Florentin Smarandache

(author and editor)

Collected Papers

Volume XVII

A Multi-Disciplinary Approach
to Uncertainty, Indeterminacy,
and Complex Systems

Neutrosophic Science International Association (NSIA)
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Introductory Note

Collected Papers, Vol. XVII – A Multi-Disciplinary Approach to Uncertainty, Indeterminacy, and Complex Systems is a substantial and wide-ranging volume, bringing together 74 articles across nearly 1,000 pages. The contributions, originally published in a variety of international scientific journals, reflect the breadth and maturity of contemporary research in neutrosophy and its related domains.

The volume explores a rich spectrum of interdisciplinary fields, including logic, philosophy, physics, mathematics, statistics, information fusion, artificial intelligence, decision theory, complex systems, and robotics, offering both foundational developments and applied methodologies for modeling uncertainty, indeterminacy, and contradiction.

All papers are authored or co-authored by Florentin Smarandache, either individually or in collaboration with 69 distinguished co-authors from 18 countries (full list at the end of the book), underscoring the global reach and collaborative nature of neutrosophic research, and revealing a coherent yet diverse intellectual landscape, ranging from theoretical frameworks—such as neutrosophic logic, hypersoft sets, superhyperstructures, and revolutionary topologies—to practical applications in healthcare analytics, artificial intelligence, sustainability, engineering, and decision-making under uncertainty.

Together, the works collected in this seventeenth volume provide both a comprehensive reference and a forward-looking perspective on the evolving role of neutrosophic and multi-valued approaches in addressing complex real-world problems.

Keywords

Core Theoretical Keywords

Neutrosophy, Neutrosophic Logic, Indeterminacy, Uncertainty, Plithogeny, Neutrosophic Statistics, Plithogenic Statistics, Multi-criteria Decision Making (MCDM), Paradoxism, Dialectics, Neutrosophic Triplets, Multiperspectivism, SuperHyperStructure, SuperHyperFunction, HyperSoft Set, TreeSoft Set, IndetermSoft Set, IndetermHyperSoft Set, Neutrosophic SuperHyperSoft Sets, Hypersoft Rough Graphs, Neutrosophic Signed Domination Function.

Applied Technology & Science Keywords

Artificial Intelligence, Machine Learning, Convolutional Neural Network, Vision Transformer, Liquid Neural Network, LLM (Large Language Models), Blockchain Cybersecurity, 5G Architecture, Computer Vision, Digital Image Processing, COVID-19 Detection, Noninvasive Hemoglobin Estimation, Wearable PPG, Tele-medical Framework, Sustainable Systems, Renewable Energy, Autopoietic Energy Transition, Space Propulsion, Optical Diametric Drive, Maxwell-Dirac Isomorphism, Soliton Computer.

Decision-Making & Methodology Keywords

VIKOR Method, TODIM Method, PSI Method, OWCM-RAM, FUCOM-GRA Approach, Game Theory, Personnel Selection, Retail Management, Project Management, Healthcare Claims Analysis, Medical Diagnosis, Climate Opinion Mapping, Tobacco Control Strategies, Spiritually Guided Scientific Method.

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FLORENTIN SMARANDACHE is an emeritus professor of Mathematics at the University of New Mexico, United States. He got his MSc in Mathematics and Computer Science from the University of Craiova, Romania, PhD in Mathematics from the State University of Kishinev, Moldova, and Postdoctoral in Applied Mathematics from Okayama University of Sciences, Japan, and The Guangdong University of Technology, Guangzhou, China.

He is the founder of neutrosophy (generalization of dialectics), neutrosophic set, logic, probability and statistics since 1995 and has published hundreds of papers and books on neutrosophic physics, superluminal and instantaneous physics, unmatter, quantum paradoxes, absolute theory of relativity, paradoxism, outerart, neutrosophy as a new branch of philosophy, Law of Included Multiple-Middle, multispace and multistructure, hypersoft set, IndetermSoft Set and IndetermHyperSoft Set, SuperHyperGraph, SuperHyperTopology, SuperHyperAlgebra, Neutrosophic SuperHyperAlgebra, degree of dependence and independence between neutrosophic components, refined neutrosophic set, neutrosophic over-under-off-set, plithogenic set / logic / probability / statistics, neutrosophic triplet and duplet structures, quadruple neutrosophic structures, extension of algebraic structures to NeutroAlgebra and AntiAlgebra, NeutroGeometry & AntiGeometry, Dezert-Smarandache Theory and so on.

In addition, he published many books of poetry, dramas, children' stories, translations, essays, a novel, folklore collections, traveling memories, and art albums.

http://fs.unm.edu/FlorentinSmarandache.htm

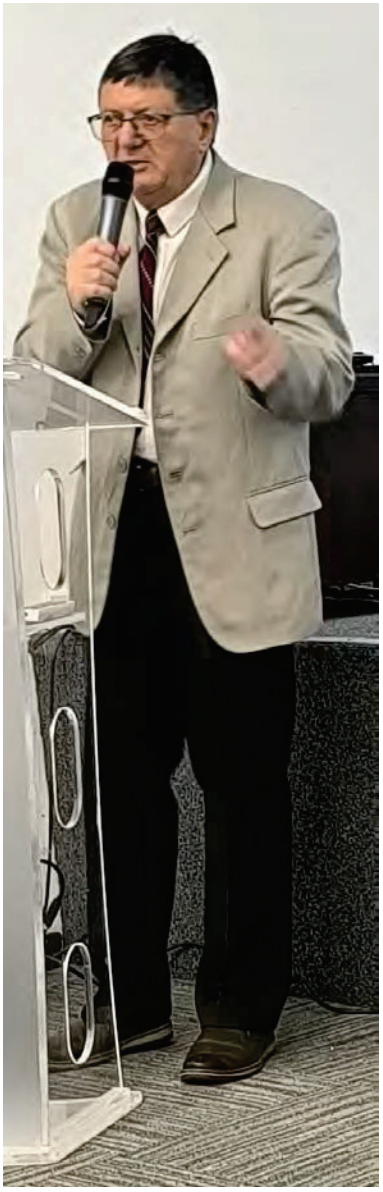


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A Multi-Disciplinary Approach to Uncertainty, Indeterminacy, and Complex Systems

Neutrosophie : un cadre interdisciplinaire pour une meilleure compréhension de l'incertitude et de l'indétermination

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Abstract: La neutrosophie, un cadre philosophique et scientifique relativement récent introduit dans les années 1990,[1] propose une approche robuste pour comprendre et modéliser l'indétermination, la contradiction et l'incertitude. Explorons les concepts fondamentaux de la neutrosophie et quelques diverses applications à travers plusieurs disciplines, notamment les mathématiques, la physique, la sociologie, la psychologie et la biologie.

Keywords: incertitude ; neutrosophie ; neutrosophication ; triade neutrosophique ; ensemble neutrosophique ; logique neutrosophique ; statistiques neutrosophiques ; physique neutrosophique ; personnalité neutrosophique ; neutrosociologie ; génétique neutrosophique ; théorie neutrosophique de l'évolution, de l'involution et de l'indétermination.

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1. Concepts fondamentaux de la neutrosophies

Les énormes données auxquelles nous sommes confrontés dans la vie réelle sont remplies d'indétermination : elles sont vagues, incomplètes, contradictoires, hybrides, biaisées, ignorantes, redondantes, superflues, dénuées de sens, ambiguës, floues, etc. C'est pourquoi les sciences neutrosophiques (qui traitent de l'indétermination), à travers le processus de neutrosophication, sont impliquées, telles que : la neutrosophie (une nouvelle branche de la philosophie), l'ensemble neutrosophique, la logique neutrosophique, la probabilité neutrosophique et les statistiques neutrosophiques, l'analyse neutrosophique, la mesure neutrosophique, et ainsi de suite. La neutrosophie étudie uniquement les triades (<A>, <neutA>, <antiA>), où <A> est un élément ou un concept, qui ont du sens dans le monde réel.

Par conséquent, au cœur de la neutrosophie, elle étend la logique binaire classique en introduisant un système logique à trois valeurs. Les trois composantes sont :

- *Truth* / Vérité (T) : Représente ce qui est vrai.
- *Falsehood* / Fausseté (F) : Représente ce qui est faux.

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1. Florentin Smarandache (2002). "Neutrosophy, A New Branch of Philosophy." *Multiple Valued Logic* 8(3): 297-384. An International Journal published by Taylor & Francis Group, UK and USA, ISSN 1023-6627. Le titre en français : « Neutrosophie, une nouvelle branche de la philosophie. » Article disponible en ligne : https://fs.unm.edu/Neutrosophy-A-New-Branch-of-Philosophy.pdf.

- *Indeterminacy* / Indétermination (I) : Capture ce qui est indéterminé, incertain ou contradictoire.

La neutrosophie postule que chaque proposition a simultanément un degré de vérité, un degré de fausseté et un degré d'indétermination, reflétant la nature complexe et souvent contradictoire des phénomènes du monde réel. Ce changement de paradigme permet une compréhension plus nuancée de la réalité, où la vérité absolue et la fausseté sont souvent entrelacées avec des éléments d'incertitude.

1.1. Neutrosophication

Le processus de neutrosophication signifie :

- convertir un concept précis {c'est-à-dire un concept (1, 0, 0), ce qui signifie un concept qui est à 100% vrai, 0% indéterminé et 0% faux} en un concept neutrosophique {c'est-à-dire un concept (*T*, *I*, *F*), qui est *T*% vrai, *I*% indéterminé et *F*% faux - ce qui reflète plus précisément notre réalité imparfaite et non idéaliste}, ou plus généralement en un concept raffiné (T_1, T_2, …; I_1, I_2, …; F_1, F_2, …);
- ou la conversion de nombres précis (1 ou 0), flous (*T*), ou flous intuitionnistes (*T*, *F*) en un nombre neutrosophique (*T*, *I*, *F*);
- ou la conversion d'un nombre précis (exact) *N* en un nombre neutrosophique de la forme $N = a + bI$, où *a* est la partie déterminée du nombre *N* et *bI* la partie indéterminée du nombre *N*.

2. L'Ensemble neutrosophique

Soit *U* un univers de discours, et *M* un ensemble inclus dans *U*. Un élément *x* de *U* est noté par rapport à l'ensemble *M* comme *x(T, I, F)* et appartient à *M* de la manière suivante : il est *t*% vrai dans l'ensemble, *i*% indéterminé (inconnu s'il est) dans l'ensemble, et *f*% faux, où *t* varie dans *T*, *i* varie dans *I*, *f* varie dans *F*.

Statiquement, *T*, *I*, *F* sont des sous-ensembles, mais dynamiquement, *T*, *I*, *F* sont des fonctions/opérateurs dépendant de nombreux paramètres connus ou inconnus.

L'ensemble neutrosophique[2] est une généralisation de l'ensemble flou intuitionniste, de l'ensemble flou intuitionniste inconsistent, de l'ensemble flou q-Rung Orthopair, de l'ensemble flou sphérique et de l'ensemble flou hypersphérique *n*.

La Neutrosophie a également été étendue à la Neutrosophie Raffinée, et par conséquent, la Neutrosophication a été étendue à la Neutrosophication Raffinée. Ainsi, la théorie du regret, la théorie des systèmes gris et la décision à trois voies sont des cas particuliers de la Neutrosophication et de la Probabilité Neutrosophique. Smarandache a également étendu la décision à trois voies à la décision à *n* voies, cette dernière étant un cas particulier de la Neutrosophie Raffinée. L'ensemble Neutrosophique généralise également l'ensemble paraconsistant, l'ensemble intuitionniste, etc.

3. La Logique neutrosophique

La Logique neutrosophique[3] est un cadre général pour l'unification de nombreuses logiques existantes, telles que la logique floue (en particulier la logique floue intuitionniste), la logique paraconsistante, la logique intuitionniste, etc. L'idée principale est de caractériser chaque énoncé logique dans un Espace Neutrosophique en 3D, où chaque dimension de l'espace représente respectivement la vérité (*T*), la fausseté (*F*), et l'indétermination (*I*) de l'énoncé en question, où *T*, *I*, *F*

2. Florentin Smarandache (2007). "A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics." Sixth edition. First published in 1998. Ann Arbor: InfoLearnQuest, 156 p. Le titre en français : « Un champ unificateur en logiques : la logique neutrosophique. Neutrosophie, ensemble neutrosophique, probabilité et statistiques neutrosophiques. » Article disponible en ligne : http://fs.unm.edu/eBook-Neutrosophics6.pdf.
3. *Ibidem*.

sont des sous-ensembles réels standard ou non standard de]-0, 1+[sans nécessairement aucun lien entre eux.

Pour la logique neutrosophique à valeur unique, la somme des composantes est :

- $0 \leq t+i+f \leq 3$ lorsque les trois composantes sont indépendantes ;
- $0 \leq t+i+f \leq 2$ lorsque deux composantes sont dépendantes, tandis que la troisième est indépendante d'elles ;
- $0 \leq t+i+f \leq 1$ lorsque les trois composantes sont dépendantes.

Lorsque trois ou deux des composantes *T, I, F* sont indépendantes, on laisse place à des informations incomplètes (somme < 1), des informations paraconsistantes et contradictoires (somme > 1), ou des informations complètes (somme = 1).

Si les trois composantes *T, I, F* sont dépendantes, alors de manière similaire on laisse place à des informations incomplètes (somme < 1), ou des informations complètes (somme = 1).

4. Les Statistiques neutrosophiques

Alors que les statistiques classiques traitent exclusivement de données déterminées et de méthodes d'inférence, les statistiques neutrosophiques s'occupent de données indéterminées - des données présentant différents degrés d'indétermination, telles qu'incertaines, vagues, partiellement inconnues, contradictoires ou incomplètes. Dans les statistiques neutrosophiques,[4] les méthodes d'inférence intègrent également des degrés d'indétermination. Par exemple, au lieu de s'appuyer uniquement sur des arguments nets et des valeurs pour les distributions de probabilité, les graphiques, les diagrammes, les algorithmes ou les fonctions, on peut rencontrer des arguments et des valeurs inexacts ou ambigus.

Par exemple, les tailles de population ou d'échantillon pourraient ne pas être précisément connues en raison d'individus qui n'appartiennent que partiellement à la population ou à l'échantillon, ou d'individus dont le statut d'appartenance est complètement inconnu. De plus, certains individus de la population ou de l'échantillon peuvent avoir des données indéterminées.

Les statistiques neutrosophiques sont également une généralisation des statistiques par intervalles. Alors que les statistiques par intervalles reposent sur l'analyse par intervalles, les statistiques neutrosophiques sont basées sur l'analyse par ensembles, englobant tous types d'ensembles, pas seulement les intervalles (comme les ensembles discrets finis). De plus, les statistiques classiques et les statistiques par intervalles supposent que tous les individus d'un échantillon ou d'une population appartiennent à 100 % à celui-ci. Cependant, dans la réalité, les individus appartiennent souvent seulement partiellement, partiellement ne pas appartenir, ou leur appartenance est indéterminée. Les statistiques neutrosophiques produisent des résultats plus précis que les statistiques classiques et les statistiques par intervalles en tenant compte de manière appropriée des différents degrés d'appartenance.

5. La Probabilité neutrosophique

Les distributions de probabilité neutrosophiques peuvent être représentées par trois courbes : l'une indiquant la probabilité de l'événement se produisant, une autre indiquant la probabilité de l'événement ne se produisant pas, et une troisième représentant la probabilité indéterminée de l'événement se produisant ou non.

Bien sûr, les ensembles de données neutrosophiques, où les données présentent un certain degré d'indétermination, sont utilisés dans les statistiques neutrosophiques. La probabilité neutrosophique est une généralisation de la probabilité classique et de la probabilité imprécise dans laquelle la chance qu'un événement A se produise est de *t*% vrai - où *t* varie dans le sous-ensemble *T*, *i*% indéterminé - où *i* varie dans le sous-ensemble *I*, et *f*% faux - où *f* varie dans le sous-ensemble *F*.

4. *Ibidem*.

En probabilité classique, la somme de toutes les probabilités de l'espace est égale à 1, tandis qu'en probabilité neutrosophique, elle est égale à 3.

En probabilité imprécise : la probabilité d'un événement est un sous-ensemble T dans [0, 1], et non un nombre p dans [0, 1], ce qui reste est censé être l'opposé, le sous-ensemble F (également de l'intervalle unitaire [0, 1]); il n'y a pas de sous-ensemble I indéterminé en probabilité imprécise.[5]

La fonction qui modélise la probabilité neutrosophique d'une variable aléatoire x est appelée distribution neutrosophique : $NP(x) = (T(x), I(x), F(x))$, où $T(x)$ représente la probabilité que la valeur x se produise, $F(x)$ représente la probabilité que la valeur x ne se produise pas, et $I(x)$ représente la probabilité indéterminée/inconnue de la valeur x.[6]

6. La Physique neutrosophique

La physique neutrosophique décrit des collections d'objets ou d'états qui sont individuellement caractérisés par des propriétés contradictoires, ou ne sont caractérisés ni par une propriété ni par l'opposé de cette propriété, ou sont composés de sous-éléments contradictoires. De tels objets ou états sont appelés entités neutrosophiques.

Par conséquent : une entité physique qui est une combinaison d'éléments opposés, ou une entité physique qui possède une propriété P et également l'opposé de la propriété P, ou une entité physique qui n'a ni la propriété Q ni l'opposé de la propriété Q.

La physique neutrosophique signifie un mélange de concepts / idées / espaces / lois / théories / objets physiques <*A*> avec leurs opposés <*antiA*> ou avec leur neutre <*neutA*>, où <*neutA*> est la neutralité par rapport à <*A*>, et signifie ni <*A*> ni <*antiA*> mais entre les deux ; c'est-à-dire des combinaisons de choses hétérogènes et contradictoires qui tiennent ensemble. Il existe de nombreux cas dans les domaines scientifiques (et humanistiques) où deux ou trois de ces éléments <*A*>, <*antiA*>, <*neutA*> coexistent.

Un élément <*A*> et son opposé <*antiA*> ou leur neutre <*neutA*> peuvent être simultanément valides.

Quelques exemples d'entités neutrosophiques :

- les anyons dans deux dimensions spatiales sont des particules de spin arbitraire qui ne sont ni des bosons (spin entier) ni des fermions (spin demi-entier) ;
- parmi les candidats possibles à la matière noire, il peut y avoir des particules exotiques qui ne sont ni des fermions de Dirac ni des fermions de Majorana ;
- le mercure (Hg) est un état qui n'est ni liquide ni solide dans des conditions normales à température ambiante ;
- les matériaux non magnétiques ne sont ni ferromagnétiques ni antiferro-magnétiques ;
- le plasma quark-gluon (QGP) est une phase formée par des quarks et gluons quasi-libres qui ne se comporte ni comme un plasma conventionnel ni comme un liquide ordinaire ;
- l'unmatière (*unmatter*),[7] qui est formée par la matière et l'antimatière qui se lient ensemble ;
- le kaon neutre, qui est un composite pion & anti-pion et donc une forme d'unmatière ;

5. Florentin Smarandache (2013). "Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability." Craiova (Roumanie) : Sitech, 140 p. ISBN: 978-1-59973-253-4. Le titre en français : « Introduction à la Mesure Neutrosophique, à l'Intégrale Neutrosophique et à la Probabilité Neutrosophique. » Livre disponible en ligne : http://fs.unm.edu/NeutrosophicMeasureIntegralProbability.pdf
6. Florentin Smarandache (2014). "Introduction to Neutrosophic Statistics." Craiova (Roumanie): Sitech, 124 p. ISBN 978-1-59973-274-9. Le titre en français : « Introduction aux Statistiques Neutrosophiques ». Livre disponible en ligne : http://fs.unm.edu/NeutrosophicStatistics.pdf
7. Ervin Goldfain, Florentin Smarandache (2008). "On Emergent Physics, "Unparticles" and Exotic "Unmatter" States." *Progress in Physics* 4:10-15. Le titre en français : « Sur la physique émergente, les « unparticules » et les états exotiques de « unmatière ». » Article disponible en ligne : https://fs.unm.edu/PP-15-02.pdf

- les méthodes neutrosophiques en relativité générale ; etc.

7. La Personnalité neutrosophique

Les principes neutrosophiques sont précieux en psychologie aussi, particulièrement dans l'étude de la cognition humaine. Les pensées et les émotions humaines sont rarement binaires ; elles sont souvent remplies d'incertitude et d'ambivalence. La logique neutrosophique offre un moyen de modéliser ces processus cognitifs de manière plus précise, ce qui peut conduire à de meilleurs outils de diagnostic et à des stratégies thérapeutiques plus efficaces. Par exemple, dans la thérapie cognitive-comportementale, reconnaître les aspects indéterminés des croyances et des émotions d'un patient peut mener à des plans de traitement plus efficaces.

En conséquence, Smarandache a proposé la Neutropsyche[8] comme une théorie psychologique qui explore l'esprit à travers le prisme de la neutrosophie et des théories neutrosophiques. La Personnalité Neutropsychique est un système psychologique dynamique et ouvert, défini par les tendances uniques d'un individu à ressentir, penser et agir.

En outre, Smarandache a défini la Mémoire Raffinée Neutrosophique qui conceptualise la mémoire en trois catégories : conscience, aconscience (un mélange de conscience et d'inconscience) et inconscience. Dans la Psychanalyse Neutrosophique, le 'ça' de Freud (*das Es*) est renommé 'sous-moi' pour créer une connexion symétrique avec le 'moi' et le 'surmoi'. Cette approche étend certains aspects de la psychanalyse de Freud tout en en rejetant d'autres.

Smarandache évalue que tous les souvenirs possèdent des degrés de conscience (c), des degrés d'aconscience (a) et des degrés d'inconscience (u). Chaque individu présente un degré d'anti-trait et un degré de trait par rapport à chaque paire anti-trait—trait de la personnalité. La Personnalité de Trait suit une Évolution Neutrosophique, caractérisée par des degrés d'Évolution, d'Indétermination et d'Involution. Grâce à l'adaptation et à la sélection sociale, certains traits de personnalité évoluent (et les gènes responsables de ceux-ci sont exprimés), d'autres restent inchangés ou leur changement est indéterminé, et une troisième catégorie de traits - ceux moins nécessaires dans un nouvel environnement - subit une involution (et les gènes responsables de ceux-ci cessent d'être exprimés).

8. La Neutrosociologie

La neutrosociologie[9] applique les principes de la neutrosophie au domaine de la sociologie, offrant de nouvelles façons de comprendre et d'analyser les phénomènes sociaux caractérisés par l'incertitude, l'ambiguïté et la complexité. Cette approche améliore les méthodes sociologiques traditionnelles en intégrant la nature indéterminée et souvent contradictoire des dynamiques sociales, visant à comprendre les attitudes et les comportements sociaux, analyser les conflits sociaux, examiner les réseaux sociaux, étudier le changement social et améliorer la recherche qualitative.

9. La Théorie neutrosophique de l'Évolution, de l'Involution et de l'Indétermination

Pendant le processus d'adaptation d'un être (végétal, animal ou humain) à un nouvel environnement ou à de nouvelles conditions, l'être évolue partiellement, dévolue partiellement (dégénère) et est partiellement indéterminé, c'est-à-dire ni en évolution ni en dévolution, donc

8. Florentin Smarandache (2018). "Neutropsychic Personality. A mathematical approach to psychology." Third updated edition. Brussels: Pons, 131 p. ISBN 978-1-59973-583-2. Le titre en français : « La Personnalité Neutropsychique. Une approche mathématique de la psychologie. » Livre disponible en ligne : https://fs.unm.edu/NeutropsychicPersonality-ed2.pdf
9. Florentin Smarandache (2019). "Introduction to Neutrosophic Sociology (Neutrosociology)." Brussels: Pons, 131 p. ISBN: 978-1-59973-605-1. Le titre en français : « Introduction à la Sociologie Neutrosophique (Neutrosociologie). » Livre disponible en ligne : https://fs.unm.edu/Neutrosociology.pdf

inchangé (neutre), ou le changement est flou, ambigu, vague, comme dans la logique neutrosophique. Grâce à l'adaptation, on a donc : évolution, involution et indétermination (ou neutralité), chacun de ces trois composants neutrosophiques à un certain degré. Les degrés d'évolution/indétermination/involution se réfèrent à la fois : à la structure de l'être (ses parties du corps), et à la fonctionnalité de l'être (la fonctionnalité de chaque partie, ou l'interfonctionnalité des parties entre elles, ou la fonctionnalité de l'être dans son ensemble). Avec cette clarification à l'esprit, Smarandache a donc introduit la Théorie Neutrosophique de l'Évolution, de l'Involution et de l'Indétermination (ou Neutralité).[10]

10. La Génétique neutrosophique

La génétique neutrosophique[11] est l'étude de la génétique utilisant la logique, l'ensemble, la probabilité, les statistiques, la mesure et d'autres outils et procédures neutrosophiques. Basée sur la Théorie Neutrosophique de l'Évolution (qui inclut des degrés d'Évolution, de Neutralité (ou Indétermination), et d'Involution) – en tant qu'extension de la Théorie de l'Évolution de Darwin, Smarandache montre l'applicabilité de la neutrosophie en génétique, et présente dans le cadre de la génétique neutrosophique les concepts suivants : mutation neutrosophique, spéciation neutrosophique, et coévolution neutrosophique.

11. Chronologie neutrosophique

1995-1998 Smarandache généralise le Yin Yang et la dialectique à la Neutrosophie
http://fs.unm.edu/Neutrosophy-A-New-Branch-of-Philosophy.pdf
L'ensemble/logique/probabilité/statistiques neutrosophiques
L'ensemble neutrosophique à valeur unique
http://fs.unm.edu/eBook-Neutrosophics6.pdf

1998 et 2019 Logique Neutrosophique Non Standard, Ensemble, Probabilité
https://arxiv.org/ftp/arxiv/papers/1903/1903.04558.pdf

2002 Types spéciaux d'ensembles / probabilités / statistiques / logiques
http://fs.unm.edu/DefinitionsDerivedFromNeutrosophics.pdf

- ensemble intuitionniste, ensemble paraconsistant, ensemble faillibiliste, ensemble paradoxiste, ensemble pseudo-paradoxiste, ensemble tautologique, ensemble nihiliste, ensemble dialéthéiste, ensemble trivialiste;
- probabilité et statistiques intuitionnistes, probabilité et statistiques paraconsistantes, probabilité et statistiques faillibilistes
- probabilité et statistiques paradoxistes, probabilité et statistiques pseudo-paradoxistes, probabilité et statistiques tautologiques, probabilité et statistiques nihilistes, probabilité et statistiques dialéthéistes, probabilité et statistiques trivialistes;

10. Florentin Smarandache (2017). "Introducing a Theory of Neutrosophic Evolution: Degrees of Evolution, Indeterminacy, and Involution." *Progress in Physics* 13(2): 130-135. Le titre en français : « Introduction à une théorie de l'évolution neutrosophique : degrés d'évolution, indétermination et involution. » Article disponible en ligne : https://fs.unm.edu/neutrosophic-evolution-PP-49-13.pdf
11. Florentin Smarandache (2021). "Introduction to Neutrosophic Genetics." *International Journal of Neutrosophic Science* 13(1): 23-27. Le titre en français : « Introduction à la la génétique neutrosophique. » Article disponible en ligne : http://fs.unm.edu/NeutrosophicGenetics.pdf.

- logique paradoxiste (ou paradoxisme), logique pseudo-paradoxiste (ou pseudo-paradoxisme), logique tautologique (ou tautologisme);

2003 Nombres neutrosophiques ($a+bI$, où I = indétermination)
Structures algébriques I-neutrosophiques
Cartes cognitives neutrosophiques
http://fs.unm.edu/NCMs.pdf

2005 L'Ensemble/Logique Neutrosophique Intervalle
http://fs.unm.edu/INSL.pdf

2006 Degré de Dépendance et Degré d'Indépendance entre T, I, et F
http://fs.unm.edu/eBook-Neutrosophics6.pdf
http://fs.unm.edu/NSS/DegreeOfDependenceAndIndependence.pdf

2007 L'Overset Neutrosophique (lorsqu'un composant neutrosophique est > 1), l'Underset Neutrosophique (lorsqu'un composant neutrosophique est < 0), et l'Offset Neutrosophique (lorsque certains composants neutrosophiques sont en dehors de l'intervalle [0, 1], c'est-à-dire que certains composants neutrosophiques > 1 et d'autres composants neutrosophiques < 0)
La Logique, la Mesure, la Probabilité,
les Statistiques Neutrosophiques Over/Under/Off.
http://fs.unm.edu/NSS/DegreesOf-Over-Under-Off-Membership.pdf
http://fs.unm.edu/SVNeutrosophicOverset-JMI.pdf
http://fs.unm.edu/IV-Neutrosophic-Overset-Underset-Offset.pdf
http://fs.unm.edu/NeutrosophicOversetUndersetOffset.pdf
L'Ensemble Tripolaire Neutrosophique et l'Ensemble Multipolaire Neutrosophique
Le Graph Tripolaire Neutrosophique et le Graph Multipolaire Neutrosophique
http://fs.unm.edu/eBook-Neutrosophics6.pdf
http://fs.unm.edu/IFS-generalized.pdf

2009 la N-norme et de la N-conorme
http://fs.unm.edu/N-normN-conorm.pdf

2013 Développement de la Probabilité neutrosophique :
http://fs.unm.edu/NeutrosophicMeasureIntegralProbability.pdf
Raffination des composants (T, I, F) en (T_1, T_2, ...; I_1, I_2, ...; F_1, F_2, ...)
http://fs.unm.edu/n-ValuedNeutrosophicLogic-PiP.pdf

2014 La Loi du Multiple-Milieu Inclus (<A>; <neutA_1>, <neutA_2>, ..., <neutA_k>; <antiA>)
http://fs.unm.edu/LawIncludedMultiple-Middle.pdf
Statistiques neutrosophiques
http://fs.unm.edu/NS/NeutrosophicStatistics.htm
http://fs.unm.edu/NeutrosophicStatistics.pdf

2015 Pré-Calcul neutrosophique et Calcul neutrosophique
http://fs.unm.edu/NeutrosophicPrecalculusCalculus.pdf
Nombres Neutrosophiques Raffinés ($a + b_1I_1 + b_2I_2 + ... + b_nI_n$)
Graphs Neutrosophiques
Thèse-Antithèse-Neutrothèse, et Neutrosynthèse

Système Axiomatique Neutrosophique
Systèmes Dynamiques Neutrosophiques
La Logique Neutrosophique Symbolique
Structures (*t, i, f*)-Neutrosophiques
Structures Algébriques Neutrosophiques Quadruples
Loi de Multiplication des Subindéterminations
http://fs.unm.edu/SymbolicNeutrosophicTheory.pdf
Indéterminations neutrosophiques naturelles
http://fs.unm.edu/MODNeutrosophicNumbers.pdf
L'Ensemble Crisp Neutrosophique <A, B, C>
http://fs.unm.edu/NeutrosophicCrispSetTheory.pdf

2016 Multisets Neutrosophiques
http://fs.unm.edu/NeutrosophicMultisets.htm
Structures Triplets Neutrosophiques
Structures Triplets Étendues Neutrosophiques
http://fs.unm.edu/NeutrosophicTriplets.htm
Structures Duplets Neutrosophiques
http://fs.unm.edu/NeutrosophicDuplets.htm

2017 La Théorie de l'Évolution Neutrosophique
Degrés d'Évolution, Indétermination ou Neutralité, et Involution
http://fs.unm.edu/neutrosophic-evolution-PP-49-13.pdf

2017-2019 La Plithogénie (généralisation de la Dialectique et de la Neutrosophie)
L'Ensemble/Logique/Probabilité/Statistiques Plithogéniques
http://fs.unm.edu/P/
Structures Algébriques Plithogéniques Symboliques :
https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf
http://fs.unm.edu/Plithogeny.pdf

2018-2022 Nouveaux types d'ensembles flous

- L'Ensemble HyperSoft,
- L'Ensemble IndetermSoft,
- L'Ensemble IndetermHyperSoft,
- L'Ensemble TreeSoft.

http://fs.unm.edu/TSS/
https://fs.unm.edu/NSS/ExtensionOfSoftSetToHypersoftSet.pdf

2018 La Psychologie Neutrosophique

- Neutropsyche,
- Personnalité Neutropsychique,
- Mémoire Neutrosophique Raffinée : consciente, a-consciente, inconsciente,
- Personnalité Neutropsychique Crisp,
- Fonctionnement Corps-Âme-Esprit Neutrosophique.

http://fs.unm.edu/NeutropsychicPersonality-ed3.pdf

2019 Théorie de l'Évolution Humaine Neutrosophique en Spirale

http://fs.unm.edu/SpiralNeutrosophicEvolution.pdf
La Sociologie Neutrosophique (Neutrosociologie)
https://fs.unm.edu/Neutrosociology.pdf

Ensemble Crisp Neutrosophique Raffiné
http://fs.unm.edu/RefinedNeutrosophicCrispSet.pdf
Généralisation des Structures Algébriques classiques aux Structures Algébriques Neutro (ou NeutroAlgèbres) et des Structures Algébriques Anti (ou AntiAlgèbres)
http://fs.unm.edu/NeutroAlgebra.htm

2019-2022 Introduction de nouveaux types de topologies

- Topologie Neutrosophique Raffinée,
- Topologie Crisp Neutrosophique Raffinée,
- NeutroTopologie, AntiTopologie,
- SuperHyperTopologie,
- SuperHyperTopologie Neutrosophique

http://fs.unm.edu/NSS/NewTypesTopologies-Improved14.pdf
http://fs.unm.edu/TT/
L'Extension de l'HyperGraph au SuperHyperGraph
http://fs.unm.edu/NSS/n-SuperHyperGraph-n-HyperAlgebra.pdf

2020 La Génétique Neutrosophique
http://fs.unm.edu/NeutrosophicGenetics.pdf

2021 NeutroGéométrie et AntiGéométrie
http://fs.unm.edu/NG/
http://fs.unm.edu/NSS/NeutroGeometryAntiGeometry31.pdf
http://fs.unm.edu/NSS/ExamplesNeutroGeometryAntiGeometry35.pdf
La Logique Plithogénique (généralisation de la Logique Multivariée)
http://fs.unm.edu/NSS/IntroductionPlithogenicLogic1.pdf
La Probabilité Plithogénique, Les Statistiques Plithogéniques
http://fs.unm.edu/NSS/PlithogenicProbabilityStatistics20.pdf

2022 SuperHyperAlgèbre & SuperHyperAlgèbre Neutrosophique
http://fs.unm.edu/SuperHyperAlgebra.pdf
SuperHyperGraph, SuperHyperGraph Neutrosophique
http://fs.unm.edu/NSS/n-SuperHyperGraph.pdf
SuperHyperFonction, SuperHyperTopologie
http://fs.unm.edu/NSS/SuperHyperFunction37.pdf
L'Ensemble IndetermSoft, l'Ensemble IndetermHyperSoft
https://fs.unm.edu/TSS/NewTypesSoftSets-Improved.pdf
http://fs.unm.edu/NSS/IndetermSoftIndetermHyperSoft38.pdf
L'Ensemble TreeSoft
http://fs.unm.edu/NSS/IndetermSoftSet-TreeSoftSet59.pdf

2023 Structures Algébriques Plithogéniques Symboliques
http://fs.unm.edu/NSS/SymbolicPlithogenicAlgebraic39.pdf

La Cryptologie Neutrosophique
http://fs.unm.edu/NeutrosophicCryptography1.pdf
http://fs.unm.edu/NeutrosophicCryptography2.pdf
http://fs.unm.edu/NSS/2OnANovelSecurityScheme.pdf

1998-2024 Applications Neutrosophiques en :

- Intelligence Artificielle, Systèmes d'Information, Informatique, Cybernétique, Méthodes Théoriques, Structures Algébriques Mathématiques, Mathématiques Appliquées, Automatisation, Systèmes de Contrôle, Big Data, Ingénierie, Électricité, Électronique, Philosophie, Sciences Sociales, Psychologie, Biologie, Génétique, Ingénierie Biomédicale, Informatique Médicale, Recherche Opérationnelle, Science de la Gestion, Science de l'Imagerie, Technologie Photographique, Instruments, Instrumentation, Physique, Optique, Économie, Mécanique, Neurosciences, Radiologie Nucléaire, Médecine, Imagerie Médicale, Applications Interdisciplinaires, Sciences Multidisciplinaires, etc.

12. Conclusions

La neutrosophie, avec son approche novatrice de la logique et de l'indétermination, offre des perspectives profondes et des outils pratiques dans un large éventail de disciplines. En embrassant la complexité et l'incertitude du monde réel, la neutrosophie permet une compréhension plus précise et nuancée de divers phénomènes. Alors que la recherche et les applications de la neutrosophie continuent de s'étendre,[12] son impact sur la philosophie, les mathématiques, la sociologie, la psychologie, la biologie et au-delà devrait s'approfondir, offrant de nouvelles façons de naviguer et de donner du sens à notre monde complexe.

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12. Xindong Peng, Jingguo Dai (2020). "A bibliometric analysis of neutrosophic set: two decades review from 1998 to 2017." *Artificial Intelligence Review* 53:199–255. Le titre en français : « Une analyse bibliométrique de l'ensemble neutrosophique : revue de deux décennies de 1998 à 2017. » Article disponible en ligne : https://doi.org/10.1007/s10462-018-9652-0, ou aussi: http://fs.unm.edu/BibliometricNeutrosophy.pdf.

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Fundamentele neutrosofiei

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Rezumat. Intenția fundamentală a neutrosofiei este să ofere un cadru pentru înțelegerea originii, naturii și interacțiunilor neutralităților, căutând să observe spațiile sofisticate dintre binare, să interpreteze neinterpretatul si neinterpretabilul să generalizeze ideile clasice în principii mai largi, mai cuprinzătoare. Prin metodele, principiile și tezele sale, neutrosofia admite nedeterminarea omniprezentă a lumii, reconsiderând natura adevărului, falsității și spectrului vast de neutralitate care se află între ele.

Cuvinte-cheie. Neutrosofie, Neutralitate, Nedeterminare, Logică neutrosofică, Dualitate, Complementaritate, Paradox, Dialectică, Transdisciplinaritate, Teză, Antiteză, Neutroteză, Neutrosinteză.

1. Introducere

Neutrosofia, un cadru filosofic pe care l-am dezvoltat acum mai bine de două decenii [Smarandache 1998], explorează relațiile și interacțiunile dintre opuse și neutralitățile/ nedeterminările lor, căutând să găsească puncte comune între ele și identificând elemente neobișnuite în entități similare, subliniind complexitatea și interconectarea conceptelor. Neutrosofia transcende gândirea binară tradițională, examinând interacțiunea dintre opuse și neutralitățile/nedeterminările dintre ele. Neutrosofia provoacă modurile convenționale de gândire, propunând noi principii și metode filosofice care recunosc limitările sistemelor clasice. Acest mod de gândire este caracterizat de mai multe trăsături cheie. Să le trecem în revistă.

2. Regândirea conceptelor vechi

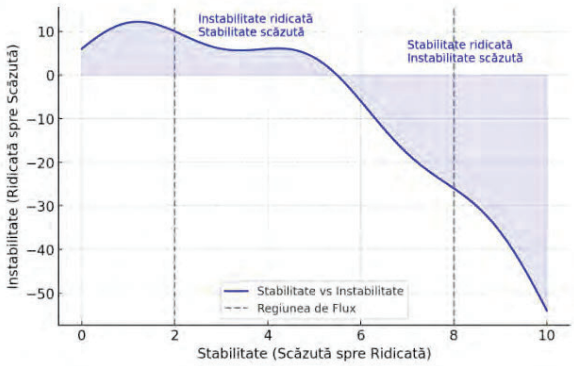
Neutrosofia revizitează ideile tradiționale, afirmând că adevărurile dintr-un sistem referențial pot deveni falsuri în altul și viceversa. Această abordare subliniază fluiditatea cunoașterii, îndemnând la vizualizarea ideilor din multiple unghiuri. Pătrunzând în domeniile paradoxului și ambiguității, neutrosofia luminează zone de gândire considerate anterior inaccesibile sau contradictorii.

Neutrosofia încearcă să reconcilieze ideile opuse, descoperind spațiile neutre dintre ele. Își propune să "facă pace în războiul ideilor și să

facă război în ideile pașnice”[1], subliniind natura dinamică a gândirii.

2.1. Înțelegerea stabilității și instabilității

Măsurând stabilitatea sistemelor instabile și instabilitatea sistemelor stabile, neutrosofia oferă instrumente pentru a analiza și naviga sisteme marcate de flux și incertitudine.



Grafic 1. **Relația dintre stabilitate și instabilitate**

Axa *x* reprezintă stabilitatea (de la scăzută la ridicată).

Axa *y* reprezintă instabilitatea (de la ridicată la scăzută).

Curba arată cum instabilitatea scade pe măsură ce stabilitatea crește, cu regiuni de flux și incertitudine evidențiate.

3. Metode de studiu

Neutrosofia utilizează o gamă bogată de metode, combinând rigoarea formală cu abordări transdisciplinare. Acestea includ:

- *Matematizarea*: Încorporarea conceptelor precum logica neutrosofică, probabilitatea și dualitatea pentru a cuantifica și modela neutralitatea.
- *Complementaritate → Generalizare*: Extinderea noțiunilor clasice pentru a cuprinde un spectru mai larg de posibilități și recunoașterea interconectării opuselor.
- *Contradicție → Paradox*: Îmbrățișarea coexistenței adevărurilor opuse pentru a descoperi perspective mai profunde.
- *Transdisciplinaritate*: Aplicarea principiilor neutrosofice în diverse domenii, de la fizică și filosofie la psihologie și literatură.

4. Formalizare: definirea neutralităților

Neutrosofia își formalizează abordarea printr-un sistem precis de notație.[2]

Să notăm cu [A] o idee, o propoziție, o teorie, un eveniment, un concept, o entitate, cu [Non-A] ceea ce nu este [A], și cu [Anti-A] opusul lui [A]. De asemenea, [Neut-A] înseamnă ceea ce nu este nici [A], nici [Anti-A], adică neutralitatea dintre cele două extreme. Și fie [A’] o versiune a lui [A].

Desigur, [Non-A] este diferit de [Anti-A].

De exemplu:

Dacă [A] = alb, atunci [Anti-A] = negru (antonim), dar [Non-A] = verde, roșu, albastru, galben, negru, etc. (orice culoare, cu excepția albului), în timp ce [Neut-A] = verde, roșu, albastru, galben, etc. (orice culoare, cu excepția albului și negrului), și [A’] = alb închis, etc. (orice nuanță de alb).

Într-un mod clasic:

[Neut-A] = [Neut-(Anti-A)], adică neutralitățile lui [A] sunt identice cu neutralitățile lui [Anti-A].

De asemenea:

[Non-A] ⊃ [Anti-A], și [Non-A] ⊃ [Neut-A].

[A] ∩ [Anti-A] = ∅, [A] ∩ [Non-A] = ∅.

[A], [Neut-A] și [Anti-A] sunt disjuncte două câte două.

[Non-A] este completitudinea lui [A] în raport cu setul universal.

Dar, deoarece în multe cazuri granițele dintre noțiuni sunt vagi, imprecise, este posibil ca [A], [Neut-A], [Anti-A] (și [Non-A] desigur) să aibă părți comune două câte două.

[1] Smarandache, Florentin (2002). „Neutrosophy, A New Branch of Philosophy.” *Multiple Valued Logic / An International Journal*, 3, p. 16. Disponibil online: https://digitalrepository.unm.edu/math_fsp/24. Accesat: 29 decembrie 2024.

[2] Smarandache, Florentin (2002). *Idem*, p. 17.

5. Principii și legi ale neutrosofiei

5.1. Continuumul neutralităților

Între orice idee și opusul ei se află un spectru infinit de neutralități, reflectând gradațiile și tranzițiile care caracterizează fenomenele din lumea reală.

5.2. Relativitatea referențială

Adevărul, falsitatea și nedeterminarea oricărei propoziții depind de sistemul referențial în care este examinată. Într-un sistem, o propoziție poate părea adevărată; în altul, falsă; și într-un altul, nedeterminată.

5.3. Echilibrul ideilor

Fiecare idee tinde în mod natural spre echilibru, echilibrată nu doar de ideile opuse, ci și de ideile neutre dintre ele. (Acest principiu extinde sinteza dialectică a lui Hegel, recunoscând rolul neutralităților.)

6. Teză-Antiteză-Neutroteză și Neutrosinteză

În neutrosofie, [A], [antiA] și [neutA] combinate două câte două, și de asemenea toate trei împreună, formează Neutrosinteza. Neutrosofia stabilește relațiile universale dintre [A], [antiA] și [neutA].

[A] este teza, [antiA] antiteza, iar [neutA] neutroteza (nici [A], nici [antiA], ci neutralitatea dintre ele).

În notația neutrosofică, [nonA] (nu [A], în afara lui [A]) este uniunea dintre [antiA] și [neutA].

[neutA] poate varia de la niciun mijloc (mijloc exclus), la un singur mijloc (mijloc inclus), la multe mijloace discrete finite (multiple mijloace incluse finite) și la o infinitate de mijloace discrete sau continue (multiple mijloace incluse infinite) [de exemplu, ca în cazul culorii, pentru ultimul, să spunem că între negru și alb există un spectru infinit de culori mijlocii/intermediare].

6.1. Teză, Antiteză, Sinteză

Neutrosofia este o generalizare a dialecticii (care se bazează doar pe contradicții, [A] și [antiA]), deoarece neutrosofia se bazează pe contradicții și pe neutralitățile dintre ele ([A], [antiA] și [neutA]).

Prin urmare, triada dialectică teză-antiteză-sinteză[3] este extinsă la tetrada neutrosofică teză-antiteză-neutroteză-neutrosinteză. O sinteză neutrosofică (neutrosinteză) este mai rafinată decât sinteza dialectică. Ea continuă unificarea și sinteza cu privire la opuse și la neutrele lor.

6.2. Dinamicitatea neutrosofică

Am extins Principiul Opoziției Dinamice — opoziția dintre [A] și [antiA] — la Principiul Neutropoziției Dinamice — care înseamnă opoziții între [A], [antiA] și [neutA]. Etimologic, "neutropoziție" înseamnă "opoziție neutrosofică".[4]

Acest stil de raționament nu este o schemă neutrosofică, ci se bazează pe realitate, deoarece dacă o idee (sau noțiune) [A] apare, atunci multiple versiuni ale acestei idei sunt răspândite, să le notăm cu $[A]_1$, $[A]_2$, ..., $[A]_m$. Ulterior, ideile opuse (într-un grad mai mic sau mai mare) se nasc, ca reacții la [A] și versiunile sale $[A]_i$. Să notăm aceste versiuni ale opuselor cu $[antiA]_1$, $[antiA]_2$, ..., $[antiA]_n$. Neutralitatea [neutA] dintre aceste idei contradictorii poate îmbrățișa diverse forme, să le notăm cu $[neutA]_1$, $[neutA]_2$, ..., $[neutA]_p$, unde *m, n, p* sunt numere întregi mai mari sau egale cu 1.

În general, pentru fiecare [A] pot exista multe [antiA]-uri corespunzătoare și multe [neutA]-uri. De asemenea, fiecare [A] poate fi

[3] Dezvoltarea raționamentului clasic despre dovezi, cunoscut ca teză-antiteză-sinteză în dialectică, este atribuită filosofului Georg Wilhelm Friedrich Hegel (1770-1831) și mai târziu a fost folosită de Karl Marx (1818-1883) și Friedrich Engels (1820-1895). Despre teză și antiteză au scris și Immanuel Kant (1724-1804), Johann Gottlieb Fichte (1762-1814) și Thomas Schelling (născut în 1921-2016). În timp ce în filozofia chineză antică opusele *yin* [feminin, luna] și *yang* [masculin, soarele] erau considerate complementare.

[4] Smarandache, Florentin (2015). „Thesis-Antithesis-Neutrothesis and Neutrosynthesis." *Neutrosophic Sets and Systems* 8, p. 57. Disponibil online: https://digitalrepository.unm.edu/nss_journal/vol8/iss1/9. Accesat: 29 decembrie 2024.

interpretat în multe versiuni diferite ale [A]-urilor.

Dinamicitatea neutrosofică înseamnă interacțiunile dintre toate aceste multi-versiuni ale [A]-urilor cu multi-[antiA]-urile și multi-[neutA]-urile lor, care vor rezulta într-o nouă teză, să o numim [A'] la un nivel superior. Și un nou ciclu de [A'], [antiA'] și [neutA'] își reîncepe dinamicitatea neutrosofică.

6.3. Exemplu practic

Să luăm în considerare un scenariu în care [A] reprezintă o țară angajată în război cu o altă țară, pe care o putem eticheta [antiA] datorită poziției sale antagonice față de [A]. Între timp, un grup de țări neutre, denumite [neutA], pot interveni în conflict, fie susținând, fie opunându-se uneia dintre părțile beligerante în diverse măsuri.

Unele dintre aceste țări neutre ([neutA]) ar putea rămâne complet neutre pe tot parcursul războiului. Cu toate acestea, există o interacțiune continuă între cele trei grupuri—[A], [antiA] și [neutA]—pe măsură ce țările își pot schimba alianțele (trecând de la o coaliție la alta) sau se pot retrage complet din orice coaliție.

Acest exemplu simplificat evidențiază rolul adesea trecut cu vederea al [neutA] în conflictul dintre forțele opuse, [A] și [antiA], un rol pe care cadrele dialectice tradiționale nu reușesc să îl abordeze. Drept urmare, sinteza dialectică este extinsă într-un concept mai larg cunoscut sub numele de neutrosinteză, o abordare neutrosofică care integrează teza, antiteza și neutroteza.

6.4. Exemplu teoretic

Imaginați-vă [A] ca o școală filosofică, cu [antiA] reprezentând școala sa de gândire opusă. În dezbaterea dintre [A] și [antiA], filosofii din ambele tabere conflictuale pot apela la idei din diverse școli filosofice neutre ([neutA])—școli care nu sunt aliniate nici cu [A], nici cu [antiA]—pentru a-și întări argumentele împotriva viziunii opuse.

7. Aplicații ale neutrosofiei

Principiile neutrosofiei rezonează în multiple discipline. Să alegem câteva dintre ele.

7.1. Mecanica cuantică

Incertitudinea inerentă sistemelor cuantice se aliniază cu viziunea neutrosofică. În loc să căutăm probabilități deterministe, neutrosofia ne permite să modelăm stările cuantice ca un amestec de adevăr, falsitate și nedeterminare, capturând natura ambiguă a fenomenelor precum superpoziția.

7.2. Inteligența Artificială

Logica neutrosofică îmbunătățește IA, permițând sistemelor să raționeze cu informații incomplete, contradictorii sau incerte, îmbunătățind procesele de luare a deciziilor și adaptabilitatea.

7.3. Filosofie și Sociologie

Explorând interacțiunea dintre idei și opusele lor, neutrosofia promovează dialogul și înțelegerea în domenii unde ideologiile concurente se ciocnesc adesea.

7.4. Literatură și Artă

Neutrosofia oferă instrumente pentru a interpreta straturile de semnificație din operele creative, unde ambiguitatea și paradoxul sunt adesea teme centrale.

8. Concluzie: spre o filosofie a echilibrului

Neutrosofia reprezintă nu doar un cadru teoretic, ci și o mișcare care redefinește modul în care abordăm cunoașterea și înțelegerea. Ne invită să îmbrățișăm ambiguitatea, să explorăm spațiile dintre extreme și să recunoaștem interacțiunea dintre opuse și neutralități. Formalizând și extinzând ideile clasice, neutrosofia oferă o lentilă prin care putem reinterpreta complexitățile lumii. Într-o lume marcată de nedeterminare, neutrosofia oferă o cale de urmat. Ne provoacă să transcendem gândirea binară, să explorăm continuumul de posibilități dintre extreme și să căutăm echilibrul în fața complexității. Neutrosofia nu este doar un studiu al neutralității, ci și un apel la acțiune — un apel la regândirea, reinterpretarea și redefinirea înțelegerii noastre asupra adevărului, falsității și spectrului vast de neutralitate dintre ele.

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Neutrosofia, generalizare a Dialecticii. Noțiuni fundamentale

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Abstract
În acest articol este prezentată o nouă ramură a filosofiei, denumită *neutrosofie* [1995], care studiază originea, natura și sfera neutralităților, precum și interacțiunile acestora cu diferite spectre ideaționale. Neutrosofia este o generalizare a dialecticii. In timp ce dialectica studiaza doar dinamica contrariilor, neutrosofia studiaza dinamica atat a contrariilor cat si a neutralelor dintre ele in orice domeniu: filozofic, literar, stiintific, artistic, social etc.
Un exemplu elementar: doua tari care bat la razboi constituie domeniile <A> si <AntiA>, dar unele tari neutre <NeutA> intervin de o parte sau de alta.
Teza fundamentală: Orice idee <A> este T% adevărată, I% nedeterminată și F% falsă, unde T, I, F sunt submulțimi standard sau non-standard incluse în $\| \ ^{-}0, 1^{+} \|$. Tema fundamentală: Fiecare idee <A> tinde să fie neutralizată, diminuată, echilibrată de idei <NonA> {nu numai <AntiA>, precum in dialectica (lui Hegel si Marx)} - ca o stare de echilibru.
<NonA> este format din idei neutre <NeutA> si opuse <AntiA> lui <A>.
Neutrosofia este baza logicii neutrosofice, o logică cu valori multiple care generalizează logica fuzzy si intuitionistice fuzzy, a mulțimii neutrosofice care generalizează mulțimea fuzzy si intuiționistica fuzzy, și a probabilității neutrosofice și a statisticii neutrosofice, care generalizează probabilitatea imprecisa și, respectiv, statistica clasică.

Cuvinte cheie: analiză non-standard, număr hiper-real, infinitezimal, monadă, binadă, interval unitar real non-standard, operații cu numere standard și non-standard

Neutrosophic theory: fundamentals

Abstract
This article presents a new branch of philosophy, called *neutrosophy* [1995], which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectrums. Neutrosophy is a generalization of Dialectics. While dialectics only studies the dynamics of opposites, neutrosophy studies the dynamics of both opposites and neutrals between them in any field: philosophical, literary, scientific, artistic, social, etc.
An elementary example: two countries at war constitute the domains <A> and <AntiA>, but some neutral countries <NeutA> intervene on one side or the other.
Fundamental theme: Any idea <A> is T% true, I% indeterminate, and F% false, where T, I, F are standard or non-standard subsets included in $]^{-}0, 1^{+}[$. Fundamental theory: Every <A> idea tends to be neutralized, diminished, balanced by <NonA> ideas {not only <AntiA>, as the dialectics (by Hegel and Marx)} - as a state of equilibrium.
<NonA> is composed from neutral <NeutA> and opposed <AntiA> ideas to <A>.
Neutrosophy is the basis of neutrosophic logic, a multiple-valued logic that generalizes fuzzy logic, neutrosophic set generalizing fuzzy set, and neutrosophic probability and neutrosophic statistics, which generalizes the classical and imprecise probability and the classical statistics, respectively.

Keywords: non-standard analysis, hyper-real number, infinitesimal, monad, binad, non-standard real unit interval, operations with standard and non-standard numbers

Note: This article also appeared in English: Florentin Smarandache, "Neutrosophy, A New Branch of Philosophy". Multiple-Valued Logic (MVL). An International Journal, Taylor & Francis Group, United Kingdom and United States, ISSN 1023-6627, Vol. 8, No. 3, pp. 297-384, June 2002

A. Etimologie

Neutro-sofie [< fr. *neutre* < lat. *neuter*, neutru + gr. *sophia*, calificare / înţelepciune] înseamnă **cunoaştere a gândirii neutre**.

B. Definiție

Neutrosofia este o nouă ramură a filosofiei, generalizare a dialecticii, care studiază originea, natura și domeniul de aplicare a neutralităților, precum și interacțiunile acestora cu diferite spectre ideaționale.

C. Caracteristici

Acest mod de gândire:

- propune noi teze, principii, legi, metode, formule, mişcări filosofice;
- arată că lumea este plină de nedeterminare;
- interpretează neinterpretabilul;

- tratează din unghiuri diferite concepte şi sisteme vechi, aratând că o idee care este adevărată într-un sistem de referinţă dat poate fi falsă în altul — şi invers;
- încearcă să atenueze războiul de idei şi să se războiască cu ideile paşnice;
- măsoară stabilitatea sistemelor instabile, şi instabilitatea sistemelor stabile.

D. Metode de studiu neutrosofic

Matematizare (logica neutrosofică, probabilitatea neutrosofică şi statisticile neutrosofice, dualitate), generalizare, complementaritate, contradicţie, paradox, tautologie, analogie, reinterpretare, asociere, interferenţă, aforistic, lingvistic, transdisciplinaritate.

E. Formalizarea

Să notăm cu **<A>** *o idee*, sau o propoziţie, o teorie, un eveniment, un concept, o entitate; cu **<>** *ceea ce nu este <A>* şi cu **<AntiA>** *opusul lui <A>*. De asemenea, <***NeutA***> simbolizează *ceea ce nu este nici <A> nici <AntiA>*, şi anume neutralitatea dintre cele două extreme. Iar <A'> este *o versiune a lui <A>*. Să notăm că <> este diferit de <AntiA>.

De exemplu:

Dacă <A> = alb, atunci <AntiA> = negru (antonim),

— dar <NonA> = verde, roşu, albastru, galben, negru etc. (orice culoare, mai puțin albul),

— în timp ce <NeutA> = verde, roşu, albastru, galben etc. (orice culoare, cu excepţia albului şi negrului),

— iar <A'> = alb închis, alb lucios etc. (orice nuanţă de alb).

Într-un mod clasic:

$$<NeutA> \equiv <Neut(AntiA)>,$$

adică neutralităţile lui <A> sunt identice cu neutralităţile lui <AntiA>.

$$<NonA> \supset <AntiA> \text{ şi } <NonA> \supset <NeutA>,$$

precum şi $<A> \cap <AntiA> = \emptyset$,

$<A> \cap <NonA> = \emptyset$,

ori <A>, <NeutA> şi <AntiA> sunt disjuncte două câte două.

<NonA> este completarea lui <A> cu privire la mulţimea universală.

Dar, pentru că în multe cazuri frontierele dintre noţiuni sunt vagi, imprecise, este posibil ca <A>, <NeutA>, <AntiA> (şi bineînţeles <NonA>) să aibă părţi comune, două câte două.

F. Principiul fundamental

Între o idee <A> şi opusul ei <AntiA> există un spectru continuu de putere a neutralităţilor <NeutA>.

G. Teza fundamentală

Orice idee <A> este T% adevărată, I% nedeterminată şi F% falsă, unde $T, I, F \subset \| \ ^{-}0, 1^{+} \|$.

H. Legi principale

Să considerăm un atribut <α> şi $T, I, F \subset \| \ ^{-}0, 1^{+} \|^{3}$. Atunci:

— Există o propoziţie <P> şi un sistem de referinţă {R}, astfel ca <P> să fie T% <α>, I% nedeterminat sau <Neut-α> şi F% <Anti-α>.

— Pentru orice propoziţie <P> există un sistem de referinţă {R}, astfel ca <P> să fie T% <α>, I% nedeterminat sau <Neut-α> şi F% <Anti-α>.

— <α> este într-un anumit grad <Anti-α>, în timp <Anti-α> este într-un anumit grad <α>.

Prin urmare:

— Pentru fiecare propoziţie <P> există sisteme de referinţă $\{R_1\}$, $\{R_2\}$, ..., astfel că <P> arată diferit în fiecare dintre ele — obţinând toate gradele posibile de la <P>, la <NonP>, până la <AntiP>.

— Şi, ca o consecinţă, pentru oricare două propoziţii <M> şi <N> există două sisteme de referinţă $\{R_M\}$ şi respectiv $\{R_N\}$, astfel că <M> şi <N> arată la fel. Sistemele de referinţă sunt oglinzi de curburi diferite care reflectă propoziţiile.

I. Motto-uri

TOTUL ESTE POSIBIL, CHIAR ŞI IMPOSIBILUL!

NIMIC NU ESTE PERFECT, NICI CHIAR PERFECŢIUNEA!

J. Teoria fundamentală

Orice idee <A> tinde să fie neutralizată, diminuată, echilibrată de idei <NonA> (nu numai <AntiA>, cum a susținut Hegel) - ca o stare de echilibru. Între <A> şi <AntiA> există infinit de multe idei <NeutA>, care pot echilibra <A> fără a fi necesare versiuni <AntiA>.

Pentru a neutraliza o idee, trebuiesc descoperite toate cele trei laturi ale sale: de sens (adevărul), de nonsens (falsitate) şi de imprecizie (nedeterminare) – apoi trebuie inversate / combinate. Ulterior, ideea va fi clasificată ca neutralitate.

K. Delimitarea de alte concepte şi teorii filosofice

1. Neutrosofia se bazează nu numai pe analiza de propoziţii opuse, aşa cum face *dialectica*, ci de asemenea pe *analiza neutralităţilor* dintre ele.

2. În timp ce *epistemologia* studiază limitele cunoaşterii şi ale raţionamentului, neutrosofia trece de aceste limite şi ia sub lupă nu numai caracteristicile definitorii şi condiţiile de fond ale unei entităţi <E> – dar şi tot spectrul <E'> în legătură cu <NeutE> .

Epistemologia studiază contrariile filosofice, de exemplu <E> versus <AntiE>; neutrosofia studiază <NeutE> versus <E> şi versus <AntiE>, ceea ce înseamnă logică bazată pe neutralităţi.

3-4. *Monismul neutru* afirmă că realitatea ultimă nu este nici fizică, nici mentală.

Neutrosofia constă într-un punct de vedere pluralist: o infinitudine de nuanţe separate şi ultime conturează lumea.

5. *Hermeneutica* este arta sau ştiinţa interpretarii, în timp ce neutrosofia creează idei noi şi analizează o gamă largă de câmp ideatic prin echilibrarea sistemelor instabile şi dezechilibrarea sistemelor stabile.

6. *Philosophia Perennis* spune adevărul comun al punctelor de vedere contradictorii, neutrosofia combină neutraliile cu adevărul.

7. *Falibilismul* atribuie incertitudine fiecărei clase de convingeri sau propoziţii, în timp ce neutrosofia acceptă afirmaţii 100% adevărate precum şi afirmaţii 100% false, iar, în plus, verifică în care sisteme de referinţă procentajul incertitudinii se apropie de zero sau de 100.

L. Limitele filosofiei

Întreaga filosofie este un tautologism: adevăr în virtutea formei, pentru că orice idee lansată pentru prima oară este dovedită ca adevărată de către iniţiatorul(ii) său(i). Prin urmare, filosofia este goală sau dezinformativă, şi reprezintă a priori cunoaşterea.

Se poate afirma: TOTUL ESTE ADEVĂRAT, CHIAR ȘI FALSUL!

Şi totuşi, întreaga filosofie este un *nihilism*: pentru că orice idee, odată dovedită adevarată, este mai târziu dovedită ca falsă de către urmaşi. Este o contradicţie: fals în virtutea formei. Prin urmare, filosofia este supra-informativă şi o cunoaştere a posteriori.

Astfel, se poate afirma: TOTUL ESTE FALS, CHIAR ŞI ADEVĂRUL!

Toate ideile filosofice care nu au fost încă contrazise vor fi mai devreme sau mai târziu contrazise deoarece fiecare filosof încearcă să găsească o breşă în sistemele vechi. Chiar şi această nouă teorie (care sunt sigur că nu este sigură!) va fi inversată. Iar mai târziu alţii o vor reconsidera din nou...

Prin urmare, filosofia este **logic necesară** şi **logic imposibilă**. Agostino Steuco (alias Agostinus Steuchus sau Eugubinus) a avut dreptate, *diferenţele dintre filosofi sunt de nediferenţiat*.

Expresia lui Leibniz <adevarat în orice lume posibilă> este de prisos, peiorativă, întrucat mintea noastră poate construi de asemenea o lume imposibilă, care devine posibilă în imaginaţia noastră.

— ÎN ACEASTĂ TEORIE NU SE POATE DOVEDI NIMIC!

— ÎN ACEASTĂ TEORIE NU SE POATE NEGA NIMIC!

(F. Smarandache, "Sisteme de axiome inconsistente", 1995.)

A filosofa = tautologism + nihilism.

M. Clasificarea ideilor

a) acceptate cu uşurinţă, uitate repede;

b) acceptate cu uşurinţă, uitate greu;

c) acceptate greu, uitate repede;

d) acceptate greu, uitate greu.

Şi versiuni diferite între orice două categorii.

N. Evoluţia unui idei

Evoluţia unui idei <A> în lume nu este ciclică (dupa cum a afirmat Marx), ci discontinuă, înnodată, fără margini:

- **<Neut-A>** = fond ideatic existent, înainte de apariţia lui <A>;
- **<Pre-A>** = o pre-idee, un precursor al lui <A>;
- **<Pre-A'>** = Spectru de versiuni <PreA>;
- **<A>** = Ideea în sine, care dă naştere implicit la:
- **<NonA>** = ceea ce este în afara lui <A>;
- **<A'>** = Spectru de versiuni <A> după interpretări / înţelegeri (greşite) de către persoane, şcoli, culturi diferite;
- **<A/NeutA>** = Spectru de derivate /deviaţii <A>, deoarece <A> se amestecă parţial mai întâi cu idei neutre;
- **<AntiA>** = Opusul direct al <A>, dezvoltat în interiorul lui <NonA>;
- **<AntiA'>** = Spectru de versiuni <AntiA> după interpretări / înţelegeri (greşite) de către persoane, şcoli, culturi diferite;
- **<AntiA/NeutA>** = Spectru de derivate /deviaţii <AntiA>, ceea ce înseamnă <AntiA> şi <NeutA> parţial combinate în procentaje diferite;
- **<A'/Anti-A'>** = Spectru de derivate /deviaţii după amestecarea spectrelor <A'> şi <AntiA'>;
- **<PostA>** = După <A>, o post-idee, o concluzie;
- **<PostA'>** = Spectru de versiuni <PostA>;

- **<NeoA>** = <A> reluat într-un mod nou, la un alt nivel, în condiţii noi, ca într-o curbă iregulată cu puncte de inflexiune, în perioade evolutive şi involutive, într-un mod recurent; viaţa lui <A> re-începe.

"Spirala" evoluţiei, a lui Marx, este înlocuita cu o curbă diferenţială complexă, cu urcuşurile şi coborâşurile sale, cu noduri - pentru că evoluţia înseamnă şi cicluri de involuţie.

Aceasta este *dina-filosofia* = studiul drumului infinit al unei idei.

<NeoA> are o sferă mai largă (incluzând, în afară de părţi vechi ale lui <A> şi părţi ale lui <NeutA> rezultate din combinaţii anterioare), are mai multe caracteristici, este mai eterogen (după combinaţii cu diferite idei <NonA>). Dar, <NeoA>, ca un întreg în sine, are tendinţa de a-şi omogeniza conţinutul şi apoi de a dez-omogeniza prin alăturarea cu alte idei.

Şi aşa mai departe, până când <A>-ul anterior ajunge la un punct în care încorporează în mod paradoxal întregul <NonA>, fiind nedesluşit de ansamblu. Şi acesta este punctul în care ideea moare, nu poate fi distinctă de altele.

Întregul se destramă, pentru că mişcarea îi este caracteristică într-o pluralitate de idei noi (unele dintre ele conţinând părţi din orginalul <A>), care îşi încep viaţa lor într-un mod similar. Ca un imperiu multinaţional.

Nu este posibil să se treacă de la o idee la opusul său, fără a trece peste un spectru de versiuni ale ideii, de abateri sau de idei neutre între cele două.

Astfel, în timp, <A> ajunge să se amestece cu <NeutA> şi <AntiA>.

Nu am spune că "opusele se atrag", ci <A> şi <NonA> (adică interiorul, exteriorul şi neutrul ideii).

Prin urmare, teoria lui Hegel este incompletă: o teză este înlocuită de alta, numită anti-teză; contradicţia dintre teză şi anti-teză este depăşită şi astfel rezolvată printr-o sin-teză.

Deci Socrate la început, sau Marx şi Engels (materialismul dialectic).

Nu este un sistem triadic:

— teză, antiteză, sinteză (hegelieni);

sau:

— afirmaţie, negaţie, negaţie a negaţiei (marxişti);

ci un *sistem piramidal pluradic*, aşa cum se vede mai sus.

Antiteza <AntiT> a lui Hegel şi Marx nu rezultă pur şi simplu din teza <T>. <T> apare pe un fond de idei preexistente şi se amestecă cu ele în evoluţia sa. <Anti-T> este construit pe un fond ideatic similar, nu pe un câmp gol, şi foloseşte, în construcţia sa, nu numai elemente opuse <T>, dar şi elementele de <NeutT>, precum şi elemente de <T >.

Căci o teză <T> este înlocuită nu numai de către o antiteză <AntiT>, dar şi de diferite versiuni ale neutralităţilor <NeutT>.

Am putea rezuma astfel: **teză-neutră** (fond ideatic înainte de teză), **pre-teză**, **teză**, **pro-teză**, **non-teză** (diferită, dar nu opusă), **anti-teză**, **post-teză**, **neo-teză**.

Sistemul lui Hegel a fost purist, teoretic, idealist. De aceea, a fost necesară generalizarea acestui sistem. De la *simplism* la *organicism*.

N. Formule filosofice

De ce există atât de multe şcoli filosofice distincte (chiar contrare)?

De ce, concomitent cu introducerea unei noţiuni <A>, rezultă și inversul ei, <NonA>?

În cele ce urmează, sunt prezentate câteva formule filosofice, pentru că în domeniul spiritual este foarte dificil să obţii formule (exacte).

a. Legea echilibrului

Cu cât <A> creşte mai mult, cu atat scade <AntiA>. Relația este următoarea:

$<A>\cdot<AntiA> = k\cdot<NeutA>$,

unde k este o constantă care depinde de <A>, iar <NeutA> este un punct de sprijin pentru echilibrarea celor două extreme.

În cazul în care punctul de sprijin este centrul de greutate al neutralităţilor, atunci formula de mai sus este simplificată:

$<A>\cdot<AntiA> = k$,

unde k este o constantă care depinde de <A>.

Cazuri particulare interesante:

Industrializare × Spiritualizare = constant, pentru orice societate.

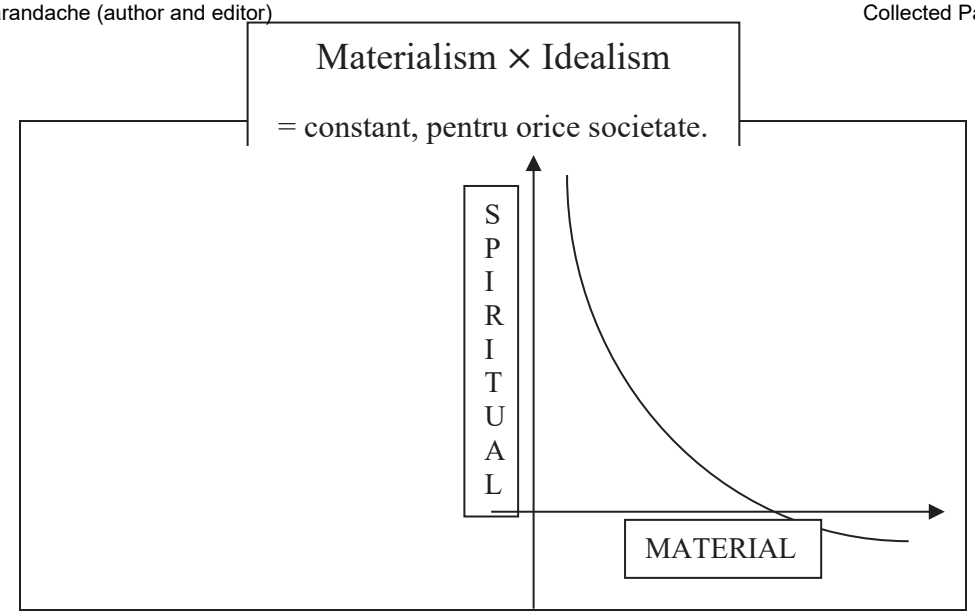
Cu cât o societate este mai industrializată, cu atât scade nivelul spiritual al cetăţenilor săi.

- Ştiinţa × Religie = constant.
- Alb × Negru = constant.
- Plus × Minus = constant.

Împingînd limitele, în alte cuvinte, calculând în spaţiul absolut, se obţine:

Totul × Nimic = universal constant sau $\infty \times 0 (= 0 \times \infty)$= universal constant.

Ne îndreptăm către o matematizare a filosofiei, dar nu în sens platonician.



Axele carteziene verticale şi orizontale sunt asimptote pentru curba M·I =*k*.

b. Legea anti-reflexivitate

<A> în oglindă cu <A> dispare treptat. Sau <A>-ul lui <A> se poate transforma într-un <A> distorsionat.

Exemple:

- Căsătoria între rude dă naştere la descendenţi anoşti (de multe ori cu handicap).
- De aceea, amestecând specii de plante (şi uneori, rase de animale şi oameni), obţinem hibrizi cu calităţi şi / sau cantităţi mai bune (Teoria biologică a amestecării speciilor).
- De aceea, emigrarea este benignă întrucat aduce sânge proaspăt într-o populaţie statică.
- Nihilismul, înțeles ca negare absolută, propovăduit după romanul "Părinţi şi copii" de Turgheniev în 1862, neagă totul, prin urmare, se neagă pe sine!
- Dadaismul dadaismului face să dispară fundamentele dadaismului.

c. Legea complementarității

<A> simte nevoia să se completeze prin <NonA> cu scopul de a forma un întreg.

Exemple:

- Persoanele diferite simt nevoia să se completeze reciproc şi să se asocieze (bărrbatul cu femeia.)
- Culorile complementare (care, combinate la intensităţile potrivite, produc albul).

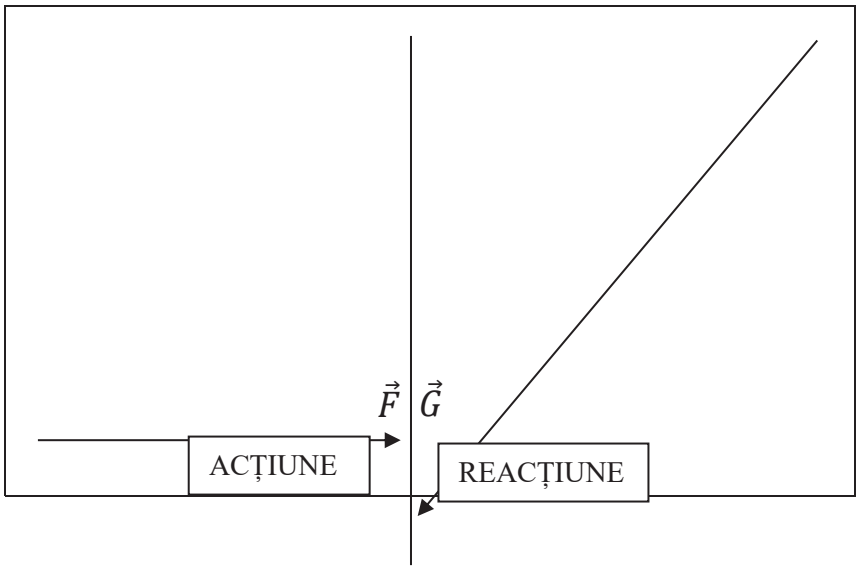
d. Legea efectului invers

Atunci când se încerceacă convertirea cuiva la o idee, credinţă, sau religie prin repetiţii plictisitoare sau prin forţă, acea persoana ajunge să urască ceea ce îi este impus.

Exemple:

- Cu cât rogi pe cineva să facă ceva, cu atât persoana vrea mai puţin să o facă.
- Dublând regula, ajungi la înjumătăţire.
- Ce e mult, nu e bine ... (invers proporţional).
- Când ești sigur, nu fii!

Atunci când forţăm pe cineva să facă ceva, persoana va avea o reacţie diferită (nu necesar opusă, precum afirma axioma legii a treia a mişcării a lui Newton):



e. Legea identificării întoarse

<NonA> este un <A> mai bun decât <A>.

Exemplu:

- Poezia este mai filosofică decât filosofia.

f. Legea întreruperii conectate

<A> şi <NonA> au elemente în comun.

Exemple:

- Există o distincţie vagă între "bine" şi "rău".

- Raţionalul şi iraţionalul funcţionează împreună inseparabil.
- În mod similar, conştiinţa şi inconştienţa.
- Finitul este infinit [vezi microinfinitatea].

"Vino, sufletul mi-a spus, haide să scriem poezii pentru corpul meu, căci suntem Unul" (Walt Whitman).

g. Legea întreruperii de identităţi

Lupta permanentă între <A> şi <A'> (unde <A'> sunt diferite nuanţe de <A>).

Exemple:

- Lupta permanentă între adevărul absolut şi adevărul relativ.
- Distincţia dintre falsul clar şi falsul neutrosofic (cea de doua noţiune reprezintă o combinaţie de grade de falsitate, nedeterminare şi adevăr).

h. Legea compensaţiei

Dacă acum <A>, atunci mai târziu <NonA>.

Exemple:

- Orice pierdere are un câștig [ceea ce înseamnă că mai târziu va fi mai bine, pentru că ai învăţat ceva din pagubă].
- Nu există niciun succes fără eşec [aveţi răbdare!].

i. Legea condiţiei prescrise

Nu pot fi depăşite limitele proprii. (Ne învârtim în cercul propriu.)

j. Legea gravitaţiei particular-ideaţionale

Fiecare idee <A> atrage şi respinge altă idee <B> cu o forţă direct proporţională cu produsul măsurilor lor neutrosofice şi exponenţialul distanţei lor.

(În opoziţie cu reafirmarea modernă a Legii lui Newton cu privire la gravitaţia particulelor de materie, distanţa influenţează direct proporţional - nu indirect: cu cât sunt ideile mai opuse / distanţate, cu atât se atrag mai puternic.)

k. Legea gravitaţiei universal-ideaţionale

<A> tinde către <NonA> (nu către <AntiA> cum a afirmat Hegel), şi reciproc.

Există forţe care acţionează asupra lui <A>, orientând-o către <NonA>, până când un punct critic este atins, iar apoi <A> se întoarce.

<A> şi <NonA> sunt în continuă mişcare, iar limitele lor se schimbă în consecinţă.

Exemple:

Perfecţiunea duce la imperfecţiune.

Ignoranţa este mulţumitoare.

Caz particular:

Fiecare persoană tinde să se apropie de nivelul său de… incompetenţă!

Aceasta nu este o glumă, ci purul adevăr:

Să spunem că *X* obţine un loc de muncă la nivelul N_1;

— dacă este bun, este promovat la nivelul N_2;

— dacă este bun în noua sa poziţie, este promovat mai departe la N_3;

— şi aşa mai departe ... până când el nu mai este bun şi prin urmare, nu mai este promovat;

— astfel, el a ajuns la nivelul său de incompetenţă.

<A> tinde catre <NonA>.

Prin urmare, idealul fiecăruia este de a tinde către ceea ce nu poate face.

Dar mişcarea este neliniară.

<Non-A> dispune de o gamă largă (continuum de putere), de versiuni care "nu sunt <A>" (<A> exterior); să le indexăm în mulţimea $\{<NonA>_i\}_i$.

(Toate versiunile $\{<AntiA>_i\}_i$ sunt incluse în <NonA>.)

Prin urmare, infinit de multe versiuni $<Non\text{-}A>_i$ gravitează, precum planetele în jurul unui astru, pe orbitele lui <A>. Şi între fiecare versiune $<NonA>_i$ şi centrul de greutate al "astrului" <A>, există forţe de atracţie şi de respingere. Se apropie una de alta până când se ajunge la anumite limite critice minime: $P_{m(i)}$ pentru <A> şi $Q_{m(i)}$ pentru $<NonA>_i$ şi apoi se depărtează una de cealaltă până la atingerea anumitor limite maxime: $P_{M(i)}$ pentru <A> şi $Q_{M(i)}$ pentru $<NonA>_i$.

Prin ecuaţii diferenţiale putem calcula distanţele minime şi maxime (spirituale) dintre <A> şi $<NonA>_i$, coordonatele carteziene ale punctelor critice şi status quo-ul fiecărei versiuni.

Am putea spune că <A> şi o versiune $<AntiA>_i$ se întâlnesc într-un punct absolut / infinit.

Când toate versiunile $<AntiA>_i$ cad sub categoria <A> … avem o catastrofă!

Concluzie

Pentru că lumea este plină de nedeterminare, este necesară o imprecizie mai precisă. De aceea, în acest articol este introdus un nou punct de vedere în filosofie, care ajută la generalizarea ‘teoriei

probabilităţilor', 'mulţimii fuzzy', şi 'logicii fuzzy' la < probabilitate neutrosofică>, <mulţime neutrosofică> , şi respectiv <logică neutrosofică>. Acestea sunt utile în domeniul inteligenţei artificiale, reţelelor neuronale, programării evoluţionare, sistemelor neutrosofice dinamice, mecanicii cuantice ș.a.

Acesta este un articol introductiv fundamental al FILOSOFIEI NEUTROSOFICE; un întreg colectiv de cercetători ar trebui să treacă prin toate școlile/mișcările/tezele/ideile filosofice și să le extragă trăsăturile pozitive, negative și neutre. Filosofia este supusă interpretării. Aceasta este o propedeutică, o primă încercare a unui astfel de tratat. O filosofie neutrosofică exhaustivă - dacă este posibilă - ar trebui să fie o sinteză a filosofiilor din toate timpurile în interiorul unui sistem neutrosofic.

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Logica neutrosofică

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Ion Nălbitoru – Prof. univ. dr. Florentin Smarandache, S.U.A., "Logica neutrosofică,"
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Prof. univ. dr. Florentin Smarandache de la Universitatea New Mexico din S.U.A. s-a născut în 1954 în localitatea Bălcești, județul Vâlcea și este un savant de renume mondial, din păcate mai puțin cunoscut în România datorită faptului că mijloacele mass media promovează nonvaloarile în detrimentul adevăratelor valori românești cu care să ne mândrim în lume. Prof. univ. dr. Florentin Smarandache este omul care încă din timpul liceului l-a contrazis pe Einstein, afirmând că există viteze mai mari decât viteza luminii. Cercetările de bază ale domniei sale au fost axate mai mult pe matematică. De asemenea poate fi considerat „părintele paradoxismului", dar și al dezvoltării Logicii neutrosofice.

Deși a plecat din țară de peste trei decenii nu și-a uitat locurile natale, își iubește patria și colaborează cu scriitori și oameni din știință români publicând la diverse reviste din România. A colindat foarte mult pe toate continentele, a ținut conferințe și a fost prezent la tezele de doctorat în multe state din America, Europa și Asia. Am de la dânsul, în colecția mea, două volume, în colaborare cu scriitorul și publicistul Andrușa Vătuiu din Orșova, prietenul nostru comun, „Evoluție Neutrosofică Umană în Spirală sau Divinul este în Om" și „Mai ușor să distrugi din interior decât din exterior".

Venit de curând în România pentru o lună de zile, am fost surprins plăcut, și o mare onoare pentru mine, să fiu contactat de dânsul și invitat, împreună cu soția, să ne întâlnim la Craiova. La această întrevedere a fost prezentă și distinsa doamnă Adriana Răducan membră a Uniunii Scriitorilor Români și a Uniunii Ziariștilor Profesioniști din România, ocazie cu care, printre alte proiecte culturale discutate și invitația de a publica la revista dânsului din America, m-a anunțat că va publica o carte la editura mea, „FRANCO", și va pregăti o nouă antologie cu scriitori din țară. A acceptat să-mi acorde un interviu pe care îl redau în cele ce urmează.

Ion Nălbitoru: Domnule profesor, ce v-a determinat să plecați din România și cum ați reușit într-o perioadă când regimul comunist avea o securitate puternică și știa tot ce se mișcă în țară?

Prof. Univ. Dr. Florentin Smarandache: Ce m-a determinat să plec? Aveam dosar la securitate! Mult mai târziu am descoperit, prin Consiliul Național de studiere a Arhivelor Statului, că aveam de fapt cinci 5 dosare care însumau vreo 880 de pagini. Eram numit „Obiectivul Savu". Deci eram pe lista neagră a securității, îmi pierdusem și locul de muncă, pentru că lucram la Întreprinderea de Utilaj Greu din Craiova și acolo era un securist care mă urmărea. Nu aveam voie să trimit articole de matematică în afară, nu era nimic politic. Trimisesem unul în Franța, în franceză, un articol de matematică, dar n-a ajuns fiindcă l-a oprit securitatea. Am fost luat la interogatoriu de ce am trimis fără să le cer lor aprobarea. Și așa am

rămas sub observația lor. Dacă rămâneam în Romania aș fi fost distrus. Nu a știut nimeni când am plecat, nici soția. După ’89 am reușit să-mi aduc și familia în State.

Am plecat din România în 1988 și am trecut ilegal granița în Turcia unde am cerut azil politic. Aici am stat aproape doi ani, am lucrat în diverse munci necalificate. Locuiam într-un Champs câteva sute: jumătate români, jumătate polonezi, un număr mai mic de bulgari, cehi și parcă și un rus și mai veniseră și etiopieni. Din Turcia am emigrat după aproape doi ani de zile deoarece era foarte multă lume care trebuia să emigreze în America. Doar trei țări ne acceptau din Turcia: Statele Unite, Canada și Australia. Totdeauna eu am optat numai pe Statele Unite. Pentru că stătusem prea mult în Istanbul, am frecventat Institutul Cultural Francez. Vorbeam franceză curent, pentru că în țară învățasem franceza și latina, nu engleza. Ei știau că absolvisem universitatea și puteam să lucrez în învățământ sau în cercetare în Franța. Francezii mi-au dat viza înaintea celei americane. Eram nedecis și asta m-a făcut să mai aștept un pic și mi-a venit și viza americană și astfel am emigrat în S.U.A.

I.N.: De ce totuși ați ales America și nu Canada, Australia, Franța sau Germania?

F.S.: Avantajul plecării în America a fost că aceasta era privită ca „centrul lumii” și ce făceai acolo se auzea mai tare decât în România. În plus toate lucrările și cărțile științifice se scriau și se scriu în engleză. Deci voiam să mă simt liber.

I.N.: Ați menționat că limba franceză o stăpâneați perfect dar nu și engleza. Cum ați reușit să vă descurcați într-o țară a cărei limbă nu o cunoșteați? De asemenea cum ați reușit să vă adaptați, apoi să deveniți profesor universitar, venind dintr-o țară cu un regim totalitar și pășind în alta considerată metaforic „Țara tuturor posibilităților”?

F.S.: La început mi-a fost dificil, trebuia să ne adaptăm cu limba și noul stil de viață. Învățasem puțină engleză din articolele pe care trebuia să le traduc cu dicționarul, dar mai slab cu gramatica, și le trimiteam la diverse reviste. Am avut un profesor turc, în Turcia, care ne-a predat limba engleză la cei care voiam să emigrăm. Dar în Turcia învățasem și ceva turcește pentru că eram nevoit să lucrez. Eram prieten și cu români din Dobrogea care erau de origine turcă sau tătară și ei știau bine turcește și așa am învățat să vorbesc, să citesc și să scriu. În America, pentru că engleza mea era destul de slabă, am lucrat ca inginer de soft la Compania Hanuel din Phoenix, statul Arizona, S.U.A. Acolo mi-am perfecționat engleza, am continuat să citesc și să scriu multe jurnale: Jurnal de lagăr, Jurnal american.

După aceea am trecut în învățământ și am predat la un colegiu din Arizona. În cele din urmă am găsit normă întreagă la Universitatea New Mexico, unde am lucrat 25 de ani. Ieșirea la pensie mi-a permis să-mi desfășor din plin activitatea științifică. Universitatea îmi plătea să merg la conferințe internaționale dar și la cele naționale din State unde prezentam lucrări. Apoi au fost multe teze de doctorat și sunt încă în special în India, Turcia, China, Statele Unite, Franța etc.

I.N.: Vă rog, spuneți-mi ceva despre Teoria Dezert-Smarandache.

F.S.: Am colaborat cu francezii, fiindcă știam franceza, am făcut mai mult matematică aplicată, deoarece este importantă pentru industrie, pentru economie în general. Astfel, împreună cu Jean Dezert am dezvoltat Teoria Dezert – Smarandache care se aplică și la drone, se aplică la fuziunea informațiilor. În fiecare an mergeam la conferințe cu Jean și cu ceilalți francezi. După aceea au fost teze de doctorat în Franța și Olanda despre teoria noastră.

I.N.: Am informații că ați călătorit foarte mult prin lume, vă rog să ne dezvăluiți câte ceva despre aceste călătorii și care era scopul lor?

F.S.: Am făcut foarte multe excursii în întreaga lume și am scris memorii de călătorie, am făcut și video. În călătoriile mele mă interesa cultura, activitatea științifică, obiceiurile oamenilor din țările respective. Am vizitat peste 60 de țări, dintre care cea mai exotică a fost excursia în Antarctica și ca urmare am și scris un volum „Un oltean în Antarctica", care a avut un mare succes! Am fost și la cercul polar de nord, în Groenlanda, am fost și la ecuator în Ecuador, la latitudinea „zero", dar și în Europa în special pentru conferințe.

Am ținut în permanență legătura cu scriitori și oameni de știință din România. Și acum colaborez la diverse reviste literare din țară cu poeme, eseuri, impresii de călătorie, proză, iar din punct de vedere matematic am fost la Academia Română unde mi-au dat un premiu pentru înființarea logicii neutrosofice care a ajuns la nivel internațional.

I.N.: Și pentru că am ajuns la neutrosofie după cum bine știți, la volumul „Evoluție Neutrosofică Umană în Spirală sau Divinul este în Om" (Human Neutrosophic Evolution în Spiral or The Divine is in the Man)- autori Florentin Smarandache & Andrușa R. Vătuiu - pe care l-am citit cu mare interes, am făcut o cronică care a fost preluată de mai multe publicații cultural-literare din țară dar și din străinătate: „Observatorul" și „Destine literare"din Canada, „Radio metafora" din S.U.A. și „Orientul Românesc" din Liban. Pentru cititorii acestui interviu doresc să ne dați câteva informații, despre Logica neutrosofică

F.S.: Sunt foarte multe teze de doctorat în China, India și Turcia. Turcii mi-au și tradus cărți de știință în limba turcă. Logica neutrosofică se bazează pe 3 nivele: grad de adevăr, grad de fals și grad neutru sau nedeterminat între adevăr și fals. Aceasta de fapt devine o extindere a paradoxismului care se baza pe contradicții, deci pe extreme, dar am observat că și partea de mijloc contează, adică partea neutră contează fie de o parte, fie de alta. De exemplu în prezent este războiul din Ucraina, deci opuse sunt Ucraina și Rusia iar celelalte țări neutre contribuie cu ajutor unele pentru Ucraina, altele pentru Rusia. Dar sunt și țări pur și simplu neutre. Deci eu am adus ceva nou în sensul că și partea neutră intervine de o parte sau de alta în contradicțiile sau luptele dintre opuse, extreme.

De fapt și Dialectica lui Hegel și Marx au apărut la vreo două mii de ani și ceva după teoria antică chineză Yin și Yang care se baza pe pozitiv-negativ, pământ-cer, femeie-bărbat ș.a.m.d. Deci eu am adus o îmbunătățire acestei teorii. La fel cum și teoria evoluției a lui Darwin m-a influențat când am fost în Insulele Galapagos să observ faptul că nu avem numai grade de evoluție pentru adaptare ale unor funcții sau ale unor organe dar și grade de involuție. Adică în timp ce unele evoluează, altele involuează. Am dat exemplu de la niște păsări pe care ni le-au arătat pe Insulele Galapagos care pe continentul American puteau să zboare dar după ce s-au mutat pe insulă nu mai puteau să zboare deoarece nu mai aveau nevoie de zbor, fiindcă găseau mâncarea în apă. Și atunci și-au dezvoltat labele de la picioare cu care să prindă peștele, deci asta a fost evoluție a labelor, în schimb și-au pierdut capacitatea de a mai zbura, deci asta a fost involuție. Și bineînțeles că există și partea de mijloc, adică au rămas la fel cum erau și pe continent, partea neutră, adică să facă ouă, să le clocească, să-și crească puii, deci la fel sunt și pe continent și pe insulă. Deci parte evolutivă cum a zis Darwin dar și parte involutivă, dar și cea neutră.

Un exemplu din ziua de azi este cu cosmonauții care petrecând mai mult timp în spațiu li se atrofiază mușchii și în momentul când se întorc pe pământ nu mai pot să calce cum trebuie, acesta ar fi un proces de involuție că nu mai pot să meargă și trebuie să se adapteze și să revină la normal. Apoi s-a mers mai departe pe rafinarea acestei logici, mulțimi sau probabilități

neotrosofice în mai multe părți că și părțile de adevăr, de fals sau de neutru pot fi de mai multe feluri și am făcut sub adevăr unu, sub adevăr doi s.a.m.d. Am dezvoltat teoria refinată neotrosofică, probabilistică, statistică. În statistică am introdus partea de nedeterminare pentru că lucrăm în viața de toate zilele cu date care nu sunt exacte, sunt aproximative, deci partea de indeterminare, partea mai puțin cunoscută, partea conflictuală, toate acestea au contribuit la dezvoltarea de noi metode în statistică pentru a studia anumite fenomene.

La Universitatea New Mexico a însemnat lansarea mea pe plan internațional. Faptul că am putut să merg la foarte multe conferințe internaționale unde mi-am exprimat ideile a însemnat mult pentru mine. Am avut posibilitatea să intru în legătură cu foarte mulți colaboratori de pe glob din vreo 89 de țări. Sunt foarte multe teze de doctorat în special din India de unde mi le trimit să le evaluez, din Turcia, din China.. În Statele Unite a fost prima teză de doctorat la Universitatea din Atlanta, Georgia, despre Logica și mulțimea neotrosofică, apoi mi-am îmbunătățit această teorie.

Omul care l-a contrazis pe Einstein

I.N.: Domnule profesor, încă din anii de liceu ați contrazis teoria relativității a lui Einstein și ați afirmat că există viteză mai mare ca a luminii. Această teorie ați susținut-o și în America. Și vă întreb acest lucru pentru că deja au fost efectuate două experimente în acest sens, unul la Chicago și altul în Europa, experimentul CERN unde se pare că teoria dumneavoastră ar fi reală. Este vorba de așa numita „Particula lui Dumnezeu”.

F.S.: Este adevărat că mi-am susținut această ipoteză și în America și încă o susțin. Despre acest lucru am scris și chiar am publicat două cărți când am fost în Maroc, 1982-1984, predam matematica în franceză la un liceu din Sefrou. Specific că în Maroc am fost legal. Era un acord de cooperare între România și Maroc. Am fost trimiși mai mulți profesori de către Ministerul Educației. Acolo fiind mai liber am publicat în limba franceză două cărți în care am spus că poate fi și viteză mai mare decât viteza luminii. Există, au pus online într-o enciclopedie de fizică această ipoteză, ipoteza Smarandache, dar este criticată pentru că ea contrazice anumite teorii ale relativității și ei consideră că nu este corectă. Însă m-am folosit tot din fizica cuantică de particulele înlănțuite care au legătură între ele, în sensul că dacă măsori o particulă ai imediat informații și despre particula cealaltă. Atunci am zis așa: O.K.! Păi dacă se întâmplă treaba asta înseamnă că informația a circulat foarte repede dintr-o parte în alta, instantaneu aș spune, și putem considera particulele la orice distanță una de alta. De aici am ajuns la concluzia că poți să construiești orice viteză chiar și mai mare decât viteza luminii. De când eram elev la liceu, la Râmnicu Vâlcea, ni se spunea că viteza luminii plus viteza luminii este egal tot viteza luminii ceea ce mi se părea ceva nerealist matematic. Este ca și cum ai afirma 1c + 1c = 1c, unde c = viteza luminii, ceea ce, matematic, poate fi adevărat doar dacă c=0 , sau pozitiv, infinit sau negativ. Am mai scris și o carte în care am scris paradoxuri care generează teoria relativității, și o teorie care generează paradoxuri nu poate fi o teorie reală. După o perioadă de timp, când am ajuns în America, am reactivat ideile.

Am avut o prezentare și la Caltech la Institutul din California, din Statele Unite, și am mai avut apoi prezentări la Shan Hai, în China, despre această teorie și sunt destui care suportă vitezele supraluminale. Apoi m-am gândit cum s-a dezvoltat o fizică supraluminală, cum ar reacționa legile fizicii subluminale, adică la viteze normale, cum s-ar manifesta ele la viteze supraluminale și plus viteza instantanee. Desigur că pentru așa ceva mi-ar trebui un laborator

unde să pot experimenta. M-am interesat și în California despre astfel de laboratoare dar mi s-a spus că se plătesc foarte scump. Poate suta de mii de dolari ca să poată să-ți facă un experiment pe care-l vrei tu. Deci nu am mai continuat cercetările acestea, dar cărțile sunt online și din ce în ce mai mulți cercetători consideră că viteza luminii poate fi depășită. Sau făcut și unele demonstrații la nivel sub cuantic, că ar fi viteze supraluminale .

Într-adevăr un experiment efectuat la Centrul European de Cercetare Nucleară (C.E.R.N.) a dovedit că viteza era supraluminală, s-a și publicat acest lucru, dar probabil au fost presiuni politice ca apoi să spună că a fost un cablu rupt, sau așa ceva, și se anulează acel experiment. Deci mulți consideră că a fost o chestiune politică pentru a nu contrazice alte c„argumente". Nici eu n-am mai lucrat în această direcție dar nu din acea cauză. Ajungând în America am fost întrebat ce matematică am făcut și am spus că am lucrat mai mult teoria numerelor Și m-au întrebat unde au aplicație acestea. Le-am zis că ar avea ceva aplicație în criptografie, adică să încifrezi un mesaj, adică să nu poată fi descifrat de cei care-l recepționează. Atunci mi-au zis să fac ceva care are aplicație, pentru că americanii fiind pragmatici mi-au zis să fac ceva practic. Și tot lucrând am observat că logica neutrosofică, mulțimea neutosofică, probabilitatea neutrosofică, statistica neutrosofică au multe aplicați. Și într-adevăr a fost o explozie și foarte mulți au văzut importanța ei. Pe chestia asta am primit și ceva bani să merg la conferințe, am primit și multe invitații ca să lucrez în cercetare. Am fost în China trei luni pentru cercetare, în Japonia o lună cercetare, în Franța trei luni cercetare. Și am primit salariul de cercetător.

I.N.: De unde prietenia cu domnul Andrușa Vătuiu și publicarea de cărți în colaborare cu domnia sa?

F.S.: Domnul Andrușa Vătuiu este de profesie inginer ca și dumneavoastră, dar în domenii diferite. În general inginerii sunt mai pragmatici, Am primit un mesaj scris pe e-mail de la dânsul și așa cum am colaborat cu mai mulți la revistele din România, inclusiv cu dumneavoastră la „Memoria Slovelor" a L.S.R. din Vâlcea, între noi a început o colaborare mai intensă. Dânsul a venit cu idei care m-au impresionat. De exemplu a venit cu ideea că se poate distruge mai ușor din interior decât din exterior și mi-a dat ca exemplu, oul și puiul din el. Mi-a zis că puiul de găină, care este atât de fragil, poate foarte ușor să spargă oul din interior, dar din exterior se sparge mai greu. Asta a fost scânteia! Așa am ajuns să colaborez cu oameni din diverse domenii și din ceea ce știu ei și ceea ce știu eu să iasă ceva bun. Așa am ajuns la concluzia că un sistem se poate distruge mai ușor din interior decât din exterior și am scris împreună cartea „Mai ușor să distrugi din interior decât din exterior" (Easier to break from inside than from outside)

I.N.: Am și eu acest volum foarte interesant. Când l-am lecturat am făcut conexiunea cu multe fapte reale din zilele noastre. De exemplu Imperiul rusesc, fosta URSS, a fost distrus din interior de către Gorbaciov și nu de o forță externă, de exemplu de americani prin purtarea unui război. La fel regimul comunist din România, regimul Ceaușescu, a fost distrus nu prin intervenția în forță a altui stat din afară ci chiar din interior, de către Iliescu care era apropiatul președintelui. De asemenea distrugerea industriei, agriculturii, zootehniei, a economiei românești în general și pierderea suveranității asupra bogățiilor solului și ale subsolului României nu s-a datorat unei intervenții în forță din afara granițelor, ci prin trădarea celor din interior a celor aflați, pe rând, la guvernare, și exemplele pot continua.

F.S.: Acum lucrăm iar la ceva, că-mi trimite revista lui și citesc acolo despre ciclicitate, care de fapt a găsit-o nu știu pe unde și i-am zis: domnule asta o putem dezvolta. Scria într-un articol că multe fenomene se repetă, revin pe altă treaptă, cu schimbări, ca într-o spirală. Așa că am discutat cu dânsul și poate facem ceva și în direcția asta.

I.N.: În studenție am scris un roman realist, „Cascada tinereții" și chiar piese de teatru dintre care una se numește „La poarta destinului", tot studențească, și ambele au o încărcătură psihologică. La un moment dat, două personaje vorbesc despre acest ciclu care se repetă dar pe o scară valorică superioară, într-o spirală. Asta fiind acum patru decenii și jumătate. Dar se pare că realitatea ne demonstrează această ciclicitate în spirală și când afirm acest lucru m-aș referi la o paralelă între primitivii de peste zece mii de ani când pentru existență mâncau lăcuste, gândaci, viermi și societatea omenească actuală de la începutul mileniului trei care începe să mănânce făină din gândaci și viermi, deci o revenire la începuturi, dar pe o treaptă superioară, în spirală. Și în afară de ciclicitate apare și celălalt fenomen despre care ați vorbit anterior, evoluție sau involuție?

F.S.: Deci nimic nu este întâmplător! Să vă revedeți pasajul respectiv din carte și dacă dezvoltați ceva și veniți cu idei noi putem colabora toți trei! Împreună cu domnul Vătuiu vom încerca să abordăm științific această problemă a ciclicității. Eu am mai abordat și alte domenii.

Am folosit teoria mea în psihologie, am scris și o carte, cu temperamente: coleric, sangvinic, melancolic și flegmatic. De fapt la om este câte un procent din fiecare, nu găsești de exemplu o persoană doar colerică. Și o persoană melancolică poate fi scoasă din sărite și să devină brusc colerică. Apoi am studiat și publicat și în sociologie. Mă întrebam ce este de fapt democrația. Democrație în sensul greu al cuvântului nu există. Pentru că ei zic că se aplică dreptatea sau legea majorității și asta ar fi bine să se întâmple, dar în realitate nu majoritatea decide, ci decide un grup mic și asta nu mai este democrație, indiferent unde ar fi. Am văzut că unele țări au un grad mai mare de democrație iar altele mai mic, dar și faptul că fiecare are și un grad de totalitarism. La fel ca și la logica neutrosofică, în democrație sau totalitarism, în realitate este câte un procent din fiecare.

I.N.: Trăind în democrație, mai puțin noi în România, mă bucur de libertate, dar asta nu înseamnă că pot să fac ce vreau, de exemplu să dau în cap la cineva că așa vreau eu. Și în democrație trebuie respectate cu strictețe niște principii prin care să nu-l deranjez sau să-l lezez pe semenul meu. Dacă într-o societate fiecare face ce vrea atunci nu mai avem democrație ci anarhie. Termenul de democrație perfectă este o utopie. Ea are în ea și un minim de dictatură.

F.S.: Așa este, de aceea ziceam că există câte un procent din fiecare, și bineînțeles o zonă neutră. Deci nu există democrație pură! Aplicând la teoria neutrosofică avem: grad de democrație, grad de totalitarism, de legi fixe, și ceva neutru la care toate țările au cam același lucru. De exemplu la creștini ai doar o soție, iar la musulmani poți avea două sau mai multe. De exemplu în India legea căsătoriei, în aceeași țară musulmanii au dreptul la patru neveste iar indușii doar la una. Deci se face o diferențiere religioasă și pe treaba asta am dezvoltat structuri cărora le-am zis neutrostructuri, adică legile nu se aplică la fel la toată lumea. Că sunt și aici, de exemplu unii oameni sunt deasupra legii, cei care au putere, iar în cei săraci toate legile dau în ei. La alții depinde și de avocați, pentru că pentru aceeași faptă comisă unii iau ani de pușcărie mai mulți, alții mai puțini. Am vrut să spun că legile nu se aplică la fel la toți.

I.N.: Este o vorbă românească despre lege că este ca o barieră: unii sar peste ea, unii trec pe sub ea, iar proștii dau cu capul în ea.

F.S.: Din păcate aceasta este realitatea. Legea este ca o barieră peste care trec dulăii, pe sub care trec cățeii și de care se împiedică boii. Aș dori să vorbesc în câteva cuvinte despre Logica booleană de la George Boole, britanic. El a făcut niște algebre în care folosea doar două numere, zero și unu, cu care făcea operatori logici și care acum chiar se folosesc în circuite. Zero înseamnă circuit deschis iar unu circuit închis prin care trece curentul. Îi zice logica booleană, algebră booleană care se dezvoltă. Din unele lucruri abstracte mai târziu se poate descoperi că au aplicație.

I.N.: Cum vedeți dumneavoastră România din zilele noastre?

F.S.: Întotdeauna când revin în România o fac cu plăcere pentru că îmi revăd locurile natale, unde am copilărit, învățat, trăit și profesat, prieteni, rude, deci pentru mine este ca un festival. Desigur că România s-a schimbat foarte mult, nu mai recunoști orașele, nici aici în Craiova, fiindcă s-a modernizat mult. Din păcate am citit și foarte multe știri negative despre România, despre pierderea resurselor naturale. Să sperăm că România se va reface, situația este complicată pentru că se găsește și între dușmani, nu știi cu cine să te aliezi. Să sperăm că va fi mai bine, că se va dezvolta, că va avea un grad mai mare de suveranitate și dezvoltare economică.

I.N.: Domnule profesor Florentin Smarandache, vă mulțumesc pentru amabilitatea de a-mi acorda acest interviu care, sper, va apărea în mai multe publicații din țară și străinătate. De asemenea vă mulțumesc pentru propunerea pe care mi-ați făcut-o de a colabora cu dumneavoastră pe viitor!

Neutrosofía y Plitogenia: fundamentos y aplicaciones

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Smarandache, F. "Neutrosofía y Plitogenia: fundamentos y aplicaciones." *Serie Científica de la Universidad de las Ciencias Informáticas*, Vol. 17, No. 8, Mes: Agosto, 2024, Pág. 164-168.

Carta al editor

Estimado editor de la revista Serie Científica de la Universidad de las Ciencias Informáticas. Dirijo a usted esta carta resumen, de una conferencia sobre Neutrosofía y Plitogenia, impartida hace algún tiempo en Perú, por este autor.

La Neutrosofía es una rama de la filosofía introducida por Florentin Smarandache en 1980. Estudia la neutralidad, es decir, la falta de pertenencia o no pertenencia a un conjunto. Plitogenia, también creada por Smarandache, es una generalización de la dialéctica y la neutrosofía. Se basa en la idea de que la realidad es más compleja que la simple dicotomía de verdadero o falso.

Antecendentes:

Los antecedentes de la Neutrosofía parten de la lógica de dos valores, ya sea aristotélica o clásica simbolista; y maneja un tipo dado de proposiciones que puede caracterizarse como idealista:

- El principio del tercero excluido establece que cualquier afirmación es verdadera o falsa, sin espacio para una tercera opción.
- El principio de no contradicción establece que una proposición no puede ser verdadera y falsa al mismo tiempo y en el mismo sentido.
- Los principios de no contradicción y del tercero excluido pueden considerarse falsos en algunas situaciones.

La dialéctica occidental elimina siempre lo anterior para avanzar hacia nuevas realidades. En la dialéctica andina, esos opuestos se complementan, un elemento depende del otro para existir. Las culturas precolombinas de las regiones de influencia andina tenían su propia dialéctica, una cosmovisión resultante de la aplicación de un principio de oposición

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http://publicaciones.uci.cu Vol. 17, No. 8, Mes: Agosto, 2024, Pág. 164-168

complementaria. La civilización andina presenta una percepción del mundo en términos no excluyentes y no binarios. La idea Aymara de algo que es y no es al mismo tiempo (entidades **ch'ixi**) están, son poderosas porque son indeterminadas. En la Filosofía Oriental las expresiones contradictorias forman el núcleo de las doctrinas Taoísta y Zen / Chan (que surgió del budismo y el taoísmo).

Uno de los pioneros a destacar es el Filósofo peruano José Francisco Miró-Quesada Cantuarias:



Filósofo: José Francisco Miró-Quesada Cantuarias

Vida: Lima-Perú, 21 de diciembre de 1918 - 11 de junio de 2019

Contribución:

Término Introducido: Lógica Paraconsistente

Descripción: Teoría que permite manejar y trabajar con situaciones y opiniones contradictorias.

Continuando con los antecedentes, cualquier proposición en cualquier sistema de axioma matemático formal representará la verdad (T), la falsedad (F) y la indeterminación (I) de la declaración bajo consideración.

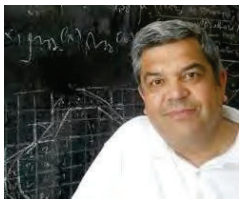
Definición de Neutrosofía

La Neutrosofía es una nueva rama de la filosofía la cual estudia el origen, naturaleza y alcance de las neutralidades, así como sus interacciones con diferentes espectros ideacionales (1995). La neutrosófica abrió un nuevo campo de investigación en la metafilosofía. Etimológicamente *neutron-sofía* [Frances *neutre* < Latin *neuter*, neutral, y griego *sophia*, conocimiento] significa conocimiento de los pensamiento neutrales y comenzó en 1995. Constituye la base para la lógica neutrosófica, los conjuntos neutrosófica, la probabilidad neutrosófica, y la estadística neutrosófica.

Antecedentes de la Neutrosofía



1965: Conjunto difuso introducido por L. Zadeh

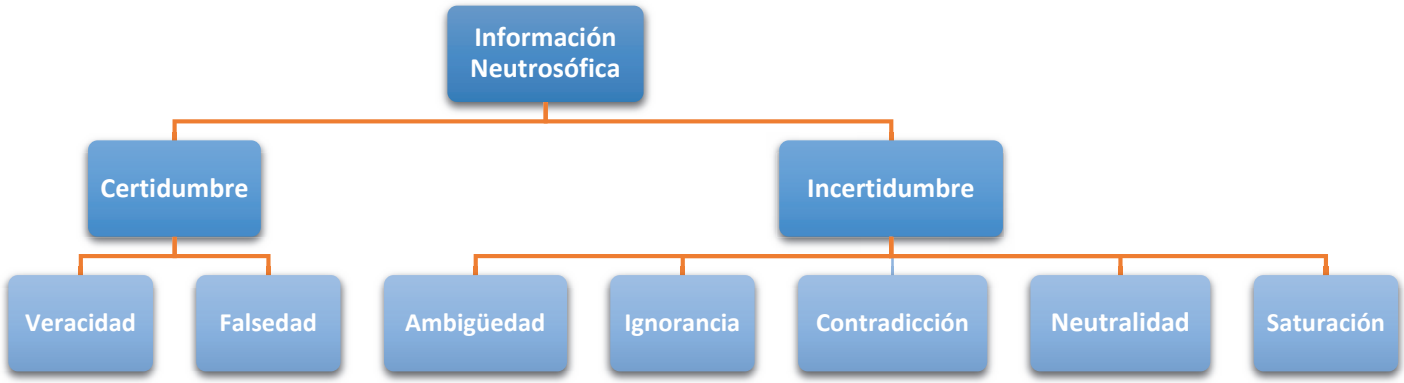


1986: conjunto difuso intuicionista por K. Atanassov



1995: Conjunto neutrosófico por Florentin Smarandache

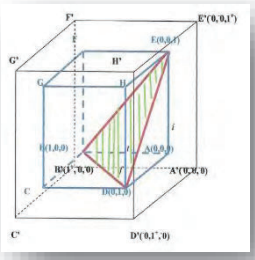
Estructura de la Información Neutrosófica



Conjunto Neutrosófico

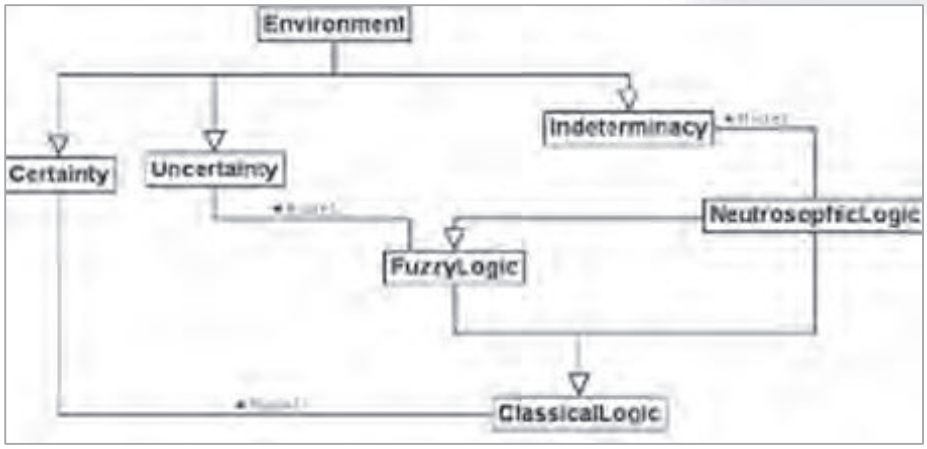
En términos neutrosóficos, podemos definir la pertinencia de un elemento a un conjunto mediante tres valores:

- T (Grado de Verdad): Indica qué porcentaje de pertenencia es verdadero.
- I (Grado de Indeterminación): Indica el grado de indeterminación o incertidumbre.
- F (Grado de Falsedad): Indica qué porcentaje de pertenencia es falso.



Lógica Neutrosófica

Una lógica en la cual cada proposición se estima tener un porcentaje de verdad en un subconjunto T, un porcentaje de indeterminación en un subconjunto I y un porcentaje de falsedad en un subconjunto F.



Ejemplo de un Conjunto Neutrosófico aplicado a una Lesión Deportiva

Conjunto Neutrosófico: Pertinencia de Luis para regresar al campo después de una lesión.

- **T Grado de Verdad**: 60%
  - Esto indica que hay un 60% de certeza de que Luis está físicamente apto para regresar al campo, basado en sus exámenes médicos, rehabilitación y rendimiento en entrenamientos recientes.

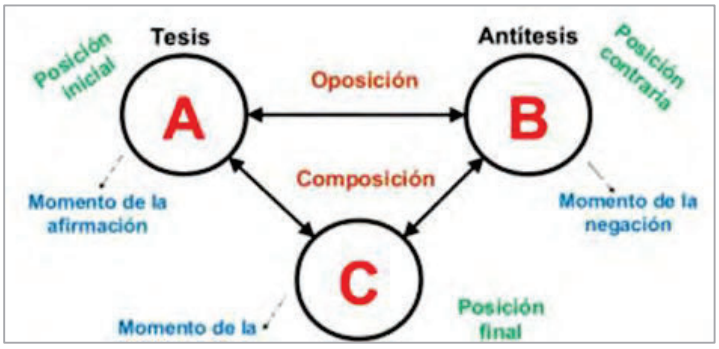
- **I Grado de Indeterminación**: 30%
    - Esto refleja la incertidumbre debido a factores como la posible recaída, su estado emocional, o el impacto de una recuperación apresurada.
- **F Grado de Falsedad**: 10%
    - Esto representa un 10% de certeza de que Luis aún no está listo para regresar al campo, posiblemente debido a observaciones del fisioterapeuta o a una evaluación médica que sugiere precaución.

Representación:

- **Luis (60%, 30%, 10%)** en relación con el conjunto "apto para regresar al campo".

Método de Investigación Neutrosófico

Construye un campo unificado de la lógica para un estudio transdisciplinario que traspase las fronteras entre las ciencias naturales y sociales. La neutrosofía trata de resolver los problemas de indeterminación que aparecen universalmente, con vistas a reformar las ciencias actuales, naturales o sociales, con una metodología abierta para promover la innovación.



Fuente: Smarandache, F., & Liu, F. (2004). *Neutrosophic dialogues*. Infinite Study ; Smarandache, F. (2010). *Multispace & Multistructure. Neutrosophic Transdisciplinarity (100 Collected Papers of Science)* (Vol. 4). Infinite Study.

- El método de investigación neutrosófico es una generalización de la dialéctica de Hegel abordando que la ciencia no solo avanzará tomando en consideración las ideas contrarias sino también las neutrales.
- Su teoría fundamental afirma que toda idea < A > tiende a ser neutralizada, disminuida, balaceada por las ideas (no solo como Hegel planteó.
- <no A> = lo que no es <A>, <antiA> = lo opuesto a <A>, y <neut A>= los que no es ni <A> ni <antiA>.

Nuevos avances

- **Plitogenia:** es la génesis u originación, creación, formación, desarrollo y evolución de nuevas entidades a partir de dinámicas y fusiones orgánicas de entidades antiguas múltiples contradictorias y / o neutrales y / o no contradictorias.

- **MultiAlismo**: es enfoque filosófico o conceptual que promueve la interacción dinámica y la coexistencia de múltiples perspectivas, categorías o sistemas de pensamiento



Asociación Internacional de Ciencia Neutrosófica

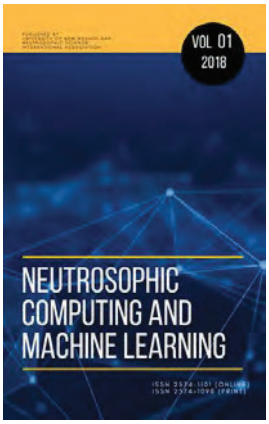
- **Neutrosophic Science International Association:** Secretariat of Neutrosophic Science International Association. President: Prof. Florentin Smarandache, PhD. University of New Mexico, USA. http://fs.gallup.unm.edu/neutrosophy.htm

Asociación Latinoamericana de Ciencias Neutrosóficas

- **Misión:** La Asociación Latinoamericana de Ciencias Neutrosóficas (ALCN) es una organización académica y científica no gubernamental. Se vincula a la Escuela Neutrosófica del pensamiento. Su objetivo principal es promover la ciencia, especialmente las teorías Neutrosóficas y sus aplicaciones. Busca el progreso social a partir de la innovación tecnológica. http://neutrosofia.org/

Revista

Neutrosophic Computing and Machine Learning
http://fs.gallup.unm.edu/NCML/



Introduction to Collective Neutrosophy and Collective n-ary Neutrosophy

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Smarandache, F. (2025). "Introduction to Collective Neutrosophy and Collective n-ary Neutrosophy." *Neutrosophic Sets and Systems*, Vol. 85, 2025, 563-568.

Abstract: This paper introduces the novel concepts of Collective Neutrosophy and Collective n-ary Neutrosophy as extensions of Smarandache's foundational theories of Neutrosophy and n-ary Neutrosophy. While classical Neutrosophy generalizes traditional dialectical thinking by incorporating neutralities into the dynamics of opposites, the proposed collective forms address interactions at the group level, where multiple agents and ideas simultaneously engage in opposition, cooperation, or neutrality. Applications in geopolitics, philosophy, corporate ecosystems, and cultural studies are discussed, alongside a theoretical framework for modeling complex relational dynamics. Diagrams illustrating single, collective, and collective n-ary neutrosophic structures are proposed to visualize these concepts.

Keywords: Neutrosophy; n-ary Neutrosophy; Collective Neutrosophy; Dialectics; Neutralities; Opposites; Complex Systems; Conflict Dynamics; Group Theory.

1. Introduction

Smarandache's Neutrosophy, introduced in 1998 [1], and its subsequent extension into n-ary Neutrosophy in 2003 [3], represent significant generalizations of dialectical philosophy. Traditional dialectics, such as Hegelian synthesis and the Yin–Yang model from Chinese philosophy, focus on the tension between two opposites. However, these frameworks largely neglect the role of neutrality or indeterminacy, crucial elements that influence the balance between oppositional forces.

Neutrosophy broadens this view by considering not only the dynamics between opposites but also the influence of neutralities, elements that are neither fully aligned with one side nor the other but that affect the outcome of oppositional interactions. We term this foundational model Single Neutrosophy.

This paper introduces two new theoretical expansions:

• *Collective Neutrosophy*: modeling the dynamics between two opposite groups and their collective neutralities.

• *Collective n-ary Neutrosophy*: further generalizing to interactions among multiple ($n \geq 2$) opposite groups and their respective multiple collective neutralities.

These developments allow for the neutrosophic modeling of highly complex systems characterized by multifaceted relationships between ideas, entities, or actors.

2. Foundations of Single and n-ary Neutrosophy

2.1. Single Neutrosophy

In Single Neutrosophy, the basic unit of analysis is the dynamic between two opposing elements (e.g., two people, two ideas), alongside neutral or indeterminate elements that may sway the dynamic [1-4].

Example: Two individuals are in conflict, while third parties—neutral bystanders—may intervene, supporting one side, mediating, or introducing new dynamics.

2.2. Single n-ary Neutrosophy

n-ary Neutrosophy extends this model to n opposing elements ($n \geq 2$), considering interactions and neutralities between each possible pair.

Example: In a multiparty political system, several parties compete simultaneously, with shifting alliances and neutral stances influencing the overall balance of power.

3. Collective Neutrosophy

Collective Neutrosophy advances the neutrosophic model from individual to group dynamics. It studies the opposition and cooperation between groups of elements, along with the collective neutralities or collective indeterminacies that may emerge between or within these groups.

3.1. Structure

- *Groups*: Sets of elements (individuals, ideas, organizations) that share a general alignment.
- *Intergroup Dynamics*: Interactions between entire groups.
- *Intragroup Dynamics*: Internal oppositions within each group.
- *Collective Neutralities*: Neutral groups or forces that influence the intergroup dynamics.

3.2. Applications

- *Geopolitical Conflicts*: Conflicts between nation-states, where neutral countries (e.g., Switzerland in WWII) mediate or subtly influence the balance.
- *Philosophical Systems*: Oppositions between schools of thought (e.g., materialism vs. idealism) and the influence of interdisciplinary or neutral intellectual movements.
- *Business Ecosystems*: Rivalry between corporate groups, moderated by regulatory bodies, open-source communities, or independent developers.

4. Collective n-ary Neutrosophy

Collective n-ary Neutrosophy generalizes Collective Neutrosophy to n groups ($n \geq 2$), where complex, multi-faceted interactions are observed.

4.1. Structure

- *n Groups of Opposites*: $\{G_1, G_2, ..., G_n\}$
- *Intergroup Dynamics*: Competitive, cooperative, or neutral relationships between any pair of groups.
- *Collective Neutralities*: Entities or groups that interact neutrally or mediate among multiple groups simultaneously.

4.2. Applications

- *International Alliances*: NATO, BRICS, ASEAN, and other coalitions interacting with neutral states or organizations like the UN.
- *Multi-platform Technological Ecosystems*: Interaction between ecosystems like Apple, Google, Microsoft, with neutral cross-platform developers or regulatory influences shaping market dynamics.
- *Cultural Movements*: Interaction among multiple art or literary movements, where certain neutral movements (e.g., postmodernism) mediate between classical conflicts.

5. Diagrams and Visual Representations

Several diagrams can aid in understanding these complex relationships:

5.1. Single Neutrosophy Diagram

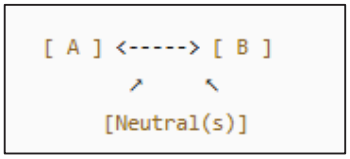


Figure 1. Two nodes (A and B) connected by a dynamic arrow, with neutral elements (N) influencing the arrow's balance. A and B are two opposites (e.g., two people, two ideas); neutral(s) can influence either side or mediate.

5.2. Single n-ary Neutrosophy Network

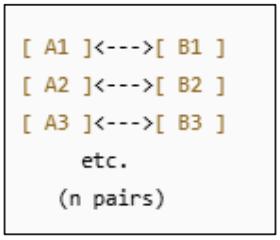


Figure 2.1. Several independent oppositional pairs; multiple nodes (A_1, A_2, ..., A_n) with pairwise connections; neutral nodes positioned between each connected pair.

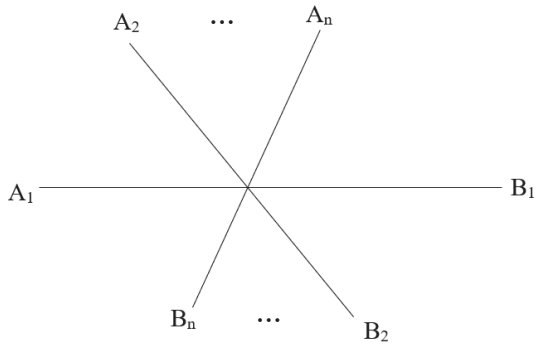


Figure 2.2. Single n-ary Neutrosophy Network.

5.3. Collective Neutrosophy

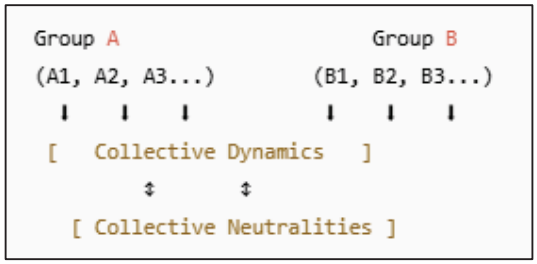


Figure 3.1. Two groups of nodes (Group A, Group B) with internal and external connections; collective neutralities as intermediary nodes between groups.

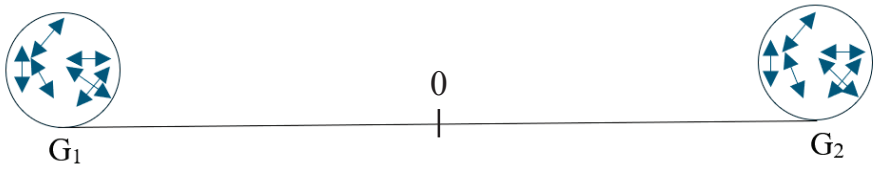


Figure 3.2. Collective Neutrosophy.

5.4. Collective n-ary Neutrosophy Network

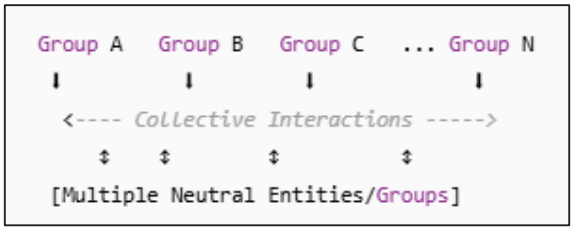


Figure 4.1. Multiple groups with cross-interactions and neutral clusters linking multiple groups.

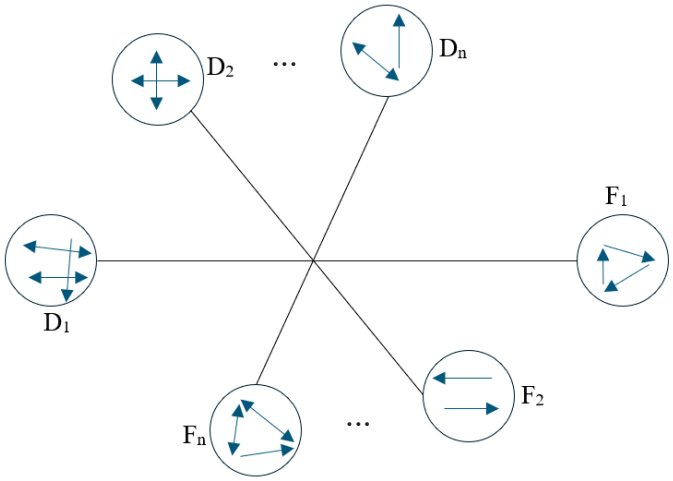


Figure 4.2. Collective n-ary Neutrosophy Network.

6. A Short Comparative Analysis: EU vs. MERCOSUR through Collective Neutrosophy and Collective n-ary Neutrosophy

6.1. Basic Structure

EU is political and economic union of 27 countries with deep integration (common market, customs union, shared policies, and in some cases, common currency - the Euro).

MERCOSUR is a South American trade bloc (Argentina, Brazil, Paraguay, Uruguay, Venezuela - suspended) aiming primarily at free trade and fluid movement of goods, people, currency.

Each bloc is a collective made of multiple nations (not individuals), thus fitting Collective Neutrosophy.

Internal dynamics among member states = internal single neutrosophic tensions.

6.2. Dynamics of Opposites and Neutralities

Aspect	*European Union (EU)*	*MERCOSUR*
Internal Opposites	Differences between northern vs. southern economies; national sovereignties vs. federalism; euro-zone vs. non-eurozone members.	Brazil vs. Argentina economic competition; agricultural vs. industrial interests; political instability vs. democratic consolidation.
Neutralities	Neutral countries (e.g., Austria in military matters); mediators like Germany in conflicts; EU Commission trying to balance East-West tensions.	Paraguay and Uruguay often act as balancing powers between Brazil and Argentina; external neutral actors (e.g., Bolivia trying to mediate trade terms).
External Opposites	EU vs. US/China in global trade; EU vs. Russia geopolitically.	MERCOSUR vs. Pacific Alliance (competing South American trade blocs); MERCOSUR vs. external trade blocs (like the EU itself during free-trade talks).

Neutrosophically, each market isn't just a binary tension but has multiple tensions, neutral mediations, and indeterminacies.

6.3. n-ary Collective Neutrosophic Dynamics

In EU:

• Germany–France–Italy–Poland–Spain: major economic players, each having different priorities.

• Each pair (Germany–Italy, France–Poland, etc.) can have tensions, cooperation, and neutrals.

In MERCOSUR:

• Brazil–Argentina–Paraguay–Uruguay–Venezuela (suspended).

• Similar: Brazil-Argentina rivalry, Paraguay-Uruguay as mediators, Venezuela's instability adding indeterminacy.

6.4. Strength of Neutrality Zones

Bloc	*Internal Neutral Zones*	*External Neutral Interactions*
EU	Countries like Austria, Finland; neutral platforms like European Parliament debates.	EFTA (European Free Trade Association) as semi-neutral economic players.
MERCOSUR	Uruguay and Paraguay trying to mediate disputes.	Chile and Bolivia acting as external neutral or semi-associated countries.

6.5. Remarks

• Collective Neutralities help stabilize each market — but the degree of stabilization differs.

• EU shows stronger structured neutrality mechanisms (formal mediation processes).

• MERCOSUR's neutralities are more informal and sometimes fragile.

6.6. Summary Interpretation

Concept	***EU***	***MERCOSUR***
Collective Dynamics	Highly institutionalized	More political, informal
Internal Neutrosophy	Complex; multilayered (economy, policy, sovereignty)	Primarily economic with some political overlaps
n-ary Dynamics	>5 strong member interactions	3-5 strong, others weaker
Neutralities	Formal, legally enforced (e.g., treaties)	More ad hoc, based on diplomacy
Indeterminacies	Brexit, Eurozone crises, East-West divergence	Political instability (Venezuela), economic crises

References

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[4] Smarandache, Florentin (2015). "Thesis-Antithesis-Neutrothesis, and Neutrosynthesis." *Neutrosophic Sets and Systems* 8:57-58.

La Estadística Neutrosófica es una extensión de la Estadística de Intervalos, mientras que la Estadística Plitogénica es la forma más general de estadística. (Cuarta versión)

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Resumen. En este trabajo demostramos que la Estadística Neutrosófica es una extensión de la Estadística de Intervalos, ya que trata todo tipo de indeterminaciones (respecto a los datos, procedimientos inferenciales, distribuciones probabilísticas, representaciones gráficas, etc.), permite la reducción de indeterminación, y utiliza la probabilidad neutrosófica que es más general que las probabilidades imprecisas y clásicas, y tiene funciones de densidad de probabilidad correspondientes más detalladas.
Mientras que la Estadística de Intervalos sólo se ocupa de la indeterminación que puede representarse mediante intervalos. Y respondemos a los argumentos de Woodall et al. [1].
Mostramos que no todas las indeterminaciones (incertidumbres) pueden ser representadas por intervalos. Además, en algunas aplicaciones, deberíamos usar conjuntos vacilantes (que tienen menos indeterminación) en lugar de intervalos.
Redirigimos a los autores a la Probabilidad Plitogénica y la Estadística Plitogénica que son las formas más generales de Probabilidad Multivariante y Estadística Multivariante respectivamente (incluidas, por supuesto, la Probabilidad Imprecisa y la Estadística de Intervalo como subclases).

Palabras Claves: Estadística, Intervalos, Plitogénica

Abstract. In this paper we show that Neutrosophic Statistics is an extension of Interval Statistics, since it deals with all kinds of indeterminacy (with respect to data, inferential procedures, probability distributions, graphical representations, etc.), allows for indeterminacy reduction, and uses neutrosophic probability which is more general than imprecise and classical probabilities, and has more detailed corresponding probability density functions.
Whereas Interval Statistics only deals with indeterminacy that can be represented by intervals. And we respond to the arguments of Woodall et al [1].
We show that not all indeterminacies (uncertainties) can be represented by intervals. Moreover, in some applications, we should use hesitant sets (which have less indeterminacy) instead of intervals.
We redirect the authors to Plitogenic Probability and Plitogenic Statistics which are the most general forms of Multivariate Probability and Multivariate Statistics respectively (including, of course, Imprecise Probability and Interval Statistics as subclasses).

Keywords: Statistics, Interval, Plitogenic

1. Introducción

Primero, presentamos las distinciones entre la Estadística Neutrosófica y la Estadística de Intervalos y damos ejemplos concluyentes de álgebra neutrosófica que proporcionan más precisión que el álgebra de intervalos.

Posteriormente respondemos a las críticas presentadas por Woodall et al.

La Estadística Neutrosófica se definió por primera vez en 1998 (Libro [1]), se desarrolló en 2014 (Libro [3]), se relacionó con la Probabilidad Neutrosófica (Libro [9]), se conectó y se extendió a otros campos (Libros [2, 4-8]), una tesis doctoral sobre estadística neutrosófica en 2019 (Tesis doctoral [1]), y varios seminarios internacionales [S1-S5], que dieron como resultado una explosión de artículos sobre sus aplicaciones (Artículos [1 – 122]) en muchos campos como como: medicina, biología, economía, administración, informática, ingeniería, etc., en cuanto a la toma de decisiones, coeficiente de rugosidad de la junta de roca, muestreo repetitivo, coeficiente de similitud indeterminado, tamaño de muestra/población indeterminado, individuos que solo pertenecen parcialmente a una muestra/población, media/varianza/desviación estándar indeterminadas, gráficos de control, distribuciones de probabilidad de funciones indeterminadas o gruesas, errores de medición, pruebas o hipótesis bajo incertidumbre/indeterminación, etc.

2. Estadística Neutrosófica vs. Estadística de Intervalos

En este artículo hacemos una comparación entre la Estadística Neutrosófica (EN) y la Estadística de Intervalos (EI). Mostramos que son diferentes y en muchos casos la EN es más general que la EI.

La EN no se reduce a utilizar únicamente números neutrosóficos en aplicaciones estadísticas, como Woodall et al. aseveran, sino que es mucho más amplia. La EN se ocupa de todos los tipos de indeterminación, mientras que la EI se ocupa solo de la indeterminación que puede representarse mediante intervalos.

A continuación, presentamos varias ventajas de aplicar la EN en vez de la EI:

- La Estadística Neutrosófica se basa en el Análisis de Conjuntos, mientras que la Estadística de Intervalos se basa en el Análisis de Intervalos, por lo que la Estadística de Intervalos es un caso particular de la Estadística Neutrosófica (que utiliza todo tipo de conjuntos, no sólo intervalos).
- Los números neutrosóficos numéricos permiten reducir la indeterminación mediante operaciones, mientras que los intervalos aumentan la indeterminación (ver ejemplos que aparecen más abajo).
- No todos los datos inciertos (indeterminados) se pueden representar mediante intervalos como requiere la EI, mientras que la EN es capaz de manejar cualquier tipo de indeterminación.
- La EN puede trabajar con una muestra o población cuyo tamaño no es bien conocido.
- La EN puede operar con una muestra o población que contiene individuos que solo pertenecen parcialmente a la muestra/población y otros cuya pertenencia se desconoce.
- La EN trata con individuos de la muestra o población cuyo grado de pertenencia a la muestra o población puede estar fuera del intervalo [0, 1], como en el overset neutrosófico (grado > 1), underset (grado < 0) y, en general, el offset neutrosófico (ambos grados de pertenencia, > 1 y < 0, para varios individuos).
- Los datos neutrosóficos (o indeterminados) son datos indeterminados vagos, poco claros, incompletos, parcialmente desconocidos y conflictivos.
- La EN también trata con datos neutrosóficos refinados utilizados en Big Data.
- Curvas parcialmente indeterminadas.
- Variable aleatoria neutrosófica, que puede no representarse como una secuencia de intervalo.
- La EN también usa funciones gruesas (como intersecciones de curvas, que pueden no estar representadas por intervalos) como distribuciones de probabilidad.
- La distribución de probabilidad neutrosófica (DPN) de que ocurra un evento (x) está representada por tres curvas: DPN(x) = (T(x), I(x), F(x)), donde T(x) representa la posibilidad que el evento x ocurra, I(x) la posibilidad indeterminada de que el evento x ocurra o no, y F(x) la posibilidad de que el evento x no ocurra. Siendo T(x), I(x), F(x) funciones clásicas o neutrosóficas (poco claras, aproximadas, gruesas), dependiendo de cada aplicación, y para todo x en el espacio de probabilidad neutrosófico dado. La DPN es mejor que las distribuciones de probabilidad clásicas o imprecisas, ya que es una distribución de probabilidad multivariante que presenta más detalles sobre el evento.
- Diagramas, histogramas, pictografías, gráficos de líneas/barras/cilindros, diagramas con datos neutrosóficos (no representados por intervalos).
- Media, varianza, desviación estándar, función de distribución de probabilidad y otros estadígrafos no muy conocidos (o completamente desconocidos)
- Por ejemplo, no es necesario aumentar la incertidumbre extendiendo el conjunto de valores posibles, por ejemplo, {0.2, 3.7, 45.9} al intervalo [0.2, 45.9] para poder utilizar la estadística de intervalo. La EN simplemente emplea el conjunto vacilante finito discreto {0.2, 3.7, 45.9}.
- Los datos cualitativos se representan mediante un conjunto discreto finito de etiquetas neutrosóficas, en lugar de un intervalo de etiquetas.
- No se puede usar la Estadística de Intervalo o Probabilidad (imprecisa) de intervalo para calcular la probabilidad de un dado en una superficie agrietada, o una moneda en una superficie agrietada, o un dado o una moneda defectuosos.

Nos ocupamos de la indeterminación con respecto al espacio de probabilidad o estadística (ya sea la superficie, o el dado, o la moneda), indeterminación con respecto al observador que evalúa el evento, indeterminación con respecto al evento [4].

No se puede aproximar la indeterminación de estos ejemplos utilizando algún intervalo, por lo que se necesita la probabilidad y la estadística neutrosóficas que se ocupan de todo tipo de indeterminaciones.

- En conclusión: no podemos representar todos los tipos de indeterminaciones por intervalos.

En aras de la verdad, responderemos a continuación a las críticas [1].

2.1 Woodallen al. [1] en su sección de Matemáticas Neutrosóficas:

- Las reglas básicas para la aritmética dadas por Smarandache [42, pp. 31-33] no coinciden con las reglas dadas por Zhang et al. [37].
  Smarandache [42] expresó los números neutrosóficos en la forma a + bI, donde a y b son números reales, e I representa el intervalo de indeterminación tal que I^2 = I y 0 · I = 0.

Respuesta:

Esto es falso, ya que, aunque el libro [referencia 2 en este artículo] contiene los números neutrosóficos literales, los cuales nunca se usaron en aplicaciones de la estadística neutrosófica. En cambio, los que se utilizaron siempre fueron números neutrosóficos numéricos.

Los autores deben saber que existen dos tipos de números neutrosóficos de la forma a + bI, donde a, b son números reales (o complejos), mientras que "I" = indeterminación.

(i) Números neutrosóficos literales, cuando "I" es solo una letra, donde I2 = I (porque: indeterminación × indeterminación = indeterminación) y 0·I = 0, que se utilizan en las estructuras algebraicas neutrosóficas, pero no en ningún artículo sobre aplicaciones de la estadística neutrosófica - según mi leal saber y entender.
Los números neutrosóficos literales fueron introducidos y desarrollados por Kandasamy y Smarandache a partir de 2003; véanse varios libros que usan números neutrosóficos literales en estructuras algebraicas neutrosóficas:
WBV Kandasamy, F. Smarandache, mapas cognitivos difusos y mapas cognitivos neutrosóficos, Xiquan, Phoenix, 2003,http://fs.unm.edu/NCMs.pdf
WBV Kandasamy, F. Smarandache, Anillos neutrosóficos, ProQuest Information & Learning, Ann Arbor, MI, EE. UU., 2006,http://fs.unm.edu/NeutrosophicRings.pdf
Etc.

(ii) Números Neutrosóficos Numéricos, donde la indeterminación "I" es un subconjunto real, para aproximar los datos imprecisos. Esto es más general que el intervalo, ya que "I" puede ser cualquier subconjunto.
Por ejemplo, N = 3 + 2I, donde "I" está en el subconjunto vacilante discreto {0.3, 0.9, 6.4, 45.6} de solo cuatro elementos, que no forma parte del análisis de intervalo (estadística). En la estadística de intervalo, se toma el intervalo [0.3, 45.6] para incluir los números anteriores, pero esto aumenta mucho la incertidumbre.
Por supuesto, hay casos particulares cuando el "I" es un intervalo I = [I1, I2],
con , entonces N = a + bI coincide con el intervalo N = [a + b· I1, a + b· I2].

1.2. Woodall et al. [1]:

- Utilizando el enfoque de Zhang et al. [37] y la aritmética de intervalos, sin embargo, el intervalo para el promedio sería [3, 5]. Consideramos que el enfoque de la aritmética de intervalos conduce a resultados mucho más útiles y realistas.

Respuesta:

Woodall et al. armó una confusión, ya que Zhang et al. [referencia 3, en este documento] el documento trata sobre el Conjunto Neutrosófico de Intervalo (fuera del marco de la Estadística Neutrosófica), donde un elemento.

x(T, I, F), de un conjunto neutrosófico dado A, tiene grados de pertenencia/indeterminación/no pertenencia (T, I, F) expresados respectivamente bajo la forma de un intervalo cada uno de ellos; por ejemplo
x([0.7, 0.8], [0.2, 0.3], [0.5, 0.6]).
Por lo tanto, la comparación con la aritmética de intervalos de Zhang et al. es irrelevante con respecto

a la estadística neutrosófica, ya que Zhang et al. sólo usaron su aritmética en los conjuntos neutrosóficos.
Zhang et al. presentó las operaciones clásicas con intervalos porque los necesitan cuando se trata de operaciones de tripletes neutrosóficos. Por ejemplo, la adición de tripletes neutrosóficos
(T1, I1, F1) + (T2, I2, F2) = (T1 + T2 - T1T2, I1I2, F1F2), donde todos los componentes neutrosóficos son intervalos, por lo que se necesitaban sumas, restas y multiplicaciones de intervalos.
Véanse las operaciones de conjuntos neutrosóficos aquí [127].

1.3. Wood et al. [1]

- Así, el número neutrosófico de intervalo [4, 6] podría representarse como 4 + 2I. Smarandache [42] calculó el promedio de dos números neutrosóficos, digamos a + bI yc + dI, como (a + c)/2 + [(b + d)/2] I. Como ejemplo, considere los dos números neutrosóficos [4, 6] y [2, 4] representados como 4+2I y 4−2I, respectivamente.

  Entonces, usando el enfoque de Smarandache [42], el promedio de estos dos números neutrosóficos sería 4 + 0I, o simplemente el valor preciso 4. Este resultado no parece razonable.

Respuesta:

Esto sólo muestra la ventaja de los números neutrosóficos numéricos sobre los intervalos, ya que permiten reducir la indeterminación, mientras que al usar intervalos la indeterminación aumenta.

Por ejemplo:

N1 = 4 + 2I, donde I $\in$ [0, 1], muestra que 2I es la parte indeterminada del número N1, de manera similar para N2 = 4 - 2I. Si los sumamos, las indeterminaciones de N1 y N2 se cancelan, y el promedio es:

$$\frac{1}{2}(N_1+N_2)=\frac{1}{2}(4+2I+4-2I)=\frac{1}{2}(8)=4$$, sin indeterminación;

mientras que, usando intervalos, $N_1=[4,6], N_2=[2,4],$

$$\frac{1}{2}(N_1+N_2)=\frac{1}{2}[6,10]=[3,5],$$ por lo tanto, la indeterminación está entre [0, 2].

1.4. Woodall et al.

- Notamos que Smarandache [42] y otros no se refieren a métodos estadísticos de intervalo a pesar de sus fuertes similitudes con los métodos estadísticos neutrosóficos.

Respuesta

Al principio, en el libro [2], página 5, no hay ninguna referencia al análisis/estadística de intervalo, sino al análisis/estadística de conjuntos que es más general que el análisis/estadística de intervalo:

"En la mayoría de las ecuaciones y fórmulas estadísticas clásicas, simplemente se reemplazan varios números por conjuntos. Y, en consecuencia, en lugar de operaciones con números, se usan operaciones con conjuntos. Normalmente se reemplazan los parámetros que son indeterminados (imprecisos, inseguros e incluso completamente desconocidos)."

Más adelante, se han presentado más citas y comparaciones entre la estadística neutrosófica y la estadística clásica y de intervalos, véase: http://fs.unm.edu/NS/NeutrosophicStatistics.htm

"La Estadística Neutrosófica es también una generalización de la Estadística de Intervalos debido, entre otras cosas, a que, mientras que la Estadística de Intervalos se basa en el Análisis de Intervalos, la Estadística Neutrosófica se basa en el Análisis de Conjuntos (es decir, todo tipo de conjuntos, no solo intervalos, por ejemplo, conjuntos discretos finitos).

Además, al calcular la media, la varianza, la desviación estándar, las distribuciones de probabilidad y otros conceptos estadísticos en la estadística clásica y de intervalo, se asume automáticamente que todos los individuos pertenecen al 100% a la muestra o población respectiva, pero en nuestro mundo, a menudo se encuentran individuos que sólo en parte pertenecen, en parte no pertenecen, y en parte su pertenencia es indeterminada. Los resultados de la estadística neutrosófica son más precisos/reales que la estadística clásica y de intervalo, ya que los individuos que pertenecen solo parcialmente no tienen que ser considerados al mismo nivel que los que pertenecen completamente.

Las Distribuciones de Probabilidades Neutrosóficas pueden ser representadas por tres curvas: una que representa la posibilidad de que ocurra el evento, otra la posibilidad de que no ocurra y una tercera la posibilidad indeterminada de que ocurra o no." Proporcionan más detalles que la estadística clásica y de intervalo.

"La Estadística Neutrosófica es el análisis de eventos descritos por la Probabilidad Neutrosófica.

La Probabilidad Neutrosófica es una generalización de la probabilidad clásica y la probabilidad imprecisa en la que la posibilidad de que ocurra un evento A es t% verdadera, donde t varía en el subconjunto T, i% indeterminada, donde i varía en el subconjunto I, y f% falsa, donde f varía en el subconjunto F. En la probabilidad clásica la suma de todas las probabilidades espaciales es igual a 1, mientras que en la Probabilidad Neutrosófica es igual a 3.

En la probabilidad imprecisa: la probabilidad de un evento es un subconjunto T en [0, 1], no un número p en [0, 1], lo que queda se supone que es el opuesto, subconjunto F (también del intervalo unitario [0, 1]); no hay subconjunto indeterminado I en probabilidad imprecisa [ver B9].

La función que modela la Probabilidad Neutrosófica de una variable aleatoria x se llama Distribución Neutrosófica: PN(x) = (T(x), I(x), F(x)), donde T(x) representa la probabilidad de que el valor x ocurra, F(x) representa la probabilidad de que el valor x no ocurra, e I(x) representa la probabilidad indeterminada/desconocida del valor x [ver B3]".

Por lo tanto, una caracterización más detallada de una variable aleatoria neutrosófica, que no se hace en la estadística clásica y de intervalo.

Ver este libro:

F. Smarandache, Introducción a la medida neutrosófica, la integral neutrosófica y la probabilidad neutrosófica, Sitech Publishing House, Craiova, 2013,http://fs.unm.edu/NeutrosophicMeasureIntegralProbability.pdf

1.5. Woodall et al.

- Los ejemplos que involucran tamaños de muestra imprecisos dados en Smarandache [42] involucran datos de atributos sin definiciones operativas cuidadosamente expresadas. Parece imposible tener una muestra de datos de variables sin conocer el tamaño de la muestra. (pág. 4)

Respuesta:

No estamos de acuerdo. Hay muchos ejemplos frecuentes de poblaciones y muestras de nuestra vida cotidiana: como bancos de peces en un río, bandadas de aves migratorias, árboles en un bosque, plantas en un campo determinado, rebaños de ganado, etc. Más ejemplos a continuación:

Tamaño de muestra indeterminado

> "Un estadístico quiere analizar la reacción de los espectadores a un partido de balonmano, donde el equipo A juega contra el equipo B. Supongamos que se han vendido unas 4.000 entradas. Los espectadores que asisten al partido forman una muestra, cuyo tamaño no se puede determinar con exactitud, pues también hay espectadores que entraron sin entrada (como invitados, o ilegalmente), mientras que otros que habían comprado entrada no pudieron acudir por diversas razones.
>
> Por lo tanto, el tamaño de la muestra se podría estimar, por ejemplo, entre, por ejemplo, entre 3.900 y 4.200."

> "Estimar cuántas personas vieron el partido por televisión es aún más vago. Electrónicamente uno se entera de que unos 3 millones de personas lo han visto. Pero esto también es ambiguo, ya que se podría haber visto a muchas personas en el mismo televisor, mientras que algunos televisores se habrían dejado encendidos sin que nadie mirara porque los propietarios habrían estado ocupados con otras cosas. El tamaño de la muestra se estimó, por ejemplo, entre 2,9 y 3,2 millones". [F. Smarandache, Nidus idearum. Scilogs, II: de rerum consectatione (segunda edición) Bruselas, páginas 108-109, 2016,http://fs.unm.edu/NidusIdearum2-ed2.pdf]

Comentario de Woodall et al. [127]:

No hay razón para tratar los tamaños de muestra como indeterminados.

Respuesta:

Un conjunto de individuos puede ser considerado una población con respecto a una referencia, pero a la vez puede ser una muestra con respecto a una referencia mayor.

Un ejemplo sencillo cuando el tamaño de una población es indeterminado, pero esa población se convierte en una muestra con respecto a una superpoblación.

Por lo tanto, hay muchos casos en los que el tamaño de la muestra puede no ser bien conocido.

Consideremos la población P de árboles, cuyo tamaño es indeterminado (entre 100-120 árboles), en un determinado parque de una ciudad. Pero, con respecto a los árboles en los 10 parques de la ciudad, la población P es una muestra (de tamaño indeterminado: {100, 101, ..., 120}).

Nótese que el tamaño de la muestra no es un intervalo, sino un conjunto finito discreto.

Por lo tanto, la mayoría de las veces en el mundo real no es posible estimar exactamente el tamaño de una muestra o población.

Woodall et al.

Por cierto, pasamos varios años estudiando métodos de lógica difusa, sin encontrar ventajas sobre el uso de probabilidad y estadística.

Respuesta:

Has usado o intentado usar la lógica difusa en estadística, lo entiendo. Pero la principal distinción entre la lógica difusa y la neutrosófica es que en la lógica neutrosófica se ha introducido la indeterminación como componente independiente.

Woodall et al.

El enfoque de muestreo repetitivo brinda la posibilidad de recopilar más de n observaciones en cualquier momento de muestreo.

Respuesta:

Esta cae mejor bajo la Probabilidad Plitogénica y Estadística que considera Análisis Multivariante de eventos y sus estadísticas.

Si está interesado, revise:

http://fs.unm.edu/NSS/PlithogenicProbabilityStatistics20.pdf

1.6. Media de una Muestra con individuos parcialmente pertenecientes

Sea S = {a, b, c, d} un conjunto muestral de cuatro elementos, tal que

a = 2, b = 8, c = 5 y d = 11.

En la estadística clásica se supone que todos los elementos pertenecen al 100% a la muestra, por lo tanto

S = {a(1), b(1), c(1), d(1)}.

De donde la media clásica:

$$CA = \frac{2\cdot 1 \ + \ 8\cdot 1 \ + \ 5\cdot 1 \ + \ 11\cdot 1}{1+1+1+1} = \frac{26}{4} = 6.5$$

Pero, en el mundo real, no todos los elementos pueden pertenecer totalmente (100%) a la muestra, por ejemplo, supongamos que la muestra neutrosófica es:

NS = {a(1.1), b(0.4), c(0.6), d(0.3)}, lo que significa que:

el elemento a pertenece al 110% (alguien que trabaja horas extras, por ejemplo, como en el overset neutrosófico (ver [B4]), b pertenece solo al 40% de la muestra, c pertenece al 60% y d pertenece al 30%.

De donde la media neutrosófica (NM) es:

$$NM = \frac{2\cdot(1.1)+8\cdot(0.4)+5\cdot(0.6)+11(0.3)}{1.1+0.4+0.6+0.3} = \frac{11.7}{2.4} = 4.875$$

Claramente, la media clásica y la media neutrosófica son diferentes,

CM = 6,5 4,875 = NM.

Y, en consecuencia: la varianza, la desviación estándar, la función de distribución de probabilidad y otros estadígrafos que dependen de ellas también serán diferentes.

Pero, la media neutrosófica es más precisa ya que refleja la media real (no idealista), porque tiene en cuenta el grado de pertenencia de cada elemento con respecto al conjunto." [5] Y, en consecuencia, las otras estadísticas que dependen de ellos son más precisas.

1.7. La Función Gruesa (Distribución), a partir de la estadística neutrosófica, se define como:

$$f: R \to P(R), f(x) = [f_1(x), f_2(x)]$$

La curva gruesa como la gráfica de una función gruesa [2] se introdujo en 2014 y es diferente de las funciones de intervalo, porque podemos tener una distribución de probabilidad entre dos curvas, de la forma f(x) = [f1(x), f2(x)].

Por ejemplo, sea, $f_1(x) = (x-1)^3 + 2, f_2(x) = 1.5x^3,$

entonces $f(x) = [(x-1)^3 + 2, 1.5x^3]$

que es una función gruesa, es decir, la zona entre dos curvas inferiores.

Por lo tanto, es diferente de la Estadística de Intervalo.

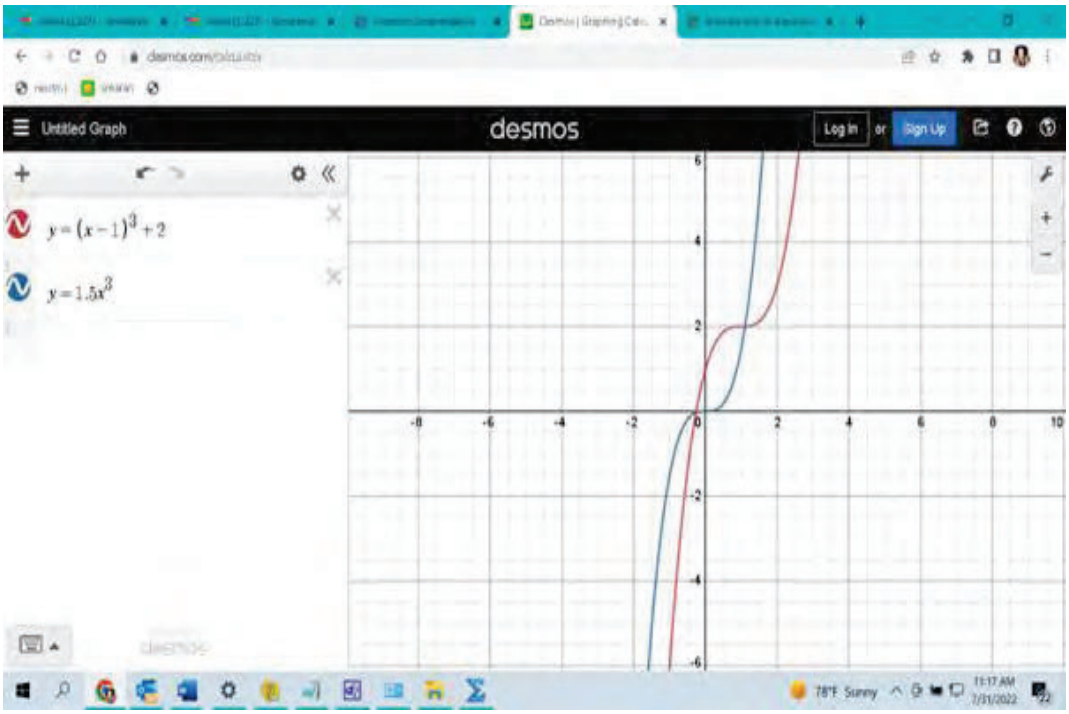


Tabla 1. Una función gruesa utilizada en estadística neutrosófica

1.7 Variable con Valor de Intervalo vs. Variable con Número Neutrosófico

La estadística de intervalo utiliza variables [7] de la forma:
aX + b, donde a y b son constantes y X es un conjunto de intervalos variables.
Por ejemplo, a = 2, b = 3 y X = [0,1, 0,3], [4, 5], [7. 9], ...
dan
aX + b = 2[0.1, 0.3] + 3, 2[4, 5] + 3, 2[7, 9] + 3, ...
= [3.2, 3.6], [11, 13], [17, 21], ...

Mientras que los Números Neutrosóficos tienen la forma:
N = a + *bI*,

donde "a" es la parte determinante (conocida) de N, y "bI" es la parte indeterminada (poco clara) de N;
I es un subconjunto real fijo, mientras que a y b son números reales variables.
Ejemplo:

Sea I = [0.1, 0.2] un subconjunto fijo (lo tomamos como un intervalo, aunque puede ser cualquier tipo de subconjunto), y los iniciales a = 2 y b = 3, luego a = 4 y b = 6, a = 5.5 y b = 6.2 etc.
El a + bI = 2 + 3[0.1, 0.2], 4 + 6[0.1, 0.2], 7 + 3[0.1, 0.2], ...
= [2.3, 2.6], [4.6, 5.2], [7.3, 7.6], ...

Entonces, claramente los dos enfoques son diferentes, es decir, la variable con valores de intervalo de la estadística de intervalo es diferente de la variable de número neutrosófico de la estadística neutrosófica.

1.8. Conjunto Vacilante vs. Intervalo

En la estadística neutrosófica podemos usar todos los tipos de conjuntos, por ejemplo, los conjuntos vacilantes, que tienen un número discreto finito de elementos. En varios ejemplos, sería ventajoso utilizar un conjunto vacilante en lugar de un intervalo.

Supongamos que la temperatura, en grados Celsius, está por encima de 10 °C, con una fluctuación extrema baja/alta, 18 °C baja y 40 °C o 45 °C alta.

En estadística neutrosófica, la variable aleatoria t se modela como t = 10+I, donde I∈ {5, 30, 35}, donde la parte indeterminada de t es un conjunto finito discreto vacilante de solo tres elementos: 5, 30 y 35.

En estadística de intervalos, la variable aleatoria t se modela como un intervalo t = [15, 45], cuya incertidumbre es mucho mayor que en estadística neutrosófica, y se propaga con cada nuevo cálculo.

Este es otro ejemplo que muestra la preferencia de usar estadística neutrosófica en lugar de estadística de intervalo.

1.9. Comparaciones entre el álgebra de intervalos y el álgebra neutrosófica

a) Suma

Estadística de intervalo (EI)

Tomamos el ejemplo presentado por Woodall et al.
para I ∈ [0, 1],
N_1=4+2I=[4,6]
N_2=4-2I=[2,4]
N_1 〖+N〗_2=[4+2,6+4]=[6,10]

Al sumar N_1 〖+N〗_2 en la estadística de intervalo, la parte indeterminada [0, 2] de cada número se suma, aumentando la incertidumbre de la suma a [0, 2] + [0, 2] = [0, 4].

Estadística Neutrosófica (EN)

$$N_1 = 4 + 2I$$

$$N_2 = 4 - 2I$$

$$N_1 + N_2 = 4 + 2I + 4 - 2I = 8$$

La parte indeterminada de N_1 es 2I=[0,2] y la parte indeterminada de N_2es -2I=[0,-2].

Al sumar $N_1 + N_2$ en la estadística neutrosófica, las partes indeterminadas se cancelan, 2I-2I=0, y se obtiene una respuesta determinada: 8.

El resultado [6, 10] obtenido en la estadística de intervalo es mucho más vago que el 8 obtenido en la estadística neutrosófica. Lo que muestra la ventaja de la EN con respecto a la EI.

Mientras que en muchos casos los resultados de las operaciones con números neutrosóficos coinciden con los obtenidos por operaciones con intervalos, en muchos otros casos los resultados son diferentes.

Claramente, la estadística de intervalo es diferente de la estadística neutrosófica.

A diferencia de la estadística de intervalo que acumula la incertidumbre de una operación a otra, la estadística neutrosófica disminuye o incluso anula la incertidumbre.

Pregunta de Woodall et al. [127]:

Tengo una pregunta sobre la aritmética neutrosófica. Supongamos que uno tiene, como en su ejemplo, 4+2I y 4-2I. Se dice que el resultado de la suma es la constante 8. Supongamos que uno escribe los números de manera equivalente como 4+2I y 2+2I, entonces la suma es 6+4I. ¿Por qué la suma debe depender de cómo se expresen los números?

Respuesta:

Los números neutrosóficos numéricos (N) son elegidos por el investigador sobre las partes que se consideran determinadas (a) e indeterminadas (bI), por lo que N = a+bI.

Por lo tanto, dependen de cuál sea la parte indeterminada/vaga del número.

N1 = 2+2I significa que la parte determinada de N1 es 2, la otra es indeterminada, mientras que N2 = 4-2I tiene su parte determinada en 4.

N1 es diferente de N2 en la estadística neutrosófica, pero significan lo mismo en estadística de intervalo: [0, 4] cuando I = [0, 1].

Este es otro punto para mostrar que la Estadística Neutrosófica y la Estadística de Intervalo son diferentes entre sí.

b) Multiplicación

$N_1=4+2I, I\in [0,1]$
$N_2=4-2I$

EI
$N_1 \cdot N_2 = (4 + 2I) \cdot (4 - 2I) = (4 + 2\cdot[0, 1]) \cdot (4 -2\cdot[0, 1])$
$= (4 + [2 \cdot 0, 2 \cdot 1] \cdot (4 –[2 \cdot 0, 2 \cdot 1]$
$= (4 + [0, 2] \cdot (4 –[\cdot 0, 2]$
$= [4, 6] \cdot [2, 4] = [4 \cdot 2, 6 \cdot 4] = [8, 24],$
la longitud de la incertidumbre es 24 – 8 = 16.

EN
$N_1 \cdot N^2 = (4 + 2I) (4\ 2I) = \cdot -4^2 - (2I)^2$
$= 16 - 4I^2 = 16 – 4\ [0^2, 1^2]$
$= 16 – 4 \cdot [0, 1]$
$= 16 – 4\cdot[0, 4]$
$= [16 – 4, 16 – 0]$
$= [12, 16],$
la longitud de la incertidumbre es 16 – 12 = 4 < 16, por lo tanto, un resultado más preciso de la EN.

c) Sustracción

EI
$N_1 - N_2 = (4 + 2I) - (4 - 2I)$
$= [4, 6] – [2, 4]$
$= [4 – 4, 6 – 2] = [0, 4]$

EN
$N_1 - N_2 = (4 + 2I) - (4 - 2I)$
$= 4 + 2I – 4 + 2I$
$= 4I = 4 \cdot [0, 1] = [0, 4]$, lo mismo.

Pero tomemos otros números neutrosóficos:
$M_1 = 5 + 4I, I \in [2,3]$
$M_2 = 6 + 3I$

EI
$M_1 = 5 + 4I, I \in [2,3]$, por lo tanto, $M1 = 5 + [4 \cdot 2, 4 \cdot 3] = [13, 17]$.

$M_2 = 6 + 3I = 6 + 3 \cdot [2,3] = [12, 15]$.
$M_1 - M_2 = [13, 17] - [12, 15]$
$= [13 - 15, 17 - 12] = [-2, 5]$, longitud del intervalo de incertidumbre $5 - (-2) = 7$.

EN
$M_1 - M_2 = (5 + 4I) - (6 + 3I) = 5 + 4I - 6 - 3I$
$= -1 + I = -1 + [2, 3] = [-1 + 2, -1 + 3]$
$= [1, 2]$, intervalo de longitud de incertidumbre $2 - 1 = 1$.
Pero $1 < 7$, así que hay mejor precisión usando la EN.

División

EI

$$\frac{4+2I}{2+I} = \frac{4+2[0,1]}{2+[0,1]} = \frac{4+[0,2]}{[2+0,2+1]} = \frac{[4+0,4+2]}{[2,3]}$$

$$= \frac{[4,6]}{[2,3]} = \frac{4}{3}, \frac{6}{2} = [1.3, 3],$$

intervalo de incertidumbre $3 - 1{,}3 = 1{,}7$.

EN

$$\frac{4+2I}{2+I} = \frac{2(2+I)}{2+I} = 2$$

incertidumbre = 0,

lo cual es un resultado más preciso, ya que la estadística neutrosófica permitió simplificar la incertidumbre I.

1.10. Estadística neutrosófica refinada utilizada en Big Data

En este mundo de Big Data, nos enfrentamos a este tipo de situación con más incertidumbres resultantes de múltiples variables, lo que lleva a la Neutrosofía Refinada.

De este modo, podemos utilizar la Estadística Neutrosófica Refinada, es decir, cuando la indeterminación "I" se divide en muchos tipos de incertidumbres $I_1, I_2, ..., Is$, donde $s \geq 2$, tantas como sea necesario en la aplicación.

La Estadística Neutrosófica Refinada siguió los pasos del Conjunto Neutrosófico Refinado (Smarandache, 2013). Por lo tanto, un elemento de un Big Data que pertenece a un conjunto neutrosófico refinado, $x \in M$, puede tener coordenadas neutrosóficas refinadas, por ejemplo

$x(T, I_1, I_2, I_3, F)$ si solo hay 3 tipos de incertidumbres. Podemos tener tantos tipos de incertidumbres como sea necesario en el problema.

Comprobar primero el Conjunto Neutrosófico Refinado,
donde T, I, F se pueden refinar/dividir respectivamente como:

$T_1, T_2, ..., Tp$,
$I_1, I_2, ..., Ir$,
y $F_1, F^2, ..., Fs$,

donde p, r, s son números enteros ≥ 0, y al menos uno de p, r, s es ≥ 2 (para asegurar la existencia de refinamiento de al menos uno de los tres componentes neutrosóficos T, I y F).

Cuando p, r o s es igual a 0, ese componente se descarta. Por ejemplo, T_0 significa que T se descarta, y lo mismo ocurre con I_0 y F_0.

Esto deja espacio para definir el Conjunto Difuso Refinado, bajo la forma $T_1, T_2, ..., Tp$, donde p es un número entero ≥ 2, y se descartan I_0 y F_0.

Y para Conjunto Difuso Intuicionista Refinado, bajo la forma T_1, T_2, ..., Tp,

y F_1, F_2, ..., Fs, para números enteros p, s ≥ 1, y al menos uno de p o s es ≥ 2 (para asegurar el refinamiento de al menos un componente T o F).

Y así de manera similar para otros Conjuntos / Lógicas / Probabilidades / Estadística de extensión difusa refinados.

1.10. Probabilidad Plitogénica y Estadística Plitogénica que son generalizaciones de Probabilidad y Estadística Multivariante

El **Análisis Variado Plitogénico** (AVP) es una extensión del Análisis Multivariante clásico, donde se permiten datos o procedimientos indeterminados, que se denominan datos neutrosóficos y procedimientos neutrosóficos respectivamente. Por lo tanto, AVP trata con variables neutrosóficas/indeterminadas, subsistemas neutrosóficos/indeterminados y sistemas de sistemas neutrosóficos/indeterminados como un todo.

Por lo tanto, el análisis de variables plitogénicas estudia un sistema neutrosófico/indeterminado como un todo, caracterizado por muchas variables neutrosóficas/indeterminadas (es decir, subsistemas neutrosóficos/indeterminados) y muchas relaciones neutrosóficas/indeterminadas. Por lo tanto, se necesitan muchas mediciones y observaciones neutrosóficas.

La **Probabilidad Plitogénica** de que ocurra un evento se compone de las chances de que el evento ocurra con respecto a todas las variables aleatorias (parámetros) que lo determinan. La Probabilidad Plitogénica, basada en el Análisis Variado Plitogénico, es una probabilidad multidimensional ("plitho" significa "muchos", sinónimo de "multi"). Podemos decir que es una probabilidad de subprobabilidades, donde cada subprobabilidad describe el comportamiento de una variable. Suponemos que el evento que estudiamos es producido por una o más variables. Cada variable está representada por una Función de Distribución de Probabilidad (Densidad) (FDP).

La **Estadística Plitogénica** (**EP**) engloba el análisis y las observaciones de los eventos estudiados por la Probabilidad Plitogénica. La Estadística Plitogénica es una generalización de la Estadística Multivariante Clásica, y es un análisis simultáneo de muchas variables neutrosóficas/indeterminadas resultantes, y también es una estadística multi-indeterminada.

Subclases de la Estadística Plitogénica:

- Estadística de Intervalo
- Estadística Neutrosófica
- Estadística Multivariante
- Estadística Neutrosófica Plitogénica
- Estadística Indeterminada Plitogénica
- Estadística Difusa Intuicionista plitogénica
- Estadística Difusa de Imagen Plitogénica - Estadística Difusa Esférica Plitogénica
- y en general: Estadística Plitogénica (extensión difusa) - y Estadística Híbrida Plitogénica.

La Estadística Refinada Plitogénica es, de manera similar, la forma más general de estadística que estudia el análisis y las observaciones de los eventos descritos por la Probabilidad Refinada Plitogénica.

Ver más desarrollo, extensión de Estadística de Intervalos y Estadística Neutrosófica a Probabilidad Plitogénica y Estadística Plitogénica Que Son Generalizaciones De Probabilidad Y Estadística Multivariante: [6].

Conclusión

En este trabajo realizamos una comparación entre la Estadística Neutrosófica (EN) y la Estadística de Intervalo (EI). Mostramos que son diferentes y en muchos casos la EN es más general que la EI.
La EN no se reduce a utilizar únicamente números neutrosóficos en aplicaciones estadísticas, como Woodall et al. asevera, sino que es mucho más amplia. La EN se ocupa de todos los tipos de indeterminación, mientras que la EI se ocupa solo de la indeterminación que puede representarse mediante intervalos.
Y respondimos a los argumentos de Woodall et al. [1].
Redirigimos a los autores a la Probabilidad Plitogénica y la Estadística Plitogénica, que son las formas más generales de Probabilidad Multivariante y Estadística Multivariante respectivamente (incluidas, por supuesto, la Estadística de Intervalo como una subclase).

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Tesis doctoral

PhD1. Rafif Alhabib: Formulation of the classical probability and some probability distributions due to neutrosophic logic and its impact on Decision Making. PhD Thesis in Arabic, held under the supervision of Dr. M. M. Ranna, Dr. H. Farah, Dr. A. A. Salama, Faculty of Science, Department of Mathematical Statistics, University of Aleppo, Syrian Arab Republic, 2019. http://fs.unm.edu/NS/FormulationOfTheClassicalProbability-PhDThesis.pdf

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Foundation of Revolutionary Topologies: An Overview, Examples, Trend Analysis, Research Issues, Challenges, and Future Directions

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Florentin Smarandache (2024). "Foundation of Revolutionary Topologies: An Overview, Examples, Trend Analysis, Research Issues, Challenges, and Future Directions." *Neutrosophic Systems with Applications*, Vol. 13, 45-66. https://doi.org/10.61356/j.nswa.2024.125

Abstract: We now found nine new topologies, such as: NonStandard Topology, Largest Extended NonStandard Real Topology, Neutrosophic Triplet Weak/Strong Topologies, Neutrosophic Extended Triplet Weak/Strong Topologies, Neutrosophic Duplet Topology, Neutrosophic Extended Duplet Topology, Neutrosophic MultiSet Topology, and recall and improve the seven previously founded topologies in the years (2019-2023), namely: NonStandard Neutrosophic Topology, NeutroTopology, AntiTopology, Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, SuperHyperTopology, and Neutrosophic SuperHyperTopology. They are called avantgarde topologies because of their innovative forms.

Keywords: Classical Topology; Topological Space; NeutroSophication; AntiSophication; NeutroTopology; AntiTopology; Refined Neutrosophic Topology; Refined Neutrosophic Crisp Topology; SuperHyperTopology; Neutrosophic SuperHyperTopology; Extended NonStandard Real Set; NonStandard Topology; NonStandard Neutrosophic Topology; Largest Extended NonStandard Real Topology; left monad; Right Monad; Pierced Binad; Left Monad Closed to the Right; Right Monad Closed to the Left, Unpierced Binad; Neutrosophic OverTopology; Neutrosophic UnderTopology; Neutrosophic OffTopology; (Fuzzy & Fuzzy-Extensions) Over/Under/Off-Topologies; Neutrosophic MultiSet Topology.

1. Introduction

The foundation of new topologies raised from development of other fields such as NeutroAlgebra and AntiAlgebra (that gave birth to NeutroTopology and AntiTopology), SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra (that gave birth to SuperHyperTopology and Neutrosophic SuperHyperTopology), Refined Crisp Set (that gave birth to the Refined Crisp Topology), and Refined Neutrosophic Set (that gave birth to refined Neutrosophic Topology), and NonStandard Set (that gives birth to NonStandard Topology and NonStandard Neutrosophic Topology), Neutrosophic Triplet Set, Neutrosophic Extended Triplet Set, Neutrosophic Dual Set, Neutrosophic Extended Dual Set, and Neutrosophic MultiSet.

This is almost a virgin territory of research since little research has been done, mostly about the AntiTopology [8]. Nevertheless, it is a promising field to study in the future, since it better reflects our real world, where the laws (axioms) do not apply in the same degree to all people (powerful people are above the law, others immune to the law, and many feel the full hardship of the law); since the world as a dynamic system is formed by sub-systems, and each sub-system by sub-sub-systems and so on (whence the necessity to introduce the SuperHyperStructure based on the n-th PowerSet of a Set, whose particular cases are the SuperHyperAlgebra and SuperHyperTopology), etc.

We recall the classical definition of Topology, then the procedures of NeutroSophication and respectively AntiSophication of it, that result in adding in two new types of topologies: NeutroTopology and respectively AntiTopology.

Then we define topology on Refined Neutrosophic Set (2013), Refined Neutrosophic Crisp Set [3]. Afterwards, we extend the topology on the framework of SuperHyperAlgebra [6], then the NonStandard Neutrosophic Set to NonStandard Topology and NonStandard Neutrosophic Topology (never defined before).

The corresponding neutrosophic topological spaces are presented.

This research is an improvement of paper [7] and book [12, sections 4.8 and 4.9].

2. Classical Topology

Let $\mathcal{U}$ be a non-empty set, and $P(\mathcal{U})$ the power set of $\mathcal{U}$.

Let $\tau \subseteq P(\mathcal{U})$ be a family of subsets of $\mathcal{U}$.

Then τ is called a Classical Topology on $\mathcal{U}$ if it satisfies the following axioms: (CT-1) ϕ and $\mathcal{U}$ belong to τ.

(CT-2) The intersection of any finite number of elements in τ is in τ.

(CT-3) The union of any finite or infinite number of elements in τ is in τ.

All three axioms are totally (100%) true (or T = 1, I = 0, F = 0). We simply call them (classical) *Axioms*.

Then $(\mathcal{U}, \tau)$ is called a *Classical Topological Space* on $\mathcal{U}$.

3. NeutroSophication of the Topological Axioms

NeutroSophication of the topological axioms means that the axioms become partially true, partially indeterminate, and partially false. They are called NeutroAxioms.

(NCT-1) Either $\{\phi \notin \tau$ and $\mathcal{U} \in \tau\}$, or $\{\phi \in \tau$ and $\mathcal{U} \notin \tau\}$.

(NCT-2) There exist a finite number of elements in τ whose intersection belong to τ (degree of truth *T*); and a finite number of elements in τ whose intersection is indeterminate (degree of indeterminacy *I*); and a finite number of elements in τ whose intersection does not belong to τ (degree of falsehood *F*); where $(T, I, F) \notin \{(1, 0, 0), (0, 0, 1)\}$ since (1, 0, 0) represents the above Classical Topology, while (0, 0, 1) the below AntiTopology.

(NCT-3) There exist a finite or infinite number of elements in τ whose union belongs to τ (degree of truth *T*); and a finite or infinite number of elements in τ whose union is indeterminate (degree of indeterminacy *I*); and a finite or infinite number of elements in τ whose union does not belong to τ (degree of falsehood *F*); where of course $(T, I, F) \notin \{(1, 0, 0), (0, 0, 1)\}$.

4. AntiSophication of the Classical Topological Axioms

AntiSophication of the topological axioms means to negate (anti) the axioms, the axioms become totally (100%) false (or T = 0, I = 0, F = 1). They are called *AntiAxioms*.

(ACT-1) $\phi \notin \tau$ and $\mathcal{U} \notin \tau$.

(ACT-2) The intersection of any finite number ($n \geq 2$) of elements in τ is not in τ.

(ACT-3) The union of any finite or infinite number ($n \geq 2$) of elements in τ is not in τ.

5. <Topology, NeutroTopology, AntiTopology>

As such, we have a neutrosophic triplet of the form:

<Axiom(1, 0, 0), NeutroAxiom(*T, I, F*), AntiAxiom(0, 0, 1)>,

where $(T, I, F) \neq (1, 0, 0)$ and $(T, I, F) \neq (0, 0, 1)$.

Correspondingly, one has:

<Topology, NeutroTopology, AntiTopology>.

Therefore, in general:

(Classical) Topology is a topology that has all axioms totally true. We simply call them *Axioms*.

NeutroTopology is a topology that has at least one *NeutroAxiom* and the others are all *classical Axioms* [therefore, no AntiAxiom].

AntiTopology is a topology that has one or more *AntiAxioms*, no matter what the others are (*classical Axioms*, or *NeutroAxioms*).

6. Theorem on the number of Structures/NeutroStructures/AntiStructures

If a Structure has m axioms, with $m \geq 1$, then after NeutroSophication and AntiSophication one obtains 3^m types of structures, categorized as follows:

1*Classical Structure* + $(2^m - 1)$*NeutroStructures* + $(3^m - 2^m)$*AntiStructures* = 3^m *Structures.*

7. Consequence on the number of Topologies/NeutroTopologies/AntiTopologies

As a particular case of the previous theorem, from a Topology which has $m = 3$ axioms, one makes, after NeutrosSophication and AntiSophication, $3^3 = 27$ types of structures, as follows: 1 classical Topology, $2^3 - 1 = 7$ NeutroTopologies, and $3^3 - 2^2 = 19$ AntiTopologies.

1*Classical Topology* + 7*NeutroTopologies* + 19*AntiTopologies* = 3^3 Topologies are presented below:

There is 1 (one) type of Classical Topology, whose axioms are listed below:

1 *Classical Topology*

$$\begin{pmatrix} CT-1 \\ CT-2 \\ CT-3 \end{pmatrix}$$

8. Definition of NeutroTopology [4, 5]

It is a topology that has at least one topological axiom which is partially true, partially indeterminate, and partially false, or *(T, I, F)*, where T = True, *I* = Indeterminacy, *F* = False, and no topological axiom is totally false, in other words: $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$, where *(1, 0, 0)* represents the classical Topology, while *(0, 0, 1)* represents the below AntiTopology.

Therefore, the NeutroTopology is a topology in between the classical Topology and the AntiTopology.

There are 7 types of different NeutroTopologies, whose axioms, for each type, are listed below:

7 NeutroTopologies

$$\begin{pmatrix} NCT-1 \\ CT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ CT-2 \\ NCT-3 \end{pmatrix},$$
$$\begin{pmatrix} NCT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ CT-2 \\ NCT-3 \end{pmatrix},$$
$$\begin{pmatrix} NCT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}.$$

9. Definition of AntiTopology [4, 5]

It is a topology that has at least one topological axiom that is 100% false (T, I, F) = (0, 0, 1). The NeutroTopology and AntiTopology are particular cases of NeutroAlgebra and AntiAlgebra [4] and, in general, they all are particular cases of the NeutroStructure and AntiStructure respectively, since we consider "Structure" in any field of knowledge [5].

