g_M : \mathcal{A} \to [0,1]^2$, typically (membership, non-membership).
Neutrosophic Measure [741–743]	3-component	$g_M : \mathcal{A} \to [0,1]^3$, typically (T, I, F).
Plithogenic Measure [10, 18, 31]	attribute-based / multi-component	Typically described by appurtenance degrees together with contradiction information relative to attribute values; for example, $$g_M : \mathcal{A} \times Pv \to [0,1]^s,$$ with an associated contradiction map on the attribute-value set Pv.

7.2 Uncertain Integral

Fuzzy integral aggregates measurable values relative to a fuzzy measure, modeling nonadditive uncertainty and interaction through Choquet integration or Sugeno max-min aggregation schemes in practice [744–746].

Definition 7.2.1 (Fuzzy integral)**.** Let $(X, \mathcal{A})$ be a measurable space, let

$$g : \mathcal{A} \to [0,1]$$

be a fuzzy measure, and let

$$f : X \to [0, \infty]$$

be an $\mathcal{A}$-measurable function.

1. The *Choquet fuzzy integral* of f with respect to g is defined by

$$\int_X f\,dg := \int_0^\infty g\big(\{x \in X : f(x) \geq \alpha\}\big)\,d\alpha,$$

   provided the right-hand side is well defined.

2. The *Sugeno fuzzy integral* of f with respect to g is defined by

$$\int_X^{\mathrm{S}} f\,dg := \sup_{\alpha \in [0,\infty)} \min\Big(\alpha,\ g\big(\{x \in X : f(x) \geq \alpha\}\big)\Big).$$

Uncertain integrals aggregate uncertainty-valued measurable information relative to an uncertain measure by applying fuzzy-integral constructions to their score-realizations.

Definition 7.2.2 (Uncertain measurable function)**.** Let $(X, \mathcal{A})$ be a measurable space, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Fix a score map

$$S_M : \mathrm{Dom}(M) \to [0,1].$$

A mapping

$$f_M : X \to \mathrm{Dom}(M)$$

is called an *uncertain measurable function of type* M if its score-realization

$$\overline{f}_M : X \to [0,1], \qquad \overline{f}_M(x) := S_M\big(f_M(x)\big),$$

is $\mathcal{A}$-measurable.

Definition 7.2.3 (Uncertain integral)**.** Let $(X, \mathcal{A})$ be a measurable space, let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

let

$$g_M : \mathcal{A} \to \mathrm{Dom}(M)$$

be an uncertain measure of type M, and let

$$f_M : X \to \mathrm{Dom}(M)$$

be an uncertain measurable function of type M.

Fix a score map

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

and define the score-realizations

$$\overline{g}_M(A) := S_M\big(g_M(A)\big) \qquad (A \in \mathcal{A}),$$

and

$$\overline{f}_M(x) := S_M\big(f_M(x)\big) \qquad (x \in X).$$

1. The *Choquet uncertain integral* of f_M with respect to g_M is defined by

$$\int_X f_M \, dg_M := \int_0^1 \overline{g}_M\big(\{x \in X : \overline{f}_M(x) \geq \alpha\}\big) \, d\alpha.$$

2. The *Sugeno uncertain integral* of f_M with respect to g_M is defined by

$$\int_X^{\mathrm{S}} f_M \, dg_M := \sup_{\alpha\in[0,1]} \min\Big(\alpha, \, \overline{g}_M\big(\{x \in X : \overline{f}_M(x) \geq \alpha\}\big)\Big).$$

Remark 7.2.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

then

$$\overline{f}_M = f_M \qquad \text{and} \qquad \overline{g}_M = g_M,$$

and Definition 7.2.3 reduces exactly to the usual fuzzy Choquet and Sugeno integrals for $[0,1]$-valued measurable functions.

Theorem 7.2.5 (Well-definedness of uncertain integrals)**.** *Let $(X, \mathcal{A})$ be a measurable space, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

let

$$g_M : \mathcal{A} \to \mathrm{Dom}(M)$$

be an uncertain measure of type M, and let

$$f_M : X \to \mathrm{Dom}(M)$$

be an uncertain measurable function of type M. Then:

1. *the score-realizations*

$$\overline{g}_M : \mathcal{A} \to [0,1] \qquad \textit{and} \qquad \overline{f}_M : X \to [0,1]$$

are well-defined;

2. *for every $\alpha \in [0,1]$, the level set*

$$E_\alpha := \{x \in X : \overline{f}_M(x) \geq \alpha\}$$

belongs to $\mathcal{A}$, and hence

$$\overline{g}_M(E_\alpha)$$

is well-defined;

3. *the mapping*

$$\alpha \longmapsto \overline{g}_M(E_\alpha)$$

is a bounded function from $[0,1]$ into $[0,1]$, so the Choquet uncertain integral

$$\int_0^1 \overline{g}_M(E_\alpha) \, d\alpha$$

is well-defined as a real number in $[0,1]$;

4. *the set*

$$\{\min(\alpha, \overline{g}_M(E_\alpha)) \mid \alpha \in [0,1]\}$$

is a nonempty subset of $[0,1]$*, so its supremum exists in* $[0,1]$*, and therefore the Sugeno uncertain integral is well-defined.*

Hence the notion of uncertain integral is mathematically well-defined.

Proof. Since

$$S_M : \mathrm{Dom}(M) \to [0,1], \qquad g_M : \mathcal{A} \to \mathrm{Dom}(M), \qquad f_M : X \to \mathrm{Dom}(M),$$

the compositions

$$\overline{g}_M = S_M \circ g_M \qquad \text{and} \qquad \overline{f}_M = S_M \circ f_M$$

are well-defined. Moreover,

$$\overline{g}_M(A) \in [0,1] \qquad (A \in \mathcal{A}),$$

and

$$\overline{f}_M(x) \in [0,1] \qquad (x \in X).$$

This proves (1).

Because f_M is uncertain measurable, its score-realization $\overline{f}_M$ is $\mathcal{A}$-measurable. Therefore, for every $\alpha \in [0,1]$, the superlevel set

$$E_\alpha = \{x \in X : \overline{f}_M(x) \geq \alpha\}$$

belongs to $\mathcal{A}$. Since $\overline{g}_M$ is defined on $\mathcal{A}$, the value

$$\overline{g}_M(E_\alpha)$$

is well-defined and belongs to $[0,1]$. This proves (2).

