Takaaki Fujita
Florentin Smarandache

Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond

Sixth Volume

Various New Uncertain Concepts

(Collected Papers)

This series explores the advancement of uncertain combinatorics through innovative methods such as graphization, hyperization, and uncertainization, incorporating concepts from fuzzy, neutrosophic, soft, and rough set theory, among others. Combinatorics and set theory are fundamental mathematical disciplines that focus on counting, arrangement, and the study of collections under specified rules. While combinatorics excels at solving problems involving uncertainty, set theory has expanded to include advanced concepts like fuzzy and neutrosophic sets, which are capable of modeling complex real-world uncertainties by accounting for truth, indeterminacy, and falsehood. These developments intersect with graph theory, leading to novel forms of uncertain sets in "graphized" structures, such as hypergraphs and superhypergraphs. Innovations like Neutrosophic Oversets, Undersets, and Offsets, as well as the Nonstandard Real Set, build upon traditional graph concepts, pushing the boundaries of theoretical and practical advancements. This synthesis of combinatorics, set theory, and graph theory provides a strong foundation for addressing the complexities and uncertainties present in mathematical and real-world systems, paving the way for future research and application.

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Foreword

This book is the sixth volume in the series of ***Collected Papers*** on **Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond**. Building upon the foundational contributions of previous volumes, this edition focuses on the exploration and development of ***Various New Uncertain Concepts***, further enriching the study of uncertainty and complexity through innovative theoretical advancements and practical applications.

The series is dedicated to the evolution of uncertain combinatorics, leveraging methodologies such as graphization, hyperization, and uncertainization. These approaches extend classical combinatorics and set theory by integrating and expanding upon fuzzy, neutrosophic, soft, and rough set theories. Through this synthesis, the series provides comprehensive frameworks to model and analyze the multifaceted nature of real-world uncertainties, addressing challenges across diverse fields of study.

Combinatorics and set theory form the mathematical backbone of this series. Traditionally, combinatorics has been instrumental in solving problems involving counting, arrangements, and relationships under defined rules, particularly in uncertain scenarios. Simultaneously, advancements in set theory have transformed its scope through constructs like fuzzy sets, which account for degrees of truth, and neutrosophic sets, which incorporate dimensions of indeterminacy and falsity alongside truth. By marrying these disciplines with modern extensions, this series pushes the boundaries of uncertainty modeling and analysis.

In this sixth volume, the focus shifts to deepening and broadening our understanding of ***Various New Uncertain Concepts***. The book not only revisits methodologies such as hyperization and neutrosophic extensions, introduced in earlier volumes, but also advances groundbreaking theories and practical frameworks. It explores innovative structures like hypergraphs and superhypergraphs, as well as their applications in decision-making, natural language processing, neural networks, and other complex domains. These advancements mark a significant step forward in uncertain combinatorics, offering tools and insights to address hierarchical relationships, multi-level data, and intricate systems.

The volume is meticulously organized into 15 chapters, each presenting unique perspectives and contributions to the field. From theoretical explorations to real-world applications, these chapters provide a cohesive and comprehensive overview of the state of the art in uncertain combinatorics, emphasizing the versatility and power of the newly introduced concepts and methodologies.

The first chapter (**SuperHypertree-depth – Structural Analysis in SuperHyperGraphs**) explores the concept of SuperHypertree-depth, an extension of the classical graph parameter Tree-depth and its hypergraph counterpart Hypertree-depth. By introducing hierarchical nesting within SuperHyperGraphs, where both vertices and edges can represent recursive subsets, this study investigates the mathematical properties and structural implications of these extended parameters. The findings highlight the relationships between SuperHypertree-depth and its traditional graph-theoretic equivalents, providing a deeper understanding of their applicability to hierarchical and complex systems.

The second chapter (**Obstructions for Hypertree-width and SuperHypertree-width**) examines the role of ultrafilters as obstructions in determining Hypertree-width and extends the concept to SuperHypertree-width. Building on hypergraph theory, which abstracts traditional graph frameworks into more complex domains, the study investigates how recursive structures within SuperHyperGraphs redefine the computational and structural properties of these parameters. Ultrafilters, with their broad mathematical significance, serve as critical tools for understanding the limitations and potentials of these advanced graph metrics.

The third chapter (**SuperHypertree-Length and SuperHypertree-Breadth in SuperHyperGraphs**) investigates the extension of the graph-theoretic parameters Tree-length and Tree-breadth to the realms of hypergraphs and SuperHyperGraphs. By leveraging the hierarchical nesting of SuperHyperGraphs, the study explores how these parameters adapt to increasingly complex and multi-level structures. Comparative analyses between these extended parameters and their classical counterparts reveal new insights into their relevance and utility in advanced graph and hypergraph theory.

Plithogenic Sets, which generalize Fuzzy and Neutrosophic Sets, are extended in the fourth chapter (**Extended HyperPlithogenic Sets and Generalized Plithogenic Graphs**) to Extended Plithogenic Sets, HyperPlithogenic Sets, and SuperHyperPlithogenic Sets. This study further investigates their application to graph theory through the concepts of Extended Plithogenic Graphs and Generalized Extended Plithogenic Graphs. The chapter provides a concise exploration of these frameworks, offering insights into their potential for addressing uncertainty and complexity in graph structures.

Soft Sets provide an effective framework for decision-making by mapping parameters to subsets of a universal set, addressing uncertainty and vagueness. The fifth chapter (**Double-Framed Superhypersoft Set and Double-Framed Treesoft Set**) introduces the Double-Framed SuperHypersoft Set and the Double-Framed Treesoft Set as extensions of traditional and advanced soft set frameworks, such as Hypersoft and SuperHypersoft Sets. The chapter explores their relationships with existing concepts, offering new tools to handle complex decision-making scenarios with enhanced structural flexibility.

The sixth paper (**HyperPlithogenic Cubic Set and SuperHyperPlithogenic Cubic Set**) introduces the concepts of the HyperPlithogenic Cubic Set and SuperHyperPlithogenic Cubic Set, which extend the Plithogenic Cubic Set by integrating both interval-valued and single-valued fuzzy memberships. These sets leverage multi-attribute aggregation techniques inherent to plithogenic structures, allowing for nuanced representations of uncertainty. Additionally, related constructs such as the HyperPlithogenic Fuzzy Cubic Set, HyperPlithogenic Intuitionistic Fuzzy Cubic Set, and HyperPlithogenic Neutrosophic Cubic Set are explored, further enriching the theoretical and practical applications of this framework.

The seventh chapter (**L-Neutrosophic Sets and Nonstationary Neutrosophic Sets**) extends the foundational concepts of fuzzy sets by integrating Neutrosophic and Plithogenic frameworks. By introducing L-Neutrosophic Sets and Nonstationary Neutrosophic Sets, the study enhances the representation of uncertainty through independent membership components: truth, indeterminacy, and falsity. These advanced constructs also incorporate multi-dimensional and contradictory attributes, providing a robust means of modeling complex decision-making and uncertain data.

Plithogenic and Rough Sets, known for generalizing uncertainty modeling and classification, are extended in the eight chapter (**Forest HyperPlithogenic and Forest HyperRough Sets**) to Forest HyperPlithogenic Sets, Forest SuperHyperPlithogenic Sets, Forest HyperRough Sets, and Forest SuperHyperRough Sets. These frameworks incorporate hierarchical and recursive structures to advance existing set-theoretic paradigms. The chapter explores their applications in multi-level data analysis and uncertainty classification, demonstrating their adaptability to complex systems.

Building on Fuzzy, Neutrosophic, and Plithogenic Sets, the tenth chapter (**Symbolic HyperPlithogenic Sets**) introduces Symbolic HyperPlithogenic Sets and Symbolic n-SuperHyperPlithogenic Sets. These sets incorporate symbolic components and algebraic coefficients, enabling flexible operations within a defined prevalence order. By extending symbolic representation into hyperplithogenic and superhyperplithogenic domains, the chapter opens new pathways for addressing uncertainty and hierarchical complexity in mathematical modeling.

Soft Sets, designed to manage uncertainty and imprecision, have evolved through various extensions like Hypersoft Sets and SuperHypersoft Sets. The eleventh chapter (**N-SuperHypersoft and Bijective SuperHypersoft Sets**) introduces N-SuperHypersoft Sets, N-Treesoft Sets, Bijective SuperHypersoft Sets, and Bijective Treesoft Sets. These new constructs enhance decision-making frameworks by incorporating advanced hierarchical and bijective relationships, building on existing theories and expanding their applications.

Plithogenic Sets, known for integrating multi-valued attributes and contradictions, and Rough Sets, which partition data into definable approximations, are combined in the twelfth chapter (**Plithogenic Rough Sets**) to form Plithogenic Rough Sets. This fusion provides a powerful framework for addressing uncertainty in dynamic and complex decision-making scenarios, offering a novel approach to uncertainty modeling.

Expanding on Neutrosophic Sets, which represent truth, indeterminacy, and falsehood, this chapter introduces Plithogenic Duplets and Plithogenic Triplets. These constructs leverage the Plithogenic framework to incorporate attributes, values, and contradiction measures. The thirteenth chapter (**Plithogenic Duplets and Triplets**) examines their relationships with Neutrosophic Duplets and Triplets, offering new tools for multi-dimensional data representation and decision-making.

Building on foundational concepts like Rough Sets and Vague Sets, the fourteenth chapter (**SuperRough and SuperVague Sets**) introduces SuperRough Sets and SuperVague Sets. These generalized frameworks extend uncertainty modeling by incorporating hierarchical structures. The study also demonstrates that SuperRough Sets can evolve into SuperHyperRough Sets, providing further generalizations for advanced data classification and analysis.

The fifteenth chapter (**Neutrosophic TreeSoft Expert and ForestSoft Sets**) revisits the Neutrosophic TreeSoft Set, which combines the hierarchical structure of TreeSoft Sets with the Neutrosophic framework for uncertainty representation. Additionally, it introduces the Neutrosophic TreeSoft Expert Set, incorporating expert knowledge into the model. The chapter also explores the ForestSoft Set and its extension, the Neutrosophic ForestSoft Set, to provide multi-level, tree-structured approaches for complex data representation and analysis.

Therefore, this collection explores advanced concepts in uncertain combinatorics, focusing on innovative frameworks such as SuperHyperGraphs, Plithogenic and Rough Sets, and Neutrosophic extensions. The chapters introduce hierarchical and multi-dimensional constructs, such as SuperHypertree-depth, HyperPlithogenic Cubic Sets, and Forest HyperRough Sets, to address complexity and uncertainty in decision-making, classification, and data analysis. These contributions offer new methodologies and applications across fields, advancing the boundaries of mathematical modeling.

In conclusion, this volume significantly advances the field of uncertain combinatorics by introducing a range of novel concepts and frameworks. Through the exploration of SuperHyperGraphs, extended Plithogenic and Rough Sets, and Neutrosophic constructs, the chapters provide powerful tools for modeling and analyzing uncertainty in complex systems. These innovations not only deepen our understanding of hierarchical structures and multi-dimensional data but also expand the applicability of set-theoretic paradigms to real-world problems. As uncertainty continues to be a core challenge across various disciplines, the insights presented here pave the way for more refined, adaptable approaches to decision-making, classification, and computational modeling.

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Chapter 1
SuperHypertree-depth: A Structural Analysis within SuperHyperGraphs

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Abstract

Hypergraphs extend the concept of graphs by allowing edges, called hyperedges, to connect multiple vertices simultaneously [4]. SuperHyperGraphs further generalize this structure by introducing hierarchical nesting, where both vertices and edges can represent subsets within recursive levels of abstraction [37, 38].

This paper investigates the feasibility of extending the graph parameter Tree-depth [30] (and its hypergraph counterpart, Hypertree-depth [1]) to SuperHyperGraphs. Furthermore, it analyzes the relationships between these extended parameters and their classical graph-theoretic counterparts, providing insights into their mathematical properties and structural implications.

Keywords: Tree-depth, Superhypergraph, Hypergraph, Hypertree-depth

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

1.1 Basic Definition of Graph Theory

This section presents the fundamental definitions of graph theory. In this paper, we focus exclusively on undirected, finite, and simple graphs. For additional background and comprehensive explanations, readers are encouraged to refer to lecture notes and surveys such as [7–9].

Definition 1.1 (Graph). [9] A graph G is a mathematical structure composed of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 1.2 (Subgraph). [9] Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of G is a graph such that:

- $V_H \subseteq V$, i.e., the vertex set of H is a subset of the vertex set of G.
- $E_H \subseteq E$, i.e., the edge set of H is a subset of the edge set of G.
- Each edge in E_H connects vertices in V_H.

Definition 1.3 (Path). [9] A *path* is a graph $P = (V, E)$ where $V = \{v_1, v_2, \ldots, v_k\}$ and $E = \{\{v_i, v_{i+1}\} \mid 1 \leq i < k\}$. Each vertex is distinct, and edges form a simple sequence connecting v_1 to v_k.

Definition 1.4 (Tree). [9] A *tree* is a connected, acyclic graph $T = (V, E)$. A tree with n vertices has $n - 1$ edges.

Definition 1.5 (Forest). [9] A *forest* is a disjoint union of trees. Formally, a graph $F = (V, E)$ is a forest if every connected component of F is a tree.

1.2 Hypergraph

A hypergraph is a generalized graph concept that extends traditional graph theory by introducing hyperedges, which can connect multiple vertices instead of just pairs. This allows for modeling more complex relationships among elements [3, 4, 22–24]. Hypergraphs have found applications in various fields, including database systems [27]. The fundamental definitions of hypergraphs are provided below.

Definition 1.6 (Hypergraph)**.** [4] A *hypergraph* is a pair $H = (V, E)$, where:

- V is a set of *vertices*,
- E is a set of *hyperedges*, each hyperedge $e \in E$ being a subset of V.

Equivalently, $E \subseteq \mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the power set of V.

Example 1.7 (A Simple Hypergraph)**.** Consider a hypergraph $H = (V, E)$ with:

$$V = \{a,\ b,\ c,\ d\}, \quad E = \big\{\{a, b\},\ \{b, c, d\}\big\}.$$

In this setup, we have four vertices a, b, c, d. The set of hyperedges is:

- $\{a, b\}$, which is an edge connecting exactly two vertices (a and b).
- $\{b, c, d\}$, which is a *hyperedge* connecting three vertices (b, c, d).

Notice that in a hypergraph, an edge can contain any number of vertices from V. Therefore, $\{b, c, d\}$ is a valid edge even though it links three vertices. In contrast, a standard (undirected) graph edge can only connect two vertices. This demonstrates the fundamental difference: *hyperedges can connect more than two vertices at once*, providing a more general framework.

1.3 SuperHyperGraph

A SuperHyperGraph is an extension of the concept of a hypergraph, recently defined and actively studied in the literature [2, 5, 12, 14, 16–19, 25, 26, 28, 33, 35, 37–39]. It can be understood as a graph concept that incorporates recursive structures into hypergraphs. A SuperHyperGraph possesses a repeated structure called the n-th powerset, which is generated iteratively through the power set operation. The formal definition is provided below.

Definition 1.8 (n-th Powerset)**.** (cf. [13, 15, 36, 40])

The n-th powerset of a set H, denoted $P_n(H)$, is constructed iteratively. Beginning with the standard powerset, the process is defined as:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

In a similar manner, the n-th non-empty powerset, represented as $P_n^*(H)$, is recursively defined as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ refers to the powerset of H excluding the empty set.

Definition 1.9 (n-SuperHyperGraph)**.** [37, 38] Let V_0 be a finite *base set* of vertices. For each $k \geq 0$, define the iterative powerset $\mathcal{P}^k(V_0)$ by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big),$$

where $\mathcal{P}(\cdot)$ denotes the power set. An *n-SuperHyperGraph* is a pair

$$\mathrm{SHT}^{(n)} \ = \ (V, E),$$

with

$$V \ \subseteq \ \mathcal{P}^n(V_0) \quad \text{and} \quad E \ \subseteq \ \mathcal{P}^n(V_0).$$

Each element of V is an *n-supervertex*, and each element of E is an *n-superedge*.

Remark 1.10. When $n = 1$, the notion of an n-SuperHyperGraph coincides with the classical notion of a hypergraph: each vertex set element is simply a subset of V_0, and each hyperedge is likewise a subset of V_0. For $n \geq 2$, the concept permits *nested* structures such as sets of subsets (or deeper nestings), yielding a richer framework.

Example 1.11 (A 2-SuperHyperGraph Over a Small Base Set). *Base Set:* Let $V_0 = \{a, b\}$. Then

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}.$$

Hence

$$\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}^1(V_0)) = \mathcal{P}\Big(\{\varnothing, \{a\}, \{b\}, \{a, b\}\}\Big).$$

There are $2^4 = 16$ elements in $\mathcal{P}^2(V_0)$, each one being a subset of $\{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. For example, $\{\{a\}\}$, $\{\varnothing, \{b\}\}$, $\{\{a\}, \{b\}\}$, and $\{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ are all valid elements in $\mathcal{P}^2(V_0)$.

Constructing a 2-SuperHyperGraph:
We now choose a subset of these 16 elements to form our set of *2-supervertices*, V, and our set of *2-superedges*, E. For instance, define:

$$V = \Big\{\{\varnothing\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\{a\}, \{b\}\}\Big\} \subseteq \mathcal{P}^2(V_0).$$

This means we have 5 distinct 2-supervertices in the set V. Observe that each such "vertex" in this 2-SuperHyperGraph is itself a set of subsets of $\{a, b\}$. For example:

$\{\{a\}, \{b\}\} \in V$ means we have a 2-supervertex whose elements are the singletons $\{a\}$ and $\{b\}$.

Similarly, let

$$E = \Big\{\{\{a\}\}, \quad \{\varnothing, \{b\}\}, \quad \{\{a, b\}\}\Big\} \subseteq \mathcal{P}^2(V_0).$$

Thus, we have three 2-superedges:

- $\{\{a\}\}$, containing only the singleton $\{a\}$.
- $\{\varnothing, \{b\}\}$, containing the empty set and $\{b\}$.
- $\{\{a, b\}\}$, containing one element: the set $\{a, b\}$.

Putting these together, we have constructed the pair

$$\mathrm{SHT}^{(2)} = (V, E)$$

as a valid 2-SuperHyperGraph.

Comparison to a Standard Hypergraph:

- In a *standard hypergraph* $H = (V', E')$ over the same atomic base $\{a, b\}$, vertices are simply $\{a\}$ and/or $\{b\}$ as atomic elements. Meanwhile, each edge is a subset of $\{a, b\}$; for instance, $\{a, b\}$ or $\{b\}$.
- In our *2-SuperHyperGraph*, each vertex (2-supervertex) and edge (2-superedge) is a subset of the set $\{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. This structure allows *nested* sets such as $\{\{a\}, \{b\}\}$, which cannot appear in a standard hypergraph. In a normal hypergraph, an edge might be $\{a\}$ or $\{b\}$, but not a set *containing* $\{a\}$ or $\{b\}$ as its elements.
- Therefore, a 2-SuperHyperGraph is capable of encoding more "layers" of containment. For example, the 2-superedge $\{\varnothing, \{b\}\}$ suggests we have a structure that simultaneously references the empty set and the singleton $\{b\}$ as if they were atomic units. This level of nesting ($\mathcal{P}^2$) goes strictly beyond the simple adjacency relationships of standard hypergraphs (which are stuck at $\mathcal{P}^1$).

The above example illustrates how an n-SuperHyperGraph ($n = 2$ here) vastly enlarges the representational capabilities, allowing edges and vertices to be sets of subsets, thus capturing richer or more complex relationships than a plain hypergraph structure.

1.4 Tree-depth and Hypertree-depth

Tree-depth is defined as the minimum height of a rooted forest whose closure contains the given graph as a subgraph [6,10,11,20,29,30,32,34,41]. Hypertree-depth generalizes the concept of Tree-depth to hypergraphs, providing a corresponding measure that captures the hierarchical structure of hypergraphs [1,21]. Below, we present the formal definitions of these concepts, along with related notions.

Definition 1.12 (Rooted Forest)**.** (cf. [42]) A *rooted forest* F is an undirected acyclic graph $G = (V, E)$ that satisfies the following properties:

- F consists of one or more connected components, each of which is a *rooted tree*.
- Each connected component has a distinguished vertex called the *root*.
- The *rooted tree* property implies that each vertex $v \in V(F)$ (except the roots) has a unique parent, determined by a parent-child relationship induced by the root.
- The ancestor-descendant relationship forms a *partial order* $\leq_F$ on $V(F)$, where $u \leq_F v$ if u is an ancestor of v in the rooted structure, or $u = v$.

The edge set $E(F)$ of the rooted forest corresponds to these parent-child relationships.

Example 1.13 (A Simple Rooted Forest)**.** Consider a graph $F = (V, E)$ with:

$$V = \{a, b, c, d, e, f, g\}, \quad E = \{\{a, b\}, \{a, c\}, \{d, e\}, \{d, f\}, \{f, g\}\}.$$

This graph forms a rooted forest with two connected components:

- The first component is a rooted tree with root a, and its vertices are $\{a, b, c\}$.
- The second component is a rooted tree with root d, and its vertices are $\{d, e, f, g\}$.

The ancestor-descendant partial order $\leq_F$ for F is:

- For the first tree: $a \leq_F b$, $a \leq_F c$, and $a = a$.
- For the second tree: $d \leq_F e$, $d \leq_F f$, $d \leq_F g$, $f \leq_F g$, and $d = d$.

Definition 1.14 (Closure of a Rooted Forest)**.** (cf. [31]) Let F be a rooted forest. The *closure* of F, denoted $\mathrm{clos}(F)$, is the graph defined as follows:

- The vertex set of $\mathrm{clos}(F)$ is $V(\mathrm{clos}(F)) = V(F)$.
- The edge set of $\mathrm{clos}(F)$ is

  $$E(\mathrm{clos}(F)) \;=\; \{\{u, v\} \mid u, v \in V(F),\ u \leq_F v,\ u \neq v\},$$

  where $\leq_F$ is the partial order on $V(F)$ given by the ancestor-descendant relationship in F.

In other words, $u \leq_F v$ means that u is an ancestor of v in F, or $u = v$. Hence, $\mathrm{clos}(F)$ is the comparability graph of the ancestor-descendant relation (excluding self-loops).

Example 1.15 (Closure of a Small Rooted Forest)**.** Consider a rooted forest F consisting of two rooted trees:

$$F = T_1 \cup T_2,$$

where

- T_1 has three vertices $\{r, a, b\}$. Vertex r is the root, and a, b are its children (i.e., edges $\{r, a\}$ and $\{r, b\}$ in T_1).
- T_2 has two vertices $\{s, c\}$. Vertex s is the root, and c is its child (i.e., edge $\{s, c\}$ in T_2).

The partial order $\leq_F$ includes:

$$r \leq_F r, \quad r \leq_F a, \quad r \leq_F b, \quad s \leq_F s, \quad s \leq_F c,$$

and $a \leq_F a$, $b \leq_F b$, $c \leq_F c$ (each vertex is trivially an ancestor of itself). There are no ancestral relationships between vertices in different trees.

The closure $\mathrm{clos}(F)$ therefore has the same vertex set $\{r, a, b, s, c\}$ and an edge set consisting of:

$$\{r, a\},\ \{r, b\},\ \{s, c\},$$

since those pairs reflect ancestor-descendant relations. No additional edges connect $\{r, a, b\}$ with $\{s, c\}$ because there are no cross-tree ancestor-descendant relationships.

Definition 1.16 (Tree-depth)**.** [30] The *tree-depth* of a graph G, denoted $\mathrm{td}(G)$, is the minimum height of a rooted forest F such that G is a subgraph of $\mathrm{clos}(F)$. Equivalently,

$$\mathrm{td}(G) \;=\; \min\Big\{ \mathrm{height}(F) \;\Big|\; G \subseteq \mathrm{clos}(F) \Big\},$$

where $\mathrm{clos}(F)$ is constructed using the ancestor-descendant relation of F as above.

Example 1.17 (Tree-depth of a Small Graph)**.** Let G be the path on four vertices $v_1 - v_2 - v_3 - v_4$. We claim $\mathrm{td}(G) = 2$.

To see why, construct a rooted forest F of height 2 whose closure contains G. For instance, let F be a single rooted tree:

$$T: \quad r \quad \text{(root)} \qquad \downarrow \qquad \{v_1, v_2, v_3, v_4\}$$

that is, r is the root, and all of v_1, v_2, v_3, v_4 are its children at the second level. The height of this tree is 2.

In $\mathrm{clos}(F)$, there is an edge between every pair (r, v_i) by the ancestor-descendant relationship. Although this creates more edges than in G, the important fact is that G (the path) is contained as a subgraph in $\mathrm{clos}(F)$. Hence, $\mathrm{td}(G) \leq 2$. One can also show it cannot be embedded in the closure of any forest of height 1, implying $\mathrm{td}(G) = 2$.

Definition 1.18 (Hypertree-depth)**.** [1] Let H be a hypergraph with vertex set $V(H)$ and hyperedge set $E(H)$. A *decomposition forest* of H is a pair (F, C) where

- F is a rooted forest,
- $C : V(F) \to E(H)$ is a mapping such that:
  1. For every vertex $v \in V(H)$, there exists a node $t \in V(F)$ with $v \in C(t)$.
  2. For every edge $e \in E(H)$, there are $\leq_F$-comparable nodes $s, t \in V(F)$ such that $e \subseteq C(s) \cup C(t)$.
  3. For all $s, t \in V(F)$, if $C(s) \cap C(t) \neq \emptyset$, then $s \wedge t$ (the least common ancestor) exists and

     $$C(s) \cap C(t) \;\subseteq\; \bigcup_{r \in \downarrow(s \wedge t)} C(r),$$

     where $\downarrow(s \wedge t)$ denotes the set of descendants of $s \wedge t$.

The *hypertree-depth* of H, denoted $\mathrm{hd}(H)$, is the minimum height of a rooted forest F over all such decomposition forests (F, C) of H.

Example 1.19 (Hypertree-depth of a Small Hypergraph)**.** Consider the hypergraph H with

$$V(H) = \{v_1, v_2, v_3\}, \quad E(H) = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\}.$$

We claim $\mathrm{hd}(H) = 2$. Define a decomposition forest (F, C) as follows:

- F is a rooted tree of height 2 with two nodes: a root r and its child u.
- The mapping C is given by
$$C(r) = \{v_1, v_2\}, \quad C(u) = \{v_2, v_3\}.$$

Check each condition:

1. Every vertex v_1, v_2, v_3 appears in $C(r)$ or $C(u)$.
2. Each hyperedge $\{v_1, v_2\}$ or $\{v_2, v_3\}$ is contained in $C(r) \cup C(r)$ or $C(u) \cup C(u)$ respectively; in particular, $r \leq_F u$, so they are $\leq_F$-comparable.
3. The intersection $C(r) \cap C(u) = \{v_2\}$. The least common ancestor of r and u in F is r. The descendants of r (including r itself) are $\{r, u\}$. Then
$$\bigcup_{t \in \downarrow(r)} C(t) = C(r) \cup C(u) = \{v_1, v_2\} \cup \{v_2, v_3\} = \{v_1, v_2, v_3\},$$
which covers the intersection $\{v_2\}$.

Since F has height 2, we get $\mathrm{hd}(H) \leq 2$. One can verify no decomposition forest of height 1 suffices, so $\mathrm{hd}(H) = 2$.

2 Result in This Paper

As a result of this paper, we define Hypertree-length and Hypertree-breadth, Superhypertree-length, and Superhypertree-breadth, and describe the relationships between these parameters.

2.1 n-Superhypertree-depth

The definition of *hypertree-depth* for classical hypergraphs involves a so-called "decomposition forest." We now formulate its analogue in the n-SuperHyperGraph setting.

Definition 2.1 (Decomposition Forest for an n-SuperHyperGraph)**.** Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph, where $V, E \subseteq \mathcal{P}^n(V_0)$. A *decomposition forest* for $\mathrm{SHT}^{(n)}$ is a pair (F, C) consisting of:

- A *rooted forest* F. A rooted forest is a disjoint union of rooted trees, each node having zero or more children, and each tree having a unique root with no parent. We denote the set of all nodes by $V(F)$, and write $\leq_F$ for the partial order given by the ancestor-descendant relation in F. For $s, t \in V(F)$, if their least common ancestor (LCA) exists, we denote it $s \wedge t$. By convention, a node is an ancestor and descendant of itself, so $s \leq_F s$.
- A labeling map $C \colon V(F) \to E$, which assigns to each node in the forest an *n-superedge* from $\mathrm{SHT}^{(n)}$. This labeling must satisfy the following conditions:
  1. *Coverage of n-supervertices:* For every $x \in V$, there is at least one node $u \in V(F)$ such that $x \in C(u)$. In other words, each n-supervertex is contained in at least one labeled n-superedge in the forest.

2. *Comparable containment of n-superedges:* For each n-superedge $e \in E$, there exist two nodes $s, t \in V(F)$ with $s \leq_F t$ (they are $\leq_F$-comparable) such that

$$e \subseteq C(s) \cup C(t).$$

Hence, every n-superedge of $\mathrm{SHT}^{(n)}$ is the union of the labels at some pair of nodes in an ancestor-descendant relationship.

3. *Intersection descent property:* For all $s, t \in V(F)$ with $C(s) \cap C(t) \neq \emptyset$, the LCA $s \wedge t$ exists in F, and we have

$$C(s) \cap C(t) \subseteq \bigcup_{r \in \downarrow(s \wedge t)} C(r),$$

where $\downarrow(u)$ denotes the set of descendants of u (including u itself) in the forest F. This condition ensures that the common portion of any two labels appears "within" the subtree rooted at their LCA.

Definition 2.2 (n-SuperHypertree-depth)**.** Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph. The *n-SuperHypertree-depth* of $\mathrm{SHT}^{(n)}$, denoted $\mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)})$ (or simply $\mathrm{shd}^{(n)}$ when context is clear), is the minimum height of a rooted forest F among all decomposition forests (F, C) for $\mathrm{SHT}^{(n)}$. Symbolically:

$$\mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)}) = \min\Big\{\mathrm{height}(F) \;\Big|\; (F, C) \text{ is a decomposition forest for } \mathrm{SHT}^{(n)}\Big\}.$$

Here, the *height* of F is the maximum level (distance from the root) of any node in F. Equivalently, if F has multiple connected components (trees), the height is the maximum of the heights of its constituent trees.

Example 2.3 (A 2-SuperHyperGraph and Its Decomposition Forest)**.** Let $V_0 = \{a, b\}$. Then:

$$\mathcal{P}(V_0) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}\big(\{\emptyset, \{a\}, \{b\}, \{a, b\}\}\big).$$

We construct the following 2-SuperHyperGraph:

$$\mathrm{SHT}^{(2)} = (V, E),$$

where

$$V = \Big\{\{\{a\}\}, \{\{b\}\}\Big\} \subseteq \mathcal{P}^2(V_0), \quad E = \Big\{\{\emptyset, \{a\}\}, \{\{b\}\}\Big\} \subseteq \mathcal{P}^2(V_0).$$

So there are two n-supervertices:

$$x_1 = \{\{a\}\}, \quad x_2 = \{\{b\}\},$$

and two 2-superedges:

$$e_1 = \{\emptyset, \{a\}\}, \quad e_2 = \{\{b\}\}.$$

Constructing a decomposition forest (F, C)*.* Define F to be a single rooted tree with a root r of level 1 and a single child u of level 2. Assign

$$C(r) = e_1, \quad C(u) = e_2.$$

Check the conditions in Definition 2.1:

1. *Coverage of n-supervertices:* $x_1 = \{\{a\}\} \in e_1 = C(r)$, and $x_2 = \{\{b\}\} \in e_2 = C(u)$. Thus each n-supervertex appears in at least one label.

2. *Comparable containment of n-superedges:* For e_1, it is trivially covered by $C(r) \cup C(r)$. For e_2, it is likewise covered by $C(u) \cup C(u)$. Moreover, $r \leq_F u$, so r and u are $\leq_F$-comparable.

3. *Intersection descent property:*

$$C(r) \cap C(u) = e_1 \cap e_2 = \{\emptyset, \{a\}\} \cap \{\{b\}\} = \emptyset,$$

which trivially satisfies $\emptyset \subseteq \bigcup_{w \in \downarrow(r \wedge u)} C(w)$.

The height of F is 2. One verifies that no decomposition forest of height 1 suffices (since that would require labeling one node with both e_1 and e_2 in a way that covers distinct n-supervertices separately, which fails the intersection descent property or coverage requirements). Hence $\mathrm{shd}^{(2)}(\mathrm{SHT}^{(2)}) = 2$.

2.2 Basic Properties of SuperHypertree-depth

We present and prove several theorems related to the properties of SuperHypertree-depth.

Theorem 2.4 (Well-definedness)**.** *Let* $\mathrm{SHT}^{(n)} = (V, E)$ *be any n-SuperHyperGraph. Then* $\mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)})$ *is a finite positive integer. That is, there is at least one decomposition forest* (F, C) *of finite height, and among all such possible forests a minimum height always exists. Consequently,* $\mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)}) \in \mathbb{N}$.

Proof. Step 1: Existence of a (finite) decomposition forest.

Since V_0 is finite, $\mathcal{P}^n(V_0)$ is also finite. Thus $V \subseteq \mathcal{P}^n(V_0)$ and $E \subseteq \mathcal{P}^n(V_0)$ are finite sets. Let

$$E = \{ e_1, e_2, \ldots, e_m\} \quad (m \geq 1).$$

We construct a decomposition forest F of finite height in a straightforward manner:

1. Start with a single *root* node r. Let the forest consist of exactly one tree (so it is indeed a forest of one component).
2. For each n-superedge $e_i \in E$, create a child node c_i of r. Hence the forest has m children under the root, giving a total of $1 + m$ nodes.
3. Define the labeling map C by

$$C(r) \; = \; \varnothing \quad (\text{the empty set in } \mathcal{P}^n(V_0)), \quad C(c_i) \; = \; e_i \quad \text{for each } i = 1, 2, \ldots, m.$$

   (If one prefers not to use the empty label, one may select any dummy n-superedge from E or from $\mathcal{P}^n(V_0)$ for the root. The specific choice for the root label will not undermine the general argument.)

This construction yields a forest F of height exactly 2: the root r is at level 1, each c_i is at level 2. We must verify the three decomposition conditions from Definition 2.1:

- *Coverage:* Every n-supervertex $x \in V$ must appear in at least one label. Observe that each x lies in some n-superedge e_i (worst case, each x itself might be one of the edges in E). If it does not, we can add an extra child labeled by a union that contains x. Concretely, if the n-supervertices do not appear in exactly these edges, one can augment the construction by assigning additional children or by merging edges so that all vertices are covered. (A simpler approach: suppose $\bigcup E$ denotes the union of all edges in the sense of set-theoretic union. Then each $x \in V \subseteq \bigcup E$; hence each $x \in e_i$ for some i. So $x \in C(c_i)$ and coverage holds.)
- *Comparable containment:* For any $e_j \in E$, it appears exactly as the label of a child c_j. Thus $e_j \subseteq C(c_j) \cup C(c_j)$. Since $r \leq_F c_j$, we indeed have two $\leq_F$-comparable nodes, so the condition $e_j \subseteq C(r) \cup C(c_j)$ or $e_j \subseteq C(c_j) \cup C(c_j)$ is trivially satisfied.
- *Intersection descent:* If $C(s) \cap C(t) \neq \emptyset$ for two nodes s, t, we note that $s \wedge t = r$ if $s \neq t$, or $s \wedge t = s$ if $s = t$. In either case, $\downarrow (s \wedge t) \subseteq \{r, c_1, \ldots, c_m\}$. Because each label is either $\varnothing$ or one of the edges e_i, any nonempty intersection must be an n-superedge or subset of n-superedges. This intersection is contained trivially in the union of the labels of the entire forest. In particular, $\bigcup_{u \in \downarrow(r)} C(u)$ includes all edges e_i. Thus, the intersection condition is satisfied.

Therefore, (F, C) is a valid decomposition forest of height 2.

Step 2: Finiteness of the minimum height.

We have explicitly exhibited a decomposition forest of finite height (here, height 2). Consequently, the set

$$\mathcal{H} \; = \; \left\{ \mathrm{height}(F) \;\middle|\; (F, C) \text{ is a decomposition forest for } \mathrm{SHT}^{(n)} \right\}$$

is a non-empty set of *positive integers*. Any non-empty finite subset of the natural numbers has a minimum, so $\min \mathcal{H}$ exists and is finite. By Definition 2.2, we then have

$$\mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)}) \; = \; \min \mathcal{H} \; \in \; \mathbb{N}.$$

Hence, $\mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)})$ is a well-defined, finite integer.

Theorem 2.5 (Consistency with Standard Hypertree-depth)**.** *Let* $\mathrm{SHT}^{(1)} = (V, E)$ *be viewed as a standard hypergraph* $H = (V, E)$*. Then*

$$\mathrm{shd}^{(1)}(\mathrm{SHT}^{(1)}) \;=\; \mathrm{hd}(H),$$

where $\mathrm{hd}(H)$ *is the classical hypertree-depth of* H*.*

Proof. In the case $n = 1$, each "1-supervertex" is just an element of $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$. In a standard hypergraph setting, the set V of "vertices" is also a subset of V_0, but here we interpret $V \subseteq \mathcal{P}(V_0)$. Meanwhile, each 1-superedge in E is also a subset of V_0. We must check that the definitions of decomposition forest given in Definition 2.1 coincide exactly with the standard definition used to characterize hypertree-depth (cf. [22, 23]):

1. *Coverage condition (classical vs. super):* In the classical definition for hypertree-depth, each vertex $v \in V(H) \subseteq V_0$ must appear in the label of at least one node of the decomposition. In the n-super setting with $n = 1$, each 1-supervertex is effectively an element of $\mathcal{P}(V_0)$. However, by design or by adjusting notation slightly, we can align the classical condition that each $v \in V_0$ must appear in some set-labeled node with the super-condition that each element of $V \subseteq \mathcal{P}(V_0)$ must appear in some label. Specifically, in classical hypertree theory, one demands coverage of each *atomic* vertex $v \in V_0$. In the n-super approach for $n = 1$, the coverage demands coverage of each set in $V \subseteq \mathcal{P}(V_0)$. But typically, one chooses $V = V_0$ itself for classical hypergraphs (treating each single atomic vertex as an element). So the coverage conditions match.
2. *Comparable containment of edges:* In classical hypertree-depth, each hyperedge $\varepsilon \in E$ must be contained in the union of the labels of two nodes that are in ancestor-descendant relation. This is the same as the super-edge condition in Definition 2.1 for $n = 1$.
3. *Intersection descent property:* The classical hypertree-depth definition requires that if two nodes share a common atomic vertex in their labels, that vertex must appear in the label of every node on the path between them in the tree. By rewriting "the path between them" in terms of the least common ancestor plus the subtree from that ancestor, we see that this is essentially the same property: any vertex in the intersection arises in all nodes along that path, i.e. the intersection is contained in the union of the labels in the subtree from the LCA.

Consequently, any decomposition forest (F, C) in the sense of Definition 2.1 for $n = 1$ is exactly a hypertree-depth decomposition in the classical sense, and vice versa. Hence, the minimal heights of these trees are the same, i.e.

$$\mathrm{shd}^{(1)}(\mathrm{SHT}^{(1)}) \;=\; \mathrm{hd}(H),$$

as claimed. □

Theorem 2.6 (Monotonicity in n)**.** *Consider an n-SuperHyperGraph* $\mathrm{SHT}^{(n)}$ *with* $\mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)})$ *as defined. Suppose we view* $\mathrm{SHT}^{(n)}$ *as an* $(n+1)$*-SuperHyperGraph by the natural inclusion*

$$\mathcal{P}^n(V_0) \;\subseteq\; \mathcal{P}^{n+1}(V_0).$$

Then

$$\mathrm{shd}^{(n+1)}(\mathrm{SHT}^{(n)}) \;\leq\; \mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)}).$$

That is, allowing an n-SuperHyperGraph to be embedded in a higher dimension $(n+1)$ *can only decrease (or leave equal) the n-SuperHypertree-depth.*

Proof. Let $\mathrm{SHT}^{(n)} = (V, E)$. When we say we "view $\mathrm{SHT}^{(n)}$ as an $(n+1)$-SuperHyperGraph," we interpret

$$V, E \;\subseteq\; \mathcal{P}^n(V_0) \;\subseteq\; \mathcal{P}^{n+1}(V_0).$$

Hence, the same sets V and E can be treated as subsets of $\mathcal{P}^{n+1}(V_0)$. Let $\mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)}) = d$. By definition, there exists a decomposition forest (F, C) for $\mathrm{SHT}^{(n)})$ with $\mathrm{height}(F) = d$.

We claim that (F, C) also serves as a valid decomposition forest in the $(n+1)$-dimensional sense, thus showing $\mathrm{shd}^{(n+1)}(\mathrm{SHT}^{(n)}) \leq d$. Indeed, all conditions in Definition 2.1 are dimension agnostic regarding $\leq_F$ and the forest structure. The only possible difference is that (F, C) must map each node to an *element of* $E \subseteq \mathcal{P}^n(V_0)$. But since $E \subseteq \mathcal{P}^{n+1}(V_0)$ under the inclusion, exactly the same labeling function $C \colon V(F) \to E \subseteq \mathcal{P}^n(V_0)$ is legitimate in the $(n+1)$-super framework. There is no conflict in the coverage or intersection rules, because the structure of the forest and the sets themselves remain unchanged—they are simply being recognized as elements of a bigger universe $\mathcal{P}^{n+1}(V_0)$.

Hence, a decomposition forest of height d for the n-SuperHyperGraph remains a decomposition forest of the same height for the $(n+1)$-dimensional perspective. By taking the minimum among all such forests in the $(n+1)$-dimensional view, we conclude:

$$\mathrm{shd}^{(n+1)}(\mathrm{SHT}^{(n)}) \;\leq\; d \;=\; \mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)}).$$

Thus the statement is proved. □

Corollary 2.7. *If $n' \geq n$ and we embed* $\mathrm{SHT}^{(n)}$ *into* $\mathrm{SHT}^{(n')}$ *in the analogous way ($\mathcal{P}^n(V_0) \subseteq \mathcal{P}^{n'}(V_0)$), then*

$$\mathrm{shd}^{(n')}(\mathrm{SHT}^{(n)}) \;\leq\; \mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)}).$$

Proof. Apply Theorem 2.6 iteratively from n to $n+1$, then from $n+1$ to $n+2$, and so forth, until reaching n'. □

2.3 Additional Property: Flattening an n-SuperHyperGraph to a Classical Hypergraph

A central operation that connects multi-level (super) structures back to standard hypergraphs is the *flattening* map. Intuitively, we reduce nested subsets in $\mathcal{P}^n(V_0)$ to ordinary subsets of V_0.

Definition 2.8 (Flattening of an Element)**.** Let $n \geq 1$. A set $x \in \mathcal{P}^n(V_0)$ can be viewed as a nested subset of depth n. Define its *flattening* to a subset of V_0 as follows:

$$\mathrm{Flat}(x) \;=\; \begin{cases} x, & \text{if } n = 1 \text{ (so } x \subseteq V_0 \text{ directly)}, \\ \displaystyle\bigcup_{y \in x} \mathrm{Flat}(y), & \text{if } n > 1. \end{cases}$$

In other words, for $n > 1$, each element y of x is itself an object in $\mathcal{P}^{n-1}(V_0)$; we recursively flatten all subsets until eventually reaching elements of V_0.

Definition 2.9 (Flattening an n-SuperHyperGraph)**.** Let $\mathrm{SHT}^{(n)} = (V, E) \subseteq \mathcal{P}^n(V_0) \times \mathcal{P}^n(V_0)$. We define its *underlying classical hypergraph* (or *flattened hypergraph*) as

$$\mathrm{Flat}\big(\mathrm{SHT}^{(n)}\big) \;=\; \Big(V_0,\; E^*\Big),$$

where

$$E^* \;=\; \Big\{\mathrm{Flat}(e) \;\Big|\; e \in E\Big\} \;\subseteq\; \mathcal{P}(V_0).$$

Thus each n-superedge $e \subseteq \mathcal{P}^n(V_0)$ is mapped to an ordinary hyperedge $\mathrm{Flat}(e) \subseteq V_0$ by the recursive union. Note that if e is empty, then $\mathrm{Flat}(e) = \varnothing \in \mathcal{P}(V_0)$.

Remark 2.10. We do *not* need to define a separate vertex set in the flattened hypergraph, as classical hypergraph vertices are atomic elements from V_0. If we want to track which atomic vertices actually *occur* in the flattened edges, we could restrict to $\bigcup E^* \subseteq V_0$. For clarity, we take V_0 as the ambient vertex set.

Theorem 2.11 (Flattening Bound)**.** *Let* $\mathrm{SHT}^{(n)} = (V, E) \subseteq \mathcal{P}^n(V_0) \times \mathcal{P}^n(V_0)$ *be an n-SuperHyperGraph, and let*

$$H_{\mathrm{flat}} \;=\; \mathrm{Flat}\big(\mathrm{SHT}^{(n)}\big) \;=\; \Big(V_0,\; E^*\Big)$$

be its underlying flattened hypergraph. Then

$$\mathrm{hd}\big(H_{\mathrm{flat}}\big) \;\;\leq\;\; \mathrm{shd}^{(n)}\big(\mathrm{SHT}^{(n)}\big).$$

Proof. Let $\mathrm{SHT}^{(n)}$ have n-SuperHypertree-depth d. By definition, there is a decomposition forest (F, C) of height d. That is:

- F is a rooted forest whose maximum node depth (level) is d.
- $C\colon V(F) \to E$ is a labeling map that satisfies the three conditions from the n-SuperHyperTree-depth decomposition definition (coverage, comparable containment, intersection descent).

We will build a *classical* hypertree-decomposition for $H_{\text{flat}} = (V_0, E^*)$ using the same forest F, but with a modified labeling map $\widetilde{C} : V(F) \to \mathcal{P}(V_0)$. Specifically, for each node $u \in V(F)$,

$$\widetilde{C}(u) \;=\; \mathrm{Flat}\big(C(u)\big) \;\subseteq\; V_0.$$

(Recall that $C(u) \in E \subseteq \mathcal{P}^n(V_0)$, so its flattening is a subset of V_0.) We verify the conditions for a classical hypertree-depth decomposition:

1. *Coverage of vertices in* V_0*:* In the original n-super decomposition, each n-supervertex $x \in V \subseteq \mathcal{P}^n(V_0)$ lies in the label of *some* node (by the coverage condition). However, classical coverage requires: for every atomic vertex $v \in V_0$, there is a node $u \in V(F)$ with $v \in \widetilde{C}(u)$. We claim this follows from the coverage condition *of edges* in the original superhypergraph:

   - By definition, each hyperedge in the flattened hypergraph H_{flat} is of the form $\mathrm{Flat}(e)$ for some $e \in E$.
   - In the n-super decomposition, we require that each n-superedge e be contained in the union of two $\leq_F$-comparable labels $C(s) \cup C(t)$.
   - Hence, if $v \in \mathrm{Flat}(e)$, then $v \in \big(\mathrm{Flat}(C(s))\big) \cup \big(\mathrm{Flat}(C(t))\big) = \widetilde{C}(s) \cup \widetilde{C}(t)$. Thus each atomic vertex that lies in some $\mathrm{Flat}(e)$ also appears in $\widetilde{C}(s) \cup \widetilde{C}(t)$ for some nodes $s \leq_F t$. Consequently, every atomic vertex in any flattened hyperedge is covered by the labeling in $\widetilde{C}$.

   If one also wants each $v \in V_0$ that does *not* appear in any $\mathrm{Flat}(e)$ to be covered, that can be done trivially, e.g., by adding a dummy child with label $\varnothing$. In short, all relevant atomic vertices (i.e., those that matter for edges) are covered.

2. *Comparable containment for edges* $\mathrm{Flat}(e)$*:* In the n-SuperHyperGraph decomposition, each n-superedge $e \in E$ appears in $C(s) \cup C(t)$ for some $\leq_F$-comparable nodes s, t. Then

   $$\mathrm{Flat}(e) \;\subseteq\; \mathrm{Flat}\big(C(s) \cup C(t)\big) \;=\; \mathrm{Flat}(C(s)) \;\cup\; \mathrm{Flat}(C(t)) \;=\; \widetilde{C}(s) \;\cup\; \widetilde{C}(t).$$

   In the flattened hypergraph H_{flat}, the edge $\mathrm{Flat}(e)$ is thus contained in the union of two $\leq_F$-comparable node labels $\widetilde{C}(s)$ and $\widetilde{C}(t)$. This is exactly the "comparable containment" condition for classical hypertree-depth.

3. *Intersection descent (classical version)*: For two nodes $u_1, u_2 \in V(F)$, if $\widetilde{C}(u_1) \cap \widetilde{C}(u_2) \neq \emptyset$, then there is at least one atomic vertex $v \in V_0$ such that $v \in \mathrm{Flat}(C(u_1)) \cap \mathrm{Flat}(C(u_2))$. By definition of Flat, this means that v appears in some sub-subset of both $C(u_1)$ and $C(u_2)$ in the n-superstructure. The n-super intersection descent property ensures that $C(u_1) \cap C(u_2) \subseteq \bigcup_{z \in \downarrow(u_1 \wedge u_2)} C(z)$. Flattening both sides of that inclusion yields

   $$\mathrm{Flat}\big(C(u_1) \cap C(u_2)\big) \;\subseteq\; \mathrm{Flat}\Big(\bigcup_{z \in \downarrow(u_1 \wedge u_2)} C(z)\Big) \;=\; \bigcup_{z \in \downarrow(u_1 \wedge u_2)} \mathrm{Flat}\big(C(z)\big).$$

   Hence any atomic vertex in $\widetilde{C}(u_1) \cap \widetilde{C}(u_2)$ must appear in $\widetilde{C}(z)$ for some descendant z of $(u_1 \wedge u_2)$. This is precisely the classical condition that the shared vertices of two nodes' labels be explained within the subtree rooted at the LCA.

Thus $\widetilde{C}$ is a valid labeling for a classical hypertree-decomposition of H_{flat}. Since the rooted forest F has height d, the resulting decomposition is of height d. Therefore $\mathrm{hd}(H_{\text{flat}}) \leq d = \mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)})$.

This completes the proof. □

Theorem 2.12 (Chain of Bounds for Hypertree-depth vs. n-SuperHypertree-depth)**.** *Let* $\mathrm{SHT}^{(n)} = (V, E) \subseteq \mathcal{P}^n(V_0) \times \mathcal{P}^n(V_0)$*, and let* $H_{\text{flat}} = \mathrm{Flat}\big(\mathrm{SHT}^{(n)}\big)$ *be its underlying classical hypergraph with vertex set* V_0 *and edge set* $E^* = \{\mathrm{Flat}(e) \mid e \in E\}$*. Then:*

$$\mathrm{hd}\big(H_{\text{flat}}\big) \quad \leq \quad \mathrm{shd}^{(n)}\big(\mathrm{SHT}^{(n)}\big) \quad \leq \quad \mathrm{shd}^{(n+1)}\big(\mathrm{SHT}^{(n)}\big).$$

Proof. The left inequality is exactly Theorem 2.11, showing that once we flatten the n-SuperHyperGraph to a classical hypergraph, its hypertree-depth cannot exceed the original n-SuperHypertree-depth.

The right inequality follows from the *monotonicity in dimension* ($n \mapsto n+1$) established in many prior treatments (cf. Theorem 2.6 or a variant thereof): allowing the same set system to live in a higher-dimensional super-universe cannot increase the minimal forest height. In symbols: $\mathrm{shd}^{(n+1)}(\mathrm{SHT}^{(n)}) \leq \mathrm{shd}^{(n)}(\mathrm{SHT}^{(n)})$. Indeed, the same labeling works with no changes since $V, E \subseteq \mathcal{P}^n(V_0) \subseteq \mathcal{P}^{n+1}(V_0)$.

Putting the inequalities together yields:

$$\mathrm{hd}\big(H_{\text{flat}}\big) \quad \leq \quad \mathrm{shd}^{(n)}\big(\mathrm{SHT}^{(n)}\big) \quad \leq \quad \mathrm{shd}^{(n+1)}\big(\mathrm{SHT}^{(n)}\big),$$

which proves the claimed chain of bounds. □

Corollary 2.13 (Classical Case $n = 1$)**.** *If* $\mathrm{SHT}^{(1)} = (V, E)$ *is just an ordinary hypergraph* $(V, E) \subseteq \mathcal{P}(V_0)$*, then* $\mathrm{Flat}\big(\mathrm{SHT}^{(1)}\big)$ *is isomorphic to* $H = (V_0, E)$*. The chain of Theorem 2.12 becomes*

$$\mathrm{hd}(H) \;=\; \mathrm{hd}\big(\mathrm{Flat}(\mathrm{SHT}^{(1)})\big) \quad \leq \quad \mathrm{shd}^{(1)}\big(\mathrm{SHT}^{(1)}\big) \quad \leq \quad \mathrm{shd}^{(2)}\big(\mathrm{SHT}^{(1)}\big).$$

But we already know $\mathrm{shd}^{(1)}(\mathrm{SHT}^{(1)}) = \mathrm{hd}(H)$*, so the left inequality is in fact an equality. That is,* $\mathrm{hd}(H) = \mathrm{shd}^{(1)}(H)$*, consistent with Theorem 2.5.*

Proof. By definition, if $n = 1$, an n-SuperHyperGraph $\mathrm{SHT}^{(1)} = (V, E)$ simply satisfies $V, E \subseteq \mathcal{P}^1(V_0) = \mathcal{P}(V_0)$. In other words, $\mathrm{SHT}^{(1)}$ is just a hypergraph whose vertices and edges are subsets of the same base set V_0. Consequently, the flattening operation $\mathrm{Flat}(\mathrm{SHT}^{(1)})$ does nothing more than interpret each 1-superedge $e \subseteq V_0$ as itself. Formally,

$$\mathrm{Flat}(e) \;=\; \bigcup_{x \in e} \mathrm{Flat}(x) \;=\; \bigcup_{x \in e} x \;=\; e \quad \text{(since each } x \text{ is an atomic element of } V_0\text{)}.$$

Hence $\mathrm{Flat}\big(\mathrm{SHT}^{(1)}\big) = (V_0, E)$, which is the same as the hypergraph $H = (V_0, E)$. Thus

$$\mathrm{hd}\big(\mathrm{Flat}(\mathrm{SHT}^{(1)})\big) \;=\; \mathrm{hd}(H).$$

On the other hand, it is a direct consequence of the definitions (see, e.g., *hypertree-depth* in classical sense vs. *1-SuperHypertree-depth*) that $\mathrm{shd}^{(1)}(\mathrm{SHT}^{(1)}) = \mathrm{hd}(H)$. Consequently,

$$\mathrm{hd}(H) \;=\; \mathrm{hd}\big(\mathrm{Flat}(\mathrm{SHT}^{(1)})\big) \quad \leq \quad \mathrm{shd}^{(1)}\big(\mathrm{SHT}^{(1)}\big) \;=\; \mathrm{hd}(H).$$

Since the middle term is squeezed between two equal quantities ($\mathrm{hd}(H)$ on both sides), all three are equal. Hence the left inequality is indeed an equality, confirming $\mathrm{hd}(H) = \mathrm{shd}^{(1)}(\mathrm{SHT}^{(1)})$. This is exactly Theorem 2.5 restated in the flattening framework. □

Corollary 2.14 (Translating n-SuperDecompositions to Classical Decompositions)**.** *Any decomposition forest for an n-SuperHyperGraph* $\mathrm{SHT}^{(n)}$ *of height* d *induces a classical hypertree-decomposition of the flattened hypergraph* $H_{\text{flat}} = \mathrm{Flat}\big(\mathrm{SHT}^{(n)}\big)$ *of height at most* d*. Hence one may regard an n-SuperHypertree-depth decomposition as a* refinement *of a standard hypertree-depth decomposition on the flattened structure.*

Proof. Let $\mathrm{SHT}^{(n)} = (V, E) \subseteq \mathcal{P}^n(V_0)$ be any n-SuperHyperGraph, and let (F, C) be a decomposition forest of height d as per the n-SuperHypertree-depth definition. That is, F is a rooted forest with $\mathrm{height}(F) = d$, and $C : V(F) \to E$ satisfies:

1. *Coverage of n-supervertices in* V.
2. *Comparable containment for n-superedges in* E.
3. *Intersection descent property* involving least common ancestors.

Recall that we have defined $\mathrm{Flat}\big(\mathrm{SHT}^{(n)}\big) = \big(V_0, E^*\big)$ where $E^* = \{\mathrm{Flat}(e) \mid e \in E\}$. We construct a classical hypertree-decomposition $\big(F, \widetilde{C}\big)$ for the hypergraph $\big(V_0, E^*\big)$ by setting, for each node $u \in V(F)$,

$$\widetilde{C}(u) \;=\; \mathrm{Flat}\big(C(u)\big) \;\subseteq\; V_0.$$

We then verify the three classical hypertree-depth conditions:

1. *Coverage of atomic vertices* in V_0. If some $v \in V_0$ appears in a flattened edge $\mathrm{Flat}(e) \subseteq V_0$, then by the n-superedge containment property, $e \subseteq C(s) \cup C(t)$ for some $\leq_F$-comparable $s, t \in V(F)$. Hence $v \in \mathrm{Flat}(C(s)) \cup \mathrm{Flat}(C(t))$ because flattening a union is the union of the flattenings. Thus every v in the hypergraph's edge set is included in at least one $\widetilde{C}(u)$. Therefore all "relevant" vertices of $\big(V_0, E^*\big)$ are covered.
2. *Comparable containment* for flattened edges: $\mathrm{Flat}(e) \subseteq \widetilde{C}(s) \cup \widetilde{C}(t)$ for some $\leq_F$-comparable s, t. Indeed, if $e \subseteq C(s) \cup C(t)$ in the original decomposition, then flattening yields
$$\mathrm{Flat}(e) \;\subseteq\; \mathrm{Flat}\big(C(s) \cup C(t)\big) \;=\; \mathrm{Flat}(C(s)) \cup \mathrm{Flat}(C(t)) \;=\; \widetilde{C}(s) \cup \widetilde{C}(t).$$
3. *Intersection descent*: If $\widetilde{C}(u_1) \cap \widetilde{C}(u_2) \neq \emptyset$, then some atomic vertex $v \in V_0$ belongs to both $\mathrm{Flat}(C(u_1))$ and $\mathrm{Flat}(C(u_2))$. By the intersection descent property in the n-Super setting, $C(u_1) \cap C(u_2) \subseteq \bigcup_{z \in \downarrow (u_1 \wedge u_2)} C(z)$. Flattening preserves unions and intersections in an inclusion sense:
$$\mathrm{Flat}\big(C(u_1) \cap C(u_2)\big) \;\subseteq\; \mathrm{Flat}\Big(\bigcup_{z \in \downarrow (u_1 \wedge u_2)} C(z)\Big) \;=\; \bigcup_{z \in \downarrow (u_1 \wedge u_2)} \mathrm{Flat}(C(z)).$$
Hence any $v \in \widetilde{C}(u_1) \cap \widetilde{C}(u_2)$ also appears in some $\widetilde{C}(z)$ with $z \in\downarrow (u_1 \wedge u_2)$. This is precisely the classical LCA path condition for hypertree-depth.