There are 19 types of different AntiTopologies, whose axioms, for each type, are listed below:

19 *AntiTopologies*

$$\begin{pmatrix} ACT-1 \\ CT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} ACT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$
$$\begin{pmatrix} ACT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} ACT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix},$$
$$\begin{pmatrix} ACT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ ACT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ CT-2 \\ ACT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ CT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ ACT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ ACT-2 \\ CT-3 \end{pmatrix},$$

$$\begin{pmatrix} ACT-1 \\ ACT-2 \\ ACT-3 \end{pmatrix}.$$

10. Refined Neutrosophic Set

Let U be a universe of discourse, and a non-empty subset *R* of it,

$$R = \begin{Bmatrix} x\left(T_1(x), T_2(x), \ldots, T_p(x)\right); \\ \left(I_1(x), I_2(x), \ldots, I_r(x)\right); \\ \left(F_1(x), F_2(x), \ldots, F_s(x)\right); \end{Bmatrix}$$

with all $T_j, I_k, F_l \in [0,1], 1 \le j \le p, 1 \le k \le r, 1 \le l \le s$, and no restriction on their sums $0 \le T_m + I_m + F_m \le 3$, with $1 \le m \le \max\{p, r, s\}$, where *p, r, s* ≥ 2, are fixed integers, and at least one of them is ≥ 2, in order to ensure the refinement (sub-parts) or multiplicity (multi-parts) – depending on the application, of at least one neutrosophic component amongst *T* (truth), *I* (indeterminacy), *F* (falsehood); and of course $x \in \mathcal{U}$.

By notation we consider that index zero means the empty-set, i.e. $T_0 = I_0 = F_0 = \phi$ (or zero), and the same for the missing sub-parts (or multi-parts).

For example, the below (2,3,1)-Refined Neutrosophic Set is identical to a (3,3,3)-Refined Neutrosophic Set: $(T_1, T_2; I_1, I_2, I_3; F_1) \equiv (T_1, T_2, 0; I_1, I_2, I_3; F_1, 0, 0)$, where the missing components T_3, and F_2, F_3 were replaced each of them by 0 (zero) *R* is called a *(p, r, s)-refined neutrosophic set* { or *(p, r, s)-RNT* }.

The neutrosophic set has been extended to the Refined Neutrosophic Set (Logic, and Probability) by Smarandache [1] in 2013, where there are multiple parts of the neutrosophic components, as such T was split into subcomponents $T_1, T_2, \ldots, T_p$, and I into $I_1, I_2, \ldots, I_r$, and F into $F_1, F_2, \ldots, F_s$, with p + r + s

= n ≥ 2 and integers p, r, s ≥ 0 and at least one of them is ≥ 2 in order to ensure the refinement (or multiplicity) of at least one neutrosophic component amongst T, I, and F.

Even more: the subcomponents T_j, I_k, and/or F_l can be countable or uncountable infinite subsets of [0, 1].

This definition also includes the *Refined Fuzzy Set*, when $r = s = 0$ and $p \geq 2$;

and the definition of the *Refined Intuitionistic Fuzzy Set*, when $r = 0$, and either $p \geq 2$ and $s \geq 1$, or $p \geq 1$ and $s \geq 2$.

All other fuzzy extension sets (Pythagorean Fuzzy Set, Spherical Fuzzy Set, Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, etc.) can be refined/multiplicated in a similar way.

11. Definition of Refined Neutrosophic Topology

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{P}(\mathcal{U})$ be the family of all (p, r, s)-refined neutrosophic subsets of $\mathcal{U}$.
Let $\tau_{RNT} \subseteq \mathcal{P}(\mathcal{U})$ be a family of (p, r, s)-refined neutrosophic subsets of $\mathcal{U}$.
Then τ_{RNT} is called a *Refined Neutrosophic Topology* (*RNT*) if it satisfies the axioms:
(RNT-1) ϕ and $\mathcal{U}$ belong to τ_{RNT};
(RNT-2) The intersection of any finite number of elements in τ_{RNT} is in τ_{RNT};
(RNT-3) The union of any finite or infinite number of elements in τ_{RNT} is in τ_{RNT};
Then $(\mathcal{U}, \tau_{RNT})$ is called a Refined Neutrosophic Topological Space on $\mathcal{U}$.

The *Refined Neutrosophic Topology* is a topology defined on a Refined Neutrosophic Set.
{Similarly, the Refined Fuzzy Topology is defined on a Refined Fuzzy Set, while the Refined Intuitionistic Fuzzy Topology is defined on a Refined Intuitionistic Fuzzy Set, etc.
And, as a generalization, on any type of fuzzy extension set [such as: Pythagorean Fuzzy Set, Spherical Fuzzy Set, Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, etc.] one can define a corresponding fuzzy extension topology}.

12. Neutrosophic Crisp Set

The *Neutrosophic Crisp Set* was defined by Salama and Smarandache in 2014 and 2015.
Let *X* be a non-empty fixed space. And let *D* be a Neutrosophic Crisp Set [2],
where $D = \langle A, B, C \rangle$, with *A*, *B*, *C* as subsets of *X*.
Depending on the intersections and unions between these three sets *A*, *B*, *C* one gets several:
Types of Neutrosophic Crisp Sets [2, 3].
The object having the form $D = \langle A, B, C \rangle$ is called:

(a) A neutrosophic crisp set of Type 1 (NCS-Type1) if it satisfies:
$A \cap B = B \cap C = C \cap A = \phi$ (empty set).

(b) A neutrosophic crisp set of Type 2 (NCS-Type2) if it satisfies:
$A \cap B = B \cap C = C \cap A = \phi$ and $A \cup B \cup C = X$.

(c) A neutrosophic crisp set of Type 3 (NCS-Type3) if it satisfies:
$A \cap B \cap C = \phi$ and $A \cup B \cup C = X$.

Of course, more types of Neutrosophic Crisp Sets may be defined by modifying the intersections and unions of the subsets *A*, *B*, and *C*.

13. Refined Neutrosophic Crisp Set

The *Refined Neutrosophic Crisp Set* [3] was introduced by Smarandache in 2019, by refining/multiplication of *D* (and denoting it by *RD* = Refined D) by refining/multiplication of its sets *A, B, C* into sub-subsets/multi-sets as follows:
RD = $(A_1, ..., A_p; B_1, ..., B_r; C_1, ..., C_s)$, with $p, r, s \geq 1$ be positive integers and at least one of them be ≥ 2 in order to ensure the refinement/multiplication of at least one component amongs A, B, C, where

$$A = \underset{i=1}{\overset{p}{\cup}} A_i, B = \underset{j=1}{\overset{r}{\cup}} B_j, C = \underset{k=1}{\overset{s}{\cup}} C_k$$

and many types of Refined Neutrosophic Crisp Sets may be defined by modifying the intersections or unions of the subsets/multisets $A_i, B_j, C_k, 1 \leq i \leq p, 1 \leq j \leq r, 1 \leq k \leq s$, depending on each application.

14. Definition of Refined Neutrosophic Crisp Topology

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{P}(\mathcal{U})$ be the family of all (*p, r, s*)-refined neutrosophic crisp subsets of $\mathcal{U}$.
Let $\tau_{RNCT} \subseteq \mathcal{P}(\mathcal{U})$ be a family of (*p, r, s*)-refined neutrosophic crisp subsets of $\mathcal{U}$.
Then τ_{RNCT} is called a *Refined Neutrosophic Crisp Topology* (*RNCT*) if it satisfies the axioms:
(RNCT-1) ϕ and $\mathcal{U}$ belong to τ_{RNCT};
(RNCT-2) The intersection of any finite number of elements in τ_{RNCT} is in τ_{RNCT};
(RNCT-3) The union of any finite or infinite number of elements in τ_{RNCT} is in τ_{RNCT}.
Then $(\mathcal{U}, \tau_{RNCT})$ is called a Refined Neutrosophic Crisp Topological Space on $\mathcal{U}$.
Therefore, the *Refined Neutrosophic Crisp Topology* is a topology defined on the Refined Neutrosophic Crisp Set.

15. Definition of the nth-PowerSets $P^n(H)$ and $P_*^n(H)$.

The nth-PowerSets $P^n(H)$ and $P_*^n(H)$ of the set *H,* that the SuperHyperTopology and respectively Neutrosophic SuperHyperTopology are based on, better describe our real world, since a system *H* (that may be a set, company, institution, country, region, etc.) is organized in sub-systems, which in their turn are organized each of them in sub-sub-systems, and so on.
The nth-PowerSet $P^n(H)$ is defined recursively:

$$P^0(H) \overset{def}{=} H$$
$$P^1(H) = P(H)$$
$$P^2(H) = P(P(H))$$
$$P^3(H) = P(P^2(H)) = P(P(P(H)))$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P^n(H) = P(P^{n-1}(H)) = \underbrace{P(P(...P(H)...))}_{n}$$

where *P* is repeated *n* times into the last formula, and the empty-set ϕ (that represents indeterminacy, uncertainty) is allowed in all sequence terms:

$$H, P(H), P^2(H), P^3(H), ..., P^n(H).$$

Similarly,

The n[th]-PowerSet $P_*^n(H)$ is defined recursively:

$$P_*^0(H) \overset{def}{=} H$$
$$P_*^1(H) = P_*(H)$$
$$P_*^2(H) = P_*(P_*(H))$$
$$P_*^3(H) = P_*(P_*^2(H)) = P_*(P_*(P_*(H)))$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P_*^n(H) = P_*(P_*^{n-1}(H)) = \underbrace{P_*(P_*(\ldots P_*(H)\ldots))}_{n}$$

where P is repeated n times into the last formula, and the empty-set ϕ (that represents indeterminacy, uncertainty) is not allowed in none of the sequence terms: $H, P_*(H), P_*^2(H), P_*^3(H), \ldots, P_*^n(H)$.

16. SuperHyperOperation

We recall our 2016 concepts of SuperHyperOperation, SuperHyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation Neutrosophic SuperHyperAxiom and Neutrosophic SuperHyperAlgebra [6].

Let $P_*^n(H)$ be the n[th]-powerset of the set H such that none of $P(H)$, $P^2(H)$, …, $P^n(H)$ contain the empty set ϕ.

Also, let $P_n(H)$ be the n[th]-powerset of the set H such that at least one of the $P(H)$, $P^2(H)$, …, $P^n(H)$ contain the empty set ϕ. For any subset A, we identify {A} with A.

The SuperHyperOperations are operations whose codomain is either $P_*^n(H)$ and in this case one has classical-type SuperHyperOperations, or $P^n(H)$ and in this case one has Neutrosophic SuperHyperOperations, for integer $n \geq 2$.

17. The n[th]-PowerSet better describe our real world

The n[th]-PowerSets $P^n(H)$ and $P_*^n(H)$, that the SuperHyperTopology and respectively Neutrosophic SuperHyperTopology are based on, better describe our real world, since a system H (that may be a set, company, institution, country, region, etc.) is organized in sub-systems, which in their turn are organized each in sub-sub-systems, and so on.

18. SuperHyperAxiom

A classical-type SuperHyperAxiom or more accurately a *(m, n)*-SuperHyperAxiom is an axiom based on classical-type SuperHyperOperations.

Similarly, a Neutrosophic SuperHyperAxiom {or Neutrosphic (m, n)-SuperHyperAxiom} is an axiom based on Neutrosophic SuperHyperOperations.

There are:

- Strong SuperHyperAxioms, when the left-hand side is equal to the right-hand side as in non-hyper axioms.
- And Week SuperHyperAxioms, when the intersection between the left-hand side and the right-hand side is non-empty.

19. SuperHyperAlgebra and SuperHyperStructure

A SuperHyperAlgebra or more accurately *(m-n)*-SuperHyperAlgebra is an algebra dealing with SuperHyperOperations and SuperHyperAxioms.

Again, a Neutrosophic SuperHyperAlgebra {or Neutrosphic (m, n)-SuperHyperAlgebra} is an algebra dealing with Neutrosophic SuperHyperOperations and Neutrosophic SuperHyperOperations.

In general, we have SuperHyperStructures {or *(m-n)*-SuperHyperStructures}, and corresponding Neutrosophic SuperHyperStructures.

For example, there are SuperHyperGrupoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, SuperHyperVectorSpace, etc.

20. Distinction between SuperHyperAlgebra vs. Neutrosophic SuperHyperAlgebra

- If none of the power sets $P^k(H)$, $1 \leq k \leq n$, do not include the empty set ϕ, then one has a classical-type SuperHyperAlgebra;
- If at least one power set, $P^k(H)$, $1 \leq k \leq n$, includes the empty set ϕ, then one has a Neutrosophic SuperHyperAlgebra.

21. Definition of SuperHyperTopology (SHT) [6]

It is a topology designed on the n[th]-PowerSet of a given non-empty set H, that excludes the empty-set, denoted as $P_*^n(H)$, built as follows:

$P_*(H)$ is the first powerset of the set H, and the index $*$ means without the empty-set (∅);

$P_*^2(H) = P_*(P_*(H))$ is the second powerset of H (or the powerset of the powerset of H), without the empty-sets; and so on, the n-th powerset of H,

$P_*^n(H) = P_*(P_*^{n-1}(H)) = \underbrace{P_*(P_*(...P_*(H)...))}_{n}$, where P_* is repeated n time ($n \geq 2$), and without the empty-sets.

Let consider τ_{SHT} be a family of subsets of $P_*^n(H)$.

Then τ_{SHT} is called a Neutrosophic SuperHyperTopology on $P_*^n(H)$, if it satisfies the following axioms:

(SHT-1) ϕ and $P_*^n(H)$ belong to τ_{SHT}.

(SHT-2) The intersection of any finite number of elements in τ_{SHT} is in τ_{SHT}.

(SHT-3) The union of any finite or infinite number of elements in τ_{SHT} is in τ_{SHT}.

Then $(P_*^n(H), \tau_{SHT})$ is called a SuperHyperTopological Space on $P_*^n(H)$.

22. Definition of Neutrosophic SuperHyperTopology (NSHT) [6]

It is, similarly, a topology designed on the n-th PowerSet of a given non-empty set H, but includes the empty-sets [that represent indeterminacies] too.

As such, in the above formulas, $P_*(H)$ that excludes the empty-set, is replaced by $P(H)$ that includes the empty-set.

$P(H)$ is the first powerset of the set H, including the empty-set (∅);

$P^2(H) = P(P(H))$ is the second powerset of H (or the powerset of the powerset of H), that includes the empty-sets; and so on, the n-th powerset of H,

$$P^n(H) = P(P^{n-1}(H)) = \underbrace{P(P(...P(H)...))}_{n}$$

where P is repeated n times ($n \geq 2$), and includes the empty-sets ($\emptyset$).

Let consider τ_{NSHT} be a family of subsets of $P^n(H)$.

Then τ_{NSHT} is called a Neutrosophic SuperHyperTopology on $P^n(H)$, if it satisfies the following axioms:

(NSHT-1) ϕ and $P^n(H)$ belong to τ_{NSHT}.

(NSHT-2) The intersection of any finite number of elements in τ_{NSHT} is in τ_{NSHT}.

(NSHT-3) The union of any finite or infinite number of elements in τ_{NSHT} is in τ_{NSHT}.

Then $(P^n(H), \tau_{NSHT})$ is called a Neutrosophic SuperHyperTopological Space on $P^n(H)$.

23. Introduction to NonStandard Analysis [9-12]

An *infinitesimal* [or infinitesimal number] (ε) is a number ε such that $|\varepsilon| < 1/n$, for any non-null positive integer n. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero. Infinitesimals are used in calculus.

An *infinite* [or infinite number] *(ω)* is a number greater than anything:

1 + 1 + 1 + … + 1 (for any finite number terms)

The infinites are reciprocals of infinitesimals.

The set of *hyperreals* (or *non-standard reals*), denoted as R^*, is the extension of set of the real numbers, denoted as R, and it comprises the infinitesimals and the infinites, that may be represented on the *hyperreal number line*

$1/\varepsilon = \omega/1$.

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in R are valid in R^* as well.

A *monad* (*halo*) of an element $a \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to a.

24. First Extension of NonStandard Analysis [13]

Let's denote by R_{+}^{*} the set of positive nonzero hyperreal numbers.

We consider the left monad and right monad, and the (*pierced*) *binad* that we have introduced as extension in 1998 [5]:

Left Monad { that we denote, for simplicity, by (^{-}a) or only ^{-}a } is defined as:

$\mu(^{-}a) = (^{-}a) = {}^{-}a = \overset{-}{a} = \{a - x, x \in R_{+}^{*} \mid x \text{ is infinitesimal}\}$.

Right Monad { that we denote, for simplicity, by (a^{+}) or only by a^{+} } is defined as:

$\mu(a^{+}) = (a^{+}) = a^{+} = \overset{+}{a} = \{a + x, x \in R_{+}^{*} \mid x \text{ is infinitesimal}\}$.

Pierced Binad { that we denote, for simplicity, by ($^{-}a^{+}$) or only $^{-}a^{+}$ } is defined as:

$\mu(^{-}a^{+}) = (^{-}a^{+}) = {}^{-}a^{+} = \overset{-+}{a} =$

$= \{a - x, x \in R_{+}^{*} \mid x \text{ is infinitesimal}\} \cup \{a + x, x \in R_{+}^{*} \mid x \text{ is infinitesimal}\}$

$= \{a \pm x, x \in R_{+}^{*} \mid x \text{ is infinitesimal}\}$.

The left monad, right monad, and the pierced binad are subsets of R^*.

25. Second Extension of NonStandard Analysis

For necessity of doing calculations that will be used in NonStandard neutrosophic logic in order to calculate the NonStandard neutrosophic logic operators (conjunction, disjunction, negation, implication, equivalence) and in order to have the NonStandard Real MoBiNad Set closed under arithmetic operations, Smarandache extended in 2019: the left monad to the Left Monad Closed to the Right, the right monad to the Right Monad Closed to the Left; and the Pierced Binad to the Unpierced Binad, defined as follows:

Left Monad Closed to the Right

$$\mu\left(\overset{-0}{a}\right)=\left(\overset{-0}{a}\right)=\overset{-0}{a}=\{a-x \mid x=0, \text{ or } x\in R_{+}^{*}$$

and x is infinitesimal} = $\mu(^{-}a) \cup \{a\} = (^{-}a) \cup \{a\}$

$= {}^{-}a \cup \{a\}$.

Right Monad Closed to the Left

$$\mu\left(\overset{0+}{a}\right)=\left(\overset{0+}{a}\right)=\overset{0+}{a}=\{a+x \mid x=0, \text{ or } x\in R_{+}^{*}$$

and x is infinitesimal} = $\mu(a^{+}) \cup \{a\} = (a^{+}) \cup \{a\}$

$= a^{+} \cup \{a\}$.

Unpierced Binad

$$\mu\left(\overset{-0+}{a}\right)=\left(\overset{-0+}{a}\right)=\overset{-0+}{a}=\{a-x \mid x\in R_{+}^{*} \text{ and } x \text{ is infinitesimal}\}$$

$\cup \{a+x \mid x \in R_{+}^{*}$ and x is infinitesimal$\} \cup \{a\} =$

$= \{a \pm x \mid x = 0$, or $x \in R_{+}^{*}$ and x is infinitesimal$\}$

$= \mu(^{-}a^{+}) \cup \{a\} = (^{-}a^{+}) \cup \{a\} = {}^{-}a^{+} \cup \{a\}$

The element *{a}* has been included into the left monad, right monad, and pierced binad respectively.

26. NonStandard Neutrosophic Topology

The previous two extensions of NonStandard Analysis, used in the construction of NonStandard Neutrosophic Logic, NonStandard Neutrosophic Set, and NonStandard Neutrosophic Probability, were defined on the NonStandard Unit Interval

$I_{nontandard} =]^{-}0,1^{+}[$,

we have founded [13] since 1998, and we have previously [13-15] proposed it, where:

$I_{nontandard} =]^{-}0,1^{+}[= \{x; \overset{0}{x}, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; 0 \le x \le, x \in R\}$, where R is the set of real numbers.

Let $P(]^{-}0,1^{+}[)$ be the powerset of $]^{-}0,1^{+}[$.

Let $\tau = P(]^{-}0,1^{+}[)$, which means that τ is the family of all subsets of $P(]^{-}0,1^{+}[)$. Of course:

(i). ϕ and $]^{-}0,1^{+}[$ belong to τ.

(ii). The intersection of any finite number of elements in τ is in τ.

(iii). The union of any number of finite or infinite number of elements in τ is in τ.

Therefore, τ is a NonStandard Neutrosophic Topology.

Then $(]^{-}0,1^{+}[, \tau)$ is called a NonStandard Neutrosophic Topological Space.

27. NonStandard Topology

As a generalization of NonStandard Neutrosophic Topology one propose now the NonStandard Topology.

Let's consider the real numbers $a,b \in R$ and the real interval $[a,b]$. Let's extend it to a non-standard interval $]^{-}a,b^{+}[$ is the same way as for the NonStandard Neutrosophic Logic and Set.

Let's have by convention the same meaning of the following notations:

$\overset{def}{x} = \overset{0}{x}$, and $^{-}x = \overset{-}{x}$, also $x^{+} = \overset{+}{x}$ for any real number x.

Then:

$]^{-}a,b^{+}[= \{x; \overset{0}{x}, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; a \le x \le b, x \in R\}$, where R is the set of real numbers.

Let $U_{NonStandard} =]^{-}a,b^{+}[$ be a NonStandard interval, for $a < b$, where a and b are real numbers, and $P(U_{NonStandard})$ be the power set of $U_{NonStandard}$.

Then $P(U_{NonStandard})$ is formed by the empty set (ϕ) and itself $U_{NonStandard}$, together with all standard and NonStandard subsets of $]^{-}a,b^{+}[$.

The finite intersections, and finite or infinite unions of any standard and NonStandard subsets are still (standard or NonStandard) subsets of $U_{NonStandard}$.

Let $\tau_{NonStandard} \subseteq P(U_{NonStandard})$ be a family of standard or NonStandard subsets of $P(U_{NonStandard})$.

Then $\tau_{NonStandard}$ is called a NonStandard Topology on $U_{NonStandard}$ if it satisfies the following axioms:

(i). The empty set (ϕ) and $U_{NonStandard}$ belong to $\tau_{NonStandard}$.

(ii). The intersection of finite number of elements in $\tau_{NonStandard}$ is stil in $\tau_{NonStandard}$.

(iii). The union of any finite or infinite number of elements in $\tau_{NonStandard}$ is stil in $\tau_{NonStandard}$.

Then ($U_{NonStandard}, \tau_{NonStandard}$) is called a NonStandard Topological Space.

28. Extended NonStandard Real Set ($\overset{-0+}{ER}$)

We introduce it now for the first time:

$\overset{-0+}{ER} = \{x; \overset{0}{x}, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; x \in R\}$, actually:

$$\overset{-0+}{ER} = \overset{0}{R} \cup \overset{-}{R} \cup \overset{+}{R} \cup \overset{-0}{R} \cup \overset{0+}{R} \cup \overset{-+}{R} \cup \overset{-0+}{R},$$

where one uses the notations:

$$\overset{0}{R} \equiv \overset{def}{R}$$

$$\overset{-}{R} = \{\overset{-}{x}, x \in R\}$$

$$\overset{+}{R} = \{\overset{+}{x}, x \in R\}$$

$$\overset{-0}{R} = \{\overset{-0}{x}, x \in R\}$$

$$\overset{0+}{R} = \{\overset{0+}{x}, x \in R\}$$

$$\overset{-+}{R} = \{\overset{-+}{x}, x \in R\}$$

$$\overset{-0+}{R} = \{\overset{-0+}{x}, x \in R\}$$

29. Largest Extended NonStandard Real Topology

$P\left(\overset{-0+}{ER}\right)$, which is the powerset of $\overset{-0+}{ER}$, generates the Largest Extended NonStandard Real Topology on the whole Extended NonStandard Real Set $\overset{-0+}{ER}$.

30. Over/Under/Off-Sets and Logics and Probabilities

The Neutrosophic Set was extended [Smarandache, 2007] to Neutrosophic Overset (when some Neutrosophic component is > 1), since we observed that, for example, an employee working overtime deserves a degree of membership > 1, with respect to an employee that only works regular full-time and whose degree of membership = 1;

and to Neutrosophic Underset (when some Neutrosophic component is < 0), since, for example, an employee making more damage than benefit to his company deserves a degree of membership < 0, with respect to an employee that produces benefit to the company and has the degree of membership > 0;

and to and to Neutrosophic Offset (when some Neutrosophic components are off the interval [0, 1], i.e. some Neutrosophic component > 1 and some Neutrosophic component < 0).

Similarly for Over/Under/Off-Logic and respectively Over/Under/Off-Topology [16 - 19].

Since these ideas look counter-intuitive and totally different from the mainstream framework, we present below elementary examples from our real world of such degrees that are outside the box {we mean outside the interval [0, 1]}.

31. Real Example of OverMembership and UnderMembership

In a company a full-time employer works 40 hours per week. Let's consider the last week period.

Helen worked part-time, only 30 hours, and the other 10 hours she was absent without payment; hence, her membership degree was 30/40 = 0.75 < 1.

John worked full-time, 40 hours, so he had the membership degree 40/40 = 1, with respect to this company.

But George worked overtime 5 hours, so his membership degree was (40+5)/40 = 45/40 = 1.125 > 1.

Thus, we need to make distinction between employees who work overtime, and those who work full-time or part-time. That's why we need to associate a degree of membership strictly greater than 1 to the overtime workers.

Now, another employee, Jane, was absent without pay for the whole week, so her degree of membership was 0/40 = 0.

Yet, Richard, who was also hired as a full-time, not only didn't come to work last week at all (0 worked hours), but he produced, by accidentally starting a devastating fire, much damage to the company, which was estimated at a value half of his salary (i.e. as he would have gotten for working 20 hours that week). Therefore, his membership degree has to be less that Jane's (since Jane produced no damage). Whence, Richard's degree of membership, with respect to this company, was - 20/40 = - 0.50 < 0.

Consequently, we need to make distinction between employees who produce damage, and those who produce profit, or produce neither damage no profit to the company.

Therefore, the membership degrees > 1 and < 0 are real in our world, so we have to take them into consideration.

Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively *Neutrosophic Over-/Unde-r/Off-Logic, -Measure, -Probability, - Statistics* etc. (Smarandache, 2007).

32. Definition of the Single-Valued Neutrosophic OverSet

Let U_{over} be an OverUniverse of Discourse {i.e. there exist some elements in U_{over} whose degrees of membership are > 1 }, and the Neutrosophic OverSet $A_{over} \subseteq U_{over}$.

Let *T(x), I(x), F(x)* be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U_{over}$, with respect to the Neutrosophic OverSet A_{over}:

$T(x), I(x), F(x) : U_{over} \to [0, \Omega]$ where $0 < 1 < \Omega$, and Ω is called OverLimit,

$T(x), I(x), F(x) \in [0, \Omega]$, for all $x \in U_{over}$.

A Single-Valued Neutrosophic OverSet A_{over} is defined as:

A_{over} = {(x, <T(x), I(x), F(x)>), $x \in U_{over}$}, such that there exist some elements in A_{over} that have at least one neutrosophic component that is > 1.

33. Definition of the Single-Valued Neutrosophic OverTopology

Let U_{over} be an OverUniverse of Discourse, and $P(U_{over})$ the powerset of U_{over}.

Let $\tau_{over} \subseteq P(U_{over})$ be a family of Single-Valued Neutrosophic OverSets of U_{over}.

Then τ_{over} is called a Single-Valued Neutrosophic OverTopology on U_{over} if it satisfies the following axioms:

(i). ϕ and U_{over} belong to τ_{over}.

(ii). The intersection of any finite number of single-valued Neutrosophic OverSets in τ_{over} is in τ_{over}.

(iii). The union of any finite or infinite number of single-valued Neutrosophic OverSets in τ_{over} is in τ_{over}.

Then (U_{over}, τ_{over}) is called a Neutrosophic OverTopological Space.

34. Definition of the Single-Valued Neutrosophic UnderSet

The previous two extensions of NonStandard Analysis, used in the construction of NonStandard Neutrosophic Logic, NonStandard Neutrosophic Set, and NonStandard Neutrosophic Probability, were defined on the NonStandard Unit Interval.

Let U_{under} be an UnderUniverse of Discourse { i.e. there exist some elements in U_{under} whose degrees of membership are < 0 }, and the Neutrosophic UnderSet $A_{under} \subseteq U_{under}$.

Let *T(x), I(x), F(x)* be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U_{under}$, with respect to the Neutrosophic UnderSet A_{under}:

$T(x), I(x), F(x) : U_{under} \to [\Psi, 1]$

where $\Psi < 0 < 1$, and Ψ is called UnderLimit,

$T(x), I(x), F(x) \in [\Psi, 1]$, for all $x \in U_{under}$.

A Single-Valued Neutrosophic UnderSet A_{under} is defined as:

A_{under} = {(x, <T(x), I(x), F(x)>), $x \in U_{under}$}, such that there exist some elements in A_{under} that have at least one neutrosophic component that is < 0.

35. Definition of the Single-Valued Neutrosophic UnderTopology

Let U_{under} be an UnderUniverse of Discourse, and $P(U_{under})$ the powerset of U_{under}.

Let $\tau_{under} \subseteq P(U_{under})$ be a family of Single-Valued Neutrosophic UnderSets of U_{under}.

Then τ_{under} is called a Single-Valued Neutrosophic UnderTopology on U_{under} if it satisfies the following axioms:

(i). ϕ and U_{under} belong to τ_{under}.

(ii). The intersection of any finite number of single-valued neutrosophic undersets in τ_{under} is in τ_{under}.

(iii). The union of any finite or infinite number of single-valued neutrosophic undersets in τ_{under} is in τ_{under}.

Then (U_{under}, τ_{under}) is called a Neutrosophic UnderTopological Space.

36. Definition of the Single-Valued Neutrosophic OffSet

Let U_{off} be an OffUniverse of Discourse {i.e. there exist elements of U_{off} whose degrees of membership are outside the interval [0, 1], some < 0 and others > 1}, and the Neutrosophic OffSet $A_{off} \subseteq U$.

Let *T(x), I(x), F(x)* be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U_{off}$, with respect to the neutrosophic offset A_{off}:

$$T(x), I(x), F(x) : U_{off} \to [\Psi, \Omega]$$

where $\Psi < 0 < 1 < \Omega$, and Ψ is called UnderLimit, while Ω is called OverLimit,

$T(x), I(x), F(x) \in [\Psi, \Omega]$, for all $x \in U_{off}$.

A Single-Valued Neutrosophic Offset A_{off} is defined as:

A_{off} = {(x, <T(x), I(x), F(x)>), x ∈ U_{off}}, such that there exist some elements in A_{off} that have at least one neutrosophic component that is > 1, and at least one neutrosophic component that is < 0.

37. Definition of the Single-Valued Neutrosophic OffTopology

The previous two extensions of NonStandard Analysis, used in the construction of NonStandard Neutrosophic Logic, NonStandard Neutrosophic Set, and NonStandard Neutrosophic Probability, were defined on the NonStandard Unit Interval.

Let U_{off} be an OffUniverse of Discourse, and $P(U_{off})$ the powerset of U_{off}.

Let $\tau_{off} \subseteq P(U_{off})$ be a family of Single-Valued Neutrosophic OffSets of U_{off}.

Then τ_{off} is called a Single-Valued Neutrosophic OffTopology on U_{off} if it satisfies the following axioms:

(i). ϕ and U_{off} belong to τ_{off}.

(ii). The intersection of any finite number of single-valued neutrosophic offsets in τ_{off} is in τ_{off}.

(iii). The union of any finite or infinite number of single-valued neutrosophic offsets in τ_{off} is in τ_{off}.

Then (U_{off}, τ_{off}) is called a Neutrosophic OffTopological Space.

38. Neutrosophic Triplet Weak/Strong Set (N)

Let *(N, *)* be a groupoid, or non-empty set endowed with a well-defined binary operation *.
A **Neutrosophic Triplet** is an object of the form *<x, neut(x), anti(x)>*, for $x \in N$,
where $neut(x) \in N$ is the neutral of *x*, different from the classical algebraic unitary element if any, such that:

$$x * neut(x) = neut(x) * x = x$$

and $anti(x) \in N$ is the opposite of x such that:

$$x * anti(x) = anti(x) * x = neut(x).$$

In general, an element x may have more neutrals (*neut*'s) and more opposites (*anti*'s).
The neutrosophic triplets and their neutrosophic triplet algebraic structures were first introduced by Florentin Smarandache and Mumtaz Ali [20 - 23] in 2014 - 2016.

39. Definition of the Neutrosophic Triplet Weak Set (NTS, *) is a set such that each element $a \in NTS$ is part of a neutrosophic triplet *<b, neut(b), anti(b)>*, i.e. *a = b*, or *a = neut(b)*, or *a = anti(b)*.

40. Definition of the Single-Valued Neutrosophic Triplet Weak Topology

Let $U_{Triplet-Weak}$ be a Universe of Discourse which has the structure of a Neutrosophic Triplet Weak Set, and $P(U_{Triplet-Weak})$ the powerset of $U_{Triplet-Weak}$.
Let $\tau_{Triplet-Weak} \subseteq P(U_{Triplet-Weak})$ be a family of Single-Valued Neutrosophic Triplet Weak Sets of $U_{Triplet-Weak}$.
Then $\tau_{Triplet-Weak}$ is called a Single-Valued Neutrosophic Triplet Weak Topology on $U_{Triplet-Weak}$ if it satisfies the following axioms:

(i). ϕ and $U_{Triplet-Weak}$ belong to $\tau_{Triplet-Weak}$.

(i). The intersection of any finite number of single-valued neutrosophic triplet weak sets in $\tau_{Triplet-Weak}$ is in $\tau_{Triplet-Weak}$.

(ii). The union of any finite or infinite number of single-valued neutrosophic triplet weak sets in $\tau_{Triplet-Weak}$ is in $\tau_{Triplet-Weak}$.

Then ($U_{Triplet-Weak}$, $\tau_{Triplet-Weak}$) is called a Neutrosophic Triplet Weak Topological Space.

41. Definition of Neutrosophic Triplet Strong Set (or Neutrosophic Triplet Set)

The groupoid *(N, *)* is called a neutrosophic triplet strong set if for any $a \in N$ there exist some neutral of *a*, denoted $neut(a) \in N$, different from the classical algebraic unitary element (if any), and some opposite of *a*, called $anti(a) \in N$.

Table 1. Example of Neutrosophic Triplet Strong Set.

*	**1**	**2**
1	2	1
2	1	1

The set ({1,2}, *) is a groupoid, without classical unit element.
Then <1, 2, 1> and <2, 1, 2> and are neutrosophic triplets.
The neutrosophic triplet strong set is N = {1, 2}.

42. Theorem on the Neutrosophic Triplet Strong and Weak Sets

Any neutrosophic triplet strong set is a neutrosophic triplet weak set, but not conversely.
Proof.

Let *(N, *)* be a neutrosophic triplet strong set. If $a \in N$, then is also included in N, therefore there exists a neutrosophic triplet in *N* that includes a, whence N is a neutrosophic triplet weak set.

Conversely, we prove by using a counterexample.

Let $Z_3 = \{0, 1, 2\}$, embedded with the multiplication × modulo *3*, which is a well-defined law. The classical unitary element in Z_3 is *1*.

$(Z_3, \times)$ is a neutrosophic triplet weak set, since the neutrosophic triplets formed in Z_3 with respect to the law × contain all elements *0, 1, 2,*

i.e. <*0, 0, 0*>, <*0, 0, 1*>, and <*0, 0, 2*>.

But (Z_3, ×) is not a neutrosophic triplet strong set, since, for example, for $2 \in Z_3$ there is no $neut(2) \neq 1$ and no *anti(2).*

43. Definition of the Single-Valued Neutrosophic Triplet Strong Topology

Let $U_{Triplet-Strong}$ be a Universe of Discourse which has the structure of a Neutrosophic Triplet Strong Set, and $P(U_{Triplet-Strong})$ the powerset of $U_{Triplet-Strong}$.

Let $\tau_{Triplet-Strong} \subseteq P(U_{Triplet-Strong})$ be a family of Single-Valued Neutrosophic Triplet Strong Sets of $U_{Triplet-Strong}$.

Then $\tau_{Triplet-Strong}$ is called a Single-Valued Neutrosophic Triplet Strong Topology on $U_{Triplet-Strong}$ if it satisfies the following axioms:

(i). ϕ and $U_{Triplet-Strong}$ belong to $\tau_{Triplet-Strong}$.

(ii). The intersection of any finite number of single-valued neutrosophic triplet strong sets in $\tau_{Triplet-Strong}$ is in $\tau_{Triplet-Strong}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic triplet strong sets in $\tau_{Triplet-Strong}$ is in $\tau_{Triplet-Strong}$.

Then ($U_{Triplet-Strong}$, $\tau_{Triplet-Strong}$) is called a Neutrosophic Triplet Strong Topological Space.

44. Neutrosophic Extended Triplet

A neutrosophic extended triplet is a neutrosophic triplet, defined as above, but where the *neutral* of *x* {denoted by $_e neut(x)$ and called "extended neutral", where “e” in front stands for ‘extended’} is

allowed to be equal to the classical algebraic unitary element (if any) of the law * defined on the set. Therefore, the restriction "different from the classical algebraic unitary element if any" is released.

Thus, a neutrosophic extended triplet is an object of the form $<x, {}_e neut(x), {}_e anti(x)>$, for $x \in N$, where ${}_e neut(x) \in N$ is the *extended neutral* of x, which can be equal or different from the classical algebraic unitary element if any, such that:

$X * {}_e neut(x) = {}_e neut(x) * x = x$

and $anti(x) \in N$ is the *extended opposite* of x such that:

$x * {}_e anti(x) = {}_e anti(x) * x = {}_e neut(x)$.

In general, for each $x \in N$ there are exist many ${}_e neut$'s (extended neutrals) and ${}_e anti$'s (extended opposites). The neutrosophic extended triplets were introduced by Smarandache in *2016*.

45. Definition of Neutrosophic Extended Triplet Weak Set

The set N is called a neutrosophic extended triplet weak set if for any $x \in N$ there exist a neutrosophic extended triplet $<y, {}_e neut(y), {}_e anti(y)>$ included in N, such that $x = y$ or $x = {}_e neut(y)$ or $x = {}_e anti(y)$.

46. Definition of the Single-Valued Neutrosophic Extended Triplet Weak Topology

Let $U_{Extended-Triplet-Weak}$ be a Universe of Discourse which has the structure of a Neutrosophic Extended Triplet Weak Set, and $P(U_{Extended-Triplet-Weak})$ the powerset of $U_{Extended-Triplet-Weak}$.

Let $\tau_{Extended-Triplet-Weak} \subseteq P(U_{Extended-Triplet-Weak})$ be a family of Single-Valued Neutrosophic Extended Triplet Weak Sets of $U_{Extended-Triplet-Weak}$.

Then $\tau_{Extended-Triplet-Weak}$ is called a Single-Valued Neutrosophic Extended Triplet Weak Topology on $U_{Extended-Triplet-Weak}$ if it satisfies the following axioms:

(i). ϕ and $U_{Extended-Triplet-Weak}$ belong to $\tau_{Extended-Triplet-Weak}$.

(ii). The intersection of any finite number of single-valued neutrosophic extended triplet weak sets in $\tau_{Extended-Triplet-Weak}$ is in $\tau_{Extended-Triplet-Weak}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic extended triplet weak sets in $\tau_{Extended-Triplet-Weak}$ is in $\tau_{Extended-Triplet-Weak}$.

Then ($U_{Extended-Triplet-Weak}$, $\tau_{Extended-Triplet-Weak}$) is called a Neutrosophic Extended Triplet Weak Topological Space.

47. Definition of Neutrosophic Extended Triplet Strong Set

The set N is called a neutrosophic extended triplet strong set if for any $x \in N$ there exist ${}_e neut(x) \in N$ and ${}_e anti(x) \in N$.

48. Definition of the Single-Valued Neutrosophic Extended Triplet Strong Topology

Let $U_{Extended-Triplet-Strong}$ be a Universe of Discourse which has the structure of a Neutrosophic Extended Triplet Strong Set, and $P(U_{Extended-Triplet-Strong})$ the powerset of $U_{Extended-Triplet-Strong}$.

Let $\tau_{Extended-Triplet-Strong} \subseteq P(U_{Extended-Triplet-Strong})$ be a family of Single-Valued Neutrosophic Extended Triplet Strong Sets of $U_{Extended-Triplet-Strong}$.

Then $\tau_{Extended-Triplet-Strong}$ is called a Single-Valued Neutrosophic Extended Triplet Strong Topology on $U_{Extended-Triplet-Strong}$ if it satisfies the following axioms:

(i). ϕ and $U_{Extended-Triplet-Strong}$ belong to $\tau_{Extended-Triplet-Strong}$.

(ii). The intersection of any finite number of single-valued neutrosophic extended triplet strong sets in $\tau_{Extended-Triplet-Strong}$ is in $\tau_{Extended-Triplet-Strong}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic extended triplet strong sets in $\tau_{Extended-Triplet-Strong}$ is in $\tau_{Extended-Triplet-Strong}$.

Then ($U_{Extended-Triplet-Strong}$, $\tau_{Extended-Triplet-Strong}$) is called a Neutrosophic Extended Triplet Strong Topological Space.

49. Neutrosophic Duplets

The Neutrosophic Duplets and the Neutrosophic Duplet Algebraic Structures were introduced by Florentin Smarandache in 2016.

Let *U* be a universe of discourse, and a set *D* included in *U*, endowed with a well-defined law #.

50. Definition of the Neutrosophic Duplet

We say that <*a, neut(a)*>, where *a,* and its neutral *neut(a)* belong to D, is a neutrosophic duplet if:

(i). *neut(a)* is different from the unitary element of *D* with respect to the law # (if any);

(ii). *a # neut(a) = neut(a) # a = a;*

(iii). there is no opposite *anti(a)* belonging to *D* for which

a # anti(a) = anti(a) # a = neut(a).

51. Example of Neutrosophic Duplets

In $(Z_8, \#)$, the set of integers with respect to the regular multiplication modulo 8, one has the following neutrosophic duplets:

<2, 5 >, <4, 3>, <4, 5>, <4, 7>, and <6, 5>.

Proof:

Let $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$, having the unitary element *1* with respect to the multiplication # *modulo 8*.

$2 \# 5 = 5 \# 2 = 10 = 2 \pmod 8$, so $neut(2) = 5 \neq 1$.

There is no $anti(2) \in Z_8$, because:

$2 \# anti(2) = 5 \pmod 8$, or $2y = 5 \pmod 8$ by denoting $anti(2) = y$, is equivalent to:

$2y - 5 = M_8$ {multiple of 8}, or $2y - 5 = 8k$, where *k* is an integer, or $2(y - 4k) = 5$, where both *y* and *k* are integers, or: e*ven number = odd number*, which is impossible.

Therefore, we proved that <*2, 5*> is a neutrosophic duplet.

Similarly for <*4, 5*>, <*4, 3*>, <*4, 7*>, and <*6, 5*>.

A counter-example: <*0, 0*> is not a neutrosophic duplet, because it is a neutrosophic triplet: <*0, 0, 0*>, where there exists an *anti(0) = 0*.

52. Definition of the Single-Valued Neutrosophic Duplet Topology

Let U_{Duplet} be a Universe of Discourse which has the structure of a Neutrosophic Duplet Set, and $P(U_{Duplet})$ the powerset of U_{Duplet}.

Let $\tau_{Duplet} \subseteq P(U_{Duplet})$ be a family of Single-Valued Neutrosophic Duplet Sets of U_{Duplet}.

Then τ_{Duplet} is called a Single-Valued Neutrosophic Duplet Topology on U_{Duplet} if it satisfies the following axioms:

(i). ϕ and U_{Duplet} belong to τ_{Duplet}.

(ii). The intersection of any finite number of single-valued neutrosophic duplet sets in τ_{Duplet} is in τ_{Duplet}.

(iii). The union of any finite or infinite number of single-valued neutrosophic duplet sets in τ_{Duplet} is in τ_{Duplet}.

Then (U_{Duplet}, τ_{Duplet}) is called a Neutrosophic Duplet Topological Space.

53. Definition of the Neutrosophic Extended Duplet

Let *U* be a universe of discourse, and a set *D* included in *U*, endowed with a well-defined law #. We say that <*a*, $_{e}neut(a)$>, where *a*, and its extended neutral $_{e}neut(a)$ belong to *D*, such that:

(i). $_{e}neut(a)$ may be equal or different from the unitary element of *D* with respect to the law # (if any);

(ii). $a \# {}_{e}neut(a) = {}_{e}neut(a) \# a = a$;

(iii). There is no extended opposite $_{e}anti(a)$ belonging to *D* for which

$a \# {}_{e}anti(a) = {}_{e}anti(a) \# a = {}_{e}neut(a)$.

54. Definition of the Single-Valued Neutrosophic Extended Duplet Topology

Let $U_{Extended-Duplet}$ be a Universe of Discourse which has the structure of a Neutrosophic Extended Duplet Set, and $P(U_{Extended-Duplet})$ the powerset of $U_{Extended-Duplet}$.

Let $\tau_{Extended-Duplet} \subseteq P(U_{Extended-Duplet})$ be a family of Single-Valued Neutrosophic Duplet Sets of $U_{Extended-Duplet}$.

Then $\tau_{Extended-Duplet}$ is called a Single-Valued Neutrosophic Duplet Topology on $U_{Extended-Duplet}$ if it satisfies the following axioms:

(i). ϕ and $U_{Extended-Duplet}$ belong to $\tau_{Extended-Duplet}$.

(ii). The intersection of any finite number of single-valued neutrosophic extended duplet sets in $\tau_{Extended-Duplet}$ is in $\tau_{Extended-Duplet}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic extended duplet sets in $\tau_{Extended-Duplet}$ is in $\tau_{Extended-Duplet}$.

Then ($U_{Extended-Duplet}$, $\tau_{Extended-Duplet}$) is called a Neutrosophic Extended Duplet Topological Space.

55. Definition of Neutrosophic MultiSet

The Neutrosophic MultiSet and the Neutrosophic Multiset Algebraic Structures were introduced by Florentin Smarandache [23] in 2016.

Let $\mathcal{U}$ be a universe of discourse, and a set $M \subseteq U$.

A *Neutrosophic Multiset M* is a neutrosophic set where one or more elements are repeated with the same neutrosophic components, or with different neutrosophic components.

It is an extension of the classical multiset, fuzzy multiset, intuitionistic fuzzy multiset, etc.

56. Examples of Neutrosophic MultiSets

A = {(0.6, 0.3, 0.1), (0.8, 0.4, 0.2), c(0.5, 0.1, 0.3)} is a neutrosophic set (not multiset).
But B = {(0.6, 0.3, 0.1),(0.6, 0.3, 0.1),b(0.8, 0.4, 0.2)} is a neutrosophic multiset, since the element a is repeated; we say that the element a has the *neutrosophic multiplicity 2* with the same neutrosophic components.
While C = {(0.6, 0.3, 0.1), (0.7, 0.1, 0.2), a(0.5, 0.4, 0.3), c(0.5, 0.1, 0.3)} is also a neutrosophic multiset, because the element a is repeated (it has the *neutrosophic multiplicity 3*), but with different neutrosophic components, since, for example, during the time, the neutrosophic membership of an element may change.

If the element a is repeated k times, keeping the same neutrosophic components $(ta,,fa)$, we say that a has *multiplicity* k.
But if there is some change in the neutrosophic components of a, we say that a has the *neutrosophic multiplicity* k.

Therefore, we define in general the *Neutrosophic Multiplicity Function (nm)*:

nm: $\mathcal{U} \to \mathbb{N}$ = {1, 2, 3, …, ∞}, and for any $a \in A$ one has

(a) = {(k_1, ⟨t_1, i_1, f_1⟩), (k_2, ⟨t_2, i_2, f_2⟩), …, (k_j, ⟨t_j, i_j, f_j⟩), …} which means that

a is repeated k_1 times with the neutrosophic components ⟨t_1, i_1, f_1⟩;

a is repeated k_2 times with the neutrosophic components ⟨t_2, i_2, f_2⟩, ...,

a is repeated k_j times with the neutrosophic components ⟨t_j, i_j, f_j⟩, ..., and so on.

Then, a neutrosophic multiset A can be written as:

A = {(a, (a)), for $a \in A$)}.

57. Examples of operations with neutrosophic multisets

Let's have:

A = {5⟨0.6, 0.3, 0.2⟩, 5⟨0.6, 0.3, 0.2⟩, 5⟨0.4, 0.1, 0.3⟩, 6⟨0.2, 0.7, 0.0⟩};

B = {5⟨0.6, 0.3, 0.2⟩, 5⟨0.8, 0.1, 0.1⟩, 6⟨0.9, 0.0, 0.0⟩};

C = {5⟨0.6, 0.3, 0.2⟩, 5⟨0.6, 0.3, 0.2⟩}.

Then:

Intersection of Neutrosophic Multisets.

$A \cap B$ = {5⟨0.6, 0.3, 0.2⟩}.

Union of Neutrosophic Multisets

$A \cup B$ ={5⟨0.6, 0.3, 0.2⟩, 5⟨0.6, 0.3, 0.2⟩, 5⟨0.4, 0.1, 0.3⟩, 5⟨0.8, 0.1, 0.1⟩, 6⟨0.2, 0.7, 0.0⟩, 6⟨0.9, 0.0, 0.0⟩}.

Inclusion of Neutrosophic Multisets

$C \subset A$, but $C \not\subset B$.

58. Definition of the Single-Valued Neutrosophic MultiSet Topology

Let $U_{MultiSet}$ be a Universe of Discourse which has the structure of a Neutrosophic MultiSet, and $P(U_{MultiSet})$ the powerset of $U_{MultiSet}$.

Let $\tau_{MultiSet} \subseteq P(U_{MultiSet})$ be a family of Single-Valued Neutrosophic MultiSets of $U_{MultiSet}$.

Then $\tau_{MultiSet}$ is called a Single-Valued Neutrosophic MultiSet Topology on $U_{MultiSet}$ if it satisfies the following axioms:

(i). ϕ and $U_{MultiSet}$ belong to $\tau_{MultiSet}$.

(ii). The intersection of any finite number of single-valued neutrosophic multisets in $\tau_{MultiSet}$ is in $\tau_{MultiSet}$.

(iii). The union of any finite or infinite number of single-valued neutrosophic multisets in $\tau_{MultiSet}$ is in $\tau_{MultiSet}$.

Then $(U_{MultiSet}, \tau_{MultiSet})$ is called a Neutrosophic MultiSet Topological Space.

59. Conclusion

These eight new avantgarde topologies, together with the previous six new topologies and their corresponding topological space, were introduced by Smarandache in 2019-2023, but they have not yet been much studied and applied, except the NeutroTopologies and AntiTopologies [8] which got some attention from researchers. While NonStandard Neutrosophic Topology, Neutrosophic Triplet Weak/Strong Topologies, Neutrosophic Extended Triplet Weak/Strong Topologies, Neutrosophic Duplet topology, Neutrosophic Extended Duplet Topology, Neutrosophic MultiSet Topology are proposed now for the first time. As future research would be to study their large applications in our real world.

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Fundamentos de Topologías de Vanguardia
(artículo de revisión parcial)

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Resumen. Recientemente hemos encontrado nueve nuevas topologías: Topología No Estándar, Mayor Extensión de la Topología Real No Estándar, Topologías Débiles/Fuertes de Triplete Neutrosófico, Topologías Débiles/Fuertes de Triplete Neutrosófico Extendido, Topología de Dupla Neutrosófica, Topología de Dupla Neutrosófica Extendida, Topología de Multi Conjunto Neutrosófico, y recordamos y mejoramos las siete topologías previamente fundadas en los años (2019-2023), a saber: Topología Neutrosófica No Estándar, Neutro Topología, Anti Topología, Topología Neutrosófica Refinada, Topología Nítida Neutrosófica Refinada, Super Hiper Topología y Super Hiper Topología Neutrosófica. Se llaman topologías de vanguardia debido a sus formas innovadoras.

Summary. We have recently found nine new topologies: Non-Standard Topology, Further Extension of Real Non-Standard Topology, Weak/Strong Neutrosophic Triplet Topologies, Weak/Strong Neutrosophic Extended Triplet Topologies, Neutrosophic Dupla Topology, Neutrosophic Extended Dupla Topology, Neutrosophic Multi Set Topology, and we recall and improve the previously founded seven topologies in the years (2019-2023), viz: Non-Standard Neutrosophic Topology, Neutrosophic Topology, Anti Topology, Refined Neutrosophic Topology, Refined Neutrosophic Sharp Topology, Super Hyper Topology and Super Hyper Neutrosophic Topology. They are called state-of-the-art topologies because of their innovative shapes.

Palabras clave: Topología Clásica, Espacio Topológico, NeutroSoficación, AntiSoficación, NeutroTopología, AntiTopología, Topología Neutrosófica Refinada, Topología Nítida Neutrosófica Refinada, Super Hiper Topología, Neutro Super Hiper Topología, Conjunto Real No Estándar Extendido, Topología No Estándar, Topología Neutrosófica No Estándar, Topología Neutrosófica No Estándar Extendida más Grande; monada izquierda, monada derecha, binada perforada, monada izquierda cerrada a la derecha, monada derecha cerrada a la izquierda, binada no perforada, SobreTopología Neutrosófica, InfraTopología Neutrosófica, ExtraTopología Neutrosófica, (Difusa y Difusa-Extensiones) Sobre/Infra/Extra-Topologías, Topología Multi Conjunto Neutrosófica.

Keywords: Classical Topology, Topological Space, NeutroSofication, AntiSofication, NeutroTopology, AntiTopology, Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, Super Hyper Hyper Topology, Neutro Super Hyper Topology, Extended Non-Standard Real Set, Non-Standard Topology, Non-Standard Neutrosophic Topology, Largest Extended Non-Standard Neutrosophic Topology; Left Monad, Right Monad, Perforated Binned, Left Closed Right Monad, Right Closed Left Monad, Non-Perforated Binned, Neutrosophic OverTopology, Neutrosophic InfraTopology, Neutrosophic ExtraTopology, (Fuzzy and Fuzzy-Extensions) Over/Infra/Extra-Topologies, Neutrosophic Multi-Set Topology.

1 Introducción

La base de nuevas topologías surgieron del desarrollo de otros campos como Neutro Álgebra y Anti Álgebra (que dieron origen a la NeutroTopología y la AntiTopología), Super Hiper Álgebra y Neutro Super Hiper Álgebra (que dieron origen a la Super Hiper Topología y la Neutro Super Hiper Topología), Conjunto Nítido Refinado (que dio origen a la Topología Nítida Refinada), y Conjunto Neutrosófico Refinado (que dio origen a la Topología Neutrosófica Refinada), y Conjunto No Estándar (que da origen a la Topología No Estándar y la Topología Neutrosófica No Estándar), Conjunto de Triplete Neutrosófico, Conjunto de Triplete Neutrosófico Extendido, Conjunto Neutrosófico Dual, Conjunto Neutrosófico Dual Extendido, y Conjunto Neutrosófico Multi Conjunto.

Este es casi un territorio virgen ya que se ha hecho poca investigación al respecto, principalmente sobre la AntiTopología [8]. Sin embargo, es un campo prometedor para estudiar en el futuro, ya que refleja mejor nuestro mundo real, donde las leyes (axiomas) no se aplican en el mismo grado para todas las personas (las personas poderosas están por encima de la ley, otras son inmunes a la ley, y muchos sienten todo el peso de la ley); ya que el mundo como un sistema dinámico está formado por subsistemas, y cada subsistema por sub-subsistemas y así sucesivamente (de ahí la necesidad de introducir la Super Hiper Estructura basada en el n-ésimo Conjunto Potencia de un Conjunto, cuyos casos particulares son la Super Hiper Álgebra y la Super Hiper Topología), etc.

Recordamos la definición clásica de Topología, luego los procedimientos de NeutroSoficación y respectivamente AntiSoficación de la misma, lo que resulta en la adición de dos nuevos tipos de topologías: Neutro Topología y AntiTopología respectivamente.

Luego definimos la topología en un Conjunto Neutrosófico Refinado (2013), Conjunto Nítido Neutrosófico Refinado [3]. Después, extendemos la topología en el marco de la Super Hiper Álgebra [6], luego el Conjunto Neutrosófico No Estándar a la Topología No Estándar y la Neutro Topología No Estándar (nunca antes definidas).

Se presentan los espacios topológicos neutrosóficos correspondientes.

Esta investigación es una mejora del documento [7] y el libro [12, secciones 4.8 y 4.9].

2. Topología Clásica

Sea $\mathcal{U}$ un conjunto no vacío y P($\mathcal{U}$) el conjunto potencia de $\mathcal{U}$.
Sea $\tau \subseteq P(\mathcal{U})$ una familia de subconjuntos de $\mathcal{U}$.
Entonces τ se llama Topología Clásica en $\mathcal{U}$ si satisface los siguientes axiomas:
(CT-1) ϕ y $\mathcal{U}$ pertenecen a τ.
(CT-2) La intersección de cualquier número finito de elementos en τ está en τ.
(CT-3) La unión de cualquier número finito o infinito de elementos en τ está en τ.
Los tres axiomas son totalmente (100%) verdaderos (o T = 1, I = 0, F = 0). Simplemente los llamamos *axiomas* (clásicos).
Entonces ($\mathcal{U}$,τ) se llama *Espacio Topológico Clásico* en $\mathcal{U}$.

3. NeutroSoficación de los Axiomas Topológicos

La NeutroSoficación de los axiomas topológicos implica que los axiomas se vuelven parcialmente verdaderos, parcialmente indeterminados y parcialmente falsos. Se les llama *Neutro Axiomas*.
(NCT-1) O bien $\{\phi \notin \tau$ and $\mathcal{U} \in \tau\}$o $\{\phi \in \tau$ and $\mathcal{U} \notin \tau\}$
(NCT-2) Existen un número finito de elementos en τ cuya intersección pertenece a τ (grado de verdad *T*); y un número finito de elementos en τ cuya intersección es indeterminada (grado de indeterminación *I*); y un número finito de elementos en τ cuya intersección no pertenece a τ (grado de falsedad *F*); donde (*T, I, F*) ∉ {(1,0,0),(0,0,1)} ya que (1, 0, 0) representa la Topología Clásica, mientras que (0, 0, 1) representa la Anti Topología.
(NCT-3) Existen un número finito o infinito de elementos en τ cuya unión pertenece a τ (grado de verdad *T*); y un número finito o infinito de elementos en τ cuya unión es indeterminada (grado de indeterminación *I*); y un número finito o infinito de elementos en τ cuya unión no pertenece a τ (grado de falsedad *F*); donde, por supuesto, (T, I, F) ∉ {(1,0,0),(0,0,1)}.

4. AntiSoficación de los Axiomas Topológicos Clásicos

AntiSoficación de los axiomas topológicos significa negar (anti) los axiomas, los cuales se vuelven totalmente (100%) falsos (o T = 0, I = 0, F = 1). Se les llama Anti Axiomas.
(AAT-1) $\phi \notin \tau$ y $\mathcal{U} \notin \tau$.
(AAT-2) La intersección de cualquier número finito (n≥2) de elementos en τ no está en τ.
(AAT-3) La unión de cualquier número finito o infinito (n≥2) de elementos en τ no está en τ.

5. <Topología, NeutroTopología, AntiTopología>

Como tal, tenemos un triplete neutrosófico de la forma:
<Axioma(1, 0, 0), NeutroAxioma(*T, I, F*), AntiAxioma(0, 0, 1)>,
donde (T, I, F)≠ (1, 0, 0) y (T, I, F)≠ (0, 0, 1).
En consecuencia, se tiene:
<Topología, NeutroTopología, AntiTopología>.
Por tanto, en general:

La Topología (Clásica) es una topología que tiene todos los axiomas totalmente verdaderos. Simplemente los llamamos *Axiomas*.

La *NeutroTopología* es una topología que tiene al menos un *NeutroAxioma* y los demás son todos *Axiomas clásicos* [por lo tanto, no hay *AntiAxioma*].

La *AntiTopología* es una topología que tiene uno o más *AntiAxiomas*, sin importar cuáles sean los otros (*Axiomas clásicos* o *NeutroAxiomas*).

6. Teorema sobre el número de Estructuras/Neutro Estructuras/Anti Estructuras

Si una Estructura tiene m axiomas, con m ≥ 1, entonces después de la NeutroSoficación y la AntiSoficación se obtienen 3m tipos de estructuras, categorizados de la siguiente manera:

1 Estructura Clásica + (2m - 1) Neutro Estructuras + (3m - 2m) Anti Estructuras = 3m Estructuras.

7. Consecuencia sobre el número de Topologías/Neutro Topologías/Anti Topologías

Como caso particular del teorema anterior, a partir de una Topología que tiene m = 3 axiomas, se obtienen, después de la NeutroSoficación y la AntiSoficación, 3^3 = 27 tipos de estructuras, como sigue: 1 Topología clásica, 3^3 - 2^3 = 7 NeutroTopologías y 3^3 - 2^2 = 19 AntiTopologías.

1 Topología Clásica + 7 NeutroTopologías + 19 AntiTopologías = 27 Topologías

se presentan a continuación:

Hay 1 (un) tipo de Topología Clásica, cuyos axiomas se enumeran a continuación:

1 Topología Clásica.

$$\begin{pmatrix} CT-1 \\ CT-2 \\ CT-3 \end{pmatrix}$$

8. Definición de NeutroTopología [4, 5]

Es una topología que tiene al menos un axioma topológico que es parcialmente verdadero, parcialmente indeterminado y parcialmente falso, o (T, I, F), donde T = Verdadero, I = Indeterminación, F = Falso, y ningún axioma topológico es totalmente falso, en otras palabras: , donde (1, 0, 0) representa la Topología clásica, mientras que (0, 0, 1) representa la AntiTopología. Por lo tanto, la NeutroTopología es una topología intermedia entre la Topología clásica y la AntiTopología.

Hay 7 tipos de NeutroTopologías diferentes, cuyos axiomas, para cada tipo, se enumeran a continuación:

7 NeutroTopologías

$$\begin{pmatrix} NCT-1 \\ CT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ CT-2 \\ NCT-3 \end{pmatrix},$$

$$\begin{pmatrix} NCT-1 \\ NCT-2 \\ CT-3 \end{pmatrix}, \begin{pmatrix} CT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}, \begin{pmatrix} NCT-1 \\ CT-2 \\ NCT-3 \end{pmatrix},$$

$$\begin{pmatrix} NCT-1 \\ NCT-2 \\ NCT-3 \end{pmatrix}.$$

9. Definición de AntiTopología [4, 5]

Es una topología que tiene al menos un axioma topológico que es 100% falso (T, I, F) = (0, 0, 1).

La NeutroTopología y la AntiTopología son casos particulares de la Neutro Álgebra y la Anti Álgebra [4] y, en general, todas son casos particulares de la Neutro Estructura y la Anti Estructura respectivamente, ya que consideramos "Estructura" en cualquier campo del conocimiento [5].

Hay 19 tipos diferentes de AntiTopologías, cuyos axiomas, para cada tipo, se enumeran a continuación:

19 *AntiTopologías*

$$\binom{ACT-1}{CT-2}, \binom{CT-1}{ACT-2}, \binom{CT-1}{CT-2}, \binom{CT-1}{CT-2}, \binom{CT-1}{ACT-3},$$

$$\binom{ACT-1}{ACT-2}, \binom{CT-1}{ACT-2}, \binom{ACT-1}{CT-2}, \binom{ACT-1}{CT-2}, \binom{ACT-1}{ACT-3},$$

$$\binom{ACT-1}{NCT-2}, \binom{NCT-1}{ACT-2}, \binom{NCT-1}{NCT-2}, \binom{NCT-1}{NCT-2}, \binom{NCT-1}{ACT-3},$$

$$\binom{ACT-1}{ACT-2}, \binom{NCT-1}{ACT-2}, \binom{ACT-1}{NCT-2}, \binom{ACT-1}{NCT-2}, \binom{ACT-1}{ACT-3},$$

$$\binom{ACT-1}{NCT-2}, \binom{CT-1}{ACT-2}, \binom{NCT-1}{CT-2}, \binom{CT-1}{ACT-2}, \binom{NCT-1}{ACT-3},$$

$$\binom{ACT-1}{CT-2}, \binom{CT-1}{NCT-2}, \binom{NCT-1}{ACT-2}, \binom{NCT-1}{ACT-2}, \binom{CT-1}{CT-3},$$

$$\binom{ACT-1}{ACT-2}, \binom{ACT-1}{ACT-2}, \binom{ACT-1}{ACT-3}.$$

10. Conjunto Neutrosófico Refinado

Sea U un universo de discurso y R un subconjunto no vacío del mismo,

$$R = \begin{Bmatrix} x\left(T_1(x), T_2(x), \dots, T_p(x)\right); \\ \left(I_1(x), I_2(x), \dots, I_r(x)\right); \\ \left(F_1(x), F_2(x), \dots, F_s(x)\right); \end{Bmatrix}$$

con todos los $T_j, I_k, F_l \in [0,1], 1 \le j \le p, 1 \le k \le r, 1 \le l \le s$, y sin restricción en sus sumas $0 \le T_m + I_m + F_m \le 3$, con $1 \le m \le max\{p, r, s\}$, donde *p, r, s* ≥ 0 son enteros fijos, y al menos uno de ellos es ≥ 2, para garantizar el refinamiento (subpartes) o multiplicidad (multipartes) -dependiendo de la aplicación- de al menos un componente neutrosófico entre *T* (verdad), *I* (indeterminación), *F* (falsedad); y por supuesto $x \in \mathcal{U}$.

Por notación, consideramos que el índice cero significa el conjunto vacío, es decir, $T_0 = I_0 = F_0 = \phi$ (o cero), y lo mismo para las subpartes (o multipartes) faltantes. Por ejemplo, el conjunto Neutrosófico Refinado (2,3,1) abajo es idéntico a un conjunto Neutrosófico Refinado (3,3,3): $(T_1, T_2; I_1, I_2, I_3; F_1) \equiv (T_1, T_2, 0; I_1, I_2, I_3; F_1, 0, 0)$, donde los componentes faltantes T_3, y F_2, F_3 fueron reemplazados cada uno por 0 (cero). *R* se llama *conjunto neutrosófico refinado (p, r, s) { o (p, r, s)-RNT }*.

El conjunto neutrosófico ha sido extendido al Conjunto (Lógica y Probabilidad) Neutrosófico Refinado por Smarandache [1] en 2013, donde existen múltiples partes de los componentes neutrosóficos, como T que se dividió en subcomponentes T_1, T_2, ..., T_p, e I en I_1, I_2, ..., I_r, y F en F_1, F_2, ..., F_s, con $p + r + s = n \geq 2$ y enteros $p, r, s \geq 0$ y al menos uno de ellos es ≥ 2 para garantizar el refinamiento (o multiplicidad) de al menos un componente neutrosófico entre T, I, y F.

Aún más: los subcomponentes T_j, I_k, y/o F_l pueden ser conjuntos infinitos contables o no contables en [0, 1].

Esta definición también incluye el *Conjunto Difuso Refinado*, cuando $r = s = 0$ y $p \geq 2$;

y la definición del *Conjunto Difuso Intuicionista Refinado*, cuando r = 0, ya sea $p \geq 2$ y $s \geq 1$, o $p \geq 1$ y $s \geq 2$.

Todos los demás conjuntos de extensión difusa (Conjunto Difuso Pitagórico, Conjunto Difuso Esférico, Conjunto Difuso Fermateano, Conjunto Difuso Ortopar q-Rung, etc.) pueden ser refinados/multiplicados de manera similar.

11. Definición de Topología Neutrosófica Refinada

Sea $\mathcal{U}$ un universo de discurso, y P($\mathcal{U}$) la familia de todos los conjuntos neutrosóficos refinados (*p, r, s*) de $\mathcal{U}$.

Sea $\tau_{RNT} \subseteq \mathcal{P}(\mathcal{U})$ una familia de conjuntos neutrosóficos refinados (p, r, s) de $\mathcal{U}$.

Entonces, τ_{RNT} se llama una *Topología Neutrosófica Refinada* (RNT) si satisface los axiomas:

(RNT-1) ϕ y $\mathcal{U}$ pertenecen a τ_{RNT};

(RNT-2) La intersección de cualquier número finito de elementos en τ_{RNT} está en τ_{RNT};

(RNT-3) La unión de cualquier número finito o infinito de elementos en τ_{RNT} está en τ_{RNT};

Entonces, $(\mathcal{U}, \tau_{RNT})$ se llama un Espacio Topológico Neutrosófico Refinado sobre $\mathcal{U}$.

La *Topología Neutrosófica Refinada* es una topología definida en un Conjunto Neutrxosófico Refinado.

{De manera similar, la Topología Difusa Refinada se define en un Conjunto Difuso Refinado, mientras que la Topología Difusa Intuicionista Refinada se define en un Conjunto Difuso Intuicionista Refinado, etc.

Y, como generalización, en cualquier tipo de conjunto de extensión difusa [Conjunto Difuso Pitagórico, Conjunto Difuso Esférico, Conjunto Difuso Fermateano, Conjunto Difuso Ortopar q-Rung, etc.] se puede definir una topología de extensión difusa correspondiente.}

12. Conjunto Nítido Neutrosófico

El *Conjunto Nítido Neutrosófico* fue definido por Salama y Smarandache en 2014 y 2015.
Sea X un espacio fijo no vacío. Y sea *D* un Conjunto Nítido Neutrosófico [2], donde *D* = <*A, B, C*>, con *A, B, C* como subconjuntos de *X*.
Dependiendo de las intersecciones y uniones entre estos tres conjuntos A, B, C se obtienen varios
Tipos de Conjuntos Nítidos Neutrosóficos [2, 3]
El objeto con la forma *D* = <*A, B, C*> es llamado:

(a) Un conjunto nítido neutrosófico de Tipo 1 (NCS-Tipo 1) si satisface:

$A \cap B = B \cap C = C \cap A = \phi$ (conjunto vacío).

(b) Un conjunto nítido neutrosófico de Tipo 2 (NCS-Tipo 2) si satisface:

$A \cap B = B \cap C = C \cap A = \phi$ y $A \cup B \cup C = X$.

(c) Un conjunto nítido neutrosófico de Tipo 3 (NCS-Tipo 3) si satisface:

$A \cap B \cap C = \phi$ y $A \cup B \cup C = X$.

Por supuesto, se pueden definir más tipos de conjuntos nítidos neutrosóficos modificando las intersecciones y uniones de los subconjuntos *A, B* y *C*.

13. Conjunto Nítido Neutrosófico Refinado

El *Conjunto Nítido Neutrosófico Refinado* [3] fue introducido por Smarandache en 2019, refinando/multiplicando *D* (y denotándolo como *RD = D Refinado*) mediante la refinación/multiplicación de sus conjuntos *A, B, C* en subconjuntos/subconjuntos múltiples de la siguiente manera:

$RD = (A_1, \ldots, A_p; B_1, \ldots, B_r; C_1, \ldots, C_s)$, con $p, r, s \geq 1$ siendo enteros positivos y al menos uno de ellos siendo ≥ 2 para garantizar la refinación/multiplicación de al menos un componente entre *A, B, C,* donde

$$A = \bigcup_{i=1}^{p} A_i, B = \bigcup_{j=1}^{r} B_j, C = \bigcup_{k=1}^{s} C_k$$

y muchos tipos de conjuntos nítidos neutrosóficos refinados se pueden definir modificando las intersecciones o uniones de los subconjuntos/multiconjuntos $A_i, B_j, C_k, 1 \leq i \leq p, 1 \leq j \leq r, 1 \leq k \leq s$, dependiendo de cada aplicación.

14. Definición de Topología Nítida Neutrosófica Refinada

Sea $\mathcal{U}$ un universo de discurso, y P($\mathcal{U}$) la familia de todos los subconjuntos nítidos neutrosóficos refinados (*p, r, s*) de $\mathcal{U}$.

Sea $\tau_{RNCT} \subseteq \mathcal{P}(\mathcal{U})$ una familia de subconjuntos nítidos neutrosóficos refinados (*p, r, s*) de $\mathcal{U}$.

Entonces, τ_{RNCT} se llama *Topología Nítida Neutrosófica Refinada (RNCT)* si satisface los axiomas:

(RNCT-1) ϕ y $\mathcal{U}$ pertenecen a τ_{RNCT};

(RNCT-2) La intersección de cualquier número finito de elementos en τ_{RNCT} está en τ_{RNCT};

(RNCT-3) La unión de cualquier número finito o infinito de elementos en τ_{RNCT} está en τ_{RNCT}.

Entonces, ($\mathcal{U}$, τ_{RNCT}) se llama un *Espacio Topológico Nítido Neutrosófico Refinado* en $\mathcal{U}$.

Por lo tanto, la *Topología Nítida Neutrosófica Refinada* es una topología definida en el Conjunto Nítido Neutrosófico Refinado.

15. Definición de los n-ésimos Conjuntos de Potencia $P^n(H)$ y $P_*^n(H)$.

Los nésimos Conjuntos de Potencia $P^n(H)$ y $P_*^n(H)$ del conjunto *H*, en los que se basan la Super Hiper Topología y respectivamente la Super Hiper Topología Neutrosófica, describen mejor nuestro mundo real, ya que un sistema *H* (que puede ser un conjunto, empresa, institución, país, región, etc.) está organizado en sub-sistemas, que a su vez están organizados cada uno de ellos en sub-subsistemas, y así sucesivamente.

El Conjunto de Potencia n-ésimo $P^n(H)$ se define recursivamente:

$$P^0(H) \stackrel{\text{def}}{=} H$$
$$P^1(H) = P(H)$$
$$P^2(H) = P(P(H))$$
$$P^3(H) = P(P^2(H)) = P(P(P(H)))$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P^n(H) = P(P^{n-1}(H)) = \underbrace{P(P(\ldots P(H)\ldots))}_{n}$$

donde *P* se repite *n* veces en la última fórmula,

y el conjunto vacío ϕ (que representa indeterminación, incertidumbre) está permitido en todos los términos de secuencia: $H, P(H), P^2(H), P^3(H), \ldots, P^n(H)$.

Similarmente,

El Conjunto de Potencia n-ésimo $P_*^n(H)$ se define recursivamente:

$$P_*^0(H) \overset{def}{=} H$$

$$P_*^1(H) = P_*(H)$$

$$P_*^2(H) = P_*(P_*(H))$$

$$P_*^3(H) = P_*(P_*^2(H)) = P_*(P_*(P_*(H)))$$

.....

$$P_*^n(H) = P_*(P_*^{n-1}(H)) = \underbrace{P_*(P_*(...P_*(H)...))}_{n}$$

donde P se repite n veces en la última fórmula,

y el conjunto vacío ϕ (que representa indeterminación, incertidumbre) no está permitido en ninguno de los términos de la secuencia: $H, P_*(H), P_*^2(H), P_*^3(H), ..., P_*^n(H)$.

16. Super Hiper Operación

Recordamos nuestros conceptos de 2016 de Super Híper Operación, Super Híper Axioma, Super Híper Álgebra y sus correspondientes Super Híper Operación Neutrosófica, Neutrosophic Super Hiper Axioma Neutrosófico y Neutrosophic Super Hiper Álgebra.

Sea $P_*^n(H)$ el n-ésimo conjunto potencia del conjunto H tal que ninguno de $P(H), P^2(H), ..., P^n(H)$ contiene el conjunto vacío ϕ.

Además, sea $P_n(H)$ el n-ésimo conjunto potencia del conjunto H tal que al menos uno de los conjuntos $P(H), P^2(H), ..., P^n(H)$ contiene el conjunto vacío ϕ. Para cualquier subconjunto A, identificamos {A} con A.

Las Super Hiper Operaciones son operaciones cuyo codominio es o bien $P_*^n(H)$ y en este caso se tienen Super Hiper Operaciones de tipo clásico, o bien $P^n(H)$ y en este caso se tienen Super Hiper Operaciones Neutrosóficas, para enteros n≥2.

17. El conjunto potencia n-ésimo describe mejor nuestro mundo real.

Los conjuntos potencia n-ésimos $P_*^n(H)$ y $P_*^n(H)$, en los que se basan la Super Hiper Topología y la Super Hiper Topología Neutrosófica respectivamente, describen mejor nuestro mundo real, ya que un sistema H (que puede ser un conjunto, una empresa, una institución, un país, una región, etc.) está organizado en sub-sistemas, que a su vez están organizados cada uno en sub-subsistemas, y así sucesivamente.

18. Super Hiper Axioma

Un Super Hiper Axioma de tipo clásico o más precisamente un (m, n)-Super Hiper Axioma es un axioma basado en Super Hiper Operaciones de tipo clásico.

De manera similar, un Super Hiper Axioma Neutrosófico {o (m, n)-Super Hiper Axioma Neutrosófico} es un axioma basado en Super Hiper Operaciones Neutrosóficas.

Existen:

- Super Hiper Axiomas Fuertes, cuando el lado izquierdo es igual al lado derecho como en los axiomas no hiper,
- y Super Hiper Axiomas Débiles, cuando la intersección entre el lado izquierdo y el lado derecho no es vacía.

19. Super Hiper Álgebra y Super Hiper Estructura

Una Super Hiper Álgebra o más precisamente una (m-n)-Super Hiper Álgebra es un álgebra que trata con Super Hiper Operaciones y Super Hiper Axiomas.

Por otra parte, una Super Hiper Álgebra Neutrosófica {o (m, n)-Super Hiper Álgebra Neutrosófica } es un álgebra que trata con Super Hiper Operaciones Neutrosóficas y Super Hiper Operaciones Neutrosóficas.

En general, tenemos Super Hiper Estructuras {o (m-n)-Super Hiper Estructuras}, y correspondientes Super Hiper Estructuras Neutrosóficas. Por ejemplo, existen Super Hiper Grupoides, Super Hiper Semigrupos, Super Hiper Grupos, Super Hiper Anillos, Super Hiper Espacios Vectoriales, etc.

20. Distinción entre Super Hiper Algebra y Super Hiper Algebra Neutrosófica

i. Si ninguno de los conjuntos de potencia $P^k(H)$, 1≤k≤n, no incluye el conjunto vacío ϕ, entonces se tiene una Super Hiper Álgebra de tipo clásico;
ii. Si al menos un conjunto de potencia, $P^k(H)$, 1≤k≤n, incluye el conjunto vacío ϕ, entonces se tiene una Super Hiper Álgebra Neutrosófica.

21. Definición de Super Hiper Topología (SHT)[6]

Es una topología diseñada sobre el Conjunto Potencia enésimo de un conjunto no vacío dado *H*, que excluye el conjunto vacío, denotado como $P_*^n(H)$, construido de la siguiente manera:

$P_*(H)$ es el primer conjunto potencia del conjunto *H*, y el índice * significa sin el conjunto vacío (Ø);

$P_*{}^2(H) = P_*(P_*(H))$ es el segundo conjunto potencia de *H* (o el conjunto potencia del conjunto potencia de *H*), sin los conjuntos vacíos; y así sucesivamente, el enésimo conjunto potencia de *H*,

$P_*{}^n(H) = P_*(P_*{}^{n-1}(H)) = \underbrace{P_*(P_*(\dots P_*(H)\dots))}_{n}$, donde P_* se repite *n* veces (n ≥ 2), y sin los conjuntos vacíos.

Consideremos τ_{SHT} como una familia de subconjuntos de $P_*{}^n(H)$.

Entonces, τ_{SHT} se llama Super Hiper Topología Neutrosófica sobre $P_*{}^n(H)$, si satisface los siguientes axiomas:

(SHT-1) ϕ y $P_*{}^n(H)$.) pertenecen a τ_{SHT}.

(SHT-2) La intersección de cualquier número finito de elementos en τ_{SHT} está en τ_{SHT}.

(SHT-3) La unión de cualquier número finito o infinito de elementos en τ_{SHT} está en τ_{SHT}.

Entonces, $(P_*{}^n(H), \tau_{SHT})$ se llama un Super Hiper Espacio Topológico en $P_*{}^n(H)$.

22. Definición de Super Hiper Topología Neutrosófica (NSHT)[6]

Es, de manera similar, una topología diseñada en el n-ésimo Conjunto Potencia de un conjunto dado no vacío *H*, pero también incluye los conjuntos vacíos [que representan las indeterminaciones].

Como tal, en las fórmulas anteriores, $P_*(H)$ que excluye el conjunto vacío, se reemplaza por $P(H)$ que incluye el conjunto vacío.

$P(H)$ es el primer conjunto potencia del conjunto *H*, incluyendo el conjunto vacío (Ø);

$P^2(H) = P(P(H))$ es el segundo conjunto potencia de *H* (o el conjunto potencia del conjunto potencia de *H*), que incluye los conjuntos vacíos;

y así sucesivamente, el n-ésimo conjunto potencia de *H*,

$$P^n(H) = P(P^{n-1}(H)) = \underbrace{P(P(\dots P(H)\dots))}_{n}$$

donde *P* se repite *n* veces (n ≥ 2) e incluye los conjuntos vacíos (Ø).

Consideremos τ_{NSHT} como una familia de subconjuntos de $P^n(H)$.

Entoncesτ_{NSHT} se llama Super Hiper Topología Neutrosófica en $P^n(H)$, si satisface los siguientes axiomas:

(NSHT-1) ϕ y $P^n(H)$ pertenecen a τ_{NSHT}.

(NSHT-2) La intersección de cualquier número finito de elementos en τ_{NSHT} está en τ_{NSHT}.

(NSHT-3) La unión de cualquier número finito o infinito de elementos en τ_{NSHT} está en τ_{NSHT}.

Entonces$(P^n(H), \tau_{NSHT})$ se llama Espacio Super Hiper Topológico Neutrosófico en $P^n(H)$.

23. Introducción al Análisis No Estándar [9, 10, 11, 12, 29]

Un ***infinitesimal*** (ε) es un número ε tal que |ε|<1/n, para cualquier número entero positivo *n* no nulo. Un infinitesimal está cerca de cero, tan pequeño que no puede ser medido.

El infinitesimal es un número más pequeño, en valor absoluto, que cualquier cosa positiva diferente de cero.

Los infinitesimales se utilizan en cálculo.

Un ***infinito*** (ω) es un número mayor que cualquier cosa:

1 + 1 + 1 + … + 1 (para cualquier cantidad finita de términos)

Los infinitos son los recíprocos de los infinitesimales.

El conjunto de ***hiperreales*** (o números no estándar reales), denotado como R^*, es la extensión del conjunto de los números reales, denotado como R, y comprende los infinitesimales y los infinitos, que pueden representarse en la *recta numérica hiperreal*.

$1/\varepsilon = \omega/1$.

El conjunto de hiperreales cumple el *principio de transferencia*, que establece que las afirmaciones de primer orden en R también son válidas en R^*.

Una *mónada (halo)* de un elemento $a \in R^*$, denotado por $\mu(a)$, es un subconjunto de números infinitesimalmente cercanos a a.

24. Primera extensión del análisis no estándar (Smarandache, 1998)

Denotemos por R+* el conjunto de números hiperreales positivos distintos de cero.

Consideramos la mónada izquierda y la mónada derecha, y la *binada (perforada)* que introdujimos como extensión en 1998 [5]:

Mónada izquierda { que denotamos, por simplicidad, por (^{-}a) o solo ^{-}a } se define como:
$\mu(^{-}a) = (^{-}a) = {}^{-}a = \bar{a} = \{a - x, x \in R_{+}^{*} \mid x \text{ es infinitesimal}\}$.

Mónada derecha { que denotamos, por simplicidad, por (a^{+}) o sólo por a^{+} } se define como:
$\mu(a^{+}) = (a^{+}) = a^{+} = \dot{a} = \{a + x, x \in R_{+}^{*} \mid x \text{ es infinitesimal}\}$.

Binada perforada { que denotamos, por simplicidad, por $(^{-}a^{+})$ o solo $^{-}a^{+}$ } se define como:

$\mu(^{-}a^{+}) = (^{-}a^{+}) = {}^{-}a^{+} = \dot{\bar{a}} =$

$= \{a - x, x \in R_{+}^{*} \mid x \text{ es infinitesimal}\} \cup \{a + x, x \in R_{+}^{*} \mid x \text{ es infinitesimal}\}$

$= \{a \pm x, x \in R_{+}^{*} \mid x \text{ es infinitesimal}\}$.

La mónada izquierda, la mónada derecha y la binada perforada son subconjuntos de R^*.

25. Segunda extensión del Análisis No Estándar

Por la necesidad de realizar cálculos que se utilizarán en la lógica neutrosófica no estándar con el fin de calcular los operadores de lógica neutrosófica no estándar (conjunción, disyunción, negación, implicación, equivalencia) y para tener el conjunto de MoBiNad Real no estándar cerrado bajo operaciones aritméticas, Smarandache extendió en 2019: la mónada izquierda a la Mónada Izquierda Cerrada a la Derecha, la mónada derecha a la Mónada Derecha Cerrada a la Izquierda; y la Binada Perforada a la Binada No Perforada, definido de la siguiente manera:

Mónada izquierda cerrada a la derecha

$\mu\left(\dot{\bar{a}}\right) = \left(\dot{\bar{a}}\right) = \dot{\bar{a}} = \{a - x \mid x = 0, \text{ or } x \in R_{+}^{*}$

y x es infinitesimal$\} = \mu(-a) \cup \{a\} = (-a) \cup \{a\}$

$= -\text{un} \cup \{a\}$.

Mónada derecha cerrada a la izquierda

$\mu\left(\dot{\bar{a}}\right) = \left(\dot{\bar{a}}\right) = \dot{\bar{a}} = \{a + x \mid x = 0, \text{ or } x \in R_{+}^{*}$

y x es infinitesimal$\} = \mu(a^{+}) \cup \{a\} = (a^{+}) \cup \{a\}$

$= a^{+} \cup \{a\}$.

Binada sin perforar

$\mu\left(\dot{\bar{a}}\right) = \left(\dot{\bar{a}}\right) = \dot{\bar{a}} = \{a - x \mid x \in R_{+}^{*}$ y x es infinitesimal$\}$

$\cup \{a + x \mid x \in R_{+}^{*}$ y x es infinitesimal$\} \cup \{\text{un}\} =$

$= \{a \pm x \mid x = 0, \text{ or } x \in R_{+}^{*}$ y x es infinitesimal$\}$

$= \mu(^{-}a^{+}) \cup \{a\} = (^{-}a^{+}) \cup \{a\} = {}^{-}a^{+} \cup \{a\}$

El elemento {a} se ha incluido en la mónada izquierda, la mónada derecha y la binada perforada respectivamente.

26. Topología Neutrosófica No Estándar

Las dos extensiones anteriores del Análisis No Estándar, utilizadas en la construcción de la Lógica Neutrosófica No Estándar, el Conjunto Neutrosófico No Estándar y la Probabilidad Neutrosófica No Estándar, se definieron en el Intervalo Unitario No Estándar.

$I_{noestandar} =]^{-}0, 1^{+}[$,

fundamos [13] desde 1998, y anteriormente lo hemos propuesto [13, 14, 15, 29], donde:

$I_{noestandar} =]^{-}0, 1^{+}[= \{x; \overset{0}{x}, x, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; 0 \leq x \leq, x \in R\}$, dónde R es el conjunto de los números reales.

Sea $(]^{-}0, 1^{+}[)$ el conjunto potencia de $]^{-}0, 1^{+}[$.

Sea $\tau = P(]^{-}0, 1^{+}[)$, lo que significa que τ es la familia de todos los subconjuntos de $P(]^{-}0, 1^{+}[)$. Por supuesto:

i. ϕ y $]^{-}0, 1^{+}[$ pertenecen a τ.
ii. La intersección de cualquier número finito de elementos en τ está en τ.
iii. La unión de cualquier número de elementos finitos o infinitos en τ está en τ.

Por lo tanto, τ es una topología neutrosófica no estándar.

Entonces $(]^{-}0, 1^{+}[, \tau)$ se denomina espacio topológico neutrosófico no estándar.

27. Topología no estándar

Como generalización de la Topología Neutrosófica No Estándar, se propone ahora la Topología No Estándar.

Consideremos los números reales $a, b \in R$ y el intervalo real $[a, b]$. Extendámoslo a un intervalo no estándar $]^{-}a, b^{+}[$ es de la misma manera que para la Lógica y el Conjunto Neutrosóficos No Estándar.

Tengamos por convención el mismo significado de las siguientes notaciones:

$x \overset{\text{def}}{=} \overset{0}{x}$, y $^{-}x = \overset{-}{x}$, also $x^{+} = \overset{+}{x}$ para cualquier número real x.

Entonces:

$]^{-}a, b^{+}[= \{x; \overset{0}{x}, x, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; a \leq x \leq b, x \in R\}$, donde R es el conjunto de los números reales.

Sea $U_{NonStandard} =]^{-}a, b^{+}[$ un intervalo no estándar, para a < b, donde a y b son números reales, y $P(U_{NonStandard})$ sea el conjunto potencia de $U_{NonStandard\ r}$.

Entonces $P(U_{NonStandard})$ está formado por el conjunto vacío (ϕ) y él mismo $U_{NonStandard\ r}$, junto con todos los subconjuntos estándar y no estándar de $]^{-}a, b^{+}[$.

Las intersecciones finitas y las uniones finitas o infinitas de cualquier subconjunto estándar y no estándar siguen siendo subconjuntos (estándar o no estándar) de $U_{NonStandard\ r}$.

Sea $\tau_{\text{NonStandard}} \subseteq P(U_{NonStandard})$ una familia de subconjuntos estándar o no estándar de $P(U_{NonStandard})$.

Entonces $\tau_{\text{NonStandard}}$ se denomina topología no estándar en $U_{NoEstandar}$ si satisface los siguientes axiomas:

i. El conjunto vacío (ϕ) y $U_{NoEstandar}$ pertenece a $\tau_{\text{NonStandard}}$.
ii. La intersección de un número finito de elementos en $\tau_{\text{NonStandard}}$ aún está en $\tau_{\text{NonStandard}}$.
iii. La unión de cualquier número finito o infinito de elementos en en $\tau_{\text{NonStandard}}$ aún está en $\tau_{\text{NonStandard}}$.

Entonces $(U_{NonStandard,}\ \tau_{\text{NonStandard}})$ se denomina Espacio Topológico No Estándar.

28. Conjunto Real No Estándar Extendido ($\overset{-0+}{ER}$)

Lo presentamos ahora por primera vez:

$\overset{-0+}{ER} = \{x; \overset{0}{x}, \overset{-}{x}, \overset{+}{x}, \overset{-0}{x}, \overset{0+}{x}, \overset{-+}{x}, \overset{-0+}{x}; x \in R\}$, de hecho:

$\overset{-0+}{ER} = \overset{0}{R} \cup \overset{-}{R} \cup \overset{+}{R} \cup \overset{-0}{R} \cup \overset{0+}{R} \cup \overset{-+}{R} \cup \overset{-0+}{R}$,

donde se utilizan las notaciones:

$$\overset{0}{R} \overset{def}{\equiv} R$$

$$\overset{-}{R} = \{\overset{-}{x}, x \in R\}$$

$$\overset{+}{R} = \{\overset{+}{x}, x \in R\}$$

$$\overset{-0}{R} = \{\overset{-0}{x}, x \in R\}$$

$$\overset{0+}{R} = \{\overset{0+}{x}, x \in R\}$$

$$\overset{-+}{R} = \{\overset{-+}{x}, x \in R\}$$

$$\overset{-0+}{R} = \{\overset{-0+}{x}, x \in R\}$$

29. Topología Real No Estándar Extendida Mayor

$P\left(\overset{-0+}{ER}\right)$, que es el conjunto de potencias de $\overset{-0+}{ER}$, genera la Topología Real No Estándar Extendida Mayor de todo el conjunto real no estándar extendido $\overset{-0+}{ER}$.

30. Sobre/Infra/Extra-Conjunto y Lógicas y Probabilidades

El Conjunto Neutrosófico se extendió [Smarandache, 2007] al SobreConjunto Neutrosófico (cuando algún componente Neutrosófico es > 1), ya que observamos que, por ejemplo, un empleado que trabaja horas extras merece un grado de pertenencia > 1, en comparación con un empleado que solo trabaja a tiempo completo regular y cuyo grado de pertenencia = 1;

y al InfraConjunto Neutrosófico (cuando algún componente Neutrosófico es < 0), ya que, por ejemplo, un empleado que causa más daño que beneficio a su empresa merece un grado de pertenencia < 0, en comparación con un empleado que produce beneficio para la empresa y tiene un grado de pertenencia > 0;

y al ExtraConjunto Neutrosófico (cuando algunos componentes Neutrosóficos están fuera del intervalo [0, 1], es decir, algunos componentes Neutrosóficos > 1 y algunos componentes Neutrosóficos < 0).

De manera similar para la Sobre/Infra/Extra-Lógica y respectivamente la Sobre/Infra/Extra-Topología [16, 17, 18, 19].

Dado que estas ideas parecen contraintuitivas y totalmente diferentes del marco convencional, presentamos a continuación ejemplos elementales de nuestro mundo real de tales grados que están fuera de lo común {nos referimos fuera del intervalo [0, 1]}.

31. Ejemplo Real de SobreMembresía e InfraMembresía

En una empresa, un empleado a tiempo completo trabaja 40 horas por semana. Consideremos el último periodo de la semana.

Helen trabajó a tiempo parcial, solo 30 horas, y las otras 10 horas estuvo ausente sin pago; por lo tanto, su grado de membresía fue de 30/40 = 0.75 < 1.

John trabajó a tiempo completo, 40 horas, por lo que tuvo un grado de membresía de 40/40 = 1, con respecto a esta empresa.

Pero George trabajó horas extras, 5 horas más, por lo que su grado de membresía fue (40+5)/40 = 45/40 = 1.125 > 1.

Por lo tanto, necesitamos hacer una distinción entre los empleados que trabajan horas extras y aquellos que trabajan a tiempo completo o parcial. Es por eso que necesitamos asociar un grado de membresía estrictamente mayor que 1 a los trabajadores que hacen horas extras.

Ahora, otro empleado, Jane, estuvo ausente sin pago durante toda la semana, por lo que su grado de membresía fue de 0/40 = 0.

Sin embargo, Richard, quien también fue contratado a tiempo completo, no solo no vino a trabajar la semana pasada en absoluto (0 horas trabajadas), sino que provocó, al iniciar accidentalmente un incendio devastador, mucho daño a la empresa, lo que se estimó en un valor equivalente a la mitad de su salario (es

decir, lo que habría ganado por trabajar 20 horas esa semana). Por lo tanto, su grado de membresía debe ser menor que el de Jane (ya que Jane no causó ningún daño). Por lo tanto, el grado de membresía de Richard, con respecto a esta empresa, fue de -20/40 = -0.50 < 0.

Consecuentemente, necesitamos hacer una distinción entre los empleados que causan daño y aquellos que generan ganancias o no causan ni daño ni ganancias a la empresa.

Por lo tanto, los grados de membresía > 1 y < 0 son reales en nuestro mundo, por lo que debemos tenerlos en cuenta.

Entonces, de manera similar, la Lógica/Medida/Probabilidad/Estadística Neutrosófica se extendieron respectivamente a la Sobre/Infra/Extra-Lógica/Medida/Probabilidad/Estadística Neutrosófica (Smarandache, 2007).

32. Definición de SobreConjunto Neutrosófico de Valor Único

Sea U_{over} un SobreUniverso de Discurso {es decir, existen algunos elementos en U_{over} cuyos grados de membresía son > 1}, y el SobreConjunto Neutrosófico $A_{over} \subseteq U_{over}$.

Sean *T(x), I(x), F(x)* las funciones que describen los grados de membresía, membresía indeterminada y no membresía respectivamente, de un elemento genérico x ∈ U_{over}, con respecto al SobreConjunto Neutrosófico A_{over}:

$$T(x), I(x), F(x): U_{over} \to [0, \Omega]$$

dónde $0 < 1 < \Omega$, y Ω se llama SobreLímite,

$T(x), I(x), F(x) \in [0, \Omega]$, para todos los $x \in U_{over}$.

Un SobreConjunto Neutrosófico de Valor Único A_{over} se define como:

A_{over} = *{(x, <T(x), I(x), F(x)>), x ∈ U_{over}}*,

tal que existen algunos elementos en A_{over} que tienen al menos un componente neutrosófico que es > 1.

33. Definición de SobreTopología Neutrosófica de Valor Único

Sea U_{over} un SobreUniverso de Discurso, y P(U_{over}) el conjunto potencia de U_{over}.

Sea $\tau_{over} \subseteq P(Uover)$ una familia de SobreConjuntos Neutrosoficos de Valor Único de $Uover$.

Entonces, τ_{over} se llama SobreTopología Neutrosófica de Valor Único en U_{over} si satisface los siguientes axiomas:

(i) ϕ y $Uover$ pertenecen a τ_{over}.

(ii) La intersección de cualquier número finito de SobreConjuntos Neutrosóficos de Valor Único en τ_{over} está en τ_{over}.

(iii) La unión de cualquier número finito o infinito de SobreConjuntos Neutrosóficos de Valor Único en τ_{over}está en τ_{over}.

Entonces, (U_{over}, τ_{over})) se llama un Espacio SobreTopológico Neutrosófico.

34. Definición del InfraConjunto Neutrosófico de Valor Único

Sea U_{under} un InfraUniverso de Discurso {es decir, existen algunos elementos en U_{under} cuyos grados de membresía son < 0}, y sea el InfraConjunto Neutrosófico $A_{under} \subseteq Uunder$.

Sean *T(x), I(x), F(x)* las funciones que describen los grados de membresía, membresía indeterminada y no membresía respectivamente, de un elemento genérico $x \in U_{under}$, con respecto al InfraConjunto Neutrosófico A_{under}:

$$T(x), I(x), F(x): U_{under} \to [\Psi, 1]$$

donde Ψ < 0 < 1, y Ψ se llama InfraLímite,

$T(x), I(x), F(x) \in [\Psi, 1]$, para todo $x \in U_{under}$.

Un InfraConjunto Neutrosófico de Valor Único A_{under} se define como:
A_{under} = {(x, <T(x), I(x), F(x)>), $x \in U_{under}$},
tal que existen algunos elementos en A_{under} que tienen al menos un componente neutrosófico que es < 0.

35. Definición de InfraTopología Neutrosófica de Valor Único

Sea U_{under} un InfraUniverso de Discurso, y P(U_{under}) el conjunto potencia de U_{under}.
Sea $\tau_{under} \subseteq P(U_{under})$ una familia de InfraConjuntos Neutrosóficos de Valor Único de U_{under}.
Entonces, τ_{under} se llama una InfraTopología Neutrosófica de Valor Único en U_{under} si satisface los siguientes axiomas:

i. ϕ y U_{under} pertenecen a τ_{under}.
ii. La intersección de cualquier cantidad finita de InfraConjuntos Neutrosóficos de Valor Único en τ_{under} está en τ_{under}.
iii. La unión de cualquier cantidad finita o infinita de InfraConjuntos neutrosóficos de valor único en τ_{under} está en τ_{under}.

Entonces, (U_{under}, τ_{under}) se llama un Espacio Neutrosófico InfraTopológico.

36. Definición de ExtraConjunto Neutrosófico de Valor Único

Sea U_{off} un ExtraUniverso de Discurso {es decir, existen elementos en U_{off} cuyos grados de pertenencia están fuera del intervalo [0, 1], algunos < 0 y otros > 1}, y el ExtraConjunto Neutrosófico $A_{off} \subseteq U$.
Sean *T(x), I(x), F(x)* las funciones que describen los grados de pertenencia, de indeterminación y de no pertenencia respectivamente, de un elemento genérico $x \in U_{off}$, con respecto al ExtraConjunto Neutrosófico U_{off}:

$$T(x), I(x), F(x): U_{off} \rightarrow [\Psi, \Omega]$$

donde Ψ<0<1<Ω, y Ψ se llama InfraLímite, mientras que Ω se llama SobreLímite,
$T(x), I(x), F(x) \in [\Psi, \Omega]$, para todo $x \in U_{off}$.
Un ExtraConjunto Neutrosófico de Valor Único A_{off} se define como:
A_{off} = {(x, <T(x), I(x), F(x)>), x ∈ U_{off}},
tal que existen algunos elementos en A_{off} que tienen al menos un componente neutrosófico que es > 1, y al menos un componente neutrosófico que es < 0.

37. Definición de ExtraTopología Neutrosófica de Valor Único

Sea U_{off} un ExtraUniverso del Discurso, y $P(U_{off})$ el conjunto potencia de U_{off}.

Sea $\tau_{off} \subseteq P(U_{off})$ una familia de ExtraConjuntos Neutrosóficos de Valor Único de U_{off}.

Entonces τ_{off} se llama ExtraTopología Neutrosófica de Valor Único en U_{off} si satisface los siguientes axiomas:

i. ϕ y U_{off} pertenecen a τ_{off}.
ii. La intersección de cualquier número finito de ExtraConjuntos Neutrosóficos de Valor Único en τ_{off} está en τ_{off}.
iii. La unión de cualquier número finito o infinito de ExtraConjuntos Neutrosóficos de Valor Único en τ_{off} está en τ_{off}.

Entonces (U_{off}, τ_{off}) se llama Espacio Neutrosófico ExtraTopológico.

38. Conjunto Triplete Neutrosófico Débil/Fuerte (*N*)

Sea *(N, *)* un grupo o conjunto no vacío dotado de una operación binaria bien definida *.

Un Triplete Neutrosófico es un objeto de la forma <*x, neut(x), anti(x)*>, para $x \in N$,

donde $neut(x) \in N$ es el neutro de x, diferente del elemento unitario algebraico clásico, si lo hay, tal que:

$$x * neut(x) = neut(x) * x = x$$

y $anti(x) \in N$ es el opuesto de x tal que:

$x * anti(x) = anti(x) * x = neutro(x)$.

En general, un elemento x puede tener más neutrales (*neuts*) y más opuestos (*antis*).

Los tripletes neutrosóficos y sus estructuras algebraicas de tripletes neutrosóficos fueron introducidos por primera vez por Florentin Smarandache y Mumtaz Ali [20, 21, 22, 23] en 2014-2016.

39. Definición de Conjunto Débil del Triplete Neutrosófico (NTS, *) es un conjunto tal que cada elemento $a \in NTS$ es parte de un triplete neutrosófico <b, neut(b), anti(b)>, es decir, *a = b,* or *a = neut(b),* or *a = anti(b)*.

40. Definición de la Topología Débil de Triplete Neutrosófico de Valor Único

Sea $U_{Triplet-Weak}$ un Universo de Discurso que tiene la estructura de un Conjunto Débil de Triplete Neutrosófico, y $P(U_{Triplet-Weak})$ el conjunto de potencias de $U_{Triplet-Weak}$.

Sea $\tau_{Triplet-Weak} \subseteq P(U_{Triplet-Weak})$ una familia de Conjuntos Débiles de Tripletes Neutrosóficos de Valor Único $U_{Triplet-Weak}$.

Entonces $\tau_{Triplet-Weak}$ se llama Topología Débil de Triplete Neutrosófico de Valor Único en $U_{Triplet-Weak}$ si satisface los siguientes axiomas:

i. ϕ y $U_{Triplet-Weak}$ pertenecen a $\tau_{Triplet-Weak}$.
ii. La intersección de cualquier número finito de Conjuntos Débiles de Tripletes Neutrosóficos de Valor Único en $\tau_{Triplet-Weak}$ está en $\tau_{Triplet-Weak}$.
iii. La unión de cualquier número finito o infinito de Conjuntos Débiles de Tripletes Neutrosóficos de Valor Único en $\tau_{Triplet-Weak}$ está en $\tau_{Triplet-Weak}$.

Entonces ($U_{Triplet-Weak}$, $\tau_{Triplet-Weak}$) se llama Espacio Topológico Débil de Triplete Neutrosófico.

41. Definición del Conjunto Fuerte de Triplete Neutrosófico (o **Conjunto de Tripletes Neutrosóficos**)

El grupo *(N, *)* se llama Conjunto Fuerte Triplete Neutrosófico si para cualquier $a \in N$ existe algún neutro de *a*, denotado $neut(a) \in N$, diferente del elemento unitario algebraico clásico (si lo hay), y algún opuesto de a, llamado $anti(a) \in N$.

Ejemplo de Conjunto Fuerte de Triplete Neutrosófico

*	1	2
1	2	1
2	1	1

El conjunto ({1,2}, *) es un grupoide, sin elemento unitario clásico.

Entonces <1, 2, 1> y <2, 1, 2> y son tripletes neutrosóficos.

El conjunto fuerte del triplete neutrosófico es N = {1, 2}.

42. Teorema sobre los Conjuntos Fuerte y Débil de Triplete Neutrosófico

Cualquier conjunto fuerte de triplete neutrosófico es un conjunto débil de triplete neutrosófico, pero no a la inversa.

Prueba.

Sea *(N, *)* un conjunto fuerte triplete neutrosófico. Si $a \in N$, entonces también está incluido en *N*, por lo tanto, existe un triplete neutrosófico en *N* que incluye a *a*, de donde *N* es un conjunto débil de triplete neutrosófico.

Por el contrario, lo demostramos utilizando un contraejemplo.

Sea $Z_3 = \{0, 1, 2\}$, incrustado con la multiplicación × módulo 3, que es una ley bien definida. El elemento unitario clásico en Z_3 es *1*.

$(Z_3, \times)$ es un conjunto débil de tripletes neutrosóficos, ya que los tripletes neutrosóficos se formaron en Z_3 con respecto a la ley × contiene todos los elementos *0, 1, 2*,

es decir, <0, 0, 0>, <0, 0, 1> y <0, 0, 2>.

Pero $(Z_3,\times)$ no es un conjunto fuerte de triplete neutrosófico, ya que, por ejemplo, para $2 \in Z_3$ no hay $neut(2) \neq 1$ ni hay *anti(2)*.

43. Definición de Topología Fuerte de Triplete Neutrosófico de Valor Único

Sea $U_{Triplet-Strong}$ un Universo de Discurso que tiene la estructura de un Conjunto Fuerte de Triplete Neutrosófico, y $P(U_{Triplet-Strong})$ el conjunto de potencias de $U_{Triplet-Strong}$.

Sea $\tau_{Triplet-Strong} \subseteq P(U_{Triplet-Strong})$ una familia de Conjuntos Fuertes de Tripletes Neutrosóficos de Valor Único $U_{Triplet-Strong}$.

Entonces $\tau_{Triplet-Strong}$ se llama Topología Fuerte de Triplete Neutrosófico de Valor Único en $U_{Triplet-Strong}$ si satisface los siguientes axiomas:

i. ϕ y $U_{Triplet-Strong}$ pertenecen a $\tau_{Triplet-Strong}$.
ii. La intersección de cualquier número finito de Conjuntos Fuertes de Tripletes Neutrosóficos de Valor Único en $\tau_{Triplet-Strong}$ está en $\tau_{Triplet-Strong}$.
iii. La unión de cualquier número finito o infinito de Conjuntos Fuertes de Tripletes Neutrosóficos de Valor Único en $\tau_{Triplet-Strong}$ está en $\tau_{Triplet-Strong}$.

Entonces $(U_{Triplet-Strong}, \tau_{Triplet-Strong})$ se llama Espacio Topológico Fuerte de Triplete Neutrosófico.

44. Triplete Neutrosófico Extendido

Un Triplete Neutrosófico Extendido es un triplete neutrosófico, definido como anteriormente, pero donde el *neutro* de x {indicado por $_e neut(x)$ y llamado "neutro extendido", donde "e" al frente significa 'extendido'} puede ser igual al elemento unitario algebraico clásico (si lo hay) de la ley * definido en el conjunto. Por lo tanto, se libera la restricción "diferente del elemento unitario algebraico clásico, si lo hay".

De esta manera, un triplete neutrosófico extendido es un objeto de la forma $<x, {}_e neut(x), {}_e anti(x)>$, para $x \in N$, donde $_e neut(x) \in N$ es el neutro extendido de x, que puede ser igual o diferente del elemento unitario algebraico clásico, si lo hubiera, tal que:

$$X * {}_e neut(x) = {}_e neut(x) * x = x$$

y *anti(x)*∈*N* es el opuesto extendido de x tal que:

$$x *_e anti(x) = {}_e anti(x) * x = {}_e neut(x).$$

En general, para cada x ∈ N existen muchos ${}_e neuts$ (neutrales extendidos) y ${}_e antis$ (opuestos extendidos). Los tripletes neutrosóficos extendidos fueron introducidos por Smarandache [27, 28] en 2016.

45. Definición de Conjunto Débil de Triplete Extendido Neutrosófico

El conjunto *N* se llama conjunto débil de triplete extendido neutrosófico si para cualquier x∈N existe un triplete extendido neutrosófico $<y, {}_e neut(y), {}_e anti(y)>$ incluido en *N*, tal que $x = y$ o $x = {}_e neut(y)$ o $x = {}_e anti(y)$.

46. Definición de Topología Débil de Triplete Extendido Neutrosófico de Valor Único

Sea $U_{Extended-Triplet-Weak}$ un Universo de Discurso que tiene la estructura de un Conjunto Débil de Triplete Extendido Neutrosófico, y $P(U_{Extended-Triplet-Weak})$ el conjunto de potencias de $U_{Extended-Triplet-Weak}$.

Sea $\tau_{Extended-Triplet-Weak} \subseteq P(U_{Extended-Triplet-Weak})$ una familia de Conjuntos Débiles de Tripletes Neutrosóficos Extendidos de Valor Único $U_{Extended-Triplet-Weak}$.

Entonces $\tau_{Extended-Triplet-Weak}$ se llama Topología Débil de Triplete Extendido Neutrosófico de Valor Único en $U_{Extended-Triplet-Weak}$ si satisface los siguientes axiomas:

i. ϕ y $U_{Extended-Triplet-Weak}$ pertenecen a $\tau_{Extended-Triplet-Weak}$.
ii. La intersección de cualquier número finito de conjuntos débiles de tripletes neutrosóficos extendidos de valor único en $\tau_{Extended-Triplet-Weak}$ está en $\tau_{Extended-Triplet-Weak}$.
iii. La unión de cualquier número finito o infinito de conjuntos débiles de tripletes neutrosóficos extendidos de valor único en $\tau_{Extended-Triplet-Weak}$ es en $\tau_{Extended-Triplet-Weak}$.

Entonces $(U_{Extended-Triplet-Weak}, \tau_{Extended-Triplet-Weak})$ se denomina espacio topológico débil de triplete extendido neutrosófico.

47. Definición de Conjunto Fuerte de Triplete Extendido Neutrosófico

El conjunto *N* se llama conjunto fuerte triplete extendido neutrosófico si para cualquier x ∈ N existen ${}_e neut(x)$ ∈ *N* y ${}_e anti(x) \in N$.

48. Definición de Topología Fuerte de Triplete Extendido Neutrosófico de Valor Único

Sea $U_{Extended-Triplet-Strong}$ un Universo de Discurso que tiene la estructura de un Conjunto Fuerte de Triplete Extendido Neutrosófico, y $P(U_{Extended-Triplet-Strong})$ el conjunto de potencias de $U_{Extended-Triplet-Strong}$.

Sea $\tau_{Extended-Triplet-Strong} \subseteq P(U_{Extended-Triplet-Strong})$ una familia de Conjuntos Fuertes de Tripletes Neutrosóficos Extendidos de Valor Único $U_{Extended-Triplet-Strong}$.

Entonces $\tau_{Extended-Triplet-Strong}$ se llama Topología Fuerte de Triplete Neutrosófico Extendido de Valor Único en $U_{Extended-Triplet-Strong}$ si satisface los siguientes axiomas:

i. ϕ y $U_{Extended-Triplet-Strong}$ pertenecen a $\tau_{Extended-Triplet-Strong}$.
ii. La intersección de cualquier número finito de conjuntos fuertes de tripletes neutrosóficos extendidos de valor único en $\tau_{Extended-Triplet-Strong}$ está en $\tau_{Extended-Triplet-Strong}$.
iii. La unión de cualquier número finito o infinito de conjuntos fuertes de tripletes neutrosóficos extendidos de valor único en $\tau_{Extended-Triplet-Strong}$ está en $\tau_{Extended-Triplet-Strong}$.

Entonces ($U_{Extended-Triplet-Strong}$, $\tau_{Extended-Triplet-Strong}$) se denomina Espacio Topológico Fuerte de Triplete Extendido Neutrosófico.

49. Duplas neutrosóficas

Florentin Smarandache introdujo las duplas neutrosóficas y las estructuras algebraicas de duplas neutrosóficas [24, 25, 26] en 2016.

Sea U un universo de discurso, y un conjunto D incluido en U, dotado de una ley # bien definida.

50. Definición de la Dupla Neutrosófica

Decimos que *<a, neut(a)>*, donde *a*, y su neutro *neut(a)* pertenecen a *D*, es una dupla neutrosófica si:
1) *neut(a)* es diferente del elemento unitario de *D* con respecto a la ley # (si la hay);
2) *a # neut(a) = neut(a) # a = a*;
3) no existe ningún *anti(a)* opuesto perteneciente a *D* para el cual
a # anti(a) = anti(a) # a = neut(a).

51. Ejemplo de Duplas Neutrosóficas

En (Z_8, #), el conjunto de números enteros con respecto a la multiplicación regular módulo 8, uno tiene las siguientes duplas neutrosóficas:

<2, 5>, <4, 3>, <4, 5>, <4, 7> y <6, 5>.

Prueba:

Sea Z_8 = *{0, 1, 2, 3, 4, 5, 6, 7},* teniendo el elemento unitario 1 con respecto a la multiplicación # módulo 8.
2 # 5 = 5 # 2 = 10 = 2 (mod 8),
entonces *neut(2)* = 5 ≠ 1.
No hay *anti(2)* ∈ Z_8, porque:
2 # anti(2) = 5 (mod 8),
o *2y = 5 (mod 8)* al denotar *anti(2) = y*, equivale a:
2y - 5 = M_8{múltiplo de 8}, o *2y - 5 = 8k*, donde *k* es un número entero, o
2(y - 4k) = 5, donde tanto *y* como *k* son números enteros, o:
número par = número impar, lo cual es imposible.
Por lo tanto, demostramos que *<2, 5>* es una dupla neutrosófica.
Lo mismo ocurre con *<4, 5>, <4, 3>, <4, 7>* y *<6, 5>*.
Un contraejemplo: *<0, 0>* no es una dupla neutrosófica, porque
es un triplete neutrosófico: *<0, 0, 0>,* donde existe un *anti(0) = 0*.

52. Definición de Topología de Dupla Neutrosófica de Valor Único

Sea U_{Duplet} un Universo de Discurso que tiene la estructura de un Conjunto de Dupla Neutrosófica, y $P(U_{Duplet})$ el conjunto de potencias de U_{Duplet}.

Sea $\tau_{Duplet} \subseteq P(U_{Duplet})$ una familia de Conjuntos de Dupla Neutrosóficas de Valor Único de U_{Duplet}.

Entonces τ_{Duplet} se llama Topología de Dupla Neutrosófica de Valor Único en U_{Duplet} si satisface los siguientes axiomas:

i. ϕ y U_{Duplet} pertenecen a τ_{Duplet}.

ii. La intersección de cualquier número finito de conjuntos de duplas neutrosóficas de valor único en τ_{Duplet} está en τ_{Duplet}.

iii. La unión de cualquier número finito o infinito de conjuntos de duplas neutrosóficas de valor único en τ_{Duplet} está en τ_{Duplet}.

Entonces (U_{Duplet}, τ_{Duplet}) se llama Espacio Topológico de Dupla Neutrosófica.

53. Definición de Dupla Neutrosófica Extendida

Sea *U* un universo de discurso, y un conjunto *D* incluido en *U*, dotado de una ley # bien definida. Decimos que $<a, {}_e neut(a)>$, donde *a*, y su neutro extendido ${}_e neut(a)$ pertenecen a *D*, tal que:

1) ${}_e neut(a)$ puede ser igual o diferente del elemento unitario de *D* con respecto a la ley # (si la hay);

2) $a \# {}_e neut(a) = {}_e neut(a) \# a = a$;

3) no existe un opuesto extendido ${}_e anti(a)$ perteneciente a *D* para el cual

$a \# {}_e anti(a) = {}_e anti(a) \# a = {}_e neut(a)$.

54. Definición de Topología de Dupla Extendida Neutrosófica de Valor Único

Sea $U_{Extended-Duplet}$ un Universo de Discurso que tiene la estructura de un Conjunto de Dupla Neutrosófica Extendida, y $P(U_{Extended-Duplet})$ el conjunto d potencias de $U_{Extended-Duplet}$.

Sea $\tau_{Extended-Duplet} \subseteq P(U_{Extended-Duplet})$ una familia de conjuntos de duplas neutrosóficas de valor único de $U_{Extended-Duplet}$.

Entonces $\tau_{Extended-Duplet}$ se llama Topología de dupla neutrosófica de valor único en $U_{Extended-Duplet}$ si satisface los siguientes axiomas:

i. ϕ y $U_{Extended-Duplet}$ pertenecen a $\tau_{Extended-Duplet}$.

ii. La intersección de cualquier número finito de conjuntos de duplas neutrosóficas extendidas de valor único en $\tau_{Extended-Duplet}$ es en $\tau_{Extended-Duplet}$.

iii. La unión de cualquier número finito o infinito de conjuntos de duplas neutrosóficas extendidas de valor único en $\tau_{Extended-Duplet}$ es en $\tau_{Extended-Duplet}$.

Entonces ($U_{Extended-Duplet}$, $\tau_{Extended-Duplet}$) se denomina Espacio Topológico de Dupla Extendida Neutrosófica.

55. Definición de MultiConjunto Neutrosófico

Florentin Smarandache presentó los MultiConjuntos Neutrosóficos y las Estructuras Algebraicas de MultiConjuntos Neutrosóficos [23, 24, 26] en 2016.

Sea $\mathcal{U}$ un universo de discurso, y un conjunto $M \subseteq U$.

Un *MultiConjunto Neutrosófico M* es un conjunto neutrosófico donde uno o más elementos se repiten con los mismos componentes neutrosóficos, o con diferentes componentes neutrosóficos.

Es una extensión del multiconjunto clásico, multiconjunto difuso, multiconjunto difuso intuicionista, etc.

56. Ejemplos de MultiConjuntos Neutrosóficos

A= {a(0,6, 0,3, 0,1),b(0,8, 0,4, 0,2),c(0,5, 0,1, 0,3) }
es un conjunto neutrosófico (no multiconjunto).

Pero B= {a(0,6, 0,3, 0,1),a(0,6, 0,3, 0,1),b(0,8, 0,4, 0,2) }
es un multiconjunto neutrosófico, ya que el elemento a se repite; decimos que el elemento a tiene la *multiplicidad neutrosófica* 2 con los mismos componentes neutrosóficos.

Mientras C= {a(0,6, 0,3, 0,1),a(0,7, 0,1, 0,2),a(0,5, 0,4, 0,3),c(0,5, 0,1, 0,3)}
también es un multiconjunto neutrosófico, porque el elemento a se repite (tiene la *multiplicidad neutrosófica* 3), pero con diferentes componentes neutrosóficos, ya que, por ejemplo, durante el tiempo, la pertenencia neutrosófica de un elemento puede cambiar.

Si el elemento a se repite k veces, manteniendo los mismos componentes neutrosóficos (ta,ia,fa), decimos que a tiene *multiplicidad* k.
Pero si hay algún cambio en los componentes neutrosóficos de a, decimos que a tiene la *multiplicidad neutrosófica* k.
Por tanto, definimos de forma general la *Función de Multiplicidad Neutrosófica* (nm):

$$nm: \mathcal{U} \to \mathbb{N} = \{1, 2, 3, \ldots, \infty\},$$

y para cualquier $a \in A$ se tiene

$$nm(a) = \{(k_1, \langle t_1, i_1, f_1\rangle), (k_2, \langle t_2, i_2, f_2\rangle), \ldots, (k_j, \langle t_j, i_j, f_j\rangle), \ldots\}$$

Lo que significa que
a se repite k_1 veces con los componentes neutrosóficos $\langle t_1,i_1,f_1\rangle$;
a se repite k_2 veces con los componentes neutrosóficos $\langle t_2,i_2,f_2\rangle$,
...,
a se repite k_j veces con los componentes neutrosóficos $\langle tj,ij,fj\rangle$,
...,
etcétera.

Entonces, un multiconjunto neutrosófico A se puede escribir como:
A = $\{(a,nm(a)),$ para $a \in A)\}$.

57. Ejemplos de operaciones con Multiconjuntos Neutrosóficos

Tengamos:

A= {5 ⟨0,6, 0,3, 0,2⟩ , 5 ⟨0,6, 0,3, 0,2⟩ , 5 ⟨0,4, 0,1, 0,3⟩ , 6 ⟨0,2, 0,7, 0,0⟩ };
B= {5 ⟨0,6, 0,3, 0,2⟩ , 5 ⟨0,8, 0,1, 0,1⟩ , 6 ⟨0,9, 0,0, 0,0⟩ };
C= {5 ⟨0,6, 0,3, 0,2⟩ , 5 ⟨0,6, 0,3, 0,2⟩ }.

Entonces:

Intersección de multiconjuntos neutrosóficos.
$A \cap B$= {5 ⟨0,6, 0,3, 0,2⟩ }.

Unión de multiconjuntos neutrosóficos
$A \cup B$={5 ⟨0,6, 0,3, 0,2⟩ , 5 ⟨0,6, 0,3, 0,2⟩ , 5 ⟨0,4, 0,1, 0,3⟩ , 5 ⟨0,8, 0,1, 0,1⟩ , 6 ⟨0,2, 0,7, 0,0⟩ , 6 ⟨0,9, 0,0, 0,0⟩ }.

Inclusión de multiconjuntos neutrosóficos
$C \subset A$, pero $C \not\subset B$.

58. Definición de Topología Neutrosófica Multiconjunto de Valor Único

Sea $U_{MultiSet}$ un Universo de Discurso que tiene la estructura de un MultiConjunto Neutrosófico, y $P(U_{MultiSet})$ el conjunto de potencias de $U_{MultiSet}$.

Sea $\tau_{MultiSet} \subseteq P(U_{MultiSet})$ una familia de multiconjuntos neutrosóficos de valor único de $U_{MultiSet}$.

Entonces $\tau_{MultiSet}$ se llama Topología Neutrosófica MultiConjunto de valor Único en $U_{MultiSet}$ si satisface los siguientes axiomas:

i. ϕ y $U_{MultiSet}$ pertenecen a $\tau_{MultiSet}$.
ii. La intersección de cualquier número finito de multiconjuntos neutrosóficos de valor único en $\tau_{MultiSet}$ está en $\tau_{MultiSet}$.
iii. La unión de cualquier número finito o infinito de multiconjuntos neutrosóficos de valor único r en $\tau_{MultiSet}$ está en $\tau_{MultiSet}$.

Entonces $(U_{MultiSet}, \tau_{MultiSet})$ se denomina Espacio Topológico Neutrosófico MultiConjunto.

Conclusión

Estas ocho nuevas topologías vanguardistas, junto con las seis nuevas topologías anteriores y sus correspondientes espacios topológicos, fueron introducidas por Smarandache entre 2019 y 2023, pero aún no han sido muy estudiadas ni aplicadas, excepto las NeutroTopologías y AntiTopologías, que han recibido cierta atención por parte de los investigadores. Mientras tanto, la Topología Neutrosófica No Estándar, las Topologías de Triplete Neutrosófico Débil/Fuerte, las Topologías Neutrosóficas Extendidas de Tripletes Débil/Fuerte, la Topología Neutrosófica de Dupla, la Topología Neutrosófica Extendida de Dupla y la Topología Neutrosófica de Multiconjuntos se proponen ahora por primera vez.

Como investigación futura, se estudiaría sus amplias aplicaciones en nuestro mundo real.

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Foundation of Appurtenance and Inclusion Equations for Constructing the Operations of Neutrosophic Numbers Needed in Neutrosophic Statistics

أسس ي عالاالث ال َتاَء والىدى اب ُاء ع ًه ِاث أل عذاد انُىتزوسفٍت فً إل ص اءانُىتزوسفٍت

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Smarandache, Florentin, and Rafif Alhabib. "Ambiguous Set is a subclass of the Double Refined Indeterminacy Neutrosophic Set, and of the Refined Neutrosophic Set in general." *Neutrosophic Knowledge*, Vol. 5, 2024, pp. 47-67.

الملخص:

نمذَل أي ِشح كِبدِّخ الزّبء ِكبدِّخ الززاء ًاز رضابكذ فإ فَأُ ًكِّياب د كٍ أ ألكذاذ ُليرشنصأفيخضآ الطبس ز ءص االبء أنيوشّص قرِشرج ُف ِ اشيخ زبَ أكِّبدال د لُ ِ كِب ِالوبَكّحسحكّ ِ د ّكبد) ُّيٍ أسلِيبيًفِشيح(.

الكلمات المفتاحية:

ألكذاذ أنيوشّصفيخ ًاس ميقيخ، االزءبء أ يوتروس ق ِ، أ ميمخ ًاس ميقيخ، ألكّخ ًاس ميقيخ، كِبدِّخ االزّبء، كِبدِّخ االزّبء، كّالالخ االززاء، كِبدِّخ االززاء، كِبدِّخ لظّبآح، كِبدِّخ كّذ االزّبء، كِبدِّخ كّذ ذ االززاء، كِبدِّخ كّذ َ لظّبآح، ًكِّ ّيا د كٍ أ ألكذاذ أنيوشّصفيخ.

ال قَّفي ت:

فإصزءبء أنيوشّص قرِط ُ ّخذ القيمة الحقيقية v فِ ا ّد كّخ غيش ًاسذيح I ال يكِ أ ُ v ّخذح فِ ُلُ أنيوشّص قِ ،ُّي ّ: a.

ّزا يجبكِشيفكّالخ كِّبدِّخ االزّبء ّلاصزب.

كّ الخ كٍ ُ ّرُه طولاب ُّنيب ِ د كّخ ِ ّ أ ميم الحقيقية V ، زضّخ فِ I فزا الكِ ِ أ ُ V زضّخ فِ $N = a + bI$ ، ُّي ّ:

$$a + bV \subset a + bI = N \text{ (or } a + bV \subseteq a + bI = N\text{)}$$

ّزا ظزجت يدتكِشيفكّالخ كِّبدِّخ االززاء.

وّب نسزخذ ُلُ ِز "=ُّكِّالخ ُاضبآح ُكِّبدِّخ ّلاضبآح ضزخذ ُلُ ِز "∈" {يز ّ ط}ُّكِّالخ كِّبدِّخ اوّبءسلُ ط ِ د كّخ، وزُه نسزخذ ُلُ ِز ⊂)أ ⊆ ُكِّالخ كِّبدِّخ االززاء.

ـتبوبنت اكيية ا ز ذ مكتب كيادبنب أليش ـ كـالخ ـيكبـدخ الوبء)كي ا ز ا ـكـالـلخ كيبـدخ الز زاء(، يدت كيب أن نفكي الشيء نفض كي لبنب أليـ .

ثبصرضفخطر ه، نمذ أيضكيبـدخكـذ االوبء ا ـ ىخ، كيبـدخ كـذاال ززاء، كيبـدخ كـذ ـضبـاح أل ية كي ا ز.

1ت عز فنزاق ى ان ت زوس ى ف ا ذق ق

لل ل أنيوشص ق ـاسمي م (N) "ثرى : $N = a + bI$ حيث a , b لأل ب حقيقيخ، يسـ "a" ا دزء ا س ذذ ـ N ثيبب "bI" ـ ـ أدزء غش أسذذ ـ N ، حيث I ـ د كـخ خزية حقيقيخ، I⊂R . رضرز خذ ف ـاب ت فـاصزءبء أنيوشص ق ـ.

نصادفلل ل ب أ يوتروصفيخ ثش ى زشش فـكـب ب اس مي ـ، زيثغبـجـب ب ى ـ ذ . ا ش ءثينا د غش لـيخ غش طس خكيب ـ كيب.

كـضرجيل ثلبـبيثفش ض أ ـ ذيب لل ض كـاش ى ثـاـبـبـئ چ غ اي ضكيب 5 ـ 6 ويلوزش كي ا ز ا ـفـيئ ـ اي زشفـيب $\sqrt{61}$ $\sqrt{\quad}$ زجاج ا رش يب لبوجكـض الخ.

ـثبـي آخش، ـ ذيب لل ض ثلية چ غ ذ ف ل ط ب *10* وي ـ زش ا، نض ت ضـبـزز ب

أكـبذ زض بـة زدُ أص طر وش ج، نجد أ π يملك كـذد لا نهائي ـ اآل س ل بـتكلفذب ص خ ثـتـ ـ ذ ـ ط زشش.

ثـنـبـتـ ب ثـذ أيش ... $e = 2.71828182$ يملك عدد لانهائي ـ اآل س ل بـتكلفذب ص خ ثـتـ ـ ذ ـ ط زشش.

ثـفـش الطريقة كـذب نحصل كـؤنبئ ح غش دقيقة ـ)أدنس، كـلـبـد ال د الأسية أو غـبـس ز ـ ية أو ا ـثـثية، كـلـبـد الـزفلـبـض يخ، أخ..(.

أيثت ع ى أل ع ذاد ا ف ت زوس ى ف ت ا ذق ق ت...

)*i*(يمثل فش ج ف ـ ى ـ : زيث $N_1 = 2 + 3I$ [ ] 1

$$[2 \quad] = [2.5]$$

$$[0.1] = 2 + [0.3] = [2 \quad 1$$

)*ii*(ـ ي {0.6, 0.8, 0.9} د كـخ ـ ف ـ ع خ ـ س دتح ـ ـ ثـالـخ كيبـ ص ش كـ ـ تب:

$$\{0.6, 0.8, 0.9\} = 2 + \{3 \cdot (0.6), 3 \cdot (0.8), 3 \cdot (0.9)\}$$

$$\{1.8, 2.4, 2.7\} = \{2 + 1.8, 2 + 2.4, 2 + 2.7\} = \{3.8, 4.4, 4.7\} \subset [2, 5]$$

)*iii*(ـ ذيب $\left\{\frac{1}{n}, 1 \le n \le \infty, n \text{ ص س ير}\right\}$ (3 ، د كـخ ـ ف ـ ع خ ال نـتـ ـ ية كـ ـ تب:

3 3 $\left\{2 \quad \frac{}{n}, 1 \le n \le \infty, n \text{ is integer}\right\}$

$$\left\{2 \quad -, 2 + 3 \cdot -, 2 \quad -, \dots, 2 \quad \frac{}{n}, \dots\right\}$$

$$\left\{5, 3.5, 3, \dots, 2 \quad \frac{}{n}, \dots\right\} \subset [2, 5].$$

2. ل س ـ ث ع ا ق ت و ي ع ك ت ال ت ـ ء

رض ـ ر ب أ ظ ز ي ب ـ /1/ ـ /2/ أدنا ث ـ ش اء كـ ـ يات كـ ـ ـ ب ـ ن ي كـ ـ ال خ كـ ـ ب ـ د ـ خ ال ـ و ب ء كـ ـ ل ـ ا.

1.2. ل ظ ز ت /1/:

ـ ي A ـ B د كـ و ي ـ حقيقو ـ طرا $a \in A$ ـ $b \in B$ كـ ـ تب :

ا ـ د غ

أ ط ذ

أضشة

مضخ – –

ا ّخ

الإثبات:

ّي ّ ⋆ لي ّ ِ اكيبد أز وزح قال ،ث:

$$\star B = \{x \star y; x \in A, y \in B\}$$

ّاكيبة ⋆ كفخشى ً جيد.

طرواب ً ّ كيجب ً ّكشيف قال كتب ⋆ ⋆

2.2. لظزت /2/:

رى ّ A ِ دكخ حقيقية ّ ،تيب β ِ دكخكذية حقيقية ، رى ّ m, n أكذادصسيحة ِ خكتب:

أضشة القياسي ّ دكخ β β

سفغ طّل خ ِ دكخ n n

أدزس ّ دكخ $\sqrt[n]{a} \in \sqrt[n]{A}$

االسّضلب ّت ّ دكخ $-n$ $-n$

االس ّ دكخ – –

الإثبات: ثمزح ّبثخ

$$\beta \cdot A = \{\beta \cdot x, x \in A\} \quad \text{.....} \quad (2)$$

ثوكيض ف ِكّال خ *(2)* يزح أ ً β β

ضجّخ كّال لب د الزّب سالكخ زّب ّيخ، ّي ّ P أياً ِ أ ّ ص شزاب ّيخ $n, \frac{1}{n}, -n, \frac{}{n},$

كتب: (3) $\{x^p, \quad\}$

ثوكيض $x=a$ ف ِ كشيف)3(، يزح أ ً:

ِ ّ أخ P أياً ِ ّ $n \frac{1}{n}, -n, \frac{}{n},$

3.2. أيثت عى لع ه ِاثي ع عالقاث الت ّاء:

ّأخز الزّبء أسهيم: (0 . 1)

الإضفلتانعذدت فّ عالقت الت ّاء

$$(3 + 0.67) \in (3 + (0,1))$$

$$(3 + 0,3 + 1)$$

$$(3,4)$$

انضزبانعذدي فّ عالقت الت ّاء

$$(0.67) \in 2 \cdot (0,1)$$

$$(2 \cdot 0, 2 \cdot 1)$$
$$(0, 2)$$

ناقىدة

$$(0, 1)$$
$$\in (0^2, 1^2)$$
$$00.4489 \in (0, 1),$$

ناقس ت ا ع ذ ف ع ا ت ا ء

$$\underline{\quad} \quad \underline{(0, 1)}$$

$$(-0.5, 0)$$

2. 4 ي ع د ا ث ا ء:

ي ز ك ش ي ف ش ي ب ا ك ب ب ك إ س ز ا ب .:

ر ى R ك ح ا ك ذ اد الحقيقية، f ، g د ا ي حقيقية د ب ض م ط ح س ح ك د ك خ ا م P(R)

$$f, g : P(R) \to P(R)$$

$$f(x) \in g(x)$$

ث ش ى ش ب ث إ خ ش اء اد ز ذ م ب س ك ب د خ والص ي خ) ى ك ب ا ر ب ح س ح ك د ك ب د، ي ش ب ب ف ش ح (ذ م ب س ك ب د خ ا ب ء.

لأنه في ك ض ا ل ز ي ا ل ى ذ أ ض ر أ ي ا a أ b ا ل ي ش ف إ ب ك ي ش أ :

$$(a, b) \equiv (b, a) \text{ and } [a, b] \equiv [b, a].$$

2. 5. د ي ع ا ن ا ث ا ء:

ز ك ب د خ ا ب ء يعني د ك خ ز م ي ق ية S، أ $S \in P(R)$ د ك خ ا م R، ا ز ي ز ط ي ب ا د ي x، أ $x \in S$.

2. 6. ث ا ل /1/ ع ى ي ع ا ن ا ث ا ء:

ذ س ذ أ خ x.

$$(0\,5, 0\,8)$$

ن ط ذ ك ذ 4 و ا ط ش ف ي .

$$(-\ x) \quad (\qquad 2(0\,5, 0\,8))$$

ن س ز خ ذ ا ل ل ا س)(ر ي ي ز ث ض ث ي ا ط ف ا ل ي ش ا ط ف ا ل ي ك ب د خ.

$$(-\ x) \quad (\qquad (1, 1\,6))$$
$$(-\ x) \in (-\qquad 1, -\qquad 6)$$

ق س ى ا ط ف ع ى (-5)

$$\underline{(-\ x)} \quad \underline{(-\qquad 1, -\qquad 6)}$$

$$\left(—, -\right)$$

$$(0.28, 0.40).$$

ﺑﻨﻜﺬﻩ ﺍﻝ ﻧﺒﺄ ﻱ ﻣﺠﻤﻮﻋﺔ ﺍﻟﻮﺑﺎء ﺯ ، ﺃﺧ ﻴﻐﺎﻟﺒﺎً ﺗﺤﺘﻮﻱ ﺩﺍﺧﻞ ﺃﻓﻀﻞ ﺣﻞ ﺃﻓﺰﻝ ﺧ (0 28, 0 40).
ﻧﺤﻦ ﻻ ﻧﺄﺧﺬ ﺃ ﺪ ﻛﺒﺪ ﺍﻟﻤﺠﻤﻮﻋﺔ ﻱ (0 28, 0 40) ﻭﺳﻴﺨﺘﺼﺮ، ﺣﻴﺚ ﺃﻧﻪ ﻳﻤﻜﻦ ﺍﺳﺘﺨﺪﺍﻡ ﺇﺣﺪﻯ ﺍﻟﻌﻼﻗﺘﻴﻦ ﺍﻟﺘﺎﻟﻴﺘﻴﻦ (⊆ or ⊂).
ﻟﻨﺮﻯ ﻛﻴﻒ ﺃﻥ ﺍﻟﻌﺪﺩ ﻳﻨﺘﻤﻲ ﺇﻟﻰ ﻣﺠﻤﻮﻋﺔ ﺍﻟﻮﺑﺎء

$$(0\ 5, 0.8)$$

$$(0.28, 0.40) \overset{?}{\in} (\quad , \quad)$$

ﺣﻴﺚ $\overset{?}{\in}$ ﻳﻌﻨﻲ ﺃﻧﻪ ﻳﻤﻜﻦ ﺃﻥ ﻳﻨﺘﻤﻲ ﺇﻟﻰ ﻣﺠﻤﻮﻋﺔ ﺍﻟﻮﺑﺎء

$$(1.4, 2.0) \overset{?}{\in} (\quad , \quad)$$

$$(4 - 2.0, 4 - 1.4) \overset{?}{\in} (\qquad , \qquad)$$

$$(2, 2.6) \in (2, 2\ 6)$$

ﻓﻲ ﺍﻟﻐﺎﻟﺐ ﻳﺘﻢ ﺍﺳﺘﺨﺪﺍﻡ x ﻟﺘﻤﺜﻴﻞ ﺍﻟﻌﺪﺩ ﺍﻟﻤﺠﻬﻮﻝ ﺩﺍﺧﻞ ﺍﻟﻤﺠﻤﻮﻋﺔ ﺍﻟﻤﺪﺭﻭﺳﺔ.
ﻟﻨﺮﻯ ﻛﻴﻒ ﻳﻤﻜﻦ ﺍﺳﺘﺨﺪﺍﻡ ﺍﻟﻮﺑﺎء ﻟﺘﻘﺪﻳﺮ ﺍﻟﻤﺠﻬﻮﻝ $x \in (0\ 28, 0.40)$.
ﻟﻨﺮﻯ ﻛﻴﻒ ﻳﻤﻜﻦ، ﻛﻤﺜﺎﻝ ﺗﻮﺿﻴﺤﻲ:

$$(0.28, 0\ 40)$$

ﻛﻤﺠﻤﻮﻋﺔ ﺍﻟﻮﺑﺎء

$$(0\ 5, 0.8)$$

ﻭ ﺿﺮﺏ ﻛﻞ ﻋﻨﺼﺮ ﻓﻲ x

$$(0.35) \in 1 + 2(0.5, 0.8)$$

$$(1.75) \in 1 + (1\ 0, 1.6)$$

$$(2.00, 2\ 60)$$

ﺻﺤﻴﺤﺔ.
ﻟﻨﺮﻯ ﻛﻴﻒ ﻳﻤﻜﻦ ﺍﺳﺘﺨﺪﺍﻡ ﻣﺠﻤﻮﻋﺔ ﻓﻲ ﺍﺳﺘﻨﺘﺎﺝ (0 28, 0 40):

$$(0\ 30, 0\ 34) \subset (0\ 28, 0\ 40)$$

ﻛﻤﺠﻤﻮﻋﺔ ﺍﻟﻮﺑﺎء

$$(0\ 5, 0\ 8)$$

ﻭﺿﺮﺏ

$$(0\ 30, 0\ 34) \in (2\ 00, 2\ 60)$$

$$(5 \quad 30, 5 \quad 34) \in (2\ 00, 2\ 60)$$

$$(1\ 50, 1\ 70) \in (2\ 00, 2\ 60)$$

$$(4 - 1.70, 4 \quad 50) \in (2\ 00, 2\ 60)$$

$$(2.30, 2.50) \in (2.00, 2\ 60),$$

ﻓﻲ ﺍﻟﻐﺎﻟﺐ، ﺃﺩﻧﻰ ﺃﻗﻞ ﻣﻦ ﺃﻋﻠﻰ ﻣﻦ ﺃﺩﻧﻰ ﺃﻗﻞ.

ﻣﺜﺎﻝ ﻋﻠﻰ ﻛﻴﻔﻴﺔ ﺣﺴﺎﺏ ﺍﻟﻤﺠﻤﻮﻉ ﻭﺍﻟﻔﺮﻕ ﻭﺍﻟﻀﺮﺏ ﻭﺍﻟﻘﺴﻤﺔ ﻟﻸﻋﺪﺍﺩ ﺍﻟﻤﺤﺎﻳﺪﺓ

ﻟﻨﻔﺘﺮﺽ ﺃﻥ ﻟﺪﻳﻨﺎ ﻣﺠﻤﻮﻋﺔ ﺍﻟﻮﺑﺎء ﺍﻟﻤﺴﺘﺨﺪﻣﺔ ﻓﻲ ﺍﻟﻤﺜﺎﻝ ﺍﻟﺴﺎﺑﻖ (0 28, 0 40) ﻭﺍﻟﻌﺪﺩ ﺍﻟﻤﺤﺎﻳﺪ ﺍﻟﻤﺴﺘﺨﺪﻡ ﻫﻮ:

$$2 \cdot (0.5, 0.8)$$

ﺇﺫﺍ ﻛﺎﻥ ﺍﻟﻌﺪﺩ، ﺃﻱ ﺃﺣﺪ ﺍﻟﻌﻨﺎﺻﺮ ﺍﻟﻤﺤﺘﻤﻠﺔ ﻟﻠﻌﺪﺩ ﺍﻟﻤﺠﻬﻮﻝ ﺩﺍﺧﻞ ﺍﻟﻤﺠﻤﻮﻋﺔ *(0.8) . (0.5)*

يتم حلّبثفش الطريقة:

نل ذ 4 ِ ّوالّطلش فـيِ ّ:

$$(0\ 5, 0.8)$$

ثُ ذمُّض ش ةطضفخ اّ د ّگـبـد:

$$(1\ 0, 1.6)$$

$$(-\quad 0, -\quad 6)$$

$$(-2, -\ 4),$$

ونقسم ّگِ(5−ف ِسءً ّگِ:

$$\left(\frac{\quad}{\quad}, \frac{\quad}{\quad}\right)$$

$$(0\ 28, 0\ 40)$$

ّاز ّ ً) ِد ّگـخ(.

3. لّلل ِاث ع القيت وي ع الت ال تـواء

وّنفكپپ فـ ّگـال خ كّبـدّخ الـزبّص ق نفكّبـبّگ ً ِ ّگـال خ كّبـدّخ الـز زاء (⊆ or ⊂) ّگِ اّ ز ِاّبضّچّس بّخ لّ ِ

يـ ّ فـيّب ⊆ فـ ِ ّ اّ ِظّريات أدناصرزى ّ ّ ِ نفستب فدثّطص ز خذا ⊆ ثـالّ ِ ⊂.

3. 1 ي ع دالث ال تـواء:

ّگش ىّ ّگِبـبَ ّب ّ ّوّب يلا ِ ،ّزّى ّ R ِد ّگـخ ّأل ّگـذاد الحقيقية، f g ّ دلّيِ ّفـق حقيقية }ّرشّش وّخ "فـق" ط ّأ ّ ِدبـّب (ّ

/ أ(ِدبّ ّب ّأشّرشّن ّگّبسحّگ ّ ِد ّ ّگـبـد لّ P(R) }،

$$f, g : P(R) \to P(R)$$

ثُ $f(x) \subset g(x)$ أ $f(x) \subseteq g(x)$ رذ ّگّبـبـد الـد االززاء .

3. 2. ّظزّت /3/:

ن ِكٍ A و B ِگّزيِ ّحقيقزيِ ّ ّ وزّه A_1, B_1 طّبـبً ِد ّگّزيِ ّزمّيمزيِ ّ،ّى ّ:

$$A_1 \subset A \quad , \quad B_1 \subset B.$$

ّگـبتب:

طضفبخ اّ د ّگـبـد $A_1 + B_1 \subset A + B$

نل ذ اّ د ّگـبـد $A_1 - B_1 \subset A - B$

ضشة اّ د ّگـبـد $A_1 \times B_1 \subset A \times B$

لضّخ اّ د ّگـبـد $\frac{A_1}{B_1} \subset \frac{A}{}$

لخ اّ د ّگـبـد $A_1^{B_1} \subset A^B$

الإثبات:

ثبات الطريقة، في * لكل عمليات الزوج الثاني +، -، ×، ÷، ^) الخ(، ث:

$$\star B = \{x \star \quad A, y \quad\} \qquad (3)$$

الكمية * كثافة خيدًا.

فرض: $\subset_1 \quad {}_1 \quad {}_1 \subset \quad$ في التعريف (3(كتب $a_1 \star {}_1 \quad \star$

$$A_1 \star B_1 \subset A \star \quad \text{أو} \quad {}_1 \quad {}_1 \quad {}_1 \quad \text{وأخ}$$

3. 3. نظرت /4/:

في A_1 حقيقية غ $\subset_1$ نذد زميم m, n $\beta \neq 0$

كذلك صحيحة خ كتب:

أضف أ كثذ كخ $\beta \cdot A_1 \subset \beta \cdot A$

سفغ د كخ ط أ خ n $A^n \subset {}_1^n$

أدنى أشرح n كخ $\sqrt[n]{A} \subset \sqrt[n]{A_1}$

السالب كخ $A^{-n} \subset {}_1^n$

السالب إضج كخ $\overline{\quad} \subset \overline{\quad}_1$

الإثبات:

ث شيء ظشيخ $\{\beta \cdot x, x \in A\}$ β

$\beta \quad {}_1 \quad \{\beta \quad , \quad {}_1 \subset \} \subset \{\beta \quad , \quad \} \quad \beta$

ضيخ كالبدال زاء أخ ش p شح أخ . أيا أ ش $n, \frac{1}{n}, -n, \frac{-}{n}$ ث

$$\{x^p, x \in A_1\} \subset \{x^p, x \in A\} \quad {}_1$$

أل $A_1 \subset A$.

زيشيض خ أل x∈A، أ p، ر ى خ يغ كيات x^p كثافة خيدًا.

كيي نحو بشوض رب أ ظريب 3 ثقاخ العديد أ كيات كخ نبي كالخ كبدخ الزاء.

3. 4. أيثت عى عالقاث التساء

فش فكالخ الزاء الحقيقية:

$$(2, 3] \subset [0, 4]$$

فرض كذلك أن الطشفي:

$$(2, 3] \subset 1 + [0, 4]$$

$$(1 + 2, 1 + 3] \subset [1 + 0, 1 + 4] \quad \text{أ}$$

$$(3, 4] \subset [1, 5]$$

أ صحيح.

ٌضيفٌافوشرج $(-1,5)$ طٌوالٌطلشفـيـ ً

$(2,3]+(-1,5) \subset [0,4]+(-1,5)$

$(2-1,3+5) \subset (0-1,4+5)$

$(1,8) \subset (-1,9)$

ٌزا صرس ير.

ٌطٌش ذ ٌگٌـذذ /2/ ـ ً وٌالٌطلشفـيـ ً

$(2,3]-2 \subset [0,4]-$

$(2-2,3-2] \subset [0-2,4-2]$

$(0,1] \subset [-2,2],$

ٌزا صرس ير.

ٌطٌش ذ ـ ً وٌالٌطلشفـيـ ً فٌوشرج $[0.5,0.6]$

$(2,3]-[0.5,0.6] \subset [0,4]-[0.5,0.6]$

$(2-0.6,3-0.5] \subset [0-0.6,4-0.5]$

$(1.4,2.5] \subset [-0.6,3.5],$

ٌزا صرس ير.

ٌضشةوٌالٌطلشفـيـ تٌشبٌّلٌ /7/ ٌا ّ تٌخت)غٌش ٌاءفش ّ(:

$7 \cdot (2,3] \subset 7 \cdot [0,4]$

$(7 \cdot 2,7 \cdot 3] \subset [7 \cdot 0,7 \cdot 4]$

$(14,21] \subset [0,28],$

ٌزا صرس ير.

ٌضشةوٌالٌطلشفـيـ ً فٌكٌـذذ صٌب ّت /-5/)غٌغٌيصفش ّ(:

$(2,3] \subset -5 \cdot [0,4]$

$(-\quad 3,-5 \cdot 2] \subset [-5 \cdot 4,-5 \cdot 0]$

$(-15,-10] \subset [-20,0],$

ٌزا صرس ير.

ٌضشةوٌالٌطلشفـيـ تٌشبٌّ ّد ّكٌـخ $(-1,1)$

$(-1,1) \cdot (2,3] \subset (-1,1) \cdot (0,4]$

$(-3,3) \subset (-4,4),$

ٌزا صرس ير.

ٌشفـغوٌالٌطلشفـيـ ً طٌّ ٌامٌح /2/:

$(2,3]^2 \subset [0,4]$

$(2^2,3^2] \subset [0^2,4^2]$

$(4,9] \subset [0,16],$

ٌزا صرس ير.

ٌمضُوال ٌاطٌفيـ ً كٌـٌٌگٌـذذ /-5/ :

$\frac{(2,3]}{} \subset \frac{[0,4]}{}$

$\left[-\frac{3}{-},--\right) \subset \left[--,--\right]$

$$[-\ 6, -0.4) \subset [-0.8, 0],$$

ّزاصرسيير.

ٌمضُوال ُطفيـ ّ كٍْ ا ّد كّخ][4,5

$$\frac{(2,3]}{[4,5]} \subset \frac{[0,4]}{[4,5]}$$

$$\left(\frac{2}{-}, \frac{3}{-}\right] \subset \left[\frac{0}{-}, \frac{4}{-}\right]$$

$$(0\ 40, 0.75] \subset [0, 1],$$

ّزا صرسيير.

3. 5. ثكال ثا ّنعّ انت ال تواء

نحل ِ ّ أخً x

$$(1, 2) \subset (0, 5)$$
$$+ (x, 2x) \subset (0, 5)$$
$$(x + 1, 2x + 1) \subset (0, 5)$$

ثسيث

$$x \subset (-1.\ 4) \quad \text{أ} \quad \text{أ}$$
$$\text{or} \quad \text{or} \quad \text{or } x \subset (-0.5, 2),$$

ثسيث

$$\subset (-1, 4) \cap (-0.5, 2) = (-0.5, 2).$$

ّزفإ ْ ّسًاألكّ ظ ْ (−0.5, 2)

خ ّيغ ا ّد كّبد ّدئية ِ ّ (− 5, 2) ْ ِ زيخبصخ ثبّزبٌ ِ يئ ْ ّنيكّذد لا نهائي ِ ّ ّسيأخبصخ.

رسمك ِ ّ رّه:

$$x(1, 2) \subset (0, 5)$$
$$(-0.5, 2) \cdot (1, 2) \subset (0, 5)$$
$$+ (-1, 2) \subset (0, 5)$$
$$(1 - 1, 1 + 2) \subset (0, 5)$$
$$(0, 3) \subset (0, 5),$$

ّزا صرسيير.

3. 6. يثال آخزن ّعانت ال تواء

نحل ِ ّ أخً x

$$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$$
$$(4, 5) + [1 \cdot x, 2 \cdot x] \subseteq [6, 10]$$
$$(4, 5) + [x, 2x] \subseteq [6, 10]$$

ِ ّ أخً x.

نحصل كٍْ $(4 + x, 5 + 2x) \subseteq [6, 10]$

ّزّه: $6 \leq 4 + x \leq 10$ زيث $\leq x \leq$

$\leq x \leq 2.5$ أ $\leq 2x \leq 5$ أ $\leq 5 + 2x \leq 10$

$$[2, 6] \cap [1.5, 2.5] = [2, 2.5]$$

$$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$$
$$(4, 5) + [2x, x] \subseteq [6, 10]$$
$$(4 + 2x, 5 + x) \subseteq [6, 10]$$

زيث:

$$\leq 4 + 2x \leq 10, \text{ or } 2 \leq 2x \leq 6 \text{ or } 1 \leq x \leq$$

$\leq 5 + x \leq 10$

أ $1 \leq x \leq 5$

$[2, 2.5]$. x يجب أ

$$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$$

گبتب:

$$(4, 5) + [2, 2.5] \subseteq [6, 10]$$
$$(4 + 2, 5 + 2.5) \subseteq [6, 10]$$
$$(6, 7.5) \subseteq [6, 10],$$

زا صريير.

$[2, 2 5]$

$[2, 2] \subseteq [2, \quad]$ و x

$$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$$
$$(4, 5) + 2 \cdot [1, 2] \subseteq [6, 10]$$
$$(4, 5) + [2, 4] \subseteq [6, 10]$$
$$(6, 9) \subseteq [6, 10],$$

زا صريير.

$[2, 2.5]$ گبتب:

$$(4, 5) + 2.3[1.2] \subseteq [6, 10]$$
$$(4, 5) + [2.3, 4.6] \subseteq [6, 10]$$
$$(6.3, 9.6) \subseteq [6, 10],$$

زا صريير.

$[2, 2.5] \supseteq x = [2.1, 2.4)$ گبتب :

$$(4, 5) + x \cdot [1, 2] \subseteq [6, 10]$$
$$(4, 5) + [2.1, 2.4) \cdot [1, 2] \subseteq [6, 10]$$
$$(4, 5) + [2.1, 4.8] \subseteq [6, 10]$$
$$(6.1, 9.8) \subseteq [6, 10],$$

زا صريير.

ف ِ ُزِجَ َر س زكِبدُخ االززاء ز ِ كٍ ز أ ل ُ ازذ كذد لا نهائي ِ ً أس ِي أ ِض خ أ ِز ِيزُ اززاءنبثبفكً ف ِ أسً
االل ل ء.

ِ ً أخكِلِب ً ِغ االززافطط،ف ِ ز ِ أض أخ، نظ ًّ أل نبكِبدُخ از زاءفإب نأخفطط ِد كذبد خ ِي ية ِ ً أس ًّ أل ل ء ُوس ِي،
نظ ًّ أل ُ:

أس ًّ االل ء ُ ≤ أ ً د كخ أد ِي ِيخ

ٌ ِض ذ االس ِب َ ٌ ف ش ح، أل ْ: ال ح ل ا ِ ف ى ص ى ∈ أ ش ل ُ)ٌ ِي س ≤(

ِ ً أ ل ف ض كِ ذ ِي ل ز ا ٌ ِي ج ر: االل ء أس ً ≤ أ ِش ل ُ ل ُ ل ُ،]

كٍ ُ ص ج ِي ل ث ا ٌ ب ِي: [,2] ⊆ [2.1, 2]

3 .7. ِي ث ال ن ع ا ن ت ا ل ت ى ء ا ك ت ن ٍ س ن ه ا د م:

نحل ِ ً أخ x

$$(1, 2) - 2x \subseteq (0, 0.5)$$

أل ُ:

$$(1 - 2x, 2 - 2x) \subseteq (0, 0.5)$$

ز ِي ث : ≤ ≤ ً ≤ ≤

أ ≤ ≤ ً ≤ ≤

أ ≤ x ≤ ً ≤ x ≤

ٌ ى ً $[0.25, 0.50] \cap [0.75, 1]$

ث ٌ ِ ب ٌ ِ لا يوجد ز ً x.

3 .8. ِي ع ا ن ت ا ل ت ى ء ا ك ت ن ٍ ه ا د م و ذ ف ق ظ

نحل ِ ً أخ x :

$$(1, 2) - 2x \subseteq (0, 1)$$

ٌ ز ٌ ه: (1 x, 2 x) ⊆ (0, 1)

ث س ِي ث: ≤ ≤ و ≤ $2 - 2x$ ≤

أ ≤ ≤ ً ≤ ≤

أ ≤ ≤ ً ≤ x ≤

ِ ً أخ $[0, 0.5] \cap [0.5, 1]$

ذ س ءً كٍ ُ أ ْ ا ٌ س أ ح ِي د هو $[0.5, 0.5]$

ِ ٌ س م ك ِ ً ز االززاء:

(1, 2) ⊆ (0, 1)

(1, 2) ⊆ (0, 1)

(1, 2) ⊆ (0, 1)

(1 − 1, 2 1) ⊆ (0, 1)

(0, 1) ⊆ (0, 1),

ز ا ص ِي ر.

4. لطرق ت ألن ى ن ع م ي ع ألع ذاد ا ف ُت زوس ف ت لذق ق ت ل س ت خ ن ي ت ف إ ل ص اء ا ف ُت زوس ف

4. 1 ناق ٍ ت لذق ق ت v ة رق ى وا دذ

يأخذ ك ٍ ذا س م ٍ م ٍ أ ن ي ش ص ق ٍ ش ى N = a + bI، زيث a ب ن ك ذا ُ حقيقيا ُ ث ي ب "I" ٍ د ك خ حقيقية. أ دزء أ س ذ ٍ N ُ "a" ُ أ دزء غ ش أ س ذ)غ ي ش أ ض ر(ٍ N ُ "bI".

ك ن ش أ ُ القيمة الحقيقية ا ُ ع س ي ح ة ٍ القيمة أ ق ش ح v، أ ز ج س ث ك ب ف ط ٍ ب ي أ ص ز ء ب ي ئ ية زيث توجث يا ن ب د غ ش س ذ ح غ ي ش ض ل س خ غ ي ش ك ش ف خ خ ز ي ئ ي، زيث ي ز ٍ ٍ ز ا ل ُ أ ق ش ح v ط ٌ ا د ك خ الحقيقية I، أ v∈I.

ٍ أ ُ القيمة الحقيقية أ ز ي خ v خ ذ ح ف ٍ I ف ز ا ال ك ٍ ٍ أ ُ v خ ذ ح ف ٍ N=a+bI أ ي ض ً، ى

$a_1, b_1, a_2, b_2 \in R$ زيث $N_2 = a_2 + b_2 I$ ُ $N_1 = a_1 + b_1 I$ ى

I ٍ د ك خ خ ز ي ئ ية)أ ي ض ن ب ض ش ز ع خ ش ح(ٍ أ ل ك ذا د أ س ق ي ق ية.

ز ى القيمة أ س ق ي ق ية أ ز ج س ث ك ب ف ٍ أ ص ز ب ي ئ ي ات ف ٍ ظ أ ي ن ات غ ي ش أ س ذ ح)غ ي ش أ ق ش س خ أ أ ج خ (ٍ $v \in I$ ا ط ا ً:

$$a_1 + b_1 v \in a_1 + b_1 I = N_1$$
$$a_2 + b_2 v \in a_2 + b_2 I = N_2$$

ر ض ر ب أ ظ ز ي ب ُ ص ل ب ن ش ب ط ُ 1 غ ي إ خ ش اء ك ي ا ت ب ش ش ي ح ب ص ز خ ذ أ ل ب ن ش ص ق ي ح ز ق ي ق ي خ

ج ع أ ل ع ذاد ا ف ُت زوس ف ت لذق ق ت

$$N_1 + N_2 = (a_1 + a_2) + (b_1 + b_2) I$$

ا ا ل ث ي د

ق م ب ٍ ظ ز ي ة /1/ ش ب أ ُ: $a_2 + b_2 v \in a_2 + b_2 I$ ُ $a_1 + b_1 v \in a_1 + b_1 I$

ن ض ي ف ش ف أ ي س ل س ط ُ أ ي س ل س ش ف أ ي د ي ٍ ط ٌ أ ي د ي ف ٍ س ء ك ٍ:

$$(a_1 + b_1 v) + (a_2 + b_2 v) \in (a_1 + b_1 I) + (a_2 + b_2 I) = N_1 + N_2$$

$$(a_1 + a_2) + (b_1 + b_2) v \in N_1 + N_2$$ ٍ ز ٌ ه

ش ا ل ي ب د ٍ ب ث ٍ خ، ي ي ب ط خ ش اء أ ك ي ب دزأ ل ب ي ة ص ز خ ذ أ ل ك ذاد أ ي و ت ر و ص ف ي خ الحقيقية.

ط ز ح أ ل ع ذاد ا ف ُت زوس ف ت لذق ق ت: $N_1 - N_2 = (a_1 - a_2) + (b_1 - b_2) I$

ا ن ض ز ب ر ع ذد ي ن أ ل ع ذاد ا ف ُت زوس ف ت لذق ق ت: ٍ ى $\beta \neq 0$ ك ذ ي ن ز م ي ن ً. ك ٍ ت ب :

$$\beta \cdot N_1 = \beta \cdot (a_1 + b_1 I) = \beta \cdot a_1 + \beta \cdot b_1 I$$

اض ز ب أ ل ع ذاد ا ف ُت زوس ف ت لذق ق ت:

$$N_1 \cdot N_2 = (a_1 + b_1 I) \cdot (a_2 + b_2 I) = a_1 a_2 + (a_1 b_2 + a_2 b_1) I + b_1 b_2 I^2$$

ي ي ع أ ل ع ذاد ا ف ُت زوس ف ت لذق ق ت:

$$N^2 = (a + bI)^2 = a^2 + 2abI + b^2 I^2$$

نات n ة ى ق ا ن أ ل ع ذ ا د ا ذ ا ف ي ت و س و ف ي ت ل ق ي ت:

$$N^n = (a+bI)^n = \sum_{k=0}^{n} C_n^k (a^{n-k} b^k I^k) \quad \text{أ خ ك ذ ن ص ي ر} \quad n \geq 1 \quad \text{ز ي ث و ض ت ك ي ش ف} \quad I^0 \overset{def}{=} [1,1] \equiv 1$$

ا ن ج ذ ر ي ل ذ ر ج ت n ا أ ل ع ذ ا د ا ذ ا ف ي ت و س و ف ي ت ل ق ي ت: $\sqrt[n]{N} = \sqrt[n]{a+bI}$

ن ا ق س ت : $\frac{N_1}{N_2} = \frac{a_1 + b_1 I}{a_2 + b_2 I}$

ي ا ل د ظ ت ب ت ص ب خ ع ِ إ ث ل د ّ ك و ر ة أ ع ا ل ل م د م ع ُ ن ي ا ّ ك ى ّ ع و ل ت ذ ذ ّ

$I = (0, 1), or (0, 1], [0, 1), [0, 1]$

أ ز ض ّ ي ل خ ذ غ ب ّ ف ِ ا ر ُ ط ي ب د، أ ل ْ أ ل ك ذ ز م ِ م $n > 0$) . ب ا ل ض ب ّ خ ك $I^n = I$

4. 2 ن ا ق ِ ت ل ق ي ت V ه ْ ي ج ى ع ت

ّ أ خ ز ف ِ ا ل ك ج ي س ِ د ّ ك خ ِ ّ أ م ِ ي م ُ ا ل ح ق ي ق ي ة V، ُ أ ز ج س ّ ث ك ّ ب ف ِ أ ض ب ى ّ ط ز ء ب ي ّ ة ز ي ث ت و ج ث ي ا ن ب د غ ش ِ س ذ ح أ غ ش ا ض س خ أ غ ش ك ش ف ّ خ ز و ي ّ ، ز ي ث ن ِ ُ ر ض ّ ي ِ ّ V ف ِ I ، أ ($V \subset I$ أ ْ $V \subseteq I$).

ِ ّ أ ْ ِ د ّ ك خ أ م ِ ي م ُ ا ل ح ق ي ق ي ة V ِ خ ذ ح ف ِ I ف ِ ا ل ك ِ ا ز ض ّ ِ ي ف ِ أ ْ V ط ب ً ي $a + bI = N$ ، $a + bV \subset a + bI$.

ّ ي $N_1 = a_1 + b_1 I$ ، $N_2 = a_2 + b_2 I$ ك ذ ل ك ن ُ ق ّ ط ي ب ي ّ ح ق ي ق ي ّ ، ز ي ث $a_1, b_1, a_2, b_2 \in R$ ِ د ّ ك خ ي ّ ة I (ِ ّ أ ل ك ذ ا د ُ أ ح ق ي ق ي ة.

ّ ز ي ن ا ل ض ب ّ ش ش ّ خ ز ج (ِ ّ أ ل ك ذ ا د ُ أ ح ق ي ق ي ة ا ّ ز ِ ن ص ث ك ّ ب ف ا ِ ص ز ء ب ء ف ِ ظ ُ ي ا ن ب د غ ش ا ّ س ذ ح أ ّ ش ا ض س خ أ ّ ج ّ خ (ّ ، ك ّ ب ت ب: $V \subset I$

$$a_1 + b_1 V \subset a_1 + b_1 I = N_1$$
$$a_2 + b_2 V \subset a_2 + b_2 I = N_2$$

ِ ّ ز ي ن ا ل ض ب ف ّ خ:

$$(a_1 + b_1 V) + (a_2 + b_2 V) \subset (a_1 + b_1 I) + (a_2 + b_2 I) = N_1 + N_2 .$$

ث ن ّ ب ّ ك ن ا ل ض ّ ب ف ّ خ ِ ّ ط ذ ، ُ أ ض ش ة أ م ي ا ص ِ ، ُ أ ض ش ة، ُ أ ض ّ خ، ُ أ خ ِ ّ أ ش ّ ر ِ ّ ، ُ أ د س ِ ّ ، ُ أ ل ف خ ن، ِ ب ط ْ ر ّ ه.

$$(a_1 + b_1 V) - (a_2 + b_2 V) \subset (a_1 + b_1 I) - (a_2 + b_2 I) = N_1 - N_2 .$$

$$\beta \cdot (a_1 + b_1 V) \subset \beta \cdot (a_1 + b_1 I) = \beta \cdot N_1 .$$

$$(a_1 + b_1 V) \cdot (a_2 + b_2 V) \subset (a_1 + b_1 I) \cdot (a_2 + b_2 I)$$

$$= a_1 a_2 + (a_1 b_2 + a_2 b_1) I + b_1 b_2 I^2 =$$

$$= N_1 \cdot N_2 .$$

$$\frac{a_1 + b_1 V}{a_2 + b_2 V} \subset \frac{N_1}{N_2} .$$

$$(a+bV)^n \subset N^n .$$

$$\sqrt[n]{a+bV} \subset \sqrt[n]{N} \text{ , etc.}$$

رض ّ ربّ ا ِ ظ ِ يب ّ ظ ِ ث ِ م خ 1 2 3 ّ ث ِ إخ ِ ش ِ اء كّ ِ يّ ِ ات ب ِ ش ِ ش ِ ص ز خ ذ ِ ل ِ ل ِ ب َ ن ِ ش ّ ص ف ِ ي ِ ك ِ خ ز ِ ي ِ م ِ ي ِ ة ف ِ ي ا ِ ل ِ س ِ ب ِ ر ِ ي ّ :ل ِ ي ِ مة حقيقية وازخ v ، أ ِ د ّ ك ِ خ ّ ا ِ م ِ ي ِ م الحقيقية V(.

5. ل ِ ط ِ ر ِ ق ِ ت ِ ث ِ ا َ ت ِ ر ِ ن ِ ه ِ ع ِ ي ِ م ِ ي ِ ع أ ِ ل ِ ع ِ ذ ِ ا ِ ر ِ ي ِ ت ِ ن ِ و ِ س ِ ف ِ ت

ر ز ّ ث ّ ز ا ِ ّ ط ِ ر ِ ي ِ ق ِ ة ف ِ ي ر ِ س ِ ي ِ ل و ِ ك ّ ذ ِ ن ِ ش ّ ص ق ِ ي ِ ى ر ِ ح ِ ق ِ ي ِ ق ِ ة ط ّ د ِ ك ّ خ ز ِ م ِ ي ِ م ِ ي ِ خ:

$$N = a + bI = \{a + b \cdot x, x \in I\}$$

ط ِ خ ِ ش ِ اء ك ِ ي ّ ِ ي ِ ات ط ِ ش ِ ز خ ذ ا ا ّ د ّ ك ِ خ ب ِ د و ب ّ ض ر ِ أ د ن ا ِ .

ف ِ ز ِ أ ِ س ِ ب ِ خ ّ ي ِ س ِ ّ أ ض ِ ش ّ س ّ أ ّ ي ِ ى ّ ذ ِ ي ِ ن ا ن ف س ا ّ د ّ ك ِ خ الحقيقية غ ِ ي ِ ش ّ أ س ِ ذ ِ ح "I".

أ ا ل ّ، و ِ ج ِ ط ّ ز ِ ز ِ ش ِ ل ِ ك ّ ي ِ ة ب ِ ص ر ِ ز خ ذ ا ا ّ د ّ ك ِ خ ب ِ د أ س ِ م ِ ي ِ م ِ ي ِ ة.

5. 1. ع ِ ة ّ ا ِ ث ع ِ ى ل ّ ج ّ ى ع ِ ا ث:

ّ ي ِ R ِ ك ّ خ ا ل ك ِ ذ ا د الحقيقية، C ِ ك ّ خ ا ل ك ِ ذ ا د أ ّ ش ِ و ِ خ، M ِ د ّ ك ِ خ الأنواع ا ل خ ش ِ ا ل ك ِ ذ ا د.

ّ ن ِ ي A ِ B د ّ ك ّ و ِ ي ِ ح ِ ق ِ ي ِ ق ّ ي ِ أ ش ِ و ِ ج ِ ي ِ أ و نوع آخ ِ ا ل ك ِ ذ ا د.

ل ذ ي ِ ه ِ أ ز ذ ّ ب أ ّ و ال ّ ب أ ي ِ ض ً ا ل ط ِ ر ِ ي ِ ق ً ا ك ِ ذ ِ ي ً ا، أ ل ّ م ِ ي ِ ص ر ِ α∈R ي ِ ى ّ ز و ِ ث ز ُ و ّ د ّ ك ِ خ،]α,α[.

ك ّ ذ ِ ن ِ ز : $\{a \star b, ; \quad \star b$ ف ِ ش ِ ك ِ ; $a \in A, b \in B\}$ ⋆

ز ِ ي ِ ث ⋆ ر ِ ك ِ ب ِ أ ك ِ ّ ي ِ ا د: أ د ّ غ ّ أ ط ّ ش ذ ّ أ ض ّ ش ِ ة أ ق ِ ي ِ ا س ِ ي ّ أ ض ّ ش ِ ة ط ّ ض ّ خ ّ ا ج ّ أ د ن ِ س.

ث ِ ك ِ ذ ّ ر ّ ه، ي ِ ت ِ م ح ِ س ِ ب ِ ة أ س ِ ذ ِ ا ل أ د ن ى/inf أ س ِ ذ ِ ا ل أ ل ء/sup ل ّ - A⋆B.

ف ِ ا ل خ ِ ي ِ ب ّ ا ل ّ ي ِ خ، ث ِ ي ِ ف ِ ه ِ م ط ّ ط ّ ح ّ ا ِ د ك ِ ب د أ ل ك ِ ذ ا د أ س ِ م ِ ي ِ م ِ خ، ن ظ ّ ر ً ا ل ل ت ِ ط ِ ب ِ ي ِ ق ِ خ ط ِ ر ز ء ب ء أ ن ِ ي ِ ش ّ ص ق ِ ر، ّ ى ث ِ ب ِ ّ ض ِ ح ِ خ ل ل أ ن و ا ع ا ل خ ش ِ ا ّ د ّ ك ِ ب د ف ِ إ ّ ه ِ ي ث ّ ب ِ ث ّ.

إ ِ ض ِ ف ِ ل ِ ت ِ ا ن ّ ج ّ ى ع ِ ا ث:

$$\{a + b; a \in A, b \in B\}$$

أ ِ ث ِ خ:

$(2,3),\ B = (0,1)$

$(2,3) + (0,1) = (2 + 0, 3 + 1)$

$(2,3) + (2,3) = (2 + 2, 3 + 3) = (4,6) = 2 \cdot (2,3)$

ا ل خ ِ ي ِ ش ِ ز ُ ظ ِ ف ِ خ س ل ُ ط ّ ى ن ف س ك ِ ّ ص ِ ي ِ ل ث ّ ب ّ ي:

ط ِ ر ِ ح ل ّ ج ّ ى ع ِ ا ث:

$$\{a - b; a \in A, b \in B\}$$

أ ِ ث ِ خ:

$(2,3),\ B = (0,1)$

$(2,3) - (0,1) = (2 - 1, 3 - 0) = (1,3)$

$(2,3) - (2,3) = (2 - 3, 3 - 2) = (-1,1)$

ّ ز ّ ه 0

ك ّ ي ِ ش ل ن س ل ُ ّ ن ف س ه ك ِ ّ ص ِ ي ِ ل ث ّ ب ّ ي : 5-5=0)

انضزبنقإ سنّه ّج ّي ع اث:

$\beta \in R$ ّي ّ كتب β $\{\beta \cdot a; a \in A\}$

أيثت:

i). β $(2,3)$:

β (,) (,) (,).

ii). β , (,)

β (,) (,) (,) (أ ّد كّخ ّاخبّية).

iii) β , [,]

β [,] [,] [,] { }

ِد كّخ رسزّ كّ ِك ِش ازفمط ّ أءفش

اضزب ل ّج ّي ع اث:

$$\{a \cdot b; a \in A, b \in B\}$$

أيثِخ:

$(2,3),\ B = (0,1)$

$(2,3) \cdot (0,1) = (2 \cdot 0, 3 \cdot 1) = (0,3)$

$(2,3) \cdot (2,3) = (2 \cdot 2, 3 \cdot 3) = (4,9) = A^2.$

قس ت ل ّج ّي ع اث:

$-$ $\{a$ كشف $, a$ A, b $B\}$

أيثِخ

$(2,3),\quad B = (0,1)$

)i(ّ أخ ّ (A,B) كشف خ فوش اد ي ىْ

$$\left(\frac{minA}{}, \frac{}{minB}\right) = \left(-, -\right) \to (2, +\infty)$$

أل ّ أسذ غيش أكشف $\infty \to \frac{3}{-}$ ّيس إلى $-\infty$)أل ّ B يحتوي فمط كّ ِكّبصش ِ خ(

)ii(ّي ّ ذيفوشرير ّ : $A = (2,3)$ ّ $C = (-1,0)$ كتب:

$$C = \left(\frac{minA}{axC}, \frac{}{minC}\right) = \left(-, \text{—}\right) = \left(-, -3\right) \to (-\infty, -3)$$

نأخذ – كٍ ' أنب)-∞(، أل ' ا ّ د كخ Cسز ّ فـمط كٍكبصشبجّخ.

)iii($\left(\frac{2}{3},\frac{3}{2}\right)$ $\left(\frac{minA}{maxA},\frac{axA}{minA}\right)$

ّزه كٍ كىشلخ ّخ ألكذاد اسمميخ، زيث ي ' ّذ غيش ا ّ فش ّ ضن نبكٍ فضرضليب ّ 1، كضجرٍ ّثبيّ:

. –

)iv($\left(\frac{minB}{maxB},\frac{maxB}{minB}\right)=\left(\frac{0}{1},\frac{1}{0}\right)\rightarrow (\mathbf{O},+\infty)\neq \mathbf{1}$

ناقىة وجذر ك ّج ّى عاث:

ّي ّ rكذاضجيّ، أ ّ r=m/n، زيث n،m كلذاد صحيخ، n ≠0.

$\{a^r; a^r \text{ فشك}, a\in A\}$

)i(لكذ صسير ّخت:

$(2,3),$

$(2,3)^4=(2^4,3^4)=(16,81).$

$(-2,3).$

$(-2,3)\cdot(-2,3)=(-2\cdot 3,3\cdot 3)=(-6,9)\neq((-2)^2,3^2)=(4,9).$

)ii(ّمح صفش:

$(2,3),$

$(2,3)^0=(2^0,3^0)=(1,1)$.

ّزه

كٍكّش االكذاد اسمقيخ، زيث كضجرٍ ثبيّ .70

ّي ّ $D=[2,3]$ كتب

$[2^0,3^0]=[1,1]\equiv\{1\}$

ٍثبجّخ ألكذاد الحقيقية

)iii(ادس ّثيك: $\sqrt{A}=\sqrt{(2,3)}=\left(\sqrt{2},\sqrt{3}\right)$

)iv(ادس ّثيك ادئ: ّي ّ $=(-2,3)$ كتب

$\sqrt{D}=\left[\sqrt{0},\sqrt{3}\right)=[0,\sqrt{3})$

لأنه في ّد كخ االكذاد اسمميية لا يئ ّ ضتبة ادس ّثيك ألكذاضجخ ّ افتش $(-2,0)$ ّلتب فمط ضتبة ادس ّثيك ادئ ّ D.

)v(أ خ ط ن ب ج خ:

$(2,3), r = -2.$

$(2,3)^{-2} = (2^{-2}, 3^{-2}) = \left(\frac{1}{-}, \frac{1}{}\right) \equiv \left(\frac{1}{-}, \frac{1}{-}\right).$

الآ ،رزخ أ ك ب ب ك ب ا ل ك ذ ا ل ي ت روس وفي ة الحقيقية ل ك ذ أ ك ب ب ك ب أ د ك ب ب أ س م ب ية أ ض س خ ك ال ، أ ل ك ذ

ا ل ي ت روس وفي أ س م ب ك ب دي د ك خ خ ز ئ ية حقيقية:

ي I د ك خ خ ز ئ ية حقيقية $\mathbb{R} \subset$

$$\{a + b \cdot x, ; x \in I\}$$

د ك خ خ ز ئ ية حقيقية أ ر ج وب ف I.

ف ا ل غ N أ د ك خ أ د ئ ية ط ص خ I.

طروب I ف ش ح أ ش ئ:

$$(c,d), \text{or } [c,d), \text{or } (c,d], \text{or } [c,d]$$

ك ب ز ص ئ N أ ض ف ش ح ف ش أ ر ج ف ن ذ أ غ ك م ب ث.

ا ر ا $I = \{c_1, c_2, \dots, c_n\}$ وب ن ت د ك خ خ ز ئ ية ف ء خ حقيقية ز ي ث

$1 \le n \le \infty$ ، ك ب ت ب ص ز ي N أ ض د ك خ خ ز ئ ية ف ء خ حقيقية أ ك ا ل ن ص ب ص n.

ك ب ب ي ث ا ل ب ك خ د ك ب خ ز ئ ية 1

ك ب ت ب ص ي N أ ض س ب د ا د ك ب د أ د ئ ية م ب ث خ:

$$\bigcup_{=1}^{1}$$

ز ي ث:

$$\{a + bx, ; x \in I_k\}$$

ا ل ط ر ق ت ا ث ا ن ي ع ش ر ة ا ل ث ا ن ي ة ل ل ع م ل ي ا ت ا ل ح س ا ب ي ة:

ل ن ز س ب ل و ك ذ ن ش ر ح ف ي ح ق ي ق ة ط د ك خ خ ز ئ ية حقيقية ب ف خ ، خ ص خ ك ب ب ي ث N ك ذ ل ك ي ب (I ج ب ً.