Now define

$$\varphi(\alpha) := \overline{g}_M(E_\alpha) \qquad (\alpha \in [0,1]).$$

By (2), φ is a well-defined function on $[0,1]$, and since $\overline{g}_M$ takes values in $[0,1]$, one has

$$0 \leq \varphi(\alpha) \leq 1 \qquad (\alpha \in [0,1]).$$

Hence φ is bounded on the compact interval $[0,1]$. Therefore the integral

$$\int_0^1 \varphi(\alpha)\, d\alpha = \int_0^1 \overline{g}_M(\{x \in X : \overline{f}_M(x) \geq \alpha\})\, d\alpha$$

is well-defined as a real number. Since $0 \leq \varphi(\alpha) \leq 1$, one also has

$$0 \leq \int_0^1 \varphi(\alpha)\, d\alpha \leq 1.$$

This proves (3).

For each $\alpha \in [0,1]$, the quantity

$$\min(\alpha, \overline{g}_M(E_\alpha))$$

is well-defined and belongs to $[0,1]$, because both arguments of min lie in $[0,1]$. Hence the set

$$\{\min(\alpha, \overline{g}_M(E_\alpha)) \mid \alpha \in [0,1]\}$$

is a nonempty subset of $[0,1]$. Since $[0,1]$ is order-complete, its supremum exists and belongs to $[0,1]$. Therefore

$$\sup_{\alpha\in[0,1]} \min\Big(\alpha,\, \overline{g}_M(\{x \in X : \overline{f}_M(x) \geq \alpha\})\Big)$$

is well-defined. This proves (4).

Therefore both the Choquet uncertain integral and the Sugeno uncertain integral are mathematically meaningful, and the notion of uncertain integral is well-defined. □

For reference, representative uncertainty-integral families and their degree representations are presented in Table 7.3.

Table 7.3: Representative uncertainty-integral families and their degree representations.

Uncertainty-integral family	**Degree form**	**Typical degree-domain / representation**
Fuzzy Integrals	scalar	$f_M : X \to [0,1]$ or $f_M : X \to [0,\infty)$, with a fuzzy measure $g_M : \mathcal{A} \to [0,1]$.
Intuitionistic Fuzzy Integrals [747, 748]	2-component	$f_M : X \to [0,1]^2$ and/or $g_M : \mathcal{A} \to [0,1]^2$, typically for membership and non-membership degrees.
Hesitant Fuzzy Integrals [749]	set-valued	$f_M : X \to \mathcal{P}_{\mathrm{fin}}([0,1]) \setminus \{\varnothing\}$, where each value is a finite set of possible membership degrees.
Picture Fuzzy Integrals [750]	3-component	$f_M : X \to [0,1]^3$ and/or $g_M : \mathcal{A} \to [0,1]^3$, typically for positive, neutral, and negative grades.
Neutrosophic Integrals [751–754]	3-component	$f_M : X \to [0,1]^3$ and/or $g_M : \mathcal{A} \to [0,1]^3$, typically with components (T, I, F).
Plithogenic Integrals [755, 756]	multi-component	Typically based on appurtenance degrees together with contradiction information on attribute values, e.g. $f_M : X \times Pv \to [0,1]^s$.

7.3 Uncertain Optimization

Fuzzy optimization studies optimization problems whose objectives, constraints, or both are fuzzy, seeking best decisions under vagueness through ranking, satisfaction levels, or alpha-cut methods effectively [757–760].

Definition 7.3.1 (Fuzzy optimization)**.** Let X be a nonempty decision set, let $\tilde{C}$ be a fuzzy set on X with membership function

$$\mu_{\tilde{C}} : X \to [0,1],$$

and let

$$\tilde{f} : X \to \mathcal{F}(\mathbb{R})$$

be a fuzzy-valued objective function, where $\mathcal{F}(\mathbb{R})$ denotes a prescribed class of fuzzy numbers (or fuzzy subsets of $\mathbb{R}$). Assume also that $\preceq$ is a ranking relation on $\mathcal{F}(\mathbb{R})$.

For a fixed level $\alpha \in (0,1]$, define the α-feasible set by

$$C_\alpha := \{x \in X \mid \mu_{\tilde{C}}(x) \geq \alpha\}.$$

The corresponding *fuzzy optimization problem* is

$$\text{maximize } \tilde{f}(x) \qquad \text{subject to } x \in C_\alpha.$$

An element $x^* \in C_\alpha$ is called an *α-optimal solution* if

$$\tilde{f}(x) \preceq \tilde{f}(x^*) \qquad \text{for all } x \in C_\alpha.$$

In general, fuzzy optimization refers to optimization problems in which the objective function, the feasible region, or both are described by fuzzy sets.

Uncertain optimization studies optimization problems in which the feasible region, the objective function, or both are represented by uncertain sets or uncertain numbers.

Definition 7.3.2 (Uncertain optimization)**.** Let X be a nonempty decision set.

Let M_C be an uncertain model with degree-domain

$$\mathrm{Dom}(M_C) \subseteq [0,1]^{k_C},$$

and let

$$\widetilde{C}_{M_C} = (X, \mu_{\widetilde{C}_{M_C}})$$

be an uncertain set of type M_C on X, where

$$\mu_{\widetilde{C}_{M_C}} : X \to \mathrm{Dom}(M_C).$$

Fix a score map

$$S_C : \mathrm{Dom}(M_C) \to [0,1].$$

Let M_f be an uncertain model with degree-domain

$$\mathrm{Dom}(M_f) \subseteq [0,1]^{k_f},$$

and let

$$\mathcal{UN}_{M_f}(\mathbb{R})$$

denote a prescribed class of uncertain numbers of type M_f on $\mathbb{R}$. Let

$$\widetilde{f}_{M_f} : X \to \mathcal{UN}_{M_f}(\mathbb{R})$$

be an uncertain-valued objective function. Assume also that

$$\preceq_{M_f}$$

is a ranking relation on $\mathcal{UN}_{M_f}(\mathbb{R})$.

For a fixed level $\alpha \in (0,1]$, define the score-induced α-feasible set by

$$C_\alpha^{S_C} := \{\, x \in X \mid S_C(\mu_{\widetilde{C}_{M_C}}(x)) \geq \alpha \,\}.$$

The corresponding *uncertain optimization problem* is

$$\text{maximize } \widetilde{f}_{M_f}(x) \qquad \text{subject to } x \in C_\alpha^{S_C}.$$

An element $x^* \in C_\alpha^{S_C}$ is called an *α-optimal solution* if

$$\widetilde{f}_{M_f}(x) \preceq_{M_f} \widetilde{f}_{M_f}(x^*) \qquad \text{for all } x \in C_\alpha^{S_C}.$$

In general, uncertain optimization refers to optimization problems in which the feasible region, the objective function, or both are described by uncertain sets or uncertain numbers.