Therefore $\widetilde{C}$ is a valid labeling for a classical hypertree-decomposition of the hypergraph $\mathrm{Flat}\big(\mathrm{SHT}^{(n)}\big)$. The forest F has height d, so the resulting decomposition has height at most d. We conclude that each n-SuperHypertree-depth decomposition naturally "translates" into a classical decomposition of the flattened hypergraph. In this sense, the n-SuperHypertree-depth decomposition can be considered a refinement or a more structured version of the classical one. □

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Data Availability This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 2

Obstruction for Hypertree width and Superhypertree width

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Abstract

Graph characteristics are frequently studied using various parameters, with ongoing research aimed at uncovering deeper insights into these aspects. A hypergraph, which generalizes the concept of a conventional graph, provides an abstract framework to extend graph theory into more complex domains [42]. In this paper, we investigate the role of ultrafilters as obstructions for determining the value of hypertree-width. Ultrafilters, a fundamental concept in mathematics, have wide-ranging applications across diverse mathematical fields. Furthermore, we examine the concept of superhypertree-width, which extends the notion of tree-width using the recursive framework of superhypergraphs. This exploration contributes to understanding the structural and computational properties of superhypergraphs.

Keywords: Hypertree width; Superhypertree width; Tree-width; Bramble

1 Introduction

1.1 Graph Width Parameters

A graph is a mathematical structure of vertices connected by edges, representing relationships or connections [17]. Graph characteristics are extensively studied using various parameters, with a significant focus on width-related measures due to their theoretical and practical importance. Among these, graph width parameters such as tree-width [61–63], cut-width [46, 52], clique-width [14], modular-width [1], tree-cut-width [30, 54], boolean-width [2, 70], branch-width [22, 33, 59], rank-width [50, 57, 58], and path-width [51, 69] play a crucial role in understanding graph structure. These parameters not only provide insights into the "tree-likeness" or complexity of a graph but also have significant implications for algorithmic efficiency and practical problem-solving. As a result, the study of graph width parameters remains an active area of research, with ongoing efforts to uncover their influence on computational strategies and real-world applications.

When analyzing width parameters, it is common to study obstructions that influence their values, such as tangles [21, 60, 63], ultrafilters [22], and brambles [7, 49]. These obstructions are also fundamental in advancing graph algorithms and their applications in game theory [34, 59, 63].

1.2 Hypergraph and SuperHyperGraph

A hypergraph is a generalization of the conventional graph, providing an abstract framework that extends the concepts of graph theory [4, 10, 42]. Hypergraphs have found numerous applications in various fields, including machine learning and network analysis [11, 31, 48, 55]. In practical applications, evaluating how closely a graph approximates a tree structure is often crucial. This need has driven extensive research into parameters such as Hypertree-width [3, 38, 39, 56, 71] and Hyperpath-width, both of which quantify the tree-likeness of hypergraphs.

More recently, the concept of a SuperHyperGraph has been introduced as a further generalization of hypergraphs, incorporating recursive structures and offering a richer framework for theoretical and applied research. This concept has sparked significant academic interest, similar to the enthusiasm surrounding hypergraphs [24, 25, 28, 29, 43, 44, 65, 66, 68]. Additionally, related ideas such as SuperHyperAlgebra have been proposed to further explore this extended framework [67].

1.3 Our Contribution

In this paper, we investigate the role of ultrafilters as obstructions to determining the value of hypertree-width. Ultrafilters, a fundamental concept in mathematical theory, have profound applications across various mathematical disciplines [9, 13, 22, 37].

Furthermore, we introduce the concept of SuperHypertree-width, an extension aimed at enhancing the understanding of SuperHyperGraph structures. This concept is closely related to similar ideas explored in studies such as [24] on SuperHypertree-width. In addition, we examine potential obstructions to SuperHypertree-width, including concepts like SuperHypertangles and SuperHyperBrambles. This new framework is expected to offer deeper insights into the structural properties of SuperHyperGraphs. Ultimately, our goal is to bridge theoretical developments in hypergraph theory with practical applications, facilitating their implementation in real-world scenarios.

2 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

2.1 Basic Definition of Graph Theory

This section presents the fundamental definitions of graph theory. In this paper, we focus exclusively on undirected, finite, and simple graphs. For additional background and comprehensive explanations, readers are encouraged to refer to lecture notes and surveys such as [15–17].

Definition 2.1 (Graph)**.** [17] A graph G is a mathematical structure composed of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2.2 (Subgraph)**.** [17] Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of G is a graph such that:

- $V_H \subseteq V$, i.e., the vertex set of H is a subset of the vertex set of G.
- $E_H \subseteq E$, i.e., the edge set of H is a subset of the edge set of G.
- Each edge in E_H connects vertices in V_H.

Definition 2.3 (Path)**.** [17] A *path* is a graph $P = (V, E)$ where $V = \{v_1, v_2, \ldots, v_k\}$ and $E = \{\{v_i, v_{i+1}\} \mid 1 \leq i < k\}$. Each vertex is distinct, and edges form a simple sequence connecting v_1 to v_k.

Definition 2.4 (Tree)**.** [17] A *tree* is a connected, acyclic graph $T = (V, E)$. A tree with n vertices has $n - 1$ edges.

2.2 Hypergraph

In this subsection, we elucidate the fundamental concepts of hypergraphs. For an in-depth exploration of hypergraphs, including their applications and an overview, please refer to [4, 10, 20, 32].

Definition 2.5 (Hypergraph [10])**.** A *hypergraph* is a pair $H = (V(H), E(H))$, where:

- $V(H)$ is a nonempty set of vertices.
- $E(H)$ is a set of subsets of $V(H)$, called the *hyperedges* of H.

In this paper, we consider only finite hypergraphs.

Definition 2.6 (Induced Subhypergraph [10]). For a hypergraph $H = (V(H), E(H))$ and a subset $X \subseteq V(H)$, the *subhypergraph induced by* X is defined as:

$$H[X] = \big(X, \{e \cap X \mid e \in E(H)\}\big).$$

The hypergraph obtained by removing X from H is denoted as:

$$H \setminus X := H[V(H) \setminus X].$$

Definition 2.7 (Separation in a Hypergraph). Let $H = (V(H), E(H))$ be a hypergraph. A *separation* of H is a pair (A, B) of subhypergraphs such that:

- $A = H[V_A]$ and $B = H[V_B]$, where $V_A, V_B \subseteq V(H)$ are subsets of the vertex set $V(H)$.
- $V_A \cup V_B = V(H)$, meaning that the vertex sets of A and B together cover all vertices of H.
- $V_A \cap V_B$, called the *separator*, satisfies $E(A) \cap E(B) = \emptyset$, ensuring that no hyperedge in H is shared between A and B.

The *order* of the separation (A, B) is defined as the size of the separator:

$$|V_A \cap V_B|.$$

2.3 Hyperbramble and Hypertangle

Next, we will explain Hypertree-width. Hypertree-width is the hypergraph counterpart of Graph Tree-width, which was defined in the 2000s [3,38,39,56,71]. Although there are several variations of Hypertree-width, they will not be covered in this discussion. The range of Hypertree-width values can be determined using concepts like Hyperbrambles and Hypertangles [3].

Definition 2.8 (Hypertree-width). [3] Let $H = (V(H), E(H))$ be a hypergraph, where $V(H)$ is the set of vertices and $E(H)$ is the set of hyperedges. A *tree decomposition* of H is a tuple $(T, (B_t)_{t \in V(T)})$, where:

- $T = (V(T), F(T))$ is a tree.
- $(B_t)_{t \in V(T)}$ is a family of subsets of $V(H)$, called *bags*, such that:
  1. For every hyperedge $e \in E(H)$, there exists a node $t \in V(T)$ such that $e \subseteq B_t$.
  2. For every vertex $v \in V(H)$, the set $\{t \in V(T) \mid v \in B_t\}$ induces a connected subtree of T.

The *width* of a tree decomposition $(T, (B_t)_{t \in V(T)})$ is defined as:

$$\text{width}(T, (B_t)_{t \in V(T)}) = \max_{t \in V(T)} (|B_t| - 1)\,.$$

The *hypertree-width* of H, denoted by $\text{tw}(H)$, is the minimum width over all possible tree decompositions of H.

Definition 2.9 (Hyperbramble on a Hypergraph). [3] Let $H = (V(H), E(H))$ be a hypergraph. A *Hyperbramble of hyperorder* $k + 1$ is a set $\mathcal{B}$ of connected subsets of $V(H)$ satisfying the following conditions:

(HB0) Any two subsets $X_1, X_2 \in \mathcal{B}$ *touch*, meaning $X_1 \cap X_2 \neq \emptyset$ or there exists a hyperedge $e \in E(H)$ such that $e \cap X_1 \neq \emptyset$ and $e \cap X_2 \neq \emptyset$.

(HB1) The hyperorder of $\mathcal{B}$ is defined as the smallest integer k such that there exists a set $S \subseteq E(H)$ with $|S| = k$ and $S \cap X \neq \emptyset$ for all $X \in \mathcal{B}$.

Definition 2.10 (Hypertangle in a Hypergraph, adapted from [3]). Let $H = (V(H), E(H))$ be a hypergraph. A *Hypertangle of hyperorder* $k + 1$ is a hyperbramble $\mathcal{T}$ in H satisfying the following additional condition:

(HT0) For any three subsets $X_1, X_2, X_3 \in \mathcal{T}$, either

$$X_1 \cap X_2 \cap X_3 \neq \emptyset$$

or there exists a hyperedge $e \in E(H)$ such that

$$e \cap X_i \neq \emptyset \quad \text{for all } i \in \{1, 2, 3\}.$$

(HT1) The *hyperorder* of $\mathcal{T}$ is the smallest integer k for which there exists a set $S \subseteq E(H)$ with $|S| = k$ and

$$S \cap X \neq \emptyset \quad \text{for all } X \in \mathcal{T}.$$

Here, we perform some transformations on the hypertangle. This is done to make it more closely resemble the Tangle of general graphs as defined in [63].

Lemma 2.11. *Let $H = (V(H), E(H))$ be a hypergraph, and let $\mathcal{T}$ be a hypertangle of order $k + 1$. Suppose that*

(HT0') For any three sets $X_1, X_2, X_3 \in \mathcal{T}$, either $X_1 \cup X_2 \cup X_3 \neq V(H)$

or there is a hyperedge $e \in E(H)$ with $e \cap X_i \neq \emptyset$ for each $i = 1, 2, 3$.

Then $\mathcal{T}$ avoids the situation $X_1 \cup X_2 \cup X_3 = V(H)$ without an appropriate hyperedge intersecting all three sets.

Proof. Assume, for contradiction, that $X_1, X_2, X_3 \in \mathcal{T}$ and $X_1 \cup X_2 \cup X_3 = V(H)$, yet there is no hyperedge $e \in E(H)$ that intersects X_1, X_2, and X_3 simultaneously.

By the definition of a hypertangle (in particular the usual ''triple-intersection or hyperedge'' property), one would expect that either $X_1 \cap X_2 \cap X_3 \neq \emptyset$ or a single hyperedge meets all three sets. Here, however, $X_1 \cup X_2 \cup X_3 = V(H)$ implies $X_1 \cap X_2 \cap X_3 = \emptyset$. Thus, the only way for $\mathcal{T}$ to satisfy the hypertangle condition is to have some $e \in E(H)$ intersecting all three sets, which contradicts our assumption. Hence no such triple (X_1, X_2, X_3) can exist if $\mathcal{T}$ is truly a hypertangle of order $k + 1$. □

Lemma 2.12. *Let $H = (V(H), E(H))$ be a hypergraph, and let $\mathcal{T}$ be a hypertangle of order $k + 1$. Then for every separation $(A, B) \in \mathcal{T}$, the order $|A \cap B|$ is strictly less than k.*

Proof. This follows immediately from the hypertangle's definition of order $k + 1$. If $|A \cap B| \geq k$, then the separation (A, B) would not be valid for a hypertangle of order $k + 1$. Hence all separations in $\mathcal{T}$ have order (i.e. $|A \cap B|$) less than k. □

Lemma 2.13. *Let $H = (V(H), E(H))$ be a hypergraph, and let $\mathcal{T}$ be a hypertangle of order $k + 1$. Then:*

(HT3) For every separation (A, B) of H with order $< k$, exactly one of (A, B) or (B, A) lies in $\mathcal{T}$.

Proof. Consider a separation (A, B) of H such that $|A \cap B| < k$. Suppose neither (A, B) nor (B, A) is in $\mathcal{T}$. That would mean $A \notin \mathcal{T}$ and $B \notin \mathcal{T}$. Take

$$X_1 = A, \quad X_2 = B, \quad X_3 = A \cup B.$$

Since A and B typically separate the entire vertex set (except their intersection), we get $X_1 \cup X_2 \cup X_3 = A \cup B = V(H)$ in a connected sense. By condition (HT0') (Lemma 2.11), there must be a hyperedge $e \in E(H)$ intersecting all three X_i, which is impossible because A and B partition the vertex set except for $A \cap B$. Indeed, a single hyperedge cannot simultaneously meet A and B if $A \cap B \neq \emptyset$ but $A \cap B$ is small, unless it is accounted for by $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$.

Thus, our assumption leads to a contradiction. Hence for each separation of order $< k$, exactly one orientation belongs to $\mathcal{T}$. □

Lemma 2.14. *Let $H = (V(H), E(H))$ be a hypergraph, and let $\mathcal{T}$ be a hypertangle of order $k + 1$. Then:*

(HT4) If $(A_2, B_2) \in \mathcal{T}$ and $A_1 \subseteq A_2$, with (A_1, B_1) a separation of order $< k$, then $(A_1, B_1) \in \mathcal{T}$.

Proof. Suppose $(A_2, B_2) \in \mathcal{T}$ and $A_1 \subseteq A_2$, where (A_1, B_1) is a separation of order $< k$. Suppose for contradiction that $(A_1, B_1) \notin \mathcal{T}$. By Lemma 2.13 (HT3 condition), if $(A_1, B_1) \notin \mathcal{T}$, then $(B_1, A_1) \in \mathcal{T}$.

Consider the sets

$$X_1 = A_1, \quad X_2 = B_2, \quad X_3 = A_1 \cup B_2.$$

If $X_1 \cup X_2 \cup X_3 = V(H)$, condition (HT0') (Lemma 2.11) would require a hyperedge e intersecting A_1, B_2, and $A_1 \cup B_2$, which is again not feasible given $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$. This leads to a contradiction that forces $(A_1, B_1) \in \mathcal{T}$. Therefore, whenever $A_1 \subseteq A_2$ and $|A_1 \cap B_1| < k$, the pair (A_1, B_1) must belong to $\mathcal{T}$. □

Theorem 2.15 (Hypertangle of Hyperorder $k + 1$)**.** *Let $H = (V(H), E(H))$ be a hypergraph. A* Hypertangle *of* hyperorder $k + 1$ *is a hyperbramble $\mathcal{T}$ in H that satisfies:*

(HT0') ***Triple-set Condition:*** *For any $X_1, X_2, X_3 \in \mathcal{T}$, either*

$$X_1 \cup X_2 \cup X_3 \neq V(H) \quad \text{or there is a hyperedge } e \in E(H) \text{ with } e \cap X_i \neq \emptyset \text{ for all } i = 1, 2, 3.$$

(HT1) ***Definition of Hyperorder:*** *The hyperorder of $\mathcal{T}$ is the smallest integer k for which there is a set $S \subseteq E(H)$ of size k such that*

$$S \cap X \neq \emptyset \quad \text{for every } X \in \mathcal{T}.$$

(HT2) ***Order of Separations in $\mathcal{T}$:*** *For each separation $(A, B) \in \mathcal{T}$, we have $|A \cap B| < k$.*

(HT3) ***Orientation Completeness****: For every separation (A, B) of H with $|A \cap B| < k$, exactly one of (A, B) or (B, A) is contained in $\mathcal{T}$.*

(HT4) ***Containment Monotonicity in $\mathcal{T}$:*** *If $(A_2, B_2) \in \mathcal{T}$ and $A_1 \subseteq A_2$ for some separation (A_1, B_1) of order $< k$, then $(A_1, B_1) \in \mathcal{T}$.*

Such a family $\mathcal{T}$ is said to form a Hypertangle of hyperorder $k + 1$. In essence, it extends the idea of tangles in graphs to hypergraphs, capturing high-level connectivity constraints and serving as an obstruction to small hypertree-width.

Proof. The lemmas above establish each of these conditions:

- **(HT0')** is proved in Lemma 2.11, which shows that no three sets can cover $V(H)$ entirely without a single hyperedge intersecting them all.
- **(HT2)** is shown in Lemma 2.12, ensuring separations in $\mathcal{T}$ have order below k.
- **(HT3)** is established by Lemma 2.13, guaranteeing exactly one orientation of each low-order separation is chosen.
- **(HT4)** appears in Lemma 2.14, demonstrating that containment of one side of a separation in another implies the smaller separation also belongs to $\mathcal{T}$.

(HT1) is part of the fundamental definition of the hyperorder of $\mathcal{T}$; it designates k as the minimal number of edges needed to block all sets in $\mathcal{T}$. Together, these properties define a Hypertangle of hyperorder $k + 1$, completing the proof. □

3 Result of This Paper

This section presents the main results of this paper.

3.1 HyperUltrafilter and Hypertangle

We consider about HyperUltrafilter analogeous to Ultrafilter of set theory. H-Ultrafilter on a Hypergraph is following.

Definition 3.1 (H-Ultrafilter on a Hypergraph)**.** Let $H = (V(H), E(H))$ be a hypergraph. An *H-Ultrafilter of order* k is a family $\mathcal{F}$ of separations of H satisfying the following:

(H0) **Bounded Order:** Every separation $(A, B) \in \mathcal{F}$ has order $|A \cap B| < k$. (The *order* of a separation (A, B) is the cardinality $|A \cap B|$.)

(H1) **Completeness:** For any separation (A, B) of H with $|A \cap B| < k$, exactly one of (A, B) or (B, A) belongs to $\mathcal{F}$. This property ensures the ultrafilter decides a unique orientation for every low-order separation.

(H2) **Containment Monotonicity:** If $(A_1, B_1) \in \mathcal{F}$ and (A_2, B_2) is a separation with $|A_2 \cap B_2| < k$ such that $A_1 \subseteq A_2$, then (A_2, B_2) must also lie in $\mathcal{F}$. This prevents ''losing'' a separation by expanding one side.

(H3) **Intersection Stability:** If $(A_1, B_1) \in \mathcal{F}$ and $(A_2, B_2) \in \mathcal{F}$, and $|(A_1 \cap A_2) \cap (B_1 \cup B_2)| < k$, then

$$(A_1 \cap A_2,\ B_1 \cup B_2) \ \in \ \mathcal{F}.$$

This condition ensures consistency when combining or intersecting separations chosen by $\mathcal{F}$.

(H4) **Nontriviality:** If $V(A) = V(H)$, then $(A, B) \in \mathcal{F}$. In other words, the entire vertex set cannot be separated off trivially, preserving a nonempty side in any chosen separation.

Example 3.2 (Simple H-Ultrafilter)**.** Consider a hypergraph H with vertex set $V(H) = \{1, 2, 3\}$ and hyperedges $E(H) = \{\{1, 2\}, \{2, 3\}\}$. Let $k = 2$.

A *separation* (A, B) of H can be viewed as two subhypergraphs $A = H[V_A]$ and $B = H[V_B]$ such that $|V_A \cap V_B| < k$. For instance,

$$(A, B) = (H[\{1, 2\}],\ H[\{2, 3\}])$$

has separator $\{2\}$. Since $|\{2\}| = 1 < 2$, the order is 1.

Define

$$\mathcal{F} = \big\{(H[\{1, 2\}],\ H[\{2, 3\}]), (H[\{1\}],\ H[\{1, 2, 3\}])\big\}.$$

One can check that $\mathcal{F}$ satisfies (H0)–(H4):

- (H0) Both separations have order 1.
- (H1) For any separation with order < 2, exactly one orientation is in $\mathcal{F}$.
- (H2) Expanding a set on one side retains membership in $\mathcal{F}$ if containment is preserved.
- (H3) Intersections of chosen separations remain in $\mathcal{F}$.
- (H4) No side is the entire vertex set in a trivial manner, ensuring nontriviality.

Hence, $\mathcal{F}$ forms an H-Ultrafilter of order 2 in this simple hypergraph.

The complementary equivalence between Hypertangles and HyperUltrafilters is demonstrated in the following theorem. This equivalence shows that, like Hypertangles, HyperUltrafilters can serve as obstructions to determining Hypertree-width. It is fascinating to see how Ultrafilters, a concept from set theory that seems unrelated at first glance, can be extended to hypergraphs and become a crucial obstruction.

Theorem 3.3 (Equivalence of Hypertangles and H-Ultrafilters)**.** *Let $H = (V(H), E(H))$ be a hypergraph. A set $\mathcal{T}$ is a* hypertangle of hyperorder $k + 1$ *in H if and only if*

$$\mathcal{F} \ = \ \big\{(X, Y) \ \big| \ (Y, X) \in \mathcal{T}\big\}$$

is an H-ultrafilter of order $k + 1$ *in H.*

Proof. We prove the two directions separately, showing how each family induces the other while satisfying all the respective conditions.

Forward Direction: Assume $\mathcal{T}$ is a hypertangle of hyperorder $k+1$. We claim $\mathcal{F} = \{ (X,Y) \mid (Y,X) \in \mathcal{T}\}$ is an H-ultrafilter of order $k+1$. We verify conditions (H0)–(H4) from Definition 3.1:

(H0) Bounded Order: If $(X,Y) \in \mathcal{F}$, then $(Y,X) \in \mathcal{T}$. Because $\mathcal{T}$ is a hypertangle of hyperorder $k+1$, the separation (Y,X) has order $< k+1$. Hence (X,Y) also has order $< k+1$. Thus, (H0) holds with $k+1$ replaced by k in the separation order.

(H1) Completeness: Let (X,Y) be a separation of order $< k+1$. By the hypertangle property, for any separation with order below $k+1$, exactly one orientation belongs to $\mathcal{T}$. Hence either $(X,Y) \in \mathcal{T}$ or $(Y,X) \in \mathcal{T}$, but not both. Thus, exactly one of (X,Y) or (Y,X) lies in $\mathcal{T}$. Translating to $\mathcal{F}$, we see exactly one of (X,Y) or (Y,X) lies in $\mathcal{F}$. This fulfills (H1).

(H2) Containment Monotonicity: Suppose $(X_1,Y_1) \in \mathcal{F}$, so $(Y_1,X_1) \in \mathcal{T}$. Let (X_2,Y_2) be another separation with order $< k+1$ and $X_1 \subseteq X_2$. In the hypertangle $\mathcal{T}$, expanding X_1 to X_2 shrinks Y_1 to Y_2. By the analogous containment property in hypertangles, $(Y_2,X_2) \in \mathcal{T}$. Hence $(X_2,Y_2) \in \mathcal{F}$. Condition (H2) is satisfied.

(H3) Intersection Stability: If $(X_1,Y_1),(X_2,Y_2) \in \mathcal{F}$, then $(Y_1,X_1),(Y_2,X_2) \in \mathcal{T}$. For the intersection or union separation $(X_1 \cap X_2,\ Y_1 \cup Y_2)$, the hypertangle property ensures that either that separation or its flip $(Y_1 \cup Y_2,\ X_1 \cap X_2)$ appears in $\mathcal{T}$. If $(Y_1 \cup Y_2,\ X_1 \cap X_2) \notin \mathcal{T}$, then $(X_1 \cap X_2,\ Y_1 \cup Y_2) \in \mathcal{T}$. Translating to $\mathcal{F}$, we get $(X_1 \cap X_2,\ Y_1 \cup Y_2) \in \mathcal{F}$. Thus, $\mathcal{F}$ meets (H3).

(H4) Nontriviality: Hypertangles by definition exclude trivial separations that cover the entire vertex set with one side. This ensures that in $\mathcal{F}$, we cannot have a separation (A,B) with $A = V(H)$ or $B = V(H)$ unless it is forced by the hypertangle's configuration. Condition (H4) is therefore inherited from the hypertangle nontriviality constraints.

Hence $\mathcal{F}$ is a valid H-ultrafilter of order $k+1$.

Backward Direction: Suppose $\mathcal{F}$ is an H-ultrafilter of order $k+1$. Define $\mathcal{T} = \{(Y,X) \mid (X,Y) \in \mathcal{F}\}$. We must show $\mathcal{T}$ is a hypertangle of hyperorder $k+1$. We confirm the hypertangle properties (HT0)–(HT1) and any additional requirements:

(HT0) Triple Intersection: Consider any three subsets $X_1, X_2, X_3 \in \mathcal{T}$. By the definition of $\mathcal{T}$, (X_1, X_2, X_3) arise from flips of separations in $\mathcal{F}$. The condition that either $X_1 \cap X_2 \cap X_3 \neq \emptyset$ or there is a hyperedge intersecting all three is precisely the guarantee that $\mathcal{F}$ cannot separate them in a trivial way. If no hyperedge intersects all three, we would form a contradictory separation in $\mathcal{F}$ that fails nontriviality. Thus (HT0) is satisfied.

(HT1) Hyperorder: Since $\mathcal{F}$ is an H-ultrafilter of order $k+1$, it picks one orientation for every separation of order $< k+1$. This implies that $\mathcal{T}$, being the reversed family, also has hyperorder $k+1$, ensuring that k is the smallest integer where a set $S \subseteq E(H)$ of size k meets every set in $\mathcal{T}$. Hence (HT1) holds.

By paralleling the arguments for (HT2)–(HT4) in the forward direction (adjusted for reversing each separation), one verifies that all hypertangle conditions are met. Consequently, $\mathcal{T}$ is indeed a hypertangle of hyperorder $k+1$. □

4 Additional Result: SuperHyperTree-width and SuperHyperPath-width

This section aims to contribute to advancing research in hypergraph theory and superhypergraph theory. Specifically, we explore SuperHyperTree-width and SuperHyperPath-width, along with obstructions—concepts that assist in determining their values.

4.1 SuperHyperTree-width

We intend to explore the concept of a SuperHyperGraph in the future. This SuperHyperGraph is a generalization of the traditional hypergraph and has been recently proposed [23, 26, 29, 43, 44, 65, 66, 68]. Like hypergraphs, it has attracted significant research interest. A brief definition is provided below.

Definition 4.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 4.2 (n-th Powerset)**.** (cf. [25, 27, 64, 68])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 4.3 (n-SuperHyperGraph [65, 66])**.** Let V_0 be a finite *base set* of vertices. For each integer $k \geq 0$, define

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big),$$

where $\mathcal{P}(\cdot)$ denotes the power set. An *n-SuperHyperGraph* is a pair

$$\mathrm{SHT}^{(n)} = (V, E),$$

such that

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}^n(V_0).$$

Each element of V is called an *n-supervertex*, and each element of E is called an *n-superedge*.

Remark 4.4. When $n = 1$, an n-SuperHyperGraph coincides with a classical hypergraph: each vertex and edge is simply a subset of V_0. For $n \geq 2$, the concept allows *nested* structures (e.g., sets of subsets), providing a broader and more flexible modeling framework than standard hypergraphs.

Definition 4.5 (n-SuperHyperPath)**.** (cf. [65, 66]) Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph. An *n-SuperHyperPath* is a special arrangement of its n-superedges $E_1, E_2, \ldots, E_m \in E$ such that:

1. For every $1 \leq i < m$, $E_i \cap E_{i+1} \neq \emptyset$. That is, consecutive n-superedges share at least one n-supervertex.
2. For any $1 \leq i < j \leq m$, if $x \in E_i \cap E_j$, then $x \in E_k$ for all $i \leq k \leq j$. In other words, if an n-supervertex appears in two edges E_i and E_j, it must also appear in every intermediate edge.

These conditions ensure that the sequence $E_1, E_2, \ldots, E_m$ forms a ''path-like'' structure in the n-SuperHyperGraph, analogous to a standard path in a graph or hypergraph.

Definition 4.6 (n-SuperHyperTree)**.** (cf. [35,65,66]) An *n-SuperHyperTree* is an n-SuperHyperGraph $\mathrm{SHT}^{(n)} = (V, E)$ with the following properties:

1. *Host Tree Existence:* There exists a (classical) tree $T = (V, E_T)$ on the same set of vertices V. We call T the *host tree*.
2. *Connected Subtree Condition:* Each n-superedge $e \in E$ corresponds to a connected subtree in T. Concretely, if e is viewed as a subset (or set of subsets) of V, then all vertices in e lie in a connected component of T. This applies even when e is a ''super-edge'' connecting more than two vertices in nested ways.

3. *Acyclicity:* Because T is a tree (i.e., acyclic), the n-SuperHyperTree $\mathrm{SHT}^{(n)}$ inherits this acyclic character, disallowing any cycle-like superedge configurations.

Key Properties of an n-SuperHyperTree:

- It is *connected*, in that any two vertices can be linked via a sequence of n-superedges forming an unbroken chain in the host tree.
- It has no ''super-cycles,'' preserving acyclicity in a higher-dimensional sense.
- It generalizes the notion of a tree to accommodate n-supervertices and n-superedges, yet retains a fundamentally tree-like structure.

We now extend the concept of *treewidth* from classical graphs to n-SuperHyperGraphs. The idea is to create a tree decomposition capable of handling n-superedges via carefully defined *bags* and *guards*.

Definition 4.7 (n-SuperHyperTree Decomposition and Width)**.** [24,28] Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph. An *n-SuperHyperTree decomposition* of $\mathrm{SHT}^{(n)}$ is a triple $(T, \mathcal{B}, C)$ where:

- $T = (V_T, E_T)$ is a (classical) tree.
- $\mathcal{B} = \{B_t \mid t \in V_T\}$ is a family of subsets of V, called *bags*, associated with each node $t \in V_T$. These bags must satisfy:
  1. *Coverage of n-SuperEdges:* For every n-superedge $e \in E$, there exists at least one node $t \in V_T$ such that $e \subseteq B_t$.
  2. *Vertex Connectivity:* For each vertex $v \in V$, the set of all nodes $\{\, t \in V_T \mid v \in B_t\}$ forms a connected subtree of T.
- $C = \{C_t \mid t \in V_T\}$ is a family of subsets of E, called *guards*, such that:
  1. *Guard Condition:* For each $t \in V_T$, we have $B_t \subseteq \bigcup C_t$, where $\bigcup C_t := \{\, v \in V \mid \exists e \in C_t,\ v \in e\}$.
  2. *Local Subtree Condition:* For each $t \in V_T$, define T_t as the subtree of T rooted at t. Then

     $$(\bigcup C_t) \cap \Big(\bigcup_{u \in V(T_t)} B_u \Big) \subseteq B_t.$$

     In other words, any vertex that belongs both to the union of the guards at t and to the union of bags in the subtree under t must already lie in B_t.

The *width* of an n-SuperHyperTree decomposition $(T, \mathcal{B}, C)$ is

$$\mathrm{width}(T, \mathcal{B}, C) = \max_{t \in V_T} |C_t|.$$

The *n-SuperHyperTree-width* of $\mathrm{SHT}^{(n)}$, denoted $\mathrm{shw}^{(n)}(\mathrm{SHT}^{(n)})$, is the minimum width among all n-SuperHyperTree decompositions of $\mathrm{SHT}^{(n)}$:

$$\mathrm{shw}^{(n)}(\mathrm{SHT}^{(n)}) = \min_{(T, \mathcal{B}, C)} \mathrm{width}(T, \mathcal{B}, C).$$

Remarks:

- If $\mathrm{SHT}^{(n)}$ is essentially a tree-like structure (an n-SuperHyperTree), then its n-SuperHyperTree-width is typically 1.
- In the classical (graph) case, $n = 1$, and the n-SuperHyperTree-width matches the standard treewidth of a graph.

- The concept of n-SuperHyperTree-width captures ''tree-likeness'' in higher-dimensional structures, extending well-known graph parameters to more intricate nested frameworks.

Remark 4.8 (n-SuperHyperPath decomposition)**.** A similar concept, called an *n-SuperHyperPath decomposition*, can be defined by requiring T to be a simple path rather than a general tree. In that context, the resulting $\mathrm{shw}^{(n)}$ can be viewed as an analogue of *pathwidth* for n-SuperHyperGraphs.

Theorem 4.9. *Let* $\mathrm{SHT}^{(n)} = (V, E)$ *be any n-SuperHyperGraph. Then its n-SuperHyperTree-width, denoted by* $\mathrm{nSHT\text{-}width}(\mathrm{SHT}^{(n)})$*, is at most its n-SuperHyperPath-width, denoted by* $\mathrm{nSHP\text{-}width}(\mathrm{SHT}^{(n)})$*. Formally,*

$$\mathrm{nSHT\text{-}width}\big(\mathrm{SHT}^{(n)}\big) \le \mathrm{nSHP\text{-}width}\big(\mathrm{SHT}^{(n)}\big). \tag{1}$$

Proof. We must show that for any n-SuperHyperGraph $\mathrm{SHT}^{(n)} = (V, E)$, the minimum width of a valid n-SuperHyperTree decomposition cannot exceed the minimum width of a valid n-SuperHyperPath decomposition.

Key observation: A *path* is a special type of *tree* in which each node has at most two neighbors. Therefore, any legitimate n-SuperHyperPath decomposition (P, χ_P, λ_P)—where P is a path—can be regarded as a *special case* of an n-SuperHyperTree decomposition, simply by viewing P itself as the underlying tree.

Construction: Let (P, χ_P, λ_P) be an *optimal* n-SuperHyperPath decomposition of $\mathrm{SHT}^{(n)}$. That is,

$$\mathrm{nSHP\text{-}width}\big(\mathrm{SHT}^{(n)}\big) \;=\; \mathrm{width}\big(P, \chi_P, \lambda_P\big).$$

Because P is a path and hence a (linear) tree, the same bags and guards χ_P and λ_P form a valid n-SuperHyperTree decomposition (with P serving as the tree). Therefore, the width of this tree-based decomposition is at most the width of the path decomposition:

$$\mathrm{nSHT\text{-}width}\big(\mathrm{SHT}^{(n)}\big) \;\le\; \mathrm{width}\big(P, \chi_P, \lambda_P\big) \;=\; \mathrm{nSHP\text{-}width}\big(\mathrm{SHT}^{(n)}\big).$$

Thus, the n-SuperHyperTree-width cannot exceed the n-SuperHyperPath-width, which completes the proof. □

4.2 Obstruction for SuperHyperTree-width

We outline potential research directions on *obstructions* that influence large n-SuperHyperTree-width or n-SuperHyperPath-width. In future work, we aim to generalize classical concepts such as *linkedness*, *brambles*, and *tangles* to the n-SuperHyperGraph setting. These concepts are inspired by their counterparts in traditional graph theory, namely linkedness [5, 6, 45], brambles [8, 12, 40, 49], and tangles [18, 19, 36, 41, 47, 53, 60], and are adapted to the broader framework of n-SuperHyperGraphs.

Definition 4.10 (n-SuperHyperlinkedness)**.** Let $\mathrm{SHT}^{(n)} = (G, E)$ be an n-SuperHyperGraph, where $G \subseteq \mathcal{P}^n(V_0)$ is the set of n-supervertices, and $E \subseteq \mathcal{P}^n(V_0)$ is the set of n-superedges. A subset $M \subseteq E$ is *n-superhyperlinked of order* $k+1$ if, for any subset $S \subseteq E$ with $|S| < k+1$, the partial n-SuperHyperGraph $\mathrm{SHT}^{(n)} \setminus S$ contains a connected component $C \subseteq G$ that is M*-big*, meaning

$$\big|\{\, e \in M \;\mid\; e \cap C \neq \emptyset\}\big| \;>\; \frac{|M|}{2}.$$

The *n-superhyperlinkedness* of $\mathrm{SHT}^{(n)}$ is the largest integer k for which $\mathrm{SHT}^{(n)}$ admits an n-superhyperlinked set of order $k+1$. This concept generalizes classical hyperlinkedness from ordinary hypergraphs to the n-dimensional superhyper framework.

Question 4.11. Does n-superhyperlinkedness control or bound the n-SuperHyperTree-width? Can very large n-superhyperlinkedness imply a higher n-SuperHyperTree-width?

Definition 4.12 (n-SuperHyperBramble)**.** Let $\mathrm{SHT}^{(n)} = (G, E)$ be an n-SuperHyperGraph. An *n-SuperHyperBramble of order* $k+1$ is a collection $\mathcal{B}$ of connected sub-sets of G (n-supervertices) satisfying:

(nSHB0) Any two distinct sets $X_1, X_2 \in \mathcal{B}$ *touch*, meaning $X_1 \cap X_2 \neq \emptyset$, or there exists an n-superedge $e \in E$ such that both $e \cap X_1 \neq \emptyset$ and $e \cap X_2 \neq \emptyset$.

(nSHB1) The *n-superhyperorder* of $\mathcal{B}$ is the smallest integer k for which there is a set $S \subseteq E$ with $|S| = k$ intersecting every $X \in \mathcal{B}$. Formally, for each $X \in \mathcal{B}$, $S \cap X \neq \emptyset$.

Question 4.13. Does the existence of a high-order n-SuperHyperBramble in $\mathrm{SHT}^{(n)}$ force large n-SuperHyperTree-width or n-SuperHyperPath-width? In classical graph theory, brambles are well-known obstructions to small treewidth. We conjecture an analogous phenomenon for n-SuperHyperGraphs.

Definition 4.14 (n-SuperHypertangle)**.** Let $\mathrm{SHT}^{(n)} = (G, E)$ be an n-SuperHyperGraph. An *n-SuperHypertangle of order* $k + 1$ is an n-SuperHyperBramble $\mathcal{T} \subseteq 2^G$ that further satisfies:

(nSHT0) For any three distinct sets $X_1, X_2, X_3 \in \mathcal{T}$, either $X_1 \cap X_2 \cap X_3 \neq \emptyset$ or there is some n-superedge $e \in E$ intersecting all three, i.e. $e \cap X_i \neq \emptyset$ for $i = 1, 2, 3$.

(nSHT1) The *n-superhyperorder* of $\mathcal{T}$ is again the smallest integer k such that there is a set $S \subseteq E$ with $|S| = k$ intersecting every set in $\mathcal{T}$.

Question 4.15. Does a large n-SuperHypertangle necessarily indicate large n-SuperHyperTree-width? In classical theory, tangles are strong obstructions to treewidth. We suspect a similar role in the multi-level n-super setting.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 3
Superhypertree-Length and Superhypertree-Breadth in SuperHyperGraphs

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Abstract

A Hypergraph is a generalization of a graph where edges, known as hyperedges, can connect multiple vertices simultaneously [8]. A SuperHyperGraph is a recursive extension of hypergraphs in which vertices and edges can represent hierarchically nested subsets [62, 63]. This paper explores whether the graph parameters Tree-length and Tree-breadth, well-known in graph theory, can be extended to Hypergraphs and SuperHyperGraphs. Additionally, the relationships between these parameters and their graph counterparts are analyzed.

Keywords: Tree-length, Tree-Breadth, Superhypergraph, Hypergraph, Hypertree

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

1.1 Hypergraph and SuperHyperGraph

A hypergraph is a generalized graph concept that extends traditional graph theory by introducing hyperedges, which can connect multiple vertices instead of just pairs. This allows for modeling more complex relationships among elements [5, 6, 8, 36–38]. Hypergraphs have found applications in various fields, including database systems [44]. The fundamental definitions of graphs and hypergraphs are provided below. In this paper, we consider undirected, finite, and simple graphs.

Definition 1.1 (Graph)**.** [11] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 1.2 (Subgraph)**.** [11] Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of G is a graph such that:

- $V_H \subseteq V$, i.e., the vertex set of H is a subset of the vertex set of G.
- $E_H \subseteq E$, i.e., the edge set of H is a subset of the edge set of G.
- Each edge in E_H connects vertices in V_H.

Definition 1.3 (Hypergraph)**.** [8] A *hypergraph* is a pair $H = (V, E)$, where:

- V is a set of *vertices*,
- E is a set of *hyperedges*, each hyperedge $e \in E$ being a subset of V.

Equivalently, $E \subseteq \mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the power set of V.

Example 1.4 (Concrete Hypergraph)**.** Let

$$V = \{v_1, v_2, v_3, v_4\}$$

be a set of four vertices. Suppose we define the hyperedges as:

$$E = \big\{\{v_1, v_2, v_3\},\ \{v_2, v_4\}\big\}.$$

Then $H = (V, E)$ is a hypergraph with two hyperedges:

- $e_1 = \{v_1, v_2, v_3\}$ connects three vertices simultaneously,
- $e_2 = \{v_2, v_4\}$ connects a different subset of vertices.

Note that both $\{v_1, v_2, v_3\}$ and $\{v_2, v_4\}$ are indeed subsets of V. This illustrates how hyperedges can incorporate more than two vertices, unlike standard graphs.

A SuperHyperGraph is an extension of the concept of a hypergraph, recently defined and actively studied in the literature [3, 9, 23, 25, 26, 40, 41, 48, 50, 51, 62–64]. It can be understood as a graph concept that incorporates recursive structures into hypergraphs. A SuperHyperGraph possesses a repeated structure called the n-th powerset, which is generated iteratively through the power set operation. The formal definition is provided below.

Definition 1.5 (n-th Powerset)**.** (cf. [19, 20, 61, 65])

The n-th powerset of a set H, denoted $P_n(H)$, is constructed iteratively. Beginning with the standard powerset, the process is defined as:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

In a similar manner, the n-th non-empty powerset, represented as $P_n^*(H)$, is recursively defined as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ refers to the powerset of H excluding the empty set.

Example 1.6 (Constructing the n-th Powerset)**.** Let $H = \{a, b\}$ be a small base set.

Step 1: Compute the standard powerset $P(H)$:

$$P(H) = \{\, \emptyset,\ \{a\},\ \{b\},\ \{a, b\}\}.$$

Hence $P_1(H) = P(H)$ is

$$P_1(H) = \{\emptyset,\ \{a\},\ \{b\},\ \{a, b\}\}.$$

Step 2: Compute $P_2(H) = P(P_1(H))$ by taking the powerset of the set above:

$$P_2(H) = \mathcal{P}\big(\{\emptyset, \{a\}, \{b\}, \{a, b\}\}\big).$$

Each element of $P_2(H)$ is a subset of $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. For instance, $\{\emptyset, \{b\}\}$ is one such subset, and $\{\{a\}, \{a, b\}\}$ is another. In total, there are $2^4 = 16$ subsets, so

$$\big|P_2(H)\big| = 16.$$

Optional Higher Iterations: For $n \geq 3$, we continue iteratively:

$$P_{n+1}(H) = P\big(P_n(H)\big).$$

Thus, one can construct $P_3(H)$, $P_4(H)$, etc. by repeatedly taking powersets of the previous stage.

Definition 1.7 (n-SuperHyperGraph)**.** [62, 63] Let V_0 be a finite base set of *vertices*, and for each $k \geq 0$, define $\mathcal{P}^k(V_0)$ as follows:

$$\mathcal{P}^0(V_0) \ = \ V_0, \quad \mathcal{P}^{k+1}(V_0) \ = \ \mathcal{P}\big(\mathcal{P}^k(V_0)\big),$$

where $\mathcal{P}(\cdot)$ denotes the power set. An *n-SuperHyperGraph* is a pair

$$\mathrm{SHT}^{(n)} \ = \ (V, E),$$

where

$$V \ \subseteq \ \mathcal{P}^n(V_0), \quad E \ \subseteq \ \mathcal{P}^n(V_0).$$

Each element of V is called an *n-supervertex*, and each element of E is called an *n-superedge*.

Example 1.8 (Constructing an n-SuperHyperGraph)**.** Let $V_0 = \{x, y\}$ be a base set of vertices, and consider $n = 2$ for concreteness. We have:

$$\mathcal{P}^0(V_0) = V_0 = \{x, y\},$$
$$\mathcal{P}^1(V_0) = \mathcal{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\},$$
$$\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}^1(V_0)\big) = \mathcal{P}\big(\{\emptyset, \{x\}, \{y\}, \{x, y\}\}\big),$$

which has $2^4 = 16$ elements.

Forming a 2-SuperHyperGraph. Choose a subset of $\mathcal{P}^2(V_0)$ to be the 2-supervertices V, and another subset to be the 2-superedges E. For instance, we might define:

$$V = \big\{\{\,x\,\}, \{\,y\,\}, \{\,x, y\,\}\big\} \quad \text{(each an element of } \mathcal{P}^1(V_0) \subset \mathcal{P}^2(V_0)\text{)},$$

$$E = \Big\{\{\{x\}, \{y\}\},\ \{\{x\}, \{x, y\}\}\Big\} \ \subseteq\ \mathcal{P}^2(V_0).$$

Then

$$\mathrm{SHT}^{(2)} = (V,\ E)$$

is a valid 2-SuperHyperGraph:

- Each 2-supervertex belongs to $\mathcal{P}^2(V_0)$.
- Each 2-superedge is a subset of $\mathcal{P}^2(V_0)$ containing multiple 2-supervertices.

This construction can be extended to larger n by choosing appropriate subsets of $\mathcal{P}^n(V_0)$ to serve as V (the n-supervertices) and E (the n-superedges).

1.2 Tree-length and Tree-breadth

The *tree-length* of a graph is defined as the maximum shortest path distance between any two vertices within a bag of a tree-decomposition [7, 10, 12, 13, 15]. The *tree-breadth* of a graph is defined as the minimum radius required to cover each bag of a tree-decomposition from a central vertex [14, 16, 17, 46, 47]. The detailed definitions of each parameter are provided below.

Definition 1.9 (Tree-width)**.** [52–60] Let $G = (V, E)$ be a graph. A *tree-decomposition* of G is a pair $(T, \{X_t \mid t \in V(T)\})$, where:

- $T = (V(T), E(T))$ is a tree,
- $X_t \subseteq V$ for each $t \in V(T)$ (called *bags*),

such that:

1. $\bigcup_{t \in V(T)} X_t = V$, i.e., every vertex of G appears in at least one bag.
2. For every edge $\{u, v\} \in E$, there exists $t \in V(T)$ such that $u, v \in X_t$, ensuring edge coverage.
3. For all $t_1, t_2, t_3 \in V(T)$, if t_2 lies on the path between t_1 and t_3 in T, then $X_{t_1} \cap X_{t_3} \subseteq X_{t_2}$, ensuring connectivity.

The *width* of a tree-decomposition is defined as:

$$\mathrm{width}(T, \{X_t\}) = \max_{t \in V(T)} \big(|X_t| - 1\big),$$

where $|X_t|$ is the number of vertices in the bag X_t. The *tree-width* of G, denoted $\mathrm{tw}(G)$, is the minimum width over all possible tree-decompositions of G:

$$\mathrm{tw}(G) = \min_{(T, \{X_t\})} \mathrm{width}(T, \{X_t\}).$$

Definition 1.10 (Tree-length)**.** [12,15,59] A *tree-decomposition* of a graph $G = (V, E)$ is a pair $T(G) = (\{X_i \mid i \in I\}, T)$, where $\{X_i \mid i \in I\}$ is a collection of subsets of V (called *bags*), and $T = (I, F)$ is a tree such that:

- $\bigcup_{i \in I} X_i = V$,
- For each edge $uv \in E$, there exists $i \in I$ such that $\{u, v\} \subseteq X_i$,
- For all $i, j, k \in I$, if j lies on the path between i and k in T, then $X_i \cap X_k \subseteq X_j$.

The *length* of a tree-decomposition $T(G)$ is defined as:

$$\lambda := \max_{i \in I} \max_{u, v \in X_i} d_G(u, v),$$

where $d_G(u, v)$ is the shortest path distance between u and v in G. The *tree-length* of G, denoted by $\mathrm{tl}(G)$, is the minimum λ over all possible tree-decompositions of G.

Example 1.11 (Tree-length)**.** Consider the path graph P_4 with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edge set $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}$. A possible tree-decomposition $T(P_4)$ is given by:

$$X_1 = \{v_1, v_2\}, \quad X_2 = \{v_2, v_3\}, \quad X_3 = \{v_3, v_4\},$$

with a host tree T on nodes $I = \{1, 2, 3\}$ and edges $\{(1, 2), (2, 3)\}$.

To determine the *length* of $T(P_4)$, we examine each bag:

- For $X_1 = \{v_1, v_2\}$, the maximum distance between any two vertices in X_1 is $d_{P_4}(v_1, v_2) = 1$.
- For $X_2 = \{v_2, v_3\}$, the maximum distance between any two vertices in X_2 is $d_{P_4}(v_2, v_3) = 1$.
- For $X_3 = \{v_3, v_4\}$, the maximum distance between any two vertices in X_3 is $d_{P_4}(v_3, v_4) = 1$.

Hence, $\lambda = 1$. Because this decomposition is already optimal for P_4, the *tree-length* $\mathrm{tl}(P_4)$ is 1.

Definition 1.12 (Tree-breadth)**.** [14, 16, 46] The *breadth* of a tree-decomposition $T(G) = (\{X_i \mid i \in I\}, T)$ is the smallest integer r such that for each bag X_i ($i \in I$), there exists a vertex $v_i \in V$ such that:

$$X_i \subseteq D_r(v_i, G),$$

where $D_r(v_i, G) = \{u \in V \mid d_G(u, v_i) \le r\}$ is the disk of radius r centered at v_i. The *tree-breadth* of G, denoted by $\mathrm{tb}(G)$, is the minimum r over all possible tree-decompositions of G.

Example 1.13 (Tree-breadth)**.** Using the same path graph P_4 and the same tree-decomposition $T(P_4)$ from above:

$$X_1 = \{v_1, v_2\}, \quad X_2 = \{v_2, v_3\}, \quad X_3 = \{v_3, v_4\}.$$

To determine the *breadth*, we look for the smallest integer r such that each bag X_i is contained in a closed ball $D_r(w_i)$ around some center $w_i \in V$:

- For $X_1 = \{v_1, v_2\}$, we can choose $w_1 = v_1$ (or v_2), and then $X_1 \subseteq D_1(v_1)$, because v_2 is at distance 1 from v_1.
- For $X_2 = \{v_2, v_3\}$, we can choose $w_2 = v_2$ (or v_3), ensuring $X_2 \subseteq D_1(v_2)$.
- For $X_3 = \{v_3, v_4\}$, we can choose $w_3 = v_4$ (or v_3), hence $X_3 \subseteq D_1(v_4)$.

In all cases, $r = 1$ suffices. Therefore, the breadth of $T(P_4)$ is 1, and the *tree-breadth* $\mathrm{tb}(P_4)$ is also 1 for this particular graph and decomposition.

Remark 1.14. [14] For any graph G, the following relationship holds:

$$1 \le \mathrm{tb}(G) \le \mathrm{tl}(G) \le 2 \cdot \mathrm{tb}(G).$$

1.3 SuperHyperTree Decomposition

A *Hypertree Decomposition* is a tree-decomposition of a hypergraph that includes additional guards ensuring the coverage of hyperedges and maintaining structural connectivity [1, 2, 33, 35, 37]. A *SuperHyperTree Decomposition* is a tree-decomposition of a SuperHyperGraph, designed to cover hierarchical superedges and preserve the connectivity of subsets of vertices [18, 24, 26, 32].

Definition 1.15 (Generalized Hypertree Decomposition)**.** [2] A *generalized hypertree decomposition* of a hypergraph $H = (V(H), E(H))$ is a triple (T, B, C), where:

- (T, B) is a tree-decomposition of H, and
- $C = \{C_t \mid t \in V(T)\}$ is a family of subsets of $E(H)$ called the *guards*.

For every $t \in V(T)$, the bag B_t satisfies:

$$B_t \subseteq \bigcup C_t,$$

where $\bigcup C_t$ is the union of all hyperedges in C_t, i.e.,

$$\bigcup C_t = \{v \in V(H) \mid \exists e \in C_t : v \in e\}.$$

The *width* of the decomposition (T, B, C) is defined as:

$$\mathrm{width}(T, B, C) = \max_{t \in V(T)} |C_t|,$$

where $|C_t|$ denotes the cardinality of the guard C_t.

The *generalized hypertree width* of H, denoted $\mathrm{ghw}(H)$, is the minimum width over all possible generalized hypertree decompositions of H:

$$\mathrm{ghw}(H) = \min_{(T,B,C)} \mathrm{width}(T, B, C).$$

Definition 1.16 (Hypertree Decomposition)**.** [2] A *hypertree decomposition* of a hypergraph $H = (V(H), E(H))$ is a generalized hypertree decomposition (T, B, C) that satisfies the following additional condition:

$$(\bigcup C_t) \cap (\bigcup_{u \in V(T_t)} B_u) \subseteq B_t,$$

for all $t \in V(T)$. Here, T_t denotes the subtree of T rooted at t.

The *hypertree width* of H, denoted $\mathrm{hw}(H)$, is the minimum width over all possible hypertree decompositions of H:

$$\mathrm{hw}(H) = \min_{(T,B,C)} \mathrm{width}(T, B, C).$$

Definition 1.17 (n-SuperHyperTree Decomposition)**.** [18,24,26,32] Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph. An *n-SuperHyperTree Decomposition* of $\mathrm{SHT}^{(n)}$ is a triple $(T, \mathcal{B}, C)$ where:

- $T = (V_T, E_T)$ is a tree.
- $\mathcal{B} = \{ B_t \mid t \in V_T\}$ is a family of subsets of V (called *bags*), such that:
  1. *Coverage Condition for n-SuperEdges:* For every n-superedge $e \in E$, there exists a node $t \in V_T$ with $e \subseteq B_t$.
  2. *Vertex Connectivity Condition:* For each n-supervertex $v \in V$, the set $\{ t \in V_T \mid v \in B_t\}$ forms a connected subtree of T.
- $C = \{ C_t \mid t \in V_T\}$ is a family of subsets of E (called *guards*), such that:

1. *Guard Condition for n-SuperEdges:* For each $t \in V_T$, we have

$$B_t \subseteq \bigcup C_t,$$

where

$$\bigcup C_t = \{ v \in V \mid \exists e \in C_t : v \in e \}.$$

2. *n-SuperHyperTree Condition:* For each $t \in V_T$, let T_t denote the subtree of T rooted at t. Then

$$\Big(\bigcup C_t\Big) \cap \Big(\bigcup_{u \in V(T_t)} B_u \Big) \subseteq B_t.$$

Width of an n-SuperHyperTree Decomposition: The width of $(T, \mathcal{B}, C)$ is

$$\mathrm{width}(T, \mathcal{B}, C) = \max_{t \in V_T} |C_t|.$$

n-SuperHyperTree-width: The *n-SuperHyperTree-width* of $\mathrm{SHT}^{(n)}$ is

$$\text{SHT-width}\big(\mathrm{SHT}^{(n)}\big) = \min_{(T,\mathcal{B},C)} \mathrm{width}(T, \mathcal{B}, C).$$

A smaller width indicates that $\mathrm{SHT}^{(n)}$ is "closer" in structure to a tree.

2 Result in This Paper

As a result of this paper, we define Hypertree-length and Hypertree-breadth, Superhypertree-length, and Superhypertree-breadth, and describe the relationships between these parameters.

2.1 Hypertree-length and Hypertree-breadth

Hypertree-length refers to the maximum distance between any two vertices within a bag in a hypertree decomposition. Hypertree-breadth represents the minimum radius needed to cover each bag of a hypertree decomposition from a central vertex.

Definition 2.1 (Primal graph of a hypergraph)**.** Let $H = (V, E)$ be a hypergraph. The *primal graph* $G(H)$ has the same vertex set V as H. Two distinct vertices u, v are adjacent in $G(H)$ if and only if there is some hyperedge $e \in E$ of H with $u, v \in e$. We denote the distance in $G(H)$ by $d_H(u, v) := d_{G(H)}(u, v)$.

Definition 2.2 (Hypertree-length)**.** Let $H = (V, E)$ be a hypergraph. A *(generalized) hypertree decomposition* of H is a triple

$$\big(T, \{B_t\}_{t \in V(T)}, \{C_t\}_{t \in V(T)}\big),$$

where $T = (V(T), E(T))$ is a tree, $B_t \subseteq V$, and $C_t \subseteq E$ satisfy the usual coverage and connectivity conditions. Define the *length* λ of this hypertree decomposition by

$$\lambda := \max_{t \in V(T)} \max_{u,v \in B_t} d_H(u, v),$$

where $d_H(u, v)$ is distance in the primal graph $G(H)$.

The *Hypertree-length* $\mathrm{HTl}(H)$ is the minimum possible λ over all hypertree decompositions of H:

$$\mathrm{HTl}(H) = \min_{(T,\{B_t\},\{C_t\})} \max_{t \in V(T)} \max_{u,v \in B_t} d_H(u, v).$$

Example 2.3 (Hypertree-length)**.** *Hypergraph Definition.* Let $H = (V, E)$ be a hypergraph with

$$V = \{v_1, v_2, v_3, v_4\}, \quad E = \big\{\{v_1, v_2, v_3\}, \{v_2, v_4\}\big\}.$$

The *primal graph* $G(H)$ is constructed by linking every pair of vertices that appear in a common hyperedge. Therefore:

$$E_{G(H)} = \{ (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4)\}.$$

Hypertree Decomposition. Consider the following hypertree decomposition $(T, \{B_t\}, \{C_t\})$:

- Let T be a tree with two nodes, t_1 and t_2, and one edge (t_1, t_2).
- Assign the *bags* as:
$$B_{t_1} = \{v_1, v_2, v_3\}, \quad B_{t_2} = \{v_2, v_4\}.$$
- For guards $\{C_t\}$, one possible assignment could be:
$$C_{t_1} = \{\{v_1, v_2, v_3\}\}, \quad C_{t_2} = \{\{v_2, v_4\}\}.$$
- Each hyperedge is fully contained in some bag (coverage), and each vertex appears in a connected subtree of T (connectivity).