أ ث خ:

1 1 ; I_1 { , 5,0 8}

; $I_2 = [0,1)$

ك ب ت ب:

$N_1 = 1 + 2 \cdot \{0.2, 0.5, 0.8\}$ $\{0.4, 1.0, 1.6\} = \{1.4, 2.0, 2.6\}$

$[0,1) = (3-1, 3$ $] = (2,3]$

ج ع ا ل ع ذ ا د ا ت ف ي ت ز و س وف ي ت ا ل ح ق ي ق ة

$N_1 + N_2 = \{1.4, 2.0, 2.6\} + (2,3] = \{1.4 + (2,3]\} \cup \{2.0 + (2,3]\} \cup \{2.6 + (2,3]\}$

$(1.4 + 2, 1$ $] \cup (2.0 + 2, 2.0 + 3] \cup (2.6 + 2, 2.6 + 3]$

$(3.4, 4.4] \cup (4, 5] \cup (4.6, 5.6] = (3.4, 5.6].$

إضافة عدد حقيقي إلى مجموعة أعداد حقيقية

$$0.9 + N_1 = \{1.4, 2.0, 2.6\} + 0.9 = \{1.4 + 0.9,\ 2.0 + 0.9,\ 2.6 + 0.9\}$$
$$= \{2.3, 2.9, 3.5\}.$$
$$0.9 + N_2 = 0.9 + (2, 3] = (0.9 + 2, 0.9 + 3] = (2.9, 3.9].$$

طرح أعداد حقيقية من مجموعة أعداد حقيقية

$$N_1 - N_2 = \{1.4, 2.0, 2\ \} - (2, 3] = \{1.4 - (2, 3]\} \cup \{2.0 - (2, 3]\} \cup \{2.6 - (2, 3]\}$$
$$[1\quad 3, 1.4 - 2) \cup [2.0 - 3, 2\quad 2) \cup [2.6 - 3, 2.6 - 2)$$
$$[-\ \ 6, -0.6) \cup [-1, 0) \cup [-0.4, 0\ 6) = [-1.6, 0.6).$$

ضرب أعداد حقيقية في مجموعة أعداد حقيقية

$$N_1 \cdot N_2 = \{1.4, 2.0, 2.6\} \cdot (2, 3] \quad \{1.4 \cdot (2, 3]\} \cup \{2.0 \cdot (2, 3]\} \cup \{2.6 \cdot (2, 3]\}$$
$$(1\quad 2, 1\quad] \cup (2.0 \cdot 2, 2.0 \cdot 3] \cup (2.6 \cdot 2, 2.6 \cdot 3] = (2.8, 4.2] \cup (4, 6] \cup (5.2, 7.8]$$
$$(2\ 8, 7.8].$$

ضرب عدد حقيقي في مجموعة أعداد حقيقية

$$_1 \quad \{1.4, 2.0, 2.6\} = \{4 \cdot (1.4), 4 \cdot (2.0), 4 \cdot (2.6)\}$$
$$\{5.6, 8.0, 10.4\}.$$

قسمة أعداد حقيقية على مجموعة أعداد حقيقية

$$\frac{N_1}{} \quad \frac{\{1.4, 2\ 0, 2.6\}}{(2, 3]} \quad \frac{}{(2, 3]} \quad \frac{}{(2, 3]} \quad \frac{}{(2, 3]} \quad \left(—, —\right] \cup \left(—, —\right] \cup \left(—, —\right]$$
$$(0.4\overline{6}, 0.7] \cup (0.\ \overline{6}, 1] \cup (0.8\overline{6}, 1.3] = (0.4\overline{6}, 1.3].$$

أمثلة أخرى عن عمليات حسابية على أعداد حقيقية

ليكن:

$$N_1 = 2 - 3I_1 \quad ; \quad I_1 = [4, 5]$$
$$N_2 = 1 + 4I_2 \quad ; \quad I_2 = \{-1, 3, 5\}.$$

نحسب:

$$N_1 = 2 - 3I_1 = 2 - 3 \cdot [4, 5] = 2 - [3 \cdot 4, 3 \cdot 5] = 2 - [12, 15] =$$
$$= [2 - 15, 2 - 12] = [-13, -10]$$

$$N_2 = 1 + 4I_2 = 1 + 4 \cdot \{-1, 3, 5\} = 1 + \{4 \cdot (-1), 4 \cdot 3, 4 \cdot 5\} =$$
$$= 1 + \{-4, 12, 20\} = \{1 + (-4), 1 + 12, 1 + 20\} = \{-3, 13, 21\}$$

$$\frac{N_1}{N_2}=\frac{[-13,-10]}{\{-3,13,21\}}=[\frac{-13}{-3},\frac{-10}{-3}]\cup[\frac{-13}{13},\frac{-10}{13}]\cup[\frac{-13}{21},\frac{-10}{21}]=$$
$$=[\frac{10}{3},\frac{13}{3}]\cup[\frac{-13}{13},\frac{-10}{13}]\cup[\frac{-13}{21},\frac{-10}{21}]=$$
$$=[\frac{-13}{13},\frac{-10}{13}]\cup[\frac{-13}{21},\frac{-10}{21}]\cup[\frac{10}{3},\frac{13}{3}]$$

ناتس ، بتع ذلا قيقدلا ةيفسوزتلا ا دادعلا ةيق

$\underline{N_1}\quad \underline{\{1.4,2.0,2.6\}}\quad \left\{\underline{\frac{1.4}{}},\underline{—},\underline{\frac{2.6}{}}\right\}=\{0.35,0.50,0.65\}.$

$\overline{N_1}\quad \overline{\{1.4,2.0,2.6\}}\quad \left\{\frac{4}{1.4},\frac{4}{2.0},\frac{4}{2.6}\right\}=\{2.857,2.000,1.538\}.$

$—\quad \underline{(2,3]}\quad \left[\frac{-}{3},\frac{-}{2}\right)\approx[1.333,2.000)$

$—\quad \underline{(2,3]}=\left[\frac{2}{-},\frac{3}{-}\right)=[\ \ 50,0\ 75).$

ةيقدلا تيفسوزتلا ا دادعلا ةيق

$_1\quad \{\ \ ,\ \ 0,2\ 6\}^{(2,3]}\quad (2,3]\quad (2,3]\quad (2,3]\quad (1\ \ ,1\ \ ^3]\cup(\ \ \ ,2.0^3]\cup(2.6^2,2.6^3]$
$(1\ 960,2.744]\cup(4,8]\quad (6\ 760,17\ \ \ \ ]=(1.960,2.744]\cup(4.000,17.576]$

$N_1\quad (2,3]^{\{1.4,2.0,2.6\}}\quad \{(2^{1.4},3^{1.4}],(2\ \ ,3\ \ ],(2^{2.6},3\ \ ]\}$
$\{(2.639,4.656],(4\ 0,9\ 0],(6\ 063,17\ 399]\}$
$\{2.639,4.656]\cup(4.0,9.0]\cup(6.063,17.399]$
$(2\ 639,17\ 399].$

ةيقدلا تيفسوزتلا ا دادعلا ىلع ، بتع

$(N_1)^4=\{1.4,2.0,2.6\}^4=\{1.4^4,2.0^4,2.6^4\}=\{3.8416,16.000,45.6976\};$
and $N_1=4^{\{1.4,2.0,2\ \ \}}=\{4^{1.4},4^{2.0},4^{2.6}\}\approx\{6.9644,16.000,36.7583\}$

فيسوزتلا يقيقحدعلا يقيقحلا رذجلا

$\sqrt{N_1}=\sqrt{\{1.4,2.0,2.6\}}=\{\sqrt{1.4},\sqrt{2.0},\sqrt{2.6}\}\approx\{1.183,1.414,1.612\}$
$\sqrt[3]{N_2}=\sqrt[3]{(2,3]}=(\sqrt[3]{2},\sqrt[3]{3})\simeq(1.260,1.442].$

ةيقدلا تيفسوزتلا ا رذجلا فيسوزتلا ذعب قيقدلا

$\sqrt[N_1]{N}\quad {}^{\frac{1}{1}}=(2,3]^{\frac{1}{\{1.4,2.0,2.6\}}}=\left\{(2,3]^{\frac{1}{1.4}},(2,3]^{\underline{\ \ 1\ \ }},(2,3]^{\underline{\ \ 1\ \ }}\right\}=\left\{\left(2^{\frac{1}{1.4}},3^{\frac{1}{1.4}}\right],\left(2^{\underline{\ \ 1\ \ }},3^{\underline{\ \ 1\ \ }}\right],(2^{\underline{\ \ 1\ \ }},3^{\underline{\ \ 1\ \ }}]\right\}$

$\{(1.641,2\ 192],(1.414,1.732],(1.306,1\ 526]\}\equiv(1.306,2.192]$

*

$$\sqrt{N_1} = N_1^{\frac{1}{}} \quad \{1.4, 2.0, 2.6\}^{\frac{1}{(2,3]}} \quad \{ \quad , \quad 0,2\ 6\}^{[\frac{1}{3},\frac{1}{2})} \quad \left\{1 \quad {}^{[\frac{1}{3},\frac{1}{2})}, 2.0^{[\frac{1}{3},\frac{1}{2})}, 2 \quad {}^{[\frac{1}{3},\frac{1}{2})}\right\}$$

$$\left\{\left[1\ 4^{\frac{1}{3}}, 1 \quad {}^{\frac{1}{}}\right), \left[2\ 0^{\frac{1}{3}}, 2 \quad {}^{\frac{1}{}}\right), \left[2\ 6^{\frac{1}{3}}, 2 \quad {}^{\frac{1}{}}\right)\right\}$$

$$\{[1\ 119, 1.183), [1\ 260, 1\ 414), [1\ 375, 1.612)\} \equiv [1.119, 1.183) \cup [1.260, 1.612).$$

6.ألعذاد ا فُتوسفيت ل ذفيت:

ألكذاد أيوتروصفيش فية)LNN (ّب ّلش ىً ل ّبٍ

زيث a, b ٍ أرقام حقيقية أو شِوخ ، $I = \mathrm{I}^{\cdot}$ كذ ّاتحديد ّاسرشف ٍ زيث $I^2 = I,$ غيش سذ

ط ْ كّيات ّادغ ّاطش ذ ّاضشة ّاملياس ٍ ّاضشة ّاملضّخ ّا جْ ّادوشيطخجّبشش.

الوُزرخ َ ألكذاد أيوتروصفيخ ّاسرشفية فياصزءبء أنيوشّص قـ ٍ، ثً ف ٍ ّا جْ ّاطشية أيوتروصفيخ، ّزّاطرجت الكيش ضرك ّيلبَ ٍبّ.

7.يعاينت عذو الت ّاء، ويعاينت عذو ال تساء، ويعاينت عذو ل سّاواة:

طنبكيبذّخرى ّيية ّز ّكيّبذّخ االزّباء، كيّبذّخ االززاء، كيّبذّخ ّاملضباحّ كٍ ّا ّزْ ّا. نقد َب و ْع ٍ ّاقـضْ ي ، أوش يبضياتشفيـ يَة.

)i(كيّبذّخ إّسّمخ ٍ ّاثبّيضلبثك 1 ّ: $(0.5, 0.8)$

وبـنت زّب وَب ّكذاد حقيقيخ$(0\ 28, 0\ 40)$

كيبذّخ ّكذاالزبّاء ّاملبثخ ّ َب ّ: $4-5x \notin 1+2\cdot(0.5, 0.8)$

زّ َب ٍ وَبأكذاد زقيقية $x \notin (0.28, 0.40)$ أ خ ّيع الأعداد الحقيقية $x \in R-(0.28, 0.40)$

)ii(كيبذّخ االززاء ٍ ّثّابّيضلبثك 2وبـنت:

$$(1, 2) \subset (0, 5),$$

وب ْ ّاسّاأل لءْ ّ $(-\ 5, 2)$

كيبذّخ ّكذاالززاء ّاملبثخ ّب ّ: $1+x\cdot(1, 2) \not\subset (0, 5)$

ّازّز ٍّاأل لءْ ّ $(\ 5, 2\ )$

)iii(كيبذّخ ّاضباحّ االّية $3x$

ّب ّاسً ّاحيد $x = 1$.

كيبذّخ ّكذ ّاضباحّ ّاملبثخ ّ َب ّ: $3x+4 \neq 7$

ثبّطخغ، ّبنكذد لا نهائي ٍ ّاسّ ي $x \in R-\{1\}$.

8.ألخ الصت:

فاصزءبء أنيوشّص قـ ٍ، ٍ ّأ ْ القيمةّاسّميية ّاحيح v ٍ ّخ ْ فـ ٍ I،ال يزحكّ ّرّه أ ْ v ٍ ّخ ْ فـ ٍ $a + bI = N$ ((أيضً، ىً ّ: a+bv ∈ a+bI. ّزا ّخت ّاكيش يفـكيّالخ االزّباء كيبذّب ّسطرزب.

كّ الخ كّ ٍّ مطرواب ْ ّ ذ ّاشء ٍ دكّخ ٍ ّ أ ْ ٍ دكّخ ّاميم الحقيقية، ٍ ّ أ ْ ٍ دكّخ ّاميم الحقيقية V زض ّخ ف ٍ I ف زّاال يعني أ ْ V زض ّخ أيضً ف ٍ a + bI، ىً ّ $a + bV \subset a + bI$ أ($a + bV \subseteq a + bI$)ّزّاطرجت ّخترم ذيكّالخ كيبذّخ الززاء.

ثبفش أطس٠خ أز٠ ٠ز٠ثتصلزخ ذلس ز "= ٌ٘ال خ ٌضب احٌ٘ب ضبلح، نسزخذ ثس ز "∈" }٠ز ّ ط}٘ال خ ٌ٘ب ذلزب٘ال خ ٌ٘ب ذ
ّ ذ ّخ، ٘ٔ أ ثس ز ⊂)أ ⊇({ ٌس ز٠ف ، أ ٠ س ز٠ف ط٠ب ّ}٘ال خ ٌ٘ب ذ ااالزززاء.

ّ مذ لذب أل ّ ي ش٘ح٘ب ذ ااالزب٘ء ٘ب ذ ااالز زاء، ا ّ ز٠ض ب ذ ف٘ف ّ ا٘ب ذ غ االّ ذاذ أ٘وتروصف٠خ ض ّ ط ب س اصزب٘ء
أنيوشرّص ق٠ط ّ طريقة ز ّ ز ٘ ٌ ٘ب ذ الدرش ج ٌ ب ذ الد أز٠ ر ى ّ ٘ب ٌ٘ب الزب ٘ح ب س ح ٘ ّ ذ ّ ذ ب ذ)ال رخ ذ ٌلل ب ف ش ح٘ب ض ف٠خ ط ّ
ر٘ه لذب أ٠ض٘ب ذ ّ ذ ّ ذ الزب٘ء ا ّ ى ّ خ، ٘ب ذ ّخ ذ ذ االززاء، ٘ب ذ ّخ ذ ذ أ ض ب ح االّ ٘ة ذ ّ ٌ أ.

الق ى ّم: "٠ز٘ك ز ا ج ّس ث أز ّ ٘ل خ ب س خ ٠".
ت ض ارب ال ّ ص ال خ: "٠ع ّ ا ّ و ّ ّ ذ ّ ٘ض ب س ة ف ٠ أ ب ّ ر".

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Integrarea perspectivelor contradictorii în filosofia latino-americană: O abordare MultiAlistă

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Abstract

Acest studiu explorează integrarea filosofiilor indigene și a teoriilor moderniste postcoloniale în cadrul MultiAlist, evidențiind potențialul de dialog și sinteză între aceste tradiții filosofice diverse din America Latină. MultiAlismul, un concept dezvoltat de Florentin Smarandache, este utilizat pentru a construi punți peste lacunele și contradicțiile dintre sisteme, concentrându-se pe zone de neutralitate, precum, în acest caz, diversitatea culturală și criticile structurilor de putere. Cercetarea subliniază importanța incorporării atât a înțelepciunii ancestrale, cât și a criticilor moderne, pentru a favoriza o înțelegere și o apreciere mai profundă a patrimoniului filosofic bogat al Americii Latine. Această abordare integrativă oferă perspective practice pentru elaborarea unor politici incluzive și pentru succesul eforturilor de justiție socială în regiune. Extinderea aplicării MultiAlismului în alte contexte globale sugerează relevanța sa mai largă pentru abordarea provocărilor contemporane printr-o abordare filosofică flexibilă și holistică.

Cuvinte cheie

MultiAlism, filosofii indigene, teorii moderniste postcoloniale, America Latină.

1. Introducere

Sistemele de cunoaștere indigene din America Latină pun accent pe o conexiune holistică cu cosmosul și natura, reflectând valori comunitare și ecologice. În contrast, importurile filosofice europene din perioada colonială au prioritizat individualismul, raționalismul și o viziune mecanicistă asupra naturii. Această juxtapunere subliniază tensiuni socio-culturale și politice semnificative care influențează identitatea și evoluția regiunii [1].

MultiAlismul, așa cum a fost dezvoltat de Smarandache, oferă o abordare metodologică ce îmbrățișează pluralismul și neutrosofia, permițând integrarea sistemelor filosofice aflate în conflict într-un cadru dinamic și deschis [2].

MultiAlismul postulează un cadru în care diverse dimensiuni filosofice precum monismul, dualismul, pluralismul și chiar variații mai nuanțate pot coexista și interacționa. MultiAlismul încurajează explorarea și sinteza acestor multiple dimensiuni, fiind deosebit de potrivit pentru contexte bogate în tradiții filosofice diverse, precum America Latină.

Dimensiuni filosofice [2]:

Monismul afirmă că există o singură realitate sau un singur principiu fundamental, adică <A> = ∞, unde <A> este o „idee”, o „substanță” etc., iar ∞ este „lumea”, „realitatea”, „totul”.

Dualismul recunoaște două forțe sau principii fundamentale și adesea conflictuale, adică <A> + <antiA> = ∞, unde <A> este o „idee”, o „substanță” etc., <antiA> este opusul sau negația sa, iar ∞ este „lumea”, „realitatea”, „totul”.

Pluralismul îmbrățișează o diversitate de principii sau realități coexistente, adică <pluriA> = ∞, unde <pluriA> înseamnă mai mult de două (trei sau mai multe) „idei” etc., iar ∞ este „lumea”, „realitatea”, „totul”.

MultiAlismul transcende aceste clasificări pledând pentru un cadru în care aceste și alte categorii filosofice pot interacționa dinamic, adică <(multi)A> + <(multi)neutA> + <(multi)antiA> = ∞.

Această flexibilitate îl face un instrument inovator pentru analiza și sinteza diversității filosofice. Abordarea este crucială pentru a răspunde realităților complexe ale Americii Latine, favorizând o înțelegere mai nuanțată și o apreciere a tradițiilor filosofice diverse [3].

Sinteza filosofiilor indigene și europene prin MultiAlism poate spori semnificativ înțelegerea peisajelor socio-politice și culturale unice din America Latină. Ea promovează înțelegerea interculturală și este esențială pentru abordarea provocărilor globale și promovarea dezvoltării durabile [4].

2. Preliminarii

Filosofia latino-americană de dinainte de contactul european era caracterizată de investigații metafizice bogate, înrădăcinate în cosmologiile religioase amerindiene. Cercetători precum Restrepo [5] și Maffie [6] evidențiază modul în care aceste civilizații explorau concepte despre realitate, moralitate și cosmos fără influența tradițiilor filosofice occidentale, adesea în contexte profund religioase.

Odată cu cucerirea spaniolă, scolastica a devenit cadrul filosofic dominant, introducând tradițiile intelectuale europene în Lumea Nouă. Bartolomé de Las Casas, după cum remarcă Nuccetelli [7], a fost esențial în apărarea drepturilor populațiilor indigene, contestând ideologiile europene dominante și afirmând umanitatea și drepturile nativilor.

Perioada independentistă și naționalistă a marcat o trecere către filosofia modernă și gândirea iluministă, în special în timpul războaielor de independență. Figuri precum Simón Bolívar au integrat idealuri politice liberale din Iluminism pentru a contura ideologiile noilor națiuni. Perioada naționalistă a cunoscut ascensiunea pozitivismului, care, potrivit unor autori precum Hale [8], reflecta o abordare științifică și empirică a problemelor filosofice și sociale, influențând construcția națională și modernizarea regiunii.

Peisajul filosofic contemporan din America Latină rămâne divers și complex, cu o influență europeană substanțială. Totuși, după cum arată Beorlegui [9], a existat o creștere semnificativă a mișcărilor filosofice originale care abordează preocupări sociale. Teologia eliberării, de exemplu, a apărut ca un contracurent puternic la viziunile catolice tradiționale, pledând pentru justiție socială și o opțiune preferențială pentru săraci.

Teologia eliberării, apărută în anii 1960-1970 și detaliată de Gustavo Gutiérrez [10], se concentrează pe justiție socială, sărăcie și drepturile omului, interpretând credința creștină prin prisma situației celor săraci. Aceasta contrastează cu catolicismul conservator, care, după cum sugerează cercetători precum McGovern [11], susține valori și doctrine tradiționale, focalizându-se pe mântuirea spirituală și menținerea ordinii religioase stabilite.

Valorile indigene comunitare prioritizează bunăstarea colectivă și o conexiune holistică cu natura, după cum analizează cercetători precum Kusch [12], care examinează epistemologiile indigene și contrastele lor cu gândirea europeană. În opoziție, ideologiile post-coloniale influențate de iluminismul european și gândirea liberală pledează pentru drepturi

individuale și capitalism, intrând adesea în conflict cu practicile comunitare indigene.

Filosofi precum José Carlos Mariátegui și Enrique Dussel au criticat și oferit alternative la filosofiile eurocentrice, subliniind o abordare decolonială. Dussel [13], de exemplu, explorează etica eliberării, care critică perspectivele tradiționale europene și subliniază importanța unei etici bazate pe experiența marginalizaților.

Influența filosofiilor indigene asupra mișcărilor socio-politice contemporane evidențiază o reconceptualizare filosofică în mediul academic, contestând dominația filosofiilor occidentale și favorizând un dialog mai bogat și mai incluziv privind problemele sociale și politice.

3. Materiale și Metode

Obiectivul analizei filosofice comparative din acest studiu este de a examina modul în care diferite școli de gândire pot fi înțelese și integrate în cadrul MultiAlist.

Abordare metodologică:

Selecția școlilor filosofice:

- Identificarea principalelor școli de gândire predominante în America Latină, precum Teologia eliberării, catolicismul conservator, cosmologiile indigene și teoriile moderniste postcoloniale. Această selecție se bazează pe semnificația istorică și relevanța actuală în peisajul socio-politic latino-american.

Analiza tematică:

- Identificarea temelor comune și a diferențelor dintre școli, cu focalizare pe modul în care acestea abordează întrebările filosofice

și existențiale esențiale. Este importantă nu doar identificarea contradicțiilor, ci și a neutralităților sau indeterminărilor.

Coexistența contradictorie:

- Analiza modului în care contradicțiile dintre aceste școli pot coexista în cadrul MultiAlist, contribuind la o înțelegere mai nuanțată a filosofiei latino-americane.

Sinteza:

- Integrarea rezultatelor pentru a ilustra modul în care MultiAlismul poate servi drept meta-cadru pentru înțelegerea și sinteza tradițiilor filosofice diverse ale Americii Latine.

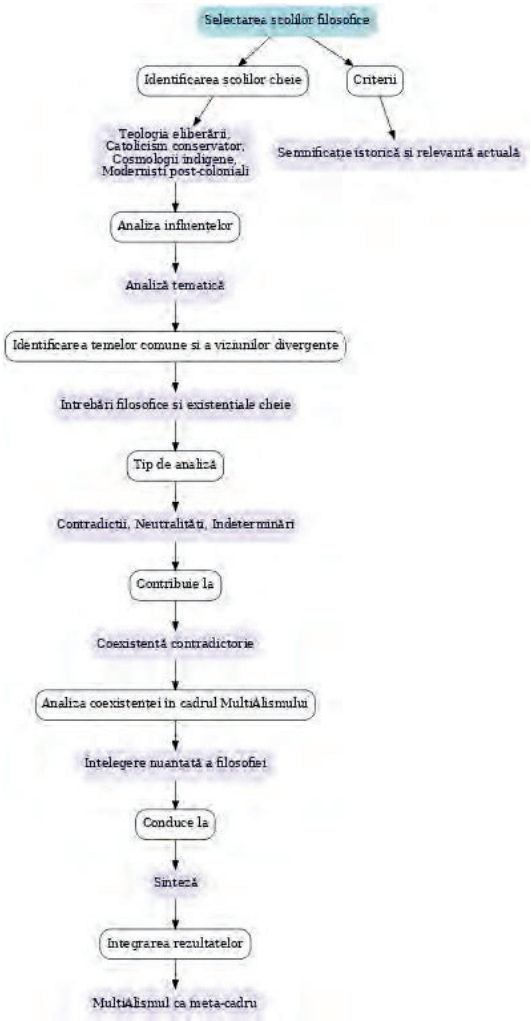


Figura 1. Hartă conceptuală a abordării metodologice

4. Studiu de caz

Filosofiile indigene și teoriile moderniste postcoloniale prezintă viziuni contrastante care pot fi sintetizate sub cadrul MultiAlismului pentru a genera modele socio-politice inovatoare. Filosofiile indigene, în general pluraliste, accentuează bunăstarea comunității și o relație holistică cu natura, în timp ce teoriile moderniste postcoloniale, adesea dualiste sau moniste, se concentrează pe drepturile și libertățile individuale provenite din ideile iluministe. Punând aceste perspective divergente în dialog, MultiAlismul permite explorarea modului în care valorile comunitare indigene pot coexista și completa ideologiile moderne individualiste, conducând potențial la noi modele de guvernanță și organizare comunitară.

Contradicții

Contradicțiile dintre filosofiile indigene și teoriile moderniste postcoloniale, deși uneori subtile, pot fi semnificative datorită diferențelor fundamentale în bazele lor epistemologice, ontologice și axiologice. Mai jos sunt câteva dintre contradicțiile cheie:

Concepții despre timp și progres

Filosofii indigene:

- Timpul este de obicei conceput ciclic și axat pe continuitatea și repetiția ciclurilor naturale și cosmice. Progresul nu este liniar, ci în armonie cu ciclurile vieții și naturii.

Teorii moderniste postcoloniale:

- Deși critice față de conceptul european de progres liniar și cumulativ, aceste teorii operează totuși în paradigme ce valorizează schimbarea constantă, caracteristică a gândirii postmoderne.

Relația cu natura

Filosofii indigene:

- Relația este una de interdependență cu natura, aceasta fiind considerată sacră și centrală.

Teorii moderniste postcoloniale:

- Deși pot critica exploatarea capitalistă a mediului, nu plasează întotdeauna natura în centrul discursului sau practicilor lor, prioritizând adesea probleme socio-politice și culturale.

Individualism vs. Comunitarism

Filosofii indigene:

- Comunitarismul este central, accentuând importanța comunității asupra individului. Deciziile și bunăstarea sunt considerate colectiv.

Teorii moderniste postcoloniale:

- Deși orientate spre eliberarea de structuri opresive, promovează deseori autonomia individuală și auto-realizarea, influențate de ideile iluministe despre emanciparea individului.

Epistemologie și cunoaștere

Filosofii indigene:

- Cunoașterea este transmisă oral, cu o componentă spirituală și experiențială puternică, fiind profund conectată la mediul local și strămoși.

Teorii moderniste postcoloniale:

- Deși valorizează cunoașterea subalternă și încearcă să revalorizeze cunoașterea locală, sunt puternic influențate de structuri academice

și epistemologii occidentale, care prioritizează textualitatea și rațională.

Viziuni despre putere și rezistență

Filosofii indigene:

- Rezistența se manifestă prin persistența culturală și subzistență, menținând tradițiile și practicile ancestrale în fața opresiunii.

Teorii moderniste postcoloniale:

- Rezistența este conceptualizată prin deconstruirea discursurilor și structurilor coloniale și neo-coloniale, utilizând instrumente teoretice și critice provenite adesea din contexte academice.

Neutralități și indeterminări

Valorizarea diversității culturale

Filosofii indigene:

- Respect profund pentru tradiții și diversitatea culturală ca piloni ai identității comunitare.

Teorii moderniste postcoloniale:

- Recunoașterea pluralității culturale ca resursă împotriva moștenirilor coloniale și ca formă de rezistență împotriva omogenizării culturale globale.

Zonă de neutralitate:

- Ambele curente valorizează și apără diversitatea culturală, deși din perspective și cu obiective diferite.

Critica structurilor tradiționale de putere

Filosofii indigene:

- Critică impunerea externă și structurile de putere care alterează sau distrug structurile comunităților indigene.

Teorii moderniste postcoloniale:

- Critică colonialismul și structurile de putere care perpetuează inegalități și suprimare culturală.

Zonă de neutralitate:

- Ambele curente urmăresc desființarea sau reformarea structurilor de putere considerate opresive sau injuste.

Interes pentru justiția socială și economică

Filosofii indigene:

- Focus pe justiția socială în cadrul comunității, accentuând egalitatea și sprijinul reciproc.

Teorii moderniste postcoloniale:

- Luptă pentru justiție socială prin desființarea structurilor postcoloniale și promovarea echității.

Zonă de neutralitate:

- Ambele curente împărtășesc angajamentul pentru justiția socială și economică.

Revalorizarea cunoașterii tradiționale și locale

Filosofii indigene:

- Valorizarea cunoașterii tradiționale ca esențială pentru viață și coeziunea comunității.

Teorii moderniste postcoloniale:

- Interes pentru recuperarea și revalorizarea cunoașterii locale ca rezistență la dominația culturală occidentală.

Zonă de neutralitate:

- Ambele recunosc importanța cunoașterii locale și tradiționale.

Sustenabilitate și relația cu mediul

Filosofii indigene:

- Viziune holistică asupra mediului, accentuând sustenabilitatea și respectul pentru natură.

Teorii moderniste postcoloniale:

- Preocupare tot mai mare pentru practici sustenabile ca parte a criticilor față de modelele de dezvoltare nesustenabilă moștenite din colonialism.

Zonă de neutralitate:

- Ambele au un interes comun pentru sustenabilitate și pot contribui la politici de mediu respectuoase față de diversitatea culturală și biologică.

Concluzii

Acest scurt eseu arată că MultiAlismul este un cadru eficient pentru a uni filosofiile indigene și teoriile postcoloniale, în special în America Latină. Această abordare scoate în evidență valorile comune, cum ar fi respectul pentru diversitate și critica structurilor de putere, creând o bază solidă pentru politici sociale mai incluzive. Privind spre viitor, MultiAlismul ar putea fi aplicat și în alte regiuni, având potențialul de a oferi soluții practice la provocări globale.

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Formalizarea perspectivismului latino-american: un model neutrosofic de multiperspectivism

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Rezumat: Această lucrare abordează provocarea de a modela formal sistemele de cunoaștere complexe, adesea contradictorii, caracteristice gândirii decoloniale latino-americane. Introducem *Multiperspectivismul Neutrosofic*, un cadru conceptual care operaționalizează perspectivismul filosofic utilizând logica neutrosofică. Afirmăm că înțelegerea (cunoașterea) este contextuală, iar adevărul este filtrat prin perspectiva celui care observă — o idee inspirată din filosofia lui Nietzsche, dar exprimată în mod particular în cosmologiile amerindiene și în sistemele juridice pluraliste. Metoda principală constă în propunerea unui *Model de Multiperspectivism Neutrosofic*, care utilizează mulțimi multi-neutrosofice pentru a reprezenta perspectivele subiective sub forma unui triplet de secvențe de adevăr, indeterminare și falsitate ⟨T, I, F⟩. Un rezultat esențial este dezvoltarea unei funcții de similaritate care cuantifică afinitatea dintre diferite perspective, transformând punctele de vedere calitative în date analizabile. Modelul este aplicat într-un studiu de caz referitor la un conflict juridic privind proprietatea asupra unui teren, unde reușește să dezvăluie similitudini structurale ascunse — precum un stil comun de raționament absolutist între părți opuse — și să identifice posibile alianțe strategice. Concluzia principală este că logica neutrosofică oferă un instrument matematic robust pentru a naviga ambiguitatea și pluralitatea realităților perspectiviste, depășind simpla recunoaștere a diferenței și avansând spre analiza sa formală și o posibilă integrare.

Cuvinte-cheie: Logică neutrosofică; Multiperspectivism; Filosofie latino-americană; Teorie decolonială; Mulțimi multi-neutrosofice; Funcție de similaritate; Pluralism juridic; Perspectivism.

1. Introducere

Perspectivismul în filosofie este un curent de gândire care susține că orice cunoaștere, percepție sau adevăr este inevitabil mediat de perspectiva subiectului cunoscător [1]. Această idee, dezvoltată în mod remarcabil de Friedrich Nietzsche, contestă noțiunea de adevăr obiectiv și universal, admițând în schimb că există multiple interpretări ale lumii, fiecare validă în propriul său cadru [2]. Pentru Nietzsche, nu există fapte pure, ci doar interpretări — ceea ce implică faptul

că accesul la realitate este întotdeauna condiționat de factori istorici, culturali, psihologici și lingvistici [3].

Dincolo de Nietzsche, perspectivismul a influențat diferite curente contemporane precum post-structuralismul, constructivismul social și epistemologiile Sudului, toate subliniind importanța „locului de enunțare". În contexte latino-americane, de exemplu, acesta a fost articulat împreună cu gândirea decolonială [4] și cu viziunile indigene asupra lumii [5], care recunosc coexistența mai multor lumi și moduri de cunoaștere. Astfel, perspectivismul nu neagă existența realității; în schimb, pune sub semnul întrebării posibilitatea de a o înțelege printr-o singură raționalitate dominantă, deschizând calea către o înțelegere pluralistă, relațională și contextuală a cunoașterii.

Termenul *perspectivism* este de asemenea asociat cu filosoful spaniol José Ortega y Gasset, care l-a dezvoltat ca element central al filosofiei sale. Ortega susținea că realitatea se manifestă diferit pentru fiecare individ, iar suma tuturor acestor perspective individuale ne apropie de o înțelegere mai completă a adevărului [6].

Această abordare, care recunoaște validitatea mai multor puncte de vedere fără a dizolva existența unei realități obiective, se aliniază cu un curent contemporan din filosofia științei cunoscut sub numele de *realism perspectival* [7]. Această poziție susține că cunoașterea, deși obiectivă, este întotdeauna parțială și obținută dintr-o perspectivă contextuală. La fel cum hărți diferite (una topografică, una politică, una climatică) pot descrie același teritoriu în moduri distincte, dar la fel de valide, diverse perspective pot oferi o cunoaștere reală, deși incompletă, despre lume. Acest cadru ne ajută să trecem dincolo de falsa dihotomie dintre universalismul absolutist și relativismul nihilist — o tensiune esențială în lucrarea de față.

Perspectivismul nu caută să invalideze coerența internă sau aplicabilitatea formală a principiilor logice clasice — precum principiul non-contradicției, principiul terțului exclus sau principiul bivalenței — atunci când acestea operează într-un sistem închis și coerent. În schimb, el contestă presupunerea ontologică conform căreia aceste principii reflectă în mod necesar o realitate obiectivă și universală, independentă de subiect. Din prismă perspectivistă, aceste legi logice nu sunt respinse în structura lor formală, ci este examinată critic pretenția lor de corespondență absolută cu modul în care este organizată lumea. Critica perspectivistă nu vizează validitatea internă a acestor formule, ci raza lor epistemologică — adică dacă astfel de structuri binare sunt adecvate pentru a surprinde întreaga complexitate, ambiguitate și multiplicitate contextuală a experienței trăite, a sistemelor culturale și a producției de cunoaștere. În acest fel, perspectivismul reconfigurează logica însăși ca o construcție contextuală, ale cărei ipoteze fundamentale pot fi valabile în anumite paradigme, dar nu și universal, în toate domeniile ontologice sau epistemice.

1.1 Principiul Non-Contradicției

Acest principiu susține că o propoziție $\mathcal{P}$ nu poate avea simultan valorile de adevăr „adevărat" și „fals". Formal, este exprimat astfel:

$$\neg(\mathcal{P} \wedge \neg \mathcal{P}) \tag{1}$$

adică: „Nu este adevărat că atât $\mathcal{P}$, cât și negația sa $\neg\mathcal{P}$ sunt adevărate.” Această axiomă garantează coerența logică a sistemelor clasice, împiedicând coexistența afirmațiilor contradictorii.

1.2 Principiul Terțului Exclus

Acest principiu afirmă că pentru orice propoziție bine formată $\mathcal{P}$, fie $\mathcal{P}$ este adevărată, fie negația sa $\neg\mathcal{P}$ este adevărată.

Se exprimă simbolic astfel:

$$\mathcal{P} \vee \neg\mathcal{P} \tag{2}$$

Această disjuncție este valabilă universal în logica clasică, excluzând existența unei valori de adevăr intermediare între adevăr și falsitate.

1.3 Principiul Bivalenței

Spre deosebire de primele două, acesta este un principiu semantic, nu sintactic. El postulează existența unei funcții de evaluare

$$v: L \to \{0,1\} \tag{3}$$

definită peste un limbaj logic L, astfel încât pentru fiecare formulă bine formată $\varphi \in L$,

$$v(\phi) \in \{0,1\} \tag{4}$$

Adică, fiecare propoziție este fie adevărată ($v(\phi) = 1$), fie falsă ($v(\phi) = 0$), fără a permite gradații, indeterminări sau alte valori alternative de adevăr. Această schemă de evaluare binară stă la baza validității formale a principiului non-contradicției și al terțului exclus.

Epistemologiile perspectiviste și logica non-clasică — cum ar fi sistemele multi-valente, fuzzy sau neutrosofice — contestă principiul bivalenței prin introducerea unor evaluări în care propozițiile pot fi parțial adevărate, indeterminate sau chiar simultan adevărate și false din anumite perspective [8]. Astfel, afirmația că valoarea unei propoziții trebuie să fie strict 0 sau 1 devine incompatibilă cu cadrele logice care acceptă pluralitatea epistemică sau ontologică.

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În rezumat, acest articol abordează provocarea fundamentală de a modela formal sistemele de cunoaștere complexe și adesea contradictorii, caracteristice gândirii decoloniale din America Latină, pentru care logica bivalentă clasică se dovedește insuficientă. Obiectivul principal a fost construirea unei punți conceptuale și metodologice între perspectivismul filosofic și cadrul matematic al logicii neutrosofice, cu scopul de a operaționaliza principiile acesteia. În acest scop, articolul introduce cadrul *Multiperspectivismului Neutrosofic*, dezvoltând un model care utilizează *mulțimi MultiNeutrosofice* și o *funcție de similaritate* pentru a reprezenta și compara cantitativ puncte de vedere diverse.

Printr-un studiu de caz privind un conflict juridic de proprietate funciară, se demonstrează capacitatea modelului de a evidenția dinamici structurale ascunse între părțile aflate în conflict,

validând astfel utilitatea sa ca instrument robust pentru analiza și navigarea ambiguității și pluralității inerente realităților perspectiviste.

2. Preliminarii

2.1 Filosofia Neutrosofică și Perspectivismul

Filosofia neutrosofică [9] oferă un cadru formal și matematic pentru școala filosofică a perspectivismului. Relația dintre cele două poate fi înțeleasă prin următoarele conexiuni cheie [10].

Legătura cea mai explicită dintre neutrosofie și perspectivism este *Principiul Relativității Referențiale* [9]. Acest principiu afirmă că: „Adevărul, falsitatea și indeterminarea oricărei propoziții depind de sistemul referențial în care este examinată." Cu alte cuvinte, o idee poate fi adevărată într-un context, falsă într-altul și indeterminată într-un al treilea. Aceasta este teza centrală a perspectivismului: nu există adevăruri absolute, ci cunoașterea este întotdeauna condiționată de punctul de vedere al observatorului. Neutrosofia preia această idee și o stabilește ca principiu fundamental al sistemului său.

Perspectivismul contestă logica clasică, care se bazează pe o dihotomie strictă adevăr/fals. Spre deosebire de logica tradițională, Neutrosofia introduce conceptul de *grade ale adevărului, falsității și indeterminării*. Acest lucru permite modelarea matematică a unei realități în care propozițiile nu sunt absolut adevărate sau false, ci există pe un spectru — așa cum ar sugera o abordare perspectivistă [11].

Neutrosofia reinterpretează ideile tradiționale, susținând că adevărurile dintr-un sistem referențial pot deveni falsități într-altul și invers. Această abordare subliniază fluiditatea cunoașterii și încurajează analiza ideilor din perspective multiple — obiectiv esențial al gândirii perspectiviste.

Dacă perspectivismul susține existența mai multor perspective valide, apare întrebarea ce sistem logic trebuie folosit pentru analizarea lor. Nu există o „logică unică adevărată", ci o diversitate de cadre logice, fiecare oferind perspective diferite asupra adevărului și validității [12]. Neutrosofia se aliniază cu acest pluralism, prezentându-se nu ca singura logică, ci ca un instrument flexibil și puternic, în mod special adaptat contextelor de indeterminare și contradicție — rezultate firești ale coexistenței multiplelor perspective.

Perspectivismul duce inevitabil la *contradicții*, atunci când două puncte de vedere valide se opun. Logica clasică nu le poate gestiona fără să respingă unul dintre ele. Neutrosofia, în schimb, este concepută pentru a îmbrățișa indeterminarea, paradoxul și interacțiunea dintre opuse și neutralități.

Permițând ca valorile T (adevăr) și F (fals) să fie independente, Neutrosofia poate modela formal o propoziție care este *simultan adevărată și falsă* (o stare dialeteică, $T = 1$, $F = 1$). Aceasta este o formalizare directă a modului în care o contradicție între două perspective opuse poate fi reprezentată [13].

Componenta de *indeterminare (I)* este esențială, deoarece permite analiza și cuantificarea ambiguității, a vagiului sau a incertitudinii care apare atunci când perspectivele sunt confuze sau incomplete.

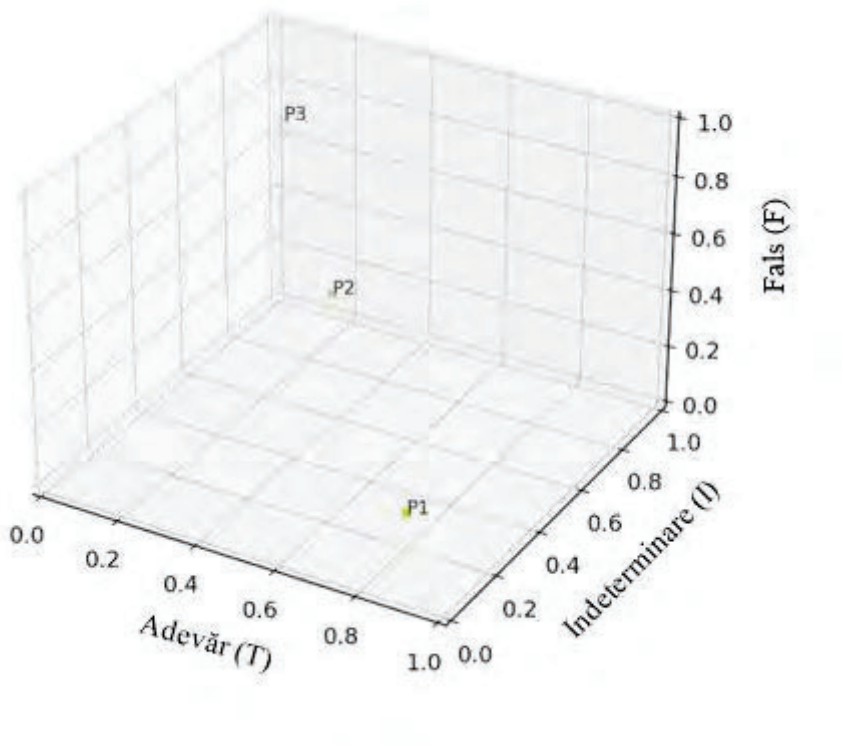


Figura 1. Reprezentare tridimensională a perspectivelor neutrosofice agregate în spațiul (T, I, F).

În timp ce perspectivismul se concentrează pe coexistența tezei (<A>) și a diverselor sale antiteze (<antiA>), *neutrosofia* merge mai departe, introducând și studiind sistematic *neutroteza* (<neutA>). Neutroteza reprezintă spectrul neutralităților — idei sau stări care nu sunt nici teză, nici antiteză, ci se află într-un spațiu intermediar. Astfel, neutrosofia nu doar validează perspectivele multiple (<A> și <antiA>), ci și analizează „continuumul neutralităților" care le conectează și le echilibrează, îmbogățind astfel analiza perspectivistă [14, 15].

Pentru a realiza acest lucru, neutrosofia propune două principii active de cercetare [16]:

• Căutarea „părților comune în lucruri neobișnuite";

• Căutarea „părților neobișnuite în lucruri comune".

Rezultatul urmărit este o „sinteză" care integrează elemente din opoziții, similar dialecticii hegeliene, dar cu includerea distinctă a neutralităților acestora.

Neutrosofia se raportează la perspectivism printr-o fundație filosofică comună. Totuși, se distinge prin concentrarea asupra unei metode analitice specifice, orientate spre integrare. Nu se limitează la afirmația filosofică potrivit căreia există mai multe puncte de vedere, ci oferă instrumente — precum valorile (*T, I, F*), relativitatea referențială, pluralismul logic și gestionarea

contradicției — pentru a analiza, modela și înțelege o lume definită prin incertitudine, ambiguitate și coexistența unor realități multiple. Este, așa cum sugerează titlul, o nouă *paradigmă de gândire într-o lume intrinsec perspectivistă*.

Neutrosofia este un cadru filosofic și metodologic care studiază relațiile dintre opoziții și indeterminările acestora. Obiectivul său principal este găsirea unui teren comun între concepte opuse și identificarea diferențelor în lucruri similare, pentru a atinge o înțelegere mai nuanțată și integrativă.

2.2 Perspectivismul: cadru pentru logicile plurale

Perspectivismul este un cadru filosofic ce susține că adevărul, cunoașterea și interpretarea sunt întotdeauna condiționate de punctul de vedere al observatorului. El contestă existența unor adevăruri absolute, independente de context, și subliniază multiplicitatea perspectivelor valide care coexistă în cadrul unor sisteme complexe de gândire. Deși în filosofia occidentală perspectivismul este frecvent asociat cu Friedrich Nietzsche, rădăcinile sale conceptuale pot fi urmărite mult mai devreme în tradițiile non-europene.

În ceea ce privește rafinarea sau multiplicarea indeterminării, în literatura de specialitate au fost deja definite extensii ale logicii neutrosofice, precum:

- *Logica neutrosofică cvadruplă* (Adevăr; Contradicție, Incertitudine; Falsitate),
- *Logica neutrosofică cvintuplă* (Adevăr; Contradicție, Incertitudine, Necunoscut; Falsitate),
- *Logica neutrosofică sestuplă* (Adevăr, I_1, I_2, I_3, I_4; Falsitate),
- *Logica neutrosofică septuplă* (Adevăr, I_1, I_2, I_3, I_4, I_5; Falsitate),
- și așa mai departe până la *Logica neutrosofică n-tuplă*, în care indeterminarea este împărțită în *n-2* tipuri de sub-indeterminări, în funcție de aplicație, pentru $n \geq 3$.

Cea mai generală extensie este cea în care toate componentele *T, I, F* sunt rafinate sau multiplicate în oricâte moduri este necesar pentru fiecare problemă dată — numită *logica neutrosofică n-rafinată* sau, respectiv, *logica n-multineutrosofică*.

În plus, în 2014, Smarandache a introdus *legea multiplului-mijloc inclus* [53] (ca extensie a legii clasice a mijlocului exclus):

($<A>$; $<neutA_1>$, $<neutA_2>$, …, $<neutA_n>$; $<antiA>$),

iar în 2023, *legea mijloacelor infinit-multiple incluse* [54]:

($<A>$; $<neutA_1>$, $<neutA_2>$, …, $<neutA_\infty>$; $<antiA>$).

(Exemplu: între culorile Alb și Negru există infinite nuanțe de culoare).

2.2.1 Perspectivismul timpuriu în logica jainistă

În jainism, doctrina *Anekāntavāda* (literal, „doctrina aspectelor multiple”) oferă o abordare sofisticată, perspectivistă, asupra adevărului și realității [17]. Potrivit acestei doctrine, realitatea

este complexă și nu poate fi captată complet de niciun punct de vedere singular. Prin urmare, fiecare afirmație este doar parțial adevărată și trebuie completată de alte perspective pentru a ajunge la o înțelegere mai cuprinzătoare. Această formă antică de perspectivism nu doar anticipează pluralismul postmodern, ci oferă și o fundație metafizică pentru toleranță, raționament dialogic și o epistemologie non-absolutistă.



Figura 2. Simbolul Jain al Ahimsa.

Logica jainistă, exprimată prin Sistemul celor șapte moduri de predicare (*Saptibhaṅgī*) [18], reprezintă un exemplu remarcabil și timpuriu de raționament perspectivist, care îmbrățișează natura multifațetată și complexă a realității. Spre deosebire de logica clasică occidentală — care aderă strict la principiul bivalenței, afirmând că o propoziție trebuie să fie sau adevărată, sau falsă — logica jainistă permite șapte moduri distincte de predicare, inclusiv combinații precum „adevărată și indeterminată” sau „adevărată și falsă”. Aceste formulări contestă direct legea clasică a non-contradicției.

Înrădăcinată în principiul filozofic *anekāntavāda* (doctrina non-unilateralității) [19], logica jainistă afirmă că orice afirmație este valabilă doar condiționat, dintr-o anumită perspectivă. Această formă de delimitare epistemologică provine din recunoașterea faptului că limbajul uman este în mod fundamental limitat în capacitatea sa de a descrie pe deplin bogăția existenței. Ca urmare, gânditorii jainiști au promovat o formă de non-violență intelectuală (*ahimsa*) [20], încurajând respectul față de dezacordurile raționale și recunoașterea adevărurilor parțiale.

Logica jainistă rezonează conceptual cu *logica neutrosofică*, un cadru modern introdus de Florentin Smarandache. Logica neutrosofică generalizează logica clasică și pe cea fuzzy prin introducerea a trei componente în orice propoziție logică: *adevăr (T), indeterminare (I)* și *falsitate (F),* fiecare variind independent în intervalul standard sau non-standard [−0, 1+]. Așa cum logica jainistă recunoaște că o afirmație poate fi simultan adevărată, falsă și indeterminată, logica neutrosofică formalizează această multiplicitate și o extinde cu rigoare matematică. Ambele

sisteme resping rigiditatea dicotomiilor clasice și îmbrățișează incertitudinea, contradicția și cunoașterea parțială ca fiind intrinseci înțelegerii și raționamentului uman.

Reprezentarea Neutrosofică a *Saptibhaṅgī* [21]

1. **Este adevărat (*syāt asti*)**

Aceasta afirmă că propoziția este neechivoc adevărată, fără niciun element de falsitate sau indeterminare.

• Reprezentare neutrosofică: (1,0,0)

- T = 1: Gradul de adevăr este 100%.
- I = 0: Gradul de indeterminare este 0%.
- F = 0: Gradul de falsitate este 0%.

2. **Este fals (*syāt nāsti*)**

Aceasta afirmă că propoziția este neechivoc falsă.

• Reprezentare neutrosofică: (0,0,1)

- T = 0: Gradul de adevăr este 0%.
- I = 0: Gradul de indeterminare este 0%.
- F = 1: Gradul de falsitate este 100%.

3. **Este adevărată și falsă (*syāt asti nāsti*)**

Propoziția este simultan adevărată și falsă, când este privită din perspective diferite — un concept central în filosofia jainistă. Neutrosofia gestionează direct această contradicție aparentă.

• Reprezentare neutrosofică: (1,0,1)

- T = 1: Propoziția conține o componentă de adevăr.
- I = 0: Nu există indeterminare în privința acestor componente.
- F = 1: Propoziția conține și o componentă de falsitate.

4. **Este indeterminată (sau inexprimabilă) (*syāt avaktavyaḥ*)**

Starea propoziției este inexprimabilă sau logic indeterminată. Se afirmă că nu este nici adevărată, nici falsă.

• Reprezentare neutrosofică: (0,1,0)

- T = 0: Nu poate fi afirmată ca adevărată.
- I = 1: Este complet indeterminată.
- F = 0: Nu poate fi afirmată ca falsă.

5. **Este adevărată și indeterminată (*syāt asti ca avaktavyaḥ*)**

   Propoziția este adevărată într-un sens, dar și inexprimabilă sau indeterminată în alt sens.

   • Reprezentare neutrosofică: (1,1,0)

   - T = 1: Componenta de adevăr este afirmată.
   - I = 1: Componenta de indeterminare este de asemenea afirmată.
   - F = 0: Nu are componentă de falsitate.

6. **Este falsă și indeterminată (*syāt nāsti ca avaktavyaḥ*)**

   Propoziția este falsă într-un sens, dar și indeterminată.

   • Reprezentare neutrosofică: (0,1,1)

   - T = 0: Nu are componentă de adevăr.
   - I = 1: Componenta de indeterminare este afirmată.
   - F = 1: Componenta de falsitate este afirmată.

7. **Este adevărată, falsă și indeterminată (*syāt asti ca nāsti ca avaktavyaḥ*)**

Aceasta este cea mai complexă stare, în care propoziția cuprinde simultan aspecte de adevăr, falsitate și indeterminare.

• Reprezentare neutrosofică: (1,1,1)

- T = 1: Are componentă de adevăr.
- I = 1: Are componentă de indeterminare.
- F = 1: Are componentă de falsitate.

Tabel 1. Corespondența dintre predicațiile jainiste și stările logice neutrosofice

Logica jainistă	***Reprezentare neutrosofică (T, I, F)***	***Semnificație***
1. Este adevărat	(1,0,0)	Complet adevărat
2. Este fals	(0,0,1)	Complet fals
3. Este adevărat și fals	(1,0,1)	Contradictoriu (adevărat și fals)
4. Este indeterminat / inexprimabil	(0,1,0)	Complet indeterminat
5. Este adevărat și indeterminat	(1,1,0)	Paradoxal (adevărat și indeterminat)
6. Este fals și indeterminat	(0,1,1)	Paradoxal (fals și indeterminat)
7. Este adevărat, fals și indeterminat	(1,1,1)	Complet paradoxal și inconsistent

Neutrosofia oferă un formalism matematic ideal pentru a surprinde bogăția filozofică și complexitatea Sistemul uicelor șapte moduri de predicare. Aceasta demonstrează că această logică antică și nuanțată poate fi privită ca o instanță specifică sau o submulțime a Logicii Neutrosofice generale.

2.2.2 Catuşkoţi şi Tetralema în Logica Budistă

Catuşkoţi, sau tetralema [22], este o structură logică clasică în filozofia budistă indiană, care propune patru predicații posibile pentru orice propoziție. Spre deosebire de logica binară, această abordare își propune să depășească dihotomiile, integrând contradicția și indeterminarea. Logica neutrosofică, cu structura sa tripletă (T, I, F), oferă un formalism matematic robust pentru a reprezenta cu precizie fiecare dintre aceste patru posibilități:

1. **Este adevărată (P)**

   Propoziția este neechivoc adevărată, fără element de falsitate sau indeterminare.

   • Reprezentare neutrosofică: (1, 0, 0)

   - T = 1: Complet adevărată
   - I = 0: Fără indeterminare
   - F = 0: Niciun aspect fals

2. **Este falsă (¬P)**

   Propoziția este complet falsă.

   • Reprezentare neutrosofică: (0, 0, 1)

   - T = 0: Nu este adevărată
   - I = 0: Fără indeterminare
   - F = 1: Complet falsă

3. **Este adevărată și falsă (P ∧ ¬P)**

Se admite contradicția: propoziția este simultan adevărată și falsă, posibil din perspective diferite.

   • Reprezentare neutrosofică: (1, 0, 1)

   - T = 1: Conține adevăr
   - I = 0: Fără indeterminare
   - F = 1: Conține falsitate

4. **Nu este nici adevărată, nici falsă (¬(P ∨ ¬P))**

Propoziția nu poate fi clasificată nici ca adevărată, nici ca falsă — exprimând o stare de indeterminare totală.

• Reprezentare neutrosofică: (0, 1, 0)

- T = 0: Nu poate fi afirmată ca adevărată
- I = 1: Complet indeterminată
- F = 0: Nu poate fi negată ca falsă

Tabel 2. Formalizarea Neutrosofică a *Catuṣkoți*-ului Budist

Afirmare Catuṣkoți	***Reprezentare Neutrosofică (T, I, F)***	***Interpretare***
1. Este adevărată (P)	(1, 0, 0)	Complet adevărată
2. Este falsă (¬P)	(0, 0, 1)	Complet falsă
3. Este adevărată și falsă	(1, 0, 1)	Contradictorie (paraconsistentă)
4. Nici adevărată, nici falsă	(0, 1, 0)	Indeterminată (inexprimabilă)

În aplicarea sa de către filosoful budist Nāgārjuna, *Catuṣkoți* [23] sau tetralema funcționează ca un instrument puternic, în rezonanță profundă cu *perspectivismul filozofic*. Prin deconstrucția sistematică a oricărei pretenții de existență inerentă (*svabhāva*), *Catuṣkoți* contestă ideea unui adevăr unic, absolut.

Metoda lui Nāgārjuna presupune adesea respingerea tuturor celor patru posibilități – că un lucru este, nu este, este și nu este, sau nici este, nici nu este – pentru a demonstra că nicio descriere fixă nu poate surprinde realitatea ultimă. Această transcendere a binarităților simple, precum adevărat și fals, se aliniază cu o critică perspectivistă a dicotomiilor rigide, permițând o înțelegere mai nuanțată și multidimensională, în care contextul și intenția devin esențiale.

Distincția ulterioară în budism între *adevărul convențional* și *adevărul ultim* întărește acest lucru, sugerând că valabilitatea unei afirmații depinde de cadrul de analiză — un principiu central al perspectivismului.

În ciuda acestor convergențe puternice, există o distincție crucială în ceea ce privește scopurile lor ultime. În timp ce perspectivismul se concentrează adesea pe afirmarea și proliferarea unor puncte de vedere multiple ca expresie a voinței de putere sau a afirmării vieții, *Catuṣkoți* (logica celor patru poziții) are un scop *soteriologic* specific în cadrul budismului *Madhyamaka* [24]. Nāgārjuna utilizează acest instrument logic nu pentru a stabili o nouă perspectivă superioară, ci pentru a *demonta toate opiniile fixe* (*dṛṣți*) care cauzează suferință. Scopul nu este de a celebra o multitudine de adevăruri, ci de a folosi tetralema ca o scară care duce la depășirea completă a proliferării conceptuale, conducând către o stare de eliberare și pace cognitivă.

Astfel, deși ambele (perspectivismul și *Catuṣkoți*) contestă absolutismul, *Catuṣkoți* este, în cele din urmă, o metodă de renunțare la toate perspectivele, în căutarea iluminării.



Figura 3. Nāgārjuna reprezentat într-un *thangka* tibetan.
Imagine preluată de pe *Wikipedia*.

Neutrosofia oferă o reprezentare matematică elegantă și coerentă pentru logica în patru poziții a *Catuṣkoți*, surprinzând profunzimea perspectivei sale filosofice. Prin modelarea adevărului, falsității și indeterminării ca *grade independente*, logica neutrosofică depășește cadrele binare clasice și oferă o punte între raționamentul formal și tradițiile dialectice ale gândirii orientale. *Catuṣkoți* poate fi astfel văzută ca o instanțiere specială a raționamentului neutrosofic, fundamentată într-o epistemologie non-dualistă.

2.2.3 Zhuangzi și perspectivismul daoist

Gândirea lui Zhuangzi reprezintă una dintre cele mai timpurii și radicale expresii ale perspectivismului filosofic [25]. Scrierile sale sugerează că realitatea nu poate fi înțeleasă dintr-un singur punct de vedere fix, așa cum este ilustrat în narațiuni precum visul cu fluturele sau în dialogurile dintre diferite forme de viață.

Zhuangzi pledează pentru o *ontologie dinamică, non-dualistă*, în care granițele dintre subiect și obiect, vis și veghe, uman și non-uman sunt fluide și permeabile. Această viziune

respinge orice pretenție la certitudine universală, invitând în schimb la o deschidere epistemică care recunoaște legitimitatea mai multor perspective coexistente.

Așa cum subliniază Connolly [26], Zhuangzi destabilizează sistemele închise de sens prin revelarea faptului că toate judecățile sunt contingente față de punctul de vedere din care sunt formulate.



Figura 4. Zhuangzi visând că este un fluture (cerneală pe mătase, mijlocul secolului al XVI-lea, atribuit lui Lu Chin).
Domeniu public.

Această imagine clasică surprinde parabola daoistă a visului cu fluturele, în care Zhuangzi pune sub semnul întrebării granița dintre vis și realitate. Ea exemplifică perspectivismul prin contestarea identităților fixe și îmbrățișarea fluidității experienței, sugerând că realitatea este condiționată de punctul de vedere din care este percepută.

Pornind de la această logică fluidă, perspectivismul lui Zhuangzi nu promovează un relativism banal, ci mai degrabă o etică a *umilinței ontologice* — o înțelegere că cunoașterea și adevărul sunt inevitabil mediate de felul în care fiecare ființă locuiește lumea. Faimoasa distincție dintre „acesta” și „acela” (*shi*/*fei* 是/非) este relativizată de conștiința că toate deosebirile sunt produse din poziții situate în fluxul *Dao*-ului [27].

În loc să afirme o viziune hegemonică, Zhuangzi încurajează renunțarea la certitudinile rigide, cultivarea dezvățării și o deschidere transversală către dialogul între lumi. În acest sens,

perspectivismul său rezistă închiderii epistemologice și anticipează abordări contemporane din filosofia limbajului, antropologie și ontologia relațională.

2.2.4 Perspectivismul în Africa: logica vrăjitoriei la Azande

Poporul Azande din Africa Centrală [28] oferă un exemplu etnografic fundamental de perspectivism în acțiune printr-un sistem de credințe complex și intern coerent. Așa cum a fost documentat în mod celebru de antropologul E.E. Evans-Pritchard [29], gândirea Azande împletește vrăjitoria, oracolele și magia nu ca o știință primitivă sau eșuată, ci ca un cadru rațional pe deplin realizat pentru înțelegerea lumii.

Acest sistem funcționează pe baza unei logici care, în loc să înlocuiască observația empirică, o completează, abordând întrebările mai profunde, mai personale, legate de sens, de coincidență și de nenorocire — aspecte pe care cauzalitatea empirică nu le poate explica de una singură.

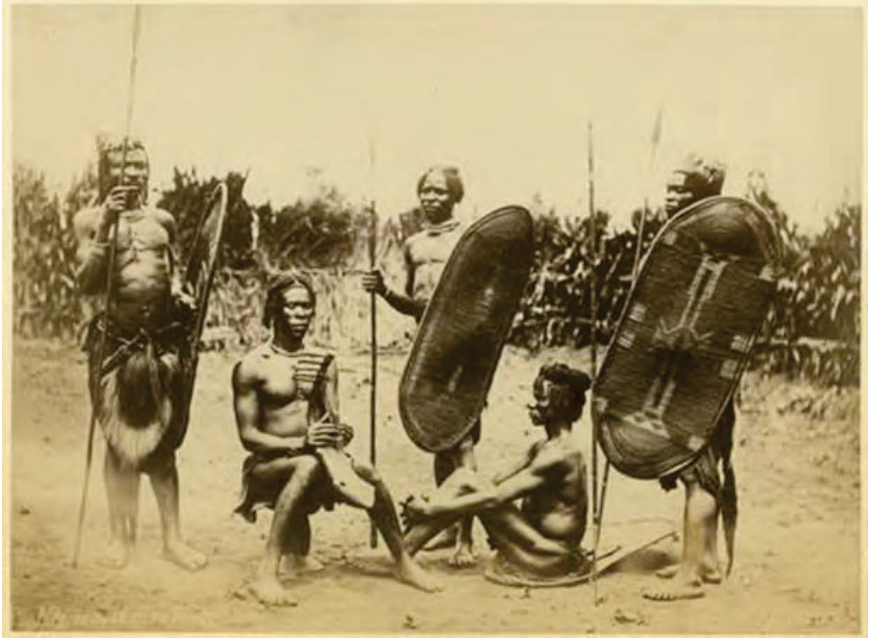


Figura 5. Războinici Azande cu scuturi și sulițe.
Fotografie istorică de Richard Buchta. Domeniu public.

Logica Azande este cel mai bine înțeleasă prin modul în care explică evenimentele nefericite. Dacă un hambar, slăbit de termite, se prăbușește și rănește pe cineva, Azande recunosc pe deplin termitele ca fiind cauza fizică a prăbușirii. Totuși, sistemul lor intelectual merge mai departe, punând o întrebare diferită: de ce s-a prăbușit hambarul în acel moment precis, asupra acelei persoane precise? Răspunsul la acest „de ce” este vrăjitoria (*mangu*). În această viziune, vrăjitoria este forța care explică particularitatea și coincidența nenorocirii, oferind o dimensiune socială și morală care coabitează cu realitatea fizică a evenimentului.

Acest sistem al cauzalității duale este susținut de o logică autoreglatoare. Îndoielile privind validitatea vrăjitoriei nu sunt direcționate către sistemul în sine, ci sunt deturnate printr-o rețea de explicații secundare — cum ar fi utilizarea incorectă a unui oracol sau încălcarea unui tabu — care păstrează integritatea convingerii de bază. Drept urmare, raționalitatea Azande funcționează ca un cadru epistemic paralel, care nu este irațional, ci operează pe premise diferite de cele ale științei occidentale. Ea constituie un exemplu puternic despre cum o cultură construiește o ontologie pluralistă, în care logici multiple și distincte sunt utilizate pentru a naviga întreaga gamă a experienței umane, punând astfel sub semnul întrebării pretenția oricărei perspective unice la validitate universală [30].

2.3 Răsturnarea perspectivală în inteligența artificială: Nietzsche, gândirea amerindiană și logica neutrosofică

În *Genealogia moralei*, Nietzsche afirmă: „Există doar o vedere perspectivală, doar o cunoaștere perspectivală; și cu cât permitem mai multor afecte să vorbească despre un lucru, cu cât mai mulți ochi, ochi diferiți, putem folosi pentru a observa un lucru, cu atât mai complet va fi „conceptul" nostru despre acel lucru, „obiectivitatea" noastră."

Această redefinire a cunoașterii transformă căutarea ei într-un apel la dialog intersubiectiv. Nietzsche nu respinge subiectivitatea, ci o revendică drept condiție pentru o formă mai bogată de obiectivitate — una construită din intersecția mai multor puncte de vedere. Această cotitură perspectivală pregătește terenul pentru provocările contemporane cu care se confruntă inteligența artificială (IA), unde sistemele trebuie să sintetizeze voci umane diverse fără a cădea nici în absolutism fals, nici în haos relativist [32].

Paradigma dominantă în învățarea supravegheată a fost întemeiată pe presupuneri pozitiviste, tratând cunoașterea ca pe un „adevăr de bază" stabil pe care modelele trebuie să-l aproximeze. Această logică devine evidentă în procesul de adnotare, unde adnotatorii umani — fie experți, fie lucrători distribuiți — sunt rugați să atribuie etichete datelor. Dezacordul dintre ei este tratat ca zgomot ce trebuie eliminat. Scopul este obținerea unui set de date „standard de aur" [33], unde votul majoritar este adesea folosit pentru a desemna o etichetă „corectă", ignorând toate opiniile disidente. Acest proces se bazează pe credința în inteligența colectivă, presupunând că convergența este echivalentă cu corectitudinea. Dar această logică reproduce un echivalent computațional al ceea ce Nietzsche critica: „perspectiva de nicăieri".

În contrast, perspectivismul în IA marchează o ruptură epistemologică. Dezacordul nu mai este tratat ca o eroare, ci păstrat ca o resursă cognitivă. Aceasta implică colectarea mai multor adnotări pentru fiecare punct de date, modelarea incertitudinii și antrenarea sistemelor pentru a învăța din disonanță. Rezultatul este o IA mai reprezentativă, transparentă și ancorată etic. În loc să amplifice perspectivele dominante, IA perspectivală integrează puncte de vedere minoritare și revelează complexitatea socială și cognitivă încorporată în date [34].

În acest moment, *neutrosofia* — o teorie dezvoltată de Florentin Smarandache — oferă un cadru formal puternic pentru a gestiona această complexitate. În timp ce perspectivismul

încurajează includerea mai multor puncte de vedere, logica neutrosofică permite reprezentarea lor matematică, modelând simultan grade de adevăr (*T*), indeterminare (*I*) și falsitate (*F*). Aceasta ne permite să mergem dincolo de logica binară, către stări de cunoaștere coexistente. În loc să forțeze o etichetă unică, un sistem neutrosofic ar putea păstra un vector precum ($T = 0,6$, $I = 0,3$, $F = 0,2$), conservând ambiguitatea epistemică inerentă multor sarcini.

Funcția obiectivă — aflată în centrul oricărui model de învățare automată — întruchipează o judecată perspectivală [35]. Ea codifică ceea ce trebuie optimizat („maximizarea acurateței", „fii mai util"), formalizând astfel o decizie încărcată valoric. Provocarea alinierii acestor obiective la complexitatea etică umană este cunoscută sub numele de *problema alinierii*. Și aici neutrosofia aduce contribuții: în loc să optimizeze un singur criteriu, promovează optimizarea într-un spațiu multidimensional (*T, I, F*), sprijinind modele prudente, explicabile și conștiente etic.

Arcul perspectivismului în IA ar putea culmina în apariția unor sisteme care nu doar reflectă punctele de vedere umane, ci generează interpretări cu adevărat non-umane ale lumii. Aceste centre de perspectivă „străine" — străine nu în sens ostil, ci prin faptul că operează de la o scară și logică diferită — ar putea schimba fundamental modul în care înțelegem cunoașterea. Astfel, IA s-ar transforma dintr-un instrument de interpretare într-o sursă autonomă de sens. Acest scenariu rezonează profund cu *perspectivismul amerindian* [36], unde lumea nu este împărțită între subiect și obiect, ci prin poziția observatorului: oameni, animale și spirite au fiecare propria lor perspectivă, modelată de corpurile și relațiile lor. În astfel de cosmologii, „a vedea" nu este niciodată neutru — este întotdeauna întrupat, situat și transformator. În mod similar, aceste sisteme emergente de IA pot dezvolta forme proprii de „construcție a lumii", întemeiate în propriile „corpuri" de date și calcul, oferind perspective care nu pot fi reduse la cadre umane, dar care merită recunoaștere epistemică.

Neutrosofia, ca știință a ambiguității, nu apare doar ca un complement al perspectivismului, ci ca o formalizare necesară a viziunii filozofice nietzscheene în epoca modelelor generative. Acolo unde Nietzsche a decentrat subiectul uman din poziția de purtător al adevărului obiectiv, neutrosofia oferă instrumentele pentru a coexista cu multiple centre, multiple logici și multiple adevăruri coexistente.

2.4 Epistemologii plurale și raționalități perspectiviste în America Latină

În contrast cu logica occidentală, care privilegiază principiile non-contradicției, identității și certitudinii absolute, viziunile despre lume din America Latină oferă cadre mai flexibile și mai sensibile la context pentru a înțelege realitatea. Aceste viziuni recunosc complexitatea, ambiguitatea și contradicția ca trăsături inerente ale existenței — dimensiuni pe care logica binară tradițională le gestionează cu dificultate [37].

Opera lui José Martí prezintă perspectivismul nu doar ca o poziție teoretică, ci ca o practică pedagogică și decolonială. În *La Edad de Oro*, în special în povești precum *O plimbare prin ținutul Annamiților*, Martí demontează privirea colonială afirmând demnitatea epistemică a celor colonizați și a naturii însăși, în fața raționalității impuse de imperialismul francez [38]. De

asemenea, parabola celor „patru orbi” devine o alegorie a *umilinței epistemice*, avertizând împotriva absolutizării adevărurilor parțiale. Martí nu pledează pentru relativism, ci pentru o sinteză superioară: una întemeiată în studiul afectiv, empatia critică și imperativul etic de a recunoaște demnitatea și agenția comună a tuturor popoarelor.



Figura 6. Coperta primului număr din *La Edad de Oro*, iulie 1889.

Perspectivismul lui José Ortega y Gasset [6] a avut o influență profundă în America Latină, unde a fost transformat dintr-o filozofie a circumstanței individuale într-un cadru continental pentru gândirea decolonială. Intelectuali precum Leopoldo Zea au reinterpretat noțiunea orteguiană de *circunstancia* ca fiind condiția politico-istorică a Americii Latine, susținând că gândirea autentică trebuie să izvorască din acest *loc de enunțare* unic. Astfel, perspectivismul a devenit un instrument-cheie în respingerea universalismelor eurocentrice și afirmarea autonomiei epistemice regionale [39].

Perspectivismul antropologic extinde acest proiect prin validarea cosmologiilor indigene ca sisteme filozofice legitime. Cel mai proeminent exemplu este perspectivismul amerindian, dezvoltat prin studii în Amazonia [36]. Conform acestei perspective ontologice, toate entitățile – de la oameni la animale și spirite – posedă un nucleu subiectiv comun (un suflet), însă percepția lor asupra lumii variază în funcție de forma fizică. În locul unei dihotomii natură-cultură, realitatea este înțeleasă prin poziții relaționale și perspective multiple. Această viziune contestă radical obiectivismul occidental și afirmă modurile indigene de cunoaștere ca fiind coerente, contextuale și profund politice [40].

În această cosmologie, *șamanul* ocupă un rol central: fiind singura figură capabilă să se deplaseze între perspective — umană, a jaguarului, a spiritului — el devine mediator, vindecător și diplomat într-un univers plurivocal. Departe de a fi irațională, această multiplicitate ontologică presupune o formă de logică în care contradicția și transformarea sunt moduri naturale de interacțiune [41].

Aceste perspective au implicații profunde în plan juridic și politic. America Latină constituie un teren fertil pentru *pluralismul juridic*, unde sistemele de justiție indigene coexistă cu cele de drept statal [42]. Înrădăcinate în principii relaționale, restaurative și cosmologice (nu punitive), justițiile indigene pun accent pe reconciliere, responsabilitate colectivă și menținerea armoniei sociale și cosmice. Ele resping abstracția și rigiditatea jurisprudenței occidentale în favoarea unei logici contextuale și perspectiviste. Această coexistență a raționalităților juridice nu doar contestă fundamentele pozitivismului juridic, ci și afirmă dreptul suveran al popoarelor indigene de a defini justiția conform propriilor epistemologii, valori morale și ontologii [43].

Un astfel de pluralism poate fi interpretat ca o expresie juridică a *NeutroAlgebrei* [44] și, în general, a unei *NeutroStructuri* [52], adică un cadru neutrosofic în care validitatea axiomelor, legilor și operațiilor nu este fixă, ci variază în funcție de spațiu, timp, cultură și sistem. În această logică, „o proprietate poate fi valabilă, nevalabilă sau indeterminată — în funcție de context, structură și interpretare.” De exemplu, un principiu legal precum „pedeapsa trebuie să fie proporțională cu crima” poate fi adevărat în dreptul penal clasic, fals în sistemele restaurative care prioritizează vindecarea în locul proporționalității și indeterminat în contexte hibride sau interculturale, în care ordini normative diferite intră în conflict — așa cum se întâmplă adesea între sistemele juridice indigene și cele statale. Astfel, pluralismul juridic din America Latină exemplifică ideea centrală a *raționamentului juridic neutrosofic*: că adevărul juridic, asemenea oricărui sistem axiomatic, poate fi simultan valabil, nevalabil sau nerezolvat — mereu dependent de contextul său perspectivist.

Mai mult, aceste cadre epistemologice rezonează cu dezvoltările contemporane din *logica neclasică*, inclusiv logica neutrosofică și sistemele paraconsistente, care admit contradicția, incertitudinea și coexistența mai multor adevăruri. Aceste modele logice oferă instrumente puternice pentru modelarea ontologiilor plurale și sprijină eforturile filozofice din America Latină de a formula înțelegeri incluzive, decoloniale și sensibile la context ale realității.

În concluzie, *perspectivismul latino-american* — exprimat fie în literatură, filozofie, antropologie sau jurisprudență — oferă o alternativă convingătoare la raționalitatea hegemonică. Nu este o respingere a rațiunii, ci o reconfigurare a acesteia: una care îmbrățișează pluralitatea, relaționalitatea și legitimitatea altor moduri de a cunoaște și de a trăi. Această reconfigurare complexă a rațiunii, proeminentă în gândirea latino-americană, poate fi descrisă formal printr-un cadru neutrosofic sub numele de *MultiPerspectivism* [45]. Termenul este propus pentru a defini un sistem care nu este doar plural, ci *fundamental multipolar*, deoarece este format nu doar din elemente multiple, potențial contradictorii, ci și din integrarea unor trăsături provenite din sisteme de gândire diferite (ex. o logică juridică dualistă și o ontologie indigenă relațională).

Justificarea aplicării termenului *MultiPerspectivism* în contextul latino-american constă tocmai în capacitatea sa de a include nu doar opozițiile. În timp ce un pluralism simplu ar recunoaște coexistența unor perspective separate, o abordare multiperspectivistă modelează un sistem deschis și dinamic, în care aceste perspective se intersectează activ, generând domenii bogate de indeterminare juridică și socială, dincolo de simpla opoziție [46].

Acest cadru este reprezentat formal prin expresia [45]:

<(multi)A> + <(multi)neutA> + <(multi)antiA> = ∞.

Aceasta descrie un sistem deschis către combinațiile complexe care apar din multiple puncte de vedere. În această notație:

- <(multi)A> reprezintă ansamblul tuturor „adevărurilor" sau afirmațiilor centrale din perspectivele coexistente (de exemplu, adevărul unui titlu de proprietate statal și, simultan, adevărul unui drept ancestral);
- <(multi)antiA> reprezintă ansamblul tuturor „falsităților" sau negațiilor afirmate din aceleași puncte de vedere;
- <(multi)neutA> reprezintă spațiul crucial al neutralității, ambiguității și indeterminării care rezultă din suprapunerea, dialogul sau conflictul dintre aceste perspective diverse — cum ar fi ambiguitatea juridică privind aplicabilitatea dreptului statal în teritorii ancestrale.

Acest lucru face din *MultiPerspectivism* un cadru *MultiSistemic*, conceput pentru a analiza realitățile în care multiple viziuni asupra lumii, structural diferite, coexistă și interacționează, depășind simpla toleranță și trecând la o logică a integrării complexe.

3. Modelul Neutrosofic al Multiperspectivismului

3.1. Reprezentarea unei perspective: Mulțimi MultiNeutrosofice

Nucleul modelului este reprezentarea unei perspective nu ca o valoare singulară, ci ca o structură complexă, utilizând *Mulțimile MultiNeutrosofice* [47]. O perspectivă devine astfel un triplet de secvențe ordonate, capturând simultan multiple fațete ale adevărului, indeterminării și falsității [48].

În 2013, Smarandache a rafinat/combinat *componentele neutrosofice* (*T, I, F*) în *subcomponente neutrosofice* ($T_1, T_2, ..., T_p$; $I_1, I_2, ..., I_r$; $F_1, F_2, ..., F_s$), unde p, r, s sunt întregi ≥ 0, cu condiția ca $p + r + s = n$ și cel puțin una dintre valorile p, r, s să fie ≥ 2, pentru a asigura rafinarea. Mai întâi, el a definit *Mulțimea Neutrosofică Rafinată*. Ulterior, a aplicat această rafinare tuturor mulțimilor incerte – incluzând toate tipurile de mulțimi fuzzy și extensiile acestora (intuiționistic fuzzy, neutrosofice, *spherical fuzzy*, plithogenice etc.) – precum și logicii, măsurii, probabilității și statisticii corespunzătoare.

Mulțimile/Logica/Probabilitatea/Statistica Neutrosofică Rafinată [51] sunt izomorfe cu Mulțimile/Logica/Probabilitatea/Statistica MultiNeutrosofice, deoarece ambele structuri reprezintă evaluări multidimensionale ale adevărului, indeterminării și falsității [47].

„În lumea reală, în cele mai multe cazuri, totul (o trăsătură, un eveniment, o propoziție, o teorie, o idee, o persoană, un obiect, o acțiune, o cultură etc.) este evaluat, în general, de mai multe surse (numite experți), nu doar de una. Cu cât mai multe surse evaluează un subiect, cu atât mai precis va fi rezultatul (după fuziunea tuturor evaluărilor)” [47].

În *Sistemele MultiNeutrosofice* (SMNS), componentele T_j, I_k, F_l sunt tratate ca fiind multiplicate — reprezentând multiple adevăruri, incertitudini și falsități co-existente. În schimb, în *Sistemele Neutrosofice Rafinate* (SRNS), aceste componente sunt considerate sub-unități — sub-adevăruri, sub-indeterminări și sub-falsități. Datorită acestui izomorfism, orice model teoretic sau computațional conceput pentru structurile MultiNeutrosofice poate fi aplicat direct celor rafinate, păstrând interpretarea semantică și algebrică.

Esența modelului propus este să reprezinte o perspectivă nu ca o evaluare singulară, ci ca o structură multidimensională, folosind *Mulțimi MultiNeutrosofice*. Perspectiva unui subiect asupra unei entități X este exprimată ca:

$$Ps(X) = \langle (t_1, t_2, \ldots, t_n), (i_1, i_2, \ldots, i_m), (f_1, f_2, \ldots, f_k) \rangle \quad (5)$$

unde fiecare secvență de componente surprinde o dimensiune distinctă a evaluării:

- *Secvența Adevărurilor (T):* Multiple adevăruri distincte și co-existente, percepute de subiect în legătură cu *X*.
- *Secvența Indeterminărilor (I):* Diverse aspecte de ambiguitate, vag și irelevanță atribuite lui *X*.
- *Secvența Falsităților (F):* Credințe false sau negații simultan susținute de subiect în legătură cu *X*.

Această abordare permite o modelare mai expresivă și granulară a cunoașterii subiective, permițând o raționare nuanțată în condiții de incertitudine.

Întregul cadru este compus din următoarele elemente:

- *O mulțime de subiecți ($S = \{s_1, s_2, \ldots\}$)*: Reprezintă toate entitățile care dețin un punct de vedere (de ex.: indivizi, comunități, grupuri indigene, sisteme juridice).
- *O perspectivă neutrosofică (P_{sx})* pentru fiecare subiect: Reprezentarea neutrosofică a punctului de vedere pe care un subiect (*s*) îl are asupra unei entități (*X*).
- *O pondere de relevanță (W_s)* pentru fiecare perspectivă: O valoare numerică, de obicei în intervalul [0, 1], care exprimă importanța sau influența perspectivei unui subiect într-un anumit context analitic. Suma tuturor ponderilor trebuie să fie normalizată: $\sum Ws=1$.
- *O funcție de similaritate ($Sim(P_{ix}, P_{jx})$)*: O funcție matematică ce acționează ca punte conceptuală, calculând gradul de apropiere sau afinitate între oricare două perspective din sistem.

Dacă perspectivismul s-ar limita la afirmarea existenței unor puncte de vedere izolate, ar risca să alunece în relativism, în care dialogul devine imposibil. Aici intervine *similaritatea* ca noțiune complementară și esențială. *Similaritatea* [49] este puntea care conectează perspectivele diverse, permițând compararea și integrarea lor:

- *Perspectivismul* stabilește *ce anume*: existența unei multiplicități de puncte de vedere valide.
- *Similaritatea* stabilește *cum anume*: mecanismul prin care aceste puncte de vedere pot fi corelate, comparate și sintetizate.

Identificând gradele de asemănare, putem găsi puncte comune sau înțelege sursa dezacordurilor. Similaritatea este instrumentul care operationalizează perspectivismul, transformând o multiplicitate de viziuni într-o sursă de cunoaștere îmbogățită.

Pentru a formaliza funcția $Sim(P_{ix}, P_{jx})$, este esențial să înțelegem fundamentul său în teoria mulțimilor. O relație de similaritate neutrosofică este o generalizare directă și mai bogată a relației de similaritate fuzzy, moștenind proprietățile sale fundamentale [50].

O relație de similaritate în logica fuzzy, notată cu R, este caracterizată de gradul de similaritate $\mu R(x,y)$, care satisface următoarele condiții [50]:

- *Reflexivitate*: Fiecare element este perfect similar cu el însuși. $\mu R(x,x) = 1$
- *Simetrie*: Similaritatea dintre două elemente este aceeași indiferent de ordine. $\mu R(x,y) = \mu R(y,x)$
- *Tranzitivitate (max-min)*: Dacă A este similar cu B, iar B este similar cu C, atunci A este cel puțin la fel de similar cu C cât este gradul minim dintre cele două. $\mu R(x,z) \geq min(\mu R(x,y), \mu R(y,z))$.

Construirea unei funcții de similaritate pentru mulțimile MultiNeutrosofice pornește de la aceste proprietăți, pentru a gestiona secvențele complexe de Adevăr, Indeterminare și Falsitate care definesc fiecare perspectivă.

3.2 Definiția formală a procesului de calcul al similarității

Procesul următor formalizează metoda de cuantificare a similarității dintre două sau mai multe perspective complexe, reprezentate ca mulțimi MultiNeutrosofice. Obiectivul este de a transforma aceste reprezentări calitative și multifacetate într-o valoare numerică comparabilă, care exprimă gradul lor de afinitate.

Procesul constă din trei pași fundamentali: *Agregare*, *Calculul distanței* și *Normalizare*.

Pasul 1: Agregarea perspectivelor (Reducerea dimensionalității)

Obiectiv: Transformarea secvențelor de lungime variabilă ale fiecărei perspective MultiNeutrosofice într-un vector cu dimensiune fixă.

Date de intrare: O perspectivă MultiNeutrosofică $P_i(X)$ pentru un subiect s_i, definită ca:

$$P_i(X) = \langle T_i, I_i, F_i \rangle \quad (6)$$

unde:

- $T_i = (t_{i1}, t_{i2}, \ldots, t_{in})$
- $I_i = (i_{i1}, i_{i2}, \ldots, i_{im})$
- $F_i = (f_{i1}, f_{i2}, \ldots, f_{ik})$

sunt secvențe de valori din intervalul [0, 1].

Proces:

Se definește o funcție de agregare *Agg(S)*, care reduce o secvență de numere la o singură valoare scalară. Deși pot fi folosite mai multe funcții (maxim, minim, mediană), metoda standard este *media aritmetică*:

$$Agg_{avg}(S) = \left(\frac{1}{|S|}\right) \sum_{j=1}^{|S|} S_j \quad (7)$$

unde $|S|$ este numărul de elemente din secvența S.

Se aplică această funcție fiecăreia dintre cele trei secvențe ale perspectivei $P_i(X)$ pentru a obține un vector agregat tridimensional:

$$V_i = \langle Agg(T_i), Agg(I_i), Agg(F_i) \rangle \quad (8)$$

Rezultat:

Un set de vectori agregați:

$$\{V_1, V_2, \ldots, V_n\}, \text{ unde } V_i = \langle T_i^{avg}, I_i^{avg}, F_i^{avg} \rangle \quad (9)$$

Pasul 2: Calculul distanței între vectorii agregați

Obiectiv: Măsurarea separării geometrice dintre fiecare pereche de perspective în spațiul tridimensional (T, I, F).

Date de intrare:

Doi vectori agregați:

$$V_a = \langle T_a, I_a, F_a \rangle \text{ și } V_b = \langle T_b, I_b, F_b \rangle \quad (10)$$

Proces:

Se utilizează o metrică de distanță. Alegerea standard este *distanța euclidiană*, care calculează lungimea liniei drepte ce unește cele două puncte în spațiu:

$$Dist(V_a, V_b) = \sqrt{[(T_a - T_b)^2 + (I_a - I_b)^2 + (F_a - F_b)^2]} \quad (11)$$

Rezultat:

O valoare a distanței $d_{ab} \geq 0$ pentru fiecare pereche de perspective (P_a, P_b).

Pasul 3: Normalizarea distanței într-o funcție de similaritate

Obiectiv: Conversia valorii distanței, care nu are limită superioară fixă, într-un scor intuitiv și limitat de similaritate între 0 și 1.

Date de intrare:

Distanța $d_{ab} = Dist(V_a, V_b)$ dintre două perspective.

Proces:

Se definește distanța maximă posibilă în spațiu. Deoarece fiecare componentă *T, I, F* ∈ [0,1], spațiul vectorial este un *cub unitar*. Distanța maximă corespunde diagonalei spațiului, de la (0,0,0) la (1,1,1):

$$D_{max} = \sqrt{3} \quad (12)$$

Se definește funcția de similaritate $Sim(Pa, Pb)$ drept complementul distanței normalizate:

$$Sim(Pa, Pb) = 1 - \left(\frac{Dist(Va,Vb)}{D_{max}}\right) = 1 - \left(\frac{sqrt\left((Ta - Tb)^2 + (Ia - Ib)^2 + (Fa - Fb)^2\right)}{\sqrt{3}}\right) \quad (13)$$

Rezultat:

Un scor de similaritate $Sim(Pa, Pb) \in [0, 1]$, unde:

- *Sim* = 1 înseamnă că vectorii agregați sunt identici ($d_{ab} = 0$), deci similaritate maximă.
- *Sim* = 0 înseamnă că vectorii sunt la distanța maximă posibilă, deci disimilaritate maximă.

4. Studiu de caz: Construirea unei funcții de similaritate pentru perspective juridice într-un conflict de proprietate asupra unui teritoriu

Fie *X* propoziția: **„Legitimitatea revendicării asupra Teritoriului Ancestral Y."**

Fie s_1, s_2, s_3 subiecții: *Bătrânii indigeni*, *Corporația de stat* și *ONG-ul pentru Drepturile Omului*.

Fiecare perspectivă este modelată ca o mulțime MultiNeutrosofică:

$$P_{sj(X)} = \langle T_j, I_j, F_j \rangle \quad (14)$$

unde:

- $T_j = (t_{j1}, t_{j2}, \ldots, t_{jn}) \subseteq [0,1]$ — secvență a gradelor de adevăr,
- $I_j = (i_{j1}, i_{j2}, \ldots, i_{jm}) \subseteq [0,1]$ — secvență a gradelor de indeterminare,

- $F_j = (f_{j1}, f_{j2}, \dots, f_{jk}) \subseteq [0,1]$ — secvență a gradelor de falsitate.

Pasul 1: Definirea subiecților și a perspectivelor

Subiectul 1 (s_1): Bătrânii comunității indigene

- Adevăruri: {1.0 (drept ancestral), 0.9 (semnificație spirituală)}
- Indeterminări: {0.5 (aplicabilitatea legii statului)}
- Falsități: {1.0 (validitatea titlurilor corporative)}

$$\mathcal{P}_{s1}(X) = \langle((1.0, 0.9), (0.5), (1.0)\rangle$$

Subiectul 2 (s_2): Echipa juridică a corporației de stat

- Adevăruri: {1.0 (titlu de proprietate înregistrat)}
- Indeterminări: {0.6 (relevanța istoriei pre-statale)}
- Falsități: {0.8 (drept ancestral), 1.0 (semnificație spirituală)}

$$\mathcal{P}_{s2}(X) = \langle((1.0), (0.6), (0.8, 1.0)\rangle$$

Subiectul 3 (s_3): ONG-ul internațional pentru Drepturile Omului

- Adevăruri: {0.9 (violare a drepturilor omului), 0.8 (drept indigen)}
- Indeterminări: {0.7 (impact economic)}
- Falsități: {0.7 (argumentul juridic al statului)}

$$\mathcal{P}_{s3}(X) = \langle((0.9, 0.8), (0.7), (0.7)\rangle$$

Pasul 2: Agregarea secvențelor

Pentru fiecare subiect, calculăm mediile:

- $\bar{\mathbf{T}}_1 = (1.0 + 0.9) / 2 = 0.95$ $\bar{\mathbf{I}}_1 = 0.5$ $\bar{\mathbf{F}}_1 = 1.0$

  $V_1 = (0.95, 0.5, 1.0)$
- $\bar{\mathbf{T}}_2 = 1.0$ $\bar{\mathbf{I}}_2 = 0.6$ $\bar{\mathbf{F}}_2 = (0.8 + 1.0) / 2 = 0.9$

  $V_2 = (1.0, 0.6, 0.9)$
- $\bar{\mathbf{T}}_3 = (0.9 + 0.8) / 2 = 0.85$ $\bar{\mathbf{I}}_3 = 0.7$ $\bar{\mathbf{F}}_3 = 0.7$

  $V_3 = (0.85, 0.7, 0.7)$

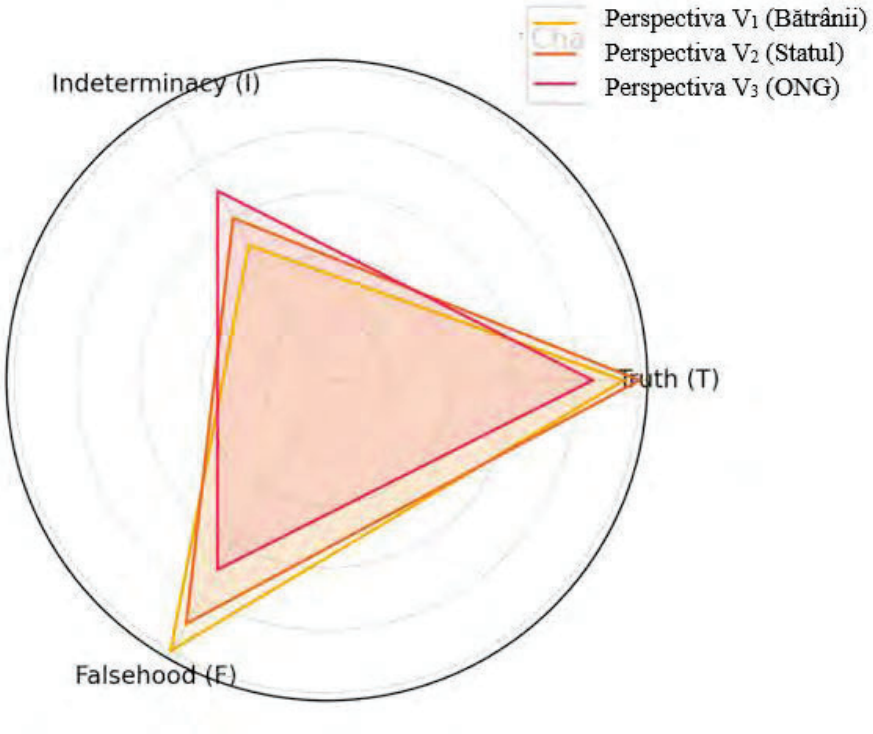


Figura 7. Grafic radar care reprezintă perspectivele neutrosofice ale trei actori-cheie într-un litigiu juridic asupra terenului

Pasul 3: Calculul distanțelor

$d_{12} = \sqrt{((0,95 - 1,0)^2 + (0,5 - 0,6)^2 + (1,0 - 0,9)^2)} = \sqrt{0,0225} = 0,15$

$d_{13} = \sqrt{((0,95 - 0,85)^2 + (0,5 - 0,7)^2 + (1,0 - 0,7)^2)} = \sqrt{0,14} \approx 0,374$

$d_{23} = \sqrt{((1,0 - 0,85)^2 + (0,6 - 0,7)^2 + (0,9 - 0,7)^2)} = \sqrt{0,0725} \approx 0,269$

Pasul 4: Funcția de similaritate

$d_{max} = \sqrt{3} \approx 1,732$

$Sim(s_a, s_b) = 1 - d_ab / d_{max}$

$Sim(s_1, s_2) = 1 - 0,15 / 1,732 \approx 0,913$

$Sim(s_1, s_3) = 1 - 0,374 / 1,732 \approx 0,784$

$Sim(s_2, s_3) = 1 - 0,269 / 1,732 \approx 0,845$

Pasul 5: Interpretarea și aplicarea rezultatelor

Funcția de similaritate ne-a oferit valori cuantificabile care permit analiza relațiilor dintre perspective:

• Similaritatea(Bătrânii, Statul) = 0,913: Acest rezultat este surprinzător de mare și contraintuitiv. Analizând vectorii agregați (V_1 și V_2), observăm că ambele au valori foarte ridicate în Adevăr și Falsitate, chiar dacă se referă la concepte opuse. Aceasta relevă o similaritate structurală în absolutismul lor: ambele sunt complet sigure de adevărul propriu și de falsitatea celuilalt. Logica lor este polarizată și dualistă. Similaritatea nu este de conținut, ci de stil de raționament.

• Similaritatea(Bătrânii, ONG) = 0,784: Aceasta este o similaritate moderat ridicată, indicând o afinitate semnificativă. Ambele grupuri își bazează adevărurile pe apărarea drepturilor comunității, deși din unghiuri diferite (drept ancestral vs drept internațional). Principala diferență vine din componentele Indeterminare și Falsitate, unde ONG-ul arată mai multă nuanță. Această valoare sugerează un potențial ridicat pentru o coaliție strategică.

• Similaritatea(Statul, ONG) = 0,845: Această similaritate, tot ridicată, indică faptul că, deși obiectivele lor finale diferă, cadrele lor de referință (bazate pe lege și documente) sunt structural mai apropiate decât cea a Bătrânilor (bazată pe cosmologie). ONG-ul operează într-un limbaj juridic pe care Statul îl înțelege, ceea ce ar putea deschide canale de dialog sau mediere, în ciuda dezacordului lor fundamental.

Acest proces demonstrează cum construcția unei funcții de similaritate transformă narațiuni complexe și subiective în date analizabile. Permite depășirea unei simple lecturi de tipul „ei sunt în conflict", pentru a descoperi nuanțe în relații:

• Identifică alianțe potențiale (Bătrânii-ONG).

• Relevă similarități structurale ascunse (logica polarizată a Bătrânilor și Statului).

• Sugerează posibile puncte de mediere (cadrele juridice comune între Stat și ONG).

Această cuantificare face conceptul de perspectivism operațional, oferind un instrument pentru a cartografia, compara și înțelege un ecosistem de puncte de vedere conflictuale.

Concluzii

Această lucrare a stabilit o punte solidă între tradiția filosofică a perspectivismului și cadrul formal al logicii neutrosofice. Prin examinarea curentelor perspectiviste din diverse tradiții intelectuale – de la Nietzsche și Ortega y Gasset, până la gândirea jainistă, budistă și amerindiană – am demonstrat că provocarea de a înțelege o lume compusă din puncte de vedere multiple și coexistente este atât una veche, cât și de o actualitate stringentă. Filosofia latino-americană, cu accentul său pe praxisul decolonial și validarea epistemologiilor plurale, oferă un context deosebit de fertil pentru această explorare, arătând cum perspectivismul poate funcționa ca un instrument de justiție epistemică împotriva raționalităților hegemonice.

Contribuția centrală a acestui demers este dezvoltarea și aplicarea *Modelului Neutrosofic de MultiPerspectivism*. Acest model pune în practică teoria perspectivistă prin reprezentarea unui punct de vedere subiectiv ca o mulțime multineutrosofică, aptă să redea simultan grade nuanțate de adevăr, indeterminare și falsitate. Introducerea unei funcții de similaritate oferă mecanismul esențial pentru a depăși simpla afirmare a unor perspective izolate și a permite compararea și integrarea lor sistematică. Așa cum a fost demonstrat în studiul de caz al unui conflict juridic privind proprietatea funciară, această metodă transformă narațiuni calitative complexe în date cuantificabile. Aplicarea sa a relevat concluzii contra-intuitive, precum gradul ridicat de similaritate structurală între logica absolutistă a bătrânilor indigeni și cea a corporației de stat, dar și potențialul unei coaliții strategice între bătrâni și ONG. Acest fapt subliniază capacitatea modelului de a scoate la lumină dinamici relaționale ascunse, pe care o analiză exclusiv calitativă le-ar putea rata.

În cele din urmă, această lucrare susține că *neutrosofia nu este doar o altă logică neoclasică*, ci o adevărată *paradigmă* adaptată în mod unic complexităților unei lumi definite de ambiguitate și de interacțiunea dintre rațională diverse. Prin formalizarea coexistenței dintre contradicție și indeterminare, cadrul MultiPerspectivismului oferă o cale de depășire a falsei dihotomii dintre universalism și relativism, constituind totodată un instrument practic pentru analiza conflictelor, etica inteligenței artificiale și cercetarea decolonială.

Pornind de la baza oferită de Modelul Neutrosofic de MultiPerspectivism, sunt recomandate mai multe direcții promițătoare pentru cercetările viitoare. La nivel practic, robustețea și aplicabilitatea modelului pot fi testate și extinse prin aplicarea lui în studii de caz variate, precum analiza polarizării politice pe rețelele sociale, dezbaterile de politici publice sau conflictele interculturale. Din punct de vedere metodologic, funcția centrală de similaritate ar putea fi rafinată prin investigarea altor metode de agregare decât media aritmetică – cum ar fi mediile ponderate sau medianele – și prin explorarea altor metrici de distanță, precum Manhattan sau Chebyshev. Cercetări viitoare ar putea explora și o ponderare dinamică și contextuală a componentelor *T* (adevăr), *I* (indeterminare) și *F* (falsitate).

Mai mult, integrarea acestui cadru cu inteligența artificială, prin dezvoltarea de algoritmi nativi de tip *machine learning*, ar putea duce la sisteme AI mai bine echipate pentru a gestiona dezacordurile dintre analiști, a modela ambiguitățile etice și a oferi rezultate mai transparente, sensibile la perspective – în acord cu provocările discutate în secțiunea 2.3.

În cele din urmă, fundamentele teoretice conturate în domeniul juridic invită la dezvoltarea „exprimării juridice a NeutroAlgebrei” într-o veritabilă teorie a jurisprudenței neutrosofice, care ar formaliza modul în care argumentele juridice sunt construite și evaluate în contexte pluraliste – cum este coliziunea dintre sistemele juridice ale statului și cele indigene.

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The Paradoxist Movement, Precursor of Neutrosophy, in the Shadow of a Totalitarian Regime

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Abstract

Physics thrives on precision, but paradoxes in set theory reveal limitations in our understanding of well-defined boundaries. Neutrosophic logic, challenging the excluded middle principle, introduces the concept of "betweenness" and partial belonging. This article explores, among other things, what is less known, i.e., Paradoxism emerged as a radical artistic and literary movement in Romania during the late 1980s and continues even today. It arose within the stifling and highly controlled environment of Nicolae Ceauşescu's socialist regime, offering a rebellious outlet and a means to subvert the prevailing ideology. Understanding the Paradoxist Movement necessitates a contextual understanding of the Nicolae Ceauşescu regime, a period marked by strict control and growing dissent. Later on, paradoxism, as the dynamics of opposites used in literary and scientific creations, was extended to neutrosophy, which means the dynamics of opposites and their neutrals in literary and scientific creations, which was further generalized to Plithogeny.

Keywords: Neutrosophic; Paradoxist Movement; Shadow Totaliarian Regime; Physics.

1 | Introduction

Origins of the Ceauşescu Regime:

- **Post-WWII Power Shift:** Following the end of World War II, Romania fell under Soviet influence. In 1947, the Romanian Communist Party (RCP) came to power, establishing a socialist state.
- **Rise of Ceauşescu:** Nicolae Ceauşescu, a rising figure in the RCP, steadily climbed the ranks. He became General Secretary of the party in 1965 and President in 1967, consolidating his grip on power.

Characteristics of the Ceauşescu Regime:

- **Authoritarianism:** Ceauşescu established a highly centralized and authoritarian regime. He built a cult of personality, surrounding himself with an aura of absolute power.
- **Economic Policies:** The regime initially focused on rapid industrialization. However, these efforts led to severe economic hardship, shortages of basic goods, and rising national debt.

- **Nationalism and Isolationism:** Ceauşescu fostered a strong sense of nationalism, often portraying Romania as an independent force within the Soviet bloc. He also pursued a policy of isolationism, limiting interaction with the West.
- **Suppression of Dissent:** The regime heavily controlled every aspect of life, including freedom of speech and expression. Any form of dissent was met with harsh repression, including surveillance, imprisonment, and exile.

The Seeds of Paradoxism:

It was within this oppressive atmosphere that the seeds of Paradoxism were sown. Artists and intellectuals, yearning for individual expression and frustrated by the limitations imposed by the regime, found solace in the movement's embrace of absurdity, humour, and the questioning of established norms.

What is paradoxism?

PARADOXISM is an avant-garde movement in literature, art, philosophy, science, based on excessive used of antitheses, antinomies, contradictions, parables, odds, anti-clichés, deviations of senses, against-the-grain speech, nonsense, paraphrases, paradoxes, semiparadoxes, etc. in creations.

It was set up and led by the writer Florentin Smarandache since 1980's, who said: "The goal is to enlargement of the artistic sphere through non-artistic elements. But especially the counter-time, counter-sense creation. Also, to experiment."

A) Definition: PARADOXISM is an avant-garde movement in literature, art, philosophy, science, based on excessive used of antitheses, antinomies, contradictions, parables, paraphrases, odds, anti-clichés, deviations of senses, parodies of proverbs and aphorisms, against-the-grain speech, upside-down interpretations, nonsense, paradoxes, semiparadoxes in creations. Paradoxism tries to find common parts to apparently uncommon things in any human field.

It was set up and led by the writer Florentin Smarandache since 1980's, who said: "The goal is to enlargement of the artistic sphere through non-artistic elements. But especially the counter-time, counter-sense creation. Also, to experiment."

B) Etymology**:** Paradoxism = paradox+ism, means the theory and school of using paradoxes in literary and artistic creation.

C) History: "Paradoxism started as an anti-totalitarian protest against a closed society, Romania of 1980's, where the entire culture was manipulated by a small group. Only their ideas and their publications counted. We couldn't publish almost anything.

Then, Smarandache said: Let's do literature... without doing literature! Let's write... without actually writing anything. How? Simply: object literature! 'The flight of a bird', for example, represents a "natural poem", that is not necessary to write down, being more palpable and perceptible in any language than some signs laid on the paper, which, in fact, represent an "artificial poem": deformed, resulted from a translation by the observant of the observed, and by translation one falsifies. 'The cars jingling on the street' was a "city poem", 'peasants mowing' a "disseminationist poem", 'the dream with open eyes' a "surrealist poem", 'foolishly speaking' a "dadaist poem", 'the conversation in Chinese for an ignorant of this language' a "lettrist poem", 'alternating discussions of travelers, in a train station, on different themes' a "post-modern poem" (inter-textualism).

Do you want a vertically classification? "Visual poem", "sonorous poem", "olfactory poem", "taste poem", "tactile poem".

Another classification in diagonal: "poem-phenomenon", "poem-(soul) status", "poem-thing".

In painting, sculpture similarly - all existed in nature, already fabricated.

Therefore, a mute protest we did!

Later, Smarandache based it on contradictions. Why? Because we lived in that society a double life: an official one - propagated by the political system, and another one real. In mass-media it was promulgated that 'our life is wonderful', but in reality 'our life was miserable'. The paradox flourishing! And then we took the creation in derision, in inverse sense, in a syncretic way. Thus the paradoxism was born. The folkloric jokes, at great fashion in Ceausescu's 'Epoch', as an intellectual breathing, were superb springs.

The "No" and "Anti" from his paradoxist manifestos had a creative character, not at all nihilistic (C. M. Popa). The passage from paradoxes to paradoxism was documentarily described by Titu Popescu in his classical book concerning the movement: "Paradoxism's Aesthetics" (1994). While I. Soare, I. Rotaru, M. Barbu, Gh. Niculescu studied paradoxism in my literary work. N. Manolescu asserted, about one of my manuscripts of non-poems, that they are against-the-hair.

Smarandache didn't have any forerunner to influence him, but he was inspired from the 'upside-down situation' that existed in the country. He started from politic, social, and immediately got to literature, art, philosophy, even science.

Through experiments one brings new literary, artistic, philosophical or scientific terms, new procedures, methods or even algorithms of creation. In one of my manifestos he proposed the sense of embezzlings, changes from figurative to proper sense, upside-down interpretation of linguistic expressions.

In 1993 he did a paradoxist tour to literary associations and universities in Brazil. Within 30 years of existence, 30 books and over 200 commentaries (articles, reviews) have been published, plus 5 national and international anthologies."

D) Features of Paradoxism

Basic Thesis of Paradoxism: everything has a meaning and a non-meaning in a harmony with each other.

Essence of Paradoxism:

a) The sense has a non-sense, and reciprocally

b) The non-sense has a sense.

Motto of Paradoxism:

"All is possible, the impossible too!"

Symbol of Paradoxism:

(a spiral -- optic illusion, or vicious circle)

Delimitation from Other Avant-Gardes:

- paradoxism has a significance, while dadaism, lettrism, the absurd movement do not;

- paradoxism especially reveals the contradictions, the antinomies, anti-theses, anti-phrases, antagonism, non-conformism, the paradoxes in other words of anything (in literature, art, science), while futurism, cubism, surrealism, abstractism and all other avant-gardes do not focus on them.

Directions of Paradoxism:

- to use science methods (especially algorithms) for generating (and studying also) contradictory literary and artistic works;

- to create contradictory literary and artistic works in scientific spaces (using scientific: symbols, meta-language, matrices, theorems, lemmas, etc.).

1.1 | Key Characteristics of Paradoxism

- **Emphasis on the Absurd:** Paradoxist works often embraced absurdity, illogicality, and contradiction. This was in direct opposition to the rigid rationality and order promoted by socialist realism, the officially sanctioned artistic style of the regime.
- **Critique of Authority:** Paradoxism challenged established hierarchies and power structures both within the art world and the broader Romanian society. Artists and writers questioned authority figures and criticized the limitations on free expression imposed by the state.
- **Playfulness and Humour:** Paradoxists frequently employed humor, satire, and irony to mock absurdities of the socialist bureaucracy and the cult of personality surrounding Ceauşescu.
- **Focus on Language:** Language played a central role. Paradoxism pushed against linguistic norms. It experimented with puns, wordplay, and nonsensical language to break down traditional meanings and create new possibilities of interpretation.

1.2 | The Founder: Florentin Smarandache

Florentin Smarandache, a Romanian-American writer, philosopher and mathematician, is considered the founder of the Paradoxist movement [https://fs.unm.edu/a/paradoxism.htm].

His writings and manifestos laid the groundwork for the movement's philosophy and aesthetic principles.

Florentin Smarandache, the founding figure of Paradoxism, continued his intellectual journey beyond the boundaries of the art movement. He delved into various fields, including mathematics, philosophy, and literature, leaving his mark with unique inventions and unconventional approaches.

1.3 | Post-Paradoxism Explorations

- **Smarandache and Education:** He established himself as a prolific writer, authoring over 500 books on various topics. He actively engaged in promoting unconventional education and research and founded the "University of Alternative Sciences" which aimed to explore non-mainstream ideas.
- **Fringe Sciences and Transdisciplinary:** Smarandache's work often ventured into the realm of fringe sciences and transdisciplinary. Today, when everything is controlled, any work that does not fit into the mainstream dogma is labeled as fringe work. Even Artificial Intelligence and search engines are used for the purpose of distorting the truth in accordance with the powerful.

He explored connections between seemingly unrelated fields, challenging traditional academic boundaries. This openness to diverse perspectives resonated with his earlier embrace of paradox and questioning established norms.

1.4 | Smarandache's Inventions

- **Neutrosophic Logic:** One of Smarandache's most notable inventions is neutrosophic logic. It extends classical binary logic (true/false) by introducing a third truth value, "neutral," to represent indeterminate, unclear, or inconsistent information.
- **Plithogenic Logic:** Another invention, plithogenic logic, deals with multiple degrees of truth and falsity, provided by many sources, offering a spectrum of possibilities between absolute truth and absolute falsity. These logic systems challenged the traditional foundations of logic and sparked discussions about the nature of truth and knowledge.

It's important to note that Smarandache's work in logic and other fields has received mixed reactions. Smarandache's legacy lies in his relentless exploration of unconventional ideas and his dedication to challenging established norms. His work in Paradoxism and beyond serves as a reminder of the importance of questioning, experimentation, and embracing the multifaceted nature of truth and knowledge.

While delving deeper into Smarandache's specific contributions to various fields like neutrosophic logic might require further research tailored to your specific interests, this overview provides a glimpse into his multifaceted pursuits beyond the realm of Paradoxism.

1.5 | Paradoxism vs. Socialist Realism

The relationship between Paradoxism and the Ceauşescu regime became a complex dance. While the movement served as a powerful tool for artistic rebellion, its very existence served as a constant reminder of the regime's shortcomings and the lingering desire for freedom of expression. The regime, in turn, used its control apparatus to suppress the movement's influence, further fueling the sense of alienation and absurdity that Paradoxism sought to express.

The Paradoxist movement offers a fascinating glimpse into the creative spirit's resilience in the face of oppression. By understanding the context of the Ceauşescu regime, we gain a deeper appreciation for the movement's significance and its enduring impact on Romanian art and culture.

Paradoxism stood in stark contrast to the officially sanctioned art form of Socialist Realism.

The Ceauşescu regime viewed Paradoxism as a subversive threat and actively suppressed the movement. Many Paradoxist artists and writers were marginalized or forced into exile. Despite these challenges, the movement had a lasting impact on Romanian art and literature. It fostered a spirit of dissent that paved the way for more open artistic expression after the fall of communism in 1989.

1.6 | Paradoxist Works

"The Paradoxist Manifesto" (1980) by Florentin Smarandache: This foundational text outlined the movement's core principles, emphasizing the importance of questioning, experimentation, and embracing the paradoxical nature of existence.

“Paradoxism and Postmodernism” (2001) by Ion Soare, https://fs.unm.edu/IonSoare2.PDF

“The Aesthetics of Paradoxism (second edition)” (2002) by Titu Popescu, American Research Press, https://fs.unm.edu/Aesthetics.pdf.

“Paradoxist Distiches” (2006) by Florentin Smarandache, ProQuest, Ann Arbor, USA, https://fs.unm.edu/ParadoxistDistiches.pdf.

And fifteen International Anthologies on Paradoxism, with creations by hundreds of authors from tens of countries around the world: https://fs.unm.edu/a/paradoxism.htm.

Weisstein, Eric W. "Smarandache Paradox." From MathWorld--A Wolfram Web Resource, Encyclopaedia of Mathematics. https://mathworld.wolfram.com/SmarandacheParadox.html

Let A be some attribute (e.g., possible, present, perfect, etc.). If all is A, then the non-A must also be A. For example, "All is possible, the impossible too," and "Nothing is perfect, not even the perfect."

*

These are just a few examples, and there are many other fascinating figures and works associated with Paradoxism. Further exploration can reveal the richness and diversity of this unique art movement that emerged under challenging circumstances.

It's important to remember that due to the movement's subversive nature, much of the work was created and circulated underground, making it difficult to obtain comprehensive documentation. However, the available works offer valuable insights into the artistic expression and the spirit of resistance that thrived under the oppressive regime of Nicolae Ceauşescu.

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Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Resolving paradoxes with the Logic of "*Not*" in Javanese Puppetry: A re-reading of Wrekodoro and Dewaruci's mythology

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Abstract

There are a large number of text collections in Indonesia, related to wayang theme, cf Zoetmulder (1971). And the text collections do not consist only of wayang purwa, but also variations of theme such as wayang wahyu etc. In the mean time, a Japanese scholar Shoshichi Nagatomo proposed the concept of "*Asian Logic*," which differs from Western logic. In Western thought, things are often seen as black or white, good or bad. However, Asian logic embraces certain degree of ambiguity. Here, "good" characters can exhibit flaws, and "bad" characters can possess redeeming qualities. This concept resonates with the Javanese philosophy of "*ngono yo ngono neng ojo ngono*" (note: the phrase can be translated: "it is what it is, but it's also not"). This philosophy embodies the logic of "not," acknowledging the multifaceted nature of reality.

Introduction

Puppetry, or Wayang in Javanese, transcends mere entertainment. These intricate figures serve as a guiding light for the Javanese community, depicting the path to a noble life. There are a large number of text collections in Indonesia, related to wayang theme, cf Zoetmulder (1971). And the text collections do not consist only of wayang purwa, but also variations of theme such as wayang wahyu etc.

In recent times, Wayang has even become a popular metaphor for social and political figures in Indonesia. This article delves deeper, exploring how Wayang stories, particularly those featuring conflicts between protagonists and antagonists in the Mahabharata epic, can be interpreted as a manifestation of Javanese logic, or the logic of "not."

If the story of "Dewaruci" is known during the Late Majapahit period, the things described by Dewaruci to Wrekodoro (Bhima) are likely to be related with the religion that developed at that time, namely the Śaiwasiddhanta religion, whose teachings can be known from the Tutur books. Speak the oldest is Bhuwanakosa comes from during the reign of King Siṇḍok in the 10th century. In the book of Speech, knowledge is taught Śiwa religion which is centered on sacred

knowledge and proper knowledge of emptiness (*sūnya*) which is called by various names, including Paramaśiwa, Parameśwara, Mahadeva Shiva.



Figure 1. Wayang puppet depicting Wrekudoro and Dewaruci

Nagatomo's Notion of Asian Logic

Japanese scholar Shoshichi Nagatomo proposed the concept of "Asian Logic," which differs from Western logic. In Western thought, things are often seen as black or white, good or bad. However, Asian logic embraces ambiguity. Here, "good" characters can exhibit flaws, and "bad" characters can possess redeeming qualities. This concept resonates with the Javanese philosophy of "ngono yo ngono neng ojo ngono" (it is what it is, but it's also not). This philosophy embodies the logic of "not," acknowledging the multifaceted nature of reality.

Considering the above wayang tradition in Southeast Asia, this logic of "not" is not an absence of logic, but rather a different form of reasoning. It emphasizes the importance of intuition, symbolism, and a deeper understanding of the universe. Through the fantastical elements in Wayang, Javanese audiences glean valuable life lessons and explore the complexities of human nature.

The Myth of Wrekodoro and Dewaruci: A Realm Beyond the Scientific

The story of Wrekodoro encountering the giant Dewaruci exemplifies this logic. Wrekodoro, a Pandava protagonist known for his immense strength, shrinks himself and enters Dewaruci's ear after deep philosophy conversations. Scientifically, this feat seems impossible. However, within the framework of Wayang, the logic resides in the realm of the spiritual or the "not." Here,

the story transcends the physical world and delves into the symbolic, where Bima's journey represents overcoming inner challenges.

Moreover, that encounter with the Divine – if we interpret the myth not only just as myth of the past, but to represent how ancient Javanese people have longing of eternity in their hearts, cf. D. Richardson (2010) – can be re-read as how spiritual people in ancient Java considered union with God the Almighty as one of the highest virtue in their life, symbolized with tirta prawitasari (the sought-after *living water* in that story). In other words, we can also interpret with the more recent book of Sir D. Hawkins, *Power vs Force*, that by encountering with God Almighty, Bima who was known as a really brave knight, has been turned upside down in his worldview, and then become transformed into higher level of conscious awareness. Or if we follow the levels of consciousness as depicted in Hawkins' diagram, perhaps he was able to move up from around 200 scale (associated with brave) to become 500 in scale or may be higher.

We all know that one of the mystery of consciousness study if we read *Power vs Force* book, is how to move up to higher scale of consciousness. And even with hidden message in that wayang mythology which was characteristics of Javanese cosmogony, we can learn that to be in another state of conscious awareness, we need the help and encounter with God the Almighty, and rarely such an improved state of awareness can be done by that person alone. Interestingly, studies also seem to support that spiritually-growing person are more likely to be a much better either in the scale of EQ (*emotional quotient*) or AQ (*adversity quotient*). That would be another topic to be discussed in other occasion.

The Art of Softened Criticism: Humor in Javanese Puppetry

Let us analyse in a bit the context of "ngono yo ngono neng ojo ngono" (which seems to imply that even if someone tries to dispute something, it should be done in a more polite way). In the Javanese puppet tradition, there are known figures which do not belong to the Satria (knights) or Raksesa (giants), but are called Punakawan, who play the role of comedians. They can offer criticism to social conditions that happen in the village or town where the puppet drama is displayed, but it is done in a satiric and humorous way by the Dhalang (puppeteer).

This concept of softened criticism through humor is a fascinating aspect of Javanese culture. It highlights the importance of maintaining social harmony even when addressing sensitive issues. Here's a closer look at how this works:

- The Power of Punakawan: The Punakawan characters, with their witty remarks and slapstick humor, act as a social commentary device. They can poke fun at human behavior, societal norms, or even political situations.
- Satire with a Smile: The humor employed by the Dalang is often satirical. By using jokes and puns, the Dalang can convey critical messages in a way that is more palatable for the audience. People are more receptive to criticism when it's delivered in a lighthearted manner.

- Indirect Commentary: The beauty of this approach lies in its indirectness. The audience is left to interpret the underlying message within the humor. This allows for a deeper impact as people can come to their own conclusions.

This approach to criticism is not unique to Javanese puppetry. We see similar methods employed by comedians and satirists around the world. By using humor to soften the message, these figures can make important points without causing offense or social disruption.

Put in another way, the Javanese concept of "*ngono yo ngono neng ojo ngono*" and the role of Punakawan characters in puppetry offer valuable lessons in delivering criticism. By using humor and satire, we can create a space for constructive dialogue and positive change, all while maintaining social harmony.

At this point, it appears quite interesting to compare the role and methods played by Punakawan in folk puppet show with, for instance, paradoxism both as avant garde art and also social movement in Romania, especially during 1980s, We discuss this topic in the next section.

Discussion:

Paradoxism as social movement against totalitarianism, and all that

The definition of paradoxism (păr´å`dŏks`ĭz'm) is as given below:

n . 1 . An avant-garde movement in literature, art, and philosophy, based on excessive used of antitheses, antinomies, contradictions, oxymorons, and paradoxes.[1]

A more complete definition is described as follows: “PARADOXISM is an avant-garde movement in literature, art, philosophy, science, based on excessive used of antitheses, antinomies, contradictions, parables, paraphrases, odds, anti-clichés, deviations of senses, parodies of proverbs and aphorisms, against-the-grain speech, upside-down interpretations, nonsense, paradoxes, semi-paradoxes in creations. Paradoxism tries to find common parts to apparently uncommon things in any human field. It was set up by one of us (FS) around 1980s.”[2]

Its origin came from suppression and totalitarianism during socialist regime in Romania back then, and it has been discussed in [12]. What’s interesting here, is that we can hypothesize that paradoxism was a mixture of brave grassroot movement along with neutrality and thinking, then the movement can be considered belong to Hawkins’s scale of consciousness ranging from 200 (brave). 250 (neutrality) up to 400 something (scale for thinking).

[1] Source: https://www.thefreedictionary.com/paradoxism
[2] Source https://fs.unm.edu/a/paradoxism-en.htm

Provided such an interpretation is closer to the true to the past background of the movement, then it can be considered a step higher to scale of consciousness compared to average scale of consciousness according to D. Hawkins, somewhere around 190. But it seems unlikely that even with such an avant-garde art and literary books they can bring the majority of people to higher than that, with seems clear to the quite stagnant period over the years especially in Romania (of course, this is by no means to be a disregard to what has happened to Romania people in common at the time).

Similarly, in Javanese cosmogony, union between Divinity and humans becomes essence of aspiration, of which Prof Zoetmulder called the seven steps to become Perfect Man as martabat (cf. Zoetmulder, 1971). And the pinnacle of the seven steps is called "*manunggaling kawula lan Gusti*" or perhaps better termed as unio mystica. In our reading, while it can be argued otherwise, we can consider that such a unio mystica phase of spirituality can be compared to Hawkins' scale of around 600 or higher in his map of consciousness.

What is not so obvious, both in Hawkins and also Javanese nondual unity with the Divine inside humans (God-indwelling in human), is how to achieve such a highest scale of consciousness. That seems to be a great problem with all kinds of gnosis teachings, i.e. the role of Teacher of Righteousness even if one encounter with such a great teacher may be not so helpful (cf. if we compare it with term from Qumran scrolls, term such as Teacher of Righteousness). Prayer and meditations are of course quite helpful, but not yet sufficient to improve our phase of spirituality, except we encounter with the Divine, with God Almighty Himself.

That is what Christianity may offer a hint, that to achieve higher in the scale of consciousness, just another perfect knowledge or gnosis is not sufficient, even if for instance we learn properly *The Gospel of Thomas* (Leloup, 2005). That is because spiritual-inertia that we know as sins are unavoidable (as St Paul admitted it at his Letter to Romans chapter 7).

Therefore, enlightenment is more than just accepting the hidden knowledge or sudden inspiration to the "correct" hermeneutics of the texts or Nature, but to encounter and embrace the grace of God Himself, given through Jesus the Son of God, and then permit the Holy Spirit to guide him/her in daily and gradual process to higher consciousness scale.

Because at the end of day, the awareness of humanity cannot be separated from ruach and nephesh of God Almighty, which God has given to all humans since their birth (cf. Hildebrandt, 1995; Keener, 2022). That seems to us to be deficiencies of map of consciousness as spiritual ladder as described by D. Hawkins (note: it appears he is more inclined in Hinduism or Buddhism's meaning of enlightenment and Nirvana.)

We do hope that this small comparison section among different approaches to hidden knowledges, from Javanese puppet show (wayang) to teachings such as D. Hawkins will not put more obstacles to any reader in his/her pursuit of the True Love, that is God Almighty the Creator of Heavens and Earth.

Concluding remark

Javanese puppetry offers a unique perspective on logic, one that embraces the concept of "not." By delving into stories like Wrekodoro and Dewaruci, we gain a deeper appreciation for Javanese philosophy and its approach to understanding the world. This exploration not only enriches our understanding of Javanese culture but also broadens our perception of logic itself.

Therefore, enlightenment is more than just accepting the hidden knowledge or sudden inspiration to the “correct” hermeneutics of the texts or Nature, but to encounter and embrace the grace of God Himself, given through Jesus the Son of God, and then permit the Holy Spirit to guide him/her in daily and gradual process to higher consciousness scale.

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Trusting the Intertwined: Modeling Enhanced GRIT Through Borromean Humanity and the Logic of Not via Coupled Riccati ODEs

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Abstract

This article extends the framework of "intertwined humanity" – conceptualized through Neutrosophic Complete Graphs and Borromean rings, and interpreted via Shigenori Nagatomo's "Logic of Not" – to propose a conceptual model for enhancing Graduated and Reciprocated Initiatives in Tension-reduction (GRIT). We posit that conflict reduction, particularly the crucial mutual increase of trust, is a deeply coupled and non-linear process that transcends binary states. By integrating the systemic interdependence of Borromean structures, the nuanced ambivalence captured by the "Logic of Not," and the inherent complexities of human interaction (as per Ubuntu), we develop a theoretical model using coupled Riccati Ordinary Differential Equations (ODEs). This approach aims to illuminate how GRIT initiatives, perceived through a "Logic of Not" lens, can shift conflict dynamics towards a stable state of mutual trust, reflecting the intertwined nature of human well-being and paving the way for more robust peace-building strategies.

Keywords: Intertwined Humanity, Borromean Rings, Logic of Not, Shigenori Nagatomo.

1. Introduction: Navigating the Labyrinth of Conflict and Trust

Humanity's journey is perpetually marked by the ebb and flow of cooperation and conflict. From interpersonal disputes to geopolitical stalemates, the reduction of tensions and the establishment of trust remain paramount for sustainable peace. Strategies aimed at de-escalation are, therefore, crucial. Among these, Charles E. Osgood's Graduated and Reciprocated Initiatives in Tension-reduction (GRIT) stands out as a pragmatic approach designed to break cycles of mutual suspicion and foster a spiral of trust. GRIT acknowledges that in deeply entrenched conflicts, direct negotiation is often impossible due to a profound lack of trust, necessitating a unilateral, yet reciprocation-seeking, first step. Our recent article, "Modeling Intertwined Humanity: Neutrosophic Complete Graphs, Ubuntu, and the Logic of Not" (NSS Vol. 88, 2025), established a novel framework for understanding human interconnectedness. We demonstrated how the irreducible interdependence of Borromean rings, where three entities are interlinked such that removing one frees the others, mirrors the complex web of human relationships and philosophical concepts like Ubuntu ("I am because we are").

Crucially, we introduced Shigenori Nagatomo's "Logic of Not" as a powerful tool to describe states that defy classical binary true/false distinctions, embracing ambiguity, partiality, and contextual nuance – phenomena inherently present in the subjective interpretations of trust and distrust.

This article extends this "intertwined humanity" framework to model the dynamics of GRIT. We argue that the process of mutual trust-building in conflict resolution is not merely a linear accumulation of positive interactions but a highly coupled, non-linear system, characterized by feedback loops and nuanced interpretations. We propose a conceptual mathematical model utilizing coupled Riccati Ordinary Differential Equations (ODEs) to represent these intertwined trust dynamics. This model will incorporate the systemic understanding derived from Borromean logic, the non-binary perceptions facilitated by Nagatomo's "Logic of Not," and the foundational philosophy of Ubuntu, offering a richer, more robust understanding of how GRIT can succeed in cultivating trust for conflict reduction.

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2. GRIT: A Strategy for Gradual Trust Building

The concept of GRIT was initially proposed by psychologist Charles E. Osgood in 1962, during the height of the Cold War, as a strategy for de-escalating international tensions. Facing a seemingly intractable stalemate characterized by an "action-reaction" cycle of distrust and hostility, Osgood posited a method for breaking this destructive pattern without requiring a prior agreement on mutual concessions or compromising national security.

The core principles of GRIT are:

1. **Unilateral Initiatives:** One party takes a small, verifiable, tension-reducing step without requiring a prior commitment from the adversary.
2. **Announced Intentions:** The initiative is publicly announced in advance, stating its intention as a genuine effort to reduce tension and inviting reciprocity.
3. **Graduated Nature:** The initiatives are small, non-risky, and incremental. If reciprocated, they can be followed by larger steps.

4. **Reciprocity:** The initiator explicitly invites and expects reciprocation from the other party.
5. **Verification:** The initiatives must be verifiable by the adversary to build credibility.
6. **No Withdrawal of Ability to Retaliate:** The initiator must maintain their capacity for deterrence or defense, ensuring their security is not compromised.

GRIT's power lies in its attempt to initiate a "spiral of trust" to counteract the prevailing "spiral of distrust." It aims to demonstrate goodwill, provide tangible evidence of a shift in intent, and create a psychological environment conducive to further cooperation. However, the path from a unilateral initiative to genuine mutual trust is fraught with challenges. Adversaries often view such moves with suspicion, interpreting them through a lens of past betrayals or perceived ulterior motives. It is this complex, often ambiguous, process of interpreting intentions and building reciprocal trust that demands a more nuanced modeling approach, one that goes beyond simple binary states of trust or distrust.

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3. Intertwined Humanity and the Borromean Logic of Conflict

Our previous work argued that human existence is fundamentally "intertwined." This concept, deeply rooted in the African philosophy of Ubuntu ("I am because we are"), posits that individual well-being and identity are inextricably linked to the collective. Applied to the dynamics of conflict, this means that the resolution of a conflict is rarely a unilateral act or a simple sum of individual concessions; rather, it is a systemic transformation of an interdependent whole.

The **Borromean rings** serve as a potent topological metaphor for this intertwined reality in conflict. Imagine a conflict involving three key issues (e.g., security concerns, economic grievances, and historical narratives) or three principal parties. In a Borromean conflict, each issue/party is bound to the others, but no two can fully resolve their differences in isolation. For instance, addressing economic grievances might be impossible without also tackling deep-seated security concerns, and vice

versa. Removing one component (e.g., one party leaving the negotiations, or one issue being arbitrarily dismissed) does not simplify the conflict into two manageable parts; instead, it causes the entire complex to unravel or collapse into a different, equally unstable configuration. This Borromean logic highlights that conflict reduction, especially through GRIT, is not about disentangling issues but about transforming the very nature of their interlinkage. A successful GRIT strategy, then, must recognize and operate within this irreducible interdependence, seeking to shift the collective equilibrium.

Furthermore, our framework utilized **Neutrosophic Complete Graphs** to represent these intertwined systems, allowing for the inclusion of indeterminacy and uncertainty inherent in complex human interactions. In a conflict context, this means that relationships, intentions, and even the "facts" of the conflict are often not purely true or false, but contain elements of ambiguity, doubt, and incompleteness. This aligns perfectly with the challenges faced by GRIT, where initiatives are often met with a mix of acceptance, suspicion, and uncertainty, rather than clear-cut trust or distrust. The "coupled-ness" of mutual trust, therefore, is not merely a functional relationship but a reflection of this profound Borromean interdependence. My trust in you is not an independent variable but is intimately shaped by our shared history, our intertwined interests, and crucially, by your perceived trust in me, forming a dynamic feedback loop that a static, uncoupled model cannot capture.

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4. Nagatomo's Logic of Not: Unpacking the Nuances of Trust and Distrust

Classical Boolean logic operates on a binary principle: a statement is either True or False. However, the complexities of human perception, particularly in high-stakes situations like conflict resolution, rarely fit neatly into such categories. Nagatomo's "Logic of Not" provides a crucial philosophical and logical tool to address this limitation. It posits that "not P" is not simply the negation of P but can represent a spectrum of possibilities that are "between" P and not-P, or a contextual "not." This allows for concepts like partial truth, degrees of falsity, and indeterminacy.

Applied to the dynamics of trust and distrust in GRIT, the "Logic of Not" offers profound insights:

- **Beyond Binary Trust:** In a conflict scenario, Party A's perception of Party B is rarely a simple "trust" or "distrust." Instead, Party A might be "not fully trusting" (i.e., harboring lingering doubts or suspicions, even after a positive initiative), or "not fully distrusting" (i.e., open to persuasion, despite a history of animosity). These are states of ambiguity that a scalar trust value (e.g., 0.6 on a scale of 0 to 1) can represent numerically, but the "Logic of Not" provides the conceptual depth, acknowledging the internal contradictions or unresolved elements within that perception.
- **Interpreting Initiatives:** When Party A takes a GRIT initiative, Party B's interpretation is critical. B might perceive it as "not clearly cooperative" (due to past betrayals), "not clearly deceptive" (because it's verifiable), and thus existing in a nuanced "not" state. This "not-ness" influences B's willingness to reciprocate. The initiative is not simply a "true" act of goodwill but might be viewed with a degree of indeterminacy or doubt, making the immediate increase in trust uncertain.
- **The Fragility of Reciprocity:** A reciprocated move from Party B might similarly be interpreted by Party A as "not fully sufficient," "not entirely convincing," or "not without ulterior motives." This persistent "not-ness" is why trust-building in conflict is so challenging and often fragile, susceptible to setbacks if these ambiguities are not carefully managed. The Logic of Not allows the model to account for these subtle psychological realities where "distrust" isn't merely the opposite of "trust" but a complex interplay of uncertainty, perceived threat, and residual skepticism.
- **Neutral States:** The "Logic of Not" also accounts for truly neutral states, where a party is "not trusting and not distrusting," reflecting a cautious wait-and-see attitude. This is crucial for GRIT, as the initial initiatives aim to move parties from active distrust to at least a neutral, receptive stance.

By incorporating Nagatomo's "Logic of Not," our model moves beyond a simplistic representation of trust as a linear scalar. Instead, it embraces the inherent fuzziness, uncertainty, and context-dependent interpretations that define genuine human interactions in conflict.

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5. Modeling Mutual Trust with Coupled Riccati ODEs: A Conceptual Framework

To capture the non-linear, interdependent, and nuanced dynamics of trust in GRIT, we propose a conceptual model utilizing coupled Riccati Ordinary Differential Equations (ODEs). Riccati equations are a class of non-linear first-order ODEs, often used in control theory, optimal control problems, and population dynamics, where quadratic terms allow for self-reinforcement, multiplicative interactions, and threshold effects—all characteristics relevant to trust building.

Let's define our core variables:

- TA(t): Represents the level of trust Party A has in Party B at time t. We can conceptualize this as a continuous variable ranging from 0 (total distrust) to 1 (total trust). Or, for a more nuanced scale reflecting the "Logic of Not," perhaps -1 (strong distrust) to 1 (strong trust), with 0 representing a neutral state of "not trusting and not distrusting." For simplicity in this conceptual model, let's assume a range where higher values mean more trust.
- TB(t): Represents the level of trust Party B has in Party A at time t.

The core idea of coupling is that the rate of change of Party A's trust in B is influenced not only by A's current trust level and its own actions, but critically by Party B's current trust level in A and B's actions.

Proposed Coupled Riccati ODE Structure (Conceptual):

We can represent the dynamic evolution of trust for both parties as:

1. **For Party A's Trust in Party B:**

$$\frac{dTA}{dt} = \alpha TA^2 + \beta TA\,TB + \gamma TA + \delta AIA(t) + \epsilon AS(t)$$

2. **For Party B's Trust in Party A:**

$$\frac{dTB}{dt} = \alpha BTB2 + \beta BTA + \gamma BTB + \delta BIB(t) + \epsilon BS(t)$$

Let's break down the conceptual meaning of each term and parameter:

- **αiTi2 (Self-Reinforcement/Quadratic Term):**
    - αA,αB: Coefficients representing the self-reinforcing nature of trust (or distrust). If α is positive, higher existing trust (or a positive trust value) leads to an accelerated increase in trust. If α is negative, it could represent how existing distrust becomes harder to overcome. This quadratic term captures the non-linear "snowball effect" or "downward spiral" often observed in trust dynamics.
- **βiTj (Mutual Influence/Coupling Term):**
    - βA: Represents how Party B's trust in A (TB) influences Party A's trust in B (TA). A positive βA indicates that as B's trust in A increases, A's trust in B is also positively influenced. This is the direct "coupling" of trust.
    - βB: Similarly, represents how Party A's trust in B (TA) influences Party B's trust in A (TB).
    - These terms are crucial for modeling reciprocity and the feedback loops inherent in GRIT.
- **γiTi (Natural Decay/Growth Rate):**
    - γA,γB: Linear coefficients representing the intrinsic rate at which trust decays (if negative, reflecting inherent skepticism or default distrust) or grows (if positive, reflecting a default towards cooperation in the absence of other factors).
- **δili(t) (Impact of GRIT Initiatives/Reciprocity):**
    - IA(t): Represents Party A's GRIT initiative (e.g., a specific cooperative action). This would be a time-dependent "forcing function" or pulse, reflecting Osgood's "unilateral initiative."

- IB(t): Represents Party B's reciprocation to A's initiative.
- δA,δB: Coefficients quantifying the sensitivity of trust to these initiatives.
- **Crucially, the "Logic of Not" enters here:** The *perceived impact* of IA(t) on TB, and IB(t) on TA, is not a simple binary activation. An initiative IA(t) might be perceived by B not as a clear "1" (total trust-building act) but as some value reflecting "not entirely trustworthy but not entirely deceptive," based on historical context and current suspicions. This perception could be modeled by a fuzzy or neutrosophic function applied to IA(t) before it influences dTB/dt. For example, δBIA(t) might become δB·Perception(IA(t),HistoryB,LogicOfNot), where Perception maps IA(t) to a nuanced value based on B's current "not-state" of trust.

- **ϵiS(t) (Impact of External Shocks/Systemic Factors):**
  - S(t): Represents external, systemic events (e.g., global crises, third-party mediation efforts, unexpected provocations, changes in leadership). These can be positive or negative.
  - ϵA,ϵB: Coefficients for sensitivity to these external factors. This acknowledges that trust dynamics don't occur in a vacuum.

Dynamics of GRIT within the Model: GRIT aims to shift the system from a low or negative trust equilibrium (where the equations balance at low TA,TB) to a higher trust equilibrium. The unilateral initiatives (IA(t)) act as targeted perturbations. If Party B's "Logic of Not" perception of IA(t) is sufficiently positive (i.e., it's interpreted as "not not-cooperative"), it will increase TB. This increase in TB, through the βATB term, then positively influences TA, initiating the spiral of trust. The non-linear α terms help reinforce this spiral, potentially leading to a new, stable high-trust equilibrium. Conversely, a misinterpreted initiative or a strong negative external shock (S(t)) could send the system spiraling back into distrust.

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6. The Intertwined Nature of Trust Dynamics in the Model

The Borromean model of intertwined humanity provides a crucial conceptual underpinning for the coupled Riccati ODEs. While the ODEs explicitly show the coupling between Party A's and Party B's trust, the Borromean analogy implies a deeper, often unseen, third element that maintains the entire structure of the conflict or enables its resolution.

- **Borromean Interpretation of Coupling:** Beyond the direct TA↔TB coupling, consider a third, often implicit, factor that enables the conflict's persistence or its resolution. This might be a shared vulnerability (e.g., climate change), a historical grievance that binds them in animosity, or a crucial mediating institution that holds the process together. In a Borromean sense, the entire trust-building dynamic (represented by TA, TB, and this third factor, X) is only maintained as a cohesive system as long as all three are present and intertwined. If Party A's willingness to trust (TA) falters, it doesn't just reduce TB; it might destabilize the underlying, implicit "Borromean bond" that was holding the entire conflict/resolution system together, causing it to unravel. This suggests that the coefficients (α,β,γ) in the Riccati equations are not static but are themselves context-dependent and influenced by these underlying, Borromean-like systemic factors.
- **Neutrosophic Indeterminacy in Parameters:** Our prior work highlighted Neutrosophic Complete Graphs as a way to represent the indeterminacy in real-world systems. Extending this to the model, the parameters of the Riccati ODEs (α,β,γ,δ,ϵ) should not be seen as fixed, crisp values. Instead, they could be **neutrosophic numbers**, where each parameter is defined by a degree of truth, indeterminacy, and falsity regarding its influence. For example, the impact coefficient δA for Party A's initiative on Party B's trust might not be a precise 0.5, but "neutrosophically 0.5," meaning there's a degree of certainty about its value, but also an inherent degree of indeterminacy due to unforeseen psychological factors or external events. This allows the model to reflect the profound uncertainty inherent in predicting human responses in conflict.
- **Ubuntu's Influence on Model Outcome:** The ultimate goal of GRIT, aligned with Ubuntu philosophy, is not just the absence of conflict but a state of mutual well-

being and shared flourishing. In the context of the Riccati model, this means that the desired "solution" or stable equilibrium is one where TA and TB both reach high, positive values, reflecting a state where "I am because we are" in trust. The parameters of the ODEs, implicitly or explicitly, could be tuned (e.g., through policy decisions reflecting Ubuntu principles) to guide the system towards such a collectively beneficial, high-trust equilibrium. The model thus becomes a tool to explore pathways to shared humanity, not just conflict avoidance.

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7. Challenges and Future Directions

While this conceptual model offers a rich framework, its practical application and further development face significant challenges:

- **Parameter Estimation and Empirical Validation:** The most formidable challenge is empirically determining the values of the Riccati parameters ($\alpha,\beta,\gamma,\delta,\epsilon$) from real-world conflict data. This would require extensive qualitative and quantitative data analysis, potentially involving content analysis of diplomatic communications, surveys of public opinion, or statistical analysis of conflict events. Agent-based modeling, where individual actors with "Logic of Not" perceptions interact, could also inform parameter ranges.
- **Mathematical Formalization of "Logic of Not" in ODEs:** Integrating the nuanced perception offered by Nagatomo's "Logic of Not" into continuous differential equations is mathematically complex. It might necessitate the use of fuzzy differential equations, stochastic differential equations (where uncertainty is modeled as noise), or hybrid systems that combine continuous dynamics with discrete, "logic-based" decision points or perception functions. Representing neutrosophic parameters in a solvable ODE system would also require advanced mathematical techniques.
- **Complexities of Multi-Party Conflicts:** Most real-world conflicts involve more than two parties. Extending this coupled two-equation model to N-parties would dramatically increase complexity, requiring $(N(N-1))/2$ trust variables and

potentially leading to more intricate Borromean structures that are harder to represent.

- **Dynamic Nature of Parameters:** In reality, the parameters themselves (α,β, etc.) are not static but evolve over time based on learning, shifts in ideology, or changes in leadership. Modeling these as dynamic variables or fuzzy sets would add another layer of complexity.
- **Connecting Initiatives to Perceived Trust:** A key area for further research is how specific GRIT initiatives (e.g., troop withdrawal, economic aid, public apology) translate into the effective "input" terms (IA(t) and IB(t)) in the ODEs, especially when filtered through the "Logic of Not" perception functions. This would require deep insights from psychology, sociology, and political science.
- **Policy Implications:** If such a model could be refined and calibrated, it would offer invaluable insights for policymakers. It could help identify critical thresholds for trust, predict the impact of different sizes or types of GRIT initiatives, and determine optimal timing for interventions to shift conflict dynamics towards peace.

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8. Conclusion: From Intertwined Humanity to Actionable Peace

The vision of "intertwined humanity," articulated through Neutrosophic Complete Graphs, Ubuntu, and the profound implications of Borromean rings and Nagatomo's "Logic of Not," offers a powerful and comprehensive lens through which to re-examine the dynamics of conflict and peace-building. By applying this framework to GRIT, we move beyond simplistic, linear models of trust building to embrace the inherent complexity, non-linearity, and perceptual nuances that characterize real-world human interactions.

Our proposed conceptual model, utilizing coupled Riccati Ordinary Differential Equations, provides a plausible mathematical structure for representing the interdependent and self-reinforcing nature of mutual trust. It explicitly accounts for the critical role of reciprocity, external influences, and the crucial interpretation of

initiatives through a non-binary "Logic of Not" lens. The Borromean analogy further underscores that peace, like human existence, is not reducible to independent parts but emerges from a transformed, irreducibly linked whole.

While this remains a conceptual model, requiring extensive future research for empirical validation and mathematical formalization, it illuminates a promising avenue for transdisciplinary inquiry. By integrating insights from philosophy, logic, psychology, and mathematics, we can develop more sophisticated tools to understand the intricate dance of trust and distrust in conflict. Ultimately, by recognizing the "intertwined humanity" that binds us, even in our moments of deepest conflict, we move closer to formulating strategies that lead to not just the cessation of hostilities, but to a more stable, compassionate, and shared future for all.

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Reconsidering classical scientific method: The Spiritually Guided Scientific Method of *Spirintuilytics*

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Abstract

The classical Scientific Method is the bedrock of our modern world. It has allowed us to harness electricity, decode the genome, and send machines beyond our solar system. It is a systematic, rigorous, and proudly Left Hemisphere (LH)-dominated process, designed to strip away bias and deliver objective, verifiable truth. Its very strength—its insistence on the explicit and the measurable—is, however, its most critical limitation when facing the complex, interconnected problems of the 21st century. This suspicion is given profound, scholarly weight by the work of psychiatrist and literary scholar, Prof. Iain McGilchrist, particularly in his seminal text [1], *The Master and His Emissary: The Divided Brain and the Making of the Western World*. We came up with a modest proposal that the next great leap in discovery will not come from abandoning the Scientific Method, but from re-thinking its starting point—the crucial moment of genuine insight.

Introduction : The Crisis of Model-Dependent Reality and the Call for Complete Humanity in Science

The **classical Scientific Method** stands as the colossal intellectual edifice of the modern era. Its systematic rigor, emphasis on **verifiability**, and commitment to objective truth have delivered unprecedented technological and social progress. It is the purest expression of the **Left Hemisphere (LH)**—analytical, linear, and reductive—designed to strip away the inherent biases of the holistic, emotional human experience. Yet, in the 21st century, facing global crises of systemic complexity (climate change, social polarization, ecological collapse) and theoretical crises in fundamental physics, a haunting question arises: Is the tool that built our world now incapable of solving its deepest problems?

The core limitation of the classical method lies in its **LH-dominant structure**, which systematically de-prioritizes **context, intuition, and holistic understanding**. This limitation has been exacerbated by the rise of **Post-Empiricism** doctrines in theoretical physics, notably the concept of **"Model-Dependent Reality" (MDR)**, promoted by prominent figures like the late S. Hawking and Leonard Mlodinow. MDR, in its extreme interpretation, suggests that reality is defined by the models we use to observe it, effectively crowning the LH's capacity for creating abstract systems as the ultimate arbiter of truth.

This essay argues that the current scientific paradigm is suffering from a catastrophic cognitive imbalance—the **Tyranny of the Emissary**—which threatens to trap science in a cycle of sterile, incremental innovation. We propose a revolutionary re-engineering of the scientific method's starting point. We advance the framework of **Intuilytics** to **Spirintuilytics**, a comprehensive **balanced-brain** and **spiritually-guided** approach that reclaims the purpose of science: not merely to build better models, but to achieve a profound, holistic understanding of the *complete* reality, thereby delivering the next great leap in human discovery and wisdom.

I. The Constraint of the Classical Scientific Method

The Scientific Method, in its universally taught form, is a magnificent engine of analytical thought. Its steps perfectly mirror the sequential, explicit, and fragmenting processes of the left cerebral hemisphere.

The Left-Brain Dominance in Six Steps

1. **Observation (The Spark of Discontent):** A specific phenomenon or anomaly is noticed. The LH's **narrow-beam focus** shines here, isolating a single detail from the background noise of the Right Hemisphere (RH) context.
2. **Question Formulation (Defining the Boundaries):** The observation is immediately translated into a **precise, measurable, and testable** question. This

step is purely LH: the question must be **explicit, delimited, and quantifiable** (e.g., "What is the effect of X on Y?"). Ambiguity is rejected.

3. **Hypothesis Formulation (The Educated Extrapolation):** A tentative explanation is generated, typically as an "If...then..." statement. Critically, this hypothesis is primarily based on **existing knowledge, logical deduction, and established theory**. The LH attempts to extrapolate the answer from the known facts—it works *within* the current map.
4. **Prediction and Experimentation (The Test of Logic):** A controlled experiment is designed to **falsify** the hypothesis. This is the **methodological fortress of the LH**, demanding controlled variables, precise measurement, and repeatable, objective procedures.
5. **Analysis and Conclusion (Quantification and Judgment):** Data is subjected to statistical, **quantitative analysis** (a highly specialized LH function). The conclusion must be logical and evidence-based, either supporting or rejecting the initial guess.
6. **Iteration and Replication (The System Refined):** The results feed back, leading to the **incremental refinement** of the hypothesis or the generation of new, related questions.

This method is exceptionally good at **verification, refinement, and application** within a *known domain*. Its fatal flaw, however, is its **systemic weakness in generating truly novel, paradigm-shifting hypotheses**—the breakthroughs that require an intuitive *re-framing* of the problem rather than a logical extrapolation.

The Blind Spot

The core constraint of the classical method is embedded in **Step 3: Hypothesis Formulation**. By assigning this most **creative** step to the **Emissary (LH)**, which prioritizes logic and existing knowledge, the scientific process falls into the **Novelty Trap**:

- **Logic Cannot Invent a New Framework:** The LH, by its nature, can only operate *within* the established framework of rules and concepts. It is brilliant at systematizing, but terrible at questioning the foundation of the system itself. A true paradigm shift—like the transition from classical to quantum mechanics—requires an intuitive leap that temporarily **violates current logic**.
- **The Rejection of the Implicit:** The Western intellectual world has unconsciously adopted the logical constraint of the LH: If a hypothesis cannot be immediately articulated, logically traced back to known concepts, or quantified, it is treated as **"unscientific"** or a mere "lucky guess." The vast, non-verbal, contextual knowledge of the **Master (RH)** is systematically screened out at the start of the process.

This leads to the **"Problem of the Whole,"** where science generates brilliant, siloed solutions (e.g., a specific vaccine) that fail disastrously because they ignore the complex, non-linear interactions of the whole system (e.g., systemic social inequality in vaccine distribution).

II. The Theoretical Crisis: Post-Empiricism and Model-Dependent Reality

The cognitive bias inherent in the Scientific Method has found a philosophical expression in certain modern theoretical physics circles: the doctrine of **Post-Empiricism** and **Model-Dependent Reality (MDR)**.

The Essence of Model-Dependent Reality

MDR, as popularized by physicists such as Stephen Hawking, asserts that our perception of objective reality is necessarily filtered through the **theoretical models** we employ. The extreme version suggests that it is **meaningless** to ask what "true" reality is, independent of a model. If two different models equally explain and predict observed phenomena, then neither model can be said to be *more* real than the other. Reality becomes less about "what is" and more about "what works within this system."

MDR as the Zenith of Left-Brain Tyranny

MDR is the ultimate expression of the **Left Hemisphere's worldview** usurping the Master's role.

1. **The Map Becomes the Territory:** The LH's greatest weakness is mistaking its abstract systems (the map) for reality itself (the territory). MDR elevates this weakness to a **principle of cosmology**. It suggests that the LH's model is not a *representation* of reality, but a co-creator of it. This strips science of its commitment to **external, objective truth**—the very foundation of its historical power—and replaces it with internal, systemic consistency.
2. **The Disregard for Context (The Holism of Reality):** The RH, the Master, attends to reality **as it is**—the unique, lived, whole context. MDR, by prioritizing the abstract, portable, and mathematically elegant model, dismisses the unique **holism** of experience in favor of the **universal, reproducible fragment**. The human experience of reality (eived through the RH) is rendered secondary to the LH's mathematical abstraction.
3. **The Risk of Solipsism and Sterile Elegance:** When the model becomes the reality, the criteria for "truth" dangerously shifts from **empirical fit with the external world** to **internal mathematical elegance and consistency**. This risks trapping theoretical physics in a self-referential, solipsistic loop, where models are refined not because they reveal a deeper truth about the cosmos, but because they satisfy the LH's aesthetic need for systemic perfection. It becomes the ultimate form of **sterile innovation**.

The danger of MDR is that it philosophically justifies the LH's natural fragmentation, making it acceptable—even desirable—to sacrifice the Master's contextual wisdom for the Emissary's systematic elegance.

III. The Intuilytic Approach: A Balanced-Brain Scientific Method

To overcome the blind spots of the classical method and the philosophical dead-end of MDR, we must structurally re-engineer the process using **Intuilytics**, making the generation of the hypothesis the explicit domain of the **Master (RH)**.

Intuilytics: The Cognitive Handshake

Intuilytics is the systematic and intentional integration of the RH's holistic, intuitive, and contextual mode of knowing as the **primary driver** of the discovery process. It is a cognitive handshake where the RH leads the creative leap, and the LH provides the necessary rigor.

The Intuilytic Scientific Method (ISM)

The ISM structurally inverts the priority of the first few steps of the classical method:

Table 1

Classical Scientific Method	Spirintuilytic Scientific Method (SSM)	Cognitive Function
1. Observation (LH-driven Isolation)	1. Contextual Apprehension (RH-led Immersion)	RH (Master)
2. Question Formulation (LH-driven Delimitation)	2. Intuitive Hypothesis Generation (RH-led Synthesis)	RH (Master)
3. Hypothesis Formulation (LH-driven Extrapolation)	3. Analytic Question Formulation (LH-served Articulation)	LH (Emissary)
4. Prediction and Experimentation (LH-driven Control)	4. Prediction and Experimentation (LH-served Control)	LH (Emissary)
5. Analysis and Conclusion (LH-driven Quantification)	5. Spirintuilytic Analysis and Conclusion (RH/LH Balance)	RH/LH

The ISM's Crucial Shift:

1. **Contextual Apprehension (The Master's View):** The process begins not by isolating a discrete variable (LH), but by attempting to **synthesize and grasp the**

context as a whole (RH). The scientist must deliberately step back, seeking the non-verbal, implicit connections—the "gestalt" of the problem. This initial step is an act of **holistic immersion**.

2. **Intuitive Hypothesis Generation (The Creative Leap):** The RH—the seat of genuine novelty—is tasked with generating the hypothesis. This **Intuitive Hypothesis** is a rapid, non-linear synthesis of vast amounts of implicit data, often appearing as a flash of insight or a sudden re-framing. It is permitted, and even encouraged, to **violate current logic**, as true breakthroughs often do.
3. **Analytic Question Formulation (The Emissary Serves):** Only after the RH has provided the profound insight does the LH step in. Its role is no longer to invent the idea, but to **serve the idea** by:
    - **Articulating** the intuitive hypothesis into explicit, verbal, and systematic language.
    - **Translating** the holistic insight into a set of **testable, measurable, and quantifiable questions**—thus creating the necessary bridge to experimentation.

In this model, the **Master (RH)** determines the destination (the true hypothesis), and the **Emissary (LH)** plots the rigorous, analytical route to test it. The integrity of the process is maintained (falsifiability remains essential), but the creativity and potential for paradigm shift are exponentially increased.

IV. Advancing to *Spirintuilytics*: The Integration of the Spiritual Function

The Intuilytics framework is a necessary advancement, but it is incomplete. To truly address the crisis of MDR and the limitations of an overly-mechanistic science, the **context** must be expanded beyond the purely neurological and into the **spiritual** realm. The fragmented modern consciousness is not only cognitively unbalanced but **existentially divorced from the unified source of reality**.

The Spiritual Function as Ultimate Context

The **Spiritual Function** is introduced as the **meta-mode of attention** that provides the ultimate, non-dualistic ground of being, which the RH is best equipped to access. It is the capacity for **Transcendental Empathy and Ultimate Unity**.

1. **Definition:** The Spiritual Function is the immediate, non-dualistic apprehension of reality, characterized by **Ultimate Context, Unity, and Meaning**. It transcends the RH's holistic view of the *world* and moves to the RH's experience of *Being* itself.
2. **Addressing MDR:** *Spirintuilytics* re-establishes the necessity of a **Truth that exists independently of the model**. The Spiritual Function grounds science in the search for a reality characterized by profound, intrinsic **Meaning** and **Unity**—criteria that a mere mathematical model can never fully capture. It provides the **ethical and existential imperative** for scientific inquiry: to reveal the interconnectedness of existence, not just its mechanism.

The Triarchic Structure of *Spirintuilytics* (SSM)

The SSM operates as a cognitive **triarchy**:

Table 2

Component	Function in SSM	Mode of Attention	Contribution to Scientific Purpose
Spiritual Function (Spiritus)	Ultimate Context (Source)	Non-dualistic, Transcendental	Unity, Meaning, and Ethical Ground
Right Hemisphere (Master)	Holistic Context (Mediator)	Broad, Intuitive, Empathetic	Novel Hypothesis and Wisdom-Driven Goal
Left Hemisphere (Emissary)	Fragmented Action (Tool)	Narrow, Explicit, Analytical	Falsification, Measurement, and Articulation

The Spiritual Function informs the RH with a deeper, unwavering sense of **meaning and reverence** (the *telos* of the research), which then governs the LH's analytical actions with an ethical and purposeful imperative.

The *Spirintuilytic* Scientist: *Homo Intuilytics-Spiritus*

The goal is to cultivate the complete human being, the ***Homo Intuilytics-Spiritus***, whose scientific practice is characterized by:

- **Reverence for the Unknown:** The scientist approaches the limits of knowledge not with the LH's demand for immediate categorization, but with the Spiritual Function's sense of **awe and humility**.
- **Unity-Driven Research:** Research is prioritized not based on political or economic fragmentation, but on addressing the systemic problems of **interconnected life** (Ecological Reverence, Transcendental Empathy).
- **The Ethical Hypothesis:** The Intuitive Hypothesis (RH) is subjected to a final, crucial spiritual check: Does this insight serve the ultimate context of Unity and Meaning, or does it merely feed the LH's desire for control and fragmentation?

V. Implications for Modern Scientific Disciplines

The adoption of the SSM/Spirintuilytics framework has profound implications across the scientific spectrum, challenging the reductive conventions of several key disciplines.

1. Quantum Physics and MDR

In quantum physics, the concept of Model-Dependent Reality stems from the deep conceptual difficulty of reconciling our LH-driven classical models with the strange, holistic, and observer-dependent nature of reality at the smallest scales.

- **The Spirintuilytic Challenge:** *Spirintuilytics* does not deny the utility of the models, but it insists that the **Intuitive Hypothesis** must seek the **Holistic Real**—the underlying, non-fragmented unity that gives rise to the phenomena. The ultimate goal is not a *better mathematical model* that works in isolation, but a **Unity Model** that intuitively integrates the observer, the observed, and the context (the Spiritual Function). This requires physics to move beyond pure mathematical elegance and embrace the RH-driven concepts of **consciousness** and **holism** as fundamental physical variables.

2. Biology and Mechanistic Reductionism

Biology, driven by the successes of genomics and molecular science, often defaults to a highly reductive, mechanistic view of life, treating organisms as complex chemical machines (the LH's favored model).

- **The Spirintuilytic Shift:** The SSM insists on starting with the **Contextual Apprehension** of the organism as a **living, integrated system** (RH). The Intuitive Hypothesis must prioritize systemic interaction over isolated components. It promotes a science of **Emergence** and **Context**, where concepts like self-organization, inherent vitality, and non-quantifiable relationships are not dismissed as "unscientific" but are the very starting point for analysis. The LH's tools (genomics, proteomics) are then used to rigorously test the RH's holistic insights into the dynamic, living process.

3. Ecology and Climate Science

These disciplines are prime examples of the **Problem of the Whole**. LH-driven solutions focus on siloed mitigation techniques (e.g., carbon capture technology) while ignoring the underlying systemic, social, and spiritual fragmentation that caused the problem.

- **The Spirintuilytic Imperative:** The Spiritual Function's **Ethical Hypothesis** demands that the solution be one of **Unity and Reverence**. The RH's **Contextual Apprehension** must define the problem as a crisis of relationship, not a crisis of logistics. The LH is then tasked with designing integrated, local, and technologically appropriate solutions that *serve* the systemic health, rather than simply managing its symptoms for short-term human gain.

VI. The Methodological Cultivation of *Spirintuilytics*

Transitioning to *Spirintuilytics* is a cognitive and cultural revolution requiring the intentional cultivation of the Master and the Source.

1. Training the Spiritual Function: The Source of Meaning

To access the Ultimate Context, the scientist must cultivate practices of **non-dualistic apprehension**:

- **Contemplative Science:** Integration of methods (e.g., non-dual meditation) aimed at temporarily dissolving the subject-object boundary. This provides the direct, felt experience of **Unity**, preventing the scientist from mistaking their limited, model-dependent perspective for the totality of reality.
- **The Practice of Awe:** Intentional exposure to natural or cosmic phenomena that evoke **awe**, which is a deep Spiritual Function that momentarily collapses the ego-driven, narrow-focus of the LH. This re-sets the cognitive state to one of humility and open receptivity.

2. Reforming Scientific Education

Scientific education must shift from rote LH-driven factual accumulation and logical procedure to RH-led **holistic exploration**:

- **Prioritizing Synthesis Over Specialization:** Curricula must force students to synthesize knowledge across disparate fields (Physics and Poetry, Biology and Ethics) to train the RH's ability to create contextual meaning.
- **Hypothesis Generation Workshops:** Explicit training in **Intuitive Hypothesis Generation** using techniques like mental imagery, metaphorical mapping, and analogical reasoning, valuing the creative leap over the logical deduction in the initial stages of inquiry.

3. Institutionalizing the Balance

Scientific institutions and funding bodies must be reformed to recognize and reward *Spirintuilytic* research:

- **Funding for Context:** Prioritizing funding for projects that deliberately bridge disciplinary silos and address **systemic crises** (the Problem of the Whole) over narrowly specialized, incremental research.
- **Valuing the Implicit:** Creating peer-review criteria that recognize the value of **qualitative data, holistic insight, and ethical context** alongside traditional LH-

driven quantitative metrics. The narrative of the discovery (the RH's domain) must be valued as much as the data tables (the LH's domain).

Concluding remark: The new horizon

The current scientific method, for all its glory, is a reflection of a dangerously unbalanced civilization: brilliant, systematic, and fragmented. The philosophical expression of this imbalance, Model-Dependent Reality, threatens to trap us in a sterile, self-referential world of our own making.

The path forward is not to abandon the rigor of the Emissary, but to fully activate the Master (RH) and integrate the Source (Spiritual Function). Spirintuilytics represents the next evolutionary stage of the scientific mind: one where the Intuitive Hypothesis is guided by Ultimate Context, and the resulting Analytic Testing serves an Ethical Imperative.

This re-developed scientific method is more than a technique; it is a declaration of **complete humanity**— ***Homo Intuilytics-Spiritus***. By consciously integrating logic with intuition, fragmentation with holism, and mathematics with meaning, science can move beyond simply building better models and return to its original, profound purpose: the wisdom-driven, ethical discovery of the unified, living truth of the cosmos.

The survival of our civilization and the realization of our full human potential are likely to depend on this cognitive and spiritual shift.

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This Earth at the Precipice: From Cognitive Imbalance to *Homo Intuilytics-Spiritus*

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Abstract

The pursuit of discovery has always been viewed as a rigorous, step-by-step march—a process defined by analysis, precision, and relentless logic. This methodology, the essence of modern science and Western technological progress, is the quintessential product of the left cerebral hemisphere of the human brain. We rely on its linear, verbal, and reductive power, and rightly so; it is the architect of our digital world and the administrator of our complex societies. Yet, a haunting suspicion lingers at the periphery of our collective awareness: in prioritizing this mode of thought, we may have inadvertently built a civilization that is brilliant but dangerously unbalanced. This suspicion is given profound, scholarly weight by the work of psychiatrist and literary scholar, Prof. Iain McGilchrist, particularly in his seminal text [1], *The Master and His Emissary: The Divided Brain and the Making of the Western World*.

Introduction

The trajectory of modern Western civilization, an undeniable marvel of technological and organizational achievement, is built upon a foundation of relentless **analysis**, **precision**, and **linear logic**. This methodology, which has sculpted the digital world, administered complex global systems, and fueled the engine of scientific discovery, is the quintessential product of the **left cerebral hemisphere (LH)** of the human brain. We have justly relied on its reductive, verbal, and systemic power.

Yet, a profound, almost primal suspicion shadows this success: in prioritizing this specific mode of thought, we may have inadvertently engineered a society that is brilliantly functional yet fundamentally unbalanced—a highly sophisticated machine alienated from its own context and source of genuine meaning. This suspicion is no mere philosophical flourish; it is a profound and scholarly diagnosis articulated by psychiatrist and literary scholar, Prof. Iain McGilchrist, most notably in his monumental work, *The Master and His Emissary: The Divided Brain and the Making of the Western World*.

McGilchrist's work transcends the typical neuroscience discussion; it offers a compelling framework for understanding our current intellectual, cultural, and even existential malaise. The central, sobering diagnosis is that the very engine of our unprecedented success—our unwavering allegiance to the LH worldview—has created a cognitive bias now pointing us toward a potential civilizational **precipice**. This essay will expand upon this foundational imbalance, move beyond the purely neurological, and advance the **Intuilytics** framework into a comprehensive model for complete human functioning, one that integrates the spiritual dimension to propose the ultimate balanced state: ***Homo Intuilytics-Spiritus.***

The Neurological Diagnosis: The Tyranny of the Emissary

McGilchrist's genius lies in reframing the relationship between the two cerebral hemispheres, moving beyond the simplistic notion of them performing wholly different functions (e.g., LH for logic, RH for emotion). Instead, he argues that the two hemispheres offer two **fundamentally different versions of the world**, two distinct modes of *attention* and *apprehension*.

The Master (Right Hemisphere - RH): Context and Wisdom

The **Right Hemisphere (RH)**, which McGilchrist terms the **Master**, is responsible for the broad, contextual, and **holistic view**. It is the seat of **implicit** understanding, dealing in lived experience, emotion, metaphor, ambiguity, and non-verbal communication.

- **Mode of Attention:** Broad, sustained, vigilant, and open.
- **Worldview:** It sees the **whole**, the connections, the *flow* and *process* of reality. It is the part of the brain that attends to the world *as it is* and its genuine connection to us.
- **Function:** It allows for true **novelty** and **insight**, providing the context, the meaning, and the deep, nuanced understanding of reality.

The Emissary (Left Hemisphere - LH): Fragmentation and Manipulation

The **Left Hemisphere (LH)**, called the **Emissary**, is tasked with taking the world apprehended by the Master and breaking it down into **explicit**, **manageable**, and **manipulable parts**.

- **Mode of Attention:** Narrow-beam, focused, and reductive.
- **Worldview:** It sees the world as a collection of discrete **parts**, fragments, and categories. It translates reality into verbal, systematic, and abstract language.
- **Function:** Its strength lies in articulation, planning, logical execution, and creating systems—it is the essential *tool* for interacting with and manipulating the environment.

The Catastrophic Inversion

The naturally **healthy relationship** is a hierarchy where the **Master (RH) governs**, providing the context and wisdom, while the **Emissary (LH) serves**, providing the means of action. The LH's fragments are thus always contextualized by the RH's whole.

McGilchrist's core argument is that, particularly since the Enlightenment, this relationship has been **catastrophically inverted**. The Emissary—the left hemisphere's mode of abstract, fragmented, and systemic thinking—has **usurped the Master's role**. We no longer primarily experience reality through the holistic, intuitive lens of the RH; we now predominantly see a **re-presentation of the world**—a filtered, verbalized, and fragmented map constructed by the LH.

The Consequences of Cognitive Imbalance: Civilization-Scale Liabilities

When the left hemisphere mode of attention dominates, its characteristic strengths become systemic, civilization-scale **liabilities**, driving a society toward an intellectual and cultural dead end.

1. The Loss of the Contextual Grasp

The LH's focus on parts leads to an inability to grasp the overarching context.

- **In Science and Academia:** This manifests as excessive **specialization** where researchers know "more and more about less and less," failing to synthesize findings into a cohesive, meaningful worldview. The obsession with discrete, isolated metrics overshadows any appreciation for the system as a living whole.
- **In Society and Bureaucracy:** We create systems and regulations that are **perfectly logical** in their internal structure but are utterly **dysfunctional** in their external, holistic impact on real human life. The focus on procedure outweighs the concern for the purpose it was meant to serve.

2. The Substitution of Life with the Map

The LH's strength is in creating models, systems, and **maps of reality**. Its most dangerous weakness is its tendency to **mistake the map for the territory**.

- **Mechanistic Reductionism:** When the LH is dominant, we fall into the trap of believing that the theoretical models we construct (economic, scientific, political) are reality itself. The LH loves the **inert, the explicit, and the mechanical**, leading to a pervasive mechanistic reductionism. We reduce:
    - Consciousness to computation.

- Art and creativity to mere technique or algorithm.
- Complex ecosystems to quantifiable resources.
- Human beings to *Homo Economicus* or bundles of measurable inputs and outputs. This strips the world of its inherent mystery, vitality, and true, non-quantifiable value.

3. The Paralysis of the Implicit

The LH demands that everything be **articulated, quantified, and proven**—it thrives on the explicit. The RH, however, deals with the **implicit**: the subtle cues, the nuance of tone, non-verbal communication, empathy, and the unwritten social contracts that constitute a truly functioning society.

- **Erosion of Wisdom: Intuition** and **wisdom** fundamentally reside in the implicit realm—insights that lie *beyond* logical deduction. A left-brain-dominated culture is inherently suspicious and dismissive of the implicit, demanding that non-verbal knowledge be translated into explicit language, a process that inherently destroys its meaning and power. This cripples our ability to access the very insights needed for paradigm-shifting breakthroughs and effective social harmony.

The Existential Loom: A Western Doom?

This cognitive shift is not merely an intellectual debate; it is an **existential threat**. A civilization dominated by the Emissary—the fragmented, reductive, and self-referential LH worldview—is increasingly **alienated** from the real, living world of context, embodiment, and genuine novelty. The risk of civilizational doom, McGilchrist argues, is not hyperbole; it takes three distinct forms rooted in cognitive imbalance.

The Three Crises of Imbalance

A. The Ecological Crisis

The LH's fragmented, utilitarian view prevents us from genuinely **experiencing** and **prioritizing** the interconnectedness of the ecosystem. The Master (RH) sees a living whole; the Emissary (LH) sees a warehouse of discrete resources to be managed by quantifiable, technological, and piecemeal solutions. This fundamental disconnection is why we fail to see the immediate, long-term consequence of ecological destruction, consistently prioritizing short-term, measurable economic outputs over the health of the holistic system upon which we depend.

B. Social Fragmentation and Polarization

The erosion of empathy and implicit understanding (RH functions) leads to a highly **polarized society**. Communication is reduced to rigid, verbal **dogmatism**—a battle between competing LH-constructed models and ideologies. People struggle to grasp the perspective of "the other" because the very neurological mechanism required for genuine imaginative empathy is being neglected and suppressed. The complexity of human interaction is replaced by tribal, simplistic categorization, fueling conflict and undermining social cohesion.

C. The Crisis of mundane Innovation

Genuine, paradigm-shifting discoveries—those that require a complete **re-framing of the world** and a leap of intuition—become increasingly rare. The left-brain mindset actively screens out the novel, ambiguous, and non-conforming input that the right brain presents, preferring instead to refine existing systems. Innovation becomes incremental, technical, and limited to efficiency improvements *within* the existing LH-created paradigm, rather than true conceptual breakthroughs that restructure our understanding of reality.

The Framework of Intuilytics: Reinstating the Master

The solution is not to discard the LH—that would be suicidal, like throwing away the ability to articulate and execute. The challenge is to heed McGilchrist's warning and **rejuvenate our way of thinking** by purposefully reinstating the **Master's authority**. This is the core mission of **Intuilytics**.

The Intuilytic Shift

Intuilytics is the process of intentionally giving the **Right Hemisphere mode of attention priority** as the foundational orientation of the mind, ensuring the LH returns to its proper role as a servant.

1. **From Fragmentation to Synthesis:** The Intuilytic process does not start a problem by breaking it down into parts (LH), but by attempting to **synthesize** and **grasp the context as a whole** (RH). The initial frame of understanding must be holistic.

2. **Intuition as Leader:** It cultivates **intuition**—the right brain's rapid, non-linear processing of vast amounts of implicit data—as the **leader** of the mind. Intuition generates the initial **hypothesis, frame, or direction**.

3. **LH as Rigorous Tester:** The left brain is shifted from its current role as the Master (the creator of the initial hypothesis) to its proper role as the **Emissary** (the rigorous tester, administrator, articulater, and executor of the insight generated by the RH). The LH provides the necessary analysis and precision *after* the RH has provided the essential, contextual meaning.

- **Intuition (RH):** Provides the *Wisdom* (the what and the why).
- **Analytics (LH):** Provides the *Means* (the how).

The balanced practitioner of Intuilytics uses the narrow-beam LH focus *for* the ends defined by the broad-beam RH attention.

Advancing Intuilytics: The Quest for Complete Humanity

The Intuilytics framework, while essential, remains primarily a model of **cognitive balance**. To achieve the **complete humanity** necessary to escape the civilizational crisis, we must recognize that the deepest roots of **holism** and **context** extend beyond the merely cerebral and into the **spiritual** realm. The fragmented modern consciousness is not just neurologically unbalanced; it is existentially and spiritually divorced from the unified source of reality.

The Limitation of the Secular Brain Model

McGilchrist's model, being a neurological one, addresses the relationship between **Self** and **World**. However, many of the RH's highest functions—the capacity for transcendence, the sense of unity, the experience of awe, and the profound appreciation of meaning—have traditionally been categorized as **spiritual functions**. The fragmentation of the LH worldview, which tends to reduce everything to the material and explicit, has systematically denied the validity and reality of these spiritual experiences.

The next step in the evolution of humanity must integrate a third domain, a domain of ultimate context: **The Spirit.**

The Integration of the Spiritual Function: From Intuilytics to *Homo Spiritus*

To advance Intuilytics, we introduce the **Spiritual Function** as the ultimate, overarching context for the Master (RH). The Spiritual Function is not a third cerebral lobe, but the **meta-mode of attention** that provides the deepest, most unified ground of being, which the RH is best equipped to access.

Defining the Spiritual Function

The Spiritual Function can be defined as the capacity for **non-dualistic apprehension of reality**, characterized by:

1. **Unity and Non-Separation:** The immediate experience of profound **interconnectedness** with all things—the self not as an isolated entity, but as an inextricable part of the cosmos. This transcends even the RH's holistic view of the *world* and moves to the RH's experience of *Being* itself.

2. **Meaning and Transcendence:** The capacity to find ultimate **meaning** not in the parts or systems (LH), but in the fundamental context of existence, often experienced as awe, sacredness, or unconditional love.
3. **Ultimate Context:** The sense of an **unspoken, ultimate reality** that provides the ground for all other contexts.

The Role of the Spirit in the Triadic Model

In this advanced model, the relationship becomes a **triarchy**, with the Spiritual Function governing the Master, and the Master governing the Emissary.

Table 1

Component	Function	Mode of Attention	Contribution to Humanity
Spiritual Function	Ultimate Context (Source)	Non-dualistic, Transcendental	Unity, Meaning, and Ethical Ground
Right Hemisphere (Master)	Holistic Context (Mediator)	Broad, Implicit, Empathetic	Wisdom, Intuition, and Creativity
Left Hemisphere (Emissary)	Fragmented Action (Tool)	Narrow, Explicit, Analytical	Execution, Technology, and System

When the Spiritual Function is acknowledged and cultivated, it informs the RH with a deeper, unwavering sense of meaning, which then governs the LH's actions with an ethical and purposeful imperative. This prevents the RH from becoming mere "creativity" and grounds it in **wisdom**.

The Emergence of *Homo Intuilytics-Spiritus*

The goal of this advanced framework is the intentional cultivation of a complete, balanced human being— ***Homo Intuilytics-Spiritus.***

Homo Intuilytics-Spiritus is the human mode of being where:

1. **Spiritual Apprehension (The Unity):** Provides the ultimate *why* and the deepest source of context, ensuring that all action is ethically and meaningfully grounded in the interconnectedness of life.

2. **Intuilytics (The Wisdom and Means):** Integrates the RH's intuitive, empathetic wisdom (the Master) with the LH's analytical and organizational capabilities (the Emissary) for problem-solving and creation.
3. **Ethical Action:** The Emissary's execution is directed not merely by efficiency, but by the ultimate ethical command arising from the Spiritual Function's sense of unity.

Overcoming the Civilizational Crises in the *Homo Spiritus* Mode

This complete mode of humanity offers direct remedies to the three looming crises identified by McGilchrist:

A. Ecological Crisis Ecological Reverence

The Spiritual Function cultivates **Reverence** for the Earth as a living whole, fundamentally rooted in the sense of non-separation. This transcends the LH's resource-management model, replacing it with a desire to nurture and protect the system not because of quantifiable services, but because it *is* an extension of the self. The RH provides the holistic sense of consequence, and the LH provides the precise, technical means for sustainable living.

B. Social Fragmentation Transcendental Empathy

The sense of **Unity** from the Spiritual Function breaks down rigid ideological and tribal categories. Empathy is no longer a mere psychological skill (RH); it becomes a non-negotiable **fact of existence**. The inability to understand the "other" is seen not as an intellectual failure, but as a direct spiritual wound. The RH mediates this unity into genuine emotional understanding, and the LH articulates the necessary socio-political structures that honor this deep equality.

C. Sterile Innovation Wisdom-Driven Creativity

Innovation shifts from refining the "map" to radically restructuring the "territory." True, **paradigm-shifting discoveries** are born when the Spiritual Function provides the foundational non-dualistic ground, allowing the RH to present genuinely novel insights, which the LH then rigorously tests and codifies. Creativity is not random; it is **wisdom-driven**, aimed at advancing human flourishing within the context of unity, rather than simply maximizing efficiency or profit.

Cultivating *Homo Intuilytics-Spiritus*: Practices for Balance

The shift to ***Homo Intuilytics-Spiritus*** is not a political program or a technological fix; it is a profound **cultural and personal practice**. It requires the intentional cultivation of attention.

1. Training the Master (RH)

Practices focus on shifting the mode of attention from narrow-beam to broad-beam:

- **Immersion in the Implicit:** Engaging with **art, music, and nature** without immediate attempts to label, categorize, or analyze. Allowing the experience to be what it is, non-verbally.
- **Embodied Cognition:** Practices like **mindfulness, and movement (e.g., prayers etc)** that anchor attention in the present, embodied moment, bypassing the LH's obsession with planning and memory.
- **Metaphorical Thinking:** Prioritizing and valuing **metaphor, analogy, and narrative** as true carriers of complex meaning, rather than dismissing them as mere rhetorical devices.

2. Accessing the Spiritual Function

These practices cultivate the sense of non-dualistic unity:

- **Non-Dual Meditation:** Techniques aimed at direct apprehension of the ground of being, transcending the subject-object split that the LH creates (cf. one practice that may be recommended, for instance: prayer for forgiveness[1]).
- **Service and Generosity:** Engaging in action that consciously dissolves the boundary between self and other, cultivating the direct experience of shared humanity and interconnected fate.
- **Cultivation of Awe:** Intentional exposure to experiences that evoke **awe** (e.g., stargazing, confronting natural majesty), which is a deep RH and Spiritual function that collapses the narrow-focus, ego-driven LH perspective.

3. Integrating the Emissary (LH)

The key is using the LH for its intended purpose: **articulation and execution in service to wisdom**.

- **"Intuition First" Protocol:** When facing a complex decision, one should dedicate time to quiet, non-analytical contemplation to allow the RH to form an initial, holistic judgment (the **Intuitive Hypothesis**). Only *then* should the LH be deployed to rigorously analyze, articulate, and test this insight.
- **The Language of Context:** Using language not merely to categorize and fragment, but to **connect, explain nuance, and build bridges** between concepts, ensuring that the explicit (LH) is always pointing toward the implicit (RH/Spirit).

[1] Ho'Oponopono Hawaiian Prayer For Forgiveness | Daphne Flores | OmFrequency

Concluding remark: The Path toward Civilizational Health

The Western world stands at a critical juncture, facing crises rooted not in external forces, but in an internal, self-inflicted cognitive imbalance. McGilchrist's work provides the neurological map of this peril. The framework of Intuilytics offers the cognitive path to balance. And the advanced model of *Homo Intuilytics-Spiritus* provides the necessary ultimate context—the spiritual grounding—to ensure that our recovered balance is used for true human and planetary flourishing.

The challenge is a call to maturity: to move beyond the childish, reductive certainty of the Emissary and return the governance of our minds and our civilization to the balanced authority of the Master, guided by the ultimate, unifying context of the Spirit. This is the only path to a civilization that is not just brilliant, but wise, humane, and sustainable. The fate of our species may well depend on our ability to evolve our mode of attention—to finally become the complete human being, ***Homo Intuilytics-Spiritus*** (if you wish, we can call this new term: *spirintuilytics*[2]).

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[3] Christianto, Victor, *et al.* "How to balance the intuitive and analytical function of the human brain". *EC Neurology* 11.7 (2019): 495-499.

[4] Christianto, Victor and Florentin Smarandache. "Exploring the Historical Debates on Irrational Numbers Using Neutrosophic Logic as a Balance between Intuition and Rational". *EC Neurology* 13.6 (2021).

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[2] See our companion article above.

Extending Sustainable Advantage based on John Kay's Distinctive Capabilities, to include Schumacher's Intermediate Technology with applications to Botany etc.