Remark 7.3.3. If both M_C and M_f are the fuzzy model, so that

$$\mathrm{Dom}(M_C) = \mathrm{Dom}(M_f) = [0,1], \qquad S_C = \mathrm{id}_{[0,1]},$$

then Definition 7.3.2 reduces to the usual fuzzy optimization framework with

$$C_\alpha^{S_C} = \{\, x \in X \mid \mu_{\widetilde{C}}(x) \geq \alpha \,\}.$$

Thus fuzzy optimization is a special case of uncertain optimization.

Theorem 7.3.4 (Well-definedness of uncertain optimization)**.** *Let X, M_C, M_f, $\widetilde{C}_{M_C}$, S_C, $\widetilde{f}_{M_f}$, and $\preceq_{M_f}$ be as in Definition 7.3.2. Then:*

1. *for every $\alpha \in (0,1]$, the score-induced α-feasible set*

$$C_\alpha^{S_C} = \{\, x \in X \mid S_C(\mu_{\widetilde{C}_{M_C}}(x)) \geq \alpha \,\}$$

is a well-defined subset of X;

2. *for every $x \in X$, the value*

$$\widetilde{f}_{M_f}(x)$$

is a well-defined uncertain number in $\mathcal{UN}_{M_f}(\mathbb{R})$;

3. *for every $x, x^* \in C_\alpha^{S_C}$, the comparison*

$$\widetilde{f}_{M_f}(x) \preceq_{M_f} \widetilde{f}_{M_f}(x^*)$$

is meaningful;

4. *consequently, the notion of α-optimal solution in Definition 7.3.2 is mathematically well-defined.*

Proof. Let $\alpha \in (0,1]$. Since

$$\mu_{\widetilde{C}_{M_C}} : X \to \mathrm{Dom}(M_C)$$

and

$$S_C : \mathrm{Dom}(M_C) \to [0,1],$$

the composition

$$x \longmapsto S_C(\mu_{\widetilde{C}_{M_C}}(x))$$

is a well-defined map from X to $[0,1]$. Therefore, for each $x \in X$, the inequality

$$S_C(\mu_{\widetilde{C}_{M_C}}(x)) \geq \alpha$$

is meaningful. Hence

$$C_\alpha^{S_C} = \{\, x \in X \mid S_C(\mu_{\widetilde{C}_{M_C}}(x)) \geq \alpha \,\}$$

is a well-defined subset of X. This proves (1).

Next, since

$$\widetilde{f}_{M_f} : X \to \mathcal{UN}_{M_f}(\mathbb{R}),$$

for every $x \in X$ one has

$$\widetilde{f}_{M_f}(x) \in \mathcal{UN}_{M_f}(\mathbb{R}).$$

Thus each objective value is a well-defined uncertain number of type M_f. This proves (2).

Now let $x, x^* \in C_\alpha^{S_C}$. By (2),

$$\widetilde{f}_{M_f}(x),\ \widetilde{f}_{M_f}(x^*) \in \mathcal{UN}_{M_f}(\mathbb{R}).$$

Since

$$\preceq_{M_f}$$

is a ranking relation on $\mathcal{UN}_{M_f}(\mathbb{R})$, the statement

$$\widetilde{f}_{M_f}(x) \preceq_{M_f} \widetilde{f}_{M_f}(x^*)$$

is meaningful. This proves (3).

Therefore all objects and conditions appearing in Definition 7.3.2 are mathematically meaningful, and the notion of α-optimal solution is well-defined. This proves (4). □

For reference, representative uncertainty-optimization families and their degree representations are presented in Table 7.4.

Table 7.4: Representative uncertainty-optimization families and their degree representations.

Uncertainty-optimization family	**Degree form**	**Typical degree-domain / representation**
Fuzzy Optimization	scalar	Typically formulated with a fuzzy feasible set $$\widetilde{C} : X \to [0, 1]$$ and/or a fuzzy-valued objective $$\widetilde{f} : X \to \mathcal{F}(\mathbb{R}),$$ with optimization carried out by α-cuts, ranking relations, or satisfaction levels.
Intuitionistic Fuzzy Optimization [761, 762]	2-component	Typically formulated with intuitionistic fuzzy constraints and/or objectives, e.g. $$\widetilde{C} : X \to [0, 1]^2, \qquad \widetilde{f} : X \to \mathcal{IF}(\mathbb{R}),$$ where the two components usually represent (membership, non-membership).
Hesitant Fuzzy Optimization [763, 764]	set-valued	Typically formulated with hesitant fuzzy decision information, e.g. $$\widetilde{C} : X \to \mathcal{P}_{\text{fin}}([0, 1]) \setminus \{\varnothing\}, \qquad \widetilde{f} : X \to \mathcal{HF}(\mathbb{R}),$$ where each decision or objective value is associated with a finite set of possible membership degrees.
Neutrosophic Optimization [765–767]	3-component	Typically formulated with neutrosophic feasible regions and/or objectives, e.g. $$\widetilde{C} : X \to [0, 1]^3, \qquad \widetilde{f} : X \to \mathcal{NS}(\mathbb{R}),$$ where the three components usually represent (T, I, F).

7.4 Uncertain Cardinality

Fuzzy cardinality models the size of a fuzzy set by a fuzzy number, where each integer receives a degree determined from alpha-cut cardinalities across thresholds (cf. [768–770]).

Definition 7.4.1 (Fuzzy Cardinality)**.** Let X be a finite nonempty set, and let A be a fuzzy set on X with membership function

$$\mu_A : X \to [0, 1].$$

For each $\alpha \in (0, 1]$, define the α-cut of A by

$$A_\alpha := \{\, x \in X \mid \mu_A(x) \geq \alpha \,\}.$$

The *fuzzy cardinality* of A is the fuzzy set

$$\text{Card}(A) : \{0, 1, \ldots, |X|\} \to [0, 1]$$

given by

$$\mu_{\text{Card}(A)}(k) := \sup\{\, \alpha \in (0, 1] \mid |A_\alpha| = k \,\}, \qquad k = 0, 1, \ldots, |X|,$$

with the convention $\sup \varnothing = 0$. Thus $\mu_{\text{Card}(A)}(k)$ expresses the degree to which the cardinality of A can be regarded as k.

Uncertain cardinality models the size of an uncertain set by an uncertainty-valued set on $\{0, 1, \dots, |X|\}$, obtained from the cardinalities of score-induced α-cuts.