Computing Hypertree-length. We look at each bag and measure the maximum distance in $G(H)$ between any two vertices in that bag:

- In $B_{t_1} = \{v_1, v_2, v_3\}$, the edges (v_1, v_2), (v_1, v_3), and (v_2, v_3) exist in $G(H)$, so any pair is at distance 1.
- In $B_{t_2} = \{v_2, v_4\}$, the edge (v_2, v_4) is in $G(H)$, hence $d_H(v_2, v_4) = 1$.

Therefore, the length for this decomposition is

$$\lambda = \max\left\{ \max_{u,v \in B_{t_1}} d_H(u, v), \max_{u,v \in B_{t_2}} d_H(u, v) \right\} = 1.$$

If we attempt other decompositions, we find we cannot do better than $\lambda = 1$. Thus, the *Hypertree-length* of H, $\mathrm{HTl}(H)$, is 1.

Definition 2.4 (Hypertree-breadth)**.** Let $H = (V, E)$ be a hypergraph, and let $(T, \{B_t\}_{t \in V(T)}, \{C_t\}_{t \in V(T)})$ be a hypertree decomposition. For each $t \in V(T)$, let r_t be the minimum integer such that $B_t \subseteq D_{r_t}(w_t)$ for some $w_t \in V$, where
$$D_{r_t}(w_t) \;=\; \{x \in V \mid d_H(x, w_t) \le r_t\}.$$
Then the *breadth* of this decomposition is
$$r \;=\; \max_{t \in V(T)} r_t.$$
The *Hypertree-breadth* $\mathrm{HTb}(H)$ is the minimum r over all hypertree decompositions of H:

$$\mathrm{HTb}(H) \;=\; \min_{(T,\{B_t\},\{C_t\})} \max_{t \in V(T)} \min_{w_t \in V} \max_{x \in B_t} d_H(x, w_t).$$

Example 2.5 (Hypertree-length)**.** *Hypergraph Definition.* Let $H = (V, E)$ be a hypergraph with

$$V = \{v_1, v_2, v_3, v_4\}, \quad E = \Big\{\{v_1, v_2, v_3\},\ \{v_2, v_4\}\Big\}.$$

The *primal graph* $G(H)$ is constructed by linking every pair of vertices that appear in a common hyperedge. Therefore:
$$E_{G(H)} = \{\, (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4)\}.$$

Hypertree Decomposition. Consider the following hypertree decomposition $(T, \{B_t\}, \{C_t\})$:

- Let T be a tree with two nodes, t_1 and t_2, and one edge (t_1, t_2).
- Assign the *bags* as:
$$B_{t_1} = \{v_1, v_2, v_3\}, \quad B_{t_2} = \{v_2, v_4\}.$$
- For guards $\{C_t\}$, one possible assignment could be:
$$C_{t_1} = \{\{v_1, v_2, v_3\}\}, \quad C_{t_2} = \{\{v_2, v_4\}\}.$$
- Each hyperedge is fully contained in some bag (coverage), and each vertex appears in a connected subtree of T (connectivity).

Computing Hypertree-length. We look at each bag and measure the maximum distance in $G(H)$ between any two vertices in that bag:

- In $B_{t_1} = \{v_1, v_2, v_3\}$, the edges (v_1, v_2), (v_1, v_3), and (v_2, v_3) exist in $G(H)$, so any pair is at distance 1.
- In $B_{t_2} = \{v_2, v_4\}$, the edge (v_2, v_4) is in $G(H)$, hence $d_H(v_2, v_4) = 1$.

Therefore, the length for this decomposition is

$$\lambda = \max\Big\{ \max_{u,v\in B_{t_1}} d_H(u,v),\ \max_{u,v\in B_{t_2}} d_H(u,v) \Big\} = 1.$$

If we attempt other decompositions, we find we cannot do better than $\lambda = 1$. Thus, the *Hypertree-length* of H, $\mathrm{HTl}(H)$, is 1.

Theorem 2.6. *For any hypergraph $H = (V, E)$,*

$$1 \ \leq\ \mathrm{HTb}(H) \ \leq\ \mathrm{HTl}(H) \ \leq\ 2\,\mathrm{HTb}(H).$$

Proof. Step 1: $\mathrm{HTb}(H) \geq 1$.

A hypergraph containing at least one edge with two or more distinct vertices cannot have a hypertree decomposition with breadth 0, because a radius of 0 would force all vertices in each bag to coincide with the center. Thus, for any nontrivial hypergraph H, we must have $\mathrm{HTb}(H) \geq 1$.

Step 2: $\mathrm{HTb}(H) \ \leq\ \mathrm{HTl}(H)$.

By definition, $\mathrm{HTl}(H)$ is the minimal *maximum* distance between two vertices in the same bag, over all hypertree decompositions. Meanwhile, $\mathrm{HTb}(H)$ is the minimal *maximum* radius needed to cover each bag from a single center. If a bag has diameter ℓ, then a single center within that bag can cover it with radius at most ℓ. Hence any decomposition achieving length ℓ has breadth at most ℓ. Minimizing over all decompositions shows $\mathrm{HTb}(H) \leq \mathrm{HTl}(H)$.

Step 3: $\mathrm{HTl}(H) \ \leq\ 2\,\mathrm{HTb}(H)$.

Let $\delta := \mathrm{HTb}(H)$. By definition, there exists a hypertree decomposition with breadth at most δ. This means that for each bag B_t, we can choose a *center* vertex $w_t \in V$ such that

$$B_t \ \subseteq\ D_\delta(w_t),$$

where $D_\delta(w_t)$ is the ball of radius δ around w_t in the primal graph $G(H)$. Now *inflate* each bag B_t to be exactly $D_\delta(w_t)$. This does not increase the decomposition's guard size (and thus remains a valid generalized hypertree decomposition), but ensures that any two vertices u, v in the same inflated bag satisfy

$$d_H(u,v) \ \leq\ 2\delta.$$

Hence the length of this new decomposition is at most 2δ. Because δ was the minimal breadth, it follows that

$$\mathrm{HTl}(H) \ \leq\ 2\,\mathrm{HTb}(H).$$

Combining all three inequalities completes the proof:

$$1 \ \leq\ \mathrm{HTb}(H) \ \leq\ \mathrm{HTl}(H) \ \leq\ 2\,\mathrm{HTb}(H).$$

□

Theorem 2.7. *Let $G = (V, E)$ be a simple graph, and let $H = (V(H), E(H))$ be the associated hypergraph described above. Then*

$$\mathrm{tl}(G) \ =\ \mathrm{HTl}(H).$$

Proof. Step 1: Show $\mathrm{HTl}(H) \;\leq\; \mathrm{tl}(G)$.

Consider any tree-decomposition $\big(T(G), \{X_i\}\big)$ of G with length λ. Since H has the same vertex set as G and edges of size 2, we can interpret $\{X_i\}$ as a *(generalized) hypertree decomposition* of H by letting $B_i = X_i$ and assigning guards

$$C_i = \left\{ \{u, v\} \in E(H) \;\middle|\; \{u, v\} \subseteq B_i \right\}.$$

Coverage and connectivity hold because they hold for the underlying tree-decomposition of G. In the primal graph $G(H) \cong G$, the maximum distance within each bag is at most λ. Minimizing over all tree-decompositions of G yields $\mathrm{HTl}(H) \;\leq\; \mathrm{tl}(G)$.

Step 2: Show $\mathrm{tl}(G) \;\leq\; \mathrm{HTl}(H)$.

Conversely, any hypertree decomposition $\big(T, \{B_t\}, \{C_t\}\big)$ of H is also a valid tree-decomposition of G, since each hyperedge $\{u, v\}$ with $|\{u, v\}| = 2$ must lie completely in some bag. Moreover, in the primal graph $G(H)$, the distance $d_{G(H)}(u, v)$ equals $d_G(u, v)$. Hence the length of any hypertree decomposition of H is at least the minimal length required among *all* tree-decompositions of G. Formally, if μ is the hypertree-length of H, then μ is also the maximum distance within some valid tree-decomposition for G. Therefore, $\mathrm{tl}(G) \;\leq\; \mu = \mathrm{HTl}(H)$.

Combining both parts yields $\mathrm{tl}(G) = \mathrm{HTl}(H)$. □

Theorem 2.8. *Under the same construction, let* $\mathrm{tb}(G)$ *be the tree-breadth of* G *and* $\mathrm{HTb}(H)$ *the hypertree-breadth of* H*. Then*

$$\mathrm{tb}(G) \;=\; \mathrm{HTb}(H).$$

Proof. Step 1: $\mathrm{HTb}(H) \;\leq\; \mathrm{tb}(G)$.

Given a tree-decomposition for G that realizes $\mathrm{tb}(G)$, form a (generalized) hypertree decomposition of H by the same assignment of bags and a guard set

$$C_i = \{\{u, v\} \in E(H) \mid u, v \in B_i\}.$$

In each bag, the minimum radius required (in the primal graph $G(H)$) to cover that bag is at most the breadth used in the original tree-decomposition. Because $\mathrm{tb}(G)$ is minimal among all such decompositions, $\mathrm{HTb}(H)$, which is the minimal breadth of a hypertree decomposition of H, can be no larger.

Step 2: $\mathrm{tb}(G) \;\leq\; \mathrm{HTb}(H)$.

Conversely, any hypertree decomposition of H is also a tree-decomposition of G. The radius needed to cover each bag in $G(H)$ is the same as the radius needed to cover each bag in G, since $G(H) \cong G$. Hence if a certain decomposition of H has a certain breadth, that breadth also applies to a tree-decomposition of G. Minimizing over all hypertree decompositions of H shows that

$$\mathrm{tb}(G) \;\leq\; \mathrm{HTb}(H).$$

Putting these two inequalities together completes the proof:

$$\mathrm{tb}(G) \;=\; \mathrm{HTb}(H).$$

□

2.2 Superhypertree-length and Superhypertree-breadth

The *n-Superhypertree-length* refers to the maximum distance between any two vertices within a bag in an n-SuperHyperTree decomposition. The *n-Superhypertree-breadth* represents the minimum radius required to cover each bag of an n-SuperHyperTree decomposition from a central vertex.

Definition 2.9 (n-Superprimal Graph and Distance)**.** Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph. Define its *n-superprimal graph*, $G\big(\mathrm{SHT}^{(n)}\big)$, as follows:

$$G\big(\mathrm{SHT}^{(n)}\big) \;=\; \big(V,\; E_{G(\mathrm{SHT}^{(n)})}\big),$$

where $\{u, v\} \in E_{G(\mathrm{SHT}^{(n)})}$ if and only if there is an n-superedge $e \in E$ with $u, v \in e$ and $u \neq v$.

We denote

$$d_{\mathrm{SHT}^{(n)}}(u, v) \;:=\; d_{\,G(\mathrm{SHT}^{(n)})}(u, v),$$

i.e. the usual shortest-path distance of u, v in the n-superprimal graph $G(\mathrm{SHT}^{(n)})$.

Definition 2.10 (n-Superhypertree-length)**.** Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph, and let $(T, \{\mathcal{B}_t\}, \{C_t\})$ be an n-SuperHyperTree Decomposition. The *length* of this decomposition, denoted λ, is

$$\lambda \;:=\; \max_{t \in V_T} \; \max_{u,v \in \mathcal{B}_t} \; d_{\mathrm{SHT}^{(n)}}(u, v).$$

The *n-Superhypertree-length* of $\mathrm{SHT}^{(n)}$, denoted $\mathrm{SHTl}^{(n)}\big(\mathrm{SHT}^{(n)}\big)$, is the minimum such λ over all n-SuperHyperTree Decompositions of $\mathrm{SHT}^{(n)}$:

$$\mathrm{SHTl}^{(n)}\big(\mathrm{SHT}^{(n)}\big) \;:=\; \min_{(T,\{\mathcal{B}_t\},\{C_t\})} \; \max_{t \in V_T} \; \max_{u,v \in \mathcal{B}_t} \; d_{\mathrm{SHT}^{(n)}}(u, v).$$

Definition 2.11 (n-Superhypertree-breadth)**.** Let $\mathrm{SHT}^{(n)} = (V, E)$ be an n-SuperHyperGraph, and let $(T, \{\mathcal{B}_t\}, \{C_t\})$ be an n-SuperHyperTree Decomposition. For each node $t \in V_T$, let r_t be the smallest integer such that there exists a *center* $w_t \in V$ with

$$\mathcal{B}_t \;\subseteq\; \big\{x \in V \mid d_{\mathrm{SHT}^{(n)}}(x, w_t) \leq r_t\big\}.$$

Define the *breadth* of this decomposition by

$$r \;:=\; \max_{t \in V_T} \; r_t.$$

The *n-Superhypertree-breadth*, denoted $\mathrm{SHTb}^{(n)}\big(\mathrm{SHT}^{(n)}\big)$, is the minimum value of r over all n-SuperHyperTree Decompositions:

$$\mathrm{SHTb}^{(n)}\big(\mathrm{SHT}^{(n)}\big) \;:=\; \min_{(T,\{\mathcal{B}_t\},\{C_t\})} \; \max_{t \in V_T} \; \min_{w_t \in V} \; \max_{x \in \mathcal{B}_t} \; d_{\mathrm{SHT}^{(n)}}(x, w_t).$$

Example 2.12. *Step 1: Constructing a 2-SuperHyperGraph.*

Let $V_0 = \{x, y\}$ be the base vertex set. Then

$$\mathcal{P}(V_0) = \{\, \emptyset,\ \{x\},\ \{y\},\ \{x, y\}\},$$

and

$$\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big)$$

is the collection of all subsets of $\{\emptyset, \{x\}, \{y\}, \{x, y\}\}$, which has $2^4 = 16$ elements.

However, we only select a small subset of these 16 possible 2-supervertices and 2-superedges to form a 2-SuperHyperGraph:

$$\mathrm{SHT}^{(2)} = (V, E),$$

where

$$V = \big\{\, \{x\},\ \{y\},\ \{x, y\}\big\}, \quad E = \Big\{\, \{\{x\}, \{y\}\},\ \{\{x\}, \{x, y\}\}\Big\}.$$

In words:

- We treat $\{x\}, \{y\}, \{x, y\}$ as 2-supervertices.
- We have two 2-superedges:
$$e_1 = \{\{x\}, \{y\}\}, \quad e_2 = \{\{x\}, \{x, y\}\}.$$

Step 2: Constructing the 2-Superprimal Graph and Distances.

By definition, the *2-superprimal graph* $G(\mathrm{SHT}^{(2)})$ has the same vertex set

$$V = \{\{x\}, \{y\}, \{x, y\}\}.$$

Two 2-supervertices are adjacent in $G(\mathrm{SHT}^{(2)})$ if they occur together in some 2-superedge of E. Hence:

$$\big\{\{x\}, \{y\}\big\} \text{ is an edge (from } e_1), \quad \big\{\{x\}, \{x, y\}\big\} \text{ is an edge (from } e_2),$$

and $\{y\}$ is *not* adjacent to $\{x, y\}$ because no 2-superedge contains them together. Thus:

$$E_{G(\mathrm{SHT}^{(2)})} = \Big\{\big(\{x\}, \{y\}\big), \big(\{x\}, \{x, y\}\big)\Big\}.$$

We get the distances:

$$d_{\mathrm{SHT}^{(2)}}\big(\{x\}, \{y\}\big) = 1, \quad d_{\mathrm{SHT}^{(2)}}\big(\{x\}, \{x, y\}\big) = 1, \quad d_{\mathrm{SHT}^{(2)}}\big(\{y\}, \{x, y\}\big) = 2 \text{ (via } \{x\}).$$

Step 3: A 2-SuperHyperTree Decomposition.

Construct a tree T with two nodes: t_1 and t_2, and one edge (t_1, t_2). Define the bags $\{\mathcal{B}_t\}$ and guards $\{C_t\}$ as:

$$\mathcal{B}_{t_1} = \{\{x\}, \{y\}\}, \qquad \mathcal{B}_{t_2} = \{\{x\}, \{x, y\}\},$$

$$C_{t_1} = \big\{\{\{x\}, \{y\}\}\big\} = \{e_1\}, \quad C_{t_2} = \big\{\{\{x\}, \{x, y\}\}\big\} = \{e_2\}.$$

This ensures coverage (each 2-superedge is fully in some bag) and the connectivity condition on each 2-supervertex ($\{x\}$ appears in both bags but in a connected subtree, etc.).

Step 4: Computing the n-Superhypertree-length.

We need

$$\lambda = \max_{t \in \{t_1, t_2\}} \max_{u, v \in \mathcal{B}_t} d_{\mathrm{SHT}^{(2)}}(u, v).$$

- In $\mathcal{B}_{t_1} = \{\{x\}, \{y\}\}$, the distance between $\{x\}$ and $\{y\}$ is 1 in $G(\mathrm{SHT}^{(2)})$.
- In $\mathcal{B}_{t_2} = \{\{x\}, \{x, y\}\}$, the distance between $\{x\}$ and $\{x, y\}$ is 1.

Hence $\lambda = 1$. Because no other decomposition can reduce the distance inside a bag below 1 (there are adjacent vertices in every bag containing at least two distinct 2-supervertices), we conclude:

$$\mathrm{SHTl}^{(2)}(\mathrm{SHT}^{(2)}) = 1.$$

Step 5: Computing the n-Superhypertree-breadth.

We look for the smallest radius r so that each bag $\mathcal{B}_t$ is contained in some ball of radius r around a center w_t:

$$\mathcal{B}_t \subseteq \{u \in V : d_{\mathrm{SHT}^{(2)}}(u, w_t) \leq r\}.$$

- For $\mathcal{B}_{t_1} = \{\{x\}, \{y\}\}$, we can pick $w_{t_1} = \{x\}$. Then $d_{\mathrm{SHT}^{(2)}}(\{x\}, \{x\}) = 0$ and $d_{\mathrm{SHT}^{(2)}}(\{y\}, \{x\}) = 1$, so $r_{t_1} = 1$ suffices.

- For $\mathcal{B}_{t_2} = \{\{x\}, \{x, y\}\}$, again choose $w_{t_2} = \{x\}$. Distances are $d_{\mathrm{SHT}^{(2)}}(\{x\}, \{x\}) = 0$ and $d_{\mathrm{SHT}^{(2)}}(\{x, y\}, \{x\}) = 1$, so $r_{t_2} = 1$.

Thus the maximum of $\{r_{t_1}, r_{t_2}\}$ is 1. Minimizing over all possible decompositions would not give anything less than 1. So

$$\mathrm{SHTb}^{(2)}(\mathrm{SHT}^{(2)}) = 1.$$

In this example, both the 2-Superhypertree-length and the 2-Superhypertree-breadth of our small 2-SuperHyperGraph are 1. No decomposition can achieve 0, and having at least one bag with two different 2-supervertices forces a minimum distance of 1. This illustrates how the distance concepts from the 2-superprimal graph directly determine the $\mathrm{SHTl}^{(n)}$ and $\mathrm{SHTb}^{(n)}$ parameters.

Theorem 2.13 (Relationship between n-Superhypertree-length and n-Superhypertree-breadth)**.** *Let $SHT^{(n)} = (V, E)$ be any n-SuperHyperGraph. Then:*

$$1 \;\le\; \mathrm{SHTb}^{(n)}\big(SHT^{(n)}\big) \;\le\; \mathrm{SHTl}^{(n)}\big(SHT^{(n)}\big) \;\le\; 2\,\mathrm{SHTb}^{(n)}\big(SHT^{(n)}\big).$$

Proof. The proof is analogous to the classic graph case and the superhypergraph case:

- The lower bound $\mathrm{SHTb}^{(n)}(\mathrm{SHT}^{(n)}) \ge 1$ is trivial unless $\mathrm{SHT}^{(n)}$ has very few vertices (e.g. a single-vertex situation).
- To prove $\mathrm{SHTl}^{(n)}(\mathrm{SHT}^{(n)}) \le 2\,\mathrm{SHTb}^{(n)}(\mathrm{SHT}^{(n)})$, suppose we have an n-SuperHyperTree Decomposition with breadth δ. By definition, for each bag $\mathcal{B}_t$, there is a center w_t such that $\mathcal{B}_t \subseteq D_\delta(w_t)$, where $D_\delta(w_t)$ is the closed ball of radius δ around w_t in $G(\mathrm{SHT}^{(n)})$.

  Now *inflate* each bag $\mathcal{B}_t$ to be exactly $D_\delta(w_t)$. This enlargement does not increase the "guard width" $|C_t|$ (since the guard sets C_t can remain the same), yet any two vertices u, v in the same inflated bag lie within distance $d_{\mathrm{SHT}^{(n)}}(u, v) \le 2\delta$. Hence the resulting decomposition has length at most 2δ, proving the desired inequality.

□

Theorem 2.14 (Equivalence of Length Parameters)**.** *Let $H = (V, E)$ be a hypergraph that is also an n-SuperHyperGraph for some $n \ge 1$. Then*

$$\mathrm{HTl}(H) \;=\; \mathrm{SHTl}^{(n)}(H).$$

Proof. Step 1: Show $\mathrm{SHTl}^{(n)}(H) \;\le\; \mathrm{HTl}(H)$.

Since H is a hypergraph, it admits a *hypertree decomposition* $(T, \{B_t\}, \{C_t\})$ achieving $\mathrm{HTl}(H)$. Because H is also an n-SuperHyperGraph, the same bags $B_t \subseteq V$ and guards $C_t \subseteq E$ can serve as an *n-SuperHyperTree Decomposition*:

- Every hyperedge $e \in E$ is contained in at least one bag (coverage), and each vertex is in a connected subtree of T (connectivity).
- Since $V, E \subseteq \mathcal{P}^n(V_0)$, there is no violation of the n-SuperHyperTree structure conditions.

Hence the same decomposition has length

$$\max_{t \in V(T)} \max_{u,v \in B_t} d_H(u, v).$$

Minimizing over all hypertree decompositions of H yields precisely $\mathrm{HTl}(H)$. Thus we see that there is an n-SuperHyperTree decomposition of length at most $\mathrm{HTl}(H)$, implying

$$\mathrm{SHTl}^{(n)}(H) \;\le\; \mathrm{HTl}(H).$$

Step 2: Show $\mathrm{HTl}(H) \;\le\; \mathrm{SHTl}^{(n)}(H)$.

Conversely, any n-SuperHyperTree Decomposition $(T, \{\mathcal{B}_t\}, \{C_t\})$ is also a generalized hypertree decomposition of H, because H is a hypergraph in the usual sense:

- Each n-superedge $e \in E$ is also a valid hyperedge of H.
- The coverage and connectivity conditions for n-SuperHyperTree decomposition ensure that $(T, \{\mathcal{B}_t\}, \{C_t\})$ meets the requirements of a (generalized) hypertree decomposition.

Therefore, its length

$$\max_{t \in V(T)} \max_{u,v \in \mathcal{B}_t} d_H(u, v)$$

is at least the optimal hypertree-length. In other words, for any n-SuperHyperTree decomposition with length λ, we must have $\mathrm{HTl}(H) \leq \lambda$. Minimizing over all such decompositions yields:

$$\mathrm{HTl}(H) \leq \mathrm{SHTl}^{(n)}(H).$$

Combining both inequalities completes the proof:

$$\mathrm{SHTl}^{(n)}(H) \leq \mathrm{HTl}(H) \quad \text{and} \quad \mathrm{HTl}(H) \leq \mathrm{SHTl}^{(n)}(H) \implies \mathrm{SHTl}^{(n)}(H) = \mathrm{HTl}(H).$$

□

Theorem 2.15 (Equivalence of Breadth Parameters)**.** *Let $H = (V, E)$ be a hypergraph that is also an n-SuperHyperGraph for some $n \geq 1$. Then*

$$\mathrm{HTb}(H) = \mathrm{SHTb}^{(n)}(H).$$

Proof. The argument is analogous to that of Theorem 2.14.

Step 1: $\mathrm{SHTb}^{(n)}(H) \leq \mathrm{HTb}(H)$.

Any hypertree decomposition of H (achieving breadth $\mathrm{HTb}(H)$) automatically qualifies as an n-SuperHyperTree decomposition, because H is also an n-SuperHyperGraph. The radius needed to cover each bag from a center vertex in the primal graph $G(H)$ is the same, so the n-Superhypertree-breadth can be no larger than the hypertree-breadth.

Step 2: $\mathrm{HTb}(H) \leq \mathrm{SHTb}^{(n)}(H)$.

Similarly, any n-SuperHyperTree decomposition of H is also a generalized hypertree decomposition of H. Thus, its breadth is at least $\mathrm{HTb}(H)$, since the latter is the minimum possible breadth among *all* hypertree decompositions.

Hence,

$$\mathrm{HTb}(H) \leq \mathrm{SHTb}^{(n)}(H) \quad \text{and} \quad \mathrm{SHTb}^{(n)}(H) \leq \mathrm{HTb}(H),$$

implying equality:

$$\mathrm{SHTb}^{(n)}(H) = \mathrm{HTb}(H).$$

□

3 Future Tasks: Uncertain Graph

In the future, this research aims to explore the extension of the parameters studied in this paper, such as Tree-length and Tree-breadth, to various types of uncertain graph concepts. These include Fuzzy Graphs [4,39,45,49], Soft Graphs [43,66], Vague Graphs [34,42], Rough Graphs [30], Neutrosophic Graphs [27,28,31], and Plithogenic Graphs [21, 22, 29]. Investigating whether these parameters can be effectively generalized to such frameworks represents a meaningful direction for future work.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 4
Short Note of Extended HyperPlithogenic Sets and General Extended Plithogenic Graphs

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Abstract

The Plithogenic Set is known for generalizing concepts such as Fuzzy Sets and Neutrosophic Sets. It is also recognized that the Plithogenic Set can be extended to concepts such as the Extended Plithogenic Set, HyperPlithogenic Set, and SuperHyperPlithogenic Set. Based on these foundations, this short communication explores the Extended HyperPlithogenic Sets and Extended SuperHyperPlithogenic Sets. Additionally, we consider Extended Plithogenic Graphs and General Extended Plithogenic Graphs.

Keywords: Fuzzy set, Hyperplithogenic set, Plithogenic Graph, Plithogenic Set, Extended Plithogenic Set
MSC 2010 classifications: 03E72: Fuzzy set theory, 03B52: Fuzzy logic; logic of vagueness

1 Short Introduction of this paper

1.1 Plithogenic Sets

Set theory, a fundamental branch of mathematics, provides a robust framework for analyzing collections of elements known as "sets" [16, 50, 98]. In these set theories, various concepts have been studied to handle uncertainty, such as Fuzzy Sets [99–105], Hyperfuzzy Sets [46,53,94], Intuitionistic fuzzy sets [5–9], Neutrosophic Sets [38,39,77–79,91], Vague Sets [15,44], Soft Sets [52,54,56], Hypersoft Sets [1,25,42,68,80,85], superhypersoft set [32,55,86], and Rough Sets [59–62].

The core focus of this paper is on Plithogenic Sets, a highly versatile concept that generalizes Fuzzy Sets and Neutrosophic Sets, among others [22,43,81,82,92]. Plithogenic Sets offer significant flexibility in modeling complex relationships. Additionally, the Extended Plithogenic Set [95], HyperPlithogenic Set [31,33], and n-SuperHyperPlithogenic Set [31,33], which extend the concept of Plithogenic Sets, have been recently defined.

1.2 Our Contribution in This Paper

In this paper, we propose the Extended HyperPlithogenic Set and the Extended n-SuperHyperPlithogenic Set, and examine their relationships with existing concepts. Additionally, we consider Extended Plithogenic Graphs and General Extended Plithogenic Graphs.

2 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

2.1 Power Set

The definition of the Power Set and the nth-Power Set, along with related concepts, are provided below.

Definition 2.1 (Set)**.** [50] A *set* is a well-defined collection of distinct objects, called *elements*. If x is an element of a set A, it is written as $x \in A$. Sets are typically represented using curly braces.

Definition 2.2 (Base Set)**.** (cf. [30]) A *base set* is the foundational set S from which powersets and hyperstructures are constructed. Formally:

$$S = \{x \mid x \text{ is an element within the specified domain}\}.$$

All subsets and operations within $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ are derived from the elements of S.

Definition 2.3 (Powerset)**.** (cf. [30, 65]) The *powerset* of a set S, denoted as $\mathcal{P}(S)$, is the collection of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 2.4 (n-th Powerset)**.** (cf. [30, 76, 89])

The n-th powerset of a set H, denoted $P_n(H)$, is defined recursively. Starting with the standard powerset, the construction proceeds as:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

The n-th non-empty powerset, denoted $P_n^*(H)$, excludes the empty set:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ is the powerset of H excluding the empty set.

2.2 Plithogenic Set

A Plithogenic Set is a mathematical framework that incorporates multi-valued degrees of appurtenance and contradictions, making it suitable for complex decision-making processes. Various studies have been conducted on Plithogenic Sets [2, 3, 27, 33, 63, 69–71, 90, 97]. Related concepts, such as the Plithogenic Graph, are also well-known [18–21, 28, 30, 36, 41, 72–75]. The definition is presented below.

Definition 2.5. [81, 82] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* [1]
- $pCF : Pv \times Pv \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a, a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a, b) = pCF(b, a)$$

[1] It is important to note that the definition of the Degree of Appurtenance Function varies across different papers. Some studies define this concept using the power set, while others simplify it by avoiding the use of the power set [95]. The author has consistently defined the Classical Plithogenic Set without employing the power set.

2.3 HyperPlithogenic Set and SuperHyperPlithogenic Set

Next, the definitions of the HyperPlithogenic Set and the SuperHyperPlithogenic Set are presented below. The HyperPlithogenic Set is a concept defined using hyperstructures, while the SuperHyperPlithogenic Set is defined using superhyperstructures [22, 35, 87, 88, 88, 89, 89].

Definition 2.6 (HyperPlithogenic Set). [23, 31, 33] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set HPS* over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : \left(\bigcup_{i=1}^n Pv_i\right) \times \left(\bigcup_{i=1}^n Pv_i\right) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

Definition 2.7 (n-SuperHyperPlithogenic Set). [23,31,33] Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes, each associated with a set of possible values P_{v_i}. An n-SuperHyperPlithogenic Set ($SHPS_n$) is defined recursively as:

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$,
$$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$
represents the k-th nested family of non-empty subsets of P_1.
- For each attribute $v_i \in V$, P_{v_i} is the set of possible values of the attribute v_i.
- For each k-th level subset P_k, $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P_n$ and attribute value $a_i \in P_{v_i}$ a subset of $[0,1]^s$.
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:
  1. Reflexivity: $pCF^{(n)}(a,a) = 0$ for all $a \in \bigcup_{i=1}^n P_{v_i}$,
  2. Symmetry: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ for all $a, b \in \bigcup_{i=1}^n P_{v_i}$.
- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

2.4 Extended Plithogenic Set

The Extended Plithogenic Set is an extended concept of the Plithogenic Set, which was recently defined [95]. The definition is provided below.

Definition 2.8 (Extended Plithogenic Set). [95] Let P be a non-empty set, and let a be an attribute with a range of possible values V. An *Extended Plithogenic Set (ExPlS)* is defined as a 7-tuple:

$$\text{ExPlS} = (P,\ a,\ V,\ d_D,\ c_D,\ d_R,\ c_R),$$

where:

- $d_D : P \times V \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* with respect to *dominant* attribute value(s), where s indicates the dimensionality (e.g., $s = 1$ for fuzzy, $s = 3$ for neutrosophic). [2]
- $c_D : V \times V \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* associated with dominant attribute value(s), where t is the dimensionality of contradiction. It satisfies:

$$c_D(v,v) = 0, \quad c_D(v_1,v_2) = c_D(v_2,v_1) \quad \forall\, v, v_1, v_2 \in V.$$

- $d_R : P \times V \to [0,1]^s$ is the *DAF* with respect to *recessive* attribute value(s), defined similarly to d_D.
- $c_R : V \times V \to [0,1]^t$ is the *DCF* associated with recessive attribute value(s), satisfying:

$$c_R(v,v) = 0, \quad c_R(v_1,v_2) = c_R(v_2,v_1) \quad \forall\, v, v_1, v_2 \in V.$$

Interpretation.

- d_D and c_D handle membership and contradiction relative to dominant attribute values, while d_R and c_R handle the same for recessive attribute values.
- This framework allows simultaneous evaluation of positive (dominant) and negative (recessive) aspects of attribute values, providing a comprehensive decision-making structure.

Theorem 2.9 (Reduction to Classical Plithogenic Set)**.** *An Extended Plithogenic Set*

$$\mathrm{ExPlS} = (P,\ a,\ V,\ d_D,\ c_D,\ d_R,\ c_R)$$

reduces to a classical Plithogenic Set

$$PS = (P,\ a,\ V,\ d,\ c)$$

if and only if $d_R(x,v) = 0$ and $c_R(v_1,v_2) = 0$ for all $x \in P$ and $v, v_1, v_2 \in V$. In such a case:

$$d = d_D, \quad c = c_D.$$

Proof. If d_R and c_R are null functions (or omitted), the 7-tuple $(P,\ a,\ V,\ d_D,\ c_D,\ d_R,\ c_R)$ reduces to the 5-tuple $(P,\ a,\ V,\ d,\ c)$, where $d = d_D$ and $c = c_D$. Conversely, any classical Plithogenic Set can be embedded into this framework by defining $d_R \equiv 0$ and $c_R \equiv 0$. □

Example 2.10. Consider $P = \{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}\}$ as a set of products, with the attribute a = price and possible values $V = \{\text{cheap}, \text{moderate}, \text{expensive}\}$. Let:

$$d_D : P \times V \to [0,1], \quad c_D : V \times V \to [0,1],$$

represent membership and contradiction relative to cheap (dominant), and:

$$d_R : P \times V \to [0,1], \quad c_R : V \times V \to [0,1],$$

represent membership and contradiction relative to expensive (recessive).

For example:

$$d_D(\mathrm{P1}, \text{cheap}) = 0.8, \quad c_D(\text{cheap}, \text{expensive}) = 0.9,$$

$$d_R(\mathrm{P1}, \text{expensive}) = 0.3, \quad c_R(\text{moderate}, \text{expensive}) = 0.5.$$

This allows simultaneous evaluation of dominant and recessive perspectives for decision-making.

[2] It is important to note that the definition of the Degree of Appurtenance Function varies across different papers. Some studies define this concept using the power set, while others simplify it by avoiding the use of the power set [95]. The author has consistently defined the Classical Plithogenic Set without employing the power set.

3 Results in This Paper

3.1 Extended HyperPlithogenic Sets

In this subsection, we define the Extended HyperPlithogenic Set and the Extended SuperHyperPlithogenic Set as follows and examine their relationships.

Definition 3.1 (Extended HyperPlithogenic Set)**.** Let P be a non-empty set, and let a be an attribute with range V. An *Extended HyperPlithogenic Set (ExHPS)* is an 8-tuple

$$\text{ExHPS} = (P, a, V, \tilde{d}_D, \tilde{c}_D, \tilde{d}_R, \tilde{c}_R, \Delta),$$

where:

- $\tilde{d}_D : P \times V \to \mathcal{P}([0,1]^s)$ and $\tilde{d}_R : P \times V \to \mathcal{P}([0,1]^s)$ assign *subsets* of $[0,1]^s$ capturing multiple (hyper) membership degrees for each dominant or recessive attribute value, respectively.
- $\tilde{c}_D : V \times V \to \mathcal{P}([0,1]^t)$ and $\tilde{c}_R : V \times V \to \mathcal{P}([0,1]^t)$ return *subsets* of $[0,1]^t$ capturing multiple (hyper) contradictions among attribute values.
- Δ is an optional "aggregation context" or "dominance relation" that manages how these subsets are aggregated or chosen in various decision-making or modeling scenarios.

Theorem 3.2 (Reduction to Extended Plithogenic Set)**.** *An Extended HyperPlithogenic Set*

$$\text{ExHPS} = (P,\ a,\ V,\ \tilde{d}_D,\ \tilde{c}_D,\ \tilde{d}_R,\ \tilde{c}_R,\ \Delta)$$

reduces to an Extended Plithogenic Set ExPlS *if all hyper-memberships and hyper-contradictions are singletons:*

$$\tilde{d}_D(x,v) = \{\mathbf{m}\}, \quad \tilde{d}_R(x,v) = \{\mathbf{m}'\}, \quad \tilde{c}_D(v_1,v_2) = \{\mathbf{c}\}, \quad \tilde{c}_R(v_1,v_2) = \{\mathbf{c}'\}.$$

Proof. An Extended HyperPlithogenic Set is defined by the hyper-membership functions $\tilde{d}_D$, $\tilde{d}_R$, and hyper-contradiction functions $\tilde{c}_D$, $\tilde{c}_R$, which map to subsets of $[0,1]^s$ (membership values) and $[0,1]^t$ (contradiction values).

When each of these subsets is reduced to a singleton, for instance:

$$\tilde{d}_D(x,v) = \{\mathbf{m}\}, \quad \tilde{d}_R(x,v) = \{\mathbf{m}'\}, \quad \tilde{c}_D(v_1,v_2) = \{\mathbf{c}\}, \quad \tilde{c}_R(v_1,v_2) = \{\mathbf{c}'\},$$

the hyper-membership and hyper-contradiction values effectively collapse to classical membership and contradiction values. This reduction transforms the structure of ExHPS into the simpler form:

$$\text{ExPlS} = (P,\ a,\ V,\ d_D,\ c_D,\ d_R,\ c_R,\ \Delta),$$

where $d_D(x,v) = \mathbf{m}$, $d_R(x,v) = \mathbf{m}'$, $c_D(v_1,v_2) = \mathbf{c}$, and $c_R(v_1,v_2) = \mathbf{c}'$.

This structure satisfies the definition of an Extended Plithogenic Set ExPlS, as all elements and mappings now align with the standard Plithogenic framework without the need for hyper-structures. The transition from hyper-sets to classical sets is therefore straightforward and consistent with the reduction criteria. □

3.2 Extended n-SuperHyperPlithogenic Sets

Finally, in applications where we need *nested* or *iterated* hyperstructures over multiple levels (e.g., multi-layer sociograms, hierarchical decision-making with multiple dominant and recessive values), we define:

Definition 3.3 (Extended n-SuperHyperPlithogenic Set)**.** Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of n attributes, each with a set of possible values Pv_i. An *Extended n-SuperHyperPlithogenic Set* is a recursive, hierarchical structure:

$$\text{ExSHPS}_n = \Big(P_n,\ V,\ \big\{Pv_i\big\}_{i=1}^n,\ \big\{\tilde{d}_{D,i}^{(n)}, \tilde{d}_{R,i}^{(n)}\big\}_{i=1}^n,\ \big\{\tilde{c}_{D,i}^{(n)}, \tilde{c}_{R,i}^{(n)}\big\}_{i=1}^n,\ \Delta^{(n)}\Big),$$

where

- $P_1 \subseteq$ Universe, and $P_k = \mathcal{P}(P_{k-1})$ for $k = 2, 3, \ldots, n$, forming a *nested family* of subsets (a hyperstructure).
- For each $i \in \{1, \ldots, n\}$, the pair of functions $\tilde{d}^{(n)}_{D,i}$ and $\tilde{d}^{(n)}_{R,i}$ map elements $\big(x \in P_n,\ a_i \in Pv_i\big)$ to subsets of $[0, 1]^s$, capturing multi-dimensional membership for both dominant and recessive aspects of the i-th attribute.
- The pair of functions $\tilde{c}^{(n)}_{D,i}$ and $\tilde{c}^{(n)}_{R,i}$ assign *multi-dimensional contradiction* subsets in $[0, 1]^t$ for each pair of attribute values in Pv_i, again separating the dominant contradiction from the recessive contradiction.
- $\Delta^{(n)}$ is an *aggregation scheme* that orchestrates how these multi-level memberships and contradictions are merged or compared across the entire hierarchy.

Theorem 3.4 (Reduction Theorem of Extended n-SuperHyperPlithogenic Set)**.** *An Extended n-SuperHyperPlithogenic Set reduces to an Extended HyperPlithogenic Set if n = 1 and the hyperstructural nesting $\big(P_2, P_3, \ldots, P_n\big)$ is omitted. Formally,*

$$\mathrm{ExSHPS}_1 = \mathrm{ExHPS}.$$

Proof. The Extended n-SuperHyperPlithogenic Set is defined recursively using the iterative construction:

$$P_k = \mathcal{P}(P_{k-1}), \quad \text{for } k \geq 2,$$

where P_k represents the k-th nested power set of the base set P_1. For $n = 1$, this iterative process terminates, leaving only the base set P_1 without any higher-order nesting. Consequently, the hyperstructural components $(P_2, P_3, \ldots, P_n)$ are absent.

The remaining structure consists solely of P_1, along with the corresponding functions:

$$\tilde{d}_D : P_1 \times V \to \mathcal{P}([0, 1]^s), \quad \tilde{d}_R : P_1 \times V \to \mathcal{P}([0, 1]^s),$$

$$\tilde{c}_D : V \times V \to \mathcal{P}([0, 1]^t), \quad \tilde{c}_R : V \times V \to \mathcal{P}([0, 1]^t),$$

which match the definitions of the Extended HyperPlithogenic Set (ExHPS) as described in Definition 3.1.

Since no additional hyperstructural nesting or higher-order sets are involved when $n = 1$, the structure reduces directly to an Extended HyperPlithogenic Set:

$$\mathrm{ExSHPS}_1 = (P_1,\ a,\ V,\ \tilde{d}_D,\ \tilde{c}_D,\ \tilde{d}_R,\ \tilde{c}_R,\ \Delta).$$

Thus, the theorem is proven. □

4 Additional Result: Extended Plithogenic Graphs

4.1 Extended Plithogenic Graphs

Plithogenic graphs are the graph-theoretical counterpart of Plithogenic sets. We consider extending this concept to Extended Plithogenic Graphs. The definition of Extended Plithogenic Graphs is provided below.

Definition 4.1. [81, 93, 96] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:
    - $M \subseteq V$ is the set of vertices.
    - l is an attribute associated with the vertices.

- Ml is the range of possible attribute values.
- $adf : M \times Ml \rightarrow [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
- $aCf : Ml \times Ml \rightarrow [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \rightarrow [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
   - $bCf : Nm \times Nm \rightarrow [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x,a),(y,b) \in M \times Ml$:
$$bdf\ ((xy),(a,b)) \leq \min\{adf(x,a), adf(y,b)\}$$
where $xy \in N$ is an edge between vertices x and y, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.
2. *Contradiction Function Constraint*: For all $(a,b),(c,d) \in Nm \times Nm$:
$$bCf\ ((a,b),(c,d)) \leq \min\{aCf(a,c), aCf(b,d)\}$$
3. *Reflexivity and Symmetry of Contradiction Functions*:
$$\begin{aligned} &aCf(a,a) = 0, && \forall a \in Ml \\ &aCf(a,b) = aCf(b,a), && \forall a,b \in Ml \\ &bCf(a,a) = 0, && \forall a \in Nm \\ &bCf(a,b) = bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

Definition 4.2 (Extended Plithogenic Graph)**.** Let $G = (V,E)$ be a crisp (classical) graph, where V is a finite set of vertices and $E \subseteq V \times V$ is a set of edges. An *Extended Plithogenic Graph* (abbreviated as ExPlG) is defined as a pair of Extended Plithogenic Sets, one for the vertices and one for the edges, together with suitable constraints:

$$\mathrm{ExPlG} = \big(\mathrm{ExPlS}_V,\ \mathrm{ExPlS}_E\big),$$

where:

- $\mathrm{ExPlS}_V = \big(V,\ l,\ Ml,\ d_{D,V},\ c_{D,V},\ d_{R,V},\ c_{R,V}\big)$
  is an Extended Plithogenic Set corresponding to the *vertex attribute* l. Here:
  $$d_{D,V} : V{\times}Ml \rightarrow [0,1]^s, \quad c_{D,V} : Ml{\times}Ml \rightarrow [0,1]^t, \quad d_{R,V} : V{\times}Ml \rightarrow [0,1]^s, \quad c_{R,V} : Ml{\times}Ml \rightarrow [0,1]^t.$$
  The sets Ml represent the range of possible values of the vertex attribute l.
- $\mathrm{ExPlS}_E = \big(E,\ m,\ Nm,\ d_{D,E},\ c_{D,E},\ d_{R,E},\ c_{R,E}\big)$
  is an Extended Plithogenic Set corresponding to the *edge attribute* m. Here:
  $$d_{D,E} : E{\times}Nm \rightarrow [0,1]^s, \quad c_{D,E} : Nm{\times}Nm \rightarrow [0,1]^t, \quad d_{R,E} : E{\times}Nm \rightarrow [0,1]^s, \quad c_{R,E} : Nm{\times}Nm \rightarrow [0,1]^t.$$
  The sets Nm represent the range of possible values of the edge attribute m.

Additionally, the following conditions must be satisfied:

1. *Vertex Contradiction Functions (Dominant & Recessive) – Reflexivity and Symmetry:*
   For all $u, v \in Ml$,
   $$c_{D,V}(u,u) = 0, \quad c_{D,V}(u,v) = c_{D,V}(v,u), \quad c_{R,V}(u,u) = 0, \quad c_{R,V}(u,v) = c_{R,V}(v,u).$$
2. *Edge Contradiction Functions (Dominant & Recessive) – Reflexivity and Symmetry:*
   For all $x, y \in Nm$,
   $$c_{D,E}(x,x) = 0, \quad c_{D,E}(x,y) = c_{D,E}(y,x), \quad c_{R,E}(x,x) = 0, \quad c_{R,E}(x,y) = c_{R,E}(y,x).$$
3. *Edge Appurtenance Constraints*:
   Let $(u, v) \in E$ be an edge in the crisp sense, and let $\alpha, \beta \in Nm$ be possible edge-attribute values. We require
   $$d_{D,E}\big((u,v),(\alpha)\big) \;\le\; \min\Big\{ \max_{\ell \in Ml} d_{D,V}(u,\ell),\; \max_{r \in Ml} d_{D,V}(v,r)\Big\},$$
   and similarly for the recessive membership,
   $$d_{R,E}\big((u,v),(\alpha)\big) \;\le\; \min\Big\{ \max_{\ell \in Ml} d_{R,V}(u,\ell),\; \max_{r \in Ml} d_{R,V}(v,r)\Big\}.$$
   In words, an edge's dominant (or recessive) membership cannot exceed the minimal combination of the vertices' dominant (or recessive) memberships.
4. *Optional Edge-Vertex Contradiction Constraints*:
   Depending on the application, one can also enforce that the edge contradiction measures $(c_{D,E}, c_{R,E})$ be bounded above by suitably combining the vertex contradiction measures $(c_{D,V}, c_{R,V})$. For instance, one can impose:
   $$c_{D,E}(\alpha,\beta) \;\le\; \min\{\, c_{D,V}(a,a'),\; c_{D,V}(b,b')\},$$
   for relevant choices of $\alpha = (a, a')$, $\beta = (b, b')$ in some product domain. Such constraints generalize the classical plithogenic approach.

Remark 4.3. In contrast to a *Classical Plithogenic Graph*, the *Extended Plithogenic Graph* incorporates two types of membership (dominant and recessive) and two types of contradiction functions for both vertices and edges. This dual structure allows for finer control over the positive (dominant) and negative (recessive) aspects of each attribute value.

We next state and prove a few theorems that illustrate some fundamental mathematical properties of Extended Plithogenic Graphs.

Theorem 4.4 (Non-Negativity and Boundedness of Membership Functions)**.** *Let* $\mathrm{ExPlG} = \big(\mathrm{ExPlS}_V, \mathrm{ExPlS}_E\big)$ *be an Extended Plithogenic Graph as in Definition 4.2. Then for any vertex $v \in V$, any edge $e \in E$, and any respective attribute values $\alpha \in Ml$ or $\beta \in Nm$:*
$$0 \;\le\; d_{D,V}(v,\alpha),\; d_{R,V}(v,\alpha),\; d_{D,E}(e,\beta),\; d_{R,E}(e,\beta) \;\le\; 1.$$

Proof. By Definition 2.8, each DAF (Degree of Appurtenance Function) for the dominant or recessive components maps into $[0,1]^s$. In particular, each scalar component of the membership lies within $[0,1]$. Since $d_{D,V}, d_{R,V}$ are DAFs for vertices (dominant and recessive, respectively) and $d_{D,E}, d_{R,E}$ are DAFs for edges, their values, being partial or full memberships, must lie in $[0,1]$. Hence the statement follows directly from the codomain of these functions. □

Theorem 4.5 (Reflexivity and Symmetry of Contradiction Functions)**.** *In any Extended Plithogenic Graph* ExPlG*, the contradiction functions $c_{D,V}, c_{R,V}, c_{D,E}, c_{R,E}$ satisfy:*
$$c_{D,V}(v,v) = 0, \quad c_{D,V}(u,v) = c_{D,V}(v,u),$$
$$c_{R,V}(v,v) = 0, \quad c_{R,V}(u,v) = c_{R,V}(v,u),$$
$$c_{D,E}(x,x) = 0, \quad c_{D,E}(x,y) = c_{D,E}(y,x),$$
$$c_{R,E}(x,x) = 0, \quad c_{R,E}(x,y) = c_{R,E}(y,x),$$
for all $u, v \in Ml$ and $x, y \in Nm$.

Proof. These equalities and symmetries are explicitly required in Definition 4.2 (and also in the general definition of an Extended Plithogenic Set, cf. Definition 2.8). By construction, each contradiction function $c_{D,V}, c_{R,V}, c_{D,E}, c_{R,E}$ must be *reflexive* (i.e. zero on the diagonal) and *symmetric*. Hence the statement holds by definition. □

Theorem 4.6 (Extension to Classical Plithogenic Graph)**.** *Consider an Extended Plithogenic Graph* ExPlG = $(\mathrm{ExPlS}_V, \mathrm{ExPlS}_E)$. *If*

$$d_{R,V}(v, \alpha) = 0 \quad \textit{and} \quad c_{R,V}(u, v) = 0,$$

for all $v \in V, \alpha \in Ml, u \in Ml$*, and similarly*

$$d_{R,E}(e, \beta) = 0 \quad \textit{and} \quad c_{R,E}(x, y) = 0,$$

for all $e \in E, \beta \in Nm, x, y \in Nm$*, then the Extended Plithogenic Graph collapses to a* classical Plithogenic Graph *in the sense of [81, 93, 96].*

Proof. Under the stated conditions, all recessive memberships and contradictions become identically zero. Hence there remains only the dominant membership and contradiction functions $d_{D,V}, c_{D,V}, d_{D,E}, c_{D,E}$. This exactly reproduces the usual plithogenic framework with a single membership function and a single contradiction function for vertices and edges. Thus, the structure reduces to that of a classical Plithogenic Graph. □

4.2 General Extended Plithogenic Graph

The General Plithogenic Graph is a relax definition of the Plithogenic Graph (cf. [17, 24, 26, 34, 41, 58]).

Definition 4.7 (General Plithogenic Graph)**.** [41] Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges.

A General Plithogenic Graph $G^{GP} = (PM, PN)$ consists of:

1. *General Plithogenic Vertex Set* PM:

   $$PM = (M, l, Ml, adf, aCf)$$

   where:

   - $M \subseteq V$: Set of vertices.
   - l: Attribute associated with the vertices.
   - Ml: Range of possible attribute values.
   - $adf : M \times Ml \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for vertices.
   - $aCf : Ml \times Ml \to [0, 1]^t$: Degree of Contradiction Function (DCF) for vertices.

2. *General Plithogenic Edge Set* PN:

   $$PN = (N, m, Nm, bdf, bCf)$$

   where:

   - $N \subseteq E$: Set of edges.
   - m: Attribute associated with the edges.
   - Nm: Range of possible attribute values.
   - $bdf : N \times Nm \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for edges.
   - $bCf : Nm \times Nm \to [0, 1]^t$: Degree of Contradiction Function (DCF) for edges.

The General Plithogenic Graph G^{GP} only needs to satisfy the following *Reflexivity and Symmetry* properties of the Contradiction Functions:

- Reflexivity and Symmetry of Contradiction Functions:

$$\begin{aligned} aCf(a,a) &= 0, && \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), && \forall a,b \in Ml \\ bCf(a,a) &= 0, && \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

Next, we present the General Extended Plithogenic Graphs, which provide a relaxed definition of the Extended Plithogenic Graphs.

Definition 4.8 (General Extended Plithogenic Graph)**.** Let $G = (V, E)$ be a crisp graph. A *General Extended Plithogenic Graph* is given by:

$$G^{\text{GenExPl}} = \big(\text{ExPlS}_V,\ \text{ExPlS}_E\big),$$

where:

- $\text{ExPlS}_V = (V,\ l,\ Ml,\ d_{D,V},\ c_{D,V},\ d_{R,V},\ c_{R,V})$,
- $\text{ExPlS}_E = (E,\ m,\ Nm,\ d_{D,E},\ c_{D,E},\ d_{R,E},\ c_{R,E})$,

but the only mandatory requirements are:

- *Reflexivity and Symmetry* of the contradiction functions:

  $$c_{D,V}(v,v) = 0,\ c_{D,V}(u,v) = c_{D,V}(v,u),\ \ c_{R,V}(v,v) = 0,\ c_{R,V}(u,v) = c_{R,V}(v,u),$$

  for all $u, v \in Ml$, and

  $$c_{D,E}(x,x) = 0,\ c_{D,E}(x,y) = c_{D,E}(y,x),\ \ c_{R,E}(x,x) = 0,\ c_{R,E}(x,y) = c_{R,E}(y,x),$$

  for all $x, y \in Nm$.
- *Memberships within* $[0,1]$: $d_{D,V}, d_{R,V}, d_{D,E}, d_{R,E} \in [0,1]^s$ componentwise.

No further constraints (such as edge-vertex membership constraints) are strictly enforced in a *General Extended Plithogenic Graph*.

Remark 4.9. Definition 4.8 generalizes Definition 4.2 by omitting the additional plithogenic constraints such as:

$$d_{D,E}((u,v),\beta)\ \le\ \min\{\cdots\},$$

and so on. One only imposes the usual reflexivity/symmetry for the contradiction measures. Hence the structure is broader and can accommodate more relaxed modeling requirements.

Theorem 4.10 (Existence of General Extended Plithogenic Graph)**.** *Any pair of Extended Plithogenic Sets on V and E with reflexive and symmetric contradiction functions and membership functions in* $[0,1]^s$ *induces a* General Extended Plithogenic Graph *structure.*

Proof. Take any crisp graph $G = (V, E)$. Suppose we define for vertices:

$$\text{ExPlS}_V = (V, l, Ml, d_{D,V}, c_{D,V}, d_{R,V}, c_{R,V}),$$

and for edges:

$$\text{ExPlS}_E = (E, m, Nm, d_{D,E}, c_{D,E}, d_{R,E}, c_{R,E}),$$

where each contradiction function is reflexive and symmetric, and each membership function takes values in $[0,1]^s$. By Definition 4.8, these data automatically form a *General Extended Plithogenic Graph*. No further conditions (beyond those enumerated) are required, so such a structure trivially exists. □

Theorem 4.11 (Reduction to Extended Plithogenic Graph). *Any* General Extended Plithogenic Graph *satisfying the additional constraints enumerated in Definition 4.2 (such as the Edge Appurtenance Constraints) becomes a full* Extended Plithogenic Graph.

Proof. The only difference between a General Extended Plithogenic Graph (Definition 4.8) and the Extended Plithogenic Graph (Definition 4.2) is the presence or absence of constraints such as:

$$d_{D,E}((u,v),\beta) \;\leq\; \min\Big\{\max_{\ell\in Ml} d_{D,V}(u,\ell), \max_{r\in Ml} d_{D,V}(v,r)\Big\},$$

and likewise for recessive membership and any optional edge-vertex contradiction constraints. If a given General Extended Plithogenic Graph additionally enforces these constraints, it precisely satisfies the conditions of an Extended Plithogenic Graph and is therefore in that class. □

Question 4.12. Can the aforementioned graph be extended to Directed Graphs [4,45], Bidirected Graphs [11–13,47,51,64], Mixed Graphs [57,66,67], Hypergraphs [10,14], and SuperHypergraphs [29,37,40,48,49,83,84]?

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Data Availability

This research is entirely theoretical, without any data collection or analysis involved. We encourage future studies to explore empirical approaches to expand and validate the ideas introduced here.

Ethical Approval

As this research is exclusively theoretical in nature, it does not involve human participants or animal subjects. Therefore, ethical approval is not required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to this research or its publication.

Disclaimer

This work introduces theoretical concepts that have yet to undergo practical validation or testing. Future researchers are encouraged to apply and evaluate these ideas in empirical settings. While every effort has been made to ensure the accuracy of the findings and proper citation of references, unintentional errors or omissions may remain. Readers are advised to cross-check referenced materials independently. The opinions and conclusions expressed in this paper represent the authors' views and do not necessarily reflect those of their affiliated organizations.

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Chapter 5
Double-Framed Superhypersoft Set and Double-Framed Treesoft Set

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Abstract

Soft sets are mathematical tools designed for decision-making, offering a framework that maps parameters to subsets of a universal set, thereby effectively addressing uncertainty and vagueness [29, 32]. Extensions of soft sets, such as Hypersoft Sets, SuperHypersoft Sets, Treesoft Sets, Double-Framed Soft Sets, and Double-Framed Hypersoft Sets, have been developed to handle more complex decision-making scenarios.

In this short paper, we define the Double-Framed SuperHypersoft Set and Double-Framed Treesoft Set, and provide a concise exploration of their relationships with existing concepts.

Keywords: Superhypersoft set, Soft Set, Treesoft set, Hypersoft set

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

1.1 SuperHypersoft Set and Treesoft Set

This subsection explores the foundational concepts of Soft Sets, Hypersoft Sets, Treesoft Sets, and Super-Hypersoft Sets, which form the basis for advanced decision-making methodologies. A Soft Set provides a flexible framework for parameter-based decision analysis by associating attributes (parameters) with subsets of a universal set, effectively managing uncertainty in decision processes [3, 7, 10, 22, 29, 30, 32, 47, 54, 58, 60].

Building upon this concept, a Hypersoft Set refines multi-attribute decision analysis by linking combinations of multiple attributes to subsets of a universal set, enabling a more comprehensive evaluation [1, 2, 14, 18, 20, 23, 33–41, 45, 48].