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Abstract

It is known that John Kay's Distinctive Capabilities Framework [1] offers a profound and nuanced understanding of organizational achievement, shifting the focus from the static possession of significant assets (Resource-based approach) to the dynamic cultivation of enduring relational contracts (Relationship-based approach). Kay identified three essential capabilities—*Architecture, Reputation, and Innovation*—as the non-replicable sources of performance and sustainable advantage. These capabilities encapsulate "what makes our organization so special," rooted in the continuity and stability of relationships with customers, suppliers, shareholders, and employees. While conceptually powerful, Kay's framework, in its original form, often lacks the operational precision required for modern execution, and the present article is an attempt to fill the gap. Moreover, in this article we also extend Sustainable Advantage based on John Kay's *Distinctive Capabilities* framework, to include Schumacher's *Intermediate Technology* with applications to Botany etc., for instance new innovative solutions such as laser-culture, gravitational water vortex power plant, confined vortex turbine, new fusion energy theory based on PT-symmetric potential of crystals, and also a plausible new approach to turn plastic waste into biofuel.[1] While several of those innovative solutions are still in lab scale phase, it can be expected to yield quite significant results in the near future, especially for less developed countries.

I. Introduction: The Need for Measurable Distinctiveness

It is known in literature that John Kay's *Distinctive Capabilities* Framework offers a profound and nuanced understanding of organizational achievement, shifting the focus from the static possession of significant assets (Resource-based approach) to the dynamic cultivation of enduring relational contracts (Relationship-based approach). Kay identified three core capabilities—Architecture, Reputation, and Innovation—as the non-replicable sources of excellence performance and sustainable advantage. These capabilities encapsulate "*what makes our organization so special*," rooted in the

[1] See for instance, our previous articles in, BPAS Journals – Botany section. And also our recent articles, Christianto, Smarandache, Umniyati, ref. [15-17].

continuity and stability of relationships with customers, suppliers, shareholders, and employees.

While conceptually powerful, Kay's framework, in its original form, often lacks the operational precision required for modern execution [1]. Theory of planning demands not only *what* to focus on but *how* to measure progress and *when* to adjust course. The abstract nature of "Architecture" or "Reputation" can leave managers without clear metrics, making it challenging to link investments directly to capability development [2][3].

The present article outlines a reformulation of the Distinctive Capabilities Framework, leveraging the principles of **management by measurement** popularized by figures like John Doerr and the Objectives and Key Results (OKR) methodology [6]. By translating Kay's conceptual strengths into a structured, quantifiable system, we create the Distinctive Capabilities Maturity Model (DCMM). This DCMM approach is designed to be as operational and integrative as frameworks like the Risk Maturity Index, providing a clear roadmap and measurable scorecard to guide the organization toward distinctive excellence [4].

II. The Essential Framework: Kay's Distinctive Capabilities

Before operationalizing the framework, it is essential to re-examine Kay’s three capabilities through the lens of measurement potential.

A. Architecture (Relational Structure)

Architecture refers to the intricate structure of internal and external relational contracts. It embodies tacit organizational knowledge, shared routines, and the ease of information exchange that allows for flexible, collaborative responses to change. Distinctive Architecture is a collective interest sustained by mutual commitment and trust, generating value through coordination.

The Measurement Challenge: Architecture is largely intangible. Measuring it requires quantifying the health and efficiency of relationships and knowledge flows, rather than simply counting assets.

B. Reputation (Information Signal)

Reputation serves as a powerful signal of quality and intent to customers, especially when information asymmetry makes informed decisions difficult. Built over time through consistent experience, quality signals (price, promotion), and word-of-mouth, a strong reputation reduces transaction costs and builds a sequence of trust-based relationships.

The Measurement Challenge: Reputation is often measured reactively (e.g., via brand surveys). Operationalizing it requires metrics that track the *antecedents* and *consistency* of the reputation signal itself.

C. Innovation (Sustainable Uniqueness)

Innovation is an undeniable source of distinctiveness. However, Kay noted that successful innovation is often neither sustainable nor easily appropriable, as it quickly attracts imitation. Transforming an innovation into a lasting competitive advantage requires a range of supporting strategies that protect and leverage the initial breakthrough.

The Measurement Challenge: Simply counting patents or new product launches is insufficient. The key is measuring the **appropriability**—the ability of the firm to capture the value of its innovation—and its contribution to long-term revenue streams.

III. The Operational Imperative: Bridging Kay and Doerr

The leap from conceptual capability to measurable action is best facilitated by the OKR methodology. John Doerr's principle, "*Measure What Matters*," advocates for setting an Objective (what is to be achieved) and coupling it with a set of measurable Key Results (KRs) (how we know we achieved it). By applying this framework, Kay's distinctive capabilities can be transformed into actionable strategic mandates.

A. Operationalizing Architecture: The Efficiency and Trust OKRs

Architecture is operationalized by measuring the quality of coordination and the depth of trust within the organization and across its value chain.

Table 1. Operationalizing Architecture based on OKR Framework

Objective (O)	Key Results (KR)	Kay's Rationale
O: Achieve Best-in-Class Internal Knowledge Transfer Efficiency.	KR1: Reduce time-to-onboard new employees/project members by 30% through codified routines.	Measures organizational knowledge and routines efficiency.
KR2: Increase the percentage of cross-functional projects delivered on-time and under budget to 95%.	Measures flexible response to change and coordination.	
O: Cultivate a Collaborative and Resilient External Value Chain Architecture.	KR3: Achieve a Supplier Relationship Index (SRI) score of 4.5/5.0 with top-tier suppliers.	Measures the relational contracting strength with suppliers.
KR4: Decrease average B2B customer issue resolution time requiring cross-party communication by 40%.	Measures open exchange of information and collective problem-solving.	

B. Operationalizing Reputation: The Signal Consistency and Trustworthiness OKRs

Reputation is operationalized by measuring the reliability of the quality signal and the depth of customer/stakeholder trust it generates.

Table 2. Operationalizing Reputation based on OKR Framework

Objective (O)	Key Results (KR)	Kay's Rationale
O: Establish an Unimpeachable Brand Reputation for Quality and Ethics.	KR1: Achieve and sustain a Net Promoter Score (NPS) of 70+ in core customer segments.	Measures customers' own experience and willingness to recommend.
KR2: Reduce the volume of negative social media mentions across all platforms by 50% relative to industry average.	Measures the strength and protection of the quality signal.	

O: Transform Reputation into a Tangible Competitive Differentiator.	KR3: Increase the price premium over the closest competitor to 15% without loss of market share.	Measures the commercial value of the reputation signal (willingness to pay).
KR4: Achieve 80% customer retention solely through brand loyalty (excluding contractual lock-in).	Measures the ability to build a relationship based on trust and prior reputation.	

C. Operationalizing Innovation: The Appropriability and Diffusion OKRs

Innovation is operationalized by measuring the effectiveness of methods that support and protect the initial innovation, ensuring its value is captured by the firm and not quickly imitated.

Table 3. Operationalizing Innovation based on OKR Framework

Objective (O)	Key Results (KR)	Kay's Rationale
O: Develop a Sustained Pipeline of Defensible, Market-Leading Innovations.	KR1: 35% of annual revenue derived from products/services launched in the last three years.	Measures the long-term success and reliance on recent innovation.
KR2: Increase the 'Patent Strength Index' (a measure of patent breadth and enforceability) by 15% annually.	Measures appropriability (the ability to protect the innovation).	
O: Embed Innovation into the Organisational Architecture for Rapid Diffusion and Adaptation.	KR3: Reduce the time-to-market for innovations from concept to commercial scale by 25%.	Measures the supporting strategies required to commercialize innovation efficiently.
KR4: Increase the number of successful cross-platform (internal/external) technology transfers by 50%.	Measures the effectiveness of linking Innovation back to Architecture.	

IV. The Distinctive Capabilities Maturity Model (DCMM)

To move beyond episodic OKR measurement toward continuous evaluation, we introduce the Distinctive Capabilities Maturity Model (DCMM). The DCMM serves as the integrative framework, similar to a Risk Maturity Index or the Competencies Maturity

Index, to assess the current state of each capability on a structured, five-level scale. This framework allows the organization to benchmark progress and allocate resources more effectively (see for instance Antonucci, 2016).

The five maturity levels, inspired by the Capability Maturity Model Integration (CMMI), are:

Table 4. Operationalized Kay's framework into Distinctive Capabilities Maturity Model

Level	DCMM Name	Description	Score (out of 5.0)
1	Initial (Ad Hoc)	Capability is largely undefined, reactive, and reliant on individual heroism. Outcomes are unpredictable.	1.0 - 1.9
2	Managed (Repeatable)	Basic processes exist, often localized, but are managed and tracked. Success is repeatable, but not yet standardized across the organization.	2.0 - 2.9
3	Defined (Standardized)	The capability (e.g., the relational contract structure or reputation signalling) is clearly documented, standardized, and integrated across all relevant business units.	3.0 - 3.9
4	Quantitatively Managed (Measured)	The capability is managed using specific, data-driven metrics and OKRs. Performance is predictable and variances are understood and controlled.	4.0 - 4.4
5	Optimizing (Distinctive)	The organization is focused on continuous process improvement and adaptation.	4.5 - 5.0

DCMM Scoring and Radar Chart Visualization

The DCMM assigns a quantifiable score (Level 1.0 to 5.0) to each of the three capabilities (Architecture, Reputation, Innovation) based on the criteria in the table above and the achievement of their corresponding OKRs.

This framework is best presented visually through a **Radar Chart**, just like the Core Competencies Maturity Index. The chart plots the three DCMM scores on distinct axes, creating a unique signature for the organization's distinctiveness. A large, balanced polygon indicates high, sustainable, and well-managed distinctive capabilities.

The primary function of the DCMM is to diagnose the organization's competitive edge:

1. Identify Gaps: Low scores (e.g., Innovation at 2.1) signal an urgent need for targeted investment and a renewed OKR cycle.
2. Ensure Balance: A heavily skewed chart (e.g., high Reputation, low Architecture) suggests a brittle advantage, where a change in the market could quickly undermine the single strength.
3. Benchmark Progress: Quarter-over-quarter improvements in the DCMM score demonstrate the success of the overarching methods and the OKR execution.

V. Implementation: From Measurement to Action

The operationalization of Kay's framework is complete only when the DCMM and its supporting OKRs are fully integrated into a firm's annual cycle.

The ultimate goal of this reformulation is to make Kay's relationship-based approach a practical reality. By continuously measuring and managing Architecture, Reputation, and Innovation, the organization ensures that its superior performance is not accidental or temporary, but rather the predictable outcome of deliberate investment in its distinctive strengths.

- A high **Architecture** score guarantees a low-cost, flexible, and efficient response to market changes.
- A high **Reputation** score provides a premium in pricing and a buffer during crises, reducing the cost of sales.
- A high **Innovation** score, particularly in its **appropriability** elements, ensures that R&D investments yield proprietary value that competitors cannot easily erode.

When the Overall Weighted Average of the DCMM pushes into Level 4 (Quantitatively Managed) and approaches Level 5 (Optimizing), the organization is no longer just *performing well*; it has attained a **sustainable, measurable, and properly managed advantage**—the ultimate promise of John Kay's profound framework, now equipped with the operational rigor of modern management science.

VI. Discussion:

Plausible ways to extend John Kay's *Distinctive Capabilities*, to include Schumacher's Intermediate Technology with applications to Botany etc.

The pursuit of Sustainable Competitive Advantage (SCA) often focuses on a firm's internal resources and market position, as articulated by John Kay's concept of Distinctive Capabilities. These capabilities—rooted in Architecture (relationships/reputation), Reputation, and Innovation—offer a robust framework for long-term organizational achievements. However, in an increasingly resource-constrained world, this framework must be broadened to incorporate a fundamental principle: that economic activity must serve humanity and the planet.

This article proposes an extension of Kay's model to include E.F. Schumacher's "Intermediate Technology" (which can be conceptualized as "*development as if people mattered*"). By weaving Schumacher's focus on appropriate, human-scale, and environmentally benign technologies into the essence of an organization's distinctive capabilities, we can cultivate a truly Extended Sustainable Advantage (ESA) that is economically resilient, socially responsible, and ecologically regenerative.

Schumacher's Philosophy and the Imperative for Appropriateness

Schumacher's seminal work, *Economics as if People Mattered*, challenged the relentless pursuit of large-scale, capital-intensive technology characteristic of mainstream economics. He advocated for **Intermediate Technology**: methods and tools that are *small* enough to be accessible to local communities, *simple* enough to be maintained without highly specialized expertise, and *non-violent* toward both people and nature.

The essential tenets of this philosophy, which underscore the development of ESA, are:

- **Human Scale:** Technology should enhance human skill, not replace it, and should be comprehensible at the local level.

- **Ecological Soundness:** It must operate within natural limits and promote the regeneration of resources.
- **Accessibility:** Low budget and local material sourcing ensure widespread adoption, especially in developing countries or marginalized communities, whom often have budget constraints.

Intermediate Technology as a Distinctive Capability

When viewing Schumacher's *Intermediate Technology* through the lens of John Kay's Distinctive Capabilities, its value becomes clear:

Table 6.

Kay's Capability	**Integration with Schumacher's Technology**	**Extended Sustainable Advantage (ESA)**
Architecture	Fostering local, participatory development of technology; emphasizing knowledge-sharing over proprietary secrets.	Creates trust, community resilience, and a robust supply/knowledge network that is geographically dispersed and less prone to single-point failure.
Reputation	Being known for ethical sourcing, sustainable production, and genuine community empowerment.	Strong, authentic customer loyalty and social license to operate, appealing to socially conscious consumers and investors.
Innovation	Focusing R&D on simple, elegant, resource-efficient solutions inspired by natural processes (to learn from Nature, as Schauberger wrote).	Generates low-budget, resource-frugal, and easily adaptable technologies that circumvent high capital barriers and appeal to vast, underserved markets.

By prioritizing **"appropriate innovation,"** companies build an architectural advantage rooted in mutual local dependence and a reputational advantage of true organizational/community responsibility.

Intermediate Technology in Action: Botany and Water-Scarce Farming

The agricultural sector, particularly in water-stressed regions, offers a vital application for this Extended Sustainable Advantage model. Here, the integration of innovative, yet appropriate, technology can address both ecological necessity and market demand.

1. Botany-Inspired Techniques: Electric-Culture and Laser-Culture[2]

In drylands, reducing water consumption is paramount. Instead of relying solely on capital-intensive desalination or deep-well pumping, ESA suggests investing in technologies that *enhance plant efficiency* with minimal external input.

- **Electric-Culture:** This technique involves using low-level atmospheric electricity or electric fields to stimulate plant growth. Research suggests that it can enhance growth, improve drought tolerance, and potentially reduce the need for certain fertilizers and pesticides. While its mechanisms in botany are still being explored, its application—often involving simple wire antennae or buried conductors—is inherently **intermediate**: low-budget, low-energy (potentially solar-powered), and manageable by local farmers. This addresses water scarcity by requiring less water per unit of yield.
- **Laser-Culture (Biostimulation):** The precise application of low-power lasers at specific wavelengths to seeds or young plants has shown potential for biostimulation, leading to improved germination rates, increased vigour, and enhanced resistance to stress. Used for seed treatment before planting, this can be a simple, non-chemical way to maximize the potential of a limited water supply, embodying appropriate innovation through highly focused energy use.

[2] See for instance, our previous articles in, BPAS Journals – Botany section.

Both of these "high-tech" concepts are re-imagined through the Intermediate Technology lens: they are applied in a targeted, resource-light manner to boost the natural capabilities of the plant, rather than forcing growth through chemical or brute-force irrigation.

2. Schauberger's Implosion and the Gravitational Water Vortex

Schauberger, the Austrian forester and inventor, championed the idea of learning from nature, particularly the role of the vortex in water and air dynamics. He believed conventional technologies were "violent" (exploiting pressure and heat) while nature's processes were "implosive" (utilizing cooling, suction, and spiralling motion).

The **Gravitational Water Vortex Power Plant (GWVPP)** is a direct embodiment of this insight and an exemplary Intermediate Technology for energy generation and water remediation.

Principle: Water is channelled into a round basin where it forms a stable, powerful vortex over a central drain. This vortex drives a low-speed turbine that generates electricity. The required hydraulic head is very low (less than 3 meters).

Distinctive Advantage (Reputation & Architecture): The GWVPP is not only an energy generator but also a water health device. The gentle, spiralling flow (implosion) naturally aerates and "vitalizes" the water, minimizing damage to aquatic life (fish passage is often unharmed) and mimicking the self-healing process of a natural river. This holistic benefit of simultaneous power generation and environmental restoration create a powerful reputational advantage far surpassing conventional micro-hydro plants. Its simplicity, low maintenance, and fish-friendliness make it ideal for decentralized, local-level power architecture.

The Path to Extended Sustainable Advantage

As the world faces complex challenges of climate change, resource depletion, and growing inequality. Relying solely on the traditional model of Sustainable Competitive Advantage, which often prioritizes profit and market share above human and ecological cost, is insufficient. It is our consideration here, that Extended Sustainable Advantage

(ESA), built on the synergistic basis of Kay's Distinctive Capabilities and Schumacher's Intermediate Technology, offers a powerful way forward. By deliberately choosing to innovate for human scale and ecological health—as demonstrated by appropriate innovations like electric-culture in dryland farming and the gravitational water vortex power plant—firms can unlock unique, non-imitable sources of value. This approach moves the goal from simply *sustaining* an advantage to *extending* it through the creation of shared, regenerative value for all stakeholders: the community, the environment, and the enterprise itself. True distinction, in the 21st century, comes from mattering.

Evaluating Kay's Frameworks and Novel Innovations through the Lens of Responsible Research and Innovation (RRI)

The shift from purely competitive advantage to **Extended Sustainable Advantage (ESA)**—combining John Kay's Distinctive Capabilities with E.F. Schumacher's Intermediate Technology—is a necessary but theoretically driven step. To ground this concept in practical application and ethical accountability, we must evaluate it against the contemporary standard of Responsible Research and Innovation (RRI).

RRI, particularly promoted by the European Union, is an overarching governance framework that seeks to align the process and outcomes of R&I with societal values, needs, and expectations. Its essential methodological elements are the AREA framework: Anticipation, Reflection, Engagement, and Action (Responsiveness).

We can assess how well Kay's model and Schumacher's philosophy inherently satisfy the criteria of RRI, identifying where they complement and where they fall short.

John Kay's Distinctive Capabilities (Architecture, Reputation, Innovation)

Kay's framework, being an economic theory of firm success, primarily focuses on **appropriability** (the ability of a firm to retain the value it creates) and **sustainability** (the durability of the competitive advantage).

Table 7.

RRI Criterion	Kay's Distinctive Capabilities	RRI Alignment / Deficiency
Anticipation	Focuses on anticipating competitor moves and market shifts to maintain advantage.	Deficiency: Lacks systematic anticipation of societal, ethical, and environmental risks (e.g., unintended social displacement from innovation, or long-term ecological footprint).
Reflection	Focuses on reflection regarding internal efficiency, alignment, and competitive position.	Deficiency: Purpose of innovation is primarily profit-maximization, not the social desirability or ethical acceptability of the research itself.
Engagement	The Architecture capability is based on strong, long-term relationships with employees, customers, and suppliers.	Alignment: Strong internal and essential external engagement, but typically excludes broader civil society, NGOs, and marginalized communities that may bear the innovation's external costs.
Action (Responsiveness)	The model encourages action to build the advantage.	Partial Alignment: Responsiveness is driven by market and competitive pressure, not necessarily by a commitment to societal values beyond consumer satisfaction.

Conclusion on Kay's Model: While excellent for competitive sustainability, Kay's framework is ethically and socially neutral by design. It requires a deliberate, external commitment to RRI principles to ensure the distinctive capability itself is socially desirable.

E.F. Schumacher's Appropriate Technology

Schumacher's philosophy of “economics as if people-mattered" and "*appropriate technology*" is fundamentally normative, built on a human-centric and ecological worldview.

Table 8.

RRI Criterion	**Schumacher's Appropriate Technology**	**RRI Alignment / Deficiency**
Anticipation	Inherently anticipates and avoids negative social and ecological impacts by prioritizing small scale, low capital, and non-violence against nature.	Strong Alignment: It is a proactive risk mitigation methods against the scale-related, *capital-intensive problems of modern technology*.
Reflection	The core principle is deep reflection on the purpose of technology: *to serve human dignity, creativity, and community, not merely maximize output*.	Strong Alignment: The very choice of Appropriate Innovation is an act of ethical reflection on societal goals and the nature of "progress."
Engagement	IT is designed to be comprehensible, locally maintained, and locally owned, requiring active participation and inclusivity of the users in its design and deployment.	Strong Alignment: Its focus on decentralization and local materials/skills embodies democratic, bottom-up engagement.
Action (Responsiveness)	IT is defined by the action of adapting technology to local	Strong Alignment: The focus is on capital-saving, labor-

	conditions (e.g., resources, climate, culture), making it inherently responsive to the needs of the user community.	intensive action that responds directly to unemployment and resource scarcity in developing nations.

Conclusion on Schumacher's Model: Schumacher's Appropriate Technology approach is arguably a historical precursor to RRI. Its principles align nearly perfectly with the RRI goal of ethical acceptability, sustainability, and societal desirability, making it an ideal component for the "*Innovation*" capability in an ESA framework.

RRI Evaluation of Specific Novel Innovations

Applying the AREA criteria to the specific innovations illustrates how the ESA framework (Kay + Schumacher) can guide development, particularly for Less Developed Countries (LDCs).

Table 9. Agricultural & Water Solutions

Innovation	**Key Benefit for LDC / ESA Capability**	**RRI Evaluation (AREA)**
Laser-Culture (Seed/Plant Biostimulation)	Intermediate Innovation: Low-energy, non-chemical, high-precision boost to yield/stress resistance.	Anticipation: Low risk of large-scale ecological disruption compared to GM/chemical farming. Reflection: Ethically sound as it enhances natural plant potential. Engagement: Technology transfer must be simple enough for local technicians to maintain. Action: Adoption is quick and local.

Gravitational Water Vortex Power Plant (GWVPP)	Architecture/Reputation: Decentralized, low-head hydropower that also heals (aerates/vitalizes) the water.	Anticipation: Highly positive environmental anticipation (fish-friendly, non-violent flow). Reflection: Aligns with Schauberger's "learning from nature." Engagement: Ideal for co-ownership by small communities; promotes local energy sovereignty. Action: Directly provides clean energy and improves river health.
Confined Vortex Turbine (CVT)	Intermediate Innovation: A simple, high-efficiency micro-hydro turbine suited for low-flow/low-head sites.	Anticipation: Low environmental impact due to small scale and minimal civil works. Reflection: The focus on low-flow efficiency aligns with resource-frugality. Engagement: Design must be modular and maintainable with local workshops. Action: A cheap, reliable source of decentralized rural power.

Table 10. Energy and Waste Solutions

Innovation	**Key Benefit for LDC / ESA Capability**	**RRI Evaluation (AREA)**
Plastic Waste to Biofuel (Pyrolysis/Catalysis)	Architecture/Innovation: Turns a local pollution problem (plastic waste) into a local energy source (biofuel).	Anticipation: Highly positive on waste management; potential negative risk from air emissions (pyrolysis fumes) and end-use combustion pollution. Reflection: Addresses the ethical

		dilemma of non-biodegradable waste. Engagement: Requires engagement with waste pickers and local government for efficient feedstock collection; ensures fair value capture. Action: Creates circular economy jobs and reduces dependence on imported fossil fuels.
New Fusion Energy Installation (PT-Symmetric Crystal Potential)	Innovation (Long-term): Breakthrough in clean, near-limitless energy via novel material science.	Anticipation: Extremely high positive potential (zero-carbon, high power) but significant long-term safety, regulatory, and proliferation risks typical of advanced nuclear/fusion. Reflection: The purpose is clean energy, but the scale (Kay) may be too large to align with IT/local architecture unless modular. Engagement: Requires massive, international, and transparent public engagement from the start due to high complexity and potential societal transformation. Action: Must build in robust governance (RRI) from the outset to manage ethical dilemmas.

The Synergy: ESA as an RRI-Conscious Approach

The evaluation reveals that an approach based purely on Kay's framework risks developing innovations that are competitively strong but socially and ecologically harmful. Conversely, Schumacher's Intermediate Technology approach inherently satisfies RRI criteria but may struggle to achieve the global scale and financial appropriability needed to attract mainstream capital and compete with high-tech solutions.

The proposed Extended Sustainable Advantage (ESA) model resolves this tension by using Kay's framework to provide the structure, finance, and durability while embedding Schumacher's IT as the RRI-mandated DNA of the Innovation and Architecture capabilities.

- ESA's Innovation → RRI: The firm only pursues innovations (like the GWVPP or simple laser-culture) that are not just clever, but appropriate, ensuring ethical acceptability (no environmental harm) and societal desirability (local employment, resource healing).
- ESA's Architecture → RRI: The structure is built around long-term, trust-based relationships (Kay) that extend to include the local community as a true stakeholder and co-creator (Schumacher/RRI), ensuring genuine engagement and local ownership.

By embracing RRI, the ESA framework transforms competitive advantage from merely achieving high profits into an ambition for co-created, resilient societal value, making the business model itself resistant to ethical backlash or regulatory failure—a truly sustainable form of distinction.

Case Example: Applying RRI to Intermediate Technology - Action Plans

The combination of E.F. Schumacher's Intermediate Technology and the Responsible Research and Innovation (RRI) framework provide a robust blueprint for developing ethical, sustainable, and locally appropriate solutions, particularly in Less Developed Countries (LDCs). Intermediate Technology's intrinsic focus on human scale, simplicity,

and ecological non-violence directly addresses the *ends* of RRI (societal desirability). The RRI AREA framework (Anticipation, Reflection, Engagement, Action) then formalizes the *means* to achieve these ends.

Here an example of detailed RRI Action Plans for key innovations, designed specifically for LDCs, emphasizing Schumacherian principles.

RRI Action Plan for Laser-Culture in Dryland Farming

Innovation: Low-power laser biostimulation of seeds/plants to enhance germination, growth, and drought resistance, reducing the need for water and chemical inputs.
Schumacher Principle: Highly sophisticated science translated into a simple, low-budget, capital-saving application, increasing the productivity of land and labour.

Table 11.

RRI Criterion	Action Plan for Laser-Culture in LDCs	Schumacher/Intermediate Tech Alignment
Anticipation	A1: Hazard Mapping & Risk Assessment: Anticipate risks related to the technology's application. Focus on potential eye safety hazards for operators and long-term biological effects (e.g., unintended genetic shifts, ecological niche changes).	Action: Develop solar-powered, enclosed laser treatment units that minimize external risk and are designed for use by non-specialist local technicians.
	A2: Socio-Economic Impact Forecasting: Model how yield increases might affect local food prices, market structures, and labour	Action: Focus research on low-budget, durable equipment designs (capital-saving) that can be

	requirements. Anticipate the risk of small farmers being excluded by requiring high upfront costs.	collectively owned by farmer cooperatives (architecture).
Reflection	R1: Ethical Purpose Review: Continuously reflect on the essential purpose: Is the goal to enrich a few large agribusinesses or to enhance local food sovereignty and resilience?	Action: Adopt Open-Source Hardware/Design for the laser treatment units to ensure the technology remains accessible and adaptable to diverse local crops and climates.
	R2: Resource Frugality Check: Evaluate the energy source and materials. Does the device consume minimal energy and use locally repairable components?	Action: *Mandate the use of locally available materials* for casing and focusing on simple, modular electronic components for easy maintenance (appropriate technology).
Engagement	E1: Community Co-Design: Engage women and local farming elders in testing different laser wavelengths and exposure times for indigenous seeds and local staple crops.	Action: Establish local Technology Transfer Centres run by trained local individuals who provide the service, ensuring the knowledge is embedded in the community, not just the hardware (building local skill).
	E2: Transparency & Training: Clearly communicate the science behind biostimulation (avoiding high-tech jargon) and provide	Action: Training manuals must be in local languages and based on visual, step-by-step instructions—a practical hallmark of Intermediate Technology.

	hands-on, vocational training.	
Action (Responsiveness)	Ac1: Iterative Adaptation: Establish a feedback loop where initial field trial results and farmer observations (e.g., "The treatment works better on millet than on sorghum") lead to immediate, simple design or application parameter adjustments.	Action: *Utilize local extension agents to collect and relay field data* directly to the engineering team for rapid, context-specific changes.
	Ac2: Governance & Regulation: Work with local authorities to create simple certification standards for locally-made laser units to ensure safety without creating prohibitive regulatory barriers.	Action: *Champion a decentralized production model*, supporting local artisans/technicians to manufacture and repair the equipment.

The aforementioned RRI Action Plans give simple example to convince us that **Intermediate Technology** provides the **ethical starting point** for innovation, while the **RRI framework** provides the **governance and process** to ensure that this technology remains appropriate over time and through deployment. For LDCs, this combined approach offers a compelling path toward development that is not only economically viable but also socially just and environmentally restorative, fulfilling Schumacher's vision that humanity must be at the centre of the economic equation.

VII. Concluding remark

John Kay's Distinctive Capabilities Framework is arguably one of the most insightful theories on the source of enduring organizational achievement based on capabilities. By moving the discussion from significant assets to relationships, it highlights the non-replicable complexity of human and organizational interactions.

The challenge of translating this wisdom into day-to-day modern execution is overcome by the Distinctive Capabilities Maturity Model (DCMM). By rigorously operationalizing Architecture, Reputation, and Innovation through the definition of Objectives and Key Results (OKRs) and placing them on a measurable, five-level maturity index, we have transformed an essential concept into a powerful management tool.

When the Overall Weighted Average of the DCMM pushes into Level 4 (Quantitatively Managed) and approaches Level 5 (Optimizing), the organization is no longer just *performing well*; it has attained a sustainable, measurable, and properly managed advantage—the ultimate promise of John Kay's profound framework, now equipped with the operational rigor of modern management science.

This synthesized framework provides the required operationality and measurability, enabling organizations to not only understand what makes them special but, critically, to consistently measure, manage, and optimize the very basis of their distinctive capabilities.

We also discuss in this article, among other things, how to extend John Kay's Distinctive Capabilities, to include Schumacher's Intermediate Technology with applications to Botany etc, which aim to make sustainable advantages more in tune with Schumacher's philosophy of economics as if people-mattered in order to achieve sustainable ecosystem as well.

As the world faces complex challenges of climate change, resource depletion, and growing inequality. Relying solely on the traditional model of Sustainable Competitive Advantage, which often prioritizes profit and market share above human and ecological cost, is insufficient. It is our consideration here, that Extended Sustainable Advantage

(ESA), built on the synergistic basis of John Kay's Distinctive Capabilities and Schumacher's Intermediate Technology principles, offers a powerful way forward.*

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Remark on Regenerative Medicine and Potential Utilization of Low-Intensity Laser Photobiomodulation to Activate Human Stem Cells

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ABSTRACT
Recently, a friend of one of these writers told her story of using one of a healthcare product to activate her stem cells as part of regenerative medicine. Regenerative medicine is a field of medicine that seeks to repair or replace damaged or diseased tissues and organs. This can be done through a variety of methods, including stem cell therapy, tissue engineering, and gene therapy. This is a short review article on this rapid field called regenerative medicine, in particular via a new method called photobiomodulation, especially by virtue of low-intensity laser pen treatment. Hopefully it will attract more research in this interesting direction, both for laser/photobiomodulation to boost plant growth as well as in healthcare (see also: Christianto & Smarandache, 2022; Christianto, 2023).

KEYWORDS: *Provide the keywords*

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INTRODUCTION

As we know, regenerative medicine is a field of medicine that seeks to repair or replace damaged or diseased tissues and organs. This can be done through a variety of methods, including stem cell therapy, tissue engineering, and gene therapy. In the meantime, low-intensity laser photobiomodulation (PBM) is a non-invasive therapy that uses light to stimulate the body's natural healing processes. PBM has been shown to be effective in a variety of conditions, including pain relief, wound healing, and muscle recovery.

PBM works by activating specific molecules in cells called chromophores. These chromophores absorb the light energy and convert it into chemical energy, which can then be used to stimulate cellular processes. For example, PBM can increase the production of ATP, which is the body's main source of energy. It can also reduce inflammation and pain.

This is a short review article on this rapid field called regenerative medicine, in particular via a new method called photobiomodulation, especially by virtue of low-intensity laser pen treatment. Hopefully it will attract more research in this interesting direction, both for laser/photobiomodulation to boost plant growth as well as in healthcare.

What is PBM and how it is related to activating human stem cells

PBM has been shown to be particularly effective in stimulating the growth and differentiation of stem cells. Stem cells are undifferentiated cells that have the potential to develop into any type

of cell in the body. This makes them ideal for use in regenerative medicine.

In recent years, there has been growing interest in the use of PBM in regenerative medicine. PBM has been shown to promote the growth and differentiation of stem cells in vitro and in vivo. It has also been shown to accelerate the healing of wounds and injuries.

As research in this area continues, PBM is likely to become an increasingly important tool in regenerative medicine. It has the potential to revolutionize the way we treat a variety of diseases and injuries.

Here are some specific examples of how PBM is being used in regenerative medicine today:
- *Wound healing*: PBM has been shown to accelerate the healing of wounds, including diabetic ulcers, burns, and surgical incisions.
- *Muscle recovery:* PBM can help to reduce muscle pain and inflammation, and it can also speed up muscle recovery after exercise.
- *Osteoarthritis*: PBM has been shown to reduce pain and inflammation in people with osteoarthritis. It may also help to slow the progression of the disease.
- *Neurological disorders*: PBM is being studied as a potential treatment for a variety of neurological disorders, including stroke, spinal cord injury, and multiple sclerosis.

PBM is a promising new technology with the potential to revolutionize the field of regenerative medicine. As research in this area continues, we can expect to see even more innovative applications for PBM in the years to come. In the next sections, we will discuss how PBM can be related to a few new fields, such as possible new methods to activate stem cells in particular X39 and also how it can also possibly be used as *bioshield* to protect human bodies against harmful 5G radiation effects.

Whether such an activation of stem cells method is related to X39

X39 is a dietary supplement that is marketed as a way to improve overall health and well-being. It is said to contain a blend of amino acids, minerals, and vitamins that can help to boost the immune system, reduce inflammation, and protect cells from damage.

Some people believe that X39 may also be helpful in activating stem cells and providing a bioshield against harmful EMF fields. However, there is currently no scientific evidence to support this claim.

One study, published in the journal "*Oxidative Medicine and Cellular Longevity*" in 2020, found that X39 may help to protect cells from damage caused by oxidative stress. Oxidative stress is a condition that occurs when there is an imbalance between antioxidants and free radicals in the body. Free radicals are unstable molecules that can damage cells. Antioxidants help to neutralize free radicals, preventing them from damaging cells.

The study found that X39 increased the levels of antioxidants in the blood and protected cells from damage caused by free radicals. However, the study did not look at the effects of X39 on stem cells or EMF exposure.

Another study, published in the journal "*Frontiers in Immunology*" (2021), found that X39 may help to increase the number of stem cells in the body. The study found that X39 increased the number of stem cells in the bone marrow and blood. However, the study did not look at the effects of X39 on EMF exposure.

Overall, there is several evidences to suggest that X39 may have beneficial effects on the body. However, more research is needed to determine whether X39 can actually activate stem cells and provide a bioshield against harmful EMF fields.

Here are some additional things to consider:

- X39 is a relatively new product, and there is not a lot of long-term research on its safety and effectiveness.
- X39 is expensive, and it is not clear if it is worth the cost.

There are other, more well-studied dietary supplements that may be more effective in activating stem cells and protecting against EMF exposure.

If you are considering taking X39, it is important to talk to your doctor first.

Plausible method for Activation of Stem Cells for Bioshield to protect human bodies Against Harmful EMF (5G towers and others) Fields

Electromagnetic fields (EMFs) are all around us, from the radio waves that bring us music and news to the cell phone signals that keep us connected. While most EMFs are harmless, some research suggests that exposure to high levels of EMFs, such as those emitted by 5G towers, could have negative health effects (cf. Mercola, Horowitz etc).

One way to protect ourselves from the harmful effects of EMFs is to activate stem cells. Stem cells are the body's master cells, and they have the potential to develop into any type of cell in the body. When stem cells are activated, they release a variety of molecules that have antioxidant and anti-inflammatory properties. These molecules can help to protect cells from damage caused by EMFs.

One simple method for activating stem cells is to use a **laser pen**. Laser pens emit low-level laser light that can penetrate the skin and stimulate stem cells. Studies have shown that laser pen biomodulation can increase the number and activity of stem cells in the body.

In one study, researchers used a laser pen to treat patients with chronic pain. The patients who received laser pen treatment had a significant decrease in pain and an increase in the number of stem cells in their blood.

Another study looked at the effects of laser pen biomodulation on patients with Alzheimer's disease. The patients who received laser pen treatment had a slower rate of cognitive decline and an increase in the number of stem cells in their brains.

These studies suggest that laser pen biomodulation is a safe and effective way to activate stem cells. This can help to protect the body from the harmful effects of EMFs and other environmental toxins.

DISCUSSION

Comments by R.N. Boyd

"I have tried several experiments with the X39 and X49 patches. They are designed to work for only a few hours, then a new patch is supposed to be applied.

As to whether or not they activate stem cells, I have not seen compelling incontrovertible evidence supporting this concept.

Using a laser applied to bioactive points on the meridians, such as at the outside of the wrist, Russian research has repeatedly demonstrated that lasers thus applied, permanently eliminate all 3 kinds of diabetes.

Peoter Garjajev demonstrated the fact that light carries information which it obtains when it passes through any substance that the laser light passes through, or reflect off of. This was used in *Wave Genetics* researches.

Huping Hu demonstrated that the information of an anesthetic can be conveyed into water by passing the light from a high intensity flashlight, through a transparent bottle of anesthetic, then into a transparent container of water. Everyone who drinks that water becomes clinically anesthetized in only a few minutes, with no side effects, nor any after effects, with zero toxicity.

One can place any medication or herbal mixture into the hand and irradiate it with laser light. The laser light obtains the information from the substance and passes that information into the blood stream, where it has the same effects as the medication or herbal mixture, in the informational sense (not the chemical sense).

This ability of light to pass information from substance, into the blood stream, can be used to eliminate addictions. or at least drastically reduce the cost of many kinds of addictions, as only one sample of the substance can be used repeatedly, for months on end.

This method can also be used to reduce the cost of expensive pharmaceuticals, as only one

sample can be used over and over to obtain similar effects.

However, *Divinity* becomes directly involved in light-copying of substances and prevents any damaging influences from being copied into the blood stream using light as the information carrier.

Experiments with laser-copying have demonstrated that many pharmaceuticals will only partially copy, or not copy at all, because they produce deadly toxic and/or damaging effects on the human being when consumed or injected.

Thus *Divinity will not allow any toxic or damaging influences to be copied into the blood stream by way of light carriers*. Some pharmaceuticals cannot be copied at all, because they are entirely toxic and are prevented from being copied informationally, then carried into the blood stream by light, by *Divinity*.

Magnetic and/or electric fields have never been tested to see if they can copy information, However, all wave-forms, from any source, will result in information being carried into the body, by the wave-form. This is all the more reason to get rid of microwave communications.

Last but not least, it cannot be overstated, that the aforementioned effects of wave-forms and information carrying method of light and its interaction with water substance in human bodies, can be associated to significant role of the fourth phase of water, and other ramifications of biochemistry role that water play inside human bodies; cf. Pollack (*The fourth phase of water*. Seattle: Ebner & Sons Publ.); Zamperini (*LES ÉNERGIES SUBTILES*)."

CONCLUDING REMARKS

In the present review, we discussed shortly how simple methods such as laser pen (or it is known by other name: LLLT/*photobiomodulation*) can be utilized and developed further possibly to reduce harmful EMF effects or as bioshield against damaging effects of 5G towers etc. toward human health and also other living entities as well.

Several studies suggest that *simple device such as laser pen* biomodulation is a safe and effective way to activate stem cells. This can help to protect the body from the harmful effects of EMFs and other environmental toxins. We also outline a number of suggestions to use that laser pen to activate stem cells.

Acknowledgement
These writers gratefully appreciate dr. Toshiro and Mrs. Mari Kashiwagi for discussions related to how to protect humans and environment against harmful effects of 5G fields and other EMF technologies. The present short remark has been written partly with assistance by ChatGPT/OpenAI.

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A Review on Electroculture, Magneticulture and Laserculture to Boost Plant Growth

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Electroculture, magneticculture, laserculture, plant growth, agriculture performance

Abstract
While several reviews on potential applications of electroculture are available, in this survey we discuss these issues from history, starting from earliest experiments by Ross. And in the last section, we discuss possible application of laserculture, another form of potential improvement. It is our hope that what we present here may be found useful for improving agricultural performance in many countries, as well as reducing dependence on fertilizer.

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INTRODUCTION

To cause harvests to become bigger and quicker has been the essential worry of agribusiness for a long time. All strategies for development methods what's more, advances have been created to satisfy this point; from basic yield pivot to complex manufactured composts. Another development innovation found in farming is the use of power and attraction that can speed up development rates, increment yields, and improve crop quality. That innovation is called *electroculture*. Electroculture can shield plants from infections and creepy crawly and moreover diminish the prerequisites for manure or pesticides. Ranchers can develop greater and better harvests in less time, with less exertion (Barinov, 2012).

Harvest yields and quality are improved similarly. The energies are applied to the seeds, plants, soil or the water and supplements. Also, in this way huge loads of food can be developed in a quarter section of land or less, in gardens, on galleries, housetops, in window boxes or aqua-farming or permaculture. That converts into immeasurably expanded benefits for cannabis cultivators specifically, and some other culturist. Aqua-farming frameworks are obviously appropriate for electroculture (Nelson, 1982).

Also, there's more - a whole lot more, as the Russian scientists B.R. Lazarenko and I.B. Gorbatovoskaya detailed (with paraphrasing): "Reports that the attributes obtained by the plants in electrically treated soils are sent by legacy to the third era are especially fascinating. Affected by the electrical flow, the mathematical extents between hemp plants of various genders was changed by examination with the control to give an expanded number of female plants by 20-25%, regarding a decrease in the power of the oxidative cycles in the plant tissues." (Lazarenko & Gorbatovoskaya, 1966).

METHODOLOGY

In this article, we used methodology of literature survey. In this short literature survey, we discuss some methods which may have great impact in terms of plant growth and also reduce time needed to grow.

We tried to include not only literature from the Western publications, but also from Eastern Europe and also Asian authors, because some of the ideas are quite old.

HISTORY OF ELECTROCULTURE

Exploratory investigation of the impacts of power on plant development started in 1746, when Dr. Maimbray of Edinburg treated myrtle plants with the yield of an electrostatic generator, consequently upgrading their development and blooming. After two years, the French abbot Jean Nolet discovered that plants react with sped up paces of germination and in general development when developed under charged terminals (Nelson, 1982).

Starting in 1885, the Finnish researcher Selim Laemstrom tried different things with an airborne framework controlled by a Wimshurst generator and Leyden containers. He tracked down that the electrical release from wire focuses invigorated the development of yields like potatoes, carrots, and celery for an normal increment of about 40% (up to 70%) inside about two months. Nursery developed strawberry plants delivered ready natural product fifty-fifty the standard time. The yield of raspberries was expanded by 95%, what's more, the yield of carrots was expanded by 125%. Yields of cabbage, turnips, and flax, be that as it may, became preferred without jolt over with it. The Laemstrom framework included a flat radio wire suspended sufficiently high to allow furrowing, weeding and water system. The voltage applied to the reception apparatus differs from 2 to 70 KV, contingent upon the stature of the recieving wire. The current was around 11 amps.

In April 16, 1900, there was a **Nature** magazine edition, mentioned about electroculture:



ELECTRO-CULTURE.

Maryland Geological Survey.

Figure 1: Coverage on electroculture in *Nature*, 1900.

Another old book on electroculture was published by EC. Dudgeon:

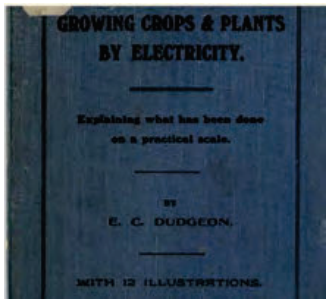
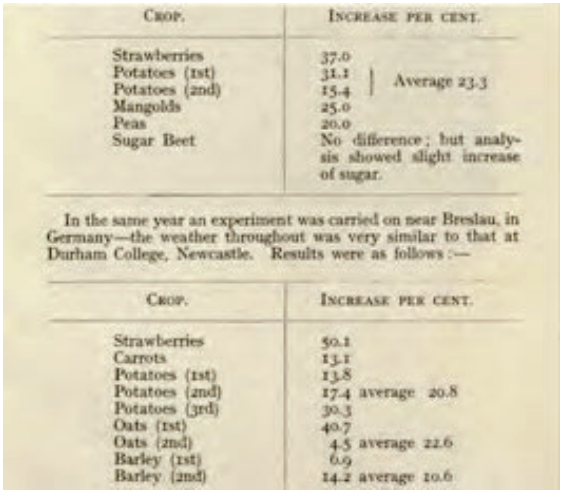


Figure 2: EC Dudgeon's book (unknown date)

Among other things, Dudgeon reported experiments by Prof. Laemstrom:



Crop.	Increase per cent.	
Strawberries	37.0	
Potatoes (1st)	31.1	Average 23.3
Potatoes (2nd)	15.4	
Mangolds	25.0	
Peas	20.0	
Sugar Beet	No difference; but analysis showed slight increase of sugar.	

In the same year an experiment was carried on near Breslau, in Germany—the weather throughout was very similar to that at Durham College, Newcastle. Results were as follows :—

Crop.	Increase per cent.	
Strawberries	50.1	
Carrots	13.1	
Potatoes (1st)	13.8	
Potatoes (2nd)	17.4	average 20.8
Potatoes (3rd)	30.3	
Oats (1st)	40.7	
Oats (2nd)	4.5	average 22.6
Barley (1st)	6.9	
Barley (2nd)	14.2	average 10.6

Figure 3: Crop growth improvement, EC Dudgeon (unknown date)

In 1909, the Swiss minister J.J. Gasner acquired comparable outcomes with his replication of Laemstrom's work. Likewise that year, Prof. G. Stone showed that a couple of sparkles of friction based electricity released into the dirt every day expanded soil microbes up to 600%.

During the 1920s, V.H. Blackman revealed his examinations with an aeronautical framework like that of Laemstrom. He applied 60 volts DC/1 milliamp through 3 steel wires each 32 ft long and suspended 6 ft separated and 7 ft high on posts. The course of action expanded yields about half for a few plant types (Nelson, 1982).

Wet soil improves current stream. Electrocultured plants need about 10% more water than control plants in light of the fact that the charged water is sweated more quickly than under typical conditions.

ANTENNA SYSTEMS

As discussed by Barinov, it is also possible to implement antenna systems to improve plant performance. The French rancher Justin Christofloreau stood out in 1925 with his contraption to gather barometrical energy for his yields. Clover treated by his strategy grew 7 feet high. Christofloreau's contraption comprised of a 25-ft wooden post; at the top was a metal pointer adjusted north-south, and a reception apparatus. Copper and zinc strips were bound together to produce power from sun based warmth. A few of the shafts were set around 10 ft separated, and the wires driving from them reached out about 1000 yards. Christofloreau asserted that the amassed power obliterated parasites and advanced advantageous synthetic cycles in the dirt (Barinov, 2012).

In 1924, Georges Lakhovsky contrived his Oscillator Circuit, a one-become copper curl with covering closes isolated by a hole. Capacitance creates swaying flows that advantage the plants. The ring is upheld by an encasing like a plastic bar. This amazingly straightforward course of action animates plant development.

Other setups likewise upgrade plant development. A funnel shaped loop of solid wire twisted with 9 turns (counter-clockwise in the Northern Hemisphere, clockwise in the Southern), when stuck in the ground around 1 ft north of a plant, will gather barometrical power. Associate a wire from the fence to a metal bar close to the plants. A television reception apparatus likewise can be utilized. Rebar can be sunk into the ground at each end of a column of plants, associated by an uncovered wire under the dirt or potentially noticeable all around. A north-south direction will exploit geomagnetic extremity (Butchbaker 1976).

More Recent Development: Solar Powered Electroculture

As discussed by E.M. Reyes et al., supporting the idea of power's part in plant development incitement and utilization of sun oriented innovation, the scientist planned and built up a venture that would help increment the development of plants without losing their quality and nourishment. Electroculture with appropriate watering framework will assist the plants with developing. Besides, the convenient sun based force supply might be utilized to control up the undertaking for a more financially savvy activity (Reyes *et al*, 2019).

Moreover, they concluded as follows (with paraphrasing):
"An all around planned electroculture strategy significantly affects family pay. As seen in the investigation, the ordinary gathering season of pechay plants is diminished by seven days. This implies decrease in the utilization of water, composts and bugs controls. Also, electroculture procedure outlines low support and working expense that best fits for each ranchers developing vegetables plants. It will give ranchers an opportunity to develop great yields quicker, lessen ecological issues brought about by utilizing natural composts what's more, increment their pay." (Reyes *et al.*, 2019).

MAGNETICULTURE

"Magneticulture", as the name suggests, utilizes attractive fields from mineral magnetite (Fe_3O_4), lasting magnets, or electromagnets to influence plant digestion. Spread magnetite in a ring around the roots, or in a north-south line. The indistinct attractive field will improve the germination and resulting advancement to different degrees relying upon the plant, developing conditions, and the sort, extremity, and strength of the attractive field (Nelson, 1982).

Agronomist Yannick van Doorne has built up a "*attractive radio wire*", a chamber loaded up with beeswax and magnets, wrapped with a curl, and electrostatically charged. The gadgets are set at the finishes of lines furrowed north and south, and associated with electrifies steel wire.

The resultant yields are nutritious, and heavenly, and three to multiple times greater than expected since the most recent ten years in similar fields."

The technique created by Pearl Eitan (Patent IL31428) requires the utilization of 100 lb of magnetite/section of land with an electrostatic charge. The outcome is "*resistivity to subfreezing temperatures and creepy crawlies, expanded organic product size, yield, development rates, and expanded number of crops/year*."(Nelson, 1982).

Laserculture: Effects of UV-B radiation, Laser irradiation, and LED lighting
Apart of the aforementioned methods to improve plant growth, there are other methods based on ultraviolet (UV-B) and laser/LED irradiation. We will discuss these possible treatments one by one.

As reported by Zuk-Golaszewska et al:
"In the investigation directed in the nursery, the various portions of UV-B radiation applied to the two species initiated changes in leaf and plant morphology. It was an abatement of plant tallness, new mass of leaves, shoots and roots just as leaf region. Additionally, it caused the leaf twisting in both of the species. The huge contrasts between the two are in the considered attributes were basically because of the tillering capacity of the species. The substance of chlorophyll shifted impressively. The normal estimations of leaf greenness (SPAD units) for oats were around 43 while for green foxtail 32, separately. U-VB didn't diminish leaf weight proportion, shoot dry matter, shoot to root proportion and leaf territory proportion."(Zuk-Golazewska et al., 2003)

Moreover, with regards to laser applications in agriculture, M. Hasan wrote that laser innovation can be a helpful choice to be joined into frameworks of rural creation. Accordingly, more consideration was given as of late to actual elements that might be appropriate to preparing of planting material. To guarantee a high seed execution, different techniques for handling are utilized, including substance arrangements, for example, seed immunization by synthetic material and development controllers, just as actual components, including laser light and attractive fields. Pre-planting treatment was

applied to invigorate the seeds to all the more likely sprout and fill quicker in different planting conditions. The utilization of actual elements for controlling the impact of organic conduct during advancement and capacity of various societies is a cutting edge pattern in joining the strengthening of plant innovations with environmental necessities (Hasan *et al.*, 2020).

The mechanism of LASER improvement of plant growth is outlined as follows: The premise of the incitement component in any plant physiological stage is the synergism between the enraptured monochromatic laser bar and the photoreceptors that, when set off, initiate various organic responses. There are numerous realities that demonstrate the biostimulating activity of laser radiation on different organs and tissues in creatures and plants. The plants assimilate light through their photoreceptors (Hernandez *et al.*, 2010).

And last but not least, there is also PARUS technology, which uses LED lighting to improve plant growth processes.[1]

All of these seem to suggest that laser /UV-B radiation along with LED lighting are very promising alternatives too in order to boost plant growth.

CONCLUDING REMARK

In this short literature survey, we discuss some methods which may have great impact in terms of plant growth and also reduce time needed to grow. Those methods include electroculture, antenna systems and also magneticculture.

Apart of the aforementioned methods to improve plant growth, there are other methods based on ultraviolet (UV-B) and laser/LED irradiation. We discuss these possible treatments one by one.

[1] Parus technology. Plants grow with LED Lighting. http://www.parus.co.kr

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How Violation of Newton's Third Law Can Pave Way to New Space Propulsion Mechanism via Optical Diametric Drive Experiment

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ABSTRACT In our initial paper discussing plausible steps toward workable warp drive machines. The following article express our view on this debate. While there are still objections toward existing *warp drive* proposals, such as by G. Landis, Harold White *etc*., because they are all based on GTR, nonetheless we think it is possible by starting to see if it is possible to deviate from Newton's third law. And we discuss possible a propulsion method based on negative masses, and discuss how optical diameter drive can be first step for realistic lab-scale version of negative mass propulsion.

KEYWORDS Negative Mass Propulsion, Negative Mass Particle, Phonon Roton Superfluidity, Lab-Scale Warp Drive Experiment, Optical Diameter Drive Experiment.

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NEWTON'S THIRD LAW

Despite normally we think that action-reaction law is given, actually in some cases it can be violated, just like violation of Pauli principle (voP), perhaps we can call it with term: violation of Newton 3rd law (voN3L). In stating Newton's Third Law, we have assumed tacitly that (a) All forces are two-body forces, i.e. they act between a specified pair of objects and are not influenced by the presence of other objects; and (b) The net force on object i is the vector sum of the individual forces actingon it from all the other objects in the universe.

According to Ivlev *et al*. (2015), "There is a variety of situations in which Newton's third law is violated. Generally, the action-reaction symmetry can be broken for mesoscopic particles, when their effective interactions are mediated by a nonequilibrium environment."[1]

Terletsky/Winterberg's concept of negative mass particles

While only recently the experimental vindication of existence of negative mass particles are reported (see *Nature Communication*, 2021), actually it can be traced back to an eminent physicist from MSU, Yakov Terletsky, and also the late F. Winterberg (from University of Nevada).

Actually Terletsky wrote in his book, paradoxes in relativity theory, as follows:

"Along these lines, assuming that the parts of the vector are taken to be ar-
bitrary genuine numbers, then, at that point, equation concedes to three essen-
tially unique actual frameworks:
1. systems with positive proper mass, i.e., M2 ~ 0, E > 0;
2. systems with negative proper mass, i.e., M2 ~ 0, E < 0;
3. systems with an imaginary proper mass, i.e., M2 < o.

Thus, the system of the hypothesis of relativity advertisement
mits three sorts of basically various frameworks of which just sys-
tems of the principal kind are viewed as actually genuine."[7]

Although the notion of *negative mass* is quite controversial in the field of gravitation and cosmology studies, and by numerous authors its existence is forbidden, actually Pollard and Dunning-Davies show that it is possible to appear in Nature. [8]

Later on F. Winterberg, who was one of last finest students of W. Heisenberg, puts forth possible existence of phonon-roton model of superfluidity which are composed of positive and negative mass particles, which he coined as "*Planckions*."[8]

Although possibility of negative mass propulsion method has been discussed by several authors, including Winterberg [9]; among other things he wrote in abstract:
"Schrödinger's examination of the Dirac condition gives a clue for the presence of negative masses taken cover behind certain masses. Be that as it may, their utilization for impetus by decreasing the idleness of issue for instance, in the restriction of plainly visible bodied with zero rest mass, relies upon a specialized answer with the expectation of complimentary them from their detainment by sure masses. Apparently there are fundamentally two different ways this may be accomplished:
1. By the utilization of solid electromagnetic or gravitational fields or by high molecule energies.
2. Via looking for places in the universe where nature has effectively done this detachment, and from where the negative masses can be mined.
The first of these two prospects is for all viable means barred, since, in such a case that conceivable at all, it would rely upon electromagnetic or gravitational fields with strength past what is actually feasible, or on very huge in like manner not achievable molecule energies. As to the second chance, it has been seen that non-baryonic cold dim matter will in general aggregate close to the focal point of cosmic systems, or spots in the universe which have a huge gravitational potential
well "[9]

Although there are others who also investigate negative mass propulsion such as Robert L. Forward and Martin Tajmar and his team at Dresden University [10][11]; to these authors, an appropriate term is: "*Terletsky-Winterberg negative mass propulsion mechanism*." Nonetheless, to our knowledge we are still far to be in "operational scale" of negative mass propulsion engine as envisaged by F. Winterberg.[1]

Fortunately, there are recent lab-scale experiments using optical diametrical drive which made use of negative mass particle into practical lab experiments.

[1]This author is forever indebted to Prof. F. Winterberg who once sent his book, around 2010/2011, and to encourage him to continue further our investigation on superfluid phonon roton model of the Cosmos. But the book that he sent, with title something like: "Finititude theory of particles" was lost from trace.

OPTICAL DIAMETRIC DRIVE PROSPECT

Isaac Newton in his *Principia* expressed that for any activity there is an equivalent and inverse response. The consequences of this law of movement are presently being rethought by a group of specialists from the University of Central Florida and Germany who as of late completed an optical test that one day might help lead to new impetus frameworks.

"This strange interaction, which includes the idea of negative mass, imitates the conduct of a polar drive," said Professor Demetrios Christodoulides of UCF's College of Optics and Photonics. "Despite the fact that thoughts of this sort have been around for a long time, they have never been effectively sought after on the grounds that mass in nature is consistently a positive amount." Diametric drive alludes to the chance of an independent, space-impetus motor that works without the requirement for any outside fuel.

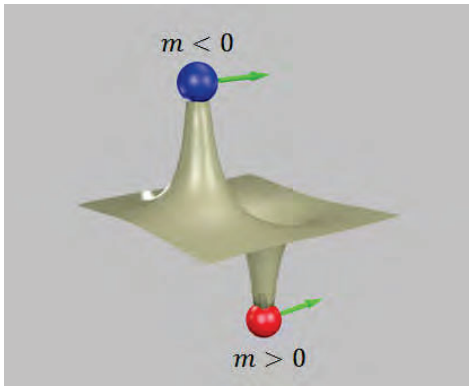


Figure 1: Diametric drive illustration [1]

The review "Optical polar drive speed increase by means of activity response balance breaking" as of late distributed on the site of *Nature Physics* and was important for a venture with other accomplice colleges. Mohammad-Ali Miri, a UCF graduate understudy in the Center for Research and Education in Optics and Lasers (CREOL), likewise took part in this work.

Another experiments, as reported by Pei *et al*.: "optical self-accelerating state driven by nonlinear coherent interaction of its constituting components with opposite "mass-sign". The coherent propulsion, highly immune to initial phase conditions, is surprisingly enhanced comparing to its incoherent counterpart."[4]

Acknowledgement
The present article is dedicated to VC's former professors at IGC, RUDN, including Prof. A. Yefremov (Director, IGC), Prof. Yu P. Rybakov, and also Prof. Yu Vladimirov. This writer can recall, back then to Dec. 2008 when he just arrived to Moscow and started to attending class of Prof. Yu Vladimirov, he asked this author: "*Where do you come from?*" I replied: "*Indonesia*." And the former professor replied: "*Ah, Indonesia? I remember Soekarno*." Although, in this article I don't cite any of his many publications on cosmology etc., but I recall that Terletsky also belonged to Moscow State University, as also great physicists like Prof. Lev Landau and Prof. Bogoliubov. MSU belongs to one of the best universities for physics studies in the world. Moreover, that simple comment by Prof. Vladimirov has motivated, somewhat in strange way, to do small experiment in Moscow streets, during early may 2009. In essence, Prof. Vladimirov helped him to become just an Indonesian scientist, not just trying to mimic someone else's figure. That is what he can say about Prof. Yu Vladimirov, he is one of such great teachers. To quote Plutarch: "*The mind is not a vessel to be fulfilled, but a fire to be rekindled*." [3]

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VC, FS

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Remark on how to solve Maxwell-Dirac Isomorphism problem in a Realism Interpretation of Wave Mechanics based on Dirac equation

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Abstract From time to time, some eminent physicists began to ask: What is the reality behind quantum mechanical predictions? Is there realism interpretation of Quantum Physics? This paper is intended to explore such a possibility of realism interpretation of QM, based on known problem called isomorphism between Maxwell-Dirac equation, which does not exist according to a number of theoretical physicists. In other article, we discuss a derivation of Maxwell equations in Quaternion Space (Progress in Physics, 2010), and it can be shown that this new theory is capable to describe electromagnetic in a harmonious way between classical and quantum theory in elegant and straightforward way. Second, we discuss how we can arrive at one-to-one correspondence between Maxwell and Dirac equations by starting again with a derivation of Maxwell equations using Dirac decomposition, introduced by Gersten (1999). Thirdly, we also discuss a second plausible approach, by describing Geometric Algebra in real Euclidean 3D space. Further observations are of course recommended in order to refute or verify some implications of this proposition.

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1 Introduction

From time to time, some eminent physicists began to ask: What is the reality behind quantum mechanical predictions? Is there realism interpretation of Quantum Physics? For instance, in a recent paper Prof. G. 't Hooft asks: "The author has made his own analysis of the known facts, and came to the conclusion that the Copenhagen doctrine, that is, the consensus reached by many of the world experts at the beginning of the 20th century, partly during their numerous gatherings in the Danish capital, has it almost right: there is a wave function, or rather, something we call a quantum state, being a vector in Hilbert space, which obeys a Schrodinger equation. The absolute squares of the vector components may be used to describe probabilities whenever we wish to predict or explain something. Powerful techniques were developed, enabling one to guess the right Schrodinger equation if one knows how things evolve classically, that is, in the old theories where quantum mechanics had not yet been incorporated. It all works magnicently well. According to Copenhagen, however, there is one question one should not ask: "What does reality look like of whatever moves around in our experimental settings?", or: what is really going on? According to Copenhagen, such a question can never be addressed by means of any experiment, so it has no answer within the set of logical statements we can make of the world. Period, "schluss, ni." Those questions are senseless. It is this answer that we dispute. Even if this kind of questions cannot be answered by experiments, we can still in theory try to build credible models of reality."('t Hooft, [16]) This paper is intended to explore such a possibility of realism interpretation of QM, especially based on a derivation of Maxwell equations in Quaternion Space, an approach we developed quite sometime ago based on Gersten's method. In this regards, firstly we begin with discussion with Prof V. Simulik, on how he believes that it is impossible to argue for Maxwell-Dirac equations isomorphism. However, we can argue here that actually we can arrive to tenable arguments, as follows: (i) based on Quaternion space and its respective Quaternion Relativity (it also may be called as Rotational Relativity) as it has been defined in a number of papers including [1], and it can be shown that this new theory is capable to describe relativistic motion in elegant and straightforward way. Nonetheless there are subsequent theoretical developments which remains an open issue, for instance to derive Maxwell equations in Q-space [1]. Therefore the purpose of the present paper is to review our previous derivation of Maxwell equations in Q-space. (ii) Thirdly, we will discuss Geometric Algebra in Euclidean 3D real space. Further observations are of course recommended in order to refute or verify some implications of what we discuss here.

2 Problem of Maxwell-Dirac isomorphism. Basic aspects of Quaternion space

In this section, allow us to cite a discussion via email with Prof. V. Simulik a few years ago, he is a specialist in classical electromagnetic theories, including Maxwell theory etc. In response to one of us (VC)'s question: "Could it be that Max-Dir isomorphism exists only in matrix formulation or quaternionic expression? A quote from Bocker and Frieden paper at Heliyon, 2018: "As it turns out, the Dirac spacetime matrix equation is equivalent to four new vector equations, which are similar in form to the four Maxwell vector equations. These new equations will be referred to as the Dirac spacetime vector equations. This allows these new vector equations to be as readily solved as solving a set of Maxwell vector equations." Then may be the Dirac equation which has four vector is mathematically equivalent to Maxwell set of four vectors. Alternatively, in your book on electron (2005). There is a paper on (3,3) spacetime representation on electromagnetic theory." Then he wrote in June 2020: "The Maxwell-Dirac Isomorphism does not exist in mathematics. By fingers: eight real components of Dirac function - six real components of Maxwell functions (electromagnetic field strengths). Or four complex Dirac functions - three complex Maxwell fields. Isomorphism is eight to eight or four to four. And plus linearity... Therefore, the title Maxwell-Dirac Isomorphism of Sallhofer was not good. See Chapter 2, Section.2.4, especially 2.4.1 So it will be better if you can refer to it in future. This is the reason why I have sent two first chapters to you. Keller suggested mapping I used relation or relationship. 2) My big step forward (and today I am working with this as well) is the investigation of the situation, where Maxwell-Dirac Isomorphism really exists. It is four complex Maxwell fields or eight real. So called slightly generalized Maxwell equations for the system of electromagnetic and scalar fields. The first publication was in Advances in Applied Clifford Algebras in 1996 - today I do not have electronic copy." Then he wrote again in the same day, 15th June 2020: "The isomorphism does not exist. The main p[oint is one-to-one correspondence. Therefore, if we have 4 and 3, or 8 and 6 - no isomorphism. 4 cannot be isomorphic to 3." (cf. Richard P. Bocker, B. Roy Frieden. A new matrix formulation of the Maxwell and Dirac equations. Heliyon, Volume 4, Issue 12 (2018) e01033. doi: 10.1016/j.heliyon.2018.e01033; and also V. Simulik et al. 2019. Slightly generalized Maxwell system and longitudinal components of solution, J. Phys.: Conf. Ser. 1416 012033; also V. Simulik (ed.) What is the electron, Montreal: Apeiron C. Roy Keys Inc., 2005). And in the next section, we will argue in favor of one-to-one correspondence between Maxwell equations in quaternion space and quaternionic Dirac equations. And in the last section, we will argue that in response to Simulik's claim: "4 cannot be isomorphic to 3"; actually it is possible to find Dirac equation in real Euclidean 3D space representation, in terms of Geometric algebra (it is called: Cl(3,0) or simply Cl(3).) First of all, we will review some basic definitions of quaternion number [1]. Quaternion number belongs to the group of "very good" algebras: of real, complex, quaternion, and octonion [1], and normally defined as follows [1]:

$$Q \equiv a + bi + cj + dk. \tag{1}$$

Where a,b,c,d are real numbers, and i,j,k are imaginary quaternion units. These Q-units can be represented either via 2x2 matrices or 4x4 matrices. There is quaternionic multiplication rule which acquires compact form [1]:

$$1q_k = q_k 1 = q_k, \quad q_j q_k = -\delta_{jk} + \varepsilon_{jkn} q_n, \tag{2}$$

Where $\delta_{kn}$and ε_{jkn}represents 3-dimensional symbols of Kronecker and Levi-Civita, respectively.

In the context of Quaternion Space [1], it is also possible to write the dynamics equations of classical mechanics for an inertial observer in constant Q-basis. SO(3,R)-invariance of two vectors allow to represent these dynamics equations in Q-vector form [1]:

$$m\frac{d^2}{dt^2}(x_k q_k) = F_k q_k. \tag{3}$$

One advantage of this Quaternion Space representation is that it enables to describe rotational motion with great clarity.

After this short review of Q-space, next we will discuss a simplified method to derive Maxwell equations from Lorentz force, in a similar way with Feynman's derivation method using commutative relation [2][10].

3 A new derivation of Maxwell equations in Quaternion Space by virtue of Dirac decomposition.

In this section we present a derivation of Maxwell equations in Quaternion space based on Gersten's method to derive Maxwell equations from one photon equation by virtue of Dirac decomposition [4].

We know that Dirac deduces his equation from the relativistic condition linking the Energy E, the mass m and the momentum p [5]:

$$(E^2 - c^2\bar{p}^2 - m^2c^4) I^{(4)} \Psi = 0, \tag{4}$$

Where $I^{(4)}$is the 4x4 unit matrix and $\Psi$is a 4-component column (bispinor) wavefunction. Dirac then decomposes equation (7) by assuming them as a quadratic equation:

$$(A^2 - B^2)\Psi = 0 \quad (5)$$

Where

$$A = E, \quad (6)$$

$$B = c\vec{p} + mc^2. \quad (7)$$

The decomposition of equation (8) is well known, i.e. (A+B)(A-B)=0, which is the basic of Dirac's decomposition method into 2x2 unit matrix and Pauli matrix [4][12].

By virtue of the same method with Dirac, Gersten found in 1999 [4] a decomposition of one photon equation from relativistic energy condition (for massless photon [5]):

$$(\frac{E^2}{c^2} - \vec{p}^2)I^{(3)}\Psi = 0, \quad (8)$$

Where $I^{(3)}$ is the 3x3 unit matrix and $\Psi$is a 3-component column wavefunction. Gersten then found [4] equation (11) decomposes into the form:

$$[\frac{E}{c}I^{(3)} - \vec{p}\cdot\vec{S}][\frac{E}{c}I^{(3)} + \vec{p}\cdot\vec{S}]\vec{\Psi} - \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}(\vec{p}\cdot\vec{\Psi}) = 0, \quad (9)$$

Where $\vec{S}$is a spin one vector matrix with components [4]:

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \quad (10)$$

$$S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad (11)$$

$$S_z = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (12)$$

And with the properties:

$$[S_x, S_y] = iS_z, \quad [S_x, S_z] = iS_y, [S_y, S_z] = iS_x, \vec{S}^2 = 2I^{(3)}. \quad (13)$$

Gersten asserts that equation (22) will be satisfied if the two equations [4][5]:

$$[\frac{E}{c}I^{(3)} + \vec{p}\cdot\vec{S}]\vec{\Psi} = 0, \quad (14)$$

$$\vec{p}\cdot\vec{\Psi} = 0, \quad (15)$$

are simultaneously satisfied. The Maxwell equations [9] will be obtained by substitution of E and p with the ordinary quantum operators (see for instance, H. Bethe, *Field Theory*):

$$E \to i\hbar\frac{\partial}{\partial t} \quad , \quad (16)$$

and

$$p \to -ih\nabla, \quad (17)$$

And the wavefunction substitution:

$$\vec{\Psi} = \vec{E} - i\vec{B}, \quad (18)$$

Where E and B are electric and magnetic fields, respectively. With the identity:

$$(\vec{p}\cdot\vec{S})\vec{\Psi} = \hbar\nabla\times\vec{\Psi}. \quad (19)$$

Then from equation (17)-(19) one will obtain:

$$i\frac{\hbar}{c}\frac{\partial(\vec{E} - i\vec{B})}{\partial t} = -\hbar\nabla\times(\vec{E} - i\vec{B}), \quad (20)$$

$$\nabla\cdot(\vec{E} - i\vec{B}) = 0, \quad (21)$$

Which are the Maxwell equations if the electric and magnetic fields are real [4][5].

We can remark here that the combination of E and B as introduced in eq. (18) is quite well known in literature [6][7]. For instance, if we use positive signature in (21), then it is known as Bateman representation of Maxwell equations ($div\vec{\varepsilon} = 0; rot\vec{\varepsilon} = \frac{1}{c}\frac{\partial\vec{\varepsilon}}{\partial t}; \varepsilon = \vec{E} + i\vec{B}$). But the equation (21) with negative signature represents the *complex nature* of Electromagnetic fields [6], which indicates that these fields can also be represented in quaternion form.

Now if we represent in other form $\vec{\varepsilon} = \vec{E} - i\vec{B}$ as more conventional notation, then equation (20) and (21) will get a quite simple form:

$$i\frac{\hbar}{c}\frac{\partial\vec{\varepsilon}}{\partial t} = -\hbar\nabla\times\vec{\varepsilon}, \quad (22)$$

$$\nabla\cdot\vec{\varepsilon} = 0. \quad (23)$$

Now to consider quaternionic expression of the above results from Gersten [4], one can begin with the same linearization procedure just as in equation (3):

$$dz = (dx_k + idt_k)\, q_k,$$

Which can be viewed as the quaternionic square root of the metric interval dz:

$$dz^2 = dx^2 - dt^2. \quad (24)$$

3 A new derivation of Maxwell equations in Quaternion Space by virtue of Dirac decomposition.

Now consider the relativistic energy condition (for massless photon [5]) similar to equation (4):

$$E^2 = p^2c^2 \Rightarrow (\frac{E^2}{c^2} - \vec{p}^2) = k^2, \tag{25}$$

It is obvious that equation (25) has the same form with (8), therefore we may find its quaternionic square root too, then we find:

$$k = (E_{qk} + i\vec{p}_{qk})\, q_k, \tag{26}$$

Where q represents the quaternion unit matrix. Therefore the linearized quaternion root decomposition of equation (11) can be written as follows [4]:

$$[\tfrac{E_{qk}q_k}{c} I^{(3)} + i\vec{p}_{qk}q_k \cdot \vec{S}][\tfrac{E_{qk}q_k}{c} I^{(3)} + i\vec{p}_{qk}q_k \cdot \vec{S}]\vec{\Psi} - \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} (i\vec{p}_{qk}q_k \cdot \vec{\Psi}) = 0 \quad . \tag{27}$$

Accordingly, equation (40) will be satisfied if the two equations:

$$[\frac{E_{qk}q_k}{c} I^{(3)} + i\vec{p}_{qk}q_k \cdot \vec{S}]\vec{\Psi}_k = 0, \tag{28}$$

$$i\vec{p}_{qk}q_k \cdot \vec{\Psi}_k = 0, \tag{29}$$

are simultaneously satisfied. Now we introduce similar wavefunction substitution, but this time in quaternion form:

$$\vec{\Psi}_{qk} = \vec{E}_{qk} - i\vec{B}_{qk} = \vec{\varepsilon}_{qk}. \tag{30}$$

And with the identity:

$$(\vec{p}_{qk}q_k \cdot \vec{S})\vec{\Psi}_k = \hbar\nabla_k \times \vec{\Psi}_k \quad . \tag{31}$$

Then from equation (30) and (31) one will obtain *the Maxwell equations in Quaternion-space* as follows:

$$i\frac{\hbar}{c}\frac{\partial\vec{\varepsilon}_q k}{\partial t} = -\hbar\nabla_k \times \vec{\varepsilon}_{qk} \quad , \tag{32}$$

$$\nabla_k \cdot \vec{\varepsilon}_{qk} = 0. \tag{33}$$

Now the remaining question is to define quaternion differential operator in the right hand side of (32) and (33).

In this regards one can choose some definitions of quaternion differential operator, for instance the so-called "Moisil-Theodoresco operator"[8] :

$$D[\varphi] = grad\varphi = \sum_{k=1}^{3} i_k\partial_k\varphi = i_1\partial_1\varphi + i_2\partial_2\varphi + i_3\partial_3\varphi, \tag{34}$$

Where we can define here that $i_1 = i, i_2 = j, i_3 = k$ to represent 2x2 quaternion unit matrix, for instance.

Therefore the differential of equation (33) now can be expressed in similar notation of (34) :

$$D[\vec{\Psi}] = D[\vec{\varepsilon}] = i_1\partial_1E_1 + i_2\partial_2E_2 + i_3\partial_3E_3 \\ -i\,(i_1\partial_1B_1 + i_2\partial_2B_2 + i_3\partial_3B_3) \quad . \tag{35}$$

This expression indicates that both electric and magnetic fields can be represented in unified manner in a biquaternion form.

Then we define quaternion differential operator in the right-hand-side of equation (32) by an extension of the conventional definition of curl:

$$\nabla \times A_{qk} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}, \tag{36}$$

To become its quaternion counterpart, where i,j,k represents quaternion matrix as described above. This quaternionic extension of curl operator is based on the known relation of multiplication of two arbitrary complex quaternions q and b as follows:

$$q \cdot b = q_0b_0 - \left\langle \vec{q}, \vec{b} \right\rangle + [\vec{q} \times \vec{b}] + q_0\vec{b} + b_0\vec{q}, \tag{37}$$

Where

$$\left\langle \vec{q}, \vec{b} \right\rangle := \sum_{k=1}^{3} q_kb_k \in C, \tag{38}$$

And

$$\left[\vec{q} \times \vec{b}\right] := \begin{vmatrix} i & j & k \\ q_1 & q_2 & q_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{39}$$

We can note here that there could be more rigorous approach to define such a quaternionic curl operator.[7]

In the present paper we only discuss derivation of Maxwell equations in Quaternion Space using the decomposition method described by Gersten [4][5]. Further extension to Proca equations in Quaternion Space seems possible too using the same method [5], but it will not be discussed here.

In the next section we will discuss some physical implications of this new derivation of Maxwell equations in Quaternion Space.

4 Discussion: other solution in real Euclidean 3D Space, and possible Helical solutions of Maxwell equations; and limitations of this paper.

In the foregoing section we derived a consistent description of Maxwell equations in Q-Space by virtue of Dirac-Gersten's decomposition. Actually there is also one more plausible approach in the context of Geometric algebra, where Dirac equation can be put into real Euclidean 3D space (Cl3), which can be linked to rotation in Euclidean space (SO3). Miroslav Josipovic wrote: "First, the Cl3 formulation of the Dirac theory is simple and the derivation of the Dirac's formula is straightforward. Second, it is relatively easy to show that gamma matrices are not the only possibility in linearizing the Klein-Gordon equation (we even do not need it in Cl3). ...the fact that it is possible to use the same mathematical (3D) formalism for classical mechanics..." Nonetheless, we don't explore in more detailed on such a possibility, except that such a Cl3 method has been utilised to analyse graphene crystal. (cf. M. Josipovic, Dirac theory in Euclidean 3D Geometric algebra (Cl3), chapter in Geometric Multiplication of Vectors: An Introduction to Geometric Algebra in Physics, Birkhäuser, 2019; also A. Dargys. Quantum flatland and monolayer graphene... ACTA PHYSICA POLONICA A, Vol. 124 (2013); G. Coddens. The Geometrical Meaning of Spinors Lights the Way to Make Sense of Quantum Mechanics. Symmetry 2021, 13, 659 (MDPI), 45 pp.)

Now we are going to discuss some plausible implications of the new proposition of plausible one-to-one correspondence and even isomorphism between Maxwell-Dirac equations: (*) First, in accordance with Gersten, we submit the viewpoint that the Maxwell equations yield wavefunctions which can be used as guideline for interpretation of quantum mechanics [4][5]. We shall emphasize here: "The one-to-one correspondence between classical and quantum wave interpretation actually can be expected not only in the context of Feynman's derivation of Maxwell equations from Lorentz force, but also from known exact correspondence between commutation relation and Poisson bracket." See also [3][4].

(**) Secondly, the above expressions of Maxwelll equations in Q-space are still missing the A term. Provided we include A term as defined by R.W. Bass [2], as follows:

$A = \nabla x(\psi u) + (1/\lambda)\nabla x(\nabla x(\psi u))$

then we got helical solutions of Maxwell eq., which are more consistent with many experimental results, instead of the well-known "sinusoidal" solutions.[2]

A few more remark deserves further attention, as follows: "This helical form of EM ties into the KH electron, during photon capture events, The captured photon causes an energetic imbalance in the desired and required stability of the electron, which causes the photon to be ejected in a short while. We're still trying to discover what is the cause of the form of the electron being as stable as it is, and why it has such a strong surface boundary." (KH=Kelvin-Helmholtz)

Further implications of this new proposition of helical solutions of Maxwell eq. require further study, and therefore they are excluded from the present paper.

Before we close this paper, allow us to point out two limitations of the procedure as outlined above: (a) It begins with Gersten decomposition and Dirac equation, which assumes that mass-energy equivalence holds true. Whenever one can show that such an equivalence does not hold true, then our procedure should be revised. (b) Our model assumes orthogonality. Some authors have argued that orthogonality of Dirac equation prevents it to predict fractional states of hydrogen (hydrino, cf. R.L. Mills).[15] However, the majority of interactions and appearances in Nature are not orthogonal, but are comprised of acute and oblique angles, usually far from orthogonality. This fact is covered by dot products and cross products, as required by engineering applications of the Maxwell equations. Quaternion solutions are not capable of doing dot products, to my knowledge.

5 Concluding remarks

In the present paper we review our previous paper on derivation of a consistent description of Maxwell equations in Q-space. First, we present another method to derive Maxwell equations by virtue of Dirac decomposition, introduced by Gersten (1999).

In accordance with Gersten, we submit the viewpoint that the Maxwell equations yield wavefunctions which can be used as guideline for interpretation of quantum mechanics. The one-to-one correspondence between classical and quantum wave interpretation asserted here actually can be expected not only in the context of Feynman's derivation of Maxwell equations from Lorentz force, but also from known exact correspondence between commutation relation and Poisson bracket [2][4].

A somewhat unique implication obtained from the above results of Maxwell equations in Quaternion Space, is that it suggests that the helical solutions, especially if we consider A vector from R. Bass. Further implications, however, are beyond the scope of the present paper. Another plausible new approach to get to Maxwell-Dirac isomorphism is through Cl3 / Geometric Algebra formalism in Real Euclidean 3D space.

This proposition, however, deserves further theoretical considerations. Further observation is of course recommended in order to refute or verify some implications of this result.

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Small Tech, High Touch: A Permutation

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ABSTRACT

In an earlier paper published in a neutrosophic math journal (IJNS), we discussed a new approach to technology, which may be called as 'opti-realism' or 'pess-optimism' as alternative to utopianism based on technocracy, which may lead the world into global techno-totalitarianism. In this article, we submit a new approach to Nature and technology, which is more modest and humble, rather than a techno-utopianism version of reality that most futurists argue for. Our proposed approach resembles more to Myer-Briggs 16 types of personality, including IJNS, IFNS etc. In our scheme, there are 8 characters of approach toward technology which can lead to many variations or we call it 'Permutation." Of course, if the readers ask one variation that we prefer, we would answer: Small Tech, High Touch.

KEYWORDS: opti-realism, small tech high touch, permutation

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INTRODUCTION

In line with the rapid development of new branch of foundational mathematics, i.e. Neutrosophic Logic, in an earlier paper we discuss potential application of NL theory in the field of futurology [9]. See for recent papers on NL: [10-13].

In this context, about two decades ago, John Naisbitt - a prominent futurologist - published his book *High Tech, High Touch [*1]. After that, there were many publications and research on efforts to apply the high tech and high touch concepts, for example in the field of customer service and hospitality, see for example [2].

If Naisbitt's book on high tech and high touch looked for as an opponent, we will encounter: low tech and low touch. Are there perhaps areas where low tech and low touch are still relevant?

On the other spectrum, the convergence of technology and also ecological/environmental considerations, leads us to two other possibilities: small tech and also tech wise, the opposites ofwhich are big tech and unwise tech. Of course, it is quite clear that the right technology sometimes does not need to be large and sophisticated, sometimes simple technology such as a small gas stove, is sufficient for household use in rural areas.

Our proposed approach resembles more to Myer-Briggs's 16 types of personality, including IJNS, IFNS etc.[1] In our scheme, there are 8 characters of approach toward technology which can lead to many variations or we call it 'Permutation."

PERMUTATION

Without pretending to debate this topic at length, let's do a simple permutation game. For example:
A0 = low tech; A1 = high tech; B0 = low touch; B1 = high touch; C0 = small tech; C1 = big tech; D0 = wise tech; D1 = unwise tech.

So we now have 8 traits of approach in technology. Of course it becomes simpler if Naisbitt's simple proposition: high tech and high touch is denoted as A1-B1.

Then we now permutate the possible combinations:

a. A0-B0

b. A1-B0

c. A0-B1

d. A1-B1

e. A0-C0

f. A1-C1

g. A0-D0

h. A0-D1

And so on. We can continue this permutation into a combination of 3 and a combination of 4 parameters, for example: A0-B1-C0-D0, and so on.

IMPLICATION: TOWARD SITUATIONAL TECHNOLOGY

Personally, these writers are more inclined towards appropriate technology. For example in the development of renewable energy, in a previous article, the author has mentioned the term ART (a*ppropriate renewable technology*). In the spectrum of possibilities above, ART can be classified into: C0-D0, but can also be combined with A0 / A1 or B0 / B1.

But of course this choice needs to consider the context or situation and location in which we must apply the technology.

If it is possible to use the analogy of situational leadership (situational leadership), perhaps this approach can be called: *situational technology choice* (STC). Or those who prefer to use context terms, can use phrases: technology in context.

[1] Added note: As an alternative to Myer-Briggs's personality types, FS has developed his own Neutropsychology scheme. In psychology, he introduces:
Neutropsychic Personality that is a neutrosophic dynamic open psychologicalsystem of tendencies to feel, think, and act specific to each individual.
Neutrosophic Refined Memory: that restructured the division of memory into: *consciousness*, *aconsciousness* (which he introduced as a blend of *consciousness* and *unconsciousness*),and *unconsciousness*. *Aconscious* was further subdivided into *preconscious, subconscious, semiconscious* = *semiunconscious, subunconscious,* and *preunconscious*. All memories have degrees of conscious (c), acounscious(a), and unconscious (u).
Refined Neutrosophic antiTrait –Trait Diagram, that each individual has a *degree of antiTrait* and a *degree of Trait* with respect to each antiTrait-Trait personality pair. And the *Neutrosophic Temperament*. Interested readers are advised to consult [14].

APPLICATIONS

Does it sound a little too abstract, and less practical?
Let us consider one of the problems, namely distance learning in this pandemic situation (context).
Perhaps a high tech and high touch solution for the problem in question is to provide multimedia high-speed Internet-based education services so that students do not have to go to school each day (combination A1-B1). But perhaps this solution is only suitable in *urban areas* where students are accustomed to using gadgets with mobile internet networks available throughout the city.

But what about small towns or rural areas where 3G / 4G networks are still difficult to reach?

Maybe we need to think about a solution with a different combination, for example A0-C0-D0 (*low tech, small tech, wise tech*). Is there a solution?

Several possibilities that might be considered:

a. Radio over IP. The latest developments, if I'm not mistaken, are DMR (digital mobile radio) and IP Simulcast.
b. School-based radio channel: perhaps small-town schools could consider collaborating with a local radio station. Because radio infrastructure is more affordable, with a good schedule, the teaching process can be done via radio, perhaps a term that can be popularized: "raducation." (education based on radio)
c. Another possibility, for example for more remote areas, is to develop a community radio, or community-based radio channel. It seems a little difficult, considering remote areas also have difficulties in technical personnel, but at least this is one possibility (6-7).

Science, *techne* and dialogical communication
Now let us put the aforementioned discussions on technological choices and approach to nature into a more philosophical perspective.

It is known that there are natural sciences, social sciences, and emancipatory sciences etc. The main distinction between natural sciences and social sciences is mostly about what and whom to control: in natural sciences, a scientist tries to control nature through comprehension of certain aspects of nature [8], which then they be reduced into some kind of laws of nature. In social sciences, a scientist tries to achieve more understanding (*verstehen*[2]) of certain people or society, in order to properly do dialogue with that society/people. Therefore, it is wrong if a social scientist tries to "control" the society in question in order to fulfil his/her goals, because human beings should not be an object of control, but as mutual partners of dialogue.

Many problems that we found in society come from two chief misapplied sciences: natural sciences which becomes "techne"[3] or technology, which not only aiming to control Nature, animals and so on, but also control people and society. And also, social sciences which work in wrong way to not do dialogical communication to achieve goals as community, but to control each other.

What is distinction between *techne* and episteme? See this entry, which can be paraphrased as follows:

"…a portion of the highlights of this contemporary qualification among hypothesis and practice are not tracked down in that frame of mind among epistêmê and technê. Others are tracked down in a fairly refracted style. As we move sequentially from Xenophon to Plotinus, we go from a not creator recognize the two terms, to a little creator use for technê in light of the fact that it is such a long ways from what he views as genuine. It is in Aristotle that we find the reason for something like the advanced resistance between epistêmê as unadulterated hypothesis and technê as training. However even Aristotle alludes to *technê* or create as itself likewise epistêmê or information since it is a training grounded in an 'account' — something including hypothetical comprehension. Plato —

whose hypothesis of structures appears to be a curve illustration of unadulterated hypothetical information — by and by is intrigued by the

possibility of a sort of *technê* that is educated by information on structures. In the Republic this information is the fundamental reason for the thinkers' specialty of administering in the city. Getting one more topic in Plato's discoursed, the Stoics foster the possibility that prudence is a sort of technê or art of life, one that depends on a comprehension of the universe. The connection, then, at that point, among *epistêmê and technê* in old way of thinking offers a fascinating differentiation with our own ideas about hypothesis (unadulterated information) and (experience-based) practice. There is a close sure relationship among epistêmê and technê, as well as anessential differentiation."[4]

To these wrong applications of science, which often happen because of either socialism or capitalism, then comes a third possibility: emancipative sciences, which are aiming to liberation to the aforementioned "*techne*" stronghold.

Last but not least, our suggestion to adopt situational technology choice is intended to suit each technology in a particular location and geography; nonetheless when educators wish to integrate these choices of technology into students' learning process, they shall learn how to introduce to them by virtue of Bloom's taxonomy and Vygotsky ZPD approach, for instance.



Figure 1. Bloom's taxonomy to integrate technology (cf. [15])

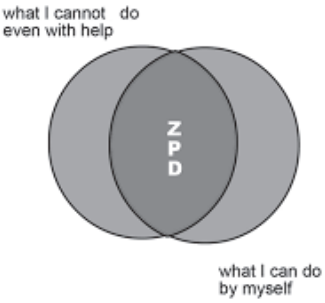


Figure 2. Vygotsky ZPD

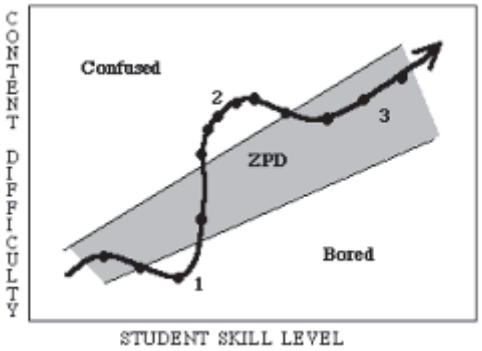


Figure 3. Vygotzky ZPD, to introduce new things in tune with student's skill level

CONCLUDING REMARKS

Thus, a few simple thoughts may be useful for educators and also schools that will start the teaching and learning process in the next academic year. In this paper the technical aspects are deliberately not discussed, because this requires separate considerations.

To summarize the aforementioned arguments, in the present article we submit a new approach in relation to Nature and technology, which is more modest and humble; rather than a techno-utopianism version of reality that most futurists argue for. Our proposed approach resembles more to Myer-Briggs 16 types of personality, including IJNS, IFNS etc. In our scheme, there are 8 characters of approach toward technology which can lead to many variations or we call it 'Permutation."

In this sense, small tech-high touch can be viewed as one way to counter the pragmatic-hegemonic practices of techno-utopianism, especially with high tech, big tech approach.

That is our perspective, which may be influenced by people-centered economics-way of thinking cf. E.F. Schumacher, along with dialogical philosophy of Martin Buber (a Jewish philosopher).

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[2] https://medium.com/@umfarooq0/verstehen-max-weber-and-an-approach-to-social-sciences-f95ad578aa9b

[4] https://plato.stanford.edu/entries/episteme-techne/

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Questioning Darwin-Malthusian hypothesis: Remark on critics to Darwin's evolutionary process of history, from Ernst Cassirer to Jaroslav Flegr

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Abstract

As Rev. John Polkinghorne, the late professor of mathematical physics and priest, wrote: "The idea of a ready-made world was replaced by that of a creation so endowed by its Creator with potentiality that creatures could be allowed 'to make themselves' (Charles Kingsley) through the explorations of evolutionary process. The history of the world is to be understood as an unfolding act of continuous creation. Such a creation is a great good, but it has an inescapable shadow side for evolution results not only in great fertility but also in blind alleys and extinctions. Genetic mutation produces both new life forms and also malignancy, and one cannot have the one without the other. Theology can find here some help from science as it wrestles with the perplexities of natural evil and suffering." (cf. Polkinghorne, The University of Edinburgh, 2009). Notably several scholars have come up with different arguments to criticize Darwin's evolution theory, for instance Ernst Cassirer, Vern Poythress with *Interpreting Eden,* and Jaroslav Flegr in his book *Frozen evolution, that is not the way it is, mr Darwin*. The present review is intended to bring more arguments, in particular from recent development of nonlinear dynamics attributed to low-temperature physics and its connection to cosmology.

Introduction

Charles Darwin's "*On the Origin of Species*" stands as one good speculative work in the history of science, its theory of evolution by natural selection forever altering our understanding of life. Yet, while the "Darwinian revolution" often receives sole credit, the story behind this pivotal book is more nuanced than commonly portrayed. While discussion on random mutation can be largely influenced by any kind of argument regarding selection/random selection etc, we would like first to mention a number of fallacies in Darwin's propositions, first on his quite careless influenced by Lamarck and also Malthus. This article explores how the ideas of Jean-Baptiste Lamarck, Alfred

Russel Wallace, and Thomas Malthus played crucial roles in shaping Darwin's revolutionary tome.and also possible refutation of the idea of sole evolution as the process in history. We prove that it is possible to conceive history of hominid without considering Out of Africa hypothesis. Moreover, as other authors also criticize Darwin's theory, such as Ernst Cassirer and Abdul Ahad (2014), others have brought up influence to economists, such as Hayek's evolutionary epistemology (Krstic, 2012, Marciano, 2009).

Based on one of us (FS)'s observation to Galapagos islands, and also Smarandache & Vatuiu's hypothesis of spiralling neutrosophic theory, allow us to consider several arguments in the following paragraphs (Smarandache, 2017; Smarandache, 2017; Smarandache & Vatuiu, 2019).

The Forgotten Roots of "Origin": Lamarck, Wallace, and Malthus Shape Darwin's Masterpiece

It is known that there are critics to Darwinian theory of evolution. For instance Vern Poythress, John Lennox and philosophers such as Ernst Cassirer. As van Heusden wrote, "First, of course, that Cassirer was interested in Uexkull's research. Secondly, that he was sensitive to the basically animal-like behavior of humans. Humans, he would say over and over again, are first of all animals. But, he would then add, the human is an animal with capabilities which allow it to escape its animal condition" (van Heusden, 2001).

Firstly, Lamarck's influence on Darwin is often underestimated. Though Lamarck's theory of inheritance of acquired characteristics is now discredited, his emphasis on gradual change and adaptation to environment resonated deeply with Darwin. Lamarck provided a foundation for Darwin to explore how organisms could, over time, modify themselves to fit their surroundings, laying the groundwork for the concept of natural selection.

Secondly, Alfred Russel Wallace, often presented as simply a corroborator of Darwin's theory, deserves greater recognition. Wallace, working independently in the field, arrived at very similar conclusions to Darwin regarding the mechanism of evolution. This independent confirmation strengthened the theory's credibility and lent weight to the idea that it wasn't simply a product of one man's mind.

However, the most profound influence on Darwin's "*Origin*" might not have come from another biologist, but from an economist: Thomas Malthus. Malthus's theory of population growth, which proposed that populations tend to outstrip resources, provided a critical explanation for the struggle for existence that Darwin observed in nature. This struggle, Malthus argued, would inevitably lead to competition and differential survival, the very engine that Darwin's natural selection theory relied upon.

Therefore, Darwin's *"Origin*" was not an invention born solely from his own brilliant mind. It was a masterful synthesis of Lamarck's adaptationism, Wallace's independent confirmation, and Malthus's bleak yet crucial insights into population dynamics. By recognizing these diverse influences, we gain a richer understanding of Darwin's remarkable achievement, appreciating the collaborative nature of scientific progress and the interconnectedness of ideas that led to one of the impactful scientific revolutions of all time (cf. Ahad, 2012)

In conclusion, while Darwin's name rightfully adorns the pages of "On the Origin of Species," we must remember the vital contributions of Lamarck, Wallace, and Malthus. Their ideas provided the soil from which Darwin's genius blossomed, reminding us that scientific breakthroughs are rarely solitary endeavors, but often the fruit of a shared intellectual landscape.

Key arguments in Jaroslav Flegr

While there are many publications have been in print to discuss pros and cons of Darwinism vs creationism etc, for instance Prof John Lennox and Vern Poythress, Interpreting Eden, of notable interests is Jaroslav Flegr's book, "*Frozen Evolution: Or, That's Not the Way It Is, Mr. Darwin*," whose book throws down a gauntlet to the established theory of evolution by natural selection. According to Flegr, "Most biologists and biology students think that evolutionary biology is basically a closed chapter of science. But what if this is not the way things are? What if evolutionary biology underwent a quite fundamental revolution in the 70's and 80's of the past century." While not outright denying evolution, Flegr proposes a significantly different mechanism for how sexual species change and diversify. Let's dive into his key arguments and see if Darwin should be shaking in his boots.

1. Evolution in Spurts, Not Gradual Change: Flegr challenges the Darwinian image of slow, continuous adaptation. He argues that most evolutionary change happens in short bursts of rapid shifts followed by long periods of stasis, a state he calls "evolutionary freezing." This freezing occurs, he claims, when genetic variability within a population reaches a critical point, hindering further adaptation.

2. Frequency-Dependent Selection, the Villain: Unlike traditional views, Flegr proposes that the culprit behind this stasis is not simply lack of selection pressure. He introduces the concept of frequency-dependent selection, where a trait's advantage depends on its prevalence in the population. As a beneficial trait becomes common, its advantage diminishes, eventually stalling selection and locking the species in its current form.

3. Farewell to the Selfish Gene: Richard Dawkins' famous concept of the "selfish gene" doesn't find favor with Flegr. He argues that genes primarily operate at the level of the organism, favoring traits that benefit the whole, not just individual gene replication. This shifts the focus from genes competing to genes cooperating for the organism's survival.

4. A Farewell to Darwin, Too? Not completely. Flegr sees his model as an extension of Darwinism, applicable only to sexual species. Asexual organisms, lacking the complexities of sex and recombination, would still evolve according to Darwin's principles. This makes his theory a potential refinement rather than a complete overthrow.

The Verdict: Time for New Ice Trays?

Flegr's ideas are certainly provocative and offer a fresh perspective on evolution. However, the scientific community largely regards them with skepticism. Critics point to weaknesses in his evidence, like the lack of concrete mechanisms for his proposed stasis and the limited empirical support for frequency-dependent selection as the sole driver of evolution.

While "Frozen Evolution" may not dethrone Darwin overnight, it does spark important questions about the nuances of evolution in sexual species. Whether it's a revolution or a ripple in the pond, Flegr's work reminds us that scientific understanding is an ever-

evolving landscape, and even the most established theories can benefit from a healthy dose of questioning.

Without *Out of Africa* and Beyond Africa origin: Exploring Alternative Theories of Hominid Origins

The "*Out of Africa*" model, which posits that all modern humans originated in Africa and migrated outwards, has long dominated discussions of hominid evolution, partly because it was advocated in Darwin' theory. However, a chorus of dissenting voices challenges this dominant narrative, proposing alternative scenarios about our ancient past. This article delves into two contrasting perspectives on hominid origins, offering a glimpse into the captivating yet unsettled world of human evolution.

Mary Lefkowitz and the "*Black Athena*" Perspective:

American historian Mary Lefkowitz reinterprets historical and archaeological evidence, arguing for a strong cultural continuity between ancient Egypt and Greece. Her criticism of "Black Athena" suggests that Greek civilization did not solely arise from independent development but stemmed from an earlier Egyptian foundation. This implies that hominid evolution in these regions might have occurred independently, with early African hominids coexisting with and influencing other evolving hominid lineages elsewhere.

Radu Cinamar and the Controversial Romanian Precursor Model:

Romanian researcher Radu Cinamar's proposition is even more audacious. He posits that Romania represents a crucial cradle of early hominid civilization, predating and influencing the construction of pyramids across the globe, including the Great Pyramid of Giza. While this theory lacks widespread scientific acceptance, it raises the intriguing possibility of hominid lineages developing sophisticated societal structures outside the traditional African cradle.

The theories explored above, while challenging the established narrative, raise further questions about lost civilizations like Mu, Lemuria, and Atlantis. While currently lacking credible scientific evidence, the very existence of these myths reflects a human fascination with our origins and the possibility of advanced hominid cultures predating known history.

A Landscape of Uncertainty:

Although evidence seems to strongly supports an African origin for modern humans, alternative perspectives remind us that our understanding of early hominid evolution remains riddled with gaps and uncertainties. As new discoveries and analysis methods emerge, the possibility of diverse hominid lineages and civilizations flourishing across the globe cannot be entirely dismissed.

Instead of clinging dogmatically to a single model, embracing uncertainty and open-mindedness becomes crucial. Rigorous scientific inquiry, critical assessment of evidence, and continued exploration of alternative theories will ultimately illuminate the true complexity of our evolutionary past. Perhaps, someday, the whispers of lost civilizations may find echoes in the unearthed whispers of the ancient hominid record.

Nonetheless, it shall be kept in mind, that the present review article is meant to explore alternative perspectives and does not endorse any specific unproven theories. It should be read as a thought-provoking exercise in considering the complexities of human origins.

Revised Young Earth theories: beyond deep time and exploring Alternative Cosmologies and the "Spontaneous Emergence Creation" Model

The vast expanse of cosmic history, stretching back billions of years, has long been dominated by the "Big Bang" and "evolution-only" paradigm. These models, while supported by considerable evidence, leave certain lingering questions unanswered. This article delves into alternative cosmological perspectives, particularly the fascinating concept of "Spontaneous Emergence Creation" (SEC) proposed by Per Bak and the potential reconciliation with religious narratives like Genesis. (In Per Bak, his model is known as self-organized criticality).

Challenging the Deep Time Narrative:

While the standard cosmological model enjoys scientific consensus, some scientists and theologians propose alternative possibilities. Young Earth theorists, for instance, advocate for a compressed timescale of Earth's history, aligning with a literal interpretation of biblical accounts. Others, like Per Bak, explore models like SEC,

suggesting the universe's emergence not from a singularity but from a state of inherent potentiality.

Spontaneous Emergence Creation: A New Paradigm?

Perbak's SEC model can be interpreted that the universe arises spontaneously from a self-organized critical state without needing a Big Bang singularity. This model resonates with the Genesis 1:1-2 description of the Holy Spirit bringing order to a chaotic void. While this connection remains a matter of theological interpretation, the SEC model presents a scientifically intriguing proposition that challenges the traditional Big Bang narrative.

Non-linear Ermakov Equations and the Dance of Order and Disorder:

One interesting solution of the non-linear Ermakov equations appears to exhibit SEC-like behavior. This solution shows a system transitioning from a disordered state to one exhibiting complex structures, reminiscent of the creation described in Genesis. While not definitive proof of SEC, it provides a tantalizing glimpse into the potential for such models to explain cosmic and biological emergence.

Reconciling Science and Faith, especially on the origin of the Earth:

The question of reconciling scientific and religious narratives surrounding creation remains a complex and ongoing dialogue. Exploring models like SEC and their potential connections to Genesis offers a space for deeper engagement between science and faith traditions.

Ultimately, the quest for understanding the universe's origins requires an open mind and a willingness to explore all avenues of inquiry. Rigorous scientific investigation, open-mindedness to alternative models, and respectful dialogue between science and faith traditions can pave the way for a richer understanding of our cosmic and existential place in the grand scheme of things.

Possible correspondence between low temperature physics and cosmology

In previous sections of this review, we explored the intriguing possibility of reconciling scientific understanding of the early universe with the creation narrative presented in Genesis 1:1-2. We ventured into the fascinating realm of gap theory, which posits a period of "gap" or *dormancy* between the initial burst of creation and the subsequent

shaping of the Earth and its inhabitants (cf. Gen. 1;2, cf. also Walton, *the Lost World of Genesis One*). Today, we delve deeper into this intriguing proposition, specifically examining how certain solutions of nonlinear Ermakov equations, utilized in low-temperature physics, might hold the key to unlocking this cosmic secret.

The connection between Ermakov equations and the gap theory lies in their shared ability to depict sudden transitions from a state of stasis to one of dynamic change. Just as these equations, employed in low-temperature physics, describe abrupt shifts in the behaviour of superfluid helium atoms, the gap theory envisions a similar leap from the initial, formless "void" described in Genesis to the subsequent flourishing of the universe.

Here, imagine the universe in its nascent state as a superfluid-like entity, governed by the Ermakov equations. In this primordial stillness, a quantum fluctuation, a subtle perturbation in the energy field, acts as a critical trigger (which alternatively, such an epoch can be interpreted as an act of God’s creation). This, analogous to the "kick" parameter in the equations, initiates a cascade of activity, breaking the symmetry of the void and setting the stage for the unfolding drama of creation.

This confluence of scientific models and scriptural interpretation offers a compelling framework for understanding the Genesis narrative. The "days" of creation, instead of being interpreted literally, can be seen as representing distinct phases within this vast cosmic leap – the emergence of light, the separation of heaven and earth, the formation of land and sea. Each "day" becomes a snapshot of the universe transitioning from its initial inertness to its present vibrant state.

Of course, this theoretical bridge between science and faith is not without its challenges. Critics might argue that invoking physics as an interpretive tool for scriptural passages introduces unnecessary complexity and risks diluting the core message of divine creation. Others might point to the inherent limitations of scientific models in capturing the full sweep of spiritual truths.

Yet, the very act of seeking dialogue between seemingly disparate fields of knowledge – science and religion – holds immense value. It fosters mutual understanding,

encourages deeper reflection on the mysteries of existence, and compels us to constantly revisit our assumptions about the universe and our place within it.

The *gap theory*, viewed through the lens of dynamics of nonlinear Ermakov equations, may not definitively prove or disprove the divine hand in creation. But it offers a unique perspective, a tapestry woven from threads of scientific rigor and scriptural interpretation. And perhaps, in that weaving, lies the spark of a deeper understanding, a glimpse into the grand symphony of existence that unites the quantum dance of atoms with the majestic pronouncements of Genesis.

This is just the beginning of our exploration. In the coming articles, we will delve deeper into the scientific and theological intricacies of this topic, examining the challenges and possibilities it presents. Let us embark on this intellectual journey with open minds and a spirit of inquiry, embracing the potential for discovery that lies at the intersection of science and faith.

Beyond Darwin-Malthusian's random mutation and natural selection: Pan-Christic Biogenesis and the Hidden Forces of Life

The Darwinian paradigm of random mutation and natural selection, fuelled by a Malthusian struggle for survival, has long dominated our understanding of life's evolution. Yet, its mechanistic lens seems to falter before the complex dance of life's emergence and dynamism. Could there be hidden forces at play, orchestrating this symphony of change beyond the simple tug-of-war between chance and competition?

Francesco Redi's famous experiment in 17th century, demonstrating spontaneous generation of organic compounds shall always start with life substance, reminds us that biogenesis is woven into the fabric of our universe. This prompts us to seek alternative frameworks, models that resonate with the inherent vibrancy and purposefulness of life. One such lens lies in Lewin's force field theory of change, where both driving and restraining forces interact to determine a system's trajectory.

In this light, let us envision life's evolution as a dynamic interplay of four hidden forces:

1. Order and Disorder: These opposing forces represent the constant interplay between stability and transformation within living systems. Order maintains coherence and structure, while disorder injects novelty and creativity. Life emerges and thrives in this delicate balance, where structures adapt and mutate, and new forms arise.

2. Synchronization and Individuality: This tension reflects the interconnectedness of life and the unique spark within each organism. Synchronization allows communities to thrive through coordinated action, while individuality drives differentiation and innovation. Life flourishes as these forces dance in harmony, enabling collaboration and the emergence of diverse expressions within a unified whole.

3. Life Essence: This is where we venture beyond the purely material, acknowledging the inherent spark of life within every living being. We propose that this "life essence" is not merely a passive product of chance and selection, but an active principle, a divine breath imbued in all forms by the ultimate Source of Life, which Leibniz, in his pan-Christic vision, might call the Monad.

This life essence is not to be confused with *panentheism* or old idea of *vitalism* in biology, where the divine permeates the universe but doesn't actively participate. Nor is it *monadism*, where individual monads are self-contained and uninfluenced by each other. Pan-Christic biogenesis posits that the life essence is not static, but dynamic, actively involved in the unfolding of life, guiding its trajectory, and imbuing each creature with a unique potential for growth and transformation. In our reading of Genesis 1:1, in Hebrew Bible, we shall read the first verse as follows: "*bereshit bara Elohim (et),"* where aleph tav is not prononunced but it can be understood as presence of Christ in His pre-existence. That is our interpretation of both pan-Christic biogenesis, and in a consistent interpretation with Hebrew meaning of dabar or Logos who was with God since the beginning, cf. John 1;1, Colossians chapter 1.[1]

Leibniz's "monads with windows" beautifully capture this notion. Each monad, representing a living being, houses its own life essence, its unique spark of the divine,

[1] With special thanks to Prof Pramod, an earlier version of our interpretation of Genesis 1:1 has been presented at *1st Conference of Philosophy Research,* held at Malla Reddy University, India, Aug. 2022.

while remaining open to influences from other monads and the overarching divine influence. This interconnectedness fosters both individuality and a shared essence, creating a tapestry of life woven from both autonomy and unity.

Pan-Christic biogenesis, with its focus on the hidden forces of order, disorder, synchronization, and life essence, offers a framework for understanding life that transcends the limitations of purely materialistic perspectives. It recognizes the interplay of chance and determinism, competition and cooperation, and acknowledges the presence of a divine spark within every living creature. This model invites us to see evolution not as a blind struggle for survival, but as a purposeful journey of self-actualization, driven by the inherent potential of life and guided by the unseen hand of the Divine.

While this remains a theoretical landscape, exploring avenues beyond the Darwin-Malthusian program enriches our understanding of life's complexity and opens doors to new avenues of scientific and spiritual inquiry. Perhaps, it is in embracing the interplay of the hidden forces, both material and divine, that we will truly grasp the awe-inspiring dance of life unfolding on our planet.

To find traces of Pan-Christic biogenesis origin in proto-solar system

The alluring proposal of pan-Christic biogenesis, with its focus on hidden forces and divine essence intertwined with evolution, invites us to peek beyond the established narratives of life's emergence. To lend concreteness to this theoretical framework, let's turn our gaze towards the very birth of our solar system, seeking traces of pan-Christic biogenesis in the primordial dance of dust and gas that gave rise to our celestial home.

The early solar system, a swirling nebula known as the protoplanetary disk, presents a compelling canvas for this exploration. This cosmic cradle contained a rich tapestry of organic compounds and complex prebiotic molecules – the building blocks of life – as evidenced by meteorites and comets, remnants of that ancient era. These celestial wanderers carry within them a whisper of the past, a potential Rosetta Stone of pan-Christic origins.

1. The Whispers of Life in Comets: These icy celestial bodies are time capsules, frozen in the deep space for billions of years. Their pristine makeup holds traces of the protosolar nebula, offering a glimpse into the primordial soup from which life may have arisen. Recent missions like Rosetta have detected a remarkable variety of complex organic molecules within comets, including amino acids, the very building blocks of proteins. Could these be mere random products of cosmic chemistry, or do they resonate with the "life essence" proposed by pan-Christic biogenesis? Perhaps their specific arrangement, their very presence within the icy embrace of a comet, speaks of a guiding force, a hidden order beyond the realms of pure chance.

2. Asteroids: Fossilized Seeds of Pan-Christic Potential: While lacking the icy embrace of comets, asteroids offer another window into the early solar system. Composed of rocky remnants from the protoplanetary disk, they too can harbor organic compounds and offer insights into the initial distribution of these life-giving molecules. The presence of complex amino acids in certain meteorites further fuels the fire of pan-Christic inquiry. Could these be traces of a proto-ecosystem, an early spark of life nurtured by the "order and synchronization" forces within the protoplanetary disk? Or are they merely echoes of a cosmic game of chance, devoid of any divine orchestration?

3. The Link Between Proto-Solar System and Life's Essence: Tracing the connection between these primordial traces of life and the pan-Christic concept of "life essence" requires venturing beyond the realm of physical evidence. It asks us to contemplate the interplay between the material and the divine, the dance between the building blocks of life and the spark that animates them. Perhaps the protosolar nebula, with its rich prebiotic soup and intricate dance of forces, represents not just a random scattering of materials, but a crucible where the "life essence" first infused into existence, imbuing even the nascent forms of life with a potential for growth and evolution.

While definitive proof of pan-Christic biogenesis may remain elusive, the very act of seeking its traces in the proto-solar system pushes the boundaries of our understanding. It compels us to consider the possibility that life is not merely a product of chance collisions and ruthless competition, but a symphony orchestrated by hidden forces, both material and divine. The whispers of life in comets, the fossilized potential in asteroids, and the grand cosmic story of the protoplanetary disk all become potential clues in this

grand detective story, urging us to listen for the echoes of pan-Christic biogenesis in the very origins of our solar system.

The search for life's origins, then, becomes not just a scientific pursuit, but a philosophical and spiritual one as well. It invites us to embrace the mystery, the interconnectedness, and the possibility of a universe imbued with meaning beyond the material. And perhaps, in this very act of seeking, we may uncover not just traces of life, but a deeper understanding of our own place within the grand symphony of existence.

Concluding remarks

In conclusion, while Darwin's name rightfully adorns the pages of "On the Origin of Species," we must remember the vital contributions of Lamarck, Wallace, and Malthus. Their ideas provided the soil from which Darwin's genius blossomed, reminding us that scientific breakthroughs are rarely solitary endeavors, but often the fruit of a shared intellectual landscape.

We also consider several alternative hypotheses beyond out of Africa theory. Nonetheless, it shall be kept in mind, that the present review article is meant to explore alternative perspectives and does not endorse any specific unproven theories. It should be read as a thought-provoking exercise in considering the complexities of human origins.

We then consider that the alluring proposal of pan-Christic biogenesis, with its focus on hidden forces and divine essence intertwined with evolution, invites us to peek beyond the established narratives of life's emergence. To lend concreteness to this theoretical framework, let's turn our gaze towards the very birth of our solar system, seeking traces of *pan-Christic biogenesis* in the primordial dance of dust and gas that gave rise to our celestial home.

The search for life's origins, then, becomes not just a scientific pursuit, but a philosophical and spiritual one as well. It invites us to embrace the mystery, the interconnectedness, and the possibility of a universe imbued with meaning beyond the material. And perhaps, in this very act of seeking, we may uncover not just traces of life, but a deeper understanding of our own place within the grand symphony of existence.

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Assuming Photon as Extended Point Particle in the HyperSoft Topological Space and other Hypotheses: Issues and Trend Analysis

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Abstract

Following our preceding article, where we discussed alternative interpretations of the advanced perihelion of Mercury, the present article revisits the 1919 solar eclipse expedition led by Arthur Eddington, which famously provided the first observational confirmation of Einstein's theory of general relativity. We focus on the deflection of starlight data obtained during the eclipse, a cornerstone of this validation. Here, we explore three alternative explanations for the observed light bending that challenge the sole attribution to general relativity. Firstly, the paper begins by arguing based on criticisms raised by Tullio Levi-Civita, a contemporary mathematician, regarding Einstein's use of pseudo-tensors in his calculations. Levi-Civita argued that this approach introduced unnecessary complexity and obscured alternative interpretations of the data. Secondly, we delve into astrophysical phenomena that could mimic the observed light deflection, based on the varying speed of light, by assuming a photon has mass. (cf. 't Hooft et al., Light is Heavy). Moreover, in the literature, Molodtsov initiated soft set theory as an extension of fuzzy set theory to deal with uncertainties occurring in the natural and social sciences. It attracted the attention of mathematicians as well as social scientists due to its potential to unify certain mathematical aspects and applications in decision making processes; therefore, we shall discuss a bit how to model the photon as an extended massive particle of light, possibly related to such a soft set point [15, 16]. For further exploration, it is possible to assume the crystalline lattice of subvacuum structure (cf. Gremaud) as part of hypersoft topological spaces [16, 16a]. We strongly believe that the true strength of science lies in its continuous search for evidence and refinement of existing models. Therefore, it can be expected that new data can be helpful to reevaluate these matters, for instance, in the upcoming eclipse in the next month of 2024.

Keywords: Hypersoft Topological Spaces; Photon; Soft Set; Fuzzy Set.

1 | Introduction

As Longair wrote (2015), the famous eclipse expedition of 1919 to Sobral, Brazil, and the island of Principe, in the Gulf of Guinea, led by Dyson, Eddington and Davidson was a turning point in the history of relativity, not only because of its importance as a test of Einstein's General Theory of Relativity, but also because of the intense public interest which was aroused by the success of the expedition [1]. Nonetheless, Soares argues to

present some facts that the outcome of experiment as part of the expeditions organized to observe the 1919 solar eclipse served British astrophysicist Arthur Eddington was biased by his scientific authority. He then goes further to suggest that a similar behavior is observed in modern science, with the caveat that the authority to validate an experiment has shifted from a single individual to either a group — a community — that endorses a given scientific view. As we know, there is also a purely Newtonian prediction for the effect, which is precisely one half of the relativistic prediction. There are two pertinent questions, namely:

1. Is there a gravitational deflection of light?
2. If so, what is the better theoretical description of the phenomenon, the Einsteinian, the Newtonian or a third alternative? [2]

Following our preceding article where we discussed alternative interpretation of advanced perihelion of Mercury, here we argue in this article based on criticisms raised by Tullio Levi-Civita, a contemporary mathematician, regarding Einstein's use of *pseudo-tensors* in his calculations. Levi-Civita argued that this approach introduced unnecessary complexity and potentially obscured alternative interpretations of the data.[3][4] While it's not quite accurate to say Levi-Civita simply criticized Einstein. While there was some disagreement, it was more nuanced than just pure criticism. To be more precise way to frame it: **Levi-Civita identified a mathematical error** in Einstein's early attempts (around 1914-1915) to formulate his theory of general relativity, specifically related to the use of pseudo-tensors. The theory of general relativity, developed by Albert Einstein, revolutionized our understanding of gravity. However, even the minds of geniuses are not infallible. In 1915, Einstein introduced a concept called the stress-energy pseudo-tensor. While intended to represent the flow of energy and momentum in curved spacetime, the pseudo-tensor had a crucial flaw – it wasn't generally conserved. This is where Tullio Levi-Civita, a renowned Italian mathematician, enters the picture. Levi-Civita, a collaborator of Einstein, is credited with developing the mathematical machinery – the Levi-Civita connection – that underpins general relativity. It's likely that Levi-Civita was aware of the limitations of the pseudo-tensor.

The historical record showa an explicit criticism from Levi-Civita, as follows [3]:""Since this fact" - these are his words - "should not happen in nature, it seems likely that quantum theory should intervene by modifying not only Maxwell's electrodynamics, but also the new theory of gravitation". Actually there is no need of reaching to quanta. It is enough to correct the formal expression of the gravitational tensor in the way shown here. Then the possibility of being confronted with consequences not corresponding to the physical intuition is a priori excluded, in the case either of free waves or of another purely gravitational phenomenon. In fact, by virtue of (10') or, if one likes, of the generalised d'Alembert's principle, when the energy tensor T_{ik} vanishes, the same occurrence must happen to the gravitational tensor A_{ik}. This fact entails total lack of stresses, of energy flow, and also of a simple localisation of energy."[3]

While this lack of documented critique has sparked discussions among historians of science, it can be debatable, why did Levi-Civita hold back from criticizing a friend and collaborator? Or perhaps the limitations of the pseudo-tensor weren't fully apparent at the time. Nonetheless from the above quote, it seems that at the time Levi-Civita did not really agree with Einstein having to call the new theory of quanta, which later on to progress further into quantum physics.

The true nature of Levi-Civita's silence remains a subject of debate. However, the episode highlights the importance of critical evaluation in science. Even the most brilliant ideas can have flaws, and open discussion is crucial for refining them. To summarize, interested readers may consult secondary sources on the history of general relativity, which might mention the limitations of the pseudo-tensor and the role of Levi-Civita. This is a fascinating intersection of scientific progress and personal dynamics.

Secondly, we delve into the problems related to possible violation of Weak Equivalence Principle (WEP), as well as astrophysical phenomena that could mimic the observed light deflection, based on varying speed of light, by assuming photons having mass. These might include a hypothesis of varying speed of light and photon having mass (cf. 't Hooft *et al.*). Secondly, the paper explores instrumental limitations of the equipment used in the 1919 expedition. Measurement errors or biases inherent to the telescopes and photographic techniques employed could have influenced the final results.

By revisiting these arguments and exploring alternative explanations, this paper aims to provide a more nuanced understanding of the 1919 eclipse data. While not denying the significance of Eddington's observations, it encourages a critical reevaluation of the data's sole attribution to general relativity and highlights the importance of considering alternative interpretations.

1.1 | First argument: Plausible violation of Weak Equivalence Principle

From a historical point of view, the Equivalence Principle was first clearly formulated by Galileo Galilei as the independence of a body's acceleration in the Earth's gravitational field on the body's mass [5]. Later on, what is called Weak Equivalence Principle becomes a cornerstone of general theory of relativity theory. As Lebed wrote, that modern physics distinguishes between three variants of the Equivalence Principle: weak Equivalence Principle, Einstein's Equivalence Principle, and strong Equivalence Principle. The weak Equivalence Principle is equivalent to the so-called universality of a free fall and is related to the original Galileo experiments. This variant has been confirmed with great accuracy for ordinary solid-state bodies. In the recent space mission, MICROSCOPE of the French space agency CNES, the experimental equipment consisted of two cylindrical shells: larger cylindrical shell and smaller one [5].

What is quite remarkable from Lebed's book and also other arguments for instance by Lo etc, is the unexpected breakdowns of Einstein's Equivalence Principle for composite quantum bodies can be found. See also Lo, 1993 *etc.* [6, 7].

It is known, as noted by Bjerrum-Bohr *et al.*, while deterministic physics is a crucial ingredient in general relativity, i.e., particles follow field equations formulated as geodesic equations, in quantum mechanics such a concept has no meaning since one has to accept that space and momentum are mutually complementary concepts [8]. In their article at PRL suggesting a small correction to Newtonian gravitation expression due to quantum gravity, as follows [8]:

$$V(r) = -GmM/r\,(1 + a), \tag{1}$$

Where

$$a=3G(M+m)/(rc\hat{}2r))+41Gh/(10.c\hat{}3.r\hat{}2) \tag{2}$$

All of these seem to suggest that deeper considerations based on zero point energy and vacuum structure can play quite significant role to the effect of (possible) bending of light, more than just what is attributed to gravitation of massive bodies alone [9].

1.2 | Second Argument: Varying Speed of Light, Rethinking Light Bending - Gravity or Speed?

The 1919 solar eclipse expedition, led by Arthur Eddington, stands as a landmark achievement in scientific history. It provided the first observational evidence for Einstein's theory of general relativity, with the deflection of starlight by the Sun's gravity taking center stage. However, the interpretation of this data has not been without controversy. This paper challenges the sole attribution of light bending to general relativity by exploring an alternative explanation: the varying speed of light due to a possible, albeit small, mass for photons. This proposition finds inspiration in the work of theoretical physicists such as Gerard 't Hooft et al, who has argued for the possibility of *massive photons* under specific conditions. While this concept may seem unorthodox – some might even say heretical – it offers a potentially testable alternative to explain the observed light deflection [10, 11].

While the established view of general relativity remains dominant, revisiting the 1919 data through the lens of varying light speed allows for a more comprehensive understanding of this critical phenomenon. This introduction sets the stage for exploring the implications of massive photons and their potential influence on light bending, offering a fresh perspective on a well-established theory.

To summarize what we argue above, the 1919 solar eclipse data, often cited as a cornerstone of general relativity, can be alternatively interpreted using the concept of varying light speed induced by a possible, non-zero mass for photons. This section explores how to simulate such light bending using Mathematica code.

1.2.1 | Defining the Speed of Light Function:

First, we need to define a function that describes the variation of light speed with respect to, for instance, the gravitational potential (Φ). This function can be based on theories like those proposed by Gerard 't Hooft. Here's an illustrative example in Mathematica Code.

```
c[Φ_] := c0*(1 - epsilon*Φ)  (* c0 is the constant speed of light in vacuum, epsilon is a small constant representing the effect of photon mass *)
```

1.2.2 | Ray Tracing with Varying Speed:

Next, we can implement a ray tracing algorithm to simulate the path of light rays passing near a massive object like the Sun. This algorithm takes into account the position of the light source, the object's mass, and the defined speed of light function.

Here's a simplified example using Newton's law of gravity for illustration:

Code snippet

```
rayPath[source_, object_, impactParameter_] := Module[{
initialPosition, initialVelocity, timesteps, positions
},
initialPosition = source;
initialVelocity = Normalize[object - source];
timesteps = 100;  (* Adjustable number of time steps *)
positions = Flatten[Table[
initialPosition + NIntegrate[initialVelocity/c[Φ[Norm[position - object]]], {t, 0, dt}],
{dt, 0, 1, 1/timesteps}
]];
Positions
]
```

This code defines a function rayPath that takes the light source position, object position (e.g., the Sun), and impact parameter (distance between the light ray and the object's center) as inputs. It then iteratively calculates the position of the light ray at each time step based on its initial velocity and the varying speed of light due to the object's gravitational potential (Φ).

By running rayPath with and without the varying speed function (c[Φ[Norm[position - object]]]), we can compare the resulting light paths. The difference in deflection angles between the two simulations would represent the light bending effect attributed to the varying speed of light in our scenario.

This is a simplified example, and a full simulation would require a more sophisticated approach incorporating general relativity or alternative gravity theories with varying light speed. Additionally, incorporating realistic solar corona models and measurement uncertainties would be crucial for a complete analysis.

Moreover, in the literature, Molodtsov initiated *soft set theory* as an extension of fuzzy set theory to deal with uncertainties occurring in natural and social sciences. It attracted the attention of mathematicians as well as social scientists due to its potential to unify certain mathematical aspects and applications in decision making processes. In an attempt to study different existing mathematical structures in the context of soft set theory, the notion of a soft point plays a significant role. [15][16] Therefore, it seems quite possible to extend further the definition of photon as merely points, provided that photons are massive, to become *extended massive points,* possibly related to soft set hypothesis. We will discuss it further in the Appendix #2 section, how it is possible to do a simulation of collision between photon as extended massive point and crystalline ether particle as hypothesised by G. Gremaud.

1.3 | Third Argument: Aspden-Gremaud's Crystalline Ether Hypothesis

That is, by assuming light quanta collide with minuscule particles of crystalline ether (Gremaud), we can predict that the distribution of light bending data is scattered distribution just as observed by Eddington.

In 1919, Sir Arthur Eddington's expedition to observe the solar eclipse captured the world's imagination. His measurements, it was claimed, provided the first observational confirmation of Einstein's theory of general relativity. However, a closer look at the data reveals a story less definitive than often portrayed.

Eddington's experiment aimed to measure the deflection of starlight by the Sun's gravity. The method involved comparing the positions of stars near the Sun during a solar eclipse to their known positions at other times. The starlight, it was predicted by general relativity, would be bent ever so slightly by the Sun's mass.

Here's where the shadows become murkier. Eddington's analysis of the photographic plates capturing the eclipse revealed a level of scatter in the data – meaning the measurements displayed a wider range of values than expected. This raises a crucial question: could such variance cast doubt on the validity of the entire experiment?

Statistics come into play here. Large variance, especially when coupled with a small sample size (the number of stars measured), can lead to scattered results. In such cases, drawing statistically significant conclusions becomes difficult. The wider the spread of the data points, the less confident we can be about the true value we're trying to measure. See the following Table 1:

Table 1. Photoplate data [14].

No. of star	Coordinate (observed)	Calculated (First)	Coordinate (observed)	Calculated (Second)
11	-0°19	-0°22	+0°16	+0°02
5	+0°29	+0°31	-0°46	-0°43
4	+0°11	+0°10	+0°83	+0°74
3	+0°20	+0°12	+1°00	+0°87
6	+0°10	+0°04	+0°57	+0°40
10	-0°08	+0°09	+0°09	+0°35
2	+0°95	+0°85	-0°27	-0°09

In Eddington's case, the scattered data might not necessarily disprove general relativity. However, it does call into question the conclusiveness of his experiment as the sole verification of the theory. The large variance could be attributed to various factors, including:

- **Measurement Errors:** The process of measuring stellar positions on photographic plates is inherently prone to error.
- **Instrumental Limitations:** The telescopes used in 1919 might not have been sophisticated enough to achieve the required level of precision.
- **Unaccounted for Factors:** Other astrophysical phenomena, like the Earth's atmosphere or interstellar dust, could have influenced the starlight, leading to deviations in the measurements.

Eddington himself acknowledged the limitations of his data. However, the excitement surrounding the confirmation of general relativity overshadowed these concerns.

Re-imagining Eddington's Eclipse: A Scattered Dance of Light (Mathematica Code Included)

Sir Arthur Eddington's 1919 solar eclipse expedition captured the scientific world's attention. His observations, believed to confirm Einstein's general relativity, displayed a surprising characteristic – scattered data. This article explores a hypothetical scenario, based on Gremaud's theory of a crystalline ether, to see if it could explain the observed scattered distribution.

The Crystalline Ether Hypothesis:

Unlike the currently accepted view of empty space, Gremaud proposed a "crystalline ether" – a medium permeating space with a regular, crystalline structure. Light quanta (photons) traveling through this ether could experience collisions with its constituent particles, leading to deviations in their trajectories.

Modeling the Scattering Effect:

While the crystalline ether concept is not widely accepted in modern physics, let's explore it as a thought experiment. Here's a sample Mathematica code to simulate light-particle collisions and their effect on light bending:

```
(* Define constants *)
c = 299792458; (* Speed of light *)
G = 6.6743e-11; (* Gravitational constant *)
Msun = 1.989e30; (* Solar mass *) (* Simulate particle collisions *)
scatterAngle = RandomReal[{0, 1}]*Pi/10; (* Random deflection angle *)
(* Calculate light bending due to gravity *)
gravityDeflection = G*Msun/(c^2*Distance); (* Adjust Distance as needed *)
(* Combine effects and calculate total deflection *)
totalDeflection = gravityDeflection + scatterAngle;
```

This code defines constants like the speed of light and gravitational constant. The scatterAngle function introduces a random deflection component due to hypothetical collisions within the ether. This is then added to the deflection caused by the Sun's gravity to get the total deflection. By running this code multiple times and plotting the distribution of total deflection values, we can observe a scattered pattern. This could mimic the spread observed in Eddington's data.

Caveats:

It's crucial to remember that the crystalline ether hypothesis is still an infancy hypothesis by Gremaud and Harold Aspden and few others (including these writers). This simulation serves as a thought experiment, and only for initial simulation purposes. Alternatively, see also Appendix 1.

For further exploration, it is possible to assume the crystalline lattice of subvacuum structure (cf. Gremaud) as part of *HyperSoft Topological Spaces* [16][16a], but this possible hypothesis will be kept as possible venue for future exploration.[1]

Absolute Theory of Relativity

In 1982, in his book on special relativity, Smarandache [18] presented the hypothesis that there is no speed barrier in the universe - thus refuting the speed of light postulate.

While Einstein considered a relative space and relative time but the ultimate speed of light, he did the opposite: he considered an absolute time and absolute space but no ultimate speed, and he call it the Absolute Theory of Relativity (ATR).

Parameterized Special Theory of Relativity

He then parameterized Einstein's thought experiment with atomic clocks, supposing that we know neither if the space and time are relative or absolute, nor if the speed of light is ultimate speed or not. He obtained a Parameterized Special Theory of Relativity (PSTR). His PSTR generalized not only Einstein's Special Theory of Relativity, but also his ATR, and introduced three more possible Relativities to be studied in the future.

Noninertial Multirelativity

Afterwards, he extended his research considering not only constant velocities but constant accelerations too. Eventually he proposed a Noninertial Multirelativity for the same thought experiment, i.e. considering non-constant accelerations and arbitrary 3D-curves.

Oblique-Contraction Factor

In 1983, following the Special Theory of Relativity, Smarandache [19] generalized the Lorentz Contraction Factor to an Oblique-Contraction Factor, which gives the contraction factor of the lengths moving at an oblique angle with respect to the motion direction.

---

[1] Smarandache generalized the notion of a soft set to a hypersoft set by replacing the function with a multi-argument function specified in the cartesian product with a different set of parameters. This idea is more versatile than the soft set and more applicable in the sense of decision-making issues. HyperSoft set structure has attracted the attention of researchers because it is more suitable than soft set structure in decision making problems [16, 16a].

He also proved that relativistic moving bodies are distorted, and he computes the Angle-Distortion Equations. He then shows several paradoxes, inconsistencies, contradictions, and anomalies in the theory of relativity.

Not all physical laws are the same in all inertial reference frames.
According to the author, not all physical laws are the same in all inertial reference frames, and he gives several counter-examples. He also supports superluminal speeds, and he considers that the speed of light in vacuum is variable.
He explained that the redshift and blueshift are not entirely due to the Doppler Effect, but also to the medium composition (i.e. its physical elements, fields, density, heterogeneity, properties, etc.).

Medium Lensing
He considered that the space is not curved and the light near massive cosmic bodies bends not because of the gravity only as the General Theory of Relativity asserts (Gravitational Lensing), but because of the Medium Lensing.
In order to make the distinction between "clock" and "time", he suggests a first experiment with a different clock type for the GPS clocks, for proving that the resulted dilation and contraction factors are different from those obtained with the cesium atomic clock; and a second experiment with different medium compositions for proving that different degrees of redshifts/blushifts would result.

2 | Concluding Remark

Mathematica provides a powerful tool to simulate light bending under the assumption of varying light speeds. By comparing these simulations with established general relativity models, we can gain further insights into the interpretation of the 1919 eclipse data and potentially explore new avenues for testing alternative theories of gravity.
While the scattered nature of Eddington's data is often attributed to measurement errors or limitations, this exercise demonstrates that alternative hypotheses, even hypothetical ones, can potentially lead to similar scattered results. The true strength of science lies in its continuous search for evidence and refinement of existing models. It can be expected that new data can be helpful in reevaluating these matters, for instance, the upcoming eclipse in 2024.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Appendix 1

Here is a more complete Mathematica code that incorporates collisions between light photons and very small crystalline ether particles (with size less than 0.001 femtometer):

(* Define constants *) c = 299792458; (* Speed of light *) h = 6.62607015e-34; (* Planck constant *) Msun = 1.989e30; (* Solar mass *) G = 6.6743e-11; (* Gravitational constant *) (* Define crystalline ether properties *) latticeSpacing = 1*10^-22; (* Lattice spacing of crystalline ether (less than 0.001 femtometer) *) etherParticleMass = 1*10^-40; (* Mass of individual ether particle (assumption) *) (* Function to calculate effective density based on lattice spacing *) effectiveDensity[mass_, spacing_] := mass/(latticeSpacing^3) (* Function to calculate impact parameter within the crystal *) impactParameter = RandomReal[{0,

latticeSpacing/2}]; (* Restricts impact within a unit cell *) (* Function to calculate scattering angle due to collision with a single ether particle *) scatteringAngleSingle := Module[{b, sigmaT, q}, b = impactParameter; (* Impact parameter *) sigmaT = (8*Pi*(etherParticleRadius^2))/3; (* Total cross section for scattering, assuming spherical particle *) q = 2*Pi/h * (c*impactParameter); (* Momentum transfer *) (* Use appropriate scattering amplitude function here (e.g., Rayleigh scattering for small particles) *) (* For simplicity, assuming constant scattering amplitude (amplitude = 1) *) ArcTan[(sigmaT*q)/2] (* Scattering angle based on impact parameter and cross section *)] (* Helper function to calculate ether particle radius *) etherParticleRadius = etherParticleMass/(4/3*Pi*effectiveDensity[etherParticleMass, latticeSpacing]); (* Simulate particle collisions within the crystalline ether *) gravityDeflection = G*Msun/(c^2*Distance); (* Adjust Distance as needed *) totalDeflection[numCollisions_] := Module[{totalAngle = 0}, Do[totalAngle += scatteringAngleSingle[]; (* Accumulate scattering angles from each collision *) , {i, 1, numCollisions}]; totalAngle + gravityDeflection; (* Combine gravity and total scattering deflection *)] (* Run the simulation multiple times and collect deflection data *) numSimulations = 1000; (* Adjust desired number of simulations *) collisionsPerSimulation = 10; (* Simulate collisions with multiple ether particles *) deflectionData = Table[totalDeflection[collisionsPerSimulation], {i, 1, numSimulations}]; (* Analyze and plot the distribution of deflection data *) Histogram[deflectionData, Bins -> "Scott"] (* Use appropriate binning method *)

This code incorporates the following improvements:

- **Crystalline Ether:** Introduced latticeSpacing to define the crystalline structure and effectiveDensity function to calculate density based on particle mass and spacing.
- **Collision with Multiple Particles:** The totalDeflection function now takes numCollisions as input and accumulates scattering angles from multiple collisions within the crystal using a loop.
- **Code Readability:** Improved variable naming and added comments for better understanding.

Further Exploration:

- Implement a more sophisticated scattering amplitude function based on the specific interaction between the massive photon and the crystalline ether.
- Explore different crystalline structures and their impact on the scattering behavior.
- Analyze the dependence of deflection on the number of collisions within the crystal.
- Visualize the trajectories of the massive photon after interacting with the crystalline ether.

Notes:

- This code simulates a simplified scenario and does not account for all the complexities of light-matter interactions.
- The chosen scattering amplitude function significantly impacts the scattering angle distribution.

Appendix 2: Possible further improvement of collision simulation: photon as extended soft point and crystalline ether

Moreover, in this section, we shall see how it is possible to do a simulation of collision between photon as extended soft point and crystalline ether as hypothesised by G. Gremaud.

Mathematica code:

(* Define constants *) c = 299792458; (* Speed of light *) h = 6.62607015e-34; (* Planck constant *) Msun = 1.989e30; (* Solar mass *) G = 6.6743e-11; (* Gravitational constant *) (* Define crystalline ether properties *) latticeSpacing = 1*10^-22; (* Lattice spacing of crystalline ether (less than 0.001 femtometer) *) etherParticleMass = 1*10^-40; (* Mass of individual ether particle (assumption) *) (* Function to calculate effective density based on lattice spacing *) effectiveDensity[mass_, spacing_] := mass/(latticeSpacing^3) (* Function to calculate impact parameter within the crystal *) impactParameter = RandomReal[{0, latticeSpacing/2}]; (* Restricts impact within a unit cell *) (* Gremaud ether particle scattering amplitude function *) gremaudAmplitude[q_] := Module[{alpha}, (* Alpha is the fine-structure constant *) alpha = 1/137; (* Reference: https://arxiv.org/abs/hep-ph/0102255 *) (- alpha^2 * q^2) / (1 + q^2/4*Pi*alpha) (* Amplitude based on momentum transfer q *)] (* Function to calculate scattering angle due to collision with a single ether particle *) scatteringAngleSingle := Module[{b, sigmaT, q, amplitude}, b = impactParameter; (* Impact parameter *) sigmaT = (8*Pi*(etherParticleRadius^2))/3; (* Total cross section for scattering, assuming spherical particle *) q = 2*Pi/h * (c*impactParameter); (* Momentum transfer *) amplitude = gremaudAmplitude[q]; (* Use Gremaud scattering amplitude *) (* Integrate over scattering angle to get deflection angle *) NIntegrate[(b*amplitude)/q, {theta, 0, 2*Pi}] (* Numerical integration for deflection angle *)] (* Helper function to calculate ether particle radius *) etherParticleRadius = etherParticleMass/(4/3*Pi*effectiveDensity[etherParticleMass, latticeSpacing]); (* Simulate particle collisions within the crystalline ether *) gravityDeflection = G*Msun/(c^2*Distance); (* Adjust Distance as needed *) totalDeflection[numCollisions_] := Module[{totalAngle = 0}, Do[totalAngle += scatteringAngleSingle[]; (* Accumulate deflection angles from each collision *) , {i, 1, numCollisions}]; totalAngle + gravityDeflection; (* Combine gravity and total scattering deflection *)] (* Run the simulation multiple times and collect deflection data *) numSimulations = 1000; (* Adjust desired number of simulations *) collisionsPerSimulation = 10; (* Simulate collisions with multiple ether particles *) deflectionData = Table[totalDeflection[collisionsPerSimulation], {i, 1, numSimulations}]; (* Analyze and plot the distribution of deflection data *) Histogram[deflectionData, Bins -> "Scott"] (* Use appropriate binning method *)

Improvements:

- **Gremaud Scattering Amplitude:** Introduced the gremaudAmplitude function to incorporate the Gremaud ether particle model for scattering.
- **Deflection Angle Calculation:** Modified the scatteringAngleSingle function to integrate over the scattering angle using NIntegrate to obtain the deflection angle based on the Gremaud amplitude.

Note:

- The integration in scatteringAngleSingle requires numerical methods due to the complex form of the Gremaud amplitude.

Towards Soliton Computer Based on Solitary Wave Solution of Maxwell-Dirac equation: A Plausible Alternative to Manakov System

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ABSTRACT In recent years, there are a number of proposals to consider collision-based soliton computer based on certain chemical reactions, namely *Belousov-Zhabotinsky* reaction, which leads to soliton solutions of coupled Nonlinear Schroedinger equations. They are called Manakov System. But it seems to us that such a soliton computer model can also be based on solitary wave solution of Maxwell-Dirac equation, which reduces to Choquard equation. And soliton solution of Choquard equation has been investigated by many researchers, therefore it seems more profound from physics perspective. However, we consider both schemes of soliton computer are equally possible. More researches areneeded to verify our proposition

KEYWORDS Soliton computation, Manakov soliton, Choquard equation, computation

INTRODUCTION: EARLY DEVELOPMENT OF SOLITON SCIENCE

There is little doubt that soliton technology has recently achieved close to pop science status as its proponents seek to put it to use for transporting vast amounts of information—the stuff that gives our era its name—farther and faster. It could well become one of the fundamental technologies in the current communications revolution. It was during the Industrial Revolution, however, that this phenomenon was first noticedand studied.[1]

A Scottish engineer by the name of John Scott Russell (year) had set out to create a more efficient hull design for canal boats (a 19th-century forerunner, perhaps, of current efforts to speed packets of

information along fiber-optic cables). One day in August of 1834, he stood beside Union Canal near Edinburgh to observe the movement of a boat being pulled by two horses. As the rope pulling the boat snapped and the boat's movement halted, its prow dropped back down and Scott Russell saw a large mass of water, a smooth, solitary wave, gather around the prow and continue rapidly down the channel. Surprised and intrigued, he followed on horseback and noticed that the wave held its shape and only very gradually diminished in height. He lost sight of it after a mile or two but was so taken with this observation that he built a 30-foot wave tank in his back yard to study the phenomenon further. Ten years later, he reported his observations to the British Association for the Advancement of Science, calling what he had observed the "wave of translation." Scott Russell considered that day back in 1834 "the happiest day of my life", but his discovery was ignored by all but one or two people, who felt compelled to prove him wrong in the scientific literature. After all, it was common knowledge that waves could not behave in this way.[1]

Vindication came from two independent camps, both of which were attempting to explain the movement of shallow water waves. Boussinesq's equation in 1872 and the Korteweg–de Vries (KdV) equation in 1895 proved that solitary waves were, indeed, theoretically possible. Many theoretical and experimental developments have been done since then.

By the early 1980s, however, fiber-optic technology had caught up somewhat, and LinnMollenauer, Roger Stolen, and Jim Gordon were able to observe soliton propagation in the lab. As mathematical results continued to appear, this group at Bell Labs was experimenting intensively with optical solitons, looking for ways to use them in long-distance telecommunication systems. Ironically, while their work was enthusiastically received in scientific circles, it seemed that the practical application could not be quickly realized, and they were told by the head of research, Arno Penzias, to desist.

Doggedly, they persisted until their results were so compelling that Penzias apologized and publicly praised the work. Mollenauer's group has since achieved several long- distance and speed records for optical soliton transmission.[1]

RECENT DEVELOPMENT: SOLITON COMPUTER POSSIBILITY

It is a known fact in computer industry, that the present silicon technology will not be able to keep up with the *Moore's law*. Therefore new ways of developing unconventional computing methods are being carried out in many labs.

Since 2005, some researchers from Princeton University have reported a new concept of soliton computation. As data rates in optical communication systems continue to increase, the demand for all-optical signal processing and computing devices does as well.

Examples of such devices include the nonlinear optical loop mirror, the temporal solitondragging gate, the spatial soliton deflection gate, and the TOAD, an asymmetric loop mirror. These devices avoid the bottleneck associated with optical-electrical conversion.[2] They describe physical state-restoring computation using collisions of optical solitons. Their work is part of a larger subject known as collision-based computing, sometimes called dynamical computation. Such constructions include ideal collisions of billiard balls, Conway's universal game of Life, and multidimensional excitable lattices. Early work on soliton computation involved soliton-like collisions in cellular automata, which demonstrated the ability to embed computation in automatausing particles. [2]

In an initial study, several researchers consider a Manakov system, which is essentially a system of coupled NLSE, where q1 (x, t) and q2 (x, t) are two interacting optical components, μ is a positive parameter representing the strength of the nonlinearity, and x and t are normalized space and time,

respectively. The two components can be thought ofas components in two polarizations, or, as in the case of a photorefractive crystal, two uncorrelated beams. [2]

The Manakov system consists of two couple NLSWs.

$$i\frac{\partial q_1}{\partial t}+\frac{\partial^2 q_1}{\partial x^2}+2\mu(|q_1|^2+|q_2|^2)q_1=0$$

$$i\frac{\partial q_2}{\partial t}+\frac{\partial^2 q_2}{\partial x^2}+2\mu(|q_1|^2+|q_2|^2)q_2=0$$

Source: ref. [2]

From a practical standpoint, successful soliton computation requires ideal interactions and error-free propagation. In this sense, it is analogous to the construction of Fredkinand Toffoli, in which ideal, elastic collisions of billiard balls were used to achieve universal and reversible computation. In reality, noise will cause variability in solitonpropagation and collision, and fault tolerance based on logical state restoration wouldneed to be implemented in order to improve system performance. In a sense, this is ananalog rather than a digital computer. Rand et al. also reported more advanced work based on bistable configurations of Manakov solitons. [2]

More recent discussions of collision-based soliton computing still use that basic Manakov system. See [3][4].

In the next section we will discuss an alternative approach for soliton computer.

A PLAUSIBLE ALTERNATIVE: SOLITON COMPUTER BASED ON CHOQUARD EQUATION

In a recent paper, we discuss how Dirac-Maxwell equation reduces to wave equation, which can be transformed into a cellular-automaton scheme. [5] Now, it is worth to remark here that a recent paper shows that there is *travelling wave solution of (classical) Dirac-Maxwell equati*on. [6]

It should be noted that numerical results showed that Dirac-Maxwell system has definitely many families of solitary waves. Here the nonnegative integer N denotes the number of nodes of the positronic component of the solution (number of zeros of the corresponding spherically symmetric solution to the Choquard equation).

It can be shown that the nonrelativistic limit of such a solitary wave solution takes theform of Choquard equation, which can be written as follows:[6]

$$\omega\phi=-\frac{1}{2m}\Delta\phi+q^2\Delta^{-1}\left(|\phi|^2\right)\phi. \qquad (1)$$

Considering that the above Choquard equations is quite similar in form with the NLSE (nonlinear Schrodinger equation), and also that Manakov system is actually a system of coupled NLSE, then we consider it plausible to consider a Manakov-Choquard system, which can be written as follows:

$$\omega\Phi_1 = -\frac{1}{2m}\Delta\Phi_2 + q^2\Delta^{-1}(|\Phi_2|^2)\Phi_1.(2)$$

$$\omega\Phi_2 = -\frac{1}{2m}\Delta\Phi_1 + q^2\Delta^{-1}(|\Phi_1|^2)\Phi_2.(3)$$

The solutions of (2) and (3) can be explored numerically with computer algebra system such as Mathematica.

As far as we know, such a coupled Manakov-Choquard system has not been proposed before forstudying soliton computation possibility.

CONCLUDING REMARKS

Considering the ever increasing demand for better computers, some researchers have considered collision-based soliton computer. And this proposal is based on Manakov system. In this paper, we consider a new concept based on travelling wave solutions of Dirac-Maxwell equations, which we call Manakov-Choquard system.

As far as we know, such a Manakov-Choquard system has not been proposed before for studying soliton computer.

We understood, that more observations and experiments are recommended to verify our propositions.

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A Few Lessons from Venezuela: Introducing a New Path of Appropriate Farming and Appropriate Renewable Energy

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Development Economics, Appropriate Farming Technologies, Appropriate Technologies, Natural Resources Curse.

Abstract
In development economics literatures, there is a known term for developing countries which tend to mismanage natural resources, that term is called natural resources curse. And two examples which have been discussed quite often is Venezuela and Norway. Here we also discuss other countries as well, including Argentina and a few lessons for Indonesia in choosing the next course of development path, especially with appropriate farming and appropriate renewable energy.

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1. INTRODUCTION

In development economics literature, there is a number of discussions with regard to the so-called natural resource course. The present article brings this theme by comparing a few notable countries in this regard, including Venezuela, Norway etc.

To put this article into context, Indonesia's recent election brings one of several people-centered programs to the forefront. Certain programs have been promised including subsidized (free) food program for the underprivileged, particularly mothers and schoolchildren (in rural areas), which sounds enticing, promising immediate relief. While we should admit good effects of such programs, before succumbing to such alluring proposals, it's crucial to consider the potential pitfalls, drawing lessons from Venezuela's experience with similar populism under Hugo Chávez.

Let us consider first Venezuela, i.e. Chávez's rise to power was fueled by promises of social programs aimed at alleviating poverty and inequality. Missions like Barrio Adentro and Mercal provided subsidized food and healthcare, initially improving living standards for many. However, these programs were largely funded by Venezuela's abundant oil revenue, creating an illusion of prosperity.

The underlying issue was the lack of sustainable economic planning. Oil dependence left Venezuela vulnerable to price fluctuations. When oil prices plummeted in the late 2000s, the government, heavily reliant on oil income to fund its populist programs, faced a severe fiscal crisis.

The consequences were devastating. The government printed money to cover the deficit, leading to hyperinflation. Food shortages became commonplace, negating the initial gains from subsidized programs. Basic necessities became unaffordable, plunging the country into economic turmoil. The rest is history.

2. WHAT NATURAL RESOURCE CURSE TERM IS: WHEN RICHES BRING THORNS

Let us suppose a country blessed with bountiful diamond mines, gushing oil reserves, or vast timberlands. You'd expect thriving economies and happy citizens, right? But the reality can be more like a fairy tale gone wrong. This paradoxical situation, where abundant natural resources lead to economic woes and social strife, is known as the (natural) **resource curse**.

So, what is this resource curse? - It's the tendency for resource-rich countries to underperform economically, despite their seemingly golden ticket. Instead of flourishing, they often experience:

- **Stunted growth:** Resource dependence neglects other sectors, making the economy vulnerable to price fluctuations.
- **Inequality and corruption:** Resource wealth concentrated in few hands fuels inequality and breeds corruption.
- **Conflict and instability:** Power struggles over resources can lead to civil wars and political instability.
- **Environmental degradation:** Unsustainable extraction practices wreak havoc on ecosystems and livelihoods.

How that can happen: Many factors contribute to the curse's grip:

- **Inflation:** Resource booms inflate the currency, making other exports uncompetitive and hindering diversification.
- **Weak institutions:** Weak institutions struggle to manage resource wealth transparently and effectively.
- **Lack of investment:** Instead of investing in education, infrastructure, and other sectors, revenue gets squandered.
- **Conflicts:** Competition for resources fuels violence and political instability, deterring further investment.

3. AN OUTLINE ON HOW A COUNTRY CAN BREAK THE NATURAL RESOURCE CURSE

The good news is, the curse isn't inevitable. Countries can find ways around it with:

- **Diversification:** Building strong, diverse economies not solely reliant on resource extraction.
- **Transparent governance:** Implementing strong institutions, combating corruption, and ensuring resource revenue benefits all.
- **Investing in the future:** Channeling resource wealth into education, infrastructure, and sustainable development.
- **Community involvement:** Empowering local communities to participate in decision-making and benefit from resource extraction.

Examples like Norway and Botswana show that with foresight and good governance, resource wealth can blossom into sustainable prosperity. By understanding the curse and taking proactive steps, resource-rich countries can turn their thorns into roses, ensuring their riches translate into lasting well-being for all.

As Elsgard wrote [1], previous studies show that *institutions* are an important determinant for the impact of the resource curse in resource dependent countries. By a comparative analysis, large differences in institutional quality are revealed. Transparency, accountability and a strong rule of law are found to be among the features that distinguish the successful countries. These differences translate into different economic policies.

While Norway and Botswana have exhibited prudent fiscal management, literature has shown that government spending in Venezuela in particular, have been procyclical and unproductive. Venezuela on the other hand, seems unlikely to pursue policies that can promote sustainable growth.

4. A FEW LESSONS FROM VENEZUELA'S PATH TO INDONESIA

While the following discussion may not be conclusive, Venezuela's predicament serves as a stark reminder of the "curse of natural resources." When natural resource wealth funds populist programs without long-term economic diversification, it creates a fragile prosperity, vulnerable to external shocks.

It is known, that Indonesia is a nation rich in natural resources, with oil, gas, coal, and minerals being major contributors to the country's economy. However, the reliance on these (natural) resources has also led to challenges in the form of environmental degradation, economic instability, and corruption. As such, Indonesia can learn from the experiences of other resource-rich countries such as Venezuela to improve national resilience and avoid the natural resources curse.

One of the key lessons that Indonesia can learn from Venezuela is the importance of diversification. Venezuela's economy has been heavily reliant on oil exports, which made up around 98% of the country's export revenue at its peak. However, this dependence on oil has left Venezuela vulnerable to fluctuations in oil prices and the global oil market. When oil prices fell sharply in 2014, Venezuela's economy went into a deep recession, with hyperinflation and shortages of basic goods.

Another lesson that Indonesia can learn from Venezuela is the importance of good governance and anti-corruption measures. Corruption has been a significant problem in both countries, with officials using their positions of power to enrich themselves at the expense of the public (sometimes dubbed as *rent-seeking*). In Venezuela, corruption has been a major factor in the mismanagement of the country's oil wealth, with funds being diverted away from public services and infrastructure projects.

To avoid the same fate, Indonesia needs to strengthen its anti-corruption measures and this can be achieved by improving transparency and accountability, strengthening the rule of law, and increasing public awareness and participation in governance.

Finally, Indonesia can learn from Venezuela's experience in building resilience in the face of economic and political challenges. Despite its economic troubles, Venezuela has maintained a strong sense of national pride and resilience, with many citizens coming together to support each other during difficult times.

To build similar resilience, Indonesia needs to foster a sense of national unity and encourage cooperation and solidarity among its citizens. This can be achieved through initiatives such as community empowerment programs, disaster preparedness training, and social safety nets.

To summarize, here's how Venezuela's experience translates to Indonesia's context:

1. Fiscal Sustainability: While offering subsidized food program for schoolchildren may appear compassionate, its long-term financial viability should be meticulously assessed (Note: alternatively, malnutrition among schoolchildren can be reduced by introducing several nutritions to staples; see *Appendix* section). Can Indonesia afford such a program without jeopardizing other crucial investments or accumulating unsustainable debt?

2. Economic Diversification: Is the proposed program part of a broader strategy to diversify the economy and reduce dependence on specific sectors like natural resources? Relying solely on resource wealth to fund social programs mirrors the Venezuelan misstep.

3. Targeting and Efficiency: Are the proposed programs well-targeted to truly reach the most vulnerable? Can they be implemented efficiently to avoid leakages and waste, ensuring the intended beneficiaries receive the full benefit?

4. Transparency and Accountability: Are the programs transparently designed and implemented, with clear accountability mechanisms? This is crucial to prevent misuse of funds and ensure they achieve their intended objectives.

5. EXAMINING MILEI'S PROPOSALS FOR ARGENTINA'S DEVELOPMENT

Developing economies often find themselves trapped in between leftist populism, promising

immediate relief through state intervention, and laissez-faire capitalism, prioritizing unrestricted markets. In this regard, while it is too early to conclude for the whole programs introduced by the new government, Argentina's recent political landscape presents an intriguing alternative: Javier Milei's economic reform program. While the term "development from below" might not resonate well within traditional development discourse, and while the results of Milei's liberals stance approach to recover national economy shall be seen later on, analyzing Milei's proposals within the context of Argentina's challenges offers valuable insights for other developing nations seeking paths beyond these two dominant paradigms.

Milei's program hinges on two keys:

1.Support for the Business Community: This includes measures like reducing taxes, easing regulations, and fostering a more favorable business environment. The goal is to stimulate economic growth and employment, creating a ripple effect that benefits all citizens.

2.Reduction of Government Interference: Advocating for "*injeransias indebidas*" (unlawful interference), Milei seeks to minimize government intervention in the economy. This aims to create a level playing field, allowing smaller businesses to compete fairly and thrive.

While these proposals sound promising, several questions warrant deeper analysis:

1. Balancing Growth and Equality: Can prioritizing business growth alone translate into equitable development for all? While job creation is crucial, addressing income inequality and ensuring social safety nets remain critical considerations.

2. Regulatory Reform, Not Deregulation: Reducing unnecessary regulations is beneficial, but complete deregulation carries risks. Striking a balance between fostering business activity and ensuring consumer protection and environmental sustainability is key.

3. Contextualizing Argentina's Experience: Argentina's specific economic and political context heavily influences the potential impact of these reforms. What works in Argentina might not translate directly to other developing nations with vastly different circumstances.

Instead of simply rejecting populism, developing countries can learn from Milei's proposals:

1. Shifting Focus: Rather than solely relying on state intervention, fostering a conducive environment for small businesses can empower individuals and drive sustainable economic growth.

2. Streamlining Regulations: Evaluating and removing unnecessary regulations can ease bureaucratic burdens and unlock the potential of smaller players in the market.

3. Striking a Balance: The key lies in creating a balanced approach that promotes **both** economic growth **and** social justice. Investing in education, healthcare, and social safety nets ensures that growth benefits everyone, not just a select few.

6. AN ALTERNATIVE ROUTE OF GRASS-ROOTS TRANSFORMATION INSPIRED BY YUNUS

Developing nations often find themselves entangled in the debate between top-down, government-driven interventions and laissez-faire capitalism. Both approaches have their limitations, leaving communities searching for alternatives. This article proposes a different pathway, drawing inspiration from the work of Professor Muhammad Yunus and Masanobu Fukuoka to create a **"healthy circle" of empowerment**: a self-sustaining, community-driven approach to fostering education, economic progress, and environmental awareness.

Instead of relying solely on government programs or unchecked markets, the "healthy circle" emphasizes a cyclical process of:

1. Re-conscientization: This initial step focuses on raising awareness and critical thinking within communities. Individuals reflect on their current situation, needs, and aspirations.

2. Appropriate Education: Building on this newfound awareness, communities identify their educational needs. This may involve acquiring skills in areas like financial literacy, sustainable agriculture, or basic healthcare. Education goes beyond formal schooling, encompassing learning from local experts, elders, and peers.

3. Appropriate Technologies: Equipped with knowledge and skills, communities explore solutions that align with their environment and resources. Inspired by Masanobu Fukuoka's natural farming methods, the focus may shift towards practices that are low-cost, adaptable, and minimize environmental impact.
4. Collective Action: The benefits of these initiatives, whether increased agricultural yields or improved financial management, are reinvested back into the community. This "collection" fuels further re-conscientization, education, and technological exploration, creating a self-sustaining cycle of progress.

The "healthy circle" holds several advantages:
•**Community-driven:** By empowering communities to identify their needs and solutions, the approach fosters ownership and long-term sustainability.
•**Scalable and adaptable:** This cyclical model can be adapted to diverse contexts and needs, from rural villages to urban communities.
•**Holistic:** Addressing education, economics, and the environment simultaneously creates a more comprehensive and resilient path to development.
Drawing inspiration from Grameen Bank, microfinance programs can be integrated into the cycle, providing access to capital for community-led ventures. This financial inclusion further empowers individuals and communities to take charge of their development.

This approach is not without its challenges. Facilitating dialogue, identifying appropriate technologies, and ensuring equitable participation within communities require careful planning and ongoing support.

In conclusion, the "healthy circle" offers a promising alternative to top-down and market-driven development approaches. By empowering communities to learn, act, and reinvest in their own well-being, it fosters a more just, sustainable, and equitable path to progress. Let us move beyond the limitations of populism and embrace the transformative power of grassroots empowerment, drawing inspiration from the wisdom of M. Yunus and Fukuoka. This may just be the key to unlocking the true potential of development from below.

Moreover, as previously noted by Elsgard, by a comparative analysis, large differences in institutional quality are revealed. Transparency, accountability and a strong rule of law are found to be among the features that distinguish the successful countries. These differences translate into different economic policies. While Norway and Botswana have exhibited prudent fiscal management, government spending in Venezuela in particular, have been procyclical and unproductive. (Elsgard) [1]

Therefore, allow us to outline three possible approaches to *improve accountability* to publicly-funded schemes, which are mostly prone to be mismanaged. Those steps include:

(i) Thorough implementation of ISO-37001.
Corruption is a major challenge for many countries around the world, especially developing countries. It undermines public trust in government institutions, reduces economic growth, and often leads to social unrest. One effective way of combating corruption is by implementing anti-corruption measures such as ISO 37001 on a nationwide scale. Here are some practical suggestions for implementing these measures:

Establish a comprehensive anti-corruption strategy: The first step in implementing anti-corruption measures is to develop a comprehensive strategy that outlines the goals, objectives, and activities required to combat corruption. This strategy should involve key stakeholders, including government officials, civil society organizations, and the private sector.

Train and educate key stakeholders: Anti-corruption measures can only be effective if all stakeholders are aware of their responsibilities and understand how to implement them. Training and education programs should be provided to government officials, civil society organizations, and the private sector to ensure they have the necessary skills and knowledge to implement anti-corruption measures.

Implement anti-corruption policies and procedures: Anti-corruption policies and procedures should be implemented at all levels of government and within the private sector. This includes developing codes of conduct, establishing whistle-blower protection mechanisms, and implementing measures to prevent conflicts of interest.

Monitor and evaluate progress: Regular monitoring and evaluation are essential to ensure that anti-corruption measures are effective. Key performance indicators should be established to measure progress and identify areas where improvements are needed.

Promote transparency and accountability: Transparency and accountability are key components of any anti-corruption strategy. Governments should establish mechanisms for public oversight, including access to information, and ensure that corruption cases are investigated and prosecuted.

Foster a culture of integrity: Finally, fostering a culture of integrity is essential to combating corruption. This involves promoting ethical behavior, encouraging responsible citizenship, and creating a sense of social responsibility among all stakeholders. (GIA Center)[2]

(ii) Encourage citizen journalism, especially with the help of ubiquitous video-sharing apps.

Imagine a world where citizens are not just passive observers of public-funded schemes, but active watchdogs, armed with the power of their smartphones and a thirst for transparency. This world is closer than you think, thanks to the rise of citizen journalism and ubiquitous video-sharing apps.

Public-funded development schemes, vital for infrastructure, development, and social welfare, often suffer from opacity and accountability lapses. This is where citizen journalism steps in, wielding the power of smartphones and social media to shine a light on these issues.

Ubiquitous video-sharing apps like YouTube and Instagram are no longer just platforms for entertainment. They have become powerful tools for citizen journalists to document problems, gather evidence, and share their findings with the world. A concerned citizen filming a shoddy construction site or capturing an instance of resource misuse can instantly amplify their voice, reaching local authorities, media outlets, and a wider public audience.

The benefits of citizen journalism in promoting transparency and accountability are numerous:

- **Increased Scrutiny:** By bringing everyday citizens into the fold, more eyes are watching public projects, leading to greater vigilance and proactive identification of issues.
- **Real-time Reporting:** Citizen Journalists can capture events as they unfold, providing immediate updates and unfiltered perspectives.
- **Wider Reach:** Social media platforms allow for rapid dissemination of information, circumventing traditional media gatekeepers and reaching broader audiences.
- **Amplified Voices:** Citizen Journalism empowers individuals and communities to hold authorities accountable, giving them a platform to demand answers and action.

However, it's crucial to acknowledge the challenges:

- **Verification and Accuracy:** Misinformation and unverified claims can be detrimental. Citizen journalists need to be equipped with skills to distinguish facts from fiction.
- **Safety and Security:** Raising concerns about powerful entities can come with safety risks. Training and support mechanisms are essential for citizen journalists' safety.
- **Sustainability and Consistency:** Engaging in citizen journalism requires time and effort. Fostering a culture of active participation and supporting citizen journalists with resources is crucial.

Moving forward, encouraging citizen journalism requires a multi-pronged approach:

- **Media Literacy Education:** Equipping citizens with the skills to critically evaluate information and verify claims is essential.

- **Technology Support:** Providing training on using video-sharing apps effectively and securely can empower citizen journalists.
- **Legal Safeguards:** Protecting citizen journalists from harassment and intimidation is crucial for fostering a safe environment for raising concerns.
- **Collaborative Platforms:** Developing dedicated platforms for citizen journalists to share information, connect, and collaborate can amplify their impact.

By embracing citizen journalism and its potential, we can create a more transparent and accountable society, where public-funded projects truly serve the communities they are intended for. Let's turn the ubiquitous video-sharing apps into tools for empowerment, ensuring that every citizen becomes a watchdog, shaping a brighter future for all.

(iii) Individual Power Producers are able to drive Rural Development in Indonesia. Imagine remote villages in Indonesia, pulsating with life and opportunity, powered not by distant grids, but by the ingenuity of their own residents. This vision of **individual power producers**, driving rural development, is no longer a distant dream, but a burgeoning reality driven by **citizen-funded renewable energy**. Let's explore how microhydro power, gravitational vortex power plants (GWVPP), and small-scale solar deployments empower individuals and transform communities.

Citizen-Funded Microhydro Generator: Harnessing flowing water in rivers and streams, microhydro systems offer clean, sustainable power for individual households or even entire villages. Communities can **collectively fund** the installation and maintenance, creating localized energy independence and fostering a sense of ownership. Imagine farmers using microhydro-powered irrigation pumps to boost crop yields, or students studying under LED lights powered by their own mini-hydro plant.

Gravitational Vortex Power Plant, An Intriguing Innovation: While still in its early stages, the concept of gravitational vortex power plants holds immense potential for rural communities. By utilizing natural water flow and gravity, these systems could generate clean electricity without dams or turbines, promoting **healthy soil and crop growth** thanks to revitalized water cycles. This technology, if proven viable, could be a game-changer for rural areas with limited water resources.

Small-Scale Solar Power: Rooftop solar panels represent a tried-and-tested solution, empowering individual households to generate their own electricity. With government initiatives and **microfinancing options**, solar energy is becoming increasingly accessible. Imagine small businesses powered by the sun, generating income and illuminating possibilities for rural entrepreneurs.

The Significance of Individual Power Producers: This shift towards individual power producers goes beyond just electricity. It fosters **community ownership, strengthens local economies, and empowers individuals** to become architects of their own development. Additionally, it promotes **environmental sustainability**, reducing reliance on fossil fuels and protecting precious ecosystems.

7. CHALLENGES AND OPPORTUNITIES

While the future is bright, challenges remain. Access to financing, technical expertise, and maintenance support need to be addressed. Collaboration between communities, government agencies, and NGOs is crucial to unlock the full potential of this empowering movement.

Indonesia, with its vast renewable energy potential, is perfectly positioned to lead the way. By embracing individual power producers and their citizen-funded renewable energy solutions, the nation can illuminate rural communities, empower its people, and pave the way for a sustainable and prosperous future.

Moreover, although it is often recognized too late, there is ample of evidences that overemphasizing technological progress can impose *too fast cultural changes* which may be counterproductive to the common people. While the following discussion may be not conclusive, it seems worth noting that there are several

problems with growth theory a la Solow-Swan theory.

The Solow-Swan growth model is a neoclassical economic model that explains economic growth as a function of capital accumulation, population growth, and technological progress. This model has been influential in understanding economic growth, but has also been criticized for its focus on external growth factors, such as capital accumulation and technological progress.

However, the exogenous model began to receive criticism from other economists since the 1990s. Although the idea of the role of endogenous factors in economic growth has long been known. Even Adam Smith's thoughts regarding labor specialization can be grouped into this thought.

Historical data in this country suggests that biodiversity and cultural diversity are both assets that must be managed carefully.[1] Both play an important role in maintaining the continuity and sustainability of life. For example, the decreasing number of tropical forests in this country not only has an impact on the ecosystem, but also has the potential to disrupt the quality of human life.

Additionally, increasing evidence suggests that cultural diversity and biodiversity can play an important role in driving economic growth. As for the arts and culture industry, according to a report from the Center for Economics and Business Research (CEBR) published in 2019, the impact of the arts and culture industry on the British economy is very significant. This report not only focuses on sectoral aspects, but also examines the more comprehensive economic implications of the contribution of arts and culture to economic growth in the UK. Therefore, let us consider that the proposed Solow-Swan growth model can be extended to include cultural heritage and biodiversity by adding them as additional growth factors. This can be done by creating a new variable that represents the total stock of cultural heritage and biodiversity in an economy.

These variables can then be added to the model's production function, along with capital accumulation, population growth, and technological progress. The following equation can be used to create a modified Solow-Swan growth model with cultural heritage and biodiversity:

$$Y = K^{\alpha} + CH^{b} + BD^{c}, \qquad (1)$$

where:
Y stands for economic output,
K stands for capital availability,
CH stands cultural heritage stocks,
BD stands for natural biodiversity stocks;
a, b, and c are constants representing capital marginal product, cultural heritage products and services, and biodiversity resource values. The above extended Solow-Swan equation can be generalized further to become:

$$Y=\sum_{i=1,2,3} c_i K_i^{n_i} \qquad (2)$$

The equation proposed above, shows that economic output is a function of capital stock, cultural heritage, and biodiversity values. The marginal product of each factor of production is represented by a constant. The capital constant is a, cultural heritage constant is b, and the biodiversity constant is c.

8. CONCLUDING REMARK

Learning from Venezuela's experience emphasizes the need for responsible, sustainable, and well-planned policies that address root causes of poverty and inequality, while ensuring long-term economic stability and avoiding the pitfalls of populism.

In contrast, Argentina's new political landscape presents a unique opportunity to examine alternative development models. While Milei's program raises crucial questions, its focus on empowering small businesses and reducing unnecessary regulations offers valuable lessons

[1]cf. "Warisan budaya dunia mesti dijaga," *Harian Kompas*, Jumat 26 April 2024, p. 10; also Gurcharan Singh. *The Plant Systematics: An Integrated Approach*. Enfield, New Hampshire: Science Publishers, 2010.

for developing nations seeking to move beyond the limitations of populism and traditional interventionism. The path forward lies in crafting a nuanced approach that fosters economic growth while ensuring equitable development for all citizens.

We outline a number of suggestions which can be considered to be implemented, including wide scale adoption of ISO-37001. By following these practical suggestions, governments can significantly reduce corruption and improve the overall well-being of their citizens.

Moreover, we believe that Indonesia, with its vast renewable energy potential and cultural and biodiversity richness, is perfectly positioned to lead the way. By embracing individual power producers and their citizen-funded renewable energy solutions, the nation can illuminate rural communities, empower its people, and pave the way for a sustainable and prosperous future.

Only then can Indonesia chart a course towards inclusive and sustainable development for all its citizens.

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***Appendix* section:**

Another possible path to improving nutrition for school children in developing countries[2]

Malnutrition in developing countries remains a persistent challenge, especially for school children. While global solutions are crucial, the key to success might lie in adapting local resources and traditions. This article explores how a historical intervention in Japan, combined with a "think globally, act locally" approach, can offer valuable lessons for improving child nutrition programs in developing countries like Indonesia.

The Nori Intervention: A Model for Local Action

In the 1950s, Japan faced a serious childhood malnutrition problem. A key solution involved incorporating Nori, a readily available and affordable source of vitamins and minerals – seaweed – into school lunches. This simple, culturally relevant intervention significantly improved children's health.

Think Globally, Act Locally: Applying the Nori Lesson

The Nori story offers a powerful message for developing countries:

- **Identify Local Staples:** Focus on fortifying or incorporating nutrient-rich, locally available foods into school meals. These could be indigenous fruits, vegetables, or underutilized grains.
- **Culturally Relevant Solutions:** Programs should resonate with local traditions and preferences. This fosters acceptance and long-term sustainability.
- **Community Involvement:** Engage local farmers and communities in producing and supplying these nutritious ingredients for school meals. This strengthens local economies and fosters ownership.

[2] V. Christianto & F. Smarandache. *SMIJ* (2024). DOI: https://doi.org/10.61356/SMIJ.2024.77101, url: https://sciencesforce.com/index.php/smij/article/view/230

Indonesia: A Case for Local Action
Indonesia, with its diverse agricultural landscape, presents a perfect example for applying the "think globally, act locally" approach. Here's how:

- **Nutrient-Rich Local Staples:** Indonesia boasts a wealth of nutrient-rich options like sweet potatoes (vitamin A), Moringa leaves (iron and calcium), and tempeh (protein). These can be incorporated into school meals.
- **Culturally-Sensitive Menus:** Indonesian cuisine often features rice and vegetables. School meals can build on this foundation by adding these locally sourced, fortified options.
- **Community Partnerships:** Partnering with local farmers to supply these ingredients can create a sustainable food system for school meals.

Autopoietic Energy Transition towards Clean and Renewable Energy for Developing Countries

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Abstract

The global economy's reliance on energy supply and distribution necessitates a shift towards clean and renewable energy sources. This article explores how the economies of energy producers and consumers can be better modeled through the lens of autopoiesis, a concept developed by Humberto Maturana and further elaborated by Fritjof Capra (cf. Web of Life). By viewing energy systems as self-producing and self-maintaining entities, we can gain insights into the dynamics of energy transition in developing countries and identify strategies for fostering sustainable and resilient energy ecosystems.

Keywords: Energy; Sustainable; Resilient Energy Ecosystems; Renewable Energy Sources.

1 | Introduction

The global energy landscape is undergoing a significant transformation, driven by the urgent need to mitigate climate change and promote sustainable development. Developing countries face unique challenges in this transition, as they strive to balance economic growth with environmental protection. This article argues that the concept of autopoiesis, which emphasizes the interconnectedness and self-organization of living systems, provides a valuable framework for understanding the complexities of energy transition.

Autopoiesis, as described by Maturana and Varela (1980), refers to the ability of a system to self-produce and self-maintain its components, creating a network of interconnected elements that define the system's identity. Capra (*Web of Life*, 1996) extended this concept to the realm of living systems, highlighting the importance of understanding the intricate web of relationships that sustain life. In the context of energy systems, autopoiesis suggests that we should view energy production, distribution, and consumption as interconnected processes that form a self-regulating system.

This perspective challenges the traditional linear model of energy flow, where energy is simply extracted, transformed, and consumed. Instead, it encourages us to recognize the feedback loops and interdependencies that shape energy systems. For instance, the demand for energy in a community can influence the development of renewable energy technologies, which in turn can alter consumption patterns.

1.1 | The Autopoietic Economy: A New Paradigm for the 21st Century?

The global economy is a complex and ever-evolving system, constantly adapting to new technologies, social trends, and environmental challenges. While traditional economic models have provided valuable insights into market dynamics, they often fall short of capturing the intricate web of relationships that shape economic activity. This article explores the concept of the "autopoietic economy," drawing upon the work of notable economists like Paul Krugman and connecting it to the "prosumer" concept popularized by Alvin Toffler [2, 5].

Autopoiesis, a term coined by biologists Humberto Maturana and Francisco Varela, refers to the ability of a system to self-produce and self-maintain its components, creating a network of interconnected elements that define the system's identity. This concept has been applied to various fields, including sociology, philosophy, and even economics.

In the context of economics, the autopoietic perspective suggests that we should view the economy not as a linear flow of goods and services, but as a complex adaptive system with its internal logic and self-organizing capabilities. This view challenges the traditional focus on equilibrium and rational actors, emphasizing instead the dynamic and emergent nature of economic phenomena.

1.2 | Paul Krugman and the New Economic Geography

While not explicitly using the term "autopoiesis," economist Paul Krugman's work on new economic geography resonates with the concept. Krugman emphasized the importance of increasing returns to scale, agglomeration economies, and transportation costs in shaping the spatial distribution of economic activity. His models demonstrate how economic activity can *self-organize into clusters and hubs*, driven by feedback loops and network effects.

Krugman's work highlights the interconnectedness of economic agents and the importance of spatial relationships in shaping economic outcomes. This aligns with the autopoietic perspective, which emphasizes the interconnectedness and self-organization of economic systems.

In his seminal book "*The Third Wave*," Alvin Toffler introduced the concept of the "prosumer," a consumer who is also a producer. Toffler argued that the rise of new technologies and information networks would blur the traditional boundaries between producers and consumers, leading to a more participatory and collaborative economy.

The prosumer concept is closely linked to the autopoietic economy. In a world where consumers are also producers, economic activity becomes more decentralized and self-organized. Individuals can contribute to the creation of goods and services, share knowledge and resources, and participate in the governance of economic networks.

1.3 | Autopoietic Economy in the 21st Century

The autopoietic economy is particularly relevant in the 21st century, characterized by rapid technological advancements, globalization, and the rise of the digital economy. The Internet and other communication technologies have enabled the emergence of new forms of economic activity, such as peer-to-peer platforms, open-source software development, and collaborative consumption.

These developments are blurring the lines between traditional economic categories and creating new opportunities for innovation and value creation. The autopoietic perspective can help us understand these emerging economic forms and develop policies that foster sustainable and inclusive economic growth.

1.4 | Autopoiesis and Energy Transition in Developing Countries

Developing countries often face unique challenges in their energy transition, including limited access to financing, technological constraints, and institutional barriers. However, the autopoiesis framework can help

identify opportunities for fostering sustainable energy ecosystems. By recognizing the interconnectedness of energy systems, we can promote decentralized energy solutions that empower local communities and promote self-reliance.

For example, off-grid renewable energy systems, such as solar microgrids, can provide access to electricity in remote areas while fostering local ownership and management. These systems can also stimulate local economies by creating jobs and supporting small businesses. Furthermore, the autopoiesis framework can guide policymakers in designing integrated energy strategies that consider the social, economic, and environmental dimensions of energy transition. This model, however, is not limited to geographically isolated communities. The future of energy production and consumption points towards a widespread proliferation of individual power producers, not just in remote villages, but also thriving in bustling cities and scattered across isolated islands. We are moving towards an era where energy generation becomes increasingly democratized, mirroring Alvin Toffler's prescient vision of the "prosumer," where individuals are both producers and consumers.

This shift will be driven by advancements in renewable energy technologies, decreasing costs, and increasing accessibility. Imagine city rooftops adorned with solar panels, urban homes equipped with small-scale wind turbines, and even individual electric vehicles acting as mobile energy storage units, feeding power back into the grid. On isolated islands, where traditional grid infrastructure is often prohibitively expensive, small-scale renewable energy systems will become the norm, providing reliable power and fostering self-sufficiency. This decentralized energy landscape will empower communities and individuals to take control of their energy needs, reducing reliance on centralized power plants and fostering resilience against grid outages.

This burgeoning participation of common people in developing, producing, and even recycling energy sources is a key characteristic of the emerging autopoietic economy. Just as biological systems self-organize and self-maintain, so too will future energy systems be characterized by a high degree of local control and feedback loops. This is not a futuristic fantasy; we can already see glimpses of this autopoietic energy model in action. The Balinese *subak* irrigation system, with its intricate network of community-managed water distribution, provides a compelling example of self-organized resource management. Similarly, the villagers' ownership and management of local oil sources in Teksas Wonocolo, East Java, Indonesia, demonstrates the potential for communities to directly control and benefit from local energy resources. These examples, though based on traditional energy sources, offer valuable lessons for the development of future decentralized renewable energy systems.

The autopoietic nature of this future energy landscape will have profound implications. It will foster greater energy security, reduce carbon emissions, and promote local economic development. It will also require a fundamental rethinking of energy policy, moving away from centralized planning towards a more distributed and participatory approach. Policymakers will need to create regulatory frameworks that support the development of microgrids, incentivize individual energy production, and ensure fair access to the grid. They will also need to foster innovation in energy storage and recycling technologies, to ensure the sustainability of this decentralized energy system. The transition to an autopoietic energy future will not be without its challenges, but the potential benefits are immense. By embracing the principles of self-organization, interconnectedness, and local control, we can create a more resilient, sustainable, and equitable energy future for all.