Definition 7.4.2 (Uncertain cardinality)**.** Let X be a finite nonempty set, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Let

$$A_M = (X, \mu_{A_M})$$

be an uncertain set of type M on X, where

$$\mu_{A_M} : X \to \mathrm{Dom}(M).$$

Assume that the following data are fixed:

- a score map

$$S_M : \mathrm{Dom}(M) \to [0, 1];$$

- a lift map

$$L_M : [0, 1] \to \mathrm{Dom}(M)$$

  such that

$$S_M\big(L_M(r)\big) = r \qquad \text{for all } r \in [0, 1].$$

For each $\alpha \in (0, 1]$, define the *score-induced α-cut* of A_M by

$$(A_M)^{S_M}_\alpha := \{\, x \in X \mid S_M(\mu_{A_M}(x)) \geq \alpha \,\}.$$

The *score-cardinality profile* of A_M is the mapping

$$c_{A_M} : \{0, 1, \dots, |X|\} \to [0, 1]$$

defined by

$$c_{A_M}(k) := \sup\{\, \alpha \in (0, 1] \mid |(A_M)^{S_M}_\alpha| = k \,\}, \qquad k = 0, 1, \dots, |X|,$$

with the convention

$$\sup \varnothing = 0.$$

The *uncertain cardinality of type M* of A_M is the uncertain set

$$\mathrm{Card}_M(A_M) = \big(\{0, 1, \dots, |X|\}, \mu_{\mathrm{Card}_M(A_M)}\big)$$

defined by

$$\mu_{\mathrm{Card}_M(A_M)}(k) := L_M\big(c_{A_M}(k)\big), \qquad k = 0, 1, \dots, |X|.$$

Thus the score

$$S_M\big(\mu_{\mathrm{Card}_M(A_M)}(k)\big) = c_{A_M}(k)$$

expresses the degree to which the cardinality of A_M can be regarded as k.

Remark 7.4.3. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1], \qquad S_M = \mathrm{id}_{[0,1]}, \qquad L_M = \mathrm{id}_{[0,1]},$$

then Definition 7.4.2 reduces exactly to the usual fuzzy cardinality:

$$\mu_{\mathrm{Card}(A)}(k) = \sup\{\, \alpha \in (0,1] \mid |A_\alpha| = k \,\}.$$

Thus fuzzy cardinality is a special case of uncertain cardinality.

Theorem 7.4.4 (Well-definedness of uncertain cardinality)**.** *Let X be a finite nonempty set, let M be an uncertain model with degree-domain*

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

let

$$A_M = (X, \mu_{A_M})$$

be an uncertain set of type M on X, and let

$$S_M : \mathrm{Dom}(M) \to [0,1], \qquad L_M : [0,1] \to \mathrm{Dom}(M)$$

satisfy

$$S_M\big(L_M(r)\big) = r \qquad (r \in [0,1]).$$

Then:

1. *for every $\alpha \in (0,1]$, the score-induced α-cut*

$$(A_M)_\alpha^{S_M} = \{\, x \in X \mid S_M(\mu_{A_M}(x)) \geq \alpha \,\}$$

is a well-defined subset of X;

2. *for every $k \in \{0,1,\ldots,|X|\}$, the quantity*

$$c_{A_M}(k) = \sup\{\, \alpha \in (0,1] \mid |(A_M)_\alpha^{S_M}| = k \,\}$$

is a well-defined element of $[0,1]$;

3. *the mapping*

$$\mu_{\mathrm{Card}_M(A_M)} : \{0,1,\ldots,|X|\} \to \mathrm{Dom}(M), \qquad \mu_{\mathrm{Card}_M(A_M)}(k) = L_M(c_{A_M}(k)),$$

is well-defined, and hence $\mathrm{Card}_M(A_M)$ is a well-defined uncertain set of type M;

4. *the score-realization of $\mathrm{Card}_M(A_M)$ is exactly the fuzzy cardinality of the score-realized fuzzy set*

$$\overline{A}_M : X \to [0,1], \qquad \overline{A}_M(x) := S_M(\mu_{A_M}(x)).$$

Consequently, the notion of uncertain cardinality is mathematically well-defined.

Proof. Let $\alpha \in (0,1]$. Since

$$\mu_{A_M} : X \to \mathrm{Dom}(M)$$

and

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

the composition

$$x \longmapsto S_M(\mu_{A_M}(x))$$

is a well-defined map from X to $[0,1]$. Hence, for each $x \in X$, the inequality

$$S_M(\mu_{A_M}(x)) \geq \alpha$$

is meaningful. Therefore

$$(A_M)_\alpha^{S_M} = \{\, x \in X \mid S_M(\mu_{A_M}(x)) \geq \alpha \,\}$$

is a well-defined subset of X. This proves (1).

Now fix

$$k \in \{0, 1, \ldots, |X|\}.$$

Consider the set

$$E_k := \{\, \alpha \in (0,1] \mid |(A_M)_\alpha^{S_M}| = k \,\}.$$

Since each $(A_M)_\alpha^{S_M}$ is a subset of the finite set X, its cardinality is an integer between 0 and $|X|$. Thus $E_k \subseteq (0,1]$. If $E_k = \varnothing$, then by convention

$$c_{A_M}(k) = \sup \varnothing = 0.$$

If $E_k \neq \varnothing$, then E_k is a nonempty subset of $(0,1] \subseteq \mathbb{R}$, hence it is bounded above by 1, so

$$\sup E_k$$

exists in $[0,1]$. Therefore in either case,

$$c_{A_M}(k) \in [0,1]$$

is well-defined. This proves (2).

Since

$$L_M : [0,1] \to \mathrm{Dom}(M),$$

for each $k \in \{0, 1, \ldots, |X|\}$ the value

$$L_M(c_{A_M}(k))$$

belongs to $\mathrm{Dom}(M)$. Hence

$$\mu_{\mathrm{Card}_M(A_M)}(k) := L_M(c_{A_M}(k))$$

defines a well-defined mapping

$$\mu_{\mathrm{Card}_M(A_M)} : \{0, 1, \ldots, |X|\} \to \mathrm{Dom}(M).$$

Therefore

$$\mathrm{Card}_M(A_M) = \bigl(\{0, 1, \ldots, |X|\}, \mu_{\mathrm{Card}_M(A_M)}\bigr)$$

is a well-defined uncertain set of type M. This proves (3).

Finally, for every $k \in \{0, 1, \ldots, |X|\}$,

$$S_M\bigl(\mu_{\mathrm{Card}_M(A_M)}(k)\bigr) = S_M\bigl(L_M(c_{A_M}(k))\bigr) = c_{A_M}(k),$$

because $S_M \circ L_M = \mathrm{id}_{[0,1]}$. But $c_{A_M}(k)$ is precisely the fuzzy-cardinality membership degree obtained from the score-realized fuzzy set

$$\overline{A}_M(x) := S_M(\mu_{A_M}(x)).$$

Thus the score-realization of $\mathrm{Card}_M(A_M)$ coincides exactly with the usual fuzzy cardinality of $\overline{A}_M$. This proves (4).