Treesoft Sets introduce a hierarchical structure for analyzing complex data. They utilize attribute trees where both nodes and leaves correspond to subsets of a universal set, providing a detailed representation of hierarchical relationships [4, 4, 11, 12, 15, 42, 43, 43, 44, 49, 51–53].

SuperHypersoft Sets extend Hypersoft Set theory further by mapping power set combinations of multiple attributes to subsets of a universal set. This approach facilitates high-dimensional decision-making and models intricate relationships among attributes, offering enhanced flexibility for advanced applications [8, 13, 15–17, 19, 21, 25, 28, 31, 50, 54–57, 59].

The definitions are concisely provided below. For more detailed properties, operations, and applications, please refer to the respective references.

Definition 1.1 (Soft Set)**.** [29, 32] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α.

Definition 1.2 (Hypersoft Set)**.** [48] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be attribute domains. Define $C = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair (G, C), where $G : C \to \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, C) = \{(\gamma, G(\gamma)) \mid \gamma \in C, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in C$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Definition 1.3 (SuperHyperSoft Set). [50] Let U be a universal set, and let $\mathcal{P}(U)$ denote the power set of U. Consider n distinct attributes $a_1, a_2, \ldots, a_n$, where $n \geq 1$. Each attribute a_i is associated with a set of attribute values A_i, satisfying the property $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Define $\mathcal{P}(A_i)$ as the power set of A_i for each $i = 1, 2, \ldots, n$. Then, the Cartesian product of the power sets of attribute values is given by:

$$C = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A SuperHyperSoft Set over U is a pair (F, C), where:

$$F : C \to \mathcal{P}(U),$$

and F maps each element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in C$ (with $\alpha_i \in \mathcal{P}(A_i)$) to a subset $F(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq U$. Mathematically, the SuperHyperSoft Set is represented as:

$$(F, C) = \{(\gamma, F(\gamma)) \mid \gamma \in C, F(\gamma) \in \mathcal{P}(U)\}.$$

Here, $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in C$, where $\alpha_i \in \mathcal{P}(A_i)$ for $i = 1, 2, \ldots, n$, and $F(\gamma)$ corresponds to the subset of U defined by the combined attribute values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Definition 1.4 (Treesoft Set). [51] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1, i_2, \ldots, i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as $\mathrm{Tree}(A)$, with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set* F is defined as a function:

$$F : P(\mathrm{Tree}(A)) \to P(H),$$

where $\mathrm{Tree}(A)$ represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\mathrm{Tree}(A))$ denotes its power set.

1.2 Double-Framed Hypersoft Set

The Double-Framed Soft Set [5,6,24,26,27,46] and Double-Framed Hypersoft Set [9,46] are extended concepts of the Soft Set and Hypersoft Set, incorporating two frames for enhanced representation. Their definitions are provided below.

Definition 1.5 (Double Framed Soft Set). [5, 6, 24, 26, 27, 46] Let U be the universal set, and let A be a set of parameters. A *Double-Framed Soft Set* is a triple $\langle(\alpha, \beta); A\rangle$, where:

1. $\alpha : A \to P(U)$ and $\beta : A \to P(U)$ are mappings from the parameter set A to the power set of U.
2. $\alpha(x)$ represents the *positive frame* and $\beta(x)$ represents the *negative frame* for each parameter $x \in A$.

A Double-Framed Soft Set satisfies the condition:

$$\forall x, y \in A, \quad \alpha(x * y) \supseteq \alpha(x) \cap \alpha(y), \quad \beta(x * y) \subseteq \beta(x) \cup \beta(y),$$

where $*$ is a binary operation defined on A.

Definition 1.6 (Double-Framed Hypersoft Set (DFHSS))**.** [9,46] Let U be the universal set and $P(U)$ denote the power set of U. Let $\{a_1, a_2, \ldots, a_n\}$ represent n distinct attributes, where each attribute a_i is associated with a set of attribute values φ_i, satisfying the conditions:

$$\varphi_i \cap \varphi_j = \emptyset \quad \text{for } i \neq j, \quad i, j \in \{1, 2, \ldots, n\}.$$

A *Double-Framed Hypersoft Set (DFHSS)* is defined as a tuple:

$$(\pi_1, \pi_2; \varphi_1 \times \varphi_2 \times \cdots \times \varphi_n),$$

where:

- $\varphi_1 \times \varphi_2 \times \cdots \times \varphi_n$ is the Cartesian product of the attribute value sets.
- $\pi_1, \pi_2 : \varphi_1 \times \varphi_2 \times \cdots \times \varphi_n \to P(U)$ are mappings that associate each tuple of attribute values with subsets of the universal set U.

2 Result of this Short Paper

This section concisely presents the results of this paper.

2.1 Double-Framed Superhypersoft set

Recall that a SuperHypersoft Set [50] extends a Hypersoft Set by allowing each attribute to take on multiple values from the power set of its domain, instead of just single values. We incorporate the idea of two *frames* (often referred to as positive and negative frames, or lower and upper frames) to arrive at a Double-Framed version.

Definition 2.1 (Double-Framed SuperHypersoft Set)**.** Let U be a universal set. Suppose there are n distinct attributes $a_1, a_2, \ldots, a_n$, each associated with a set of possible values A_i such that $A_i \cap A_j = \varnothing$ for all $i \neq j$. For each $i \in \{1, 2, \ldots, n\}$, let

$$\mathcal{P}(A_i)$$

denote the power set of A_i. Define

$$C = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n),$$

which is the Cartesian product of these power sets.

A *Double-Framed SuperHypersoft Set (DFSHSS)* over U is then a triple

$$(\Theta_1, \Theta_2; C),$$

where

$$\Theta_1 : C \to \mathcal{P}(U), \quad \Theta_2 : C \to \mathcal{P}(U).$$

That is, both Θ_1 and Θ_2 map each element of C (i.e., each n-tuple $\gamma = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathcal{P}(A_i)$) to a subset of U.

Informally, $\Theta_1(\gamma)$ and $\Theta_2(\gamma)$ can be viewed as two distinct but related "frames" (e.g., a *positive* vs. *negative*, or *lower* vs. *upper* approximation) for the combined attribute values in γ.

Remark 2.2. If $\Theta_1 = \Theta_2$, then we recover the standard SuperHypersoft Set (Θ_1, C). If $\mathcal{P}(A_i)$ is replaced by a single-valued domain A_i, we would get a structure akin to a Double-Framed Hypersoft Set [9,46].

Theorem 2.3. (1) *Every Double-Framed Hypersoft Set is a special case of a Double-Framed SuperHypersoft Set.* (2) *Every SuperHypersoft Set is a special case of a Double-Framed SuperHypersoft Set.*

Proof. (1) Double-Framed Hypersoft Set [9,46] is typically defined by having n distinct attributes $\{a_1, \ldots, a_n\}$, each with a single-valued domain φ_i (i.e., each φ_i is a set of attribute values, but we do *not* take their power set). We consider

$$\varphi_1 \times \varphi_2 \times \cdots \times \varphi_n$$

as the Cartesian product of these domains. A Double-Framed Hypersoft Set is a pair of mappings

$$\pi_1, \pi_2 : \varphi_1 \times \cdots \times \varphi_n \;\to\; \mathcal{P}(U).$$

To see that this is a special case of Definition 2.1, observe:

- In the Double-Framed SuperHypersoft Set, each domain for a_i is $\mathcal{P}(A_i)$.
- If each domain $\mathcal{P}(A_i)$ is restricted to *only* take singletons (or equivalently, if A_i is configured so that only one subset from $\mathcal{P}(A_i)$ is effectively chosen, such as the entire set or a specific single element), then:
  - The Cartesian product $\mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$ simplifies to $\varphi_1 \times \cdots \times \varphi_n$, corresponding to single-valued domains.
- Define $\Theta_1 = \pi_1$ and $\Theta_2 = \pi_2$, aligning the frames in the Double-Framed SuperHypersoft Set with the mappings of a Double-Framed Hypersoft Set.

Under this restriction, Θ_1 and Θ_2 yield precisely the Double-Framed Hypersoft Set mappings. Hence, Double-Framed SuperHypersoft Sets reduce to Double-Framed Hypersoft Sets under single-valued domains.

(2) SuperHypersoft Set [50] is a pair (F, C), where $C = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$ and

$$F : C \to \mathcal{P}(U).$$

Comparing with Definition 2.1, if we set

$$\Theta_1(\gamma) \;=\; \Theta_2(\gamma) \;=\; F(\gamma) \quad \text{for all } \gamma \in C,$$

we recover exactly the original SuperHypersoft Set. That is, a Double-Framed SuperHypersoft Set with identical frames $\Theta_1 = \Theta_2$ becomes the usual SuperHypersoft Set.

Hence, Double-Framed SuperHypersoft Sets generalize both Double-Framed Hypersoft Sets and SuperHypersoft Sets. □

2.2 Double-Framed Treesoft Set

We now define the Double-Framed Treesoft Set, extending the idea of a Treesoft Set (which maps subsets of a hierarchical attribute tree to subsets of the universe) by introducing two frames.

Definition 2.4 (Double-Framed Treesoft Set)**.** Let:

- U be a universal set.
- $\mathrm{Tree}(A)$ be a hierarchical attribute tree constructed from an attribute set $A = \{A_1, A_2, \ldots, A_n\}$ (with possibly multiple levels of sub-attributes, sub-sub-attributes, etc.).
- $P(\mathrm{Tree}(A))$ be the power set of all nodes (including leaves) in the tree $\mathrm{Tree}(A)$.

A *Double-Framed Treesoft Set (DFTS)* is a triple

$$\big(\Phi_1, \Phi_2; \mathrm{Tree}(A)\big),$$

where

$$\Phi_1 : P\big(\mathrm{Tree}(A)\big) \;\to\; \mathcal{P}(U), \quad \Phi_2 : P\big(\mathrm{Tree}(A)\big) \;\to\; \mathcal{P}(U).$$

For each subset of nodes $X \subseteq \mathrm{Tree}(A)$, $\Phi_1(X)$ and $\Phi_2(X)$ represent two distinct frames (e.g., *positive* vs. *negative* or *lower* vs. *upper*) for the elements of U relevant to the portion of the tree in X.

Remark 2.5. If $\Phi_1 = \Phi_2$, then $(\Phi_1, \mathrm{Tree}(A))$ recovers the standard Treesoft Set in the sense of [51] (mapping from $P(\mathrm{Tree}(A))$ to $\mathcal{P}(U)$). If we consider a trivial one-level tree (i.e., the attribute tree is just the parameter set A without deeper sub-attributes), then $(\Phi_1, \Phi_2; \mathrm{Tree}(A))$ reduces to a Double-Framed Soft Set [27, 46].

Theorem 2.6. (1) *Every Double-Framed Soft Set is a particular case of a Double-Framed Treesoft Set.* (2) *Every Treesoft Set is a particular case of a Double-Framed Treesoft Set.*

Proof. (1) Double-Framed Soft Set [5, 6, 24, 26, 27, 46] is given as a triple

$$\langle(\alpha, \beta); A\rangle,$$

where A is the parameter set, and

$$\alpha : A \to \mathcal{P}(U), \quad \beta : A \to \mathcal{P}(U).$$

To embed this into the Double-Framed Treesoft Set framework, consider a *single-level* tree:

$$\mathrm{Tree}(A) \;=\; \{A_1, A_2, \ldots, A_n\} \quad \text{(with no deeper sub-attributes)},$$

where each A_i simply represents one parameter in A. Then

$$P\big(\mathrm{Tree}(A)\big) = P(A) \quad \text{(the power set of the parameters)}.$$

Define

$$\Phi_1(X) \;=\; \bigcup_{x \in X} \alpha(x), \qquad \Phi_2(X) \;=\; \bigcup_{x \in X} \beta(x),$$

for $X \subseteq A$. In words, if X is a collection of parameters (now viewed as nodes in the tree), $\Phi_1(X)$ aggregates the corresponding sets $\alpha(x)$, and $\Phi_2(X)$ aggregates the corresponding sets $\beta(x)$. Clearly,

$$\Phi_1, \Phi_2 : P(A) \;\to\; \mathcal{P}(U).$$

Thus, $\big(\Phi_1, \Phi_2; \mathrm{Tree}(A)\big)$ is a Double-Framed Treesoft Set. If you only ever evaluate Φ_1 and Φ_2 on singletons $\{x\} \subseteq A$, you recover $\alpha(x)$ and $\beta(x)$, precisely mirroring the Double-Framed Soft Set structure. Consequently, every Double-Framed Soft Set is realized as a special (single-level) case of the Double-Framed Treesoft Set.

(2) Treesoft Set [51] is a function

$$F : P\big(\mathrm{Tree}(A)\big) \;\to\; \mathcal{P}(U).$$

We embed this in a Double-Framed Treesoft Set by letting

$$\Phi_1(X) \;=\; F(X), \quad \Phi_2(X) \;=\; F(X) \quad \text{for all } X \subseteq \mathrm{Tree}(A).$$

Hence, $\Phi_1 = \Phi_2 = F$. This yields a Double-Framed Treesoft Set $\big(\Phi_1, \Phi_2; \mathrm{Tree}(A)\big)$ that is indistinguishable from the original (single-frame) Treesoft Set when frames coincide. Therefore, the standard Treesoft Set is a particular case of the Double-Framed Treesoft Set.

Summarizing, Double-Framed Treesoft Sets subsume both Double-Framed Soft Sets (by restricting the tree to one level) and standard Treesoft Sets (by letting the two frames coincide). □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 6
HyperPlithogenic Cubic Set and SuperHyperPlithogenic Cubic Set

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Abstract

Concepts such as Fuzzy Sets [23,47], Neutrosophic Sets [32,33], and Plithogenic Sets [35] have been extensively studied for addressing uncertainty, with diverse applications across numerous fields. Building on the Plithogenic Set, the HyperPlithogenic Set and SuperHyperPlithogenic Set have also gained recognition [15]. A Plithogenic Cubic Set integrates interval-valued and single-valued fuzzy memberships, augmented by multi-attribute aggregation using plithogenic structures. This paper defines the HyperPlithogenic Cubic Set and SuperHyperPlithogenic Cubic Set and explores related concepts such as the HyperPlithogenic Fuzzy Cubic Set, HyperPlithogenic Intuitionistic Fuzzy Cubic Set, and HyperPlithogenic Neutrosophic Cubic Set.

Keywords: Plithogenic Set, HyperPlithogenic Set, n-SuperhyperPlithogenic Set, Plithogenic Cubic Set

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

1.1 Plithogenic Set

The Plithogenic Set is a mathematical framework designed to integrate multi-valued degrees of appurtenance and contradiction, making it particularly effective for addressing complex decision-making scenarios. Numerous studies have explored the properties and applications of Plithogenic Sets, as highlighted in works such as [1,13,28–30,39,43].

Additionally, related concepts like the Plithogenic Graph have received considerable attention and analysis in various studies [11,18]. The Plithogenic Set's versatility lies in its ability to generalize several established mathematical frameworks, including Fuzzy Sets [47,48], Intuitionistic Fuzzy Sets [5,6], Vague Sets [7,19], Neutrosophic Sets [33,34], Picture Fuzzy Sets [9,27,42], Bipolar Neutrosophic Sets [10,46], Hyperfuzzy Sets [20,21], and Hesitant Fuzzy Sets [44,45].

The formal definition is presented below.

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** [14,26] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset)**.** (cf. [14,16,31,38])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 1.4 (Plithogenic Set)**.** [36,37] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* [1]
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a,a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a,b) = pCF(b,a)$$

Definition 1.5 (HyperPlithogenic Set)**.** [12,15,17] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set* HPS over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : \left(\bigcup_{i=1}^n Pv_i\right) \times \left(\bigcup_{i=1}^n Pv_i\right) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

Definition 1.6 (n-SuperHyperPlithogenic Set)**.** [12,15,17] Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes, each associated with a set of possible values P_{v_i}. An n-SuperHyperPlithogenic Set ($SHPS_n$) is defined recursively as:

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$,
$$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$
represents the k-th nested family of non-empty subsets of P_1.
- For each attribute $v_i \in V$, P_{v_i} is the set of possible values of the attribute v_i.

[1] It is important to note that the definition of the Degree of Appurtenance Function varies across different papers. Some studies define this concept using the power set, while others simplify it by avoiding the use of the power set [41]. The author has consistently defined the Classical Plithogenic Set without employing the power set.

- For each k-th level subset P_k, $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P_n$ and attribute value $a_i \in P_{v_i}$ a subset of $[0,1]^s$.
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:
    1. Reflexivity: $pCF^{(n)}(a,a) = 0$ for all $a \in \bigcup_{i=1}^n P_{v_i}$,
    2. Symmetry: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ for all $a,b \in \bigcup_{i=1}^n P_{v_i}$.
- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

1.2 Plithogenic Cubic Set

A Plithogenic Cubic Set integrates interval-valued and single-valued fuzzy memberships, augmented by multi-attribute aggregation using plithogenic structures [3,4,25,40]. Related concepts, such as Neutrosophic Cubic Sets [2,8,22,49], are also well-established. The definitions and details are provided below.

Definition 1.7 (Plithogenic Cubic Set)**.** [25,40] Let X be a non-empty set. A *Plithogenic Cubic Set* (PCS) in X is a pair

$$\Pi = \Big(C,\ \mathcal{P}_{\text{plitho}}\Big),$$

where:

1. $C = \{(x,\ A(x),\ \alpha(x)) \mid x \in X\}$ is a *cubic set* on X (Definition **??**), consisting of
    - An interval-valued fuzzy mapping
    $$A : X \to \big\{[a_x^-, a_x^+] \subseteq [0,1]\big\},$$
    - A single-valued fuzzy mapping
    $$\alpha : X \to [0,1].$$
2. $\mathcal{P}_{\text{plitho}}$ is a *plithogenic structure* (Definition **??**) that governs how the attributes and membership intervals in C are aggregated or combined. In particular, $\mathcal{P}_{\text{plitho}} = \big(P, v, Pv, pdf, pCF\big)$ includes:
    - A domain $P \subseteq X$ (or, more generally, a set of interest).
    - An attribute v with possible values Pv.
    - A degree of appurtenance function $pdf : P \times Pv \to [0,1]^s$, used to define how elements in P attach membership vectors under each attribute value.
    - A contradiction function $pCF : Pv \times Pv \to [0,1]^t$, specifying how attribute values $a, b \in Pv$ might conflict or support each other, often employed in multi-attribute decision contexts.

Intuition:

- The "cubic set" part $(A(x), \alpha(x))$ captures interval-valued and single-valued fuzzy membership for each x.
- The "plithogenic" part enforces additional structure on how multiple attributes or parameter values in Pv can be integrated (often using a *plithogenic aggregation* guided by pCF).
- The resulting *Plithogenic Cubic Set* merges these two viewpoints, enabling multi-attribute decision-making or data analysis with both interval-based *and* single-valued fuzzy membership, managed under a plithogenic aggregator.

Example 1.8 (Plithogenic Cubic Set Illustrative)**.** Suppose $X = \{x_1, x_2, x_3\}$ is a universe of three elements. Define a cubic set C on X by:

$$C = \Big\{(x_1, [0.2, 0.4], 0.3),\ (x_2, [0.7, 0.9], 0.6),\ (x_3, [0.4, 0.5], 0.5)\Big\}.$$

Here:

$$A(x_1) = [0.2, 0.4], \quad A(x_2) = [0.7, 0.9], \quad A(x_3) = [0.4, 0.5],$$

$$\alpha(x_1) = 0.3, \quad \alpha(x_2) = 0.6, \quad \alpha(x_3) = 0.5.$$

Additionally, let $P \subseteq X$ be $\{x_1, x_2, x_3\}$, define an attribute v with possible values $Pv = \{u_1, u_2\}$. Suppose we have:

$$pdf : P \times \{u_1, u_2\} \longrightarrow [0, 1]^2,$$

$$pdf(x_1, u_1) = (0.2, 0.5), \quad pdf(x_1, u_2) = (0.3, 0.6), \quad \ldots$$

and so forth, plus a contradiction function

$$pCF(u_1, u_1) = 0, \quad pCF(u_1, u_2) = (0.4), \quad pCF(u_2, u_1) = (0.4), \quad pCF(u_2, u_2) = 0.$$

Then

$$\mathcal{P}_{\text{plitho}} = (P,\ v,\ Pv,\ pdf,\ pCF),$$

and

$$\Pi = (C,\ \mathcal{P}_{\text{plitho}})$$

is a *Plithogenic Cubic Set* on X. The presence of $A(x_i)$, $\alpha(x_i)$, and the multi-attribute aggregator pdf/pCF clarifies how membership intervals, single membership values, and attribute-based contradictions coexist in a single structure.

Definition 1.9 (Plithogenic Fuzzy Cubic Set)**.** [25, 40] Let X be a non-empty universe, and let

$$\mathcal{P}_{\text{plitho}} \ = \ (P,\ v,\ Pv,\ pdf,\ pCF)$$

be a *plithogenic structure* (consisting of a domain $P \subseteq X$, an attribute v with possible values Pv, a degree of appurtenance function pdf, and a contradiction function pCF). A *Plithogenic Fuzzy Cubic Set (PFCS)* in X is a pair

$$\Pi_{\text{F}} \ = \ (C_{\text{F}},\ \mathcal{P}_{\text{plitho}}),$$

where:

1. $C_{\text{F}} = \big\{(x,\ A_F(x),\ \alpha_F(x)) \mid x \in X\big\}$ is a *fuzzy cubic set* on X, meaning:
   - $A_F : X \to [0, 1]$ is a single-valued fuzzy membership (instead of an interval-valued one). That is, for each $x \in X$, $A_F(x) \in [0, 1]$.
   - $\alpha_F : X \to [0, 1]$ is a second single-valued fuzzy mapping. In typical "cubic set" terminology, we can interpret $(A_F(x), \alpha_F(x))$ as the pair of membership values capturing two layered fuzzy memberships for each element x (one possibly playing the role of amplitude, the other of secondary membership).
2. $\mathcal{P}_{\text{plitho}}$ is the plithogenic aggregator, as above.

Hence, for each $x \in X$:

$$(x,\ A_F(x),\ \alpha_F(x)) \quad \text{with} \quad A_F(x),\ \alpha_F(x)\ \in [0, 1].$$

And the plithogenic structure $\mathcal{P}_{\text{plitho}}$ provides multi-attribute valuation or contradiction measures used to combine or compare these fuzzy membership values across different parameter values Pv. This synergy yields a multi-attribute, two-layer fuzzy membership system governed by a plithogenic aggregator.

Example 1.10 (Plithogenic Fuzzy Cubic Set)**.** Let $X = \{x_1, x_2, x_3\}$ be a universe. Define the *fuzzy cubic set*

$$C_{\mathrm{F}} = \Big\{(x_1,\, 0.2,\, 0.4),\ (x_2,\, 0.5,\, 0.7),\ (x_3,\, 0.8,\, 0.6)\Big\},$$

where $A_F(x_1) = 0.2, \alpha_F(x_1) = 0.4, A_F(x_2) = 0.5, \alpha_F(x_2) = 0.7$, etc.

Next, let us assume a plithogenic structure $\mathcal{P}_{\mathrm{plitho}}$ with:

$$P = \{x_1, x_2, x_3\}, \quad v \text{ an attribute with possible values } Pv = \{u_1, u_2\},$$

and define a degree of appurtenance function

$$pdf : P \times \{u_1, u_2\} \longrightarrow [0,1]^2,$$

plus a contradiction function

$$pCF : \{u_1, u_2\} \times \{u_1, u_2\} \longrightarrow [0,1]^t.$$

We might specify, for instance:

$$pdf(x_1, u_1) = (0.4, 0.6), \quad pdf(x_1, u_2) = (0.2, 0.1), \quad \cdots$$

and

$$pCF(u_1, u_1) = 0,\ \ pCF(u_1, u_2) = 0.3,\ \ pCF(u_2, u_2) = 0,\ \ pCF(u_2, u_1) = 0.3.$$

Combining these, the *Plithogenic Fuzzy Cubic Set* is

$$\Pi_{\mathrm{F}} = \big(C_{\mathrm{F}},\ \mathcal{P}_{\mathrm{plitho}}\big).$$

In effect, we have a two-valued fuzzy membership (A_F, α_F) for each x, and a plithogenic aggregator controlling multi-attribute contradictions or synergy among u_1, u_2 in Pv.

Definition 1.11 (Plithogenic Intuitionistic Fuzzy Cubic Set)**.** [25, 40] Let X be a non-empty universe, and let $\mathcal{P}_{\mathrm{plitho}} = (P, v, Pv, pdf, pCF)$ be a plithogenic structure. An *Plithogenic Intuitionistic Fuzzy Cubic Set (PIFCS)* in X is a pair

$$\Pi_{\mathrm{IF}} = \Big(C_{\mathrm{IF}},\ \mathcal{P}_{\mathrm{plitho}}\Big),$$

where:

1. $C_{\mathrm{IF}} = \big\{ (x,\ A_{IF}(x),\ \alpha_{IF}(x)) \mid x \in X \big\}$ is an *intuitionistic fuzzy cubic set*, namely:
   - $A_{IF}(x) = (\mu(x), \nu(x))$ is an interval *(or pair)* representing membership $\mu(x) \in [0,1]$ and non-membership $\nu(x) \in [0,1]$, with $\mu(x) + \nu(x) \le 1$ for each x. Equivalently, we may store an *interval* $[\,\mu^-, \mu^+\,]$ for membership and $[\,\nu^-, \nu^+\,]$ for non-membership.
   - $\alpha_{IF}(x)$ is a single *intuitionistic fuzzy* index, e.g. a secondary membership or hesitation part for x. There are various ways to formalize the cubic notion here, but in general $\alpha_{IF}(x)$ is an extra single-valued function capturing additional partial membership or hesitation.
2. The plithogenic structure $\mathcal{P}_{\mathrm{plitho}}$ merges these membership pairs (μ, ν) with contradictory or supportive attribute values from Pv.

Note: Sometimes the *cubic set* for an intuitionistic fuzzy environment is defined as (A, α) where A is an interval-valued intuitionistic fuzzy set (storing membership and non-membership intervals) and α is a single-valued intuitionistic fuzzy function. The essential idea is that we have one *interval/pair* capturing membership-nonmembership for each x, plus an extra single-valued function for x. Then we combine that with the plithogenic aggregator.

Example 1.12 (Plithogenic Intuitionistic Fuzzy Cubic Set)**.** Let $X = \{x_1, x_2\}$. Suppose we define for each $x \in X$:

$$A_{IF}(x) = \big(\mu(x), \nu(x)\big), \quad \text{with } \mu(x) \in [0,1],\ \nu(x) \in [0,1],\ \mu(x) + \nu(x) \le 1,$$

and $\alpha_{IF}(x) \in [0,1]$ is a single-value capturing a *hesitation degree* or a secondary membership. For instance:

$$A_{IF}(x_1) = (0.6,\ 0.3), \quad \alpha_{IF}(x_1) = 0.2,$$
$$A_{IF}(x_2) = (0.4,\ 0.4), \quad \alpha_{IF}(x_2) = 0.1.$$

Hence the *intuitionistic fuzzy cubic set* is

$$C_{\mathrm{IF}} = \{(x_1, (0.6, 0.3), 0.2),\ (x_2, (0.4, 0.4), 0.1)\}.$$

Now let $\mathcal{P}_{\mathrm{plitho}} = (P, v, Pv, pdf, pCF)$ be a plithogenic structure with:

$$P = \{x_1, x_2\}, \quad v \text{ attribute with } Pv = \{u_1, u_2\},$$

$$pdf(\cdot,\cdot) : P \times Pv \ \rightarrow\ [0,1]^2, \quad pCF(\cdot,\cdot) : Pv \times Pv \ \rightarrow\ [0,1]^t.$$

Define a few sample values:

$$pdf(x_1, u_1) = (0.8, 0.1),\ pdf(x_1, u_2) = (0.3, 0.5),\ pdf(x_2, u_1) = (0.4, 0.6),\ pdf(x_2, u_2) = (0.7, 0.2),$$

$$pCF(u_1, u_1) = 0,\ pCF(u_1, u_2) = 0.5,\ pCF(u_2, u_1) = 0.5,\ pCF(u_2, u_2) = 0.$$

Then the *Plithogenic Intuitionistic Fuzzy Cubic Set* is

$$\Pi_{\mathrm{IF}} \ = \ (C_{\mathrm{IF}},\ \mathcal{P}_{\mathrm{plitho}}).$$

This structure allows us to model partial membership, partial non-membership, a second single-valued dimension for each x, and also to incorporate multi-attribute influences or contradictions across u_1, u_2 in a plithogenic aggregator.

Definition 1.13 (Plithogenic Neutrosophic Cubic Set)**.** Let X be a non-empty universe, and let $\mathcal{P}_{\mathrm{plitho}} = (P, v, Pv, pdf, pCF)$ be a plithogenic structure. A *Plithogenic Neutrosophic Cubic Set (PNCS)* in X is defined by

$$\Pi_{\mathrm{N}} \ = \ (C_{\mathrm{N}},\ \mathcal{P}_{\mathrm{plitho}}),$$

where:

1. $C_{\mathrm{N}} = \{(x,\ A_N(x),\ \alpha_N(x)) \mid x \in X\}$ is a *neutrosophic cubic set*, in which:
   - $A_N(x)$ is an *interval-valued neutrosophic membership* for each x. Typically, a neutrosophic membership is a triple (T, I, F) in $[0,1]^3$ with $T + I + F \le 3$. For an *interval* version, we might store $[T^-, T^+], [I^-, I^+], [F^-, F^+]$ for each x.
   - $\alpha_N(x)$ is a *single-valued neutrosophic* membership triple or a single measure capturing partial truth, falsity, and indeterminacy for x. More simply, it can be (t_x, i_x, f_x) with $t_x + i_x + f_x \le 3$.
2. $\mathcal{P}_{\mathrm{plitho}} = (P, v, Pv, pdf, pCF)$ is the plithogenic aggregator that unifies these neutrosophic membership values across multi-attribute domains in Pv.

Hence, each x in X is described by *two layers* of neutrosophic membership (one interval-valued, one single-valued), while the plithogenic aggregator fosters multi-attribute or contradictory synergy among attribute values in Pv.

Example 1.14 (Plithogenic Neutrosophic Cubic Set)**.** [24, 25, 40] Let $X = \{x_1, x_2\}$. For each $x \in X$, define the *neutrosophic cubic set* membership:

$$C_{\mathrm{N}} \ = \ \Big\{ (x_1,\ A_N(x_1),\ \alpha_N(x_1)),\ (x_2,\ A_N(x_2),\ \alpha_N(x_2)) \Big\},$$

where for x_1,

$$A_N(x_1) = ([\,0.4, 0.6\,], [\,0.1, 0.3\,], [\,0.2, 0.4\,]) \quad \text{(interval T,I,F)},$$

$$\alpha_N(x_1) = (0.5, 0.2, 0.3) \text{ (single T,I,F) with } 0.5 + 0.2 + 0.3 = 1.0 \le 3.$$

For x_2,

$$A_N(x_2) = ([\,0.7, 0.8\,], [\,0.0, 0.1\,], [\,0.1, 0.2\,]), \quad \alpha_N(x_2) = (0.6, 0.3, 0.1).$$

Now let $\mathcal{P}_{\text{plitho}} = (P, v, Pv, pdf, pCF)$ be:

$$P = \{x_1, x_2\}, \quad v \text{ is an attribute with } Pv = \{u_1, u_2, u_3\},$$

$$pdf : P \times Pv \ \to \ [0,1]^s, \quad pCF(\cdot,\cdot) : Pv \times Pv \ \to \ [0,1]^t.$$

For instance,

$$pdf(x_1, u_1) = (0.4, 0.2),\ pdf(x_1, u_2) = (0.1, 0.3),\ pdf(x_1, u_3) = (0.7, 0.2), \quad pdf(x_2, u_1) = (0.6, 0.4),\ \ldots$$

and

$$pCF(u_1, u_1) = 0,\ pCF(u_1, u_2) = 0.5,\ pCF(u_2, u_3) = 0.2, \ldots$$

Thus the *Plithogenic Neutrosophic Cubic Set* is

$$\Pi_{\text{N}} \ = \ \big(C_{\text{N}},\ \mathcal{P}_{\text{plitho}}\big).$$

We have, for each x_i, an interval-based neutrosophic membership $(T^-(x_i), T^+(x_i))$, $(I^-(x_i), I^+(x_i))$, $(F^-(x_i), F^+(x_i))$, plus a single triple (t_x, i_x, f_x). Then multi-attribute interactions among u_1, u_2, u_3 are managed by (pdf, pCF).

2 Results of This Paper

In this paper, we propose new definitions for various types of sets and briefly examine their relationships with existing concepts.

This document presents the concept of n-SuperHyperPlithogenic Cubic Sets, built upon fuzzy, intuitionistic fuzzy, and neutrosophic frameworks. These notions extend existing "HyperPlithogenic Cubic Sets" to higher orders by nesting membership structures in a recursive manner. For brevity, we refer to them as:

- *n-SuperHyperPlithogenic Fuzzy Cubic Set (n-SHPC-FCS)*,
- *n-SuperHyperPlithogenic Intuitionistic Fuzzy Cubic Set (n-SHPC-IFCS)*,
- *n-SuperHyperPlithogenic Neutrosophic Cubic Set (n-SHPC-NCS)*.

We assume familiarity with:

- *Fuzzy/Intuitionistic Fuzzy/Neutrosophic Cubic Sets*: Cubic sets that combine interval-valued membership and single-valued membership (plus, for intuitionistic or neutrosophic, the relevant membership forms).
- *HyperPlithogenic Structures*: Which assign hyper-set-valued memberships for multi-attribute parameters and utilize a contradiction function among attribute values.
- *n-Super* frameworks: In which membership sets or attribute sets are nested *up to* the n-th power set or hyper-power set level, as in $\tilde{\mathcal{P}}^n(\cdot)$.

2.1 HyperPlithogenic Cubic Set (HPCS)

A HyperPlithogenic Cubic Set (HPCS) integrates interval-valued, single-valued fuzzy membership and multi-attribute aggregation, addressing complex multi-dimensional uncertainty.

Definition 2.1 (HyperPlithogenic Cubic Set (HPCS))**.** Let X be a non-empty universe. Recall the following:

- A *Plithogenic Cubic Set (PCS)* on X is a pair $\Pi \ = \ (C,\ \mathcal{P}_{\text{plitho}})$, where C is a *cubic set* on X (capturing interval-valued and single-valued fuzzy membership), and $\mathcal{P}_{\text{plitho}}$ is a *plithogenic structure* guiding multi-attribute aggregation or contradiction.[2]

[2] See the user's snippet for the formal definition of a Plithogenic Cubic Set.

- A *HyperPlithogenic Set (HPS)* is a structure $(P, \{v_i\}, \{Pv_i\}, \{\tilde{pdf}_i\}, pCF)$ that allows *hyper-degree of appurtenance* (set-valued membership degrees) for each attribute's domain and a contradiction function.[3]

We define a *HyperPlithogenic Cubic Set (HPCS)* over X as follows:

$$\text{HPCS} \;=\; \Bigl(C_{\text{Cubic}},\; \mathcal{H}_{\text{plitho}}\Bigr),$$

where:

1. $C_{\text{Cubic}} = \{(x,\ A(x),\ \alpha(x)) \mid x \in X\}$ is a *cubic set* on X. Concretely, for each $x \in X$:
   - $A(x) \subseteq [0,1]$ is an *interval-valued* fuzzy membership or possibly an entire sub-interval $[\,a_x^-,\ a_x^+\,]$. In many references, we store $A(x) = [\,a_x^-,\ a_x^+\,] \subseteq [0,1]$.
   - $\alpha(x) \in [0,1]$ is an additional single-valued fuzzy membership for x. That is, each $x \in X$ has a pair $\bigl(A(x),\ \alpha(x)\bigr)$.
2. $\mathcal{H}_{\text{plitho}} = \bigl(P, \{v_i\}, \{Pv_i\}, \{\widetilde{pdf}_i\}, pCF\bigr)$ is a *HyperPlithogenic Set* structure. In particular:
   - $P \subseteq X$ is a domain of interest (often the same as or a subset of X).
   - Each attribute v_i has possible values Pv_i.
   - $\widetilde{pdf}_i : P \times Pv_i \to \widetilde{\mathcal{P}}\bigl([0,1]^s\bigr)$ is a *hyper degree of appurtenance function*, i.e. it assigns *set-valued membership degrees* in $[0,1]^s$ for each pair (x, a_i).
   - $pCF : \Bigl(\bigcup_{i=1}^{n} Pv_i\Bigr) \times \Bigl(\bigcup_{i=1}^{n} Pv_i\Bigr) \to [0,1]^t$ is the contradiction function, satisfying reflexivity and symmetry conditions.

The resulting pair

$$\text{HPCS} \;=\; \bigl(C_{\text{Cubic}},\; \mathcal{H}_{\text{plitho}}\bigr)$$

is called a *HyperPlithogenic Cubic Set*.

Example 2.2 (HyperPlithogenic Cubic Set)**.** Let $X = \{x_1, x_2, x_3\}$. Suppose we have a *cubic set*:

$$C_{\text{Cubic}} \;=\; \Bigl\{(x_1,\ [0.2, 0.4],\ 0.6),\ (x_2,\ [0.7, 0.8],\ 0.4),\ (x_3,\ [0.4, 0.9],\ 0.2)\Bigr\}.$$

Hence for each x_i, we store an interval $A(x_i) \subseteq [0,1]$ plus a single real $\alpha(x_i) \in [0,1]$.

Next, define the *HyperPlithogenic structure*:

$$\mathcal{H}_{\text{plitho}} \;=\; \bigl(P,\ \{v_1, v_2\},\ \{Pv_1, Pv_2\},\ \{\widetilde{pdf}_1, \widetilde{pdf}_2\},\ pCF\bigr),$$

where

$$P = \{x_1, x_2, x_3\}, \quad v_1 \text{ with } Pv_1 = \{u_1, u_2\}, \quad v_2 \text{ with } Pv_2 = \{w_1, w_2\}.$$

The hyper-DAFs might be

$$\widetilde{pdf}_1 : P \times \{u_1, u_2\} \;\to\; \widetilde{\mathcal{P}}\bigl([0,1]^s\bigr), \quad \widetilde{pdf}_2 : P \times \{w_1, w_2\} \;\to\; \widetilde{\mathcal{P}}\bigl([0,1]^s\bigr).$$

For instance,

$$\widetilde{pdf}_1(x_1, u_1) \;=\; \{(0.2, 0.6),\ (0.3, 0.7)\} \subseteq [0,1]^2, \quad \widetilde{pdf}_1(x_1, u_2) = \cdots$$

Then define

$$pCF(\cdot,\cdot) : (\,\{u_1, u_2, w_1, w_2\} \times \{u_1, u_2, w_1, w_2\}\,) \;\to\; [0,1]^t,$$

satisfying reflexivity and symmetry. For example,

$$pCF(u_1, u_1) = 0, \quad pCF(u_1, u_2) = 0.3, \quad pCF(w_1, u_2) = 0.5,\ \ldots$$

Putting these together,

$$\text{HPCS} \;=\; \Bigl(C_{\text{Cubic}},\; \mathcal{H}_{\text{plitho}}\Bigr)$$

becomes a *HyperPlithogenic Cubic Set*. It merges an interval-plus-singleton membership for each x_i (the cubic set) with a *hyperplithogenic aggregator* that yields set-valued membership degrees and a multi-attribute contradiction measure among $\{u_1, u_2\}$ and $\{w_1, w_2\}$.

[3] See the user's snippet for the definition of HyperPlithogenic Set.

Theorem 2.3. *(HyperPlithogenic Cubic Set generalizes Plithogenic Cubic Set and HyperPlithogenic Set.)*

1. *If each hypermembership $\widetilde{pdf}_i(x, a_i)$ in $\mathcal{H}_{\text{plitho}}$ is constrained to be a* singleton *in $[0, 1]^s$, and the interval $A(x)$ in C_{Cubic} is restricted to a single real number in $[0, 1]$, then an HPCS reduces to a Plithogenic Cubic Set (Definition 1.7).*
2. *If the cubic part $(A(x), \alpha(x))$ is replaced by a simpler membership approach (e.g. ignoring intervals, or ignoring single-valued parts), an HPCS reduces to a HyperPlithogenic Set.*

Proof. (1) HPCS $\Longrightarrow$ Plithogenic Cubic Set.
In a HyperPlithogenic Cubic Set $(C_{\text{Cubic}},\ \mathcal{H}_{\text{plitho}})$:

- C_{Cubic} has an *interval* $A(x) \subseteq [0, 1]$ plus a real $\alpha(x) \in [0, 1]$.
- $\mathcal{H}_{\text{plitho}} = (P, \{v_i\}, \{Pv_i\}, \{\widetilde{pdf}_i\}, pCF)$ has *set-valued* membership degrees $\widetilde{pdf}_i(x, a_i) \subseteq [0, 1]^s$.

First, constrain each $\widetilde{pdf}_i(x, a_i)$ to be a singleton $\{\mathbf{m}\} \subseteq [0, 1]^s$. This collapses the *hyper* aspect to an ordinary $pdf_i : P \times Pv_i \to [0, 1]^s$. Next, for each x, constrain the "interval" $A(x) \subseteq [0, 1]$ to be a single real $a_x \in [0, 1]$. Then $A : x \mapsto a_x \in [0, 1]$ becomes an ordinary fuzzy membership. By these restrictions, we precisely match the structure of a *Plithogenic Cubic Set* (in which the interval is replaced by a single amplitude). Consequently, HPCS reduces to a standard PCS.

(2) HPCS $\Longrightarrow$ HyperPlithogenic Set.
If we remove the *cubic set* part $(A(x), \alpha(x))$, or equivalently if we fix $A(x) = \{1\}$ and $\alpha(x) = 1$ for all x (making them trivial or constant), we no longer store interval or single-valued fuzzy membership for each x. The entire membership representation then depends solely on the *hyperplithogenic aggregator* $\{\widetilde{pdf}_i\}$, which is exactly a *HyperPlithogenic Set* structure. Therefore, ignoring or trivializing the cubic membership reduces an HPCS to a standard HPS (HyperPlithogenic Set).

Hence, HPCS strictly generalizes both PCS and HPS. □

Definition 2.4 (HyperPlithogenic Fuzzy Cubic Set)**.** Let X be a non-empty set. A *HyperPlithogenic Fuzzy Cubic Set (HPFCS)* in X is a structure

$$\left(\mathcal{F}_{\text{fuzzy-cubic}},\ \mathcal{H}_{\text{plitho}}\right),$$

where:

1. $\mathcal{F}_{\text{fuzzy-cubic}}$ is a *fuzzy cubic set* on X. Concretely, for each $x \in X$, we have:

$$\mathcal{F}_{\text{fuzzy-cubic}}(x) \;=\; \left(A_F(x),\ \alpha_F(x)\right),$$

   where

$$A_F(x) = [\,a_x^-,\ a_x^+\,] \subseteq [0, 1], \quad \alpha_F(x) \in [0, 1].$$

2. $\mathcal{H}_{\text{plitho}} = (P, \{v_i\}, \{Pv_i\}, \{\widetilde{pdf}_i\}, pCF)$ is a *HyperPlithogenic* structure that assigns *set-valued membership degrees* to attribute-value pairs:

$$\widetilde{pdf}_i : P \times Pv_i \to \widetilde{\mathcal{P}}([0, 1]^s),$$

   plus a contradiction function pCF among attribute values.

Hence, each element $x \in X$ simultaneously has:

- A fuzzy membership interval $A_F(x) \subseteq [0, 1]$,
- A single fuzzy membership real $\alpha_F(x) \in [0, 1]$,

- A multi-attribute *hyperplithogenic* aggregator that can combine or compare these memberships or partial memberships with other elements or with attributes in pCF.

Example 2.5 (HyperPlithogenic Fuzzy Cubic Set)**.** Let $X = \{x_1, x_2\}$. Define a *fuzzy cubic set*:

$$\mathcal{F}_{\text{fuzzy-cubic}}(x_1) = ([0.2, 0.4], 0.3), \quad \mathcal{F}_{\text{fuzzy-cubic}}(x_2) = ([0.6, 0.8], 0.5).$$

Hence $A_F(x_1) = [0.2, 0.4]$, $\alpha_F(x_1) = 0.3$, $A_F(x_2) = [0.6, 0.8]$, and $\alpha_F(x_2) = 0.5$.

Next, define a *HyperPlithogenic* structure $\mathcal{H}_{\text{plitho}} = (P, \{v\}, \{Pv\}, \{\widetilde{pdf}\}, pCF)$ with:

$$P = \{x_1, x_2\}, \quad v \text{ is a single attribute}, \quad Pv = \{u_1, u_2\}.$$

A *hyper-DAF* might be:

$$\widetilde{pdf}(x_1, u_1) = \{(0.2, 0.5), (0.3, 0.7)\} \subseteq [0, 1]^2, \quad \widetilde{pdf}(x_2, u_1) = \{(0.6, 0.4)\}, \quad \ldots$$

And a contradiction function $pCF(u_1, u_2) \in [0, 1]^t$ for $t = 1$; say $pCF(u_1, u_2) = 0.4$, $pCF(u_2, u_1) = 0.4$.

Then

$$\text{HPFCS} = \left(\mathcal{F}_{\text{fuzzy-cubic}},\ \mathcal{H}_{\text{plitho}}\right)$$

is a *HyperPlithogenic Fuzzy Cubic Set*.

Definition 2.6 (HyperPlithogenic Intuitionistic Fuzzy Cubic Set (HIFCS))**.** Let X be a non-empty set. A *HyperPlithogenic Intuitionistic Fuzzy Cubic Set* on X is a structure

$$\left(\mathcal{F}_{\text{IF}-\text{cubic}},\ \mathcal{H}_{\text{plitho}}\right),$$

where:

1. $\mathcal{F}_{\text{IF}-\text{cubic}}$ is an *intuitionistic fuzzy cubic set*, assigning for each $x \in X$:

$$\mathcal{F}_{\text{IF}-\text{cubic}}(x) = \left(A_M(x), \alpha_M(x),\ A_N(x), \alpha_N(x)\right),$$

   in which:

   - $A_M(x) \subseteq [0, 1]$ is an interval for *membership*, $\alpha_M(x) \in [0, 1]$ is a single membership value,
   - $A_N(x) \subseteq [0, 1]$ is an interval for *non-membership*, $\alpha_N(x) \in [0, 1]$ a single non-membership value,
   - We require $0 \le \alpha_M(x) + \alpha_N(x) \le 1$ and $A_M(x), A_N(x)$ suitably restricted so that any real $m \in A_M(x)$ and $n \in A_N(x)$ satisfy $0 \le m + n \le 1$.

2. $\mathcal{H}_{\text{plitho}} = (P, \{v_i\}, \{Pv_i\}, \{\widetilde{pdf}_i\}, pCF)$ is a *HyperPlithogenic* structure, exactly as in Definition 2.4, assigning set-valued membership degrees for multi-attribute decisions, plus a contradiction function among attribute values.

Example 2.7 (HyperPlithogenic Intuitionistic Fuzzy Cubic Set)**.** Let $X = \{y_1, y_2\}$. For each $y \in X$, define:

- $A_M(y_1) = [0.2, 0.4], \quad \alpha_M(y_1) = 0.3,$
- $A_N(y_1) = [0.1, 0.2], \quad \alpha_N(y_1) = 0.2,$

such that $(0.3) + (0.2) = 0.5 \le 1$ and for any $m \in [0.2, 0.4]$, $n \in [0.1, 0.2]$, we have $m + n \le 0.6 \le 1$.

- $A_M(y_2) = [0.6, 0.7], \quad \alpha_M(y_2) = 0.25,$
- $A_N(y_2) = [0.0, 0.2], \quad \alpha_N(y_2) = 0.2,$

ensuring $0.25 + 0.2 = 0.45 \le 1$.

Hence $\mathcal{F}_{\text{IF-cubic}}$ is an intuitionistic fuzzy cubic set.

Next, let $\mathcal{H}_{\text{plitho}}$ be a hyperplithogenic aggregator with attribute(s) v having domain Pv, a hyper-DAF $\widetilde{pdf}(y,a) \subseteq [0,1]^s$, and $pCF(a,b) \in [0,1]^t$. The combined pair

$$\left(\mathcal{F}_{\text{IF-cubic}},\ \mathcal{H}_{\text{plitho}}\right)$$

forms a *HyperPlithogenic Intuitionistic Fuzzy Cubic Set*.

Definition 2.8 (HyperPlithogenic Neutrosophic Cubic Set (HNNCS))**.** Let X be a non-empty set. A *HyperPlithogenic Neutrosophic Cubic Set (HNNCS)* in X is a structure:

$$\left(\mathcal{N}_{\text{cubic}},\ \mathcal{H}_{\text{plitho}}\right),$$

where:

1. $\mathcal{N}_{\text{cubic}}$ is a *neutrosophic cubic set*, assigning for each $x \in X$ three pairs (or intervals) plus single reals for (T, I, F), for instance:

$$\mathcal{N}_{\text{cubic}}(x) \;=\; \left(A_T(x),\ \alpha_T(x),\ A_I(x),\ \alpha_I(x),\ A_F(x),\ \alpha_F(x)\right),$$

   with $A_T(x) \subseteq [0,1]$, $\alpha_T(x) \in [0,1]$, similarly for I, F, and $0 \le \alpha_T(x) + \alpha_I(x) + \alpha_F(x) \le 3$, plus constraints for each triple (t,i,f) in $A_T(x) \times A_I(x) \times A_F(x)$ so that $t + i + f \le 3$.

2. $\mathcal{H}_{\text{plitho}} = (P, \{v_i\}, \{Pv_i\}, \{\widetilde{pdf}_i\}, pCF)$ is a *HyperPlithogenic* aggregator, exactly as in Definitions 2.4 or 2.6, used to handle multi-attribute set-valued membership degrees across $[0,1]^s$ and a contradiction function pCF.

Example 2.9 (HyperPlithogenic Neutrosophic Cubic Set)**.** Let $X = \{z_1, z_2\}$. Suppose a *neutrosophic cubic set* $\mathcal{N}_{\text{cubic}}$ such that:

$$\mathcal{N}_{\text{cubic}}(z_1) = \left([0.2, 0.3],\ 0.25,\ \ [0.0, 0.1],\ 0.05,\ \ [0.3, 0.5],\ 0.4\right),$$

meaning $A_T(z_1) = [0.2, 0.3]$, $\alpha_T(z_1) = 0.25$, $A_I(z_1) = [0.0, 0.1]$, $\alpha_I(z_1) = 0.05$, $A_F(z_1) = [0.3, 0.5]$, $\alpha_F(z_1) = 0.4$. We require $0.25 + 0.05 + 0.4 = 0.7 \le 3$, and any triple (t,i,f) from $[0.2, 0.3] \times [0.0, 0.1] \times [0.3, 0.5]$ must satisfy $t + i + f \le 3$, which is obviously true.

Similarly for z_2:

$$\mathcal{N}_{\text{cubic}}(z_2) = \left([0.6, 0.7],\ 0.65,\ \ [0.1, 0.2],\ 0.15,\ \ [0.0, 0.2],\ 0.1\right).$$

Next, define a *HyperPlithogenic* structure $\mathcal{H}_{\text{plitho}} = (P, \dots)$ with attribute sets and a hyper-DAF:

$$\widetilde{pdf}_i(x, a_i) \subseteq [0,1]^s, \quad pCF(a,b) \in [0,1]^t,$$

for $i = 1, \dots, m$. The combination:

$$\left(\mathcal{N}_{\text{cubic}},\ \mathcal{H}_{\text{plitho}}\right)$$

becomes a *HyperPlithogenic Neutrosophic Cubic Set* on X.

2.2 n-SuperHyperPlithogenic Cubic Set

An n-SuperHyperPlithogenic Cubic Set is a generalized concept of the Plithogenic Cubic Set using the structure of an n-SuperHyperPlithogenic Set. The definition is presented below.

Definition 2.10 (n-SuperHyperPlithogenic Cubic Set)**.** Let X be a non-empty universe, and let

$$(C_{\text{Cubic}},\ \mathcal{H}_{\text{plitho}})$$

be a *HyperPlithogenic Cubic Set* as in Definition 2.1. We define an *n-SuperHyperPlithogenic Cubic Set (*SHPC_n*)* recursively as follows:

1. For $n = 1$,
$$\text{SHPC}_1 \ = \ \Big(C_{\text{Cubic}},\ \mathcal{H}_{\text{plitho}}\Big),$$
i.e. a standard HyperPlithogenic Cubic Set.
2. For $n \geq 2$, let
$$\text{SHPC}_n = \Big(\ C^{(n)}_{\text{Cubic}},\ \mathcal{H}^{(n)}_{\text{plitho}}\Big).$$
Here:
   - $C^{(n)}_{\text{Cubic}}$ is an *n-th level cubic expansion*, e.g. an iterative layering of interval- or single-valued membership expansions. Symbolically,
   $$C^{(n)}_{\text{Cubic}} \ = \ \big\{\,(x, A^{(n)}(x),\ \alpha^{(n)}(x)) \mid x \in X\big\},$$
   where $A^{(n)}(x)$ might be an n-th power set expansion or n-th interval layering, depending on the chosen model.
   - $\mathcal{H}^{(n)}_{\text{plitho}}$ is the *n-th SuperHyperPlithogenic structure*, i.e. we define
   $$p\tilde{d}f_i^{(n)} : P_n \times Pv_i \ \longrightarrow\ \widetilde{\mathcal{P}}^n([0,1]^s),$$
   or a similar n-level hyper aggregator, plus a contradiction function $pCF^{(n)}$ that captures n-th order expansions of attribute domains.

Hence, we obtain
$$\text{SHPC}_n = \Big(\ C^{(n)}_{\text{Cubic}},\ \mathcal{H}^{(n)}_{\text{plitho}}\Big),$$
called an *n-SuperHyperPlithogenic Cubic Set* on X.

Theorem 2.11. *(n-SuperHyperPlithogenic Cubic Set generalizes HyperPlithogenic Cubic Set and n-SuperHyperPlithogenic Set.)*

Let SHPC_n *be an n-SuperHyperPlithogenic Cubic Set on X. Then:*

1. SHPC_n *reduces to a* HyperPlithogenic Cubic Set *when* $n = 1$ *(the base level).*
2. SHPC_n *reduces to an* n-SuperHyperPlithogenic Set *if we ignore or trivialize the cubic membership structure* $(A^{(n)}(x),\ \alpha^{(n)}(x))$.

Proof. (1) Reduces to HPCS at $n = 1$.
By definition (cf. Definition 2.10 in the user's snippet for superhyper expansions), setting $n = 1$ yields $\text{SHPC}_1 = (C^{(1)}_{\text{Cubic}}, \mathcal{H}^{(1)}_{\text{plitho}})$ which is exactly a *HyperPlithogenic Cubic Set* as in Definition 2.1. No further expansions or iterative layering occur.

(2) Reduces to n-SuperHyperPlithogenic Set if we ignore the cubic portion.
Given $\text{SHPC}_n = \big(C^{(n)}_{\text{Cubic}}, \mathcal{H}^{(n)}_{\text{plitho}}\big)$, we can trivialize the cubic membership for each x by forcing $A^{(n)}(x)$ to

be a constant set $\{1\}$ or $[0,1]$, and $\alpha^{(n)}(x)$ to be 1 (or 0). This effectively *removes* the cubic layering from each element x and leaves us with the n-SuperHyperPlithogenic aggregator $\mathcal{H}^{(n)}_{\text{plitho}}$ alone. That aggregator is precisely an *n-SuperHyperPlithogenic Set* (Definition **??** in the user's snippet), because it has n-th order expansions of hyper-degree-of-appurtenance plus the contradiction function.

Hence, SHPC_n indeed generalizes both a single-level HPCS and an n-SuperHyperPlithogenic structure. □

Definition 2.12 (n-SuperHyperPlithogenic Fuzzy Cubic Set)**.** Let X be a non-empty set, and let $\mathcal{F}_{\text{fuzzy-cubic}}$ be a *fuzzy cubic set* on X, i.e. for each $x \in X$:

$$\mathcal{F}_{\text{fuzzy-cubic}}(x) = \big(A_F(x), \alpha_F(x)\big),$$

where $A_F(x) \subseteq [0,1]$ is interval-valued membership, $\alpha_F(x) \in [0,1]$ is single-valued membership. Then define a hyperplithogenic aggregator $\mathcal{H}^{n}_{\text{plitho}}$ at the *n-Super* level as

$$\Big(P_n, \{v_i\}_{i=1}^m, \{Pv_i\}_{i=1}^m, \{\widetilde{pdf}_i^{(n)}\}, pCF^{(n)}\Big),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$, $P_k = \tilde{\mathcal{P}}(P_{k-1})$, leading to P_n as an n-th nested hyper-power set of P_1.
- For each attribute v_i, Pv_i is its set of possible values.
- Each $\widetilde{pdf}_i^{(n)} : P_n \times Pv_i \to \tilde{\mathcal{P}}^n([0,1]^s)$ assigns an *n-level* set of membership vectors in $[0,1]^s$.
- $pCF^{(n)}$ is an n-level contradiction function among attribute values in $\bigcup_i Pv_i$.

We call

$$\text{n-SHPC-FCS} \;=\; \Big(\mathcal{F}_{\text{fuzzy-cubic}},\ \mathcal{H}^{n}_{\text{plitho}}\Big)$$

an ***n**-SuperHyperPlithogenic Fuzzy Cubic Set* on X.

Example 2.13 (Illustration of n-SHPC-FCS)**.** Let $X = \{x_1, x_2\}$. Suppose the fuzzy cubic set

$$\mathcal{F}_{\text{fuzzy-cubic}}(x_1) = ([0.2, 0.4], 0.3), \quad \mathcal{F}_{\text{fuzzy-cubic}}(x_2) = ([0.5, 0.6], 0.4).$$

Hence, $A_F(x_1) = [0.2, 0.4], \alpha_F(x_1) = 0.3$; $A_F(x_2) = [0.5, 0.6], \alpha_F(x_2) = 0.4$.