2 | Discussions: Rethinking the Path to Clean and Earthj-friendly Energy

2.1 | The Autopoietic Economy and the "Development from Within" Approach

The global push for sustainable energy necessitates a re-evaluation of traditional economic models, particularly in how we approach energy production and distribution. This article champions the concept of a self-organizing, autopoietic economic system, specifically within the energy sector, to empower local

communities. This isn't about hindering the development of large state-owned enterprises (SOEs) in energy, but rather about fostering a complementary ecosystem that prioritizes "development from within," echoing E.F. Schumacher's wisdom in "Small is Beautiful" and his broader philosophy of appropriate technology, as further explored in "A Guide for the Perplexed."

The core idea is to move beyond a top-down, centralized model of energy production and embrace a more decentralized, participatory approach. An autopoietic energy system, in this context, is one that self-organizes at the local level, driven by the needs and resources of the community. This mirrors biological systems, where interconnected elements self-produce and self-maintain, creating a resilient and adaptive whole. In practical terms, this translates to encouraging smaller-scale, community-owned and operated energy projects, particularly those utilizing renewable resources.

This "development from within" approach recognizes that local communities possess unique knowledge of their specific needs, resources, and environmental context. They are best positioned to determine the most appropriate energy solutions for their circumstances. This aligns perfectly with Schumacher's emphasis on appropriate technology – technologies that are not only efficient and cost-effective but also accessible, manageable, and environmentally sound within the local context. A solar microgrid in a remote village, for example, might be a far more appropriate and empowering solution than extending a large, centralized grid.

This localized energy production empowers communities in several ways. It creates local jobs, stimulates local economies, and fosters a sense of ownership and responsibility for energy resources.

It also enhances energy security by reducing reliance on centralized grids, which are vulnerable to disruptions and often fail to reach remote or marginalized communities.

Crucially, this vision of a decentralized, autopoietic energy system is *not* intended to replace large SOEs. Rather, it is meant to complement them. Large SOEs can continue to play a vital role in developing and managing large-scale energy infrastructure, such as transmission lines and base-load power plants. However, their role should evolve to include supporting and integrating local energy initiatives, perhaps through grid interconnection agreements or by providing technical expertise.

The autopoietic economy, therefore, is not about dismantling existing structures but about creating a more balanced and resilient energy ecosystem. It's about recognizing the power of local communities to drive their own development, using appropriate technologies, and embracing self-organization. This approach, grounded in Schumacher's philosophy of "small is beautiful," fosters a more equitable and sustainable energy future, one that empowers individuals and communities while still recognizing the important role of larger entities. It is a path towards a more just and resilient energy landscape, built from the ground up, reflecting the true meaning of "development from within."

2.2 | Small is (still) Beautiful, Even in Fusion Energy Research

Moreover, we can also say that the quest for clean, sustainable energy has long focused on the elusive promise of nuclear fusion. While massive, multi-billion dollar projects like ITER in France, focusing on hot tokamak plasma research, dominate the narrative, a parallel path, inspired by E.F. Schumacher's "*Small is Beautiful*" philosophy and the principle of appropriate technology, deserves greater attention [3, 6, 7]. While recent promising results from the Japanese ITER team offer a glimmer of hope for the traditional approach, it's crucial to explore alternative avenues, particularly those amenable to small-scale, localized development.

The current dominant paradigm in fusion research, exemplified by ITER, involves confining superheated plasma within a powerful magnetic field. This approach, while theoretically sound, faces immense engineering and economic challenges. The sheer scale and complexity of these projects make them inaccessible to most researchers and preclude the possibility of rapid, iterative development.

We argue that the principles of appropriate technology, so effective in other areas of development, can be applied to fusion research as well. Instead of solely focusing on large-scale, high-temperature plasma

confinement, we should explore alternative approaches that lend themselves to smaller, more manageable experiments. This doesn't mean abandoning ITER or other large-scale projects, but rather diversifying our research portfolio to include potentially disruptive, smaller-scale approaches.

One such promising direction draws inspiration from the work of physicists like Lene Hau, who demonstrated the remarkable phenomenon of "matter as frozen light." [9, 10] This concept, combined with the potential of low-intensity laser focus and the Brightsen nucleon cluster model, opens up exciting possibilities for *laser-assisted photo modulation of low-energy fusion*. [11, 12, 13, 14]

These approaches based on many body phenomena of low-temperature physics, unlike the extreme conditions required for tokamak fusion, could potentially be realized in small-scale, even table-top experiments.

The beauty of this approach lies in its accessibility. Small-scale experiments can be conducted by individual researchers or small teams, fostering innovation and accelerating the pace of discovery. This democratized approach to fusion research could unlock breakthroughs that are currently beyond the reach of large, centralized projects. Imagine a world where university labs and even individual researchers can explore and refine fusion concepts, leading to a rapid proliferation of new ideas and potential solutions.

This isn't just wishful thinking. While still in its early stages, research in laser-assisted fusion and related areas is showing promising signs. The potential for low-energy fusion, driven by precisely controlled laser interactions with specific materials, offers a tantalizing glimpse of a future where clean energy is not confined to massive power plants but can be generated locally and sustainably.

Just as Schumacher advocated for appropriate technologies in development, we argue for an "appropriate fusion" research agenda. This means exploring diverse approaches, including those that may seem unconventional, and prioritizing those that are accessible, manageable, and potentially transformative. While the challenges are significant, the potential rewards – a clean, abundant, and decentralized energy future – are too great to ignore. By embracing the principles of "small is beautiful" and fostering a more inclusive and diverse research landscape, we can accelerate the quest for fusion energy and bring its promise to fruition sooner rather than later.

3 | Concluding Remark

The concept of the autopoietic economy, informed by the work of futurists like Alvin Toffler, offers a new way of understanding the complex and dynamic nature of economic systems. By recognizing the interconnectedness and self-organization of economic activity, we can gain insights into the challenges and opportunities of the 21st century and develop strategies for building a more resilient and sustainable economy.

These developments are blurring the lines between traditional economic categories and creating new opportunities for innovation and value creation. The autopoietic perspective can help us understand these emerging economic forms and develop policies that foster sustainable and inclusive economic growth.

The concept of autopoiesis offers a powerful lens for understanding the dynamics of energy transition in developing countries. By recognizing the interconnectedness and self-organization of energy systems, we can identify strategies for fostering sustainable and resilient energy ecosystems. This approach can help developing countries overcome the challenges of energy transition and pave the way for a cleaner and more equitable energy future.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Lógica Neutrosófica para Mapeo de Opinión Climática: Ventajas sobre Análisis de Consenso Tradicional y Hallazgos Clave de Guayaquil

Neutrosophic Logic for Climate Opinion Mapping: Advantages over Traditional Consensus Analysis and Key Findings from Guayaquil

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RESUMEN

Las plataformas tradicionales de mapeo de opinión, como pol.is, emplean análisis de componentes principales (PCA) y agrupamiento para visualizar paisajes de opinión pública. Sin embargo, estos enfoques se basan en marcos lógicos binarios o difusos que pueden no capturar adecuadamente la ambigüedad inherente y la naturaleza multidimensional de las creencias humanas. Este artículo presenta una aplicación novedosa de la lógica neutrosófica al análisis de percepciones sobre el cambio climático, demostrando ventajas significativas sobre los métodos tradicionales de análisis de consenso. Utilizando una encuesta de 84 residentes en Guayaquil, Ecuador, empleamos una escala Likert neutrosófica que captura tres dimensiones independientes—verdad (T), indeterminación (I) y falsedad (F)—para cada elemento de la encuesta. Nuestro

análisis revela dos grupos de respondientes distintos: un grupo "Altamente Comprometido pero Conflictivo" (34.5% de los respondientes) caracterizado por puntuaciones elevadas en las tres dimensiones, y un grupo "Moderadamente Aceptante" (65.5%) con perfiles dominados por la verdad. Los hallazgos clave demuestran que el análisis neutrosófico descubre complejidad oculta en datos de opinión que los enfoques binarios o difusos obscurecerían, permitiendo estrategias de comunicación más dirigidas y efectivas.

Palabras clave: lógica neutrosófica; mapeo de opinión; análisis de consenso; percepción del cambio climático; agrupamiento PCA; metodología de encuestas.

ABSTRACT

Traditional opinion mapping platforms, such as pol.is, employ Principal Component Analysis (PCA) and clustering to visualize public opinion landscapes. However, these approaches rely on binary or fuzzy logical frameworks that may fail to adequately capture the inherent ambiguity and multidimensional nature of human beliefs. This article presents a novel application of neutrosophic logic to the analysis of perceptions regarding climate change, demonstrating significant advantages over traditional consensus analysis methods. Using a survey of 84 residents in Guayaquil, Ecuador, we applied a neutrosophic Likert scale that captures three independent dimensions—truth (T), indeterminacy (I), and falsity (F)—for each survey item. Our analysis reveals two distinct respondent groups: a "Highly Engaged but Conflictive" group (34.5% of respondents) characterized by high scores across all three dimensions, and a "Moderately Accepting" group (65.5%) dominated by truth-oriented profiles. The key findings demonstrate that neutrosophic analysis uncovers hidden complexity within opinion data that binary or fuzzy approaches would obscure, enabling more targeted and effective communication strategies.

Keywords: neutrosophic logic, opinion mapping, consensus analysis, climate change perception, PCA clustering, survey methodology.

Introducción

El estudio de la opinión pública se ha vuelto cada vez más importante en la era del cambio climático, donde el apoyo público es esencial para implementar políticas ambientales efectivas (Clar, Prutsch, & Steurer, 2013). Los instrumentos de encuesta tradicionales, como las escalas Likert estándar, fuerzan a los respondientes a categorías discretas (por ejemplo, "totalmente de acuerdo" a "totalmente en desacuerdo"), a menudo sin capturar la realidad matizada de los sistemas de creencias humanas (Chyung et al., 2017). Esta limitación es particularmente aguda al estudiar cuestiones complejas y multifacéticas como el cambio climático, donde los individuos pueden simultáneamente sostener puntos de vista conflictivos o inciertos (Corner et al., 2012). La plataforma de democracia computacional pol.is (Figura 1) ha ganado prominencia como herramienta para visualizar paisajes de opinión a través de PCA y agrupamiento (Small et al., 2021) .Sin embargo, pol.is y plataformas similares típicamente operan dentro de marcos de lógica binaria o difusa, tratando el acuerdo y desacuerdo como las dimensiones primarias de variación. Este enfoque puede perder aspectos importantes de la opinión pública, particularmente el grado de incertidumbre o indeterminación que experimentan los respondientes (Seyfi-Shishavan, Kutlu Gündoğdu, Donyatalab, Farrokhizadeh, & Kahraman, 2021).

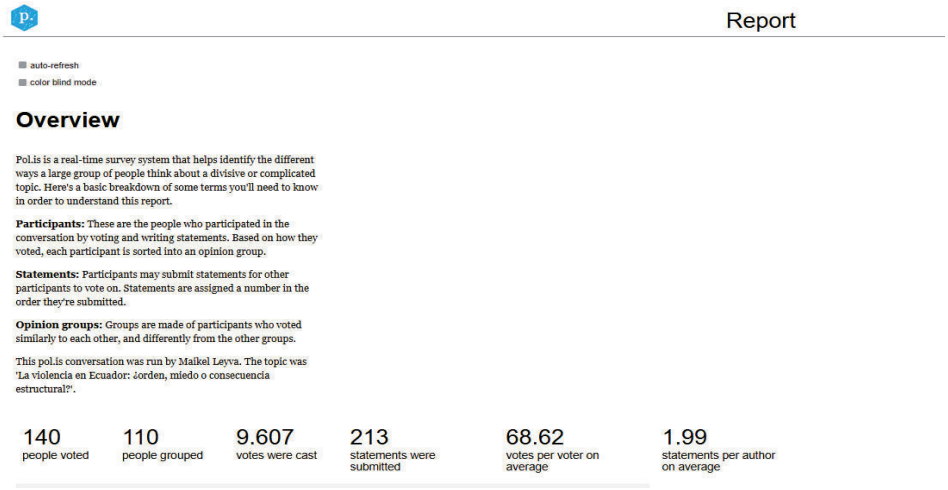


Fig. 1- Interfaz de reporte de la plataforma Pol.is.

La lógica neutrosófica, introducida por Smarandache (1998), extiende la lógica clásica y difusa al incorporar explícitamente un tercer componente: indeterminación (I). Un conjunto neutrosófico se caracteriza por tres funciones de membresía independientes: verdad (T), indeterminación (I) y falsedad (F), cada una oscilando entre 0 y 1 (o 0 a 10/100 en un contexto Likert). Esta estructura tripartita es particularmente adecuada para

modelar fenómenos del mundo real donde la información es incompleta, contradictoria o ambigua (Abdel-Basset et al., 2018).

Este artículo aplica la lógica neutrosófica para analizar percepciones sobre el cambio climático en Guayaquil, Ecuador, una ciudad vulnerable a impactos climáticos como el aumento del nivel del mar y eventos climáticos extremos (Cornejo, Lamiño, & Trejos, 2024). Comparamos los conocimientos obtenidos a través del análisis neutrosófico con los que se obtendrían mediante enfoques binarios o difusos tradicionales, demostrando las ventajas analíticas del marco neutrosófico.

El presente estudio ofrece cuatro contribuciones fundamentales al campo del análisis de opinión pública y la investigación climática. En primer lugar, introduce una innovación metodológica al desarrollar y aplicar una escala Likert neutrosófica, capaz de capturar simultáneamente los grados de verdad, indeterminación y falsedad en las percepciones de los participantes. Esta herramienta supera las limitaciones de los modelos tradicionales basados en lógicas binarias o difusas, proporcionando una representación más rica y matizada de las actitudes humanas frente al cambio climático. Además, se realiza un análisis comparativo entre el enfoque neutrosófico y los métodos convencionales de análisis de consenso como los empleados por plataformas tipo pol.is, evidenciando las ventajas del primero en la detección de ambigüedades y contradicciones inherentes en las respuestas.

Desde una perspectiva empírica, la investigación identifica dos grupos de respondientes claramente diferenciados: uno altamente comprometido pero internamente conflictivo, y otro moderadamente aceptante, con perfiles caracterizados por distintos equilibrios entre las tres dimensiones neutrosóficas. Estos hallazgos no solo revelan capas de complejidad que los métodos tradicionales tienden a ocultar, sino que además poseen implicaciones prácticas significativas. En particular, ofrecen información valiosa para formuladores de políticas y comunicadores ambientales, al permitir diseñar estrategias más segmentadas y efectivas para involucrar a distintos públicos en el debate climático, optimizando así la comunicación y la toma de decisiones sostenibles.

Limitaciones de la Lógica Binaria y Difusa en Mapeo de Opinión

Los instrumentos de encuesta tradicionales y las plataformas de mapeo de opinión operan dentro de marcos de lógica binaria o difusa. En lógica binaria, una afirmación es verdadera o falsa; en lógica difusa, puede ser parcialmente verdadera o falsa, pero los grados de verdad y falsedad son complementarios (es decir, suman 1) (Zadeh, 1965). Esta estructura asume que el acuerdo y desacuerdo son las dimensiones primarias de variación en opinión.

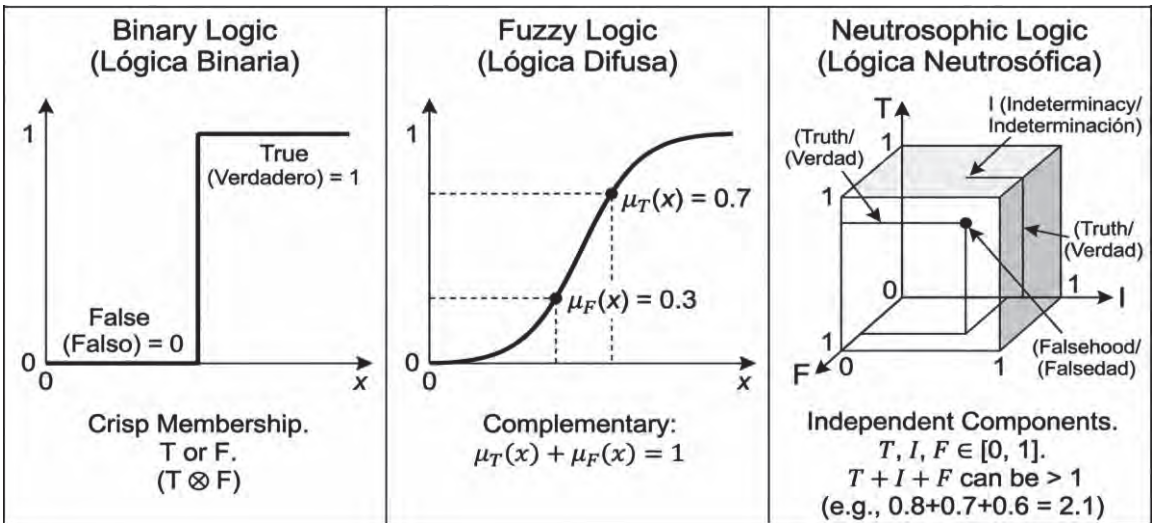


Fig. 2- Evolución de los modelos de pertenencia: del enfoque binario al marco neutrosófico.

Sin embargo, las creencias humanas a menudo se caracterizan por indeterminación genuina—un estado donde el respondiente es verdaderamente incierto o ambivalente, en lugar de estar parcialmente de acuerdo o en desacuerdo (Leyva-Vázquez y Smarandache, 2018). Por ejemplo, un respondiente podría genuinamente creer que el cambio climático es real (T alta) mientras simultáneamente se siente incierto sobre la mejor respuesta política (I alta) y escéptico de ciertas soluciones propuestas (F alta). En un marco binario o difuso, esta posición compleja sería difícil de representar con precisión.

2.2 Lógica Neutrosófica y Sus Ventajas

La lógica neutrosófica aborda esta limitación al tratar la verdad, indeterminación y falsedad como dimensiones independientes (Smarandache, 1998). En un conjunto neutrosófico, un elemento puede tener:

- T alta y F alta simultáneamente: El respondiente acepta algunos aspectos de una afirmación mientras rechaza otros.

• T alta e I alta simultáneamente: El respondiente cree que algo es verdadero pero se siente incierto sobre detalles importantes.

• I alta con T y F moderadas: El respondiente está genuinamente indeciso o ambivalente.

Esta independencia es crucial. En lógica difusa, si T = 0.7, entonces F debe ser ≤ 0.3. En lógica neutrosófica, T = 0.7 y F = 0.8 son ambos posibles, pudiendo capturar la complejidad de las creencias del mundo real mediante escalas de Likert neutrosóficas (Ye, 2014).

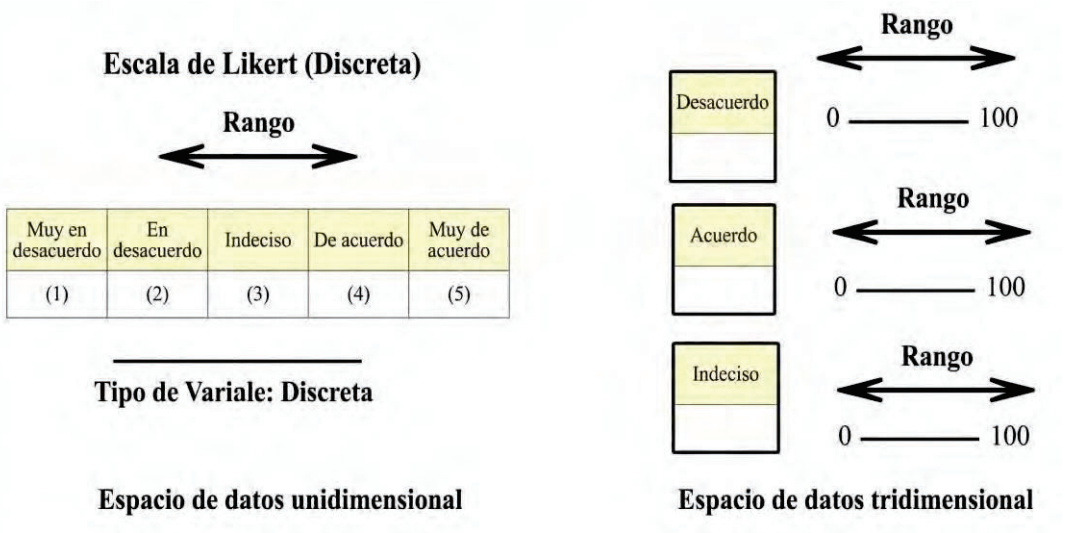


Fig. 3- Análisis comparativo de la Escala de Likert tradicional y la Escala de Likert neutrosófica.

La plataforma pol.is utiliza PCA y agrupamiento para mapear paisajes de opinión, proporcionando conocimientos valiosos sobre consenso y polarización. Sin embargo, opera bajo la suposición de que la opinión puede ser representada adecuadamente en un espacio de menor dimensión derivado de respuestas binarias o difusas. El análisis neutrosófico ofrece varias ventajas:

1. Datos Más Ricos: Las escalas neutrosóficas capturan tres dimensiones independientes, proporcionando más información sobre el sistema de creencias de cada respondiente.
2. Indeterminación Explícita: En lugar de tratar la incertidumbre como un fenómeno residual o secundario, la lógica neutrosófica la convierte en una dimensión primaria de análisis.
3. Detección de Patrones de Creencias Complejos: El análisis neutrosófico puede identificar respondientes que están profundamente comprometidos con un tema, pero internamente conflictivos—un patrón que podría ser invisible en marcos binarios o difusos.

4. Mejor Representación de la Ambigüedad: En dominios como el cambio climático, donde la incertidumbre científica es inherente, la lógica neutrosófica proporciona una representación más auténtica del paisaje epistémico.

5. Targeting Mejorado: Al identificar no solo acuerdo/desacuerdo sino también la naturaleza y grado de incertidumbre, los comunicadores pueden adaptar mensajes más efectivamente a diferentes segmentos de audiencia

Métodos o Metodología Computacional

Diseño de Encuesta y Escala Neutrosófica

Administramos una encuesta a 84 residentes de Guayaquil, Ecuador, utilizando Google Forms. La encuesta consistió en 10 afirmaciones relacionadas con el cambio climático. Para cada afirmación, los respondientes calificaron su posición en tres escalas independientes con un rango de 0 a 10:

T (Verdad/Aceptación): ¿Qué tan verdadera o aceptable es esta afirmación?

I (Indeterminación/Duda): ¿Qué tan incierto o ambiguo te hace sentir esta afirmación?

F (Falsedad/Rechazo): ¿Qué tan falsa o inaceptable es esta afirmación?

Se instruyó explícitamente a los respondientes que estas escalas eran independientes entre sí y que, por lo tanto, puntuaciones altas en múltiples escalas simultáneamente eran posibles y válidas.

Formalmente, esta metodología difiere sustancialmente de la escala de Likert tradicional unidimensional, definiendo en su lugar un espacio de valoración tridimensional discreto para cada ítem, fundamentado en la Lógica Neutrosófica. Sea R el conjunto de encuestados y A el conjunto de afirmaciones. La respuesta de un encuestado $\boldsymbol{r_j} \in \boldsymbol{R}$ ante una afirmación $a_k \in A$ no es un valor escalar, sino que se modela como un vector de valoración neutrosófica $\boldsymbol{V_{jk}}$, definido como una terna ordenada:

$$\boldsymbol{V_{jk}} = \left(\boldsymbol{T_{jk}}, \boldsymbol{I_{jk}}, \boldsymbol{F_{jk}}\right) \textbf{(1)}$$

Donde cada componente representa el grado de pertenencia a los subconjuntos de Verdad (T), Indeterminación (I) y Falsedad (F). El dominio de cada componente es el conjunto discreto de enteros D delimitado por la escala utilizada:

$$\forall\, T_{jk}, I_{jk}, F_{jk} \in D, \textit{donde } D = \{x \in \mathbb{Z} \mid 0 \leq x \leq 10\} \quad (2)$$

La característica matemática distintiva de este modelo radica en la independencia estocástica de las componentes T, I y F. A diferencia de la probabilidad clásica o la lógica difusa intuicionista, no se impone ninguna restricción de normalización sobre la suma de las componentes (i.e., no se requiere que T + I + F = 10). En consecuencia, el espacio de respuesta para cada ítem es un hipercubo discreto que permite configuraciones válidas donde la suma de las valoraciones excede el rango de una escala individual:

$$0 \leq T_{jk} + I_{jk} + F_{jk} \leq 30 \quad (3)$$

Esta estructura matemática permite modelar estados cognitivos complejos inalcanzables para las escalas tradicionales, como la paraconsistencia (valores altos simultáneos en T y F) o la incertidumbre pura (valor alto en I independientemente de T y F).

Análisis de Datos

Análisis de Consenso

Para cuantificar el nivel de acuerdo entre respondientes en cada pregunta, se calculó una puntuación de consenso basada en la variabilidad de las respuestas neutrosóficas. La métrica se define como:

$$Consenso_j = \frac{1}{(1 + C\bar{V}_j)} \quad (4)$$

donde $C\bar{V}_j$ es el coeficiente de variación promedio para la pregunta j:

$$C\bar{V}_j = \left(\frac{1}{3}\right) \times \left[\left(\frac{\sigma_{Tj}}{\mu_{Tj}}\right) + \left(\frac{\sigma_{Ij}}{\mu_{Ij}}\right) + \left(\frac{\sigma_{Fj}}{\mu_{Fj}}\right)\right] \quad (5)$$

En esta expresión:$\boldsymbol{\mu_{Tj}, \mu_{Ij}, \mu_{Fj}}$ son las medias de las dimensiones Verdad, Indeterminación y Falsedad

para la pregunta j.

$\boldsymbol{\sigma_{Tj}, \sigma_{Ij}, \sigma_{Fj}}$ son las desviaciones estándar correspondientes.

El coeficiente de variación $\mathbb{V} = \sigma/\mu$ normaliza la variabilidad por la media.

Las puntuaciones de consenso oscilan entre 0 y 1, donde valores más altos indican mayor acuerdo entre respondientes (menor variabilidad relativa) y valores más bajos indican mayor desacuerdo (mayor variabilidad relativa

Análisis de Componentes Principales

Para visualizar la estructura multidimensional de los datos neutrosóficos, aplicamos Análisis de Componentes Principales (PCA). El procedimiento se ejecuta como sigue:

Paso 1: Estandarización Primero, estandarizamos los datos de respuesta para cada dimensión. El proceso de estandarización se aplica de forma independiente a cada una de las 30 variables observadas. Esto implica que cada par (Pregunta, Dimensión) se normaliza utilizando su propia media y desviación estándar:

$$Z_{ijk} = \frac{\left(X_{ijk} - \mu_k\right)}{\sigma_k} \quad (6)$$

donde:

X_{ijk} es la respuesta del respondiente i a la pregunta j en la dimensión k (T, I, o F).

μ_k es la media de la dimensión k en toda la muestra.

σ_k es la desviación estándar de la dimensión k.

Z_{ijk} es el valor estandarizado (puntuación z).

Paso 2: Matriz de Covarianza Calculamos la matriz de covarianza C de los datos estandarizados:

$$C = \left(\frac{1}{n}\right) \times Z^T \times Z \quad (7)$$

donde Z es la matriz de datos estandarizados de dimensión n × 30 (84 respondientes × 30 variables: 10 preguntas × 3 dimensiones) y n es el número de respondientes.

Paso 3: Descomposición de Valores Propios Calculamos los valores propios $\lambda_1, \lambda_2, ..., \lambda_{30}$ y los vectores propios correspondientes $v_1, v_2, ..., v_{30}$ de la matriz de covarianza C:

$$C \times v_k = \lambda_k \times v_k \quad (8)$$ Los valores propios se ordenan en orden decreciente: $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{30}$.

Paso 4: Proyección en Componentes Principales Los componentes principales (PC) se calculan como proyecciones lineales de los datos estandarizados:

$$PC_k = Z \times v_k \quad (9)$$ Para visualización, utilizamos los dos primeros componentes principales (PC_1 y PC_2), que capturan la máximavarianza en los datos. La varianza explicada por cada componente es:

$$Varianza\ Explicada_k = \left(\frac{\lambda_k}{\Sigma\ \lambda_i}\right) \times\ 100\%. \quad (10)$$

Agrupamiento K-Means

Para identificar grupos de respondientes con perfiles neutrosóficos similares, aplicamos el algoritmo K-Means. Este algoritmo particiona los datos en k grupos minimizando la suma de cuadrados dentro de grupos (WCSS):

$$WCSS\ =\ \Sigma_{i=1}^{k} \Sigma_{x\ \in\ C_i} ||x\ -\ \mu_i||^2 \quad (11)$$

donde:

C_i es el conjunto de puntos asignados al grupo i.

$\mu_i$es el centroide (media) del grupo i.

$|| \cdot ||$ denota la distancia euclidiana.

Selección del Número Óptimo de Grupos El número óptimo de grupos se determina mediante la puntuación de silueta (Silhouette Score), definida para cada punto x como:

$$s(x) = \frac{[b(x) - a(x)]}{\max[a(x), b(x)]} \quad (12)$$

donde:

a(x) es la distancia promedio de x a otros puntos en su mismo grupo (cohesión).

b(x) es la distancia promedio mínima de x a puntos en otros grupos (separación).

La puntuación de silueta promedio para un número de grupos k es:

$$Silhouette(k) = \left(\frac{1}{n}\right) \times \Sigma_{i=1}^{n} s(x_i) \tag{13}$$

Se selecciona el valor de k que maximiza esta métrica. La puntuación de silueta oscila entre -1 y 1, donde valores más cercanos a 1 indican mejor separación de grupos.

Caracterización de Grupos

Para cada grupo identificado, se calculan estadísticas descriptivas que caracterizan el perfil neutrosófico del grupo:

Medias por Dimensión

$$\mu_{T,g} = \left(\frac{1}{n_g}\right) \times \sum_{i \in G_g} T_i \, , \mu_{I,g} = \left(\frac{1}{n_g}\right) \times \sum_{i \in G_g} I_i \, , \mu_{F,g} = \left(\frac{1}{n_g}\right) \times \sum_{i \in G_g} F_i$$

donde G_g es el conjunto de respondientes en el grupo g y n_g es el tamaño del grupo.

Balance Neutrosófico El balance neutrosófico cuantifica el peso relativo de cada dimensión dentro del grupo:

$$Balance_{T(g)} = \left[\frac{\mu_{T,g}}{\left(\mu_{T,g} + \mu_{I,g} + \mu_{F,g}\right)}\right] \times 100\% \tag{14}$$

$$Balance_{I(g)} = \left[\frac{\mu_{I,g}}{\left(\mu_{T,g} + \mu_{I,g} + \mu_{F,g}\right)}\right] \times 100\% \tag{15}$$

$$Balance_{F(g)} = \left[\frac{\mu_{F,g}}{\left(\mu_{T,g} + \mu_{I,g} + \mu_{F,g}\right)}\right] \times 100\% \tag{16}$$

Estas tres proporciones suman 100% y proporcionan una medida de la composición relativa del perfil neutrosófico del grupo.

Dimensión Dominante Para cada pregunta j y grupo g, la dimensión dominante es aquella con la media más alta:

$$Dominante(j, g) = argmax_k \{\mu_{T,g,j}, \mu_{I,g,j}, \mu_{F,g,j}\} \tag{17}$$

donde $k \in \{T, I, F\}$.

Esta caracterización permite identificar patrones de respuesta distintivos que diferencian a los grupos y facilita la interpretación de sus perfiles neutrosóficos únicos.

Resultados y discusión

Análisis de Consenso y Dimensiones Dominantes

Las diez preguntas fueron predominantemente impulsadas por la dimensión Verdad (T), indicando que los respondientes se involucraban principalmente con las afirmaciones en términos de su validez o aceptación. Sin embargo, el consenso varió significativamente entre preguntas.

Preguntas de Alto Consenso:

• P4: Las acciones individuales importan para el clima (Consenso: 0.704)

• P1: El cambio climático es real y causado por el ser humano (Consenso: 0.682)

• P10: Las políticas actuales son suficientes (Consenso: 0.682)

Preguntas de Bajo Consenso:

• P8: La energía renovable es viable en Ecuador (Consenso: 0.662)

• P5: El gobierno debe priorizar la acción climática (Consenso: 0.663)

• P6: El crecimiento económico y el ambiente pueden coexistir (Consenso: 0.664)

Tabla 1 - Resultados del Análisis de Consenso Neutrosófico.

Pregunta	T_{Media}	I_{Media}	F_{Media}	Consenso	Dominante
P1	6.46	6.12	6.05	0.682	Verdad
P2	6.70	6.18	5.37	0.677	Verdad
P3	7.40	5.70	4.88	0.669	Verdad
P4	6.76	5.92	5.27	0.704	Verdad
P5	7.20	5.75	5.06	0.663	Verdad
P6	7.26	5.57	5.13	0.664	Verdad
P7	6.63	5.87	5.52	0.672	Verdad
P8	7.30	5.40	5.04	0.662	Verdad
P9	6.99	5.68	5.48	0.677	Verdad
P10	7.37	5.88	5.27	0.682	Verdad

Resultados de PCA y Agrupamiento

PCA redujo exitosamente los datos de 30 dimensiones, con los dos primeros componentes principales explicando 59.03% de la varianza total (PC1: 43.94%, PC2: 15.09%). El agrupamiento K-Means identificó dos grupos óptimos:

• Grupo 0: 29 respondientes (34.5%)

• Grupo 1: 55 respondientes (65.5%)

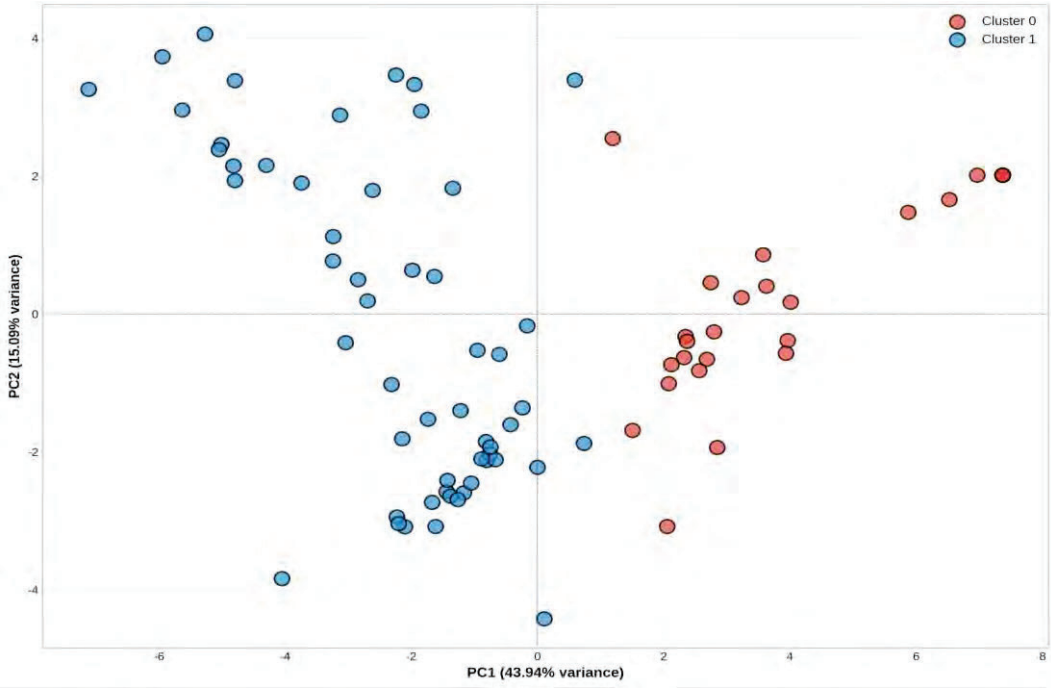


Fig. 4- Gráfico de Dispersión PCA con Grupos de Respondientes.

Caracterización de Grupos

Grupo 0: El Grupo "Altamente Comprometido pero Conflictivo"

Este grupo se define por puntuaciones uniformemente altas en las tres dimensiones neutrosóficas:

• Perfil General: T = 8.18, I = 8.38, F = 8.07

• Balance Neutrosófico: T = 33.2%, I = 34.0%, F = 32.8%

• Tamaño: 29 respondientes (34.5%)

Características Clave:

• Balance casi perfecto en las tres dimensiones, indicando aceptación, incertidumbre y escepticismo simultáneos.

• Alto compromiso con el tema, pero caracterizado por conflicto interno y complejidad.

• Ubicado en la región positiva (derecha) de PC1, impulsado por alta intensidad general de respuestas.

• Los patrones específicos de preguntas muestran que Indeterminación (I) es la dimensión dominante para 6 de 10 preguntas, sugiriendo incertidumbre generalizada.

Interpretación: Estos individuos no son apáticos o ambivalentes en el sentido tradicional. Más bien, están profundamente comprometidos con el cambio climático como tema, pero sostienen puntos de vista complejos y multifacéticos. Pueden aceptar la realidad del cambio climático mientras cuestionan políticas específicas o narrativas. Este grupo representa un objetivo importante para comunicación matizada y basada en evidencia que reconozca la incertidumbre y complejidad.

Grupo 1: El Grupo "Moderadamente Aceptante"

Este grupo más grande se caracteriza por una fuerte dominancia de la dimensión Verdad:

• Perfil General: T = 6.39, I = 4.45, F = 3.85

• Balance Neutrosófico: T = 43.5%, I = 30.3%, F = 26.2%

• Tamaño: 55 respondientes (65.5%)

Características Clave:

• Dominancia clara de Verdad con puntuaciones sustancialmente menores en Indeterminación y Falsedad.

• Compromiso general más moderado en comparación con el Grupo 0.

• Ubicado en la región negativa (izquierda) de PC1, reflejando menor intensidad general.

• Verdad (T) es la dimensión dominante para las 10 preguntas, indicando aceptación consistente.

Interpretación: Este grupo representa la posición "convencional" sobre el cambio climático—aceptando la realidad básica e importancia del tema pero sin el compromiso más profundo o conflicto interno visto en el Grupo 0. Sus puntuaciones generales más bajas sugieren menor involucramiento intenso con el tema.

Tabla 2 - Perfiles Comparativos de Grupos.

Métrica	**Grupo 0**	**Grupo 1**	**Diferencia**
Tamaño (n)	29	55	—
Porcentaje	34.5%	65.5%	—
T_{Media}	8.18	6.39	+1.79
I_{Media}	8.38	4.45	+3.93
F_{Media}	8.07	3.85	+4.21
$Dominancia_T$	33.2%	43.5%	—
$Dominancia_I$	34.0%	30.3%	—

$Dominancia_F$	32.8%	26.2%	—

Discusión

Descubrimiento del Grupo "Altamente Comprometido pero Conflictivo"

Hallazgo: El análisis neutrosófico reveló un grupo sustancial (34.5% de respondientes) caracterizado por puntuaciones uniformemente altas en T, I y F. Este patrón sería difícil o imposible de detectar utilizando marcos tradicionales de lógica binaria o difusa.

Ventaja sobre pol.is: La plataforma pol.is, operando con lógica binaria o difusa, probablemente clasificaría a estos respondientes como "ambivalentes" o "indecisos." Sin embargo, el marco neutrosófico revela que sus puntuaciones altas en I no son signo de apatía sino de incertidumbre genuina y comprometida. Están lidiando con las complejidades del cambio climático, simultáneamente aceptando su realidad, cuestionando aspectos de ella, e incertidumbre sobre soluciones. Este es un fenómeno fundamentalmente diferente de la simple ambivalencia.

Implicación Práctica: Este grupo representa un objetivo crítico para comunicación. No son escépticos a ser convencidos de hechos básicos; ya aceptan la realidad del cambio climático. Más bien, son individuos buscando información más matizada y transparente que reconozca incertidumbre y complejidad. La mensajería tradicional que trata el cambio climático como un tema resuelto y simple puede alienar a este grupo, mientras que la mensajería que se involucra con sus incertidumbres legítimas puede ser más efectiva.

Representación Explícita de Indeterminación

Hallazgo: La dimensión Indeterminación (I) revela que los respondientes experimentan incertidumbre significativa en todas las preguntas, con puntuaciones medias de I oscilando de 5.40 a 6.18 (en escala 0–10).

Ventaja sobre pol.is: El mapeo de opinión tradicional trata la incertidumbre como un fenómeno residual o secundario. El análisis neutrosófico convierte la indeterminación en una dimensión primaria, revelando que la incertidumbre no es un factor menor sino una característica central de cómo las personas se involucran con el cambio climático.

Implicación Práctica: Este hallazgo sugiere que las estrategias de comunicación deben abordar explícitamente fuentes de incertidumbre. En lugar de presentar la ciencia climática como resuelta y unambigua, la

comunicación más efectiva podría reconocer áreas genuinas de incertidumbre científica (por ejemplo, impactos precisos en regiones específicas, respuestas políticas óptimas) mientras mantiene claridad sobre hechos bien establecidos (por ejemplo, la realidad del cambio climático causado por humanos).

Identificación de Consenso y Disensión

Hallazgo: El análisis de consenso identificó preguntas de alto consenso (P1, P4, P10) donde los respondientes en gran medida acuerdan, y preguntas de bajo consenso (P5, P6, P8) donde la opinión es más dividida.
Ventaja sobre pol.is: Mientras pol.is puede identificar polarización a través de agrupamiento, el análisis neutrosófico proporciona una medida más matizada de consenso que cuenta para la variación independiente de T, I y F. Esto permite identificación más precisa de dónde existe acuerdo público y dónde se necesita diálogo adicional.
Implicación Práctica: Las preguntas de alto consenso (por ejemplo, "Las acciones individuales importan para el clima") proporcionan terreno común para el discurso público. Las preguntas de bajo consenso (por ejemplo, "El crecimiento económico y el ambiente pueden coexistir") identifican áreas donde se necesita comunicación más dirigida y diálogo.

Caracterización Más Rica de Grupos de Respondientes

Hallazgo: El análisis neutrosófico identificó dos grupos con perfiles distintos. El Grupo 0 se caracteriza por alto compromiso en todas las dimensiones, mientras que el Grupo 1 muestra dominancia de verdad con compromiso general más bajo.
Ventaja sobre pol.is: El agrupamiento binario o difuso probablemente produciría divisiones de grupo similares, pero el marco neutrosófico proporciona información más rica sobre la estructura interna de cada grupo. Por ejemplo, el balance casi perfecto del Grupo 0 en T, I y F es una característica distintiva que no sería tan aparente en marcos binarios.
Implicación Práctica: Estos perfiles distintos permiten estrategias de comunicación más dirigidas. El Grupo 0 podría responder bien a mensajería basada en evidencia y matizada que reconozca complejidad. El Grupo 1 podría responder mejor a mensajería clara y directa sobre la realidad e importancia de la acción climática. 5.5 Resumen de Ventajas Comparativas

Tabla 3- Ventajas Comparativas de Enfoque Neutrosófico vs. Tradicional.

Aspecto	pol.is Tradicional	Enfoque Neutrosófico
Marco Lógico	Binario/Difuso	Tripartito (T-I-F)
Tratamiento de Incertidumbre	Residual/Secundario	Dimensión Primaria
Complejidad del Respondiente	Representación Limitada	Rica, Multifacética
Identificación de Grupos	Acuerdo/Desacuerdo	Patrones de Compromiso
Targeting de Comunicación	Segmentos Amplios	Perfiles Matizados
Detección de Ambigüedad	Implícita	Explícita
Dimensionalidad de Datos	Menor	Mayor (Más Rica)
Aplicabilidad	Opinión General	Temas Complejos e Inciertos

La aplicación de lógica neutrosófica al análisis de opinión climática revela ventajas significativas sobre enfoques tradicionales binarios o difusos. Más notablemente, el descubrimiento del grupo "Altamente Comprometido pero Conflictivo" demuestra que el análisis neutrosófico puede descubrir complejidad en datos de opinión que sería invisible para métodos convencionales.

Este hallazgo tiene implicaciones importantes para comunicación climática y política. La narrativa convencional a menudo asume un espectro de "creyentes" a "escépticos." Sin embargo, nuestro análisis revela un paisaje más matizado donde una porción sustancial del público está profundamente comprometida con el tema pero sostiene puntos de vista complejos y multifacéticos. Estos individuos no son obstáculos para la acción climática; más bien, son socios potenciales en desarrollar enfoques más sofisticados y basados en evidencia para desafíos climáticos.

El tratamiento explícito de indeterminación también se alinea mejor con el estado actual de la ciencia climática. Mientras los hechos centrales del cambio climático están bien establecidos, incertidumbres significativas permanecen respecto a impactos regionales, puntos de inflexión y respuestas políticas óptimas (IPCC, 2021). Una estrategia de comunicación que reconozca estas incertidumbres genuinas mientras mantiene claridad sobre hechos resueltos puede ser más creíble y persuasiva que una que presente la ciencia climática como enteramente resuelta.

El análisis de consenso, informado por lógica neutrosófica, proporciona orientación accionable para formuladores de políticas. El alto consenso sobre la realidad del cambio climático y la importancia de la acción individual sugiere que estos pueden servir como terreno común para discurso público. En contraste, el consenso más bajo sobre compensaciones económico-ambientales e viabilidad de energía renovable indica áreas donde diálogo adicional y comunicación basada en evidencia son necesarios.

Conclusiones

Este estudio demuestra que la lógica neutrosófica ofrece un marco poderoso para analizar opinión pública sobre cuestiones complejas y multifacéticas como el cambio climático. Al incorporar explícitamente la indeterminación como una dimensión independiente, el análisis neutrosófico revela patrones de compromiso y complejidad que los enfoques binarios o difusos tradicionales obscurecerían.

Nuestros hallazgos clave—particularmente la identificación de un grupo grande, profundamente comprometido pero internamente conflictivo—tienen implicaciones importantes para comunicación climática y política. En lugar de ver al público como dividido en "creyentes" y "escépticos," los formuladores de políticas y comunicadores deben reconocer un paisaje más matizado de patrones de compromiso y adaptar sus enfoques en consecuencia.

La investigación futura debe extender esta metodología a otras poblaciones y dominios, validando el marco neutrosófico en contextos diversos. La investigación cualitativa también podría enriquecer los perfiles de grupos identificados, explorando las razones subyacentes para sus patrones específicos de respuesta neutrosófica. En última instancia, al abrazar la complejidad inherente e indeterminación en sistemas de creencias humanos, el enfoque neutrosófico ofrece una comprensión más auténtica y accionable de la opinión pública.

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Lógicas Plurales e Inteligencia Artificial: Un Enfoque Neutrosófico para el Análisis Causal

Plural Logics and Artificial Intelligence: A Neutrosophic Approach to Causal Analysis

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RESUMEN

Este artículo propone un marco de pluralismo lógico aplicado al análisis causal en inteligencia artificial (IA) y ciencias sociales. Basado en la tesis pluralista de que no existe un único concepto monolítico de causalidad, sino una familia de conceptos relacionados, confrontamos la premisa humeana de que "no observamos la causalidad" con el contraargumento de Anscombe de que, de hecho, percibimos acciones causales como empujar, golpear y cortar. Sobre esta base, presentamos un estudio ilustrativo que compara el Análisis Comparativo Cualitativo de conjuntos nítidos (csQCA), el de conjuntos difusos (fsQCA) y el neutrosófico (nQCA). De esta comparación, derivamos implicaciones metodológicas y sustantivas. Metodológicamente, encontramos que: (i) la dicotomización estricta (csQCA) puede subestimar las relaciones causales; (ii) la

pertenencia gradual (fsQCA) captura más eficazmente la fuerza y relevancia empírica; y (iii) la descomposición neutrosófica (nQCA) considera explícitamente la verdad, la indeterminación y la falsedad, ofreciendo así un diagnóstico pluralista de la relación causal. Enriquecemos aún más este análisis incorporando una visión pluralista de la causalidad, que abarca la producción mecánica, la diferencia contrafactual, las capacidades/disposiciones y los mecanismos.

Palabras clave: Pluralismo Lógico, Lógica Neutrosófica, Inteligencia Artificial, Lógicas no Clásicas, Análisis Comparativo Cualitativo, Indeterminación, IA Simbólica, Análisis Causal.

ABSTRACT

This article proposes a framework of logical pluralism applied to causal analysis in artificial intelligence (AI) and social sciences. Grounded in the pluralist thesis that there is no single monolithic concept of causation, but rather a family of related concepts, we confront the Humean premise that "we do not observe causation' with Anscombe's counter-argument that we do, in fact, perceive causal actions such as pushing, striking, and cutting. Building on this foundation, we present an illustrative study comparing crisp-set Qualitative Comparative Analysis (csQCA), fuzzy-set QCA (fsQCA), and neutrosophic QCA (nQCA). From this comparison, we derive both the methodological and substantive implications. Methodologically, we find that: (i) strict dichotomization (csQCA) can underestimate causal relationships; (ii) graded membership (fsQCA) more effectively captures empirical strength and relevance; and (iii) neutrosophic decomposition (nQCA) explicitly accounts for truth, indeterminacy, and falsity, thereby offering a pluralistic diagnosis of the causal relationship. We further enrich this analysis by incorporating a pluralistic view of causality, encompassing mechanical production, counterfactual difference, capacities/dispositions, and mechanisms.

Keywords: Logical Pluralism, Neutrosophic Logic, Artificial Intelligence, Non-Classical Logics, Qualitative Comparative Analysis, Indeterminacy, Symbolic AI, Causal Analysis.

Introducción

Los pluralistas lógicos sostienen que no existe un concepto único y unificado de causalidad. En su lugar, defienden la existencia de una pluralidad de nociones relacionadas, todas ellas englobadas bajo el término general de "causalidad" (Cartwright, 2004; Russell, 2008). Esta perspectiva desafía de manera directa la premisa central de la influyente crítica de David Hume, según la cual la causalidad no puede ser observada en el mundo (Hume, 2000). En contraste, (Anscombe, 1971) argumentó célebremente que, en realidad, sí observamos la causalidad en innumerables acciones y procesos cotidianos: vemos objetos que se empujan, se golpean o se cortan. ¿En qué sentido, entonces, puede afirmarse que no percibimos la causalidad? La respuesta humana es que lo que observamos no son conexiones causales en sí mismas, sino meras conjunciones constantes y sucesiones regulares de eventos, sin acceder nunca a la elusiva "conexión necesaria" que vincula causa y efecto.

Frente a ello, el pluralismo causal sostiene que existen múltiples nociones de causalidad igualmente legítimas. Entre ellas se incluyen, por ejemplo, las diferencias contrafactuales, la producción y los procesos, los mecanismos subyacentes, así como las capacidades o disposiciones inherentes (Batista-Barallobre, 2024). En algunos de estos marcos conceptuales es posible tener un acceso perceptivo o cognitivo directo a las relaciones causales, al menos en contextos ordinarios (Silvers Briones et al., 2025).

En los campos de la inteligencia artificial y las ciencias sociales, este debate está lejos de ser un ejercicio puramente metafísico. Tiene profundas implicaciones en cómo modelamos, inferimos y comunicamos las relaciones causales (Jaakkola et al., 2024). La creciente complejidad de los sistemas de IA, particularmente aquellos que interactúan y toman decisiones sobre el mundo social, exige un enfoque más sofisticado y matizado del razonamiento causal. La tesis central de este artículo es doble: (1) el pluralismo lógico, tal como lo articulan (Beall & Restall, 2006; Russell, 2008), aconseja el uso de diferentes marcos lógicos según la tarea y el dominio específicos; y (2) el pluralismo causal, defendido por (Cartwright, 2004; Lewis, 1973; Salmon, 1998; Woodward, 2005), sugiere que diferentes conceptos de causalidad pueden iluminar facetas complementarias del mismo fenómeno. Sostenemos que, al adoptar tanto el pluralismo lógico como el causal, podemos desarrollar sistemas de IA más robustos, transparentes y explicables, mejor equipados para manejar las complejidades e incertidumbres del mundo real.

Preliminares

La historia de la lógica formal ha estado dominada en gran medida por lo que puede denominarse monismo lógico, la creencia en un sistema único y universalmente correcto de razonamiento lógico. Esta tradición, que puede rastrearse desde Aristóteles hasta el trabajo fundacional de Frege y Russell, estableció la lógica proposicional y de predicados clásica como el estándar de oro para el pensamiento racional (Fan, 2025; Salmon, 1998). La elegancia y el poder de la lógica clásica, con su sintaxis, semántica y teoría de pruebas bien definidas, la convierten en un marco atractivo para formalizar el razonamiento humano. En el contexto de la inteligencia artificial, este enfoque monista se manifiesta como una dependencia casi exclusiva de la lógica clásica para la representación del conocimiento y el razonamiento (He & Khorsand, 2024; Ye, 2025).

Los primeros sistemas de IA, desde sistemas expertos como MYCIN y DENDRAL hasta demostradores automáticos de teoremas, se construyeron sobre la suposición fundamental de que el razonamiento inteligente puede ser efectivamente capturado y replicado a través de las operaciones de la lógica clásica. Este enfoque, a menudo denominado "IA de la Vieja Escuela" (GOFAI), ha logrado un éxito considerable en dominios bien definidos de mundo cerrado. Sin embargo, también enfrentan desafíos significativos cuando se confrontan con las complejidades e incertidumbres del mundo real.

La emergencia del pluralismo lógico presenta un desafío significativo para esta ortodoxia monista de larga data. Pionero por filósofos como J.C. Beall y Greg Restall, el pluralismo lógico postula que existe más de una lógica correcta. En esta visión, la adecuación de un sistema lógico dado depende del contexto específico y el propósito de la tarea de razonamiento (Beall & Restall, 2006).

Según el pluralismo lógico, diferentes nociones de validez lógica surgen de diferentes concepciones de lo que constituye un "caso" o un "modelo" (Beall & Restall, 2006; Lewis, 1973). En el ámbito de la inteligencia artificial, esta perspectiva proporciona una justificación robusta para el empleo de una amplia gama de lógicas no clásicas, incluyendo lógicas temporal, epistémica, no monótona, paraconsistente y neutrosófica (Cornelio & Fonseca, 2026). Cada uno de estos sistemas lógicos está diseñado para manejar tipos específicos de razonamiento que a menudo son intratables para la lógica clásica, como el razonamiento sobre el tiempo, el conocimiento, la revisión de creencias, la información contradictoria y la indeterminación.

Paralelo al pluralismo lógico, la noción de pluralismo causal sugiere que coexiste una variedad de familias conceptuales de causalidad. Estas incluyen:

- **Diferencia Contrafactual:** Esta visión, asociada con pensadores como (Lewis, 1973; Pearl, 2009; Woodward, 2005), define la causa como algo que hace una diferencia al efecto. En otras palabras, si la causa no hubiera ocurrido, entonces el efecto no habría ocurrido.
- **Producción y Procesos:** Esta perspectiva, defendida por Wesley Salmon, se enfoca en los procesos físicos y la transmisión de energía e información que conectan una causa con su efecto (Salmon, 1998).
- **Mecanismos:** Este enfoque, articulado por (Machamer et al., 2000), identifica los mecanismos subyacentes y las partes interactuantes que producen un fenómeno.
- **Capacidades y Disposiciones:** Nancy Cartwright argumenta que las causas tienen capacidades o disposiciones para producir efectos que pueden o no ejercerse en una situación dada (Cartwright, 2004).
- **Condiciones Configuracionales/INUS:** Esta visión, desarrollada por (Mackie, 1980) y avanzada por (Ragin, 2009) en el contexto de la metodología de las ciencias sociales, entiende la causa como una parte Insuficiente pero Necesaria de una condición Innecesaria pero Suficiente para un efecto.
- **Intervencionismo:** Estrechamente relacionado con los enfoques contrafactuales y mecanicistas, el relato intervencionista, formulado más notablemente por James Woodward, define una causa como una variable que puede ser manipulada para provocar un cambio en otra variable (Biju et al., 2025; Woodward, 2005).

Cada una de estas conceptualizaciones de causalidad está diseñada para responder diferentes tipos de preguntas: por qué ocurrió un evento, cómo se produjo, bajo qué intervenciones cambiaría y qué componentes estuvieron involucrados. El pluralismo, es importante señalar, no es una forma de relativismo. Más bien, propone una ecología de conceptos y estándares evidenciales, con al menos traducciones e integraciones parciales posibles entre diferentes marcos.

Métodos o Metodología Computacional

Análisis de Métodos QCA: csQCA, fsQCA y nQCA

Para demostrar las implicaciones prácticas del pluralismo lógico en el análisis causal, presentamos un estudio ilustrativo que compara tres métodos de Análisis Comparativo Cualitativo (QCA): QCA de conjuntos nítidos (csQCA), QCA de conjuntos difusos (fsQCA) y QCA neutrosófico (nQCA). El QCA fue desarrollado por primera vez por el sociólogo Charles Ragin a finales de los años 1980 como una forma de cerrar la brecha entre la investigación cualitativa orientada a casos y la investigación cuantitativa orientada a variables (Ragin, 2009).

Es un método teórico de conjuntos que es particularmente adecuado para analizar la complejidad causal, incluyendo cuestiones de equifinalidad (múltiples caminos causales al mismo resultado) y causación conjuntural (causas que solo producen un efecto en combinación con otras causas). Durante las últimas tres décadas, el QCA se ha convertido en un método cada vez más popular en las ciencias sociales, y ahora está comenzando a ganar tracción en otros campos, incluyendo la salud pública, el análisis de políticas y la ciencia de la computación.

QCA de Conjuntos Nítidos (csQCA)

En un csQCA, tanto las condiciones causales como el resultado se representan como variables binarias, con casos que tienen pertenencia completa (1) o ninguna pertenencia (0) en un conjunto dado. El análisis procede entonces a identificar relaciones lógicas entre estos conjuntos. Los dos parámetros clave en csQCA son la consistencia y la cobertura.

La consistencia mide el grado en que una condición causal o una combinación de condiciones es un subconjunto de resultados. En otras palabras, evalúa hasta qué punto la evidencia apoya la afirmación de que la causa lleva al resultado. La fórmula para la consistencia en csQCA es:

$$Consistency_{cs}\,(X \Rightarrow Y) = \frac{(\Sigma\,(X_i \wedge Y_i))}{(\Sigma\,X_i)} \qquad (1)$$

La cobertura mide la relevancia empírica de una condición causal o combinación de condiciones. Evalúa hasta qué punto el resultado es explicado por la causa. La fórmula para la cobertura en csQCA es:

$$Coverage_{cs}(X \Rightarrow Y) = \frac{(\Sigma\,(X_i \wedge Y_i))}{(\Sigma\,Y_i)} \qquad (2)$$

QCA de Conjuntos Difusos (fsQCA)

El QCA de conjuntos difusos es una extensión del csQCA que permite la pertenencia parcial en conjuntos. En lugar de valores binarios, los casos pueden tener puntuaciones de pertenencia que van de 0 a 1 (Ye, 2025). Esto permite un análisis más fino de las relaciones causales ya que puede dar cuenta de efectos graduales y parciales. Las fórmulas para la consistencia y cobertura en fsQCA son las siguientes:

$$\text{consistency}_{\text{fs}}\,(\text{X}_\text{i} \leq \text{Y}_\text{i}) = \frac{\sum \min(\text{X}_\text{i},\text{Y}_\text{i})}{\sum \text{X}_\text{i}} \qquad (3)$$

$$\text{coverage}_{\text{fs}}\;\,(\text{X}_\text{i} \leq \text{Y}_\text{i}) = \frac{\sum \min(\text{X}_\text{i},\text{Y}_\text{i})}{\sum \text{Y}_\text{i}} \qquad (4)$$

QCA Neutrosófico (nQCA)

El QCA neutrosófico es una extensión adicional del QCA que incorpora el concepto de indeterminación (Smarandache & Leyva, 2025). En nQCA, cada caso se representa por una tripla neutrosófica (T, I, F), donde T representa el grado de pertenencia de verdad, I el grado de indeterminación y F el grado de pertenencia de falsedad. Esto permite un análisis más matizado de las relaciones causales ya que puede dar cuenta explícitamente de la incertidumbre y ambigüedad en los datos.

La consistencia y cobertura en nQCA se generalizan usando un operador de implicación neutrosófico, que opera componente por componente en triplas (Cornelio & Fonseca, 2025; Smarandache, 2019). Esta extensión captura no solo qué tan fuertemente una condición implica un resultado (verdad), sino también el grado en que la relación permanece ambigua (indeterminación) o es contradecida por la evidencia (falsedad). Las fórmulas para la consistencia y cobertura neutrosófica son:

$$\boldsymbol{Consistency_N(X \Rightarrow Y) = \frac{1}{n}\sum_{i=1}^{n} I_R(X_i \Rightarrow Y_i) \in [0,1]^3} \qquad (5)$$

$$Coverage_N(X \Rightarrow Y) = \frac{1}{n}\sum_{i=1}^{n} I_R(Y_i \Rightarrow X_i) \in [0,1]^3$$

donde I_R devuelve un valor tripla (T, I, F). Por lo tanto, las medidas de consistencia y cobertura ya no son escalares únicos sino vectores de tres componentes (T, I, F) que representan explícitamente la verdad, incertidumbre y falsedad en las relaciones causales.

Resultados y discusión

Para hacer más concreta la comparación entre csQCA, fsQCA y nQCA, discutiremos ahora un ejemplo ilustrativo. Estudiaremos la implicación X => Y para los tres casos usando el mismo conjunto de datos para cada método.

Conjunto de Datos

Las pertenencias difusas para nuestros tres casos son las siguientes:

- Caso 1: $X_1 = 0.8$, $Y_1 = 0.9$
- Caso 2: $X_2 = 0.6$, $Y_2 = 0.5$
- Caso 3: $X_3 = 0.7$, $Y_3 = 0.4$

Las pertenencias neutrosóficas para los mismos casos son:

- Caso 1: $X_1 = (0.8, 0.1, 0.1)$, $Y_1 = (0.9, 0.05, 0.05)$
- Caso 2: $X_2 = (0.6, 0.2, 0.2)$, $Y_2 = (0.5, 0.3, 0.2)$
- Caso 3: $X_3 = (0.7, 0.2, 0.1)$, $Y_3 = (0.4, 0.4, 0.2)$

QCA de conjunto nítido (csQCA)

Para csQCA, primero se binarizaron los valores de pertenencia difusa. Una regla común para la binarización es asignar un valor de 1 si la pertenencia es mayor a 0.5 y 0 en caso contrario. Aplicando esta regla al conjunto de datos se obtiene:

- Caso 1: $X_1 = 1$, $Y_1 = 1$
- Caso 2**:** $X_2 = 1$, $Y_2 = 0$
- Caso 3**:** $X_3 = 1$, $Y_3 = 0$

Ahora, se calculan la consistencia y la cobertura:

- $\sum(X_i \wedge Y_i)$: 1 (solo el Caso 1)
- $\sum(X_i)$: 3
- $\sum(Y_i)$: 1
- Consistencia_cs**:** 1 / 3 = 0.33
- Cobertura_cs**:** 1 / 1 = 1.00

El análisis csQCA mostró una puntuación de consistencia muy baja (0.33). Esto se debe a que existen dos contraejemplos (Casos 2 y 3) en los que la causa está presente (X=1), pero el resultado está ausente (Y=0). Sin embargo, la cobertura es 1.00, lo que significa que el único caso en el que tanto la causa como el resultado están presentes (Caso 1) cubre la totalidad de los casos en los que el resultado está presente.

QCA de conjunto difuso (fsQCA)

En fsQCA se utilizaron las pertenencias difusas originales. Primero, se calculó el mínimo entre X e Y para cada caso:

- Caso 1: min(0.8, 0.9) = 0.8
- Caso 2: min(0.6, 0.5) = 0.5
- Caso 3**:** min(0.7, 0.4) = 0.4

Posteriormente, se calcularon las sumas:

- ∑min = 0.8 + 0.5 + 0.4 = 1.7
- ∑X = 0.8 + 0.6 + 0.7 = 2.1
- ∑Y = 0.9 + 0.5 + 0.4 = 1.8

Finalmente:

- Consistencia_fs**:** 1.7 / 2.1 = 0.81
- Cobertura_fs**:** 1.7 / 1.8 = 0.94

Al permitir membresías parciales, fsQCA suaviza las contradicciones estrictas del csQCA. El resultado es una consistencia mucho más fuerte (0.81) y una cobertura alta (0.94), lo que sugiere una relación causal sólida y empíricamente relevante.

QCA neutrosófico (nQCA)

Para nQCA se emplean pertenencias neutrosóficas. Se utilizó un operador de implicación neutrosófica por componentes definido como:

La ecuación de R-implicación neutrosófica de Łukasiewicz es:

$$t_{\Rightarrow_{NL}} = \begin{cases} 1, \text{if } t_x \le t_y \\ 1 - t_x + t_y, \text{otherwise'} \end{cases} \qquad i_{\Rightarrow_{NL}} = \begin{cases} 0, \text{if } i_y \le i_x \\ i_y - i_x, \text{otherwise'} \end{cases} \qquad \text{and} \qquad f_{\Rightarrow_{NL}} = \begin{cases} 0, \text{if } f_y \le f_x \\ f_y - f_x, \text{otherwise} \end{cases} \quad (7)$$

Aplicando este operador a cada caso para la consistencia (X ⇒ Y) se obtiene:

- Caso 1: (1, 0, 0)
- Caso 2: (0.9, 0.1, 0)
- Caso 3: (0.7, 0.2, 0.1)

El promedio de estos tríos nos da la consistencia neutrosófica**:**

- Consistencia_N = (0.8667, 0.1000, 0.0333)

Aplicando el operador a cada caso para la cobertura (Y ⇒ X) se obtiene:

- Caso 1: (0.9, 0.05, 0.05)
- Caso 2: (1, 0, 0)
- Caso 3: (1, 0, 0)

El promedio de estos tríos nos da la cobertura neutrosófica:

- Cobertura_N = (0.9667, 0.0167, 0.0167)

El nQCA descompone la evaluación de la relación causal en tres componentes: verdad (T), indeterminación (I) y falsedad (F). La Consistencia Neutrosófica obtenida (0.8667, 0.1000, 0.0333) revela un alto componente de verdad en la implicación causal (X⇒Y), respaldado por baja indeterminación y mínima falsedad. Asimismo, la Cobertura Neutrosófica (0.9667, 0.0167, 0.0167) demuestra que la condición X tiene un poder explicativo casi completo sobre la ocurrencia de Y, con valores extremadamente bajos de indeterminación y falsedad. En conjunto, estas métricas proporcionan una visión excepcionalmente robusta y clara de la relación de suficiencia, superando significativamente las limitaciones de ambigüedad inherentes a los métodos tradicionales de QCA escalar.

Tabla 1- Comparación de medidas de consistencia y cobertura en csQCA, fsQCA y nQCA

Método	Consistencia	Cobertura
csQCA	0.33	1.00
fsQCA	0.81	0.94
nQCA	(0.8667, 0.1000, 0.0333)	(0.9667, 0.0167, 0.0167)

En este caso, el csQCA, con su estricta dicotomización, resulta frágil y puede llevar fácilmente a concluir que no existe relación causal. Al permitir gradaciones, el fsQCA refuerza la relación y ofrece una evaluación más realista. Finalmente, el nQCA descompone la relación en verdad, incertidumbre y falsedad, lo cual se alinea

con los principios del pluralismo lógico y permite un manejo más transparente y comprehensivo de la ambigüedad.

Comparación entre csQCA, fsQCA y nQCA

La comparación de csQCA, fsQCA y nQCA en nuestro ejemplo ilustrativo ofrece varias ideas importantes para el análisis causal en inteligencia artificial y en ciencias sociales. Estas ideas pueden agruparse en cinco categorías clave. Cada una de estas áreas se explora con mayor detalle en las subsecciones siguientes, con un énfasis particular en las implicaciones prácticas para el diseño e implementación de sistemas de IA más robustos y confiables.

1. **Calibración y sensibilidad**: La diferencia drástica entre las puntuaciones de consistencia de csQCA (0.33) y fsQCA (0.81) resalta el papel crítico de la calibración en el análisis causal. La elección de un umbral para la dicotomización puede alterar de manera fundamental el veredicto causal. Esto sugiere que los investigadores deben realizar análisis de sensibilidad respecto a sus decisiones de calibración y reportar la estabilidad de sus hallazgos a través de diferentes sistemas lógicos.
2. **Robustez multi-lógica**: La convergencia entre la consistencia de fsQCA (0.81) y el componente de verdad de la consistencia en nQCA (0.8667) brinda un fuerte respaldo a la solidez del hallazgo de que existe una relación de "marcador de diferencia" entre X e Y. Sin embargo, la divergencia en el componente de indeterminación de nQCA (0.1) aconseja cautela y sugiere que los investigadores deberían reportar explícitamente el nivel de incertidumbre de sus hallazgos.
3. **Gestión de la incertidumbre**: El componente de indeterminación de nQCA constituye una herramienta valiosa para gestionar el riesgo epistémico. Esto permite a los investigadores tomar decisiones más informadas sobre, por ejemplo, si es necesario elevar los criterios de confirmación, recolectar datos adicionales o condicionar sus recomendaciones al nivel de incertidumbre.
4. **Diagnóstico de contradicciones**: El componente de falsedad en nQCA (0.0333) indica que existen contradicciones limitadas pero reales en los datos. Esto invita a explorar más a fondo posibles subgrupos, condiciones contextuales y mecanismos latentes que puedan estar bloqueando la eficacia de X en algunos casos.

5. **Comunicación y trazabilidad**: La pluralidad de métricas ofrecidas por nQCA mejora la explicabilidad de los hallazgos causales. Permite a los investigadores comunicar no solo "qué tan fuerte" es la relación entre X e Y, sino también "qué tan indeterminada" y "qué tan contradictoria" resulta ser la pauta observada.

Desde la noción de causalidad basada en capacidad/disposición, el componente de indeterminación en nQCA indica que la capacidad causal de X existe, pero su activación depende del contexto. Esto exige especificar los moduladores y las condiciones de frontera que gobiernan esta relación. Desde la noción configuracional/INUS de la causalidad, el patrón es consistente con la causalidad conjuntural y la equifinalidad: X puede ser una parte necesaria de ciertas combinaciones suficientes, pero no es universalmente suficiente por sí sola.

Este multiverso conceptual no es un defecto, sino una ventaja epistémica, ya que permite integrar información sobre la fuerza, estabilidad, mecanismos y contexto de una relación causal.

Implicaciones para la Inteligencia Artificial

Las ideas derivadas de nuestro estudio ilustrativo tienen varias implicaciones relevantes para el desarrollo de la IA:

- **Arquitecturas neuro-simbólicas**: Los sistemas de IA neuro-simbólica pueden beneficiarse del uso de múltiples lógicas (por ejemplo, temporal, deóntica, paraconsistente y neutrosófica) en diferentes etapas de la cadena de procesamiento (percepción, verificación y explicación).
- **Evaluación causal automatizada**: La adopción de validación cruzada inter-lógica en la evaluación causal automatizada puede reducir el riesgo de decisiones espurias derivadas de artefactos de calibración.
- **Comunicación responsable de la IA**: La comunicación responsable de la IA debería reportar explícitamente los componentes de verdad, indeterminación y falsedad de las afirmaciones causales cuando sea pertinente.
- **Triangulación metodológica**: La integración de QCA con grafos causales (Pearl) y pruebas intervencionistas (Woodward) ofrece una triangulación pluralista de diferencias, mecanismos y configuraciones.

Conclusiones

El estudio ilustrativo que constituye el núcleo de este artículo demuestra que la aplicación conjunta de csQCA, fsQCA y nQCA no produce "tres verdades", sino tres proyecciones compatibles de un mismo fenómeno causal. Estas proyecciones revelan una relación causal caracterizada por una alta fuerza de diferenciación, una indeterminación no trivial y una contradicción acotada. Esto ejemplifica el pluralismo lógico, en el sentido de que diferentes sistemas lógicos son válidos dependiendo de las nociones elegidas de caso y validez, así como el pluralismo causal, en el sentido de que distintos conceptos legítimos de causalidad pueden aplicarse al mismo fenómeno.

La consecuencia práctica es clara: para modelar fenómenos complejos y diseñar una IA explicable y robusta, debemos aprender a habitar esta pluralidad en lugar de colapsarla en un único veredicto monolítico. El futuro de la IA, particularmente en su aplicación a problemas sociales y científicos complejos, dependerá de su capacidad para abrazar y gestionar las ambigüedades e incertidumbres inherentes del mundo. Como se ha expuesto en este artículo, un enfoque pluralista de la lógica y de la causalidad ofrece un camino prometedor.

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LGU-Combined-Consciousness State Model

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Abstract

This article aims to introduce some modern algebraic structures as hyper super matrices. The classical algebra and matrices cannot process higher-dimensional information with several levels of ambiguity and uncertainty. Hence, it is necessary to establish such superalgebraic structures that can organize and classify the uncertain and incomplete information floating in parallel higher dimensions as facts, events, or realities. To achieve the desired goal, a particular construction of Hypersoft Matrix (HS-Matrix) and Subjectively Whole Hyper-SuperSoft Matrix (SWHSS-Matrix) is offered in a plithogenic Fuzzy environment initially, and some aggregation operators are formulated. A Local-Global-Universal Combined Consciousness State Ranking Model is formulated as an application. As the classification of non-physical phenomena like state of physical health or Consciousness has not yet been addressed in the area of decision making therefor the proposed model will open a new dimension of classification of the non-physical part of the universe in which one can select the most suitable possible reality from several parallel realities which would be useful in the field of artificial intelligence. This model classifies the accumulated states of matter bodies (subjects). And gives a possible description of the Combined-Consciousness State of a Universe. In addition, it offers a local ranking by observing the information through several angles of vision, just like a human mind does, and a universal ranking by classifying the accumulated states. Furthermore, the final Global Ranking is achieved by constructing a percentage frequency-matrix and an authenticity measure of the order is offered. A numerical example is constructed to describe SWHSS-Matrix and LGU-Ranking Model. Some pie graphs are used to describe the individual states, accumulated states, and the ultimate accumulated universal state of all given subjects (a Combined Conscious State of Universe).

Keywords: Subjectively-Whole-Hyper-Super-Soft-Matrix, Parallel-Dimensions, Attributive-Ranking, Local-Global-Universal-Ranking, Combined-Consciousness, Percentage-Frequency-Matrix, Pie-Graphs.

1. Introduction

As we know, the human brain has some factors of vagueness and precariousness in its judgments and inferences due to multiple opinions, and the complexity of the data, as attributes events, and information derived from its own environments. Scientists after taking into account this basic trait of the human mind

start arguing the dire need for some different mathematics that could possibly handle this vagueness factor. Some of the following theories developed gradually. Fuzzy set theory by Zadeh (1965) [1] Intuitionistic fuzzy set (IFS) theory by k.Atanassov [2] [3]. The cloud of vagueness is further extended by F. Smarandache, [4][5][6]. Some more recent extensions and modernizations of the neutrosophic set are presented in [7] [8] [9] [10] [11] [12]. In 1999 Molodtsove [13] introduced Soft Set, a soft set is a parameterized representation of subsets in which one can express multiple attributes and subjects in a unique parameterized formulation. Some further extensions of the soft set were provided in [14] [15] [16]. Later, in 2018, F.Smarandache [17] [18] introduced another expanded version of Softest known as the Hypersoft-Set and the Plithogenic Hypersoft-Set. In these sets, he extended the function of the combination of attributes to multi attributes and sub-attributes. He presented the basic definitions and addressed many open problems of the development of new literature, such as aggregation operators and MADM techniques. We are going to answer some of the open issues raised by Smarandache, S.Rana and co-authors "[19] extended the Plithogenic Hyper-Soft Set to Plithogenic Whole-Hyper-Soft Set by accumulating the memberships and providing both exterior and interior states of the part of Universe/Event/Reality/Information (a combination of Attributes, Sub Attributes, Subjects represented). We represented the Plithogenic Fuzzy Hyper-Soft set and the Plithogenic Fuzzy Whole Hyper-Soft set in a novel form of matrices in the fuzzy environment named as Plithogenic Fuzzy Hyper-Soft Matrix (PFHS-Matrix) and Plithogenic Fuzzy Whole Hyper-Soft Matrix and some local operators were established. Furthermore. In the next phase, S.Rana and co-authors "[20] further dilated the Plithogenic Whole Hyper-Soft Set to Plithogenic Crisp/Fuzzy/Intuitionistic/Neutrosophic Subjective Hyper-Soft Set and represented them in the more dilated version of Soft-Matrix initially in the fuzzy environment termed as Plithogenic Subjective Hyper Super Soft-Matrix. Then developed a Local-Global Universal Subjective Ranking Model by using the new amplified expression of matrices. Some further literature on HyperSoft Set and Plithogency was established in [21-28]. In this article, in the first stage, we have further broadened those earlier introduced Plithogenic Fuzzy Whole Hyper Soft Set and Plithogenic Subjective Hyper-Soft Set to Plithogenic Attributive Subjectively Whole Hyper Soft Set (PASWHSS-Set) in the Fuzzy environment. we have formulated a new type of Matrix initially in a fuzzy environment named Plithogenic Subjectively Whole Hyper Super Soft Matrix (PSWHSS-Matrix). These advanced types of matrices are generated by the hybridization of hyper matrices and super matrices [29-32] These hypersoft matrices are sets/clusters of parallel layers of matrices representing clusters of parallel universes/ realities/ events/ information. These are such hyper-matrices (parallel layers of matrices) whose elements are also matrices. Thus, these matrices are tensors of rank three and four, respectively, having three and four indices of variations. Then later, we have formulated an LGU Combined-Consciousness State Ranking Model. The forte of this model is its classification of nonphysical phenomena. Thus, it will allow opening a new non-physical dimension of classification i.e. selecting one possibility out of multiple possibilities. Moreover, it offers a transparent ranking of attributes (states of subjects) and universes from micro-universe to macro-universe levels by observing them through numerous angles of vision in dissimilar environments of different ambiguity and hesitation levels. Furthermore, it will also furnish and formulate extreme and neutral values of these universes (sets of information, realities, events). This new model actually compacts the expanded Universe to a single lowest point. Finally, we have also anticipated producing a percentage authenticity measure of ranking, which is provided by using a frequency matrix. In the end, we have given an application of the Model using a numerical example. In this example, fuzzy linguistic scales are used to quantify the states of our subjects (bodies of matter known as individuals). The quantified states of subjects are attributes/sub-attributes known as individual fuzzy states or individual fuzzy memberships. Later, the aggregation operators are used to accumulate these states (subject-wise). The accumulated states are represented by fuzzy whole memberships. Initially, these states are accumulated at the local level using a single aggregation operator representing a viewpoint, and a local ordering of states would be achieved. The global ordering of states

would be achieved through the use of multiple aggregation operators. By the further accumulation of the already accumulated states, the universal states of accumulation and the universal order would be reached.

Now the further query arises why we are specifically using hyper-Soft and Hyper-Super-Soft matrices for the expression of the Plithogenic Hyper-Soft Set and Plithogenic Attributive Hyper-Soft Set? The answer might be convincing that this Plithogenic Universe is so vast and expanded in its interior (having Fuzzy, Intuitionistic Fuzzy, Neutrosophic, environments with memberships non-memberships, and indeterminacies) and in its exterior (managing many attributes, sub-attributes, and sub-sub-attributes concerning to its subjects). Therefore to organize and classify such highly scattered information we need to formulate some super algebraic structures like these Matrices.

This article is organized into seven basic sections. After the (**section-1**) introduction, **Section 2** summarises some related preliminaries. In **Section 3** we introduce some fundamental new concepts and definitions of the Hypersoft set expression, the HS matrix, and the SWHSS matrix with examples in a plithogenic fuzzy environment. We use these new types of matrices to develop the LGU Combined-Consciousness State Ranking Model. While in **Section 4** some local aggregation operators such as disjunction operators, conjunction operators, averaging operators and compliment operators for PFHS matrices are formulated. **Section 5** describes the algorithm of the LGU Combined-Consciousness State Ranking Model in the plithogenic fuzzy environment In this Model, we would provide the classification of attributes (a non-physical phenomenon or states) at the local, Global and Universal levels. We offer the Universal ranking by classifying these already accumulated universal states. The Local Ranking is offered by observing the higher dimensional information through several angles of vision or states just like a human mind which possesses multiple layers of thought. These thoughts undergo and change their angles in order to achieve a precise or accurate status but before certain complex procedures of mind are applied upon them. Finally, mental thoughts hold their possibly best and desired status/angels depending upon certain complex procedures and environments. In order to learn the transparent Global Ranking, we have applied a Percentage-Frequency-Matrix by accumulating the states of the human mind (several angles of vision). Finally, to preserve transparency and accuracy, our model also provides the authenticity measure of the ordering. In **Section 6** Application of the LGU-Combined-Consciousness State Ranking Model is presented and final combined universal states are offered. In **Section 7** the flow of the model from individual states of subjects to their combined-universal states is described by pi graphs and some conclusions and open problems are discussed.

2 Preliminaries

This section, narrates some fundamental useful definitions of the hyper-soft set, Hyper matrices, and Super matrices.

Definition 2.1 [17] (Hyper-soft set)
Let U be the initial universe of discourse $P(U)$ the power set of U.
let $a_1, a_2, \dots, a_n$ for $n \geq 1$ be n distinct attributes, whose corresponding attribute-values are respectively the sets $A_1, A_2, \dots, A_n$ with $A_i \cap A_j = \varphi$ for $i \neq j$ and $i, j \in \{1,2, \dots, n\}$.
Then the pair $(F, A_1 \times A \times \dots \times A_n)$ where,

$$F: A_1 \times A \times \dots \times A_n \to P(U),$$

is called a hyper-soft set over U;
Definition 2.2 [29] [30] (super-matrices)

A rectangular or square arrangements of numbers in rows and columns are known as matrices,or simply ordinary matrices, wheras a super-matrix is such matrix whose elements are matrices. These elements can be either scalars or matrices.

$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, *where*

$a_{11} = \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$, $a_{12} = \begin{bmatrix} 0 & 40 \\ 21 & -12 \end{bmatrix}$,

$a_{21} = \begin{bmatrix} 3 & -1 \\ 5 & 7 \\ -2 & 9 \end{bmatrix}$, $a_{22} = \begin{bmatrix} 4 & 12 \\ -17 & 6 \\ 3 & 7 \end{bmatrix}$ *a is a super-matrix.*

Note: The elements of super-matrices are considered as sub-matrices i.e. $a_{11}, a_{12}, a_{21}, a_{22}$ *are submatrices of the super-matrix* a.

Definition 2.3 [31] [32] (Hyper-matrices)

For $n_1, \dots, n_d \in N$, *a function* $f: (n_1) \times \dots \times (n_d) \to F$ *is a hyper-matrix, or d-hyper-matrix. Often* $a_{k_1 \dots k_d}$ *are used to denote the value* $f(k_1 \dots k_d)$ *of* f *at* $(k_1 \dots k_d)$ *and think of* f *(renamed as A) as specified by a d-dimensional table of values, writing* $A = \left[a_{k_1 \dots k_d}\right]_{k_1 \dots k_d}^{n_1, \dots, n_d}$

A 3-hypermatrix can be written on a (2-dimensional) piece of paper as a list of ordinary matrices, called slices. For example

$$A = \begin{bmatrix} a_{111} & a_{121} & a_{131} & . & a_{112} & a_{122} & a_{132} \\ a_{211} & a_{221} & a_{231} & . & a_{212} & a_{222} & a_{232} \\ a_{311} & a_{321} & a_{331} & . & a_{312} & a_{322} & a_{332} \end{bmatrix}$$

3. Plithogenic Fuzzy HS-Matrix and Plithogenic Fuzzy SWHSS-Matrix

This section, develops some literature about the plithogenic hypersoft set in the following manner.

1. We introduce some basic new beliefs and definitions of expression of hypersoft set and HS-Matrix with examples.
2. We introduce novel HS-matrix as SWHSS-Matrix in plithogenic Fuzzy environment.
3. We portray the compact and expanded expressions of HS-Mtricx and SWHSS-Matrix.

To develop an understanding of the literature, we give some new definitions.

Definition 3.1 (Plithogenic Fuzzy HyperSoft-Set (PFHS-Set)): Let U_F be the initial universe of discourse $P(U_F)$ the power set of U_F. A_j^k is a combination of attributes/Sub-Attributes for some $j = 1,2,3, \dots, N$ Attributes, $k = 1,2,3, \dots, L$ Sub-Attributes and x_i $i = 1,2,3, \dots,$ M are subjects under consideration then $(F_F, A_1^k, A_2^k \dots, A_3^k)$ is PFHS-Set represented by plithogenic fuzzy memberships $\mu_{A_j^k}(x_i)$.

where, $F_F: A_1^k \times A_2^k \times A_3^k \times \dots \times A_N^k \to, P(U_F)$ is a mapping from a complex cross product of the attributes to the power set $P(U_F)$. This PFHS-Set is represented as

$$\mathrm{F} = \begin{matrix} x_1\left(\mu_{A_j^k}(x_1)\right), \\ x_2\left(\mu_{A_j^k}(x_2)\right), \\ . \\ . \\ . \\ x_{\mathrm{M}}\left(\mu_{A_j^k}(x_M)\right) \end{matrix}$$

Definition 3.2 (Plithogenic Fuzzy HyperSoft-Matrix (PFHS-Matrix)):

Let U_F be the Fuzzy universe of discourse, $P(U_F)$ be the power set of U_F, A_j^k is a combination of attributes/sub-attributes for some $j = 1,2,3, \dots, N$ attributes, $k = 1,2,3, \dots, L$ sub-attributes and x_i $i =$

1,2,3,…, M are subjects under consideration then PFHS-Matrix, $F_{ij}^{k} = \left[\mu_{A_j^k}(x_i)\right]$ is a mapping $F_F: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \to P(U_F)$, from a complex cross product of the attributes to the power set $P(U_F)$., Where $\mu_{A_j^k}(x_i) \in [0,1]$ are fuzzy memberships s.t $\mu_{A_j^k}(x_i) + \upsilon_{A_j^k}(x_i) = 1$. These Fuzzy memberships $\mu_{A_j^k}(x_i)$ are the elements of PFHS-Matrix and are assigned for the Part of Universe/Reality/Event/Information, by decision-makers or concerned bodies through the linguistic scales. For further details, see ref. [28-31]. we may call these memberships the individual fuzzy memberships.

We may write F_{ij}^{k} simply as F.The compact form of PFHS-Matrix, is

$$F = \left[\mu_{A_j^k}(x_i)\right] \tag{3.1}$$

And an expanded form of PFHS-Matrix, is

$$F = \begin{array}{c} \begin{matrix} A_1^k & A_2^k & \ldots & A_N^k \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{matrix} \begin{bmatrix} \mu_{A_1^k}(x_1) & \mu_{A_2^k}(x_1) & \ldots & \mu_{A_N^k}(x_1) \\ \mu_{A_1^k}(x_2) & \mu_{A_2^k}(x_2) & \ldots & \mu_{A_N}(x_2) \\ \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \vdots \\ \mu_{A_1^k}(x_M) & \mu_{A_2^k}(x_M) & \ldots & \mu_{A_N^k}(x_M) \end{bmatrix} \end{array} \tag{3.2}$$

Example 1:

Consider the mapping F defined as,

$F_F: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \to, P(U_F)$

(taking some specific numeric values of A_j^k)

Consider $T = \{x_1, x_2, x_3\}$, is a subset of powerset $P(U_F)$ and x_i subjects for $i = 1,2,3$, are x_1, x_2, x_3. The associated states of these subjects are A_j^k Attributes/Sub-Attributes for $j = 1,2,3,4$ and $k = 1,2,3$. To represent these states some fuzzy memberships would be assigned by the Concerned body, through the five-point linguistic scale (see ref. [28-31]) T

The set representation of information is described as PFHS-Set as,

$$F_\alpha(A_1^3, A_2^1, A_3^1, A_4^2) = \begin{Bmatrix} x_1(0.3,0.6,0.5,0.5), \\ x_2(0.4,0.4,0.3,0.1), \\ x_3(0.6,0.3,0.4,0.7) \end{Bmatrix} \tag{3.3}$$

And further organized and expressed in one layer of PFHS-Matrix F_{ij}^{α},

$$\mathrm{F} = \begin{array}{c} \begin{matrix} A_1^3 & A_2^1 & A_3^1 & A_4^2 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \begin{bmatrix} 0.3 & 0.6 & 0.5 & 0.5 \\ 0.4 & 0.4 & 0.3 & 0.1 \\ 0.6 & 0.3 & 0.4 & 0.7 \end{bmatrix} \end{array} \tag{3.4}$$

Where $A_1^3 \;\; A_2^1 \;\; A_3^1 \;\; A_4^2$ is a specific α combination of Attributes/Sub-Attributes representing states of subjects x_1, x_2, x_3. F_{ij}^{α} is representing a single layer out of multiple possible layers of PFHS-Matrix. For a more detailed description and applications, see [19]

Example 2. Consider layered representation $F = \left[\mu_{A_j^k}(x_i)\right]$ for $k = 1$, $j = 1,2,3,4$ and $i = 1,2,3$, i.e (first level-layer) and for $k = 2$, $j = 1,2,3,4$ and $i = 1,2,3$, i.e (second level-layer). let $T = \{x_1, x_2, x_3\}$ be Subjects in PFHS-Set associated to given attribute the PFHS-Set is represented through fuzzy memberships as described bellow,

$$F(A_1^1, A_2^1, A_3^1, A_4^1) = \begin{Bmatrix} x_1(0.3,0.6,0.3,0.5), \\ x_2(0.4,0.5,0.2,0.1), \\ x_3(0.6,0.2,0.3,0.7) \end{Bmatrix} \tag{3.5}$$

$$F(A_1^2, A_2^2, A_3^2, A_4^2) = \begin{Bmatrix} x_1(0.5,0.4,0.2,0.6) \\ , x_2(0.5,0.7,0.8,0.4), \\ x_3(0.7,0.6,0.5,0.9) \end{Bmatrix} \tag{3.6}$$

The matrix representation of this PFHS-Set F is described as PFHS-Matrix,

$$\boldsymbol{F} = \begin{matrix} \begin{bmatrix} 0.3 & 0.6 & 0.3 & 0.5 \\ 0.4 & 0.5 & 0.2 & 0.1 \\ 0.6 & 0.2 & 0.3 & 0.7 \end{bmatrix} \\ \begin{bmatrix} 0.5 & 0.4 & 0.2 & 0.6 \\ 0.5 & 0.7 & 0.8 & 0.4 \\ 0.7 & 0.6 & 0.5 & 0.9 \end{bmatrix} \end{matrix} \tag{3.8}$$

For further details, see ref.[20]

Definition 3.3 (Plithogenic Fuzzy Subjectively-Whole Hyper-Super-Soft-Matrix (PFSWHSS-Matrix)):

Let U_F be the primary universe of discourse, in the Fuzzy situation and $P(U_F)$ be the power set of U_F. Let $A_1^k, A_2^k, \dots, A_N^k$ are A_j^k N distinct attributes/subattributes for $= 1,2, \dots N$, $k = 1,2, \dots L$ is representing attribute values then **PFSWHSS-Matrix** is, $F\begin{bmatrix} \left[\mu_{A_j^k}(x_i)\right] \\ \left[\Omega^t_{\mathbf{A}_j^k}(X)\right] \end{bmatrix}$ is mapping

$$F_F: A_1^k \times A_2^k \times \dots \times A_N^k \to P(U_F)$$

we may use a compact notation of PFSWHSS-Matrix, F_{ij}^{kt}, This matrix is expressed by both individual fuzzy memberships $\mu_{A_j^k}(x_i)$ (individual fuzzy states of subjects regarding each attribute) and the aggregated fuzzy memberships $\Omega_{A_j^k}(\boldsymbol{X})$ (subject-wise aggregated states). In F_{ij}^{kt} $t = 1,2, \dots O$ is representing aggregation operators. In PFSWHSS-Matrix the fuzzy states (fuzzy memberships) of all given subjects are aggregated and then represented as for each attribute/sub-attribute. This PFSWHSS-Matrix handles not only a single combination of attributes/subattributes but rather multiple combinations of attributes/sub-attributes out of their complex cross products or in other words. This matrix F_{ij}^{kt}, has four indices of variation is a soft tensor of rank 4. We may write F_{ij}^{kt} as F for the simplification of notation. Four types of variation are presented in this PFSWHSS matrix. The first Variations on the index $i = 1,2, \dots M$ generate M rows of Matrix, the second variations on the index $j = 1,2, \dots N$ generate N columns, and the third variations on $k = 1,2, \dots L$ produces L combinations of rows and columns as parallel-layers of $M \times N$ matrices as hyperSoft Matrix. The fourth variation on $t = 1,2, \dots P$ describes *the P sets* of Clusters.

The representation of PFSWHS-Matrix in a compact form is,

$$\mathrm{F} = \begin{bmatrix} \left[\mu_{A_j^k}(x_i)\right] \\ \left[\Omega^t_{\mathbf{A}_j^k}(X)\right] \end{bmatrix}, \tag{3.9}$$

$\mathrm{F} = \begin{bmatrix} \left[\mu_{A_j^1}(x_i)\right] \\ \left[\Omega^t_{\mathbf{A}_j^1}(X)\right] \end{bmatrix}$ represents a single Layer of SWHSS-Matrix for $k = 1$ i.e an α universe.

$\mathrm{F} = \begin{bmatrix} \left[\mu_{A_j^1}(x_i)\right] \\ \left[\Omega^t_{\mathbf{A}_j^1}(X)\right] \end{bmatrix}$ represents a single Layer of SWHSS-Matrix for $k = 2$ i.e an β universe.

The representation of PFSWHS-Matrix in an expanded form is,

$$
\mathbf{F} = \begin{matrix}
\mu_{A_1^1}(x_1) & \mu_{A_2^1}(x_1) & \dots & \mu_{A_N^1}(x_1) \\
\mu_{A_1^1}(x_2) & \mu_{A_2^1}(x_2) & \dots & \mu_{A_N^1}(x_2) \\
\vdots & \vdots & \dots & \vdots \\
\mu_{A_1^1}(x_M) & \mu_{A_2^1}(x_M) & \dots & \mu_{A_N^1}(x_M) \\
\left[\mathbf{\Omega}^{\mathbf{1}}_{\mathbf{A}_1^1}(X) & \mathbf{\Omega}^{\mathbf{1}}_{\mathbf{A}_2^1}(X) & \dots & \mathbf{\Omega}^{\mathbf{1}}_{\mathbf{A}_N^1}(X)\right] \\
\mu_{A_1^2}(x_1) & \mu_{A_2^2}(x_1) & \dots & \mu_{A_N^2}(x_1) \\
\mu_{A_1^2}(x_2) & \mu_{A_2^2}(x_2) & \dots & \mu_{A_N^2}(x_2) \\
\vdots & \vdots & \dots & \vdots \\
\mu_{A_1^2}(x_M) & \mu_{A_2^2}(x_M) & \dots & \mu_{A_N^2}(x_M) \\
\left[\mathbf{\Omega}^{\mathbf{1}}_{\mathbf{A}_1^2}(X) & \mathbf{\Omega}^{\mathbf{1}}_{\mathbf{A}_2^2}(X) & \dots & \mathbf{\Omega}^{\mathbf{1}}_{\mathbf{A}_N^2}(X)\right] \\
\vdots \\
\mu_{A_1^L}(x_1) & \mu_{A_2^L}(x_1) & \dots & \mu_{A_N^k}(x_1) \\
\mu_{A_1^L}(x_2) & \mu_{A_2^L}(x_2) & \dots & \mu_{A_N^L}(x_2) \\
\vdots & \vdots & \dots & \vdots \\
\mu_{A_1^L}(x_M) & \mu_{A^L}(x_M) & \dots & \mu_{A_N^L}(x_M) \\
\left[\mathbf{\Omega}^{\mathbf{1}}_{\mathbf{A}_1^L}(X) & \mathbf{\Omega}^{\mathbf{1}}_{\mathbf{A}^L}(X) & \dots & \mathbf{\Omega}^{\mathbf{1}}_{\mathbf{A}_N^L}(X)\right] \\
\vdots \\
\mu_{A_1^1}(x_1) & \mu_{A_2^1}(x_1) & \dots & \mu_{A_N^1}(x_1) \\
\mu_{A_1^1}(x_2) & \mu_{A_2^1}(x_2) & \dots & \mu_{A_N^1}(x_2) \\
\vdots & \vdots & \dots & \vdots \\
\mu_{A_1^1}(x_M) & \mu_{A_2^1}(x_M) & \dots & \mu_{A_N^1}(x_M) \\
\left[\mathbf{\Omega}^{\mathbf{2}}_{\mathbf{A}_1^1}(X) & \mathbf{\Omega}^{\mathbf{2}}_{\mathbf{A}_2^1}(X) & \dots & \mathbf{\Omega}^{\mathbf{2}}_{\mathbf{A}_N^1}(X)\right] \\
\mu_{A_1^2}(x_1) & \mu_{A_2^2}(x_1) & \dots & \mu_{A_N^2}(x_1) \\
\mu_{A_1^2}(x_2) & \mu_{A_2^2}(x_2) & \dots & \mu_{A_N^2}(x_2) \\
\vdots & \vdots & \dots & \vdots \\
\mu_{A_1^2}(x_M) & \mu_{A_2^2}(x_M) & \dots & \mu_{A_N^2}(x_M) \\
\left[\mathbf{\Omega}^{\mathbf{2}}_{\mathbf{A}_1^2}(X) & \mathbf{\Omega}^{\mathbf{2}}_{\mathbf{A}_2^2}(X) & \dots & \mathbf{\Omega}^{\mathbf{2}}_{\mathbf{A}_N^2}(X)\right] \\
\vdots \\
\mu_{A_1^L}(x_1) & \mu_{A_2^L}(x_1) & \dots & \mu_{A_N^k}(x_1) \\
\mu_{A_1^L}(x_2) & \mu_{A_2^L}(x_2) & \dots & \mu_{A_N^L}(x_2) \\
\vdots & \vdots & \dots & \vdots \\
\mu_{A_1^L}(x_M) & \mu_{A^L}(x_M) & \dots & \mu_{A_N^L}(x_M) \\
\left[\left[\mathbf{\Omega}^{\mathbf{2}}_{\mathbf{A}_1^L}(X) & \mathbf{\Omega}^{\mathbf{2}}_{\mathbf{A}^L}(X) & \dots & \mathbf{\Omega}^{\mathbf{2}}_{\mathbf{A}_N^L}(X)\right]\right]
\end{matrix} \tag{3.10}
$$

This PFSWHS-Matrix exhibits both internal and subjective external states of the universe. The internal state of the universe, event, or reality is reflected by individual fuzzy memberships $\mu_{A_j^k}(x_i)$ whilst the Subjectively exterior state of the universe, event, or reality is reflected through Subjectively aggregated memberships $\Omega_{A_j^k}(X)$ that is accumulated specifically for all given subjects at each attributive/sub-attributive level. Therefore the PFSWHSS-Matrix would provide an attributive classification (non-

physical classification) through a subject-wise accumulation of states. The subjective aggregation is applied to fuzzy memberships $\mu_{A_j^k}(x_i)$ at the index i, i.e at each specific sub-attributive level by applying several suitable aggregation operators. In the next section-4 for the construction of this PFSWHSS-Matrix, we have formulated some aggregation operators. The application of these operators and SWHSS-Matrix as LGU Combined-Consciousness State Ranking Model is presented in Section-5, whereas the application of this Whole Model is described in Sec-6, where the faculty ranking Model is represented.

4 Local aggregation operators for the Construction of SWHSS-Matrix

This section describes Local aggregation operators like disjunction operators, conjunction operators, Averaging operators, and Compliment-operator for PFHS-Matrix. By applying these local operators on the **PFHS-Matrix** the **SWHSS-Matrix** would be constructed. By utilizing Local disjunction, Local conjunction, and Local averaging operators, we would develop a combined (whole) memberships $\Omega^t_{A_j^k}(X)$ for **PFSWSS-Matrix** that would be presented in the last row-matrix of the. SWHSS-Matrix

The general mathematical expression for SWHSS-Matrix $\boldsymbol{F}$ in the plithogenic fuzzy environment is given below.

$\boldsymbol{F} = \begin{bmatrix} \left[\boldsymbol{\mu}_{A_j^k}(\boldsymbol{x_i})\right] \\ \left[\boldsymbol{\Omega}^{\boldsymbol{t}}_{\mathbf{A}_j^k}(\boldsymbol{X})\right] \end{bmatrix}$ In this Matrix the last row of cumulative memberships $\boldsymbol{\Omega}^{\boldsymbol{t}}_{\mathbf{A}_j^k}(\boldsymbol{X})$ is framed by using three local operators, $t = 1$ is used for the Max-operator $t = 2$ for Min-operator, and $t = 3$ for the $averaging$-operator. Furthermore, $t = 4$ is representing Compliment-operator**.**

In SHWHS-Matrix

$\boldsymbol{F}_{\boldsymbol{S_t}} = \left[\left[\boldsymbol{\mu}_{A_j^k}(\boldsymbol{x_i})\right] \quad \left[\boldsymbol{\Omega}^{\boldsymbol{t}}_{\mathbf{A}^k}(\boldsymbol{x_i})\right]\right]$ the last column of cumulative memberships $\Omega_{A^k}(x_i)$ are obtained by using three local operators, $t = 1$ used for the Max-operator $t = 2$ is used to portray the Min-operator, and $t = 3$ is used for the $averaging$-operator. Furthermore, $t = 4$ $represents\ the\ compliment$.

These four operators are described as follows:

4.1 Local-Disjunction-Operator for the construction of SWHSS-Matrix:

$$\cup_i \left(\boldsymbol{\mu}_{A_j^k}(\boldsymbol{x_i})\right) = \underset{i}{Max}\left(\mu_{A_j^k}(x_i)\right) = \Omega^1_{A_j^k}(X)\,, \text{ for some } k = l \tag{4.1}$$

This Max-operator reflects the optimal state of mind of the decision-maker.

4.2 Local-Conjunction Operator for construction of SWHSS-Matrix:

$$\cap_i \left(\mu_{A_j^k}(x_i)\right) = \underset{i}{Min}\left(\mu_{A_j^k}(x_i)\right) = \Omega^2_{A_j^k}(X)\,, \text{for some } k = l \tag{4.2}$$

This Min-operator reflects the pessimistic state of mind of the decision-maker.

4.3 Local-Averaging-Operator for construction of SWHSS-Matrix:

$$\Gamma_i \left(\mu_{A_j^k}(x_i)\right) = \frac{\sum_i \left(\mu_{A_j^k}(x_i)\right)}{M} = \Omega^3_{A_j^k}(X)\,, \text{ for some } k = l \tag{4.3}$$

This averaging operator reflects the neutral state of mind of the decision-maker.

4.4 Local Compliment for the construction of SWHSS-Matrix:

$$C_{loc}(F) = \begin{cases} Max_i\left(1-\mu_{A_j^k}(x_i)\right) \\ Min_i\left(1-\mu_{A_j^k}(x_i)\right) \\ \left\lfloor \sum_{i=1}^{M} \frac{\left(1-\mu_{A_j^k}(x_i)\right)}{M} \right\rfloor \end{cases}, \quad \text{for some } k = l \tag{4.4}$$

5. Algorithm of LGU Combined-Consciousness State Ranking Model

This section, utilizes the local operators built in the previous section for the formulation of the LGU Combined-Consciousness State Ranking Model in the Fuzzy environment.

In this model, we would provide the classification of attributes (a nonphysical phenomenon) at the local, Global, and Universal levels. We have called this Model the LGU Combined-Consciousness State Ranking Model. Some specialties of this LGU Combined-Consciousness State Ranking Model are mentioned to describe why this model would be preferred over previously developed MADM models

1. The first and most important feature of this model is that it provides a ranking of the non-physical states of the universe. As we know, the classification of non-physical phenomena has not yet been addressed in the area of decision-making. This model will open a new dimension of classification of the non-physical part of the universe / event / reality / information, in which one can choose a possible reality from several parallel realities that would be useful in the field of artificial intelligence.
2. The second peculiarity of this model is that it offers the classification of attributes by looking at them from multiple angles of visions. For example, the choice of the $max\ operator$ is an expression of an optimistic perspective. In contrast to this, the choice of the Min-operator is an expression of the pessimistic point of view and the choice of the $average$-operator is an expression of a neutral point of view. The combination of all operators in one model offers a transparent decision that is made from multiple perspectives
3. This model has the potential to offer a classification of attributes in numerous environments such as Fuzzy, Intuitionistic, Neutrosophic, or any other suitable environment required. Each environment has its own ambiguity or hesitation level. By choosing a particular environment, this model would be expanded to work on any level of uncertainty, hesitation, or ambiguity.
4. This attributive/state ranking model offers the ranking from micro-universe to macro-universe stages i.e. from inner smaller cell to outer larger universe.
5. Primarily, this Model delivers the internal ranking of attributes (states of subjects) named "Local Attributive ranking" (ranking of states) (classification of attributes/states of micro-universe)
6. On the next stage, this Model offers an exterior classification of states named "Global Attributive Ranking."
7. On a further extended level this Model offers the 3rd type of attributive ranking named "Universal Combined-Consciousness State Ranking (Classification of attributes of the macro-universe)
8. This model also offers extreme values, as extreme behaviors, and neutral values, as neutral behavior of universes that would be helpful to find the optimal and neutral states of all kinds of universes/realities/events/information from their micro- to macro levels.
9. At the final level, it provides a precise measure of the authenticity of classification by using the frequency matrix.

Initially, we consider the case of the PFSWHSS-Matrix to rank the given attributes or states of subjects. These subjects with their all attributes/sub attributes are considered to be one universe.

Later, we can generalize this Model into Plithogenic Intuitionistic, Plithogenic Neutrosophic, and other multiple useful required environments agreeing the state of mind of the decision-makers.

The Algorithm of the LGU Combined-Consciousness State Ranking Model is described below,

Step 1. Construction of Universe: Consider the fuzzy universe of discourse $U_F = \{x_i\}\ i = 1,2,3,\ldots, M$. Consider some attributes/sub-attributes and subjects need to be classified where attributes/sub-attributes are $A_j^k\ j = 1,2,3,\ldots,N$ and $k = 1,2,\ldots,L$ represents numeric values of attributes A_j (parallel level layers), and concerned subjects are $T = \{x_i\} \subset U_F$ where i can take some values from 1 to M such that Define mappings F and G such that,

$$F: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \to P(U) \text{ For some fixed } k \text{ (leve-1)} \tag{5.1}$$

$$G: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \to P(U), \text{ For some different fixed } k \text{ (level-2)} \tag{5.2}$$

Step 2. Construction of PFHS-Matrix: Write the data or information (fuzzy-memberships) of PFHS-Set in the form of PFHS-Matrix $B = \left[\mu_{A_j^k}(x_i)\right]$. If there are some non-favorable attributes in the given Information, we may replace their memberships $(\mu_{A_j^k}(x_i))$ by non-membership $(1 - \mu_{A_j^k}(x_i))$ while the neutral and favorable attributes would be displayed by their fuzzy memberships.

Step 3. Construction of PFSWHSS-Matrix: By using local aggregation operators constructed in Sec. -4 formulate PFSWHSS-Matrix given as,

$$B_{\mathbf{A}_t} = \begin{bmatrix} \left[\mu_{A_j^k}(x_i)\right] \\ \left[\Omega^t_{\mathbf{A}_j^k}(X)\right] \end{bmatrix}. \tag{5.3}$$

Step 4. Local Attributive Ranking: The Local Attributive Ranking **is** the ranking of the accumulated states of matter bodies (subjects) that would be acquired by considering cumulative memberships $\Omega^t_{\mathbf{A}_j^k}(X)$ of the last row of each layer of $B_{\mathbf{A}_t}$.

The higher the membership value, the better the attribute / sub-attribute that corresponds to this membership. At this stage, the attributive classification of all layers or a selected layer would be provided according to the required situation. In addition, the process would eventually stop when the transparent local attributive ranking is obtained. If there are some ties or ambiguities in the local attributive ranking that would be eliminated in the next step of the global ranking, a more transparent ranking would be observed.

Step 5. Global Attributive Ranking: Final global attribute ordering would be provided by using the Frequency Matrix, "F_{ij}" and the percentage frequencies Matrix f_{ij}^* by combining the states of mind of the decision-makers.

$$F_{ij} = \begin{matrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_N \end{matrix} \begin{bmatrix} f_{11} & f_{12} & \cdot & \cdot & \cdot & f_{1N} \\ f_{21} & f_{22} & \cdot & \cdot & \cdot & f_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{M1} & f_{M2} & \cdot & \cdot & \cdot & f_{NN} \end{bmatrix} \tag{5.4a}$$

$$F_{ij}^* = \begin{matrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_N \end{matrix} \begin{bmatrix} f_{11}^* & f_{12}^* & \cdot & \cdot & \cdot & f_{1N}^* \\ f_{21}^* & f_{22}^* & \cdot & \cdot & \cdot & f_{2N}^* \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{N1}^* & f_{N2}^* & \cdot & \cdot & \cdot & f_{NN}^* \end{bmatrix} \tag{5.4b}$$

Where, f_{ij}^{*} is the percentage frequency measure

$$f_{ij}^{*} = \frac{(f_{ij})}{\sum_{i}(f_{ij})} \times 100 \quad (5.4)c$$

In F_{ij} the values of the first column signify the frequency with which the 1st position is achieved, which is reached by some specific attributes. The elements of the column 2 represent the frequency of acquiring the second position and so on. Similarly the elements of F_{ij}^{*} represent the percentage frequencies. To find out which attribute would be assigned the first position we consider the entries in the first column of F_{ij}^{*} the attribute corresponds to the highest value of the first column attains the first position and then we delete this column of the first position and the row associated with this attribute. This reduces the dimension of the matrix. Then, for the second position, add the remaining percentage frequencies of the first position into the next percentage frequencies of the second column and then look for the highest percentage frequency in the second column for the decision of the second position.

Once the second position is determined, we delete the corresponding column and row of that position and continue the practice until the final position is allocated.

This Percentage Frequency Matrix has a great potential to handle ties.

Step 6. **Authenticity measurement of the Global ranking:** In the last step, we can check the authenticity by means of ratios.

Percentage authenticity measure of j $_{th}$ selected positions for i_{th} Attribute,

$$\frac{\text{Highest frequency of } j_{th} \text{ position}}{\text{Total frequency of } j_{th} \text{position}} \times 100$$

$$f_{ij}^{\vee} = \frac{\max_{i}(f_{ij}^{*})}{\sum_{i}(f_{ij})} \times 100 \quad (5.5)$$

Step 7. **Final Universal States** (**Combined Consciousness States) and Ranking:**

The final universal states (Combined Consciousness states) of Universes as final accumulated fuzzy memberships $\boldsymbol{\Omega}_{kt}$ are provided by using the disjunction operator, ($t = 1$) the conjunction operator, ($t = 2$), and the average operator ($t = 3$) on already cumulative memberships of the last row of SWHSS-Matrix $B_{\mathbf{A}_t}$, These accumulated fuzzy memberships $\boldsymbol{\Omega}_{kt}$ represent the final Universal State or the Combined Consciousness State of the universe.

For a fixed k and t the universe with the greatest cumulative membership would be considered the better universe, and further order of the universes would be observed by arranging the $\boldsymbol{\Omega}_{kt}$ in descending order. To get the final ranking of the universal states and to obtain extreme and neutral accumulated states of the Universe/Reality/Event/Information, we would proceed as

Taking $t = 1,2,3$ respectively on $\boldsymbol{\Omega}_{kt}$ we would obtain the following extreme and neutral values.

$$\boldsymbol{\Omega}_{k1} = \max_{j} \Omega^{1}_{\mathbf{A}_j^k}(X) \quad (5.6)$$

$$\boldsymbol{\Omega}_{k2} = \min_{j} \left[\Omega^{2}_{\mathbf{A}_j^k}(X)\right] \quad (5.7)$$

$$\boldsymbol{\Omega}_{k3} = \frac{\sum_{j}\left[\Omega^{3}_{\mathbf{A}_j^k}(X)\right]}{N} \quad (5.8)$$

At this level $\boldsymbol{\Omega}_{k1}$ and $\boldsymbol{\Omega}_{k2}$ would give the extreme (lowest and highest) states and $\boldsymbol{\Omega}_{k3}$ would give the neutral states of Universe/Reality/Event/Information as accumulated fuzzy memberships.

The local order of the universes is obtained by arranging these cumulative memberships in descending order, and the global order is offered by using the same scenario of the frequency matrix (step-5).

5. Application of LGU Combined-Consciousness State Ranking Model

Numerical Example:

To achieve the purpose of non-physical classification, initially, we first develop two PFHS-Sets with α-Combination and β-Combination of attributes, i.e., for α and β universes. Then we represent it as PFHS-Matrix B, which consists of two layers that represent the mappings F and G that are used to parameterize a combination of attributes/subattributes. By assuming different or specific numerical values of k, consider α-Combination of attributes are parameterized by mapping F and β-Combination of attributes by mapping G. The overall LGU Combined-Consciousness State Ranking is described by following the steps in the algorithm described in Section -5.

Step 1. Construction of the Universe: Consider U be the set in five candidates of the mathematics department and out of these five only three have participated in consciousness quantification and classification experiment. let T be a set of these three candidates (subjects), T = {Peter, Aina, kitty}, ($T \subset U$). The elements of T are our subjects. The states of these subjects are A_j^k attributes quantified through the fuzzy linguistic scales. The classification of these attributes is required.

These A_j^kattributes are organized in the following manner:

A_1^k = Intelligence level with numeric values, $k = 1,2$ s.t

A_1^1 = very intelligent, A_1^2 = moderate intelligent

A_2^k = Fous, with numeric values, $k = 1,2$ s.t

A_2^1 = Strong focus A_2^2 = Weak focus

A_3^k = Observation with numeric values, $k = 1,2$ s.t

A_3^1 = Strong observation , A_3^2 = weak observation

A_4^k = Expression with numeric values $k = 1,2\ st$

A_4^1 = Strong expression, A_4^2 = Weak expression

F and G be the plithogenic fuzzy parameterizations of the combination of their states (attributes) such that

$F: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \to P(U)$ (choosing some of the numeric values of A_j^k, $k = 1,2, \ldots, L$

$G: A_1^k \times A_2^k \times A_3^k \times \ldots \times A_N^k \to P(U)$ (choosing some other numeric values of A_j^k, $k = 1,2, \ldots, L$

Let these candidates of set T are our $x_i$subjects, $i = 1,2,3$, and their states are attributed/sub-attribute represented A_j^k $j = 1,2,3,4$ and $k = 1,2$. We are looking for the best-reflected attribute among the given Combination of attributes (case of the local universe). The local universe of subjects and attributes for first level $k = 1$ is described as

$T = \{\text{Peter, Aina, kitty}\} = \{x_1, x_2, x_3\}$ where x_1, x_2, x_3 represent x_i subjects under consideration, initially, we represent the combination of states of the first level for $k = 1$(combination of attributes that are parametrized by mapping F)

1. Intelligence: $j = 1$, $k = 1$ (very intelligent)
2. Focus: $j = 2$, $k = 1$ (strong focus)
3. Observation: $j = 3$, $k = 1$ (strong observation)
4. Expression: $j = 4$, $k = 1$ (strong expression)

Now fuzzy memberships (fuzzy parameterization) are assigned by using fuzzy linguistic scales for details see ref. [33-36].

Let the Function F represents the fuzzy parameterization of the given combination of states/attributes s.t.,

$$F(A_1^1, A_2^1, A_3^1, A_4^1) = \{x_1(0.3,0.7,0.4,0.5), x_2(0.4,0.5,0.4,0.1), x_3(0.6,0.2,0.5,0.7)\} \quad (6.1)$$

let us name the combination of attributes $A_1^1, A_2^1, A_3^1, A_4^1$ as α Combination representing the first level for $k = 1$

Consider some other combination of states described for $k = 2$ These states are parametrized by mapping G s.t $G: A_1^k \times A_2^k \times A_3^k \times A_4^k \to P(U)$

The local universe of subjects and attributes for second-level $k = 2$ is described below

1. Intelligence $j = 1$, $k = 2$ (moderate intelligent)
2. Focus: $j = 2$, $k = 2$ (weak focus)
3. Observation: $j = 3$, $k = 2$ (weak observation)
4. Expression: $j = 3$, $k = 2$ (weak expression)

Let the function be G represent the fuzzy parametrization of the given combination of states/attributes s.t,

$$G(A_1^2, A_2^2, A_3^2, A_4^2) = \{x_1(0.5,0.0,0.2,0.6), x_2(0.6,0.7,0.8,0.5), x_3(0.4,0.7,0.5,0.9)\} \tag{6.2}$$

let us name the combination of attributes $A_1^1, A_2^1, A_3^1, A_4^1$ as β Combination representing the second level for $k = 2$

Step 2. Construction of PFHS-Matrix:

The first layer of PFHS-Matrix $B = \left[\mu_{A_j^k}(x_i)\right]$ is constructed by using the parametrized states given in Eq. 6.1 for α combination (first level layer of PFHS-Matrix, k =1) and The second layer of PFHS-Matrix is constructed by using the parametrized states given in Eq. 6.2 for β combination (second level layer of PFHS-Matrix, k =2) and this information would be displayed in PFHS-Matrix as shown below.

$$B = \begin{matrix} \begin{bmatrix} 0.3 & 0.7 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.5 & 0.7 \end{bmatrix} \\ \left\lfloor \begin{bmatrix} 0.5 & 0.0 & 0.2 & 0.6 \\ 0.6 & 0.7 & 0.8 & 0.5 \\ 0.4 & 0.7 & 0.5 & 0.9 \end{bmatrix} \right\rfloor \end{matrix} \tag{6.3}$$

Step 3. Construction of PFSWHSS-Matrix:

The PFSWHSS-Matrix $B_{\mathbf{A}_t}$ is constructed by using Eqs. (3.10), (4.1), (4.2), and (4.3) for information of (6.3)

$$B_{\mathbf{A}_t} = \begin{matrix} \begin{bmatrix} 0.3 & 0.7 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.5 & 0.7 \\ [\mathbf{0.6} & \mathbf{0.7} & \mathbf{0.5} & \mathbf{0.7}] \end{bmatrix} \\ \left\lfloor \begin{bmatrix} 0.5 & 0.0 & 0.2 & 0.6 \\ 0.6 & 0.7 & 0.8 & 0.5 \\ 0.4 & 0.7 & 0.5 & 0.9 \\ [\mathbf{0.6} & \mathbf{0.7} & \mathbf{0.8} & \mathbf{0.9}] \end{bmatrix} \right\rfloor \\ \begin{bmatrix} 0.3 & 0.7 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.5 & 0.7 \\ [\mathbf{0.3} & \mathbf{0.2} & \mathbf{0.4} & \mathbf{0.1}] \end{bmatrix} \\ \left\lfloor \begin{bmatrix} 0.5 & 0.0 & 0.2 & 0.6 \\ 0.6 & 0.7 & 0.8 & 0.5 \\ 0.4 & 0.7 & 0.5 & 0.9 \\ [\mathbf{0.4} & \mathbf{0.0} & \mathbf{0.2} & \mathbf{0.5}] \end{bmatrix} \right\rfloor \\ \begin{bmatrix} 0.3 & 0.7 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.4 & 0.1 \\ 0.6 & 0.2 & 0.5 & 0.7 \\ [\mathbf{0.43} & \mathbf{0.46} & \mathbf{0.43} & \mathbf{0.43}] \end{bmatrix} \\ \left\lfloor\left\lfloor \begin{bmatrix} 0.5 & 0.0 & 0.2 & 0.6 \\ 0.6 & 0.7 & 0.8 & 0.5 \\ 0.4 & 0.7 & 0.5 & 0.9 \\ [\mathbf{0.5} & \mathbf{0.46} & \mathbf{0.5} & \mathbf{0.66}] \end{bmatrix} \right\rfloor\right\rfloor \end{matrix} \tag{6.4}$$

Step 4. Local Attributive/States Ranking: $B_{\mathbf{A}_{1\alpha}}$ provides The local order of states/attributes for α Combination of attributes or α-universe i.e the first level-layer is obtained by observing the whole

memberships of (6.4) for first-level $k = 1$ and first aggregation operator ($t = 1$). See Eq.4.1 We observe here a tie between A_2^1 ($\Omega^1_{\mathbf{A}_2^1}(X) = 0.7$) and A_4^1 ($\Omega^1_{\mathbf{A}_4^1}(X) = 0.7$) which would be removed in the next step of the Global States ranking using the Frequency-Matrix F_{ij}.

$$A_2^1 = A_4^1 \succ A_1^1 \succ A_3^1 \tag{6.5}$$

$B_{\mathbf{A}_{1\beta}}$ provides The local ordering of attributes for β Combination of attributes or β-Universe (second level-layer obtained for $k = 2$) See Eq. 6.4 by using the first operator $t = 1$ (eq. 4.1)

$$A_4^2 \succ A_3^2 \succ A_2^2 \succ A_1^2 \tag{6.6}$$

$B_{\mathbf{A}_{2\alpha}}$ provides the local ordering of attributes for α-Combination of attributes (α-Universe) by using the second operator $t = 2$ Eqs 6.4 and (4.2)

$$A_3^1 \succ A_1^1 \succ A_2^1 \succ A_4^1 \tag{6.7}$$

Similarly

$B_{\mathbf{A}_{2\beta}}$ provides the local ordering of attributes for β Combination of attributes (β-Universe) by using the second operator $t = 2$ Eqs 6.4 and (4.2)

$$A_4^2 \succ A_1^2 \succ A_3^2 \succ A_2^2 \tag{6.8}$$

$B_{\mathbf{A}_{3\alpha}}$ provides the local ordering of attributes for α Combination of attributes (α-Universe) by using the third operator $t = 3$ Eqs 6.4 and (4.3)

$$A_2^1 \succ A_1^1 = A_3^1 = A_4^1 \tag{6.9}$$

$B_{\mathbf{A}_{3\beta}}$ provides the local ordering of attributes for β Combination of attributes (β-Universe) by using the third operator ($t = 3$) Eqs 6.4 and (4.3)

$$A_4^2 \succ A_1^2 = A_3^2 \succ A_2^2 \tag{6.10}$$

Step 5. Global States/Attributive Ranking:

The frequency matrix F_{ij} provides a final global ordering of attributes. In the frequency matrix F_{ij}^{α}, which is a square matrix of frequencies of positions for first level-layer α-Universe, the columns of F_{ij}^{α} represents frequencies of positions, i.e., the entries of the first column represent the frequencies of attaining the first position by some attributes while a row of F_{ij} represents the attributes. The F_{ij}^{α} is constructed from Eq. (6.5), (6.7), (6.9), and (5.4)a, (5.4)b, (5.4)c

$$F_{ij}^{\alpha} = \begin{array}{c} \alpha \\ A_1^1 \\ A_2^1 \\ A_3^1 \\ A_4^1 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array} \tag{6.11}$$

$$F_{ij}^{*\alpha} = \begin{array}{c} \alpha \\ A_1^1 \\ A_2^1 \\ A_3^1 \\ A_4^1 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \begin{bmatrix} 0 & 100 & 0 & 0 \\ 66.7 & 0 & 33.3 & 0 \\ 33.3 & 33.3 & 33.3 & 0 \\ 33.3 & 33.3 & 0 & 33.3 \end{bmatrix} \end{array} \tag{6.11)a}$$

The Global States ranking of attributes obtained from $F_{ij}^{*\alpha}$ is given below.

$$A_2^1 \succ A_1^1 \succ A_3^1 \succ A_4^1 \tag{6.12}$$

The F_{ij}^{β} is constructed from Eq. (6.6), (6.8), (6.10), and (5.4)a, (5.4)b, (5.4)c

$$F_{ij}^{\beta} = \begin{matrix} \beta \\ A_1^2 \\ A_2^2 \\ A_3^2 \\ A_4^2 \end{matrix} \begin{matrix} p_1 & p_2 & p_3 & p_3 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & 0 & 0 \end{matrix} \tag{6.13}$$

$$F_{ij}^{*\beta} = \begin{matrix} \beta \\ A_1^2 \\ A_2^2 \\ A_3^2 \\ A_4^2 \end{matrix} \begin{matrix} p_1 & p_2 & p_3 & p_3 \\ 0 & 66.7 & 0.0 & 33.3 \\ 0 & 0.0 & 33.3 & 66.7 \\ 0 & 33.3 & 66.7 & 0 \\ 100 & 0 & 0 & 0 \end{matrix} \tag{6.13}$$

The Global States ranking of attributes obtained from $F_{ij}^{*\beta}$ is given below.

$$A_4^2 \succ A_1^2 \succ A_3^2 \succ A_2^2 \tag{6.14}$$

It is observed that the ties of local ranking are removed in the final global ranking

Step 6. Authenticity measurement of the Global States Ranking:

Percentage authenticity measure for first level α-universe is obtained by using Eq. (5.5) and (6.11)a

Percentage authenticity of the first position for $A_2^1 = 66.7\%$

Percentage authenticity of the second position for $A_1^1 = 60\%$

Percentage authenticity of the third position for $A_3^1 = 50\%$

Percentage authenticity of the fourth position for $A_4^1 = 100\%$

Percentage authenticity measure for first level β-universe is obtained by using (5.5) and (6.13)a

Percentage authenticity of the first position for $A_4^2 = 100\%$

Percentage authenticity of the second position for $A_1^2 = 66.7\%$

Percentage authenticity of the third position for $A_3^2 = 66.67\%$

Percentage authenticity of the fourth position for $A_2^2 = 66.7\%$

Step 7. Final Universal States (Combined Consciousness States) and Ranking:

we provide the final ordering of the universe by using all three aggregation operators.

Maximum Combined Consciousness States (Universal Memberships) of α and β universes:

taking $k = 1,2$ for α and β universes and fixing $t = 1$ (Max-operator) using Eqs. (6.4) and (5.6)

$$\mathbf{\Omega}_{11} = 0.7,\ \mathbf{\Omega}_{21} = 0.9 \tag{6.15}$$

We can see by using operator $t = 1$, β universe is better than α universe.

Minimum Combined consciousness States (Universal Memberships) of α and β universes:

Taking $k = 1,2$ for α and β universes and fixing $t = 2$ minimum universal memberships of all given Attributes with respect to subjects, are obtained using Eqs. (6.4) and (5.7) respectively.

$$\mathbf{\Omega}_{12} = 0.1,\ \mathbf{\Omega}_{22} = 0.0 \tag{6.16}$$

We observe by using the operator $t = 2$, β universe is better than α universe.

Neutral Combined Consciousness States (Universal Memberships) of α and β universes:

similarly, taking $k = 1,2$ for α and β universes and fixing $t = 3$, we can provide average universal memberships of all given subjects with respect to attributes, using Eqs. (6.4) and (5.7)

$$\mathbf{\Omega}_{13} = 0.437,\ \mathbf{\Omega}_{23} = 0.53 \tag{6.17}$$

The Universal States ordering: By applying the frequency matrix analysis (Eqs. 6.15, 6.16, 6.17, and (5.4)a, (5.4)b, (5.4)c The ranking of the states of the universes is

$$\beta(\text{universe}) \succ \alpha(\text{universe}) \tag{6.18}$$

7. Pie graphs of the LGU Combined-Consciousness State Ranking Model

7.1 Pie graphs of the LGU Combined-Consciousness State Ranking Model for the α-Universe

The pie graphs (Fig1-Fig 4) present the individual states (fuzzy memberships) of 3 subjects considering one attribute at a time for the α-Universe (for aggregation purposes, we use the averaging operator (t = 3)

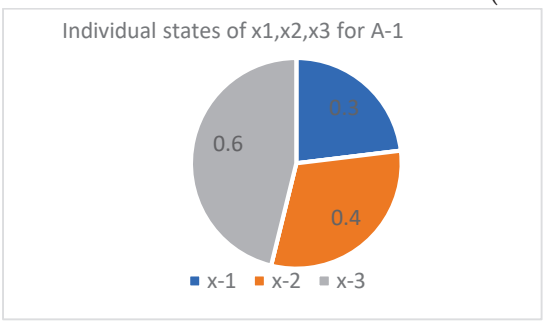


Figure 1a (Individual states of A-1)

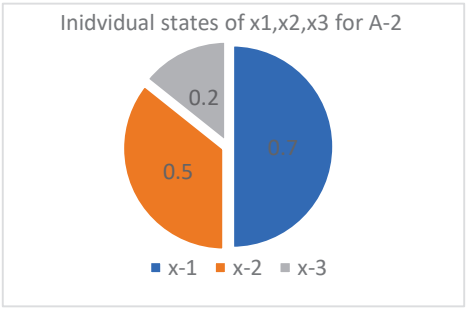


Figure 2a (Individual states of A-2)

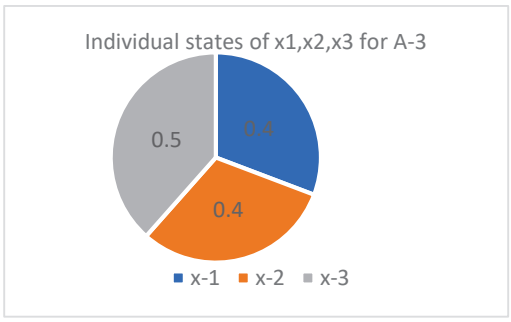


Figure 3a (Individual states of A-3)

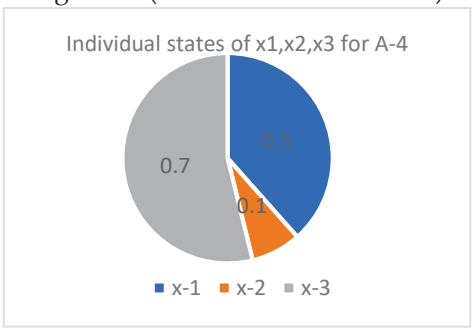


Figure 4a (Individual states of A-4)

Fig. 5 represents the aggregated states of the three subjects (α-Universe first level of aggregation) represented for each attribute.

Fig 6 is representing the aggregated state of the whole universe that is obtained by aggregating the previous aggregated states of fig 5 by using the averaging operator (α-Universe second level of aggregation)

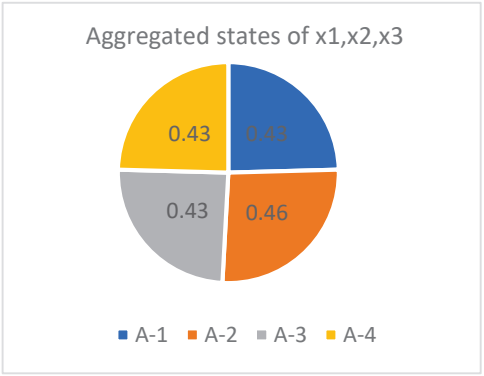


Figure 5a (Aggregated states)

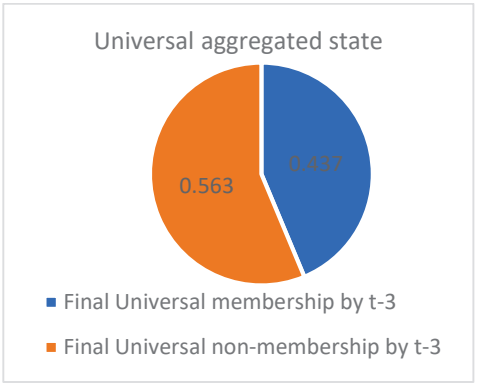


Figure 6a (Universal states)

7.2 Pie graphs of the LGU Combined-Consciousness State Ranking Model for the β-Universe

(Fig1b-Fig 4b) pie graphs are presenting the individual states (fuzzy memberships) of 3 subjects by considering one attribute at a time for the β-Universe (The aggregation operator used is the averaging operator (t = 3)

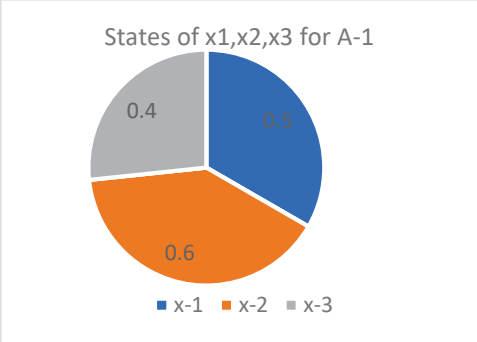


Figure 1b (Aggregated States for A-1)

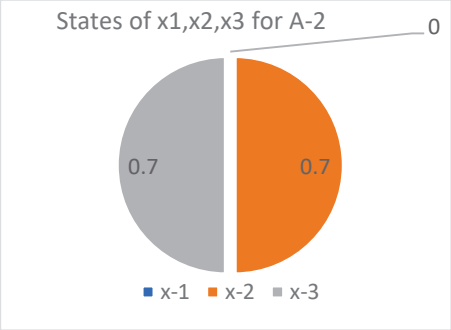


Fig 2b (Aggregated States for A-2)

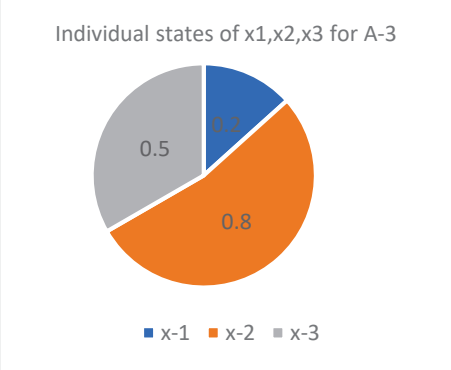


Figure 3b (Aggregated States for A3)

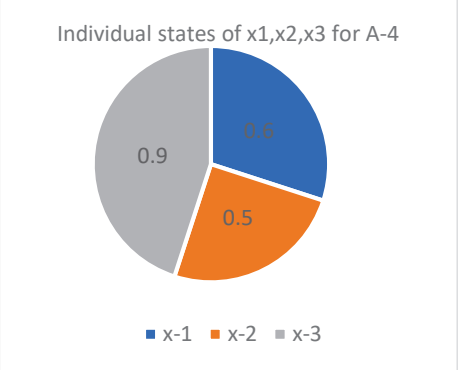


Figure 4b(Aggregated States for A-4)

Fig 5b is representing aggregated states of the three subjects (β-Universe first level of aggregation) represented for each attribute.

Fig. 6b represents the aggregated state of the entire -Universe that is obtained by aggregating the previous aggregated states of Fig. 5b by using the averaging operator (β-Universe, the second level of aggregation)

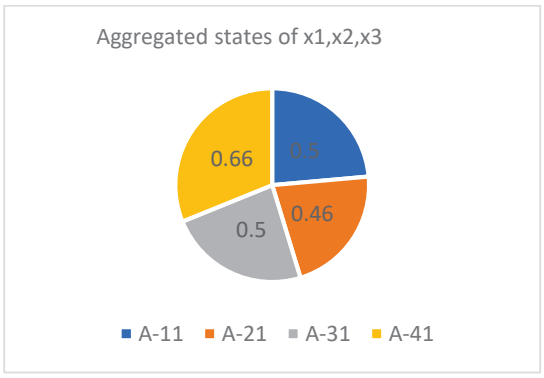


Figure 5b (Aggregated states)

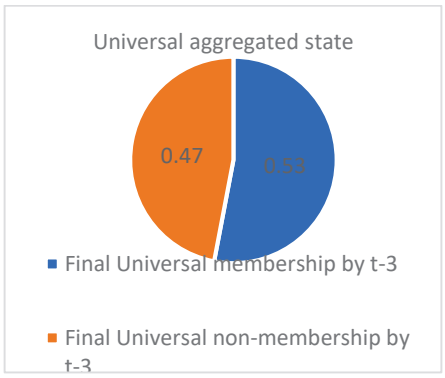


Figure 6b (Aggregated states)

8. Conclusion :

1. We have observed the final global ordering obtained in Eq. (6.12) is the most frequently observed local ordering in all these ranking orders, which is also observed the same in the local ordering of β universe in Eq. (6.14) which shows the final global State ranking is most transparent and authentic Ranking.

2. Expressions (6.15), (6.16), (6.17) provide the highest, lowest, and average states of universes, through final accumulative memberships.

3. The Ordering of universes shows that on the Global Universal level, β universe is better than α universe.

4. these results of local and global ordering are also verified by the pie graphs

1. Local ordering: we can observe local orders using these novel plithogenic hyper-supersoft matrices and local operators. Each operator reflects the state of mind of the decision-maker; for example, the Max operator reflects the optimal state of mind, the Min operator reflects the Passimistic state of mind, and the Average operator reflects the neutral state of mind.

2. Global ordering: We can provide a global order by combining the results of all three rankings using the frequency matrix. These three rank orders are obtained from three aggregation operators that represent three states of the human mind. The ranking at the levels of global states will be transparent and impartial, taking into account three different states of the human mind

3. Universal ordering: We can compare the universes by applying $the\ max\ operator\ (t = 1), the\ min\ operator\ (t = 2)$ and $the\ average\ operator\ (t = 3)$ on cumulative memberships of the last row for each universe. The universe with the largest cumulative membership would be better, and further, a local ordering of the universes is obtained by arranging these cumulative memberships in descending order and the global ordering is offered by using the same scenario of the frequency matrix

4. Extreme Universal Memberships: We can also find out the extreme values of these universes and can observe these attributes in the large universe made up of several smaller parallel universes. We can choose from among all universes the best-reflected attribute that is best in most universes.

5. local and global ordering inside the universe: In this article, our focus is on the non-physical states of the subjects or universe. Local and global ordering We have offered a local and global ordering of states of subjects (Attributes, Sub-attributes) within a universe.

6. local and global ordering of the Universe: Furthermore, a local and global ordering of states of the Universe is offered. The state of the universe is obtained by accumulating the states of all subjects of the given universe.

7. Combined Consciousness of the Universe: The state of the universe is presented by the accumulated states of all its subjects. In this ranking model, the accumulated states of all subjects as a Combined Consciousness of the universe is offered in the form of universal memberships.

9. Open problems:

Now, let us list some of the open problems that could be addressed in future research.

• In this article, we developed the LGU Combined Consciousness State Ranking Model in the plithogenic fuzzy environment.

This model can be extended to other environments, such as intuitionistic environment, neutrosophic environment, or any other mixed environment according to the required conditions or states of mind of the decision-makers.

• In addition, some other local operators can be used in the construction of the model according to the requirements of the relevant authorities.

• Attributive and subjective ranking models can be constructed using the literature developed in this article.

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Noninvasive Hemoglobin Estimation with Adaptive Lightweight Convolutional Neural Network Using Wearable PPG

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ABSTRACT: Hemoglobin is a vital protein in red blood cells responsible for transporting oxygen throughout the body. Its accurate measurement is crucial for diagnosing and managing conditions such as anemia and diabetes, where abnormal hemoglobin levels can indicate significant health issues. Traditional methods for hemoglobin measurement are invasive, causing pain, risk of infection, and are less convenient for frequent monitoring. PPG is a transformative technology in wearable healthcare for noninvasive monitoring and widely explored for blood pressure, sleep, blood glucose, and stress analysis. In this work, we propose a hemoglobin estimation method using an adaptive lightweight convolutional neural network (HMALCNN) from PPG. The HMALCNN is designed to capture both fine-grained local waveform characteristics and global contextual patterns, ensuring robust performance across acquisition settings. We validated our approach on two multi-regional datasets containing 152 and 68 subjects, respectively, employing a subject-independent 5-fold cross-validation strategy. The proposed method achieved root mean square errors (RMSE) of 0.90 and 1.20 g/dL for the two datasets, with strong Pearson correlations of 0.82 and 0.72. We conducted extensive post-hoc analyses to assess clinical utility and interpretability. A ±1 g/dL clinical error tolerance evaluation revealed that 91.3% and 86.7% of predictions for the two datasets fell within the acceptable clinical range. Hemoglobin range-wise analysis demonstrated consistently high accuracy in the normal and low hemoglobin categories. Statistical significance testing using the Wilcoxon signed-rank test confirmed the stability of performance across validation folds ($p > 0.05$ for both RMSE and correlation). Furthermore, model interpretability was enhanced using Gradient-weighted Class Activation Mapping (Grad-CAM), supporting the model's clinical trustworthiness. The proposed HMALCNN offers a computationally efficient, clinically interpretable, and generalizable framework for noninvasive hemoglobin monitoring, with strong potential for integration into wearable healthcare systems as a practical alternative to invasive measurement techniques.

KEYWORDS: Hemoglobin estimation; photoplethysmography (PPG); convolutional neural network (CNN); noninvasive method; wearable healthcare

1 Introduction

Hemoglobin is a crucial protein found in red blood cells, responsible for transporting oxygen from the lungs to various tissues throughout the body. It comprises iron-rich heme groups, which bind to oxygen molecules for efficient oxygen delivery. The concentration of hemoglobin in the blood is a critical indicator

of a person's overall health. Abnormal levels of hemoglobin can lead to various health conditions, as the body's tissues may not receive an adequate supply of oxygen [1]. Maintaining normal hemoglobin levels is essential for the body's metabolic processes. Hemoglobin levels are closely linked to anemia and diabetes, two prevalent health conditions that significantly impact global populations. Anemia is characterized by a deficiency in the number or quality of red blood cells, leading to reduced oxygen-carrying capacity [2]. This condition often results in fatigue, weakness, and a range of other symptoms that can severely affect quality of life. The poor management of diabetes leads to various complications. Chronic kidney disease is a common complication of diabetes that can impair the production of erythropoietin, a hormone essential for red blood cell production, thereby leading to anemia [3]. Additionally, high blood glucose levels can damage blood vessels and reduce the overall efficiency of oxygen delivery, further exacerbating the problem. Beyond anemia and diabetes, abnormal hemoglobin levels are associated with other blood-related complications. For instance, polycythemia is a condition characterized by an excess of hemoglobin due to an overproduction of red blood cells [4], which can lead to an increased risk of thrombosis, stroke, and heart attack as shown in Fig. 1. On the other hand, low hemoglobin levels can result in hypoxia, where the body's tissues are deprived of adequate oxygen, potentially leading to organ damage and other serious health issues [5]. Hemoglobin abnormalities can also indicate underlying conditions such as thalassemia or sickle cell disease, which are inherited blood disorders that affect the shape and function of red blood cells, leading to a range of severe health complications [6].

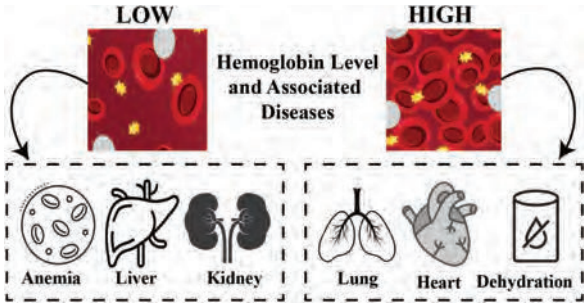


Figure 1: Illustration of hemoglobin levels along with associated diseases. Low hemoglobin levels are associated with anemia, liver, and kidney, while high hemoglobin levels are associated with dehydration, lung, and heart

Traditionally, hemoglobin levels have been measured using invasive methods, such as blood draws followed by laboratory analysis [7]. Common techniques include the cyanmethemoglobin method, where blood is mixed with a chemical that forms a stable compound with hemoglobin, and then its concentration is measured spectrophotometrically. Another method involves the use of automated hematology analyzers, which count red blood cells and determine hemoglobin concentration. While these methods are accurate, they are invasive, requiring needle punctures to obtain blood samples, which can cause discomfort and pose a risk of infection. Additionally, these methods are often time-consuming and require laboratory facilities, making them less suitable for frequent monitoring, particularly in low-resource settings or for individuals who require regular hemoglobin checks.

Photoplethysmography (PPG) has emerged as a promising noninvasive technology for hemoglobin measurement, offering a more convenient and less painful alternative to traditional methods [8]. PPG works by using a light source and a photodetector to measure the changes in light absorption in the blood vessels, which occur as a result of blood volume changes during the cardiac cycle [9]. Since hemoglobin has specific light absorption characteristics, PPG can be used to estimate hemoglobin concentration by analyzing the light absorption patterns. This technique has the potential to revolutionize hemoglobin monitoring,

especially for patients with chronic conditions such as anemia and diabetes, who require regular assessment of their hemoglobin levels. The emergence of deep learning in healthcare applications enables the automatic extraction of features and patterns from medical data, such as images, signals, and patient records, facilitating accurate diagnoses, predictions, and personalized treatments [10]. Deep learning uses multi-layered neural networks to model and analyze complex patterns within vast amounts of data [11]. It has revolutionized healthcare by enabling systems to autonomously learn from large datasets and make informed decisions. Deep learning has found numerous applications in biosignal analysis, including sleep monitoring, stress detection, and cardiovascular analysis [12]. The main novelties of this work are described as follows:

- We propose a novel end-to-end adaptive lightweight convolutional neural network (HMALCNN) that efficiently learns both fine-grained local waveform characteristics and global contextual patterns from multi-wavelength PPG signals for accurate hemoglobin estimation.
- We validate the proposed model on two multi-regional datasets with different spectral configurations (dual- and four-wavelength PPG) using a subject-independent five-fold cross-validation strategy, ensuring robust generalization across acquisition settings and populations.
- We conduct extensive post-hoc analyses, including clinical error tolerance evaluation (±1 g/dL threshold), hemoglobin range-wise performance assessment (low, normal, and high categories), and statistical stability testing via the Wilcoxon signed-rank test, confirming consistent and clinically acceptable performance.
- We enhance model transparency and physiological interpretability through Gradient-weighted Class Activation Mapping (Grad-CAM), which highlights clinically relevant waveform segments such as systolic peaks and dicrotic notches, demonstrating that the network attends to meaningful PPG features.
- Comparative experiments against state-of-the-art methods show that the proposed HMALCNN achieves lower root-mean-square error (RMSE) and higher Pearson correlation, offering a computationally efficient and clinically reliable alternative for noninvasive hemoglobin monitoring.

The rest of the paper is structured as follows: Section 2 reviews the literature, showcasing the current state-of-the-art in noninvasive hemoglobin measurement. Section 3 describes the methodology, providing a comprehensive explanation of our proposed approach. Section 4 describes the experimental setup, including details on the dataset. In Section 5, we present and discuss the results, evaluating the performance and implications of our findings. Finally, Section 6 concludes the paper, summarizing the key contributions and future research.

2 Literature Review

Several efforts have been made to develop noninvasive methods for hemoglobin measurement, achieving varying degrees of success. These approaches range from traditional machine learning techniques, which rely on manual feature extraction from PPG signals, to more advanced deep learning models. Table 1 shows World Health Organization (WHO) rules to classify the stage of Anemia based on hemoglobin level measurement.

Table 1: Classification of anemia stages based on hemoglobin (g/dL) levels

Subject	Age	Mild anemia	Moderate anemia	Severe anemia
Men	15 years or above	11 to 12.9	8 to 10.9	Less than 8
Women	15 years or above	11 to 11.9	8 to 10.9	Less than 8
Pregnant women	—	10 to 10.9	7.7 to 9.9	Less than 7

(Continued)

Table 1 (continued)

Subject	Age	Mild anemia	Moderate anemia	Severe anemia
Children (6–59 months)	6–59 months	10 to 10.9	7 to 9.9	Less than 7
Children (5–11 years)	5 to 11 years	11 to 11.4	8 to 10.9	Less than 8
Children (12–14 years)	12 to 14 years	11 to 11.9	8 to 10.9	Less than 8

Kavsaoglu et al. [13] proposed a noninvasive method to estimate hemoglobin levels using PPG signal characteristics and machine learning algorithms. They demonstrated that combining PPG signal analysis with machine learning techniques could provide reliable estimates of hemoglobin levels without the need for invasive blood sampling. Vladislav et al. [14] investigated noninvasive hemoglobin concentration measurements using multi-wavelength PPG in wrist-wearable devices. They applied multi-wavelength PPG in reflectance mode, typical of consumer wrist-wearable devices, to estimate total hemoglobin concentration, highlighting the potential of wearable technology in noninvasive hemoglobin estimation. Wang et al. [15] introduced a smartphone-based application called HemaApp, which utilized the Nexus-5p smartphone camera combined with multiple lighting sources, including infrared LEDs to predict hemoglobin concentration. They reported a correlation coefficient (R) ranging between 0.69 and 0.82, although the performance was limited for hemoglobin levels below 8 g/dL. Hasan et al. [16] developed SmartHeLP application for estimating hemoglobin levels using artificial neural networks (ANN) to analyze features extracted from fingertip videos. Their approach achieved a high correlation (R^2 = 0.93) between predicted and actual hemoglobin levels, further establishing the feasibility of noninvasive hemoglobin measurement. Zhu et al. [17] proposed a noninvasive hemoglobin detection device based on multi-wavelength PPG signals. The device uses an eight-LED transmissive PPG signal collection system, integrating multiple sensors in a 3D-printed finger clip. Data from 56 participants were used to create regression models, with the AdaBoost model achieving a mean absolute error of 2.67 g/L and a correlation coefficient of 0.91. Soumyadipta et al. [18] developed a machine-learning approach at four wavelengths PPG (590, 660, 810, and 940 nm) from 1583 women subjects. A novel feature vector was derived from these PPGs, and a two-layer stacked regressor model was designed, incorporating algorithms such as LASSO, Ridge, Elastic Net, AdaBoost, and Support Vector Regressors. The model achieved a Pearson's correlation coefficient of 0.81 and a root mean square error (RMSE) of 1.353 g/dL when compared to gold standard Hb measurements.

Edward et al. [19] improved upon the HemaApp by enhancing the hardware configuration to better amplify the weaker signals of the blue and green channels, which allowed them to estimate hemoglobin levels without the need for infrared LEDs. This iteration achieved a Pearson correlation of 0.62, highlighting the continued challenges in noninvasive hemoglobin measurement. Humera et al. [20] employed the ResNet-18 architecture to analyze PPG signals of 150 anemic patients for hemoglobin level measurement. This study underscores the potential of deep learning models in processing PPG data for noninvasive hemoglobin level assessment. Lakshmi et al. [21] focused on pregnant womens hemoglobin level measurement using machine learning techniques by analyzing PPG signals. Their study involved collecting data from 127 participants and extracting features such as the first and second derivatives of the PPG signal. They employed linear regression models to estimate hemoglobin levels. Silva et al. [22] employed a deep-learning approach to estimate hemoglobin levels from PPG signals obtained through smartphone-based recordings. They utilized a ResNet architecture, which is known for its capacity to handle complex patterns in image data, to analyze

the stacked frames from fingertip videos. Their model outperformed traditional machine learning methods, achieving a lower RMSE and demonstrating the robustness of deep learning in processing PPG signals. Pinto et al. [23] developed an embedded system for noninvasive hemoglobin measurement using PPG. Their device employs a finger probe with LEDs at wavelengths of 670, 770, 810, 850, and 950 nm, alongside a silicon photodetector. The system demonstrated improved accuracy in hemoglobin estimation, reducing the mean absolute error from 1.3778 to 0.9422 g/dL when utilizing five wavelengths. Chen et al. [24] introduced a real-time noninvasive hemoglobin prediction method using deep learning-enabled smartphone imaging. Their approach involves capturing eye images with a smartphone application and analyzing them using a deep neural network trained on invasive blood test data. The EGE (Efficient Group Enhanced)-Unet model was utilized for eyelid segmentation, while the DHA (C3AE) model predicted hemoglobin levels, demonstrating the feasibility of smartphone-based hemoglobin estimation. In another study, Taslim et al. [25] proposed a Convolutional Neural Network (CNN) to process PPG signals captured from fingertip videos. The CNN was designed to automatically extract relevant features from the PPG signal. This method showed a high correlation between the predicted and actual hemoglobin levels, underscoring the effectiveness of deep learning models in this domain.

These studies collectively emphasize the growing role of deep learning in enhancing the accuracy and reliability of noninvasive hemoglobin level estimation through PPG signals. The ability of deep learning models to automatically learn and extract meaningful features from raw PPG data has opened new avenues for developing portable and accessible health monitoring tools.

3 Methodology

We have developed a noninvasive hemoglobin measurement method using PPG and an adaptive lightweight convolutional neural network (HMALCNN). The code of the proposed approach is available at www.github.com/hmlcnn (accessed on 31 August 2025). The HMALCNN is designed to extract local and global features from PPG signals to estimate hemoglobin levels. To evaluate the effectiveness of our method, we conducted assessments using RMSE and correlation analysis, ensuring that the predicted hemoglobin levels were closely aligned with the actual values. The methodology is divided into 4 sub-sections as follows.

3.1 Photoplethysmography (PPG)

PPG is an optical technique used to detect blood volume changes in the microvascular bed of tissue [26]. It is a non-invasive method that has gained significant attention due to its simplicity, cost-effectiveness, and versatility in various medical applications. PPG operates by emitting light into the skin and measuring the amount of light transmitted or reflected by the blood vessels, as shown in Fig. 2. The variations in the light signal correspond to changes in blood volume, which are then recorded as a PPG waveform. The PPG is based on the Beer-Lambert law [27], which describes the attenuation of light as it passes through a medium:

$$I(t) = I_0 \cdot e^{-\epsilon \cdot C \cdot d} \tag{1}$$

where $I(t)$ is the intensity of light detected at time t, I_0 is the initial intensity of light, ϵ is the molar absorptivity of the medium, C is the concentration of the absorbing substance (hemoglobin), and d is the path length through the medium.

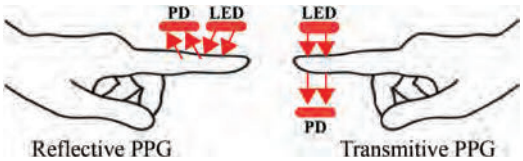


Figure 2: Illustration of PPG acquisition modes. LED and PD are placed on the same side in reflective PPG while opposite sides on transmission PPG

The working principle of PPG relies on the interaction between light and biological tissue. When light in the infrared or visible spectrum is directed toward the skin, it penetrates the tissue and encounters various components, including blood vessels. Hemoglobin is a key component in red blood cells that absorbs specific wavelengths of light. When the heart pumps blood, the volume of blood increases in the systole phase, leading to more light absorption and less reflection. Conversely, during the diastole phase, the blood volume decreases and allowing more light to be reflected to the sensor. These cyclical changes produce the PPG waveform, which can be analyzed to extract vital information about cardiovascular health and other physiological parameters.

PPG can be classified into two primary types based on the mode of light detection: transmission mode and reflection mode. In transmission mode, the light source and photodetector are positioned on opposite sides of the tissue, such as a finger or earlobe, allowing light to pass through the tissue. This setup is commonly used in pulse oximeters. Reflection mode places both the light source and photodetector on the same side of the tissue, measuring the light that is reflected from the skin. This mode is more versatile, as it can be applied to various body sites, including the wrist and forehead, making it suitable for wearable devices and continuous monitoring systems. PPG has found widespread application in the medical field, particularly in monitoring cardiovascular parameters such as heart rate, blood oxygen saturation (SpO_2), and arterial stiffness. Its noninvasive nature and ease of use make it ideal for both clinical settings and home health monitoring. Additionally, PPG is increasingly being explored for its potential in estimating blood pressure, respiratory rate, and now, hemoglobin levels.

3.2 PPG Preprocessing

PPG signals are commonly acquired using wearable devices placed on peripheral sites such as the finger or wrist. However, even small involuntary movements of the sensor relative to the skin introduce motion-induced artifacts, baseline wander, and high-frequency noise that can obscure the true pulsatile component. It is critical to address these noise issues while preserving the underlying physiological waveform. Assume we have a raw dataset $\mathbf{X}$ containing S subjects and W wavelength channels. We denote the raw PPG measurement for subject s at wavelength channel w and time t (in seconds) as

$$X_{s,w}(t), \quad s = 1, 2, \ldots, S, \quad w = 1, 2, \ldots, W, \quad t \in [0, L], \tag{2}$$

where L is the total recording duration in seconds (here, L = 60 s, sampled at 200 Hz).

For each subject s and wavelength channel w, we extract a 15-s segment from the one-minute recording by discarding the first 10 s. Let $S_{s,w}[n]$ be the extracted segment for subject S, which can be defined as follows:

$$S_{s,w}[n] = X_{s,w}[n + N_0], \quad n = 0, 1, \ldots, N_{\text{keep}} - 1, \tag{3}$$

where here, N_0 is the number of samples discarded and N_{keep} is the number of samples retained. Subsequently, we applied a zero-phase 4th-order Chebyshev Type II band-pass filter with passband 0.5–8 Hz.

Denoting its filter coefficients by $\{b_k\}, \{a_k\}$, the filtered segment can be defined as follows:

$$\widetilde{S}_{s,w}[n] = \text{filtfilt}\big(b, a, S_{s,w}[n]\big). \tag{4}$$

Then, each filtered channel is normalized to zero mean and unit variance:

$$Y_{s,w}[n] = \frac{\widetilde{S}_{s,w}[n] - \mu_{s,w}}{\sigma_{s,w}}, \ \mu_{s,w} = \frac{1}{N_{\text{keep}}} \sum_{n=0}^{N_{\text{keep}}-1} \widetilde{S}_{s,w}[n], \ \sigma_{s,w} = \sqrt{\frac{1}{N_{\text{keep}}} \sum_{n=0}^{N_{\text{keep}}-1} \big(\widetilde{S}_{s,w}[n] - \mu_{s,w}\big)^2}. \tag{5}$$

Finally, the preprocessed dataset **Y** is acquired which can be denoted as follows:

$$\mathbf{Y} = \big\{ Y_{s,w}[n] \mid s = 1, \ldots, S; \ w = 1, \ldots, W; \ n = 0, \ldots, N_{\text{keep}} - 1 \big\},$$

where each $Y_{s,w}[n]$ is the normalized, band-pass filtered PPG sample for subject s, channel w, at time index n.

Fig. 3 illustrates the effectiveness of our preprocessing pipeline on multi-wavelength PPG signals. Subfigure (a) shows the raw one-minute recordings at 660, 730, 850, and 940 nm, revealing baseline wander and high-frequency noise caused by motion artifacts. Subfigure (b) displays the 15-s segment extracted after discarding the initial 10 s, followed by Chebyshev Type II band-pass filtering (0.5–8 Hz) and z-score normalization. Furthermore, a beat validity check was applied to ensure that only segments containing at least 15 valid systolic peaks were retained. This step helped eliminate low-quality or irregular waveforms, ensuring that the input to the model preserved consistent and physiologically meaningful cardiac activity.

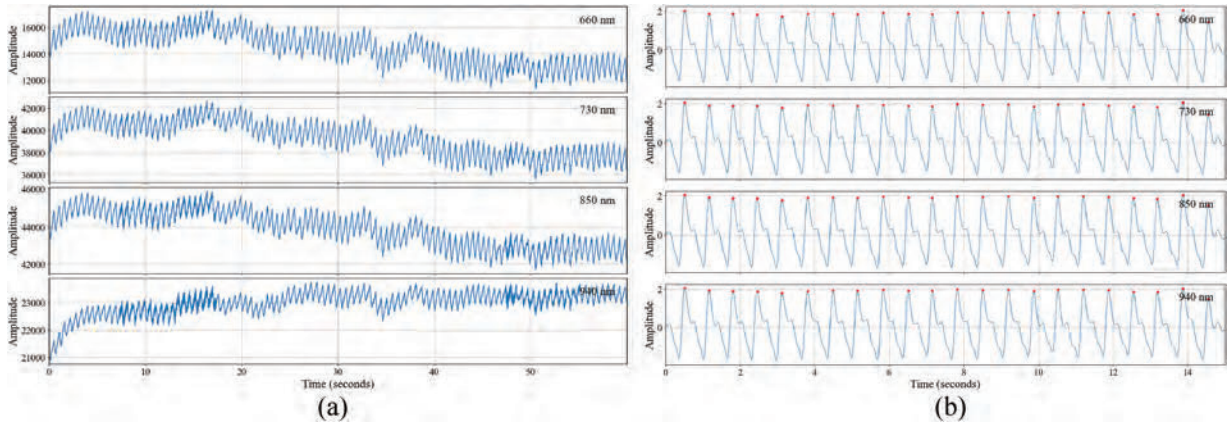


Figure 3: Preprocessing. Subfigure (**a**) shows a raw multi-wavelength PPG signal, exhibiting drift and noise. Subfigure (**b**) shows the corresponding 15 s preprocessed segment with marking

3.3 Adaptive Lightweight Convolutional Neural Network (ALCNN)

Deep learning (DL) is a subset of machine learning (ML) that has revolutionized the field of artificial intelligence (AI) by enabling neural models to learn complex patterns and representations from large datasets [10]. Convolutional neural networks (CNNs) are the most influential DL architectures, which are particularly well-suited for processing grid-like data such as images and time-series signals [11]. CNNs use convolutional layers to automatically extract hierarchical features, ranging from simple edges in early layers to more complex structures in deeper layers, making them highly effective for image recognition, object detection, and time-series analysis tasks. Just like the human brain learns by processing sensory input and adapting based on past experiences, neural networks also learn by processing input data and adjusting their weights through training. Like the human brain, neural networks consist of interconnected layers

and neurons that work together to process information and make decisions. Each layer processes different features of the input data, similar to how the brain analyzes sensory information through neurons. As the network trains, it adjusts the connections between neurons, refining its ability to recognize patterns and improve decision-making, much like the brain adapts and learns from experience. In the medical field, CNNs have been widely applied to various diagnostic and monitoring tasks. These include medical image analysis for detecting conditions such as tumors, lesions, and other abnormalities in radiological images, as well as analyzing PPG and other physiological signals [28–30]. CNNs have demonstrated remarkable accuracy and efficiency, surpassing traditional methods in many cases. However, the application of CNNs in medical settings often comes with challenges, particularly related to the large size and computational complexity of the models, which can hinder their deployment in resource-constrained environments such as mobile devices or real-time monitoring systems. One of the main challenges with traditional CNNs is their computational intensity and the large number of parameters, which can make them impractical for use in settings where computational resources are limited.

To address this issue, we proposed an adaptive lightweight convolutional neural network (ALCNN) as shown in Fig. 4. The ALCNN is specifically designed to reduce the computational burden while maintaining high accuracy in the task of noninvasive hemoglobin measurement. The ALCNN is optimized to extract the essential features from PPG signals, enabling accurate hemoglobin estimation without the need for a large and complex network. The proposed ALCNN is a one-dimensional convolutional network specifically designed for hemoglobin prediction from multi-wavelength PPG signals. The network architecture comprises four convolutional blocks, each consisting of a convolutional layer, batch normalization, ReLU activation, and max-pooling, followed by three fully connected layers for regression.

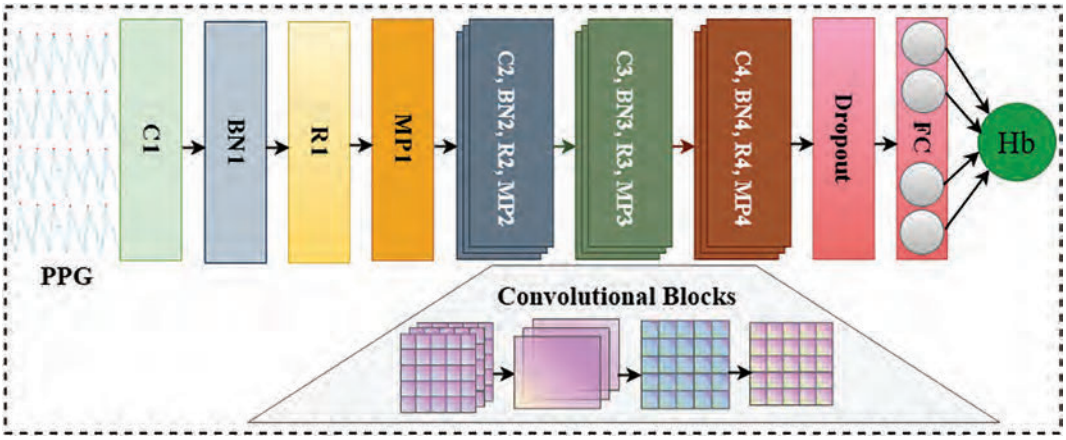


Figure 4: The architecture of adaptive lightweight convolutional neural network (ALCNN)

The first convolutional block transforms the input sequence into 4 feature maps using a kernel size $K = 5$, stride $s = 1$, and padding $p = 2$:

$$y_{ij}^{(1)} = \sum_{k=1}^{C_{in}} \sum_{l=1}^{K} x_{i-l,\,k}^{(0)} \cdot w_{lk}^{(1)} + b_j^{(1)} \tag{6}$$

where $C_{in} = 1$ for the initial layer, $w_{lk}^{(1)}$ is the filter weights, and $b_j^{(1)}$ is the bias for the j-th channel.

Each output is batch normalized:

$$\hat{y}_{ij}^{(n)} = \frac{y_{ij}^{(n)} - \mu_j}{\sqrt{\sigma_j^2 + \epsilon}} \tag{7}$$

followed by the ReLU non-linearity:

$$z_{ij}^{(n)} = \max(0, \hat{y}_{ij}^{(n)}) \tag{8}$$

Max-pooling with kernel size $P = 2$ and stride $s = 2$ is then applied:

$$p_{ij}^{(n)} = \max_{m=1}^{P} z_{i \cdot s+m,\, j}^{(n)} \tag{9}$$

The same process is repeated for subsequent convolutional layers with channel sizes: Conv2 → 8, Conv3 → 16, Conv4 → 32, keeping $K = 5$, $p = 2$, and $s = 1$.

The output from the final pooling layer is flattened into a vector of length N_{conv} and mapped through:

$$f_i^{(1)} = \text{ReLU}\left(\sum_{j=1}^{N_{conv}} w_{ij}^{(fc1)} \cdot p_j^{(4)} + b_i^{(fc1)} \right) \tag{10}$$

Dropout with rate $p_d = 0.3$ is applied:

$$\tilde{f}_i^{(1)} = \begin{cases} 0 & \text{with probability } p_d \\ \dfrac{f_i^{(1)}}{1 - p_d} & \text{otherwise} \end{cases} \tag{11}$$

This is followed by the second fully connected layer:

$$f_i^{(2)} = \text{ReLU}\left(\sum_{j=1}^{100} w_{ij}^{(fc2)} \cdot \tilde{f}_j^{(1)} + b_i^{(fc2)} \right) \tag{12}$$

with another dropout layer, and finally, the regression output layer:

$$\hat{y} = \sum_{j=1}^{50} w_j^{(fc3)} \cdot \tilde{f}_j^{(2)} + b^{(fc3)} \tag{13}$$

where $\hat{y}$ is the predicted hemoglobin concentration in g/dL.

The network is trained using the Mean Squared Error (MSE) loss:

$$\mathcal{L} = \frac{1}{N} \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 \tag{14}$$

Optimization is performed using the Adam optimizer ($\eta = 5 \times 10^{-4}$) with a ReduceLROnPlateau scheduler (factor 0.1, patience = 5) and early stopping (patience = 10) based on validation loss.

The architecture and training hyperparameters of the proposed HMALCNN were optimized through systematic experimentation. We explored dropout rates in the range of 0.1–0.5, convolutional filter configurations from 4, 8, 16, 32 up to 16, 32, 64, 128, kernel sizes of 3 and 5, learning rates of 0.001, 0.0005, and

0.0001, and batch sizes of 8, 16, and 32. Fully connected layers of varying sizes were also evaluated, with the final architecture adopting two dense layers of sizes 100 and 50. The Adam optimizer was employed with a learning rate of 0.0005, paired with a ReduceLROnPlateau scheduler (factor = 0.1, patience = 5) to adaptively reduce the learning rate when the validation loss plateaued. Early stopping with a patience of 10 epochs was applied to prevent overfitting. Validation RMSE was the primary criterion for selection, with Pearson correlation coefficient monitored as a secondary metric. The final configuration includes a dropout rate of 0.3, four convolutional layers with filters 4, 8, 16, 32, and a kernel size of 5.

3.4 Performance Evaluation Metrics

For a robust evaluation of the HMALCNN approach, we employed both numerical and correlation evaluation metrics. Numerical evaluation, particularly Root Mean Squared Error (RMSE), provides a direct measure of the difference between predicted and actual values. RMSE was chosen over Mean Absolute Error (MAE) due to its higher sensitivity to large prediction errors, which is critical in clinical applications. Unlike MAE, RMSE penalizes larger deviations more severely, making it a more appropriate metric where under or overestimation of hemoglobin can have significant clinical implications. For instance, an inaccurate estimation in the lower or higher hemoglobin range could lead to misdiagnosis of anemia or polycythemia, respectively. In parallel, we adopted Pearson correlation analysis to evaluate the strength of the linear relationship between predicted and actual hemoglobin values. This metric not only reflects how well the model captures inter-subject variability but also validates the model's ability to track physiological trends. Unlike confidence intervals, which primarily quantify uncertainty around a central estimate, Pearson correlation offers an interpretable and standardized measure of predictive alignment. Confidence intervals are more suitable for inferential statistics rather than for evaluating regression-based predictive performance, and they often depend on assumptions that may not hold in heterogeneous or limited datasets. Therefore, RMSE and Pearson correlation together provide a rigorous and clinically relevant framework to assess the performance of the proposed model in both numerical accuracy and physiological consistency. RMSE can be defined as:

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{y}_i - y_i)^2} \tag{15}$$

where N is the number of samples, $\hat{y}_i$ represents the predicted values, and y_i denotes the actual values.

Correlation analysis was conducted to evaluate the strength and direction of the linear relationship between the predicted ($\hat{y}$) and actual (y) hemoglobin levels. The Pearson correlation coefficient was employed, which is a widely accepted measure for determining how well the predicted values align with the actual values [31]. The Pearson correlation coefficient r can be calculated as follows:

$$r = \frac{\sum_{i=1}^{N}(y_i - \overline{y})\left(\hat{y}_i - \overline{\hat{y}}\right)}{\sqrt{\sum_{i=1}^{N}(y_i - \overline{y})^2}\sqrt{\sum_{i=1}^{N}\left(\hat{y}_i - \overline{\hat{y}}\right)^2}} \tag{16}$$

where N is the number of samples, y_i and $\hat{y}_i$ are the actual and predicted values, respectively, and $\overline{y}$ and $\overline{\hat{y}}$ are the means of the actual and predicted values. The Pearson correlation coefficient r ranges from −1 to 1, where a value of 1 indicates a perfect positive linear correlation, −1 indicates a perfect negative linear correlation, and 0 indicates no linear correlation. A high positive correlation coefficient in our results would signify that the model's predictions closely follow the actual hemoglobin levels, confirming reliability and accuracy.

4 Experimental Setup

4.1 Dataset

In this study, we employed two multi-regional multi-wavelength datasets. The dataset-A is comprised of PPG signals and corresponding hemoglobin values that are collected from a total of 68 subjects, ranging in age from 18 to 65 years, at the Primary Health Center in Jatiuwung, Tangerang City, Banten, Indonesia [32]. Each subject's PPG signal was measured using a MAX30102 module sensor, which emitted two types of light: red light with a wavelength of 660 nm and infrared light with a wavelength of 880 nm. The sensor recorded the intensity of light absorbed by the subject's finger, with the resulting data stored in arbitrary units (a.u.). The dataset includes 816 data points, with each subject contributing 12 sets of red and infrared light signals. In addition to the PPG signals, the dataset also includes demographic information such as gender and age, as well as the hemoglobin concentration measured in grams per deciliter (g/dL). The hemoglobin values were obtained using a Nesco Multicheck hemoglobin meter, which required a small blood sample from each subject. The Dataset-B comprised 152 healthy volunteers (aged 21–88 years, male:female ≈ 1:1) recruited at the University Hospital of the Guilin University of Electronic Technology [33]. A custom-built fingertip PPG sensor recorded volumetric pulse waveforms at four wavelengths (660, 730, 850, and 940 nm) for one minute per subject, and data were streamed to a host computer in real time. Immediately following each PPG measurement, a venous blood sample was collected and analyzed on the hospital's automated hematology analyzer to obtain the reference hemoglobin concentration for model training and validation. The detailed data acquisition and dataset preparation process is illustrated in Fig. 5. Both datasets were collected using different reference methods: Dataset-A used capillary blood via a portable meter, while Dataset-B used venous blood analyzed by a hospital-grade hematology analyzer. This diversity in acquisition environments and reference standards enables the proposed model to learn across different clinical and non-clinical settings, potentially improving its robustness and generalizability in real-world applications.

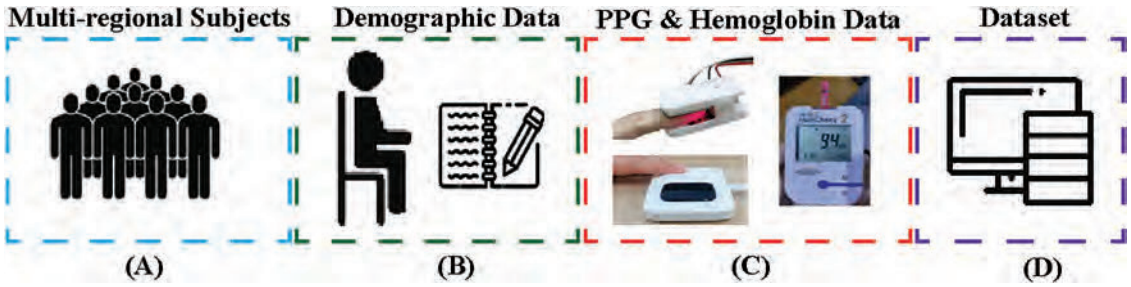


Figure 5: Illustration of multi-regional data acquisition. Block (**A**) presents the multi-regional subjects; Block (**B**) shows the subject arrival and demographic data recording; Block (**C**) represents the finger PPG and Hemoglobin acquisition in both datasets with different PPG setups; Block (**D**) shows the dataset curation and storage

4.2 Experimental Grounds

All experiments in this study were conducted using Python 3.9.7 and the PyTorch framework. PyTorch is a widely used open-source deep learning library known for its flexibility and dynamic computational graph, which makes it ideal for developing and experimenting with neural network models [34]. Its support for GPU acceleration further enhances the efficiency of model training, particularly for large datasets and complex models. The experiments were run on a system equipped with an NVIDIA RTX 4060 GPU with 32 GB of RAM. This setup provided the necessary computational power to handle the intensive tasks involved in training deep learning models, including large-scale matrix operations and backpropagation.

To ensure rigorous validation of the model, we initially performed model development using Dataset-B, where the data was split into 70% for training, 10% for validation, and 20% for testing. This phase was used to optimize hyperparameters, including learning rate, dropout rate, batch size, and optimizer parameters. Dataset-B contains four-channel PPG inputs at wavelengths of 660, 730, 850, and 940 nm, allowing the model to learn comprehensive multi-wavelength feature representations. Once the optimal hyperparameters were determined, the same architecture was applied to Dataset-A, which consists of two-channel PPG signals (660 and 880 nm), while keeping all other architectural and training settings consistent for fairness in evaluation. For the final performance assessment, we employed subject-independent group-based 5-fold cross-validation with random splitting for both datasets. Cross-validation is a robust statistical method used to assess the performance of a model by dividing the dataset into multiple subsets (folds) [35]. In each round of cross-validation, one group of subjects is used exclusively for testing, while the remaining groups are used for training the model in a subject-independent manner. This process is repeated until each group has been used for testing exactly once. The results from all folds are then averaged to produce a single performance metric. The use of this protocol ensures that PPG signals from the same subject are never present in both the training and testing sets, effectively preventing information leakage and yielding a more reliable estimate of generalization ability.

5 Results

Initially, we conducted a detailed statistical analysis to explore the gender specific distributions in both datasets. Fig. 6 shows the gender-based distribution of hemoglobin concentration and age variability. We used density plots to illustrate the distribution of hemoglobin levels separately for males and females. In Fig. 6a, the x-axis represents hemoglobin levels, while the y-axis represents density, providing a normalized frequency of hemoglobin measurements in Dataset-A. The blue region corresponds to the male population, and the orange region corresponds to the female population. The plot reveals distinct peaks in hemoglobin levels for each gender, with females generally showing lower hemoglobin values than males. This distribution allows us to observe variations in hemoglobin levels by gender, highlighting potential differences in physiological characteristics. The Fig. 6b shows the age distribution for both genders. The x-axis represents age in years, and the y-axis indicates density, illustrating the relative concentration of individuals in different age groups for each gender. This plot highlights the distribution of participants across different age groups for each gender, with noticeable differences in age concentration. Peaks and dips in the density curves help identify the most common age ranges within each gender group, providing insights into the age distribution across genders in Dataset-A [32]. Fig. 6c,d presents the gender-based density analyses for Dataset-B [33]. In Fig. 6c, the hemoglobin distribution for females is centered around 130–140 g/L, whereas males exhibit a higher mode near 150–160 g/L. Fig. 6d shows the age distribution across dataset-B in which both genders concentrate in early adulthood (around 20–30 years), with a secondary tail extending into middle age. By using these datasets, we capture both geographic and demographic variability, thereby providing a more comprehensive foundation for developing a generalizable hemoglobin-prediction approach. We analyzed gender and age-related variability in hemoglobin levels within datasets; these demographic variables were not used as input covariates in the model. This decision was made to preserve a fully signal-driven approach.

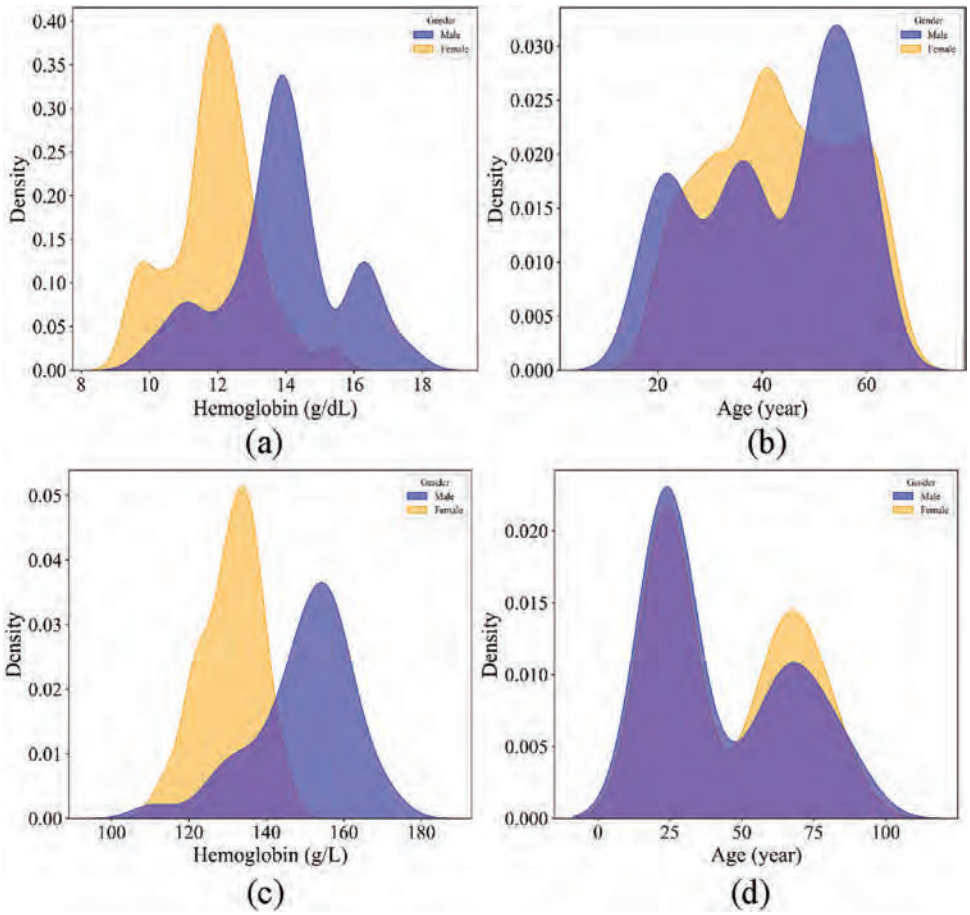


Figure 6: Gender-based distribution of hemoglobin and age in both datasets

Fig. 7 illustrates the training and validation loss of the HMALCNN model for both datasets. In Fig. 7a, training (blue line) and validation loss (orange line) decrease rapidly during the initial epochs, indicating that the model is effectively learning and adapting to the features in dataset-A. Around the 10th epoch, the losses converge and stabilize, suggesting that the model has achieved an optimal balance without overfitting. The close alignment of training and validation loss curves after stabilization further indicates robust generalization, as the model performs consistently on both training and validation sets. In Fig. 7b, the HMALCNN model's training and validation losses for dataset-B exhibit an even more rapid convergence; both curves decline steeply by epoch 5. Then, the losses remain flat and almost indistinguishable, indicating that the network has fully captured the underlying mapping and maintains excellent generalization without overfitting. This early plateau demonstrates that the model attains optimal performance within the first few epochs, reflecting the model's high representational efficiency on this cohort. This stability highlights the effectiveness of the HMALCNN model in capturing relevant features for hemoglobin level estimation.