Hence all objects and conditions in Definition 7.4.2 are mathematically meaningful, and the notion of uncertain cardinality is well-defined. □

7.5 Uncertain Quantifiers

Fuzzy quantifiers are graded linguistic operators expressing quantities like most, many, or few, assigning truth degrees to quantified statements through fuzzy cardinality and semantics formally [771, 771, 772].

Definition 7.5.1 (Fuzzy Quantifier)**.** [771, 771, 772] Let X be a nonempty finite universe, and let

$$\mathcal{F}(X) := [0,1]^X$$

be the family of all fuzzy subsets of X.

A *(monadic) fuzzy quantifier* on X is a mapping

$$Q : \mathcal{F}(X) \to [0,1].$$

For each fuzzy set $A \in \mathcal{F}(X)$, the value $Q(A)$ is interpreted as the truth degree of the quantified statement determined by Q and A.

In particular, when the semantics is based on fuzzy cardinality, one commonly distinguishes:

$$q_{\mathrm{abs}} : \{0, 1, \ldots, |X|\} \to [0,1]$$

as an *absolute fuzzy quantifier*, and

$$q_{\mathrm{rel}} : [0,1] \to [0,1]$$

as a *relative fuzzy quantifier*.

Uncertain quantifiers are uncertainty-valued linguistic operators on uncertain sets, whose score-realizations recover ordinary fuzzy quantifiers.

Definition 7.5.2 (Uncertain quantifier)**.** Let X be a finite nonempty universe, and let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Define

$$\mathcal{U}_M(X) := \{\mu : X \to \mathrm{Dom}(M)\},$$

the family of all uncertain sets of type M on X.

Assume that the following data are fixed:

- a score map
$$S_M : \mathrm{Dom}(M) \to [0,1];$$
- a lift map
$$L_M : [0,1] \to \mathrm{Dom}(M)$$
such that
$$S_M\big(L_M(r)\big) = r \qquad \text{for all } r \in [0,1].$$

For each uncertain set
$$A_M = (X, \mu_{A_M}) \in \mathcal{U}_M(X),$$
define its score-realization
$$\overline{A}_M \in [0,1]^X$$
by
$$\mu_{\overline{A}_M}(x) := S_M\big(\mu_{A_M}(x)\big) \qquad (x \in X).$$

A *(monadic) uncertain quantifier of type M* on X is a mapping
$$Q_M : \mathcal{U}_M(X) \to \mathrm{Dom}(M)$$
for which there exists a fuzzy quantifier
$$Q : [0,1]^X \to [0,1]$$
such that, for every $A_M \in \mathcal{U}_M(X)$,
$$Q_M(A_M) = L_M\big(Q(\overline{A}_M)\big).$$
Equivalently,
$$S_M\big(Q_M(A_M)\big) = Q(\overline{A}_M).$$

For each uncertain set A_M, the value $Q_M(A_M) \in \mathrm{Dom}(M)$ is interpreted as the uncertainty-valued truth degree of the quantified statement determined by Q_M and A_M.

Definition 7.5.3 (Absolute and relative uncertain quantifiers)**.** Under the same assumptions as in Definition 7.5.2, an *absolute uncertain quantifier of type M* is a mapping
$$q^M_{\mathrm{abs}} : \{0, 1, \ldots, |X|\} \to \mathrm{Dom}(M)$$
for which there exists an absolute fuzzy quantifier
$$q_{\mathrm{abs}} : \{0, 1, \ldots, |X|\} \to [0,1]$$
such that
$$q^M_{\mathrm{abs}}(k) = L_M\big(q_{\mathrm{abs}}(k)\big) \qquad (k = 0, 1, \ldots, |X|).$$

Similarly, a *relative uncertain quantifier of type M* is a mapping
$$q^M_{\mathrm{rel}} : [0,1] \to \mathrm{Dom}(M)$$
for which there exists a relative fuzzy quantifier
$$q_{\mathrm{rel}} : [0,1] \to [0,1]$$
such that
$$q^M_{\mathrm{rel}}(r) = L_M\big(q_{\mathrm{rel}}(r)\big) \qquad (r \in [0,1]).$$

Definition 7.5.4 (Cardinality-based uncertain quantifier)**.** Let $A_M \in \mathcal{U}_M(X)$, and let

$$c_{A_M} : \{0, 1, \ldots, |X|\} \to [0, 1]$$

denote the score-cardinality profile of A_M, i.e.,

$$c_{A_M}(k) = \sup\{\, \alpha \in (0, 1] \mid |(A_M)^{S_M}_{\alpha}| = k \,\}, \qquad k = 0, 1, \ldots, |X|,$$

with $\sup \varnothing = 0$.

If $q_{\text{abs}} : \{0, 1, \ldots, |X|\} \to [0, 1]$ is an absolute fuzzy quantifier, then the induced *cardinality-based uncertain quantifier* is defined by

$$Q^{\text{abs}}_M(A_M) := L_M\left(\max_{0 \le k \le |X|} \min\bigl(c_{A_M}(k),\, q_{\text{abs}}(k)\bigr) \right).$$

If $q_{\text{rel}} : [0, 1] \to [0, 1]$ is a relative fuzzy quantifier and

$$\rho(A_M) \in [0, 1]$$

is a prescribed score-based relative cardinality index of A_M, then the induced relative uncertain quantifier is defined by

$$Q^{\text{rel}}_M(A_M) := L_M\bigl(q_{\text{rel}}(\rho(A_M))\bigr).$$

Remark 7.5.5. If M is the fuzzy model, so that

$$\text{Dom}(M) = [0, 1], \qquad S_M = \text{id}_{[0,1]}, \qquad L_M = \text{id}_{[0,1]},$$

then Definition 7.5.2 reduces exactly to the usual notion of a fuzzy quantifier

$$Q : [0, 1]^X \to [0, 1].$$

Thus fuzzy quantifiers are special cases of uncertain quantifiers.