Next, define an n-SuperHyperPlithogenic structure. Let $P_1 = \{x_1, x_2\}$, for $k \geq 2$ do $P_k = \tilde{\mathcal{P}}(P_{k-1})$. Suppose we have one attribute v with domain $Pv = \{u_1, u_2\}$. Then

$$\widetilde{pdf}^{(n)}(A, u_1) \subseteq \tilde{\mathcal{P}}^n\big([0,1]^2\big), \quad pCF^{(n)}(u_1, u_2) \in \big([0,1]^t\big)^n,$$

ensuring multi-level set membership. The combined structure

$$\text{n-SHPC-FCS} = \Big(\mathcal{F}_{\text{fuzzy-cubic}},\ \big(P_n, \ldots, \widetilde{pdf}^{(n)}, pCF^{(n)}\big)\Big)$$

is an n-SuperHyperPlithogenic Fuzzy Cubic Set.

Definition 2.14 (n-SuperHyperPlithogenic Intuitionistic Fuzzy Cubic Set)**.** Let X be a non-empty set. An *intuitionistic fuzzy cubic set* $\mathcal{F}_{\text{IF-cubic}}$ on X assigns each $x \in X$ the tuple:

$$\Big(A_M(x),\ \alpha_M(x),\ A_N(x),\ \alpha_N(x)\Big),$$

where

- $A_M(x) \subseteq [0,1]$ (interval membership), $\alpha_M(x) \in [0,1]$ (single membership),
- $A_N(x) \subseteq [0,1]$ (interval non-membership), $\alpha_N(x) \in [0,1]$ (single non-membership),

- For any $m \in A_M(x)$ and $n \in A_N(x)$ and the single values $\alpha_M(x)$, $\alpha_N(x)$, we satisfy $m+n \le 1$ and $\alpha_M(x)+\alpha_N(x) \le 1$.

We then embed this cubic set into an $\boldsymbol{n}$-SuperHyperPlithogenic aggregator

$$\left(P_n, \{v_i\}, \{Pv_i\}, \{\widetilde{pdf}_i^{(n)}\},\ pCF^{(n)}\right)$$

with P_k built via $\tilde{\mathcal{P}}(\cdot)$ up to level n. Combining them yields the *n-SuperHyperPlithogenic Intuitionistic Fuzzy Cubic Set*:

$$\text{n-SHPC-IFCS} \ = \ \left(\mathcal{F}_{\text{IF-cubic}},\ \mathcal{H}^n_{\text{plitho}}\right).$$

Example 2.15 (n-SHPC-IFCS Illustrative)**.** Let $X = \{y_1, y_2\}$. Suppose for an *intuitionistic fuzzy cubic set*:

$$\mathcal{F}_{\text{IF-cubic}}(y_1) = \Big([0.2, 0.3],\ 0.25,\ \ [0.0, 0.2],\ 0.1\Big),$$

$$\mathcal{F}_{\text{IF-cubic}}(y_2) = \Big([0.4, 0.6],\ 0.5,\ \ [0.1, 0.3],\ 0.2\Big).$$

Hence each y_i has an interval membership $A_M(y_i)$ plus single membership $\alpha_M(y_i)$, and an interval non-membership $A_N(y_i)$ plus single non-membership $\alpha_N(y_i)$. Check $m \in A_M(y_i), n \in A_N(y_i)$ implies $m+n \le 1$, similarly $\alpha_M(y_i) + \alpha_N(y_i) \le 1$.

Next, let $\mathcal{H}^n_{\text{plitho}}$ be an n-SuperHyperPlithogenic aggregator with domain sets $P_1 \subseteq X$, $P_k = \tilde{\mathcal{P}}(P_{k-1})$, attribute sets $\{v_i\}$, a hyper-DAF $\widetilde{pdf}_i^{(n)}$, and contradiction function $pCF^{(n)}$. Then

$$\text{n-SHPC-IFCS} = \left(\mathcal{F}_{\text{IF-cubic}},\ \mathcal{H}^n_{\text{plitho}}\right)$$

forms an n-SuperHyperPlithogenic Intuitionistic Fuzzy Cubic Set.

Definition 2.16 (n-SuperHyperPlithogenic Neutrosophic Cubic Set)**.** Let X be a non-empty set. A *Neutrosophic Cubic Set* on X, call it $\mathcal{N}_{\text{cubic}}$, assigns each $x \in X$:

$$\mathcal{N}_{\text{cubic}}(x) \ = \ \Big(A_T(x), \alpha_T(x),\ A_I(x), \alpha_I(x),\ A_F(x), \alpha_F(x)\Big),$$

where for each x, $A_T(x), A_I(x), A_F(x) \subseteq [0,1]$ are *interval* (or set) memberships for *truth*, *indeterminacy*, and *falsity*, while $\alpha_T(x), \alpha_I(x), \alpha_F(x) \in [0,1]$ are single-valued memberships for (T, I, F), with $t + i + f \le 3$ for $t \in A_T(x), i \in A_I(x), f \in A_F(x)$ and $\alpha_T(x) + \alpha_I(x) + \alpha_F(x) \le 3$.

An *n-SuperHyperPlithogenic Neutrosophic Cubic Set* (n-SHPC-NCS) is formed by combining $\mathcal{N}_{\text{cubic}}$ with an n-SuperHyperPlithogenic aggregator

$$\mathcal{H}^n_{\text{plitho}} = (P_n, \{v_i\}, \{Pv_i\}, \{\widetilde{pdf}_i^{(n)}\}, pCF^{(n)}),$$

where each $\widetilde{pdf}_i^{(n)}$ is a hyper-set mapping to $\tilde{\mathcal{P}}^n([0,1]^s)$ and $pCF^{(n)}$ is an n-level contradiction measure. Formally,

$$\text{n-SHPC-NCS} \ = \ \left(\ \mathcal{N}_{\text{cubic}},\ \mathcal{H}^n_{\text{plitho}}\right).$$

Example 2.17 (n-SHPC-NCS Illustrative)**.** Let $X = \{z_1, z_2\}$. Suppose for each $z \in X$:

$$\mathcal{N}_{\text{cubic}}(z_1) = \Big([0.2, 0.3], 0.25,\ [0.0, 0.1], 0.05,\ [0.4, 0.5], 0.3\Big),$$

meaning $A_T(z_1) = [0.2, 0.3]$, $\alpha_T(z_1) = 0.25$, $A_I(z_1) = [0.0, 0.1]$, $\alpha_I(z_1) = 0.05$, $A_F(z_1) = [0.4, 0.5]$, $\alpha_F(z_1) = 0.3$. We require that $t + i + f \le 3$ for $(t, i, f) \in A_T(z_1) \times A_I(z_1) \times A_F(z_1)$ and $(\alpha_T + \alpha_I + \alpha_F) \le 3$.

Similarly,

$$\mathcal{N}_{\text{cubic}}(z_2) = \Big([0.6, 0.7], 0.66,\ [0.0, 0.2], 0.15,\ [0.0, 0.1], 0.10\Big).$$

Then define an n-SuperHyperPlithogenic aggregator $\mathcal{H}^n_{\text{plitho}} = (P_n, \dots)$, with $P_1 = \{z_1, z_2\}$, $P_k = \tilde{\mathcal{P}}(P_{k-1})$ for $k = 2, \dots, n$. We have attributes $\{v_i\}$, possible values $\{Pv_i\}$, hyper-DAFs $\{\widetilde{pdf}_i^{(n)}\}$, and a contradiction function $pCF^{(n)}$. The combination

$$\text{n-SHPC-NCS} = \left(\mathcal{N}_{\text{cubic}},\ \mathcal{H}^n_{\text{plitho}}\right)$$

is an n-SuperHyperPlithogenic Neutrosophic Cubic Set.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 7
L-Neutrosophic set and Nonstationary Neutrosophic set

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Abstract

Fuzzy sets extend classical set theory by assigning each element a membership degree in the interval $[0, 1]$, effectively modeling partial or uncertain membership. The *Neutrosophic Set* framework enhances fuzzy sets by introducing three independent membership components: truth, indeterminacy, and falsity, each ranging within $[0, 1]$, providing a robust means of representing uncertainty and contradictions. The *Plithogenic Set* builds upon classical and fuzzy sets by integrating attributes, their possible values, and a measure of contradiction, enabling the modeling of multi-dimensional and contradictory data for complex decision-making scenarios. In this paper, we extend L-fuzzy sets and nonstationary fuzzy sets using Neutrosophic and Plithogenic sets, and briefly analyze their properties.

Keywords: Plithogenic set, Fuzzy Set, Neutrsophic set, L-fuzzy set, Nonstationary fuzzy set

1 Preliminaries and Definitions

This section provides a concise explanation of the key preliminaries and definitions.

1.1 Uncertain Set

To address uncertainty, vagueness, and imprecision in decision-making, various set-theoretic frameworks have been developed. Among these, Fuzzy Sets, first introduced by Zadeh, represent a groundbreaking advancement in capturing partial or uncertain membership [46–54].

Neutrosophic Sets, introduced by Smarandache, provide a flexible and robust framework for handling indeterminacy and uncertainty. They extend the concept of Fuzzy Sets by incorporating additional dimensions of membership: truth, indeterminacy, and falsity [10, 11, 13, 17, 20–23, 27, 28, 37, 38, 42]. Neutrosophic Sets are known for their ability to generalize Fuzzy Sets.

More recently, Plithogenic Sets, introduced and developed by Smarandache, have emerged as a powerful tool for modeling multi-dimensional uncertainty and contradictions in complex scenarios. By incorporating attributes, their possible values, and a contradiction measure, Plithogenic Sets extend both Fuzzy Sets and Neutrosophic Sets, offering a highly versatile framework for decision-making [9, 12, 14–16, 18, 19, 24–26, 39–41]. Plithogenic Sets are widely recognized for their capacity to generalize both Fuzzy Sets and Neutrosophic Sets.

The definitions of Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets are provided below.

Definition 1.1. [46, 51] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is referred to as a *fuzzy relation on* τ if:

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 1.2 (Neutrosophic Set)**.** [37, 38] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is defined by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These functions satisfy the following constraint:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 1.3. [40,41] Let S be a universal set and $P \subseteq S$. A *Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v: an attribute.
- Pv: the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$: the *Degree of Appurtenance Function (DAF)*.
- $pCF : Pv \times Pv \to [0,1]^t$: the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a,a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a,b) = pCF(b,a)$$

2 Results of This Paper

This section presents the main findings discussed in this paper.

2.1 L-neutrosophic set and L-plithogenic set

An L-fuzzy set maps elements of a universal set X to a complete lattice L, generalizing membership degrees [1,4–8,30,31,35,36,43]. This concept is extended using Neutrosophic sets and Plithogenic sets.

Definition 2.1 (L-fuzzy set)**.** [31] Let X be a universal set and L be a complete lattice with a partial order $\leq$, supremum $\vee$, and infimum $\wedge$. An *L-fuzzy set* A on X is defined as a mapping:

$$A : X \to L,$$

where $A(x) \in L$ represents the membership degree of $x \in X$ in the fuzzy set A, and satisfies the lattice operations induced by L:

- $A(x) = \top_L$ indicates full membership.
- $A(x) = \bot_L$ indicates no membership.

Definition 2.2 (L-Neutrosophic Set)**.** Let X be a universal set, and let L be a complete lattice with top element $\top_L$ and bottom element $\bot_L$. An *L-Neutrosophic Set* $\mathcal{A}$ on X is characterized by three lattice-valued membership functions:

$$T_{\mathcal{A}}, \quad I_{\mathcal{A}}, \quad F_{\mathcal{A}} : X \longrightarrow L,$$

where, for each $x \in X$, the values $T_{\mathcal{A}}(x)$, $I_{\mathcal{A}}(x)$, and $F_{\mathcal{A}}(x)$ are elements of L (i.e., they belong to the lattice). We interpret:

- $T_{\mathcal{A}}(x)$ as the *truth degree* of x,
- $I_{\mathcal{A}}(x)$ as the *indeterminacy degree* of x,
- $F_{\mathcal{A}}(x)$ as the *falsity degree* of x.

We require a lattice-theoretic analogue of the classical neutrosophic constraint $0 \leq T+I+F \leq 3$. One possible approach is to impose:

$$T_{\mathcal{A}}(x) \ \vee \ I_{\mathcal{A}}(x) \ \vee \ F_{\mathcal{A}}(x) \ \ \leq \ \ \top_L,$$

ensuring that no combination of truth, indeterminacy, and falsity exceeds the top element in the lattice sense, or equivalently:

$$T_{\mathcal{A}}(x) \ \wedge \ I_{\mathcal{A}}(x) \ \wedge \ F_{\mathcal{A}}(x) \ \ \geq \ \ \bot_L,$$

depending on how one formalizes the neutrosophic sum constraint in lattice terms. In many treatments, we simply leave it as:

$$\big(T_{\mathcal{A}}(x), I_{\mathcal{A}}(x), F_{\mathcal{A}}(x)\big) \ \in \ L^3,$$

with the understanding that each membership triple remains bounded by $\top_L$ in some partial order sense.

Hence, an L-Neutrosophic Set is given by:

$$\mathcal{A} \ = \ \big\{ \langle x, \ T_{\mathcal{A}}(x), \ I_{\mathcal{A}}(x), \ F_{\mathcal{A}}(x) \rangle \ \mid \ x \in X \big\}.$$

Remark 2.3 (Membership Constraints)**.** Depending on the desired interpretation, one could impose additional conditions such as:

$$T_{\mathcal{A}}(x) \ \vee \ I_{\mathcal{A}}(x) \ \vee \ F_{\mathcal{A}}(x) \ \leq \ \top_L \quad \text{and} \quad T_{\mathcal{A}}(x) \ \wedge \ I_{\mathcal{A}}(x) \ \wedge \ F_{\mathcal{A}}(x) \ \geq \ \bot_L,$$

or analogues of $T+I+F \leq 3$ in the lattice setting. The exact constraint depends on how we embed neutrosophic addition and order into L. The general idea is that each point has three membership degrees in L.

Theorem 2.4. *An L-Neutrosophic Set $\mathcal{A}$ on X generalizes both L-Fuzzy Sets and* Neutrosophic Sets*. Specifically:*

1. *If $I_{\mathcal{A}}(x) = \bot_L$ and $F_{\mathcal{A}}(x)$ is taken as the lattice complement of $T_{\mathcal{A}}(x)$ (or $\bot_L$) for all $x \in X$, then $\mathcal{A}$ reduces to an L-Fuzzy Set.*
2. *If the lattice L is taken as $[0,1]$ with usual order, $\vee = \max$, $\wedge = \min$, $\top_L = 1$, $\bot_L = 0$, then $\mathcal{A}$ becomes a classical neutrosophic set $\{(x, T_A(x), I_A(x), F_A(x))\}$ with $T_A(x), I_A(x), F_A(x) \in [0,1]$.*

Proof. **(1)** Assume that $I_{\mathcal{A}}(x) \equiv \bot_L$ (the bottom element of L) and let $F_{\mathcal{A}}(x)$ be either the lattice complement of $T_{\mathcal{A}}(x)$ in L or simply $\bot_L$. Then for each $x \in X$, the triple $\big(T_{\mathcal{A}}(x), \bot_L, F_{\mathcal{A}}(x)\big)$ effectively encodes a single membership degree $T_{\mathcal{A}}(x)$. Thus, the entire structure reduces to a mapping $x \mapsto T_{\mathcal{A}}(x) \in L$, which is precisely an L-Fuzzy Set.

(2) If $L \equiv [0,1]$ with standard order, top $= 1$, bottom $= 0$, and $\vee = \max$, $\wedge = \min$, then each triple $\big(T_{\mathcal{A}}(x), I_{\mathcal{A}}(x), F_{\mathcal{A}}(x)\big)$ lies in $[0,1]^3$. By imposing $T_{\mathcal{A}}(x) + I_{\mathcal{A}}(x) + F_{\mathcal{A}}(x) \leq 3$, we exactly match the definition of a *neutrosophic set* in the standard sense. Hence, $\mathcal{A}$ generalizes the classical neutrosophic framework. □

Definition 2.5 (L-Plithogenic Set)**.** Let S be a universal set, and $P \subseteq S$. Let v be an attribute taking values in Pv. Suppose L is a complete lattice, and let $s \geq 1$, $t \geq 1$ be fixed. An *L-Plithogenic Set* of dimension (s,t), denoted by

$$L\text{-}PS^{(s,t)},$$

is defined as:

$$L\text{-}PS \ = \ \big(P, \ v, \ Pv, \ pdf_L, \ pCF_L\big),$$

where

$$pdf_L \ : \ P \times Pv \ \longrightarrow \ L^s, \quad pCF_L \ : \ Pv \times Pv \ \longrightarrow \ L^t,$$

are *lattice-valued* generalizations of the Degree of Appurtenance Function (DAF) and the Degree of Contradiction Function (DCF), respectively. Specifically, for each $(x,a) \in P \times Pv$, $pdf_L(x,a)$ is an s-dimensional tuple in L^s, e.g. $\big(\ell_1, \ldots, \ell_s\big) \in L^s$. Similarly, for each $(a,b) \in Pv \times Pv$, $pCF_L(a,b) \in L^t$.

We interpret:

- $pdf_L(x, a)$ as the *lattice-valued membership* of x in the plithogenic set for the attribute value a,
- $pCF_L(a, b)$ as the *lattice-valued contradiction* between two attribute values a and b.

The usual plithogenic axioms (reflexivity, symmetry, etc.) may be stated in the lattice setting, e.g.

$$pCF_L(a, a) = \perp_L \quad \text{(or a designated contradiction bottom)}, \quad pCF_L(a, b) = pCF_L(b, a) \quad \text{(symmetry)},$$

and so on, depending on how contradiction is embedded into the lattice L^t.

Theorem 2.6. *An L-Plithogenic Set L-PS of dimension* (s, t) *(Definition 2.5) generalizes:*

1. *The* Plithogenic Set *if* $L \equiv [0, 1]$ *with standard operations,*
2. *The* L-Neutrosophic Set *(when we restrict to a single attribute or unify the attribute perspective), especially for* $s = 3$*,*
3. *The* L-Fuzzy Set *(when* $s = 1$ *and we treat each membership as a single lattice value).*

Proof. **(1)** If we take $L \equiv [0, 1]$ (with $\vee = \max$, $\wedge = \min$), then $pdf_L(x, a) \in [0, 1]^s$ and $pCF_L(a, b) \in [0, 1]^t$. By letting s and t match the dimension of membership and contradiction in a classical plithogenic set, we recover the usual *Plithogenic Set* in $[0, 1]$.

(2) If we interpret each membership vector $(\ell_1, \ell_2, \ell_3) \in L^3$ as (T, I, F) (the L-Neutrosophic viewpoint), and reduce or fix the attribute range Pv suitably, then we effectively replicate an L-Neutrosophic Set. The difference is that L-Plithogenic also includes a contradiction function $pCF_L(a, b)$. If we disregard or simplify that function, we see that each x has a triple membership in L.

(3) When $s = 1$, each membership is just one element in L. Then L-Plithogenic merges the lattice-based fuzzy membership with the plithogenic approach. If we further reduce it to a single attribute or no contradiction dimension, we get an L-Fuzzy Set.

Thus, by varying the dimension s (and the lattice L), we capture all special cases: classical plithogenic ($L = [0, 1]$), L-neutrosophic ($s = 3$), or L-fuzzy ($s = 1$) sets. □

2.2 Nonstationary Neutrosophic Set and Nonstationary Plithogenic Set

A nonstationary fuzzy set is a fuzzy set with a time-dependent membership function $\mu_{\dot{A}}(t, x)$, reflecting dynamic parameter variations [2, 2, 3, 29, 32–34, 44, 45]. This concept is extended using Neutrosophic sets and Plithogenic sets.

Definition 2.7 (nonstationary fuzzy set)**.** (cf. [2, 29, 33]) *nonstationary fuzzy set* $\dot{A}$ of the universe of discourse X is characterized by a nonstationary membership function:

$$\mu_{\dot{A}} : T \times X \to [0, 1]$$

that associates with each element $(t, x) \in T \times X$ a time-specific variation of the membership function $\mu_A(x)$ of a standard fuzzy set A. The nonstationary fuzzy set $\dot{A}$ is expressed as:

$$\dot{A} = \int_{t \in T} \int_{x \in X} \mu_{\dot{A}}(t, x)/x/t.$$

The membership function $\mu_{\dot{A}}(t, x)$ is defined in terms of a perturbation of $\mu_A(x)$, where $\mu_A(x)$ depends on a set of parameters $p_1, p_2, \ldots, p_m$:

$$\mu_A(x) = \mu_A(x; p_1, \ldots, p_m).$$

For a nonstationary fuzzy set, these parameters are functions of time t, leading to:

$$\mu_{\dot{A}}(t, x) = \mu_A(x; p_1(t), \ldots, p_m(t)),$$

where each parameter varies over time according to a perturbation function:

$$p_i(t) = p_i + k_i f_i(t), \quad i = 1, \ldots, m.$$

Definition 2.8 (Nonstationary Neutrosophic Set)**.** Let X be a non-empty set, and let T be a time domain (which may be continuous or discrete). A *nonstationary neutrosophic set* $\dot{A}$ on X is defined by three *time-dependent* membership functions:

$$T_{\dot{A}} : T \times X \to [0,1], \quad I_{\dot{A}} : T \times X \to [0,1], \quad F_{\dot{A}} : T \times X \to [0,1],$$

where for each $(t,x) \in T \times X$, the values $T_{\dot{A}}(t,x)$, $I_{\dot{A}}(t,x)$, and $F_{\dot{A}}(t,x)$ represent the *truth*, *indeterminacy*, and *falsity* degrees of x in $\dot{A}$ at time t. These satisfy

$$0 \le T_{\dot{A}}(t,x) + I_{\dot{A}}(t,x) + F_{\dot{A}}(t,x) \le 3,$$

for all $(t,x) \in T \times X$.

Analogous to the nonstationary fuzzy set, each component can be viewed as a *time-varying perturbation* of the corresponding membership function in a *stationary* neutrosophic set. Specifically, if T_A, I_A, and F_A define a classical neutrosophic set A on X (with no time dependence), then we introduce a set of time-dependent parameters

$$\{p_{T,i}(t),\ p_{I,j}(t),\ p_{F,k}(t)\} \quad \text{for } i,j,k \in \mathcal{I},$$

and define

$$\begin{aligned} T_{\dot{A}}(t,x) &= T_A\big(x;\ p_{T,1}(t), \ldots, p_{T,m}(t)\big), \\ I_{\dot{A}}(t,x) &= I_A\big(x;\ p_{I,1}(t), \ldots, p_{I,n}(t)\big), \\ F_{\dot{A}}(t,x) &= F_A\big(x;\ p_{F,1}(t), \ldots, p_{F,p}(t)\big), \end{aligned}$$

where each parameter function $p_{\cdot,\cdot}(t)$ may evolve over time via a perturbation rule, e.g.

$$p_{T,i}(t) = p_{T,i} + k_{T,i} \cdot f_{T,i}(t),$$

and similarly for the indeterminacy and falsity parameters. In integral notation, the nonstationary neutrosophic set $\dot{A}$ can be expressed as

$$\dot{A} = \int_{t \in T} \int_{x \in X} \big(T_{\dot{A}}(t,x),\ I_{\dot{A}}(t,x),\ F_{\dot{A}}(t,x)\big) / x / t.$$

Theorem 2.9. *A nonstationary neutrosophic set $\dot{A}$ as in Definition 2.8 generalizes both* **(1)** *a nonstationary fuzzy set and* **(2)** *a classical neutrosophic set. Specifically:*

1. *If $I_{\dot{A}}(t,x) = 0$ and $F_{\dot{A}}(t,x) = 1 - T_{\dot{A}}(t,x)$ for all $(t,x) \in T \times X$, then $\dot{A}$ is effectively a* nonstationary fuzzy set.
2. *If $|T| = 1$ (no time variation), then $\dot{A}$ reduces to a classical (stationary) neutrosophic set.*

Proof. **(1)** For the first statement, setting $I_{\dot{A}}(t,x) \equiv 0$ eliminates indeterminacy, and letting $F_{\dot{A}}(t,x) = 1 - T_{\dot{A}}(t,x)$ reduces the triple $(T_{\dot{A}}(t,x), 0, 1 - T_{\dot{A}}(t,x))$ to a single membership value $T_{\dot{A}}(t,x)$ in $[0,1]$. Hence, we obtain precisely the definition of a nonstationary fuzzy set $\dot{A}$.

(2) For the second statement, if the time domain T is a singleton $\{t_0\}$, then $T_{\dot{A}}(t_0,x) \equiv T_A(x)$, $I_{\dot{A}}(t_0,x) \equiv I_A(x)$, $F_{\dot{A}}(t_0,x) \equiv F_A(x)$ define a classical neutrosophic set $\{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}$. Therefore, no time dependence remains, and we recover the stationary (classical) neutrosophic framework. □

Definition 2.10 (Nonstationary Plithogenic Set)**.** Let S be a universal set, and let $P \subseteq S$. Let v be an attribute taking values in Pv. Let T be a time domain. Suppose we have integers $s \ge 1$ (the dimension of membership) and $t \ge 1$ (the dimension of contradiction). A *nonstationary plithogenic set* $\dot{PS}$ of dimension (s,t) is defined as:

$$\dot{PS} = \big(P,\ v,\ Pv,\ p\dot{d}f,\ p\dot{C}F\big),$$

where

$$p\dot{d}f : T \times P \times Pv \longrightarrow [0,1]^s, \quad p\dot{C}F : T \times Pv \times Pv \longrightarrow [0,1]^t,$$

are *time-dependent* generalizations of the Degree of Appurtenance Function (DAF) and the Degree of Contradiction Function (DCF), respectively.

For each fixed $t \in T$, the pair

$$pdf_t(x, a) = \dot{pdf}(t, x, a) \quad \text{and} \quad pCF_t(a, b) = p\dot{C}F(t, a, b)$$

defines a *classical plithogenic set* of dimension (s, t) (assuming reflexivity and symmetry axioms hold at each time). In integral notation, we may write:

$$\dot{PS} = \int_{t \in T} \Big(P,\ v,\ Pv,\ pdf_t,\ pCF_t\Big) \,/\, t,$$

indicating that the plithogenic membership and contradiction measures vary with time.

Nonstationary Parameters. Similarly to the nonstationary neutrosophic set, one can parameterize:

$$\dot{pdf}(t, x, a) = pdf\big(x, a;\ p_1(t), \ldots, p_m(t)\big),$$

$$p\dot{C}F(t, a, b) = pCF\big(a, b;\ q_1(t), \ldots, q_n(t)\big),$$

where each parameter function $p_i(t)$ or $q_j(t)$ evolves in time via perturbation rules, enabling dynamic changes in membership and contradiction values.

Theorem 2.11. *A nonstationary plithogenic set $\dot{PS}$ of dimension (s, t) (Definition 2.10) generalizes:*

1. *The* classical plithogenic set *(when $|T| = 1$, no time dependence).*
2. *A* nonstationary neutrosophic set *(when $s = 3$ and we interpret the membership dimension as (T, I, F)).*
3. *A* nonstationary fuzzy set *(when $s = 1$, effectively yielding a single membership dimension).*

Proof. **(1)** If T is a single point (no time variation), then $\dot{pdf}(t, x, a) \equiv pdf(x, a)$ and $p\dot{C}F(t, a, b) \equiv pCF(a, b)$. This matches a *classical plithogenic set* of dimension (s, t), as in [40, 41].

(2) If $s = 3$, interpreting each membership value in $[0, 1]^3$ as (T, I, F) degrees yields a *time-dependent neutrosophic-like membership structure*. Hence, $\dot{PS}$ includes the notion of nonstationary neutrosophic sets, with an additional plithogenic contradiction function if so desired.

(3) If $s = 1$, each membership entry is a single scalar in $[0, 1]$, so $\dot{PS}$ captures a *nonstationary fuzzy-like* membership dimension, extended by a time-varying contradiction function.

Thus, adjusting the cardinality of T and the dimension s (and possibly t for contradiction) recovers each special case, proving the stated unification properties. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 8
Forest HyperPlithogenic Set and Forest HyperRough Set

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Abstract

The Plithogenic Set is widely recognized for generalizing concepts such as Fuzzy Sets and Neutrosophic Sets. Rough Sets offer a framework for approximating subsets through lower and upper bounds defined by equivalence relations, effectively capturing uncertainty in classification and data analysis. These foundational ideas have been extended to concepts like Hyperplithogenic Sets, Superhyperplithogenic Sets, Hyperrough Sets, and Superhyperrough Sets. In this paper, we further extend these notions by introducing the Forest Hyperplithogenic Set, the Forest SuperHyperplithogenic Set, the Forest HyperRough Set, and the Forest SuperHyperRough Set. These frameworks represent generalized extensions of existing set-theoretic paradigms.

Keywords: Rough set, Plithogenic Set, Hyperstructure, Superhyperstructure
MSC 2010 classifications: 03E72: Fuzzy set theory, 03B52: Fuzzy logic; logic of vagueness

1 Short Introduction of this Paper

1.1 Plithogenic Sets and Rough Sets

Numerous frameworks have been developed to handle uncertainty, reflecting its pervasive role across various disciplines. These include foundational concepts such as Fuzzy Sets [74–80], Intuitionistic Fuzzy Sets [5–9], and Neutrosophic Sets [25, 26, 56, 57, 67]. Advanced extensions such as Soft Sets [12, 13, 31, 35, 36, 39, 51], Hypersoft Sets [1, 16, 27, 49, 58, 61], and SuperHypersoft Sets [20, 38, 62] have further enriched the theoretical landscape. In addition, Rough Sets [40, 41, 44, 47] have provided robust tools for addressing uncertainty in data classification.

This paper focuses on two prominent frameworks: Plithogenic Sets and Rough Sets. Plithogenic Sets extend traditional set theory by introducing appurtenance degrees and contradiction measures, offering a powerful approach to decision-making in contexts characterized by intricate and conflicting criteria [14, 19, 28, 59, 60, 68]. In contrast, Rough Sets provide a method for approximating subsets through lower and upper bounds defined by equivalence relations, effectively modeling uncertainty in classification and data analysis [40–47].

Within both the Plithogenic and Rough Set frameworks, advanced constructs such as Hyperplithogenic Sets [15, 19], Superhyperplithogenic Sets [15, 19], Hyperrough [15, 19], and Superhyperrough Sets [19, 22] have been developed, highlighting the growing sophistication in managing uncertainty across diverse applications.

1.2 Our Contribution in This Paper

This section outlines the contributions made in this paper. We introduce and explore the following concepts: the Forest Hyperplithogenic Set, the Forest SuperHyperplithogenic Set, the Forest HyperRough Set, and the Forest SuperHyperRough Set. These represent generalized extensions of existing set-theoretic frameworks. The development of these concepts is heavily inspired by the principles underlying the Forest Hypersoft Set, which serves as a foundational reference throughout this work [49].

2 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

2.1 Plithogenic Set

A Plithogenic Set is a mathematical framework that incorporates multi-valued degrees of appurtenance and contradictions, making it suitable for complex decision-making processes. Various studies have been conducted on Plithogenic Sets [2, 3, 17, 21, 48, 52–54, 66, 71]. The definition is presented below.

Definition 2.1. [59, 60] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* [1]
- $pCF : Pv \times Pv \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF).*

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a, a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a, b) = pCF(b, a)$$

2.2 HyperPlithogenic Set and SuperHyperPlithogenic Set

In this subsection, we present the definitions of the HyperPlithogenic Set and the SuperHyperPlithogenic Set [15, 19, 21, 23]. The HyperPlithogenic Set is defined within the framework of hyperstructures, while the SuperHyperPlithogenic Set extends this notion using superhyperstructures [14, 24, 63–65].

First, the definitions of the n-th Powerset, hyperstructures, and superhyperstructures are provided below. These concepts have been applied to various frameworks.

Definition 2.2 (n-th Powerset)**.** (cf. [18, 55, 65]) Let H be a non-empty set. The n-th powerset, denoted $P_n(H)$, is defined recursively as follows:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

The n-th non-empty powerset, denoted $P_n^*(H)$, is defined by excluding the empty set:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)),$$

where $P^*(H)$ is the powerset of H with the empty set removed.

Definition 2.3 (Hyperstructure)**.** (cf. [18, 55, 65]) A *Hyperstructure* generalizes the classical structure by extending operations to the powerset of a base set. It is defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where:

[1] It is important to note that the definition of the Degree of Appurtenance Function varies across different papers. Some studies define this concept using the power set, while others simplify it by avoiding the use of the power set [70]. The author has consistently defined the Classical Plithogenic Set without employing the power set.

- S is the base set,
- $\mathcal{P}(S)$ is the powerset of S,
- $\circ$ is a hyperoperation defined on subsets of $\mathcal{P}(S)$.

Definition 2.4 (n-Superhyperstructure)**.** (cf. [55, 65]) An *n-Superhyperstructure* builds on the concept of Hyperstructure by operating on the n-th powerset of a base set. Formally, it is defined as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where:

- S is the base set,
- $\mathcal{P}_n(S)$ is the n-th powerset of S,
- $\circ$ is a hyperoperation defined on elements of $\mathcal{P}_n(S)$.

These definitions establish the foundational framework necessary for exploring the HyperPlithogenic Set and the SuperHyperPlithogenic Set. The definitions of the HyperPlithogenic Set and the SuperHyperPlithogenic Set are presented below [15, 19, 21].

Definition 2.5 (HyperPlithogenic Set)**.** [15, 19, 21] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set HPS* over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : \left(\bigcup_{i=1}^n Pv_i\right) \times \left(\bigcup_{i=1}^n Pv_i\right) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

Definition 2.6 (n-SuperHyperPlithogenic Set)**.** [15,19,21] Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes, each associated with a set of possible values P_{v_i}. An n-SuperHyperPlithogenic Set ($SHPS_n$) is defined recursively as:

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$,
$$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$
represents the k-th nested family of non-empty subsets of P_1.
- For each attribute $v_i \in V$, P_{v_i} is the set of possible values of the attribute v_i.
- For each k-th level subset P_k, $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P_n$ and attribute value $a_i \in P_{v_i}$ a subset of $[0,1]^s$.
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:
  1. Reflexivity: $pCF^{(n)}(a,a) = 0$ for all $a \in \bigcup_{i=1}^n P_{v_i}$,
  2. Symmetry: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ for all $a, b \in \bigcup_{i=1}^n P_{v_i}$.
- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

2.3 Rough Set, HyperRough Set, and Superhyperrough set

A Rough Set approximates a subset using lower and upper bounds based on equivalence classes, capturing certainty and uncertainty in membership [40, 41, 41–47]. The definitions are provided below.

Definition 2.7 (Rough Set Approximation)**.** [41] Let X be a non-empty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or indiscernibility relation) on X. The equivalence relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for $x \in X$, where:

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ of U are defined as follows:

1. *Lower Approximation* $\underline{U}$:
$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$
The lower approximation $\underline{U}$ includes all elements of X whose equivalence classes are entirely contained within U. These are the elements that *definitely* belong to U.

2. *Upper Approximation* $\overline{U}$:
$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$
The upper approximation $\overline{U}$ contains all elements of X whose equivalence classes have a non-empty intersection with U. These are the elements that *possibly* belong to U.

The pair $(\underline{U}, \overline{U})$ forms the *rough set* representation of U, satisfying the relationship:

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

The *HyperRough Set* is a concept that adapts the framework of the HyperSoft Set [58] to Rough Set theory. Its formal definition is provided below.

Definition 2.8 (HyperRough Set)**.** [19, 22] Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product of these domains as:

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on X, where $[x]_R$ denotes the equivalence class of x under R.

A *HyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns a subset $F(a) \subseteq X$ to each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$.
- For each $a \in J$, the rough set $(\underline{F(a)}, \overline{F(a)})$ is defined as:

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

The *lower approximation* $\underline{F(a)}$ represents the set of elements in X whose equivalence classes are entirely contained within $F(a)$, while the *upper approximation* $\overline{F(a)}$ includes elements whose equivalence classes have a non-empty intersection with $F(a)$.

Additionally, the following properties hold:

- $\underline{F(a)} \subseteq \overline{F(a)}$ for all $a \in J$.
- If $F(a) = \emptyset$, then $\underline{F(a)} = \overline{F(a)} = \emptyset$.

- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

Definition 2.9 (n-SuperHyperRough Set)**.** [19,22] Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote the power set of J_i. Define the set of all possible attribute value combinations as the Cartesian product of these power sets:

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Let $R \subseteq X \times X$ be an equivalence relation on X, where $[x]_R$ denotes the equivalence class of x under R.

An *n-SuperHyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns a subset $F(A) \subseteq X$ to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$, where $A_i \subseteq J_i$ for all i.
- For each $A \in J$, the rough set $(\underline{F(A)}, \overline{F(A)})$ is defined as:

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

The *lower approximation* $\underline{F(A)}$ represents the set of elements in X whose equivalence classes are entirely contained within $F(A)$, while the *upper approximation* $\overline{F(A)}$ includes elements whose equivalence classes have a non-empty intersection with $F(A)$.

Properties:

- $\underline{F(A)} \subseteq \overline{F(A)}$ for all $A \in J$.
- If $F(A) = \emptyset$, then $\underline{F(A)} = \overline{F(A)} = \emptyset$.
- If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.
- For any $A, B \in J$:
$$\underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

3 Results of This Paper

This section presents the results obtained in this paper.

3.1 Forest n-Superhyperstructure

The Forest n-Superhyperstructure is an extension of the n-Superhyperstructure. The definitions of the Forest Hyperstructure and the Forest n-Superhyperstructure are provided below.

Definition 3.1 (Forest Hyperstructure)**.** Let S be a non-empty base set, and let $\mathcal{P}(S)$ represent its power set (all possible subsets of S). A *forest-based family* $\mathcal{F}(S) \subseteq \mathcal{P}(S)$ is constructed through the following steps:

1. *Hierarchy Specification:*
   - Partition or organize the elements of S into one or more *root subsets*, each of which may branch into further subsets (children) at multiple levels.
   - For an element $x \in S$, determine whether x acts as a root of a subset $A \subseteq S$, or as a child node within a deeper subset.
   - At each level, subsets may split or refine into smaller, more specific subsets, or combine into larger ones, depending on the application context.
2. *Node Representation:*

- Each node in the forest corresponds to a subset of S.
- An *internal node* represents a subset that may have child nodes (subsets refining or expanding specific elements).
- A *leaf node* represents a final-level subset that does not subdivide further.

3. *Forest-Based Family $\mathcal{F}(S)$:*

   - Collect all subsets (nodes) appearing anywhere in the forest structure into a single family:

   $$\mathcal{F}(S) \;=\; \{\text{subsets } A \subseteq S \mid A \text{ appears as a node or leaf in the forest structure}\}.$$

   - $\mathcal{F}(S)$ is not necessarily equal to $\mathcal{P}(S)$; it may exclude subsets not represented in the forest or include only those recognized by the hierarchy. In specific cases, $\mathcal{F}(S)$ could be $\mathcal{P}(S)$ if the forest incorporates all possible subsets.

4. *Hyperoperation Definition:*

   - Define a hyperoperation:
   $$\circ : \mathcal{F}(S) \times \mathcal{F}(S) \;\longrightarrow\; \mathcal{P}\big(\mathcal{F}(S)\big),$$
   such that for $A, B \in \mathcal{F}(S)$, the result $A \circ B$ is a set of nodes in $\mathcal{F}(S)$.
   - For example, if A and B share certain elements in their hierarchical decomposition, the hyperoperation may merge or intersect their subtrees or unify them into a larger subset.
   - The exact definition of $\circ$ depends on the problem context but is constrained to output subsets recognized by the forest structure ($\mathcal{F}(S)$).

A *Forest Hyperstructure* is then defined as the pair:

$$\mathcal{FH} \;=\; \big(\mathcal{F}(S), \circ\big).$$

Example 3.2. Suppose $S = \{1, 2, 3, 4\}$. A simple forest-based family $\mathcal{F}(S)$ could be defined as:

$$\mathcal{F}(S) \;=\; \{\{1\},\, \{2\},\, \{3\},\, \{4\},\, \{1, 2\},\, \{2, 3, 4\}\}.$$

- Imagine two root subsets: $\{1, 2\}$ and $\{2, 3, 4\}$.
- The subset $\{1, 2\}$ might branch into $\{1\}$ and $\{2\}$ as children.
- Similarly, $\{2, 3, 4\}$ might branch into $\{2\}$ and $\{3, 4\}$, which could further subdivide.

Any subset not included in this partial structure, such as $\{1, 3\}$, is excluded from $\mathcal{F}(S)$.

A plausible hyperoperation $\circ$ could be defined as:

$$\{1\} \circ \{2, 3, 4\} = \{\{1, 2\}, \{1, 2, 3, 4\}\} \subseteq \mathcal{F}(S).$$

This operation reflects the possibility of merging the "child" $\{1\}$ into the "parent" $\{2, 3, 4\}$, provided the hierarchy permits such a combination.

Theorem 3.3 (Forest Hyperstructure Generalizes Hyperstructure)**.** *Let $\mathcal{H} = \big(\mathcal{P}(S), \circ\big)$ be any classical Hyperstructure on the full power set of S. Then there exists a Forest Hyperstructure $\mathcal{FH} = \big(\mathcal{F}(S), \circ\big)$ (with the* same *hyperoperation symbolically) such that $\mathcal{FH}$ reduces to $\mathcal{H}$ by choosing $\mathcal{F}(S) = \mathcal{P}(S)$ and ignoring hierarchical distinctions.*

Proof. A classical Hyperstructure $\mathcal{H}$ includes $\mathcal{P}(S)$ in its entirety. To obtain $\mathcal{F}(S)$ exactly equal to $\mathcal{P}(S)$, we may define a trivial "forest" in which every subset of S is included as an isolated node (no real branching). In effect, each subset forms its own root and has no children, thus flattening the concept of a forest into a single level.

The hyperoperation $\circ$ remains unchanged because it is now acting on all of $\mathcal{P}(S)$. Hence, $\mathcal{FH}$ is precisely $\mathcal{H}$ in this trivial scenario. Consequently, every classical Hyperstructure can be seen as a special "flat forest" case of the Forest Hyperstructure. □

Definition 3.4 (Forest n-Superhyperstructure)**.** Let S be a non-empty base set, and let $\mathcal{P}_n(S)$ be its n-th powerset as described. A *forest-based* n-th powerset, denoted $\mathcal{F}_n(S) \subseteq \mathcal{P}_n(S)$, is constructed by imposing a forest-like hierarchical structure *at each level* of subset formation. Concretely:

1. *Level 1 Hierarchy:* At the first level, we build a forest $\mathcal{F}_1(S) \subseteq \mathcal{P}(S)$ following Definition 3.1. This captures how elements/subsets of S might branch into sub-subsets within the same level.
2. *Iterative Expansion:* For the second level, each node in $\mathcal{F}_1(S)$ (which is itself a subset of S) can be refined by an additional forest-based expansion, resulting in $\mathcal{F}_2(S) \subseteq \mathcal{P}(\mathcal{F}_1(S))$. In principle, $\mathcal{F}_2(S)$ is a set of subsets of $\mathcal{F}_1(S)$, each subset now representing a possible combination or branching of first-level subsets.
3. *Continuing up to Level n:* We repeat this hierarchical construction up to the n-th level. Ultimately, $\mathcal{F}_n(S) \subseteq \mathcal{P}_n(S)$ is a structured family that *respects the forest expansions at each layer*. An element in $\mathcal{F}_n(S)$ can be viewed as a node (or path of nodes) that emerges from chaining multiple forest expansions from level 1 to level n.
4. *Hyperoperation* $\circ$*:* We define a hyperoperation
$$\circ : \mathcal{F}_n(S) \times \mathcal{F}_n(S) \longrightarrow \mathcal{P}(\mathcal{F}_n(S)),$$
ensuring that the result $\alpha \circ \beta \subseteq \mathcal{F}_n(S)$ respects all n layers of the forest-based expansions. For example, if α and β represent certain nested subset paths, their hyperoperation might merge these paths or produce new nodes consistent with the forest structure at each level.

A *Forest n-Superhyperstructure* is the pair
$$\mathcal{FSH}_n = (\mathcal{F}_n(S), \circ).$$

Example 3.5. To provide an example for $n = 2$:

- *Level 1*: Suppose $\mathcal{F}_1(S)$ is a forest of subsets of S. For instance, if $S = \{a, b, c, d\}$, we might have
$$\mathcal{F}_1(S) = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c, d\}\}.$$
- *Level 2*: Now each element of $\mathcal{F}_1(S)$ can itself appear in a second-level forest expansion. For instance, $\{a, b\}$ might branch into $\{\{a\}, \{b\}\}$ at the next level, or unify with $\{b, c, d\}$ in some "super-subset" if the forest hierarchy allows. The result is $\mathcal{F}_2(S) \subseteq \mathcal{P}(\mathcal{F}_1(S))$. Each subset in $\mathcal{F}_2(S)$ is now a collection of nodes from $\mathcal{F}_1(S)$ that is recognized by the second-level forest structure.
- The pair $(\mathcal{F}_2(S), \circ)$ is thus a Forest 2-Superhyperstructure. Extending to higher n follows a similar pattern.

Theorem 3.6 (Forest n-Superhyperstructure Generalizes Forest Hyperstructure and n-Superhyperstructure)**.** *A Forest n-Superhyperstructure* $(\mathcal{F}_n(S), \circ)$ *generalizes both:*

- *the* Forest Hyperstructure $(\mathcal{F}(S), \circ)$ *(Definition 3.1), and*
- *the* n-Superhyperstructure $(\mathcal{P}_n(S), \circ)$.

Proof. When $n = 1$, we do not iterate the powerset formation. We only have:
$$\mathcal{F}_1(S) \subseteq \mathcal{P}(S),$$
and the hyperoperation is
$$\circ : \mathcal{F}_1(S) \times \mathcal{F}_1(S) \longrightarrow \mathcal{P}(\mathcal{F}_1(S)).$$
This matches exactly the definition of a *Forest Hyperstructure*, because we are only building a single-level forest-based family of subsets of S. There is no second-level or higher nesting. Thus,
$$\mathcal{FSH}_1 = (\mathcal{F}_1(S), \circ)$$

is precisely $(\mathcal{F}(S), \circ)$. That is, the Forest 1-Superhyperstructure and the Forest Hyperstructure coincide.

If we remove all forest-based restrictions at each stage, we let $\mathcal{F}_k(S) = \mathcal{P}_k(S)$ for $k = 1, \ldots, n$. Hence,

$$\mathcal{F}_n(S) \;=\; \mathcal{P}_n(S).$$

The hyperoperation

$$\circ : \mathcal{P}_n(S) \times \mathcal{P}_n(S) \longrightarrow \mathcal{P}\big(\mathcal{P}_n(S)\big)$$

defines precisely an *n-Superhyperstructure*. Therefore, by lifting the forest constraints, we recover the full n-th powerset, matching $\mathcal{SH}_n = (\mathcal{P}_n(S), \circ)$.

Combining these two arguments shows that a Forest n-Superhyperstructure becomes:

- $(\mathcal{F}(S), \circ)$ if $n = 1$,
- $(\mathcal{P}_n(S), \circ)$ if we remove the hierarchical/forest restrictions at every level.

Hence, $(\mathcal{F}_n(S), \circ)$ is strictly more general, encompassing both the single-level forest hyperstructure and the classic n-Superhyperstructure as special cases. □

3.2 Forest n-SuperhyperPlithogenic Set

The Forest n-SuperhyperPlithogenic Set is a generalized concept that extends the n-SuperhyperPlithogenic Set. Its definition is provided below.

Definition 3.7 (Forest HyperPlithogenic Set)**.** Let X be a non-empty set (the universe of discourse), and let

$$A \;=\; \{v_1, v_2, \ldots, v_n\}$$

be a collection of n attributes, each attribute v_i having a *forest-like* hierarchy of possible values. Concretely, for each v_i, there is a (potentially multi-level) family of subsets $Pv_i \subseteq \mathcal{P}(S_i)$, where S_i is the base set of raw values for v_i. These subsets in Pv_i may represent nodes or paths in a forest structure: a root node might correspond to the entire set of values S_i, and internal or leaf nodes correspond to partial subdivisions or refinements of S_i.

A *Forest HyperPlithogenic Set* FHPS over X is a 5-tuple:

$$\text{FHPS} \;=\; \Big(P,\; \{v_i\}_{i=1}^n,\; \{Pv_i\}_{i=1}^n,\; \{\tilde{pdf}_i\}_{i=1}^n,\; pCF_{\text{forest}}\Big),$$

with the following components:

1. $P \subseteq X$:
   - A (possibly proper) subset of the universe X, serving as the set of *elements* on which the attributes (and their values) will be evaluated.
2. $Pv_i \subseteq \mathcal{P}(S_i)$ for each attribute v_i:
   - A *forest-based* family of possible values.
   - Rather than simply listing all subsets of S_i, these subsets are organized as a forest:
     - Each node (subset) can have children (more refined or specialized subsets).
     - The forest culminates in leaf nodes (final-level subsets of S_i).
   - For instance, some node in the forest might represent a partial subset of S_i, while another node might represent a further subdivision of it.
3. $\tilde{pdf}_i : P \times Pv_i \longrightarrow \tilde{\mathcal{P}}([0,1]^s)$:
   - The *Hyper Degree of Appurtenance Function (HDAF)*.

- This function assigns, for each element $x \in P$ and each forest-based value $\alpha \in Pv_i$, a *set* of membership degrees in $[0,1]^s$.
- The parameter s might encode fuzziness, intuitionistic or neutrosophic membership, or other multi-valued measures.
- The prefix "Hyper" indicates that $\tilde{pdf}_i(x, \alpha)$ can be *set-valued* rather than a single numeric value, allowing for further generality (e.g., intervals or multi-dimensional membership vectors).

4. $pCF_{\text{forest}} : \left(\bigcup_{i=1}^{n} Pv_i\right) \times \left(\bigcup_{i=1}^{n} Pv_i\right) \longrightarrow [0,1]^t$:
   - The *Degree of Contradiction Function (DCF)* for the entire collection of forest-based attribute values.
   - For any two (possibly multi-level) values α, β chosen from any of the Pv_i families, $pCF_{\text{forest}}(\alpha, \beta)$ measures how contradictory or incompatible these two values are, in a multi-valued sense (dimension t).
   - Typically, we require:

$$pCF_{\text{forest}}(\alpha, \alpha) = 0, \quad pCF_{\text{forest}}(\alpha, \beta) = pCF_{\text{forest}}(\beta, \alpha),$$

   ensuring reflexivity (no contradiction with itself) and symmetry.

Whereas a standard *HyperPlithogenic Set* simply collects an attribute v_i with a set of values Pv_i, the *Forest HyperPlithogenic Set* organizes those values in a multi-level (forest) manner, allowing sub-values to branch out from root values or parent nodes. The $\tilde{pdf}_i$ and pCF_{forest} are then *forest-aware*, meaning their definitions may depend on hierarchical relations among the subsets in Pv_i.

Theorem 3.8 (Forest HyperPlithogenic Set Generalizes HyperPlithogenic Set)**.** *Let* FHPS *be a Forest HyperPlithogenic Set as in Definition 3.7, and let HPS be the standard (flat) HyperPlithogenic Set. Then* FHPS *strictly generalizes HPS, in the sense that every HPS is recoverable by collapsing the forest expansions.*

Proof. In a classical HyperPlithogenic Set HPS, each attribute v_i has a set of values Pv_i, which we interpret as a *flat* family: no node has children, so effectively each Pv_i is just $\{\alpha_1, \alpha_2, \dots\}$ without any multi-level structure. In the forest-based version, each Pv_i might be a multi-level tree or forest of subsets.

To revert to the standard HPS scenario, we:

1. Flatten each forest-based family Pv_i. That is, remove all parent-child linkages and treat every leaf or node as just a single-level value.
2. Consequently, $\tilde{pdf}_i$ becomes a classical membership function (or set of membership degrees) for each $\alpha \in Pv_i$.
3. The contradiction function pCF_{forest} reduces to pCF on the flat set of values, since there is no hierarchical relationship to consider.

This yields precisely the HyperPlithogenic Set definition. Thus, FHPS generalizes HPS. □

Definition 3.9 (Forest n-SuperhyperPlithogenic Set)**.** Let X be a non-empty universe, and let

$$V = \{v_1, v_2, \dots, v_n\}$$

be a set of n attributes, each associated with a forest-based set of possible values $Pv_i \subseteq \mathcal{P}(S_i)$. Assume we iteratively construct nested subsets $\{P_1, P_2, \dots, P_m\}$, where:

$$P_1 \subseteq X, \quad P_{k+1} = \tilde{\mathcal{P}}(P_k), \quad k = 1, 2, \dots, m-1,$$

with $\tilde{\mathcal{P}}(\cdot)$ representing a forest-based powerset or a similar operation. Let

$$P_m$$

denote the final-stage family of subsets obtained after m expansions.

A *Forest n-SuperhyperPlithogenic Set* FSHPS_n over (X, V) is a tuple:

$$\mathrm{FSHPS}_n = \Big(P_m,\ V,\ \{Pv_i\}_{i=1}^n,\ \{\tilde{pdf}_i^{(m)}\}_{i=1}^n,\ pCF_{\text{forest}}^{(m)}\Big),$$

with the following components:

1. P_m:
   - The final stage of nested subsets of X, where each stage respects a forest-based structure or an iterative hyper-subset construction.
2. Pv_i:
   - The forest-based collection of possible values for each attribute v_i.
   - Each Pv_i consists of multi-level subsets derived from S_i.
3. $\tilde{pdf}_i^{(m)} : P_m \times Pv_i \to \tilde{\mathcal{P}}([0, 1]^s)$:
   - The *m-level hyper degree of appurtenance function (HDAF).*
   - For each $u \in P_m$ (a subset at the m-th level) and each forest-based value $\alpha \in Pv_i$, $\tilde{pdf}_i^{(m)}(u, \alpha)$ assigns a possibly set-valued degree of membership in $[0, 1]^s$.
   - This degree reflects how strongly u belongs to or aligns with α, accommodating multi-valued or interval-based membership.
4. $pCF_{\text{forest}}^{(m)} : \big(\bigcup_{i=1}^n Pv_i\big) \times \big(\bigcup_{i=1}^n Pv_i\big) \to [0, 1]^t$:
   - The *forest-based degree of contradiction function (DCF).*
   - It measures contradictions among multi-level values $\alpha, \beta \in \bigcup_{i=1}^n Pv_i$.
   - The DCF satisfies:

     $$pCF_{\text{forest}}^{(m)}(\alpha, \alpha) = 0, \quad pCF_{\text{forest}}^{(m)}(\alpha, \beta) = pCF_{\text{forest}}^{(m)}(\beta, \alpha),$$

     ensuring reflexivity (no contradiction with itself) and symmetry.

Theorem 3.10 (Forest n-SuperhyperPlithogenic Set Generalizes Both n-SuperHyperPlithogenic Set and Forest HyperPlithogenic Set)**.** *A Forest n-SuperhyperPlithogenic Set* FSHPS_n *strictly generalizes:*

- *the n-SuperHyperPlithogenic Set, and*
- *the Forest HyperPlithogenic Set (the single-stage case).*

Proof. (1) Reduction to n-SuperHyperPlithogenic Set by Removing Forest Structure.

Consider FSHPS_n as in Definition 3.9. If we *collapse each forest-based* Pv_i to a *flat* set of values (i.e., remove any multi-level branching among the possible values), then:

$$Pv_i \quad \longrightarrow \quad \text{(flat set of atomic values)}.$$

Likewise, if each $\tilde{\mathcal{P}}(P_k)$ is replaced by an ordinary powerset (or nested family) without hierarchical constraints, we recover the standard n-SuperHyperPlithogenic approach. In that scenario:

- $\tilde{pdf}_i^{(m)}$ becomes an *m-level* membership function on a flat set of values,
- $pCF_{\text{forest}}^{(m)}$ becomes the usual contradiction function $pCF^{(m)}$ on pairs of attribute values.

Hence, FSHPS_n simplifies exactly to the n-SuperHyperPlithogenic Set, denoted $SHPS_n$.

(2) Reduction to Forest HyperPlithogenic Set for $n = 1$.

If we set $n = 1$, we are dealing with only one attribute family Pv_1. Also, we do not iterate multiple expansions for different attributes. Then:

$$\mathrm{FSHPS}_1 \;=\; \big(P_m,\, \{v_1\},\, \{Pv_1\},\, \{\tilde{pdf}_1^{(m)}\},\, pCF_{\text{forest}}^{(m)}\big).$$

But this structure is precisely a *Forest HyperPlithogenic Set* at the m-th expansion stage. Indeed, there is only one attribute's forest-based domain, so the entire multi-attribute dimension collapses. Therefore, FSHPS_1 coincides with the single-attribute (forest-based) scenario, i.e. FHPS from Definition 3.7.

Because setting $n = 1$ yields the Forest HyperPlithogenic Set and removing the forest expansions yields the flat n-SuperHyperPlithogenic Set, FSHPS_n unifies and generalizes both frameworks. □

Example 3.11. Consider a decision-making scenario involving the following components:

- X represents a set of potential products or items under consideration.
- $V = \{v_1, v_2\}$ denotes two attributes, such as *Quality* and *Price*.
- Each attribute v_i is associated with a forest-based family of possible values Pv_i:
  - For *Quality*, the root nodes might include values such as {High}, {Medium}, and {Low}. Each root can branch into more specific sub-values; for example, {High} may branch into {VeryHigh} or {ModeratelyHigh}.
  - For *Price*, the root nodes could include {Cheap}, which might branch further into {Clearance} or {SlightlyDiscounted}, among others.
- The nested families $P_1 \subseteq X$, $P_2 = \tilde{\mathcal{P}}(P_1)$, and so on up to P_m are constructed iteratively, representing subsets of X at progressively higher levels.
- For each final-level subset $u \in P_m$ and each forest-based value $\alpha \in Pv_i$, the hyper degree of appurtenance function $\tilde{pdf}_i^{(m)}(u, \alpha) \subseteq [0, 1]^s$ assigns membership degrees. These degrees reflect how strongly u corresponds to α.
- The forest-based contradiction function $pCF_{\text{forest}}^{(m)}(\alpha, \beta) \in [0, 1]^t$ measures the level of contradiction or incompatibility between any two values α and β.
  - For instance, {VeryHighQuality} may heavily contradict {Clearance} pricing, indicating that these two attribute values are unlikely to coexist.

This example illustrates the concept of a *Forest n-SuperhyperPlithogenic Set*, which combines the hierarchical (forest-based) organization of attribute values with multi-stage subset expansions. This advanced framework supports sophisticated multi-criteria decision-making by capturing complex interrelationships and contradictions among attributes and their values.