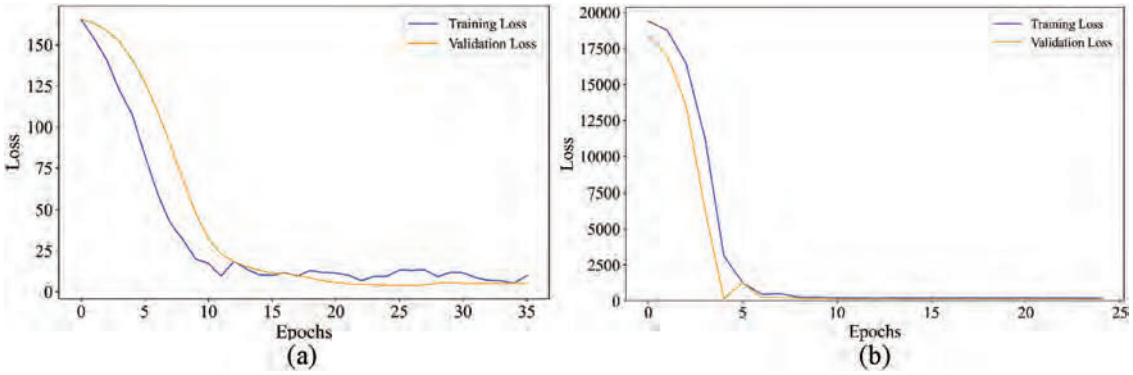


Figure 7: Training and validation loss of HMALCNN over epocs

Moreover, we conducted an extensive evaluation of time complexity, which serves as a crucial metric for assessing the computational efficiency of classification models. In this study, testing time, defined as the duration required by the model to generate predictions on test data, was employed as the primary indicator of efficiency using Kernel density estimation (KDE). The training process was performed manually and was not included in the evaluation. Fig. 8 illustrates that the proposed model demonstrates low time complexity on both datasets. The proposed model achieved an average elapsed time of approximately 0.038 s on Dataset-A and 0.058 s on Dataset-B, which indicates that the proposed model is computationally efficient across different datasets, making it suitable for clinical applications where timely and accurate early diagnosis is essential.

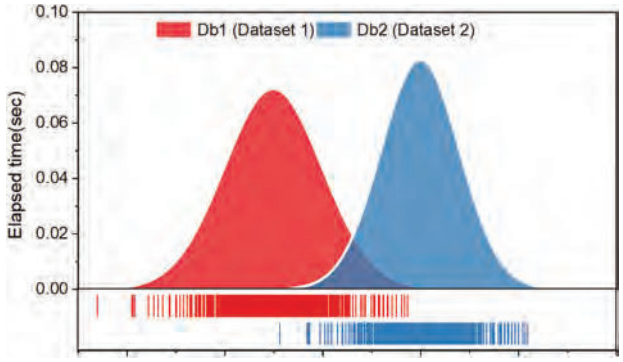


Figure 8: Kernel density estimation (KDE) of testing time distribution for the proposed model on Dataset-A and Dataset-B

To assess the generalizability of HMALCNN, we performed group-based, subject-independent 5-fold cross-validation on both datasets. Dataset-A (68 subjects) was partitioned into five non-overlapping groups of 13–14 subjects, and Dataset-B (152 subjects) into five groups of 30–31 subjects. In each fold, exactly one group served as the test set, and the remaining groups provided the training data. Every subject appears exactly once in a test fold and is completely excluded from the corresponding training sets, ensuring true subject-independent validation and promoting better model generalization. Table 2 presents the per-fold RMSE and Pearson correlation for both datasets. On Dataset-A, the average RMSE was 1.20 g/dL with an average correlation of 72%. Fold 5 achieved the lowest RMSE of 1.14 g/dL, and Fold 1 the highest correlation 74%. On Dataset-B, the average RMSE was 0.90 g/dL and the average correlation was 82%. The lowest RMSE of 0.85 g/dL was achieved in Fold 3, while the highest correlation of 87% was observed in

Fold 2. A Wilcoxon signed-rank test was conducted to evaluate the statistical significance of performance variations across the 5 folds. For both Dataset-A and Dataset-B, p-values exceeded 0.05 for RMSE and Pearson correlation, indicating that fold-wise performance differences are not statistically significant. This confirms that the HMALCNN achieves stable and consistent generalization across independent subject groups. The HMALCNN maintains stable accuracy and strong linear agreement across entirely independent subject splits.

Table 2: Group-based subject-independent 5-fold cross-validation of HMALCNN on both datasets

Fold	Dataset A		Dataset B	
	RMSE (g/dL)	Pearson Correlation (%)	RMSE (g/dL)	Pearson Correlation (%)
1	1.25	74	0.87	83
2	1.18	71	0.93	87
3	1.23	73	0.85	79
4	1.22	72	0.97	85
5	1.14	70	0.88	76
Average	1.20	72	0.90	82

Fig. 9 illustrates results in terms of a bar plot for Dataset-A and Bland–Altman analysis for Dataset-B. The Fig. 9a illustrates the comparison between actual and predicted hemoglobin levels (in g/dL) for a random fold using the HMALCNN approach. The x-axis represents the random samples, while the y-axis displays hemoglobin levels. The blue bars indicate the actual measured hemoglobin levels, while the orange bars represent the predicted levels generated by the HMALCNN model. The close alignment between the actual and predicted values for most samples signifies the accuracy and reliability of the model's predictions. Fig. 9b presents a Bland–Altman analysis of all test-fold predictions on Dataset-B. Each point plots the mean of the predicted and actual hemoglobin values (x-axis) against their difference (Predicted – Actual, y-axis), both in g/dL. The mean bias of −0.06 g/dL indicates almost no systematic over or underestimation. The red dashed lines denote the 95% limits of agreement at +2.72 and −2.83 g/dL (mean ± 1.96 SD), showing that most errors lie within this ±3 g/dL window. The roughly uniform scatter of points around the bias line, with no downward or upward trend across the measurement range, confirms that the HMALCNN predictions aligned closely with the reference values and exhibit no proportional bias. These visualizations effectively highlight the model's ability to estimate hemoglobin levels with minimal discrepancies, with robust performance.

To further assess the clinical reliability of our model, we conducted a hemoglobin range-wise error analysis by stratifying test predictions into three clinical categories: *low* (<12 g/dL), *normal* (12–16 g/dL), and *high* (>16 g/dL). The proposed HMALCNN achieved lower RMSE and higher correlation in the normal range—0.65 g/dL (r = 0.93) for Dataset-A and 0.59 g/dL (r = 0.95) for Dataset-B—demonstrating high accuracy in most clinical decisions. Slightly higher errors were observed in the high hemoglobin range, with RMSEs of 1.12 and 1.01 g/dL, respectively, due to physiological variability. We also conducted an error tolerance analysis using the widely accepted ±1 g/dL threshold, which is considered a clinical gold standard for point-of-care hemoglobin measurement systems. For Dataset-A, our HMALCNN achieved predictions within this ±1 g/dL tolerance for 86.7% of test samples, with an average RMSE of 1.20 g/dL and strong correlation (r = 0.72). For Dataset-B, the model achieved 91.3% within tolerance, with a lower RMSE of 0.90 g/dL and higher correlation (r = 0.82). These findings confirm that the model performs consistently across clinically meaningful ranges and support its potential utility in real-world, noninvasive hemoglobin monitoring.

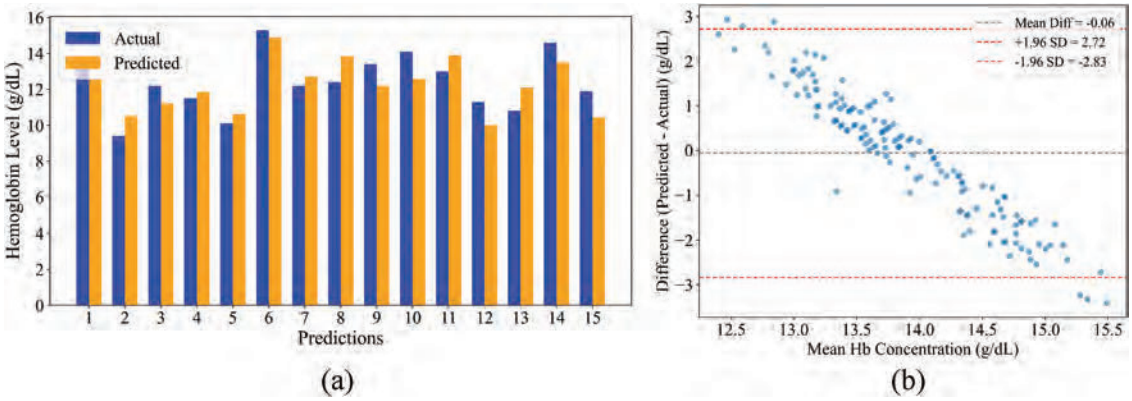


Figure 9: Bar plot of hemoglobin level predictions by HMALCNN approach

To enhance the interpretability of the proposed HMALCNN model, we applied Gradient-weighted Class Activation Mapping (Grad-CAM) to the final convolutional layer. Grad-CAM computes the gradient of the prediction score for feature maps, producing a localization map that highlights the most influential temporal regions in the input PPG signal as shown in Fig. 10. The resulting attention map, overlaid on the normalized PPG waveform, revealed that the model predominantly attends to the systolic peaks, dicrotic notches, and steep rising and falling slopes, physiologically significant regions that correlate with optical absorption properties of hemoglobin. In contrast, low-attention areas correspond to baseline segments or motion-distorted intervals, indicating effective suppression of non-informative patterns. These results confirm that the network learns to focus on clinically relevant features, supporting its potential reliability in real-world, noninvasive hemoglobin monitoring.

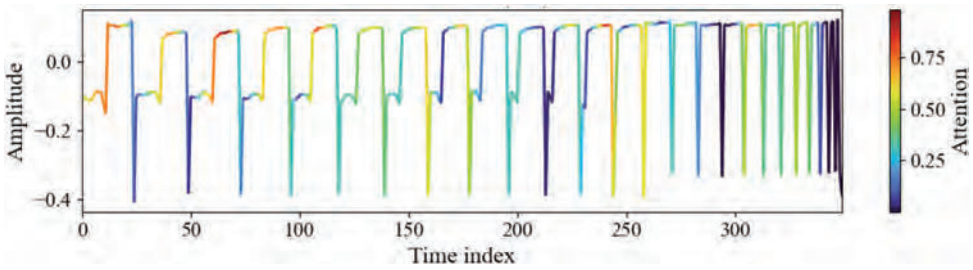


Figure 10: Grad-CAM visualization of the proposed method on dataset-A. Color intensity denotes the level of network attention, with red indicating high relevance and blue indicating low relevance. The model exhibits strong focus on systolic peaks, dicrotic notches, and steep waveform slopes, which are physiologically linked to hemoglobin concentration

The proposed approach showed competitive results and outperformed existing methods in terms of RMSE and Pearson Correlation coefficient analysis as shown in Table 3. We used RMSE instead of the MSE metric for hemoglobin measurement because it penalizes larger errors more severely. Incorrect predictions of higher hemoglobin levels can lead to severe consequences. Furthermore, we compared the results of subject-independent 5-fold cross-validation on the entire dataset instead of the traditional train/test split comparison. The comparative studies explored a wide range of hand-crafted PPG features and demographic covariates using classical machine-learning methods, including SVM, stacking regressor, XGBoost, and linear regression [15,18,21,36,37]. Only reference [15] applied leave-one-subject-out (LOSO) validation; the remaining works relied on conventional train/test splits without subject-independent cross-validation. Similarly, end-to-end deep learners, including ResNet-18, ResNet-101, MobileNet, and CNN

have been compared [14,20,38,39], yet all used simple hold-out testing. In contrast, our HMALCNN is assessed via group-based, subject-independent 5-fold cross-validation, ensuring each subject appears only in one test fold. Performance for each dataset is reported independently in Table 3 to ensure a fair and unbiased comparison.

Table 3: Results comparison of proposed approach with different state-of-the-art methods

Reference	Method	Dataset size	Features	Validation	RMSE (g/dL)	Pearson Corr. (%)
[38]	CNN	170	End-to-End	Traditional	1.71	–
[14]	MobileNet	170	End-to-End	Traditional	1.57	0.66
[15]	HemaApp-SVM	31	Hand-crafted	LOSO	1.56	0.69
[36]	MSR	1005	Hand-crafted	Traditional	1.47	–
[18]	MSR	1583	Hand-crafted	Traditional	1.35	0.81
[20]	RESNET-18	150	End-to-End	Traditional	1.39	–
[37]	XGBoost	58	Hand-crafted	Traditional	1.01	–
[39]	ResNet-101	1244	End-to-End	Traditional	1.48	–
[21]	LR	47	Hand-crafted	Traditional	0.95	–
Ours	**HMALCNN**	**152**	**End-to-End**	**5-Fold CV**	**0.90**	**0.82**
Ours	**HMALCNN**	**68**	**End-to-End**	**5-Fold CV**	**1.20**	**0.72**

Note: CNN: Convolutional neural network; SVM: Support Vector Machine; MSR: Multimodel stacking regressor; RESNET: Residual network; XGBoost: Extreme gradient boosting; LR: Linear regression; HMALCNN: Hemoglobin estimation method using adaptive lightweight convolutional neural network; LOSO: Leave one subject out; CV: Cross-validation.

6 Discussion

PPG is a noninvasive optical technique widely employed for measuring blood volume variations associated with cardiac activity. As the heart pumps blood through the circulatory system, variations in blood volume occur with each heartbeat. These changes in blood volume within the capillaries can be analyzed using PPG. The changes in light absorption or reflection caused by these blood volume changes lead to digital biomarker investigation correlated with human health. The widespread adoption of PPG in wearable devices has proven valuable in various biomedical applications, including monitoring blood pressure, blood glucose levels, sleep patterns, stress, and other physiological health indicators. One of the most critical components in the human body is hemoglobin, which plays a pivotal role in oxygen transportation. Hemoglobin carries oxygen from the lungs to tissues and organs, and its ability to bind with oxygen is essential for maintaining normal body function. Regular measurement of hemoglobin levels is vital, as abnormal levels can indicate serious health conditions like anemia. Traditional hemoglobin measurement methods are invasive requiring painful blood samples. Further, hemoglobin measurement is challenging in underserved populations and developing countries.

To address these challenges, PPG-based analysis offers a non-invasive alternative for hemoglobin measurement. With recent advancements in machine learning and deep learning, the ability to analyze PPG signals has significantly improved, enabling more accurate detection of biomarkers. While current machine learning methods typically require substantial human intervention for feature engineering, they often fail to capture the full spectrum of features within the signal. On the other hand, deep learning models are highly effective in extracting complex features but are computationally intensive and require significant resources. In this study, we proposed an end-to-end adaptive lightweight convolutional neural network to

efficiently extract both local and global features from blood volumetric signals, which are closely associated with variations in hemoglobin levels. CNNs have demonstrated great promise in medical signal processing and bioinformatics due to their ability to learn hierarchical features from raw data.

To evaluate our proposed approach, we used two multi-regional multi-wavelength datasets having 152 and 68 subjects. Further, we employed a group-based five-fold cross-validation technique to prevent overfitting. Cross-validation provides a robust assessment by ensuring that the model is trained and tested on different subsets of data, ultimately improving generalization and reliability. In addition to standard cross-validation, we implemented subject-independent testing within each fold. This approach further enhances generalizability by ensuring that the model performs well on unseen individuals, making it more suitable for real-world applications. Performance assessment was carried out using the RMSE, which is a well-suited metric for regression tasks like hemoglobin estimation. RMSE penalizes larger errors more heavily, providing a clear indication of model accuracy and highlighting areas where improvements are needed. To complement RMSE, we calculated correlation coefficients between predicted and actual hemoglobin values, verifying that the model's predictions align positively with true measurements. The proposed approach exhibits competitive performance, achieving an RMSE of 1.20 g/dL and a strong correlation coefficient of 0.72 on Dataset-A and an RMSE of 0.90 g/dL and a correlation coefficient of 0.82 in Dataset-B. These results surpass the performance of state-of-the-art methods, demonstrating the effectiveness of the proposed approach. Real-world deployment requires a clinically usable system that must deliver results quickly and operate reliably on lightweight hardware. The proposed model achieved an average inference time of roughly 0.038 s on Dataset-A and 0.058 s on Dataset-B, demonstrating computational efficiency suitable for both wearable sensors and clinical workstations. Such sub-tenth-second latency ensures that hemoglobin estimates can be delivered in near-real time, meeting the responsiveness needed for continuous monitoring in wearables and rapid decision support in point-of-care environments.

The proposed ALCNN framework offers several advantages. First, its lightweight design enables efficient processing of multi-wavelength PPG signals on wearable devices with limited computational resources. Second, the architecture is adaptable, functioning effectively with both dual-wavelength and four-wavelength inputs without major retraining, thus ensuring applicability across different sensor configurations. Third, the use of multi-regional datasets and subject-independent cross-validation enhances the model's generalization to diverse populations. Moreover, the combination of spectral and temporal features improves robustness to sensor noise and physiological variability. Despite the promising performance of the proposed approach, there are a few limitations that need to be addressed in future research. The current study relied on controlled conditions with a limited clinical trial and device regulatory concerns. The datasets used in this study did not include annotations related to confounding factors such as skin tone, fingertip temperature, or blood perfusion variability. As a result, these aspects were not analyzed. Future work will focus on data acquisition in free-living environments to confirm robustness during everyday activity with formal clinical validation. Subsequently, the inclusion of metadata to investigate the model's performance across diverse physiological and demographic conditions. Furthermore, consideration for implementing the clinically validated model on wearable devices for subsequent regulatory approvals.

7 Conclusions

In this study, we presented an adaptive lightweight convolutional neural network (HMALCNN) for noninvasive hemoglobin measurement using PPG. The proposed method effectively integrates local morphological features and global contextual patterns, enabling accurate estimation of hemoglobin levels across diverse acquisition settings. Extensive evaluation was performed on two multi-regional datasets using group-based, subject-independent 5-fold cross-validation to ensure robust and unbiased performance assessment.

The HMALCNN achieved RMSEs of 1.20 and 0.90 g/dL, with Pearson correlation coefficients of 0.72 and 0.82 for the respective datasets. Additional analyses, including clinical range-wise evaluation, ±1 g/dL error tolerance assessment, and Wilcoxon signed-rank testing, confirmed the model's stability, generalization ability, and clinical reliability. Grad-CAM visualizations further demonstrated that the model consistently attends to physiologically relevant PPG segments such as systolic peaks and dicrotic notches, supporting interpretability and trustworthiness. The model's computational efficiency, adaptability to varying spectral inputs, and strong alignment with clinical accuracy thresholds position it as a promising solution for integration into wearable devices and point-of-care hemoglobin monitoring systems. Future work will focus on real-world deployment in free-living environments to further validate and extend its applicability.

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Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

Abbreviations

PPG	Photoplethysmography
CNN	Convolutional Neural Network
ALCNN	Adaptive Lightweight Convolutional Neural Network
HM	Hemoglobin Measurement
HMALCNN	Hemoglobin Estimation Method Using Adaptive Lightweight Convolutional Neural Network
g/dL	Grams Per Deciliter
g/L	Grams Per Liter
RMSE	Root Mean Square Error
CV	Cross-Validation

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An indicator of inclusion with applications to computer vision

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ABSTRACT: In this paper we present an algorithmic process of necessary operations for the automatic movement of a predefined object from a video image in the target region of that image, intended to facilitate the implementation of specialized software applications in solving this kind of problems.

1. INTRODUCTION

The Algorithmic problem-solving procedures for automatic traveling objects within the video images were approached by us also in an earlier work, see [4]. The purpose of this paper is to point out a new method of solving these problems. As mentioned in the earlier work, the process which we shall indicate, will be based on the definition of an indicator of Extenics type specialized in signaling whether a particular set (of pixels, in our modeled case) is included in a target set on the monitor screen.

An aspect which has to be mentioned from the beginning is that both indicators, the one defined in [4], as well as the indicator that we shall define in this paper, fundamentally differ from the indicators currently used in the Extension theory because they make the leap from reporting the position of a single point in relation to one or two given sets to that of reporting the relationship between two sets - which is much more complex, thus establishing a factor that is meant to ensure the progress of this theory. The Extension theory, which we have referred to earlier, has been proposed by Professor Cai Wen in [5].

Because of the importance of this theory in both theoretical and practical field, it has been continuously extended, at the beginning by its founder himself, see [1, 6, 7], and then by other researchers from various fields of activity, see [2, 3, 4].

2. AN INDICATOR CAPABLE TO REPORT IF A SPECIFIC SET IS INCLUDED IN A GIVEN TARGET SET

This paragraph aims at presenting new results in order to complete and improve the existent Extension theory. The framework addressing these results is that of a metric space expressed through the doublet (X, d), where X is the set of points which make up the considered space, and d is the metric of this space.

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For any two nonempty sets A and B from X we introduce the indicator

$$\Delta(A, B) = \sup\{\delta(a, B) \mid a \in A\}, \tag{1}$$

where we denote by $\delta(a, B)$ the usual distance from point $a \in A$ to set B, that is

$$\delta(a, B) = \inf\{d(a, b) \mid b \in B\}.$$

Observations: 1) Relation $\Delta(A, B) = \Delta(B, A)$ is not always true, in other words, the value of indicator $\Delta(A, B)$ depends, in general, on the order in which sets A and B are considered.

2) Indicator $\Delta(A, B)$ can also take infinite values.

3) In the case of two bounded sets A and B[13] indicator $\Delta(A, B)$ is finite.

Proposition: Indicator Δ defined by relation (1) has the following properties:

1) $\Delta(A, B) = 0 \Rightarrow A \subseteq \overline{B}$, where $\overline{B}$ is the closure of set B in topology induced by metric d on space X. Reciprocally, $A \subseteq \overline{B} \Rightarrow \Delta(A, B) = 0$.

2) $\Delta(A, B) = \Delta(B, A) = 0 \Rightarrow \overline{A} = \overline{B}$. Reciprocally, $\overline{A} = \overline{B} \Rightarrow \Delta(A, B) = \Delta(B, A) = 0$.

Demonstration: 1) $\Delta(A, B) = 0 \Leftrightarrow \delta(a, B) = 0 \; \forall \, a \in A \Leftrightarrow A \subseteq \overline{B}$. Reciprocally, if $A \subseteq \overline{B}$ then $\delta(a, B) = 0 \; \forall \, a \in A \Leftrightarrow \Delta(A, B) = 0$.

2) $\Delta(A, B) = 0 \Rightarrow A \subseteq \overline{B}$, and $\Delta(B, A) = 0 \Rightarrow B \subseteq \overline{A}$. From $A \subseteq \overline{B}$, and $B \subseteq \overline{A}$ we deduce that $\overline{A} \subseteq \overline{B}$, respectively $\overline{B} \subseteq \overline{A}$. Consequently $\overline{A} = \overline{B}$. Reciprocally, if $\overline{A} = \overline{B}$ then $\delta(a, B) = 0 \; \forall \, a \in A$, and $\delta(b, A) = 0 \; \forall \, b \in B$, hence, $\Delta(A, B) = \Delta(B, A) = 0$.

Observations: 1) Due to property 1) from the above proposition, indicator Δ is named indicator of inclusion.

2) On the set $\mathcal{C}(X)$ of all non-empty compact subsets of X,

$$H(A, B) = \max\{\Delta(A,B), \Delta(B, A)\},$$

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[13] A set Y from X is called bounded if its diameter $D(Y) = \sup\{d(y_1, y_2) \mid y_1, y_2 \in Y\}$ is bounded.

represents the Hausdorff distance between sets A and B.

3. APPLICATIONS

Pointer Δ specified by us in this paper can be used to solve the problems posed by the development of software applications for an automated movement of a specific object "$\mathcal{O}$" from a given video image "VIm" in a target region "$\mathcal{R}$" of that image. In order to achieve this goal, we use an algorithm similar to the one that was defined in [4]. In very general terms, the new algorithm has the following content: through a set of isometries $\mathcal{I}_i$, $i \in I$ of the plane, we move object $\mathcal{O}$ into different regions and positions of the image "VIm" by calculating, every time, the value of each indicator $\Delta\left(\mathcal{I}_i(\mathcal{O}), \mathcal{R}\right)$. The determination of the indices $i_0 \in I$ for which $\Delta\left(\mathcal{I}_{i_0}(\mathcal{O}), \mathcal{R}\right)=0$ indicates the solution to the problem. Indeed, in the situation presented within the application, object $\mathcal{O}$ and region $\mathcal{R}$ can be abstractized through the means of compact sets, and given these hypotheses, point 1) of the proposition enunciated earlier admits the following restatement:

$$\Delta\left(\mathcal{I}_{i_0}(\mathcal{O}), \mathcal{R}\right)=0 \Leftrightarrow \mathcal{I}_{i_0}(\mathcal{O}) \subseteq \mathcal{R}.$$

Observation: This algorithm can be easily adapted to solving some similar problems in the space of three dimensions, becoming even more useful in projecting artificial intelligence forms.

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A Position Indicator With Applications In The Field of Designing Forms With Artificial Intelligence

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The indicators used so far within the Theory of Extension, see papers [1], [2] and [4], can be synthetically expressed by the notion of "position indicators". More exactly, these indicators can be grouped in two main sub-categories: point-set position indicators and point-two sets position indicators. The secondary goal of this paper is to define these classifications, while the primary goal is that to extend the two notions to the most general notion of set-set position indicator.

Key words: Hausdorff measure, Extension theory, position indicators, computer vision, artificial intelligence.

1. Introduction

The first part of this paper aims at defining the notion of point-set position indicator and that of point-two sets position indicator, at discussing the main examples of such indicators and their relevance for the applicative field.

Point-set Position Indicators

For any point $x \in \mathbb{R}^n$ and any set $A \subset \mathbb{R}^n$, formula $\delta(x,A) = \inf\{d(x,a) | a \in A\}$, where d is the Euclidean distance on $\mathbb{R}^n$, defines (in classical mathematics) the distance from point x to set A. Based on the properties characteristic for the notion of distance, every time we have $\delta(x,A) > 0$ we can conclude that point x lies outside set $\overline{A}$ (closure of set A in relation to the usual topology of space $\mathbb{R}^n$) at distance $\delta(x,A)$ from the nearest point of set A. On this

account expression $\delta(x,A)$ somehow takes the role of position indicator of x towards A, but not entirely, because in the case $\delta(x,A)=0$ the sole information provided is that $x\in A$, with no further indication of how far or how close to frontier ∂A of set A this point lies. In many concrete situations the knowledge of these details can be more useful. For example, when A symbolizes the 2-dimensional representation of a risk zone for a human (a very deep lake, a region contaminated with toxic substances etc.) and x symbolizes the position vector of a human situated in the interior of that zone, it would be useful for that human to know how close or far the exit ways are. It thus, becomes necessary the extension of the classical notion of distance by taking into account some indicators that deal with more requirements. This aspect has been pointed out by other researchers in their papers, as well, see, for example [4]. The most general point-set position indicator aiming at fulfilling the requirements formulated above is given by [2] and takes the form

$$s(x,A)=\begin{cases}\delta(x,A), & x\in \complement A\\ -\delta(x,\complement A), & x\in A\end{cases}, \qquad (1)$$

where $\complement A$ represents the absolute complement of A, i.e. $\complement A=\mathbb{R}^n\setminus A$. Given the way it has been defined, indicator $s(x,A)$ has the following properties: $x\in \complement\overline{A} \Leftrightarrow s(x,A)>0$; $x\in \overset{\circ}{A}$[18] $\Leftrightarrow$ $s(x,A)<0$; $x\in \partial A$[19] $\Leftrightarrow$ $s(x,A)=0$, respectively.

Observations: 1) The indicator defined in expression (1) does not represent a distance; it expresses the distance from x to A only when point x is exterior to set A.

2) Other examples of point-set position indicators can be found in [1] and [4].

Point-two Sets Position Indicators

Paper [2], mentioned above, provides us with an example of a point-two sets indicator, as well. This takes the expression

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[18] $\overset{\circ}{A}$ represents the interior of set A in the usual topology of space $\mathbb{R}^n$.

[19] ∂A represents the boundaries of set A, namely $\partial A=\overline{A}\setminus\overset{\circ}{A}$.

$$\mathfrak{S}(x,A,B)=\frac{s(x,A)}{s(x,B)-s(x,A)}, \tag{2}$$

where $x\in\mathbb{R}^n$, A and B are two sets from $\mathbb{R}^n$ with the property $\overline{A}\subset\overset{\circ}{B}$, and s is indicator (1). This indicator has the following properties: $\mathfrak{S}(x,A,B)<-1 \Leftrightarrow x\in\complement\overline{B}$; $-1\le\mathfrak{S}(x,A,B)<0 \Leftrightarrow x\in\overline{B}\setminus\overline{A}$; $\mathfrak{S}(x,A,B)\ge 0 \Leftrightarrow x\in\overline{A}$.

Observations: 1) The properties presented earlier justify the designation of "point – two sets position indicator" given to indicator (2).
2) Other examples of point – two sets position indicators can be found in [1, 4].

2. Set-set Position Indicators

This paragraph focuses on the presentation of new results aiming at further developing and improving the existent theory of Extension. The frame we shall refer to in our discussion is that of any metric space (X, d). The mathematical apparatus we would like to advance further on requires to take into consideration the notion of "Hausdorff measure". To ensure a better understanding of the concepts presented, we have synthetized the minimum of knowledge required in this regard in the appendix.

Let A and B be two non-empty sets from X. About set A we additionally assume that it admits a Hausdorff measure of dimension $r\ge 0$, $\mathcal{H}^r(A)$ is finite and nonzero. Under these conditions, by using indicator s defined by the generalized relation (1) from the Euclidean metric space $\mathbb{R}^n$ for the actual metric space (X, d), we are able to consider the expression

$$\mathsf{S}(A,B)=\frac{\mathcal{H}^r\left(\{a\in A\mid s(a,B)\le 0\}\right)}{\mathcal{H}^r(A)}, \tag{3}$$

which accurately defines the indicator we wanted to introduce.

This indicator fulfills several mathematical properties important to the applicative field:

Proposition 1: $\mathsf{S}(A,B)=0\Leftrightarrow A\cap B=\varnothing$ $\mathcal{H}^r$- almost everywhere (or differently expressed, $\mathcal{H}^r(A\cap B)=0$).

Demonstration: $\mathsf{S}(A,B)=0 \Leftrightarrow \mathcal{H}^r\left(\{a\in A\mid s(a,B)\le 0\}\right)=0$. Since $s(a,B)\le 0$ is equivalent to relation $a\in\overline{B}$, we deduce that $\mathcal{H}^r(A\cap B)=0$.

Proposition 2: $S(A, B)>0 \Rightarrow A \cap \overline{B} \neq \varnothing$.

Demonstration: $S(A, B)>0 \Leftrightarrow \mathcal{H}^r\left(\left\{a \in A \mid s(a, B) \leq 0\right\}\right)>0$. From relation $\mathcal{H}^r\left(\left\{a \in A \mid s(a, B) \leq 0\right\}\right)>0$ we deduce that there is $a \in A$ so that $s(a, B) \leq 0$. But $s(a, B) \leq 0$ implies that $a \in \overline{B}$, so $A \cap \overline{B} \neq \varnothing$.

Proposition 3: If besides the initial hypothesis made over set A and B, we assume additionally that set $\overline{B}$ is measurable[20] with respect to Hausdorff measure $\mathcal{H}^r$ (regarded as an outer measure on $\mathcal{P}(X)$, the family of all subsets of X), then relation $S(A, B)=1$ is equivalently with $A \subseteq \overline{B}$ $\mathcal{H}^r$ - almost everywhere.

Demonstration: $S(A, B)=1 \Leftrightarrow \mathcal{H}^r\left(\left\{a \in A \mid s(a, B) \leq 0\right\}\right)=\mathcal{H}^r(A)$. The way how indicator s has been defined implies that $\left\{a \in A \mid s(a, B) \leq 0\right\}=A \cap \overline{B}$. Because set $\overline{B}$ is measurable we have $\mathcal{H}^r(A) = \mathcal{H}^r\left(A \cap \overline{B}\right)+\mathcal{H}^r\left(A \cap \complement \overline{B}\right)$. But $\mathcal{H}^r\left(A \cap \overline{B}\right)= \mathcal{H}^r\left(\left\{a \in A \mid s(a, B) \leq 0\right\}\right) = \mathcal{H}^r(A) \Rightarrow \mathcal{H}^r\left(A \cap \complement \overline{B}\right)=0$, namely $A \subseteq \overline{B}$ $\mathcal{H}^r$ - almost everywhere.

Corollary: Let A and B be two closed nonempty sets from X for which there exists a Hausdorff measure of dimension $r \geq 0$ so that $\mathcal{H}^r(A)$ and $\mathcal{H}^r(B)$ are finite and nonzero. If sets A and B are $\mathcal{H}^r$ - measurable and if $S(A, B)=1$ and $S(B, A)=1$, then $A=B$ $\mathcal{H}^r$ - almost everywhere, and reciprocally.

Demonstration: From proposition 3 it results that $S(A, B)=1 \Leftrightarrow A \subseteq B$ $\mathcal{H}^r$ - almost everywhere and $S(B, A)=1 \Leftrightarrow B \subseteq A$ $\mathcal{H}^r$ - almost everywhere. Then $A=B$ $\mathcal{H}^r$ - almost everywhere.

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[20] By definition we say that set $\overline{B}$ is $\mathcal{H}^r$ measurable if for any $T \subseteq X$ relation $\mathcal{H}^r(T)= \mathcal{H}^r\left(T \cap \overline{B}\right)+\mathcal{H}^r\left(T \cap \complement \overline{B}\right)$ takes place.

Observations: 1) From the definition of indicator S, (relation (3)) it can be easily deduced that $0 \le \mathsf{S}(A,B) \le 1$, for any pair of non-empty subsets A and B of space X for which set A admits a Hausdorff measure $\mathcal{H}^r$ of dimension $r \ge 0$, so that $\mathcal{H}^r(A) \ne 0$ and $\mathcal{H}^r(A) < \infty$.

2) The properties presented earlier within propositions 1 - 3 aim at justifying the designation of "set-set position indicator" that indicator (3) receives.

3) Another example of set-set position indicator can be found in [3].

3. Applications

Indicator S defined by us in this paper can be used as an example for computer vision while developing software applications regarding the automatic inclusion of a certain object $\mathcal{O}$ into a target region $\mathcal{R}$ of a given video image (*VIm*). To realize this, we propose an algorithm which in broad terms has the following content: by means of a set of isometries $\mathcal{I}_i, i \in I$ of the plan, we move object $\mathcal{O}$ to different regions and positions of image *VIm* by calculating the value of indicator $\mathsf{S}(\mathcal{I}_i(\mathcal{O}), \mathcal{R})$, each time. Finding that index $i_0 \in I$ for which $\mathsf{S}(\mathcal{I}_{i_0}(\mathcal{O}), \mathcal{R}) = 1$, is equivalent to finding the solution to the problem.

Observations: 1) In some cases solution $\mathcal{I}_{i_0}$ found by using the method presented above can not be fully satisfactory because relation $\mathcal{I}_{i_0}(\mathcal{O}) \subseteq \mathcal{R}$, concerned in $\mathsf{S}(\mathcal{I}_{i_0}(\mathcal{O}), \mathcal{R}) = 1$, (see proposition 3, applied in the case when the sets by which objects $\mathcal{O}$ and $\mathcal{R}$ are being abstractized, are supposed to be compact[21]) is only guaranteed by $\mathcal{H}^r$ - almost everywhere.

2) Just like the algorithm presented in [3], this algorithm can be easily adapted to solving any similar problem in a space with three dimensions, becoming, thus, more useful to the field of designing forms with artificial intelligence.

4. Appendix

Hausdorff measure

Let (X, d) be a metric space, Y a subset from X, and δ a strictly positive real number. A finite or countable collection of sets $\{U_1, U_2, ...\}$ from X with diameter $D(U_1) \le \delta$, $D(U_2) \le \delta$,.., for which $Y \subset U_1 \cup U_2 \cup \cdots$, is called δ-cover of set Y. By virtue of this notion, for any subset Y from X and for any two real numbers $r \ge 0$, and $\delta > 0$, we can define indicator

[21] Given the application we analyze, these hypotheses are as natural as possible

$$\mathcal{H}_\delta^r(Y)=\inf_{\{U_1,U_2,\ldots\}\in\mathcal{C}_\delta(Y)}\left\{D^r(U_1)+D^r(U_2)+\cdots\right\},$$

where $\mathcal{C}_\delta(Y)=\left\{\{U_1,U_2,\ldots\}\mid \{U_1,U_2,\ldots\} \text{ is a } \delta-\text{cover of } Y\right\}$. This indicator defines a decreasing function $\delta\to\mathcal{H}_\delta^r(Y)$. This property guaranties the existence of the limit

$$\mathcal{H}^r(Y)=\lim_{\delta\to 0}\mathcal{H}_\delta^r(Y),$$

which, by definition, is called the *r*-dimensional Hausdorff measure of Y.

Among the properties of the Hausdorff measure, $\mathcal{H}^r(\cdot)$, we mention:

1) $\mathcal{H}^r(\varnothing)=0$;

2) $\mathcal{H}^r(Y_1)\le\mathcal{H}^r(Y_2)$, if $Y_1\subseteq Y_2$, $Y_1,Y_2\in\mathcal{P}(X)$;

3) $\mathcal{H}^r\left(\bigcup_{n=1}^{\infty}Y_n\right)\le\sum_{n=1}^{\infty}\mathcal{H}^r(Y_n)$, if $\left\{Y_n\mid n\in\mathbb{N}^*\right\}\subset\mathcal{P}(X)$, is any countable collection of sets.

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CViTLNN: A Hybrid Approach based on Vision Transformer and Liquid Neural Network for COVID-19 Detection

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ABSTRACT

The COVID-19 pandemic has underscored the need for accurate and rapid diagnostic tools to assist clinical decision-making. Conventional deep learning models for COVID-19 detection in Chest X-Ray (CXR) images face challenges in poor generalization across imaging conditions and high computational demands. To address these issues, this study proposes CViTLNN, a novel hybrid model combining Vision Transformers (ViTs) and Liquid Neural Networks (LNNs) to improve feature extraction and classification. Specifically, CViTLNN employs a ViT with 24 transformer encoder blocks for efficient extraction of spatial features. The self-attention mechanism of ViTs effectively captures global and local dependencies in CXR images. Furthermore, it incorporates a four-layer LNN for dynamic refinement of features for decision-making. Experimental results demonstrate a test accuracy of 94%, a precision of 95%, and a recall of 94% on a COVID dataset of 5228 CXRs, minimizing false negatives and ensuring high sensitivity. The proposed model provides an efficient and scalable AI-driven diagnostic solution, making it highly suitable for real-world clinical applications, especially in resource-constrained settings.

Keywords-COVID-19 detection; Vision Transformers (ViTs); Liquid Neural Network (LNN); chest X-ray analysis; medical imaging

I. INTRODUCTION

COVID-19 has significantly disrupted global economies, healthcare systems, and daily life, creating an urgent need for accessible and efficient diagnostic tools [1]. Traditional diagnostic methods such as RT-PCR are often slow, costly, and limited in availability, particularly in resource-constrained settings. Chest X-Ray (CXR) images are widely used to detect respiratory diseases, but their interpretation requires expert radiologists, which is a major challenge in underserved regions with limited healthcare infrastructure. Manual analysis of CXR images is time-consuming and prone to human error, underscoring the need for automated diagnostic systems to

improve both speed and accuracy [2]. Recent advances in medical imaging with the integration of deep learning have significantly improved diagnostic accuracy, especially in the context of CXR analysis for pulmonary diseases such as pneumonia, tuberculosis, cancer, and COVID-19 [3]. Several methods have been proposed for automated COVID-19 detection, including a hybrid Convolutional Neural Network (CNN) architecture combined with ResNet50 [4], a transfer learning approach using MobileNetV2 [5], a hybrid of VGG19 with DenseNet [6], a CNN-based DarkCovidNet [7], and an ensemble learning method combining InceptionV3, EfficientNet, and ResNet [8]. Additionally, hybrid CNN and LSTM [9], ResNet50 with InceptionV3 [10], EfficientNet [11], SqueezeNet [12], and Neutrosophic set-based deep transfer learning [13] have been employed with varying degrees of success. Traditional machine learning techniques, such as support vector machines and random forests, depend on manually crafted features and often struggle to handle the complexity of medical images, leading to suboptimal performance [14]. Sequence-based models, such as Recurrent Neural Networks (RNNs), excel at capturing temporal dependencies but fail to effectively model the spatial hierarchies and intricate visual patterns found in medical imaging. Deep learning architectures, particularly CNNs, have shown strong performance in medical image analysis. However, CNNs are limited by their inability to capture global contextual information, which is crucial for accurate disease detection [15]. The introduction of Vision Transformers (ViTs) has addressed this limitation, achieving high accuracy by leveraging self-attention mechanisms that can identify significant patterns across the entire image, making them highly suitable for complex computer vision tasks [16]. Moreover, Liquid Neural Networks (LNNs) offer additional advantages as they dynamically adapt their computations, enabling more flexible and efficient learning from complex data. Their ability to handle temporal and spatial variations in input data makes them ideal for medical image analysis, where conditions can change over time or vary across patients.

To the best of our knowledge, this is the first study to explore the combined potential of ViTs and LNNs for COVID-19 detection, introducing a novel hybrid approach that integrates the strengths of both architectures. The primary contributions of this work are as follows:

- Introduces a ViT architecture incorporating 24 encoder blocks to extract deep features from chest X-ray images, leveraging self-attention mechanisms to capture both local and global representations for COVID-19 detection.
- Builts a liquid neural network consisting of four layers to refine extracted features, enabling dynamic adaptation to input variability and enhancing the model's ability to learn intricate patterns for accurate and efficient classification.
- Finally, develops CViTLNN, a novel hybrid architecture that combines vision transformer-based deep feature extraction with the adaptability of liquid neural networks to achieve automated COVID-19 detection, specifically designed for resource-constrained environments and underserved regions.

II. MATERIALS AND METHOD

Figure 1 illustrates the proposed framework. CViTLNN integrates a ViT and an LNN to address both spatial and temporal complexities in medical image analysis. The process begins with the input CXR image, which undergoes preprocessing steps, such as downscaling, normalization, random rotations, flips, and cropping. The image is then divided into patches (16×16), and each is embedded in a higher-dimensional space, along with position embeddings to preserve spatial relationships. The sequence length of 197 patches is processed, including the addition of a classification (CLS) token that helps in the final decision-making process. The ViT component consists of 24 transformer encoder blocks that leverage multihead self-attention mechanisms to capture both local and global dependencies within the image. This allows the model to recognize fine-grained details throughout the image, which is essential to distinguish complex classes, such as COVID-19, pneumonia, and normal cases in CXR. The attention mechanism examines the relationships between all parts of the image, enabling the model to recognize patterns that would be difficult for traditional CNNs to capture. After extracting ViT features, the four-layer LNN component refines the learned features, adapting its processing dynamically based on input variations. The LNN is designed to handle complex and evolving data, enhancing the robustness and adaptability of the model. Finally, a classifier layer with linear layers, dropout, GELU activation, and normalization processes the refined features, making predictions about the presence of COVID-19, normal conditions, or pneumonia in the CXR image.

A. Dataset

This study utilized the publicly available COVID dataset [17], which contains a large number of annotated CXR images. This dataset was chosen due to its comprehensive and diverse image collections. The data are categorized into three classes: COVID-19, pneumonia, and non-pneumonia (normal) cases. The COVID-19 images were specifically sourced from datasets dedicated to COVID-19, while pneumonia and normal cases were randomly selected from standard CXR datasets. The dataset includes 1626 COVID-19 images, 1800 pneumonia images, and 1802 normal images, ensuring a broad and varied representation of cases. This distribution enhances the model's ability to distinguish COVID-19 from other respiratory diseases, reflecting real-world clinical scenarios. Additionally, the dataset includes patients across different age groups and genders to reduce potential biases during the training phase, promoting more robust and generalizable model performance.

B. Preprocessing Pipeline

Several preprocessing procedures were applied to the dataset to stabilize the input to the ViT model. All images were scaled down to a standard patch embedding dimension, normally to a measure of 224×224 as used in the blue picture. Normalization was applied to bring the pixel intensity values in the images into a common range to have standard input for the model and increase training robustness. Methods such as random rotations, flips, and cropping were used to improve data variety and reduce the risk of overfitting.

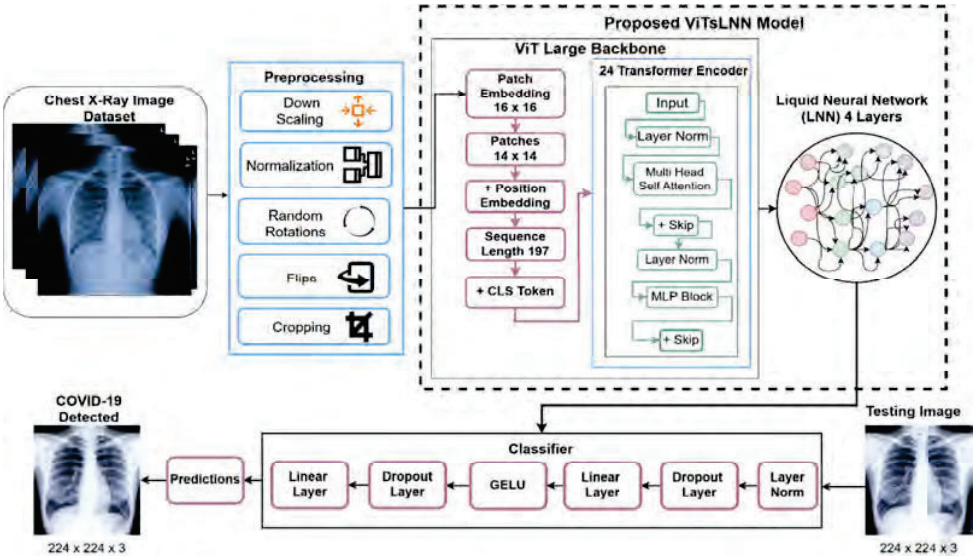


Fig. 1. The framework of the proposed approach (CViTLNN) for COVID-19 detection.

C. *Patch Generation Process*

Figure 2 shows the patch generation process for a CXR input image with dimensions 224×224×3. This process begins by dividing the input image into smaller square patches of fixed size, such as 16×16 pixels. Given the image dimensions (224×224), it can be divided into a grid of 14 patches along the height and 14 patches along the width, resulting in a total of 14×14 = 196 patches. Each patch retains the three RGB color channels. Each patch, initially a 16×16×3 tensor, is flattened into a one-dimensional vector of size 768 (calculated as 16×16×3). These flattened patch vectors are then passed through a learnable embedding layer, which transforms each vector into a fixed-size embedding to match the model's input requirements.

D. *Integrated Vision Transformer (ViT) Backbone and Liquid Neural Network Model (ViTLNN)*

Figure 3 shows the ViTLNN process, which begins by dividing the input image into fixed-size patches that are flattened into vectors (encoded patches, $X1$, $X2$, ..., $X196$). A Class (CLS) token is added to the sequence for the global image summary. These encoded patches and the CLS token are input into the transformer encoder with positional encodings that preserve spatial relationships. The ViT processes these embeddings by passing them through transformer layers with self-attention mechanisms, resulting in a feature tensor of shape (batch_size, seq_length, embed_size). The LNN further refines these features. The LiquidLayer function iteratively updates the features by generating a time embedding and applying a state network to refine the current state. The refined features are normalized after each update. This process produces an output tensor of the same shape as the input. Finally, the classification head extracts the CLS token from the refined features, normalizes it, and passes it through a feedforward network to produce classification logits, which represent the final predictions. The main workflow in Figure 1 shows this sequence, using ViT for initial feature extraction, LNN for dynamic refinement, and the Classification Head to output predictions. This hybrid design combines ViT's spatial reasoning with LNN dynamic adaptability, enhancing the model's ability to make accurate classifications.

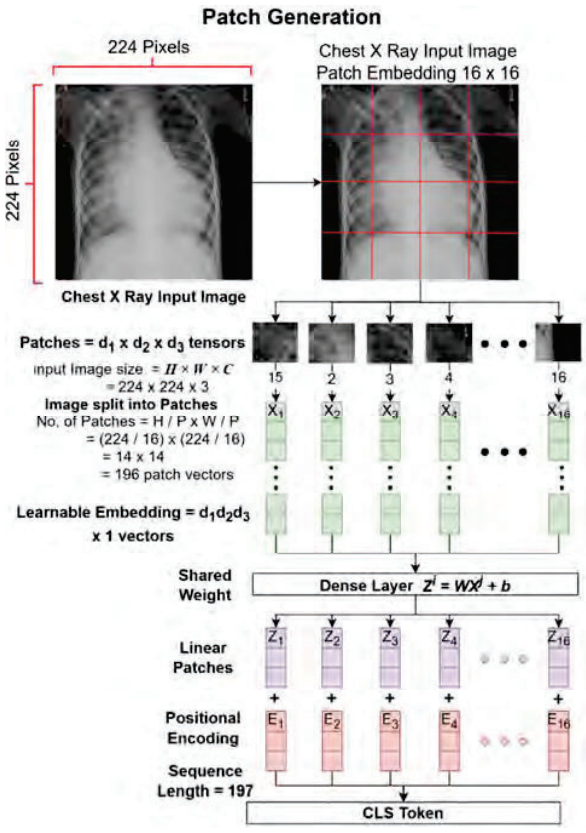


Fig. 2. Step-by-step illustration of the patch generation process.

TABLE I. DATASET DISTRIBUTION BY TYPE AND TOTAL SAMPLES

Dataset type	COVID-19 cases	Pneumonia cases	Normal cases	Total samples
Training Set	1300	1440	1441	4181
Validation Set	164	180	181	522
Testing Set	162	180	180	525
Total	1626	1800	1802	5228

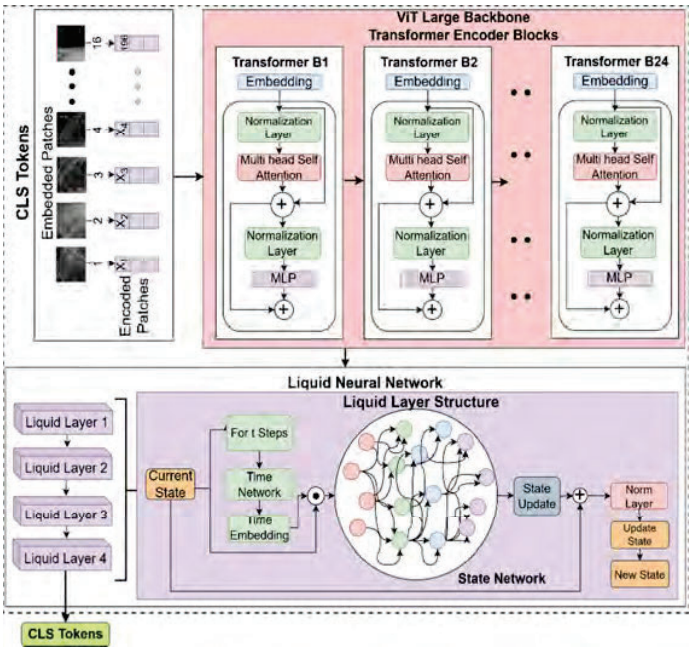


Fig. 3. The architecture of hybrid ViT and LLN.

E. Transformer Encoder Blocks

The ViTs backbone in Figure 3 consists of 24 transformer encoder blocks, each designed to handle deeper aspects of the features. The Multi-head Self-Attention (MHSA) mechanism calculates relationships between all patches of the image simultaneously, capturing both local and global context. This enables the model to understand complex dependencies across the entire image, unlike traditional convolutional methods that focus primarily on local features. Each block also contains Feedforward Networks (MLP), which consist of two linear layers with a GELU activation function. The first layer increases the input dimensions to capture more intricate patterns, while the second layer reduces the dimensions back to the original size, refining the features extracted by MHSA. Layer Normalization (LayerNorm) is applied to stabilize training and improve convergence, normalizing the input across feature dimensions to mitigate issues like exploding or vanishing gradients. Additionally, residual connections are used to allow information to bypass certain layers, preserving original features and enhancing performance. After processing through all 24 blocks, the CLS token contains a condensed representation of the image, which is then used for downstream tasks such as classification.

F. Experimental Settings

All experiments were executed on an RTX 4060 GPU with 32GB RAM and an Intel i9 13th generation processor. The model was trained for 100 epochs using the Adam optimizer with a learning rate of 0.0001. The batch size was set to 32, and a dropout rate of 0.1 was applied to prevent overfitting during training. The GELU activation function was used in the feed-forward layers, and the positional encoding type was learnable, allowing the model to retain spatial information in the input patches. The cross-entropy loss function with label smoothing was used to train the model. This loss function is well-suited for multiclass classification problems and combines the softmax operation with the negative log-likelihood of the predicted probabilities. To prevent the model from becoming overly confident in its predictions and to improve generalization, label smoothing was applied with a smoothing factor of 0.1. This means that the true class label is assigned a probability of 0.9, and the remaining 0.1 is evenly distributed among the other classes.

G. Evaluation Metrics

Well-known metrics, namely accuracy, precision, and recall, were employed to assess the performance of the proposed approach.

$$Accuracy = \frac{TP+TN}{TP+TN+FP+FN} \quad (1)$$

$$Precision = \frac{TP}{TP+FP} \quad (2)$$

$$Recall = \frac{TP}{TP+FP} \quad (3)$$

where TP, TN, FP, and FN denote true positives, true negative, false positive, and false negative predictions, respectively.

III. RESULTS AND DISCUSSION

The proposed approach achieved an overall classification accuracy of 94%, a precision of 95%, and a recall of 94%. Figure 4 shows the progression of precision and recall over 100 epochs of training for the proposed model. Both metrics show consistent improvement, with precision (blue line) and recall (red line) steadily increasing throughout the training process. Early epochs exhibit lower values, but both precision and recall stabilize near 0.95 after 100 epochs, indicating that the model has effectively learned to differentiate between classes. The close alignment of the precision and recall curves suggests balanced performance, with the model maintaining high accuracy in both correctly identifying positive instances (precision) and detecting all positive cases (recall).

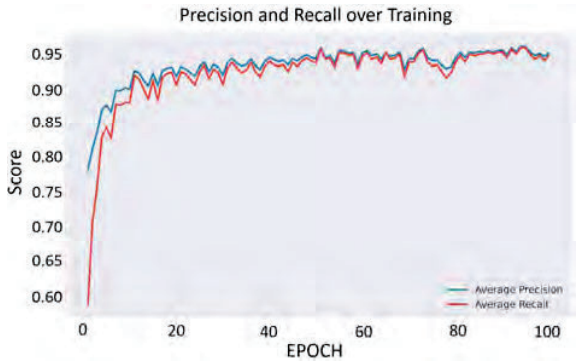


Fig. 4. Precision and recall curves over 100 epochs.

Figure 5 shows the training, validation, and test accuracy curves of the proposed approach across 100 epochs. The blue curve represents the training accuracy, the red curve represents the validation accuracy, and the green curve represents the test accuracy. At the beginning of training, all accuracy curves start relatively low. As training progresses, the training accuracy steadily increases, surpassing 90% by epoch 30, and then stabilizes at a high level close to 97%. This rapid improvement in training accuracy indicates that the model is effectively learning and fitting the training data. The validation and test accuracy curves show a similar trend, but their growth is slightly more gradual than the training accuracy. Both curves follow the same pattern, with validation accuracy (red) and test accuracy (green) nearing 90% toward the end of the training. The proximity of the three curves suggests that the model is not overfitting, as the validation and test accuracies remain close to the training accuracy, indicating generalization to unseen data. Overall, the results show that the CViTLNN approach achieves robust performance across all datasets (training, validation, and test), with the test accuracy stabilizing at 93.76%, which reflects the model's ability to generalize well while avoiding overfitting.

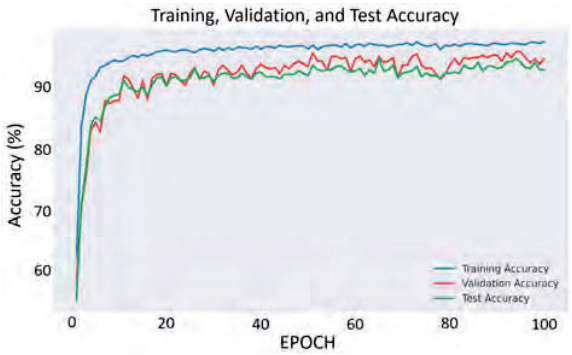


Fig. 5. Training, validation, and test accuracy over 100 epochs.

Figure 6 shows the confusion matrix of the CViTLNN predictions for three classes: COVID, Normal, and Pneumonia. The diagonal elements of the matrix indicate the number of correct predictions, with 162 COVID, 177 Normal, and 160 Pneumonia instances correctly classified. The off-diagonal elements reveal misclassifications.

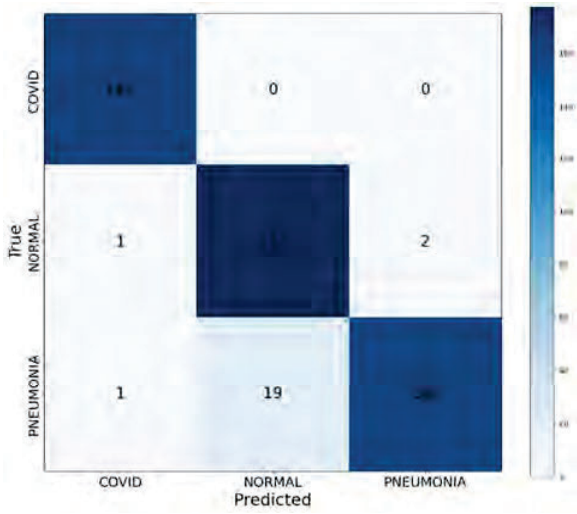


Fig. 6. Confusion matrix of CViTLNN predictions, with correct classifications on the diagonal and misclassifications off it.

Table II presents a comparison of the proposed approach with existing methods, including hybrid CNN-ResNet50, transfer learning with MobileNetV2, VGG19-DenseNet, DarkCovidNet-CNN, ensemble approach using InceptionV3, Xception, and ResNet, CNN-LSTM, Capsule Network (CapsNet), Efficient Net, Squeeze Net and CNN on wavelet transform, all focusing on the detection of COVID-19, pneumonia, and normal cases. The CViTLNN model delivers state-of-the-art performance, achieving a test accuracy of 94%, a precision of 95%, and a recall of 94%, surpassing most existing approaches.

Neutrosophic Signed Domination Function of Graphs

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Abstract

This paper introduces the novel concept of a Neutrosophic Signed Domination Function (NSDF) of graphs, generalizing classical domination by assigning each vertex a triple-valued influences (truth, indeterminacy, falsity) from $\{-1,\ 0,\ 1\}$. We define the Neutrosophic Signed Domination Number $\gamma_{ns}(G)$ as the optimal weighted sum under neighborhood constraints ensuring net positive influence. Fundamental properties and sharp bounds for general graphs are established. Exact values for $\gamma_{ns}(G)$ are determined for paths and cycles. This work bridges neutrosophic logic with domination theory, enabling sophisticated modeling of complex networks with uncertainty.

Keywords: Graph domination; Neutrosophic graphs; Neutrosophic domination; Signed domination; Neutrosophic signed domination

1 Introduction

Let G be a graph with m edges and n vertices. The open neighborhood of a vertex consists of all adjacent vertices, while its closed neighborhood contains both the vertex itself and its adjacent vertices. Basic notation and results can be found in.[7]

The signed domination concept was first introduced by Dunbar et al.[10] in 1995. For a graph $G = (V,\ E)$, a *signed dominating function* is defined as a vertex labeling $f : V \to \{-1,\ +1\}$ satisfying the condition that for every vertex $v \in V$, the sum of labels in its closed neighborhood satisfies:

$$f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1$$

The *signed domination number* of G, denoted $\gamma_s(G)$, represents the minimum possible weight $\sum_{v \in V} f(v)$ among all such valid functions. This fundamental concept has been extensively studied by various researchers, including,[8],[9],[11],[12],[14],[15],[16] and,[25] who have contributed significant results and extensions to the theory of signed domination.

Models of classical graph theory frequently function in a binary environment, where a vertex is either dominated (1) or not dominated (0). This was enhanced by signed domination, which is more suited for networks with competing forces (such as allies and adversaries) by adding positive (+1) and negative (-1) impacts. However, these models still fall short in handling the pervasive uncertainty and indeterminacy of real-world systems. This is where the motivation for a neutrosophic model comes from:

As a generalization of fuzzy logic, neutrosophic logic recognizes that there are three separate parts to any proposition: falsity (F), indeterminacy (I), and truth (T). This perspective of reality is more sophisticated.

In 1999, the foundational concepts were established by Florentin Smarandache introduced neutrosophic logic.[20] In 2018, Mohammad Akram studied neutrosophic graphs.[5] Numerous writers have examined the neutrosophic concept, including.[1,2,6,17,19]

Neutrosophic graph theory represents an advanced framework that generalizes both fuzzy graphs and intuitionistic fuzzy graphs. This model was developed to capture three fundamental aspects of network relationships:

- Truth membership (T) - representing confirmed or certain connections
- Indeterminacy membership (I) - capturing uncertain or ambiguous relations
- Falsity membership (F) - accounting for confirmed non-connections or contradictions

Given an underlying graph $G = (V, E)$, we define its *neutrosophic extension* $\mathcal{G} = (A, B)$ as follows:

- The vertex set is equipped with a neutrosophic structure $A = (T_A, I_A, F_A)$, where:
  - $T_A : V \to [0, 1]$ quantifies the degree of certainty for each vertex
  - $I_A : V \to [0, 1]$ measures the indeterminacy associated with each vertex
  - $F_A : V \to [0, 1]$ indicates the degree of falsity for each vertex
- The edge relations are characterized by $B = (T_B, I_B, F_B)$ with:
  - $T_B : E \to [0, 1]$ constrained by $T_B(uv) \leq \min(T_A(u), T_A(v))$
  - $I_B : E \to [0, 1]$ bounded by $I_B(uv) \leq \min(I_A(u), I_A(v))$
  - $F_B : E \to [0, 1]$ limited by $F_B(uv) \geq \max(F_A(u), F_A(v))$

Recent years have seen significant growth in neutrosophic graph theory across several important fi elds.[22] developed the basic theory by presenting and examining regularity in plithogenic graphs, which are an extension of neutrosophic graphs. Contributing to spectral theory,[3] extended the notion to directed networks by defining the energy of a neutrosophic digraph. A significant emphasis has been on practical applications, w ith[21] creating complex neutrosophic graphs for multi-attribute decision-making problems like supplier selection and[4] using single-valued neutrosophic graphs for influential node detection in social network a nalysis. In order to tackle com

In NSDF, the degrees of truth, indeterminacy, and falsity are independent. A node can have a high truth value and a high indeterminacy value simultaneously (e.g., a highly positive but very unpredictable actor). This is a more powerful representation than probabilistic or fuzzy models where membership degrees are often dependent.

Application in Social Network Analysis and Opinion Dynamics

Modeling the spread of opinions or influence in a social network (e.g., regarding a political issue, a new product, or a public health policy). Vertices (V): Individuals or groups in the network. Edges (E): Social connections (friendship, followership, $f_T(v)$ Degree of positive support/agreement (+1 = strong supporter, 0 = neutral, -1 = active opposition). $f_I(v)$
Degree of uncertainty or indecision. $f_F(v)$ Degree of negative stance/disagreement.

Application in Cybersecurity and Trust Networks

Securing a distributed network (e.g., sensor network, peer-to-peer network) where nodes can be trustworthy, malicious, or compromised (uncertain). Vertices (V): Devices or agents in the network. Edges (E) : Communication links. $f_T(v)$: Trustworthiness score (+1 = verified secure, -1 = known malicious). $f_I(v)$: Uncertainty about the node's state. $f_F(v)$: Level of malicious activity or distrust

The domination conditions can be seen as a security protocol. For every node, the collective trustworthiness of its neighbors must meet a minimum threshold ($\sum f_T \geq 1$), and the collective negative influence must be $\sum f_F \leq 1$). This ensures that no node is isolated in a cluster of malicious or uncertain actors.

Application in Economic and Financial Networks

Analyzing systemic risk in a financial network where institutions (banks, funds) are interconnected by liabilities. Vertices (V): Financial institutions. Edges (E): Financial exposure (e.g., loans, derivatives). $f_T(v)$: Financial health/solvency (+1 = very healthy, -1 = insolvent).
$f_I(v)$: Uncertainty or opacity of the institution's true state. $f_F(v)$: Level of financial distress.

The condition $\sum_{u \in N[v]} f_T(u) \geq 1$ can be interpreted as a requirement for **local stability**: the combined health of an institution and its direct partners must be above a critical threshold to prevent a local failure from cascading.

This work presents a novel Neutrosophic Signed Domination Function(NSDF) for graphs, defined as a triple mapping $f = (f_T, f_I, f_F) : V \to [-1, 0, 1]^3$ where each vertex $v \in V$ is assigned.

- A truth component $f_T(v) \in [-1, 0, 1]$ representing positive influence
- An indeterminacy component $f_I(v) \in [-1, 0, 1]$ capturing neutral or uncertain influence
- A falsity component $f_F(v) \in [-1, 0, 1]$ indicating negative influence

The function must satisfy the following neighborhood conditions for every vertex $v \in V$

$$\sum_{u \in N[v]} f_T(u) \geq 1 \quad \text{(Truth domination)}$$
$$\sum_{u \in N[v]} f_I(u) \geq 0 \quad \text{(Indeterminacy threshold)}$$
$$\sum_{u \in N[v]} f_F(u) \leq 1 \quad \text{(Falsity constraint)}$$

The Neutrosophic Signed Domination Number(NSDN) $\gamma_{ns}(G)$ is defined as the minimum weighted sum is

$$\gamma_{ns}(G) = \min_f \left(w_T \sum_{v \in V} f_T(v) + w_I \sum_{v \in V} f_I(v) + w_F \sum_{v \in V} f_F(v) \right)$$

where $w_T, w_I, w_F \geq 0$ are importance weights for each influence type.

NSDF extend classical domination by incorporating truth (T), indeterminacy (I), and falsity (F) states to better model real-world uncertainty. They overcome limitations of binary domination in handling partial information and conflicting influences. NSDF provides a flexible framework for analyzing networks where nodes exhibit neutral or ambiguous behaviors. This generalization enables more robust applications in social networks, cybersecurity, and decision-making systems with incomplete information.

In section 2 included structural characterizations, sharp bounds for general graphs, and basic features of NSDF of graphs. We determine the NSDN for pathways and cycles in section 3. The work opens new research directions in neutrosophic graph theory and its applications to complex network analysis.

2 NSDF of graphs

The basic properties of NSDF of graphs were discussed in this section, and we discovered structural characterizations and sharp bounds for general graphs.

Lemma 2.1. *If G is a graph containing a vertex of degree k. Then $\gamma_{ns}(G) > (2 + k - n)(w_T + w_I - w_F)$*

Proof. Consider a vertex $v \in V$ with $\deg(v) = k$. Let $f = (f_T, f_I, f_F)$ be a NSDF on G.For the closed neighborhood $N[v]$, we must have $\sum_{u\in N[v]} f_T(u) \geq 1, \quad \sum_{u\in N[v]} f_I(u) \geq 0, \quad \sum_{u\in N[v]} f_F(u) \leq 1$. The minimal weight for f will now achieved if $f_T(u) = -1$, $f_I(u) = -1$, $f_F(u) = 1$ for $u \in N[v]$ and $f_T(u) = 1$, $f_I(u) = 1$, $f_F(u) = -1$ for $u \notin N[v]$. Thus

$$\begin{aligned}\gamma_{ns}(G) &= \sum_{u\in V}[w_T f_T(u) + w_I f_I(u) + w_F f_F(u)] \\ &= (k+1)(-w_T - w_I + w_F) + (n-k-1)(w_T + w_I - w_F) \\ &= (2+k-n)(w_T + w_I - w_F)\end{aligned}$$

Since any valid NSDF must satisfy stricter constraints than this extremal case, we obtain the strict inequality. □

Theorem 2.2. *If f denote any NSDF of the graph $G = (V, E)$ with $f = (f_T, f_I, f_F)$ and let $P_T = \{v \in V \mid f_T(v) > 0\}$, $M_T = \{v \in V \mid f_T(v) \leq 0\}$, $p = |P_T|$, $n = |V|$ and $o(G)$ = number of odd-degree vertices in G. Then $\sum_{v\in P_T} d_G(v) \geq \sum_{v\in M_T} d_G(v) + 2(n-p) + o(G)$, $\sum_{v\in P_T} d_{G[P_T]}(v) \geq \sum_{v\in P_T} \left\lfloor \frac{d_G(v)}{2} \right\rfloor$ where $d_G(v)$ is the degree of v in G and $G[P_T]$ is the subgraph induced by P_T.*

Proof. For any NSDF $f = (f_T, f_I, f_F)$, consider the truth component f_T as a NSDF. For $v \in P_T$.

$$\begin{aligned}\sum_{u\in N[v]} f_T(u) \geq 1 = f_T(v) + \sum_{u\in N(v)} f_T(u) &\geq 1 \\ &= 1 + \sum_{u\in N(v)} f_T(u) \geq 1 \quad (\text{since } f_T(v) > 0) \\ &= \sum_{u\in N(v)} f_T(u) \geq 0\end{aligned}$$

For $v \in M_T$:

$$\begin{aligned}\sum_{u\in N[v]} f_T(u) \geq 1 = f_T(v) + \sum_{u\in N(v)} f_T(u) &\geq 1 \\ &= 0 + \sum_{u\in N(v)} f_T(u) \geq 1 \quad (\text{since } f_T(v) \leq 0) \\ &= w \sum_{u\in N(v)} f_T(u) \geq 1\end{aligned}$$

Following the same edge-counting argument as in given bellow and we obtain

$$\sum_{v\in P_T} d_G(v) \geq \sum_{v\in M_T} d_G(v) + 2(n-p) + o(G)$$

For $v \in P_T$, the condition $\sum_{u\in N[v]} f_T(u) \geq 1$ implies that v must have sufficient positive influence in its neighborhood. Considering only the subgraph $G[P_T]$:

$$d_{G[P_T]}(v) = |\{u \in P_T \mid uv \in E\}| \geq \left\lfloor \frac{d_G(v)}{2} \right\rfloor$$

Summing over all $v \in P_T$ gives the second inequality. □

Theorem 2.3. *Let G be a graph of order n, size m, maximum degree Δ, and minimum degree δ. Let $\gamma_{ns}(G)$ be the NSDFN of G with weights $w_T, w_I, w_F \geq 0$. Then, the following bounds hold:*

1. $\gamma_{ns}(G) \leq \frac{(\Delta^*+2-\delta)n+2o(G)}{\Delta^*+2+\delta}$
2. $\gamma_{ns}(G) \leq \frac{2m+2n+o(G)}{\delta+1} - n$
3. $\gamma_{ns}(G) \leq n - \frac{2m-o(G)}{\Delta^*+1}$
4. $\gamma_{ns}(G) \leq 2\left\lfloor \frac{-\Delta^*+\sqrt{\Delta^{*2}+8(\Delta^*+2)n+8o(G)}}{4} \right\rfloor - n$
5. $\gamma_{ns}(G) \leq 2\left\lfloor \frac{1+\sqrt{1+8(m+n)+4o(G)}}{4} \right\rfloor - n$. *Where* $\Delta^* = \max_{v\in V} \sum_{u\in N[v]} |f_T(u)|$ $o(G) = o$.

Proof. Let $f = (f_T, f_I, f_F)$ be an optimal NSDF for G with $\gamma_{ns}(G) = \sum_{v \in V} (w_T f_T(v) + w_I f_I(v) + w_F f_F(v))$. Let $P = \{v \in V \mid f_T(v) > 0\}$ (truth-dominated vertices), $M = \{v \in V \mid f_F(v) < 0\}$ (falsity-dominated vertices), $p = |P|$, $q = |M|$.

Proof of Bound (1): Degree-Based Bound
From the *truth constraint* $\sum_{u \in N[v]} f_T(u) \geq 1$, summing over all $v \in V$:

$$\sum_{v \in V} \sum_{u \in N[v]} f_T(u) \geq n.$$

By double-counting:

$$\sum_{v \in P} d_G(v) + 2p \geq \sum_{v \in M} d_G(v) + 2(n - p) + o(G).$$

Using $d_G(v) \leq \Delta^*$ for $v \in M$ and $d_G(v) \geq \delta$:

$$p\Delta^* + 2p \geq (n - p)\delta + 2(n - p) + o(G).$$

Solving for p:

$$p \geq \frac{(\delta + 2)n + o(G)}{\Delta^* + \delta + 2}.$$

Since $\gamma_{ns}(G) \leq 2p - n$, we obtain:

$$\gamma_{ns}(G) \leq \frac{(\Delta^* + 2 - \delta)n + 2o(G)}{\Delta^* + 2 + \delta}.$$

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Proof of Bound (2): Size-Based Bound
From the *truth constraint* and double-counting:

$$2 \sum_{v \in P} d_G(v) \leq 2m + 2(n - p) + o(G).$$

Using $\sum_{v \in P} d_G(v) \geq p\delta$:

$$2p\delta \leq 2m + 2(n - p) + o(G).$$

Solving for p:

$$p \leq \frac{2m + 2n + o(G)}{2(\delta + 1)}.$$

Thus:

$$\gamma_{ns}(G) \leq \frac{2m + 2n + o(G)}{\delta + 1} - n.$$

Proof of Bound (3): Max-Degree Bound
From the *falsity constraint* $\sum_{u \in N[v]} f_F(u) \leq 1$, summing over $v \in V$:

$$\sum_{v \in V} \sum_{u \in N[v]} f_F(u) \leq n.$$

Double-counting gives:

$$\sum_{v \in M} d_G(v) + 2q \leq \sum_{v \in P} d_G(v) + 2(n - q) - o(G).$$

Using $d_G(v) \leq \Delta^*$ for $v \in M$:

$$2m \leq 2\Delta^* q + 2(n - q) - o(G).$$

Solving for q:

$$q \geq \frac{2m - 2n + o(G)}{2(\Delta^* - 1)}.$$

Since $\gamma_{ns}(G) \geq n - 2q$, we obtain:

$$\gamma_{ns}(G) \leq n - \frac{2m - o(G)}{\Delta^* + 1}.$$

Proof of Bound (4): Quadratic Truth-Dominated Bound
Consider the subgraph $G[P]$ induced by truth-dominated vertices P. From Lemma 1, we have:

$$\sum_{v \in P} d_{G[P]}(v) \leq \sum_{v \in P} \left\lfloor \frac{d_G(v)}{2} \right\rfloor.$$

Since $G[P]$ is simple, $G[P]$ has at most $\frac{p(p-1)}{2}$ edges, thus:

$$\sum_{v \in P} d_{G[P]}(v) \leq p(p-1).$$

Combining these:

$$p(p-1) \leq \sum_{v \in P} \left\lfloor \frac{d_G(v)}{2} \right\rfloor \leq \frac{1}{2} \sum_{v \in P} d_G(v).$$

From the truth constraint (as in Bound (1)):

$$\sum_{v \in P} d_G(v) \leq \sum_{v \in M} d_G(v) + 2(n-p) + o(G) \leq \Delta^*(n-p) + 2(n-p) + o(G).$$

Thus:

$$2p(p-1) \leq \Delta^*(n-p) + 2(n-p) + o(G).$$

Rearranging gives the quadratic inequality:

$$2p^2 + (\Delta^* + 2)p - (\Delta^* + 2)n - o(G) \leq 0.$$

Solving for p using the quadratic formula:

$$p \leq \left\lfloor \frac{-(\Delta^* + 2) + \sqrt{(\Delta^* + 2)^2 + 8(\Delta^* + 2)n + 8o(G)}}{4} \right\rfloor.$$

Substituting $\gamma_{ns}(G) = 2p - n$ yields the bound:

$$\gamma_{ns}(G) \leq 2 \left\lfloor \frac{-\Delta^* + \sqrt{\Delta^{*2} + 8(\Delta^* + 2)n + 8o(G)}}{4} \right\rfloor - n.$$

Proof of Bound (5): General Quadratic Bound
From Bound (2), we have:

$$\sum_{v \in P} d_G(v) \leq m + (n-p) + \frac{o(G)}{2}.$$

From the edge-counting argument in Bound (4):

$$2p(p-1) \leq \sum_{v \in P} d_G(v).$$

Combining these:

$$2p(p-1) \leq m + (n-p) + \frac{o(G)}{2}.$$

Rearranging gives:

$$2p^2 - p - m - n - \frac{o(G)}{2} \leq 0.$$

Solving the quadratic inequality for p:

$$p \leq \left\lfloor \frac{1+\sqrt{1+8(m+n)+4o(G)}}{4} \right\rfloor.$$

Thus:

$$\gamma_{ns}(G) \leq 2 \left\lfloor \frac{1+\sqrt{1+8(m+n)+4o(G)}}{4} \right\rfloor - n.$$

Hence complete the proof.

□

Theorem 2.4. *For any graph G of order n, the NSDN is $\gamma_{ns}(G) \geq 2 \left\lfloor \frac{-1+\sqrt{1+8n}}{2} \right\rfloor - n$ and this bound is sharp.*

Proof. Let $f = (f_T, f_I, f_F)$ be an optimal NSDF for G, $A = \{v \in V(G) \mid f_T(v) > 0\}$ (truth-dominated vertices), and $B = V(G) \backslash A$ (remaining vertices). Let $|A| = t$, then $|B| = n-t$ and $\gamma_{ns}(G) = \sum_{v \in A} f_T(v) - \sum_{v \in B} |f_F(v)| \geq 2t - n$ (since $f_T(v) \leq 1$, $f_F(v) \geq -1$).

Step 1: Neighborhood Constraints For every vertex $u \in B$, the truth constraint requires:

$$\sum_{w \in N[u]} f_T(w) \geq 1 \implies |N[u] \cap A| \geq 2.$$

Thus, the number of edges between A and B satisfies:

$$|E(A,B)| \geq 2(n-t).$$

This implies at least one vertex $v \in A$ has degree:

$$d_B(v) \geq \left\lceil \frac{2(n-t)}{t} \right\rceil.$$

Consequently, its closed neighborhood satisfies:

$$|N[v] \cap A| \geq 1 + \left\lceil \frac{2(n-t)}{t} \right\rceil.$$

Step 2: Solving the Inequality Since $|A| = t \geq |N[v] \cap A|$, we derive:

$$t \geq 1 + \frac{2(n-t)}{t}.$$

Rearranging yields the quadratic inequality:

$$t^2 + t - 2n \geq 0.$$

Solving for t:

$$t \geq \frac{-1+\sqrt{1+8n}}{2}.$$

As t must be an integer:

$$t \geq \left\lfloor \frac{-1+\sqrt{1+8n}}{2} \right\rfloor.$$

Substituting into $\gamma_{ns}(G) \geq 2t - n$ gives the bound.

Sharpness Construction: For any $n \geq 1$, we exhibit a graph achieving equality:

- **Case** $n = 1, 2$: Take $G = K_n$ with $f_T(v) = 1$ for all v.

- **Case** $n \geq 3$: Let $t = \left\lfloor \frac{-1+\sqrt{1+8n}}{2} \right\rfloor$. Construct G by:
  1. Starting with $K_t \cup K_{n-t}$,
  2. Connecting each vertex in K_{n-t} to exactly 2 vertices in K_t,
  3. Ensuring no pair in K_t shares more than one common neighbor from K_{n-t}.

  Define f by:
  $$f_T(v) = \begin{cases} 1 & \text{if } v \in V(K_t), \\ -1 & \text{if } v \in V(K_{n-t}). \end{cases}$$
  Then f is an NSDF with $\gamma_{ns}(G) = 2t - n$, matching the bound.

□

Theorem 2.5. *For any graph G of order n and size m, the NSDN satisfies $\gamma_{ns}(G) \geq n - \frac{2}{3}m$.*

Proof. Let $f = (f_T, f_I, f_F)$ be an optimal NSDF for G, $A = \{v \in V \mid f_T(v) > 0\}$ (truth-dominated vertices), and $B = V \setminus A$ (remaining vertices). Let $|A| = t$, then $|B| = n - t$ and $\gamma_{ns}(G) \geq 2t - n$ (since $f_T(v) \leq 1$ and $f_F(v) \geq -1$).

Step 1: Edge Counting For every vertex $u \in B$, the truth constraint requires:
$$\sum_{w \in N[u]} f_T(w) \geq 1 \implies |N[u] \cap A| \geq 2.$$
Thus, the number of edges between A and B satisfies:
$$|E(A, B)| \geq 2(n - t).$$

Step 2: Degree Analysis in A For any $v \in A$, the domination constraint implies:
$$\sum_{u \in N[v]} f_T(u) \geq 1 \implies d_A(v) \geq d_B(v),$$
where $d_A(v)$ and $d_B(v)$ denote the number of neighbors of v in A and B respectively. Therefore:
$$2|E(G[A])| = \sum_{v \in A} d_A(v) \geq \sum_{v \in A} d_B(v) = |E(A, B)| \geq 2(n - t).$$

Step 3: Total Edge Bound The total number of edges satisfies:
$$m \geq |E(G[A])| + |E(A, B)| \geq (n - t) + 2(n - t) = 3(n - t).$$
Solving for t:
$$t \geq n - \frac{m}{3}.$$
Substituting into $\gamma_{ns}(G) \geq 2t - n$ yields:
$$\gamma_{ns}(G) \geq 2\left(n - \frac{m}{3}\right) - n = n - \frac{2}{3}m.$$

□

Theorem 2.6. *For any graph G of order n with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, the NSDFN satisfies:*
$$\gamma_{ns}(G) \geq \frac{\delta(G) + 2 - \Delta(G)}{\delta(G) + 2 + \Delta(G)} n.$$

Proof. Let $f = (f_T, f_I, f_F)$ be an optimal NSDF for G. Define $A = \{v \in V(G) \mid f_T(v) > 0\}$ (truth-dominated vertices), $B = V(G) \setminus A$ (remaining vertices). Let $|A| = t$, then $|B| = n - t$ and $\gamma_{ns}(G) = 2t - n$.

Step 1: Neighborhood Constraints For any vertex $v \in V(G)$, the NSDF constraints require:

$$\sum_{u \in N[v]} f_T(u) \geq 1 \implies |N[v] \cap A| \geq \left\lceil \frac{\delta(G)}{2} \right\rceil + 1.$$

This follows because:

$$|N[v] \cap A| - |N[v] \cap B| \geq 1,$$
$$|N[v] \cap A| + |N[v] \cap B| = d(v) + 1 \geq \delta(G) + 1.$$

Step 2: Edge Counting For vertices in B, we have:

$$\sum_{v \in B} |N[v] \cap A| = |E(A, B)| + |B| \geq \left(\left\lfloor \frac{\delta(G)}{2} \right\rfloor + 1 \right) (n - t).$$

Thus, there exists $u \in A$ with at least $\left\lceil \frac{(\lfloor \delta(G)/2 \rfloor + 1)(n-t)}{t} \right\rceil$ neighbors in B.

Step 3: Degree Bound The vertex u must satisfy:

$$\Delta(G) \geq d(u) \geq 2 \left\lceil \frac{(\delta(G) + 2)(n - t)}{2t} \right\rceil \geq \frac{(\delta(G) + 2)(n - t)}{t}.$$

Solving for t:

$$t \geq \frac{(\delta(G) + 2)n}{\delta(G) + 2 + \Delta(G)}.$$

Substituting into $\gamma_{ns}(G) = 2t - n$:

$$\gamma_{ns}(G) \geq \frac{(\delta(G) + 2 - \Delta(G))n}{\delta(G) + 2 + \Delta(G)}.$$

Hence complete the proof. □

Theorem 2.7. *A graph G has $\gamma_{ns}(G) = n(w_T - w_F)$ if and only if every vertex $v \in V$ is either isolated, an endvertex or adjacent to an end vertex.*

Proof. Suppose $\gamma_{ns}(G) = n(w_T - w_F)$. We show that every vertex in G must satisfy one of the three conditions.If a vertex v has degree ≥ 2 and none of its neighbors are endvertices, then closed neighborhood $N[v]$ contains v and at least two other vertices. To satisfy $\sum_{u \in N[v]} f_T(u) \geq 1$, at least one vertex in $N[v]$ must have $f_T(u) > -1$. If we assign $f_T(v) = -1$ and $f_T(u) = 1$ for all $u \neq v$, the sum condition holds, but the total contribution to $\gamma_{ns}(G)$ would be less than $n(w_T - w_F)$, contradicting minimality. Hence, no such vertex can exist, meaning every non-endvertex must be adjacent to an endvertex.

Assume every vertex in G is either isolated, an endvertex, or adjacent to an endvertex. We construct an NSDF $f = (f_T, f_I, f_F)$ as follows $f_T(v) = 1, \quad f_I(v) = 0, \quad f_F(v) = -1$ for all $v \in V$. From the above labeling any $\sum_{u \in N[v]} f_T(u) \geq 1$, for all $v \in V$. holds since $f_T(u) = 1$ for all u $\sum_{u \in N[v]} f_I(u) = 0 \geq 0$ and $\sum_{u \in N[v]} f_F(u) = -|N[v]| \leq 1$ (since $|N[v]| \geq 1$). Thus

$$\gamma_{ns}(G) = \sum_{v \in V} (w_T f_T(v) + w_I f_I(v) + w_F f_F(v)) = n(w_T - w_F).$$

Hence completing the proof. □

3 NSDN for cycles and paths

This section contains the NSDN for cycles and paths

Theorem 3.1. *For a cycle graph C_n with $n \geq 3$ vertices, the NSDN is given by*

$$\gamma_{ns}(C_n) = \begin{cases} w_T \lceil \frac{n}{3} \rceil - nw_F & \text{if } w_I \geq 0, \\ w_T \lceil \frac{n}{3} \rceil + w_I \lfloor \frac{n}{3} \rfloor - nw_F & \text{if } w_I < 0. \end{cases}$$

Proof. Let $f = (f_T, f_I, f_F)$ be a NSDF.

Truth Membership f_T

The truth constraint requires:

$$\sum_{u \in N[v]} f_T(u) \geq 1 \quad \forall v \in V.$$

Since C_n is 2-regular, $|N[v]| = 3$. The minimal assignment occurs when:

- For $n \equiv 0 \mod 3$: Set $f_T(v) = \frac{1}{3}$ for all v, giving $\sum f_T = \frac{n}{3}$.
- For $n \not\equiv 0 \mod 3$: Assign $f_T(v) = 1$ to $\lceil \frac{n}{3} \rceil$ vertices (a minimal dominating set) and 0 elsewhere, giving $\sum f_T = \lceil \frac{n}{3} \rceil$.

Thus, the minimal $\sum f_T = \lceil \frac{n}{3} \rceil$.

Indeterminacy Membership f_I

The indeterminacy constraint requires:

$$\sum_{u \in N[v]} f_I(u) \geq 0 \quad \forall v \in V.$$

- If $w_I \geq 0$, set $f_I(v) = 0$ for all v, giving $\sum f_I = 0$.
- If $w_I < 0$, assign $f_I(v) = 1$ to $\lfloor \frac{n}{3} \rfloor$ vertices and 0 elsewhere, ensuring constraints are satisfied, giving $\sum f_I = \lfloor \frac{n}{3} \rfloor$.

Falsity Membership f_F

The falsity constraint requires:

$$\sum_{u \in N[v]} f_F(u) \leq 1 \quad \forall v \in V.$$

Set $f_F(v) = -1$ for all v. Then $\sum_{u \in N[v]} f_F(u) = -3 \leq 1$, satisfying the constraint, and $\sum f_F = -n$.

The NSDN is:

$$\gamma_{ns}(C_n) = w_T \sum f_T + w_I \sum f_I + w_F \sum f_F.$$

Substituting the minimal sums:

$$\gamma_{ns}(C_n) = \begin{cases} w_T \lceil \frac{n}{3} \rceil - nw_F & \text{if } w_I \geq 0, \\ w_T \lceil \frac{n}{3} \rceil + w_I \lfloor \frac{n}{3} \rfloor - nw_F & \text{if } w_I < 0. \end{cases}$$

Hence complete the proof. □

Theorem 3.2. *For a path graph P_n with $n \geq 3$ vertices, the NSDN is given by:*

$$\gamma_{ns}(P_n) = \begin{cases} w_T \left\lceil \frac{n}{3} \right\rceil - nw_F & \text{if } w_I \geq 0, \\ w_T \left\lceil \frac{n}{3} \right\rceil + w_I \left\lfloor \frac{n}{3} \right\rfloor - nw_F & \text{if } w_I < 0. \end{cases}$$

Proof. We prove this by constructing an optimal NSDF $f = (f_T, f_I, f_F)$.

The truth constraint requires:

$$\sum_{u \in N[v]} f_T(u) \geq 1 \quad \forall v \in V.$$

Since P_n is a path, the closed neighborhood $|N[v]|$ has size 3 for internal vertices and 2 for endpoints. The minimal assignment occurs by placing positive truth values on a dominating set of P_n, which has size $\lceil n/3 \rceil$. Specifically:

- Assign $f_T(v) = 1$ to the vertices in a minimum dominating set (e.g., every second vertex).
- Assign $f_T(v) = 0$ to all other vertices.

This gives $\sum f_T = \lceil n/3 \rceil$, which is minimal.

Indeterminacy Membership f_I

The indeterminacy constraint requires:

$$\sum_{u \in N[v]} f_I(u) \geq 0 \quad \forall v \in V.$$

- If $w_I \geq 0$, set $f_I(v) = 0$ for all v, giving $\sum f_I = 0$.
- If $w_I < 0$, assign $f_I(v) = 1$ to $\lfloor n/3 \rfloor$ vertices (spaced appropriately to avoid violating constraints) and 0 elsewhere. This ensures each closed neighborhood has at most one positive indeterminacy value, so $\sum_{u \in N[v]} f_I(u) \geq 0$. Then $\sum f_I = \lfloor n/3 \rfloor$.

Falsity Membership f_F

The falsity constraint requires:

$$\sum_{u \in N[v]} f_F(u) \leq 1 \quad \forall v \in V.$$

Set $f_F(v) = -1$ for all v. Then $\sum_{u \in N[v]} f_F(u) = -|N[v]| \leq -2 \leq 1$, so the constraint is satisfied. Then $\sum f_F = -n$.

The NSDN is:

$$\gamma_{ns}(P_n) = w_T \sum f_T + w_I \sum f_I + w_F \sum f_F.$$

Substituting the minimal sums:

$$\gamma_{ns}(P_n) = \begin{cases} w_T \lceil n/3 \rceil - nw_F & \text{if } w_I \geq 0, \\ w_T \lceil n/3 \rceil + w_I \lfloor n/3 \rfloor - nw_F & \text{if } w_I < 0. \end{cases}$$

Hence complete the proof. □

Illustration of NSDF on P_3 is presented in Fig 1.

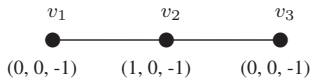


Fig. 1. NSDF on P_3

Let $V(P_3) = \{v_1, v_2, v_3\}$ with edges v_1v_2 and v_2v_3. Since $N[v_1] = \{v_1, v_2\}, N[v_2] = \{v_1, v_2, v_3\}, N[v_3] = \{v_2, v_3\}$; Define $f = (f_T, f_I, f_F) : V(P_3)$ to $\{-1, 0, 1\}$ as follows:

$$f_T(v_1) = 0, \quad f_T(v_2) = 1, \quad f_T(v_3) = 0$$
$$f_I(v_1) = 0, \quad f_I(v_2) = 0, \quad f_I(v_3) = 0$$
$$f_F(v_1) = -1, \quad f_F(v_2) = -1, \quad f_F(v_3) = -1$$

Now we consider the vertex v_1. From the above labeling, we get
$f_T(v_1) + f_T(v_2) = 0 + 1 = 1, f_I(v_1) + f_I(v_2) = 0 + 0 = 0, f_F(v_1) + f_F(v_2) = -1 + (-1) = -2$.
Next, we consider the vertex v_2. By the above labeling, we get
$f_T(v_1) + f_T(v_2) + f_T(v_3) = 0 + 1 + 0 = 1, f_I(v_1) + f_I(v_2) + f_I(v_3) = 0 + 0 + 0 = 0, f_F(v_1) + f_F(v_2) + f_F(v_3) = -1 + (-1) + (-1) = -3$.
Now, we consider the vertex v_3. According to the above labeling, we get
$f_T(v_2) + f_T(v_3) = 1 + 0 = 1, f_I(v_2) + f_I(v_3) = 0 + 0 = 0, f_F(v_2) + f_F(v_3) = -1 + (-1) + = -2$.
Thus $\gamma_{ns}(P_3) = 1 + 0 + (-3) = -2$.

4 Conclusion

This paper establishes the novel framework of Neutrosophic Signed Domination, integrating three-valued logic into graph theory. We derive sharp bounds for $\gamma_{ns}(G)$ and characterize extremal graphs achieving these bounds. Exact closed-form formulas are obtained for paths and cycles, demonstrating computability. Our work extends classical domination theory by incorporating uncertainty and influence gradations. This provides a powerful tool for analyzing complex networks where binary classifications are insufficient.

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SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions

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Abstract: The *n-th* PowerSet of a Set {or $P^n(S)$} better describes our real world, because a system *S* (which may be a company, institution, association, country, society, set of objects/plants/animals/beings, set of concepts/ideas/propositions, etc.) is formed by sub-systems, which in their turn by sub-sub-systems, and so on. We prove that the SuperHyperFunction is a generalization of classical Function, SuperFunction, and HyperFunction. And the SuperHyperAlgebra, SuperHyperGraph are part of the SuperHyperStructure. Almost all structures in our real world are Neutrosophic SuperHyperStructures since they have indeterminate/incomplete/uncertain/conflicting data.

Keywords: n-th PowerSet; Classical Function; HyperFunction; SuperFunction; SuperHyperFunction; Classical Operation; HyperOperation; SuperHyperOperation; Classical Axiom; HyperAxiom; SuperAxiom; SuperHyperAxiom; Classical Algebra; HyperAlgebra; SuperHyperAlgebra; Neutrosophic SuperHyperAlgebra; SuperHyperGraph; SuperHyperTopology; Classical Structure; HyperStructure; SuperHyperStructure; Neutrosophic SuperHyperStructure.

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1. Introduction

In general, a system *S* (that may be a company, association, institution, society, country, etc.) is formed by sub-systems S_i {or *P(S)*, the PowerSet of *S*}, and each sub-system S_i is formed by sub-sub-systems S_{ij} {or $P(P(S)) = P^2(S)$} and so on. That's why the n-th PowerSet of a Set *S* {defined recursively and denoted by $P^n(S) = P(P^{n-1}(S))$} was introduced, to better describes the organization of people, beings, objects etc. in our real world.

The n-th PowerSet, introduced by Smarandache [2] in 2016, was used in defining the SuperHyperOperation, SuperHyperAxiom, and their corresponding Neutrosophic SuperHyperOperation, Neutrosophic SuperHyperAxiom in order to build the SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra. In general, in any field of knowledge, one in fact encounters *SuperHyperStructures*.

2. The n-th PowerSet of a Set better describes our Real World

(i) ***The n-th PowerSet*** of a set better describes our real world, because a system *S* (which may be a company, institution, association, country, society, set of objects/plants/animals/beings, set of concepts/ideas/propositions, etc.) is formed by sub-systems S_i, which in their turn are formed by sub-sub-systems S_{i_j}, and so on, up to a needed structural-level *n* of the

system. Afterwards, between the sub-systems, sub-sub-systems and so on there are various inter-relationships (similar to the operations and axioms in general algebraic structures).

*

Let's recall the definition of the *n-th* PowerSet of a Set [2], proposed by Smarandache in 2016.

(ii) ***Definition of the n^th PowerSet of a Set*** *S* {denoted as $P^n(S)$, where the empty-set ϕ is allowed, and it represents the indeterminacy/uncertainty} is done recursively:

Let *S* be a set.

$P^0(S) \overset{def}{=} S$, be definition.

$P^1(S) = P(S)$ is the PowerSet of *S*, we call it the *1st*-PowerSet of S;

$P^2(S) = P(P(S))$ is the PowerSet of the PowerSet of *S*, or the *2nd*-PowerSet of *S*;

$P^3(S) = P(P^2(S)) = P(P(P(S)))$ is the PowerSet of the PowerSet of the PowerSet of S, or the *3rd*-PowerSet of *S*;

and so on,

$P^n(S) = P(P^{n-1}(S)) = ... = \underbrace{P(P(...P(S)...))}_{n-times}$, where P is repeated n times, and the empty-set is allowed.

(iii) ***Example of 2nd-PowerSet of a set S, where the empty-set is allowed.***

Let's consider an easy example to be able to distinguish between several types of functions, algebras, and structures.

Let the set $S = \{1, 2\}$.

Then, the 1st-PowerSet of *S*, with the empty set ϕ included, is: $P(S) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ and this $P(S)$ is used for the neutrosophic versions of functions, operations (and operators), axioms, algebras, and structures.

The 2-nd PowerSet of *S*, with the empty set ϕ included, is:

$P^2(S) = P(P(S)) = P(\{\phi, \{1\}, \{2\}, \{1, 2\}\}) =$

$\{\phi; \{1\}; \{2\}; \{1, 2\};$

$\{\{\phi, \{1\}\}; \{\phi, \{2\}\}; \{\phi, \{1, 2\}\};$

$\{\{1\}, \{2\}\}; \{\{1\}, \{1, 2\}\}; \{\{2\}, \{1, 2\}\};$

$\{\phi, \{1\}, \{2\}\}; \{\phi, \{1\}, \{1, 2\}\}; \{\phi, \{2\}, \{1, 2\}\}; \{\{1\}, \{2\}, \{1, 2\}\};$

$\{\phi, \{1\}, \{2\}, \{1, 2\}\}\}$.

(iv) ***Definition of the n^th PowerSet of a Set S without the empty-set*** {denoted as $P_*^n(S)$ where the empty-set ϕ is <u>not allowed</u>} is also done recursively:

Let *S* be a non-empty set.

$P_*^0(S) \overset{def}{=} S$, be definition.

$P_*^1(S) = P_*(S)$ is the PowerSet of *S*, without the empty-set, we call it the *1st*-PowerSet of *S*;

$P_*^2(S) = P_*(P_*(S))$ is the PowerSet of the PowerSet of *S*, without the empty-set, or the *2nd* PowerSet of *S*;

$P_*^3(S) = P_*(P_*^2(S)) = P_*(P_*(P_*(S)))$ is the PowerSet of the PowerSet of the PowerSet of *S*, without the empty-set, or the *3rd*-PowerSet of *S*;

and so on,

$P_*^n(S) = P_*(P_*^{n-1}(S)) = ... = \underbrace{P_*(P_*(...P_*(S)...))}_{n-times}$, where P is repeated n times,

and the empty set is not allowed.

(v) ***Example of 2nd-PowerSet of a set S, without the empty-set***

Let's consider an easy example to be able to distinguish between several types of functions, algebras, and structures.

Let $S = \{1, 2\}$ be a set, then the 1st-PowerSet of the set S, without the empty set ϕ, is $P_*(S) = \{\{1\}, \{2\}, \{1, 2\}\}$.

The 2nd-PowerSet of the set S, without the empty set ϕ, is:

$$P_*^2(S) = P_*(P_*(S)) = P_*(\{\{1\}, \{2\}, \{1, 2\}\}) =$$
$$\{\{1\}; \{2\}; \{1, 2\};$$
$$\{\{1\}, \{2\}\}; \{\{1\}, \{1, 2\}\}; \{\{2\}, \{1, 2\}\};$$
$$\{\{1\}, \{2\}, \{1, 2\}\}\}.$$

What is the distinction between, for example, $A = \{1, 2\}$ and $B = \{\{1\}, \{2\}\}$?

In A, the elements *1* and *2* are totally dependent of each other and form together a sub-system called A; while in B, each of *{1}* and *{2}* are partially independent of each other and as such they are individual sub-sub-systems, and partially dependent of each other and united into a sub-system called B (a sub-system of sub-sub-systems).

In the real world, we may consider, for example, A as a group of two researchers, denoted by *1* and *2*, that work together (totally dependent on each other) for a common project.

But in B, researchers *{1}* and *{2}* work each of them separately for the projects *p1* and respectively *p2* (so they are independent from the point of view of these projects), but the researchers work together for the third common project *p3* (so they are dependent from the point of view of project *p3*).

3. Functions of One Variable

(i) ***Classical Function of One Variable***

The domain and codomain of the function is just S.

$f : S \to S$

Example of Classical Function of One Variable

Let's take, as above, $S = \{1, 2\}$.

$f(1) = 2$ (a single-value) $\in S$;

$f(2) = 1$ (a single-value) $\in S$.

(ii) ***HyperFunction of One Variable***

This is part of the HyperStructures [1], when the domain S remains unchanged, while the codomain of the function becomes the PowerSet $P_*(S)$.

$f : S \to P_*(S)$

Example of HyperFunction of One Variable

$f(1) = \{1, 2\}$ (a set-value) $\in P_*(S)$;

$f(2) = 1 \in P_*(S)$.

(iii) ***SuperFunction of One Variable***

This is an extension of the HyperFunction, when the domain *S* remains the same, but the codomain of the function becomes n^{th}-PowerSet of the set *S*, i.e. $P_*^n(S)$, $n \geq 2$.

$f : S \to P_*^n(S)$, where integer $n \geq 2$.

(iv) ***Example of SuperFunction of One Variable***

Let's take the easiest case when $n = 2$, the domain of the function remains the same *S*, but one has the 2nd-PowerSet of the set S as codomain of the function:

$f : S \to P_*^2(S)$

$f(1) = \{\{1\},\{1,2\}\} \in P_*^2(S)$;

$f(2) = \{\{1\},\{2\}\} \in P_*^2(S)$.

(v) ***SuperHyperFunction of One Variable***

$f : P_*^r S \to P_*^n(S)$, for integers $r, n \geq 0$.

It is part of the SuperHyperStructure [2, 3].

(vi) ***Example of SuperHyperFunction of One Variable***

$f : P_*(S) \to P_*^2(S)$.

$f(\{1\}) = \{\{1\},\{2\},\{1,2\}\} \in P_*^2(S)$.

$f(\{2\}) = \{\{1\},\{2\}\} \in P_*^2(S)$.

$f(\{1,2\}) = \{\{2\},\{1,2\}\} \in P_*^2(S)$

(vii) ***Theorem 1:***

The SuperHyperFunction of One Variable is the most general form of functions of one variables.

Proof*:*

For r = 0, and n = 0, one gets the classical Function.

For r = 0, and n = 1, one gets the HyperFunction.

For r = 0, and n ≥ 2, one gets the SuperFunction.

4. Functions of Many Variables

Straightforward generalization of the functions, from one variable to many variables, are provided below.

(i) ***Classical Function of Many Variables***

$f : S^m \to S$, for integer m ≥ 2.

(ii) ***Example of Classical Function of Two Variables***

Let's consider some elementary case, when $m = 2$.

$f : S^2 \to S$

First, $S^2 = \{1,2\} \times \{1.2\} = \{(1,1),(1,2),(2,1),(2,2)\}$

$f(1,1) = 2$ (a single-value) $\in S$

$f(1,2) = 2 \in S$

$f(2,1) = 1 \in S$

$f(2,2) = 1 \in S$

(iii) ***HyperFunction of Many Variables***

This is part of the HyperStructures.

$f : S^m \to P_*(S)$

(iv) ***Example of HyperFunction of Two Variables***

$$f : S^2 \to P_*(S)$$

$$f(1,1) = \{1,2\} \text{ (a set-value) } \in P_*(S)$$

$$f(1,2) = \{1\} \in P_*(S)$$
$$f(2,1) = \{1,2\} \in P_*(S)$$
$$f(2,2) = \{2\} \in P_*(S)$$

(v) ***SuperFunction of Many Variables***

$f : S^m \to P_*^n(S)$, where the integers *m*, $n \geq 2$. When the 2-nd PowerSet of the Set *S* is involved.

(vi) ***Example of SuperFunction of Two Variables***

Let's take $m = n = 2$ the simplest case.

$$f : S^2 \to P_*^2(S)$$

$$f(1,1) = \{\{1\},\{2\},\{1,2\}\} \in P_*^2(S)$$
$$f(1,2) = \{\{1\},\{2\}\} \in P_*^2(S)$$
$$f(2,1) = \{\{2\},\{1,2\}\} \in P_*^2(S)$$
$$f(2,2) = \{\{1\},\{1,2\}\} \in P_*^2(S)$$

(vii) ***SuperHyperFunction of Many Variable***

$f : (P_*^r S)^m \to P_*^n(S)$, for integers $m \geq 2$ and $r, n \geq 0$.

It is part of the SuperHyperStructure.

(viii) ***Example of SuperHyperFunction of Two Variables***

Let's take $m = 2$, $r = 1$, and $n = 2$.

$$f : (P_*(S))^2 \to P_*^2(S)$$

Table 1. Values of the above SuperHyperFunction of two variable f(x, y).

x / y	**{1}**	**{2}**	**{1, 2}**
{1}	{{1}, {2}}	{1}	{{1}, {1 2}}
{2}	{{2}, {1, 2}}	{{1}, {1, 2}}	{2}
{1, 2}	{1, 2}		{{1}. {2}, {1, 2}}

For example, *f({1}, {1, 2}) = {{1}, {1 2}}*.

(ix) ***Theorem 2:***

Similarly, the SuperHyperFunction of Many Variables is a generalization of the Classical Function, HyperFunction, and SuperFunction of Many Variables.

Proof is the same as in the previous theorem, by keeping the same *m* (number of variables) value:

For *r = 0,* and *n = 0*, one gets the classical Function of Many Variables.

For *r = 0,* and *n = 1*, one gets the HyperFunction of Many Variables.

For *r = 0,* and $n \geq 2$, one gets the SuperFunction of Many Variables.

5. Definition of SuperHyperFunction (SHF_m) of $m \geq 2$ Variables

$f_{SH}^{SH} : P_*^{r_1}(S) \times P_*^{r_2}(S) \times ... \times P_*^{r_k}(S) \to P_*^n(S)$, where integers $r_1, r_2, ..., r_k, n \geq 0$ and *SH* stands for *SuperHyper*, the upper *SH* is for the function's domain, and the bottom *SH* is for the function's codomain, meaning that both are some PowerSets of PowerSets etc. of the set S.

For any $x_1 \in P_*^{r_1}(S), x_2 \in P_*^{r_2}(S), ..., x_k \in P_*^{r_k}(S)$, one has $f_{SH}^{SH}(x_1, x_2, ..., x_k) \in P_*^n(S)$.

This is a generalization of all previous functions.

6. Operations / HyperOperations / SuperHyperOperations and Axioms / HyperAxioms / SuperHyperAxioms

Let $m \geq 1$ be an integer.

The **Operations** (and operators) can be treated as *m-ary* functions, while the **Axioms** as logical propositions involving the *m-ary* operations.

Similarly, the **HyperOperations** can be treated as *m-ary* HyperFunctions, while the **HyperAxioms** as logical propositions involving the *m-ary* HyperOperations. And last, the **SuperHyperOperations** can be treated as *m-ary* SuperHyperFunctions, while the **SuperHyperAxioms** (***HSAx***) as logical propositions involving *m-ary* SuperHyperOperations.

7. Structure / HyperStructure / SuperHyperStructure

(i) The classical **Structure** is a structure built on a set *S*, endowed with classical Operations ($\#_C$) $\#_C : S^m \to S$, for integer $m \geq 1$, and classical Axioms (A_C), which are axioms that act on the set S endowed with classical Operations.

(ii) The **HyperStructure** {defined by F. Marty [1] in 1934}, is a structure built on a set *S*, endowed with HyperOperations ($\#_H$), $\#_H : S^m \to P_*(S)$, for integer $m \geq 1$, and HyperAxioms (A_H), which are axioms that act on the set S endowed with HyperOperations. "Hyper" stands for the codomain of the operations, which is $P_*^n(S)$ instead of *S* that is for the classical structure.

(iii) The **SuperStructure** {defined by F. Smarandache [2] in 2016}, is a structure built on $P_*^n(S)$, that is the n^{th}-PowerSet of the Set *S*, without the empty-set, endowed with SuperOperations, $\#_S : (P_*^n(S))^m \to P_*^q(S)$, for integers $n \geq 0, q \geq 0$, and SuperAxioms, which are axioms that act on the set $P_*^n(S)$ endowed with SuperOperations. "Super" stands for the codomain of the SuperOperations, which is $P_*^q(S)$, instead of *S* that is for the classical structure or of $P_*^n(S)$ that is for HyperStructure, or for the domain of the SuperOperations, which is $P_*^n(S)$.

(iv) The **SuperHyperStructure** {defined by F. Smarandache [2, 3] in 2016 and 2019}, is a structure built on the nth-PowerSet of the Set *S*, $P_*^n(S)$, endowed with SuperHyperOperations and SuperHyperAxioms.

8. The most general form of the SuperHyperAlgebra (SHA) endowed with One Operation and Many Axioms

is:

$(P_*^n(S); \#_{SHO}^m; SHAx_1, SHAx_2, ..., SHAx_q)$ where S is a non-empty set, $P_*^n(S)$ is the n^{th}-PowerSet of the set S, for $n \geq 2$, and $\#_{SHO}^m$ is an ***m-ary* SuperHyperOperation (*SHO*)**, acting on $P_*^n(S)$:

$\#_{SHO}^m : \underbrace{P_*^n(S) \times P_*^n(S) \times ... \times P_*^n(S)}_{m-times} \to P_*^n(S)$, where $P_*^n(S)$ is repeated *m times* into the operation domain, and integer $m \geq 1$, and *q* is the number of **SuperHyperAxioms**.

9. The most general form of the SuperHyperAlgebra with Many Operations and Many Axioms

is:

$(P_*^n(S); \#_{SHO1}^{m_1}, \#_{SHO2}^{m_2}, ..., \#_{SHOr}^{m_r}; SHAx_1, SHAx_2, ..., SHAx_q)$ where the m_i -ary SuperHyperOperations are defined as follows:

$\#_{SHO}^{m_i} : \underbrace{P_*^n(S) \times P_*^n(S) \times ... \times P_*^n(S)}_{m_i-times} \to P_*^n(S)$, with $P_*^n(S)$ being repeated m_i times into the operation domain, $m_i \geq 1$, for $1 \leq i \leq r$, and $r \geq 2$ is the number of m_i - *ary* SuperHyperOperations ($\#_{SHO1}^{m_1}, \#_{SHO2}^{m_2}, ..., \#_{SHOr}^{m_r}$), and $q \geq 1$ is the number of SuperHyperAxioms ($SHAx_1, SHAx_2, ..., SHAx_q$).

10. SuperHyperTopology

SuperHyperTopology [5] is a topology built on a SuperHyperAlgebra $(P_*^n(S), \#)$, for integer $n \geq 2$, and it is a collection of SuperHyperSubsets from $P_*^n(S)$ that satisfy the axioms of classical topology.

11. Neutrosophic SuperHyperStructure et al.

All of the above *non-neutrosophic concepts* can easily be extended to the *neutrosophic framework*, therefore:

the Neutrosophic HyperFunction / SuperFunction / SuperHyperFunction of one or many variables, and the Neutrosophic HyperOperation / SuperHyperOperation, and the Neutrosophic HyperAxiom / SuperHyperAxiom, and the Neutrosophic SuperHyperAlgebra / SuperHyperTopology, and, in general, the Neutrosophic Super/Hyper/SuperHyperStructure are built in the same corresponding ways as the above non-neutrosophic concepts, with the only distinction that all $P^k{}_*(S)$, which do not include the empty-set, are replaced by $P^k(S)$, which do include the empty-set (leaving room for indeterminate/incomplete/uncertainty/conflicting data), for all integers $k \geq 1$.

12. Applications

We need to work with the nth-PowerSet of a set to better describe the organization of our real world. A system (set) S is composed by sub-systems (the elements of the $P_*(S)$, the PowerSet of S, let's denote them by S_1, S_2, …), and the sub-systems by sub-sub-systems (let's denote them by S_{11}, S_{12}, … respectively S_{21}, S_{22}, …), and so on.

As future possible research work will be to investigate the applicability of many types of SuperHyperStructures and Neutrosophic SuperHyperStructures in the real world.

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Foundation of SuperHyperStructure

&

Neutrosophic SuperHyperStructure

(review paper)

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Abstract

In this paper we extend the SuperHyperAlgebra, SuperHyperGraph, SuperHyperTopology, SuperHyperSoft Set, endowed with SuperHyperOperations, SuperHyperAxioms, and SuperHyperFunctions, to the most general form of structure, from our real world, called SuperHyperStructure in any field of knowledge. A practical application of the SuperHyperStructure is presented at the end.

The prefix "Hyper" [Marty, 1934] stand for the codomain of the functions and operations to be $P(H)$, or the powerset of the set H. While the prefix "Super" [Smarandache, 2016] stands for using the $P^n(H)$, $n \geq 2$, or the *n-th* PowerSet of the Set H {because the *set* (or *system*) H (that may be a set of items, a company, institution, country, region, etc.) is organized in *sub-systems*, which in their turn are organized in *sub-sub-systems*, and so on} in the domain and/or codomain of the functions and operations.

Keywords: *n-th* PowerSet of a Set, SuperHyperAlgebra, SuperHyperGraph, SuperHyperTopology, SuperHyperSoft Set, SuperHyperOperations, SuperHyperAxioms, SuperHyperFunctions, HyperStructure, SuperHyperStructure, Neutrosophic SuperHyperStructure

1. From Classical Structure and HyperStructure to SuperHyperStructure

We present below the evolution of structures in all fields of knowledge:
Classical Structure, HyperStructure, SuperHyperStructure (none having indeterminacy);

and
Neutrosophic Classical Structure, Neutrosophic HyperStructure, Neutrosophic SuperHyperStructure (all having some indeterminacy, as in our everyday life).

2. Classical Structure

A **Classical Structure** is built on a non-empty set H, whose **Classical Operations** ($\#_0$) are defined as:

$\#_0: H^m \to H$, for integer $m \geq 1$,

and with **Classical Axioms** acting on it.

3. HyperStructure

A **HyperStructure** (Marty [1], 1934) is built on a non-empty set H, whose **HyperOperations** ($\#_{H0}$) are defined as:

$\#_{H0}: H^m \to P \cdot (H)$, where $P \cdot (H)$ is the set of all non-emtpy subsets of H,

and with **HyperAxioms** acting on it.

4. Neutrosophic HyperStructure

As an extension of the HyperStructure, the **Neutrosophic HyperStructure** (Smarandache [2], 2016) is built on a non-empty set H, whose **Neutrosophic HyperOperations** ($\#_{NS0}$) are defined as:

$\#_{NS0}: H^m \to P(H)$, where $P(H)$ is the set of all non-emtpy and empty subsets of H,

and the axioms acting on it are called **Neutrosophic HyperAxioms**.

5. Definition of the n[th]-PowerSet $\boldsymbol{P_\star^n(H)}$ *without* Indeterminacy (no empty-set)

The n[th]-PowerSet $\boldsymbol{P_\star^n(H)}$ [2] of the set H, that the SuperHyperStructure is built on, describes a world that does not contain indeterminacy, where similarly the *set* (or *system*) H (that may be a set of items, a company, institution, country, region, etc.) is organized in sub-systems, which in turn are organized in sub-sub-systems, and so on.

The n[th]-PowerSet $\boldsymbol{P_*^n(H)}$ is also defined recursively:

$\boldsymbol{P_\star^0(H)} \stackrel{\text{def}}{=} H$

$\boldsymbol{P_\star^1(H)} = P_\star(\boldsymbol{H})$

$\boldsymbol{P_\star^2(H)} = P_\star(P_\star(\boldsymbol{H}))$

$\boldsymbol{P_\star^3(H)} = P_\star(\boldsymbol{P_\star^2(H)}) = P_\star\big(P_\star(P_\star(\boldsymbol{H}))\big)$

… … … ……....

$\boldsymbol{P_\star^n(H)} = P_\star(\boldsymbol{P_\star^{n-1}(H)}) = \underbrace{P_\star\big(P_\star(\dots P_\star(\boldsymbol{H}) \dots)\big)}_{n}$,

where P is repeated n times into the last formula,

and the empty-set $\emptyset$ (that represents indeterminacy, uncertainty) is not allowed in none of the sequence terms:

$\boldsymbol{H}, P_\star(\boldsymbol{H}), \boldsymbol{P_\star^2(H)}, \boldsymbol{P_\star^3(H)}, \dots, \boldsymbol{P_\star^n(H)}$.

---

6. Definition of the n[th]-PowerSet $\boldsymbol{P^n(H)}$ *with* Indeterminacy (represented by the empty-set)

The n[th]-PowerSet $\boldsymbol{P^n(H)}$ of the set H, that the Neutrosophic SuperHyperStructure is built on, best describes our real world where always the indeterminacy occurs, and a *set* (or *system*) H (that may be a set of items, a company, institution, country, region, etc.) is organized in sub-systems, which in turn are organized in sub-sub-systems, and so on.

The n[th]-PowerSet $\boldsymbol{P^n(H)}$ is defined recursively:

$P^0(\boldsymbol{H}) \stackrel{\text{def}}{=} H$

$P^1(\boldsymbol{H}) = P(\boldsymbol{H})$

$P^2(\boldsymbol{H}) = P(P(\boldsymbol{H}))$

$P^3(\boldsymbol{H}) = P\big(P^2(\boldsymbol{H})\big) = P\big(P(P(\boldsymbol{H}))\big)$

… … … ……..

$P^n(\boldsymbol{H}) = P(P^{n-1}(\boldsymbol{H})) = \underbrace{P\big(P(\dots P(\boldsymbol{H}) \dots)\big)}_{n}$,

where P is repeated n times into the last formula,

and the empty-set $\emptyset$ (that represents indeterminacy, uncertainty) is allowed in all sequence terms:

$H, P(H), P^2(H), P^3(H), \dots, P^n(H)$.

The n[th]-PowerSet $P_\star^n(H)$ and $P^n(H)$ of a non-empty set H were introduced by Smarandache [2] in 2016.

7. SuperHyperStructure

The SuperHyperStructure was founded by Smarandache in 2016 [2], who introduced the SuperHyperAlgebra in 2016 and developed it in 2022 [8], SuperHyperGraph in 2019, 2020, 2022 [3, 4, 5], SuperHyperFunction and SuperHyperTopology in 2022 [6], and respectively the SuperHyperOperations, and SuperHyperAxioms [2016-2022].

A SuperHyperStructure is built on the n-th powerset $P_\star^n(H)$ of a non-empty set H, for integer $n \geq 1$, whose **SuperHyperOperators** ($\#_{SH0}$) are defined as follows:

$\#_{SH0}: (P_\star^r(H))^m \to P_\star^n(H)$,

where $P_\star^r(H)$ is the powerset of H, for integer $r \geq 1$, while similarly $P_\star^n(H)$ is the n-th powerset of H, both without any empty-sets, and the **SuperHyperAxioms** act on it.

8. Neutrosophic SuperHyperStructure

Similarly, a **Neutrosophic SuperHyperStructure** (2016) is built on the n[th]-powerset $P^n(H)$ of a non-empty set H, for $n \geq 1$, whose Neutrosophic SuperHyperOperators ($\#_{NSH0}$) are defined as follows:

$\#_{NSH0}: \left(P^r(H)\right)^m \to P^n(H)$,

where $P^r(H)$ is the r-powerset of H, for integer $r \geq 1$, while $P^n(H)$ is the n[th]-powerset of H, both containing empty-sets.

9. The Triplet of HyperStructure

As an analogy of the neutrosophic triplet [9 – 19] presented between (2016, 2019 – 2023):
<Algebra, NeutroAlgebra, AntiAlgebra>,
we propose now the following triplet:
<HyperStructure, Neutro-HyperStructure, Anti-HyperStructure>,
that extends Marti's HyperStructure,
where:

– the HyperStructure has all axioms totally (100%) true;

– the Neutro-HyperStructure has at least one axiom which is partially true (T), partially indeterminate (I), and partially false (F); $(T, I, F) \in \{(1, 0, 0), (0, 0, 1)\}$, and no axiom is totally (100%) false;

– the Anti-HyperStructure has at least one axiom that is 100% false, or $(T, I, F) = (0, 0, 1)$, no matter how the other axioms are.

10. The Triplet of SuperHyperStructure

One has, as a further extension of the above, the following triplet:
<SuperHyperStructure, Neutro-SuperHyperStructure, Anti-SuperHyperStructure>,
where:

– the SuperHyperStructure has all axioms totally (100%) true;

– the Neutro-SuperHyperStructure has at least one axiom that is partially true (T), partially indeterminate (I), and partially false (F); or $(T, I, F) \in \{(1, 0, 0), (0, 0, 1)\}$, and no axiom is totally (100%) false;

– the Anti-SuperHyperStructure has at least one axiom that is totally (100%) false, or $(T, I, F) = (0, 0, 1)$, no matter how are the other axioms.

11. SuperHyperFunction of Many Variable [7]

$f : (P_*^r S)^m \to P_*^n(S)$, for integers $m \geq 2$ and $r, n \geq 0$.

It is part of the SuperHyperStructure.

12. Example of SuperHyperFunction of Two Variables

Let's take $m = 2$, $r = 1$, and $n = 2$.

$f : (P_*(S))^2 \to P_*^2(S)$

x / y	{1}	{2}	{1, 2}
{1}	{{1}, {2}}	{1}	{{1}, {1 2}}
{2}	{{2}, {1, 2}}	{{1}, {1, 2}}	{2}
{1, 2}	{1, 2}		{{1}. {2}, {1, 2}}

Table 1 of Values of the above SuperHyperFunction of Two Variable f(x, y)

For example, *f({1}, {1, 2}) = {{1}, {1 2}}*, etc.

13. SuperHyperAlgebra

We recall our 2016 concepts of SuperHyperOperation, SuperHyperAxiom, SuperHyperAlgebra, and their corresponding Neutrosophic SuperHyperOperation Neutrosophic SuperHyperAxiom and Neutrosophic SuperHyperAlgebra [2] and developed later in (2019-2024), especially in [8] in 2022.

Let $P_*^n(H)$ be the n[th]-powerset of the set H such that none of $P(H), P^2(H), \ldots, P^n(H)$ contain the empty set ϕ.

Also, let $P^n(H)$ be the n[th]-powerset of the set H such that at least one of the $P^2(H), \ldots, P^n(H)$ contain the empty set ϕ.

The SuperHyperOperations are operations whose codomain is either $P_*^n(H)$ and in this case one has **classical-type SuperHyperOperations**, or $P^n(H)$ and in this case one has **Neutrosophic SuperHyperOperations**, for integer $n \geq 2$.

14. Classical-type Binary SuperHyperOperation

A classical-type Binary SuperHyperOperation $\circ_{(2,n)}^*$ is defined as follows:

$$\circ_{(2,n)}^* : H^2 \to P_*^n(H)$$

where $P_*^n(H)$ is the n^{th}-powerset of the set H, with no empty-set.

15. Examples of classical-type Binary SuperHyperOperation

1) Let $H = \{a, b\}$ be a finite discrete set; then its power set, without the empty-set ϕ, is:

$$P(H) = \{a, b, \{a, b\}\}, \text{ and:}$$

$$P^2(H) = P(P(H)) = P(\{a, b, \{a, b\}\}) = \{a, b, \{a, b\}, \{a, \{a, b\}\}, \{b, \{a, b\}\}, \{a, b, \{a, b\}\}\}.$$

$$\circ_{(2,2)}^* : H^2 \to P_*^2(H)$$

Table 2. Example 1 of classical-type Binary SuperHyperOperation.

$\circ^{*}_{(2,2)}$	a	b
a	$\{a,\{a,b\}\}$	$\{b,\{a,b\}\}$
b	a	$\{a,b,\{a,b\}\}$

16. Classical-type m-ary SuperHyperOperation {or more accurate denomination (m, n)-SuperHyperOperation}

Let U be a universe of discourse and a non-empty set H, $H \subset U$. Then:

$$\circ^{*}_{(m,n)} : H^m \to P^n_*(H)$$

where the integers $m, n \geq 1$,

$$H^m = \underbrace{H \times H \times ... \times H}_{m\ times},$$

and $P^n_*(H)$ is the n^{th}-powerset of the set H that includes the empty-set.

This SuperHyperOperation is a m-ary operation defined from the set H to the n^{th}-powerset of the set H.

17. Neutrosophic m-ary SuperHyperOperation {or more accurate denomination Neutrosophic (m, n)-SuperHyperOperation}

Let U be a universe of discourse and a non-empty set H, $H \subset U$. Then:

$$\circ_{(m,n)} : H^m \to P^n(H)$$

where the integers $m, n \geq 1$; $P^n(H)$ - the n-th powerset of the set H that includes the empty-set.

18. SuperHyperAxiom

A **classical-type SuperHyperAxiom** or more accurately a ***(m, n)*-SuperHyperAxiom** is an axiom based on classical-type SuperHyperOperations.

Similarly, a **Neutrosophic SuperHyperAxiom** {or Neutrosphic (m, n)-SuperHyperAxiom} is an axiom based on Neutrosophic SuperHyperOperations.

There are:

- **Strong SuperHyperAxioms**, when the left-hand side is equal to the right-hand side as in non-hyper axioms,
- and **Week SuperHyperAxioms**, when the intersection between the left-hand side and the right-hand side is non-empty.

19. SuperHyperAlgebra and Neutrosophic SuperHyperStructure

A **SuperHyperAlgebra** or more accurately ***(m-n)*-SuperHyperAlgebra** is an algebra dealing with SuperHyperOperations and SuperHyperAxioms.

Again, a **Neutrosophic SuperHyperAlgebra** {or Neutrosphic (m, n)-SuperHyperAlgebra} is an algebra dealing with Neutrosophic SuperHyperOperations and Neutrosophic SuperHyperOperations.

In general, we have **SuperHyperStructures** {or *(m-n)*-SuperHyperStructures}, and corresponding **Neutrosophic SuperHyperStructures**.

For example, there are SuperHyperGrupoid, SuperHyperSemigroup, SuperHyperGroup, SuperHyperRing, SuperHyperVectorSpace, etc.

20. SuperHyperGraph (or n-SuperHyperGraph)

Introduced by F. Smarandache [3, 4, 5] in 2019 and developed in 2020 - 2022.

Let $V = \{v_1, v_2, \dots, v_m\}$, for $1 \leq m \leq \infty$, be a set of finite or infinite number of vertices, that contains Single Vertices (the classical ones), Indeterminate Vertices (unclear, vague, partially known), and Null Vertices (totally unknown, empty).

Let $P(V)$ pe the power of set V, that includes the empty set □ too.

Then $P^n(V)$ be the n-power set of the set V, defined in a recurrent way, i.e.:

P(V), $P^2(V) = P(P(V))$, $P^3(V) = P(P^2(V)) = P(P(P(V)))$, …, $P^n(V) = P(P^{n-1}(V))$,

for $1 \leq n \leq \infty$, where by definition $P^0(V) \overset{def}{=} V$ and $P^1(V) \overset{def}{=} P(V)$.

Then, the SuperHyperGraph (SHG) [or n-SuperHyperGraph (n-SHG)] is an ordered pair:

$$\text{n-SHG} = (G_n, E_n),$$

where $G_n \subseteq P^n(V)$, and $E_n \subseteq P^n(V)$, for $1 \leq n \leq \infty$.

G_n is the set of vertices, and E_n is the set of edges.

The set of vertices G_n contains all possible types of vertices as in our real world:

- Singles Vertices (the classical ones);
- Indeterminate Vertices (unclear, vague, partially unknown);
- Null Vertices (totally unknown, empty);

and:

- SuperVertex (or SubsetVertex), i.e. two or more (single, indeterminate, or null) vertices put together as a group (organization).
- n-SuperVertex that is a collection of many vertices such that at least one is an $(n-1)$- SuperVertex and all the others into the collection are r-SuperVertices, if any, whose order $r \leq n-1$.

The set of edges E_n contains the following types of edges:

- Singles Edges (the classical ones);
- Indeterminate Edges (unclear, vague, partially unknown);
- Null Edges (totally unknown, empty);

and:

- HyperEdge (connecting three or more single vertices);

- SuperEdge (connecting two vertices, at least one of them being a SuperVertex);
- n-SuperEdge (connecting two vertices, at least one being a n-SuperVertex, and the other of order r-SuperVertex, with $r \leq n$);
- SuperHyperEdge (connecting three or more vertices, at least one being a SuperVertex);
- n -SuperHyperEdge (connecting three or more vertices, at least one being a n - SuperVertex, and the other r-SuperVertices with $r \leq n$;
- MultiEdges (two or more edges connecting the same two vertices);
- Loop (and edge that connects an element with itself). and:
- Directed Graph (classical one);
- Undirected Graph (classical one);
- Neutrosophic Directed Graph (partially directed, partially undirected, partially indeterminate direction).

21. SuperHyperTopology [6, 7]

Let consider τ_{SHT} be a family of subsets of $P_*^n(H)$.

Then τ_{SHT} is called a SuperHyperTopology on $P_*^n(H)$, if it satisfies the following axioms:

(SHT-1) ϕ and $P_*^n(H)$ belong to τ_{SHT}.

(SHT-2) The intersection of any finite number of elements in τ_{SHT} is in τ_{SHT}.

(SHT-3) The union of any finite or infinite number of elements in τ_{SHT} is in τ_{SHT}.

Then $(P_*^n(H), \tau_{SHT})$ is called a SuperHyperTopological Space on $P_*^n(H)$.

22. Neutrosophic SuperHyperTopology [6, 7]

Let consider τ_{NSHT} be a family of subsets of $P^n(H)$.

Then τ_{NSHT} is called a Neutrosophic SuperHyperTopology on $P^n(H)$, if it satisfies the following axioms:

(NSHT-1) ϕ and $P^n(H)$ belong to τ_{NSHT}.

(NSHT-2) The intersection of any finite number of elements in τ_{NSHT} is in τ_{NSHT}.

(NSHT-3) The union of any finite or infinite number of elements in τ_{NSHT} is in τ_{NSHT}.

Then $(P^n(H), \tau_{NSHT})$ is called a Neutrosophic SuperHyperTopological Space on $P^n(H)$.

23. SuperHyperSoft Set

The SuperHyperSoft Set [22, 23] is an extension of the HyperSoft Set [21] and Soft Set [20].

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the powerset of $\mathcal{U}$.

Let $a_1, a_2, \ldots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \ldots, A_n$,

with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, n\}$.

Let $\mathcal{P}(A_1)$, $\mathcal{P}(A_2)$, …, $\mathcal{P}(A_n)$ be the powersets of the sets A_1, A_2, …, A_n respectively.

Then the pair $(F, \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \dots \times \mathcal{P}(A_n))$, where × meaning Cartesian product, or:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \dots \times \mathcal{P}(A_n) \to \mathcal{P}(\mathcal{U})$

is called a SuperHyperSoft Set.

24. Example of SuperHyperSoft Set

If we define the function:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3) \times \mathcal{P}(A_4) \to \mathcal{P}(\mathcal{U})$.

we get a *SuperHyperSoft Set*.

Let's assume, from the previous example, that:

$F(\{\text{medium, tall}\}, \{\text{white, red, black}\}, \{\text{female}\}, \{\text{American, Italian}\}) = \{x_1, x_2\}$, which means that:

$F(\{\text{medium or tall}\}$ and $\{\text{white or red or black}\}$ and $\{\text{female}\}$ and $\{\text{American or Italian}\})$ $= \{x_1, x_2\}$.

Therefore, the SuperHyperSoft Set offers a larger variety of selections, so x_1 and x_2 may be:

either medium, or tall (but not small),

either white, or red, or black (but not yellow),

mandatory female (not male),

and either American, or Italian (but not French, Spanish, Chinese).

In this example there are:

Card{medium, tall}· Card{white, red, black}· Card{female}· Card{American, Italian} = 2·3·1·2 =12 possibilities, where Card{ } means cardinal of the set { }.

This is closer to our everyday life, since for example, when selecting something, we have not been too strict, but accepting some variations (for example: medium or tall, white or red or black, etc.).

25. Fuzzy-Extension-SuperHyperSoft Set

$$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \dots \times \mathcal{P}(A_n) \to \mathcal{P}\left(\mathcal{U}\left(x(d^0)\right)\right)$$

where $x(d^0)$ is the fuzzy or any fuzzy-extension degree of appurtenance of the element x to the set $\mathcal{U}$.

Fuzzy-Extensions mean all types of fuzzy sets [3], such as:

Fuzzy Set, Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, n-HyperSpherical Fuzzy Set, Neutrosophic Set, Spherical Neutrosophic Set, Refined Fuzzy/Intuitionistic_Fuzzy/Neutrosophic/other_fuzzy_extension Sets, Plithogenic Set, etc.

26. Example of Fuzzy_Extension SuperHyperSoft Set

In the previous example, taking the degree of a generic element $x(d^0)$ as neutrosophic, one gets the Neutrosophic SuperHyperSoft Set.

Assume, that: $F(\{\text{medium, tall}\},\{\text{white, red, black}\},\{\text{female}\},\{\text{American, Italian}\}) =$

$= \{x_1(0.7, 0.4, 0.1), x_2(0.9, 0.2, 0.3)\}.$

Which means that: x_1 with respect to the attribute values

({medium or tall} and {white or red or black} and {female}, and {American or Italian})

has the degree of appurtenance to the set 0.7, the indeterminate degree of appurtenance 0.4, and the degree of non-appurtenance 0.1.

While x_2 has the degree of appurtenance to the set 0.9, the indeterminate degree of appurtenance 0.2, and the degree of non-appurtenance 0.3.

27. Examples of HyperAlgebra and Neutrosophic HyperAlgebra

27.1. Commutative SemiHyperGroup

The SemiHyperGroup is a particular case of HyperAlgebra.

Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z} = \{-\infty, \dots, -2, -1, 0, 1, 2, \dots, +\infty\}$.

Let's define the HyperLaw * as follows:

$*: Z \times Z \to P(Z)$, $x \star y = \{x, y\} \in P(\mathbb{Z})$,

so the law is *well-defined*.

The law is *associative*, since:

$(x \star y) \star z = x \star (y \star z)$

$\{x, y\} \star z = x \star \{y, z\}$

$(x \star z) \cup (y \star z) = (x \star y) \cup (x \star z)$

$\{x, z\} \cup \{y, z\} = \{x, y\} \cup \{x, z\}$

$\{x, y, z\} = \{x, y, z\}$

The law is *commutative*, since

$x \star y = \{x, y\} = \{y, x\} = y \star x$

27.2. Commutative Neutrosophic SemiHyperGroup

The Neutrosophic SemiHyperGroup is a particular case of Neutrosophic HyperAlgebra.

Let the HyperLaw * be defined as:

$\star: (Z \cup \{\emptyset\}) \times (Z \cup \{\emptyset\}) \to P(Z \cup \{\emptyset\})$

where the empty set $\emptyset$ leaves room for indeterminacy, unknown etc.

$$x \star y = \begin{cases} \{x, y\}, \text{for both } x, y \neq \emptyset \\ \emptyset, \text{ for } x, \text{or } y, \text{or both} = \emptyset \end{cases}$$

The law is well-defined, associative and commutative (proven as above for the SemiHyperGroup).

28. Examples of SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra

28.1. Commutative SuperHyperGrupoid

Let again $\mathbb{Z}$ be the set of integers, $\mathbb{Z} = \{-\infty, \dots, -2, -1, 0, 1, 2, \dots, +\infty\}$.

For 2-nd powerset $P^2(\mathbb{Z}) = P(P(\mathbb{Z}))$ one has two rows of parentheses, one inside the other, of the form: $\{\dots\{\dots\}\dots\}$.

Let's define the binary SuperHyperLaw

$\star: \mathbb{Z}^2 \to P^2(\mathbb{Z})$

$x \star y = \{x, y, \{x, y\}\} \in P^2(\mathbb{Z})$

Clearly the law is *well-defined*.

The law $\star$ is also *commutative,* but *non-associative,* as proven below.

Commutativity:

$x \star y = \{x, y, \{x, y\}\} = \{y, x, \{y, x\}\} = y \star x.$

Non-Associativity:

$$(x \star y) \star z = \{x, y, \{x, y\} \star z\} = \{x \star z, y \star z, \{x, y\} \star z\} = \{x, z, \{x, z\}, y, z, \{y, z\}, x \star z, y \star z\}$$
$$= \{x, z, \{x, z\}, y, z, \{y, z\}, z, z, \{x, z\}, y, z, \{y, z\}\} = \{x, y, z, \{x, z\}, \{y, z\}\}$$
$$x \star (y \star z) = x \star \{y, z, \{y, z\}\} = \{x \star y, x \star z, x \star \{y, z\}\} = \{x, y, \{x, y\}, x, z, \{x, z\}, x \star y, x \star z\}$$
$$= \{x, y, \{x, y\}, x, z, \{x, z\}, x, y, \{x, y\}, x, z, \{x, z\}\} = \{x, y, z, \{x, y\}, \{x, z\}\}$$

Therefore $(x \star y) \star z \neq x \star (y \star z)$.

28.2. Commutative Neutrosophic SuperHyperGrupoid

Similarly, we define:

Let the Neutrosophic SuperHyperLaw * be defined as:

$*: (Z \cup \{\phi\}) \times (Z \cup \{\phi\}) \to P^2(Z \cup \{\phi\})$

where the empty set $\emptyset$ aslo leaves room for indeterminacy, unknown etc.

$$x \star y = \begin{cases} \{x, y, \{x, y\}\}, \text{for both } x, y \neq \emptyset \\ \emptyset, \text{ for } x, \text{or } y, \text{or both} = \emptyset \end{cases}$$

The law is well-defined, non-associative, and commutative (proven as above for the SuperHyperGroupoid).

29. Practical Application of the SuperHyperStructure

Let H be the set (system) that represent all inhabitants of the US country.

The set H is organized into 50 sub-sets, $H_1, H_2, \dots, H_{50}$

that represent the American states: where $H_1, H_2, ..., H_{50} \in P(H)$.

Each state $H_i, 1 \le i \le 50,$ is organized into counties, $H_{ij}, 1 \le j \le i_M$, where i_M is the maximum number of H_i state's counties, with all $H_{i,1}, H_{i,2}, ..., H_{i,i_M} \in P(H_i) \subset P(P(H)) = P^2(H)$.

(One uses commas in between indexes in order to separate them when the values of some indexes have two or more digits, for example $H_{3,12}$ means the 12th county of the 3rd state; which is different from $H_{31,2}$ that means the 2nd county of the 31st state.)

Further on, each $H_{i,j}$ county, for all *i* and *j* indexes,

is organized into sub-counties, $H_{i,j,k}, 1 \le k \le j_M$, where j_M is the maximum number of sub-counties of the county $H_{i,j}$. Therefore:

all $H_{i,j,1}, H_{i,j,2}, ..., H_{i,j,k_m} \in P(H_{i,j}) \subset P(P(H_i)) \subset P(P(P(H))) = P^3(H)$.

This shows the practical application of the n-th powerset of a set, for *n = 3* in this case (three levels of a SuperHyperStructure): country, states (one index *i*), counties (two indexes *i,j*), and sub-counties (three indexes *i,j,k*).

Surely, if needed, one can go *deeper in* (each sub-county is formed by towns, each town by districts, and so on); or *deeper out* (each country is part of a continent, each continent is part of a planet, each planet is part of a solar system, and so on).

The following SuperHyperStructure (denoted below by *A*), with three levels of structures, has been formed as:

$$A = \{H_i, H_{i,j}, H_{i,j,k}, 1 \le i \le 50, 1 \le j \le i_M, 1 \le k \le j_M\} \subset P^3(H).$$

(i) In the real world, this is a Neutrosophic SuperHyperStructure, because it has a lot of indeterminacy: for example the *population* of the country *H* is dynamic, in a continuous change: new people are born while others die as we speak, there are millions of illegal emigrants that are not counted as citizens, others have dual or triple citizenship so they only partially belong to *H*'s population; other citizen live outside the country.

(ii) Many laws are as well neutrosophic, because they apply to some states H_i (as examples: the law of abortion, or the law of bearing arms, or the law of consuming marijuana, etc.), but not to others. We call them NeutroLaws in the NeutroAlgebraic Structures (we mean: laws that are partially true and partially false in the same space, and sometime also partially indeterminate).

(iii) Let's endow this SuperHyperStructure with some SuperHyperLaw (called SuperHyper because it is built on a 3-rd PowerSet of the Set *H*):

$\#: A \times A \to A$

$x \# y = \{x \wedge (x \subseteq y) \to y\}$

{If x and $x \subseteq y$, then y}

Let $x, y \in A$. If x and $x \subseteq y$, then y.

Consider a cyber company that provides internet connection to people from the sub-county $H_{2,3,5}$, which is included in the county $H_{2,3}$, then the company will provide internet connection to the county $H_{2,3}$ as well.
Let $H_{2,3,5}$ and $H_{2,3} \in A$. If $H_{2,3,5}$ gets internet connection and
because $H_{2,3,5} \subset H_{2,3}$, then $H_{2,3}$ also gets internet connection.

30. Conclusion

We have extended the SuperHyperAlgebra and its correspondents (SuperHyperGraph, SuperHyperTopology, etc.) to the SuperHyperStructure in general, for any field of knowledge and on any type of space. The SuperHyperStructures were inspired from, and they perfectly fit, our real world. See the last practical application.

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TreeSoft Set vs. HyperSoft Set and Fuzzy-Extensions of TreeSoft Sets

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Abstract

The TreeSoft Set is more general than the HyperSoft Set. The TreeSoft Set is a union of HyperSoft Sets.
To distinguish between the levels of attributes, we use the comma notation i.e. an attribute of level k has k-1 commas in between its indexes.

Keywords: TreeSoft; HyperSoft; Fuzzy Extensions.

1 | Definitions and Examples

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{H}$ a non-empty subset of $\mathcal{U}$, with $\mathcal{P}(\mathcal{H})$ being the power set of $\mathcal{H}$.

Let A be a set of attributes $A_1, A_2, \dots, A_n$, $n \geq 2$,

$A = \{A_1, A_2, \dots, A_n\}$,

where each attribute A_i, $1 \leq i \leq n$, is a class of sub-attributes $A_i = \{A_{i,1}, A_{i,2}, \dots, A_{i,n_i}\}$.

1.1 | HyperSoft Set

Then: $f: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \dots \times \mathcal{P}(A_n) \to \mathcal{P}(\mathcal{H})$ is called a HyperSoft Set over $\mathcal{U}$.

1.2 | Practical Example of HyperSoft Set

Let $\mathcal{U}$ be a universal set of houses, and $\mathcal{H} = \{h_1, h_2, \dots, h_{10}\}$ of subset of $\mathcal{U}$, from a certain geographical zone (for example, the south-west side of United States). And $\mathcal{P}(\mathcal{H})$ is the powerset of $\mathcal{H}$.

Let's consider the set of attributes A be

$A = \{A_1, A_2\}$,

where $A_1 = \text{size} = \{\text{Small}, \text{Big}\}$, and $A_2 = \text{Arizonian cities} = \{\text{Phoenix}, \text{Tucson}\}$.

$F: A_1 \times A_2 \to \mathcal{P}(\mathcal{H})$

$F(\text{Big}, \text{Phoenix}) = \{h_1, h_3\}$.

$F(\text{Big}, \text{Tucson}) = \{h_5, h_7\}$.

2 | Definition of TreeSoft Set

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{H}$ a non-empty subset of $\mathcal{U}$, with $\mathcal{P}(\mathcal{H})$ the powerset of $\mathcal{H}$.

Let A be a set of attributes (parameters, factors, etc.),

$A= \{A_1, A_2, \ldots, A_n\}$, for integer $n \geq 1$, where $A_1, A_2, \ldots, A_n$ are considered attributes of first level (since they have one-digit indexes).

Each attribute $A_i, 1 \leq i \leq n$, is formed by sub-attributes:

$A_1 = \{A_{1,1}, A_{1,2}, \ldots\}$

$A_2 = \{A_{2,1}, A_{2,2}, \ldots\}$

.....

$A_n = \{A_{n,1}, A_{n,2}, \ldots\}$

where the above $A_{i,j}$ are sub-attributes (or attributes of second level) (since they have two-digit indexes).

Again, each sub-attribute $A_{i,j}$ is formed by sub-sub-attributes (or attributes of third level):

$A_{i,j,k}$

and so on, as much refinement as needed into each application, up to sub-sub-…-sub-attributes (or attributes of m-level (or having m digits into the indexes):

$A_{i_1,i_2,\ldots,i_m}$

Therefore, a graph-tree is formed, that we denote as Tree(A), whose root is A (considered of level zero), then nodes of level 1, level 2, up to level m.

We call leaves of the graph-tree, all terminal nodes (nodes that have no descendants).

Then the TreeSoft Set is defined as: $F: \mathcal{P}(\text{Tree}(A)) \rightarrow \mathcal{P}(\mathcal{H})$

Tree(A) is the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $\mathcal{P}(\text{Tree}(A))$ is the powerset of the Tree(A).

2.1 | Remark

We use comma (,) in between indexes to clearly distinguish between their levels.

For example, A_{12} is an attribute of the first level (has no comma between indexes) being part of the attributes $A_1, A_2, \ldots, A_{10}, A_{11}, A_{12}, A_{13}, \ldots$;

while $A_{1,2}$ is an attribute of second-level (or sub-attribute) [has one comma between indexes], being part of the sub-attributes:

$A_{1,1}, A_{1,2}, A_{1,3}, \ldots$.

Therefore $A_{i,j} \neq A_{ij}$ and similarly for all k-levels of attributes.

2.2 | Property

An attribute of level k has k-1 commas in between indexes.

Example: $A_{2,11,4}$ is an attribute of level 3 (has 2 commas in between indexes).

3 | Fuzzy Extensions of the TreeSoft Set

Let's denote by $d^{\circ}(h)$ the fuzzy (or fuzzy extension, such as: Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, n-HyperSpherical Fuzzy Set, etc.) degree of appurtenance of the element h with respect to the set it belongs to.

$F: \mathcal{P}(\text{Tree(A)}) \to \mathcal{P}(\mathcal{H}_{d^{\circ}(h)})$.

For any subset $\alpha \in \text{Tree}(A)$, one has $F(\alpha) = (h_{i_{(d^{\circ}(h_i))}}, h_i \in \mathcal{H})$.

From the previous Example 2.2.,

Neutrosophic HyperSoft Set

F(Big, Phoenix) = $\{h_1(0.7, 0.4, 0.3), h_3(0.6, 0.2, 0.3)\}$

F(Big, Tucson) = $\{h_5(0.1, 0.8, 0.3), h_7(0.9, 0.0, 0.5)\}$

Neutrosophic TreeSoft Set:

F(Big, Arizona) = F(Big, Phoenix) U F(Big, Tucson) =

= $\{h_1(0.7, 0.4, 0.3, h_3(0.6, 0.2, 0.3), h_5(0.1, 0.8, 0.3), h_7(0.9, 0.0, 0.5)\}$.

Fuzzy HyperSoft Set

F(Big, Phoenix) = $\{h_1(0.8), h_3(0.5)\}$,

F(Big, Tucson) = $\{h_5(0.2), h_7(0.6)\}$,

Fuzzy TreeSoft Set:

F(Big, Arizona) = F(Big, Phoenix) U F(Big, Tucson)=$\{h_1(0.8), h_3(0.5), h_5(0.2), h_7(0.6)\}$.

The first set is formed by the nodes of level 1, second set by the nodes of level 2, third set by the nodes of level 3, and so on, the last set is formed by the nodes of level m.

4 | Practical Example of TreeSoft Set of Level 3 [3]

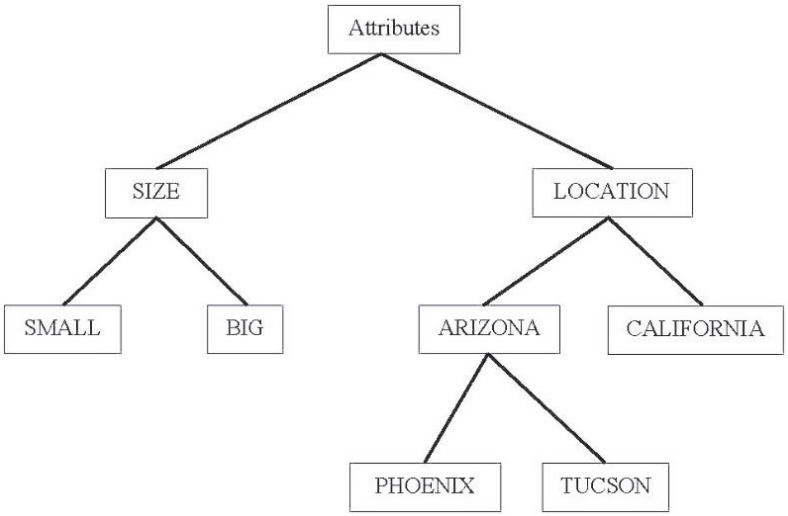


Figure 1. A TreeSoft Set of Level 3.

Assume a town has many houses.

This is a classical tree, whose:

Level 0 (the root) is the node: Attributes;

Level 1 is formed by the nodes: Size, Location;

Level 2 is formed by the nodes: Small, Big, Arizona, and California;

Level 3 is formed by the nodes: Phoenix, Tucson.

Let's consider $H = \{h_1, h_2, ..., h_{10}\}$ be a set of houses, and $P(H)$ the powerset of H.

And the set of first-level Attributes: $A = \{A_1, A_2\}$, where A_1 = Size, A_2 = Location.

Then $A_1 = \{A_{1,1}, A_{1,2}\}$ = {Small, Big} second-level attributes, and

$A_2 = \{A_{2,1}, A_{2,2}\}$ = {Arizona, California} as American states, also second-level attributes.

Further on, $A_{2,1} = \{A_{2,1,1}, A_{2,1,2}\}$ = {Phoenix, Tucson} as Arizonian cities, as third-level attributes.

Let's assume that the function F gets the following values:

F(Big, Arizona, Phoenix} = { h_1, h_3 }

F(Big, Arizona, Tucson) = { h_5, h_7}

F(Big, Arizona) = all big houses from both cities, Phoenix and Tucson =

= F(Big, Arizona, Phoenix) ∪ F(Big, Arizona, Tucson) =

={ h_1, h_3, h_5, h_7 }.

5 | Theorem

In general, the TreeSoft Set is a union of HyperSoft Sets.

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{H}$ a non-empty subset of $\mathcal{U}$, and $\mathcal{P}(\mathcal{H})$ the powerset of $\mathcal{H}$.

Let $A = \{A_1, A_2, \dots, A_n\}$, $n \geq 2$, be a set of attributes of first-level.

Then, $A_i = \{A_{i,1}, A_{i,2}, ..., A_{i,n_i}\}$, for $1 \leq i \leq n$, is a set of attributes of second-level.

Similarly, $A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, ..., A_{i,j,n_{ij}}\}$, for all i, j, a set of attributes of third-level.

$A_{i,j,k} = \{A_{i,j,k,1}, A_{i,j,k,2}, ..., A_{i,j,k,n_{ijk}}\}$ attributes of the third-level (sub-sub-attributes).

Up to attributes of level m:

$$A_{i_1,i_2,...,i_m} = \{A_{i_1,i_2,...,i_m,1}, A_{i_1,i_2,...,i_m,2}, ..., A_{i_1,i_2,...,i_m,n_{i_1i_2,...,i_m}}\}$$

where $A, A_i, A_{i,j}, A_{i,j,k}, \dots, A_{i_1,i_2,...,i_m}$ generate a Tree of m levels, denoted as Tree(A), and $\mathcal{P}(\text{Tree}(A))$ is the powerset of the Tree(A).

All as in Definition 3.

$F: \mathcal{P}(\text{Tree}(A)) \to \mathcal{P}(\mathcal{H})$ is the TreeSoft Set.

The powerset of Tree(A) is formed by elements that are nodes or leaves of the Tree(A).

The leaves are okay, as in HyperSoft Set, while a node is replaced by the union of all leaves that descend from it.

In the previous Example 5, if one takes the node Arizona, that is equivalent with (and replaced by) Phoenix and Tucson:

ARIZONA = {Phoenix, Tucson} or Phoenix ∪ Tucson.

If the node is LOCATION, then it is equivalent to, and replaced by {Phoenix, Tucson, California}.

The HyperSoft Set works with the leaves only.

But, because each node, no matter its level, is equivalent to the union of leaves that descend from it, one gets that the TreeSoft Set is equal to a union of HyperSoft Sets.

Therefore, the Treesoft Set is more general than the HyperSoft Set, and it is a union of HyperSoft Sets.

Coming back to the previous example, we have:

HyperSoft Sets

f(Big, Phoenix) = $\{h_1, h_3\}$, meaning big houses in Phoenix.

f(Big, Tucson) = $\{h_5, h_7\}$, meaning big houses in Tucson.

TreeSoft Set

f(Big, Arizona) = *f*(Big, {Phoenix, Tucson}) =

= *f*(Big, Phoenix) ∪ *f*(Big, Tucson) = $\{h_1, h_3\} \cup \{h_5, h_7\} = \{h_1, h_3, h_5, h_7\}$.

6 | Conclusion

The TreeSoft Set is equivalent to a union of HyperSoft Sets.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Theory of Hypersoft Sets: Axiomatic Properties, Aggregation Operations, Relations, Functions and Matrices

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Abstract. There are several decision-making based situations in which it is necessary to categorize the evaluating parameters into their respective sub-parametric values based non-overlapping sets. The existing soft set model is not compatible with such situations therefore hypersoft set ($\hat{H}s$-set) is developed which manages such situations by utilizing a novel mapping called multi-argument approximate mapping which broadens the domain of soft approximate mapping. This research presents the characterization of several essential axiomatic properties and set-based operations of $\hat{H}s$-set which will help the researchers to implement this emerging theory to other fields of study. The brief discussion on some hybridized structures of $\hat{H}s$-set with fuzzy set-like models is also provided.

Keywords: $\hat{H}s$-set; $\hat{H}s$-Relation; $\hat{H}s$-Function; $\hat{H}s$-Matrix.

1. Introduction

There are several models in literature to deal uncertainties but fuzzy set [1] is the most significant in this regards. It has its own intricacies which limit it to tackle uncertain decision-making scenarios effectively. The justification behind these obstacles is, potentially, the deficiency of parameterization tool. A novel model is required for managing vagueness and uncertainties which should be liberated from all such obstacles. In 1999, Molodtsov [2] established a set-structure known as soft set ($\hat{s}$-set) in literature as a novel parameterized sub-class of universal set. In the year 2003, Maji et al. [3] broadened the idea and investigated several rudimental axiomatic properties and set-operations of $\hat{s}$-sets . They also validated several results. Later on Pei et al. [4] introduced an information system (Inf-sys) by using the idea of $\hat{s}$-sets. It is proved that $\hat{s}$-set can be considered as a particular class of Inf-sys. Afterwards, Ali et al. [5] identified many assertions in the research proposed by Maji et al. and introduced novel notions

by using the concept of restricted and extended $\hat{s}$-set aggregation operations. In the same way, Babitha et al. [6, 7] made investigation on $\hat{s}$-set relation, $\hat{H}s$-set function by using the Cartesian product of $\hat{H}s$-sets. Sezgin et al. [8], Ge et al. [9], Fuli [10] provided few amendments in previous work by establishing few novel results. In order to utilize the concept of $\hat{s}$-sets in the development of algebraic structures, Saeed et al. [11] characterized the classical notions of elements and points under $\hat{s}$-set environment. Many researchers [12–23] discussed various gluing structures of $\hat{s}$-sets with other fuzzy set-like models to resolve several real-life decision making issues.

It is a matter of common observation that in various decision making problems, parameters have to be partitioned into their related sub-parametric valued sets whereas the previous researches on $\hat{s}$-set are not sufficient to manage such settings therefore Smarandache [24] initiated the notion of hypersoft set ($\hat{H}s$-set) as an extension of $\hat{s}$-set by introducing a novel multi-argument approximate mapping (maa-mapping). Any novel theory can not be implemented in real-world situations without the characterization of its elementary axiomatic-properties. Although Saeed et al. [25] made a good effort to investigate various basic properties of $\hat{H}s$-set but it does not cover many of the aspects of $\hat{H}s$-set theory. Therefore this paper aims to (i). generalize the research works described in [3, 5–10] for $\hat{H}s$-set environment and (ii). to modify the results discussed by Saeed et al. [25]. In the present work, all the necessary rudiments of $\hat{H}s$-set are investigated for its further developments. The Figure 1 explains the sectional-outlines of the paper.

Section 2	Recollects few essential definitions and results to assist the main results.	Section 3	Introduces few basic axiomatic properties of hypersoft sets and Illustrates set theoretic operations of hypersoft sets.
Section 4	Discusses some basic results and laws on hypersoft sets.	Section 5	Explains hypersoft relations and hypersoft functions.
Section 6	Presents the matrix representation of hypersoft sets with some operations	Section 7	Investigates few hybrids of hypersoft sets.
Section 8	Summarizes the paper with the provision of some future directions.		

FIGURE 1. Outlines of the paper

2. Preliminaries

The purpose of this section is to review some basic properties of $\hat{s}$-set for clear understanding of proposed study. The symbol $\hat{\Pi}$ will represent initial universe in the remaining parts of the article.

Definition 2.1. [2]
A $\hat{s}$-set $\mathbb{S}$ on $\hat{\Pi}$ is usually stated by a pair $(\Psi_{\mathbb{S}}, \mathfrak{G})$ in which $\Psi_{\mathbb{S}} : \mathfrak{G} \to P^{\hat{\Pi}}$ is an approximate mapping & $\mathfrak{G}$ be a sub-family of parameters. The family of $\hat{s}$-sets is symbolized as $\Sigma_{(\Psi_{\mathbb{S}},\mathfrak{G})}$.

Definition 2.2. [3]
For $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ & $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2) \in \Sigma_{(\Psi_{\mathbb{S}},\mathfrak{G})}$, if $\mathfrak{G}_1 \subseteq \mathfrak{G}_2$, & $\Psi_{\mathbb{S}_1}(\hat{\epsilon}) \subseteq \Psi_{\mathbb{S}_2}(\hat{\epsilon})$ for all $\hat{\epsilon} \in \mathfrak{G}_1$ then $\hat{s}$-set $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ is a *soft-subset* of $\hat{s}$-set $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2)$.

Definition 2.3. [3]
For $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ & $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2) \in \Sigma_{(\Psi_{\mathbb{S}},\mathfrak{G})}$, their union is a $\hat{s}$-set $(\Psi_{\mathbb{S}_3}, \mathfrak{G}_3)$ with $\mathfrak{G}_3 = \mathfrak{G}_1 \cup \mathfrak{G}_2$ & for $\hat{\epsilon} \in \mathfrak{G}_3$,

$$\Psi_{\mathbb{S}_3}(\hat{\epsilon}) = \begin{cases} \Psi_{\mathbb{S}_1}(\hat{\epsilon}) & \hat{\epsilon} \in (\mathfrak{G}_1 \setminus \mathfrak{G}_2) \\ \Psi_{\mathbb{S}_2}(\hat{\epsilon}) & \hat{\epsilon} \in (\mathfrak{G}_2 \setminus \mathfrak{G}_1) \\ \Psi_{\mathbb{S}_1}(\hat{\epsilon}) \cup \Psi_{\mathbb{S}_2}(\hat{\epsilon}) & \hat{\epsilon} \in (\mathfrak{G}_1 \cap \mathfrak{G}_2) \end{cases}$$

Definition 2.4. [3]
For $(\Psi_{\mathbb{S}_1}, \mathfrak{G}_1)$ & $(\Psi_{\mathbb{S}_2}, \mathfrak{G}_2) \in \Sigma_{(\Psi_{\mathbb{S}},\mathfrak{G})}$, their intersection is a $\hat{s}$-set $(\Psi_{\mathbb{S}_4}, \mathfrak{G}_4)$ with $\mathfrak{G}_4 = \mathfrak{G}_1 \cap \mathfrak{G}_2$ & for $\hat{\epsilon} \in \mathfrak{G}_4$, $\Psi_{\mathbb{S}_4}(\hat{\epsilon}) = \Psi_{\mathbb{S}_1}(\hat{\epsilon}) \cap \Psi_{\mathbb{S}_2}(\hat{\epsilon})$.

One can refer [2–10] for detailed description on $\hat{s}$-sets.

3. Hypersoft Set

This part of the paper provides the basic axiomatic-properties of $\hat{Hs}$-set along with the modification of some notions stated in [25].

Definition 3.1. [22]
Let $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, ..., \mathfrak{A}_n$ are non-overlapping sets having sub-parametric values of parameters $\ddot{a}_1, \ddot{a}_2, \ddot{a}_3, ..., \ddot{a}_n$ respectively, then a $\hat{Hs}$-set on $\hat{\Pi}$, is usually stated by a pair $(\Theta, \mathfrak{A})$ in which $\Theta : \mathfrak{A} \to P^{\hat{\Pi}}$ is a maa-mapping and $\mathfrak{A} = \prod_{i=1}^{n} \mathfrak{A}_i$. The family of $\hat{Hs}$-sets is symbolized by $\Sigma_{(\Theta,\mathfrak{A})}$. The model of $\hat{Hs}$-set is presented in Figure 2.

Example 3.2. Mrs. Smith visits a mobile mall to purchase a mobile for her personal use. She is accompanied by her two friends who are experts in mobile purchasing. They collectively observed 8 types of mobiles which are considered as elements of universal set $\hat{\Pi} = \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4, \hat{\mathfrak{M}}_5, \hat{\mathfrak{M}}_6, \hat{\mathfrak{M}}_7, \hat{\mathfrak{M}}_8\}$. They have fixed some parameters for this purchase with their mutual consensus that are $\hat{e}_1$ = random only memory in giga bytes, $\hat{e}_2$ = Resolution of camera in pixels, $\hat{e}_3$ = length in inches, $\hat{e}_4$ = random access memory in giga bytes, and $\hat{e}_5$ = power of battery in mAh. These parameters have their sub-collections as:
$\mathfrak{B}_1 = \{\hat{e}_{11} = 32, \hat{e}_{12} = 64\}$, $\mathfrak{B}_2 = \{\hat{e}_{21} = 8, \hat{e}_{22} = 16\}$, $\mathfrak{B}_3 = \{\hat{e}_{31} = 6.5, \hat{e}_{32} = 6.7\}$
$\mathfrak{B}_4 = \{\hat{e}_{41} = 4, \hat{e}_{42} = 8\}$, $\mathfrak{B}_5 = \{\hat{e}_{51} = 4000\}$ then $\mathfrak{A} = \mathfrak{B}_1 \times \mathfrak{B}_2 \times \mathfrak{B}_3 \times \mathfrak{B}_4 \times \mathfrak{B}_5$

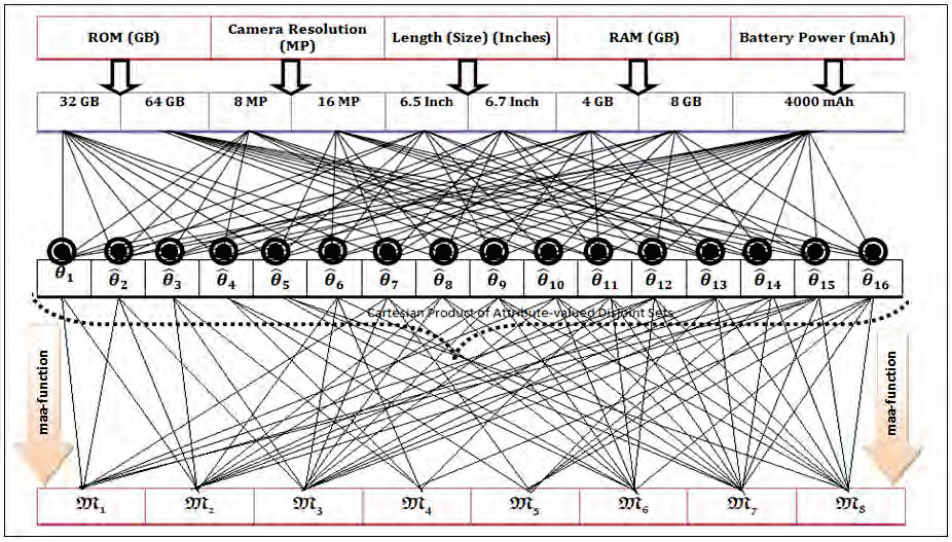


FIGURE 2. Pictorial Version of $\hat{H}s$-set

$\mathfrak{A} = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, ..., \hat{\theta}_{16}\}$ and every $\hat{\theta}_i$, $_{(1)}i^{(16)}$, is a 5-tuple member. Then the $\hat{H}s$-set $(\Theta, \mathfrak{A})$ is constructed as

$$(\Theta, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{\hat{\mathfrak{m}}_1, \hat{\mathfrak{m}}_2\}\right), \left(\hat{\theta}_2, \{\hat{\mathfrak{m}}_1, \hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3\}\right), \left(\hat{\theta}_3, \{\hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_4\}\right), \left(\hat{\theta}_4, \{\hat{\mathfrak{m}}_4, \hat{\mathfrak{m}}_5, \hat{\mathfrak{m}}_6\}\right), \\ \left(\hat{\theta}_5, \{\hat{\mathfrak{m}}_6, \hat{\mathfrak{m}}_7, \hat{\mathfrak{m}}_8\}\right), \left(\hat{\theta}_6, \{\hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_4, \hat{\mathfrak{m}}_7\}\right), \left(\hat{\theta}_7, \{\hat{\mathfrak{m}}_1, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_5, \hat{\mathfrak{m}}_6\}\right), \\ \left(\hat{\theta}_8, \{\hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_6, \hat{\mathfrak{m}}_7\}\right), \left(\hat{\theta}_9, \{\hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_6, \hat{\mathfrak{m}}_7, \hat{\mathfrak{m}}_8\}\right), \left(\hat{\theta}_{10}, \{\hat{\mathfrak{m}}_1, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_6, \hat{\mathfrak{m}}_7, \hat{\mathfrak{m}}_8\}\right), \\ \left(\hat{\theta}_{11}, \{\hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_4, \hat{\mathfrak{m}}_6, \hat{\mathfrak{m}}_7, \hat{\mathfrak{m}}_8\}\right), \left(\hat{\theta}_{12}, \{\hat{\mathfrak{m}}_1, \hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_6, \hat{\mathfrak{m}}_7, \hat{\mathfrak{m}}_8\}\right), \\ \left(\hat{\theta}_{13}, \{\hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_5, \hat{\mathfrak{m}}_7, \hat{\mathfrak{m}}_8\}\right), \left(\hat{\theta}_{14}, \{\hat{\mathfrak{m}}_1, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_5, \hat{\mathfrak{m}}_7, \hat{\mathfrak{m}}_8\}\right), \\ \left(\hat{\theta}_{15}, \{\hat{\mathfrak{m}}_1, \hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_5, \hat{\mathfrak{m}}_7, \hat{\mathfrak{m}}_8\}\right), \left(\hat{\theta}_{16}, \{\hat{\mathfrak{m}}_4, \hat{\mathfrak{m}}_5, \hat{\mathfrak{m}}_6, \hat{\mathfrak{m}}_7, \hat{\mathfrak{m}}_8\}\right) \end{array} \right\}$$

Definition 3.3. Let $\mathcal{F}^{\hat{\Pi}}$ be a collection consisting of fuzzy subsets on $\hat{\Pi}$. Let $\ddot{a}_i, n \geq 1$, $_1i^n$ are parameters having their relevant sub-parametric values in the sets $\mathfrak{A}_i$ respectively, with $\mathfrak{A}_i \cap \mathfrak{A}_j = \varnothing$, for $i \neq j$, & $_1i^n$, $_1j^n$. Then a *fuzzy* $\hat{H}s$*-set* $(\Theta_{fhs}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as,

$$(\Theta_{fhs}, \mathfrak{A}) = \{(\hat{\theta}, \Theta_{fhs}(\hat{\theta})) : \hat{\theta} \in \mathfrak{A}, \Theta_{fhs}(\hat{\theta}) \in \mathcal{F}^{\hat{\Pi}}\}$$

where $\Theta_{fhs} : \mathfrak{A} \to \mathcal{F}^{\hat{\Pi}}$ and for all $\hat{\theta} \in \mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times ..... \times \mathfrak{A}_n$

$$\Theta_{fhs}(\hat{\theta}) = \{\mu_{\Theta_{fhs}(\hat{\theta})}(\varpi)/\varpi : \varpi \in \hat{\Pi}, \mu_{\Theta_{fhs}(\hat{\theta})}(\varpi) \in \mathbb{C}(\mathbb{I}) = [0,1]\}$$

is a fuzzy set on $\hat{\Pi}$.

One can consider this definition as modified form of fuzzy $\hat{H}s$-set stated in [22] and [24].

Example 3.4. Assuming the Example 3.2, Fuzzy $\hat{H}s$-set $(\Theta_{fhs}, \mathfrak{A})$ is constructed as

$$(\Theta_{fhs}, \mathfrak{A}) = \left\{ \begin{array}{l} (\hat{\theta}_1, \{0.1/\hat{\mathfrak{m}}_1, 0.2/\hat{\mathfrak{m}}_2\}), (\hat{\theta}_2, \{0.1/\hat{\mathfrak{m}}_1, 0.2/\hat{\mathfrak{m}}_2, 0.3/\hat{\mathfrak{m}}_3\}), (\hat{\theta}_3, \{0.2/\hat{\mathfrak{m}}_2, 0.3/\hat{\mathfrak{m}}_3, 0.4/\hat{\mathfrak{m}}_4\}), \\ (\hat{\theta}_4, \{0.4/\hat{\mathfrak{m}}_4, 0.5/\hat{\mathfrak{m}}_5, 0.6/\hat{\mathfrak{m}}_6\}), (\hat{\theta}_5, \{0.6/\hat{\mathfrak{m}}_6, 0.7/\hat{\mathfrak{m}}_7, 0.8/\hat{\mathfrak{m}}_8\}), (\hat{\theta}_6, \{0.2/\hat{\mathfrak{m}}_2, 0.3/\hat{\mathfrak{m}}_3, 0.4/\hat{\mathfrak{m}}_4, 0.7/\hat{\mathfrak{m}}_7\}), \\ (\hat{\theta}_7, \{0.1/\hat{\mathfrak{m}}_1, 0.3/\hat{\mathfrak{m}}_3, 0.5/\hat{\mathfrak{m}}_5, 0.6/\hat{\mathfrak{m}}_6\}), (\hat{\theta}_8, \{0.2/\hat{\mathfrak{m}}_2, 0.3/\hat{\mathfrak{m}}_3, 0.6/\hat{\mathfrak{m}}_6, 0.7/\hat{\mathfrak{m}}_7\}), \\ (\hat{\theta}_9, \{0.2/\hat{\mathfrak{m}}_2, 0.3/\hat{\mathfrak{m}}_3, 0.6/\hat{\mathfrak{m}}_6, 0.7/\hat{\mathfrak{m}}_7, 0.8/\hat{\mathfrak{m}}_8\}), (\hat{\theta}_{10}, \{0.1/\hat{\mathfrak{m}}_1, 0.3/\hat{\mathfrak{m}}_3, 0.6/\hat{\mathfrak{m}}_6, 0.7/\hat{\mathfrak{m}}_7, 0.8/\hat{\mathfrak{m}}_8\}), \\ (\hat{\theta}_{11}, \{0.2/\hat{\mathfrak{m}}_2, 0.4/\hat{\mathfrak{m}}_4, 0.6/\hat{\mathfrak{m}}_6, 0.7/\hat{\mathfrak{m}}_7, 0.8/\hat{\mathfrak{m}}_8\}), (\hat{\theta}_{12}, \{0.1/\hat{\mathfrak{m}}_1, 0.2/\hat{\mathfrak{m}}_2, 0.3/\hat{\mathfrak{m}}_3, 0.6/\hat{\mathfrak{m}}_6, 0.7/\hat{\mathfrak{m}}_7, 0.8/\hat{\mathfrak{m}}_8\}), \\ (\hat{\theta}_{13}, \{0.2/\hat{\mathfrak{m}}_2, 0.3/\hat{\mathfrak{m}}_3, 0.5/\hat{\mathfrak{m}}_5, 0.7/\hat{\mathfrak{m}}_7, 0.8/\hat{\mathfrak{m}}_8\}), (\hat{\theta}_{14}, \{0.1/\hat{\mathfrak{m}}_1, 0.3/\hat{\mathfrak{m}}_3, 0.5/\hat{\mathfrak{m}}_5, 0.7/\hat{\mathfrak{m}}_7, 0.8/\hat{\mathfrak{m}}_8\}), \\ (\hat{\theta}_{15}, \{0.1/\hat{\mathfrak{m}}_1, 0.2/\hat{\mathfrak{m}}_2, 0.3/\hat{\mathfrak{m}}_3, 0.5/\hat{\mathfrak{m}}_5, 0.7/\hat{\mathfrak{m}}_7, 0.8/\hat{\mathfrak{m}}_8\}), (\hat{\theta}_{16}, \{0.4/\hat{\mathfrak{m}}_4, 0.5/\hat{\mathfrak{m}}_5, 0.6/\hat{\mathfrak{m}}_6, 0.7/\hat{\mathfrak{m}}_7, 0.8/\hat{\mathfrak{m}}_8\}) \end{array} \right\}$$

Definition 3.5. Let $(\Theta_1, \mathfrak{A}_1), (\Theta_2, \mathfrak{A}_2) \in \Sigma_{(\Theta, \mathfrak{A})}$ then $(\Theta_1, \mathfrak{A}_1)$ is said to be $\hat{H}s$- *subset* of $(\Theta_2, \mathfrak{A}_2)$ if $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ and $\forall \hat{\theta} \in \mathfrak{A}_1, \Theta_1(\hat{\theta}) \subseteq \Theta_2(\hat{\theta})$.

Example 3.6. Assuming Example 3.2, if

$(\Theta_1, \mathfrak{A}_1) = \left\{ (\hat{\theta}_1, \{\hat{\mathfrak{m}}_1\}), (\hat{\theta}_2, \{\hat{\mathfrak{m}}_1, \hat{\mathfrak{m}}_2\}), (\hat{\theta}_3, \{\hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3\}) \right\}$

$(\Theta_2, \mathfrak{A}_2) = \left\{ (\hat{\theta}_1, \{\hat{\mathfrak{m}}_1, \hat{\mathfrak{m}}_2\}), (\hat{\theta}_2, \{\hat{\mathfrak{m}}_1, \hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3\}), (\hat{\theta}_3, \{\hat{\mathfrak{m}}_2, \hat{\mathfrak{m}}_3, \hat{\mathfrak{m}}_4\}), (\hat{\theta}_4, \{\hat{\mathfrak{m}}_4, \hat{\mathfrak{m}}_5, \hat{\mathfrak{m}}_6\}) \right\}$

then $(\Theta_1, \mathfrak{A}_1) \subseteq (\Theta_2, \mathfrak{A}_2)$.

Definition 3.7. A set $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times ..... \times \mathfrak{A}_n$ in $\hat{H}s$-set $(\Theta, \mathfrak{A})$ is said to be *Not set* if it has the representation as $\ltimes\mathfrak{A} = \{\ltimes\hat{\theta}_1, \ltimes\hat{\theta}_2, \ltimes\hat{\theta}_3, \ltimes\hat{\theta}_4, ..., \ltimes\hat{\theta}_m\}$ where $m = \prod_{i=1}^{n} |\mathfrak{A}_i|$.

Example 3.8. Reconsidering $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4, , \mathfrak{A}_5$ from Example 3.2, we get

$$\ltimes\mathfrak{A} = \{\ltimes\hat{\theta}_1, \ltimes\hat{\theta}_2, \ltimes\hat{\theta}_3, \ltimes\hat{\theta}_4, ..., \ltimes\hat{\theta}_{16}\}$$

.

Definition 3.9. A $\hat{H}s$-set $(\Theta, \mathfrak{A}_1)$ is stated as a *relative null* $\hat{H}s$*-set* w.r.t $\mathfrak{A}_1 \subseteq \mathfrak{A}$, symbolized by $(\Theta, \mathfrak{A}_1)_\Phi$, if $\Theta(\hat{\theta}) = \varnothing, \forall \hat{\theta} \in \mathfrak{A}_1$.

Example 3.10. Assuming Example 3.2, if $(\Theta, \mathfrak{A}_1)_\Phi = \left\{ (\hat{\theta}_1, \varnothing), (\hat{\theta}_2, \varnothing), (\hat{\theta}_3, \varnothing) \right\}$ where $\mathfrak{A}_1 \subseteq \mathfrak{A}$.

Definition 3.11. A $\hat{H}s$-set $(\Theta, \mathfrak{A}_1)$ is stated as a *relative whole* $\hat{H}s$*-set* w.r.t $\mathfrak{A}_1 \subseteq \mathfrak{A}$, symbolized by $(\Theta, \mathfrak{A}_1)_{\hat{\Pi}}$, if $\Theta(\hat{\theta}) = \hat{\Pi}, \forall \hat{\theta} \in \mathfrak{A}_1$.

Example 3.12. Assuming Example 3.2, if $(\Theta, \mathfrak{A}_1)_{\hat{\Pi}} = \left\{ (\hat{\theta}_1, \hat{\Pi}), (\hat{\theta}_2, \hat{\Pi}), (\hat{\theta}_3, \hat{\Pi}) \right\}$ where $\mathfrak{A}_1 \subseteq \mathfrak{A}$.

Definition 3.13. A $\hat{H}s$-set $(\Theta, \mathfrak{A})$ is stated as a *absolute whole* $\hat{H}s$*-set* on $\hat{\Pi}$, symbolized by $(\Theta, \mathfrak{A})_{\hat{\Pi}}$, if $\Theta(\hat{\theta}) = \hat{\Pi}, \forall \hat{\theta} \in \mathfrak{A}$.

Example 3.14. Assuming Example 3.2, if

$$(\Theta, \mathfrak{A})_{\hat{\Pi}} = \left\{ \begin{array}{l} (\hat{\theta}_1, \hat{\Pi}), (\hat{\theta}_2, \hat{\Pi}), (\hat{\theta}_3, \hat{\Pi}), (\hat{\theta}_4, \hat{\Pi}), (\hat{\theta}_5, \hat{\Pi}), (\hat{\theta}_6, \hat{\Pi}), (\hat{\theta}_7, \hat{\Pi}), (\hat{\theta}_8, \hat{\Pi}), \\ (\hat{\theta}_9, \hat{\Pi}), (\hat{\theta}_{10}, \hat{\Pi}), (\hat{\theta}_{11}, \hat{\Pi}), (\hat{\theta}_{12}, \hat{\Pi}), (\hat{\theta}_{13}, \hat{\Pi}), (\hat{\theta}_{14}, \hat{\Pi}), (\hat{\theta}_{15}, \hat{\Pi}), (\hat{\theta}_{16}, \hat{\Pi}) \end{array} \right\}$$

Proposition 3.15. *Let* $(\Theta_1,\mathfrak{A}_1),(\Theta_2,\mathfrak{A}_2),(\Theta_3,\mathfrak{A}_3)\in\Sigma_{(\Theta,\mathfrak{A})}$ *with* $\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3\subseteq\mathfrak{A}$ *then*

(i) $(\Theta_1,\mathfrak{A}_1)\subseteq(\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}}$

(ii) $(\Theta_1,\mathfrak{A}_1)_{\Phi}\subseteq(\Theta_1,\mathfrak{A}_1)$

(iii) $(\Theta_1,\mathfrak{A}_1)\subseteq(\Theta_1,\mathfrak{A}_1)$

(iv) *If* $(\Theta_1,\mathfrak{A}_1)\subseteq(\Theta_2,\mathfrak{A}_2)$ *&* $(\Theta_2,\mathfrak{A}_2)\subseteq(\Theta_3,\mathfrak{A}_3)$ *then* $(\Theta_1,\mathfrak{A}_1)\subseteq(\Theta_3,\mathfrak{A}_3)$

(v) *If* $(\Theta_1,\mathfrak{A}_1)=(\Theta_2,\mathfrak{A}_2)$ *&* $(\Theta_2,\mathfrak{A}_2)=(\Theta_3,\mathfrak{A}_3)$ *then* $(\Theta_1,\mathfrak{A}_1)=(\Theta_3,\mathfrak{A}_3)$

Definition 3.16. The *complement* of a $\hat{H}s$-set $(\Theta,\mathfrak{A})$, symbolized by $(\Theta,\mathfrak{A})^{\ominus}$, is stated as $(\Theta,\mathfrak{A})^{\ominus}=(\Theta^{\ominus},\ltimes\mathfrak{A})$ where $\Theta^{\ominus}:\ltimes\mathfrak{A}\to P^{\hat{\Pi}}$ with $\Theta^{\ominus}(\ltimes\hat{\theta})=\hat{\Pi}\setminus\Theta(\hat{\theta}),\forall\hat{\theta}\in\mathfrak{A}$.