Theorem 7.5.6 (Well-definedness of uncertain quantifiers)**.** *Let X be a finite nonempty set, let M be an uncertain model with degree-domain*

$$\text{Dom}(M) \subseteq [0, 1]^k,$$

and let

$$S_M : \text{Dom}(M) \to [0, 1], \qquad L_M : [0, 1] \to \text{Dom}(M)$$

satisfy

$$S_M\bigl(L_M(r)\bigr) = r \qquad (r \in [0, 1]).$$

Then:

1. *for every uncertain set*

   $$A_M = (X, \mu_{A_M}) \in \mathcal{U}_M(X),$$

   the score-realization

   $$\overline{A}_M \in [0, 1]^X$$

   is well-defined;

2. *if*

$$Q : [0,1]^X \to [0,1]$$

is a fuzzy quantifier, then the induced mapping

$$Q_M : \mathcal{U}_M(X) \to \mathrm{Dom}(M), \qquad Q_M(A_M) := L_M\big(Q(\overline{A}_M)\big),$$

is well-defined;

3. *for every* $A_M \in \mathcal{U}_M(X)$*,*

$$S_M\big(Q_M(A_M)\big) = Q(\overline{A}_M);$$

4. *the absolute and relative uncertain quantifiers in Definition 7.5.3 are well-defined;*

5. *the cardinality-based uncertain quantifier in Definition 7.5.4 is well-defined.*

Hence the notion of uncertain quantifier is mathematically well-defined.

Proof. Let

$$A_M = (X, \mu_{A_M}) \in \mathcal{U}_M(X).$$

Since

$$\mu_{A_M} : X \to \mathrm{Dom}(M)$$

and

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

for every $x \in X$ the value

$$S_M(\mu_{A_M}(x))$$

belongs to $[0,1]$. Hence

$$\mu_{\overline{A}_M}(x) := S_M(\mu_{A_M}(x))$$

defines a well-defined mapping

$$\overline{A}_M : X \to [0,1].$$

Therefore

$$\overline{A}_M \in [0,1]^X$$

is a well-defined fuzzy set. This proves (1).

Now let

$$Q : [0,1]^X \to [0,1]$$

be a fuzzy quantifier. Since $\overline{A}_M \in [0,1]^X$, the value

$$Q(\overline{A}_M)$$

is a well-defined element of $[0,1]$. Because

$$L_M : [0,1] \to \mathrm{Dom}(M),$$

the value

$$L_M(Q(\overline{A}_M))$$

belongs to $\mathrm{Dom}(M)$. Hence

$$Q_M(A_M) := L_M(Q(\overline{A}_M))$$

defines a well-defined mapping

$$Q_M : \mathcal{U}_M(X) \to \mathrm{Dom}(M).$$

This proves (2).

For (3), by the defining relation

$$S_M \circ L_M = \mathrm{id}_{[0,1]},$$

one has

$$S_M(Q_M(A_M)) = S_M\big(L_M(Q(\overline{A}_M))\big) = Q(\overline{A}_M).$$

For (4), if

$$q_{\mathrm{abs}} : \{0, 1, \ldots, |X|\} \to [0, 1],$$

then for each integer $k \in \{0, 1, \ldots, |X|\}$ the value

$$q_{\mathrm{abs}}(k) \in [0, 1]$$

is well-defined, hence

$$q^M_{\mathrm{abs}}(k) := L_M(q_{\mathrm{abs}}(k)) \in \mathrm{Dom}(M)$$

is well-defined. Thus

$$q^M_{\mathrm{abs}} : \{0, 1, \ldots, |X|\} \to \mathrm{Dom}(M)$$

is a well-defined absolute uncertain quantifier. The proof for

$$q^M_{\mathrm{rel}}(r) := L_M(q_{\mathrm{rel}}(r))$$

is identical.

For (5), let $A_M \in \mathcal{U}_M(X)$. Since X is finite, for each

$$k \in \{0, 1, \ldots, |X|\}$$

the score-cardinality profile value

$$c_{A_M}(k) \in [0, 1]$$

is well-defined. Also,

$$q_{\mathrm{abs}}(k) \in [0, 1].$$

Hence

$$\min\big(c_{A_M}(k), q_{\mathrm{abs}}(k)\big) \in [0, 1]$$

for every k. Because the index set $\{0, 1, \ldots, |X|\}$ is finite, the maximum

$$\max_{0 \le k \le |X|} \min\big(c_{A_M}(k), q_{\mathrm{abs}}(k)\big)$$

exists and belongs to $[0, 1]$. Applying L_M, we obtain

$$Q^{\mathrm{abs}}_M(A_M) \in \mathrm{Dom}(M),$$

so Q^{abs}_M is well-defined.

Similarly, if $\rho(A_M) \in [0,1]$ is a prescribed score-based relative cardinality index, then

$$q_{\mathrm{rel}}(\rho(A_M)) \in [0,1],$$

and therefore

$$Q_M^{\mathrm{rel}}(A_M) = L_M(q_{\mathrm{rel}}(\rho(A_M))) \in \mathrm{Dom}(M)$$

is well-defined.

Thus all constructions in Definitions 7.5.2–7.5.4 are mathematically meaningful, and the notion of uncertain quantifier is well-defined. □

For reference, a catalogue of representative uncertainty-quantifier families is presented in Table 7.5.

Table 7.5: A catalogue of representative uncertainty-quantifier families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-quantifier family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy Quantifiers:* $Q_M : [0,1]^X \to [0,1]$, or equivalently $Q_M : \mathcal{F}(X) \to [0,1]$.
2	*Intuitionistic Fuzzy Quantifiers [773, 774]:* $Q_M : ([0,1]^2)^X \to [0,1]^2$, or equivalently $Q_M : \mathcal{IF}(X) \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic Quantifiers [775, 776]:* $Q_M : ([0,1]^3)^X \to [0,1]^3$, or equivalently $Q_M : \mathcal{NS}(X) \to [0,1]^3$ (e.g., (T, I, F)).
k	*k-component uncertainty quantifiers:* $Q_M : \mathrm{Dom}(M)^X \to \mathrm{Dom}(M)$, or equivalently $Q_M : \mathcal{U}_M(X) \to \mathrm{Dom}(M)$ (model-specific semantics).

7.6 Uncertain Function

A fuzzy function is a total extensional fuzzy relation between fuzzy equality spaces, assigning each input a fuzzily unique output compatible with underlying similarity structures [777–779].

Definition 7.6.1 (Fuzzy Function)**.** Let X and Y be nonempty sets, and let

$$E_X : X \times X \to [0,1], \qquad E_Y : Y \times Y \to [0,1]$$

be fuzzy equalities on X and Y, respectively. A fuzzy relation

$$f : X \times Y \to [0,1]$$

is called a *fuzzy function* from X to Y if, for all $x, x' \in X$ and $y, y' \in Y$,

1. $$\min\{f(x,y),\ E_X(x,x')\} \le f(x',y);$$
2. $$\min\{f(x,y),\ E_Y(y,y')\} \le f(x,y');$$
3. $$\min\{f(x,y),\ f(x,y')\} \le E_Y(y,y');$$

4.
$$\sup_{y \in Y} f(x, y) = 1.$$

Thus a fuzzy function is a total fuzzy relation that is extensional with respect to the given fuzzy equalities and single-valued in the fuzzy sense.