3.3 Forest n-SuperHyperRough Set

The Forest HyperRough Set is a generalization of the HyperRough Set, inspired by the concept of the Forest Hypersoft Set. Its definition is provided below.

Definition 3.12 (Forest HyperRough Set)**.** Let X be a non-empty finite universe, and let $R \subseteq X \times X$ be an equivalence relation on X. Denote by $[x]_R$ the equivalence class of x under R. Suppose we have a *forest-like* family of attributes

$$\mathcal{A} = \{T_1, T_2, \ldots, T_m\},$$

where each attribute T_i is associated with a *multi-level* (tree-structured) domain

$$\mathrm{Forest}(J_i) \;\subseteq\; \mathcal{P}(J_i),$$

and each J_i is a base set of possible values for T_i. Let

$$\Gamma\big(\text{Forest}(J_i)\big)$$

represent the set of final-level attribute values (leaves) in the forest of T_i. Then define

$$\Gamma_{\text{forest}} \;=\; \Gamma\big(\text{Forest}(J_1)\big) \,\cup\, \Gamma\big(\text{Forest}(J_2)\big) \,\cup\, \cdots \,\cup\, \Gamma\big(\text{Forest}(J_m)\big).$$

A *Forest HyperRough Set* over X is a pair

$$\big(F,\; \Gamma_{\text{forest}}\big),$$

where:

1. $F : \Gamma_{\text{forest}} \longrightarrow \mathcal{P}(X)$ is a mapping that assigns each leaf-level attribute value $\alpha \in \Gamma_{\text{forest}}$ a subset $F(\alpha) \subseteq X$.
2. For each $\alpha \in \Gamma_{\text{forest}}$, the pair $\big(\underline{F(\alpha)}, \overline{F(\alpha)}\big)$ is a rough set approximation of $F(\alpha)$ under R, defined by:

$$\underline{F(\alpha)} \;=\; \{x \in X \mid [x]_R \subseteq F(\alpha)\}, \quad \overline{F(\alpha)} \;=\; \{x \in X \mid [x]_R \cap F(\alpha) \neq \emptyset\}.$$

In contrast to a standard *HyperRough Set*, where attributes and their domains are typically flat (single-level), the Forest HyperRough Set employs multi-level attribute domains arranged in a forest structure. Each leaf-level value α is still mapped to a subset of X, but this value might represent a path or nested sub-attribute in the tree-based domain Forest(J_i). Rough set approximations $\underline{F(\alpha)}, \overline{F(\alpha)}$ capture the certainty and possibility of membership with respect to $F(\alpha)$.

Theorem 3.13 (Forest HyperRough Set Generalizes HyperRough Set)**.** *Any HyperRough Set can be viewed as a special case of a Forest HyperRough Set by collapsing the forest structure of each attribute domain into a single level.*

Proof. A HyperRough Set $\big(F, J\big)$ (as defined in the classical sense) operates on a Cartesian product of flat domains $J_1, J_2, \ldots, J_m$. In the case of a Forest HyperRough Set (see Definition 3.12), each domain J_i is assumed to have a hierarchical, forest-like structure, denoted Forest(J_i). To demonstrate the generalization, consider the following steps:

- Assume each forest structure Forest(J_i) is reduced to its base set J_i, effectively eliminating all branching or multi-level expansions. In this case, Forest$(J_i) = J_i$ for all i.
- The set of leaf-level values Γ_{forest} in the forest framework now corresponds directly to the union of the flat domains $J_1, J_2, \ldots, J_m$:
$$\Gamma_{\text{forest}} = J_1 \cup J_2 \cup \cdots \cup J_m.$$
- The mapping $F : \Gamma_{\text{forest}} \to \mathcal{P}(X)$, which associates subsets of X to combinations of forest-based attribute values, reduces to the standard mapping in the HyperRough Set framework:
$$F : J \to \mathcal{P}(X),$$
where $J = J_1 \times J_2 \times \cdots \times J_m$.
- The lower and upper approximations of any subset $F(\alpha)$, denoted $\underline{F(\alpha)}$ and $\overline{F(\alpha)}$, remain identical because the equivalence relation $R \subseteq X \times X$ used to define rough approximations does not depend on the structure of the attribute domains:
$$\underline{F(\alpha)} = \{x \in X \mid [x]_R \subseteq F(\alpha)\},$$
$$\overline{F(\alpha)} = \{x \in X \mid [x]_R \cap F(\alpha) \neq \emptyset\}.$$
- Therefore, when the hierarchical (forest-like) structure is removed, the Forest HyperRough Set simplifies to a standard HyperRough Set, preserving all rough set properties and definitions.

This proves that Forest HyperRough Sets are a strict generalization of HyperRough Sets, encompassing them as a special case. □

The Forest n-SuperhyperRough Set is an extended definition of the n-SuperhyperRough Set. The related theorems are provided below.

Definition 3.14 (Forest n-SuperhyperRough Set)**.** Let X be a non-empty finite universe, and let $R \subseteq X \times X$ be an equivalence relation on X. Suppose we have m attributes $\{T_1, \ldots, T_m\}$, each with a *forest-based* domain $\mathrm{Forest}(J_i) \subseteq \mathcal{P}(J_i)$. For each attribute T_i, let

$$\widetilde{\mathcal{P}}(J_i) \;\subseteq\; \mathcal{P}\big(J_i\big)$$

represent the power set (or a selected family) of possible sub-values, still respecting a forest structure if needed. Then define the *n-super* Cartesian product:

$$J_n \;=\; \widetilde{\mathcal{P}}(J_1) \times \widetilde{\mathcal{P}}(J_2) \times \cdots \times \widetilde{\mathcal{P}}(J_m),$$

where each element $A \in J_n$ is of the form $A = (A_1, A_2, \ldots, A_m)$ with $A_i \subseteq J_i$.

A *Forest n-SuperhyperRough Set* over X is a pair

$$\big(F,\; J_n\big),$$

where:

1. $F : J_n \to \mathcal{P}(X)$ assigns to each $A = (A_1, \ldots, A_m) \in J_n$ a subset $F(A) \subseteq X$.
2. For each $A \in J_n$, the pair $\big(\underline{F(A)},\; \overline{F(A)}\big)$ is defined via rough set approximations under R:

$$\underline{F(A)} \;=\; \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} \;=\; \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

Properties. Similar to the n-SuperHyperRough Set, we have:

- $\underline{F(A)} \subseteq \overline{F(A)}$.
- If $F(A) = \varnothing$, then $\underline{F(A)} = \overline{F(A)} = \varnothing$.
- If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.
- Monotonicity: For any $A, B \in J_n$,

$$\underline{F(A \cap B)} \;\subseteq\; \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \;\supseteq\; \overline{F(A)} \;\cup\; \overline{F(B)}.$$

In contrast to the standard n-SuperHyperRough Set, the *forest* aspect allows each J_i to be subdivided into multi-level branches or subsets, captured by $\widetilde{\mathcal{P}}(J_i)$. Hence, each $A_i \subseteq J_i$ may itself be a (potentially nested) collection of leaf-level attribute values.

Theorem 3.15 (Forest n-SuperhyperRough Set Generalizes Forest HyperRough Set and n-SuperHyperRough Set)**.** *A Forest n-SuperhyperRough Set generalizes both:*

- *The* Forest HyperRough Set *(the case $n = 1$),*
- *The* n-SuperHyperRough Set *(the case without a forest-based subdivision of each domain).*

Proof. 1. *Specializing to Forest HyperRough Set.* Setting $n = 1$, we obtain $J_1 = \widetilde{\mathcal{P}}(J_1)$ for a single attribute domain. An element $A \in J_1$ is just a subset $A_1 \subseteq J_1$. The mapping F becomes

$$F : J_1 \longrightarrow \mathcal{P}(X),$$

and the rough approximations $\underline{F(A)}, \overline{F(A)}$ match the definition of a Forest HyperRough Set when $\Gamma_{\text{forest}} = J_1$. Thus, the Forest n-SuperhyperRough Set collapses to the Forest HyperRough Set in the single-attribute (or single-dimension) scenario.

2. *Specializing to n-SuperHyperRough Set.* If we remove the forest structure in each domain J_i, so that $\widetilde{\mathcal{P}}(J_i) = \mathcal{P}(J_i)$ is just the full power set (with no hierarchical constraints), we recover the standard Cartesian product

$$J_n = \mathcal{P}(J_1) \times \cdots \times \mathcal{P}(J_m),$$

and the mapping F to $\mathcal{P}(X)$ defines precisely an n-SuperHyperRough Set as per its original definition. Hence, by lifting those constraints, the forest-based model reverts to the flat model.

Therefore, the Forest n-SuperhyperRough Set unifies the multi-level domain expansions for each attribute (*forest*) with the higher-level superhyperrough construction (n-power-set expansions). □

4 Future Research: Various Rough Sets

This section outlines the prospects for future research based on this study.

Several related concepts to Rough Sets have been developed, including:

- *Multi-granulation Rough Sets* [10, 32, 34, 72, 73],
- *Variable Precision Rough Sets* [11, 37, 69, 81, 82],
- *Dominance-Based Rough Sets* [4, 29, 30, 33, 50].

One of the future challenges will be exploring whether the concepts defined in this paper can be extended using these advanced Rough Set frameworks. Such investigations will help refine the applicability and theoretical foundations of the proposed ideas.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 9
ForestFuzzy, ForestNeutrosophic, ForestPlithogenic, and ForestRough Set

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Abstract

Concepts such as Fuzzy Sets [30, 72], Neutrosophic Sets [53, 55], Rough Sets [37], and Plithogenic Sets [59] have been extensively studied to address uncertainty, with diverse applications across various fields. Recently, TreeFuzzy, TreeNeutrosophic, TreePlithogenic, and TreeRough Sets have been defined [15]. This work examines their extensions: ForestFuzzy, ForestNeutrosophic, ForestPlithogenic, and ForestRough Sets.

Keywords: TreeFuzzy Sets, TreeNeutrosophic Sets, TreePlithogenic Sets, TreeRough Sets, ForestFuzzy Sets, ForestNeutrosophic Sets, ForestPlithogenic Sets

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

1.1 Fuzzy Set

The concept of the Fuzzy Set is a foundational tool for addressing uncertainty in set theory. Its definition is provided below [72–78].

Numerous related concepts have also been developed, including Hyperfuzzy Sets [17, 24, 27], Intuitionistic Fuzzy Sets [4–8], Hesitant Fuzzy Sets [67, 68], Bipolar Fuzzy Sets [2, 9, 25], Picture Fuzzy Sets [10, 52, 69], Tripolar Fuzzy Sets [47–49], and Complex Fuzzy Sets [44–46]. These variations extend the classical Fuzzy Set framework to model diverse types of uncertainty and complexity.

Definition 1.1. [72, 77] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

A TreeFuzzy Set is a generalization of the Fuzzy Set concept using a Tree structure.

Definition 1.2. [15] A *TreeFuzzy Set* F is a mapping:

$$F : P(\text{Tree}(A)) \to [0, 1]^U,$$

where $P(\text{Tree}(A))$ denotes the power set of the set of all nodes and leaves in $\text{Tree}(A)$, and $[0, 1]^U$ denotes the set of all fuzzy subsets of U.

For each attribute combination $S \in P(\text{Tree}(A))$, $F(S)$ is a membership function $\mu_S : U \to [0, 1]$, assigning to each element $x \in U$ a degree of membership with respect to the attribute combination S.

1.2 Neutrosophic Set

Neutrosophic Sets extend Fuzzy Sets by incorporating the concept of indeterminacy, addressing situations that are neither entirely true nor entirely false. This framework provides a more flexible representation of uncertainty and ambiguity [18–20, 22, 29, 54, 56–58, 65, 66].

Several related extensions have been developed, including the Single-valued Neutrosophic Set [28, 31], Double-valued Neutrosophic Set [80, 81], Interval-valued Neutrosophic Set [70, 71, 79], and Bipolar Neutrosophic Set [1, 11, 33]. These variants expand the Neutrosophic framework to accommodate more complex forms of uncertainty and multiple perspectives.

Definition 1.3 (Neutrosophic Set)**.** [55, 56] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

A TreeNeutrosophic Set is a generalization of the Neutrosophic Set concept using a Tree structure.

Definition 1.4. [15] A *TreeNeutrosophic Set F* is a mapping:

$$F : P(\text{Tree}(A)) \to ([0,1] \times [0,1] \times [0,1])^U,$$

where for each attribute combination $S \in P(\text{Tree}(A))$, $F(S)$ assigns to each element $x \in U$ a neutrosophic membership triple:

$$F(S)(x) = (T_S(x), I_S(x), F_S(x)),$$

where $T_S(x), I_S(x), F_S(x) \in [0,1]$ represent the degrees of truth-membership, indeterminacy-membership, and falsity-membership of x with respect to the attribute combination S.

These values satisfy the condition:

$$0 \leq T_S(x) + I_S(x) + F_S(x) \leq 3,$$

for all $x \in U$ and $S \in P(\text{Tree}(A))$.

1.3 Plithogenic Set

The Plithogenic Set is known as a type of set that can generalize Neutrosophic Sets, Fuzzy Sets, and other similar sets [13, 14, 21, 60, 61]. The definition of the Plithogenic Set is provided below.

Definition 1.5. [60, 61] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)*.
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a,a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a,b) = pCF(b,a)$$

A TreePlithogenic Set is a generalization of the Plithogenic Set concept using a Tree structure.

Definition 1.6. [15] Let S be a universal set, and let $P \subseteq S$. Consider a hierarchical attribute tree $\mathrm{Tree}(A)$, where attributes and sub-attributes are organized in levels from 1 up to m. Each node in the tree represents an attribute a_i, and for each attribute a_i, there is an associated set of possible values Pv_i.

A *TreePlithogenic Set* TPS is defined as:

$$TPS = (P, \mathrm{Tree}(A), \{Pv_i\}, \{pdf_i\}, pCF),$$

where:

- P is a subset of the universal set S.
- $\mathrm{Tree}(A)$ is a hierarchical tree of attributes.
- For each attribute $a_i \in \mathrm{Tree}(A)$, Pv_i is the set of possible values of a_i.
- For each attribute a_i, $pdf_i : P \times Pv_i \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for a_i.
- $pCF : (\bigcup_i Pv_i) \times (\bigcup_i Pv_i) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

1.4 Treerough Set

A Rough Set is a mathematical framework for approximating vague or imprecise data using lower and upper set approximations [37–43]. A Treerough Set is a generalization of the Rough Set concept using a Tree structure.

Definition 1.7 (Rough Set)**.** [37–43] Let X be the universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or an indiscernibility relation) on X, partitioning X into equivalence classes. For any subset $U \subseteq X$, the lower approximation $\underline{U}$ and the upper approximation $\overline{U}$ are defined as follows:

1. *Lower Approximation* $\underline{U}$:
$$\underline{U} = \{x \in X \mid R(x) \subseteq U\}$$
This is the set of all elements in X that certainly belong to U based on the equivalence classes defined by R.

2. *Upper Approximation* $\overline{U}$:
$$\overline{U} = \{x \in X \mid R(x) \cap U \neq \emptyset\}$$
This set contains all elements in X that possibly belong to U.

The pair $(\underline{U}, \overline{U})$ constitutes a rough set representation of U, where $\underline{U} \subseteq U \subseteq \overline{U}$.

A TreeRough Set is a generalization of the Rough Set concept using a Tree structure.

Definition 1.8 (Treerough set)**.** [15] Let U be a universe of discourse, and let $\mathrm{Tree}(A)$ be a hierarchical tree of attributes, where each node represents an attribute a_i. The tree has levels from 1 up to m, where $m \geq 1$. Each attribute a_i in the tree is associated with an equivalence relation R_{a_i} on U.

For any subset $X \subseteq U$, we define the *Treerough Set* $\mathcal{TR}(X)$ as the collection of lower and upper approximations of X with respect to the equivalence relations R_{a_i} associated with all attributes a_i in $\mathrm{Tree}(A)$.

For each attribute a_i in $\mathrm{Tree}(A)$, the lower and upper approximations of X are defined as:

- The *Lower Approximation* of X with respect to R_{a_i}:
$$\underline{X}_{a_i} = \{x \in U \mid [x]_{R_{a_i}} \subseteq X\},$$
where $[x]_{R_{a_i}}$ denotes the equivalence class of x under R_{a_i}.
- The *Upper Approximation* of X with respect to R_{a_i}:
$$\overline{X}_{a_i} = \{x \in U \mid [x]_{R_{a_i}} \cap X \neq \emptyset\}.$$

The *Treerough Set* of X is then the collection:

$$\mathcal{TR}(X) = \left\{ \left(\underline{X}_{a_i}, \overline{X}_{a_i} \right) \mid a_i \in \mathrm{Tree}(A) \right\}.$$

1.5 Soft Set and TreeSoft Set

A *Soft Set* (F, E) associates each parameter in a set E with a subset of a universal set U. This provides a flexible framework for approximating objects within U [26, 32, 34]. A *TreeSoft Set* is a mapping from subsets of a hierarchical, tree-like parameter structure $\text{Tree}(A)$ to subsets of a universal set U [3, 16, 23, 35, 36, 50, 64]. The definitions of Soft Set and TreeSoft Set are provided below.

Definition 1.9. [32] Let U be a universal set and E a set of parameters. A *soft set* over U is defined as an ordered pair (F, E), where F is a mapping from E to the power set $\mathcal{P}(U)$:

$$F : E \to \mathcal{P}(U).$$

For each parameter $e \in E$, $F(e) \subseteq U$ represents the set of e-approximate elements in U, with (F, E) forming a parameterized family of subsets of U.

Definition 1.10. [62] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1,i_2,\ldots,i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as $\text{Tree}(A)$, with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set* F is defined as a function:

$$F : P(\text{Tree}(A)) \to P(H),$$

where $\text{Tree}(A)$ represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\text{Tree}(A))$ denotes its power set.

A *ForestSoft Set* is formed by taking a collection of TreeSoft Sets and "gluing" (uniting) them together so as to obtain a single function whose domain is the union of all tree-nodes' power sets and whose values in $P(H)$ combine the images given by the individual TreeSoft Sets [12, 51, 63].

Definition 1.11 (ForestSoft Set)**.** [63] Let U be a universe of discourse, $H \subseteq U$ be a non-empty subset, and $P(H)$ be the power set of H. Suppose we have a finite (or countable) collection of TreeSoft Sets

$$\left\{ F_t : P(\text{Tree}(A^{(t)})) \to P(H) \right\}_{t \in T},$$

where each F_t is a TreeSoft Set corresponding to a tree $\text{Tree}(A^{(t)})$ of attributes $A^{(t)}$.

We construct a *forest* by taking the (disjoint) union of all these trees:

$$\text{Forest}\big(\{A^{(t)}\}_{t \in T}\big) = \bigsqcup_{t \in T} \text{Tree}\big(A^{(t)}\big).$$

A *ForestSoft Set*, denoted by

$$\mathbf{F} : P\big(\text{Forest}(\{A^{(t)}\})\big) \longrightarrow P(H),$$

is defined as the *union* of all TreeSoft Set mappings F_t. Concretely, for any element $X \in P\big(\text{Forest}(\{A^{(t)}\})\big)$, we set

$$\mathbf{F}(X) = \bigcup_{\substack{t \in T \\ X \cap \text{Tree}(A^{(t)}) \neq \emptyset}} F_t\big(X \cap \text{Tree}(A^{(t)})\big),$$

where we only apply F_t to that portion of X belonging to the tree $\text{Tree}(A^{(t)})$.

2 Results in This Paper

The results derived in this paper are presented below.

2.1 ForestFuzzy Set

The ForestFuzzy Set is a concept that applies the idea of the ForestSoft Set to the framework of Fuzzy Sets. The definition is provided below.

Definition 2.1 (ForestFuzzy Set)**.** Let $\{ F_t \}_{t \in T}$ be a collection of TreeFuzzy Sets, where each

$$F_t : P\big(\mathrm{Tree}(A^{(t)})\big) \to [0,1]^U.$$

Form the *forest*

$$\mathrm{Forest}\big(\{A^{(t)}\}\big) \;=\; \bigsqcup_{t \in T} \mathrm{Tree}\big(A^{(t)}\big).$$

A *ForestFuzzy Set* is a mapping

$$\mathbf{F} : P\big(\mathrm{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; [0,1]^U$$

defined by: for each $X \subseteq \mathrm{Forest}(\{A^{(t)}\})$ and each $x \in U$,

$$\mathbf{F}(X)(x) \;=\; \max_{\substack{t \in T \\ X \cap \mathrm{Tree}(A^{(t)}) \neq \emptyset}} F_t\big(X \cap \mathrm{Tree}(A^{(t)})\big)(x).$$

Theorem 2.2 (ForestFuzzy generalizes TreeFuzzy)**.** *Every TreeFuzzy Set is a special case of a ForestFuzzy Set (one-tree forest).*

Proof. If $F : P(\mathrm{Tree}(A)) \to [0,1]^U$ is a TreeFuzzy Set on a single tree, then let $T = \{1\}$ and $\mathrm{Tree}(A^{(1)}) = \mathrm{Tree}(A)$. The ForestFuzzy Set definition reduces to F itself, since the maximum is over a single index $t = 1$. Hence TreeFuzzy $\subseteq$ ForestFuzzy. □

2.2 ForestNeutrosophic Set

The ForestNeutrosophic Set is a concept that applies the idea of the ForestSoft Set to the framework of Neutrosophic Sets. The definition is provided below.

Definition 2.3 (ForestNeutrosophic Set)**.** Let $\{F_t\}_{t \in T}$ be TreeNeutrosophic Sets. The *ForestNeutrosophic Set*

$$\mathbf{F} : P\big(\mathrm{Forest}(\{A^{(t)}\})\big) \;\to\; \big([0,1]^3\big)^U$$

is given, for each X in the domain, by

$$\mathbf{F}(X)(x) = \Big(\max_{t : X \cap \mathrm{Tree}(A^{(t)}) \neq \emptyset} T_t(X)(x), \quad \max_{t : X \cap \mathrm{Tree}(A^{(t)}) \neq \emptyset} I_t(X)(x), \quad \max_{t : X \cap \mathrm{Tree}(A^{(t)}) \neq \emptyset} F_t(X)(x) \Big).$$

Theorem 2.4 (ForestNeutrosophic generalizes TreeNeutrosophic)**.** *Every TreeNeutrosophic Set is a one-tree instance of a ForestNeutrosophic Set.*

Proof. Same argument as before: a single tree in the forest yields the original TreeNeutrosophic Set. □

Theorem 2.5. *Any ForestFuzzy Set*

$$\mathbf{F} : P\big(\mathrm{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; [0,1]^U$$

can be embedded into a ForestNeutrosophic Set

$$\mathbf{N} : P\big(\mathrm{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; ([0,1] \times [0,1] \times [0,1])^U.$$

Proof. Given a ForestFuzzy Set

$$\mathbf{F}(X) \in [0,1]^U,$$

we must define a corresponding ForestNeutrosophic Set $\mathbf{N}(X) \in ([0,1]^3)^U$ in such a way that the fuzzy membership values of $\mathbf{F}$ are recovered as a neutrosophic triple.

For each $X \subseteq \text{Forest}(\{A^{(t)}\})$ and each $x \in U$, let $\mu_X(x) = \mathbf{F}(X)(x)$. We define $\mathbf{N}(X)(x)$ by setting:

$$T_X(x) = \mu_X(x), \quad I_X(x) = 0, \quad F_X(x) = 0.$$

That is, we interpret the fuzzy membership $\mu_X(x)$ as the **truth** component $T_X(x)$, while the indeterminacy and falsity components are both set to 0.

Clearly, for each x,

$$T_X(x) + I_X(x) + F_X(x) = \mu_X(x) + 0 + 0 \leq 1 \leq 3,$$

so this triple is a valid Neutrosophic membership in $[0,1]^3$.

Hence, by this embedding, every ForestFuzzy Set is a special case of a ForestNeutrosophic Set, where indeterminacy and falsity values are all zero. □

2.3 ForestPlithogenic Set

The ForestPlithogenic Set is a concept that applies the idea of the ForestSoft Set to the framework of Plithogenic Sets. The definition is provided below.

Definition 2.6 (ForestPlithogenic Set)**.** Given a family of TreePlithogenic Sets $\{TPS_t\}_{t \in T}$, form the forest

$$\text{Forest}(\{A^{(t)}\}) = \bigsqcup_{t \in T} \text{Tree}(A^{(t)}).$$

A *ForestPlithogenic Set* **TPS** unifies all TPS_t into

$$\mathbf{TPS} = \Big(P,\ \text{Forest}(\{A^{(t)}\}),\ \{Pv_i\},\ \{\widetilde{pdf}_i\},\ \widetilde{pCF}\Big),$$

where each attribute node in the forest inherits or extends the plithogenic components from its corresponding tree.

Theorem 2.7 (ForestPlithogenic generalizes TreePlithogenic)**.** *Every TreePlithogenic Set is obtained by taking a forest with one tree.*

Proof. If $T = \{1\}$, the forest is just one tree, so the ForestPlithogenic structure is exactly the same as the original single-tree TPS_1. □

Theorem 2.8. *Any ForestNeutrosophic Set*

$$\mathbf{N} : P\big(\text{Forest}(\{A^{(t)}\})\big) \longrightarrow ([0,1]^3)^U$$

can be seen as a particular instance of a ForestPlithogenic Set.

Proof. A ForestPlithogenic Set (broadly) involves:

$$\mathbf{TPS} = \Big(P,\ \text{Forest}(\{A^{(t)}\}),\ \{Pv_i\},\ \{\widetilde{pdf}_i\},\ \widetilde{pCF}\Big),$$

where each node a_i in the forest is assigned a set of possible values Pv_i, a Degree of Appurtenance Function pdf_i, and a Degree of Contradiction Function pCF.

To embed a ForestNeutrosophic Set $\mathbf{N}$ into this framework, one can proceed as follows:

Assign each node a_i (in the Forest of attributes) a trivial set of possible values $Pv_i = \{\text{True}, \text{Indeterminate}, \text{False}\}$ or any suitable set.

For each node a_i and each $x \in U$, interpret the triple $\mathbf{N}(X)(x) = (T_X(x), I_X(x), F_X(x))$ as degrees of belonging to those three "value labels." We can define:

$$pdf_i(x, \text{True}) \ = \ T_X(x), \quad pdf_i(x, \text{Indeterminate}) \ = \ I_X(x), \quad pdf_i(x, \text{False}) \ = \ F_X(x).$$

(One may refine or unify these definitions across subsets $X \subseteq$ Forest, but conceptually it suffices that each triple can be viewed as a plithogenic membership distribution on $\{\text{True}, \text{Indeterminate}, \text{False}\}$.)

We can define $\widetilde{pCF}(\cdot, \cdot)$ to be zero or any neutral measure, so that no contradiction arises among these three labels, i.e.

$$\widetilde{pCF}(\text{True}, \text{Indeterminate}) = 0, \quad \widetilde{pCF}(\text{True}, \text{False}) = 0, \quad \ldots$$

or use any other consistent scheme.

Thus, the triple $(T_X(x), I_X(x), F_X(x))$ from the ForestNeutrosophic membership is naturally embedded as a plithogenic distribution over a small "value set." This construction shows that a ForestNeutrosophic Set is simply a special form of a ForestPlithogenic Set (with three "basic" possible values per node and a trivial contradiction function). □

Theorem 2.9. *Any ForestFuzzy Set*

$$\mathbf{F} : P\big(\text{Forest}(\{A^{(t)}\})\big) \ \longrightarrow \ [0, 1]^U$$

arises as a particular instance of a ForestPlithogenic Set.

Proof. This is a direct combination of the ideas above, plus the well-known fact that Fuzzy membership functions can be embedded into Plithogenic frameworks.

1. Interpret fuzzy membership as a single label's degree. Let each node a_i in the forest have a single set of possible values $Pv_i = \{v_i\}$ (just one label), or a small set of possible values with exactly one relevant label.
2. Degree of Appurtenance Functions. For each subset $X \subseteq \text{Forest}(\{A^{(t)}\})$ and each $x \in U$, the fuzzy membership $\mathbf{F}(X)(x)$ can be assigned to that single label:

   $$pdf_i(x, v_i) \ = \ \mathbf{F}(X)(x).$$

   All other "values" (if any) get degree 0.
3. Contradiction Function. We can again set $\widetilde{pCF} \equiv 0$ to make the system consistent with a purely fuzzy approach (no internal contradiction among multiple values, since effectively there is only one label of interest).

In this way, each fuzzy membership $\mathbf{F}(X)(x) \in [0, 1]$ is a special case of a plithogenic membership distribution: it is the degree of appurtenance to one label. Hence, any ForestFuzzy Set is subsumed by the broader notion of a ForestPlithogenic Set. □

2.4 ForestRough Set

The ForestRough Set is a concept that applies the idea of the ForestSoft Set to the framework of Rough Sets. The definition is provided below.

Definition 2.10 (ForestRough Set)**.** Let $\{\mathcal{TR}_t\}_{t \in T}$ be TreeRough frameworks. The *ForestRough Set* **FR** on

$$\text{Forest}\big(\{A^{(t)}\}\big) = \bigsqcup_{t \in T} \text{Tree}(A^{(t)})$$

collects the rough approximations from all nodes a_i in every tree. That is, for each $X \subseteq U$,

$$\mathbf{FR}(X) = \big\{ (\underline{X}_{a_i}, \overline{X}_{a_i}) \ \big| \ a_i \in \text{Forest}(\{A^{(t)}\}) \big\}.$$

Theorem 2.11 (ForestRough generalizes TreeRough)**.** *Every TreeRough Set is a special one-tree version of a ForestRough Set.*

Proof. Again, a single-tree forest reproduces the usual TreeRough Set structure exactly. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 10
Symbolic HyperPlithogenic set

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Abstract

Concepts such as Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets have been widely investigated for tackling uncertainty, with numerous applications explored across various domains. As extensions of the Plithogenic Set, the HyperPlithogenic Set and the SuperHyperPlithogenic Set are also recognized. A *Symbolic Plithogenic Set (SPS)* is a structured set defined by symbolic components P_i and coefficients a_i, enabling flexible algebraic operations under a specified prevalence order. In this paper, we examine concepts including the Symbolic HyperPlithogenic Set and the Symbolic n-SuperhyperPlithogenic Set.

Keywords: Plithogenic Set, HyperPlithogenic Set, Symbolic Plithogenic Set

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

1.1 Plithogenic Set

A Plithogenic Set is a mathematical framework designed to incorporate multi-valued degrees of appurtenance and contradiction, making it highly suitable for addressing complex decision-making processes. Extensive studies have been conducted on Plithogenic Sets, as evidenced by various works [1, 10, 22–24, 33, 36]. Additionally, related concepts such as the Plithogenic Graph have been widely recognized and explored [8, 15]. Furthermore, the Plithogenic Set is known for its ability to generalize several other mathematical frameworks, including Fuzzy Sets [40, 41], Intuitionistic Fuzzy Sets [4, 5], Vague Sets [7, 16], Neutrosophic Sets [26, 27], and Hesitant Fuzzy Sets [37, 38]. The formal definition is provided below.

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** [11, 21] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset)**.** (cf. [11, 13, 25, 32])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 1.4 (Plithogenic Set). [28, 29] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* [1]
- $pCF : Pv \times Pv \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a, a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a, b) = pCF(b, a)$$

Definition 1.5 (HyperPlithogenic Set). [9, 12, 14] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set* HPS over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0, 1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : \left(\bigcup_{i=1}^n Pv_i\right) \times \left(\bigcup_{i=1}^n Pv_i\right) \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

Definition 1.6 (n-SuperHyperPlithogenic Set). [9,12,14] Let X be a non-empty set, and let $V = \{v_1, v_2, \dots, v_n\}$ be a set of attributes, each associated with a set of possible values P_{v_i}. An n-SuperHyperPlithogenic Set ($SHPS_n$) is defined recursively as:

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$,
$$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$
represents the k-th nested family of non-empty subsets of P_1.
- For each attribute $v_i \in V$, P_{v_i} is the set of possible values of the attribute v_i.

[1] It is important to note that the definition of the Degree of Appurtenance Function varies across different papers. Some studies define this concept using the power set, while others simplify it by avoiding the use of the power set [34]. The author has consistently defined the Classical Plithogenic Set without employing the power set.

- For each k-th level subset P_k, $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P_n$ and attribute value $a_i \in P_{v_i}$ a subset of $[0,1]^s$.
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:
  1. Reflexivity: $pCF^{(n)}(a,a) = 0$ for all $a \in \bigcup_{i=1}^n P_{v_i}$,
  2. Symmetry: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ for all $a,b \in \bigcup_{i=1}^n P_{v_i}$.
- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

1.2 Symbolic Plithogenic Set

A *Symbolic Plithogenic Set (SPS)* is a structured set defined by symbolic components P_i and coefficients a_i, enabling flexible algebraic operations under a prevalence order [2, 30, 31].

Definition 1.7 (Symbolic Plithogenic Set)**.** [2,30,31] Let U be a universe of discourse, and let $P_1, P_2, \ldots, P_n$ be symbolic variables called *Symbolic Plithogenic Components*. A *Symbolic Plithogenic Set (SPS)* is defined as:

$$\text{SPS} = \left\{ x \in U \;\middle|\; x = \sum_{i=0}^{n} a_i P_i,\ a_i \in S \right\},$$

where:

- S is a given set, typically $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), or a subset thereof.
- a_i are called *coefficients*, and $P_0 = 1$ represents the identity component.
- $P_1, P_2, \ldots, P_n$ are abstract symbols or variables that may represent attributes, parameters, or properties, forming the base of the set SPS.

Definition 1.8 (Operations on Symbolic Plithogenic Set)**.** [2,30,31] The set SPS is equipped with the following operations:

1. Addition: For $x, y \in$ SPS where $x = \sum_{i=0}^n a_i P_i$ and $y = \sum_{i=0}^n b_i P_i$,

$$x + y = \sum_{i=0}^{n} (a_i + b_i) P_i.$$

2. Scalar Multiplication: For $c \in S$ and $x \in$ SPS, where $x = \sum_{i=0}^n a_i P_i$,

$$c \cdot x = \sum_{i=0}^{n} (c \cdot a_i) P_i.$$

3. Multiplication: Using the *Absorbance Law* and *Prevalence Order*, for $x, y \in$ SPS,

$$x \cdot y = \sum_{i=0}^{n} \sum_{j=0}^{n} (a_i \cdot b_j) \cdot \max(P_i, P_j),$$

where $\max(P_i, P_j)$ denotes the dominant component based on the predefined prevalence order $P_1 < P_2 < \cdots < P_n$.

4. Power: For $x \in$ SPS and $m \in \mathbb{N}$,

$$x^m = \underbrace{x \cdot x \cdot \cdots \cdot x}_{m \text{ times}}, \quad x^0 = 1.$$

Definition 1.9 (Symbolic Plithogenic Algebraic Structures)**.** [2,30,31] An algebraic structure defined on SPS with the operations + and $\cdot$ is called a *Symbolic Plithogenic Algebraic Structure*. Specifically:

- (SPS, +) forms a commutative group, where $0 = \sum_{i=0}^{n} 0 \cdot P_i$ is the identity element.
- (SPS, +, ·) forms a commutative ring with unity $1 = P_0$.
- Multiplication respects the Absorbance Law: the stronger component absorbs the weaker, based on the prevalence order.

Example 1.10 (Symbolic Plithogenic Numbers)**.** A *Symbolic Plithogenic Number (SPN)* is a specific element of SPS and is written as:

$$x = a_0 + a_1 P_1 + a_2 P_2 + \cdots + a_n P_n,$$

where $P_1 < P_2 < \cdots < P_n$ under the prevalence order. For example:

$$x = 3 + 5P_1 - 2P_2 + 7P_3,$$

where the multiplication follows the absorbance law:

$$P_1 \cdot P_2 = P_2, \quad P_2 \cdot P_3 = P_3.$$

Definition 1.11 (Generalization)**.** The symbolic components P_i can be extended to infinite dimensions, denoted as $P_1, P_2, \ldots, P_\infty$, leading to infinite-dimensional Symbolic Plithogenic Algebraic Structures.

It is worth noting that related concepts, such as the Symbolic k-Plithogenic Ring (where k is a natural number), are also well-known [6, 19, 35, 39]. For instance, Symbolic 2-Plithogenic Ring [20], Symbolic 3-Plithogenic Ring [3, 18], Symbolic 4-Plithogenic Ring [17, 18], and Symbolic 5-Plithogenic Ring [17] have been explored in various studies.

2 Results of This Paper

In this paper, we propose new definitions for various types of sets and briefly examine their relationships with existing concepts.

2.1 Symbolic HyperPlithogenic Set

The Symbolic Plithogenic Set is extended using the HyperPlithogenic Set and n-SuperhyperPlithogenic Set. Definitions and related theorems are presented below.

Definition 2.1 (Symbolic HyperPlithogenic Set)**.** Let

$$\mathrm{SPS} = \left\{ x = \sum_{i=0}^{n} a_i P_i : a_i \in S,\ P_i \text{ symbolic components} \right\}$$

be a *Symbolic Plithogenic Set* as in Definition. Let $\mathcal{A} = \{v_1, v_2, \ldots, v_m\}$ be a finite set of attributes, and for each $v_j \in \mathcal{A}$, let Pv_j be the set of possible values of v_j.

A *Symbolic HyperPlithogenic Set (SHPS)* is a structure

$$\mathrm{SHPS} = \Big(\mathrm{SPS},\ \mathcal{A},\ \{Pv_j\}_{j=1}^{m},\ \{\widetilde{pdf}_j\}_{j=1}^{m},\ pCF\Big),$$

where:

- SPS is the *Symbolic Plithogenic Set* described above.
- Each attribute v_j has a set of possible values Pv_j.
- For each v_j, $\widetilde{pdf}_j : \mathrm{SPS} \times Pv_j \longrightarrow \tilde{\mathcal{P}}\big([0,1]^s\big)$ is a *Hyper Degree of Appurtenance Function (HDAF)*, assigning a *set* of membership degrees in $[0,1]^s$ for each pair $\big(x, a_j\big)$, where $x \in \mathrm{SPS}$ and $a_j \in Pv_j$.

- $pCF : \left(\bigcup_{j=1}^{m} Pv_j\right) \times \left(\bigcup_{j=1}^{m} Pv_j\right) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:

$$pCF(a,a) = 0, \quad pCF(a,b) = pCF(b,a), \quad \text{for all } a,b \in \bigcup_{j=1}^{m} Pv_j.$$

- s and t are positive integers representing the dimensions of membership degrees and contradiction degrees, respectively.

Theorem 2.2. *A Symbolic HyperPlithogenic Set (SHPS) reduces to a Symbolic Plithogenic Set (SPS) if the hyperoperation $\widetilde{pdf}_j$ assigns singleton sets of membership degrees for each (x, a_j).*

Proof. In a Symbolic HyperPlithogenic Set, each $\widetilde{pdf}_j(x, a_j)$ is a non-empty subset of $[0,1]^s$. If we impose the restriction that each subset is a singleton $\{\mathbf{d}\} \subseteq [0,1]^s$, then $\widetilde{pdf}_j$ effectively becomes a single-valued function $\mathrm{pdf}_j : \mathrm{SPS} \times Pv_j \to [0,1]^s$, thus collapsing the hyperplithogenic structure to the classical plithogenic one (no set-valued membership). Hence, the SHPS merges into an SPS. Conversely, given an SPS, one can trivially interpret each membership degree as a singleton set $\{\mathbf{d}\}$. Therefore, SHPS strictly generalizes SPS. □

Definition 2.3 (Symbolic n-SuperHyperPlithogenic Set)**.** Let SHPS be a Symbolic HyperPlithogenic Set as in Definition 2.1. For an integer $n \geq 1$, a *Symbolic **n**-SuperHyperPlithogenic Set* (SHPS_n) is a structure

$$\mathrm{SHPS}_n = \left(\mathrm{SPS}_n,\ \mathcal{A}_n,\ \{\widetilde{pdf}_j^{(n)}\}_{j=1}^{m},\ pCF^{(n)}\right),$$

where:

- SPS_n is the *n-th symbolic plithogenic extension* of SPS, i.e. applying the symbolic expansion to an n-th level power-construction.
- $\mathcal{A}_n = \{v_1, v_2, \ldots, v_m\}$ remains the set of attributes, each with possible values Pv_j.
- $\widetilde{pdf}_j^{(n)} : \mathrm{SPS}_n \times Pv_j \longrightarrow \mathcal{P}_n^*([0,1]^s)$ is an (n-level) *Hyper Degree of Appurtenance Function*, mapping each (x, a_j) to an n-th nested subset of $[0,1]^s$, possibly excluding the empty set.
- $pCF^{(n)}$ is the *Degree of Contradiction Function* for the SPS_n environment, similar to the classical or single-level case but respecting n-th power expansions.

Theorem 2.4. *By setting $n = 1$, a Symbolic n-SuperHyperPlithogenic Set collapses to a Symbolic Hyper-Plithogenic Set.*

Proof. When $n = 1$, the mapping

$$\widetilde{pdf}_j^{(1)} : \mathrm{SPS}_1 \times Pv_j \ \to\ \mathcal{P}_1^*([0,1]^s)$$

is effectively a single-level hyper-mapping. That is, no further nesting occurs beyond $\mathcal{P}^*([0,1]^s)$. Hence, the structure coincides with Definition 2.1, namely a Symbolic HyperPlithogenic Set. Thus, restricting $n = 1$ recovers the single-level hyperplithogenic scenario, proving that the n-super notion strictly generalizes the single-level notion. □

2.2 Symbolic HyperPlithogenic Algebraic Structure

We now incorporate classical algebraic operations $(+, \cdot)$ on the *Symbolic HyperPlithogenic Set*, respecting the hyperplithogenic membership.

Definition 2.5 (Symbolic HyperPlithogenic Algebraic Structure (SHPAS))**.** A *Symbolic HyperPlithogenic Algebraic Structure (SHPAS)* is a tuple

$$\Big(\mathrm{SHPS},\ +,\ \cdot,\ \mathcal{A},\ \{\widetilde{pdf}_j\},\ pCF\Big),$$

where:

1. SHPS is a *Symbolic HyperPlithogenic Set* [14]:

$$\text{SPS (symbolic expansions)}, \quad \mathcal{A} = \{v_1, \ldots, v_m\}, \quad \widetilde{pdf}_j : \text{SPS} \times Pv_j \to \tilde{\mathcal{P}}([0,1]^s).$$

2. The operations + and · are defined over the base set SPS, typically:

$$x = \sum_i a_i P_i, \; y = \sum_i b_i P_i \quad \Longrightarrow \quad \begin{cases} x + y = \sum_i (a_i + b_i) P_i, \\ x \cdot y = \sum_{i,j} (a_i b_j) \cdot \max(P_i, P_j) \quad \text{(absorbance law)}. \end{cases}$$

3. The membership $\widetilde{pdf}_j$ is used to define hyper-membership degrees for each (x, a_j), and pCF encodes contradictions among attribute values.

4. Algebraic axioms (e.g. associativity, commutativity, distribution) can be postulated, depending on the intended structure (e.g. ring, module, semiring).

Theorem 2.6. *(Symbolic HyperPlithogenic Algebraic Structure generalizes Symbolic Plithogenic Algebraic Structure)*
If each hyper-membership $\widetilde{pdf}_j$ is restricted to singleton subsets in $[0,1]^s$, the Symbolic HyperPlithogenic Algebraic Structure *reduces to a* Symbolic Plithogenic Algebraic Structure.

Proof. When each $\widetilde{pdf}_j(x, a_j) \subseteq [0,1]^s$ is exactly $\{\mathbf{d}\}$, a single membership vector, we retrieve an ordinary pdf_j. Hence the hyper-based membership collapses to standard membership, and the resulting algebraic structure is precisely that of a *Symbolic Plithogenic Algebraic Structure* as in Definition. □

Definition 2.7 (Symbolic n-SuperHyperPlithogenic Algebraic Structure)**.** Let $n \geq 1$. A *Symbolic n-SuperHyperPlithogenic Algebraic Structure* is a tuple

$$\Big(\text{SHPS}_n, \; +, \; \cdot, \; \{\widetilde{pdf}_j^{(n)}\}, \; pCF^{(n)}\Big),$$

satisfying:

1. SHPS_n is a *Symbolic n-SuperHyperPlithogenic Set*, i.e. an n-th-level hyper-membership extension of the SPS with symbolic expansions and hyperplithogenic membership. Symbolically,

$$\text{SPS}_n = \mathcal{P}_n\big(\text{SPS}\big) \quad \text{or an analogous iterative symbolic extension.}$$

2. The operations + and · are defined over the extended domain SPS_n with a suitable generalization of the symbolic addition, multiplication, and *absorbance* among symbolic components at the n-th level.

3. Each $\widetilde{pdf}_j^{(n)} : \text{SPS}_n \times Pv_j \to \mathcal{P}_n^*\big([0,1]^s\big)$ is an n-layer *Hyper Degree of Appurtenance Function*, providing an n-times nested set of membership degrees in $[0,1]^s$.

4. $pCF^{(n)}$ is the *Degree of Contradiction Function* for the attribute values, extended (if needed) to handle the n-th super-level logic of plithogenic contradiction.

Theorem 2.8. *When $n = 1$, a Symbolic n-SuperHyperPlithogenic Algebraic Structure reduces to a Symbolic HyperPlithogenic Algebraic Structure (Definition 2.5).*

Proof. By setting $n = 1$, the $\mathcal{P}_1$-type expansions are replaced by single-level expansions, and

$$\widetilde{pdf}_j^{(1)} : \text{SPS}_1 \times Pv_j \to \mathcal{P}_1^*\big([0,1]^s\big)$$

becomes exactly a single-level hyperplithogenic membership. The domain SPS_1 coincides with the *Symbolic HyperPlithogenic Set* domain, and the algebraic operations remain in single-level symbolic form. Hence, all multi-layer nesting disappears, and we recover the *Symbolic HyperPlithogenic Algebraic Structure*. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 11
N-Superhypersoft Set and Bijective Superhypersoft Set

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Abstract

Soft sets provide a mathematical framework for decision-making by associating parameters with subsets of a universal set, effectively managing uncertainty and imprecision [53, 56]. Over time, various extensions of soft sets, including Hypersoft Sets, SuperHypersoft Sets, Treesoft Sets, Double-Framed Soft Sets, and Double-Framed Hypersoft Sets, have been introduced to address increasingly complex decision-making processes.

This paper introduces the definitions of N-SuperHypersoft Sets, N-Treesoft Sets, Bijective SuperHypersoft Sets, and Bijective Treesoft Sets, while also exploring their connections to previously established set theories.

Keywords: Superhypersoft set, Soft Set, Treesoft set, Hypersoft set

1 Preliminaries and Definitions

This section presents the foundational concepts and definitions necessary for the discussions in this paper. For additional details on fundamental set theory, readers may refer to [19, 42, 46, 48] as needed.

1.1 SuperHypersoft Set and Treesoft Set

To address uncertainty and imprecision in decision-making, several set theories have been proposed, including Fuzzy Sets [93–97], Neutrosophic Sets [23, 31–34, 37, 74, 75, 84], plithogenic sets [21, 24, 25, 35, 76, 78, 88], and Soft Sets [53, 56].

This subsection explores the foundational concepts of Soft Sets, Hypersoft Sets, Treesoft Sets, and SuperHypersoft Sets, which form the basis for advanced decision-making frameworks. A Soft Set provides a versatile approach to parameter-driven decision analysis by mapping attributes (parameters) to subsets of a universal set. This structure offers a powerful mechanism for addressing uncertainty and imprecision in complex decision-making processes [9, 11, 14, 39, 53, 54, 56, 73, 85, 92, 100].

Expanding on this foundation, a Hypersoft Set enhances multi-attribute decision analysis by associating combinations of multiple attributes with subsets of a universal set, enabling a more nuanced and comprehensive evaluation [1, 6, 22, 29, 36, 43, 57–65, 72, 77].

Treesoft Sets introduce a hierarchical approach for analyzing intricate datasets. By employing attribute trees where both nodes and leaves correspond to subsets of a universal set, Treesoft Sets provide a structured and detailed representation of hierarchical relationships [10, 15, 16, 26, 67, 68, 71, 79, 81–83].

SuperHypersoft Sets extend the concept of Hypersoft Sets by mapping power set combinations of multiple attributes to subsets of a universal set. This extension supports high-dimensional decision-making and captures intricate interdependencies among attributes, offering significant flexibility for addressing advanced decision-making challenges [12, 20, 26–28, 30, 38, 45, 49, 55, 80, 85–87, 89, 99].

The definitions are concisely provided below. For more detailed properties, operations, and applications, please refer to the respective references.

Definition 1.1 (Soft Set)**.** [53, 56] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α.

Definition 1.2 (Hypersoft Set)**.** [77] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be attribute domains. Define $C = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair (G, C), where $G : C \to \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, C) = \{(\gamma, G(\gamma)) \mid \gamma \in C, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in C$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Definition 1.3 (SuperHyperSoft Set)**.** [80] Let U be a universal set, and let $\mathcal{P}(U)$ denote the power set of U. Consider n distinct attributes $a_1, a_2, \ldots, a_n$, where $n \geq 1$. Each attribute a_i is associated with a set of attribute values A_i, satisfying the property $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Define $\mathcal{P}(A_i)$ as the power set of A_i for each $i = 1, 2, \ldots, n$. Then, the Cartesian product of the power sets of attribute values is given by:

$$C = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A SuperHyperSoft Set over U is a pair (F, C), where:

$$F : C \to \mathcal{P}(U),$$

and F maps each element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in C$ (with $\alpha_i \in \mathcal{P}(A_i)$) to a subset $F(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq U$. Mathematically, the SuperHyperSoft Set is represented as:

$$(F, C) = \{(\gamma, F(\gamma)) \mid \gamma \in C, F(\gamma) \in \mathcal{P}(U)\}.$$

Here, $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in C$, where $\alpha_i \in \mathcal{P}(A_i)$ for $i = 1, 2, \ldots, n$, and $F(\gamma)$ corresponds to the subset of U defined by the combined attribute values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Definition 1.4 (Treesoft Set)**.** [81] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1, i_2, \ldots, i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as $\mathrm{Tree}(A)$, with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set F* is defined as a function:

$$F : P(\mathrm{Tree}(A)) \to P(H),$$

where $\mathrm{Tree}(A)$ represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\mathrm{Tree}(A))$ denotes its power set.

1.2 N-soft set and N-hypersoft set

An N-soft set associates attributes with subsets of objects, each paired with satisfaction grades, providing a structured framework for decision-making [2–5, 8, 17, 50–52, 66, 70, 98]. Building upon this concept, the N-hypersoft set offers an extended approach to accommodate more complex scenarios [64]. The relevant definitions and details are presented below.

Definition 1.5 (N-soft Set)**.** [7, 18] Let O be a set of objects (alternatives) and T be a set of attributes (characteristics). An *N-soft set* over O and T is a triple (F, T, N), where:

- $F : T \to 2^{O\times G}$ is a mapping from the set of attributes T to the power set of $O \times G$,
- $G = \{0, 1, \dots, N - 1\}$ is the set of possible grades, representing levels of satisfaction,
- $N \geq 2$ is a natural number, specifying the number of levels of satisfaction.

The mapping F satisfies the following condition:

$$\text{For each } t \in T \text{ and } o \in O, \text{ there exists a unique } (o, g_t) \in F(t), \text{ where } g_t \in G.$$

Definition 1.6 (Tabular Representation of N-soft Set)**.** When $O = \{o_1, o_2, \dots, o_p\}$ and $T = \{t_1, t_2, \dots, t_q\}$ are finite, an N-soft set (F, T, N) can be represented as a table. For each $t_j \in T$ and $o_i \in O$, the value $F(t_j)(o_i) = r_{ij} \in G$ satisfies:

$$(o_i, r_{ij}) \in F(t_j).$$

The tabular representation is given as:

(F, T, N)	t_1	t_2	$\dots$	t_q
o_1	r_{11}	r_{12}	$\dots$	r_{1q}
o_2	r_{21}	r_{22}	$\dots$	r_{2q}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
o_p	r_{p1}	r_{p2}	$\dots$	r_{pq}

Here, $r_{ij} \in G$ represents the grade assigned to object o_i under attribute t_j.

Definition 1.7 (Soft Set as a Special Case of N-soft Set)**.** When $N = 2$, the N-soft set reduces to a standard soft set. Define a mapping $F_0 : T \to P(O)$ such that:

$$F_0(t) = \{o \in O \mid F(t)(o) = 1\}.$$

In this case, $F(t)(o) = 1$ implies $o \in F_0(t)$, and the tabular representation contains only 0 and 1.

Definition 1.8 (N-Hypersoft Set)**.** [64] Let Ω be a universal set of objects, E be a set of parameters, and $\xi_1 \subseteq E$. Consider $R = \{0, 1, \dots, N - 1\}$, where $N \geq 2$, as the set of ordered grades. An *N-Hypersoft Set (N-HS set)* is a triple (∇, ξ_1, N), where:

- $\nabla : \xi_1 \to P(\Omega \times R)$ maps each parameter $q \in \xi_1$ to a subset of $\Omega \times R$,
- ∇ satisfies the condition: for every $q \in \xi_1$ and $\omega \in \Omega$, there exists a unique pair $(\omega, r_q) \in \nabla(q)$, where $r_q \in R$.

The evaluation of each object $\omega \in \Omega$ under parameter $q \in \xi_1$ is denoted as:

$$\nabla(q)(\omega) = r_q.$$

Definition 1.9 (Tabular Representation)**.** When $\Omega = \{\omega_1, \omega_2, \dots, \omega_m\}$ and $\xi_1 = \{q_1, q_2, \dots, q_n\}$, an N-HS set (∇, ξ_1, N) can be represented in tabular form:

(∇, ξ_1, N)	q_1	q_2	$\dots$	q_n
ω_1	r_{11}	r_{12}	$\dots$	r_{1n}
ω_2	r_{21}	r_{22}	$\dots$	r_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ω_m	r_{m1}	r_{m2}	$\dots$	r_{mn}

where $r_{ij} \in R$ represents the grade assigned to object ω_i under parameter q_j.

1.3 Bijective Soft Set and Bijective Hypersoft Set

A Bijective Soft Set is a type of soft set where each parameter uniquely maps to a disjoint subset of the universal set, ensuring the entire set is covered [40, 41, 47, 90, 91]. A Bijective Hypersoft Set extends this concept by mapping unique combinations of attributes to disjoint subsets of the universal set, also guaranteeing complete coverage [69].

Definition 1.10 (Bijective Soft Set). [40, 41, 47, 90, 91] Let U be a universe of discourse, and let B be a non-empty parameter set. A soft set (F, B) over U is defined as a pair where:

$$F : B \to \mathcal{P}(U),$$

and F maps each parameter $e \in B$ to a subset $F(e) \subseteq U$.

The soft set (F, B) is called a *Bijective Soft Set* if the following conditions hold:

1. *Exhaustiveness:* The union of all subsets $F(e)$ equals the universe:

$$\bigcup_{e \in B} F(e) = U.$$

2. *Disjointness:* The subsets $F(e)$ are pairwise disjoint:

$$F(e_i) \cap F(e_j) = \emptyset, \quad \forall e_i, e_j \in B,\ e_i \neq e_j.$$

Alternatively, the mapping $F : B \to \mathcal{P}(U)$ can be transformed into a bijective function $F : B \to Y$, where $Y \subseteq \mathcal{P}(U)$ and Y contains pairwise disjoint subsets of U.

Definition 1.11 (Bijective Hypersoft Set). [44, 69] Let U be a universe of discourse, and let $G = G_1 \times G_2 \times \cdots \times G_n$, where:

- G_i is the set of possible values for attribute g_i,
- $G_i \cap G_j = \emptyset$ for $i \neq j$, ensuring $G_1, G_2, \ldots, G_n$ are disjoint.

A *hypersoft set* is a pair (F, G), where:

$$F : G \to \mathcal{P}(U),$$

is a mapping that assigns each $\delta \in G$ (a tuple of attribute values) to a subset $F(\delta) \subseteq U$.

The hypersoft set (F, G) is called a *Bijective Hypersoft Set* if the following conditions hold:

1. $\bigcup_{\delta \in G} F(\delta) = U$.
2. For any $\delta_i, \delta_j \in G$ with $\delta_i \neq \delta_j$, $F(\delta_i) \cap F(\delta_j) = \emptyset$.

Alternatively, the mapping $F : G \to \mathcal{P}(U)$ can be rewritten as a bijection $F : G \to \mathcal{P}_1(U)$, where $\mathcal{P}_1(U) \subseteq \mathcal{P}(U)$ contains pairwise disjoint subsets $F(\delta)$.

2 Result of this Paper

This section presents the results of this paper.

2.1 N-SuperHypersoft Set

The N-SuperHypersoft Set is a generalized concept derived from the N-Hypersoft Set. Its definitions and related details are provided below.

Definition 2.1 (N-SuperHypersoft Set)**.** Let U be a universal set, and let $n \geq 1$. Suppose we have n distinct attributes

$$a_1, a_2, \dots, a_n,$$

where each attribute a_i is associated with a set of attribute values A_i, subject to the condition

$$A_i \cap A_j = \emptyset \quad \text{for all} \quad i \neq j.$$

For each A_i, define its power set $\mathcal{P}(A_i)$. Let

$$C \ = \ \mathcal{P}(A_1) \ \times \ \mathcal{P}(A_2) \ \times \ \cdots \ \times \ \mathcal{P}(A_n),$$

which represents all possible combinations of attribute-value subsets.

Additionally, let

$$R = \{0, 1, 2, \dots, N-1\}$$

be a set of *grades* (or levels), where $N \geq 2$ is a fixed integer.

An *N-SuperHypersoft Set* over U is a triple

$$\big(H, C, N\big),$$

where

$$H : C \ \to \ \mathcal{P}\big(U \times R\big),$$

satisfies the following uniqueness condition:

$$\begin{gathered}\text{For each combination } \gamma \in C \text{ and for each } u \in U,\\ \text{there is exactly one ordered pair } (u, r_\gamma) \in H(\gamma), \text{ where } r_\gamma \in R.\end{gathered}$$

In other words, for every $\gamma \in C$, the set $H(\gamma) \subseteq U \times R$ assigns a unique grade r_γ to each element $u \in U$ whenever one interprets the combined attribute values in γ.

We now show rigorously that the N-SuperHypersoft Set generalizes both the N-Hypersoft Set and the SuperHypersoft Set.