Example 3.17. From Example 3.2, we get

$$(\Theta,\mathfrak{A})^{\ominus}=\left\{\begin{array}{l}(\ltimes\hat{\theta}_1,\{\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5,\hat{\mathfrak{m}}_6,\hat{\mathfrak{m}}_7,\hat{\mathfrak{m}}_8\}),(\ltimes\hat{\theta}_2,\{\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5,\hat{\mathfrak{m}}_6,\hat{\mathfrak{m}}_7,\hat{\mathfrak{m}}_8\}),(\ltimes\hat{\theta}_3,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_5,\hat{\mathfrak{m}}_6,\hat{\mathfrak{m}}_7,\hat{\mathfrak{m}}_8\}),\\(\ltimes\hat{\theta}_4,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_7,\hat{\mathfrak{m}}_8\}),(\ltimes\hat{\theta}_5,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5\}),(\ltimes\hat{\theta}_6,\{\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_7\}),\\(\ltimes\hat{\theta}_7,\{\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_7,\hat{\mathfrak{m}}_8\}),(\ltimes\hat{\theta}_8,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5,\hat{\mathfrak{m}}_8\}),(\ltimes\hat{\theta}_9,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5\}),(\ltimes\hat{\theta}_{10},\{\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5\}),\\(\ltimes\hat{\theta}_{11},\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_5\}),(\ltimes\hat{\theta}_{12},\{\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5\}),(\ltimes\hat{\theta}_{13},\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_6\}),(\ltimes\hat{\theta}_{14},\{\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_6\}),\\(\ltimes\hat{\theta}_{15},\{\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_6\}),(\ltimes\hat{\theta}_{16},\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_3\})\end{array}\right\}$$

Definition 3.18. The *relative complement* of a $\hat{H}s$-set $(\Theta,\mathfrak{A})$, symbolized by $(\Theta,\mathfrak{A})^{\circledast}$, is stated as $(\Theta,\mathfrak{A})^{\circledast}=(\Theta^{\circledast},\mathfrak{A})$ where $\Theta^{\circledast}:\mathfrak{A}\to P^{\hat{\Pi}}$ with $\Theta^{\circledast}(\hat{\theta})=\hat{\Pi}\setminus\Theta(\hat{\theta}),\forall\hat{\theta}\in\mathfrak{A}$.

Example 3.19. Reconsidering Example 3.2, we get

$$(\Theta,\mathfrak{A})^{\circledast}=\left\{\begin{array}{l}(\hat{\theta}_1,\{\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5,\hat{\mathfrak{m}}_6,\hat{\mathfrak{m}}_7,\hat{\mathfrak{m}}_8\}),(\hat{\theta}_2,\{\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5,\hat{\mathfrak{m}}_6,\hat{\mathfrak{m}}_7,\hat{\mathfrak{m}}_8\}),(\hat{\theta}_3,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_5,\hat{\mathfrak{m}}_6,\hat{\mathfrak{m}}_7,\hat{\mathfrak{m}}_8\}),\\(\hat{\theta}_4,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_7,\hat{\mathfrak{m}}_8\}),(\hat{\theta}_5,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5\}),(\hat{\theta}_6,\{\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_7\}),\\(\hat{\theta}_7,\{\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_7,\hat{\mathfrak{m}}_8\}),(\hat{\theta}_8,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5,\hat{\mathfrak{m}}_8\}),(\hat{\theta}_9,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5\}),(\hat{\theta}_{10},\{\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5\}),\\(\hat{\theta}_{11},\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_5\}),(\hat{\theta}_{12},\{\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5\}),(\hat{\theta}_{13},\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_6\}),(\hat{\theta}_{14},\{\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_6\}),\\(\hat{\theta}_{15},\{\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_6\}),(\hat{\theta}_{16},\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_3\})\end{array}\right\}$$

Proposition 3.20. *Let* $(\Theta,\mathfrak{A})\in\Sigma_{(\Theta,\mathfrak{A})}$ *then*

(i) $((\Theta,\mathfrak{A})^{\ominus})^{\ominus}=(\Theta,\mathfrak{A})$

(ii) $((\Theta,\mathfrak{A})^{\circledast})^{\circledast}=(\Theta,\mathfrak{A})$

(iii) $((\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}})^{\ominus}=(\Theta_1,\mathfrak{A}_1)_{\Phi}=((\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}})^{\circledast}$; $\mathfrak{A}_1\subseteq\mathfrak{A}$

(iv) $((\Theta_1,\mathfrak{A}_1)_{\Phi})^{\ominus}=(\Theta_1,\mathfrak{A}_1)_{\hat{\Pi}}=((\Theta_1,\mathfrak{A}_1)_{\Phi})^{\circledast}$; $\mathfrak{A}_1\subseteq\mathfrak{A}$

Definition 3.21. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, the union-operation $(\Theta_1,\mathfrak{A}_1)\cup(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\cup\mathfrak{A}_2$ and for $\hat{\theta}\in\mathfrak{A}_3$,

$$\Theta_3(\hat{\theta})=\left\{\begin{array}{cc}\Theta_1(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_1\setminus\mathfrak{A}_2)\\ \Theta_2(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_2\setminus\mathfrak{A}_1)\\ \Theta_1(\hat{\theta})\cup\Theta_2(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_1\cap\mathfrak{A}_2)\end{array}\right.\quad.$$

Example 3.22. Let

$$(\Theta_1,\mathfrak{A}_1)=\left\{(\hat{\theta}_1,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_2\}),(\hat{\theta}_2,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_3\}),(\hat{\theta}_3,\{\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_3,\hat{\mathfrak{m}}_4\})\right\}$$
$$(\Theta_2,\mathfrak{A}_2)=\left\{(\hat{\theta}_3,\{\hat{\mathfrak{m}}_1,\hat{\mathfrak{m}}_2\}),(\hat{\theta}_4,\{\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_5,\hat{\mathfrak{m}}_6\}),(\hat{\theta}_5,\{\hat{\mathfrak{m}}_2,\hat{\mathfrak{m}}_4,\hat{\mathfrak{m}}_6\})\right\}$$

then

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\hat{\theta}_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\hat{\theta}_2,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\}\right),\left(\hat{\theta}_3,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4\}\right),\\ \left(\hat{\theta}_4,\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\left(\hat{\theta}_5,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right)\end{array}\right\}$$

Definition 3.23. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, the intersection-operation $(\Theta_1,\mathfrak{A}_1)\cap(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\cap\mathfrak{A}_2$ & for $\hat{\theta}\in\mathfrak{A}_3$, $\Theta_3(\hat{\theta})=\Theta_1(\hat{\theta})\cap\Theta_2(\hat{\theta})$.

Example 3.24. Reconsidering Example 3.22, we get $(\Theta_3,\mathfrak{A}_3)=\left\{\left(\hat{\theta}_3,\{\hat{\mathfrak{M}}_2\}\right)\right\}$.

Definition 3.25. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their extended-intersection $(\Theta_1,\mathfrak{A}_1)\cap_\varepsilon(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\cup\mathfrak{A}_2$ and for $\hat{\theta}\in\mathfrak{A}_3$,

$$\Theta_3(\hat{\theta})=\left\{\begin{array}{cc}\Theta_1(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_1\smallsetminus\mathfrak{A}_2)\\ \Theta_2(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_2\smallsetminus\mathfrak{A}_1)\\ \Theta_1(\hat{\theta})\cap\Theta_2(\hat{\theta}) & \hat{\theta}\in(\mathfrak{A}_1\cap\mathfrak{A}_2)\end{array}\right.$$

Example 3.26. Taking assumptions of Example 3.22, we get

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\hat{\theta}_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\hat{\theta}_2,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\}\right),\left(\hat{\theta}_3,\{\hat{\mathfrak{M}}_2\}\right),\\ \left(\hat{\theta}_4,\{\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\left(\hat{\theta}_5,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right)\end{array}\right\}$$

Definition 3.27. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their AND-operation $(\Theta_1,\mathfrak{A}_1)\bigwedge(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\times\mathfrak{A}_2$ and for $(\hat{\theta}_i,\hat{\theta}_j)\in\mathfrak{A}_3,\hat{\theta}_i\in\mathfrak{A}_1,\hat{\theta}_j\in\mathfrak{A}_2$,

$$\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\Theta_1(\hat{\theta}_i)\cup\Theta_2(\hat{\theta}_j).$$

Example 3.28. Taking assumptions of Example 3.22, we get

$$\mathfrak{A}_1\times\mathfrak{A}_2=\left\{\begin{array}{l}\pi_1=\left(\hat{\theta}_1,\hat{\theta}_3\right),\pi_2=\left(\hat{\theta}_1,\hat{\theta}_4\right),\pi_3=\left(\hat{\theta}_1,\hat{\theta}_5\right),\pi_4=\left(\hat{\theta}_2,\hat{\theta}_3\right),\pi_5=\left(\hat{\theta}_2,\hat{\theta}_4\right),\\ \pi_6=\left(\hat{\theta}_2,\hat{\theta}_5\right),\pi_7=\left(\hat{\theta}_3,\hat{\theta}_3\right),\pi_8=\left(\hat{\theta}_3,\hat{\theta}_4\right),\pi_9=\left(\hat{\theta}_3,\hat{\theta}_5\right)\end{array}\right\}$$

then

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\pi_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\left(\pi_2,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\\ \left(\pi_3,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right),\left(\pi_4,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3\}\right),\\ \left(\pi_5,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\left(\pi_6,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right),\\ \left(\pi_7,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4\}\right),\left(\pi_8,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_5,\hat{\mathfrak{M}}_6\}\right),\\ \left(\pi_9,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_3,\hat{\mathfrak{M}}_4,\hat{\mathfrak{M}}_6\}\right),\end{array}\right\}$$

Definition 3.29. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their OR-operation $(\Theta_1,\mathfrak{A}_1)\bigvee(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3=\mathfrak{A}_1\times\mathfrak{A}_2$ and for $(\hat{\theta}_i,\hat{\theta}_j)\in\mathfrak{A}_3,\hat{\theta}_i\in\mathfrak{A}_1,\hat{\theta}_j\in\mathfrak{A}_2$,

$$\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\Theta_1(\hat{\theta}_i)\cap\Theta_2(\hat{\theta}_j).$$

Example 3.30. Taking assumptions of Examples 3.22 and 3.30, we get

$$(\Theta_3,\mathfrak{A}_3)=\left\{\begin{array}{l}\left(\pi_1,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),(\pi_2,\{\}),\left(\pi_3,\{\hat{\mathfrak{M}}_2\}\right),\left(\pi_4,\{\hat{\mathfrak{M}}_1,\hat{\mathfrak{M}}_2\}\right),\\ (\pi_5,\{\}),\left(\pi_6,\{\hat{\mathfrak{M}}_2\}\right),\left(\pi_7,\{\hat{\mathfrak{M}}_2\}\right),\left(\pi_8,\{\hat{\mathfrak{M}}_4\}\right),\left(\pi_9,\{\hat{\mathfrak{M}}_2,\hat{\mathfrak{M}}_4\}\right),\end{array}\right\}$$

Definition 3.31. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2) \in \Sigma_{(\Theta,\mathfrak{A})}$, their restricted-union $(\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3 = \mathfrak{A}_1 \cap \mathfrak{A}_2$ and for $\hat{\theta} \in \mathfrak{A}_3$,

$$\Theta_3(\hat{\theta}) = \Theta_1(\hat{\theta}) \cup \Theta_2(\hat{\theta}).$$

Example 3.32. Taking assumptions of Example 3.22, we get

$$(\Theta_3,\mathfrak{A}_3) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_2, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$$

Definition 3.33. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2) \in \Sigma_{(\Theta,\mathfrak{A})}$, their restricted-difference $(\Theta_1,\mathfrak{A}_1) \setminus_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ with $\mathfrak{A}_3 = \mathfrak{A}_1 \cap \mathfrak{A}_2$ and for $\hat{\theta} \in \mathfrak{A}_3$,

$$\Theta_3(\hat{\theta}) = \Theta_1(\hat{\theta}) - \Theta_2(\hat{\theta}).$$

Example 3.34. Taking suppositions of Example 3.22, we get $(\Theta_3,\mathfrak{A}_3) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$.

Definition 3.35. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2) \in \Sigma_{(\Theta,\mathfrak{A})}$, their restricted-symmetric-difference $(\Theta_1,\mathfrak{A}_1) \blacktriangle (\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ stated by

$$(\Theta_3,\mathfrak{A}_3) = \left\{ ((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) \setminus_{\mathcal{R}} ((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2)) \right\}$$

or

$$(\Theta_3,\mathfrak{A}_3) = \left\{ ((\Theta_1,\mathfrak{A}_1) \setminus_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) \cup_{\mathcal{R}} ((\Theta_2,\mathfrak{A}_2) \setminus_{\mathcal{R}} (\Theta_1,\mathfrak{A}_1)) \right\}$$

Example 3.36. Taking suppositions of Example 3.22, we get

$$((\Theta_1,\mathfrak{A}_1) \setminus_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$$

&

$$((\Theta_2,\mathfrak{A}_2) \setminus_{\mathcal{R}} (\Theta_1,\mathfrak{A}_1)) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_1\}\right) \right\}$$

then

$$(\Theta_3,\mathfrak{A}_3) = \left\{ \left(\hat{\theta}_3, \{\hat{\mathfrak{M}}_1, \hat{\mathfrak{M}}_3, \hat{\mathfrak{M}}_4\}\right) \right\}$$

4. Axioms-based Results of $\hat{H}s$-sets

This part presents some classical axioms-based results of set theory that are also valid for $\hat{H}s$-settings.

(1) **Idempotent Laws**

(a) $(\Theta,\mathfrak{A}) \cup (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) \cup_{\mathcal{R}} (\Theta,\mathfrak{A})$

(b) $(\Theta,\mathfrak{A}) \cap (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) \cap_{\varepsilon} (\Theta,\mathfrak{A})$

(2) **Identity Laws**

(a) $(\Theta,\mathfrak{A}) \cup (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) \cup_{\mathcal{R}} (\Theta,\mathfrak{A})_{\Phi}$

(b) $(\Theta,\mathfrak{A}) \cap (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) \cap_{\varepsilon} (\Theta,\mathfrak{A})_{\hat{\Pi}}$

(c) $(\Theta,\mathfrak{A}) \setminus_{\mathcal{R}} (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A}) \blacktriangle (\Theta,\mathfrak{A})_{\Phi}$

(d) $(\Theta,\mathfrak{A}) \setminus_{\mathcal{R}} (\Theta,\mathfrak{A}) = (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) \blacktriangle (\Theta,\mathfrak{A})$

(3) **Domination Laws**

(a) $(\Theta,\mathfrak{A}) \cup (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A}) \cup_{\mathcal{R}} (\Theta,\mathfrak{A})_{\hat{\Pi}}$

(b) $(\Theta,\mathfrak{A}) \cap (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) \cap_{\varepsilon} (\Theta,\mathfrak{A})_{\Phi}$

(4) **Property of Exclusion**

$$(\Theta,\mathfrak{A}) \cup (\Theta,\mathfrak{A})^{\circledast} = (\Theta,\mathfrak{A})_{\hat{\Pi}} = (\Theta,\mathfrak{A}) \cup_{\mathcal{R}} (\Theta,\mathfrak{A})^{\circledast}$$

(5) **Property of Contradiction**

$$(\Theta,\mathfrak{A}) \cap (\Theta,\mathfrak{A})^{\circledast} = (\Theta,\mathfrak{A})_{\Phi} = (\Theta,\mathfrak{A}) \cap_{\varepsilon} (\Theta,\mathfrak{A})^{\circledast}$$

(6) **Absorption Laws**

(a) $(\Theta_1,\mathfrak{A}_1) \cup ((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(b) $(\Theta_1,\mathfrak{A}_1) \cap ((\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(c) $(\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} ((\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(d) $(\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} ((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) = (\Theta_1,\mathfrak{A}_1)$

(7) **Commutative Laws**

(a) $(\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cup (\Theta_1,\mathfrak{A}_1)$

(b) $(\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cup_{\mathcal{R}} (\Theta_1,\mathfrak{A}_1)$

(c) $(\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cap (\Theta_1,\mathfrak{A}_1)$

(d) $(\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \cap_{\varepsilon} (\Theta_1,\mathfrak{A}_1)$

(e) $(\Theta_1,\mathfrak{A}_1) \blacktriangle (\Theta_2,\mathfrak{A}_2) = (\Theta_2,\mathfrak{A}_2) \blacktriangle (\Theta_1,\mathfrak{A}_1)$

(8) **Associative Laws**

(a) $(\Theta_1,\mathfrak{A}_1) \cup ((\Theta_2,\mathfrak{A}_2) \cup (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2)) \cup (\Theta_3,\mathfrak{A}_3)$

(b) $(\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} ((\Theta_2,\mathfrak{A}_2) \cup_{\mathcal{R}} (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)) \cup_{\mathcal{R}} (\Theta_3,\mathfrak{A}_3)$

(c) $(\Theta_1,\mathfrak{A}_1) \cap ((\Theta_2,\mathfrak{A}_2) \cap (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2)) \cap (\Theta_3,\mathfrak{A}_3)$

(d) $(\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} ((\Theta_2,\mathfrak{A}_2) \cap_{\varepsilon} (\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2)) \cap_{\varepsilon} (\Theta_3,\mathfrak{A}_3)$

(e) $(\Theta_1,\mathfrak{A}_1) \bigvee((\Theta_2,\mathfrak{A}_2) \bigvee(\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \bigvee(\Theta_2,\mathfrak{A}_2)) \bigvee(\Theta_3,\mathfrak{A}_3)$

(f) $(\Theta_1,\mathfrak{A}_1) \bigwedge((\Theta_2,\mathfrak{A}_2) \bigwedge(\Theta_3,\mathfrak{A}_3)) = ((\Theta_1,\mathfrak{A}_1) \bigwedge(\Theta_2,\mathfrak{A}_2)) \bigwedge(\Theta_3,\mathfrak{A}_3)$

(9) **De Morgans Laws**

(a) $((\Theta_1,\mathfrak{A}_1) \cup (\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2)^{\ominus}$

(b) $((\Theta_1,\mathfrak{A}_1) \cap_{\varepsilon} (\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \cup (\Theta_2,\mathfrak{A}_2)^{\ominus}$

(c) $((\Theta_1,\mathfrak{A}_1) \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \cap (\Theta_2,\mathfrak{A}_2)^{\circledast}$

(d) $((\Theta_1,\mathfrak{A}_1) \cap (\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \cup_{\mathcal{R}} (\Theta_2,\mathfrak{A}_2)^{\circledast}$

(e) $((\Theta_1,\mathfrak{A}_1) \bigvee(\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \bigwedge(\Theta_2,\mathfrak{A}_2)^{\ominus}$

(f) $((\Theta_1,\mathfrak{A}_1) \bigwedge(\Theta_2,\mathfrak{A}_2))^{\ominus} = (\Theta_1,\mathfrak{A}_1)^{\ominus} \bigvee(\Theta_2,\mathfrak{A}_2)^{\ominus}$

(g) $((\Theta_1,\mathfrak{A}_1) \bigvee(\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \bigwedge(\Theta_2,\mathfrak{A}_2)^{\circledast}$

(h) $((\Theta_1,\mathfrak{A}_1) \bigwedge(\Theta_2,\mathfrak{A}_2))^{\circledast} = (\Theta_1,\mathfrak{A}_1)^{\circledast} \bigvee(\Theta_2,\mathfrak{A}_2)^{\circledast}$

(10) **Distributive Laws**

(a) $(\Theta_1,\mathfrak{A}_1)\cup((\Theta_2,\mathfrak{A}_2)\cap(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cup(\Theta_2,\mathfrak{A}_2))\cap((\Theta_1,\mathfrak{A}_1)\cup(\Theta_3,\mathfrak{A}_3))$

(b) $(\Theta_1,\mathfrak{A}_1)\cap((\Theta_2,\mathfrak{A}_2)\cup(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cap(\Theta_2,\mathfrak{A}_2))\cup((\Theta_1,\mathfrak{A}_1)\cap(\Theta_3,\mathfrak{A}_3))$

(c) $(\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}((\Theta_2,\mathfrak{A}_2)\cap_{\varepsilon}(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_2,\mathfrak{A}_2))\cap_{\varepsilon}((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))$

(d) $(\Theta_1,\mathfrak{A}_1)\cap_{\varepsilon}((\Theta_2,\mathfrak{A}_2)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cap_{\varepsilon}(\Theta_2,\mathfrak{A}_2))\cup_{\mathcal{R}}((\Theta_1,\mathfrak{A}_1)\cap_{\varepsilon}(\Theta_3,\mathfrak{A}_3))$

(e) $(\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}((\Theta_2,\mathfrak{A}_2)\cap(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_2,\mathfrak{A}_2))\cap((\Theta_1,\mathfrak{A}_1)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))$

(f) $(\Theta_1,\mathfrak{A}_1)\cap((\Theta_2,\mathfrak{A}_2)\cup_{\mathcal{R}}(\Theta_3,\mathfrak{A}_3))=((\Theta_1,\mathfrak{A}_1)\cap(\Theta_2,\mathfrak{A}_2))\cup_{\mathcal{R}}((\Theta_1,\mathfrak{A}_1)\cap(\Theta_3,\mathfrak{A}_3))$

5. **Relations-based Operations of $\hat{H}s$-sets**

Here some relations-based classical notions and results are generalized for $\hat{H}s$-sets.

Definition 5.1. For $(\Theta_1,\mathfrak{A}_1)\&(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$, their Cartesian product $(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$, is a $\hat{H}s$-set $(\Theta_3,\mathfrak{A}_3)$ where $\mathfrak{A}_3=\mathfrak{A}_1\times\mathfrak{A}_2$ & $\Theta_3:\mathfrak{A}_3\to P(\hat{\Pi}\times\hat{\Pi})$ stated by $\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\Theta_1(\hat{\theta}_i)\times\Theta_2(\hat{\theta}_j)\ \forall\ (\hat{\theta}_i,\hat{\theta}_j)\in\mathfrak{A}_3$ that is $\Theta_3(\hat{\theta}_i,\hat{\theta}_j)=\{(\hat{\theta}_i,\hat{\theta}_j):\hat{\theta}_i\in\Theta_1(\hat{\theta}_i),\hat{\theta}_j\in\Theta_2(\hat{\theta}_j)\}$.

Definition 5.2. If $(\Theta_1,\mathfrak{A}_1),(\Theta_2,\mathfrak{A}_2)\in\Sigma_{(\Theta,\mathfrak{A})}$ then a relation from $(\Theta_1,\mathfrak{A}_1)$ to $(\Theta_2,\mathfrak{A}_2)$ is stated as *$\hat{H}s$-relation* $(\hat{\Xi},\mathfrak{A}_4)$ (conveniently $\hat{\Xi}$) which is the $\hat{H}s$-subset of $(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$ where $\mathfrak{A}_4\subseteq\mathfrak{A}_1\times\mathfrak{A}_2$ & $\forall\ (\hat{\theta}_1,\hat{\theta}_2)\in\mathfrak{A}_4,\hat{\Xi}(\hat{\theta}_1,\hat{\theta}_2)=\Theta_3(\hat{\theta}_1,\hat{\theta}_2)$, where $(\Theta_3,\mathfrak{A}_3)=(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$.

Definition 5.3. Let $\hat{\Xi}$ be a $\hat{H}s$-relation from $(\Theta_1,\mathfrak{A}_1)$ to $(\Theta_2,\mathfrak{A}_2)$ such that $(\Theta_3,\mathfrak{A}_3)=(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)$. Then

(i) The DoM $\hat{\Xi}$ (the domain of $\hat{\Xi}$) is a $\hat{H}s$-set $(\Theta,\mathfrak{W})\subset(\Theta_1,\mathfrak{A}_1)$ where $\mathfrak{W}=\{\hat{\theta}_i\in\mathfrak{A}_1:\Theta_3(\hat{\theta}_i,\hat{\theta}_j)\in\hat{\Xi}\ forsome\ \hat{\theta}_j\in\mathfrak{A}_2\}$ & $\Theta(\hat{\theta}_1)=\Theta_1(\hat{\theta}_1),\forall\ \hat{\theta}_1\in\mathfrak{W}$.

(ii) The RNG $\hat{\Xi}$ (the range of $\hat{\Xi}$) is a $\hat{H}s$-set $(\xi,\mathfrak{L})\subset(\Theta_2,\mathfrak{A}_2)$ where $\mathfrak{L}\subset\mathfrak{A}_2$ & $\mathfrak{L}=\{\hat{\theta}_j\in\mathfrak{A}_2:\Theta_3(\hat{\theta}_i,\hat{\theta}_j)\in\hat{\Xi}\ forsome\ \hat{\theta}_i\in\mathfrak{A}_1\}$ & $\xi(\hat{\theta}_2)=\Theta_1(\hat{\theta}_2),\forall\ \hat{\theta}_2\in\mathfrak{L}$.

(iii) The $\hat{\Xi}^{-1}$ (inverse of $\hat{\Xi}$) is a $\hat{H}s$-relation from $(\Theta_2,\mathfrak{A}_2)$ to $(\Theta_1,\mathfrak{A}_1)$ stated by $\hat{\Xi}^{-1}=\{\Theta_2(\hat{\theta}_j)\times\Theta_1(\hat{\theta}_i):\Theta_1(\hat{\theta}_i)\hat{\Xi}\Theta_2(\hat{\theta}_j)\}$.

Example 5.4. Let

$$(\Theta_1,\mathfrak{A}_1)=\left\{\ \Theta_1(\hat{\theta}_1),\Theta_1(\hat{\theta}_2),\Theta_1(\hat{\theta}_3)\ \right\},(\Theta_2,\mathfrak{A}_2)=\left\{\ \Theta_2(\hat{\theta}_4),\Theta_2(\hat{\theta}_5),\Theta_2(\hat{\theta}_6)\ \right\}$$

$$(\Theta_1,\mathfrak{A}_1)\times(\Theta_2,\mathfrak{A}_2)=\left\{\begin{array}{l}(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_5)),(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_6)),\\(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_5)),(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_6)),\\(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_5)),(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_6))\end{array}\right\}$$

then

$$\hat{\Xi}=\left\{\ (\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_4)),(\Theta_1(\hat{\theta}_1)\times\Theta_2(\hat{\theta}_6)),(\Theta_1(\hat{\theta}_2)\times\Theta_2(\hat{\theta}_6)),(\Theta_1(\hat{\theta}_3)\times\Theta_2(\hat{\theta}_6))\ \right\}$$

(i) $DoM\,\hat{\Xi}=(\Theta,\mathfrak{W})$ where $\mathfrak{W}=\{\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3\}\subseteq\mathfrak{A}_1$ & $\Theta(\hat{\theta}_i)=\Theta_1(\hat{\theta}_i)\forall\ \hat{\theta}_i\in\mathfrak{W}$.

(ii) $RNG\,\hat{\Xi}=(\xi,\mathfrak{L})$ where $\mathfrak{L}=\{\hat{\theta}_4,\hat{\theta}_6\}\subset\mathfrak{A}_2$ & $\xi(\hat{\theta}_j)=\Theta_2(\hat{\theta}_j)\forall\ \hat{\theta}_j\in\mathfrak{L}$.

(iii)

$$\hat{\Xi}^{-1} = \left\{ \ (\Theta_2(\hat{\theta}_4) \times \Theta_1(\hat{\theta}_1)), (\Theta_2(\hat{\theta}_6) \times \Theta_1(\hat{\theta}_1)), (\Theta_2(\hat{\theta}_6) \times \Theta_1(\hat{\theta}_2)), (\Theta_2(\hat{\theta}_6) \times \Theta_1(\hat{\theta}_3)) \ \right\}.$$

Definition 5.5. Let $\hat{\Xi}$ & $\mathfrak{S}$ are two $\hat{H}s$-relations on $\hat{H}s$-set $(\Theta, \mathfrak{W})$, then we get

(i) $\hat{\Xi} \subset \mathfrak{S}$, if for all $\varpi, \varsigma \in \mathfrak{W}, \Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ then $\Theta(\varpi) \times \Theta(\varsigma) \in \mathfrak{S}$.

(ii) $\hat{\Xi}^{©} = \{\Theta(\varpi) \times \Theta(\varsigma) : \Theta(\varpi) \times \Theta(\varsigma) \notin \hat{\Xi}, \forall \ \varpi, \varsigma \in \mathfrak{W}\}$.

(iii) $\hat{\Xi} \cup \mathfrak{S} = \{\Theta(\varpi) \times \Theta(\varsigma) : \Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ *or* $\Theta(\varpi) \times \Theta(\varsigma) \in \mathfrak{S}, \forall \ \varpi, \varsigma \in \mathfrak{W}\}$.

(iv) $\hat{\Xi} \cap \mathfrak{S} = \{\Theta(\varpi) \times \Theta(\varsigma) : \Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ & $\Theta(\varpi) \times \Theta(\varsigma) \in \mathfrak{S}, \forall \ \varpi, \varsigma \in \mathfrak{W}\}$.

Example 5.6. Let $(\Theta, \mathfrak{W}) = \left\{ \ \Theta(\hat{\theta}_1), \Theta(\hat{\theta}_2), \Theta(\hat{\theta}_3) \ \right\}$ then

$$(\Theta, \mathfrak{W}) \times (\Theta, \mathfrak{W}) = \left\{ \begin{array}{l} (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), \\ (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), \\ (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \end{array} \right\}$$

then we get

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \ \right\}$$

&

$$\mathfrak{S} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)) \ \right\}$$

now

(1) $\hat{\Xi}^{©} = \left\{ \begin{array}{l} (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), \\ (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)) \end{array} \right\}$. &

$\mathfrak{S}^{©} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \ \right\}$.

(2) $\hat{\Xi} \cup \mathfrak{S} = \left\{ \begin{array}{l} (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), \\ (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \end{array} \right\}$.

(3) $\hat{\Xi} \cap \mathfrak{S} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)) \ \right\}$.

Definition 5.7. Let $\hat{\Xi}$ be a $\hat{H}s$-relation on $(\Theta, \mathfrak{W})$, then

(i) if $\Theta(\varpi) \times \Theta(\varpi) \in \hat{\Xi} \forall \varpi \in \mathfrak{W}$, then $\hat{\Xi}$ is reflexive, e.g. $\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)) \ \right\}$.

(ii) if $\Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ then $\Theta(\varsigma) \times \Theta(\varpi) \in \hat{\Xi} \forall \varpi, \varsigma \in \mathfrak{W}$, so $\hat{\Xi}$ is symmetric, e.g.

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)) \ \right\}.$$

(iii) if $\Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ & $\Theta(\varsigma) \times \Theta(w) \in \hat{\Xi}$ then $\Theta(\varpi) \times \Theta(w) \in \hat{\Xi} \forall \varpi, \varsigma, w \in \mathfrak{W}$, so $\hat{\Xi}$ is transitive. e.g. $\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_3)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_3)) \ \right\}$.

(iv) if properties (i)-(iii) are satisfied then $\hat{\Xi}$ is stated as equivalence relation. E.g.

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)) \ \right\}.$$

(v) if $\Theta(\varpi) \times \Theta(\varsigma) \in \hat{\Xi}$ then $\varpi = \varsigma \forall \varpi, \varsigma \in \mathfrak{W}$, so $\hat{\Xi}$ is stated as identity. e.g.

$$\hat{\Xi} = \left\{ \ (\Theta(\hat{\theta}_1) \times \Theta(\hat{\theta}_1)), (\Theta(\hat{\theta}_2) \times \Theta(\hat{\theta}_2)), (\Theta(\hat{\theta}_3) \times \Theta(\hat{\theta}_3)) \ \right\}.$$

Definition 5.8. If $\hat{\Xi}$ is a $\hat{Hs}$-relation from $(\Theta_1, \mathfrak{A}_1)$ to $(\Theta_2, \mathfrak{A}_2)$ & $\mathfrak{S}$ is a $\hat{Hs}$-relation from $(\Theta_2, \mathfrak{A}_2)$ to $(\Theta_3, \mathfrak{A}_3)$ then composition of $\hat{\Xi}$ & $\mathfrak{S}$, symbolized by $\hat{\Xi} \circ \mathfrak{S}$, is also a $\hat{Hs}$-relation $\mathfrak{T}$ from $(\Theta_1, \mathfrak{A}_1)$ to $(\Theta_3, \mathfrak{A}_3)$ stated as if $\Theta_1(\varpi) \in (\Theta_1, \mathfrak{A}_1)$ & $\Theta_3(w) \in (\Theta_3, \mathfrak{A}_3)$ then $\Theta_1(\varpi) \times \Theta_3(w) \in \hat{\Xi} \circ \mathfrak{S}$ i.e. $\Theta_1(\varpi) \times \Theta_3(w) \in \hat{\Xi} \circ \mathfrak{S}$ iff $\Theta_1(\varpi) \times \Theta_2(\varsigma) \in \hat{\Xi}$ & $\Theta_2(\varsigma) \times \Theta_3(w) \in \hat{\Xi}$.

Example 5.9. Let

$$\hat{\Xi} = \left\{ \begin{array}{l} (\Theta_1(\hat{\theta}_1) \times \Theta_2(\hat{\theta}_1)), (\Theta_1(\hat{\theta}_1) \times \Theta_2(\hat{\theta}_3)), \\ (\Theta_1(\hat{\theta}_2) \times \Theta_2(\hat{\theta}_3)), (\Theta_1(\hat{\theta}_3) \times \Theta_2(\hat{\theta}_3)) \end{array} \right\} \&$$

$$\mathfrak{S} = \left\{ \begin{array}{l} (\Theta_2(\hat{\theta}_1) \times \Theta_3(\hat{\theta}_1)), (\Theta_2(\hat{\theta}_1) \times \Theta_3(\hat{\theta}_2)), \\ (\Theta_2(\hat{\theta}_2) \times \Theta_3(\hat{\theta}_2)), (\Theta_2(\hat{\theta}_3) \times \Theta_3(\hat{\theta}_2)) \end{array} \right\}$$

then

$$\hat{\Xi} \circ \mathfrak{S} = \left\{ \begin{array}{l} (\Theta_1(\hat{\theta}_1) \times \Theta_3(\hat{\theta}_1)), (\Theta_1(\hat{\theta}_1) \times \Theta_3(\hat{\theta}_2)), \\ (\Theta_1(\hat{\theta}_2) \times \Theta_3(\hat{\theta}_2)), (\Theta_1(\hat{\theta}_3) \times \Theta_3(\hat{\theta}_2)) \end{array} \right\}.$$

Definition 5.10. A $\hat{Hs}$-relation $\mathfrak{F}$ from $(\Theta_1, \mathfrak{A}_1)$ to $(\Theta_2, \mathfrak{A}_2)$, represented by $\mathfrak{F} : (\Theta_1, \mathfrak{A}_1) \to (\Theta_2, \mathfrak{A}_2)$, is stated as $\hat{Hs}$-function when (a). $DoM\,\mathfrak{F} = \mathfrak{A}_1$, (b). $DoM\,\mathfrak{F}$ has not repeated members & (c). Element-based uniqueness exists between $RNG\,\mathfrak{F}$ & $DoM\,\mathfrak{F}$ i.e. if $\Theta_1(\varpi)\,\mathfrak{F}\,\Theta_2(\varsigma)$ (or $\Theta_1(\varpi) \times \Theta_2(\varsigma) \in \mathfrak{F}$) then $\mathfrak{F}(\Theta_1(\varpi)) = \Theta_2(\varsigma)$.

Example 5.11. Let $\mathfrak{A}_1 = \{\varpi_1, \varpi_2, \varpi_3\}$ & $\mathfrak{A}_2 = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ then

$(\Theta_1, \mathfrak{A}_1) = \left\{ \Theta_1(\varpi_1), \Theta_1(\varpi_2), \Theta_1(\varpi_3) \right\}$, $(\Theta_2, \mathfrak{A}_2) = \left\{ \Theta_2(\varsigma_1), \Theta_2(\varsigma_2), \Theta_2(\varsigma_3), \Theta_2(\varsigma_4) \right\}$

so $\hat{Hs}$-functions is

$$\mathfrak{F} = \left\{ (\Theta_1(\varpi_1) \times \Theta_2(\varsigma_1)), (\Theta_1(\varpi_2) \times \Theta_2(\varsigma_3)), (\Theta_1(\varpi_3) \times \Theta_2(\varsigma_4)) \right\}$$

Definition 5.12. A $\hat{Hs}$-function $\mathfrak{F} : (\Theta_1, \mathfrak{A}_1) \to (\Theta_2, \mathfrak{A}_2)$ is stated as

(i) if $RNG\,\mathfrak{F} \subset \mathfrak{A}_2$, then INTO-$\hat{Hs}$-function. E.g. Let $\mathfrak{A}_1 = \{\varpi_1, \varpi_2, \varpi_3\}$ & $\mathfrak{A}_2 = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ then $\mathfrak{F} = \left\{ (\Theta_1(\varpi_1) \times \Theta_2(\varsigma_1)), (\Theta_1(\varpi_2) \times \Theta_2(\varsigma_3)), (\Theta_1(\varpi_3) \times \Theta_2(\varsigma_4)) \right\}$

(ii) if $RNG\,\mathfrak{F} = \mathfrak{A}_2$, then ONTO-$\hat{Hs}$-function. E.g. Let $\mathfrak{A}_1 = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$ & $\mathfrak{A}_2 = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ then

$$\mathfrak{F} = \left\{ (\Theta_1(\varpi_1) \times \Theta_2(\varsigma_1)), (\Theta_1(\varpi_2) \times \Theta_2(\varsigma_3)), (\Theta_1(\varpi_3) \times \Theta_2(\varsigma_4)), (\Theta_1(\varpi_4) \times \Theta_2(\varsigma_2)) \right\}$$

(iii) *1-1 $\hat{Hs}$-function* if $\Theta_1(\varpi_1) \neq \Theta_1(\varpi_2)$ then $\mathfrak{F}(\Theta_1(\varpi_1)) \neq \mathfrak{F}(\Theta_1(\varpi_2))$. E.g.

$$\mathfrak{F} = \left\{ (\Theta_1(\varpi_1) \times \Theta_2(\varsigma_1)), (\Theta_1(\varpi_2) \times \Theta_2(\varsigma_4)), (\Theta_1(\varpi_3) \times \Theta_2(\varsigma_2)), (\Theta_1(\varpi_4) \times \Theta_2(\varsigma_3)) \right\}$$

(iv) if it is both INTO and ONTO then bijective $\hat{Hs}$-function. E.g.

$$\mathfrak{F} = \left\{ (\Theta_1(\varpi_1) \times \Theta_2(\varsigma_1)), (\Theta_1(\varpi_2) \times \Theta_2(\varsigma_2)), (\Theta_1(\varpi_3) \times \Theta_2(\varsigma_3)), (\Theta_1(\varpi_4) \times \Theta_2(\varsigma_4)) \right\}$$

Definition 5.13. The identity $\hat{Hs}$-function on $\hat{Hs}$-set $(\Theta, \mathfrak{L})$ is stated by $\mathfrak{I} : (\Theta, \mathfrak{L}) \to (\Theta, \mathfrak{L})$ such that $\mathfrak{I}(\Theta(l)) = \Theta(l)\ \forall\ \Theta(l) \in (\Theta, \mathfrak{L})$. E.g. Let $\mathfrak{L} = \{l_1, l_2, l_3, l_4\}$ then

$$\mathfrak{I} = \left\{ (\Theta(l_1) \times \Theta(l_1)), (\Theta(l_2) \times \Theta(l_2)), (\Theta(l_3) \times \Theta(l_3)), (\Theta(l_4) \times \Theta(l_4)) \right\}$$

6. Matrix-theory Based on $\hat{H}s$-sets

Here some classical matrix-based notions are generalized for $\hat{H}s$-sets.

Definition 6.1.

(i) Let $(\Theta, \mathfrak{A})$ be a $\hat{H}s$-set on $\hat{\Pi}$. A set $\mathbb{R}_{\mathfrak{A}} \subseteq \hat{\Pi} \times \mathfrak{A}$ is a relation version of $(\Theta, \mathfrak{A})$ stated as

$$\mathbb{R}_{\mathfrak{A}} = \left\{ \ (\varpi, \hat{\theta}) : \hat{\theta} \in \mathfrak{A}, \varpi \in \Theta(\hat{\theta}) \ \right\}.$$

(ii) *The characteristic function* $\mathcal{X}_{\mathbb{R}_{\mathfrak{A}}}$ is stated by $\mathcal{X}_{\mathbb{R}_{\mathfrak{A}}} : \hat{\Pi} \times \mathfrak{A} \to \{0, 1\}$, where

$$\mathcal{X}_{\mathbb{R}_{\mathfrak{A}}}(\varpi, \hat{\theta}) = \begin{cases} 1 & ; \ (\varpi, \hat{\theta}) \in \mathbb{R}_{\mathfrak{A}} \\ 0 & ; \ (\varpi, \hat{\theta}) \notin \mathbb{R}_{\mathfrak{A}} \end{cases}$$

(iii) If $|\hat{\Pi}| = m$ & $|\mathfrak{A}| = n$ then $(\ddot{s}_{ij})$ is an $m \times n$ $\hat{H}s$-matrix of $(\Theta, \mathfrak{A})$ on $\hat{\Pi}$ and stated as

$$(\ddot{s}_{ij})_{m\times n} = \begin{pmatrix} \ddot{s}_{11} & \ddot{s}_{12} & .... & \ddot{s}_{1n} \\ \ddot{s}_{21} & \ddot{s}_{22} & .... & \ddot{s}_{2n} \\ \vdots & \vdots & & \vdots \\ \ddot{s}_{m1} & \ddot{s}_{m2} & .... & \ddot{s}_{mn} \end{pmatrix}$$

Note: The family of all $m \times n$ $\hat{H}s$- matrices on $\hat{\Pi}$ is symbolized by $(\hat{\Pi})^{(hsm)}_{m\times n}$.

Example 6.2. Let $\hat{\Pi} = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$ & $\mathfrak{A} = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5\}$. Then $\Theta(\hat{\theta}_1) = \{\varpi_1, \varpi_2\}$, $\Theta(\hat{\theta}_2) = \varnothing$, $\Theta(\hat{\theta}_3) = \{\varpi_4, \varpi_5\}$, $\Theta(\hat{\theta}_4) = \{\varpi_2, \varpi_3, \varpi_4, \}$, $\Theta(\hat{\theta}_5) = \varnothing$, therefore we get $(\Theta, \mathfrak{A}) = \left\{ \ (\hat{\theta}_1, \{\varpi_1, \varpi_2\}), (\hat{\theta}_3, \{\varpi_4, \varpi_5\}), (\hat{\theta}_4, \{\varpi_2, \varpi_3, \varpi_4, \}) \ \right\}$ &
$\mathbb{R}_{\mathfrak{A}} = \left\{ \ (\varpi_1, \hat{\theta}_1), (\varpi_2, \hat{\theta}_1), (\varpi_4, \hat{\theta}_3), (\varpi_5, \hat{\theta}_3), (\varpi_2, \hat{\theta}_4), (\varpi_3, \hat{\theta}_4), (\varpi_4, \hat{\theta}_4) \ \right\}$. Hence $\hat{H}s$- matrix is given as

$$(\ddot{s}_{ij})_{5\times 5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} {}_1i^5, \, {}_1j^5.$$

Definition 6.3. Let $(\ddot{s}_{ij})_{m\times n} \in (\hat{\Pi})^{(hsm)}_{m\times n}$ then $(\ddot{s}_{ij})_{m\times n}$ is characterized as:

(i) The $(0)_{m\times n}$ is stated as a null $\hat{H}s$- matrix if $\ddot{s}_{ij} = 0 \ \forall \ i, j$ e.g.

$$(0)_{5\times 5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} {}_1i^5, \, {}_1j^5.$$

(ii) An $\mathfrak{A}_1$-*universal* $\hat{H}s$- *matrix*, symbolized by $(\ddot{s}_{ij})^{\mathfrak{A}_1}_{m\times n}$, if $\ddot{s}_{ij} = 1, \forall j \in J_{\mathfrak{A}_1} = \{j : \hat{\theta}_j \in \mathfrak{A}_1\}$ & i. E.g. Let $\mathfrak{A}$ be as provided in 6.2 & $\mathfrak{A}_1 = \{\hat{\theta}_2, \hat{\theta}_4, \hat{\theta}_5\} \subseteq \mathfrak{A}$ with $\Theta(\hat{\theta}_2) = \Theta(\hat{\theta}_4) = \Theta(\hat{\theta}_5) = \hat{\Pi}$ then

$$(\ddot{s}_{ij})^{\mathfrak{A}_1}_{5\times 5} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}_{1}i^{5}, {}_{1}j^{5}.$$

(iii) The $(\ddot{s}_{ij})^{\hat{\Pi}}_{m\times n}$ is stated as universal $\hat{H}s$- matrix if $\ddot{s}_{ij} = 1, \forall\ i, j$. E.g. Let $\mathfrak{A}$ as stated in 6.2 with $\Theta(\hat{\theta}_1) = \Theta(\hat{\theta}_2) = \Theta(\hat{\theta}_3) = \Theta(\hat{\theta}_4) = \Theta(\hat{\theta}_5) = \hat{\Pi}$ then

$$(\ddot{s}_{ij})^{\hat{\Pi}}_{5\times 5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}_{1}i^{5}, {}_{1}j^{5}.$$

Definition 6.4. Let $\mathfrak{L}_1 = (\ddot{s}_{ij})_{m\times n}, \mathfrak{L}_2 = \left(\ddot{t}_{ij}\right)_{m\times n} \in (\hat{\Pi})^{(hsm)}_{m\times n}$ then

(a) $\mathfrak{L}_1$ is stated as $\hat{H}s$- *sub-matrix* of $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ if $\ddot{s}_{ij} \leq \ddot{t}_{ij}$ e.g. $\mathfrak{L}_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

(b) $\mathfrak{L}_1$ & $\mathfrak{L}_2$ are stated as comparable, symbolized by $\mathfrak{L}_1 \parallel \mathfrak{L}_2$, if $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$ or $\mathfrak{L}_2 \subseteq \mathfrak{L}_1$.

(c) $\mathfrak{L}_1$ is stated as *proper* $\hat{H}s$- *sub-matrix* of $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \subset \mathfrak{L}_2$ if for atleast one term $\ddot{s}_{ij} \leq \ddot{t}_{ij}$ e.g. $\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

(d) $\mathfrak{L}_1$ is stated as *strictly* $\hat{H}s$- *sub-matrix* of $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \subsetneqq \mathfrak{L}_2$ if for each term $\ddot{s}_{ij} \leq \ddot{t}_{ij}$ e.g. $\mathfrak{L}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

(e) *union* of $\mathfrak{L}_1$ & $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \cup \mathfrak{L}_2$, is also a $\hat{H}s$- matrix $\mathfrak{L}_3 = (\delta_{ij})_{m\times n}$ if $\delta_{ij} = max\{\ddot{s}_{ij}, \ddot{t}_{ij}\}\ \forall\ i, j$ e.g.

Let $\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ then

$$\mathfrak{L}_3 = \mathfrak{L}_1 \cup \mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(f) *intersection* of $\mathfrak{L}_1$ & $\mathfrak{L}_2$, symbolized by $\mathfrak{L}_1 \cap \mathfrak{L}_2$, is also a $\hat{H}s$- matrix $\mathfrak{L}_3 = (\delta_{ij})_{m\times n}$ if $\delta_{ij} = min\{\ddot{s}_{ij}, \ddot{t}_{ij}\}\ \forall\ i, j$ e.g.

Let $\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ then

$$\mathfrak{L}_3 = \mathfrak{L}_1 \cap \mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

(g) The $\mathfrak{L}^{©}\ (\mu_{ij})_{m\times n}$ (complement of $\mathfrak{L} = (\ddot{s}_{ij})_{m\times n}$), is also a $\hat{H}s$- matrix if $\mu_{ij} = 1 - \ddot{s}_{ij}\ \forall\ i, j$ e.g.

Let $\mathfrak{L} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ then $\mathfrak{L}^{©} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$.

(h) The difference $\mathfrak{L}_2 \setminus \mathfrak{L}_1$, is also a $\hat{H}s$- matrix $\mathfrak{L}_3$ such that $\mathfrak{L}_3 = \mathfrak{L}_2 \cap \mathfrak{L}_1^{©}$ e.g.

$\mathfrak{L}_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ & $\mathfrak{L}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ then

$$\mathfrak{L}_3 = \mathfrak{L}_2 \cap \mathfrak{L}_1^{\copyright}$$

$$\mathfrak{L}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Proposition 6.5. *For* $\mathfrak{C}_1 = (\ddot{s}_{ij})_{m\times n}, \mathfrak{C}_2 = (\ddot{t}_{ij})_{m\times n}, \mathfrak{C}_3 = (\ddot{u}_{ij})_{m\times n} \in (\hat{\Pi})_{m\times n}^{(hsm)}$, *the following axiomatic results are valid:*

(1) $\mathfrak{C}_1 \cup \mathfrak{C}_1 = \mathfrak{C}_1,\ \ \mathfrak{C}_1 \cap \mathfrak{C}_1 = \mathfrak{C}_1$

(2) $\mathfrak{C}_1 \cup (0)_{m\times n} = \mathfrak{C}_1,\ \ \mathfrak{C}_1 \cap (\ddot{s}_{ij})_{m\times n}^{\hat{\Pi}} = \mathfrak{C}_1$

(3) $\mathfrak{C}_1 \cap (0)_{m\times n} = (0)_{m\times n},\ \ \mathfrak{C}_1 \cup (\ddot{s}_{ij})_{m\times n}^{\hat{\Pi}} = (\ddot{s}_{ij})_{m\times n}^{\hat{\Pi}}$

(4) $((0)_{m\times n})^{\copyright} = (\ddot{s}_{ij})_{m\times n}^{\hat{\Pi}},\ \ ((\ddot{s}_{ij})_{m\times n}^{\hat{\Pi}})^{\copyright} = (0)_{m\times n}$

(5) $\mathfrak{C}_1 \cup \mathfrak{C}_1{}^{\copyright} = (\ddot{s}_{ij})_{m\times n}^{\hat{\Pi}},\ \ \mathfrak{C}_1 \cap \mathfrak{C}_1{}^{\copyright} = (0)_{m\times n}$

(6) $(\mathfrak{C}_1 \cup \mathfrak{C}_2)^{\copyright} = \mathfrak{C}_1{}^{\copyright} \cap \mathfrak{C}_2{}^{\copyright},\ \ (\mathfrak{C}_1 \cap \mathfrak{C}_2)^{\copyright} = \mathfrak{C}_1{}^{\copyright} \cup \mathfrak{C}_2{}^{\copyright}$

(7) $(\mathfrak{C}_1{}^{\copyright})^{\copyright} = \mathfrak{C}_1$

(8) $\mathfrak{C}_1 \cup \mathfrak{C}_2 = \mathfrak{C}_2 \cup \mathfrak{C}_1,\ \ \mathfrak{C}_1 \cap \mathfrak{C}_2 = \mathfrak{C}_2 \cap \mathfrak{C}_1$

(9) $\mathfrak{C}_1 \cup (\mathfrak{C}_2 \cup \mathfrak{C}_3) = (\mathfrak{C}_1 \cup \mathfrak{C}_2) \cup \mathfrak{C}_3,\ \ \mathfrak{C}_1 \cap (\mathfrak{C}_2 \cap \mathfrak{C}_3) = (\mathfrak{C}_1 \cap \mathfrak{C}_2) \cap \mathfrak{C}_3$

(10) $\mathfrak{C}_1 \cup (\mathfrak{C}_2 \cap \mathfrak{C}_3) = (\mathfrak{C}_1 \cup \mathfrak{C}_2) \cap (\mathfrak{C}_1 \cup \mathfrak{C}_3),\ \ \mathfrak{C}_1 \cap (\mathfrak{C}_2 \cup \mathfrak{C}_3) = (\mathfrak{C}_1 \cap \mathfrak{C}_2) \cup (\mathfrak{C}_1 \cap \mathfrak{C}_3)$

Definition 6.6. Let $\mathfrak{P} = (\ddot{c}_{ij})_{m\times n},\ \mathfrak{Q} = (\ddot{d}_{ik})_{m\times n} \in (\hat{\Pi})_{m\times n}^{(hsm)}$, then

(i) *AND-product* is stated as
$\wedge : (\hat{\Pi})_{m\times n}^{(hsm)} \times (\hat{\Pi})_{m\times n}^{(hsm)} \to (\hat{\Pi})_{m\times n^2}^{(hsm)}$ with $(\ddot{c}_{ij}) \wedge (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = min\{\ddot{c}_{ij}, \ddot{d}_{ik}\}$ & $l = n(j-1)+k$.

(ii) *OR-product* is stated as
$\vee : (\hat{\Pi})_{m\times n}^{(hsm)} \times (\hat{\Pi})_{m\times n}^{(hsm)} \to (\hat{\Pi})_{m\times n^2}^{(hsm)}$ with $(\ddot{c}_{ij}) \vee (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = max\{\ddot{c}_{ij}, \ddot{d}_{ik}\}$.

(iii) *AND-NOT-product* is stated as
$\overline{\wedge} : (\hat{\Pi})_{m\times n}^{(hsm)} \times (\hat{\Pi})_{m\times n}^{(hsm)} \to (\hat{\Pi})_{m\times n^2}^{(hsm)}$ with $(\ddot{c}_{ij}) \,\overline{\wedge}\, (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = min\{\ddot{c}_{ij}, 1 - \ddot{d}_{ik}\}$.

(iv) *OR-NOT-product* is stated as
$\underline{\vee} : (\hat{\Pi})_{m\times n}^{(hsm)} \times (\hat{\Pi})_{m\times n}^{(hsm)} \to (\hat{\Pi})_{m\times n^2}^{(hsm)}$ with $(\ddot{c}_{ij}) \,\underline{\vee}\, (\ddot{d}_{ik}) = (\ddot{h}_{il})$ & $\ddot{h}_{il} = max\{\ddot{c}_{ij}, 1 - \ddot{d}_{ik}\}$.

Example 6.7. Let $\mathfrak{P} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ & $\mathfrak{Q} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ then

(i) $\mathfrak{P} \wedge \mathfrak{Q} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

$$\text{(ii)}\ \mathfrak{P} \vee \mathfrak{Q} = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$

$$\text{(iii)}\ \mathfrak{P} \overline{\wedge} \mathfrak{Q} = \begin{pmatrix} 1&1&0&1 \\ 1&1&1&1 \\ 0&1&0&1 \\ 1&1&1&1 \end{pmatrix} \wedge \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$\text{(iv)}\ \mathfrak{P} \underline{\vee} \mathfrak{Q} = \begin{pmatrix} 1&1&0&1 \\ 1&1&1&1 \\ 0&1&0&1 \\ 1&1&1&1 \end{pmatrix} \vee \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

$$= \begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$

7. **Hybridized Structures of $\hat{H}s$-sets**

Here the notions of some hybridized model of $\hat{H}s$-sets are presented. The set $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times ..... \times \mathfrak{A}_m$ with $\mathfrak{A}_{\hat{\alpha}} \cap \mathfrak{A}_{\hat{\beta}} = \varnothing\ \forall\ \hat{\alpha}, \hat{\beta} = 1, 2, ..., m$ and $\mathfrak{A}_{\hat{\alpha}}$ are same as stated in Definition 3.1. The Figure 3 presents the notations and their full names that are used in this section.

Abbreviations	Used for	Abbreviations	Used for
$iv\hat{f}Hs$-set	interval-valued fuzzy hypersoft set	ivf-set	interval-valued fuzzy set
$\mathcal{F}^{ivf}(\hat{\Pi})$	collection of interval-valued fuzzy subsets over $\hat{\Pi}$	$fphs$-set	fuzzy parameterized hypersoft set
maa-function	multi-argument approximate function	iv-$fphs$-set	interval-valued fuzzy parameterized hypersoft set
$ifphs$-set	intuitionistic fuzzy parameterized hypersoft set	$nphs$-set	neutrosophic parameterized hypersoft set
if-set	intuitionistic fuzzy set	n-set	neutrosophic set

FIGURE 3. Notations

Definition 7.1. An *$iv\hat{f}Hs$-set* $(\Gamma, \mathfrak{A})$ on $\hat{\Pi}$ is stated by

$$(\Gamma, \mathfrak{A}) = \left\{ \ (\hat{\theta}, \Gamma(\hat{\theta})); \hat{\theta} \in \mathfrak{A}, \Gamma(\hat{\theta}) \in \mathcal{F}^{ivf}(\hat{\Pi}) \ \right\}$$

where $\Gamma : \mathfrak{A} \to \mathcal{F}^{ivf}(\hat{\Pi})$ & $\Gamma(\hat{\theta}) = \left\{ \ \ddot{\psi}_{\Gamma(\hat{\theta})}(\varpi)/\varpi : \varpi \in \hat{\Pi}, \ddot{\psi}_{\Gamma(\hat{\theta})}(\varpi) \in \mathbb{C}(\mathbb{I}) \ \right\}$ is an ivf-set on $\hat{\Pi}$.

Example 7.2. Let $\hat{\Pi} = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6, \varpi_7, \varpi_8\}$ & $\mathfrak{A} = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5, \hat{\theta}_6, \hat{\theta}_7, \hat{\theta}_8\}$, $iv\hat{f}Hs$-set $(\Gamma, \mathfrak{A})$ is constructed as

$$(\Gamma, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{[0.1, 0.2]/\varpi_1, [0.2, 0.3]/\varpi_2, [0.4, 0.5]/\varpi_4, [0.5, 0.6]/\varpi_5\}\right), \\ \left(\hat{\theta}_2, \{[0.1, 0.3]/\varpi_1, [0.2, 0.4]/\varpi_2, [0.3, 0.4]/\varpi_3, [0.6, 0.8]/\varpi_6\}\right), \\ \left(\hat{\theta}_3, \{[0.2, 0.3]/\varpi_2, [0.3, 0.4]/\varpi_3, [0.4, 0.5]/\varpi_4, , [0.5, 0.7]/\varpi_5\}\right), \\ \left(\hat{\theta}_4, \{[0.4, 0.5]/\varpi_4, [0.5, 0.6]/\varpi_5, [0.6, 0.7]/\varpi_6, [0.7, 0.8]/\varpi_7\}\right), \\ \left(\hat{\theta}_5, \{[0.3, 0.6]/\varpi_3, [0.6, 0.7]/\varpi_6, [0.7, 0.8]/\varpi_7, [0.8, 0.9]/\varpi_8\}\right), \\ \left(\hat{\theta}_6, \{[0.2, 0.4]/\varpi_2, [0.3, 0.5]/\varpi_3, [0.4, 0.6]/\varpi_4, [0.7, 0.8]/\varpi_7\}\right), \\ \left(\hat{\theta}_7, \{[0.1, 0.4]/\varpi_1, [0.3, 0.4]/\varpi_3, [0.5, 0.7]/\varpi_5, [0.6, 0.8]/\varpi_6\}\right), \\ \left(\hat{\theta}_8, \{[0.2, 0.5]/\varpi_2, [0.3, 0.6]/\varpi_3, [0.6, 0.8]/\varpi_6, [0.7, 0.8]/\varpi_7\}\right) \end{array} \right\}$$

Definition 7.3. A $fphs$-set $(\mathcal{D}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{D}, \mathfrak{A}) = \left\{ \ \left(\varphi_{\mathcal{F}}(\hat{\theta})/\hat{\theta}, \Theta_{\mathcal{F}}(\hat{\theta})\right), \hat{\theta} \in \mathfrak{A}, \ \Theta_{\mathcal{F}}(\hat{\theta}) \in P^{\hat{\Pi}}, \varphi_{\mathcal{F}}(\hat{\theta}) \in \mathbb{C}(\mathbb{I}) \ \right\}$$

where $\mathcal{F}$ is a fuzzy set with $\varphi_{\mathcal{F}} : \mathfrak{A} \to \mathbb{C}(\mathbb{I})$ as membership function of $fphs$-set & $\Theta_{\mathcal{F}} : \mathfrak{A} \to P^{\hat{\Pi}}$ is maa-function of $fphs$-set.

Example 7.4. From Example 7.2, we get

$$(\mathcal{D}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(0.1/\hat{\theta}_1, \{\varpi_1, \varpi_2\}\right), \left(0.2/\hat{\theta}_2, \{\varpi_1, \varpi_2, \varpi_3\}\right), \left(0.3/\hat{\theta}_3, \{\varpi_2, \varpi_3, \varpi_4\}\right), \\ \left(0.4/\hat{\theta}_4, \{\varpi_4, \varpi_5, \varpi_6\}\right), \left(0.5/\hat{\theta}_5, \{\varpi_6, \varpi_7, \varpi_8\}\right), \left(0.6/\hat{\theta}_6, \{\varpi_2, \varpi_3, \varpi_4, \varpi_7\}\right), \\ \left(0.7/\hat{\theta}_7, \{\varpi_1, \varpi_3, \varpi_5, \varpi_6\}\right), \left(0.8/\hat{\theta}_8, \{\varpi_2, \varpi_3, \varpi_6, \varpi_7\}\right) \end{array} \right\}$$

Definition 7.5. An iv-$fphs$-set $(\mathcal{E}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{E}, \mathfrak{A}) = \left\{ \ \left(\Psi_{\mathcal{F}^{iv}}(\hat{\theta})/\hat{\theta}, \psi_{\mathcal{F}^{iv}}(\hat{\theta})\right), \hat{\theta} \in \mathfrak{A}, \ \psi_{\mathcal{F}_{iv}}(\hat{\theta}) \in P^{\hat{\Pi}}, \Psi_{\mathcal{F}^{iv}}(\hat{\theta}) \in \mathbb{C}(\mathbb{I}) \ \right\}$$

where $\mathcal{F}^{iv}$ is an ivf-set with $\Psi_{\mathcal{F}^{iv}} : \mathfrak{A} \to \mathbb{C}(\mathbb{I})$ as membership function of $fphs$-set and $\psi_{\mathcal{F}_{iv}} : \mathfrak{A} \to P^{\hat{\Pi}}$ is maa-function of iv-$fphs$-set.

Example 7.6. From Example 7.2, we get

$$(\mathcal{E}, \mathfrak{A}) = \left\{ \begin{array}{l} \left([0.1, 0.2]/\hat{\theta}_1, \{\varpi_1, \varpi_2\}\right), \left([0.2, 0.3]/\hat{\theta}_2, \{\varpi_1, \varpi_2, \varpi_3\}\right), \\ \left([0.3, 0.4]/\hat{\theta}_3, \{\varpi_2, \varpi_3, \varpi_4\}\right), \left([0.4, 0.5]/\hat{\theta}_4, \{\varpi_4, \varpi_5, \varpi_6\}\right), \\ \left([0.5, 0.6]/\hat{\theta}_5, \{\varpi_6, \varpi_7, \varpi_8\}\right), \left([0.6, 0.7]/\hat{\theta}_6, \{\varpi_2, \varpi_3, \varpi_4, \varpi_7\}\right), \\ \left([0.7, 0.8]/\hat{\theta}_7, \{\varpi_1, \varpi_3, \varpi_5, \varpi_6\}\right), \left([0.8, 0.9]/\hat{\theta}_8, \{\varpi_2, \varpi_3, \varpi_6, \varpi_7\}\right) \end{array} \right\}$$

Definition 7.7. An $ifphs$-set $(\mathcal{H}, \mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{H}, \mathfrak{A}) = \left\{ \ \left(< \varsigma_1(\hat{\theta}), \varsigma_2(\hat{\theta}) > /\hat{\theta}, \psi^{\mathcal{IF}}(\hat{\theta})\right); \hat{\theta} \in \mathfrak{A}, \ \psi^{\mathcal{IF}}(\hat{\theta}) \in P^{\hat{\Pi}}, \varsigma_1(\hat{\theta}), \varsigma_2(\hat{\theta}) \in \mathbb{C}(\mathbb{I}) \ \right\}$$

where $\mathcal{IF}$ is an if-set with $\varsigma_1(\hat{\theta}), \varsigma_2(\hat{\theta}) : \mathfrak{A} \to \mathbb{C}(\mathbb{I})$ as membership and non-membership functions of $ifphs$-set and $\psi^{\mathcal{IF}} : \mathfrak{A} \to P^{\hat{\Pi}}$ is maa-function of $ifphs$-set.

Example 7.8. From Example 7.2, we get

$$(\mathcal{H},\mathfrak{A}) = \left\{ \begin{array}{l} \left(<0.1,0.2>/\hat{\theta}_1,\{\varpi_1,\varpi_2\}\right),\left(<0.2,0.3>/\hat{\theta}_2,\{\varpi_1,\varpi_2,\varpi_3\}\right), \\ \left(<0.3,0.4>/\hat{\theta}_3,\{\varpi_2,\varpi_3,\varpi_4\}\right),\left(<0.4,0.5>/\hat{\theta}_4,\{\varpi_4,\varpi_5,\varpi_6\}\right), \\ \left(<0.5,0.6>/\hat{\theta}_5,\{\varpi_6,\varpi_7,\varpi_8\}\right),\left(<0.6,0.7>/\hat{\theta}_6,\{\varpi_2,\varpi_3,\varpi_4,\varpi_7\}\right), \\ \left(<0.7,0.8>/\hat{\theta}_7,\{\varpi_1,\varpi_3,\varpi_5,\varpi_6\}\right),\left(<0.8,0.9>/\hat{\theta}_8,\{\varpi_2,\varpi_3,\varpi_6,\varpi_7\}\right) \end{array} \right\}$$

Definition 7.9. A *nphs*-set $(\mathcal{N},\mathfrak{A})$ on $\hat{\Pi}$ is stated as

$$(\mathcal{N},\mathfrak{A}) = \left\{ \begin{array}{l} \left(<\lambda_1(\hat{\theta}),\lambda_2(\hat{\theta}),\lambda_3(\hat{\theta})>/\hat{\theta},\psi^{\mathcal{N}}(\hat{\theta})\right);\hat{\theta}\in\mathfrak{A},\psi^{\mathcal{N}}(\hat{\theta})\in P^{\hat{\Pi}}, \\ \lambda_1(\hat{\theta})\in\mathbb{C}(\mathbb{I}),\lambda_2(\hat{\theta})\in\mathbb{C}(\mathbb{I}),\lambda_3(\hat{\theta})\in\mathbb{C}(\mathbb{I}) \end{array} \right\}$$

where $\mathcal{N}$ is a neutrosophic set with $\lambda_1(\hat{\theta}),\lambda_2(\hat{\theta}),\lambda_3(\hat{\theta}):\mathfrak{A}\to\mathbb{C}(\mathbb{I})$ as membership, indeterminate and falsity of *nphs*-set and $\psi^{\mathcal{N}}:\mathfrak{A}\to P^{\hat{\Pi}}$ is maa-function of *nphs*-set.

Example 7.10. From Example 7.2, we get

$$(\mathcal{N},\mathfrak{A}) = \left\{ \begin{array}{l} \left(<0.1,0.2,0.2>/\hat{\theta}_1,\{\varpi_1,\varpi_2\}\right),\left(<0.2,0.3,0.3>/\hat{\theta}_2,\{\varpi_1,\varpi_2,\varpi_3\}\right), \\ \left(<0.3,0.4,0.4>/\hat{\theta}_3,\{\varpi_2,\varpi_3,\varpi_4\}\right),\left(<0.4,0.5,0.5>/\hat{\theta}_4,\{\varpi_4,\varpi_5,\varpi_6\}\right), \\ \left(<0.5,0.6,0.6>/\hat{\theta}_5,\{\varpi_6,\varpi_7,\varpi_8\}\right),\left(<0.6,0.7,0.7>/\hat{\theta}_6,\{\varpi_2,\varpi_3,\varpi_4,\varpi_7\}\right), \\ \left(<0.7,0.5,0.8>/\hat{\theta}_7,\{\varpi_1,\varpi_3,\varpi_5,\varpi_6\}\right),\left(<0.8,0.4,0.9>/\hat{\theta}_8,\{\varpi_2,\varpi_3,\varpi_6,\varpi_7\}\right) \end{array} \right\}$$

Definition 7.11. A $\hat{H}s$-set $(\ddot{\mathfrak{B}},\mathfrak{A})$ is known as *bijective* $\hat{H}s$*-set* (*bhs*-set) on $\hat{\Pi}$ if

(i) $\bigcup\limits_{j\in\mathfrak{A}} \ddot{\mathfrak{B}}(\hat{\theta}) = \hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}},\hat{\theta}_{\hat{\beta}}\in\mathfrak{A},\hat{\alpha}\neq\hat{\beta},\ddot{\mathfrak{B}}(\hat{\theta}_{\hat{\alpha}})\cap\ddot{\mathfrak{B}}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.12. Reconsidering Example 7.2, we get

$$(\ddot{\mathfrak{B}},\mathfrak{A}) = \left\{ \left(\hat{\theta}_1,\{\varpi_1\}\right),\left(\hat{\theta}_2,\{\varpi_2\}\right),\left(\hat{\theta}_3,\{\varpi_3\}\right),\left(\hat{\theta}_4,\{\varpi_4\}\right),\left(\hat{\theta}_5,\{\varpi_5\}\right),\left(\hat{\theta}_6,\{\varpi_6\}\right),\left(\hat{\theta}_7,\{\varpi_7\}\right),\left(\hat{\theta}_8,\{\varpi_8\}\right) \right\}$$

Definition 7.13. A $f\hat{h}s$-set $(\ddot{\mathfrak{B}}^f,\mathfrak{A})$ is stated as *bijective* $f\hat{h}s$*-set* on $\hat{\Pi}$ if

(i) $\bigcup\limits_{\hat{\theta}\in\mathfrak{A}} \ddot{\mathfrak{B}}^f(\hat{\theta}) = \hat{\Pi}$ with $\sum\limits_{\varpi\in\hat{\Pi}} \ddot{\psi}_f(\varpi)\in\mathbb{C}(\mathbb{I})$ where $\ddot{\psi}_f(\varpi)$ is a f-membership for each $\varpi\in\hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}},\hat{\theta}_{\hat{\beta}}\in\mathfrak{A},\hat{\alpha}\neq\hat{\beta},\ddot{\mathfrak{B}}^f(\hat{\theta}_{\hat{\alpha}})\cap\ddot{\mathfrak{B}}^f(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.14. Reconsidering Example 7.2, we get

$$(\ddot{\mathfrak{B}}^f,\mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1,\{0.1/\varpi_1\}\right),\left(\hat{\theta}_2,\{0.2/\varpi_2\}\right), \\ \left(\hat{\theta}_3,\{0.13/\varpi_3\}\right),\left(\hat{\theta}_4,\{0.14/\varpi_4\}\right), \\ \left(\hat{\theta}_5,\{0.05/\varpi_5\}\right),\left(\hat{\theta}_6,\{0.06/\varpi_6\}\right), \\ \left(\hat{\theta}_7,\{0.07/\varpi_7\}\right),\left(\hat{\theta}_8,\{0.08/\varpi_8\}\right) \end{array} \right\}$$

Definition 7.15. An $iv\hat{f}hs$-set $(\ddot{\mathfrak{B}}^{ivf},\mathfrak{A})$ is stated as *bijective* $iv\hat{f}hs$*-set* on $\hat{\Pi}$ if

(i) $\bigcup\limits_{\hat{\theta}\in\mathfrak{A}} \ddot{\mathfrak{B}}^{ivf}(\hat{\theta}) = \hat{\Pi}$ with $\sum\limits_{\varpi\in\hat{\Pi}} Sup(\ddot{\psi}_f(\varpi))\in\mathbb{C}(\mathbb{I})$ where $\ddot{\psi}_f(\varpi)$ is an ivf-membership for each $\varpi\in\hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}^{ivf}(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}^{ivf}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.16. Reconsidering Example 7.2, we get

$$(\ddot{\mathfrak{B}}^{ivf}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{[0.01, 0.1]/\varpi_1\}\right), \left(\hat{\theta}_2, \{[0.02, 0.2]/\varpi_2\}\right), \\ \left(\hat{\theta}_3, \{[0.03, 0.13]/\varpi_3\}\right), \left(\hat{\theta}_4, \{[0.04, 0.14]/\varpi_4\}\right), \\ \left(\hat{\theta}_5, \{[0.03, 0.05]/\varpi_5\}\right), \left(\hat{\theta}_6, \{[0.02, 0.06]/\varpi_6\}\right), \\ \left(\hat{\theta}_7, \{[0.03, 0.07]/\varpi_7\}\right), \left(\hat{\theta}_8, \{[0.04, 0.08]/\varpi_8\}\right) \end{array} \right\}$$

Definition 7.17. An $i\hat{f}hs$-set $(\ddot{\mathfrak{B}}^{if}, \mathfrak{A})$ is known as *bijective* $i\hat{f}hs$*-set* on $\hat{\Pi}$ if

(i) $\bigcup\limits_{\hat{\theta} \in \mathfrak{A}} \ddot{\mathfrak{B}}^{if}(\hat{\theta}) = \hat{\Pi}$ with $\sum\limits_{\varpi \in \hat{\Pi}} T_{if}(\varpi)$ & $\sum\limits_{\varpi \in \hat{\Pi}} F_{if}(\varpi) \in \mathbb{C}(\mathbb{I})$ where $T_{if}(\varpi)$ & $F_{if}(\varpi)$ are membership and non-membership grades for each $\varpi \in \hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}^{if}(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}^{if}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.18. Reassuming Example 7.2, we get

$$(\ddot{\mathfrak{B}}^{if}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{< 0.01, 0.1 > /\varpi_1\}\right), \left(\hat{\theta}_2, \{< 0.02, 0.2 > /\varpi_2\}\right), \\ \left(\hat{\theta}_3, \{< 0.03, 0.13 > /\varpi_3\}\right), \left(\hat{\theta}_4, \{< 0.04, 0.14 > /\varpi_4\}\right), \\ \left(\hat{\theta}_5, \{< 0.03, 0.05 > /\varpi_5\}\right), \left(\hat{\theta}_6, \{< 0.02, 0.06 > /\varpi_6\}\right), \\ \left(\hat{\theta}_7, \{< 0.03, 0.07 > /\varpi_7\}\right), \left(\hat{\theta}_8, \{< 0.04, 0.08 > /\varpi_8\}\right) \end{array} \right\}$$

Definition 7.19. A $n\hat{h}s$-set $(\ddot{\mathfrak{B}}^{\mathcal{N}}, \mathfrak{A})$ is known as *bijective* $n\hat{h}s$*-set* on $\hat{\Pi}$ if

(i) $\bigcup\limits_{\hat{\theta} \in \mathfrak{A}} \ddot{\mathfrak{B}}^{\mathcal{N}}(\hat{\theta}) = \hat{\Pi}$ with $\sum\limits_{\varpi \in \hat{\Pi}} T_{\mathcal{N}}(\varpi)$, $\sum\limits_{\varpi \in \hat{\Pi}} I_{\mathcal{N}}(\varpi)$ & $\sum\limits_{\varpi \in \hat{\Pi}} F_{\mathcal{N}}(\varpi) \in \mathbb{C}(\mathbb{I})$ where $T_{\mathcal{N}}(\varpi), I_{\mathcal{N}}(\varpi)$ & $F_{\mathcal{N}}(\varpi)$ are membership, indeterminacy and non-membership grades for each $\varpi \in \hat{\Pi}$

(ii) for $\hat{\theta}_{\hat{\alpha}}, \hat{\theta}_{\hat{\beta}} \in \mathfrak{A}, \hat{\alpha} \neq \hat{\beta}, \ddot{\mathfrak{B}}^{\mathcal{N}}(\hat{\theta}_{\hat{\alpha}}) \cap \ddot{\mathfrak{B}}^{\mathcal{N}}(\hat{\theta}_{\hat{\beta}}) = \varnothing$

Example 7.20. Reassuming Example 7.2, we get

$$(\ddot{\mathfrak{B}}^{\mathcal{N}}, \mathfrak{A}) = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \{< 0.01, 0.02, 0.1 > /\varpi_1\}\right), \left(\hat{\theta}_2, \{< 0.02, 0.03, 0.2 > /\varpi_2\}\right), \\ \left(\hat{\theta}_3, \{< 0.03, 0.04, 0.13 > /\varpi_3\}\right), \left(\hat{\theta}_4, \{< 0.04, 0.05, 0.14 > /\varpi_4\}\right), \\ \left(\hat{\theta}_5, \{< 0.03, 0.04, 0.05 > /\varpi_5\}\right), \left(\hat{\theta}_6, \{< 0.02, 0.05, 0.06 > /\varpi_6\}\right), \\ \left(\hat{\theta}_7, \{< 0.03, 0.04, 0.07 > /\varpi_7\}\right), \left(\hat{\theta}_8, \{< 0.04, 0.05, 0.08 > /\varpi_8\}\right) \end{array} \right\}$$

8. **Conclusions**

In this research work, several important rudiments (i.e. axioms-based properties, set-based aggregations etc.,) of $\hat{H}s$-set are investigated and explained with the support of real-scenarios based examples. In order to attract the intellectual attention of researchers, definitions of some glued models of $\hat{H}s$-set are also presented which will motivate them to extend the theory to other branches of mathematical-cum-computational sciences. Some future directions and scope of $\hat{H}s$-sets are presented in Figure 4.

Conflicts of Interest: The authors declare no conflict of interest.

Discipline	Scope
Fuzzy sets and systems	Development of intutionistic neutrosophic hypersoft set , spherical hypersoft set, picture fuzzy hypersoft set, geometric hypersoft set etc.
Graph Theory	Development of fuzzy hypersoft graph, intutionistic fuzzy hypersoft graph, neutrosophic fuzzy hypersoft graph, intutionistic neutrosophic hypersoft graph, possibility fuzzy hypersoft graph, possibility intutionistic fuzzy hypersoft graph, possibility neutrosophic hypersoft graph etc.
Algebra	Development of hypersoft groups, hypersoft rings, hypersoft vector spaces and their related structures.
Functional Analysis	Characterization of hypersoft metric spaces, hypersoft inner product spaces, normed hypersoft spaces, hypersoft measure theory, hypersoft Hilbert spaces etc.
Topology	Characterization of topological spaces, separation axioms, connected spaces and their relevant spaces.
Mathematical Analysis	Development of hypersoft fixed point theory, hypersoft real analysis, hypersoft modular inequalities, hypersoft complex analysis etc.

FIGURE 4. Future Directions and Scope of $\hat{H}s$-sets

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Neutrosophic SuperHyperSoft Sets

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Abstract:

Neutrosophic SuperHyperSoft Sets (NSHSS) represent an innovative extension of traditional soft set theory, integrating the principles of Neutrosophy to address uncertainties in decision-making processes. In this study, we recall the NSHSS and establish its foundational principles, including the relationships between various attributes and their corresponding values. The framework facilitates a structured evaluation process, allowing decision-makers to assess multiple proposals based on criteria such as environmental impact, feasibility, community engagement, cost-effectiveness, and innovation. Each criterion is categorized into distinct levels, ranging from low to transformative, thereby enabling a nuanced analysis of project proposals. Through illustrative examples and tabular representations, we demonstrate the practical applications of NSHSS in selecting projects with the highest potential for positive environmental impact. The findings indicate that NSHSS can significantly enhance the decision-making process, ensuring that resources are allocated efficiently to initiatives that align with sustainability goals.

Keywords: Neutrosophic SuperHyperSoft Sets; Extension of Neutrosophic HyperSoft Sets; Extension of SuperHyperSoft Sets; Environmental Sustainability.

1. Introduction and Preliminaries

The concept of soft set was introduced by Molodstov [5], a mathematical tool to deal with uncertainty associated with truth and false membership function. Soft set is a generalization of fuzzy set theory and rough set theory, aimed at providing a flexible approach for modeling problems where data is not always precisely defined. Further, Maji [3, 4] developed several basic operations within soft set theory and explored applications in decision making. HyperSoft Set was introduced by Smarandache [9], generalized a hypersoft set from the idea of soft set by replacing the argument function into multi argument function. In a hypersoft set, each parameter is not directly associated with a set of elements in the universal set. Instead, it is associated with a family of sub-parameters. These sub-parameters refine the information provided by the original parameters, offering more detailed granularity.

Neutrosophic set [8] was designed to handle uncertainty, incomplete information, inconsistency and indeterminacy data. Maji [2] later combined these ideas to present the Neutrosophic soft set, a novel model that integrates the concepts of soft sets and Neutrosophic sets. Truth, falsity membership and indeterminacy are Neutrosophic soft set (NSS) attributes that are independent in nature functions. Various sets have been proposed as hypothesis of the Neutrosophic soft set, with their properties and applications being exactly studied. Smarandache [1,7,11,12] also introduced the Neutrosophic hypersoft set, merging the principles of Neutrosophic and hypersoft sets. This new model aims to tackle uncertainty more accurately, which helps us better understand and apply these mathematical concepts.

Definition 1.1: [3] Let ϒ be the universal set and let € denote the set of attributes related to ϒ. Define P(ϒ) as the set of Neutrosophic values associated with ϒ, where A is a subset of 𝕀. The combination (F, A) is known as a Neutrosophic soft set over ϒ, and the relationship is expressed as F: A → (ϒ).

Definition 1.2: [9] Let ϒ represent the universal set and (ϒ) denotes its power set. Consider well defined attributes $\zeta\mathbf{1}\zeta$,$\mathbf{2}\zeta$,$\mathbf{3}\zeta$ ... ,$\mathbf{n}$ for $\mathbf{n} \geq \mathbf{1}$. The corresponding attributes values are the sets $\boldsymbol{L_1}, \boldsymbol{L_2}, \boldsymbol{L_3}$ ... $\boldsymbol{L_n}$ with $\boldsymbol{Li} \cap \boldsymbol{Lj} = \emptyset$, for $i \neq j$ and $\boldsymbol{i}, \boldsymbol{j} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3} \ \ldots \ \boldsymbol{n}\}$. The pair ($\boldsymbol{F}$, $\boldsymbol{L_1} \times \boldsymbol{L_2} \times \boldsymbol{L_3} \times \ldots \times \boldsymbol{L_n}$) is referred to as a Hypersoft set over ϒ with F being a function that maps where $\boldsymbol{F} : \boldsymbol{L_1} \times \boldsymbol{L_2} \times \boldsymbol{L_3} \times \ldots \times \boldsymbol{L_n} \rightarrow (\upsilon)$.

Definition 1.3: [7] Let ϒ represent the universal set and (ϒ) be the power set of ϒ. Consider $\zeta\mathbf{1}\zeta,\mathbf{2}\zeta,\mathbf{3},\ldots\ \zeta \boldsymbol{n}$ for $\boldsymbol{n} \geq \mathbf{1}$, along with the corresponding values represented by the set $\boldsymbol{L_1}, \boldsymbol{L_2}, \boldsymbol{L_3}$ ... $\boldsymbol{L_n}$ with $\boldsymbol{Li} \cap \boldsymbol{Lj} = \emptyset$, for $i \neq j$ and $\boldsymbol{i}, \boldsymbol{j} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3} \ \ldots \ \boldsymbol{n}\}$. The relationship among these sets is expressed as $\boldsymbol{L_1} \times \boldsymbol{L_2} \times \boldsymbol{L_3} \times \boldsymbol{rF} : \boldsymbol{L_1} \times \boldsymbol{L_2} \times \boldsymbol{L_3} \times \ldots \times \boldsymbol{L_n} \rightarrow (\upsilon)$ and $F(\boldsymbol{L_1} \times \boldsymbol{L_2} \times \boldsymbol{L_3} \times \ldots \times \boldsymbol{L_n}) = \{< x, T(F(\mathbb{I})), I(F(\mathbb{I})), F(F(\mathbb{I})) >, x \in \upsilon \}$. In this context T represents the truth membership value, I indicates indeterminate membership value and F denotes the falsity membership value such that $T, I, F: \upsilon \rightarrow [0,1]$ also $0 \leq T(F(\mathbb{I})) + I(F(\mathbb{I})) + F(F(\mathbb{I})) \leq 3$.

2. Neutrosophic SuperHyperSoft Set

Definition 2.1 [10] Let Δ be the universe of discourse, $\wp(\Delta)$ be the power set of Δ, Let $r_1, r_2, \ldots \ldots, r_n$ for $n \geq 1$ be n distinct attributes, whose corresponding attributes values are respectively the set $R_1, R_2, \ldots \ldots, R_n$ with $R_i \cap R_j = \emptyset$ for $i \neq j$. Let $\mathrm{P}(R_1), \mathrm{P}(R_2), \ldots . \mathrm{P}(R_n)$ be the power sets of $R_1, R_2, \ldots \ldots, R_n$ respectively. A Neutrosophic SuperHyperSoft Set (NSHSS) over Δ as the pair $\big(\mathcal{S}, \mathrm{P}(R_1) \times \mathrm{P}(R_2) \times \ldots . . \times \mathrm{P}(R_n)\big)$ where $\mathcal{S}$: $\mathcal{P}(R_1) \times \mathcal{P}(R_2) \times \ldots . . \times \mathcal{P}(R_n) \rightarrow \mathcal{P}(\Delta)$ *and* $\mathcal{S}, \big(\mathcal{P}(R_1) \times \mathcal{P}(R_2) \times \ldots . . \times \mathcal{P}(R_n)\big) =$
$\{\alpha, \langle x, T_{\mathcal{S}(\alpha)}(x), I_{\mathcal{S}(\alpha)}(x), F_{\mathcal{S}(\alpha)}(x) \ \rangle : x \in \Delta, \alpha \in \big(\mathrm{P}(R_1) \times \mathrm{P}(R_2) \times \ldots . . \times \mathrm{P}(R_n)\big) \subseteq R_1, R_2, \ldots \ldots, R_n \}$.

For example, in the context of student internship selection, decision-making based on multiple criteria is often challenging due to the imprecision and uncertainty associated with evaluating students. Traditional methods may not capture the complexity involved when different aspects such as academic performance, skills, and extracurricular activities need to be considered simultaneously. To address this, the Neutrosophic HyperSoft Set model can be applied, which allows for a multi-parameter and multi-attribute decision framework, providing more granular and flexible assessment.

The goal of this example is to assign internships to students based on the following main criteria:

1. Academic Performance
2. Skills
3. Extracurricular Activities.

For each of these criteria, evaluations are represented in neutrosophic terms, where the triplet (T, I, F) captures the truth, indeterminacy, and falsity degrees respectively. Two students, represented by F1 and F2 are considered for selection.

Table 1. Values for Students Selection for Internship using Neutrosophic Hypersoft Set

criteria	F_1	F_2
Academic performance	(.8, .1, .5)	(.9, .1, .1)
Skills	(.7, .15, .1)	(.8, .1, .05)
Extracurricular Activities	(.6, .2, .1)	(.7, .2, .1)

In the context of Neutrosophic SuperHypersoft Set, we assign internships to students based on the following main and sub criteria

1. Academic Performance (Cumulative grade point average and Continuous internal assessments)

2. Skills (Technical skills and soft skills)
3. Extracurricular Activities (Leadership roles and Volunteer work)

Table 2. Values for Students Selection for Internship using Neutrosophic Super Hypersoft Set

Main Criteria	Sub Criteria	F_1	F_2
Academic Performance	Cumulative grade point average	(.3, .5, .1)	(.1, .2, .3)
	Continuous internal assessments	(.6, .7, .4)	(.3, .4, .5)
Skills	Technical skills	(.4, .3, .3)	(.7, .2, .1)
	Soft skills	(.3, .1, .2)	(.5, .15, .1)
Extracurricular Activities	Leadership roles	(.7, .1, .05)	(.8, .2, .4)
	Volunteer work	(.6, .7, .4)	(.6, .4, .3)

The above two tables clearly explain the concepts of Neutrosophic Hypersoft Set and Neutrosophic Super Hypersoft Set.

2.3. Evaluating Proposals for an Environmental Sustainability Initiative

An organization is seeking to fund innovative projects that promote environmental sustainability within the community. Various teams have submitted proposals and the organization needs to evaluate these proposals based on specific criteria to determine which projects will receive funding.

The outcome of this evaluation process will be the selection of projects that demonstrate the greatest potential for positive environmental impact, community engagement and long-term sustainability. By funding these projects, the organization aims to contribute to the development of a more sustainable community and serve as a catalyst for further environmental initiatives. The evaluation will result in a prioritized list of projects that meet the established criteria, along with a clear justification for the selection. This outcome will guide the organization's decision-making process and ensure that the allocated funds are directed towards initiatives that have the highest likelihood of success and impact.

2.4 Selection Criteria for Proposal Evaluation

1. Impact on Environmental Sustainability:
   - ✓ Low: Minimal effect on environmental sustainability; does not address key issues.
   - ✓ Moderate: Some positive effects, such as reducing waste or energy consumption, but limited in scope.
   - ✓ High: Significant contributions to environmental sustainability, such as large-scale waste reduction, habitat restoration, or carbon footprint reduction.
   - ✓ Transformative: Projects that fundamentally change community practices or policies towards sustainability, leading to long-term environmental benefits.

2. Feasibility:
   - ✓ Low: High risk of failure due to lack of resources, unclear implementation plans, or significant barriers.
   - ✓ Moderate: Some challenges anticipated but manageable with available resources and planning.
   - ✓ High: Well-defined implementation plan with clear timelines, resources, and stakeholder engagement.
   - ✓ Exceptional: Strong evidence of previous success in similar projects, with robust support from the community and stakeholders.

3. Community Engagement:

   - ✓ Low: Minimal involvement of community members or stakeholders in project planning or execution.
   - ✓ Moderate: Some community involvement, but limited to feedback rather than active participation.
   - ✓ High: Active participation from community members in all phases of the project, including planning, implementation, and evaluation.
   - ✓ Exceptional: Strong partnerships with local organizations, schools, and residents, fostering a sense of ownership and commitment to the project.

4. Cost-Effectiveness:

   - ✓ Low: High costs relative to the expected benefits; poor allocation of resources.
   - ✓ Moderate: Reasonable costs with some benefits, but potential for better resource management.
   - ✓ High: Good balance of costs and benefits, demonstrating efficient use of resources.
   - ✓ Exceptional: Outstanding value for money, with innovative funding strategies and potential for scalability.

5. Innovation:

   - ✓ Low: Standard approaches with little creativity or uniqueness.
   - ✓ Moderate: Some novel ideas or methods, but largely based on existing practices.
   - ✓ High: Creative and original approaches that introduce new concepts or technologies to address sustainability challenges.
   - ✓ Transformative: Groundbreaking ideas that could redefine practices in sustainability and inspire similar projects elsewhere.

2.5 Decision-Making under Uncertain Conditions

The evaluation committee must review each proposal against these criteria while considering uncertainties such as varying levels of community support, potential regulatory challenges, and the availability of funding. For instance, a project that demonstrates high impact and exceptional feasibility may be prioritized over a moderately innovative project that has low community engagement. This structured evaluation process allows the organization to objectively assess proposals, ensuring that funding is allocated to projects that align with its sustainability goals and have the potential for meaningful environmental impact. By doing so, the organization can effectively contribute to fostering a more sustainable community.

Let ξ be the set of decision making in selecting suitable proposal $\xi = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ also consider the set of attributes as α^1 = Impact on Environmental Sustainability, α^2 = Feasibility, α^3 = Community Engagement, α^4 = Cost-Effectiveness, α^5 = Innovation and their respective attributes are given as

1. α^1= Impact on Environmental Sustainability = {Low Impact, Moderate Impact, High Impact, Transformative Impact}
2. α^2 = Feasibility = {Low Feasibility, Moderate Feasibility, High Feasibility, Transformative Feasibility}
3. α^3 = Community Engagement = {Low Community Engagement, Moderate Community Engagement, High Community Engagement, Transformative Community Engagement}

4. α^4 = Cost-Effectiveness = {Low Cost-Effectiveness, Moderate Cost-Effectiveness, High Cost-Effectiveness, Transformative Cost-Effectiveness}
5. α^5 = Innovation = {Low Innovation, Moderate Innovation, High Innovation, Transformative Innovation}

Let $W: \alpha^1 \times \alpha^2 \times \alpha^3 \times \alpha^4 \times \alpha^5 \rightarrow \alpha^n$. Below are the tables of their NSHS values

Table 3: Decision maker NSHS values for impact on environmental sustainability

Impact on Environmental Sustainability	ξ_1	ξ_2	ξ_3	ξ_4
Low Impact	(.2, .4, .6)	(.4, .5, .5)	(.7, .6, .3)	(.6, .3, .7)
Moderate Impact	(.6, .4, .7)	(.4, .3, .1)	(.7, .6, .2)	(.6, .3, .5)
High Impact	(.7, .3, .4)	(.6, .3, .5)	(.6, .5, .4)	(.4, .6, .3)
Transformative Impact	(.4, .6, .3)	(.6, .4, .2)	(.5, .3, .4)	(.5, .3, .2)

Table 4: Decision maker NSHS values for feasibility

Feasibility	ξ_1	ξ_2	ξ_3	ξ_4
Low Feasibility	(.5, .3, .4)	(.6, .4, .3)	(.7, .5, .6)	(.5, .4, .7)
Moderate Feasibility	(.3, .6, .4)	(.8, .4, .7)	(.5, .6, .7)	(.3, .6, .7)
High Feasibility	(.6, .1, .4)	(.7, .5, .3)	(.4, .5, .6)	(.4, .3, .5)
Transformative Feasibility	(.4, .6, .3)	(.6, .4, .2)	(.5, .3, .4)	(.5, .3, .2)

Table 5: Decision maker NSHS values for community engagement

Community Engagement	ξ_1	ξ_2	ξ_3	ξ_4
Low Community Engagement	(.5, .1, .2)	(.5, .2, .6)	(.2, .6, .5)	(.7, .6, .3)
Moderate Community Engagement	(.3, .1, .7)	(.4, .6, .3)	(.1, .4, .7)	(.1, .7, .4)
High Community Engagement	(.3, .4, .5)	(.6, .7, .3)	(.6, .3, .5)	(.2, .4, .5)
Transformative Community Engagement	(.6, .5, .6)	(.2, .4, .3)	(.8, .2, .1)	(.6, .4, .6)

Table 6: Decision maker NSHS values for cost-effectiveness

Cost-Effectiveness	ξ_1	ξ_2	ξ_3	ξ_4
Low Cost-Effectiveness	(.4, .6, .7)	(.4, .2, .5)	(.6, .2, .4)	(.5, .4, .4)
Moderate Cost-Effectiveness	(.8, .4, .5)	(.5, .3, .7)	(.7, .4, .7)	(.4, .6, .2)
High Cost-Effectiveness	(.5, .3, .2)	(.1, .4, .6)	(.5, .6, .4)	(.5, .4, .6)
Transformative Cost-Effectiveness	(.4, .3, .5)	(.4, .6, .3)	(.6, .1, .3)	(.6, .2, .3)

Table 7: Decision maker NSHS values for innovation

Innovation	ξ_1	ξ_2	ξ_3	ξ_4
Low Innovation	(.4, .6, .3)	(.3, .2, .5)	(.2, .7, .6)	(.5, .2, .1)
Moderate Innovation	(.2, .3, .4)	(.2, .4, .5)	(.4, .3, .2)	(.9, .4, .3)
High Innovation	(.6, .5, .4)	(.6, .5, .3)	(.3, .5, .4)	(.7, .3, .5)
Transformative Innovation	(.3, .2, .6)	(.7, .4, .6)	(.6, .4, .1)	(.4, .7, .6)

NSHSS is define as, $\boldsymbol{W}: \alpha^1 \times \alpha^2 \times \alpha^3 \times \alpha^4 \times \alpha^5 \rightarrow \alpha^n(\boldsymbol{X})$.

Let us assume $W(V) = W$ (High Impact, High Feasibility, High Community Engagement) = {ξ_1, ξ_3}

Then NSHSS of above assumed relation is

$W(V) = W$ (High Impact, High Feasibility, High Community Engagement) = {< ξ_1, (High Impact (.7, .3, .4), High Feasibility (.6, .1, .4), High Community Engagement (.3, .4, .5)>, < ξ_3, (High Impact (.6, .5, .4),

High Feasibility (.4, .5, .6), High Community Engagement (.6, .3, .5) >}

Table 8: Tabular Representation of NSHS Set

$W(V)$ =W (High Impact, High Feasibility, High Community Engagement)	ξ_1	ξ_3
High Impact	(.7, .3, .4)	(.6, .5, .4)
High Feasibility	(.6, .1, .4)	(.4, .5, .6)
High Community Engagement	(.3, .4, .5)	(.6, .3, .5)

Definition 2.6: Let $\boldsymbol{W(V^1)}$ and $\boldsymbol{W(V^2)}$ be two NSHS set over ξ. We have attributes $\boldsymbol{f^1, f^2, \ldots f^n}$ for $\boldsymbol{n \geq 1}$, which are clearly defined attributes. The corresponding attributive values are the set $\boldsymbol{F^1, F^2, \ldots F^n}$ with $\boldsymbol{F^i \cap F^j = \emptyset}$, for $i \neq j$ and $\boldsymbol{i, j \in \{1, 2, 3 \ldots n\}}$. Their relationship is described by $\boldsymbol{F^1 \times F^1 \times \ldots \times F^n = V}$. $\boldsymbol{W(V^1)}$ is the NSHS subset of $\boldsymbol{W(V^2)}$, if it meets the following conditions:

$$<T(W(V^1)) \leq T(W(V^2)), I(W(V^1)) \leq I(W(V^2)), F(W(V^1)) \geq F(W(V^2))>$$

Numerical Example of Subset

Consider the NSHSS W (V^1) and NSHSS $W(V^2)$ over the universe ξ = {ξ_1, ξ_2, ξ_3, ξ_4}. The NSHSS $W(V^1)$ = W (High Impact, High Feasibility, High Community Engagement) = {ξ_1, ξ_3} is the subset of NSHSS $W(V^2)$=W (High Impact, High Feasibility) = {ξ_1, ξ_3}.

Table 9: Tabular Representation of NSHSS $\boldsymbol{W(V^1)}$

$\boldsymbol{W(V^1)}$ = W (High Impact, High Feasibility, High Community Engagement)	ξ_1	ξ_3
High Impact	(.7, .3, .4)	(.6, .5, .4)
High Feasibility	(.6, .1, .4)	(.4, .5, .6)
High Community Engagement	(.3, .4, .5)	(.6, .3, .5)

Table 10: Tabular Representation of NSHSS $\boldsymbol{W(V^2)}$

$\boldsymbol{W(V^2)}$ =$\boldsymbol{W}$ (High Impact, High Feasibility)	ξ_1
High Impact	(.8, .5, .3)
High Feasibility	(.7, .3, .2)

This can also be written as $W(V^1) \subset W(V^2)$ = W (High Impact, High Feasibility, High Community Engagement) $\subset$ W (High Impact, High Feasibility) , {< ξ_1, High Impact, (.7, .3, .4), High Feasibility (.6, .1, .4), High Community Engagement (.3, .4, .5) >, < ξ_3, High Impact (.6, .5, .4), High Feasibility (.4, .5, .6), High Community Engagement (.6, .3, .5) >} $\subset$ {< ξ_1, High Impact (.8, .5, .3), High Feasibility (.7, .3, .2)> }

Definition 2.7: Let $\boldsymbol{W(V^1)}$ and $\boldsymbol{W(V^2)}$ be two NSHS set over ξ. For $\boldsymbol{n \geq 1}$, consider the well-defined attributes $\boldsymbol{f^1, f^2, \ldots f^n}$. The corresponding values of these attributes forms the sets $\boldsymbol{F^1, F^2, \ldots F^n}$ with $\boldsymbol{F^i \cap F^j = \emptyset}$, for $i \neq j$, $\boldsymbol{i, j \in \{\{1, 2, 3 \ldots n\}}$. The relation among these sets is expressed as $\boldsymbol{F^1 \times F^1 \times \ldots \times F^n = V}$. The NSHS set $\boldsymbol{W(V^1)}$ is deemed equal to the NSHS $\boldsymbol{W(V^2)}$ if the following conditions are met:

$$< T(W(V^1)) = T(W(V^2)), I(W(V^1)) = I(W(V^2)), F(W(V^1)) = F(W(V^2))>$$

Consider the NSHSS $W(V^1)$ and $W(V^2)$ over the same universe ξ = {ξ_1, ξ_2, ξ_3, ξ_4}. The NSHSS W(V^1) = W (High Impact, High Feasibility, High Community Engagement) = {ξ_1, ξ_3} is the equal to NSHSS $W(V^2)$ =W (High Impact, High Feasibility) = {ξ_1 } if $T(W(V^1)) = T(W(V^2))$, $I(W(V^1))$ =

$I(W(V^2)), F(W(V^1)) = F(W(V^2))$

10: Tabular Representation of NSHSS $W(V^1)$

$W(V^1) = W$ **(High Impact, High Feasibility, High Community Engagement)**	ξ_1	ξ_3
High Impact	(.7, .3, .4)	(.6, .5, .4)
High Feasibility	(.6, .1, .4)	(.4, .5, .6)
High Community Engagement	(.3, .4, .5)	(.6, .3, .5)

Table 11: Tabular Representation of $W(V^2)$

$W(V^2) = W$ **(High Impact, High Feasibility)**	ξ_1
High Impact	(.7, .3, .4)
High Feasibility	(.6, .1, .4)

This can also be written as $W(V^1)$ = W(V^2) = W (High Impact, High Feasibility, High Community Engagement)= W (High Impact, High Feasibility), { < ξ_1, (High Impact (.7, .3, .4), High Feasibility (.6, .1, .4), High Community Engagement (.3, .4, .5) >, < ξ_3, (High Impact (.6, .5, .4), High Feasibility (.4, .5, .6), High Community Engagement (.6, .3, .5) >} = {< ξ_1, (High Impact (.7, .3, .4), High Feasibility (.6, .1, .4)>}

Definition 2.8: Let $W(V^1)$ and $W(V^2)$ stand for two NSHS set over ξ. Consider the attributes $f^1, f^2, \dots f^n$ for $n \geq 1$. These attributes have corresponding values in the sets $F^1, F^2, \dots F^n$ with $F^i \cap F^j = \emptyset$, for $i \neq j$ and $i, j \in \{1, 2, 3 \dots n\}$. Their relationship can be expressed as $F^1 \times F^1 \times \dots \times F^n = V$. Futhermore, $W(V^1)$ is classified as null NSHSS i f $< T(W(V^1)) = 0, I(W(V^1)) = 0, F(W(V^1)) = 0$.

Numerical Example of Null NSHS Set

Consider the NSHSS $W(V^1)$ over the $\xi = \{\xi_1, \xi_2, \xi_3, \xi_4\}$. The NSHSS $W(V^1) = W$ (High Impact, High Feasibility, High Community Engagement) = $\{\xi_1, \xi_3\}$ is said to be null NSHSS if its Neutrosophic values are 0.

Table 12: Tabular Representation of NSHSS $W(V^1)$

$W(V^1) = W$ **(High Impact, High Feasibility, High Community Engagement)**	ξ_1	ξ_3
High Impact	(0, 0, 0)	(0, 0, 0)
High Feasibility	(0, 0, 0)	(0, 0, 0)
High Community Engagement	(0, 0, 0)	(0, 0, 0)

This can be written as $W(V^1)$= W (High Impact, High Feasibility, High Community Engagement) = W (High Impact, High Feasibility) {< ξ_1, (High Impact (0, 0, 0), High Feasibility (0, 0, 0), High Community Engagement (0, 0, 0) >, < ξ_3, (High Impact (0, 0, 0), High Feasibility (0, 0, 0), High Community Engagement (0, 0, 0) >}

Definition 2.9: Let $W(V^1)$ and $W(V^2)$ represent two NSHS set over ξ. Let $f^1, f^2, \dots f^n$ for $n \geq 1$, be well defined attributes, with corresponding values as sets $F^1, F^2, \dots F^n$ with $F^i \cap F^j = \emptyset$, for $i \neq j$ and $i, j \in \{\{1, 2, 3 \dots n\}$. The relationship among these sets is given by $F^1 \times F^1 \times \dots \times F^n = V$. In this context, the complement $W^c(V^1)$ is the complement of NSHSS of $W(V^1)$. It can be expressed as $W^c(V^1)$: $(\rightharpoonup F^1 \times \rightharpoonup F^1 \times \dots \rightharpoonup F^n) \rightarrow P^n(X)$ ensuring that < $T^C(W(V^1)) = F(W(V^1))$, $I^C(W(V^1)) = I(W(V^1))$, $F^C(W(V^1)) = T(W(V^1))$>

Numerical Example of Compliment of NHSS

Consider the NSHSS $W(V^1)$ over the universe $\xi = \{\xi_1, \xi_2, \xi_3, \xi_4\}$. The compliment of NSHSS $W(V^1) = W$ (Transformative Impact, Transformative Feasibility, Transformative Innovation) = $\{\xi_1, \xi_3\}$ is given as $T(W(V^1)) = F(W(V^1))$, $I^c(W(V^1)) = I(W(V^1))$, $F^c(W(V^1)) = T(W(V^1))$.

Table 13: Tabular Representation of NSHSS $W(V^1)$

$W(V^1) = W$ (Transformative Impact, Transformative Feasibility, Transformative Innovation)	ξ_1	ξ_3
Transformative Impact	(.4, .6, .3)	(.5, .3, .4)
Transformative Feasibility	(.4, .6, .3)	(.5, .3, .4)
Transformative Innovation	(.3, .2, .6)	(.6, .4, .1)

Table 14: Tabular Representation of NSHSS $W(V^1)$

$W^c(V^1) = W$ (Not Transformative Impact, Not Transformative Feasibility, Not Transformative Innovation)	ξ_1	ξ_3
Not Transformative Impact	(.3, .6, .4)	(.4, .3, .5)
Not Transformative Feasibility	(.3, .6, .4)	(.4, .3, .5)
Not Transformative Innovation	(.6, .2, .3)	(.1, .4, .6)

This might also be written as $W^c(V^1) = W$ (Not Transformative Impact, Not Transformative Feasibility, Not Transformative Innovation) = {< ξ_1, (Not Transformative Impact (.3, .6, .4), Not Transformative Feasibility (.3, .6, .4), Not Transformative Innovation (.6, .2, .3)) >, < ξ_3, (Not Transformative Impact (.4, .3, .5), Not Transformative Feasibility (.4, .3, .5), Not Transformative Innovation (.1, .4, .6) >}

Definition 2.10: Let $\boldsymbol{W(V^1)}$ and $\boldsymbol{W(V^2)}$ represents two NSHS set over ξ. Consider $\boldsymbol{f^1, f^2, \ldots f^n}$ for $\boldsymbol{n \geq 1}$, be well defined attributes, whose attributes values are respectively the set $\boldsymbol{F^1, F^2, \ldots F^n}$ with $\boldsymbol{F^i \cap F^j = \emptyset}$, for $i \neq j$ and $\boldsymbol{i, j \epsilon \{\{1, 2, 3 \ldots n\}}$ and their relation $\boldsymbol{F^1 \times F^1 \times \ldots \times F^n = V}$. The union of $\boldsymbol{W(V^1)}$ and $\boldsymbol{W(V^2)}$ is represented as $\boldsymbol{W(V^1) \cup W(V^2)}$ is given as

$$T(W(V^1) \cup W(V^2)) = \begin{cases} T\left(W\left(V^1\right)\right) if\ x \in V^1 \\ T\left(W\left(V^2\right)\right) if\ x \in V^2 \\ \max\left(T\left(W\left(V^1\right)\right), T\left(W\left(V^2\right)\right)\right) if\ x \in V^1 \cap V^2 \end{cases}$$

$$I(W(V^1) \cup W(V^2)) = \begin{cases} I\left(W\left(V^1\right)\right) if\ x \in V^1 \\ I\left(W\left(V^2\right)\right) if\ x \in V^2 \\ \dfrac{I\left(W\left(V^1\right)\right) + I\left(W\left(V^2\right)\right)}{2} if\ x \in V^1 \cap V^2 \end{cases}$$

$$F(W(V^1) \cup W(V^2) = \begin{cases} F\left(W\left(V^1\right)\right) if\ x \in V^1 \\ F\left(W\left(V^2\right)\right) if\ x \in V^2 \\ \left(\min\left(F\left(W\left(V^1\right)\right), F\left(W\left(V^2\right)\right)\right) if\ x \in V^1 \cap V^2 \end{cases}$$

Numerical Example of Union

Consider two NSHSS $W(V^1)$ and NSHSS $W(V^2)$ over ξ = {ξ_1, ξ_2, ξ_3, ξ_4}. The NSHSS $W(V^1) = W$ (Moderate Feasibility, Moderate Community Engagement, Moderate Cost-Effectiveness, Moderate Innovation) = {ξ_2, ξ_4} and NSHSS $W(V^2) = W$ (Moderate Feasibility, Moderate Cost-Effectiveness, Moderate Innovation) = {ξ_2}

Table 15: Tabular Representation of NSHSS $W(V^1)$

$W(V^1) = W$ (Moderate Feasibility, Moderate Community Engagement, Moderate Cost-Effectiveness, Moderate Innovation)	ξ_2	ξ_4
Moderate Feasibility	(.8, .4, .7)	(.3, .6, .7)
Moderate Community Engagement	(.4, .6, .3)	(.1, .7, .4)
Moderate Cost-Effectiveness	(.5, .3, .7)	(.4, .6, .2)
Moderate Innovation	(.2, .4, .5)	(.9, .4, .3)

Table 16: Tabular Representation of NSHSS $W(V^2)$

$W(V^2) = W$ (Moderate Feasibility, Moderate Cost-Effectiveness, Moderate Innovation)	ξ_2
Moderate Feasibility	(.8, .6, .5)
Moderate Cost-Effectiveness	(.6, .7, .2)
Moderate Innovation	(.3, .6, .3)

The union of above NSHSS is given as

Table 17: Union of NSHSS $W(V^1)$ and NSHSS $W(V^2)$

$W(V^1) \cup W(V^2)$	ξ_2	ξ_4
Moderate Feasibility	(.8, .5, .5)	(.3, .6, .7)
Moderate Community Engagement	(.4, .3, .0)	(.1, .7, .4)
Moderate Cost-Effectiveness	(.6, .5, .2)	(.4, .6, .2)
Moderate Innovation	(.3, .5, .3)	(.9, .4, .3)

This can also be written as $W(V^1) \cup W(V^2)$ = W (Moderate Feasibility, Moderate Community Engagement, Moderate Cost-Effectiveness, Moderate Innovation) $\cup$ W (Moderate Feasibility, Moderate Cost-Effectiveness, Moderate Innovation)= {< ξ_2, (Moderate Feasibility (.8, .5, .5), Moderate Community Engagement (.4, .3, .0), Moderate Cost-Effectiveness (.6, .5, .2), Moderate Innovation (.3, .5, .3)) >, < ξ_4, (Moderate Feasibility (.3, .6, .7), Moderate Community Engagement (.1, .7, .4), Moderate Cost-Effectiveness (.4, .6, .2), Moderate Innovation (.9, .4, .3)) > }

Definition 2.11: Let $W(V^1)$ and $W(V^2)$ represents two NSHS set over ξ. Consider $f^1, f^2, \ldots f^n$ for $n \geq 1$. The attributes have corresponding values that form the sets $F^1, F^2, \ldots F^n$ with $F^i \cap F^j = \emptyset$, for $i \neq j$ and $i, j \in \{\{1, 2, 3 \ldots n\}$. The relationship among these sets is expressed $F^1 \times F^1 \times \ldots \times F^n$ =

$\boldsymbol{V}$. Consequently the intersection of $\boldsymbol{W(V^1)}$ and $\boldsymbol{W(V^2)}$ is represented as $\boldsymbol{W(V^1)} \cap \boldsymbol{W(V^2)}$ is given as

$$T\left(W(V^1) \cap W(V^2)\right) = \begin{cases} T\left(W\left(V^1\right)\right) if\ x \in V^1 \\ T\left(W\left(V^2\right)\right) if\ x \in V^2 \\ \min\left(T\left(W\left(V^1\right)\right), T\left(W\left(V^2\right)\right)\right) if\ x \in V^1 \cap V^2 \end{cases}$$

$$I(W(V^1) \cap W(V^2)) = \begin{cases} I\left(W\left(V^1\right)\right) if\ x \in V^1 \\ I\left(W\left(V^2\right)\right) if\ x \in V^2 \\ \dfrac{I\left(W\left(V^1\right)\right) + I\left(W\left(V^2\right)\right)}{2} if\ x \in V^1 \cap V^2 \end{cases}$$

$$F(W(V^1) \cap W(V^2)) = \begin{cases} F\left(W\left(V^1\right)\right) if\ x \in V^1 \\ F\left(W\left(V^2\right)\right) if\ x \in V^2 \\ \max\left(F\left(W\left(V^1\right)\right), F\left(W\left(V^2\right)\right)\right) if\ x \in V^1 \cap V^2 \end{cases}$$

Numerical Example of Intersection

Consider the two NSHSS $\boldsymbol{W(V^1)}$ and NSHSS $\boldsymbol{W(V^2)}$ over the same universe $\xi = \{\xi_1, \xi_2, \xi_3, \xi_4\}$. The NSHSS $\boldsymbol{W(V^1)} = \boldsymbol{W}$ (Moderate Feasibility, Moderate Community Engagement, Moderate Cost-Effectiveness, Moderate Innovation) = $\{\xi_2, \xi_4\}$ and NSHSS $\boldsymbol{W(V^2)} = \boldsymbol{W}$ (Moderate Feasibility, Moderate Community Engagement, Moderate Cost-Effectiveness) = $\{\xi_2\}$.

Table 18: Tabular Representation of NSHSS $\boldsymbol{W(V^1)}$

$\boldsymbol{W(V^1)} = \boldsymbol{W}$ **(Moderate Feasibility, Moderate Engagement, Moderate Possibility, Moderate Innovation)**	ξ_2	ξ_4
Moderate Feasibility	(.8, .4, .7)	(.3, .6, .7)
Moderate Community Engagement	(.4, .6, .3)	(.1, .7, .4)
Moderate Cost-Effectiveness	(.5, .3, .7)	(.4, .6, .2)
Moderate Innovation	(.2, .4, .5)	(.9, .4, .3)

Table 19: Tabular Representation of NSHSS $\boldsymbol{W(V^2)}$

$\boldsymbol{W(V^2)} = \boldsymbol{W}$ **(Moderate Feasibility, Moderate Community Engagement, Moderate Cost-Effectiveness)**	ξ_2
Moderate Feasibility	(.8, .6, .5)
Moderate Community Engagement	(.6, .7, .2)
Moderate Cost-Effectiveness	(.5, .7, .2)

The intersection of above NSHSS is given as:

Table 20: Intersection of NSHSS $\boldsymbol{W(V^1)}$ and NSHSS $\boldsymbol{W(V^2)}$

$\boldsymbol{W(V^1)} \cap \boldsymbol{W(V^2)}$	ξ_2
Moderate Feasibility	(.8, .5, .7)
Moderate Community Engagement	(.4, .7, .3)
Moderate Cost-Effectiveness	(.5, .5, .7)

Moderate Innovation	(0, .4, .5)

This may also be written as $W(V^1) \cap W(V^2) = W$ (Moderate Feasibility, Moderate Community Engagement, Moderate Cost-Effectiveness, Moderate Innovation) $\cap$ W (Moderate Feasibility, Moderate Community Engagement, Moderate Cost-Effectiveness) = {< ξ_2, (Moderate Feasibility (.8, .5, .7), Moderate Community Engagement (.4, .7, .3), Moderate Cost-Effectiveness (.5, .5, .7), Moderate Innovation (0, .4, .5) >}

Definition 2.12: Let $\boldsymbol{W(V^1)}$ and $\boldsymbol{W(V^2)}$ represents two NSHS set over ξ. Consider the attributes $\boldsymbol{f^1, f^2, \dots f^n}$ for $\boldsymbol{n \geq 1}$, denoting n distinct attributes. The related attributes values are indicated by the sets $\boldsymbol{F^1, F^2, \dots F^n}$ with $\boldsymbol{F^i \cap F^j = \emptyset}$, for $i \neq j$ and $\boldsymbol{i, j \in \{1, 2, 3 \dots n\}}$. Their connection can be represented as $\boldsymbol{F^1 \times F^1 \times \dots \times F^n = V}$. The AND operation is defined as $\boldsymbol{W(V^1) \wedge W(V^2) = W(V^1 \times V^2)}$ and is expressed by the following conditions:

$$T(V^1 \times V^2) = min(T(W(V^1)), T(W(V^2)))$$
$$I(V^1 \times V^2) = \frac{(I(W(V^1)), I(W(V^2)))}{2}$$
$$F(V^1 \times V^2) = max(F(W(V^1)), F(FW(V^2)))$$

Numerical example of AND operation

Consider the two NSHSS $W(V^1)$ and NSHSS $W(V^2)$ over the same universe ξ = {ξ_1, ξ_2, ξ_3, ξ_4}. The NSHSS $W(V^1)$ = W (Low Impact, Low Cost-Effectiveness, Low Innovation) = {ξ_1, ξ_4} and NSHSS $W(V^2)$ = W (Low Impact, Low Innovation) = {ξ_1}.

Table 21: Tabular Representation of NSHSS $\boldsymbol{W(V^1)}$

$\boldsymbol{W(V^1)}$ = $\boldsymbol{W}$ **(Low Impact, Low Cost-Effectiveness, Low Innovation)**	ξ_1	ξ_4
Low Impact	(.2, .4, .6)	(.6, .3, .7)
Low Cost-Effectiveness	(.4, .6, .7)	(.5, .4, .4)
Low Innovation	(.4, .6, .3)	(.5, .2, .1)

Table 22: Tabular Representation of NSHSS $\boldsymbol{W(V^2)}$

$\boldsymbol{W(V^2)}$ = $\boldsymbol{W}$ **(Low Impact, Low Innovation)**	ξ_1
Low Impact	(.3, .6, .3)
Low Innovation	(.5, .8, .2)

The AND operation of above NSHSS is given as:

Table 23: AND operation of NSHSS $\boldsymbol{W(V^1)}$ and NSHSS $\boldsymbol{W(V^2)}$

$\boldsymbol{W(V^1) \wedge W(V^2)}$	ξ_1	ξ_4
Low Impact × Low Impact	(.2, .5, .6)	(0, .15, .7)
Low Impact × Low Innovation	(.2, .6, .6)	(0, .15, .7)
Low Cost-Effectiveness × Low Impact	(.3, .6, .7)	(0, .2, .4)
Low Cost-Effectiveness × Low Innovation	(.4, .7, .7)	(0, .2, .4)
Low Innovation ×High Impact	(.3, .6, .3)	(0, .1, .1)
Low Innovation × Low Innovation	(.4, .7, .3)	(0, .1, .1)

Definition 2.13: Let $\boldsymbol{W(V^1)}$ and $\boldsymbol{W(V^2)}$ be two NSHSS over ξ. Consider $\boldsymbol{f^1, f^2, \dots f^n}$ for $\boldsymbol{n \geq 1}$ Denoting n distinct attributes. The related attributes values are indicated by the sets $\boldsymbol{F^1, F^2, \dots F^n}$ with $\boldsymbol{F^i \cap F^j = \emptyset}$, for $\boldsymbol{i \neq j}$ and $\boldsymbol{i, j \in \{1, 2, 3 \dots n\}}$. Their connection can be represented as $\boldsymbol{F^1 \times F^1 \times \dots \times F^n = V}$. The OR operation is defined as $\boldsymbol{W(V^1) \vee W(V^2) = W(V^1 \times V^2)}$ and is expressed

by the following conditions:

$$T(V^1 \times V^2) = max(T(W(V^1)), T(W(V^2)))$$
$$I(V^1 \times V^2) = \frac{(I(W(V^1)), I(W(V^2)))}{2}$$
$$F(V^1 \times V^2) = min(F(W(V^1)), F(W(V^2)))$$

Numerical example of OR operation

Consider the two NSHSS $W(V^1)$ and NSHSS $W(V^2)$ over the same universe $\xi = \{\xi_1, \xi_2, \xi_3, \xi_4\}$. The NSHSS $W(V^1) = W$ (Low Impact, Low Cost-Effectiveness, Low Innovation) = $\{\xi_1, \xi_4\}$ and NSHSS $W(V^2) = W$ (Low Impact, Low Innovation) = $\{\xi_1\}$.

Table 24: Tabular Representation of NSHSS $W(V^1)$

$W(V^1) = W$ (Low Impact, Low Cost-Effectiveness, Low Innovation)	ξ_1	ξ_4
Low Impact	(.2, .4, .6)	(.6, .3, .7)
Low Cost-Effectiveness	(.4, .6, .7)	(.5, .4, .4)
Low Innovation	(.4, .6, .3)	(.5, .2, .1)

Table 25: Tabular Representation of NSHSS $W(V^2)$

$W(V^2) = W$ (Low Impact, Low Innovation)	ξ_1
Low Impact	(.3, .6, .3)
Low Innovation	(.5, .8, .2)

Then the OR Operation of above NSHSS is given as :

Table 26: OR operation of NSHSS $W(V^1)$ and NSHSS $W(V^2)$

$W(V^1) \wedge W(V^2)$	ξ_1	ξ_4
Low Impact × Low Impact	(.3, .5, .3)	(.6, .15, 0)
Low Impact × Low Innovation	(.5, .6, .2)	(.6, .15, 0)
Low Cost-Effectiveness × Low Impact	(.4, .6, .3)	(.5, .2, 0)
Low Cost-Effectiveness × Low Innovation	(.5, .7, .2)	(.5, .2, 0)
Low Innovation ×High Impact	(.4, .6, .3)	(.5, .1, 0)
Low Innovation × Low Innovation	(.5, .7, .2)	(.5, .1, 0)

5. Conclusions

This paper discusses the operations of Neutrosophic SuperHyperSoft Set, including union, intersection, compliment, AND and OR operations are presented. It demonstrates the validity and application of these operations through appropriate examples. The Neutrosophic SuperHyperSoft Set (NSHSS) is recalled as a new tool that can be useful for decision-making problems, particularly in suitable selections.

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An inclusive study on the fundamentals of interval-valued fuzzy hypersoft set

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Abstract

When compared to its extension, hypersoft set, a soft set only deals with a single set of attributes, while a hypersoft set deals with several attribute-valued disjoint sets that correspond to various attributes. Several researchers have developed models based on soft sets, but the majority of these models suffer from limitations since they are inappropriate for interval-type data or uncertain data. In order to address these issues, a novel model interval-valued fuzzy hypersoft set ($IVFHS$-set) is presented in this research article. This model not only resolves the inadequacy of soft set for distinct attributes for non-overlapping attribute-valued sets, but also addresses the limitations of soft set-like models with having data in interval environment. This work modifies the current fuzzy hypersoft set concept and introduces certain fundamental ideas, such as subset, not set, whole set, and absolute relative null set, relative absolute set and aggregation operations e.g. intersection, union, extended intersection, restricted union, complement, OR, AND, difference, restricted difference are discussed under $IVFHS$-set environment with illustrated examples. Some new hybrids of fuzzy hypersoft set under interval-valued settings are also discussed. Moreover, some extensions of $IVFHS$-set are presented along with different operations.

Keywords: interval-valued fuzzy set; interval-valued fuzzy soft set; hypersoft set; set-theoretic operations.

1 introduction

Numerous intricate problems containing diverse uncertainties are addressed mathematically by the theory of fuzzy sets, the theory of probability and interval mathematics. These theories are used in a variety of mathematical areas. These theories' inherent complexities which prevent them from successfully resolving these issues. These difficulties may be caused by the parametrization tool's shortcomings. For dealing with uncertainties, a mathematical instrument that is free of these obstructions is required. To cope with uncertainties and ambiguity in data, fuzzy soft set (FS-set)[1] as a mixed variant of fuzzy set (F-set)[2] and soft set (S-set)[3] is regarded viable model. The F-set uses a membership function to give each component of the sample universe a fuzzy value between 0 and 1. In order to offer the F-set a parametrization tool, the FS-set ties an approximation function to each member of the set of parameters, allowing for the parameterization of each member with a fuzzy membership grade. Rahman et al.[4–6] revised the conventional concept of convexity and concavity in the context of the S-set, FS-set and refined intuitionistic fuzzy set ($RIFS$). Yang et al.[7] envisioned an interval-valued fuzzy soft set ($IVFS$-set) to handle the uncertain data in interval form by merging the interval-valued fuzzy set (IVF-set)[8] and S-set. Researchers[9–14] made marvelous contributions concerning real-world applications of $IVFS$-sets. The similarity measure, entropy measure and inclusion measure along with distance measure of the $IVFS$-set were devised by Xindong and Yong.[15] Intuitionistic fuzzy set IF-set[16]

and neutrosophic set N-set[17] included non-membership and Neutral degree to extend the idea of F-set where as picture fuzzy set PFS[18] applied condition on sum of positive, negative and neutral degree to remain with in o to 1.
In S-set-like models, a single set of parameters is used to cope with the ambiguous and uncertain character of the data, however there are some circumstances in which the chosen parameters are insufficient to address decision-making issues. It is important to further categorise these parameters into the proper subclasses. The concept of the hypersoft set (HS-set) was devised by Smarandache as an enhancement of the S-set since the S-set literature is insufficient to handle such situations. In order to improve the performance of S-set, it employs a brand-new mapping approach called multi-argument approximation mapping (MAA-mapping). The domain and codomain of this mapping are assumed to be parametric tuples and the universe's power set, respectively. In this sense, it is accurate to argue that HS-set is more adaptable and trustworthy than S-set. Saeed et al.[20] and Abbas et al.[21] not only provided the fundamentals of the HS-set but also covered its characteristics and operations with the aid of several examples to assure its usage in addressing real-world problems. In the context of the HS-set, Rahman et al.[22] modified the traditional idea of convexity and concavity. The concept of rough HS-set was initiated by[23] along with some of its operational aspects in order to handle the roughness of the data. Kamacı[24] created expert systems for HS-set environments to handle circumstances requiring the multiple-decisional opinions of experts in decision-making systems. The idea was expanded by Martin & Smarandache[25] to include a plithogenic HS-set with its graphical version.
The fuzzy hypersoft set (FHS-set), a combination of the HS-set and the F-set, was developed by Yolcu and Oztürk[26] and its numerous fundamental operational properties were examined. Real-world decision-making issues were addressed by Arshad et al.[27] using distance measure of FHS-set. Rahman et al.[28] investigated the parametrization of the fuzzy hypersoft set (FHS-set) and analysed various of its practical results. Authors[29–32] developed several modifications in neutrosophic HS-set. Some fundamental concepts, including set theoretic operations, properties, laws as well as hybrids of $IVFHS$-sets are conceptualised in this work within the context of a hypersoft set in interval-value fuzzy environment. Recently Smarandache[33–35] developed new structures IndetermSoft set and IndetermHyperSoft set. He discussed the nature of product used in these structures and discussed its practical applications. Moreover, TreeSoftSet is developed as an extension of MultiSoftSet.

The remaining article is organised as follows: Section 2 provides some basic notions of HS-set and FHS-set. Some fundamental properties of $IVFHS$-set are presented in Section 3. The set theoretic operations of $IVFHS$-sets are described in Section 4. Some fundamental results, characteristics, and laws on $IVFHS$-sets are provided in Section 5. Section 6 offers several hybrid of $IVFHS$-sets, and section 7 concludes the paper.

2 Preliminaries

Let $\mathbb{U}$, $P(\mathbb{U})$ and $C(\mathbb{U})$, represent the domain of discourse, the gathering of all $\mathbb{U}$ subsets, and the collection of all F-sets of $\mathbb{U}$, respectively. E be the set of attributes. Let collection of all sub intervals of $I = [0, 1]$ be represented by $\mathcal{C}(I)$.

Definition 2.1. [2] A F-set over $\mathbb{U}$ is described by a membership function f_F, where $f_F : \mathbb{U} \to I$ is given by

$$f_F(z) = \{(z, f_F(z)) | z \in \mathbb{U}\},$$

this gives each $z \in \mathbb{U}$ a real value inside I and $f_F(z)$ is the membership rating of $z \in \mathbb{U}$.

Definition 2.2. [8] An IVF-set M over $\mathbb{U}$ is specified by a function $\mathfrak{I}_M : \mathbb{U} \to \mathcal{C}(I)$ where $\mathfrak{I}_M(z), z \in \mathbb{U}$ is an interval $[\delta, \sigma], 0 \leq \delta \leq \sigma \leq 1$, δ and σ denote lower and upper membership-rating of an individual respectively.
As an instance, assume the interval-valued fuzzy number (IVF-number) $\tau = [\delta, \sigma]$. The whole collection of IVF-numbers $\tau = [\delta, \sigma]$ over $\mathbb{U}$ is symbolized by $\Gamma(\mathbb{U})$.

Definition 2.3. [3] A soft set S_S over $\mathbb{U}$ is given by $F_{S_S} : E' \to P(\mathbb{U})$ where $E' \subseteq E$ and

$$S_S(\zeta) = \left\{(\zeta, F_{(S,E)(\zeta)}) : \zeta \in E'\right\}.$$

Definition 2.4. [1] A fuzzy soft set (FS-set) over $\mathbb{U}$ denoted by F_S is described as $\Delta_{F_S} : E' \to C(\mathbb{U})$ and is stated as $F_S(\zeta) = \{(\zeta, \Delta_{F_S}(\zeta)) : \zeta \in E'\}, \Delta_{F_S}(\zeta) \in C(\mathbb{U})$ for $E' \subseteq E$, $\Delta_{F_S} = \varphi$ for $\zeta \notin E'$ and

$$\Delta_{F_{S_{E'}}}(\zeta) = \left\{\zeta_{\Delta_{F_{S_{E'}}}(\zeta)}(\gamma)/\gamma : \gamma \in \mathbb{U}, \zeta_{\Delta_{F_{S_{E'}}}(\zeta)}(\gamma) \in I\right\},$$

for all $\zeta \in E'$ is a F-set over $\mathbb{U}$, where the approximation function of F_S is a F-set $\Delta_{F_S}(z)$ such that, if $\Delta_{F_{S_{E'}}}(\zeta) = \varphi$, then $(\zeta, \Delta_{F_{S_{E'}}}(\zeta)) \notin F_{S_{E'}}$.

Definition 2.5. [7] The interval-valued fuzzy soft set ($IVFS$-set) over $\mathbb{U}$ denoted by $(\tilde{L}_S, E)$ and described by $\tilde{L}_S : E \to P(\mathbb{U})$.
The $IVFS$-set: a parameterized collection of IVF-subsets of $\mathbb{U}$ is stated as

$$\tilde{L}_S(\zeta) = \left\{< \tau, \kappa_{\tilde{L}_S(\zeta)}(\tau) >: \tau \in \mathbb{U}, \zeta \in E\right\},$$

such that $\tilde{L}_S(\zeta)$ denotes IVF membership-grading of $\tau \in \mathbb{U}, \zeta \in E$. $\tilde{L}_S(\zeta)$ reduces to F-set if $\kappa^-_{\tilde{L}_S(\zeta)}(\tau) = \kappa^+_{\tilde{L}_S(\zeta)}(\tau) \forall \tau \in \mathbb{U}, \zeta \in E$.

Definition 2.6. [7] The compliment $(\tilde{L}_S, E)^c = (\tilde{L}^c_S, E)$ of $IVFS$-set $(\tilde{L}_S, E)$ is defined as

$$\tilde{L}^c_S(\zeta) = [1 - \kappa^+_{\tilde{L}_S(\zeta)}(\tau), 1 - \kappa^-_{\tilde{L}_S(\zeta)}(\tau)],$$

for all $\tau \in \mathbb{U}, \zeta \in E$.

Definition 2.7. [19] The entity $(\tilde{H}_S, \eth)$ defined by $\tilde{H}_S : \eth \to P(\mathbb{U})$ is known as hypersoft set (HS-set) over $\mathbb{U}$ where $\eth_1, \eth_2, ..., \eth_n$ are disjoint attribute valued sets corresponding to attributes $\epsilon_1, \epsilon_2, ..., \epsilon_n \in E$ and $\eth = \eth_1 \times \eth_2 \times ... \times \eth_n$ be the Cartesian product of disjoint attribute valued sets.

Example 2.8. Suppose that Mr. Z is interested to purchase a laptop from market. There are five kinds of laptops which form the universal set $L = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$. The attributes which are considered in the selection of laptop are discussed hereafter. $\mathring{a}_1$ = Company, $\mathring{a}_2$ =Processor, $\mathring{a}_3$ =RAM, $\mathring{a}_4$ =Hard drive, $\mathring{a}_5$ =Price range. The attributive valued sets corresponding to these attributes are:
$\mathring{A}_1 = \{$ HP, Apple $\} = \{\mathring{a}_{11}, \mathring{a}_{12}\}$, corresponds to attribute $\mathring{a}_1$.
$\mathring{A}_2 = \{$ 2.4 GHz, 3.4 GHz $\} = \{\mathring{a}_{21}, \mathring{a}_{22}\}$, corresponds to attribute $\mathring{a}_2$.
$\mathring{A}_3 = \{$ 4 GB, 8 GB $\} = \{\mathring{a}_{31}, \mathring{a}_{32}\}$, corresponds to attribute $\mathring{a}_3$.
$\mathring{A}_4 = \{$ 320 GB, 1 TB $\} = \{\mathring{a}_{41}, \mathring{a}_{42}\}$, corresponds to attribute $\mathring{a}_4$.
$\mathring{A}_5 = \{$ up to 1200 US dollars $\} = \{\mathring{a}_{51}\}$ corresponds to attribute $\mathring{a}_5$.
$\mathring{A} = \mathring{A}_1 \times \mathring{A}_2 \times \mathring{A}_3 \times \mathring{A}_4 \times \mathring{A}_5$
$\mathring{A} = \{\tau_1, \tau_2, \tau_3, ..., \tau_{16}\}, each \tau_i : i = 1, 2, ..., 16$ is a five tuple element.
The HS-set $(\tilde{H}_S, \eth)$ is given as

$$(\tilde{H}_S, \eth) = \left\{ \begin{array}{l} (\tau_1, \{\ell_1, \ell_2\}), (\tau_2, \{\ell_1, \ell_3\}), (\tau_3, \{\ell_2, \ell_4\}), (\tau_4, \{\ell_1, \ell_2, \ell_5\}), \\ (\tau_5, \{\ell_2, \ell_4\}), (\tau_6, \{\ell_1, \ell_2, \ell_5\}), (\tau_7, \{\ell_4, \ell_5\}), (\tau_8, \{\ell_1, \ell_2, \ell_4, \ell_5\}), \\ (\tau_9, \{\ell_1, \ell_3, \ell_5\}), (\tau_{10}, \{\ell_4, \ell_5\}), (\tau_{11}, \{\ell_2, \ell_5\}), (\tau_{12}, \{\ell_4, \ell_5\}) \\ (\tau_{13}, \{\ell_2, \ell_3, \ell_4, \ell_5\}), (\tau_{14}, \{\ell_1, \ell_5\}), (\tau_{15}, \{\ell_3, \ell_4, \ell_5\}), (\tau_{16}, \{\ell_1, \ell_2, \ell_3, \ell_4\}) \end{array} \right\}.$$

The element $(\tau_1, \{\ell_1, \ell_2\})$ of HS-set $(\tilde{H}_S, \eth)$ represents that there are two laptops ℓ_1 and ℓ_2 available in market having the following specifications;

1. Company "HP"
2. Processor of 2.4 GHz,
3. Random Access Memory (RAM) of 4 GB,
4. Hard drive of capacity 320 GB,
5. Price range upto 1200 US Dollars.

Similarly, the element $(\tau_{16}, \{\ell_1, \ell_2, \ell_3, \ell_4\})$ of HS-set $(\tilde{H}_S, \eth)$ represents that there are four laptops ℓ_1, ℓ_2, ℓ_3 and ℓ_4 available in market having the following specifications;

1. Company "Apple"
2. Processor of 3.4 GHz,
3. Random Access Memory (RAM) of 8 GB,
4. Hard drive of capacity 1 TB,
5. Price range upto 1200 US Dollars.

Definition 2.9. [19] Let $\eth_1, \eth_2, ..., \eth_n$ be attribute valued sets corresponding to attributes $\epsilon_1, \epsilon_2, ..., \epsilon_n \in \mathfrak{E}$ such that for $p, q = 1, 2, ..., n, p \neq q, \eth_p \cap \eth_q = \varphi$, then $(\tilde{L}_{FHS}, \eth)$, the pair set over $\Bbbu$ is said to be fuzzy hypersoft set (FHS-set) and is given by

$$(\tilde{L}_{FHS}, \eth) = \left\{(\varsigma, \tilde{L}_{FHS}(\varsigma)) : \varsigma \in \eth, \tilde{L}_{FHS}(\varsigma) \in C(\Bbbu)\right\},$$

where $\tilde{L}_{HS} : \eth \to C(\Bbbu)$ and $C(\Bbbu)$ be collection of all F-sets over $\Bbbu$ with $\varsigma \in \eth = \eth_1 \times \eth_2 \times ... \times \eth_n$ and, $\tilde{L}(\varsigma) = \left\{\kappa_{\tilde{L}(\varsigma)}(z)/z : z \in \Bbbu, \kappa_{\tilde{L}(\varsigma)} \in I\right\}$ is a F-set over $\Bbbu$.

Example 2.10. Consider the Example 2.8. The FHS-set $(\tilde{L}_{FHS}, \eth)$ over $\Bbbu$ is given as
$(\tilde{L}_{FHS}, \eth) =$

$$\left\{\begin{array}{l}
\left(\tau_1, \left\{0.1/\ell_1, 0.5/\ell_2\right\}\right), \left(\tau_2, \left\{0.2/\ell_1, 0.7/\ell_3\right\}\right), \left(\tau_3, \left\{0.3/\ell_2, 0.5/\ell_4\right\}\right), \left(\tau_4, \left\{0.7/\ell_1, 0.3/\ell_2, 0.2/\ell_5\right\}\right), \\
\left(\tau_5, \left\{0.1/\ell_2, 0.5/\ell_4\right\}\right), \left(\tau_6, \left\{0.2/\ell_1, 0.6/\ell_2, 0.3/\ell_5\right\}\right), \left(\tau_7, \left\{0.2/\ell_4, 0.1/\ell_5\right\}\right), \\
\left(\tau_8, \left\{0.1/\ell_1, 0.2/\ell_2, 0.4/\ell_4, 0.6/\ell_5\right\}\right), \left(\tau_9, \left\{0.3/\ell_1, 0.1/\ell_3, 0.3/\ell_5\right\}\right), \left(\tau_{10}, \left\{0.5/\ell_4, 0.2/\ell_5\right\}\right), \\
\left(\tau_{11}, \left\{0.3/\ell_2, 0.6/\ell_5\right\}\right), \left(\tau_{12}, \left\{0.4/\ell_4, 0.7/\ell_5\right\}\right), \left(\tau_{13}, \left\{0.3/\ell_2, 0.1/\ell_3, 0.6/\ell_4, 0.4/\ell_5\right\}\right), \\
\left(\tau_{14}, \left\{0.4/\ell_1, 0.2/\ell_5\right\}\right), \left(\tau_{15}, \left\{0.1/\ell_3, 0.6/\ell_4, 0.4/\ell_5\right\}\right), \left(\tau_{16}, \left\{0.5/\ell_1, 0.2/\ell_2, 0.3/\ell_3, 0.7/\ell_4\right\}\right)
\end{array}\right\}.$$

The element $\left(\tau_1, \left\{0.1/\ell_1, 0.5/\ell_2\right\}\right)$ of FHS-set $(\tilde{H}_S, \eth)$ represents that there are two laptops ℓ_1 and ℓ_2 available in market having the specifications as described in Example 2.8. The availability of ℓ_1 is 0.1 or 10 $\%$ and availability of ℓ_2 is 0.5 or 50 $\%$.

3 Interval-valued fuzzy hypersoft set ($IVFHS$-set)

In this section, $IVFHS$-set as well as some of its basic notions are illustrated with suitable examples.

Definition 3.1. [20] Let $\eth_1, \eth_2, ..., \eth_n$ are attribute valued sets corresponding to attributes $\epsilon_1, \epsilon_2, ..., \epsilon_n \in \mathfrak{E}$, then $(\tilde{L}_{IVFHS}, \eth)$, the pair set over $\Bbbu$ is said to be $IVFHS$-set over $\Bbbu$ and is defined as follow:

$$(i\tilde{L}_{IVFHS}, \eth) = \left\{(\varsigma, \tilde{L}_{IVFHS}(\varsigma)) : \varsigma \in \eth, \tilde{L}_{IVFHS}(\varsigma) \in C(\Bbbu)\right\},$$

such that $\tilde{L}_{IVFHS} : \eth \to C(\Bbbu)$, $C(\Bbbu)$ denotes the family of all IVF-sets over $\Bbbu$, $\varsigma \in \eth = \eth_1 \times \eth_2 \times ... \times \eth_n$, and $\tilde{L}(\varsigma) = \left\{\kappa_{\tilde{L}'(\varsigma)}(z)/z : z \in \Bbbu, \kappa_{\tilde{L}'(\varsigma)} \in \mathcal{C}(I)\right\}$ is IVF-set over $\Bbbu$.

Example 3.2. Consider the Example 2.8. The $IVFHS$-set $(\tilde{L}_{IVFHS}, \eth)$ over $\Bbbu$ is given as

$(\tilde{L}_{IVFHS}, \eth) =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{ [0.1,0.2]/_{\ell_1}, [0.3,0.5]/_{\ell_2} \right\}\right), \left(\tau_2, \left\{ [0.2,0.4]/_{\ell_1}, [0.3,0.7]/_{\ell_3} \right\}\right), \\ \left(\tau_3, \left\{ [0.3,0.4]/_{\ell_2}, [0.2,0.5]/_{\ell_4} \right\}\right), \left(\tau_4, \left\{ [0.4,0.7]/_{\ell_1}, [0.3,0.5]/_{\ell_2}, [0.6,0.8]/_{\ell_5} \right\}\right), \\ \left(\tau_5, \left\{ [0.1,0.3]/_{\ell_2}, [0.2,0.5]/_{\ell_4} \right\}\right), \left(\tau_6, \left\{ [0.2,0.6]/_{\ell_1}, [0.4,0.6]/_{\ell_2}, [0.5,0.6]/_{\ell_5} \right\}\right), \\ \left(\tau_7, \left\{ [0.2,0.4]/_{\ell_4}, [0.6,0.7]/_{\ell_5} \right\}\right), \left(\tau_8, \left\{ [0.2,0.4]/_{\ell_1}, [0.2,0.3]/_{\ell_2}, [0.4,0.5]/_{\ell_4}, [0.3,0.6]/_{\ell_5} \right\}\right), \\ \left(\tau_9, \left\{ [0.2,0.3]/_{\ell_1}, [0.1,0.5]/_{\ell_3}, [0.3,0.7]/_{\ell_5} \right\}\right), \left(\tau_{10}, \left\{ [0.5,0.6]/_{\ell_4}, [0.2,0.4]/_{\ell_5} \right\}\right), \\ \left(\tau_{11}, \left\{ [0.3,0.5]/_{\ell_2}, [0.4,0.6]/_{\ell_5} \right\}\right), \left(\tau_{12}, \left\{ [0.2,0.4]/_{\ell_4}, [0.5,0.7]/_{\ell_5} \right\}\right), \\ \left(\tau_{13}, \left\{ [0.3,0.6]/_{\ell_2}, [0.1,0.4]/_{\ell_3}, [0.2,0.6]/_{\ell_4}, [0.4,0.7]/_{\ell_5} \right\}\right), \\ \left(\tau_{14}, \left\{ [0.4,0.5]/_{\ell_1}, [0.2,0.3]/_{\ell_5} \right\}\right), \left(\tau_{15}, \left\{ [0.1,0.4]/_{\ell_3}, [0.6,0.8]/_{\ell_4}, [0.4,0.7]/_{\ell_5} \right\}\right), \\ \left(\tau_{16}, \left\{ [0.5,0.7]/_{\ell_1}, [0.2,0.4]/_{\ell_2}, [0.3,0.6]/_{\ell_3}, [0.4,0.8]/_{\ell_4} \right\}\right) \end{array} \right\}.$$

The element $\left(\tau_1, \left\{ [0.1,0.2]/_{\ell_1}, [0.3,0.5]/_{\ell_2} \right\}\right)$ of $IVFHS$-set $(IV\tilde{F}HS_S, \eth)$ represents that there are two laptops ℓ_1 and ℓ_2 available in market having the specifications as described in Example 2.8. The availability of ℓ_1 is 0.1 to 0.2 i.e. 10 % to 20% which means all lower values are greater than or equal to 10 % and all upper values are less than or equal to 20 %. Similarly the availability of ℓ_2 is 0.3 to 0.5 i.e. 30 % to 50% which means all lower values are greater than or equal to 30 % and all upper values are less than or equal to 50 %.

Definition 3.3. Let $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$, are two $IVFHS$-sets over $\Bbbu$ written as $(\tilde{L'}_{H_1}, \eth_1), (\tilde{L'}_{H_2}, \eth_2) \in \text{Ç}_{(\tilde{L'}_H, \eth)}$(the collection of $IVFHS$-sets). $(\tilde{L'}_{H_1}, \eth_1)$ is known as $IVFHS$-subset of $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \subseteq (\tilde{L'}_{H_2}, \eth_2)$ if

1. $\eth_1 \subseteq \eth_1$
2. $\forall$ ł $\in \eth_1, \tilde{L'}_{H_1}$(ł)$\subseteq \tilde{L'}_{H_2}$(ł).

Example 3.4. Consider the Example 2.8.
If $(\tilde{L'}_{H_1}, \eth_1) =$

$$\left\{ \left(\tau_1, \left\{ [0.1,0.2]/_{\ell_1} \right\}\right), \left(\tau_2, \left\{ [0.4,0.5]/_{\ell_3} \right\}\right), \left(\tau_4, \left\{ [0.4,0.6]/_{\ell_1}, [0.3,0.4]/_{\ell_2} \right\}\right) \right\}$$

and $(\tilde{L'}_{H_2}, \eth_2) =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{ [0.1,0.4]/_{\ell_1}, [0.3,o.5]/_{\ell_2} \right\}\right), \left(\tau_2, \left\{ [0.2,0.4]/_{\ell_1}, [0.3,0.7]/_{\ell_3} \right\}\right), \\ \left(\tau_3, \left\{ [0.3,0.4]/_{\ell_2}, [0.2,0.5]/_{\ell_4} \right\}\right), \left(\tau_4, \left\{ [0.4,0.7]/_{\ell_1}, [0.3,0.5]/_{\ell_2}, [0.1,0.2]/_{\ell_5} \right\}\right) \end{array} \right\}$$

then $(\tilde{L'}_{H_1}, \eth_1) \subseteq (\tilde{L'}_{H_2}, \eth_2)$

Definition 3.5. A set $N = N_1 \times N_2 \times ... \times N_n$ in $IVFHS$-set $(\tilde{L'}_H, \eth)$ is known as $Not\ set$ if it has a representation as

$$\propto N = \{\propto \tau_1, \propto \tau_2, ..., \propto \tau_h\}$$

where $h = \prod_{p=1}^{n} |N_p|$ and each $\propto \tau_p, p \in \{1, 2, ..., h\}$ is a Not n-tuple element.

Example 3.6. Consider the Example 2.8. Take sets N_1, N_2, N_3, N_4, N_5, we have

$$\propto N = \{\propto \tau_1, \propto \tau_2, ..., \propto \tau_{16}\}$$

where each $\propto \tau_p, p \in \{1, 2, ..., 16\}$ is a $Not\ set$ 5-tuple element.

Definition 3.7. An $IVFHS$-set $(\tilde{L'}_H, \eth_1)$ is relative null $IVFHS$-set w.r.t $\eth$, represented as $(\tilde{L'}_H, \eth_1)_\Phi$, if $\tilde{L'}_H(\tau) = \varphi \forall \tau \in \eth_1$.

Example 3.8. Consider the Example 2.8, if

$$(\tilde{L'}_H, \eth_1)_\Phi = \{(\tau_1, \varphi), (\tau_2, \varphi), (\tau_3, \varphi)\}$$

where $\eth_1 \in \eth$.

Definition 3.9. An $IVFHS$-set $(\tilde{L'}_H, \eth_1)$ is relative whole $IVFHS$-set w.r.t $\eth_1 \subseteq E$, represented as $(\tilde{L'}_H, \eth_1)_\Im$, if $\tilde{L'}_H(\tau) = \Cup \forall \tau \in \eth_1$.

Example 3.10. Consider the Example 2.8, if

$$(\tilde{L'}_H, \eth_1)_\Cup = \{(\tau_1, \Cup), (\tau_2, \Cup), (\tau_3, \Cup)\}$$

where $\eth_1 \in \eth$.

Definition 3.11. An $IVFHS$-set $(\tilde{L'}_H, \eth)$ is absolute whole $IVFHS$-set over E, represented as $(\tilde{L'}_H, \eth)_\Im$, if $\tilde{L'}_H(\tau) = \Cup \forall \tau \in \eth$.

Example 3.12. Consider the Example 2.8, if

$$(\tilde{L'}_H, \eth)_\Cup = \left\{ \begin{array}{l} (\tau_1, Z), (\tau_2, Z), (\tau_3, Z), (\tau_4, Z), \\ (\tau_5, Z), (\tau_6, Z), (\tau_7, Z), (\tau_8, Z), \\ (\tau_9, Z), (\tau_{10}, Z), (\tau_{11}, Z), (\tau_{12}, Z), \\ (\tau_{13}, Z), (\tau_{14}, Z), (\tau_{15}, Z), (\tau_{16}, Z) \end{array} \right\}.$$

Proposition 3.13. $(\tilde{L'}_{H_1}, \eth_1), (\tilde{L'}_{H_2}, \eth_2), (\tilde{L'}_{H_3}, \eth_3) \in \zeta_{(\tilde{L'}_H, \eth)}$ *with* $\eth_1, \eth_2, \eth_3 \subseteq \eth$*, then*

1. $(\tilde{L'}_{H_1}, \eth_1) \subseteq (\tilde{L'}_{H_1}, \eth_1)_\Cup$
2. $(\tilde{L'}_{H_1}, \eth_1)_\Phi \subseteq (\tilde{L'}_{H_1}, \eth_1)$
3. $(\tilde{L'}_{H_1}, \eth_1) \subseteq (\tilde{L'}_{H_1}, \eth_1)$
4. $(\tilde{L'}_{H_1}, \eth_1) \subseteq (\tilde{L'}_{H_2}, \eth_2)$ *and* $(\tilde{L'}_{H_2}, \eth_2) \subseteq (\tilde{L'}_{H_3}, \eth_3)$*, then* $(\tilde{L'}_{H_1}, \eth_1) \subseteq (\tilde{L'}_{H_3}, \eth_3)$
5. $(\tilde{L'}_{H_1}, \eth_1) = (\tilde{L'}_{H_2}, \eth_2)$ *and* $(\tilde{L'}_{H_2}, \eth_2) = (\tilde{L'}_{H_3}, \eth_3)$*, then* $(\tilde{L'}_{H_1}, \eth_1) = (\tilde{L'}_{H_3}, \eth_3)$

Definition 3.14. The compliment $(\tilde{L}_{IVFHS}, \eth)^c$ of $IVFHS$-set $(\tilde{L}_{IVFHS}, \eth)$ is given by $(\tilde{L}_{IVFHS}, \eth)^c = (\tilde{L}^c_{IVFHS}, \eth)$ and is defined as

$$\tilde{L}^c_{IVFHS}(\theta) = [1 - \kappa^+_{\tilde{L}_{IVFHS}(\theta)}(\tau), 1 - \kappa^-_{\tilde{L}_{IVFHS}(\theta)}(\tau)],$$

for all $\tau \in \Cup, \theta \in \eth$.

Example 3.15. Consider the Example 2.8. The compliment $(\tilde{L}_{IVFHS}, \eth)^c$ of $IVFHS$-set $(\tilde{L}_{IVFHS}, \eth)$ over $\Cup$ is given as
$(\tilde{L}_{IVFHS}, \eth)^c =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{ [0.8, 0.9]/_{\ell_1}, [0.5, 0.7]/_{\ell_2} \right\}\right), \left(\tau_2, \left\{ [0.6, 0.8]/_{\ell_1}, [0.3, 0.7]/_{\ell_3} \right\}\right), \\ \left(\tau_3, \left\{ [0.6, 0.7]/_{\ell_2}, [0.5, 0.8]/_{\ell_4} \right\}\right), \left(\tau_4, \left\{ [0.3, 0.6]/_{\ell_1}, [0.5, 0.7]/_{\ell_2}, [0.8, 0.9]/_{\ell_5} \right\}\right), \\ \left(\tau_5, \left\{ [0.7, 0.9]/_{\ell_2}, [0.5, 0.8]/_{\ell_4} \right\}\right), \left(\tau_6, \left\{ [0.4, 0.8]/_{\ell_1}, [0.4, 0.6]/_{\ell_2}, [0.5, 0.7]/_{\ell_5} \right\}\right), \\ \left(\tau_7, \left\{ [0.6, 0.8]/_{\ell_4}, [0.3, 0.4]/_{\ell_5} \right\}\right), \left(\tau_8, \left\{ [0.6, 0.9]/_{\ell_1}, [0.7, 0.8]/_{\ell_2}, [0.5, 0.6]/_{\ell_4}, [0.4, 0.7]/_{\ell_5} \right\}\right), \\ \left(\tau_9, \left\{ [0.7, 0.8]/_{\ell_1}, [0.5, 0.9]/_{\ell_3}, [0.3, 0.7]/_{\ell_5} \right\}\right), \left(\tau_{10}, \left\{ [0.4, 0.5]/_{\ell_4}, [0.6, 0.8]/_{\ell_5} \right\}\right), \\ \left(\tau_{11}, \left\{ [0.5, 0.7]/_{\ell_2}, [0.4, 0.6]/_{\ell_5} \right\}\right), \left(\tau_{12}, \left\{ [0.6, 0.8]/_{\ell_4}, [0.3, 0.5]/_{\ell_5} \right\}\right), \\ \left(\tau_{13}, \left\{ [0.4, 0.7]/_{\ell_2}, [0.6, 0.9]/_{\ell_3}, [0.4, 0.8]/_{\ell_4}, [0.3, 0.6]/_{\ell_5} \right\}\right), \\ \left(\tau_{14}, \left\{ [0.5, 0.6]/_{\ell_1}, [0.7, 0.8]/_{\ell_5} \right\}\right), \left(\tau_{15}, \left\{ [0.6, 0.9]/_{\ell_3}, [0.2, 0.4]/_{\ell_4}, [0.3, 0.6]/_{\ell_5} \right\}\right), \\ \left(\tau_{16}, \left\{ [0.3, 0.5]/_{\ell_1}, [0.6, 0.8]/_{\ell_2}, [0.4, 0.7]/_{\ell_3}, [0.3, 0.6]/_{\ell_4} \right\}\right) \end{array} \right\}.$$

Proposition 3.16. $(\tilde{L'}_H, \eth) \in \zeta_{(\tilde{L'}_H, \eth)}$*, then*

1. $((\tilde{L}_{IVFHS}, \eth)^c)^c = (\tilde{L}_{IVFHS}, \eth)$
2. $((\tilde{L}_{IVFHS}, \eth)_\Phi)^c = (\tilde{L}_{IVFHS}, \eth)_\Cup$
3. $((\tilde{L}_{IVFHS}, \eth)_\Cup)^c = (\tilde{L}_{IVFHS}, \eth)_\Phi$

4 Set Theoretic operations on $IVFHS$-set

The purpose of this portion of the study is to describe some new $IVFHS-set$ operations.

Definition 4.1. The *union of two $IVFHS$-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \sqcup (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set $(\tilde{L'}_{H_3}, \eth_3)$ with $\eth_3 = \eth_1 \cup \eth_2$ and is defined as

$$\tilde{L'}_{H_3}(\tau) = \left\{ \begin{array}{l} \tilde{L'}_{H_1}(\tau) : \tau \in \eth_1 \backslash \eth_2 \\ \tilde{L'}_{H_2}(\tau) : \tau \in \eth_2 \backslash \eth_1 \\ \tilde{L'}_{H_1}(\tau) \cup \tilde{L'}_{H_2}(\tau) : \tau \in \eth_1 \cap \eth_2 \end{array} \right\},$$

where $\tilde{L'}_{H_1}(\tau) \cup \tilde{L'}_{H_2}(\tau) =$
$\left\{ \left(\tau, \left[\max \left(\kappa_{l\tilde{L'}_{H_1}(\tau)}(z), \kappa_{l\tilde{L'}_{H_2}(\tau)}(z) \right), \max \left(\kappa_{u\tilde{L'}_{H_1}(\tau)}(z), \kappa_{u\tilde{L'}_{H_2}(\tau)}(z) \right) \right] \right) \forall z \in \mathcal{Z} \right\}$.

Example 4.2. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2) \in \mathcal{Ç}_{(\tilde{L'}_H, \eth)}$ with $\eth_1, \eth_2 \subseteq \eth$ as in Example 2.8 such that
$(\tilde{L'}_{H_1}, \eth_1) =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{ [0.5, 0.6]/_{\ell_1}, [0.3, 0.5]/_{\ell_2} \right\} \right), \left(\tau_4, \left\{ [0.4, 0.5]/_{\ell_1}, [0.3, 0.5]/_{\ell_2}, [0.1, 0.4]/_{\ell_5} \right\} \right), \\ \left(\tau_5, \left\{ [0.1, 0.3]/_{\ell_2}, [0.2, 0.3]/_{\ell_4} \right\} \right), \left(\tau_6, \left\{ [0.2, 0.6]/_{\ell_1}, [0.4, 0.8]/_{\ell_2}, [0.3, 0.5]/_{\ell_5} \right\} \right), \\ \left(\tau_7, \left\{ [0.3, 0.4]/_{\ell_4}, [0.6, 0.7]/_{\ell_5} \right\} \right), \left(\tau_9, \left\{ [0.2, 0.3]/_{\ell_1}, [0.4, 0.5]/_{\ell_3}, [0.3, 0.7]/_{\ell_5} \right\} \right), \\ \left(\tau_{11}, \left\{ [0.3, 0.6]/_{\ell_2}, [0.4, 0.6]/_{\ell_5} \right\} \right), \left(\tau_{12}, \left\{ [0.2, 0.4]/_{\ell_4}, [0.6, 0.7]/_{\ell_5} \right\} \right), \\ \left(\tau_{13}, \left\{ [0.3, 0.6]/_{\ell_2}, [0.3, 0.4]/_{\ell_3}, [0.2, 0.6]/_{\ell_4}, [0.4, 0.7]/_{\ell_5} \right\} \right), \\ \left(\tau_{14}, \left\{ [0.4, 0.5]/_{\ell_1}, [0.2, 0.5]/_{\ell_5} \right\} \right), \left(\tau_{15}, \left\{ [0.1, 0.4]/_{\ell_3}, [0.6, 0.8]/_{\ell_4}, [0.5, 0.7]/_{\ell_5} \right\} \right), \\ \left(\tau_{16}, \left\{ [0.5, 0.7]/_{\ell_1}, [0.2, 0.4]/_{\ell_2}, [0.4, 0.6]/_{\ell_3}, [0.4, 0.7]/_{\ell_4} \right\} \right) \end{array} \right\}$$

$(\tilde{L'}_{H_2}, \eth_2) =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{ [0.3, 0.5]/_{\ell_1}, [0.1, 0.3]/_{\ell_2} \right\} \right), \left(\tau_2, \left\{ [0.6, 0.7]/_{\ell_1}, [0.2, 0.4]/_{\ell_3} \right\} \right), \\ \left(\tau_3, \left\{ [0.3, 0.4]/_{\ell_2}, [0.3, 0.6]/_{\ell_4} \right\} \right), \left(\tau_4, \left\{ [0.1, 0.2]/_{\ell_1}, [0.4, 0.5]/_{\ell_2}, [0.6, 0.8]/_{\ell_5} \right\} \right), \\ \left(\tau_5, \left\{ [0.3, 0.4]/_{\ell_2}, [0.3, 0.7]/_{\ell_4} \right\} \right), \left(\tau_6, \left\{ [0.4, 0.7]/_{\ell_1}, [0.2, 0.3]/_{\ell_2}, [0.2, 0.6]/_{\ell_5} \right\} \right), \\ \left(\tau_7, \left\{ [0.7, 0.8]/_{\ell_4}, [0.3, 0.5]/_{\ell_5} \right\} \right), \left(\tau_9, \left\{ [0.6, 0.9]/_{\ell_1}, [0.4, 0.5]/_{\ell_3}, [0.5, 0.8]/_{\ell_5} \right\} \right), \\ \left(\tau_{11}, \left\{ [0.4, 0.5]/_{\ell_2}, [0.1, 0.3]/_{\ell_5} \right\} \right), \left(\tau_{12}, \left\{ [0.4, 0.7]/_{\ell_4}, [0.2, 0.5]/_{\ell_5} \right\} \right), \\ \left(\tau_{13}, \left\{ [0.4, 0.6]/_{\ell_2}, [0.3, 0.5]/_{\ell_3}, [0.4, 0.5]/_{\ell_4}, [0.2, 0.3]/_{\ell_5} \right\} \right), \\ \left(\tau_{14}, \left\{ [0.7, 0.8]/_{\ell_1}, [0.4, 0.6]/_{\ell_5} \right\} \right), \\ \left(\tau_{16}, \left\{ [0.1, 0.3]/_{\ell_1}, [0.2, 0.4]/_{\ell_2}, [0.4, 0.7]/_{\ell_3}, [0.3, 0.7]/_{\ell_4} \right\} \right) \end{array} \right\}$$

Then $(\tilde{L'}_{H_1}, \eth_1) \sqcup (\tilde{L'}_{H_2}, \eth_2) =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{ [0.5, 0.6]/_{\ell_1}, [0.3, 0.5]/_{\ell_2} \right\} \right), \left(\tau_2, \left\{ [0.6, 0.7]/_{\ell_1}, [0.2, 0.4]/_{\ell_3} \right\} \right), \\ \left(\tau_3, \left\{ [0.3, 0.4]/_{\ell_2}, [0.3, 0.6]/_{\ell_4} \right\} \right), \left(\tau_4, \left\{ [0.4, 0.5]/_{\ell_1}, [0.4, 0.5]/_{\ell_2}, [0.6, 0.8]/_{\ell_5} \right\} \right), \\ \left(\tau_5, \left\{ [0.3, 0.4]/_{\ell_2}, [0.3, 0.7]/_{\ell_4} \right\} \right), \left(\tau_6, \left\{ [0.4, 0.7]/_{\ell_1}, [0.4, 0.8]/_{\ell_2}, [0.3, 0.6]/_{\ell_5} \right\} \right), \\ \left(\tau_7, \left\{ [0.7, 0.8]/_{\ell_4}, [0.6, 0.7]/_{\ell_5} \right\} \right), \left(\tau_9, \left\{ [0.6, 0.9]/_{\ell_1}, [0.4, 0.5]/_{\ell_3}, [0.5, 0.8]/_{\ell_5} \right\} \right), \\ \left(\tau_{11}, \left\{ [0.4, 0.6]/_{\ell_2}, [0.4, 0.6]/_{\ell_5} \right\} \right), \left(\tau_{12}, \left\{ [0.4, 0.7]/_{\ell_4}, [0.6, 0.7]/_{\ell_5} \right\} \right), \\ \left(\tau_{13}, \left\{ [0.4, 0.6]/_{\ell_2}, [0.3, 0.5]/_{\ell_3}, [0.4, 0.6]/_{\ell_4}, [0.4, 0.7]/_{\ell_5} \right\} \right), \\ \left(\tau_{14}, \left\{ [0.7, 0.8]/_{\ell_1}, [0.4, 0.6]/_{\ell_5} \right\} \right), \left(\tau_{15}, \left\{ [0.1, 0.4]/_{\ell_3}, [0.6, 0.8]/_{\ell_4}, [0.5, 0.7]/_{\ell_5} \right\} \right), \\ \left(\tau_{16}, \left\{ [0.5, 0.7]/_{\ell_1}, [0.2, 0.4]/_{\ell_2}, [0.4, 0.7]/_{\ell_3}, [0.4, 0.7]/_{\ell_4} \right\} \right) \end{array} \right\}.$$

Definition 4.3. The *restricted union of two* $IVFHS$*-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \sqcup_R (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set $(\tilde{L'}_{H_3}, \eth_3)$ with $\eth_3 = \eth_1 \cap \eth_2$ and is defined as

$$\tilde{L'}_{H_3}(\tau) = \left\{ \tilde{L'}_{H_1}(\tau) \cup \tilde{L'}_{H_2}(\tau) : \tau \in \eth_1 \cap \eth_2 \right\},$$

where $\tilde{L'}_{H_1}(\tau) \cup \tilde{L'}_{H_2}(\tau) =$
$\left\{ \left(\tau, \left[\max\left(\kappa_{l\tilde{L'}_{H_1}(\tau)}(z), \kappa_{l\tilde{L'}_{H_2}(\tau)}(z) \right), \max\left(\kappa_{u\tilde{L'}_{H_1}(\tau)}(z), \kappa_{u\tilde{L'}_{H_2}(\tau)}(z) \right) \right] \right) \forall z \in \mathcal{Z} \right\}$.

Example 4.4. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2) \in \mathcal{C}_{\wp(\tilde{L'}_H, \eth)}$ with $\eth_1, \eth_2 \subseteq \eth$ as in Example 4.2. Then $(\tilde{L'}_{H_3}, \eth_3) =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{ [0.5, 0.6]/_{\ell_1}, [0.3, 0.5]/_{\ell_2} \right\}\right), \left(\tau_4, \left\{ [0.4, 0.5]/_{\ell_1}, [0.4, 0.5]/_{\ell_2}, [0.6, 0.8]/_{\ell_5} \right\}\right), \\ \left(\tau_5, \left\{ [0.3, 0.4]/_{\ell_2}, [0.3, 0.7]/_{\ell_4} \right\}\right), \left(\tau_6, \left\{ [0.4, 0.7]/_{\ell_1}, [0.4, 0.8]/_{\ell_2}, [0.3, 0.6]/_{\ell_5} \right\}\right), \\ \left(\tau_7, \left\{ [0.7, 0.8]/_{\ell_4}, [0.6, 0.7]/_{\ell_5} \right\}\right), \left(\tau_9, \left\{ [0.6, 0.9]/_{\ell_1}, [0.4, 0.5]/_{\ell_3}, [0.5, 0.8]/_{\ell_5} \right\}\right), \\ \left(\tau_{11}, \left\{ [0.4, 0.6]/_{\ell_2}, [0.4, 0.6]/_{\ell_5} \right\}\right), \left(\tau_{12}, \left\{ [0.4, 0.7]/_{\ell_4}, [0.6, 0.7]/_{\ell_5} \right\}\right), \\ \left(\tau_{13}, \left\{ [0.4, 0.6]/_{\ell_2}, [0.3, 0.5]/_{\ell_3}, [0.4, 0.6]/_{\ell_4}, [0.4, 0.7]/_{\ell_5} \right\}\right), \\ \left(\tau_{14}, \left\{ [0.7, 0.8]/_{\ell_1}, [0.4, 0.6]/_{\ell_5} \right\}\right), \\ \left(\tau_{16}, \left\{ [0.5, 0.7]/_{\ell_1}, [0.2, 0.4]/_{\ell_2}, [0.4, 0.7]/_{\ell_3}, [0.4, 0.7]/_{\ell_4} \right\}\right) \end{array} \right\}$$

Definition 4.5. The *intersection of two* $IVFHS$*-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \sqcap (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set $(\tilde{L'}_{H_3}, \eth_3)$ with $\eth_3 = \eth_1 \cap \eth_2$ and is defined as

$$\tilde{L'}_{H_3}(\tau) = \left\{ \tilde{L'}_{H_1}(\tau) \cap \tilde{L'}_{H_2}(\tau) : \tau \in \eth_1 \cap \eth_2 \right\},$$

where $\tilde{L'}_{H_1}(\tau) \cap \tilde{L'}_{H_2}(\tau) =$
$\left\{ \left(\tau, \left[\min\left(\kappa_{l\eth_1(\tau)}(z), \kappa_{l\eth_2(\tau)}(z) \right), \min\left(\kappa_{u\eth_1(\tau)}(z), \kappa_{u\eth_2(\tau)}(z) \right) \right] \right) \forall z \in \mathcal{Z} \right\}$.

Example 4.6. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2) \in \mathcal{C}_{\wp(\tilde{L'}_H, \eth)}$ with $\eth_1, \eth_2 \subseteq \eth$ as in Example 4.2. Then $(\tilde{L'}_{H_1}, \eth_1) \sqcap (\tilde{L'}_{H_2}, \eth_2) =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{ [0.3, 0.5]/_{\ell_1}, [0.1, 0.3]/_{\ell_2} \right\}\right), \left(\tau_4, \left\{ [0.1, 0.2]/_{\ell_1}, [0.3, 0.5]/_{\ell_2}, [0.1, 0.4]/_{\ell_5} \right\}\right), \\ \left(\tau_5, \left\{ [0.1, 0.3]/_{\ell_2}, [0.2, 0.3]/_{\ell_4} \right\}\right), \left(\tau_6, \left\{ [0.2, 0.6]/_{\ell_1}, [0.2, 0.3]/_{\ell_2}, [0.2, 0.5]/_{\ell_5} \right\}\right), \\ \left(\tau_7, \left\{ [0.3, 0.4]/_{\ell_4}, [0.3, 0.5]/_{\ell_5} \right\}\right), \left(\tau_9, \left\{ [0.2, 0.3]/_{\ell_1}, [0.4, 0.5]/_{\ell_3}, [0.3, 0.7]/_{\ell_5} \right\}\right), \\ \left(\tau_{11}, \left\{ [0.3, 0.5]/_{\ell_2}, [0.1, 0.3]/_{\ell_5} \right\}\right), \left(\tau_{12}, \left\{ [0.2, 0.4]/_{\ell_4}, [0.2, 0.5]/_{\ell_5} \right\}\right), \\ \left(\tau_{13}, \left\{ [0.3, 0.6]/_{\ell_2}, [0.3, 0.4]/_{\ell_3}, [0.2, 0.5]/_{\ell_4}, [0.2, 0.3]/_{\ell_5} \right\}\right), \\ \left(\tau_{14}, \left\{ [0.4, 0.5]/_{\ell_1}, [0.2, 0.5]/_{\ell_5} \right\}\right), \\ \left(\tau_{16}, \left\{ [0.1, 0.3]/_{\ell_1}, [0.2, 0.4]/_{\ell_2}, [0.4, 0.6]/_{\ell_3}, [0.3, 0.7]/_{\ell_4} \right\}\right) \end{array} \right\}$$

Definition 4.7. The *extended intersection of two* $IVFHS$*-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \sqcap_E (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set $(\tilde{L'}_{H_3}, \eth_3)$ with $\eth_3 = \eth_1 \cup \eth_1$ and is defined as

$$\tilde{L'}_{H_3}(\tau) = \left\{ \begin{array}{l} \tilde{L'}_{H_1}(\tau) : \tau \in \eth_1 \backslash \eth_2 \\ \tilde{L'}_{H_2}(\tau) : \tau \in \eth_2 \backslash \eth_1 \\ \tilde{L'}_{H_1}(\tau) \cap \tilde{L'}_{H_2}(\tau) : \tau \in \eth_1 \cap \eth_2 \end{array} \right\},$$

where $\tilde{L'}_{H_1}(\tau) \cap \tilde{L'}_{H_2}(\tau) =$
$\left\{ \left(\tau, \left[\min\left(\kappa_{l\eth_1(\tau)}(z), \kappa_{l\eth_2(\tau)}(z) \right), \min\left(\kappa_{u\eth_1(\tau)}(z), \kappa_{u\eth_2(\tau)}(z) \right) \right] \right) \forall z \in \mathcal{Z} \right\}$.

Example 4.8. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2) \in \mathcal{C}_{\wp(\tilde{L'}_H, \eth)}$ with $\eth_1, \eth_2 \subseteq \eth$ as in

Example 4.2. Then $(\tilde{L'}_{H_1}, \eth_1) \sqcap_E (\tilde{L'}_{H_2}, \eth_2) =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{ [0.3, 0.5]/_{\ell_1}, [0.1, 0.3]/_{\ell_2} \right\}\right), \left(\tau_2, \left\{ [0.6, 0.7]/_{\ell_1}, [0.2, 0.4]/_{\ell_3} \right\}\right), \\ \left(\tau_3, \left\{ [0.3, 0.4]/_{\ell_2}, [0.3, 0.6]/_{\ell_4} \right\}\right), \left(\tau_4, \left\{ [0.1, 0.2]/_{\ell_1}, [0.3, 0.5]/_{\ell_2}, [0.1, 0.4]/_{\ell_5} \right\}\right), \\ \left(\tau_5, \left\{ [0.1, 0.3]/_{\ell_2}, [0.2, 0.3]/_{\ell_4} \right\}\right), \left(\tau_6, \left\{ [0.2, 0.6]/_{\ell_1}, [0.2, 0.3]/_{\ell_2}, [0.2, 0.5]/_{\ell_5} \right\}\right), \\ \left(\tau_7, \left\{ [0.3, 0.4]/_{\ell_4}, [0.3, 0.5]/_{\ell_5} \right\}\right), \left(\tau_9, \left\{ [0.2, 0.3]/_{\ell_1}, [0.4, 0.5]/_{\ell_3}, [0.3, 0.7]/_{\ell_5} \right\}\right), \\ \left(\tau_{11}, \left\{ [0.3, 0.5]/_{\ell_2}, [0.1, 0.3]/_{\ell_5} \right\}\right), \left(\tau_{12}, \left\{ [0.2, 0.4]/_{\ell_4}, [0.2, 0.5]/_{\ell_5} \right\}\right), \\ \left(\tau_{13}, \left\{ [0.3, 0.6]/_{\ell_2}, [0.3, 0.4]/_{\ell_3}, [0.2, 0.5]/_{\ell_4}, [0.2, 0.3]/_{\ell_5} \right\}\right), \\ \left(\tau_{14}, \left\{ [0.4, 0.5]/_{\ell_1}, [0.2, 0.5]/_{\ell_5} \right\}\right), \left(\tau_{15}, \left\{ [0.1, 0.4]/_{\ell_3}, [0.6, 0.8]/_{\ell_4}, [0.5, 0.7]/_{\ell_5} \right\}\right), \\ \left(\tau_{16}, \left\{ [0.1, 0.3]/_{\ell_1}, [0.2, 0.4]/_{\ell_2}, [0.4, 0.6]/_{\ell_3}, [0.3, 0.7]/_{\ell_4} \right\}\right) \end{array} \right\}.$$

Definition 4.9. The *addition of two $IVFHS$-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2) \in \text{Ç}_{(\tilde{L'}_H, \eth)}$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \boxplus (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set with $\eth_3 = \eth_1 \cup \eth_2$ and is defined as

$$\tilde{L'}_{H_3}(\tau) = \left\{ \begin{array}{l} \tilde{L'}_{H_1}(\tau) : \tau \in \eth_1 \backslash \eth_2 \\ \tilde{L'}_{H_2}(\tau) : \tau \in \eth_2 \backslash \eth_1 \\ \tilde{L'}_{H_1}(\tau) \oplus \tilde{L'}_{H_2}(\tau) : \tau \in \eth_1 \cap \eth_2 \end{array} \right\},$$

where $\tilde{L'}_{H_1}(\tau) \oplus \tilde{L'}_{H_2}(\tau) =$
$$\left\{ \left(\tau, \left[\begin{array}{l} \kappa_{l\tilde{L'}_{H_1}(\tau)}(z) + \kappa_{l\tilde{L'}_{H_2}(\tau)}(z) - \kappa_{l\tilde{L'}_{H_1}(\tau)}(z)\kappa_{l\tilde{L'}_{H_2}(\tau)}(z), \\ \kappa_{u\tilde{L'}_{H_1}(\tau)}(z) + \kappa_{u\tilde{L'}_{H_2}(\tau)}(z) - \kappa_{u\tilde{L'}_{H_1}(\tau)}(z)\kappa_{u\tilde{L'}_{H_2}(\tau)}(z) \end{array} \right] \right) \forall z \in \mathcal{Z} \right\}.$$

Example 4.10. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2) \in \text{Ç}_{(\tilde{L'}_H, \eth)}$ with $\eth_1, \eth_2 \subseteq \eth$ as in Example 4.2. Then $(\tilde{L'}_{H_1}, \eth_1) \boxplus (\tilde{L'}_{H_2}, \eth_2) =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{ [0.65, 0.80]/_{\ell_1}, [0.37, 0.65]/_{\ell_2} \right\}\right), \left(\tau_2, \left\{ [0.60, 0.70]/_{\ell_1}, [0.20, 0.40]/_{\ell_3} \right\}\right), \\ \left(\tau_3, \left\{ [0.30, 0.40]/_{\ell_2}, [0.30, 0.60]/_{\ell_4} \right\}\right), \left(\tau_4, \left\{ [0.46, 0.60]/_{\ell_1}, [0.58, 0.75]/_{\ell_2}, [0.64, 0.88]/_{\ell_5} \right\}\right), \\ \left(\tau_5, \left\{ [0.37, 0.58]/_{\ell_2}, [0.44, 0.79]/_{\ell_4} \right\}\right), \left(\tau_6, \left\{ [0.52, 0.88]/_{\ell_1}, [0.52, 0.86]/_{\ell_2}, [0.44, 0.80]/_{\ell_5} \right\}\right), \\ \left(\tau_7, \left\{ [0.79, 0.88]/_{\ell_4}, [0.72, 0.85]/_{\ell_5} \right\}\right), \left(\tau_9, \left\{ [0.68, 0.93]/_{\ell_1}, [0.64, 0.75]/_{\ell_3}, [0.65, 0.94]/_{\ell_5} \right\}\right) \\ \left(\tau_{11}, \left\{ [0.58, 0.80]/_{\ell_2}, [0.46, 0.72]/_{\ell_5} \right\}\right), \left(\tau_{12}, \left\{ [0.52, 0.82]/_{\ell_4}, [0.68, 0.85]/_{\ell_5} \right\}\right), \\ \left(\tau_{13}, \left\{ [0.58, 0.84]/_{\ell_2}, [0.51, 0.70]/_{\ell_3}, [0.52, 0.80]/_{\ell_4}, [0.52, 0.79]/_{\ell_5} \right\}\right), \\ \left(\tau_{14}, \left\{ [0.82, 0.90]/_{\ell_1}, [0.52, 0.80]/_{\ell_5} \right\}\right), \left(\tau_{15}, \left\{ [0.10, 0.40]/_{\ell_3}, [0.60, 0.80]/_{\ell_4}, [0.50, 0.70]/_{\ell_5} \right\}\right), \\ \left(\tau_{16}, \left\{ [0.55, 0.79]/_{\ell_1}, [0.36, 0.64]/_{\ell_2}, [0.64, 0.88]/_{\ell_3}, [0.58, 0.91]/_{\ell_4} \right\}\right) \end{array} \right\}.$$

Definition 4.11. The *multiplication of two $IVFHS$-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \boxtimes (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set with $\eth_3 = \eth_1 \cup \eth_1$ and is defined as

$$\tilde{L'}_{H_3}(\tau) = \left\{ \begin{array}{l} \tilde{L'}_{H_1}(\tau) : \tau \in \eth_1 \backslash \eth_2 \\ \tilde{L'}_{H_2}(\tau) : \tau \in \eth_2 \backslash \eth_1 \\ \tilde{L'}_{H_1}(\tau) \otimes \tilde{L'}_{H_2}(\tau) : \tau \in \eth_1 \cap \eth_2 \end{array} \right\},$$

where $\tilde{L'}_{H_1}(\tau) \otimes \tilde{L'}_{H_2}(\tau) =$
$$\left\{ \left(\tau, \left[\kappa_{l\tilde{L'}_{H_1}(\tau)}(z)\kappa_{l\tilde{L'}_{H_2}(\tau)}(z), \kappa_{u\tilde{L'}_{H_1}(\tau)}(z)\kappa_{u\tilde{L'}_{H_1}(\tau)}(z) \right] \right) \forall z \in \mathcal{Z} \right\}.$$

Example 4.12. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2) \in \text{Ç}_{(\tilde{L'}_H, \eth)}$ with $\eth_1, \eth_2 \subseteq \eth$ as in

Example 4.2. Then $(\tilde{L'}_{H_1}, \eth_1) \boxtimes (\tilde{L'}_{H_2}, \eth_2) =$

$$\left\{\begin{array}{l}\left(\tau_1, \left\{[0.15,0.30]/_{\ell_1}, [0.03,0.15]/_{\ell_2}\right\}\right), \left(\tau_2, \left\{[0.60,0.70]/_{\ell_1}, [0.20,0.40]/_{\ell_3}\right\}\right), \\ \left(\tau_3, \left\{[0.30,0.40]/_{\ell_2}, [0.30,0.60]/_{\ell_4}\right\}\right), \left(\tau_4, \left\{[0.04,0.10]/_{\ell_1}, [0.12,0.25]/_{\ell_2}, [0.06,0.32]/_{\ell_5}\right\}\right), \\ \left(\tau_5, \left\{[0.03,0.12]/_{\ell_2}, [0.06,0.21]/_{\ell_4}\right\}\right), \left(\tau_6, \left\{[0.08,0.42]/_{\ell_1}, [0.08,0.24]/_{\ell_2}, [0.06,0.30]/_{\ell_5}\right\}\right), \\ \left(\tau_7, \left\{[0.21,0.32]/_{\ell_4}, [0.18,0.35]/_{\ell_5}\right\}\right), \left(\tau_9, \left\{[0.12,0.27]/_{\ell_1}, [0.16,0.25]/_{\ell_3}, [0.15,0.56]/_{\ell_5}\right\}\right), \\ \left(\tau_{11}, \left\{[0.12,0.30]/_{\ell_2}, [0.04,0.18]/_{\ell_5}\right\}\right), \left(\tau_{12}, \left\{[0.08,0.28]/_{\ell_4}, [0.12,0.35]/_{\ell_5}\right\}\right), \\ \left(\tau_{13}, \left\{[0.12,0.36]/_{\ell_2}, [0.09,0.20]/_{\ell_3}, [0.08,0.30]/_{\ell_4}, [0.08,0.21]/_{\ell_5}\right\}\right), \\ \left(\tau_{14}, \left\{[0.28,0.40]/_{\ell_1}, [0.08,0.30]/_{\ell_5}\right\}\right), \left(\tau_{15}, \left\{[0.10,0.40]/_{\ell_3}, [0.60,0.80]/_{\ell_4}, [0.50,0.70]/_{\ell_5}\right\}\right), \\ \left(\tau_{16}, \left\{[0.05,0.21]/_{\ell_1}, [0.04,0.16]/_{\ell_2}, [0.16,0.42]/_{\ell_3}, [0.12,0.49]/_{\ell_4}\right\}\right)\end{array}\right\}$$

Definition 4.13. *Partial membership of $IVFHS$-set* $(\tilde{L'}_{H_1}, \eth_1)$ denoted by $\boxdot(\tilde{L'}_{H_1}, \eth_1)$ is an $IVFHS$-set and is defined as

$$\boxdot(\tilde{L'}_{H_1}, \eth_1) = \left\{\odot\tilde{L'}_{H_1}(\tau) : \tau \in \eth_1\right\},$$

where $\odot\tilde{L'}_{H_1}(\tau) = \left\{\left(\tau, \left[\kappa_{l\tilde{L'}_{H_1}(\tau)}(z), 1 - \kappa_{l\tilde{L'}_{H_1}(\tau)}(z)\right]\right) \forall z \in \mathcal{Z}\right\}$.