An uncertain function is a total extensional uncertain relation between uncertainty-equality spaces, whose score-realization is a fuzzy function.

Definition 7.6.2 (Uncertain equality space)**.** Let X be a nonempty set, let M be an uncertain model with degree-domain
$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$
and fix a score map
$$S_M : \mathrm{Dom}(M) \to [0, 1].$$
An *uncertain equality* on X is a mapping
$$E_X^M : X \times X \to \mathrm{Dom}(M)$$
such that its score-realization
$$\overline{E}_X(x, x') := S_M\big(E_X^M(x, x')\big) \qquad (x, x' \in X)$$
is a fuzzy equality on X. The pair
$$(X, E_X^M)$$
is then called an *uncertain equality space of type M*.

Definition 7.6.3 (Uncertain function)**.** Let X and Y be nonempty sets, and let
$$(X, E_X^M) \qquad \text{and} \qquad (Y, E_Y^M)$$
be uncertain equality spaces of the same type M, where
$$E_X^M : X \times X \to \mathrm{Dom}(M), \qquad E_Y^M : Y \times Y \to \mathrm{Dom}(M).$$

An *uncertain relation of type M* from X to Y is a mapping
$$f_M : X \times Y \to \mathrm{Dom}(M).$$
Its score-realization is the fuzzy relation
$$\overline{f}_M(x, y) := S_M\big(f_M(x, y)\big) \qquad (x \in X,\ y \in Y).$$

The relation f_M is called an *uncertain function of type M* from X to Y if, for all $x, x' \in X$ and $y, y' \in Y$,

1.
$$\min\big\{S_M(f_M(x, y)), S_M(E_X^M(x, x'))\big\} \le S_M(f_M(x', y));$$

2.
$$\min\{S_M(f_M(x,y)), S_M(E_Y^M(y,y'))\} \le S_M(f_M(x,y'));$$

3.
$$\min\{S_M(f_M(x,y)), S_M(f_M(x,y'))\} \le S_M(E_Y^M(y,y'));$$

4.
$$\sup_{y\in Y} S_M(f_M(x,y)) = 1.$$

Equivalently, f_M is an uncertain function if and only if its score-realization

$$\overline{f}_M : X \times Y \to [0,1]$$

is a fuzzy function from the fuzzy equality space

$$(X, \overline{E}_X)$$

to the fuzzy equality space

$$(Y, \overline{E}_Y).$$

Remark 7.6.4. If M is the fuzzy model, so that

$$\mathrm{Dom}(M) = [0,1] \qquad \text{and} \qquad S_M = \mathrm{id}_{[0,1]},$$

then Definition 7.6.3 reduces exactly to the usual definition of a fuzzy function. Thus fuzzy functions are special cases of uncertain functions.

Theorem 7.6.5 (Well-definedness of uncertain functions)**.** *Let X and Y be nonempty sets, let*

$$(X, E_X^M) \qquad \textit{and} \qquad (Y, E_Y^M)$$

be uncertain equality spaces of type M, and let

$$f_M : X \times Y \to \mathrm{Dom}(M)$$

be an uncertain relation. Then:

1. *the score-realizations*

$$\overline{E}_X : X \times X \to [0,1], \qquad \overline{E}_Y : Y \times Y \to [0,1], \qquad \overline{f}_M : X \times Y \to [0,1]$$

are well-defined;

2. *all expressions appearing in Definition 7.6.3 are meaningful real numbers in $[0,1]$;*
3. *consequently, the notion of uncertain function is mathematically well-defined.*

Proof. Since

$$E_X^M : X \times X \to \mathrm{Dom}(M), \qquad E_Y^M : Y \times Y \to \mathrm{Dom}(M), \qquad f_M : X \times Y \to \mathrm{Dom}(M),$$

and

$$S_M : \mathrm{Dom}(M) \to [0,1],$$

their compositions

$$\overline{E}_X(x,x') = S_M\big(E_X^M(x,x')\big), \qquad \overline{E}_Y(y,y') = S_M\big(E_Y^M(y,y')\big),$$

and

$$\overline{f}_M(x,y) = S_M\big(f_M(x,y)\big)$$

are well-defined mappings into $[0,1]$. This proves (1).

Now let $x, x' \in X$ and $y, y' \in Y$. Then

$$\overline{f}_M(x,y), \quad \overline{f}_M(x',y), \quad \overline{f}_M(x,y'), \quad \overline{E}_X(x,x'), \quad \overline{E}_Y(y,y') \in [0,1].$$

Hence the quantities

$$\min\!\big\{\overline{f}_M(x,y), \overline{E}_X(x,x')\big\}, \qquad \min\!\big\{\overline{f}_M(x,y), \overline{E}_Y(y,y')\big\},$$

and

$$\min\!\big\{\overline{f}_M(x,y), \overline{f}_M(x,y')\big\}$$

are well-defined real numbers in $[0,1]$. Likewise,

$$\overline{f}_M(x',y), \qquad \overline{f}_M(x,y'), \qquad \overline{E}_Y(y,y')$$

are well-defined real numbers in $[0,1]$. Therefore the inequalities in Definition 7.6.3 are meaningful.

Moreover, for each fixed $x \in X$, the set

$$\{\, \overline{f}_M(x,y) \mid y \in Y \,\} \subseteq [0,1]$$

is bounded above by 1. Since $[0,1]$ is order-complete, the quantity

$$\sup_{y \in Y} \overline{f}_M(x,y)$$

exists and belongs to $[0,1]$. Hence the totality condition

$$\sup_{y \in Y} \overline{f}_M(x,y) = 1$$

is also meaningful. This proves (2).

Therefore all objects and all conditions appearing in Definition 7.6.3 are mathematically meaningful, and the notion of uncertain function is well-defined. This proves (3). □

For reference, representative uncertainty-function families and their degree representations are presented in Table 7.6.

Table 7.6: Representative uncertainty-function families and their degree representations.