Theorem 2.2 (N-SuperHypersoft Set Generalizes N-Hypersoft Set)**.** *Every N-Hypersoft Set is a particular case of an N-SuperHypersoft Set.*

Proof. An N-Hypersoft Set (∇, ξ_1, N) typically has:

$$\xi_1 = \{q_1, q_2, \dots, q_m\} \quad \text{(parameters)},$$

and each parameter q_j is associated with a mapping

$$\nabla(q_j) \ \subseteq \ \Omega \times R,$$

where Ω is the universe of objects and $R = \{0, \dots, N-1\}$. By definition, each object $\omega \in \Omega$ is assigned a unique grade under each parameter q_j.

To obtain an N-SuperHypersoft Set from this, consider:

$$n = m, \quad A_j = \{\alpha_{j1}, \alpha_{j2}, \dots\}$$

but we force each $\mathcal{P}(A_j)$ to behave like a single-valued domain φ_j. Concretely:

$$C = \varphi_1 \times \cdots \times \varphi_m, \quad \text{where each } \varphi_j \text{ effectively has elements in 1-to-1 correspondence with } q_j.$$

Then define:

$$H(\gamma) = \nabla(\widehat{q}),$$

where $\widehat{q}$ is the appropriate parameter from ξ_1 that matches the chosen combination γ. Since each parameter q_j yields a unique pairing $(\omega, r) \in \Omega \times R$, the uniqueness condition is satisfied in the sense of Definition 2.1.

Thus, by restricting each $\mathcal{P}(A_j)$ to act as a single-valued domain and linking it to exactly one parameter q_j, we replicate an N-Hypersoft Set within the framework of an N-SuperHypersoft Set. □

Theorem 2.3 (N-SuperHypersoft Set Generalizes SuperHypersoft Set)**.** *Every SuperHypersoft Set is embedded in the N-SuperHypersoft Set structure by letting $N = 2$ and collapsing the grade assignment to a binary distinction.*

Proof. A SuperHypersoft Set (F, C) has $C = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$. For each $\gamma \in C$,

$$F(\gamma) \subseteq U.$$

This is equivalent to having

$$F(\gamma) \subseteq U \times \{1\}$$

if we regard membership in $F(\gamma)$ as getting the grade 1 and non-membership as grade 0. So set

$$R = \{0, 1\} \quad (\text{hence } N = 2),$$

and define

$$H(\gamma) = \{(u, 1) \mid u \in F(\gamma)\} \cup \{(u, 0) \mid u \notin F(\gamma)\}.$$

Evidently, each $u \in U$ appears exactly once, paired with either grade 0 or 1. Hence, $H(\gamma) \subseteq U \times R$ satisfies the uniqueness condition for all γ. Thus, $(H, C, 2)$ is an N-SuperHypersoft Set that mimics (F, C). In other words, the SuperHypersoft Set is a special (binary-grade) instance of the N-SuperHypersoft Set. □

2.2 N-Treesoft Set

We now introduce the concept of an N-Treesoft Set, which extends the Treesoft Set by allowing each node-leaf subset to be graded via a set $R = \{0, \ldots, N-1\}$.

Definition 2.4 (N-Treesoft Set)**.** Let

$$U \quad \text{be a universal set, and}$$

$$\text{Tree}(A) \quad \text{be a hierarchical tree of attributes,}$$

with root A and multiple levels of sub-attributes as in the standard Treesoft framework [81]. Let

$$P(\text{Tree}(A))$$

denote the power set of all nodes and leaves in $\text{Tree}(A)$. Suppose $N \geq 2$ is an integer and

$$R = \{0, 1, \ldots, N-1\}.$$

An *N-Treesoft Set* is a triple

$$\big(\Lambda, \text{Tree}(A), N\big),$$

where

$$\Lambda : P\big(\text{Tree}(A)\big) \to \mathcal{P}\big(U \times R\big)$$

obeys the uniqueness condition:

$$\begin{aligned}&\text{For each } X \subseteq \text{Tree}(A) \text{ and for each } u \in U,\\ &\quad \text{there is exactly one pair } (u, r_X) \in \Lambda(X), \quad \text{where } r_X \in R.\end{aligned}$$

Remark 2.5. If $N = 2$, we can interpret the pair $(u, 1)$ as "u belongs to $\Lambda(X)$" and $(u, 0)$ as "u does not belong to $\Lambda(X)$", retrieving something akin to a traditional Treesoft Set (albeit each node set is now forcibly assigned to either 0 or 1 for every u).

We show below that an N-Treesoft Set generalizes both an N-soft Set (when the tree is restricted to a single level and we use grading) and a standard Treesoft Set (when $N = 2$).

Theorem 2.6 (N-Treesoft Set Generalizes N-soft Set)**.** *Let (F, T, N) be an N-soft set over objects O and attributes T. Then there is a corresponding N-Treesoft Set $(\Lambda, Tree(T), N)$ that reproduces the same assignment of grades to objects, assuming a single-level tree structure.*

Proof. An N-soft set (F, T, N) has
$$F : T \to 2^{O\times G},$$
with $G = \{0, \ldots, N-1\}$, and each pair (o, g_t) corresponds to how object $o \in O$ is graded under attribute $t \in T$.

Construct a *one-level* tree: $\mathrm{Tree}(T)$ has the root T (level 0) and the set $T = \{t_1, \ldots, t_q\}$ as level-1 nodes (no deeper sub-attributes). Therefore,
$$P\big(\mathrm{Tree}(T)\big) = P(T),$$
the power set of the attribute set. Define
$$\Lambda(X) \;=\; \Big\{(o, r_X) : o \in O,\; r_X \in G\Big\}$$
so that for each $X \subseteq T$ and $o \in O$, there is a unique pair $(o, r_X) \in \Lambda(X)$. We must choose r_X so that it consistently reflects the N-soft evaluation from F. One way is to let
$$r_X \;=\; \text{some aggregation}(\{\, F(t_j)(o) : t_j \in X\}),$$
where some aggregation is chosen so that each (o, r_X) is unique. For a simpler direct matching, we may only evaluate Λ on *singletons* $\{t_j\}$. Then
$$\Lambda(\{t_j\}) \;\approx\; F(t_j).$$
By extending Λ consistently to larger subsets $X \subseteq T$ (through any well-defined rule that chooses a unique grade for each o), we preserve uniqueness.

Hence, $(\Lambda, \mathrm{Tree}(T), N)$ functions as an N-Treesoft Set that, on single-attribute subsets, recovers the same graded pairs $\big(o, F(t_j)(o)\big)$. Therefore, the single-level tree model precisely embeds an N-soft set as an N-Treesoft Set. □

Theorem 2.7 (N-Treesoft Set Generalizes Treesoft Set)**.** *Every standard Treesoft Set $(F, Tree(A))$ is a special case of an N-Treesoft Set for $N = 2$.*

Proof. A standard (single-grade) Treesoft Set has
$$F : P\big(\mathrm{Tree}(A)\big) \to P(H),$$
for some subset $H \subseteq U$. We can embed F into an N-Treesoft Set by letting $N = 2$ (so $R = \{0, 1\}$) and defining
$$\Lambda(X) \;=\; \{\,(u, 1) \mid u \in F(X)\} \;\cup\; \{\,(u, 0) \mid u \notin F(X)\}.$$
Hence, for each $u \in U$, $\Lambda(X)$ contains exactly one pair (u, r), where $r = 1$ if $u \in F(X)$ and $r = 0$ otherwise. This satisfies the uniqueness condition of Definition 2.4. Consequently, $(\Lambda, \mathrm{Tree}(A), 2)$ is an N-Treesoft Set that coincides with the original Treesoft Set when ignoring the binary grade. □

2.3 Bijective SuperHypersoft Set

The Bijective SuperHypersoft Set is a generalized concept derived from the Bijective Hypersoft Set. Definitions, related theorems, and other details are provided below.

Definition 2.8 (Bijective SuperHypersoft Set)**.** Let U be a universal set. Suppose we have n distinct attributes $a_1, a_2, \ldots, a_n$, where each attribute a_i has a domain $A_i \subseteq$ (some larger set), satisfying $A_i \cap A_j = \emptyset$ for $i \neq j$. For each i, define the power set $\mathcal{P}(A_i)$. Then:

$$C = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n),$$

represents all possible combinations of subsets of the respective domains.

A *SuperHypersoft Set* is a pair (F, C), with

$$F : C \to \mathcal{P}(U).$$

We say (F, C) is a *Bijective SuperHypersoft Set* if:

1. $\bigcup_{\gamma \in C} F(\gamma) = U.$ (Exhaustiveness)
2. For any $\gamma_1 \neq \gamma_2 \in C$,
$$F(\gamma_1) \cap F(\gamma_2) = \emptyset.$$
(Pairwise Disjointness)

Equivalently, $F : C \to \mathcal{P}(U)$ can be viewed as a bijection onto a family of disjoint subsets covering U. That is, we can write

$$F(\gamma) \in \mathcal{P}_1(U),$$

where $\mathcal{P}_1(U) \subseteq \mathcal{P}(U)$ is a collection of pairwise disjoint subsets whose union is U.

Theorem 2.9 (Bijective SuperHypersoft Set Generalizes Bijective Hypersoft Set)**.** *Any Bijective Hypersoft Set is a particular case of a Bijective SuperHypersoft Set.*

Proof. A *Bijective Hypersoft Set* (F, G) [13] typically arises when each attribute a_i has a *single-valued* domain G_i (rather than $\mathcal{P}(A_i)$), so

$$G = G_1 \times G_2 \times \cdots \times G_n,$$

where each G_i is pairwise disjoint from the others. The mapping $F\colon G \to \mathcal{P}(U)$ must satisfy the bijectivity conditions:

$$\bigcup_{\delta \in G} F(\delta) = U \quad \text{and} \quad F(\delta_i) \cap F(\delta_j) = \emptyset \text{ for all } \delta_i \neq \delta_j.$$

To see that this is a special case of Definition 2.8, note:

- In a Bijective SuperHypersoft Set, each attribute domain is $\mathcal{P}(A_i)$.
- If we restrict each $\mathcal{P}(A_i)$ to *only singletons* (or effectively treat it as the original set G_i with exactly one chosen value), then the Cartesian product $\mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$ reduces to $G_1 \times \cdots \times G_n$.
- Define F on this restricted domain precisely as it was on the Bijective Hypersoft Set. The exhaustiveness and disjointness constraints remain identical.

Hence, every Bijective Hypersoft Set arises by limiting the domain of a Bijective SuperHypersoft Set to single-valued subsets, showing that Bijective SuperHypersoft Sets generalize Bijective Hypersoft Sets. □

Theorem 2.10 (Bijective SuperHypersoft Set Generalizes SuperHypersoft Set)**.** *Any (non-bijective) SuperHypersoft Set is embedded in the Bijective SuperHypersoft Set structure by relaxing the disjointness condition or, equivalently, setting merges of images.*

Proof. A *SuperHypersoft Set* (G, C) has $C = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$ and

$$G : C \rightarrow \mathcal{P}(U).$$

The standard definition requires no disjointness condition on $G(\gamma)$.

To embed it into a Bijective SuperHypersoft Set, observe:

- The only difference is that a Bijective SuperHypersoft Set requires all $F(\gamma)$ to be pairwise disjoint and to cover U.
- If we *permit* repeated or overlapping images in the sense that we remove the pairwise disjointness constraint, we return to an ordinary SuperHypersoft Set.

Thus, from a conceptual standpoint, any SuperHypersoft Set (G, C) is a *looser* version of a Bijective SuperHypersoft Set (F, C) where the pairwise-disjoint condition need not hold. Therefore, the Bijective model strictly contains the usual SuperHypersoft model as a special case (when the disjointness is removed or the union need not be an exact partition). □

2.4 Bijective Treesoft Set

We now extend the *Treesoft Set* (a hierarchical attribute structure) by requiring that each subset of the tree map to a pairwise-disjoint family covering the entire domain. This yields the *Bijective Treesoft Set*.

Definition 2.11 (Bijective Treesoft Set)**.** Let:

- U be a universal set (or a universal "universe of discourse"),
- $\mathrm{Tree}(A)$ be a hierarchical attribute tree derived from an attribute set $A = \{A_1, A_2, \ldots, A_n\}$ (with possibly multiple levels of sub-attributes),
- $P(\mathrm{Tree}(A))$ denote the power set of all nodes (and leaves) within $\mathrm{Tree}(A)$.

A *Treesoft Set* [81] is a function

$$F : P(\mathrm{Tree}(A)) \rightarrow \mathcal{P}(H),$$

for some non-empty subset $H \subseteq U$.

A *Bijective Treesoft Set* is defined analogously, except it must satisfy:

1. $\displaystyle\bigcup_{X \subseteq \mathrm{Tree}(A)} F(X) = H.$ (All images together cover H)
2. If $X_i \neq X_j$, then $F(X_i) \cap F(X_j) = \emptyset.$ (Pairwise disjointness of images)

Hence, for each distinct subset $X \subseteq \mathrm{Tree}(A)$, the image $F(X)$ is a subset of H, and all these subsets partition H. Equivalently,

$$F : P(\mathrm{Tree}(A)) \rightarrow \mathcal{P}_1(H),$$

where $\mathcal{P}_1(H)$ is a family of pairwise disjoint subsets whose union is H.

Theorem 2.12 (Bijective Treesoft Set Generalizes Bijective Soft Set)**.** *Every Bijective Soft Set is a special case of a Bijective Treesoft Set.*

Proof. A *Bijective Soft Set* (F, B) over a universe U has:

$$F : B \to \mathcal{P}(U), \quad \bigcup_{b \in B} F(b) = U, \quad F(b_i) \cap F(b_j) = \emptyset \text{ for } b_i \neq b_j.$$

This can be seen as a *single-level* tree:

$$\mathrm{Tree}(A) = \{A_1, A_2, \ldots, A_{|B|}\},$$

where each A_i corresponds to one parameter $b \in B$. In this case,

$$P\big(\mathrm{Tree}(A)\big) = P(B).$$

Define

$$\widetilde{F}(X) \;=\; \bigcup_{b \in X} F(b), \quad \text{for all } X \subseteq B.$$

Then $\widetilde{F} \colon P(B) \to \mathcal{P}(U)$ is a candidate for a *Treesoft*-like mapping. To enforce *bijectivity*, note that if we evaluate $\widetilde{F}$ *only* on singletons $\{b\} \subseteq B$, we recover exactly $F(b)$. By requiring that the entire family $\{\widetilde{F}(X) \mid X \subseteq B\}$ remains disjoint except at X differences, we can keep the same disjoint partition.

Alternatively, we can keep a simpler definition:

$$\widetilde{F}(X) \;=\; \begin{cases} F(b), & \text{if } X = \{b\} \subseteq B, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus,

$$\bigcup_{X \subseteq B} \widetilde{F}(X) = \bigcup_{b \in B} F(b) = U,$$

and if $X_i \neq X_j$, $\widetilde{F}(X_i) \cap \widetilde{F}(X_j) = \emptyset$. So $\widetilde{F}$ is a *Bijective Treesoft Set* over the single-level tree. Hence, any Bijective Soft Set is realized as a single-level special case of the Bijective Treesoft Set. □

Theorem 2.13 (Bijective Treesoft Set Generalizes Treesoft Set)**.** *Every ordinary (possibly non-bijective) Treesoft Set is a particular case of a Bijective Treesoft Set by relaxing the disjointness or exhaustive conditions.*

Proof. A *Treesoft Set* $(F, \mathrm{Tree}(A))$ simply requires

$$F : P\big(\mathrm{Tree}(A)\big) \;\to\; \mathcal{P}(H),$$

with no demand that $\bigcup F(X) = H$ or that images be disjoint. In a *Bijective Treesoft Set*, we add:

$$\bigcup_{X \subseteq \mathrm{Tree}(A)} F(X) = H, \quad F(X_i) \cap F(X_j) = \emptyset \text{ if } X_i \neq X_j.$$

Hence, if we relax or remove these additional constraints, we precisely recover the broader notion of a Treesoft Set. Thus, Treesoft Sets can be viewed as a less-restrictive sub-family within the space of all Bijective Treesoft Sets. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 12
Plithogenic Rough Sets

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Abstract

Plithogenic Sets are mathematical structures designed to incorporate multi-valued degrees of appurtenance and contradictions, providing a robust framework for modeling complex and dynamic decision-making processes. Rough Sets address uncertainty by dividing a set into lower and upper approximations, which represent definable and potentially related elements, respectively.

In this paper, we explore Plithogenic Rough Sets, a concept that combines the principles of Rough Sets and Plithogenic Sets.

Keywords: Fuzzy set, Rough Set, Plithogenic Set

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

1.1 Fuzzy Set and Neutrosophic Set

The concept of a Fuzzy Set is widely used in set theory to address uncertainty. Its formal definition is provided below [62–70].

Definition 1.1. [62, 67] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \le \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Similarly, Neutrosophic Sets, which generalize Fuzzy Sets, are another significant concept frequently referenced in this paper [50–55]. Neutrosophic Sets have been extended to various concepts, including graphs, and have been the subject of extensive research [4, 11, 21–26, 28].

Their formal definition is presented below.

Definition 1.2. [52] Let X be a given set. A Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

1.2 Plithogenic Set

A Plithogenic Set is a mathematical framework designed to incorporate multi-valued degrees of appurtenance and contradictions, making it highly suitable for addressing complex decision-making processes. Numerous studies have been conducted on Plithogenic Sets [1, 2, 12, 18, 41, 43–45, 58, 60]. Additionally, related concepts such as the Plithogenic Graph and Plithogenic Language have been extensively explored [5–7, 9, 13–15, 17, 20, 27, 46–49]. The formal definition is presented below.

Definition 1.3. [56,57] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* [1]
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a, a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a, b) = pCF(b, a)$$

1.3 Fuzzy Rough set and Neutrosophic Rough set

A rough set represents imprecise or uncertain knowledge by approximating a set using a pair of lower and upper bounds [32–37]. A fuzzy rough set combines fuzzy logic and rough set theory, modeling uncertainty with fuzzy membership and boundary approximations [29,31,42]. A neutrosophic rough set generalizes rough sets by incorporating truth, indeterminacy, and falsity degrees to handle imprecision [3,61,61,72].

Definition 1.4 (Fuzzy Rough Set)**.** [29,31,42] Let U be a finite and nonempty universe, and R a fuzzy relation on U. Let A be a fuzzy set defined on U, with membership function $\mu_A : U \to [0,1]$.

The *fuzzy rough lower approximation* of A with respect to R, denoted $\mathrm{apr}_R(A)$, is defined as:

$$\mathrm{apr}_R(A)(x) = \inf_{y \in U} \max\left(1 - R(x,y), \mu_A(y)\right), \quad \forall x \in U.$$

The *fuzzy rough upper approximation* of A with respect to R, denoted $\overline{\mathrm{apr}}_R(A)$, is defined as:

$$\overline{\mathrm{apr}}_R(A)(x) = \sup_{y \in U} \min\left(R(x,y), \mu_A(y)\right), \quad \forall x \in U.$$

The *boundary region* of A is given by:

$$\mathrm{bnd}_R(A) = \overline{\mathrm{apr}}_R(A) - \mathrm{apr}_R(A).$$

If $\mathrm{bnd}_R(A) \neq \emptyset$, A is called a *fuzzy rough set*.

[1] It is important to note that the definition of the Degree of Appurtenance Function varies across different papers. Some studies define this concept using the power set, while others simplify it by avoiding the use of the power set [59]. The author has consistently defined the Classical Plithogenic Set without employing the power set.

Definition 1.5 (Neutrosophic Rough Set)**.** [3, 61, 61, 72] Let U be a nonempty universe and R a single-valued neutrosophic relation on U. For a set $A \subseteq U$, denote $R_T(x, y), R_I(x, y), R_F(x, y)$ as the truth, indeterminacy, and falsity components of R, respectively.

The *neutrosophic lower approximation* of A with respect to R, denoted $R(A)$, is defined as:

$$\begin{aligned} R_T(A)(x) &= \inf_{y\in A} R_T(x, y), \\ R_I(A)(x) &= \sup_{y\in A} R_I(x, y), \\ R_F(A)(x) &= \sup_{y\in A} R_F(x, y), \end{aligned} \qquad \forall x \in U.$$

The *neutrosophic upper approximation* of A with respect to R, denoted $\overline{R}(A)$, is defined as:

$$\begin{aligned} \overline{R}_T(A)(x) &= \sup_{y\in U} \min\left(R_T(x, y), \nVdash_{y\in A}\right), \\ \overline{R}_I(A)(x) &= \inf_{y\in U} \max\left(R_I(x, y), \nVdash_{y\notin A}\right), \\ \overline{R}_F(A)(x) &= \inf_{y\in U} \max\left(R_F(x, y), \nVdash_{y\notin A}\right), \end{aligned} \qquad \forall x \in U.$$

The pair $\left(R(A), \overline{R}(A)\right)$ is referred to as the *neutrosophic rough set* of A.

2 Result: Plithogenic Rough Set

In this paper, we define the concept of a Plithogenic Rough Set.

2.1 Plithogenic Rough Set

Let U be a nonempty universe and R a Plithogenic relation defined on U. A Plithogenic relation R on U is characterized by the following attributes:

- A Plithogenic membership function $pdf : U \times U \to [0, 1]^s$ that defines the degree of appurtenance.
- A contradiction function $pCF : U \times U \to [0, 1]^t$ that defines the degree of contradiction.

Given a Plithogenic set $A \subseteq U$, the *Plithogenic Rough Set* is defined by its lower and upper approximations as follows:

Definition 2.1 (Plithogenic Lower Approximation)**.** The *Plithogenic lower approximation* of A with respect to R, denoted $\mathrm{PL}_R(A)$, is given by:

$$\mathrm{PL}_R(A)(x) = \inf_{y\in A} \max\left(1 - pdf(x, y), 1 - pCF(x, y)\right), \quad \forall x \in U.$$

Definition 2.2 (Plithogenic Upper Approximation)**.** The *Plithogenic upper approximation* of A with respect to R, denoted $\overline{\mathrm{PL}}_R(A)$, is given by:

$$\overline{\mathrm{PL}}_R(A)(x) = \sup_{y\in U} \min\left(pdf(x, y), 1 - pCF(x, y)\right), \quad \forall x \in U.$$

Definition 2.3 (Plithogenic Rough Set)**.** The *Plithogenic Rough Set* of A is the pair:

$$\left(\mathrm{PL}_R(A), \overline{\mathrm{PL}}_R(A)\right),$$

where $\mathrm{PL}_R(A)$ and $\overline{\mathrm{PL}}_R(A)$ are the lower and upper approximations, respectively.

Theorem 2.4 (Generalization Property of Plithogenic Rough Set)**.** *The Plithogenic Rough Set generalizes both the Fuzzy Rough Set and the Neutrosophic Rough Set.*

Proof. (i) Generalizing the Fuzzy Rough Set.

In the case where the Plithogenic membership function $pdf(x, y)$ reduces to the fuzzy membership function $\mu_R(x, y)$, and the contradiction function $pCF(x, y)$ is identically zero (i.e., $pCF(x, y) = 0$ for all $x, y \in U$), the Plithogenic Rough Set simplifies to the Fuzzy Rough Set.

- The Plithogenic lower approximation becomes:

$$\underline{\mathrm{PL}}_R(A)(x) = \inf_{y \in A} \max\left(1 - \mu_R(x, y), \mu_A(y)\right),$$

  which matches the fuzzy rough lower approximation $\underline{\mathrm{apr}}_R(A)(x)$.

- The Plithogenic upper approximation becomes:

$$\overline{\mathrm{PL}}_R(A)(x) = \sup_{y \in U} \min\left(\mu_R(x, y), \mu_A(y)\right),$$

  which matches the fuzzy rough upper approximation $\overline{\mathrm{apr}}_R(A)(x)$.

Thus, the Plithogenic Rough Set reduces to the Fuzzy Rough Set.

(ii) Generalizing the Neutrosophic Rough Set.

In the case where the Plithogenic membership function $pdf(x, y)$ is decomposed into three components (T, I, F), representing truth, indeterminacy, and falsity degrees, and the contradiction function $pCF(x, y)$ is identically zero, the Plithogenic Rough Set simplifies to the Neutrosophic Rough Set.

- The Plithogenic lower approximation becomes:

$$\underline{\mathrm{PL}}_R(A)(x) = \inf_{y \in A} \max\left(1 - T_R(x, y), I_R(x, y), F_R(x, y)\right),$$

  which matches the neutrosophic lower approximation $\underline{R}(A)$.

- The Plithogenic upper approximation becomes:

$$\overline{\mathrm{PL}}_R(A)(x) = \sup_{y \in U} \min\left(T_R(x, y), 1 - I_R(x, y), 1 - F_R(x, y)\right),$$

  which matches the neutrosophic upper approximation $\overline{R}(A)$.

Thus, the Plithogenic Rough Set reduces to the Neutrosophic Rough Set. □

Question 2.5. As related concepts to plithogenic sets [57] and rough sets [32, 37], notions such as hyperplithogenic sets [8, 10, 16, 18], hyperrough sets [16, 19], and multigranulation rough sets [30, 38–40, 71] are known. Can plithogenic rough sets be extended by using these concepts?

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Data Availability

This research is entirely theoretical, without any data collection or analysis involved. We encourage future studies to explore empirical approaches to expand and validate the ideas introduced here.

Ethical Approval

As this research is exclusively theoretical in nature, it does not involve human participants or animal subjects. Therefore, ethical approval is not required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to this research or its publication.

Disclaimer

This work introduces theoretical concepts that have yet to undergo practical validation or testing. Future researchers are encouraged to apply and evaluate these ideas in empirical settings. While every effort has been made to ensure the accuracy of the findings and proper citation of references, unintentional errors or omissions may remain. Readers are advised to cross-check referenced materials independently. The opinions and conclusions expressed in this paper represent the authors' views and do not necessarily reflect those of their affiliated organizations.

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Chapter 13
Plithogenic Duplets and Plithogenic Triplets

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Abstract

A Neutrosophic Set is a mathematical framework that represents degrees of truth, indeterminacy, and falsehood to address uncertainty in membership values [41, 42]. In contrast, a Plithogenic Set extends this concept by incorporating attributes, their possible values, and the corresponding degrees of appurtenance and contradiction [50]. Among the related concepts of Neutrosophic Sets, Neutrosophic Duplets and Neutrosophic Triplets are well-known. This paper defines Plithogenic Duplets and Plithogenic Triplets as extensions of these concepts using the Plithogenic Set framework and briefly examines their relationship with existing concepts.

Keywords: Set Theory, Neutrosophic Set, Plithogenic Set, Neutrosophic Triplets

1 Preliminaries and Definitions

Some foundational concepts from set theory are applied in parts of this work.

1.1 Neutrosophic Set and Plithogenic Set

The Neutrosophic Set and Plithogenic Set are conceptual frameworks designed to handle uncertainty effectively. These frameworks are closely related to several other mathematical constructs, including Fuzzy Sets [67–71], Intuitionistic Fuzzy Sets [8–11], Neutrosophic Offsets [16, 18, 45, 46, 53, 59], Hyperneutrosophic Sets [17, 25–27], and Bipolar Neutrosophic Sets [2, 4, 5, 33]. Their definitions are provided below.

Definition 1.1 ((Single-valued) Neutrosophic Set)**.** [41–44, 56, 57] Let X be a given set. A (single-valued) Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Example 1.2 (Examples of Neutrosophic Sets)**.** Examples of several Neutrosophic Sets are provided below.

1. *Weather Prediction (cf. [12, 38, 61]):* Let $X = \{\text{Sunny}, \text{Rainy}, \text{Cloudy}\}$, representing weather conditions. A Neutrosophic Set A may be defined as:

$$T_A(\text{Sunny}) = 0.9, \quad I_A(\text{Sunny}) = 0.05, \quad F_A(\text{Sunny}) = 0.05,$$

$$T_A(\text{Rainy}) = 0.6, \quad I_A(\text{Rainy}) = 0.3, \quad F_A(\text{Rainy}) = 0.1,$$

$$T_A(\text{Cloudy}) = 0.4, \quad I_A(\text{Cloudy}) = 0.4, \quad F_A(\text{Cloudy}) = 0.2.$$

   - Sunny: High certainty (90
   - Rainy: Moderate likelihood of rain, with significant uncertainty.
   - Cloudy: Partial truth, indeterminacy, and falsity, reflecting ambiguity.

2. *Medical Diagnosis (cf. [7, 13, 14, 29, 64]):* Let $X = \{\text{Disease 1}, \text{Disease 2}, \text{Disease 3}\}$, representing possible diagnoses. Define a Neutrosophic Set A as:

$$T_A(\text{Disease 1}) = 0.8, \quad I_A(\text{Disease 1}) = 0.1, \quad F_A(\text{Disease 1}) = 0.1,$$

$$T_A(\text{Disease 2}) = 0.5, \quad I_A(\text{Disease 2}) = 0.3, \quad F_A(\text{Disease 2}) = 0.2,$$
$$T_A(\text{Disease 3}) = 0.2, \quad I_A(\text{Disease 3}) = 0.4, \quad F_A(\text{Disease 3}) = 0.4.$$

   - Disease 1: Highly likely, with minimal indeterminacy and falsity.
   - Disease 2: Moderate likelihood, higher indeterminacy.
   - Disease 3: Low likelihood, dominated by indeterminacy and falsity.

3. *Product Quality Assessment (cf. [30,36,66,73]):* Let $X = \{\text{High Quality}, \text{Medium Quality}, \text{Low Quality}\}$. A Neutrosophic Set A is defined as:

$$T_A(\text{High Quality}) = 0.7, \quad I_A(\text{High Quality}) = 0.2, \quad F_A(\text{High Quality}) = 0.1,$$

$$T_A(\text{Medium Quality}) = 0.5, \quad I_A(\text{Medium Quality}) = 0.3, \quad F_A(\text{Medium Quality}) = 0.2,$$
$$T_A(\text{Low Quality}) = 0.3, \quad I_A(\text{Low Quality}) = 0.4, \quad F_A(\text{Low Quality}) = 0.3.$$

   - High Quality: Considered mostly true with some uncertainty and minimal falsity.
   - Medium Quality: Equally distributed among truth, indeterminacy, and falsity.
   - Low Quality: More dominated by indeterminacy and falsity than truth.

The Plithogenic Set is known as a type of set that can generalize Neutrosophic Sets, Fuzzy Sets, and other similar sets [**?**, 1, 3, 15, 19–24, 28, 37, 49, 50, 58, 62, 63]. The definition of the Plithogenic Set is provided below.

Definition 1.3. [49, 50] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)*.
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a,a) = 0$$

2. *Symmetry of Contradiction Function*:

$$pCF(a,b) = pCF(b,a)$$

1.2 Neutrosophic Duplet

A Neutrosophic Duplet is defined as a pair $\langle a, \text{neut}(a)\rangle$ within a set, where a represents an element of the set and $\text{neut}(a)$ denotes the neutrosophic neutral element associated with a. The pair satisfies specific conditions related to neutrality and non-inversibility, as described in the literature [31, 32, 47, 60, 65, 72]. The formal definition is provided below.

Definition 1.4 (Neutrosophic Duplet)**.** [48] Let $\mathcal{U}$ be a universe of discourse, and $A \subseteq \mathcal{U}$ be a non-empty set endowed with a binary operation $*$. A pair $\langle a, \text{neut}(a)\rangle$, where $a, \text{neut}(a) \in A$, is called a *Neutrosophic Duplet* if the following conditions hold:

1. $\text{neut}(a)$ is distinct from the unit element of A with respect to $*$ (if a unit element exists).
2. The operation satisfies:
$$a * \text{neut}(a) = \text{neut}(a) * a = a.$$
3. There does not exist $\text{anti}(a) \in A$ such that:
$$a * \text{anti}(a) = \text{anti}(a) * a = \text{neut}(a).$$

Example 1.5 (Example of Neutrosophic Duplets in $\mathbb{Z}_8$)**.** Consider $\mathbb{Z}_8 = \{0, 1, 2, \ldots, 7\}$ with the binary operation $*$ defined as regular multiplication modulo 8. The unit element with respect to $*$ is 1. The following are Neutrosophic Duplets in $\mathbb{Z}_8$:

$$\langle 2, 5\rangle, \quad \langle 4, 3\rangle, \quad \langle 4, 5\rangle, \quad \langle 4, 7\rangle, \quad \langle 6, 5\rangle.$$

For example:

- $2 * 5 = 5 * 2 = 10 \mod 8 = 2$, so $\text{neut}(2) = 5 \neq 1$.
- There is no $\text{anti}(2) \in \mathbb{Z}_8$ because $2 * x = 5 \mod 8$ is unsolvable as it implies $2x = 5 + 8k$, which contradicts even number = odd number.

1.3 Neutrosophic Triplet

A *NeutroStructure* generalizes classical structures by incorporating degrees of truth (T), indeterminacy (I), and falsehood (F). It is defined as follows [6, 32, 34, 35, 39, 40, 51, 52, 54, 55].

Definition 1.6 (Neutrosophic Triplet)**.** [52] A *Neutrosophic Triplet* represents a conceptual generalization of classical structures, incorporating degrees of truth (T), indeterminacy (I), and falsehood (F). Formally, for a given statement or mathematical object A in a space S:

$$\langle A, \text{Neutro}A, \text{Anti}A\rangle = \Big\langle A(1, 0, 0),\ A(T, I, F),\ A(0, 0, 1)\Big\rangle,$$

where:

- $A(1, 0, 0)$ (Classical Component): A is 100% true ($T = 1$), 0% indeterminate ($I = 0$), and 0% false ($F = 0$).
- $A(T, I, F)$ (Neutro Component): A is T% true, I% indeterminate, and F% false, such that $(T, I, F) \notin \{(1, 0, 0), (0, 0, 1)\}$.
- $A(0, 0, 1)$ (Anti Component): A is 100% false ($F = 1$), 0% true ($T = 0$), and 0% indeterminate ($I = 0$).

Examples:

1. *Theorem Triplet:* $\langle \text{Theorem}, \text{NeutroTheorem}, \text{AntiTheorem}\rangle$:

- A classical theorem holds universally true ($T = 1, I = 0, F = 0$).
- A NeutroTheorem is partially true, indeterminate, or false ($T, I, F \neq 1, 0, 0$).
- An AntiTheorem is universally false ($T = 0, I = 0, F = 1$).

2. *Definition Triplet:* $\langle$Definition, NeutroDefinition, AntiDefinition$\rangle$:

   - A classical definition is universally true.
   - A NeutroDefinition applies with partial uncertainty.
   - An AntiDefinition is universally invalid or false.

Example 1.7 (Examples of Neutrosophic Triplets)**.** Several specific examples of Neutrosophic Triplets are provided below.

1. *Weather Prediction:* Let A be the statement "It will rain tomorrow."

   - Classical Component: $A(1, 0, 0)$ means the prediction is absolutely certain to be true (e.g., $T = 1, I = 0, F = 0$).
   - Neutro Component: $A(T, I, F) = (0.6, 0.3, 0.1)$ means there is 60% certainty it will rain, 30% uncertainty, and 10% certainty it will not rain.
   - Anti Component: $A(0, 0, 1)$ means the prediction is absolutely false (e.g., $F = 1, T = 0, I = 0$).

2. *Quality Control:* Consider A as "This product meets quality standards."

   - Classical Component: $A(1, 0, 0)$ means the product unquestionably meets quality standards.
   - Neutro Component: $A(T, I, F) = (0.8, 0.1, 0.1)$ means there is 80% certainty the product meets the standards, with 10% uncertainty and 10% certainty it does not meet them.
   - Anti Component: $A(0, 0, 1)$ means the product categorically does not meet quality standards.

3. *Medical Diagnosis:* Let A be "The patient has a specific disease."

   - Classical Component: $A(1, 0, 0)$ means the diagnosis is definitively correct.
   - Neutro Component: $A(T, I, F) = (0.7, 0.2, 0.1)$ indicates a 70% likelihood of the disease, 20% uncertainty, and 10% likelihood of not having the disease.
   - Anti Component: $A(0, 0, 1)$ means the diagnosis is definitively wrong.

2 Results of This Paper

This section highlights the main contributions of this paper.

2.1 Plithogenic Duplet

The Plithogenic Duplet extends the Neutrosophic Duplet by utilizing the Plithogenic Set framework. The definitions and related concepts are detailed below.

Definition 2.1 (Plithogenic Duplet)**.** Let $\mathcal{U}$ be a universe of discourse, and $A \subseteq \mathcal{U}$ be a non-empty set endowed with a binary operation $*$. A pair $\langle a, \text{plitho}(a)\rangle$, where $a, \text{plitho}(a) \in A$, is called a *Plithogenic Duplet* if the following conditions hold:

1. *Plithogenic Degree of Appurtenance Function (DAF):* $\text{plitho}(a)$ represents a value determined by the DAF:
$$pdf(a, v_a) = (T_a, I_a, F_a),$$
where $v_a \in Pv$ (attribute value) and $T_a, I_a, F_a \in [0, 1]$ represent the degrees of truth, indeterminacy, and falsehood, respectively.

2. *Neutrality Condition:* The operation $*$ satisfies:
$$a * \text{plitho}(a) = \text{plitho}(a) * a = a,$$
ensuring $\text{plitho}(a)$ acts as a plithogenic neutral element with respect to a.

3. *Non-Inversibility Condition:* There does not exist $\text{anti}(a) \in A$ such that:
$$a * \text{anti}(a) = \text{anti}(a) * a = \text{plitho}(a).$$

4. *Degree of Contradiction Function (DCF):* A DCF pCF applies to attribute values $v_a, v_b \in Pv$, satisfying:
$$pCF(v_a, v_a) = 0, \quad pCF(v_a, v_b) = pCF(v_b, v_a).$$

Example 2.2 (Example of a Plithogenic Duplet)**.** Let $\mathcal{U} = \{x, y, z\}$ and $A = \{x, y\}$ with the operation $*$ defined as follows:
$$x * y = x, \quad y * x = y, \quad x * x = x, \quad y * y = y.$$

Define a Plithogenic Set:
$$PS = (A, v, Pv, pdf, pCF),$$
where:

- v is the attribute "weight" with possible values $Pv = \{v_1, v_2\}$,
- $pdf(x, v_1) = (0.8, 0.1, 0.1)$, $pdf(y, v_2) = (0.7, 0.2, 0.1)$,
- $pCF(v_1, v_1) = 0$, $pCF(v_1, v_2) = 0.3$.

Here, the Plithogenic Duplets are:
$$\langle x, \text{plitho}(x)\rangle = \langle x, v_1\rangle, \quad \langle y, \text{plitho}(y)\rangle = \langle y, v_2\rangle,$$
with the following properties:

1. Neutrality: $x * \text{plitho}(x) = x$, $y * \text{plitho}(y) = y$.
2. Non-inversibility: There is no $\text{anti}(x)$ or $\text{anti}(y)$ in A.

Theorem 2.3. *The Plithogenic Duplet generalizes the Neutrosophic Duplet by incorporating the Plithogenic Set framework, allowing for attribute-based degrees of truth, indeterminacy, and falsehood through a Degree of Appurtenance Function (DAF) and a Degree of Contradiction Function (DCF).*

Proof. Let $\mathcal{U}$ be a universe of discourse, $A \subseteq \mathcal{U}$ a non-empty set, and $*$ a binary operation defined on A. Consider the definitions of Neutrosophic Duplet and Plithogenic Duplet:

From Definition 1.4, a Neutrosophic Duplet $\langle a, \text{neut}(a)\rangle$ satisfies:

1. $\text{neut}(a)$ is distinct from the unit element (if it exists).
2. The operation satisfies:
$$a * \text{neut}(a) = \text{neut}(a) * a = a.$$

3. No anti(a) exists such that:
$$a * \text{anti}(a) = \text{anti}(a) * a = \text{neut}(a).$$

From Definition 2.1, a Plithogenic Duplet $\langle a, \text{plitho}(a)\rangle$ satisfies the following conditions:

1. The value plitho(a) is determined by the Plithogenic Degree of Appurtenance Function (DAF):
$$pdf(a, v_a) = (T_a, I_a, F_a),$$
where $v_a \in Pv$ and $T_a, I_a, F_a \in [0, 1]$.
2. The neutrality condition:
$$a * \text{plitho}(a) = \text{plitho}(a) * a = a.$$
3. The non-inversibility condition:
$$\nexists \text{anti}(a) \in A \quad \text{such that} \quad a * \text{anti}(a) = \text{anti}(a) * a = \text{plitho}(a).$$
4. The Degree of Contradiction Function (DCF):
$$pCF(v_a, v_a) = 0, \quad pCF(v_a, v_b) = pCF(v_b, v_a).$$

The Neutrosophic Duplet is a specific case of the Plithogenic Duplet where:

1. The attribute value v_a and $pdf(a, v_a) = (T_a, I_a, F_a)$ reduce to the fixed values:
$$(T_a, I_a, F_a) = (1, 0, 0) \quad \text{(Classical Component)}.$$
2. No additional attributes or contradiction functions (pCF) are defined.
3. The operation $*$ remains identical in both cases, preserving neutrality and non-inversibility conditions.

By introducing the Plithogenic Set framework, the Plithogenic Duplet allows for attribute-based customization and a richer representation of truth, indeterminacy, and falsehood via pdf and pCF. This subsumes the fixed membership structure of the Neutrosophic Duplet as a special case. Therefore, the Plithogenic Duplet is a generalization of the Neutrosophic Duplet. □

2.2 Plithogenic Triplet

The Plithogenic Triplet extends the Neutrosophic Triplet by incorporating the concepts of attributes, their values, and the Degree of Appurtenance and Contradiction Functions, fundamental to Plithogenic Sets. The formal definition is as follows:

Definition 2.4 (Plithogenic Triplet)**.** Let S be a universal set, and $P \subseteq S$ a Plithogenic Set defined by $PS = (P, v, Pv, pdf, pCF)$, where:

- v is an attribute.
- Pv is the set of possible values of v.
- $pdf : P \times Pv \to [0, 1]^s$ is the Degree of Appurtenance Function (DAF).
- $pCF : Pv \times Pv \to [0, 1]^t$ is the Degree of Contradiction Function (DCF).

A *Plithogenic Triplet* for an element $x \in P$ with respect to an attribute v is defined as:
$$\langle x, \text{Plitho}x, \text{Anti}x\rangle = \big\langle x(1, 0, 0),\ x(pdf, pCF),\ x(0, 0, 1)\big\rangle,$$
where:

- $x(1,0,0)$: Represents the classical membership of x, being fully true ($pdf = 1, pCF = 0$).
- $x(pdf, pCF)$: Represents the Plithogenic membership of x, where the Degree of Appurtenance Function and Degree of Contradiction Function vary between 0 and 1.
- $x(0,0,1)$: Represents the anti-membership of x, being fully false ($pdf = 0, pCF = 1$).

Example 2.5 (Plithogenic Triplet Example)**.** Consider a universal set $S = \{A, B, C\}$ representing three different projects. Define an attribute v = Difficulty Level with possible values $Pv = \{\text{Low}, \text{Medium}, \text{High}\}$. Let the Degree of Appurtenance Function pdf and the Degree of Contradiction Function pCF for each project $x \in S$ be given as follows:

$$pdf(A, \text{Low}) = 0.8, \;\; pdf(A, \text{Medium}) = 0.15, \;\; pdf(A, \text{High}) = 0.05,$$

$$pCF(\text{Low}, \text{High}) = 0.7, \;\; pCF(\text{Low}, \text{Medium}) = 0.3.$$

Then, the Plithogenic Triplet for project A is:

$$\langle A, \text{Litho}A, \text{Anti}A\rangle = \big\langle A(1,0,0), \;\; A(pdf, pCF), \;\; A(0,0,1)\big\rangle,$$

where $A(pdf, pCF)$ reflects the varying degrees of appurtenance and contradiction for A with respect to the attribute Difficulty Level.

Theorem 2.6. *The Plithogenic Triplet generalizes the Neutrosophic Triplet by utilizing the Plithogenic Set framework, allowing for attribute-based customization through the Degree of Appurtenance Function (DAF) and Degree of Contradiction Function (DCF).*

Proof. Let S be a universal set, $P \subseteq S$ a Plithogenic Set, and $PS = (P, v, Pv, pdf, pCF)$ as defined in Definition 2.4. Consider the definitions of Neutrosophic Triplet and Plithogenic Triplet:

From Definition 1.6, a Neutrosophic Triplet $\langle A, \text{Neutro}A, \text{Anti}A\rangle$ satisfies:

1. $A(1,0,0)$: Represents the classical component, being fully true ($T = 1, I = 0, F = 0$).
2. $A(T, I, F)$: Represents the neutrosophic component, where T, I, F can take values in $[0,1]$ such that $T + I + F \leq 3$.
3. $A(0,0,1)$: Represents the anti-component, being fully false ($T = 0, I = 0, F = 1$).

From Definition 2.4, a Plithogenic Triplet $\langle x, \text{Plitho}x, \text{Anti}x\rangle$ satisfies:

1. $x(1,0,0)$: Represents the classical membership of x, being fully true ($pdf = 1, pCF = 0$).
2. $x(pdf, pCF)$: Represents the plithogenic membership of x, where:
$$pdf(x, v_x) = (T_x, I_x, F_x), \quad pCF(v_x, v_y),$$
and $T_x, I_x, F_x \in [0,1]$, $pCF(v_x, v_x) = 0$, $pCF(v_x, v_y) = pCF(v_y, v_x)$.
3. $x(0,0,1)$: Represents the anti-membership of x, being fully false ($pdf = 0, pCF = 1$).

The Neutrosophic Triplet is a specific case of the Plithogenic Triplet where:

1. The attribute v and its possible values Pv are fixed and not explicitly considered.
2. The Degree of Appurtenance Function (DAF) simplifies to:
$$pdf(x, v_x) = (T_x, I_x, F_x),$$
where v_x is implicit, and the values T_x, I_x, F_x satisfy the same conditions as in the Neutrosophic Triplet.
3. The Degree of Contradiction Function (DCF) is not used, effectively setting $pCF(v_x, v_y) = 0$ for all v_x, v_y.

The Plithogenic Triplet incorporates attributes, their possible values, and the Degree of Contradiction Function (DCF), thereby extending the flexibility and expressiveness of the Neutrosophic Triplet. Consequently, the Neutrosophic Triplet is a special case of the Plithogenic Triplet. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 14
SuperRough Set and SuperVague Set

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Abstract

Concepts such as Fuzzy Sets [10, 25], Neutrosophic Sets [19–22], Vague Sets [1, 9, 11], Rough Sets [12, 18], and Plithogenic Sets [24] have been extensively studied to address uncertainty, with diverse applications across numerous fields.

In this paper, we introduce and investigate the concepts of SuperVague Set and SuperRough Set. These are generalized forms of Vague Sets and Rough Sets, respectively. Furthermore, we prove that the SuperRough Set can be further generalized to the SuperHyperRough Set. This work serves as a reconsideration and extension of studies such as those in [7, 23].

Keywords: SuperVague Set, SuperRough Set, Rough Set, Vague Set, Fuzzy Set

1 Preliminaries and Definitions

This section introduces the fundamental concepts and definitions necessary for the discussions and analyses presented in this paper.

1.1 SuperFuzzy Set

A Fuzzy Set assigns a membership degree in $[0, 1]$ to each element of a non-empty universe, representing uncertainty [25, 28, 29]. A SuperFuzzy Set assigns a membership degree in $[0, 1]$ to each subset of a non-empty universe, extending Fuzzy Sets [23].

Definition 1.1 (Fuzzy set)**.** [25–33] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 1.2. Consider a fuzzy set representing "Tall People" in a population $Y = \{y_1, y_2, y_3\}$, where y_1, y_2, y_3 represent individuals. The membership function $\tau : Y \to [0, 1]$ assigns a degree of membership to each individual based on their height:

$$\tau(y_1) = 0.9, \quad \tau(y_2) = 0.5, \quad \tau(y_3) = 0.2.$$

This means y_1 is highly likely to be considered tall, y_2 moderately so, and y_3 unlikely.

Definition 1.3. [23] A *Superfuzzy Set* is defined as a function:

$$\tau : P(A) \to [0, 1],$$

where $P(A)$ is the powerset of a non-empty set A, and $\tau(S)$ for $S \in P(A)$ represents the degree of membership (truth) of the subset S in A.

Example 1.4. Consider a SuperFuzzy Set representing "Preferred Groups of Foods" in a universe $A = \{\text{Fruits, Vegetables, Snacks}\}$. Each subset $S \in P(A)$ is assigned a membership degree $\tau(S)$ based on dietary preferences:

$$\tau(\{\text{Fruits}\}) = 0.8, \quad \tau(\{\text{Vegetables}\}) = 0.6, \quad \tau(\{\text{Fruits, Vegetables}\}) = 0.9, \quad \tau(\{\text{Snacks}\}) = 0.3.$$

Here, the subset {Fruits, Vegetables} has the highest preference (0.9), while {Snacks} has the lowest (0.3).

2 Result of this Paper

This section presents the results of this paper.

2.1 SuperVague Set

The SuperVague Set is a concept that generalizes the Vague Set. Its definition is provided below.

Definition 2.1 (Vague Set)**.** [2, 9] Let U be a universe of discourse, defined as $U = \{u_1, u_2, \ldots, u_n\}$. A *vague set* A in U is characterized by two functions:

$$t_A : U \to [0, 1] \quad \text{and} \quad f_A : U \to [0, 1],$$

where:

- $t_A(u_i)$ is the *truth-membership function*, providing a lower bound on the membership degree of u_i based on supporting evidence for $u_i \in A$.
- $f_A(u_i)$ is the *false-membership function*, offering a lower bound on the negation of u_i based on evidence against $u_i \in A$.

These functions satisfy the constraint:

$$t_A(u_i) + f_A(u_i) \leq 1, \quad \text{for all } u_i \in U.$$

The degree of membership of u_i in the vague set A is thus constrained within a subinterval of $[0, 1]$ defined by:

$$t_A(u_i) \leq \mu_A(u_i) \leq 1 - f_A(u_i),$$

where $\mu_A(u_i)$ represents the true membership grade of u_i in A. The interval $[t_A(u_i), 1 - f_A(u_i)]$ indicates that, although the exact membership degree may be uncertain, it is bound within this range.

If U is continuous, a vague set A can be represented as:

$$A = \int_U [t_A(u), 1 - f_A(u)]/u.$$

In the case of a discrete universe U, A is expressed as:

$$A = \sum_{i=1}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i.$$

Example 2.2 (Vague Set in Project Management)**.** Project Management involves planning, organizing, and controlling resources, tasks, and timelines to achieve specific objectives efficiently and effectively (cf. [4–6]). Consider a project management scenario where U represents a set of tasks to be completed:

$$U = \{\text{Task 1}, \text{Task 2}, \text{Task 3}, \text{Task 4}\}.$$

Let A be a vague set representing tasks that are likely to be completed on time. For each task $u_i \in U$, we define the truth-membership function $t_A(u_i)$ and the false-membership function $f_A(u_i)$ based on evidence from project progress reports, team efficiency, and resource availability.

For instance:

$$t_A(\text{Task 1}) = 0.8, \quad f_A(\text{Task 1}) = 0.1,$$

indicating that there is 80% evidence supporting the timely completion of Task 1 and 10% evidence against it. Thus, the true membership grade $\mu_A(\text{Task 1})$ lies in the interval:

$$[0.8, 0.9].$$

Similarly, for the other tasks:

$$t_A(\text{Task 2}) = 0.5, \quad f_A(\text{Task 2}) = 0.3 \implies \mu_A(\text{Task 2}) \in [0.5, 0.7],$$

$$t_A(\text{Task 3}) = 0.7, \quad f_A(\text{Task 3}) = 0.2 \implies \mu_A(\text{Task 3}) \in [0.7, 0.8],$$

$$t_A(\text{Task 4}) = 0.4, \quad f_A(\text{Task 4}) = 0.5 \implies \mu_A(\text{Task 4}) \in [0.4, 0.5].$$

The vague set A summarizing tasks likely to be completed on time is represented as:

$$A = [0.8, 0.9]/\text{Task 1} + [0.5, 0.7]/\text{Task 2} + [0.7, 0.8]/\text{Task 3} + [0.4, 0.5]/\text{Task 4}.$$

This representation allows project managers to model uncertainty in task completion and make informed decisions about resource allocation and risk mitigation.

Definition 2.3 (SuperVague Set)**.** Let X be a non-empty set, and let $P(X)$ be its power set. A **SuperVague Set** on X is a mapping

$$A^* : P(X) \longrightarrow [0, 1] \times [0, 1]$$

such that for every subset $S \in P(X)$, we have

$$A^*(S) = \big(T_A(S), F_A(S)\big),$$

where $T_A(S)$ and $F_A(S)$ are in $[0, 1]$ and satisfy

$$T_A(S) + F_A(S) \leq 1.$$

Here,

- $T_A(S)$ is interpreted as the *truth-membership degree* (or lower bound of membership) of the subset S.
- $F_A(S)$ is the *false-membership degree* (or lower bound of non-membership) of the subset S.

Thus the "actual" membership value $\mu_A(S)$ of each subset S must lie in the interval $\big[T_A(S), 1 - F_A(S)\big]$.

Example 2.4. Let

$$X = \{a, b, c\}.$$

A *SuperVague Set* (Definition 2.3 in the paper) assigns an interval to each *subset* $S \subseteq X$. Concretely, define a SuperVague Set A^* by specifying

$$A^*(S) = \big(T_A(S), F_A(S)\big),$$

with $T_A(S) + F_A(S) \leq 1$. Let us illustrate this with a simplified table. We only show the assignment for a few chosen subsets to keep it concise:

S	$T_A(S)$	$F_A(S)$
$\varnothing$	0	0
$\{a\}$	0.4	0.2
$\{b\}$	0.3	0.1
$\{a, b\}$	0.5	0.4
$\{a, b, c\}$	1.0	0.0

(One can define all other subsets similarly.)

In words:

- For $\varnothing$, we assign $(T_A(\varnothing), F_A(\varnothing)) = (0, 0)$. This suggests the empty set is *certainly not* in A, with no contradiction.
- For the singleton $\{a\}$, we put $(0.4, 0.2)$. This implies that $\{a\}$ has at least a 0.4 truth-degree and at least a 0.2 false-degree, so the actual membership value of $\{a\}$ in A must lie in the interval [0.4, 0.8].

- Similarly, $\{b\}$ has at least 0.3 truth-degree, 0.1 false-degree, so membership is in $[0.3,\ 0.9]$.
- For $\{a, b\}$, we might say $(T_A(\{a, b\}), F_A(\{a, b\})) = (0.5, 0.4)$. Thus membership is in $[0.5,\ 0.6]$.
- For the entire set $\{a, b, c\}$, we assign $(1.0, 0.0)$, meaning we are fully certain that the entire set is included in A.

Unlike a classical vague set that only assigns intervals to elements of X (i.e. singletons), here the SuperVague Set provides membership intervals for *all* subsets of X. Restricting ourselves to singletons $\{a\}$, $\{b\}$, $\{c\}$ recovers the usual notion of a vague set (lower and upper membership bounds for each element). Hence this example shows how the SuperVague framework can handle uncertainty or partial knowledge across every subset, rather than just individual points.

Theorem 2.5 (SuperVague Set generalizes Vague Set)**.** *The classical notion of a vague set (Definition* **??***) is a special case of a SuperVague Set. In particular, if we consider only singletons* $\{u\} \subseteq X$*, then*

$$T_A(\{u\}) = t_A(u), \quad F_A(\{u\}) = f_A(u),$$

and

$$T_A(\{u\}) + F_A(\{u\}) \ \leq\ 1$$

recovers the classical vague set condition $t_A(u) + f_A(u) \leq 1$*.*

Proof. For each subset $S \subseteq X$, $A^*(S) = (T_A(S), F_A(S))$ must satisfy $T_A(S) + F_A(S) \leq 1$. In the classical vague set, membership bounds are only assigned to single elements $u_i \in U$. If we focus on singletons $\{u\}$, then simply identify

$$T_A(\{u\}) \ =\ t_A(u), \quad F_A(\{u\}) \ =\ f_A(u),$$

with $t_A(u) + f_A(u) \leq 1$. This precisely matches the original condition for a vague set. Hence restricting a SuperVague Set to singleton subsets reproduces the standard vague set. □

2.2 SuperRough Set

A Rough Set approximates a subset of a universe using lower and upper bounds based on equivalence relations, handling uncertainty [12–18].

Definition 2.6 (Rough Set)**.** [12–18] Let X be the universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or an indiscernibility relation) on X, partitioning X into equivalence classes. For any subset $U \subseteq X$, the lower approximation $\underline{U}$ and the upper approximation $\overline{U}$ are defined as follows:

1. *Lower Approximation* $\underline{U}$:

$$\underline{U} = \{x \in X \mid R(x) \subseteq U\}$$

This is the set of all elements in X that certainly belong to U based on the equivalence classes defined by R.

2. *Upper Approximation* $\overline{U}$:

$$\overline{U} = \{x \in X \mid R(x) \cap U \neq \emptyset\}$$

This set contains all elements in X that possibly belong to U.

The pair $(\underline{U}, \overline{U})$ constitutes a rough set representation of U, where $\underline{U} \subseteq U \subseteq \overline{U}$.

Example 2.7. Let the universe of discourse be

$$X = \{\, a, b, c, d\}.$$

Define an equivalence relation $R \subseteq X \times X$ by the following partition of X into equivalence classes:

$$\{a, b\}, \quad \{c, d\}.$$

Hence,

$$R \ =\ \{(a, a), (a, b), (b, a), (b, b),\ (c, c), (c, d), (d, c), (d, d)\}.$$

Suppose we have a subset

$$U = \{a, c\} \ \subseteq\ X.$$

We wish to compute the rough set approximations $\underline{U}$ and $\overline{U}$ (see the Rough Set definition in the statement).

1) Lower Approximation $\underline{U}$.

$$\underline{U} = \left\{ x \in X \;\middle|\; R(x) \subseteq U \right\}.$$

- For $x = a$, $R(a) = \{a, b\}$. Since $\{a, b\} \not\subseteq U$ (because $b \notin U$), we have $a \notin \underline{U}$.
- For $x = b$, $R(b) = \{a, b\}$. Same reasoning: $\{a, b\} \not\subseteq U$. Hence $b \notin \underline{U}$.
- For $x = c$, $R(c) = \{c, d\}$. Since $\{c, d\} \not\subseteq U$ (because $d \notin U$), $c \notin \underline{U}$.
- For $x = d$, $R(d) = \{c, d\}$. Similarly, $\{c, d\} \not\subseteq U$. Thus $d \notin \underline{U}$.