Uncertainty-function family	**Degree form**	**Typical degree-domain / representation**
Fuzzy Functions	scalar	Typically represented by a fuzzy relation $$f_M : X \times Y \to [0,1],$$ or equivalently $f_M : X \to [0,1]^Y$, expressing the degree to which y is the image of x.
Intuitionistic Fuzzy Functions [780–783]	2-component	Typically represented by $$f_M : X \times Y \to [0,1]^2,$$ or equivalently $f_M : X \to ([0,1]^2)^Y$, with components such as (membership, non-membership).
Picture Fuzzy Functions [784, 785]	3-component	Typically represented by $$f_M : X \times Y \to [0,1]^3,$$ or equivalently $f_M : X \to ([0,1]^3)^Y$, with components such as (positive, neutral, negative), subject to a model-specific constraint like $$p(x,y) + n(x,y) + q(x,y) \le 1.$$
Neutrosophic Functions [786–789]	3-component	Typically represented by $$f_M : X \times Y \to [0,1]^3,$$ or equivalently $f_M : X \to ([0,1]^3)^Y$, with components such as (T, I, F).
Plithogenic Functions [790, 791]	attribute-based / multi-component	Typically described by appurtenance degrees together with contradiction information relative to attribute values; for example, $$f_M : (X \times Y) \times Pv \to [0,1]^s,$$ with an associated contradiction map on the attribute-value set Pv.

Chapter 8

Conclusion

In this book, we have presented a broad and systematic survey of extension concepts in applied mathematics developed on the basis of fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, plithogenic sets, and related uncertainty-oriented frameworks.

It is hoped that future research will further advance the application of fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, and plithogenic sets to other disciplines, as well as their extension to additional related concepts. It is also expected that further progress will be made in studies involving computational experiments and the design of algorithms based on these frameworks.

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Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this book.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer (Others)

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

Appendix (List of Tables)

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A Survey of Fuzzy and Uncertain Concepts in Applied Mathematics

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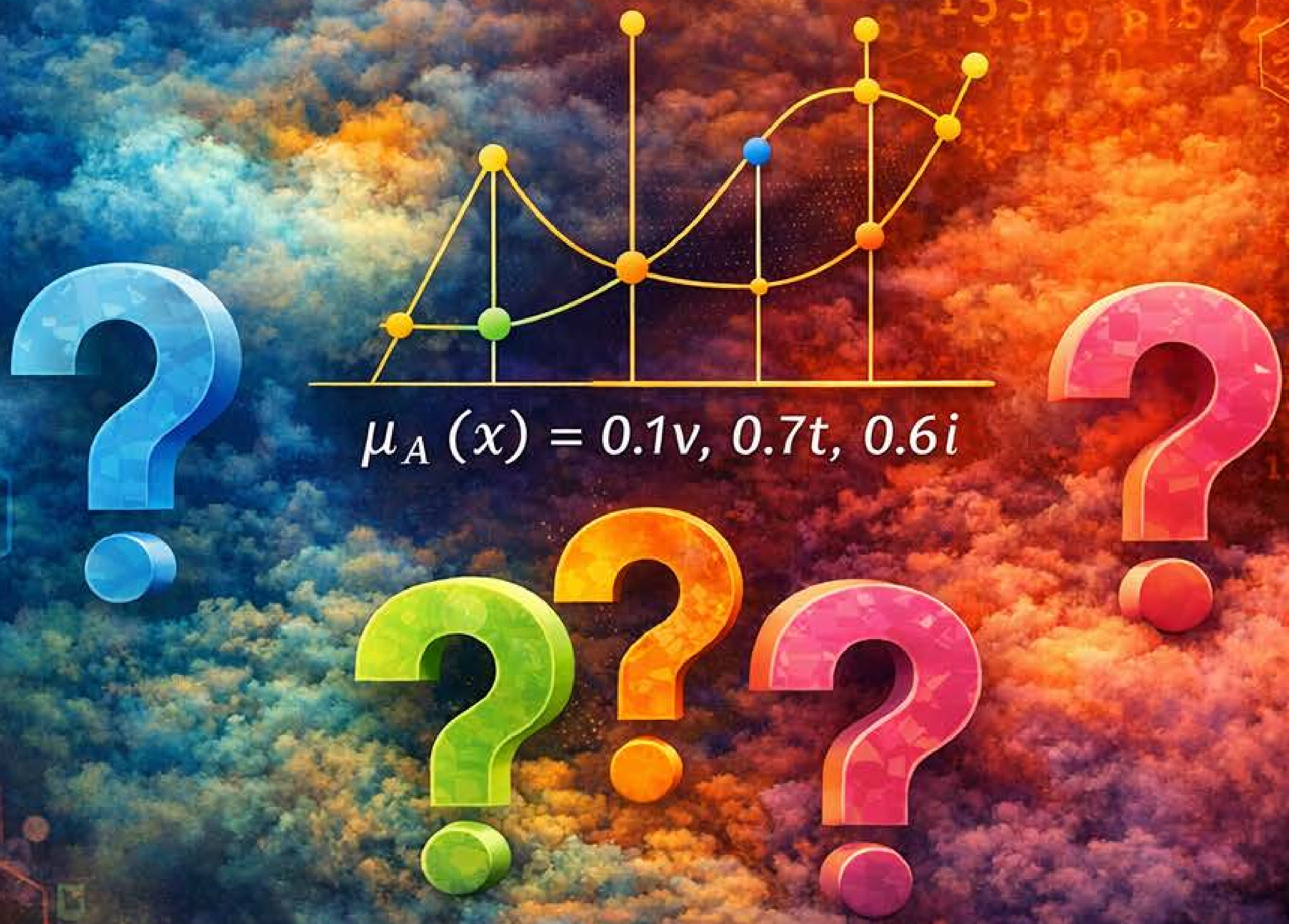
Real-world phenomena often involve vagueness, partial truth, incomplete information, and, in some cases, indeterminacy or inconsistency. To represent such uncertainty in a mathematically rigorous manner, a wide variety of generalized set-theoretic frameworks have been developed, including fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, vague sets, hesitant fuzzy sets, picture fuzzy sets, plithogenic sets, and several related extensions. These frameworks have attracted substantial attention in both theoretical studies and practical applications across many areas of applied mathematics.

This book presents a broad and systematic survey of mathematical concepts in applied mathematics that are formulated by using fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, plithogenic sets, and related uncertainty-oriented models. Its purpose is to collect these diverse topics in a unified reference, clarify their mathematical structures, and highlight their common features and differences. By organizing a wide range of fuzzy and uncertainty-related topics within a single framework, this book aims to provide readers with a clear overview of the current landscape of fuzzy mathematics and its extensions in applied mathematics.

Keywords: Fuzzy Set, Intuitionistic Fuzzy Set, Neutrosophic Set, Plithogenic Set

This book provides a comprehensive survey of fuzzy and uncertainty-oriented concepts in applied mathematics, covering fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, plithogenic sets, and related extensions. By organizing these diverse frameworks within a unified perspective, it highlights underlying mathematical structures, compares their expressive capabilities, and showcases their applications across discrete, algebraic, and analytical contexts, offering both theoretical insights and practical tools for modeling vagueness, and indeterminacy.



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