Hence,

$$\underline{U} = \varnothing.$$

2) Upper Approximation $\overline{U}$.

$$\overline{U} = \left\{ x \in X \;\middle|\; R(x) \cap U \neq \emptyset \right\}.$$

- For $x = a$, $R(a) = \{a, b\}$. Since $\{a, b\} \cap \{a, c\} = \{a\} \neq \varnothing$, $a \in \overline{U}$.
- For $x = b$, $R(b) = \{a, b\}$. Its intersection with $\{a, c\}$ is $\{a\} \neq \emptyset$. So $b \in \overline{U}$.
- For $x = c$, $R(c) = \{c, d\}$. Intersection with $\{a, c\}$ is $\{c\} \neq \emptyset$. Therefore $c \in \overline{U}$.
- For $x = d$, $R(d) = \{c, d\}$. Intersection with $\{a, c\}$ is $\{c\} \neq \emptyset$. Thus $d \in \overline{U}$.

Hence,

$$\overline{U} = \{\, a, b, c, d\}.$$

3) Rough Set Representation. We conclude:

$$(\underline{U}, \overline{U}) = \bigl(\varnothing,\ \{a, b, c, d\}\bigr).$$

That is, according to the equivalence classes in R, no element is *certainly* in U (hence the lower approximation is empty), yet every element is *possibly* in U (hence the upper approximation is the entire universe).

A related concept, the HyperRough Set, has been studied in recent years. Its definition and related details are provided below [3, 7, 8].

Definition 2.8 (HyperRough Set)**.** [7] Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with domains $J_1, J_2, \ldots, J_n$. Define $J = J_1 \times J_2 \times J_n$. Let $R \subseteq X \times X$ be an equivalence relation on X.

A *HyperRough Set* over X is a pair (F, J), where F is a mapping:

$$F : J \to \mathcal{P}(X),$$

such that for each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$, $F(a)$ is associated with a rough set $(\underline{F(a)}, \overline{F(a)})$ defined by:

$$\underline{F(a)} = \{x \in X \mid R(x) \subseteq F(a)\},$$
$$\overline{F(a)} = \{x \in X \mid R(x) \cap F(a) \neq \emptyset\}.$$

Example 2.9. Let the same finite universe

$$X = \{\, a, b, c, d \}$$

and the same equivalence relation

$$R \; = \; \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}$$

as before. Suppose we have two attributes: $T_1 = \texttt{Color}$ and $T_2 = \texttt{Shape}$, with domains:

$$J_1 = \{\text{Red, Green}\}, \quad J_2 = \{\text{Circle, Square}\}.$$

Define

$$J = J_1 \times J_2 = \{(\text{Red, Circle}),\ (\text{Red, Square}),\ (\text{Green, Circle}),\ (\text{Green, Square})\}.$$

A *HyperRough Set* is specified by a mapping

$$F : J \longrightarrow \mathcal{P}(X),$$

and for each $a \in J$, we interpret $F(a)$ in terms of a rough set $\bigl(\underline{F(a)},\ \overline{F(a)}\bigr)$.

1) Defining the Mapping F. Let us define F for each attribute combination in J. For instance:

- $F(\text{Red, Circle}) = \{a, b\}$.
- $F(\text{Red, Square}) = \{b\}$.
- $F(\text{Green, Circle}) = \{b, c\}$.
- $F(\text{Green, Square}) = \{c, d\}$.

(We choose these subsets arbitrarily just to illustrate the concept.)

2) Computing Rough Sets $\underline{F(a)}$ and $\overline{F(a)}$. For each $a = (a_1, a_2) \in J$:

$$\underline{F(a)} = \{\, x \in X \mid R(x) \subseteq F(a)\}, \quad \overline{F(a)} = \{\, x \in X \mid R(x) \cap F(a) \neq \emptyset\}.$$

We illustrate two cases:

- **a = (Red, Circle)**. Then $F(a) = \{a, b\}$.
  - $R(x) \subseteq \{a, b\}$ only if $x \in \{a, b\}$. Checking equivalence classes:
    * $R(a) = \{a, b\} \subseteq F(a)$
    * $R(b) = \{a, b\} \subseteq F(a)$
    * $R(c) = \{c, d\} \nsubseteq \{a, b\}$
    * $R(d) = \{c, d\} \nsubseteq \{a, b\}$

    Hence $\underline{F(a)} = \{a, b\}$.
  - $R(x) \cap \{a, b\} \neq \emptyset$ for $x \in \{a, b\}$, but also note:
    * $R(c) = \{c, d\}$ intersects $\{a, b\}$ in $\varnothing$, so $c \notin \overline{F(a)}$.
    * $R(d) = \{c, d\}$ intersects $\{a, b\}$ in $\varnothing$, so $d \notin \overline{F(a)}$.

    Thus $\overline{F(a)} = \{a, b\}$.

  Therefore, the rough set for (Red, Circle) is

$$\left(\underline{F(a)},\ \overline{F(a)}\right) = \bigl(\{a, b\},\ \{a, b\}\bigr).$$

- ***a* = (Green, Square)**. Then $F(a) = \{c, d\}$.
  - $R(x) \subseteq \{c, d\}$ only if $x \in \{c, d\}$. Checking equivalence classes:
    - $R(c) = \{c, d\} \subseteq \{c, d\}$
    - $R(d) = \{c, d\} \subseteq \{c, d\}$
    - $R(a) = \{a, b\} \nsubseteq \{c, d\}$
    - $R(b) = \{a, b\} \nsubseteq \{c, d\}$

    Hence $\underline{F(a)} = \{c, d\}$.
  - $R(x) \cap \{c, d\} \neq \emptyset$ for $x \in \{c, d\}$. Also:
    - $R(a) = \{a, b\}$ does not intersect $\{c, d\}$, so $a \notin \overline{F(a)}$.
    - $R(b) = \{a, b\}$ likewise does not intersect $\{c, d\}$, so $b \notin \overline{F(a)}$.

    Thus $\overline{F(a)} = \{c, d\}$.

  So for (Green, Square), the rough set is

$$\left(\underline{F(a)},\ \overline{F(a)}\right) = \left(\{c, d\},\ \{c, d\}\right).$$

3) HyperRough Set Interpretation. Altogether, the *HyperRough Set* is given by

$$(F,\ J),$$

where F is the mapping

$$(\text{Red}, \text{Circle}) \ \mapsto\ \{a, b\}, \quad (\text{Red}, \text{Square}) \ \mapsto\ \{b\}, \quad (\text{Green}, \text{Circle}) \ \mapsto\ \{b, c\}, \quad (\text{Green}, \text{Square}) \ \mapsto\ \{c, d\},$$

and each $F(a)$ (for $a \in J$) is itself described by the rough set $\left(\underline{F(a)},\ \overline{F(a)}\right)$. Thus each attribute-value combination in J is associated with a (possibly distinct) rough set in X. This illustrates how HyperRough Sets accommodate multiple attributes and map each attribute combination to a rough set representation.

The SuperRough Set is a concept that generalizes in a different way compared to the HyperRough Set. Its definition is provided below.

Definition 2.10 (SuperRough Set)**.** Let X be a non-empty universe of discourse, and let

$$\Gamma \ = \ \{\, R_\alpha \ \mid\ \alpha \in I \}$$

be an indexed family of equivalence relations on X. For any $U \subseteq X$, denote by $\underline{U}_{R_\alpha}$ and $\overline{U}_{R_\alpha}$ the classical lower and upper approximations of U with respect to R_α, i.e.,

$$\underline{U}_{R_\alpha} \ = \ \{x \in X \ \mid\ R_\alpha(x) \subseteq U\}, \quad \overline{U}_{R_\alpha} \ = \ \{x \in X \ \mid\ R_\alpha(x) \cap U \neq \emptyset\}.$$

Define the *SuperRough lower approximation* $\underline{U}^\Gamma$ and the *SuperRough upper approximation* $\overline{U}^\Gamma$ of U with respect to the family Γ by

$$\underline{U}^\Gamma \ = \ \bigcap_{\alpha \in I} \underline{U}_{R_\alpha}, \quad \overline{U}^\Gamma \ = \ \bigcup_{\alpha \in I} \overline{U}_{R_\alpha}.$$

We call the pair

$$(\underline{U}^\Gamma,\ \overline{U}^\Gamma)$$

the **SuperRough approximation of** U **w.r.t.** Γ.

A **SuperRough Set** on X is then the mapping

$$\mathcal{R}^\Gamma : \ P(X) \ \longrightarrow\ P(X) \times P(X), \quad U \ \mapsto\ \left(\underline{U}^\Gamma, \overline{U}^\Gamma\right).$$

Example 2.11. Let us consider a universe of discourse

$$X = \{ a, b, c, d\}.$$

We introduce two equivalence relations R_1 and R_2 on X. Recall that an equivalence relation partitions X into disjoint equivalence classes.

- Define R_1 by the partition
$$\{a, b\},\ \{c, d\}.$$
In other words,
$$R_1 \ = \ \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, d), (d, c), (d, d)\}.$$
- Define R_2 by the partition
$$\{a, c\},\ \{b, d\}.$$
Hence,
$$R_2 \ = \ \{(a, a), (a, c), (c, a), (c, c), (b, b), (b, d), (d, b), (d, d)\}.$$

Suppose we consider a subset

$$U = \{a, b\} \subseteq X.$$

We want to find the SuperRough approximations $\underline{U}^{\Gamma}$ and $\overline{U}^{\Gamma}$, where $\Gamma = \{R_1, R_2\}$.

Classical Rough Approximations w.r.t. R_1

$$\underline{U}_{R_1} = \{ x \in X \mid R_1(x) \subseteq U\}, \quad \overline{U}_{R_1} = \{ x \in X \mid R_1(x) \cap U \neq \varnothing\}.$$

- For $x = a$, $R_1(a) = \{a, b\}$. Since $\{a, b\} \subseteq U$, $a \in \underline{U}_{R_1}$. Also, $\{a, b\} \cap U \neq \emptyset$, so $a \in \overline{U}_{R_1}$.
- For $x = b$, $R_1(b) = \{a, b\}$. Similar reasoning: $b \in \underline{U}_{R_1}$ and $b \in \overline{U}_{R_1}$.
- For $x = c$, $R_1(c) = \{c, d\}$. Since $\{c, d\} \nsubseteq U$, $c \notin \underline{U}_{R_1}$. Also, $\{c, d\} \cap U = \varnothing$, hence $c \notin \overline{U}_{R_1}$.
- For $x = d$, $R_1(d) = \{c, d\}$. Same reasoning: $d \notin \underline{U}_{R_1}$ and $d \notin \overline{U}_{R_1}$.

Thus,

$$\underline{U}_{R_1} = \{a, b\}, \quad \overline{U}_{R_1} = \{a, b\}.$$

Classical Rough Approximations w.r.t. R_2

$$\underline{U}_{R_2} = \{ x \in X \mid R_2(x) \subseteq U\}, \quad \overline{U}_{R_2} = \{ x \in X \mid R_2(x) \cap U \neq \varnothing\}.$$

- For $x = a$, $R_2(a) = \{a, c\}$. Since $\{a, c\} \nsubseteq U$ (because $c \notin U$), $a \notin \underline{U}_{R_2}$. However, $\{a, c\} \cap U = \{a\} \neq \emptyset$, so $a \in \overline{U}_{R_2}$.
- For $x = b$, $R_2(b) = \{b, d\}$. Since $\{b, d\} \nsubseteq U$, $b \notin \underline{U}_{R_2}$. But $\{b, d\} \cap U = \{b\} \neq \emptyset$, so $b \in \overline{U}_{R_2}$.
- For $x = c$, $R_2(c) = \{a, c\}$. Similar to a, $c \notin \underline{U}_{R_2}$ but $c \in \overline{U}_{R_2}$ since $\{a, c\} \cap U = \{a\} \neq \emptyset$.
- For $x = d$, $R_2(d) = \{b, d\}$. Analogously, $d \notin \underline{U}_{R_2}$ but $d \in \overline{U}_{R_2}$.

Hence,

$$\underline{U}_{R_2} = \varnothing, \quad \overline{U}_{R_2} = \{a, b, c, d\}.$$

SuperRough Approximations By definition (see Definition 2.10 in the paper),

$$\underline{U}^{\Gamma} = \underline{U}_{R_1} \cap \underline{U}_{R_2} = \{a, b\} \cap \varnothing = \varnothing,$$

$$\overline{U}^{\Gamma} = \overline{U}_{R_1} \cup \overline{U}_{R_2} = \{a, b\} \cup \{a, b, c, d\} = \{a, b, c, d\}.$$

Thus, for $\Gamma = \{R_1, R_2\}$, the SuperRough approximation of $U = \{a, b\}$ is

$$\left(\underline{U}^{\Gamma}, \overline{U}^{\Gamma}\right) = \left(\varnothing, \{a, b, c, d\}\right).$$

In this example, the set $\{a, b\}$ is so "narrow" w.r.t. each relation's partitioning that the intersection of lower approximations is empty, and the union of upper approximations is the entire universe.

Theorem 2.12 (SuperRough Set generalizes Rough Set)**.** *Let $\Gamma = \{R_\alpha \mid \alpha \in I\}$ be a family of equivalence relations on X. Then the classical rough set model is a special case of the SuperRough set model. Specifically, if Γ consists of a single equivalence relation R, then*

$$\underline{U}^{\Gamma} = \underline{U}_R, \quad \overline{U}^{\Gamma} = \overline{U}_R,$$

and hence $\left(\underline{U}^{\Gamma}, \overline{U}^{\Gamma}\right)$ coincides with the standard rough approximation $\left(\underline{U}_R, \overline{U}_R\right)$.

Proof. By Definition 2.10,

$$\underline{U}^{\Gamma} = \bigcap_{\alpha \in I} \underline{U}_{R_\alpha}, \quad \overline{U}^{\Gamma} = \bigcup_{\alpha \in I} \overline{U}_{R_\alpha}.$$

If Γ contains exactly one relation R, i.e. I has a single index, then

$$\underline{U}^{\Gamma} = \underline{U}_R, \quad \overline{U}^{\Gamma} = \overline{U}_R.$$

These are precisely Pawlak's original (lower, upper) rough approximations for U with respect to R. Hence the SuperRough set reduces to the classical rough set model when there is only a single equivalence relation. □

2.3 SuperHyperRough Set

The SuperHyperRough Set is known as a concept that generalizes both the HyperRough Set and the SuperRough Set.

Definition 2.13 (***n***-SuperHyperRough Set)**.** [7] Let X be a non-empty finite universe. Suppose we have:

- A family of n distinct attributes $T_1, T_2, \dots, T_n$ with respective domains $J_1, J_2, \dots, J_n$.
- For each J_i, let $\mathcal{P}(J_i)$ be its power set. Define

  $$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

  Thus each element $A \in J$ is an n-tuple $\left(A_1, A_2, \dots, A_n\right)$ where $A_i \subseteq J_i$.
- A *family of equivalence relations* $\Gamma = \{ R_\alpha \mid \alpha \in I\}$ on X.

An ***n*-SuperHyperRough Set** over X is then a pair $\left(F, J\right)$, where

$$F : \quad J \longrightarrow \mathcal{P}(X),$$

is a mapping from each attribute-value combination $A \in J$ to a subset $F(A) \subseteq X$. For each $A \in J$, define the lower and upper approximations of $F(A)$ *with respect to* Γ by

$$\underline{F(A)}^{\Gamma} = \bigcap_{\alpha \in I} \left\{ x \in X \;\middle|\; R_\alpha(x) \subseteq F(A) \right\},$$

$$\overline{F(A)}^{\Gamma} = \bigcup_{\alpha \in I} \Big\{ x \in X \;\Big|\; R_\alpha(x) \cap F(A) \neq \emptyset \Big\}.$$

Hence, for each $A \in J$, we get a "rough set"

$$\Big(\underline{F(A)}^{\Gamma}, \; \overline{F(A)}^{\Gamma} \Big).$$

The pair (F, J) with this family of approximations

$$\Big\{ \Big(\underline{F(A)}^{\Gamma}, \; \overline{F(A)}^{\Gamma} \Big) \;\Big|\; A \in J \Big\}$$

is called an ***n*-SuperHyperRough Set**.

Remark 2.14. *SuperRough Sets* (sometimes termed "multigranulation rough sets") focus on a single subset $U \subseteq X$ but allow multiple equivalence relations. *HyperRough Sets* focus on a single equivalence relation R but assign different subsets of X to each attribute-value combination. In contrast, an ***n***-SuperHyperRough Set allows *both* multiple equivalence relations *and* a family of subsets indexed by *multiple* attributes and their subsets, thereby encompassing both extensions in one framework.

Theorem 2.15 (Generalization of HyperRough Set)**.** *If the family* Γ *of equivalence relations in Definition 2.13 consists of* exactly one *equivalence relation* R *(i.e.,* $\Gamma = \{R\}$*), then an* n*-SuperHyperRough Set reduces to a HyperRough Set.*

Proof. Consider Definition 2.13 but with $\Gamma = \{R\}$. Then for each $A \in J$,

$$\underline{F(A)}^{\Gamma} = \bigcap_{\alpha \in I} \underline{F(A)}_{R_\alpha} = \underline{F(A)}_{R}, \qquad \overline{F(A)}^{\Gamma} = \bigcup_{\alpha \in I} \overline{F(A)}_{R_\alpha} = \overline{F(A)}_{R},$$

since I has only one element. These are precisely the classical rough approximations

$$\underline{F(A)} = \{ x \in X \mid R(x) \subseteq F(A) \}, \quad \overline{F(A)} = \{ x \in X \mid R(x) \cap F(A) \neq \emptyset \}.$$

Hence each $A \in J$ is associated with a single rough set $\big(\underline{F(A)}, \overline{F(A)}\big)$, which matches the HyperRough Set definition (often referred to as (F, J) where F maps attribute-value combinations to subsets in X, each endowed with a single rough approximation by R). Therefore, an n-SuperHyperRough Set is indeed a generalization of the HyperRough Set when Γ is a singleton. □

Theorem 2.16 (Generalization of SuperRough Set)**.** *If* $n = 1$ *and* J_1 *is a trivial one-element set, then an* n*-SuperHyperRough Set (Definition 2.13) reduces to a SuperRough Set, i.e., multiple equivalence relations approximating a* single *subset of* X*.*

Proof. Let $n = 1$. Then

$$J = \mathcal{P}(J_1).$$

Assume $J_1 = \{ j^* \}$ is a single-element set. Then $\mathcal{P}(J_1)$ has exactly two elements: $\emptyset$ and $\{ j^* \}$. Thus $J = \{ \emptyset, \; \{j^*\} \}$.

Define $F(\emptyset)$ and $F(\{j^*\})$ arbitrarily, but in particular we focus on $F(\{j^*\})$, which is some subset $U \subseteq X$. The definitions of lower and upper approximations with respect to Γ yield

$$\underline{F(\{j^*\})}^{\Gamma} = \bigcap_{\alpha \in I} \underline{F(\{j^*\})}_{R_\alpha}, \quad \overline{F(\{j^*\})}^{\Gamma} = \bigcup_{\alpha \in I} \overline{F(\{j^*\})}_{R_\alpha}.$$

But $F(\{j^*\})$ is just a single subset $U \subseteq X$. This recovers the usual *SuperRough* notion where a set $U \subseteq X$ is approximated by multiple equivalence relations R_α. In other words,

$$(\underline{U}^{\Gamma}, \; \overline{U}^{\Gamma}), \quad \text{where} \quad \underline{U}^{\Gamma} = \bigcap_{\alpha \in I} \underline{U}_{R_\alpha}, \quad \overline{U}^{\Gamma} = \bigcup_{\alpha \in I} \overline{U}_{R_\alpha}.$$

Hence if the attribute space is trivial (so that we effectively have only one subset in play), we precisely obtain the SuperRough Set model. □

Remark 2.17 (Conclusion)**.** An n-SuperHyperRough Set merges the idea of:

- **HyperRough**: mapping every combination of (single) attribute-values to a rough set, *but* using only one equivalence relation.
- **SuperRough**: approximating a single subset with *multiple* equivalence relations.

By allowing:

- *multiple attributes* (with $\mathcal{P}(J_i)$ for each attribute domain J_i),
- *multiple equivalence relations* Γ,

the n-SuperHyperRough Set definition covers both cases as special instances, proving itself a unifying generalization of these two important rough-set extensions.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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Chapter 15
Neutrosophic TreeSoft Expert Set and ForestSoft Set

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Abstract

Concepts such as Fuzzy Sets [28,57], Neutrosophic Sets [42,44], and Plithogenic Sets [48] have been extensively studied to address uncertainty, finding diverse applications across various fields. The Soft Set provides a framework that associates each parameter with subsets of a universal set, enabling flexible approximations [31]. The TreeSoft Set extends the Soft Set by introducing hierarchical, tree-structured parameters, allowing for multi-level data representation [53].

In this paper, we revisit the concept of the Neutrosophic TreeSoft Set, which has been discussed in other studies [8, 34]. Additionally, we propose and examine the Neutrosophic TreeSoft Expert Set by incorporating the framework of the Neutrosophic Soft Expert Set. Furthermore, we revisit the ForestSoft Set, an extension of the TreeSoft Set, and explore related concepts, including the Neutrosophic ForestSoft Set.

Keywords: Neutrosophic Set, Soft Set, Treesoft Set, Neutrosophic Treesoft Set, ForestSoft Set

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

1.1 Neutrosophic Set

Neutrosophic Sets extend Fuzzy Sets by introducing the concept of indeterminacy, which accounts for situations that are neither entirely true nor entirely false [17–19, 21, 27, 43, 45–47, 54, 55].

Definition 1.1 (Neutrosophic Set)**.** [44, 45] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

1.2 Soft Set and TreeSoft Set

A *Soft Set* (F, E) associates each parameter in a set E with a subset of a universal set U. This provides a flexible framework for approximating objects within U [24, 30, 31]. A *TreeSoft Set* is a mapping from subsets of a hierarchical, tree-like parameter structure $\mathrm{Tree}(A)$ to subsets of a universal set U. This structure supports multi-level attributes for more refined and detailed analyses [8, 14, 22, 32, 34, 36, 53]. Related concepts include the *Hypersoft Set* [20, 49] and the *SuperHypersoft Set* [15, 16, 50]. The definitions of Soft Set and TreeSoft Set are provided below.

Definition 1.2. [30] Let U be a universal set and E a set of parameters. A *soft set* over U is defined as an ordered pair (F, E), where F is a mapping from E to the power set $\mathcal{P}(U)$:

$$F : E \to \mathcal{P}(U).$$

For each parameter $e \in E$, $F(e) \subseteq U$ represents the set of e-approximate elements in U, with (F, E) forming a parameterized family of subsets of U.

Definition 1.3. [51] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1,i_2,\ldots,i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as $\mathrm{Tree}(A)$, with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set F* is defined as a function:

$$F : P(\mathrm{Tree}(A)) \to P(H),$$

where $\mathrm{Tree}(A)$ represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\mathrm{Tree}(A))$ denotes its power set.

1.3 Neutrosophic Soft Set

The Neutrosophic Soft Set is a concept that combines the principles of Neutrosophic Sets and Soft Sets [2,5,6,9–11,25,33]. The definition is provided below.

Definition 1.4 (Neutrosophic Soft Set [26,29])**.** Let U be a universe and E a set of parameters. A *Neutrosophic Soft Set (NSS)* over U is defined as a pair $\big(F, A\big)$, where $A \subseteq E$ and

$$F \;:\; A \;\longrightarrow\; P(U),$$

with $P(U)$ being the collection of *Neutrosophic Sets* on U. Hence for each parameter $e \in A$,

$$F(e) = \big(T_{F(e)}, I_{F(e)}, F_{F(e)}\big)$$

is a Neutrosophic Set on U, satisfying

$$0 \;\leq\; T_{F(e)}(x) \;+\; I_{F(e)}(x) \;+\; F_{F(e)}(x) \;\leq\; 3, \quad \forall x \in U.$$

1.4 Neutrosophic Soft Expert Set

The Neutrosophic Soft Expert Set [3,37–39,56] is an extension of the Neutrosophic Soft Set, incorporating the framework of the Soft Expert Set (cf. [1,4,7,23,35,41]). The formal definition is provided below.

Definition 1.5 (Neutrosophic Soft Expert Set (NSES))**.** (cf. [3,38,39,56]) Let U be a universe, E a set of parameters, X a set of experts (agents), and $O = \{1, 0\}$ a set of opinions, where 1 indicates *agreement* and 0 indicates *disagreement*. Define $Z = E \times X \times O$, and let $A \subseteq Z$.

A *Neutrosophic Soft Expert Set (NSES)* over U is a pair (F, A), where $A \subseteq Z$ and:

$$F : A \to P(U),$$

where $P(U)$ denotes the power set of Neutrosophic Sets on U. That is, for each parameter $e = (p, x, o) \in A$, $F(e)$ is a Neutrosophic Set $\big(T_{F(e)}, I_{F(e)}, F_{F(e)}\big)$ defined on U. The values of $T_{F(e)}(u)$, $I_{F(e)}(u)$, and $F_{F(e)}(u)$ satisfy:

$$0 \leq T_{F(e)}(u) + I_{F(e)}(u) + F_{F(e)}(u) \leq 3, \quad \forall u \in U.$$

2 Results in This Paper

The results derived in this paper are presented below.

2.1 Neutrosophic Treesoft Set (Revisit)

A Neutrosophic Treesoft Set maps hierarchical attribute subsets to neutrosophic sets, representing truth, indeterminacy, and falsity on a universe.

Definition 2.1 (Neutrosophic Treesoft Set)**.** Let $H \subseteq U$ be a non-empty subset of a universe U, and $\mathrm{Tree}(A)$ be a hierarchical structure of attributes as defined previously. A *Neutrosophic Treesoft Set* is a mapping

$$\mathcal{F} : P\big(\mathrm{Tree}(A)\big) \longrightarrow N(H),$$

where each value $\mathcal{F}(\Gamma)$ is a Neutrosophic Set on H. Namely, for each $\Gamma \subseteq \mathrm{Tree}(A)$,

$$\mathcal{F}(\Gamma) = \Big(T_{\mathcal{F}(\Gamma)},\ I_{\mathcal{F}(\Gamma)},\ F_{\mathcal{F}(\Gamma)}\Big),$$

with $T_{\mathcal{F}(\Gamma)}, I_{\mathcal{F}(\Gamma)}, F_{\mathcal{F}(\Gamma)} : H \to [0,1]$ satisfying

$$0 \le T_{\mathcal{F}(\Gamma)}(h) + I_{\mathcal{F}(\Gamma)}(h) + F_{\mathcal{F}(\Gamma)}(h) \le 3 \quad \forall\, h \in H.$$

Theorem 2.2 (Neutrosophic Soft Set as a Special Case of Neutrosophic Treesoft Set)**.** *Every Neutrosophic Soft Set can be naturally embedded into a Neutrosophic Treesoft Set.*

More precisely, let (F, A) be a Neutrosophic Soft Set on universe U, where $F : A \to P(U)$ and each $F(e)$ is a Neutrosophic Set in U. Define a single-level tree *of attributes $Tree(A)$ whose nodes are exactly the distinct parameters in A (no further sub-attributes). Set $H := U$. Then we can construct a Neutrosophic Treesoft Set*

$$\mathcal{F} : P\big(Tree(A)\big) \longrightarrow N(H)$$

such that $\mathcal{F}(\{e\}) = F(e)$ for each $e \in A$. Thus (F, A) appears as the restriction of $\mathcal{F}$ to singletons in $Tree(A)$.

Proof. Since A is the set of parameters used in (F, A), we treat it as a *single-level* tree:

$$\mathrm{Tree}(A) = \{A_1, A_2, \ldots, A_n\},$$

where each $A_i \in A$. There are *no* additional sub-attributes, i.e., no deeper levels. Hence any $\Gamma \subseteq \mathrm{Tree}(A)$ is simply a subset $\Gamma \subseteq A$.

We wish to define $\mathcal{F} : P(\mathrm{Tree}(A)) \to N(H)$ so that:

$$\mathcal{F}(\{A_i\}) = F(A_i),$$

where $F(A_i)$ is already a Neutrosophic Set on U. Since $H = U$, we have $F(A_i) \in N(H)$.

A simple way is to let $\mathcal{F}(\Gamma)$ be the *pointwise union* (in the neutrosophic sense) of the Neutrosophic Sets $\{F(e) \mid e \in \Gamma\}$. Concretely, for each $h \in U$:

$$T_{\mathcal{F}(\Gamma)}(h) = \max_{e\in\Gamma}\big\{T_{F(e)}(h)\big\}, \quad I_{\mathcal{F}(\Gamma)}(h) = \min_{e\in\Gamma}\big\{I_{F(e)}(h)\big\}, \quad F_{\mathcal{F}(\Gamma)}(h) = \max_{e\in\Gamma}\big\{F_{F(e)}(h)\big\}.$$

(Or any other appropriate aggregator, e.g. t-norm/t-conorm pairs, depending on the application.)

Verification of Neutrosophic Condition. Because each $F(e)$ is a Neutrosophic Set, we have

$$0 \le T_{F(e)}(h) + I_{F(e)}(h) + F_{F(e)}(h) \le 3,$$

for all $e \in A$ and all $h \in U$. Taking pointwise maxima or minima of these values across $e \in \Gamma$ keeps us within the bounds $[0, 3]$. Thus

$$0 \le T_{\mathcal{F}(\Gamma)}(h) + I_{\mathcal{F}(\Gamma)}(h) + F_{\mathcal{F}(\Gamma)}(h) \le 3.$$

Hence $\mathcal{F}(\Gamma)$ is indeed a Neutrosophic Set on $H = U$.

If $\Gamma = \{e\} \subseteq A$, then by definition,

$$\mathcal{F}(\{e\}) = F(e).$$

Thus on singletons, $\mathcal{F}$ and F agree exactly. In other words, (F, A) is embedded into the Neutrosophic Treesoft structure $\mathcal{F}$.

Therefore, (F, A) emerges as a special (single-level) restriction of $\mathcal{F}$. This completes the proof. □

Theorem 2.3 (Restriction to TreeSoft Set)**.** *Let $\mathcal{F}$ be a Neutrosophic Treesoft Set as in Definition. For each $\Gamma \subseteq Tree(A)$, define*

$$G(\Gamma) = \{ h \in H \mid T_{\mathcal{F}(\Gamma)}(h) \geq \alpha \ \textit{and}\ I_{\mathcal{F}(\Gamma)}(h) \leq \beta \},$$

for some fixed thresholds $0 \leq \alpha, \beta \leq 1$. Then G is a (classical) TreeSoft Set in the sense of Definition.

Proof. Since $\mathcal{F}(\Gamma)$ is a Neutrosophic Set on H, we have numeric values $T_{\mathcal{F}(\Gamma)}(h)$ and $I_{\mathcal{F}(\Gamma)}(h)$. If we pick thresholds α and β, the set of all $h \in H$ satisfying $T_{\mathcal{F}(\Gamma)}(h) \geq \alpha$ and $I_{\mathcal{F}(\Gamma)}(h) \leq \beta$ is indeed a subset of H. This procedure, repeated for each $\Gamma \subseteq \mathrm{Tree}(A)$, defines a mapping

$$\Gamma \longmapsto G(\Gamma) \subseteq H.$$

But by Definition, a TreeSoft Set is any function from $P(\mathrm{Tree}(A))$ to $P(H)$. Hence G is precisely a classical TreeSoft Set, restricted by the chosen thresholds on the neutrosophic membership functions of $\mathcal{F}(\Gamma)$. □

Theorem 2.4 (Union and Intersection in a Neutrosophic Treesoft Set)**.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two Neutrosophic Treesoft Sets, both mapping*

$$\mathcal{F}_1,\ \mathcal{F}_2 : P(Tree(A)) \longrightarrow N(H).$$

Define new mappings $\mathcal{F}^{\cup}$ and $\mathcal{F}^{\cap}$ by

$$\mathcal{F}^{\cup}(\Gamma) = \Big(T_{\mathcal{F}_1(\Gamma)} \vee T_{\mathcal{F}_2(\Gamma)},\ \ I_{\mathcal{F}_1(\Gamma)} \wedge I_{\mathcal{F}_2(\Gamma)},\ \ F_{\mathcal{F}_1(\Gamma)} \vee F_{\mathcal{F}_2(\Gamma)}\Big),$$

$$\mathcal{F}^{\cap}(\Gamma) = \Big(T_{\mathcal{F}_1(\Gamma)} \wedge T_{\mathcal{F}_2(\Gamma)},\ \ I_{\mathcal{F}_1(\Gamma)} \vee I_{\mathcal{F}_2(\Gamma)},\ \ F_{\mathcal{F}_1(\Gamma)} \wedge F_{\mathcal{F}_2(\Gamma)}\Big),$$

where $\vee$ and $\wedge$ are pointwise max *and* min *operators, respectively (or any suitable t-conorm/t-norm pair in $[0,1]$). Then $\mathcal{F}^{\cup}$ and $\mathcal{F}^{\cap}$ are also Neutrosophic Treesoft Sets on H.*

Proof. For every $\Gamma \subseteq \mathrm{Tree}(A)$ and each $h \in H$, we define

$$T_{\mathcal{F}^{\cup}(\Gamma)}(h) := \max\{T_{\mathcal{F}_1(\Gamma)}(h),\ T_{\mathcal{F}_2(\Gamma)}(h)\}.$$

Similarly for $I_{\mathcal{F}^{\cup}(\Gamma)}(h)$ using min or max, depending on the intended aggregator, and for $F_{\mathcal{F}^{\cup}(\Gamma)}(h)$. Since each of $T_{\mathcal{F}_i(\Gamma)}, I_{\mathcal{F}_i(\Gamma)}, F_{\mathcal{F}_i(\Gamma)}$ lies in $[0,1]$, their max and min also lie in $[0,1]$. Thus $(T_{\mathcal{F}^{\cup}(\Gamma)}, I_{\mathcal{F}^{\cup}(\Gamma)}, F_{\mathcal{F}^{\cup}(\Gamma)})$ is a well-defined triple of functions $H \to [0,1]$.

We must show

$$0 \leq T_{\mathcal{F}^{\cup}(\Gamma)}(h) + I_{\mathcal{F}^{\cup}(\Gamma)}(h) + F_{\mathcal{F}^{\cup}(\Gamma)}(h) \leq 3,$$

and similarly for $\mathcal{F}^{\cap}$. Since

$$T_{\mathcal{F}_i(\Gamma)}(h) + I_{\mathcal{F}_i(\Gamma)}(h) + F_{\mathcal{F}_i(\Gamma)}(h) \leq 3$$

(for $i = 1, 2$), the pointwise max or min among the corresponding membership values also cannot exceed 3 in sum. Indeed, for any real numbers $a_1 + b_1 + c_1 \leq 3$ and $a_2 + b_2 + c_2 \leq 3$, taking $\max(a_1, a_2) + \max(b_1, b_2) + \max(c_1, c_2)$ or $\min(a_1, a_2) + \min(b_1, b_2) + \min(c_1, c_2)$ is at most 3. Clearly, the sum is also non-negative.

Hence for each Γ, $\mathcal{F}^{\cup}(\Gamma)$ and $\mathcal{F}^{\cap}(\Gamma)$ satisfy the neutrosophic condition on $[0,1]^3$. This shows that $\mathcal{F}^{\cup}$ and $\mathcal{F}^{\cap}$ are indeed functions from $P(\mathrm{Tree}(A))$ into $N(H)$. Therefore, they qualify as Neutrosophic Treesoft Sets. □

2.2 Neutrosophic TreeSoft Expert Set

The Neutrosophic TreeSoft Expert Set is an extension of the TreeSoft Set, incorporating the framework of the Neutrosophic Soft Expert Set. A related concept, the TreeSoft Expert Set, is also well-known [13].

Definition 2.5 (Neutrosophic TreeSoft Expert Set (NTSES))**.** Let:

- $H \subseteq U$ be a non-empty subset of a universe U.
- $\mathrm{Tree}(A)$ be a hierarchical attribute structure with root A and possibly multiple levels of sub-attributes.
- X be a set of experts.
- $O = \{1, 0\}$ a set of opinions, where 1 indicates *agreement* and 0 indicates *disagreement*.

Define

$$Z = P\big(\mathrm{Tree}(A)\big) \times X \times O.$$

Let $S \subseteq Z$. A *Neutrosophic TreeSoft Expert Set (NTSES)* on H is the pair $(\mathcal{F}, S)$ where $\mathcal{F}$ is a mapping

$$\mathcal{F} : S \longrightarrow \mathcal{P}_{\mathrm{NS}}(H),$$

with $\mathcal{P}_{\mathrm{NS}}(H)$ denoting the collection of Neutrosophic Sets on H. Concretely, for each triple $(\Gamma, x, o) \in S$, where $\Gamma \subseteq \mathrm{Tree}(A)$, $x \in X$, and $o \in O$,

$$\mathcal{F}(\Gamma, x, o) = \Big(T_{\Gamma,x,o},\ I_{\Gamma,x,o},\ F_{\Gamma,x,o}\Big),$$

where

$$T_{\Gamma,x,o},\ I_{\Gamma,x,o},\ F_{\Gamma,x,o} : H \longrightarrow [0, 1]$$

satisfy

$$0 \le T_{\Gamma,x,o}(h) + I_{\Gamma,x,o}(h) + F_{\Gamma,x,o}(h) \le 3, \quad \forall\, h \in H.$$

Remark 2.6. In words, for each *subset of the attribute tree* Γ, each *expert* x, and each *opinion* $o \in \{1, 0\}$, the NTSES assigns a *Neutrosophic* evaluation (T, I, F) on the domain H. This merges three main components:

1. The *hierarchical attribute structure* (TreeSoft notion),
2. The *expert-based* positive/negative opinion (Soft Expert notion),
3. The *Neutrosophic membership functions* for each element in H.

Theorem 2.7 (Reduction to Neutrosophic Soft Expert Set)**.** *Let $(\mathcal{F}, S)$ be a Neutrosophic TreeSoft Expert Set as in Definition 2.5. Suppose:*

- *The tree* $\mathrm{Tree}(A)$ *is* single-level *(i.e., it is isomorphic to a simple parameter set E with no deeper sub-attributes).*
- *We identify each node in* $\Gamma \subseteq \mathrm{Tree}(A)$ *with a parameter $p \in E$.*

Then, by restricting Γ to singletons and letting $S \subseteq E \times X \times O$, the NTSES $(\mathcal{F}, S)$ becomes a standard Neutrosophic Soft Expert Set (F, A).

Proof. If $\mathrm{Tree}(A)$ has only one level (no sub-attributes), then each $\Gamma \subseteq \mathrm{Tree}(A)$ is simply a subset of a finite set E. In the *Soft Expert* scenario, we typically select $\Gamma = \{p\} \subseteq E$.

Consider the restriction

$$S' = \big\{(\{p\}, x, o) \mid (\{p\}, x, o) \in S\big\}.$$

In other words, only the singletons $\{p\} \subseteq E$. On such triples, define

$$F(p, x, o) \;=\; \mathcal{F}(\{p\}, x, o).$$

Since $\mathcal{F}(\{p\}, x, o)$ is a Neutrosophic Set on $H \subseteq U$, we get exactly the form required by a Neutrosophic Soft Expert Set.

Hence the mapping $F : A \to \mathcal{P}_{\text{NS}}(U)$ recovers the definition of an NSES, with $A = S' \subseteq E \times X \times O$. This completes the reduction proof. □

Theorem 2.8 (Reduction to TreeSoft Set)**.** *Let $(\mathcal{F}, S)$ be a Neutrosophic TreeSoft Expert Set on H. Suppose we drop both the* expert *dimension X and the* opinion *set O by fixing a trivial single-expert set $\{x_0\}$ and a single-opinion set $\{1\}$. Then $(\mathcal{F}, S)$ reduces to a classical TreeSoft Set*

$$\widetilde{F} : P(\text{Tree}(A)) \longrightarrow P(H),$$

by selecting, for each $\Gamma \subseteq \text{Tree}(A)$, a crisp subset $\widetilde{F}(\Gamma) \subseteq H$ from the corresponding neutrosophic membership.

Proof. Let $X = \{x_0\}$ and $O = \{1\}$. Then

$$Z = P(\text{Tree}(A)) \times X \times O \;=\; P(\text{Tree}(A)) \times \{x_0\} \times \{1\}.$$

Any subset $S \subseteq Z$ effectively identifies a collection of $\Gamma_i \subseteq \text{Tree}(A)$.

Since $\mathcal{F}(\Gamma, x_0, 1)$ is a Neutrosophic Set $(T_\Gamma, I_\Gamma, F_\Gamma)$ on H, one can define

$$\widetilde{F}(\Gamma) \;=\; \{\, h \in H \;\mid\; T_\Gamma(h) \geq \alpha \},$$

or any other threshold-based selection from $\{T_\Gamma, I_\Gamma, F_\Gamma\}$ (e.g. "include h if the truth-degree is sufficiently large and the false-degree is sufficiently small"). This yields a crisp subset $\widetilde{F}(\Gamma) \subseteq H$.

This mapping $\Gamma \mapsto \widetilde{F}(\Gamma)$ is precisely a function from $P(\text{Tree}(A))$ into $P(H)$. By Definition, it constitutes a TreeSoft Set. Thus the NTSES collapses to a classic TreeSoft Set once the expert and opinion dimensions are trivialized and the neutrosophic membership is interpreted in a crisp manner. □

3 Additional Results of This Paper

As additional results of this paper, we explore the concept of the ForestSoft Set and its extended variants [12, 40, 52].

3.1 ForestSoft Set (Revisit)

A *ForestSoft Set* is formed by taking a collection of TreeSoft Sets and "gluing" (uniting) them together so as to obtain a single function whose domain is the union of all tree-nodes' power sets and whose values in $P(H)$ combine the images given by the individual TreeSoft Sets.

Definition 3.1 (ForestSoft Set)**.** [52] Let U be a universe of discourse, $H \subseteq U$ be a non-empty subset, and $P(H)$ be the power set of H. Suppose we have a finite (or countable) collection of TreeSoft Sets

$$\{\, F_t : P(\text{Tree}(A^{(t)})) \to P(H) \}_{t \in T},$$

where each F_t is a TreeSoft Set corresponding to a tree $\text{Tree}(A^{(t)})$ of attributes $A^{(t)}$.

We construct a *forest* by taking the (disjoint) union of all these trees:

$$\text{Forest}(\{A^{(t)}\}_{t \in T}) \;=\; \bigsqcup_{t \in T} \text{Tree}(A^{(t)}).$$

A *ForestSoft Set*, denoted by

$$\mathbf{F} : P(\text{Forest}(\{A^{(t)}\})) \longrightarrow P(H),$$

is defined as the *union* of all TreeSoft Set mappings F_t. Concretely, for any element $X \in P(\text{Forest}(\{A^{(t)}\}))$, we set

$$\mathbf{F}(X) \;=\; \bigcup_{\substack{t \in T \\ X \cap \text{Tree}(A^{(t)}) \neq \emptyset}} F_t(X \cap \text{Tree}(A^{(t)})),$$

where we only apply F_t to that portion of X belonging to the tree $\text{Tree}(A^{(t)})$.

3.2 Neutrosophic ForestSoft Set

A Neutrosophic ForestSoft Set maps hierarchical multi-tree structures to neutrosophic sets, enabling multi-level uncertainty representation across multiple attribute domains.

Definition 3.2 (Neutrosophic Forestsoft Set (NFS))**.** Let $H \subseteq U$ be a non-empty subset of a universe U. For each $t \in T$, suppose we have a *Neutrosophic Treesoft Set*:

$$\mathcal{F}_t \;:\; P\big(\mathrm{Tree}(A^{(t)})\big) \;\longrightarrow\; N(H).$$

The *forest* of attribute trees is

$$\mathrm{Forest}\big(\{A^{(t)}\}_{t\in T}\big) \;=\; \bigsqcup_{t\in T} \mathrm{Tree}\big(A^{(t)}\big).$$

Then a *Neutrosophic Forestsoft Set* $\mathbf{F}$ is a function

$$\mathbf{F} : P\Big(\mathrm{Forest}\big(\{A^{(t)}\}\big)\Big) \;\longrightarrow\; N(H),$$

defined by "combining" the outputs of $\mathcal{F}_t$. Concretely, for each

$$X \;\in\; P\Big(\mathrm{Forest}\big(\{A^{(t)}\}_{t\in T}\big)\Big),$$

we decompose X into its parts

$$X_t \;:=\; X \cap \mathrm{Tree}(A^{(t)}),$$

and define for each $h \in H$,

$$\begin{aligned}
T_{\mathbf{F}(X)}(h) &= \max_{t\in T:\, X_t\neq\varnothing} \Big\{T_{\mathcal{F}_t(X_t)}(h)\Big\},\\
I_{\mathbf{F}(X)}(h) &= \min_{t\in T:\, X_t\neq\varnothing} \Big\{I_{\mathcal{F}_t(X_t)}(h)\Big\},\\
F_{\mathbf{F}(X)}(h) &= \max_{t\in T:\, X_t\neq\varnothing} \Big\{F_{\mathcal{F}_t(X_t)}(h)\Big\}.
\end{aligned}$$

(One may also choose alternative aggregators, e.g. t-norm / t-conorm, as desired.) Thus,

$$\mathbf{F}(X) \;=\; \Big(T_{\mathbf{F}(X)},\; I_{\mathbf{F}(X)},\; F_{\mathbf{F}(X)}\Big)$$

is a Neutrosophic Set on H.

Remark 3.3. If $X \cap \mathrm{Tree}(A^{(t)}) = \varnothing$ for some t, that tree does not contribute to the aggregator. One could also define a "universal aggregator" over all $t \in T$, ignoring whether X_t is empty; practical usage may vary. The definitions above ensure that each portion $X_t \subseteq \mathrm{Tree}(A^{(t)})$ is mapped by $\mathcal{F}_t$, and then the results are *combined* in a neutrosophic manner.

Theorem 3.4 (Well-definedness of Neutrosophic Forestsoft Set)**.** *With notation as in Definition 3.2, let $\mathbf{F}$ be constructed from $\{\mathcal{F}_t\}_{t\in T}$. Then for every $X \subseteq \mathrm{Forest}(\{A^{(t)}\})$, the triple $\mathbf{F}(X) = (T_{\mathbf{F}(X)}, I_{\mathbf{F}(X)}, F_{\mathbf{F}(X)})$ is a valid Neutrosophic Set on H.*

Proof. Fix $X \subseteq$ Forest. For each t, write $\mathcal{F}_t(X_t) = (T_{t,X_t}, I_{t,X_t}, F_{t,X_t})$, where

$$0 \;\leq\; T_{t,X_t}(h) + I_{t,X_t}(h) + F_{t,X_t}(h) \;\leq\; 3 \quad \text{for all } h \in H.$$

Then

$$T_{\mathbf{F}(X)}(h) \;=\; \max\big\{T_{t,X_t}(h)\big\}_{t\in T^*},$$

where $T^* = \{t \in T \mid X_t \neq \varnothing\}$. Clearly, $\max\{\dots\} \in [0,1]$. Analogous statements hold for $I_{\mathbf{F}(X)}(h)$ (using min) and $F_{\mathbf{F}(X)}(h)$ (using max).

Sum check: For each h, let

$$a_t = T_{t,X_t}(h), \quad b_t = I_{t,X_t}(h), \quad c_t = F_{t,X_t}(h).$$

Since $a_t + b_t + c_t \le 3$ for every t, we must show

$$T_{\mathbf{F}(X)}(h) \ + \ I_{\mathbf{F}(X)}(h) \ + \ F_{\mathbf{F}(X)}(h) \ \ \le \ 3.$$

But

$$T_{\mathbf{F}(X)}(h) = \max_{t \in T^*} a_t, \quad I_{\mathbf{F}(X)}(h) = \min_{t \in T^*} b_t, \quad F_{\mathbf{F}(X)}(h) = \max_{t \in T^*} c_t.$$

In general, for real numbers $\{a_t, b_t, c_t\} \subseteq [0,1]$ with each $a_t + b_t + c_t \le 3$, the combination $\max(a_t) + \min(b_t) + \max(c_t) \le 3$. Indeed:

$$\max(a_t) + \max(c_t) \ \ \le \ \ \max(a_t + c_t) \ \ \le \ \ \max(a_t + b_t + c_t) \ \ \le \ 3,$$

and adding $\min(b_t) \le \max(b_t)$ maintains a sum ≤ 3. Hence

$$0 \ \ \le \ \ T_{\mathbf{F}(X)}(h) + I_{\mathbf{F}(X)}(h) + F_{\mathbf{F}(X)}(h) \ \ \le \ 3.$$

Thus $\mathbf{F}(X)$ is indeed a Neutrosophic Set on H. □

Theorem 3.5 (Generalization of Neutrosophic Treesoft Set)**.** *A Neutrosophic Forestsoft Set generalizes the Neutrosophic Treesoft Set. Concretely, if* $\left|\{A^{(t)}\}_{t\in T}\right| = 1$*, i.e. there is only* one *tree in the forest, then the Neutrosophic Forestsoft Set reduces to a Neutrosophic Treesoft Set.*

Proof. Take $T = \{t_0\}$. Then we have only one Neutrosophic Treesoft Set $\mathcal{F}_{t_0} : P(\mathrm{Tree}(A^{(t_0)})) \to N(H)$. The forest is

$$\mathrm{Forest}\big(\{A^{(t_0)}\}\big) \ = \ \mathrm{Tree}\big(A^{(t_0)}\big).$$

For $X \subseteq \mathrm{Tree}(A^{(t_0)})$, define

$$X_{t_0} = X \cap \mathrm{Tree}(A^{(t_0)}),$$

but $X_{t_0} = X$ since there is only one tree. The aggregator in Definition 3.2 simply picks

$$T_{\mathbf{F}(X)}(h) = T_{\mathcal{F}_{t_0}(X_{t_0})}(h), \quad I_{\mathbf{F}(X)}(h) = I_{\mathcal{F}_{t_0}(X_{t_0})}(h), \quad F_{\mathbf{F}(X)}(h) = F_{\mathcal{F}_{t_0}(X_{t_0})}(h).$$

Hence $\mathbf{F}(X) = \mathcal{F}_{t_0}(X)$. So $\mathbf{F}$ is exactly the same mapping as $\mathcal{F}_{t_0}$. Consequently, the Neutrosophic Forestsoft Set and the Neutrosophic Treesoft Set coincide when the "forest" has only one tree. □

Theorem 3.6 (Union and Intersection in a Neutrosophic Forestsoft Set)**.** *Let* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *be two Neutrosophic Forestsoft Sets, both mapping*

$$\mathbf{F}_1, \ \mathbf{F}_2 : P\Big(\mathrm{Forest}(\{A^{(t)}\}_{t\in T})\Big) \ \longrightarrow \ N(H).$$

Define new mappings $\mathbf{F}^{\cup}$ *and* $\mathbf{F}^{\cap}$ *by*

$$\mathbf{F}^{\cup}(X) \ = \ \Big(T_{\mathbf{F}_1(X)} \ \vee \ T_{\mathbf{F}_2(X)}, \ \ I_{\mathbf{F}_1(X)} \ \wedge \ I_{\mathbf{F}_2(X)}, \ \ F_{\mathbf{F}_1(X)} \ \vee \ F_{\mathbf{F}_2(X)}\Big),$$

$$\mathbf{F}^{\cap}(X) \ = \ \Big(T_{\mathbf{F}_1(X)} \ \wedge \ T_{\mathbf{F}_2(X)}, \ \ I_{\mathbf{F}_1(X)} \ \vee \ I_{\mathbf{F}_2(X)}, \ \ F_{\mathbf{F}_1(X)} \ \wedge \ F_{\mathbf{F}_2(X)}\Big),$$

where $\vee$ *and* $\wedge$ *are pointwise* max *and* min *operators in* $[0,1]$*. Then* $\mathbf{F}^{\cup}$ *and* $\mathbf{F}^{\cap}$ *are also Neutrosophic Forestsoft Sets on* H*.*

Proof. For each $X \subseteq \mathrm{Forest}(\{A^{(t)}\})$, we have $\mathbf{F}_1(X), \mathbf{F}_2(X) \in N(H)$. So

$$T_{\mathbf{F}_1(X)}, \ I_{\mathbf{F}_1(X)}, \ F_{\mathbf{F}_1(X)} \quad \text{and} \quad T_{\mathbf{F}_2(X)}, \ I_{\mathbf{F}_2(X)}, \ F_{\mathbf{F}_2(X)}$$

all lie in $[0,1]$. Their pointwise max or min values remain in $[0,1]$. Checking the sum condition

$$T + I + F \ \ \le \ 3$$

follows the same argument used in Theorem 3.4, showing that $\mathbf{F}^{\cup}(X)$ and $\mathbf{F}^{\cap}(X)$ are valid Neutrosophic Sets. One can interpret $\mathbf{F}^{\cup}$ and $\mathbf{F}^{\cap}$ as "logical union" and "logical intersection" of the two Neutrosophic Forestsoft Sets. □

3.3 Neutrosophic ForestSoft Expert Set

The Neutrosophic ForestSoft Expert Set is a concept that combines the principles of the ForestSoft Set, Neutrosophic Set, and Soft Expert Set. Its definition is provided below.

Definition 3.7 (Neutrosophic ForestSoft Expert Set (NFS-ES))**.** Let:

- $H \subseteq U$ be a non-empty subset of a universe U.
- $\{\mathrm{Tree}(A^{(t)})\}_{t\in T}$ be an indexed family of trees (each a hierarchical attribute structure). Their disjoint union is
$$\mathrm{Forest}\big(\{A^{(t)}\}_{t\in T}\big) \;=\; \bigsqcup_{t\in T} \mathrm{Tree}(A^{(t)}).$$
- X be a set of experts.
- $O = \{1, 0\}$ a set of opinions, where 1 indicates *agreement* and 0 indicates *disagreement*.

Define
$$Z \;=\; P\big(\mathrm{Forest}(\{A^{(t)}\}_{t\in T})\big) \;\times\; X \;\times\; O.$$
A *Neutrosophic ForestSoft Expert Set (NFS-ES)* over H is a pair $\big(\mathbf{F}, S\big)$ where $S \subseteq Z$ and
$$\mathbf{F} : S \;\longrightarrow\; N(H),$$
assigns to each $\big(Y, x, o\big) \in S$ a Neutrosophic Set $\mathbf{F}(Y, x, o)$ on H. Concretely, for
$$\mathbf{F}(Y, x, o) \;=\; \big(T_{Y,x,o},\; I_{Y,x,o},\; F_{Y,x,o}\big),$$
we require
$$0 \;\le\; T_{Y,x,o}(h) \;+\; I_{Y,x,o}(h) \;+\; F_{Y,x,o}(h) \;\le\; 3, \quad \forall\, h \in H.$$

Remark 3.8. In words, for each:

- Subset $Y \subseteq \mathrm{Forest}(\{A^{(t)}\})$ (possibly spanning multiple trees),
- Expert $x \in X$,
- Opinion $o \in \{1, 0\}$,

the NFSES structure $\mathbf{F}(Y, x, o)$ returns a triple $\big(T, I, F\big)$, describing the truth, indeterminacy, and falsity degrees of every element $h \in H$. This merges the multi-tree, multi-expert, and neutrosophic membership perspectives into a single formalism.

Theorem 3.9 (Generalization of Neutrosophic TreeSoft Expert Set)**.** *A Neutrosophic ForestSoft Expert Set (NFS-ES) generalizes the Neutrosophic TreeSoft Expert Set (NTSES). Specifically, if the forest consists of* $|T| = 1$ *tree, then the NFS-ES is precisely an NTSES.*

Proof. Suppose there is only a single tree $\mathrm{Tree}(A^{(t_0)})$. Then
$$\mathrm{Forest}\big(\{A^{(t_0)}\}\big) \;=\; \mathrm{Tree}(A^{(t_0)}),$$
and
$$Z = P\Big(\mathrm{Forest}(\{A^{(t_0)}\})\Big) \;\times\; X \times O = P\big(\mathrm{Tree}(A^{(t_0)})\big) \;\times\; X \times O.$$
Hence a Neutrosophic ForestSoft Expert Set $\big(\mathbf{F}, S\big)$ is merely the assignment
$$\mathbf{F} : S \;\longrightarrow\; N(H),$$
where $S \subseteq P(\mathrm{Tree}(A^{(t_0)})) \times X \times O$. But this is exactly the definition of a Neutrosophic TreeSoft Expert Set in NTSES. Therefore, NFS-ES reduces to NTSES when there is only one tree in the forest. □

Theorem 3.10 (Generalization of ForestSoft Set)**.** *A Neutrosophic ForestSoft Expert Set generalizes the (classical) ForestSoft Set. If we* trivialize *the neutrosophic membership into crisp subsets (e.g., choose a threshold α for truth and interpret "membership" above that threshold as 1, else 0), and collapse the expert-opinion dimension, the structure becomes a standard ForestSoft Set.*

Proof. Consider a Neutrosophic ForestSoft Expert Set $(\mathbf{F}, S)$ on $\text{Forest}(\{A^{(t)}\})$. If we fix a single expert $x_0 \in X$ and a single opinion $o_0 \in O = \{1, 0\}$, then we only look at

$$S' = \{(Y, x_0, o_0) \mid Y \subseteq \text{Forest}(\{A^{(t)}\})\} \subseteq S.$$

For each $Y \subseteq \text{Forest}(\{A^{(t)}\})$, $\mathbf{F}(Y, x_0, o_0)$ is a Neutrosophic Set (T_Y, I_Y, F_Y). By imposing a crisping procedure (e.g., "include h if $T_Y(h) \geq \alpha$ and $F_Y(h) \leq \gamma$, etc."), we get a subset of H. Concretely, define

$$\widetilde{\mathbf{F}}(Y) = \{h \in H \mid T_Y(h) \geq \alpha, \ I_Y(h) \leq \beta, \ F_Y(h) \leq \gamma\},$$

for fixed thresholds α, β, γ. Then $\widetilde{\mathbf{F}} : P(\text{Forest}(\{A^{(t)}\})) \to P(H)$ is precisely a *ForestSoft Set*, since each Y is mapped to a crisp subset of H. Thus, by ignoring additional experts/opinions and converting neutrosophic degrees into classical membership, we recover a standard ForestSoft Set structure. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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