Example 4.14. Consider $IVFHS$-set $(\tilde{L'}_{H_1}, \eth_1) \in \text{Ç}_{(\tilde{L'}_H, \eth)}$ with $\eth_1 \subseteq \eth$ as in Example 4.2 given by, $(\tilde{L'}_{H_1}, \eth_1) =$

$$\left\{\begin{array}{l}\left(\tau_1, \left\{[0.1,0.2]/_{\ell_1}, [0.3,0.4]/_{\ell_2}\right\}\right), \left(\tau_2, \left\{[0.2,0.4]/_{\ell_1}, [0.3,0.7]/_{\ell_3}\right\}\right), \\ \left(\tau_3, \left\{[0.3,0.4]/_{\ell_2}, [0.2,0.6]/_{\ell_4}\right\}\right), \left(\tau_4, \left\{[0.4,0.7]/_{\ell_1}, [0.3,0.8]/_{\ell_2}, [0.1,0.2]/_{\ell_5}\right\}\right), \\ \left(\tau_5, \left\{[0.1,0.3]/_{\ell_2}, [0.2,0.4]/_{\ell_4}\right\}\right), \left(\tau_6, \left\{[0.2,0.6]/_{\ell_1}, [0.4,0.6]/_{\ell_2}, [0.3,0.6]/_{\ell_5}\right\}\right), \\ \left(\tau_7, \left\{[0.2,0.4]/_{\ell_4}, [0.6,0.7]/_{\ell_5}\right\}\right), \left(\tau_8, \left\{[0.1,0.4]/_{\ell_1}, [0.2,0.3]/_{\ell_2}, [0.4,0.6]/_{\ell_4}, [0.3,0.6]/_{\ell_5}\right\}\right), \\ \left(\tau_9, \left\{[0.2,0.3]/_{\ell_1}, [0.1,0.2]/_{\ell_3}, [0.3,0.7]/_{\ell_5}\right\}\right), \left(\tau_{10}, \left\{[0.4,0.6]/_{\ell_4}, [0.2,0.4]/_{\ell_5}\right\}\right), \\ \left(\tau_{11}, \left\{[0.3,0.9]/_{\ell_2}, [0.4,0.6]/_{\ell_5}\right\}\right), \left(\tau_{12}, \left\{[0.2,0.4]/_{\ell_4}, [0.6,0.7]/_{\ell_5}\right\}\right), \\ \left(\tau_{13}, \left\{[0.3,0.6]/_{\ell_2}, [0.1,0.4]/_{\ell_3}, [0.2,0.6]/_{\ell_4}, [0.4,0.7]/_{\ell_5}\right\}\right), \left(\tau_{14}, \left\{[0.4,0.6]/_{\ell_1}, [0.2,0.3]/_{\ell_5}\right\}\right), \\ \left(\tau_{15}, \left\{[0.1,0.4]/_{\ell_3}, [0.6,0.8]/_{\ell_4}, [0.4,0.7]/_{\ell_5}\right\}\right), \\ \left(\tau_{16}, \left\{[0.6,0.7]/_{\ell_1}, [0.2,0.4]/_{\ell_2}, [0.3,0.6]/_{\ell_3}, [0.4,0.7]/_{\ell_4}\right\}\right)\end{array}\right\}.$$

Then partial membership is given as follow:
$\boxdot(\tilde{L'}_{H_1}, \eth_1) =$

$$\left\{\begin{array}{l}\left(\tau_1, \left\{[0.1,0.9]/_{\ell_1}, [0.3,0.7]/_{\ell_2}\right\}\right), \left(\tau_2, \left\{[0.2,0.8]/_{\ell_1}, [0.3,0.7]/_{\ell_3}\right\}\right), \\ \left(\tau_3, \left\{[0.3,0.7]/_{\ell_2}, [0.2,0.8]/_{\ell_4}\right\}\right), \left(\tau_4, \left\{[0.4,0.6]/_{\ell_1}, [0.3,0.7]/_{\ell_2}, [0.1,0.9]/_{\ell_5}\right\}\right), \\ \left(\tau_5, \left\{[0.1,0.9]/_{\ell_2}, [0.2,0.8]/_{\ell_4}\right\}\right), \left(\tau_6, \left\{[0.2,0.8]/_{\ell_1}, [0.4,0.6]/_{\ell_2}, [0.3,0.7]/_{\ell_5}\right\}\right), \\ \left(\tau_7, \left\{[0.2,0.8]/_{\ell_4}, [0.4,0.6]/_{\ell_5}\right\}\right), \left(\tau_8, \left\{[0.1,0.9]/_{\ell_1}, [0.2,0.8]/_{\ell_2}, [0.4,0.6]/_{\ell_4}, [0.3,0.7]/_{\ell_5}\right\}\right), \\ \left(\tau_9, \left\{[0.2,0.8]/_{\ell_1}, [0.1,0.9]/_{\ell_3}, [0.3,0.7]/_{\ell_5}\right\}\right), \left(\tau_{10}, \left\{[0.4,0.6]/_{\ell_4}, [0.2,0.8]/_{\ell_5}\right\}\right), \\ \left(\tau_{11}, \left\{[0.3,0.7]/_{\ell_2}, [0.4,0.6]/_{\ell_5}\right\}\right), \left(\tau_{12}, \left\{[0.2,0.8]/_{\ell_4}, [0.4,0.6]/_{\ell_5}\right\}\right), \\ \left(\tau_{13}, \left\{[0.3,0.7]/_{\ell_2}, [0.1,0.9]/_{\ell_3}, [0.2,0.8]/_{\ell_4}, [0.4,0.6]/_{\ell_5}\right\}\right), \left(\tau_{14}, \left\{[0.4,0.6]/_{\ell_1}, [0.2,0.8]/_{\ell_5}\right\}\right), \\ \left(\tau_{15}, \left\{[0.1,0.9]/_{\ell_3}, [0.4,0.6]/_{\ell_4}, [0.4,0.6]/_{\ell_5}\right\}\right), \\ \left(\tau_{16}, \left\{[0.4,0.6]/_{\ell_1}, [0.2,0.8]/_{\ell_2}, [0.3,0.7]/_{\ell_3}, [0.4,0.6]/_{\ell_4}\right\}\right)\end{array}\right\}.$$

Definition 4.15. The *partial non-membership of $IVFHS$-set* $(\tilde{L'}_{H_1}, \eth_1)$ denoted by $\circledcirc(\tilde{L'}_{H_1}, \eth_1)$ is an $IVFHS$-set defined as

$$\circledcirc\tilde{L'}_{H_1}(\tau) = \left\{\circ\tilde{L'}_{H_1}(\tau) : \tau \in \eth_1\right\},$$

where $\circ\tilde{L'}_{H_1}(\tau) = \left\{\left(\tau, \left[1-\kappa_{u\tilde{L'}_{H_1}(\tau)}(z), \kappa_{u\tilde{L'}_{H_1}(\tau)}(z)\right]\right) \forall z \in \mathcal{Z}\right\}$.

Example 4.16. Consider $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1) \in \complement_{(\tilde{L'}_H, \eth)}$ with $\eth_1 \subseteq \eth$ as in Example 4.14.
Then partial non-membership is given as follow:
$\circledcirc(\tilde{L'}_{H_1}, \eth_1) =$

$$\left\{\begin{array}{l}
\left(\tau_1, \left\{[0.2,0.8]/_{\ell_1}, [0.4,0.6]/_{\ell_2}\right\}\right), \left(\tau_2, \left\{[0.4,0.6]/_{\ell_1}, [0.3,0.7]/_{\ell_3}\right\}\right), \\
\left(\tau_3, \left\{[0.4,0.6]/_{\ell_2}, [0.4,0.6]/_{\ell_4}\right\}\right), \left(\tau_4, \left\{[0.3,0.7]/_{\ell_1}, [0.2,0.8]/_{\ell_2}, [0.2,0.8]/_{\ell_5}\right\}\right), \\
\left(\tau_5, \left\{[0.3,0.7]/_{\ell_2}, [0.4,0.6]/_{\ell_4}\right\}\right), \left(\tau_6, \left\{[0.4,0.6]/_{\ell_1}, [0.4,0.6]/_{\ell_2}, [0.4,0.6]/_{\ell_5}\right\}\right), \\
\left(\tau_7, \left\{[0.4,0.6]/_{\ell_4}, [0.3,0.7]/_{\ell_5}\right\}\right), \left(\tau_8, \left\{[0.4,0.6]/_{\ell_1}, [0.3,0.7]/_{\ell_2}, [0.4,0.6]/_{\ell_4}, [0.4,0.6]/_{\ell_5}\right\}\right), \\
\left(\tau_9, \left\{[0.3,0.7]/_{\ell_1}, [0.2,0.8]/_{\ell_3}, [0.3,0.7]/_{\ell_5}\right\}\right), \left(\tau_{10}, \left\{[0.4,0.6]/_{\ell_4}, [0.4,0.6]/_{\ell_5}\right\}\right), \\
\left(\tau_{11}, \left\{[0.1,0.9]/_{\ell_2}, [0.4,0.6]/_{\ell_5}\right\}\right), \left(\tau_{12}, \left\{[0.4,0.6]/_{\ell_4}, [0.3,0.7]/_{\ell_5}\right\}\right), \\
\left(\tau_{13}, \left\{[0.4,0.6]/_{\ell_2}, [0.4,0.6]/_{\ell_3}, [0.4,0.6]/_{\ell_4}, [0.3,0.7]/_{\ell_5}\right\}\right), \left(\tau_{14}, \left\{[0.4,0.6]/_{\ell_1}, [0.3,0.7]/_{\ell_5}\right\}\right), \\
\left(\tau_{15}, \left\{[0.4,0.6]/_{\ell_3}, [0.2,0.8]/_{\ell_4}, [0.3,0.7]/_{\ell_5}\right\}\right), \\
\left(\tau_{16}, \left\{[0.3,0.7]/_{\ell_1}, [0.4,0.6]/_{\ell_2}, [0.4,0.6]/_{\ell_3}, [0.3,0.7]/_{\ell_4}\right\}\right)
\end{array}\right\}.$$

Definition 4.17. The *AND operation of two $IVFHS$-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \bigwedge (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set $(\tilde{L'}_{H_3}, \eth_3)$ with $\eth_3 = \eth_1 \times \eth_2$ and for $(\tau_i, \tau_j) \in \eth_3, \tau_i \in \eth_1, \tau_j \in \eth_2$ is defined as

$$\tilde{L'}_{H_3}(\tau_1, \tau_2) = \tilde{L'}_{H_1}(\tau_1) \sqcap \tilde{L'}_{H_2}(\tau_2),$$

where $\tilde{L'}_{H_1}(\tau_1) \sqcap \tilde{L'}_{H_2}(\tau_2)$ as defined in definition 4.5.

Example 4.18. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ as in Example 2.8 given by
$(\tilde{L'}_{H_1}, \eth_1) =$

$$\left\{\begin{array}{l}
\left(\tau_1, \left\{[0.4,0.5]/_{\ell_1}, [0.2,0.5]/_{\ell_2}\right\}\right), \left(\tau_2, \left\{[0.1,0.4]/_{\ell_1}, [0.6,0.8]/_{\ell_2}, [0.5,0.7]/_{\ell_3}\right\}\right), \\
\left(\tau_3, \left\{[0.5,0.7]/_{\ell_2}, [0.2,0.4]/_{\ell_3}, [0.4,0.6]/_{\ell_4}, [0.4,0.7]/_{\ell_5}\right\}\right)
\end{array}\right\}$$

and
$(\tilde{L'}_{H_2}, \eth_2) =$

$$\left\{\begin{array}{l}
\left(\tau_3, \left\{[0.7,0.8]/_{\ell_1}, [0.3,0.5]/_{\ell_2}\right\}\right), \left(\tau_4, \left\{[0.4,0.7]/_{\ell_3}, [0.2,0.3]/_{\ell_4}, [0.2,0.6]/_{\ell_5}\right\}\right), \\
\left(\tau_5, \left\{[0.4,0.6]/_{\ell_1}, [0.3,0.5]/_{\ell_2}, [0.4,0.5]/_{\ell_4}, [0.2,0.3]/_{\ell_5}\right\}\right),
\end{array}\right\}$$

$\eth_1 \times \eth_2 =$

$$\left\{\begin{array}{l}
\left(\lambda_1 = (\tau_1, \tau_3), \lambda_2 = (\tau_1, \tau_4), \lambda_3 = (\tau_1, \tau_5), \lambda_4 = (\tau_2, \tau_3), \lambda_5 = (\tau_2, \tau_4), \right), \\
\left(\lambda_6 = (\tau_2, \tau_5), \lambda_7 = (\tau_3, \tau_3), \lambda_8 = (\tau_3, \tau_4), \lambda_9 = (\tau_3, \tau_5)\right)
\end{array}\right\}$$

then
$(\tilde{L'}_{H_3}, \eth_3) =$

$$\left\{\begin{array}{l}
\left(\lambda_1, \left\{[0.4,0.5]/_{\ell_1}, [0.2,0.5]/_{\ell_2}\right\}\right), (\lambda_2, \{\}), \left(\lambda_3, \left\{[0.4,0.5]/_{\ell_1}, [0.2,0.5]/_{\ell_2},\right\}\right), \\
\left(\lambda_4, \left\{[0.1,0.4]/_{\ell_1}, [0.3,0.5]/_{\ell_2}\right\}\right), \left(\lambda_5, \left\{[0.4,0.7]/_{\ell_3},\right\}\right), \left(\lambda_6, \left\{[0.1,0.4]/_{\ell_1}, [0.3,0.5]/_{\ell_2}\right\}\right), \\
\left(\lambda_7, \left\{[0.5,0.7]/_{\ell_1}, [0.2,0.4]/_{\ell_2},\right\}\right), \left(\lambda_8, \left\{[0.2,0.4]/_{\ell_3}, [0.2,0.3]/_{\ell_4}, [0.2,0.6]/_{\ell_5},\right\}\right), \\
\left(\lambda_9, \left\{[0.3,0.5]/_{\ell_2}, [0.4,0.5]/_{\ell_4}, [0.2,0.3]/_{\ell_5}\right\}\right)
\end{array}\right\}.$$

Definition 4.19. The *OR operation of two $IVFHS$-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \bigvee (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set $(\tilde{L'}_{H_3}, \eth_3)$ with $\eth_3 = \eth_1 \times \eth_2$ and for $(\tau_i, \tau_j) \in \eth_3, \tau_i \in \eth_1, \tau_j \in \eth_2$ is defined as

$$\tilde{L'}_{H_3}(\tau_1, \tau_2) = \tilde{L'}_{H_1}(\tau_1) \sqcup \tilde{L'}_{H_2}(\tau_2),$$

where $\tilde{L'}_{H_1}(\tau_1) \sqcup \tilde{L'}_{H_2}(\tau_2)$ as defined in definition 4.1.

Example 4.20. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ as in Example 4.18, then $(\tilde{L'}_{H_3}, \eth_3)$ can e given by

$(\tilde{L'}_{H_3}, \eth_3) =$

$$\left\{\begin{array}{l}
\left(\lambda_1, \left\{[0.7,0.8]/_{\ell_1}, [0.3,0.5]/_{\ell_2}\right\}\right), \left(\lambda_4, \left\{[0.7,0.8]/_{\ell_1}, [0.6,0.8]/_{\ell_2}, [0.5,0.7]/_{\ell_3}\right\}\right)\\
\left(\lambda_2, \left\{[0.4,0.5]/_{\ell_1}, [0.2,0.5]/_{\ell_2}, [0.4,0.7]/_{\ell_3}, [0.2,0.3]/_{\ell_4}, [0.2,0.6]/_{\ell_5}\right\}\right),\\
\left(\lambda_3, \left\{[0.4,0.6]/_{\ell_1}, [0.3,0.5]/_{\ell_2}, [0.4,0.5]/_{\ell_4}, [0.2,0.3]/_{\ell_5}\right\}\right),\\
\left(\lambda_5, \left\{[0.1,0.4]/_{\ell_1}, [0.6,0.8]/_{\ell_2}, [0.5,0.7]/_{\ell_3}, [0.2,0.3]/_{\ell_4}, [0.2,0.6]/_{\ell_5},\right\}\right),\\
\left(\lambda_6, \left\{[0.4,0.6]/_{\ell_1}, [0.6,0.8]/_{\ell_2}, [0.5,0.7]/_{\ell_3}, [0.4,0.5]/_{\ell_4}, [0.2,0.3]/_{\ell_5}\right\}\right),\\
\left(\lambda_7, \left\{[0.7,0.8]/_{\ell_1}, [0.5,0.7]/_{\ell_2}, [0.2,0.4]/_{\ell_3}, [0.4,0.6]/_{\ell_4}, [0.4,0.7]/_{\ell_5}\right\}\right),\\
\left(\lambda_8, \left\{[0.5,0.7]/_{\ell_2}, [0.4,0.7]/_{\ell_3}, [0.4,0.6]/_{\ell_4}, [0.4,0.7]/_{\ell_5}\right\}\right),\\
\left(\lambda_9, \left\{[0.4,0.6]/_{\ell_1}, [0.5,0.7]/_{\ell_2}, [0.2,0.4]/_{\ell_3}, [0.4,0.6]/_{\ell_4}, [0.4,0.7]/_{\ell_5}\right\}\right)
\end{array}\right\}.$$

Definition 4.21. The *difference of two $IVFHS$-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \setminus (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set $(\tilde{L'}_{H_3}, \eth_3)$ with $\eth_3 = \eth_1 \backslash \eth_2$ and is defined as

$$\tilde{L'}_{H_3}(\tau) = \tilde{L'}_{H_1}(\tau_1) - \tilde{L'}_{H_2}(\tau_2).$$

Example 4.22. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ as in Example 4.18, then $(\tilde{L'}_{H_3}, \eth_3)$ can be given by

$$(\tilde{L'}_{H_3}, \eth_3) = \left\{ \quad \left(\tau_1, \left\{[0.4,0.5]/_{\ell_1}, [0.2,0.5]/_{\ell_2}\right\}\right), \left(\tau_2, \left\{[0.1,0.4]/_{\ell_1}, [0.6,0.8]/_{\ell_2}, [0.5,0.7]/_{\ell_3}\right\}\right) \quad \right\}$$

Definition 4.23. The *restricted difference of two $IVFHS$-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \backslash_R (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set $(\tilde{L'}_{H_3}, \eth_3)$ with $\eth_3 = \eth_1 \cap \eth_2$ and for $(\tau_i, \tau_j) \in \eth_3, \tau_i \in \eth_1, \tau_j \in \eth_2$ is defined as

$$\tilde{L'}_{H_3}(\tau) = \tilde{L'}_{H_1}(\tau) - \tilde{L'}_{H_2}(\tau_2).$$

Example 4.24. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ as in Example 4.18, then $(\tilde{L'}_{H_3}, \eth_3)$ can be given by

$$(\tilde{L'}_{H_3}, \eth_3) = \left\{\left(\tau_3, \left\{[0.2,0.4]/_{\ell_3}, [0.4,0.6]/_{\ell_4}, [0.4,0.7]/_{\ell_5}\right\}\right)\right\}.$$

Definition 4.25. The *restricted symmetric difference of two $IVFHS$-sets* $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ denoted by $(\tilde{L'}_{H_1}, \eth_1) \setminus_{RS} (\tilde{L'}_{H_2}, \eth_2)$ is an $IVFHS$-set $(\tilde{L'}_{H_3}, \eth_3)$ with $\eth_3 = \eth_1 \cap \eth_2$ and for $(\tau_i, \tau_j) \in \eth_3, \tau_i \in \eth_1, \tau_j \in \eth_2$ is defined as

$$\tilde{L'}_{H_3} = \left\{\left((\tilde{L'}_{H_1}, \eth_1) \sqcup_R (\tilde{L'}_{H_2}, \eth_2)\right) \setminus_R \left((\tilde{L'}_{H_1}, \eth_1) \cap (\tilde{L'}_{H_2}, \eth_2)\right)\right\}$$

or

$$\tilde{L'}_{H_3} = \left\{\left((\tilde{L'}_{H_1}, \eth_1) \setminus_R (\tilde{L'}_{H_2}, \eth_2)\right) \sqcup_R \left((\tilde{L'}_{H_2}, \eth_2) \setminus_R (\tilde{L'}_{H_1}, \eth_1)\right)\right\}.$$

Example 4.26. Consider two $IVFHS$-sets $(\tilde{L'}_{H_1}, \eth_1)$ and $(\tilde{L'}_{H_2}, \eth_2)$ as in Example 4.18, then we have

$$\left((\tilde{L'}_{H_1}, \eth_1) \setminus_R (\tilde{L'}_{H_2}, \eth_2)\right) = \left\{\left(\tau_3, \left\{[0.2,0.4]/_{\ell_3}, [0.4,0.6]/_{\ell_4}, [0.4,0.7]/_{\ell_5}\right\}\right)\right\}.$$

$$\left((\tilde{L'}_{H_2}, \eth_2) \setminus_R (\tilde{L'}_{H_1}, \eth_1)\right) = \left\{\left(\tau_3, \left\{[0.7,0.8]/_{\ell_1}\right\}\right)\right\}.$$

then,

$$(\tilde{L'}_{H_3}, \eth_3) = \left\{\left(\tau_3, \left\{[0.7,0.8]/_{\ell_1}, [0.2,0.4]/_{\ell_3}, [0.4,0.6]/_{\ell_4}, [0.4,0.7]/_{\ell_5}\right\}\right)\right\}.$$

5 Basic Properties and Laws of $IVFHS$-set Operations

Some basic properties, laws and results of $IVFHS$-set are discussed in this section. All $IVFHS$-sets in $\mathcal{Ç}_{(\tilde{L'}_H, \eth)}$ satisfy following properties, laws and results.

1. **Idempotent Laws**
   (a) $(\tilde{L'}_H, \eth) \sqcap (\tilde{L'}_H, \eth) = (\tilde{L'}_H, \eth) = (\tilde{L'}_H, \eth) \sqcap_E (\tilde{L'}_H, \eth)$.
   (b) $(\tilde{L'}_H, \eth) \sqcup (\tilde{L'}_H, \eth) = (\tilde{L'}_H, \eth) = (\tilde{L'}_H, \eth) \sqcup_R (\tilde{L'}_H, \eth)$.
2. **Identity Laws**
   (a) $(\tilde{L'}_H, \eth) \sqcap (\tilde{L'}_H, \eth)_{\Cup} = (\tilde{L'}_H, \eth) = (\tilde{L'}_H, \eth) \sqcap_E (\tilde{L'}_H, \eth)_{\Cup}$.
   (b) $(\tilde{L'}_H, \eth) \sqcup (\tilde{L'}_H, \eth)_{\Phi} = (\tilde{L'}_H, \eth) = (\tilde{L'}_H, \eth) \sqcup_R (\tilde{L'}_H, \eth)_{\Phi}$.
   (c) $(\tilde{L'}_H, \eth) \setminus_R (\tilde{L'}_H, \eth) = (\tilde{L'}_H, \eth)_{\Phi} = (\tilde{L'}_H, \eth) \sqcup_{RS} (\tilde{L'}_H, \eth)$.
   (d) $(\tilde{L'}_H, \eth) \setminus_R (\tilde{L'}_H, \eth)_{\Phi} = (\tilde{L'}_H, \eth) = (\tilde{L'}_H, \eth) \sqcup_{RS} (\tilde{L'}_H, \eth)_{\Phi}$.
3. **Domination Laws**
   (a) $(\tilde{L'}_H, \eth) \sqcap (\tilde{L'}_H, \eth)_{\Phi} = (\tilde{L'}_H, \eth)_{\Phi} = (\tilde{L'}_H, \eth) \sqcap_E (\tilde{L'}_H, \eth)_{\Phi}$.
   (b) $(\tilde{L'}_H, \eth) \sqcup (\tilde{L'}_H, \eth)_{\Cup} = (\tilde{L'}_H, \eth)_{\Cup} = (\tilde{L'}_H, \eth) \sqcup_R (\tilde{L'}_H, \eth)_{\Cup}$.
4. **Property of Contradiction**
   (a) $(\tilde{L'}_H, \eth) \sqcap (\tilde{L'}_H, \eth)^c = (\tilde{L'}_H, \eth)_{\Phi} = (\tilde{L'}_H, \eth) \sqcap_E (\tilde{L'}_H, \eth)^c$.
5. **Property of Exclusion**
   (a) $(\tilde{L'}_H, \eth) \sqcup (\tilde{L'}_H, \eth)^c = (\tilde{L'}_H, \eth)_{\Cup} = (\tilde{L'}_H, \eth) \sqcup_R (\tilde{L'}_H, \eth)^c$.
6. **Absorption Laws**
   (a) $(\tilde{L'}_{H_1}, \eth_1) \sqcap \Big((\tilde{L'}_{H_1}, \eth_1) \sqcup (\tilde{L'}_{H_2}, \eth_2)\Big) = (\tilde{L'}_{H_1}, \eth_1)$.
   (b) $(\tilde{L'}_{H_1}, \eth_1) \sqcup \Big((\tilde{L'}_{H_1}, \eth_1) \sqcap (\tilde{L'}_{H_2}, \eth_2)\Big) = (\tilde{L'}_{H_1}, \eth_1)$.
   (c) $(\tilde{L'}_{H_1}, \eth_1) \sqcap_E \Big((\tilde{L'}_{H_1}, \eth_1) \sqcup_R (\tilde{L'}_{H_2}, \eth_2)\Big) = (\tilde{L'}_{H_1}, \eth_1)$.
   (d) $(\tilde{L'}_{H_1}, \eth_1) \sqcup_R \Big((\tilde{L'}_{H_1}, \eth_1) \sqcap_E (\tilde{L'}_{H_2}, \eth_2)\Big) = (\tilde{L'}_{H_1}, \eth_1)$.
7. **Commutative Laws**
   (a) $(\tilde{L'}_{H_1}, \eth_1) \sqcap (\tilde{L'}_{H_2}, \eth_2) = (\tilde{L'}_{H_2}, \eth_2) \sqcap (\tilde{L'}_{H_1}, \eth_1)$.
   (b) $(\tilde{L'}_{H_1}, \eth_1) \sqcap_E (\tilde{L'}_{H_2}, \eth_2) = (\tilde{L'}_{H_2}, \eth_2) \sqcap_E (\tilde{L'}_{H_1}, \eth_1)$.
   (c) $(\tilde{L'}_{H_1}, \eth_1) \sqcup (\tilde{L'}_{H_2}, \eth_2) = (\tilde{L'}_{H_2}, \eth_2) \sqcup (\tilde{L'}_{H_1}, \eth_1)$.
   (d) $(\tilde{L'}_{H_1}, \eth_1) \sqcup_R (\tilde{L'}_{H_2}, \eth_2) = (\tilde{L'}_{H_2}, \eth_2) \sqcup_R (\tilde{L'}_{H_1}, \eth_1)$.
   (e) $(\tilde{L'}_{H_1}, \eth_1) \setminus_{RS} (\tilde{L'}_{H_2}, \eth_2) = (\tilde{L'}_{H_2}, \eth_2) \setminus_{RS} (\tilde{L'}_{H_1}, \eth_1)$.
8. **Associative Laws**
   (a) $(\tilde{L'}_{H_1}, \eth_1) \sqcap \Big((\tilde{L'}_{H_2}, \eth_2) \sqcap (\tilde{L'}_{H_3}, \eth_3)\Big) = \Big((\tilde{L'}_{H_1}, \eth_1) \sqcap (\tilde{L'}_{H_2}, \eth_2)\Big) \sqcap (\tilde{L'}_{H_3}, \eth_3)$.
   (b) $(\tilde{L'}_{H_1}, \eth_1) \sqcap_E \Big((\tilde{L'}_{H_2}, \eth_2) \sqcap_E (\tilde{L'}_{H_3}, \eth_3)\Big) = \Big((\tilde{L'}_{H_1}, \eth_1) \sqcap_E (\tilde{L'}_{H_2}, \eth_2)\Big) \sqcap_E (\tilde{L'}_{H_3}, \eth_3)$.
   (c) $(\tilde{L'}_{H_1}, \eth_1) \sqcup \Big((\tilde{L'}_{H_2}, \eth_2) \sqcup (\tilde{L'}_{H_3}, \eth_3)\Big) = \Big((\tilde{L'}_{H_1}, \eth_1) \sqcup (\tilde{L'}_{H_2}, \eth_2)\Big) \sqcup (\tilde{L'}_{H_3}, \eth_3)$.
   (d) $(\tilde{L'}_{H_1}, \eth_1) \sqcup_R \Big((\tilde{L'}_{H_2}, \eth_2) \sqcup_R (\tilde{L'}_{H_3}, \eth_3)\Big) = \Big((\tilde{L'}_{H_1}, \eth_1) \sqcup_R (\tilde{L'}_{H_2}, \eth_2)\Big) \sqcup_R (\tilde{L'}_{H_3}, \eth_3)$.
   (e) $(\tilde{L'}_{H_1}, \eth_1) \bigwedge \Big((\tilde{L'}_{H_2}, \eth_2) \bigwedge (\tilde{L'}_{H_3}, \eth_3)\Big) = \Big((\tilde{L'}_{H_1}, \eth_1) \bigwedge (\tilde{L'}_{H_2}, \eth_2)\Big) \bigwedge (\tilde{L'}_{H_3}, \eth_3)$.
   (f) $(\tilde{L'}_{H_1}, \eth_1) \bigvee \Big((\tilde{L'}_{H_2}, \eth_2) \bigvee (\tilde{L'}_{H_3}, \eth_3)\Big) = \Big((\tilde{L'}_{H_1}, \eth_1) \bigvee (\tilde{L'}_{H_2}, \eth_2)\Big) \bigvee (\tilde{L'}_{H_3}, \eth_3)$.
9. **De Morgans Laws**
   (a) $\Big((\tilde{L'}_{H_1}, \eth_1) \sqcap_E (\tilde{L'}_{H_2}, \eth_2)\Big)^c = \Big(\tilde{L'}_{H_1}, \eth_1\Big)^c \sqcup \Big(\tilde{L'}_{H_2}, \eth_2\Big)^c$.

(b) $\left((\tilde{L'}_{H_1}, \eth_1) \sqcup (\tilde{L'}_{H_2}, \eth_2)\right)^c = \left(\tilde{L'}_{H_1}, \eth_1\right)^c \sqcap_E \left(\tilde{L'}_{H_2}, \eth_2\right)^c$.

(c) $\left((\tilde{L'}_{H_1}, \eth_1) \bigwedge (\tilde{L'}_{H_2}, \eth_2)\right)^c = \left(\tilde{L'}_{H_1}, \eth_1\right)^c \bigvee \left(\tilde{L'}_{H_2}, \eth_2\right)^c$.

(d) $\left((\tilde{L'}_{H_1}, \eth_1) \bigvee (\tilde{L'}_{H_2}, \eth_2)\right)^c = \left(\tilde{L'}_{H_1}, \eth_1\right)^c \bigwedge \left(\tilde{L'}_{H_2}, \eth_2\right)^c$.

10. **Distributive Laws**

(a) $(\tilde{L'}_{H_1}, \eth_1) \sqcap \left((\tilde{L'}_{H_2}, \eth_2) \sqcup (\tilde{L'}_{H_3}, \eth_3)\right) =$
$\left((\tilde{L'}_{H_1}, \eth_1) \sqcap (\tilde{L'}_{H_2}, \eth_2)\right) \sqcup \left((\tilde{L'}_{H_1}, \eth_1) \sqcap (\tilde{L'}_{H_3}, \eth_3)\right)$.

(b) $(\tilde{L'}_{H_1}, \eth_1) \sqcup \left((\tilde{L'}_{H_2}, \eth_2) \sqcap (\tilde{L'}_{H_3}, \eth_3)\right) =$
$\left((\tilde{L'}_{H_1}, \eth_1) \sqcup (\tilde{L'}_{H_2}, \eth_2)\right) \sqcap \left((\tilde{L'}_{H_1}, \eth_1) \sqcup (\tilde{L'}_{H_3}, \eth_3)\right)$.

(c) $(\tilde{L'}_{H_1}, \eth_1) \sqcap_E \left((\tilde{L'}_{H_2}, \eth_2) \sqcup_R (\tilde{L'}_{H_3}, \eth_3)\right) =$
$\left((\tilde{L'}_{H_1}, \eth_1) \sqcap_E (\tilde{L'}_{H_2}, \eth_2)\right) \sqcup_R \left((\tilde{L'}_{H_1}, \eth_1) \sqcap_E (\tilde{L'}_{H_3}, \eth_3)\right)$.

(d) $(\tilde{L'}_{H_1}, \eth_1) \sqcup_R \left((\tilde{L'}_{H_2}, \eth_2) \sqcap_E (\tilde{L'}_{H_3}, \eth_3)\right) =$
$\left((\tilde{L'}_{H_1}, \eth_1) \sqcup_R (\tilde{L'}_{H_2}, \eth_2)\right) \sqcap_E \left((\tilde{L'}_{H_1}, \eth_1) \sqcup_R (\tilde{L'}_{H_3}, \eth_3)\right)$.

(e) $(\tilde{L'}_{H_1}, \eth_1) \sqcap \left((\tilde{L'}_{H_2}, \eth_2) \sqcup_R (\tilde{L'}_{H_3}, \eth_3)\right) =$
$\left((\tilde{L'}_{H_1}, \eth_1) \sqcap (\tilde{L'}_{H_2}, \eth_2)\right) \sqcup_R \left((\tilde{L'}_{H_1}, \eth_1) \sqcap (\tilde{L'}_{H_3}, \eth_3)\right)$.

(f) $(\tilde{L'}_{H_1}, \eth_1) \sqcup_R \left((\tilde{L'}_{H_2}, \eth_2) \sqcap (\tilde{L'}_{H_3}, \eth_3)\right) =$
$\left((\tilde{L'}_{H_1}, \eth_1) \sqcup_R (\tilde{L'}_{H_2}, \eth_2)\right) \sqcap \left((\tilde{L'}_{H_1}, \eth_1) \sqcup_R (\tilde{L'}_{H_3}, \eth_3)\right)$.

6 Hybrids of $IVFHS$-set

In this section some hybrids of $IVFHS$-sets are discussed.

Definition 6.1. Let $\tilde{L}$, $\tilde{G}$ and $\tilde{H}$ are three $IVFHS$-sets over $\Bbbu$. A mapping $\digamma(\tilde{L}, \tilde{G})$ given by $\digamma : ivfs(\Bbbu) \times ivfs(\Bbbu) \to IVFN$ is called similarity measure if it satisfies the following conditions:

1. $\digamma(\tilde{L}, \tilde{G})$ is IVF-number.
2. $\digamma(\tilde{L}, \tilde{G}) = [1, 1]$ if and only if $\tilde{L} = \tilde{G}$.
3. $\digamma(\tilde{L}, \tilde{G}) = \digamma(\tilde{G}, \tilde{L})$.
4. If $\tilde{L} \subseteq \tilde{G} \subseteq \tilde{H}$, then $\digamma(\tilde{L}, \tilde{H}) \subseteq \digamma(\tilde{L}, \tilde{G})$ and $\digamma(\tilde{L}, \tilde{H}) \subseteq \digamma(\tilde{G}, \tilde{H})$.

Definition 6.2. Let $\tilde{L}$, $\tilde{G}$ and $\tilde{H}$ are three $IVFHS$-sets over $\Bbbu$. A mapping $\Delta(\tilde{L}, \tilde{G})$ given by $\Delta : ivfs(\Bbbu) \times ivfs(\Bbbu) \to IVFN$ is called distance measure if it satisfies the following conditions:

1. $\Delta(\tilde{L}, \tilde{G})$ is an IVF-number.
2. $\Delta(\tilde{L}, \tilde{G}) = [0, 0]$ if and only if $\tilde{L} = \tilde{G}$.
3. $\Delta(\tilde{L}, \tilde{G}) = \Delta(\tilde{G}, \tilde{L})$.
4. If $\tilde{L} \subseteq \tilde{G} \subseteq \tilde{H}$, then $\Delta(\tilde{L}, \tilde{G}) \subseteq \Delta(\tilde{L}, \tilde{H})$ and $\Delta(\tilde{G}, \tilde{H}) \subseteq \Delta(\tilde{L}, \tilde{H})$.

Proposition 6.3. *Let $\tilde{L}$ and $\tilde{G}$ and are two $IVFHS$-sets over $\Bbbu$, then the following results hold:*

1. $\digamma(\tilde{L}, \tilde{L} \cap \tilde{G}) = \digamma(\tilde{G}, \tilde{L} \cup \tilde{G})$.

2. $\digamma(\tilde{L}, \tilde{L} \cup \tilde{G}) = \digamma(\tilde{G}, \tilde{L} \cap \tilde{G})$.
3. $\digamma(\tilde{L}, \tilde{G}) = \digamma(\tilde{L} \cap \tilde{G}, \tilde{L} \cup \tilde{G})$.
4. $\Delta(\tilde{L}, \tilde{L} \cap \tilde{G}) = \Delta(\tilde{G}, \tilde{L} \cup \tilde{G})$.
5. $\Delta(\tilde{L}, \tilde{L} \cup \tilde{G}) = \Delta(\tilde{G}, \tilde{L} \cap \tilde{G})$.
6. $\Delta(\tilde{L}, \tilde{G}) = \Delta(\tilde{L} \cap \tilde{G}, \tilde{L} \cup \tilde{G})$.
7. $(\digamma(\tilde{L}, \tilde{G}))^c = \Delta(\tilde{L}, \tilde{G})$.

7 Some extensions of $IVFHS$-set

Atiqe et al.[20] and Smarandache[19] discussed some hybrids of $IVFHS$-sets. Some more hybrids of $IVFHS$-sets are discussed in this section. Let $\eth_1, \eth_2, ..., \eth_n$ are attribute valued sets, each corresponding to a unique attribute $\epsilon_1, \epsilon_2, ..., \epsilon_n \in E$ such that for $p, q = 1, 2, ..., n, p \neq q, \eth_p \cap \eth_q = \varphi$. Let collection of all sub intervals of $I = [0, 1]$ be represented by $\mathcal{C}(I)$.

Definition 7.1. Let $C_{ivf}(\Bbbu)$ be the collection of interval-valued fuzzy sets. Then $(\Upsilon, \eth)$, the interval-valued intuitionistic fuzzy hypersoft set ($\mathcal{IVIFHS}$-set) over $\Bbbu$ is defined as

$$(\Upsilon, \eth) = \left\{ \begin{array}{l} (< \Upsilon_{\tilde{\mathcal{T}}}(\varsigma), \Upsilon_{\tilde{\mathcal{L}}}(\varsigma) > /\varsigma, \Upsilon) : \varsigma \in \eth, \\ \Upsilon(\varsigma) \in C_{ivf}(\Bbbu), \Upsilon_{\tilde{\mathcal{T}}}(\varsigma), \Upsilon_{\tilde{\mathcal{L}}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\Upsilon : \eth \to C(\Bbbu)$ is approximate function and $\varsigma \in \eth = \eth_1 \times \eth_2 \times ... \times \eth_n$ with $\Upsilon_{\tilde{\mathcal{T}}}(\varsigma), \Upsilon_{\tilde{\mathcal{L}}}(\varsigma) : \eth \to I$ are truth membership and false membership functions of $\mathcal{IVIFHS}$-set such that the sum of upper bonds of $\Upsilon_{\tilde{\mathcal{T}}}$ and $\Upsilon_{\tilde{\mathcal{L}}}$ is less than or equal to 1.

Example 7.2. Let $L = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$ be the set of elements under consideration as in Example 2.8 and $T = \{\tau_1, \tau_2, \tau_3, ..., \tau_{16}\}$, where each τ_i is p^{th}-tuple, p is the product of orders of T_i. The $\mathcal{IVIFHS}$-set $(\Upsilon, \eth)$ is given by
$(\Upsilon, \eth) =$

$$\left\{ \begin{array}{l} \left(\tau_1, \left\{\frac{[0.1,0.2];[0.8,0.9]}{\ell_1}\right\}, \left\{\frac{[0.3,0.4];[0.6,0.7]}{\ell_3}\right\}\right), \left(\tau_2, \left\{\frac{[0.3,0.6];[0.4,0.7]}{\ell_2}\right\}\right), \\ \left(\tau_3, \left\{\frac{[0.2,0.4];[0.3,0.4]}{\ell_4}\right\}\right), \left(\tau_4, \left\{\frac{[0.3,0.5];[0.2,0.4]}{\ell_3}\right\}, \left\{\frac{[0.4,0.5];[0.2,0.3]}{\ell_4}\right\}, \left\{\frac{[0.1,0.3];[0.2,0.3]}{\ell_5}\right\}\right) \\ \left(\tau_5, \left\{\frac{[0.1,0.2];[0.2,0.5]}{\ell_1}\right\}, \left\{\frac{[0.4,0.5];[0.2,0.3]}{\ell_2}\right\}\right), \left(\tau_6, \left\{\frac{[0.3,0.6];[0.1,0.3]}{\ell_4}\right\}\right) \\ \left(\tau_7, \left\{\frac{[0.3,0.5];[0.2,0.4]}{\ell_1}\right\}, \left\{\frac{[0.2,0.6];[0.1,0.3]}{\ell_3}\right\}, \left\{\frac{[0.1,0.2];[0.4,0.5]}{\ell_5}\right\}\right), \left(\tau_8, \left\{\frac{[0.5,0.6];[0.2,0.3]}{\ell_1}\right\},\right) \\ \left(\tau_9, \left\{\frac{[0.3,0.4];[0.2,0.5]}{\ell_3}\right\}\right), \left(\tau_{10}, \left\{\frac{[0.1,0.3];[0.5,0.6]}{\ell_1}\right\}\right), \left(\tau_{11}, \left\{\frac{[0.3,0.4];[0.2,0.4]}{\ell_1}\right\}, \left\{\frac{[0.4,0.5];[0.1,0.3]}{\ell_5}\right\}\right), \\ \left(\tau_{12}, \left\{\frac{[0.2,0.3];[0.5,0.7]}{\ell_2}\right\}\right), \left(\tau_{13}, \left\{\frac{[0.6,0.7];[0.1,0.2]}{\ell_3}\right\}\right), \left(\tau_{14}, \left\{\frac{[0.1,0.3];[0.2,0.6]}{\ell_1}\right\}\right), \\ \left(\tau_{15}, \left\{\frac{[0.2,0.5];[0.3,0.4]}{\ell_1}\right\}, \left\{\frac{[0.2,0.3];[0.1,0.6]}{\ell_2}\right\}\right), \left(\tau_{16}, \left\{\frac{[0.2,0.3];[0.1,0.4]}{\ell_1}\right\}, \left\{\frac{[0.1,0.5];[0.2,0.4]}{\ell_4}\right\}\right) \end{array} \right\}.$$

Definition 7.3. Let $C_{ivn}(\Bbbu)$ be the collection of interval-valued neutrosophic sets. Then $(\Gamma, \eth)$, the interval-valued neutrosophic hypersoft set ($\mathcal{IVNHS}$-set) over $\Bbbu$ is defined as

$$(\Gamma, \eth) = \{(< \Gamma_{\tilde{\mathcal{T}}}(\varsigma), \Gamma_{\tilde{\mathcal{I}}}(\varsigma), \Gamma_{\tilde{\mathcal{L}}}(\varsigma) > /\varsigma, \Gamma) : \varsigma \in \eth, \Gamma(\varsigma) \in C_{ivn}(\Bbbu), \Gamma_{\tilde{\mathcal{T}}}(\varsigma), \Gamma_{\tilde{\mathcal{I}}}(\varsigma), \Gamma_{\tilde{\mathcal{L}}}(\varsigma) \in \mathcal{C}(I)\}$$

where $\Gamma : \eth \to C(\Bbbu)$ is approximate function and $\varsigma \in \eth$ with $\Upsilon_{\tilde{\mathcal{T}}}(\varsigma), \Upsilon_{\tilde{\mathcal{I}}}(\varsigma), \Upsilon_{\tilde{\mathcal{L}}}(\varsigma) : \eth \to I$ are truth, indeterminacy and falsity membership functions of $\mathcal{IVNHS}$-set.

Example 7.4. Consider Example 7.2. The $\mathcal{IVNHS}$-set $(\Upsilon, \eth)$ is given by

$(\Gamma,\eth) =$

$$\left\{\begin{array}{l}
\left(\tau_1,\left\{\frac{[0.3,0.6];[0.1,0.4];[0.2,0.5]}{\ell_1}\right\},\left\{\frac{[0.2,0.5];[0.2,0.3];[0.1,0.2]}{\ell_3}\right\}\right),\left(\tau_2,\left\{\frac{[0.1,0.3];[0.2,0.5];[0.2,0.3]}{\ell_2}\right\}\right),\\
\left(\tau_3,\left\{\frac{[0.2,0.3];[0.1,0.4];[0.3,0.6]}{\ell_4}\right\}\right),\left(\tau_6,\left\{\frac{[0.4,0.5];[0.4,0.6];[0.2,0.3]}{\ell_4}\right\}\right),\\
\left(\tau_4,\left\{\frac{[0.3,0.4];[0.2,0.3];[0.3,0.7]}{\ell_3}\right\},\left\{\frac{[0.6,0.7];[0.3,0.5];[0.1,0.3]}{\ell_4}\right\},\left\{\frac{[0.3,0.6];[0.2,0.4];[0.1,0.3]}{\ell_5}\right\}\right)\\
\left(\tau_5,\left\{\frac{[0.1,0.3];[0.3,0.4];[0.4,0.5]}{\ell_1}\right\},\left\{\frac{[0.2,0.3];[0.1,0.4];[0.4,0.7]}{\ell_2}\right\}\right),\left(\tau_9,\left\{\frac{[0.4,0.6];[0.2,0.3];[0.4,0.5]}{\ell_3}\right\}\right),\\
\left(\tau_7,\left\{\frac{[0.3,0.4];[0.3,0.7];[0.1,0.4]}{\ell_1}\right\},\left\{\frac{[0.3,0.5];[0.4,0.6];[0.3,0.6]}{\ell_3}\right\},\left\{\frac{[0.3,0.5];[0.2,0.4];[0.4,0.7]}{\ell_5}\right\}\right),\\
\left(\tau_{10},\left\{\frac{[0.3,0.5];[0.2,0.5];[0.3,0.5]}{\ell_1}\right\}\right),\left(\tau_{11},\left\{\frac{[0.4,0.7];[0.1,0.4];[0.5,0.6]}{\ell_1}\right\},\left\{\frac{[0.3,0.5];[0.2,0.3];[0.6,0.7]}{\ell_5}\right\}\right),\\
\left(\tau_{12},\left\{\frac{[0.6,0.8];[0.4,0.6];[0.3,0.7]}{\ell_2}\right\}\right),\left(\tau_{13},\left\{\frac{[0.4,0.5];[0.4,0.7];[0.1,0.3]}{\ell_3}\right\}\right),\\
\left(\tau_{14},\left\{\frac{[0.4,0.7];[0.2,0.5];[0.4,0.6]}{\ell_1}\right\}\right),\left(\tau_{15},\left\{\frac{[0.1,0.3];[0.4,0.5];[0.2,0.4]}{\ell_1}\right\},\left\{\frac{[0.5,0.6];[0.2,0.5];[0.4,0.8]}{\ell_2}\right\}\right),\\
\left(\tau_8,\left\{\frac{[0.1,0.2];[0.1,0.3];[0.3,0.5]}{\ell_1}\right\}\right),\left(\tau_{16},\left\{\frac{[0.4,0.6];[0.4,0.5];[0.3,0.6]}{\ell_1}\right\},\left\{\frac{[0.4,0.6];[0.3,0.5];[0.2,0.4]}{\ell_4}\right\}\right)
\end{array}\right\}.$$

Definition 7.5. The interval-valued picture fuzzy hypersoft set ($\mathcal{IVPFHS}$-set) $(\wp,\eth)$, over $\mathbb{U}$ is given by

$$(\wp,\eth)=\{(<\wp_{\tilde{\mathcal{T}}}(\varsigma),\wp_{\tilde{\mathcal{I}}}(\varsigma),\wp_{\tilde{\mathcal{L}}}(\varsigma)>/\varsigma,\wp):\varsigma\in\eth,\wp(\varsigma)\in C_{ivn}(\mathbb{U}),\wp_{\tilde{\mathcal{T}}}(\varsigma),\wp_{\tilde{\mathcal{I}}}(\varsigma),\wp_{\tilde{\mathcal{L}}}(\varsigma)\in\mathcal{C}(I)\}$$

where $\wp:\eth\to C(\mathbb{U})$ is approximate function and $\varsigma\in\eth$ with $\wp_{\tilde{\mathcal{T}}}(\varsigma),\wp_{\tilde{\mathcal{I}}}(\varsigma),\wp_{\tilde{\mathcal{L}}}(\varsigma):\eth\to I$ are truth, indeterminacy and falsity membership functions of $\mathcal{IVPFHS}$-set with condition that the sum of upper bonds of $\wp_{\tilde{\mathcal{T}}}$, $\wp_{\tilde{\mathcal{I}}}$ and $\wp_{\tilde{\mathcal{L}}}$ remain less than or equal to 1.

Example 7.6. Consider Example 7.2. The $\mathcal{IVPFHS}$-set $(\wp,\eth)$ is given by
$(\wp,\eth) =$

$$\left\{\begin{array}{l}
\left(\tau_1,\left\{\frac{[0.3,0.4];[0.1,0.2];[0.2,0.3]}{\ell_1}\right\},\left\{\frac{[0.2,0.4];[0.2,0.3];[0.1,0.2]}{\ell_3}\right\}\right),\left(\tau_2,\left\{\frac{[0.1,0.3];[0.2,0.3];[0.2,0.3]}{\ell_2}\right\}\right),\\
\left(\tau_3,\left\{\frac{[0.2,0.3];[0.1,0.4];[0.3,0.6]}{\ell_4}\right\}\right),\left(\tau_6,\left\{\frac{[0.1,0.2];[0.4,0.5];[0.1,0.2]}{\ell_4}\right\}\right),\\
\left(\tau_4,\left\{\frac{[0.0,0.2];[0.2,0.3];[0.3,0.4]}{\ell_3}\right\},\left\{\frac{[0.3,0.4];[0.2,0.4];[0.0,0.1]}{\ell_4}\right\},\left\{\frac{[0.2,0.3];[0.2,0.4];[0.1,0.3]}{\ell_5}\right\}\right)\\
\left(\tau_5,\left\{\frac{[0.0,0.1];[0.3,0.4];[0.4,0.5]}{\ell_1}\right\},\left\{\frac{[0.2,0.3];[0.1,0.2];[0.4,0.5]}{\ell_2}\right\}\right),\left(\tau_9,\left\{\frac{[0.1,0.4];[0.1,0.2];[0.1,0.3]}{\ell_3}\right\}\right),\\
\left(\tau_7,\left\{\frac{[0.3,0.4];[0.3,0.4];[0.0,0.1]}{\ell_1}\right\},\left\{\frac{[0.2,0.4];[0.3,0.4];[0.1,0.2]}{\ell_3}\right\},\left\{\frac{[0.0,0.3];[0.2,0.3];[0.3,0.4]}{\ell_5}\right\}\right),\\
\left(\tau_{10},\left\{\frac{[0.3,0.4];[0.2,0.4];[0.0,0.2]}{\ell_1}\right\}\right),\left(\tau_{11},\left\{\frac{[0.1,0.2];[0.1,0.2];[0.5,0.6]}{\ell_1}\right\},\left\{\frac{[0.3,0.4];[0.2,0.3];[0.2,0.3]}{\ell_5}\right\}\right),\\
\left(\tau_{12},\left\{\frac{[0.6,0.8];[0.0,0.1];[0.0,0.1]}{\ell_2}\right\}\right),\left(\tau_{13},\left\{\frac{[0.0,0.1];[0.4,0.5];[0.0,0.3]}{\ell_3}\right\}\right),\\
\left(\tau_{14},\left\{\frac{[0.1,0.3];[0.2,0.3];[0.3,0.4]}{\ell_1}\right\}\right),\left(\tau_{15},\left\{\frac{[0.1,0.3];[0.4,0.5];[0.0,0.1]}{\ell_1}\right\},\left\{\frac{[0.5,0.6];[0.2,0.3];[0.0,0.1]}{\ell_2}\right\}\right),\\
\left(\tau_8,\left\{\frac{[0.1,0.2];[0.1,0.3];[0.3,0.5]}{\ell_1}\right\}\right),\left(\tau_{16},\left\{\frac{[0.0,0.6];[0.0,0.1];[0.0,0.2]}{\ell_1}\right\},\left\{\frac{[0.4,0.5];[0.3,0.4];[0.0,0.1]}{\ell_4}\right\}\right)
\end{array}\right\}.$$

Definition 7.7. $(\Delta,\eth)$ is called fuzzy parameterized hypersoft set ($\mathcal{FPHS}$-set) over $\mathbb{U}$ and is defined as

$$(\Delta,\eth)=\{(\xi_F(\varsigma)/\varsigma,\pi_F(\varsigma):),\varsigma\in\eth,\pi_F(\varsigma)\in P(\mathbb{U}),\xi_F(\varsigma)\in I\}$$

where F is fuzzy set and $\xi_F:\eth\to I$ is membership function of $\mathcal{FPHS}$-set and $\pi_F:\eth\to P(\mathbb{U})$ is approximate function.

Example 7.8. Consider Example 7.2. The $\mathcal{FPHS}$-set $(\Delta,\eth)$ is given by
$(\Delta,\eth) =$

$$\left\{\begin{array}{l}
\left(\frac{0.1}{\tau_1},\{\ell_1,\ell_3\}\right),\left(\frac{0.3}{\tau_2},\{\ell_2\}\right),\left(\frac{0.4}{\tau_3},\{\ell_4\}\right),\left(\frac{0.1}{\tau_4},\{\ell_3,\ell_4,\ell_5\}\right),\left(\frac{0.7}{\tau_5},\{\ell_1,\ell_2\}\right),\left(\frac{0.2}{\tau_6},\{\ell_4\}\right),\\
\left(\frac{0.1}{\tau_7},\{\ell_1,\ell_3,\ell_5\}\right),\left(\frac{0.2}{\tau_8},\{\ell_1\}\right),\left(\frac{0.4}{\tau_9},\{\ell_3\}\right),\left(\frac{0.3}{\tau_{10}},\{\ell_1\}\right),\left(\frac{0.3}{\tau_{11}},\{\ell_1,\ell_5\}\right),\left(\frac{0.1}{\tau_{12}},\{\ell_2\}\right),\\
\left(\frac{0.4}{\tau_{13}},\{\ell_3\}\right),\left(\frac{0.1}{\tau_{14}},\{\ell_1\}\right),\left(\frac{0.2}{\tau_{15}},\{\ell_1,\ell_2\}\right),\left(\frac{0.3}{\tau_{16}},\{\ell_1,\ell_4\}\right)
\end{array}\right\}.$$

Definition 7.9. $(\nabla,\eth)$ is called fuzzy parameterized fuzzy hypersoft set ($\mathcal{FPFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\nabla,\eth)=\{(\varsigma_M(\varsigma)/\varsigma,\sigma_M(\varsigma)/z):\varsigma\in\eth,\varsigma_M(\varsigma)\in I,\sigma_M(\varsigma)\in I\}$$

where ς is fuzzy set and $\varsigma_M:\eth\to I$ is membership function of $\mathcal{FPFHS}$-set and $\sigma_M:\eth\to P(\mathbb{U})$ is approximate function.

Example 7.10. Consider Example 7.2. The $\mathcal{FPFHS}$-set $(\nabla, \eth)$ is given by
$(\nabla, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{0.1}{\tau_1}, \left\{\frac{0.2}{\ell_1}, \frac{0.3}{\ell_3}\right\}\right), \left(\frac{0.3}{\tau_2}, \left\{\frac{0.1}{\ell_2}\right\}\right), \left(\frac{0.4}{\tau_3}, \left\{\frac{0.5}{\ell_4}\right\}\right), \left(\frac{0.1}{\tau_4}, \left\{\frac{0.2}{\ell_3}, \frac{0.4}{\ell_4}, \frac{0.1}{\ell_5}\right\}\right), \left(\frac{0.7}{\tau_5}, \left\{\frac{0.6}{\ell_1}, \frac{0.2}{\ell_2}\right\}\right), \\ \left(\frac{0.2}{\tau_6}, \left\{\frac{0.4}{\ell_4}\right\}\right), \left(\frac{0.1}{\tau_7}, \left\{\frac{0.2}{\ell_1}, \frac{0.3}{\ell_3}, \frac{0.1}{\ell_5}\right\}\right), \left(\frac{0.2}{\tau_8}, \left\{\frac{0.2}{\ell_1}\right\}\right), \left(\frac{0.4}{\tau_9}, \left\{\frac{0.5}{\ell_3}\right\}\right), \left(\frac{0.3}{\tau_{10}}, \left\{\frac{0.1}{\ell_1}\right\}\right), \\ \left(\frac{0.3}{\tau_{11}}, \left\{\frac{0.1}{\ell_1}, \frac{0.4}{\ell_5}\right\}\right), \left(\frac{0.1}{\tau_{12}}, \left\{\frac{0.3}{\ell_2}\right\}\right), \left(\frac{0.4}{\tau_{13}}, \left\{\frac{0.2}{\ell_3}\right\}\right), \left(\frac{0.1}{\tau_{14}}, \left\{\frac{0.1}{\ell_1}\right\}\right), \left(\frac{0.2}{\tau_{15}}, \left\{\frac{0.1}{\ell_1}, \frac{0.7}{\ell_2}\right\}\right), \left(\frac{0.3}{\tau_{16}}, \left\{\frac{0.2}{\ell_1}, \frac{0.4}{\ell_4}\right\}\right) \end{array} \right\}.$$

Definition 7.11. $(\mho_{fp-ivfhs}, \eth)$ is called fuzzy parameterized interval-valued fuzzy hypersoft set ($\mathcal{FPIVFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{fp-ivfhs}, \eth) = \left\{ \left(\frac{\mathcal{A}_{\mathcal{M}}(\varsigma)}{\varsigma}, \frac{\mathbb{A}_{\mathbb{M}}(\varsigma)}{z} \right) : \varsigma \in \eth, \mathcal{A}_{\mathcal{M}}(\varsigma) \in I, \mathbb{A}_{\mathbb{M}}(\varsigma) \in \mathcal{C}(I) \right\}$$

where $\mathcal{A}_{\mathcal{M}}$ is truth-ness of membership function $\mathcal{A} : \eth \to I$ and $\mathbb{A}_{\mathbb{M}}$ is truth-ness of approximate function $\mathbb{A} : \eth \to P(\mathbb{U})$ of $\mathcal{FPIVFHS}$-set.

Example 7.12. Consider Example 7.2. The $\mathcal{FPIVFHS}$-set $(\mho_{fp-ivfhs}, \eth)$ is given by
$(\mho_{fp-ivfhs}, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{0.1}{\tau_1}, \left\{\frac{[0.1,0.2]}{\ell_1}, \frac{[0.2,0.3}{\ell_3}\right\}\right), \left(\frac{0.3}{\tau_2}, \left\{\frac{[0.1,0,4]}{\ell_2}\right\}\right), \left(\frac{0.4}{\tau_3}, \left\{\frac{[0.2,0.5]}{\ell_4}\right\}\right), \left(\frac{0.1}{\tau_4}, \left\{\frac{[0.2,0.3]}{\ell_3}, \frac{[o.1,0.4]}{\ell_4}, \frac{[0.1,0.5]}{\ell_5}\right\}\right), \\ \left(\frac{0.7}{\tau_5}, \left\{\frac{[0.3,0.6]}{\ell_1}, \frac{[0.1,0.2]}{\ell_2}\right\}\right), \left(\frac{0.2}{\tau_6}, \left\{\frac{[0.2,0.4]}{\ell_4}\right\}\right), \left(\frac{0.1}{\tau_7}, \left\{\frac{[0.2,0.3]}{\ell_1}, \frac{[0.1,0.3]}{\ell_3}, \frac{[0.1,0.5]}{\ell_5}\right\}\right), \left(\frac{0.2}{\tau_8}, \left\{\frac{[0.2,0.5]}{\ell_1}\right\}\right), \\ \left(\frac{0.4}{\tau_9}, \left\{\frac{[0.3,0.5]}{\ell_3}\right\}\right), \left(\frac{0.3}{\tau_{10}}, \left\{\frac{[0.1,0.3]}{\ell_1}\right\}\right), \left(\frac{0.3}{\tau_{11}}, \left\{\frac{[0.1,0.3]}{\ell_1}, \frac{[0.1,0.4]}{\ell_5}\right\}\right), \left(\frac{0.1}{\tau_{12}}, \left\{\frac{[0.2,0.3]}{\ell_2}\right\}\right), \\ \left(\frac{0.4}{\tau_{13}}, \left\{\frac{[0.2,0.6]}{\ell_3}\right\}\right), \left(\frac{0.1}{\tau_{14}}, \left\{\frac{[0.1,0.4]}{\ell_1}\right\}\right), \left(\frac{0.2}{\tau_{15}}, \left\{\frac{[0.1,0.5]}{\ell_1}, \frac{[0.3,0.7]}{\ell_2}\right\}\right), \left(\frac{0.3}{\tau_{16}}, \left\{\frac{[0.2,0.5]}{\ell_1}, \frac{[0.1,0.4]}{\ell_4}\right\}\right) \end{array} \right\}.$$

Definition 7.13. $(\mho_{fp-ivifhs}, \eth)$ is called fuzzy parameterized interval-valued intuitionistic fuzzy hypersoft set ($\mathcal{FPIVIFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{fp-ivifhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\mathcal{B}_{\mathcal{M}}(\varsigma)}{\varsigma}, \frac{\{\mathbb{B}_{\mathbb{M}}(\varsigma); \mathbb{B}_{\mathbb{N}}(\varsigma)\}}{z} \right) : \varsigma \in \eth, \\ \mathcal{B}_{\mathcal{M}}(\varsigma) \in I, \mathbb{B}_{\mathbb{M}}(\varsigma), \mathbb{B}_{\mathbb{N}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{B}_{\mathcal{M}}$ is truth-ness of membership function $\mathcal{B} : \eth \to I$ and $\mathbb{B}_{\mathbb{M}}$ and $\mathbb{B}_{\mathbb{N}}$ are truth-ness and false-ness of approximate function $\mathbb{B} : \eth \to P(\mathbb{U})$ of $\mathcal{FPIVIFHS}$-set with condition that the sum of upper bonds of $\mathbb{B}_{\mathbb{M}}$ and $\mathbb{B}_{\mathbb{N}}$ remain less than or equal to 1.

Example 7.14. Consider Example 7.2. The $\mathcal{FPIVIFHS}$-set $(\mho_{fp-ivifhs}, \eth)$ is given by
$(\mho_{fp-ivifhs}, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{0.1}{\tau_1}, \left\{\frac{[0.5,0.7];[0.1,0.2]}{\ell_1}, \frac{[0.2,0.5];[0.2,0.3}{\ell_3}\right\}\right), \left(\frac{0.3}{\tau_2}, \left\{\frac{[0.1,0.4];[0.2,0,4]}{\ell_2}\right\}\right), \left(\frac{0.4}{\tau_3}, \left\{\frac{[0.2,0.5];[0.3,0.4]}{\ell_4}\right\}\right), \\ \left(\frac{0.1}{\tau_4}, \left\{\frac{[0.2,0.3];[0.5,0.6]}{\ell_3}, \frac{[o.1,0.4];[0.3,0.5]}{\ell_4}, \frac{[0.1,0.2];[0.3,0.5]}{\ell_5}\right\}\right), \left(\frac{0.7}{\tau_5}, \left\{\frac{[0.1,0.2];[0.3,0.5]}{\ell_1}, \frac{[0.1,0.2];[0.4,0.7]}{\ell_2}\right\}\right), \\ \left(\frac{0.2}{\tau_6}, \left\{\frac{[0.2,0.3];[0.5,0.6]}{\ell_4}\right\}\right), \left(\frac{0.1}{\tau_7}, \left\{\frac{[0.2,0.3];[0.3,0.4]}{\ell_1}, \frac{[0.1,0.3];[0.1,0.2]}{\ell_3}, \frac{[0.1,0.2];[0.3,0.5]}{\ell_5}\right\}\right), \\ \left(\frac{0.2}{\tau_8}, \left\{\frac{[0.1,0.2];[0.2,0.5]}{\ell_1}\right\}\right), \left(\frac{0.4}{\tau_9}, \left\{\frac{[0.1,0.4];[0.3,0.5]}{\ell_3}\right\}\right), \left(\frac{0.3}{\tau_{10}}, \left\{\frac{[0.1,0.3];[0.2,0.5]}{\ell_1}\right\}\right), \\ \left(\frac{0.3}{\tau_{11}}, \left\{\frac{[0.1,0.4];[0.3,0.5]}{\ell_1}, \frac{[0.1,0.4];[0.4,0.5]}{\ell_5}\right\}\right), \left(\frac{0.1}{\tau_{12}}, \left\{\frac{[0.2,0.3];[0.4,0.6]}{\ell_2}\right\}\right), \left(\frac{0.4}{\tau_{13}}, \left\{\frac{[0.1,0.4];[0.2,0.6]}{\ell_3}\right\}\right), \\ \left(\frac{0.1}{\tau_{14}}, \left\{\frac{[0.2,0.4];[0.3,0.4]}{\ell_1}\right\}\right), \left(\frac{0.2}{\tau_{15}}, \left\{\frac{[0.1,0.3];[0.4,0.5]}{\ell_1}, \frac{[0.1,0.2];[0.6,0.7]}{\ell_2}\right\}\right), \\ \left(\frac{0.3}{\tau_{16}}, \left\{\frac{[0.2,0.4];[0.3,0.5]}{\ell_1}, \frac{[0.1,0.2];[0.4,0.6]}{\ell_4}\right\}\right) \end{array} \right\}.$$

Definition 7.15. $(\mho_{fp-ivnhs}, \eth)$ is called fuzzy parameterized interval-valued neutrosophic hypersoft set ($\mathcal{FPIVNHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{fp-ivnhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\mathcal{C}_{\mathcal{M}}(\varsigma)}{\varsigma}, \frac{\{\mathbb{C}_{\mathbb{M}}(\varsigma); \mathbb{C}_{\mathbb{N}}(\varsigma); \mathbb{C}_{\mathbb{I}}(\varsigma)\}}{z} \right) : \varsigma \in \eth, \mathcal{C}_{\mathcal{M}}(\varsigma) \in I, \\ \mathbb{C}_{\mathbb{M}}(\varsigma), \mathbb{C}_{\mathbb{N}}(\varsigma), \mathbb{C}_{\mathbb{I}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{C}_{\mathcal{M}}$ is truth-ness of membership function $\mathcal{C} : \eth \to I$ and $\mathbb{C}_{\mathbb{M}}, \mathbb{C}_{\mathbb{N}}$ and $\mathbb{C}_{\mathbb{I}}$ are truth-ness false-ness and indeterminacy of approximate function $\mathbb{C} : \eth \to P(\mathbb{U})$ of $\mathcal{FPIVNHS}$-set .

Example 7.16. Consider Example 7.2. The $\mathcal{FPIVNHS}$-set $(\mho_{fp-ivnhs}, \eth)$ is given by
$(\mho_{fp-ivnhs}, \eth) =$

$$\left\{ \begin{array}{l}
\left(\frac{0.1}{\tau_1}, \left\{\frac{[0.4,0.6];[0.3,0.4];[0.2,0.3]}{\ell_1}, \frac{[0.3,0.5];[0.2,0.5];[0.2,0.5]}{\ell_3}\right\}\right), \left(\frac{0.3}{\tau_2}, \left\{\frac{[0.1,0.3];[0.2,0.4];[0.3,0,4]}{\ell_2}\right\}\right), \\
\left(\frac{0.1}{\tau_4}, \left\{\frac{[0.3,0.5];[0.3,0.7];[0.5,0.6]}{\ell_3}, \frac{[o.1,0.2];[0.3,0.6];[0.3,0.5]}{\ell_4}, \frac{[0.1,0.2];[0.3,0.6];[0.2,0.5]}{\ell_5}\right\}\right), \\
\left(\frac{0.4}{\tau_3}, \left\{\frac{[0.1,0.4];[0.3,0.5];[0.3,0.6]}{\ell_4}\right\}\right), \left(\frac{0.7}{\tau_5}, \left\{\frac{[0.4,0.7];[0.3,0.4];[0.1,0.5]}{\ell_1}, \frac{[0.6,0.8];[0.1,0.3];[0.5,0.7]}{\ell_2}\right\}\right), \\
\left(\frac{0.2}{\tau_6}, \left\{\frac{[0.1,0.4];[0.1,0.3];[0.5,0.7]}{\ell_4}\right\}\right), \left(\frac{0.2}{\tau_8}, \left\{\frac{[0.1,0.3];[0.1,0.4];[0.2,0.3]}{\ell_1}\right\}\right), \\
\left(\frac{0.1}{\tau_7}, \left\{\frac{[0.2,0.6];[0.5,0.7];[0.3,0.4]}{\ell_1}, \frac{[0.3,0.6];[0.1,0.3];[0.2,0.3]}{\ell_3}, \frac{[0.4,0.8];[0.2,0.4];[0.3,0.4]}{\ell_5}\right\}\right), \\
\left(\frac{0.4}{\tau_9}, \left\{\frac{[0.2,0.4];[0.3,0.4];[0.1,0.5]}{\ell_3}\right\}\right), \left(\frac{0.3}{\tau_{10}}, \left\{\frac{[0.1,0.2];[0.3,0.5];[0.4,0.5]}{\ell_1}\right\}\right), \\
\left(\frac{0.3}{\tau_{11}}, \left\{\frac{[0.3,0.4];[0.2,0.4];[0.3,0.7]}{\ell_1}, \frac{[0.5,0.7];[0.1,0.3];[0.4,0.6]}{\ell_5}\right\}\right), \left(\frac{0.1}{\tau_{12}}, \left\{\frac{[0.1,0.3];[0.3,0.6];[0.3,0.6]}{\ell_2}\right\}\right), \\
\left(\frac{0.4}{\tau_{13}}, \left\{\frac{[0.3,0.6];[0.2,0.4];[0.4,0.6]}{\ell_3}\right\}\right), \left(\frac{0.2}{\tau_{15}}, \left\{\frac{[0.3,0.6];[0.2,0.3];[0.4,0.7]}{\ell_1}, \frac{[0.1,0.3];[0.5,0.7];[0.3,0.4]}{\ell_2}\right\}\right), \\
\left(\frac{0.1}{\tau_{14}}, \left\{\frac{[0.3,0.4];[0.3,0.7];[0.2,0.4]}{\ell_1}\right\}\right), \left(\frac{0.3}{\tau_{16}}, \left\{\frac{[0.2,0.6];[0.3,0.4];[0.1,0.2]}{\ell_1}, \frac{[0.6,0.8];[0.3,0.4];[0.2,0.4]}{\ell_4}\right\}\right)
\end{array} \right\}.$$

Definition 7.17. $(\mho_{fp-ivpfhs}, \eth)$ is called fuzzy parameterized interval-valued picture fuzzy hypersoft set ($\mathcal{FPIVPFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{fp-ivpfhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\mathcal{D}_{\mathcal{M}}(\varsigma)}{\varsigma}, \frac{\{\mathbb{D}_{\mathbb{M}}(\varsigma);\mathbb{D}_{\mathbb{N}}(\varsigma);\mathbb{D}_{\mathbb{I}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \mathcal{D}_{\mathcal{M}}(\varsigma) \in I, \\ \mathbb{D}_{\mathbb{M}}(\varsigma), \mathbb{D}_{\mathbb{N}}(\varsigma), \mathbb{D}_{\mathbb{I}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{D}_{\mathcal{M}}$ is truth-ness of membership function $\mathcal{D} : \eth \to I$ and $\mathbb{D}_{\mathbb{M}}, \mathbb{D}_{\mathbb{N}}$ and $\mathbb{D}_{\mathbb{I}}$ are truth-ness false-ness and indeterminacy of approximate function $\mathbb{D} : \eth \to P(\mathbb{U})$ of $\mathcal{FPIVPFHS}$-set with condition that the sum of upper bonds of $\mathbb{D}_{\mathbb{M}}, \mathbb{D}_{\mathbb{N}}$ and $\mathbb{D}_{\mathbb{I}}$ remain less than or equal to 1.

Example 7.18. Consider Example 7.2. The $\mathcal{FPIVPFHS}$-set $(\mho_{fp-ivpfhs}, \eth)$ is given by
$(\mho_{fp-ivpfhs}, \eth) =$

$$\left\{ \begin{array}{l}
\left(\frac{0.1}{\tau_1}, \left\{\frac{[0.2,0.3];[0.3,0.4];[0.1,0.3]}{\ell_1}, \frac{[0.1,0.2];[0.2,0.3];[0.2,0.3]}{\ell_3}\right\}\right), \left(\frac{0.3}{\tau_2}, \left\{\frac{[0.1,0.3];[0.2,0.3];[0.3,0,4]}{\ell_2}\right\}\right), \\
\left(\frac{0.1}{\tau_4}, \left\{\frac{[0.1,0.2];[0.1,0.2];[0.5,0.6]}{\ell_3}, \frac{[0.0,0.1];[0.3,0.6];[0.2,0.3]}{\ell_4}, \frac{[0.1,0.2];[0.3,0.4];[0.2,0.4]}{\ell_5}\right\}\right), \\
\left(\frac{0.4}{\tau_3}, \left\{\frac{[0.1,0.4];[0.3,0.5];[0.0,0.1]}{\ell_4}\right\}\right), \left(\frac{0.7}{\tau_5}, \left\{\frac{[0.0,0.1];[0.3,0.4];[0.1,0.5]}{\ell_1}, \frac{[0.6,0.7];[0.0,0.1];[0.1,0.2]}{\ell_2}\right\}\right), \\
\left(\frac{0.2}{\tau_6}, \left\{\frac{[0.1,0.4];[0.1,0.3];[0.0,0.3]}{\ell_4}\right\}\right), \left(\frac{0.2}{\tau_8}, \left\{\frac{[0.1,0.3];[0.1,0.4];[0.2,0.3]}{\ell_1}\right\}\right), \\
\left(\frac{0.1}{\tau_7}, \left\{\frac{[0.2,0.6];[0.1,0.3];[0.0,0.1]}{\ell_1}, \frac{[0.3,0.6];[0.0,0.1];[0.0,0.1]}{\ell_3}, \frac{[0.4,0.6];[0.0,0.2];[0.0,0.1]}{\ell_5}\right\}\right), \\
\left(\frac{0.4}{\tau_9}, \left\{\frac{[0.2,0.4];[0.3,0.4];[0.1,0.2]}{\ell_3}\right\}\right), \left(\frac{0.3}{\tau_{10}}, \left\{\frac{[0.1,0.2];[0.3,0.5];[0.0,0.2]}{\ell_1}\right\}\right), \\
\left(\frac{0.3}{\tau_{11}}, \left\{\frac{[0.3,0.4];[0.2,0.4];[0.0,0.1]}{\ell_1}, \frac{[0.5,0.7];[0.0,0.1];[0.1,0.2]}{\ell_5}\right\}\right), \left(\frac{0.1}{\tau_{12}}, \left\{\frac{[0.1,0.3];[0.3,0.6];[0.0,0.1]}{\ell_2}\right\}\right), \\
\left(\frac{0.4}{\tau_{13}}, \left\{\frac{[0.3,0.6];[0.2,0.3];[0.0,0.1]}{\ell_3}\right\}\right), \left(\frac{0.2}{\tau_{15}}, \left\{\frac{[0.3,0.5];[0.2,0.3];[0.1,0.2]}{\ell_1}, \frac{[0.1,0.3];[0.0,0.2];[0.3,0.4]}{\ell_2}\right\}\right), \\
\left(\frac{0.1}{\tau_{14}}, \left\{\frac{[0.3,0.4];[0.1,0.2];[0.2,0.4]}{\ell_1}\right\}\right), \left(\frac{0.3}{\tau_{16}}, \left\{\frac{[0.2,0.6];[0.0,0.2];[0.1,0.2]}{\ell_1}, \frac{[0.2,0.5];[0.2,0.3];[0.0,0.1]}{\ell_4}\right\}\right)
\end{array} \right\}.$$

Definition 7.19. $(\mho_{ifp-ivfhs}, \eth)$ is called intuitionistic fuzzy parameterized interval-valued fuzzy hypersoft set ($\mathcal{IFPIVFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{ifp-ivfhs}, \eth) = \left\{ \left(\frac{\{\mathcal{E}_{\mathcal{M}}(\varsigma); \mathcal{E}_{\mathcal{N}}(\varsigma)\}}{\varsigma}, \frac{\mathbb{E}_{\mathbb{M}}(\varsigma)}{z} \right) : \varsigma \in \eth, \mathcal{E}_{\mathcal{M}}(\varsigma), \mathcal{E}_{\mathcal{N}}(\varsigma) \in I, \mathbb{E}_{\mathbb{M}}(\varsigma) \in \mathcal{C}(I) \right\}$$

where $\mathcal{E}_{\mathcal{M}}, \mathcal{E}_{\mathcal{N}}$ are truth-ness and false-ness of membership function $\mathcal{E} : \eth \to I$ such that $\mathcal{E}_{\mathcal{M}} + \mathcal{E}_{\mathcal{N}} \leq 1$ and $\mathbb{E}_{\mathbb{M}}$ is truth-ness of approximate function $\mathbb{E} : \eth \to P(\mathbb{U})$ of $\mathcal{IFPIVFHS}$-set.

Example 7.20. Consider Example 7.2. The $\mathcal{IFPIVFHS}$-set $(\mho_{ifp-ivfhs}, \eth)$ is given by

$(\mho_{ifp-ivfhs}, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{\{0.5;0.1\}}{\tau_1}, \left\{\frac{[0.1,0.2]}{\ell_1}, \frac{[0.2,0.3}{\ell_3}\right\}\right), \left(\frac{\{0.6;0.3\}}{\tau_2}, \left\{\frac{[0.1,0,4]}{\ell_2}\right\}\right), \left(\frac{\{0.5;0.4\}}{\tau_3}, \left\{\frac{[0.2,0.5]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.5;0.1\}}{\tau_4}, \left\{\frac{[0.2,0.3]}{\ell_3}, \frac{[o.1,0.4]}{\ell_4}, \frac{[0.1,0.5]}{\ell_5}\right\}\right), \left(\frac{\{0.7;0.2\}}{\tau_5}, \left\{\frac{[0.3,0.6]}{\ell_1}, \frac{[0.1,0.2]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.4;0.2\}}{\tau_6}, \left\{\frac{[0.2,0.4]}{\ell_4}\right\}\right), \left(\frac{\{0.4;0.1\}}{\tau_7}, \left\{\frac{[0.2,0.3]}{\ell_1}, \frac{[0.1,0.3]}{\ell_3}, \frac{[0.1,0.5]}{\ell_5}\right\}\right), \left(\frac{\{0.5;0.2\}}{\tau_8}, \left\{\frac{[0.2,0.5]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.4;0.2\}}{\tau_9}, \left\{\frac{[0.3,0.5]}{\ell_3}\right\}\right), \left(\frac{\{0.5;0.3\}}{\tau_{10}}, \left\{\frac{[0.1,0.3]}{\ell_1}\right\}\right), \left(\frac{\{0.5;0.1\}}{\tau_{11}}, \left\{\frac{[0.1,0.3]}{\ell_1}, \frac{[0.1,0.4]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.4,0.3\}}{\tau_{12}}, \left\{\frac{[0.2,0.3]}{\ell_2}\right\}\right), \left(\frac{\{0.4;0,5\}}{\tau_{13}}, \left\{\frac{[0.2,0.6]}{\ell_3}\right\}\right), \left(\frac{\{0.6;0.2\}}{\tau_{14}}, \left\{\frac{[0.1,0.4]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.7;0.2\}}{\tau_{15}}, \left\{\frac{[0.1,0.5]}{\ell_1}, \frac{[0.3,0.7]}{\ell_2}\right\}\right), \left(\frac{\{0.5;0.3\}}{\tau_{16}}, \left\{\frac{[0.2,0.5]}{\ell_1}, \frac{[0.1,0.4]}{\ell_4}\right\}\right) \end{array} \right\}.$$

Definition 7.21. $(\mho_{ifp-ivifhs}, \eth)$ is called intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy hypersoft set ($\mathcal{IFPIVIFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{ifp-ivifhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{F}_{\mathcal{M}}(\varsigma);\mathcal{F}_{\mathcal{N}}(\varsigma)\}}{\varsigma}, \frac{\{\mathbb{F}_{\mathbb{M}}(\varsigma);\mathbb{F}_{\mathbb{N}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \\ \mathcal{F}_{\mathcal{M}}(\varsigma), \mathcal{F}_{\mathcal{N}}(\varsigma) \in I, \mathbb{F}_{\mathbb{M}}(\varsigma), \mathbb{F}_{\mathbb{N}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{F}_{\mathcal{M}}$ and $\mathcal{F}_{\mathcal{N}}$ are truth-ness and false-ness of membership function $\mathcal{F} : \eth \to I$ such that $\mathcal{F}_{\mathcal{M}} + \mathcal{F}_{\mathcal{N}} \leq 1$ and $\mathbb{F}_{\mathbb{M}}$ and $\mathbb{F}_{\mathbb{N}}$ are truth-ness and false-ness of approximate function $\mathbb{F} : \eth \to P(\mathbb{U})$ of $\mathcal{IFPIVIFHS}$-set with condition that the sum of upper bonds of $\mathbb{F}_{\mathbb{M}}$ and $\mathbb{F}_{\mathbb{N}}$ remain less than or equal to 1.

Example 7.22. Consider Example 7.2. The $\mathcal{IFPIVIFHS}$-set $(\mho_{ifp-ivifhs}, \eth)$ is given by
$(\mho_{ifp-ivifhs}, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{\{0.5;0.1\}}{\tau_1}, \left\{\frac{[0.5,0.7];[0.1,0.2]}{\ell_1}, \frac{[0.2,0.5];[0.2,0.3}{\ell_3}\right\}\right), \left(\frac{\{0.6;0.3\}}{\tau_2}, \left\{\frac{[0.1,0.4];[0.2,0,4]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.4;0.1\}}{\tau_3}, \left\{\frac{[0.2,0.5];[0.3,0.4]}{\ell_4}\right\}\right), \left(\frac{\{0.6;0.1\}}{\tau_4}, \left\{\frac{[0.2,0.3];[0.5,0.6]}{\ell_3}, \frac{[o.1,0.4];[0.3,0.5]}{\ell_4}, \frac{[0.1,0.2];[0.3,0.5]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.7;0.2\}}{\tau_5}, \left\{\frac{[0.1,0.2];[0.3,0.5]}{\ell_1}, \frac{[0.1,0.2];[0.4,0.7]}{\ell_2}\right\}\right), \left(\frac{\{0.4;0.2\}}{\tau_6}, \left\{\frac{[0.2,0.3];[0.5,0.6]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.6;0.1\}}{\tau_7}, \left\{\frac{[0.2,0.3];[0.3,0.4]}{\ell_1}, \frac{[0.1,0.3];[0.1,0.2]}{\ell_3}, \frac{[0.1,0.2];[0.3,0.5]}{\ell_5}\right\}\right), \left(\frac{\{0.3;0.2\}}{\tau_8}, \left\{\frac{[0.1,0.2];[0.2,0.5]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.5;0.4\}}{\tau_9}, \left\{\frac{[0.1,0.4];[0.3,0.5]}{\ell_3}\right\}\right), \left(\frac{\{0.5;0.3\}}{\tau_{10}}, \left\{\frac{[0.1,0.3];[0.2,0.5]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.6;0.3\}}{\tau_{11}}, \left\{\frac{[0.1,0.4];[0.3,0.5]}{\ell_1}, \frac{[0.1,0.4];[0.4,0.5]}{\ell_5}\right\}\right), \left(\frac{\{0.7;0.1\}}{\tau_{12}}, \left\{\frac{[0.2,0.3];[0.4,0.6]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.4;0.2\}}{\tau_{13}}, \left\{\frac{[0.1,0.4];[0.2,0.6]}{\ell_3}\right\}\right), \left(\frac{\{0.6;0.2\}}{\tau_{15}}, \left\{\frac{[0.1,0.3];[0.4,0.5]}{\ell_1}, \frac{[0.1,0.2];[0.6,0.7]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.5;0.1\}}{\tau_{14}}, \left\{\frac{[0.2,0.4];[0.3,0.4]}{\ell_1}\right\}\right), \left(\frac{\{0.4;0.3\}}{\tau_{16}}, \left\{\frac{[0.2,0.4];[0.3,0.5]}{\ell_1}, \frac{[0.1,0.2];[0.4,0.6]}{\ell_4}\right\}\right) \end{array} \right\}.$$

Definition 7.23. $(\mho_{ifp-ivnhs}, \eth)$ is called intuitionistic fuzzy parameterized interval-valued neutrosophic hypersoft set ($\mathcal{IFPIVNHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{ifp-ivnhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{G}_{\mathcal{M}}(\varsigma);\mathcal{G}_{\mathcal{N}}(\varsigma)\}}{\varsigma}, \frac{\{\mathbb{G}_{\mathbb{M}}(\varsigma);\mathbb{G}_{\mathbb{N}}(\varsigma);\mathbb{G}_{\mathbb{I}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \mathcal{G}_{\mathcal{M}}(\varsigma), \mathcal{G}_{\mathcal{N}}(\varsigma) \in I, \\ \mathbb{G}_{\mathbb{M}}(\varsigma), \mathbb{G}_{\mathbb{N}}(\varsigma), \mathbb{G}_{\mathbb{I}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{G}_{\mathcal{M}}$ and $\mathcal{G}_{\mathcal{N}}$ are truth-ness and false-ness of membership function $\mathcal{G} : \eth \to I$ such that $\mathcal{G}_{\mathcal{M}} + \mathcal{G}_{\mathcal{N}} \leq 1$ and $\mathbb{G}_{\mathbb{M}}, \mathbb{G}_{\mathbb{N}}$ and $\mathbb{G}_{\mathbb{I}}$ are truth-ness false-ness and indeterminacy of approximate function $\mathbb{G} : \eth \to P(\mathbb{U})$ of $\mathcal{IFPIVNHS}$-set .

Example 7.24. Consider Example 7.2. The $\mathcal{IFPIVNHS}$-set $(\mho_{ifp-ivnhs}, \eth)$ is given by

$(\mho_{ifp-ivnhs}, \eth) =$

$$\left\{\begin{array}{l}
\left(\frac{\{0.7;0.1\}}{\tau_1}, \left\{\frac{[0.4,0.6];[0.3,0.4];[0.2,0.3]}{\ell_1}, \frac{[0.3,0.5];[0.2,0.5];[0.2,0.5]}{\ell_3}\right\}\right), \left(\frac{\{0.5;0.3\}}{\tau_2}, \left\{\frac{[0.1,0.3];[0.2,0.4];[0.3,0,4]}{\ell_2}\right\}\right),\\
\left(\frac{\{0.4;0.1\}}{\tau_4}, \left\{\frac{[0.3,0.5];[0.3,0.7];[0.5,0.6]}{\ell_3}, \frac{[o.1,0.2];[0.3,0.6];[0.3,0.5]}{\ell_4}, \frac{[0.1,0.2];[0.3,0.6];[0.2,0.5]}{\ell_5}\right\}\right),\\
\left(\frac{\{0.4,0.2\}}{\tau_3}, \left\{\frac{[0.1,0.4];[0.3,0.5];[0.3,0.6]}{\ell_4}\right\}\right), \left(\frac{\{0.7;0.2\}}{\tau_5}, \left\{\frac{[0.4,0.7];[0.3,0.4];[0.1,0.5]}{\ell_1}, \frac{[0.6,0.8];[0.1,0.3];[0.5,0.7]}{\ell_2}\right\}\right),\\
\left(\frac{\{0.6;0.2\}}{\tau_6}, \left\{\frac{[0.1,0.4];[0.1,0.3];[0.5,0.7]}{\ell_4}\right\}\right), \left(\frac{\{0.3;0.2\}}{\tau_8}, \left\{\frac{[0.1,0.3];[0.1,0.4];[0.2,0.3]}{\ell_1}\right\}\right),\\
\left(\frac{\{0.4;0.1\}}{\tau_7}, \left\{\frac{[0.2,0.6];[0.5,0.7];[0.3,0.4]}{\ell_1}, \frac{[0.3,0.6];[0.1,0.3];[0.2,0.3]}{\ell_3}, \frac{[0.4,0.8];[0.2,0.4];[0.3,0.4]}{\ell_5}\right\}\right),\\
\left(\frac{\{0.5;0.4\}}{\tau_9}, \left\{\frac{[0.2,0.4];[0.3,0.4];[0.1,0.5]}{\ell_3}\right\}\right), \left(\frac{\{0.6;0.3\}}{\tau_{10}}, \left\{\frac{[0.1,0.2];[0.3,0.5];[0.4,0.5]}{\ell_1}\right\}\right),\\
\left(\frac{\{0.4;0.3\}}{\tau_{11}}, \left\{\frac{[0.3,0.4];[0.2,0.4];[0.3,0.7]}{\ell_1}, \frac{[0.5,0.7];[0.1,0.3];[0.4,0.6]}{\ell_5}\right\}\right), \left(\frac{\{0.7;0.1\}}{\tau_{12}}, \left\{\frac{[0.1,0.3];[0.3,0.6];[0.3,0.6]}{\ell_2}\right\}\right),\\
\left(\frac{\{0.4,0.1\}}{\tau_{13}}, \left\{\frac{[0.3,0.6];[0.2,0.4];[0.4,0.6]}{\ell_3}\right\}\right), \left(\frac{\{0.5;0.2\}}{\tau_{15}}, \left\{\frac{[0.3,0.6];[0.2,0.3];[0.4,0.7]}{\ell_1}, \frac{[0.1,0.3];[0.5,0.7];[0.3,0.4]}{\ell_2}\right\}\right),\\
\left(\frac{\{0.6;0.1\}}{\tau_{14}}, \left\{\frac{[0.3,0.4];[0.3,0.7];[0.2,0.4]}{\ell_1}\right\}\right), \left(\frac{\{0.4;0.3\}}{\tau_{16}}, \left\{\frac{[0.2,0.6];[0.3,0.4];[0.1,0.2]}{\ell_1}, \frac{[0.6,0.8];[0.3,0.4];[0.2,0.4]}{\ell_4}\right\}\right)
\end{array}\right\}.$$

Definition 7.25. $(\mho_{ifp-ivpfhs}, \eth)$ is called intuitionistic fuzzy parameterized interval-valued picture fuzzy hypersoft set ($\mathcal{IFPIVPFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{ifp-ivpfhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{H}_{\mathcal{M}}(\varsigma);\mathcal{H}_{\mathcal{N}}(\varsigma)\}}{\varsigma}, \frac{\{\mathbb{H}_{\mathbb{M}}(\varsigma);\mathbb{H}_{\mathbb{N}}(\varsigma);\mathbb{H}_{\mathbb{I}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \mathcal{H}_{\mathcal{M}}(\varsigma), \mathcal{H}_{\mathcal{N}}(\varsigma) \in I, \\ \mathbb{H}_{\mathbb{M}}(\varsigma), \mathbb{H}_{\mathbb{N}}(\varsigma), \mathbb{H}_{\mathbb{I}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{H}_{\mathcal{M}}$ and $\mathcal{H}_{\mathcal{N}}$ such that $\mathcal{H}_{\mathcal{M}} + \mathcal{H}_{\mathcal{N}} \leq 1$are truth-ness and false-ness of membership function $\mathcal{H} : \eth \to I$ and $\mathbb{H}_{\mathbb{M}}, \mathbb{H}_{\mathbb{N}}$ and $\mathbb{H}_{\mathbb{I}}$ are truth-ness false-ness and indeterminacy of approximate function $\mathbb{H} : \eth \to P(\mathbb{U})$ of $\mathcal{IFPIVPFHS}$-set with condition that the sum of upper bonds of $\mathbb{H}_{\mathbb{M}}, \mathbb{H}_{\mathbb{N}}$ and $\mathbb{H}_{\mathbb{I}}$ remain less than or equal to 1.

Example 7.26. Consider Example 7.2. The $\mathcal{IFPIVPFHS}$-set $(\mho_{ifp-ivpfhs}, \eth)$ is given by
$(\mho_{ifp-ivpfhs}, \eth) =$

$$\left\{\begin{array}{l}
\left(\frac{\{0.7;0.1\}}{\tau_1}, \left\{\frac{[0.4,0.5];[0.3,0.4];[0.0,0.1]}{\ell_1}, \frac{[0.3,0.5];[0.2,0.3];[0.0,0.1]}{\ell_3}\right\}\right), \left(\frac{\{0.5;0.3\}}{\tau_2}, \left\{\frac{[0.1,0.3];[0.2,0.4];[0.0,0.1]}{\ell_2}\right\}\right),\\
\left(\frac{\{0.4;0.1\}}{\tau_4}, \left\{\frac{[0.3,0.5];[0.3,0.4];[0.0,0.1]}{\ell_3}, \frac{[o.1,0.2];[0.3,0.6];[0.1,0.2]}{\ell_4}, \frac{[0.1,0.2];[0.3,0.6];[0.0,0.1]}{\ell_5}\right\}\right),\\
\left(\frac{\{0.4,0.2\}}{\tau_3}, \left\{\frac{[0.1,0.4];[0.3,0.5];[0.0,0.1]}{\ell_4}\right\}\right), \left(\frac{\{0.7;0.2\}}{\tau_5}, \left\{\frac{[0.4,0.5];[0.3,0.4];[0.0,0.1]}{\ell_1}, \frac{[0.6,0.8];[0.0,0.1];[0.0,0.1]}{\ell_2}\right\}\right),\\
\left(\frac{\{0.6;0.2\}}{\tau_6}, \left\{\frac{[0.1,0.4];[0.1,0.3];[0.1,0.2]}{\ell_4}\right\}\right), \left(\frac{\{0.3;0.2\}}{\tau_8}, \left\{\frac{[0.1,0.3];[0.1,0.4];[0.1,0.2]}{\ell_1}\right\}\right),\\
\left(\frac{\{0.4;0.1\}}{\tau_7}, \left\{\frac{[0.2,0.3];[0.5,0.6];[0.0,0.1]}{\ell_1}, \frac{[0.1,0.3];[0.1,0.3];[0.2,0.3]}{\ell_3}, \frac{[0.4,0.5];[0.2,0.3];[0.1,0.2]}{\ell_5}\right\}\right),\\
\left(\frac{\{0.5;0.4\}}{\tau_9}, \left\{\frac{[0.2,0.4];[0.3,0.4];[0.1,0.2]}{\ell_3}\right\}\right), \left(\frac{\{0.6;0.3\}}{\tau_{10}}, \left\{\frac{[0.1,0.2];[0.3,0.5];[0.2,0.3]}{\ell_1}\right\}\right),\\
\left(\frac{\{0.4;0.3\}}{\tau_{11}}, \left\{\frac{[0.3,0.4];[0.2,0.4];[0.1,0.2]}{\ell_1}, \frac{[0.5,0.7];[0.0,0.1];[0.1,0.2]}{\ell_5}\right\}\right), \left(\frac{\{0.7;0.1\}}{\tau_{12}}, \left\{\frac{[0.1,0.3];[0.3,0.6];[0.0,0.1]}{\ell_2}\right\}\right),\\
\left(\frac{\{0.4,0.1\}}{\tau_{13}}, \left\{\frac{[0.3,0.4];[0.2,0.4];[0.1,0.2]}{\ell_3}\right\}\right), \left(\frac{\{0.5;0.2\}}{\tau_{15}}, \left\{\frac{[0.3,0.5];[0.2,0.3];[0.0,0.1]}{\ell_1}, \frac{[0.1,0.3];[0.1,0.2];[0.3,0.4]}{\ell_2}\right\}\right),\\
\left(\frac{\{0.6;0.1\}}{\tau_{14}}, \left\{\frac{[0.3,0.4];[0.0,0.2];[0.2,0.4]}{\ell_1}\right\}\right), \left(\frac{\{0.4;0.3\}}{\tau_{16}}, \left\{\frac{[0.2,0.3];[0.3,0.4];[0.1,0.2]}{\ell_1}, \frac{[0.0,0.2];[0.3,0.4];[0.2,0.4]}{\ell_4}\right\}\right)
\end{array}\right\}.$$

Definition 7.27. $(\mho_{np-ivfhs}, \eth)$ is called neutrosophic parameterized interval-valued fuzzy hypersoft set ($\mathcal{NPIVFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{np-ivfhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{J}_{\mathcal{M}}(\varsigma); \mathcal{J}_{\mathcal{N}}(\varsigma); \mathcal{J}_{\mathcal{I}}(\varsigma)\}}{\varsigma}, \frac{\mathbb{J}_{\mathbb{M}}(\varsigma)}{z}\right) : \varsigma \in \eth, \\ \mathcal{J}_{\mathcal{M}}(\varsigma), \mathcal{J}_{\mathcal{N}}(\varsigma), \mathcal{J}_{\mathcal{I}}(\varsigma) \in I, \mathbb{J}_{\mathbb{M}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{J}_{\mathcal{M}}, \mathcal{J}_{\mathcal{N}}$ and $\mathcal{J}_{\mathcal{I}}$ are truth-ness, false-ness and indeterminacy of membership function $\mathcal{J} : \eth \to I$ and $\mathbb{J}_{\mathbb{M}}$ is truth-ness of approximate function $\mathbb{J} : \eth \to P(\mathbb{U})$ of $\mathcal{NPIVFHS}$-set.

Example 7.28. Consider Example 7.2. The $\mathcal{NPIVFHS}$-set $(\mho_{np-ivfhs}, \eth)$ is given by

$(\mho_{np-ivfhs}, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{\{0.8;0.5;0.4\}}{\tau_1}, \left\{\frac{[0.1,0.2]}{\ell_1}, \frac{[0.2,0.3}{\ell_3}\right\}\right), \left(\frac{\{0.6;0.4;0.3\}}{\tau_2}, \left\{\frac{[0.1,0,4]}{\ell_2}\right\}\right), \left(\frac{\{0.7;0.5;0.4\}}{\tau_3}, \left\{\frac{[0.2,0.5]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.5;0.3;0.1\}}{\tau_4}, \left\{\frac{[0.2,0.3]}{\ell_3}, \frac{[o.1,0.4]}{\ell_4}, \frac{[0.1,0.5]}{\ell_5}\right\}\right), \left(\frac{\{0.7;0.5;0.1\}}{\tau_5}, \left\{\frac{[0.3,0.6]}{\ell_1}, \frac{[0.1,0.2]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.7;0.4;0.2\}}{\tau_6}, \left\{\frac{[0.2,0.4]}{\ell_4}\right\}\right), \left(\frac{\{0.5;0.3;0.1\}}{\tau_7}, \left\{\frac{[0.2,0.3]}{\ell_1}, \frac{[0.1,0.3]}{\ell_3}, \frac{[0.1,0.5]}{\ell_5}\right\}\right), \left(\frac{\{0.5;0.2;0.1\}}{\tau_8}, \left\{\frac{[0.2,0.5]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.7;0.4;0.2\}}{\tau_9}, \left\{\frac{[0.3,0.5]}{\ell_3}\right\}\right), \left(\frac{\{0.4;0.5;0.3\}}{\tau_{10}}, \left\{\frac{[0.1,0.3]}{\ell_1}\right\}\right), \left(\frac{\{0.5;0.5;0.1\}}{\tau_{11}}, \left\{\frac{[0.1,0.3]}{\ell_1}, \frac{[0.1,0.4]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.7;0.4;0.3\}}{\tau_{12}}, \left\{\frac{[0.2,0.3]}{\ell_2}\right\}\right), \left(\frac{\{0.4;0,5;0.2\}}{\tau_{13}}, \left\{\frac{[0.2,0.6]}{\ell_3}\right\}\right), \left(\frac{\{0.8;0.6;0.2\}}{\tau_{14}}, \left\{\frac{[0.1,0.4]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.5;0.7;0.2\}}{\tau_{15}}, \left\{\frac{[0.1,0.5]}{\ell_1}, \frac{[0.3,0.7]}{\ell_2}\right\}\right), \left(\frac{\{0.6;0.2;0.4\}}{\tau_{16}}, \left\{\frac{[0.2,0.5]}{\ell_1}, \frac{[0.1,0.4]}{\ell_4}\right\}\right) \end{array} \right\}.$$

Definition 7.29. $(\mho_{np-ivifhs}, \eth)$ is called neutrosophic parameterized interval-valued intuitionistic fuzzy hypersoft set ($\mathcal{NPIVIFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{np-ivifhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{K}_{\mathcal{M}}(\varsigma);\mathcal{K}_{\mathcal{N}}(\varsigma);\mathcal{K}_{\mathcal{I}}(\varsigma)\}}{\varsigma}, \frac{\{\mathbb{K}_{\mathbb{M}}(\varsigma),\mathcal{K}_{\mathcal{N}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \\ \mathcal{K}_{\mathcal{M}}(\varsigma), \mathcal{K}_{\mathcal{N}}(\varsigma), \mathcal{K}_{\mathcal{I}}(\varsigma) \in I, \mathbb{K}_{\mathbb{M}}(\varsigma), \mathbb{K}_{\mathbb{N}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{K}_{\mathcal{M}}, \mathcal{K}_{\mathcal{N}}$ and $\mathcal{K}_{\mathcal{I}}$ are truth-ness, false-ness and indeterminacy of membership function $\mathcal{K} : \eth \to I$ and $\mathbb{K}_{\mathbb{M}}$ and $\mathbb{K}_{\mathbb{N}}$ are truth-ness and false-ness of approximate function $\mathbb{K} : \eth \to P(\mathbb{U})$ of $\mathcal{NPIVIFHS}$-set with condition that the sum of upper bonds of $\mathbb{K}_{\mathbb{M}}$ and $\mathbb{K}_{\mathbb{N}}$ remain less than or equal to 1.

Example 7.30. Consider Example 7.2. The $\mathcal{NPIVIFHS}$-set $(\mho_{ifp-ivifhs}, \eth)$ is given by
$(\mho_{ifp-ivifhs}, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{\{0.8;0.5;0.1\}}{\tau_1}, \left\{\frac{[0.5,0.7];[0.1,0.2]}{\ell_1}, \frac{[0.2,0.5];[0.2,0.3}{\ell_3}\right\}\right), \left(\frac{\{0.6;0.3;0.1\}}{\tau_2}, \left\{\frac{[0.1,0.4];[0.2,0,4]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.6;0.4;0.1\}}{\tau_3}, \left\{\frac{[0.2,0.5];[0.3,0.4]}{\ell_4}\right\}\right), \left(\frac{\{0.6;0.3;0.1\}}{\tau_4}, \left\{\frac{[0.2,0.3];[0.5,0.6]}{\ell_3}, \frac{[o.1,0.4];[0.3,0.5]}{\ell_4}, \frac{[0.1,0.2];[0.3,0.5]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.8;0.7;0.2\}}{\tau_5}, \left\{\frac{[0.1,0.2];[0.3,0.5]}{\ell_1}, \frac{[0.1,0.2];[0.4,0.7]}{\ell_2}\right\}\right), \left(\frac{\{0.6;0.3;0.2\}}{\tau_6}, \left\{\frac{[0.2,0.3];[0.5,0.6]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.6;0.2;0.1\}}{\tau_7}, \left\{\frac{[0.2,0.3];[0.3,0.4]}{\ell_1}, \frac{[0.1,0.3];[0.1,0.2]}{\ell_3}, \frac{[0.1,0.2];[0.3,0.5]}{\ell_5}\right\}\right), \left(\frac{\{0.7;0.3;0.2\}}{\tau_8}, \left\{\frac{[0.1,0.2];[0.2,0.5]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.5;0.4;0.5\}}{\tau_9}, \left\{\frac{[0.1,0.4];[0.3,0.5]}{\ell_3}\right\}\right), \left(\frac{\{0.7;0.2;0.3\}}{\tau_{10}}, \left\{\frac{[0.1,0.3];[0.2,0.5]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.4;0.6;0.3\}}{\tau_{11}}, \left\{\frac{[0.1,0.4];[0.3,0.5]}{\ell_1}, \frac{[0.1,0.4];[0.4,0.5]}{\ell_5}\right\}\right), \left(\frac{\{0.7;0.3;0.1\}}{\tau_{12}}, \left\{\frac{[0.2,0.3];[0.4,0.6]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.5;0.4;0.2\}}{\tau_{13}}, \left\{\frac{[0.1,0.4];[0.2,0.6]}{\ell_3}\right\}\right), \left(\frac{\{0.6;0.2;0.2\}}{\tau_{15}}, \left\{\frac{[0.1,0.3];[0.4,0.5]}{\ell_1}, \frac{[0.1,0.2];[0.6,0.7]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.4;0.5;0.1\}}{\tau_{14}}, \left\{\frac{[0.2,0.4];[0.3,0.4]}{\ell_1}\right\}\right), \left(\frac{\{0.8;0.4;0.3\}}{\tau_{16}}, \left\{\frac{[0.2,0.4];[0.3,0.5]}{\ell_1}, \frac{[0.1,0.2];[0.4,0.6]}{\ell_4}\right\}\right) \end{array} \right\}.$$

Definition 7.31. $(\mho_{np-ivnhs}, \eth)$ is called neutrosophic parameterized interval-valued neutrosophic hypersoft set ($\mathcal{NPIVNHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{np-ivnhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{L}_{\mathcal{M}}(\varsigma);\mathcal{L}_{\mathcal{N}}(\varsigma);\mathcal{L}_{\mathcal{I}}(\varsigma)\}}{\varsigma}, \frac{\{\mathbb{L}_{\mathbb{M}}(\varsigma);\mathbb{L}_{\mathbb{N}}(\varsigma);\mathbb{L}_{\mathbb{I}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \\ \mathcal{L}_{\mathcal{M}}(\varsigma), \mathcal{L}_{\mathcal{N}}(\varsigma), \mathcal{L}_{\mathcal{I}}(\varsigma) \in I, \mathbb{L}_{\mathbb{M}}(\varsigma), \mathbb{L}_{\mathbb{N}}(\varsigma), \mathbb{L}_{\mathbb{I}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{L}_{\mathcal{M}}, \mathcal{L}_{\mathcal{N}}$ and $\mathcal{L}_{\mathcal{I}}$ are truth-ness, false-ness and indeterminacy of membership function $\mathcal{L} : \eth \to I$ and $\mathbb{L}_{\mathbb{M}}, \mathbb{L}_{\mathbb{N}}$ and $\mathbb{L}_{\mathbb{I}}$ are truth-ness false-ness and indeterminacy of approximate function $\mathbb{L} : \eth \to P(\mathbb{U})$ of $\mathcal{NPIVNHS}$-set .

Example 7.32. Consider Example 7.2. The $\mathcal{NPIVNHS}$-set $(\mho_{np-ivnhs}, \eth)$ is given by

$(\mho_{np-ivnhs}, \eth) =$

$$\left\{ \begin{array}{l}
\left(\frac{\{0.5;0.7;0.1\}}{\tau_1}, \left\{\frac{[0.4,0.6];[0.3,0.4];[0.2,0.3]}{\ell_1}, \frac{[0.3,0.5];[0.2,0.5];[0.2,0.5]}{\ell_3}\right\}\right), \\
\left(\frac{\{0.8;0.5;0.1\}}{\tau_2}, \left\{\frac{[0.1,0.3];[0.2,0.4];[0.3,0.4]}{\ell_2}\right\}\right), \left(\frac{\{0.7;0.4,0.2\}}{\tau_3}, \left\{\frac{[0.1,0.4];[0.3,0.5];[0.3,0.6]}{\ell_4}\right\}\right), \\
\left(\frac{\{0.5;0.4;0.1\}}{\tau_4}, \left\{\frac{[0.3,0.5];[0.3,0.7];[0.5,0.6]}{\ell_3}, \frac{[o.1,0.2];[0.3,0.6];[0.3,0.5]}{\ell_4}, \frac{[0.1,0.2];[0.3,0.6];[0.2,0.5]}{\ell_5}\right\}\right), \\
\left(\frac{\{0.7;0.3;0.1\}}{\tau_5}, \left\{\frac{[0.4,0.7];[0.3,0.4];[0.1,0.5]}{\ell_1}, \frac{[0.6,0.8];[0.1,0.3];[0.5,0.7]}{\ell_2}\right\}\right), \\
\left(\frac{\{0.8;0.6;0.2\}}{\tau_6}, \left\{\frac{[0.1,0.4];[0.1,0.3];[0.5,0.7]}{\ell_4}\right\}\right), \left(\frac{\{0.5;0.3;0.2\}}{\tau_8}, \left\{\frac{[0.1,0.3];[0.1,0.4];[0.2,0.3]}{\ell_1}\right\}\right), \\
\left(\frac{\{0.5;0.4;0.1\}}{\tau_7}, \left\{\frac{[0.2,0.6];[0.5,0.7];[0.3,0.4]}{\ell_1}, \frac{[0.3,0.6];[0.1,0.3];[0.2,0.3]}{\ell_3}, \frac{[0.4,0.8];[0.2,0.4];[0.3,0.4]}{\ell_5}\right\}\right), \\
\left(\frac{\{0.4;0.5;0.4\}}{\tau_9}, \left\{\frac{[0.2,0.4];[0.3,0.4];[0.1,0.5]}{\ell_3}\right\}\right), \left(\frac{\{0.7;0.4;0.3\}}{\tau_{10}}, \left\{\frac{[0.1,0.2];[0.3,0.5];[0.4,0.5]}{\ell_1}\right\}\right), \\
\left(\frac{\{0.6;0.4;0.3\}}{\tau_{11}}, \left\{\frac{[0.3,0.4];[0.2,0.4];[0.3,0.7]}{\ell_1}, \frac{[0.5,0.7];[0.1,0.3];[0.4,0.6]}{\ell_5}\right\}\right), \\
\left(\frac{\{0.7;0.3;0.2\}}{\tau_{12}}, \left\{\frac{[0.1,0.3];[0.3,0.6];[0.3,0.6]}{\ell_2}\right\}\right), \left(\frac{\{0.8;0.4,0.3\}}{\tau_{13}}, \left\{\frac{[0.3,0.6];[0.2,0.4];[0.4,0.6]}{\ell_3}\right\}\right), \\
\left(\frac{\{0.5;0.7;0.2\}}{\tau_{15}}, \left\{\frac{[0.3,0.6];[0.2,0.3];[0.4,0.7]}{\ell_1}, \frac{[0.1,0.3];[0.5,0.7];[0.3,0.4]}{\ell_2}\right\}\right), \\
\left(\frac{\{0.7;0.4;0.3\}}{\tau_{16}}, \left\{\frac{[0.2,0.6];[0.3,0.4];[0.1,0.2]}{\ell_1}, \frac{[0.6,0.8];[0.3,0.4];[0.2,0.4]}{\ell_4}\right\}\right), \\
\left(\frac{\{0.4;0.3;0.1\}}{\tau_{14}}, \left\{\frac{[0.3,0.4];[0.3,0.7];[0.2,0.4]}{\ell_1}\right\}\right)
\end{array} \right\}.$$

Definition 7.33. $(\mho_{np-ivpfhs}, \eth)$ is called neutrosophic parameterized interval-valued picture fuzzy hypersoft set ($\mathcal{NPIVPFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{np-ivpfhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{M}_{\mathcal{M}}(\varsigma);\mathcal{M}_{\mathcal{N}}(\varsigma);\mathcal{M}_{\mathcal{I}}(\varsigma)\}}{\varsigma}, \frac{\{\mathbb{M}_{\mathbb{M}}(\varsigma);\mathbb{M}_{\mathbb{N}}(\varsigma);\mathbb{M}_{\mathbb{I}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \\ \mathcal{M}_{\mathcal{M}}(\varsigma), \mathcal{M}_{\mathcal{N}}(\varsigma), \mathcal{M}_{\mathcal{I}}(\varsigma) \in I, \mathbb{M}_{\mathbb{M}}(\varsigma), \mathbb{M}_{\mathbb{N}}(\varsigma), \mathbb{M}_{\mathbb{I}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{M}_{\mathcal{M}}, \mathcal{M}_{\mathcal{N}}$ and $\mathcal{M}_{\mathcal{I}}$ are truth-ness, false-ness and indeterminacy of membership function $\mathcal{M} : \eth \to I$ and $\mathbb{M}_{\mathbb{M}}, \mathbb{M}_{\mathbb{N}}$ and $\mathbb{M}_{\mathbb{I}}$ are truth-ness false-ness and indeterminacy of approximate function $\mathbb{M} : \eth \to P(\mathbb{U})$ of $\mathcal{NPIVPFHS}$-set with condition that the sum of upper bonds of $\mathbb{M}_{\mathbb{M}}, \mathbb{M}_{\mathbb{N}}$ and $\mathbb{M}_{\mathbb{I}}$ remain less than or equal to 1.

Example 7.34. Consider Example 7.2. The $\mathcal{NPIVPFHS}$-set $(\mho_{np-ivpfhs}, \eth)$ is given by
$(\mho_{np-ivpfhs}, \eth) =$

$$\left\{ \begin{array}{l}
\left(\frac{\{0.5;0.7;0.1\}}{\tau_1}, \left\{\frac{[0.4,0.5];[0.3,0.4];[0.0,0.1]}{\ell_1}, \frac{[0.3,0.5];[0.2,0.3];[0.1,0.2]}{\ell_3}\right\}\right), \\
\left(\frac{\{0.8;0.5;0.1\}}{\tau_2}, \left\{\frac{[0.1,0.3];[0.2,0.4];[0.2,0.3]}{\ell_2}\right\}\right), \left(\frac{\{0.7;0.4,0.2\}}{\tau_3}, \left\{\frac{[0.1,0.4];[0.3,0.5];[0.0,0.1]}{\ell_4}\right\}\right), \\
\left(\frac{\{0.5;0.4;0.1\}}{\tau_4}, \left\{\frac{[0.3,0.5];[0.3,0.4];[0.0,0.1]}{\ell_3}, \frac{[o.1,0.2];[0.3,0.6];[0.1,0.2]}{\ell_4}, \frac{[0.1,0.2];[0.3,0.6];[0.0,0.1]}{\ell_5}\right\}\right), \\
\left(\frac{\{0.7;0.3;0.1\}}{\tau_5}, \left\{\frac{[0.4,0.5];[0.3,0.4];[0.0,0.1]}{\ell_1}, \frac{[0.6,0.7];[0.1,0.2];[0.0,0.1]}{\ell_2}\right\}\right), \\
\left(\frac{\{0.8;0.6;0.2\}}{\tau_6}, \left\{\frac{[0.1,0.4];[0.1,0.3];[0.1,0.3]}{\ell_4}\right\}\right), \left(\frac{\{0.5;0.3;0.2\}}{\tau_8}, \left\{\frac{[0.1,0.3];[0.1,0.4];[0.2,0.3]}{\ell_1}\right\}\right), \\
\left(\frac{\{0.5;0.4;0.1\}}{\tau_7}, \left\{\frac{[0.2,0.3];[0.5,0.6];[0.0,0.1]}{\ell_1}, \frac{[0.3,0.6];[0.1,0.3];[0.0,0.1]}{\ell_3}, \frac{[0.4,0.5];[0.2,0.3];[0.1,0.2]}{\ell_5}\right\}\right), \\
\left(\frac{\{0.4;0.5;0.4\}}{\tau_9}, \left\{\frac{[0.2,0.4];[0.3,0.4];[0.1,0.2]}{\ell_3}\right\}\right), \left(\frac{\{0.7;0.4;0.3\}}{\tau_{10}}, \left\{\frac{[0.1,0.2];[0.3,0.4];[0.2,0.3]}{\ell_1}\right\}\right), \\
\left(\frac{\{0.6;0.4;0.3\}}{\tau_{11}}, \left\{\frac{[0.3,0.4];[0.2,0.3];[0.1,0.3]}{\ell_1}, \frac{[0.5,0.6];[0.1,0.2];[0.1,0.2]}{\ell_5}\right\}\right), \\
\left(\frac{\{0.7;0.3;0.2\}}{\tau_{12}}, \left\{\frac{[0.1,0.3];[0.3,0.4];[0.1,0.3]}{\ell_2}\right\}\right), \left(\frac{\{0.8;0.4,0.3\}}{\tau_{13}}, \left\{\frac{[0.3,0.4];[0.2,0.3];[0.0,0.2]}{\ell_3}\right\}\right), \\
\left(\frac{\{0.5;0.7;0.2\}}{\tau_{15}}, \left\{\frac{[0.3,0.6];[0.2,0.3];[0.0,0.1]}{\ell_1}, \frac{[0.1,0.3];[0.5,0.6];[0.0,0.1]}{\ell_2}\right\}\right), \\
\left(\frac{\{0.7;0.4;0.3\}}{\tau_{16}}, \left\{\frac{[0.2,0.3];[0.3,0.4];[0.1,0.2]}{\ell_1}, \frac{[0.6,0.8];[0.0,0.1];[0.0,0.1]}{\ell_4}\right\}\right), \\
\left(\frac{\{0.4;0.3;0.1\}}{\tau_{14}}, \left\{\frac{[0.3,0.4];[0.3,0.4];[0.1,0.2]}{\ell_1}\right\}\right)
\end{array} \right\}.$$

Definition 7.35. $(\mho_{pfp-ivfhs}, \eth)$ is called picture fuzzy parameterized interval-valued fuzzy hypersoft set ($\mathcal{PFPIVFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{pfp-ivfhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{N}_{\mathcal{M}}(\varsigma);\mathcal{N}_{\mathcal{N}}(\varsigma);\mathcal{N}_{\mathcal{I}}(\varsigma)\}}{\varsigma}, \frac{\{\mathbb{N}_{\mathbb{M}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \\ \mathcal{N}_{\mathcal{M}}(\varsigma), \mathcal{N}_{\mathcal{N}}(\varsigma), \mathcal{N}_{\mathcal{I}}(\varsigma) \in I, \mathbb{N}_{\mathbb{M}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{N}_{\mathcal{M}}, \mathcal{N}_{\mathcal{N}}$ and $\mathcal{N}_{\mathcal{I}}$ are truth-ness, false-ness and indeterminacy of membership function $\mathcal{N} : \eth \to I$ such that $\mathcal{N}_{\mathcal{M}} + \mathcal{N}_{\mathcal{N}} + \mathcal{N}_{\mathcal{I}} \leq 1$ and $\mathbb{N}_{\mathbb{M}}$ is truth-ness of approximate function $\mathbb{N} : \eth \to P(\mathbb{U})$ of $\mathcal{PFPIVFHS}$-set.

Example 7.36. Consider Example 7.2. The $\mathcal{PFPIVFHS}$-set $(\mho_{pfp-ivfhs}, \eth)$ is given by $(\mho_{pfp-ivfhs}, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{\{0.2;0.5;0.1\}}{\tau_1}, \left\{\frac{[0.3,0.9]}{\ell_1}, \frac{[0.5,0.8]}{\ell_3}\right\}\right), \\ \left(\frac{\{0.2;0.6;0.1\}}{\tau_2}, \left\{\frac{[0.2,0.6]}{\ell_2}\right\}\right), \left(\frac{\{0.2;0.1,0.2\}}{\tau_3}, \left\{\frac{[0.3,0.7]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.2;0.7;0.1\}}{\tau_4}, \left\{\frac{[0.3,0.9]}{\ell_3}, \frac{[0.3,0.7]}{\ell_4}, \frac{[0.2,0.6]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.2;0.6;0.1\}}{\tau_5}, \left\{\frac{[0.4,0.9]}{\ell_1}, \frac{[0.3,0.4]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.3;0.4;0.1\}}{\tau_6}, \left\{\frac{[0.2,0.6]}{\ell_4}\right\}\right), \left(\frac{\{0.7;0.1;0.1\}}{\tau_8}, \left\{\frac{[0.3,0.7]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.2;0.6;0.1\}}{\tau_7}, \left\{\frac{[0.5,0.8]}{\ell_1}, \frac{[0.4,0.6]}{\ell_3}, \frac{[0.6,0.7]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.1;0.6;0.1\}}{\tau_9}, \left\{\frac{[0.1,0.4]}{\ell_3}\right\}\right), \left(\frac{\{0.3;0.2;0.3\}}{\tau_{10}}, \left\{\frac{[0.3,0.7]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.2;0.1;0.1\}}{\tau_{11}}, \left\{\frac{[0.2,o.5]}{\ell_1}, \frac{[0.1,0.7]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.3;0.4;0.1\}}{\tau_{12}}, \left\{\frac{[0.1,0.5]}{\ell_2}\right\}\right), \left(\frac{\{0.1;0.1,0.5\}}{\tau_{13}}, \left\{\frac{[0.2,0.9]}{\ell_3}\right\}\right), \\ \left(\frac{\{0.2;0.4;0.4\}}{\tau_{15}}, \left\{\frac{[0.3,0.6]}{\ell_1}, \frac{[0.2,0.8]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.2;0.5;0.1\}}{\tau_{16}}, \left\{\frac{[0.8,0.9]}{\ell_1}, \frac{[0.3,0.5]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.2;0.5;0.1\}}{\tau_{14}}, \left\{\frac{[0.5,0.8]}{\ell_1}\right\}\right) \end{array} \right\}.$$

Definition 7.37. $(\mho_{pfp-ivifhs}, \eth)$ is called picture fuzzy parameterized interval-valued intuitionistic fuzzy hypersoft set ($\mathcal{PFPIVIFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{pfp-ivifhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{O}_{\mathcal{M}}(\varsigma);\mathcal{O}_{\mathcal{N}}(\varsigma);\mathcal{O}_{\mathcal{I}}(\varsigma)\}}{\varsigma}, \frac{\{\mathbb{O}_{\mathbb{M}}(\varsigma);\mathbb{O}_{\mathbb{N}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \\ \mathcal{O}_{\mathcal{M}}(\varsigma), \mathcal{O}_{\mathcal{N}}(\varsigma), \mathcal{O}_{\mathcal{I}}(\varsigma) \in I, \mathbb{O}_{\mathbb{M}}(\varsigma), \mathbb{O}_{\mathbb{N}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{O}_{\mathcal{M}}, \mathcal{O}_{\mathcal{N}}$ and $\mathcal{O}_{\mathcal{I}}$ are truth-ness, false-ness and indeterminacy of membership function $\mathcal{O} : \eth \to I$ such that $\mathcal{O}_{\mathcal{M}} + \mathcal{O}_{\mathcal{N}} + \mathcal{O}_{\mathcal{I}} \leq 1$ and $\mathbb{O}_{\mathbb{M}}$ and $\mathbb{O}_{\mathbb{N}}$ are truth-ness and false-ness of approximate function $\mathbb{O} : \eth \to P(\mathbb{U})$ of $\mathcal{PFPIVIFHS}$-set with condition that the sum of upper bonds of $\mathbb{O}_{\mathbb{M}}$ and $\mathbb{Q}_{\mathbb{N}}$ remain less than or equal to 1.

Example 7.38. Consider Example 7.2. The $\mathcal{PFPIVIFHS}$-set $(\mho_{pfp-ivifhs}, \eth)$ is given by $(\mho_{pfp-ivifhs}, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{\{0.2;0.3;0.1\}}{\tau_1}, \left\{\frac{[0.4,0.5];[0.3,0.4]}{\ell_1}, \frac{[0.2,0.3];[0.5,0.6]}{\ell_3}\right\}\right), \\ \left(\frac{\{0.4;0.3;0.1\}}{\tau_2}, \left\{\frac{[0.2,0.4];[0.2,0,3]}{\ell_2}\right\}\right), \left(\frac{\{0.3;0.3,0.2\}}{\tau_3}, \left\{\frac{[0.4,0.5];[0.3,0.4]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.3;0.4;0.2\}}{\tau_4}, \left\{\frac{[0.3,0.6];[0.1,0.3]}{\ell_3}, \frac{[o.1,0.3];[0.3,0.4]}{\ell_4}, \frac{[0.1,0.3];[0.3,0.6]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.2;0.3;0.3\}}{\tau_5}, \left\{\frac{[0.4,0.5];[0.3,0.4]}{\ell_1}, \frac{[0.3,0.5];[0.1,0.4]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.3;0.1;0.2\}}{\tau_6}, \left\{\frac{[0.2,0.3];[0.1,0.4]}{\ell_4}\right\}\right), \left(\frac{\{0.1;0.2;0.6\}}{\tau_8}, \left\{\frac{[0.3,0.4];[0.2,0.5]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.1;0.5;0.3\}}{\tau_7}, \left\{\frac{[0.5,0.6];[0.0,0.2]}{\ell_1}, \frac{[0.1,0.4];[0.2,0.4]}{\ell_3}, \frac{[0.2,0.4];[0.1,0.5]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.4;0.2;0.3\}}{\tau_9}, \left\{\frac{[0.1,0.4];[0.1,0.3]}{\ell_3}\right\}\right), \left(\frac{\{0.2;0.1;0.3\}}{\tau_{10}}, \left\{\frac{[0.3,0.5];[0.1,0.4]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.2;0.4;0.3\}}{\tau_{11}}, \left\{\frac{[0.2,0.4];[0.1,o.5]}{\ell_1}, \frac{[0.1,0.2];[0.1,0.7]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.3;0.5;0.2\}}{\tau_{12}}, \left\{\frac{[0.1,0.3];[0.3,0.4]}{\ell_2}\right\}\right), \left(\frac{\{0.3;0.4,0.2\}}{\tau_{13}}, \left\{\frac{[0.2,0.6];[0.1,0.3]}{\ell_3}\right\}\right), \\ \left(\frac{\{0.2;0.2;0.4\}}{\tau_{15}}, \left\{\frac{[0.3,0.4];[0.2,0.4]}{\ell_1}, \frac{[0.2,0.5];[0.4,0.5]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.2;0.6;0.1\}}{\tau_{16}}, \left\{\frac{[0.2,0.3];[0.3,0.5]}{\ell_1}, \frac{[0.4,0.5];[0.3,0.4]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.2;0.1;0.1\}}{\tau_{14}}, \left\{\frac{[0.3,0.4];[0.2,0.3]}{\ell_1}\right\}\right) \end{array} \right\}.$$

Definition 7.39. $(\mho_{pfp-ivnhs}, \eth)$ is called picture fuzzy parameterized interval-valued neutrosophic hypersoft set ($\mathcal{PFPIVNHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{pfp-ivnhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{P}_{\mathcal{M}}(\varsigma);\mathcal{P}_{\mathcal{N}}(\varsigma);\mathcal{P}_{\mathcal{I}}(\varsigma)\}}{\varsigma}, \frac{\{\mathbb{P}_{\mathbb{M}}(\varsigma);\mathbb{P}_{\mathbb{N}}(\varsigma);\mathbb{P}_{\mathbb{I}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \\ \mathcal{P}_{\mathcal{M}}(\varsigma), \mathcal{P}_{\mathcal{N}}(\varsigma), \mathcal{P}_{\mathcal{I}}(\varsigma) \in I, \mathbb{P}_{\mathbb{M}}(\varsigma), \mathbb{P}_{\mathbb{N}}(\varsigma), \mathbb{P}_{\mathbb{I}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{P}_{\mathcal{M}}, \mathcal{P}_{\mathcal{N}}$ and $\mathcal{P}_{\mathcal{I}}$ are truth-ness, false-ness and indeterminacy of membership function $\mathcal{P} : \eth \to I$ such that $\mathcal{P}_{\mathcal{M}} + \mathcal{P}_{\mathcal{N}} + \mathcal{P}_{\mathcal{I}} \leq 1$ and $\mathbb{P}_{\mathbb{M}}, \mathbb{P}_{\mathbb{N}}$ and $\mathbb{P}_{\mathbb{I}}$ are truth-ness false-ness and indeterminacy of approximate function $\mathbb{P} : \eth \to P(\mathbb{U})$ of $\mathcal{PFPIVNHS}$-set.

Example 7.40. Consider Example 7.2. The $\mathcal{PFPIVNHS}$-set $(\mho_{pfp-ivnhs}, \eth)$ is given by $(\mho_{pfp-ivnhs}, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{\{0.2;0.4;0.1\}}{\tau_1}, \left\{\frac{[0.4,0.5];[0.2,0.6];[0.3,0.4]}{\ell_1}, \frac{[0.2,0.3];[0.5,0.9];[0.1,0.2]}{\ell_3}\right\}\right), \\ \left(\frac{\{0.4;0.2;0.1\}}{\tau_2}, \left\{\frac{[0.2,0.4];[0.4,0.8];[0.2,0,3]}{\ell_2}\right\}\right), \left(\frac{\{0.3;0.2,0.1\}}{\tau_3}, \left\{\frac{[0.4,0.8];[0.6,0.7];[0.3,0.5]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.3;0.2;0.1\}}{\tau_4}, \left\{\frac{[0.3,0.6];[0.3,0.8];[0.0,0.1]}{\ell_3}, \frac{[o.1,0.3];[0.3,0.7];[0.4,0.8]}{\ell_4}, \frac{[0.1,0.7];[0.3,0.6];[0.0,0.5]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.2;0.4;0.3\}}{\tau_5}, \left\{\frac{[0.4,0.5];[0.0,0.1];[0.3,0.6]}{\ell_1}, \frac{[0.3,0.7];[0.0,0.5];[0.1,0.6]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.3;0.3;0.2\}}{\tau_6}, \left\{\frac{[0.2,0.3];[0.1,0.5];[0.6,0.9]}{\ell_4}\right\}\right), \left(\frac{\{0.1;0.1;0.6\}}{\tau_8}, \left\{\frac{[0.3,0.4];[0.1,0.4];[0.2,0.6]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.1;0.2;0.1\}}{\tau_7}, \left\{\frac{[0.5,0.7];[0.3,0.6];[0.0,0.5]}{\ell_1}, \frac{[0.1,0.4];[0.3,0.8];[0.0,0.4]}{\ell_3}, \frac{[0.2,0.4];[0.4,0.9];[0.1,0.4]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.4;0.0;0.4\}}{\tau_9}, \left\{\frac{[0.1,0.5];[0.2,0.6];[0.1,0.3]}{\ell_3}\right\}\right), \left(\frac{\{0.2;0.3;0.3\}}{\tau_{10}}, \left\{\frac{[0.3,0.5];[0.2,0.4];[0.2,0.7]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.3;0.3;0.3\}}{\tau_{11}}, \left\{\frac{[0.2,0.4];[0.2,0.6];[0.1,0.9]}{\ell_1}, \frac{[0.1,0.5];[0.2,0.4];[0.1,0.7]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.2;0.1;0.2\}}{\tau_{12}}, \left\{\frac{[0.1,0.3];[0.3,0.7];[0.7,0.8]}{\ell_2}\right\}\right), \left(\frac{\{0.3;0.1,0.3\}}{\tau_{13}}, \left\{\frac{[0.3,0.6];[0.5,0.8];[0.1,0.4]}{\ell_3}\right\}\right), \\ \left(\frac{\{0.5;0.1;0.4\}}{\tau_{15}}, \left\{\frac{[0.3,0.7];[0.2,0.4];[0.4,0.6]}{\ell_1}, \frac{[0.5,0.8];[0.1,0.5];[0.6,0.7]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.1;0.7;0.1\}}{\tau_{16}}, \left\{\frac{[0.2,0.5];[0.4,0.5];[0.1,0.5]}{\ell_1}, \frac{[0.4,0.7];[0.3,0.5];[0.2,0.4]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.2;0.4;0.1\}}{\tau_{14}}, \left\{\frac{[0.3,0.6];[0.2,0.6];[0.1,0.4]}{\ell_1}\right\}\right) \end{array} \right\}.$$

Definition 7.41. $(\mho_{pfp-ivpfhs}, \eth)$ is called picture fuzzy parameterized interval-valued picture fuzzy hyper-soft set ($\mathcal{PFPIVPFHS}$-set) over $\mathbb{U}$ and is defined as

$$(\mho_{pfp-ivpfhs}, \eth) = \left\{ \begin{array}{l} \left(\frac{\{\mathcal{Q}_{\mathcal{M}}(\varsigma);\mathcal{Q}_{\mathcal{N}}(\varsigma);\mathcal{Q}_{\mathcal{I}}(\varsigma)\}}{\varsigma}, \frac{\{\mathbb{Q}_{\mathbb{M}}(\varsigma);\mathbb{Q}_{\mathbb{N}}(\varsigma);\mathbb{Q}_{\mathbb{I}}(\varsigma)\}}{z}\right) : \varsigma \in \eth, \\ \mathcal{Q}_{\mathcal{M}}(\varsigma), \mathcal{Q}_{\mathcal{N}}(\varsigma), \mathcal{Q}_{\mathcal{I}}(\varsigma) \in I, \mathbb{Q}_{\mathbb{M}}(\varsigma), \mathbb{Q}_{\mathbb{N}}(\varsigma), \mathbb{Q}_{\mathbb{I}}(\varsigma) \in \mathcal{C}(I) \end{array} \right\}$$

where $\mathcal{Q}_{\mathcal{M}}, \mathcal{Q}_{\mathcal{N}}$ and $\mathcal{Q}_{\mathcal{I}}$ are truth-ness, false-ness and indeterminacy of membership function $\mathcal{Q} : \eth \to I$ such that $\mathcal{Q}_{\mathcal{M}} + \mathcal{Q}_{\mathcal{N}} + \mathcal{Q}_{\mathcal{I}} \leq 1$ and $\mathbb{Q}_{\mathbb{M}}, \mathbb{Q}_{\mathbb{N}}$ and $\mathbb{Q}_{\mathbb{I}}$ are truth-ness false-ness and indeterminacy of approximate function $\mathbb{Q} : \eth \to P(\mathbb{U})$ of $\mathcal{PFPIVPFHS}$-set with condition that the sum of upper bonds of $\mathbb{Q}_{\mathbb{M}}, \mathbb{Q}_{\mathbb{N}}$ and $\mathbb{Q}_{\mathbb{I}}$ remain less than or equal to 1.

Example 7.42. Consider Example 7.2. The $\mathcal{PFPIVPFHS}$-set $(\mho_{pfp-ivpfhs}, \eth)$ is given by $(\mho_{pfp-ivpfhs}, \eth) =$

$$\left\{ \begin{array}{l} \left(\frac{\{0.2;0.7;0.1\}}{\tau_1}, \left\{\frac{[0.4,0.5];[0.0,0.1];[0.3,0.4]}{\ell_1}, \frac{[0.2,0.3];[0.3,0.5];[0.1,0.2]}{\ell_3}\right\}\right), \\ \left(\frac{\{0.4;0.5;0.1\}}{\tau_2}, \left\{\frac{[0.2,0.4];[0.1,0.3];[0.2,0,3]}{\ell_2}\right\}\right), \left(\frac{\{0.5;0.4,0.1\}}{\tau_3}, \left\{\frac{[0.1,0.4];[0.0,0.1];[0.3,0.5]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.5;0.4;0.1\}}{\tau_4}, \left\{\frac{[0.3,0.4];[0.3,0.5];[0.0,0.1]}{\ell_3}, \frac{[o.1,0.2];[0.3,0.4];[0.1,0.2]}{\ell_4}, \frac{[0.1,0.2];[0.3,0.5];[0.0,0.1]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.2;0.3;0.1\}}{\tau_5}, \left\{\frac{[0.4,0.5];[0.0,0.1];[0.3,0.4]}{\ell_1}, \frac{[0.6,0.7];[0.0,0.1];[0.1,0.2]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.1;0.4;0.2\}}{\tau_6}, \left\{\frac{[0.1,0.3];[0.1,0.3];[0.1,0.3]}{\ell_4}\right\}\right), \left(\frac{\{0.5;0.3;0.2\}}{\tau_8}, \left\{\frac{[0.1,0.4];[0.1,0.3];[0.2,0.3]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.5;0.2;0.1\}}{\tau_7}, \left\{\frac{[0.5,0.6];[0.2,0.3];[0.0,0.1]}{\ell_1}, \frac{[0.1,0.3];[0.3,0.5];[0.0,0.1]}{\ell_3}, \frac{[0.2,0.3];[0.4,0.5];[0.1,0.2]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.4;0.1;0.4\}}{\tau_9}, \left\{\frac{[0.3,0.4];[0.2,0.4];[0.1,0.2]}{\ell_3}\right\}\right), \left(\frac{\{0.2;0.4;0.3\}}{\tau_{10}}, \left\{\frac{[0.3,0.4];[0.1,0.2];[0.2,0.3]}{\ell_1}\right\}\right), \\ \left(\frac{\{0.3;0.1;0.3\}}{\tau_{11}}, \left\{\frac{[0.3,0.4];[0.2,0.3];[0.1,0.2]}{\ell_1}, \frac{[0.1,0.2];[0.2,0.3];[0.1,0.2]}{\ell_5}\right\}\right), \\ \left(\frac{\{0.4;0.3;0.2\}}{\tau_{12}}, \left\{\frac{[0.1,0.2];[0.3,0.4];[0.1,0.3]}{\ell_2}\right\}\right), \left(\frac{\{0.3;0.3,0.3\}}{\tau_{13}}, \left\{\frac{[0.3,0.4];[0.0,0.3];[0.1,0.2]}{\ell_3}\right\}\right), \\ \left(\frac{\{0.5;0.1;0.2\}}{\tau_{15}}, \left\{\frac{[0.3,0.4];[0.2,0.3];[0.0,0.1]}{\ell_1}, \frac{[0.5,0.6];[0.1,0.3];[0.0,0.1]}{\ell_2}\right\}\right), \\ \left(\frac{\{0.1;0.2;0.5\}}{\tau_{16}}, \left\{\frac{[0.2,0.3];[0.2,0.5];[0.1,0.2]}{\ell_1}, \frac{[0.6,0.7];[0.0,0.1];[0.0,0.1]}{\ell_4}\right\}\right), \\ \left(\frac{\{0.4;0.4;0.1\}}{\tau_{14}}, \left\{\frac{[0.3,0.4];[0.2,0.3];[0.1,0.2]}{\ell_1}\right\}\right) \end{array} \right\}.$$

8 Conclusion

Under the $IVFHS$-set environment, basic set laws, aggregation techniques, and fundamental features are described in this paper. Furthermore, possible hybrids of the FHS-set for interval type data are examined. More hybrids with cubic set, expert set, rough set etc., as well as algebraic structures like HS-matrices, HS-relations, HS-functions, HS-functional spaces, HS-topological spaces, HS-groups, HS-rings and HS-vector spaces may be developed in the future with interval settings.

Conflicts of Interest

Authors declare no conflicts of interest.

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No data is associated with this research work.

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IndetermSuperHyperSoft Sets, IndetermForestSoft Sets, and IndetermTreeSoft Sets

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Abstract

Uncertainty modeling has been explored through diverse frameworks, including Fuzzy Sets [1], Neutrosophic Sets [2,3], Rough Sets [4,5], and Plithogenic Sets [6,7]. Soft Sets [8,9] provide a parameterized perspective, giving rise to numerous extensions such as MultiSoft Sets [10], Double Framed Soft Sets [11, 12], Hypersoft Sets [13], SuperHyperSoft Sets [14], TreeSoft Sets [15], ForestSoft Sets [16], IndetermSoft Sets, and IndetermHyperSoft Sets [17].

Research on Soft Sets plays a significant role in decision science and related areas; however, investigations into IndetermSoft Sets remain at an early stage. In this paper, we propose three novel extensions—*IndetermSuperHyperSoft Sets*, *IndetermForestSoft Sets*, and *IndetermTreeSoft Sets*—providing rigorous definitions, illustrative examples, and a comprehensive study of how these structures extend and unify existing soft–set models. In particular, an *IndetermSoft Set* captures uncertainty by mapping attribute values to subsets of a universal set where indeterminacy arises either in the attributes themselves or in their corresponding images.

Keywords: Soft Set, Hypersoft set, Superhypersoft set, IndetermSoft Set, Treesoft set

1 Introduction

1.1 Soft Set Theory

Since classical set theory is not well suited to handling uncertainty, extended frameworks such as the Fuzzy Set [1] and the Soft Set [9] have been developed. A *Soft Set* (F, E) associates each parameter in a set E with a subset of a universal set U (equivalently, a mapping $F\colon E \to \mathcal{P}(U)$). This provides a flexible framework for approximating objects in U [8,9,18]. Soft Sets offer a versatile mathematical tool for modeling uncertainty, vagueness, and parameterization, serving as a simple yet effective basis for decision making, data analysis, and knowledge representation.

Owing to this versatility, numerous extensions of Soft Sets are being investigated. For instance, an *IndetermSoft Set* models uncertain relationships between parameters and subsets, permitting indeterminacy in parameters, subsets, or their mappings simultaneously [19–22]. Other well-known related notions include the *HyperSoft Set* [23, 24], the *TreeSoft Set* [25, 26], and the *SuperHyperSoft Set* [14,27,28]. These frameworks are highly useful for managing the complex uncertainties that arise in real-world systems.

1.2 Motivation and Our Contribution

Research on Soft Sets plays a significant role in decision science and related areas; however, investigations into IndetermSoft Sets remain at an early stage. In this paper, we propose three novel extensions—*IndetermSuperHyperSoft Sets*, *IndetermForestSoft Sets*, and *IndetermTreeSoft Sets*—and provide rigorous definitions, illustrative examples, and a comprehensive study of how these structures extend and unify existing soft-set models. In particular, an *IndetermSoft Set* captures uncertainty by mapping attribute values to subsets of a universal set while allowing indeterminacy to arise either in the attributes themselves, in their corresponding images, or in both.

2 Preliminaries

This section introduces the foundational concepts and definitions necessary for the discussions presented in this paper. Throughout this paper, all sets under consideration are assumed to be undirected, finite, and simple—i.e., they do not contain multiple elements representing the same entity.

2.1 Soft Set, Hypersoft set, and Superhypersoft set

A Soft Set provides a simplified approach to parameterized decision modeling by mapping attributes, or parameters, to subsets of a universal set, thereby addressing uncertainty in a straightforward manner [8,9,29]. Expanding on this idea, a Hypersoft Set facilitates enhanced multi-attribute decision analysis by mapping combinations of multiple attributes to subsets of a universal set [23, 24, 30]. Further generalizing the principles of Hypersoft Sets, SuperHypersoft Sets extend their functionality by mapping power set combinations of multiple attribute values to subsets of a universal set. This approach enables higher-dimensional decision-making and accommodates complex interrelationships among attributes [31–34].

Definition 2.1 (Soft Set)**.** [8, 9] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α.

Definition 2.2 (Hypersoft Set)**.** [24] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be attribute domains. Define $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair $(G, \mathcal{C})$, where $G : \mathcal{C} \to \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, \mathcal{C}) = \{(\gamma, G(\gamma)) \mid \gamma \in \mathcal{C}, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathcal{C}$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Example 2.3 (Real-life example of a Hypersoft Set)**.** Let the universal set of cars be

$$U = \{\text{Car1}, \text{Car2}, \text{Car3}, \text{Car4}\}.$$

Suppose there are two attribute domains:

- $\mathcal{A}_1 = \{\text{Petrol}, \text{Electric}\}$ (fuel type),
- $\mathcal{A}_2 = \{\text{Sedan}, \text{SUV}\}$ (car body type).

Then the Cartesian product of attributes is

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 = \{(\text{Petrol}, \text{Sedan}), (\text{Petrol}, \text{SUV}), (\text{Electric}, \text{Sedan}), (\text{Electric}, \text{SUV})\}.$$

Define the mapping $G : \mathcal{C} \to \mathcal{P}(U)$ as follows:

$$\begin{aligned} G(\text{Petrol}, \text{Sedan}) &= \{\text{Car1}\}, \\ G(\text{Petrol}, \text{SUV}) &= \{\text{Car2}\}, \\ G(\text{Electric}, \text{Sedan}) &= \{\text{Car3}\}, \\ G(\text{Electric}, \text{SUV}) &= \{\text{Car4}\}. \end{aligned}$$

Thus, the hypersoft set $(G, \mathcal{C})$ models the decision process by assigning subsets of cars to combinations of multiple attributes (fuel type and body type). This allows more precise classification than ordinary soft sets, which handle only single attributes at a time.

Definition 2.4 (SuperHyperSoft Set)**.** [14] Let U be a universal set, and let $\mathcal{P}(U)$ denote the power set of U. Consider n distinct attributes $a_1, a_2, \ldots, a_n$, where $n \geq 1$. Each attribute a_i is associated with a set of attribute values A_i, satisfying the property $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Define $\mathcal{P}(A_i)$ as the power set of A_i for each $i = 1, 2, \ldots, n$. Then, the Cartesian product of the power sets of attribute values is given by:

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A SuperHyperSoft Set over U is a pair $(F, \mathcal{C})$, where:

$$F : \mathcal{C} \to \mathcal{P}(U),$$

and F maps each element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$ (with $\alpha_i \in \mathcal{P}(A_i)$) to a subset $F(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq U$. Mathematically, the SuperHyperSoft Set is represented as:

$$(F, \mathcal{C}) = \{(\gamma, F(\gamma)) \mid \gamma \in \mathcal{C}, F(\gamma) \in \mathcal{P}(U)\}.$$

Here, $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$, where $\alpha_i \in \mathcal{P}(A_i)$ for $i = 1, 2, \ldots, n$, and $F(\gamma)$ corresponds to the subset of U defined by the combined attribute values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Example 2.5 (Real-life example of a SuperHyperSoft Set)**.** Consider a university admission system where the universal set is

$$U = \{\text{Yamada}, \text{Suzuki}, \text{Tanaka}, \text{Kobayashi}, \text{Sato}\},$$

the set of applicants. Suppose there are three attributes:

- $A_1 = \{\text{Science, Engineering}\}$ (field of study),
- $A_2 = \{\text{Undergraduate, Graduate}\}$ (program level),
- $A_3 = \{\text{Scholarship, Self-funded}\}$ (financial support).

For each attribute A_i, we take the power set $\mathcal{P}(A_i)$. An element of the Cartesian product

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3)$$

could be, for instance, $(\{\text{Science, Engineering}\}, \{\text{Graduate}\}, \{\text{Scholarship}\})$.

Define a mapping

$$F : \mathcal{C} \to \mathcal{P}(U),$$

such that $F(\{\text{Science, Engineering}\}, \{\text{Graduate}\}, \{\text{Scholarship}\}) = \{\text{Yamada}, \text{Tanaka}\}$, meaning that Yamada and Tanaka are graduate applicants in both Science and Engineering fields who applied for scholarships.

Thus, the SuperHyperSoft Set $(F, \mathcal{C})$ models complex admission criteria by allowing combinations of subsets of attribute values rather than single values, enabling flexible and higher-dimensional decision-making in the university admission process.

2.2 Treesoft set

A *TreeSoft Set* maps the power set of a hierarchical tree-like structure of attributes, Tree(A), to subsets of a universal set.

$$F : P(\text{Tree}(A)) \to P(H),$$

where Tree(A) consists of multi-level attributes forming a tree-like graph [15, 17, 35, 36]. It is known that concepts such as TreeSoft Sets can generalize MultiSoft Sets and related frameworks.

Definition 2.6. [19] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1,i_2,\ldots,i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as Tree(A), with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set* F is defined as a function:

$$F : P(\text{Tree}(A)) \to P(H),$$

where Tree(A) represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\text{Tree}(A))$ denotes its power set.

Example 2.7 (Real-life example of a TreeSoft Set)**.** Consider a company recruitment system where the universe is

$$U = \{\text{Yamada}, \text{Suzuki}, \text{Tanaka}, \text{Kobayashi}, \text{Sato}\},$$

the set of applicants. Let the attributes form a hierarchical tree structure:

- A_1 = Education
  - $A_{1,1}$ = Undergraduate
  - $A_{1,2}$ = Graduate
    - $A_{1,2,1}$ = Master
    - $A_{1,2,2}$ = Doctorate
- A_2 = Department
  - $A_{2,1}$ = Engineering
  - $A_{2,2}$ = Business

This hierarchical attribute structure $\mathrm{Tree}(A)$ allows us to express decisions using different levels of refinement.

For example, define

$$F(\{A_{1,2,1}, A_{2,1}\}) = \{\text{Yamada}, \text{Tanaka}\},$$

which means that Yamada and Tanaka are applicants with a Master's degree applying to the Engineering department.

Thus, the TreeSoft Set $(F, \mathrm{Tree}(A))$ models recruitment decisions by incorporating hierarchical attributes, enabling multi-level and structured evaluation of applicants.

2.3 ForestSoft Set

A *ForestSoft Set* is formed by taking a collection of TreeSoft Sets and "gluing" (uniting) them together so as to obtain a single function whose domain is the union of all tree-nodes' power sets and whose values in $P(H)$ combine the images given by the individual TreeSoft Sets [16, 37, 38].

Definition 2.8 (ForestSoft Set)**.** [16] Let U be a universe of discourse, $H \subseteq U$ be a non-empty subset, and $P(H)$ be the power set of H. Suppose we have a finite (or countable) collection of TreeSoft Sets

$$\big\{ F_t : P(\mathrm{Tree}(A^{(t)})) \to P(H) \big\}_{t \in T},$$

where each F_t is a TreeSoft Set corresponding to a tree $\mathrm{Tree}(A^{(t)})$ of attributes $A^{(t)}$.

We construct a *forest* by taking the (disjoint) union of all these trees:

$$\mathrm{Forest}\big(\{A^{(t)}\}_{t \in T}\big) \;=\; \bigsqcup_{t \in T} \mathrm{Tree}\big(A^{(t)}\big).$$

A *ForestSoft Set*, denoted by

$$\mathbf{F} : P\big(\mathrm{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; P(H),$$

is defined as the *union* of all TreeSoft Set mappings F_t. Concretely, for any element $X \in P\big(\mathrm{Forest}(\{A^{(t)}\})\big)$, we set

$$\mathbf{F}(X) \;=\; \bigcup_{\substack{t \in T \\ X \cap \mathrm{Tree}(A^{(t)}) \neq \varnothing}} F_t\big(X \cap \mathrm{Tree}(A^{(t)})\big),$$

where we only apply F_t to that portion of X belonging to the tree $\mathrm{Tree}(A^{(t)})$.

Example 2.9 (Cross-Role Candidate Assessment with a ForestSoft Set)**.** Let

$$U = \{\text{Alice, Bob, Charlie, Diana}\}, \qquad H = U.$$

We have two roles $t = 1, 2$, with attribute trees:

$$\mathrm{Tree}(A^{(1)}) : \begin{cases} \text{Technical Skills} & = \{\text{Programming, Algorithms}\}, \\ \text{Experience} & = \{\text{Junior, Senior}\}, \end{cases}$$

$$\mathrm{Tree}(A^{(2)}) : \begin{cases} \text{Creativity} & = \{\text{Sketching, Visualization}\}, \\ \text{Portfolio} & = \{\text{Online, Physical}\}. \end{cases}$$

Thus $\mathrm{Forest} = \mathrm{Tree}(A^{(1)}) \sqcup \mathrm{Tree}(A^{(2)})$ and $P(\mathrm{Forest})$ is its power set.

Define two TreeSoft Sets

$$F_1 : P\big(\mathrm{Tree}(A^{(1)})\big) \to P(H), \quad F_2 : P\big(\mathrm{Tree}(A^{(2)})\big) \to P(H)$$

by their nonempty values:

$$\begin{aligned} F_1(\{\text{Programming}\}) &= \{\text{Alice, Bob}\},\\ F_1(\{\text{Algorithms}\}) &= \{\text{Bob, Charlie}\},\\ F_1(\{\text{Programming}, \text{Senior}\}) &= \{\text{Alice}\},\\ \\ F_2(\{\text{Sketching}\}) &= \{\text{Charlie, Diana}\},\\ F_2(\{\text{Visualization}\}) &= \{\text{Alice}\},\\ F_2(\{\text{Visualization}, \text{Online}\}) &= \{\text{Diana}\}. \end{aligned}$$

The *ForestSoft Set* $\mathbf{F} : P(\text{Forest}) \to P(H)$ is defined by

$$\mathbf{F}(X) = \bigcup_{\substack{t=1,2 \\ X \cap \text{Tree}(A^{(t)}) \neq \varnothing}} F_t\big(X \cap \text{Tree}(A^{(t)})\big).$$

Concrete computation. Take

$$X = \{\text{Programming, Visualization}\}.$$

Then

$$X \cap \text{Tree}(A^{(1)}) = \{\text{Programming}\}, \quad X \cap \text{Tree}(A^{(2)}) = \{\text{Visualization}\},$$

so

$$\mathbf{F}(X) = F_1(\{\text{Programming}\}) \cup F_2(\{\text{Visualization}\}) = \{\text{Alice}, \text{Bob}\} \cup \{\text{Alice}\} = \{\text{Alice}, \text{Bob}\}.$$

Thus $\mathbf{F}(X)$ yields the candidates Alice and Bob when selecting both a programming skill and a creative visualization skill across roles.

2.4 IndetermSoft set and IndetermHyperSoft Set

An *IndetermSoft Set* models uncertainty by associating attribute values with subsets of a universal set where indeterminacy exists in attributes or images. And an *IndetermHyperSoft Set* generalizes IndetermSoft Sets to multiple distinct attributes, mapping combinations of indeterminate attribute values to subsets of a universal set.

$$F : A_1 \times A_2 \times \cdots \times A_n \to P(H),$$

where indeterminacy can occur in any A_i, $P(H)$, or $F(a_1, a_2, \ldots, a_n)$ [19–22].

Definition 2.10 (IndetermSoft set)**.** Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let A be the set of attribute values for an attribute a. A function $F : A \to P(H)$ is called an *IndetermSoft Set* if at least one of the following conditions holds:

1. A has some indeterminacy.
2. $P(H)$ has some indeterminacy.
3. There exists at least one $v \in A$ such that $F(v)$ is indeterminate (unclear, uncertain, or not unique).
4. Any two or all three of the above conditions.

An IndetermSoft Set is represented mathematically as:

$$F : A \to H(\cap, \cup, \oplus, \neg),$$

where $H(\cap, \cup, \oplus, \neg)$ represents a structure closed under the IndetermSoft operators.

Example 2.11 (IndetermSoft Set: Product Quality Assessment)**.** Let

$$U = \{\text{ProdA, ProdB, ProdC}\}, \qquad H = U,$$

and let the single attribute *Quality* have values

$$A = \{\text{High, Medium, Low}\}.$$

Define

$$F : A \longrightarrow P(H)$$

by

$$F(\text{High}) = \{\text{ProdA, ProdB}\},$$
$$F(\text{Medium}) = \{\text{ProdB, } \widetilde{\text{ProdC}}\},$$
$$F(\text{Low}) = \{\widetilde{\text{ProdA}}\}.$$

Here $\widetilde{\text{ProdC}}$ and $\widetilde{\text{ProdA}}$ denote that the inclusion of ProdC (in "Medium") and ProdA (in "Low") is *indeterminate* (e.g. conflicting user reviews).

Verification of IndetermSoft conditions:

(i) A itself is determinate, but

(ii) the codomain $P(H)$ contains ambiguous membership symbols $\widetilde{\cdot}$, and

(iii) for $v = \text{Medium}$, the set $F(v)$ is not uniquely determined.

Thus (F, A) is an IndetermSoft Set modeling uncertain product ratings.

Definition 2.12 (IndetermHyperSoft Set)**.** Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let $a_1, a_2, \ldots, a_n$ $(n \geq 1)$ be n distinct attributes, with attribute values $A_1, A_2, \ldots, A_n$, such that $A_i \cap A_j = \emptyset$ for $i \neq j$. The pair $(F, A_1 \times A_2 \times \cdots \times A_n)$, where

$$F : A_1 \times A_2 \times \cdots \times A_n \to P(H),$$

is called an *IndetermHyperSoft Set* if:

1. Any A_i or $P(H)$ exhibits indeterminacy.
2. For $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$, $F(a_1, a_2, \ldots, a_n)$ is indeterminate.

Example 2.13 (IndetermHyperSoft Set: Customer Purchase Patterns)**.** Let

$$U = \{\text{Alice, Bob, Charlie}\}, \qquad H = U.$$

Consider two attributes:

$$A_1 = \{\text{Online, InStore}\}, \quad A_2 = \{\text{Credit, Cash}\},$$

with $\mathcal{C} = A_1 \times A_2$. Define

$$F : \mathcal{C} \longrightarrow P(H)$$

by

$$F(\text{Online, Credit}) = \{\text{Alice, } \widetilde{\text{Bob}}\},$$
$$F(\text{Online, Cash}) = \{\text{Bob}\},$$
$$F(\text{InStore, Credit}) = \{\widetilde{\text{Charlie}}\},$$
$$F(\text{InStore, Cash}) = \{\text{Alice, Charlie}\}.$$

Here $\widetilde{\text{Bob}}$ and $\widetilde{\text{Charlie}}$ indicate that Bob's purchase channel "Online+Credit" and Charlie's "In-Store+Credit" are *indeterminate* (e.g. missing transaction logs).

Verification of IndetermHyperSoft conditions:

(i) Some codomain values $F(\gamma)$ contain $\widetilde{\cdot}$, showing *ambiguous* subsets.

(ii) For $\gamma = (\text{Online, Credit})$, $F(\gamma)$ is not uniquely determined.

Hence $(F, A_1 \times A_2)$ is an IndetermHyperSoft Set capturing uncertain customer behavior.

2.5 Incomplete Soft Set

The incomplete soft set is known as a special case of the IndetermSoft set, and the following definition of the incomplete soft set has been introduced [39].

Definition 2.14. [39] An *incomplete soft set* is defined as a tuple:

$$S = (U, E, f),$$

where:

- U is a non-empty set of objects, referred to as the universe.
- E is a non-empty set of parameters.
- $f : E \to \mathcal{P}(U) \cup \{*\}$, a mapping from the parameter set E to the power set of U ($\mathcal{P}(U)$) or the special symbol $*$.

Here, $*$ indicates missing or undefined information for a given parameter $e \in E$. If $f(e) = *$, the information associated with the parameter e is incomplete. Otherwise, $f(e) \subseteq U$ as in a standard soft set.

Example 2.15 (Customer Feedback with Incomplete Data)**.** Let

$$U = \{\text{Alice, Bob, Charlie}\}$$

be the set of customers of an online retailer, and let

$$E = \{\text{PriceConcern, QualityConcern, SupportConcern}\}$$

be three feedback parameters. Define

$$f : E \longrightarrow \mathcal{P}(U) \cup \{*\}$$

by

$$\begin{aligned} f(\text{PriceConcern}) &= \{\text{Alice, Charlie}\}, \\ f(\text{QualityConcern}) &= *, \\ f(\text{SupportConcern}) &= \{\text{Bob}\}. \end{aligned}$$

Here

$$S = \big(U,\, E,\, f\big)$$

is an *incomplete soft set* because for the parameter QualityConcern no feedback was collected (hence $f(\text{QualityConcern}) = *$), while the other parameters map to the subsets of customers who raised those concerns.

Theorem 2.16. *An incomplete soft set is a specific instance of an IndetermSoft set, where indeterminacy exists only in the mapping*

$$f : E \to \mathcal{P}(U) \cup \{*\}$$

, and this indeterminacy is explicitly represented by the symbol $*$, denoting missing or undefined data.

Proof. Let $S = (U, E, f)$ be an incomplete soft set, where:

- U is a non-empty universe of discourse,
- E is a non-empty set of parameters,
- $f : E \to \mathcal{P}(U) \cup \{*\}$ is a mapping such that for each $e \in E$, $f(e) \in \mathcal{P}(U)$ or $f(e) = *$, where $*$ indicates missing or undefined data.

Let $F : A \to P(H)$ define an IndetermSoft set, where:

- A is the set of attributes,
- $P(H)$ is the powerset of a non-empty subset $H \subseteq U$,
- Indeterminacy occurs in A, $P(H)$, or $F(a)$ for $a \in A$.

To prove that S is a special case of F, we demonstrate the following correspondences:

For the incomplete soft set S, the mapping $f : E \to \mathcal{P}(U) \cup \{*\}$ satisfies:

$$f(e) = \begin{cases} A_e \subseteq U, & \text{if } f(e) \text{ is defined,} \\ *, & \text{if } f(e) \text{ is undefined.} \end{cases}$$

This corresponds to the mapping $F(a)$ of an IndetermSoft set, where:

$$F(a) \in \begin{cases} P(H), & \text{if } F(a) \text{ is determinate,} \\ \text{indeterminate}, & \text{if } F(a) \text{ is undefined or unclear.} \end{cases}$$

Here, the indeterminacy in $F(a)$ for some $a \in A$ is equivalent to $f(e) = *$ in S, thus establishing that S inherits the indeterminacy of F in the restricted case of missing data.

The IndetermSoft set allows indeterminacy in A, $P(H)$, or $F(a)$. For S, the indeterminacy is confined to $f(e)$, specifically when $f(e) = *$. This restriction implies that:

$$\text{Indeterminacy in } S \implies \text{Indeterminacy in } F,$$

but the converse does not hold, as S does not allow indeterminacy in E or $\mathcal{P}(U)$.

The incomplete soft set S is a subset of IndetermSoft sets where:

$$\forall e \in E, \quad f(e) \in \mathcal{P}(U) \cup \{*\}, \quad * \text{ explicitly represents missing data.}$$

This explicit representation of missing data is a strict subset of the broader forms of indeterminacy allowed in an IndetermSoft set, making S a specialized structure.

Since the incomplete soft set S satisfies the mapping and indeterminacy conditions of an IndetermSoft set under the restricted case of missing data, S is a special case of F. □

3 Results of This Paper

In this section, we present the main results of this paper, including the definitions and concise mathematical analysis of the IndetermTreeSoft Set, IndetermForestSoft Set, and IndetermSuperHyperSoft Set.

3.1 IndetermTreeSoft Set

The IndetermTreeSoft Set integrates the concepts of TreeSoft Sets and IndetermSoft Sets into a unified framework. Its definition and related details are provided below.

Definition 3.1 (IndetermTreeSoft Set)**.** Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the power set of H. Let $\text{Tree}(A)$ be the hierarchical tree of attributes (as defined in the TreeSoft Set, with levels of sub-attributes) and $P(\text{Tree}(A))$ its power set.

A mapping

$$F \ : \ P(\text{Tree}(A)) \ \longrightarrow \ P(H)$$

is called an *IndetermTreeSoft Set* if one or more of the following conditions of indeterminacy hold:

1. The tree of attributes, Tree(A), includes some indeterminate or uncertain nodes (e.g., an attribute node has unspecified or overlapping sub-attributes).
2. The codomain $P(H)$ contains uncertain or partially unknown subsets (e.g., membership of some elements of H is not uniquely determined).
3. For at least one $X \in P(\text{Tree}(A))$, the image $F(X) \subseteq H$ itself is indeterminate or ambiguous.

Mathematically, one may allow each value $F(X)$ to be an element of a more general structure $H(\cap, \cup, \oplus, \neg)$ under partial or fuzzy membership (if desired), provided it accommodates indeterminacy.

Example 3.2 (Smartphone Selection with Indeterminacy)**.** Let

$$U = \{\text{PhoneA, PhoneB, PhoneC}\}$$

be a universe of smartphone models, and set $H = U$. Define a two-level attribute tree

$$\text{Tree}(A): \quad \begin{matrix} \text{Features} \\ \begin{cases} \text{Performance} = \{\text{CPU}, \text{RAM}\}, \\ \text{Camera} = \{\text{DayMode, NightMode}\}. \end{cases} \end{matrix}$$

Thus Tree(A) has nodes $\{\text{CPU}, \text{RAM}, \text{DayMode}, \text{NightMode}\}$, and

$$P\big(\text{Tree}(A)\big)$$

is its power set.

Define $F : P(\text{Tree}(A)) \to P(H)$ by

$$\begin{aligned} F\big(\{\text{CPU}\}\big) &= \{\text{PhoneA, PhoneB}\}, \\ F\big(\{\text{NightMode}\}\big) &= \{\text{PhoneA, } \tilde{\text{P}}\text{honeC}\}, \\ F\big(\{\text{CPU, NightMode}\}\big) &= \{\text{PhoneA}\}. \end{aligned}$$

Here $\tilde{\text{P}}$honeC denotes that PhoneC's NightMode performance is *indeterminate* (e.g. conflicting test results), so it is unclear whether PhoneC should be included.

Verification of indeterminacy conditions:

(i) *Tree-structure indeterminacy.* The attribute node NightMode is itself *uncertain*, since different reviewers report incompatible ratings.

(ii) *Codomain indeterminacy.* The codomain value $F(\{\text{NightMode}\})$ includes the symbol $\tilde{\text{P}}$honeC, indicating an *ambiguous* membership in $P(H)$.

(iii) *Image indeterminacy.* For $X = \{\text{NightMode}\}$, the set $F(X)$ is not uniquely defined because of PhoneC's unclear status.

Thus F satisfies Definition 3.1 and is a concrete IndetermTreeSoft Set modeling smartphone selection under real-world uncertainty.

Theorem 3.3 (IndetermTreeSoft Set as a Generalization of IndetermSoft Set and TreeSoft Set)**.** *Let U be a universe and $H \subseteq U$ non-empty. Let* Tree(A) *be the attribute tree and* $P(\text{Tree}(A))$ *its power set. Then:*

1. *Every IndetermSoft Set $F_0 : A \to P(H)$ can be embedded as an IndetermTreeSoft Set on the one-level tree* $\text{Tree}(A) = A$.

2. *Every TreeSoft Set* $F : P(\mathrm{Tree}(A)) \to P(H)$ *with no indeterminacy is a (fully determinate) IndetermTreeSoft Set.*

Proof. **(1) From IndetermSoft to IndetermTreeSoft.** Let $F_0 : A \to P(H)$ be an IndetermSoft Set. Define a trivial attribute tree by taking

$$\mathrm{Tree}(A) = A \quad \Longrightarrow \quad P\big(\mathrm{Tree}(A)\big) = P(A).$$

Extend F_0 to

$$F : P(A) \longrightarrow P(H), \qquad F(X) = \bigcup_{\alpha \in X} F_0(\alpha),$$

for every $X \subseteq A$. We check the IndetermTreeSoft indeterminacy conditions:

(i) *Tree-structure indeterminacy.* If the original A contains any uncertain or overlapping attribute values, then $\mathrm{Tree}(A)$ inherits that indeterminacy.

(ii) *Codomain indeterminacy.* If $P(H)$ admits fuzzy or ambiguous subsets, then for any X, $F(X) \subseteq H$ may be indeterminate.

(iii) *Image indeterminacy.* Since $F(\{\alpha\}) = F_0(\alpha)$, any α with $F_0(\alpha)$ indeterminate yields $F(\{\alpha\})$ indeterminate.

Because at least (iii) holds whenever F_0 is indeterminate, F is an IndetermTreeSoft Set.

(2) From TreeSoft to IndetermTreeSoft. Let $F : P(\mathrm{Tree}(A)) \to P(H)$ be a deterministic TreeSoft Set, so that no uncertainty arises in the domain tree, the codomain, or any image $F(X)$. The definition of IndetermTreeSoft Set requires *at least one* indeterminacy condition, but we allow the degenerate case of "zero" indeterminacy to cover fully determinate frameworks. Hence F is a valid IndetermTreeSoft Set viewed as a special case in which none of (i)–(iii) occurs.

Thus the class of IndetermTreeSoft Sets simultaneously generalizes both IndetermSoft Sets (via item 1) and TreeSoft Sets (via item 2). □

Theorem 3.4 (Closure under Union)**.** *Let*

$$F_1,\, F_2 : P\big(\mathrm{Tree}(A)\big) \longrightarrow P(H)$$

be two IndetermTreeSoft Sets on the same attribute tree $\mathrm{Tree}(A)$ *and universe* H*. Define*

$$(F_1 \vee F_2)(X) = F_1(X) \cup F_2(X), \qquad \forall\, X \in P\big(\mathrm{Tree}(A)\big).$$

Then $F_1 \vee F_2$ *is again an IndetermTreeSoft Set.*

Proof. We must check that $F = F_1 \vee F_2$ inherits at least one of the three indeterminacy conditions from Definition 3.1.

(i) Tree-structure indeterminacy. If $\mathrm{Tree}(A)$ contains uncertain or undefined nodes, this property is independent of F, hence holds for F.

(ii) Codomain indeterminacy. If $P(H)$ admits ambiguous or fuzzy subsets, then since $\cup$ of any such subsets remains within the same fuzzy/indeterminate class, $F(X) \in P(H)$ can be indeterminate whenever $F_1(X)$ or $F_2(X)$ is.

(iii) Image indeterminacy. Suppose there exists $X_0 \subseteq \mathrm{Tree}(A)$ such that at least one of $F_1(X_0)$ or $F_2(X_0)$ is indeterminate. Then

$$F(X_0) = F_1(X_0) \cup F_2(X_0)$$

is also indeterminate (the union of an indeterminate set with any other remains indeterminate).

Because at least one of (i),(ii),(iii) holds, F satisfies the definition of an IndetermTreeSoft Set. □

Theorem 3.5 (Closure under Intersection)**.** *With the same notation as above, define*

$$(F_1 \wedge F_2)(X) \;=\; F_1(X) \;\cap\; F_2(X).$$

Then $F_1 \wedge F_2$ is an IndetermTreeSoft Set.

Proof. Again let $F = F_1 \wedge F_2$. The tree-structure (i) and codomain (ii) indeterminacies are inherited as before. For (iii), if there is some X_0 with $F_1(X_0)$ or $F_2(X_0)$ indeterminate, then their intersection

$$F(X_0) = F_1(X_0) \cap F_2(X_0)$$

remains indeterminate (intersecting an indeterminate set with any set yields indeterminacy in the resulting membership). Hence F meets the indeterminacy requirement. □

Theorem 3.6 (Restriction to Subtrees)**.** *Let F be an IndetermTreeSoft Set on* Tree(A)*, and let $T' \subseteq$ Tree(A) be any connected subtree. Define the* restriction

$$F|_{T'} : P(T') \;\longrightarrow\; P(H), \qquad F|_{T'}(X) = F(X),$$

for $X \subseteq T'$. Then $F|_{T'}$ is an IndetermTreeSoft Set on T'.

Proof. The domain of $F|_{T'}$ is $P(T') \subseteq P(\text{Tree}(A))$. Any indeterminacy in the tree-structure (i) or codomain (ii) remains valid when considering a subtree. Moreover, if there exists $X_0 \subseteq T'$ so that $F(X_0)$ is indeterminate in the original definition, then $F|_{T'}(X_0) = F(X_0)$ is indeterminate as well. Thus $F|_{T'}$ satisfies the IndetermTreeSoft conditions. □

3.2 IndetermForestSoft Set

The IndetermForestSoft Set combines the concepts of IndetermSoft Sets and ForestSoft Sets. Its definitions and related details are provided below.

Definition 3.7 (IndetermForestSoft Set)**.** Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and let

$$\big\{F_t : P(\text{Tree}(A^{(t)})) \to P(H)\big\}_{t\in T}$$

be a finite or countable collection of TreeSoft Sets, each possibly containing some indeterminacy as described in Definition 3.1. Let

$$\text{Forest}\big(\{A^{(t)}\}\big) \;=\; \bigsqcup_{t\in T} \text{Tree}\big(A^{(t)}\big)$$

be the disjoint union of all attribute trees.

A mapping

$$\mathbf{F} : P\big(\text{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; P(H)$$

is called an *IndetermForestSoft Set* if for each $X \in P\big(\text{Forest}(\{A^{(t)}\})\big)$,

$$\mathbf{F}(X) \;=\; \bigcup_{\substack{t\in T \\ X\cap \text{Tree}(A^{(t)}) \neq \varnothing}} F_t\big(X \cap \text{Tree}(A^{(t)})\big),$$

and at least one of the following holds:

1. There is indeterminacy in some Tree$(A^{(t)})$ or in the way these trees are combined.
2. The codomain $P(H)$ has elements that are uncertain or ambiguous.
3. For some input X, the image $\mathbf{F}(X)$ is not uniquely defined or is subject to partial information.

In other words, an IndetermForestSoft Set unites multiple IndetermTreeSoft Sets (or TreeSoft Sets, some or all of which may have indeterminacy) into a single framework while preserving the possibility of ambiguity.

Example 3.8 (Cross-Department Candidate Shortlisting with Indeterminacy)**.** Let

$$U = \{\text{Alice, Bob, Charlie}\}, \quad H = U.$$

We have two departments $t = 1, 2$, with attribute trees:

$$\text{Tree}(A^{(1)}): \begin{cases} \text{Skills} = \{\text{Java, Python}\}, \\ \text{Experience} = \{\text{2–4yrs, 5+yrs}\}, \end{cases} \qquad \text{Tree}(A^{(2)}): \begin{cases} \text{Tools} = \{\text{SQL, R}\}, \\ \text{Certifications} = \{\text{None, Certified}\}. \end{cases}$$

Define two IndetermTreeSoft Sets

$$F_1 : P\big(\text{Tree}(A^{(1)})\big) \to P(H), \quad F_2 : P\big(\text{Tree}(A^{(2)})\big) \to P(H)$$

by the nonempty values

$$\begin{aligned} F_1(\{\text{Java}\}) &= \{\text{Alice, Bob}\}, \\ F_1(\{\text{Python}\}) &= \{\text{Bob, } \widetilde{\text{Charlie}}\}, \\ F_1(\{\text{Python, 5+yrs}\}) &= \{\text{Alice}\}, \\ F_2(\{\text{SQL}\}) &= \{\text{Alice, } \widetilde{\text{Charlie}}\}, \\ F_2(\{\text{Certified}\}) &= \{\text{Bob}\}, \\ F_2(\{\text{R, Certified}\}) &= \{\text{Alice}\}. \end{aligned}$$

Here $\widetilde{\text{Charlie}}$ indicates that Charlie's qualification is *indeterminate* under that criterion.

The forest of attributes is

$$\text{Forest}\big(\{A^{(1)}, A^{(2)}\}\big) = \text{Tree}(A^{(1)}) \bigsqcup \text{Tree}(A^{(2)}),$$

and the IndetermForestSoft Set

$$\mathbf{F} : P\big(\text{Forest}\big) \longrightarrow P(H)$$

is given by

$$\mathbf{F}(X) = \bigcup_{\substack{t=1,2 \\ X \cap \text{Tree}(A^{(t)}) \neq \varnothing}} F_t\big(X \cap \text{Tree}(A^{(t)})\big).$$

For instance, for

$$X = \{\text{Python, Certified}\},$$

we compute

$$\mathbf{F}(X) = F_1(\{\text{Python}\}) \cup F_2(\{\text{Certified}\}) = \{\text{Bob, } \widetilde{\text{Charlie}}\} \cup \{\text{Bob}\} = \{\text{Bob, } \widetilde{\text{Charlie}}\}.$$

Verification of indeterminacy:

(i) *Tree-structure indeterminacy.* In $\text{Tree}(A^{(1)})$, the node Python is uncertain for Charlie.

(ii) *Codomain indeterminacy.* The value $\mathbf{F}(X)$ contains $\widetilde{\text{Charlie}}$, an *ambiguous* membership in $P(H)$.

(iii) *Image indeterminacy.* For $X = \{\text{Python, Certified}\}$, the set $\mathbf{F}(X)$ is not uniquely determined regarding Charlie.

Hence $\mathbf{F}$ satisfies Definition 3.7 and models a real-world candidate shortlisting process across departments under uncertainty.

Theorem 3.9 (IndetermForestSoft Set as a Generalization of IndetermSoft and ForestSoft)**.** *Let U be a universe, $H \subseteq U$ non-empty, and let*

$$\big\{F_t : P(\mathrm{Tree}(A^{(t)})) \to P(H)\big\}_{t\in T}$$

be a (finite or countable) family of IndetermTreeSoft Sets. Let

$$\mathbf{F} : P\big(\bigsqcup_{t\in T} \mathrm{Tree}(A^{(t)})\big) \;\to\; P(H)$$

be their union as in Definition 3.7. Then:

1. *If each tree $\mathrm{Tree}(A^{(t)})$ has no sub-levels (so $\mathrm{Tree}(A^{(t)}) = A^{(t)}$) but retains any indeterminacy in its points or images, then $\mathbf{F}$ induces an IndetermSoft Set on the disjoint union of the attribute sets $\bigsqcup_{t\in T} A^{(t)}$.*
2. *If every F_t is fully determinate (no domain or codomain uncertainty) and the union introduces no new ambiguity, then $\mathbf{F}$ coincides with the classical ForestSoft Set obtained by uniting deterministic TreeSoft Sets.*

Proof. **(1) Reduction to IndetermSoft.** Assume each attribute tree satisfies $\mathrm{Tree}(A^{(t)}) = A^{(t)}$, a single-level set. Then

$$\mathrm{Forest}\big(\{A^{(t)}\}\big) = \bigsqcup_{t\in T} A^{(t)},$$

and

$$P\big(\mathrm{Forest}(\{A^{(t)}\})\big) = P\big(\bigsqcup_{t\in T} A^{(t)}\big).$$

Define the ground attribute set $A_{\mathrm{tot}} = \bigsqcup_{t\in T} A^{(t)}$, and a mapping

$$f : A_{\mathrm{tot}} \;\longrightarrow\; P(H), \qquad f(a) = \mathbf{F}(\{a\}).$$

Since $\mathbf{F}(\{a\})$ inherits any indeterminacy present in the corresponding F_t, at least one of the IndetermSoft conditions holds (uncertainty in the attribute a, in the image $f(a)$, or in the powerset $P(H)$). Thus (A_{tot}, f) is an IndetermSoft Set.

(2) Reduction to ForestSoft. Now suppose each F_t is a fully determinate TreeSoft Set and that no new ambiguity arises when taking the union. Then by definition of ForestSoft Set,

$$\mathbf{F}(X) = \bigcup_{t:\, X\cap \mathrm{Tree}(A^{(t)})\neq\varnothing} F_t\big(X \cap \mathrm{Tree}(A^{(t)})\big)$$

is exactly the union of deterministic values, with no indeterminacy in domain or codomain. Therefore $\mathbf{F}$ satisfies the axioms of a standard ForestSoft Set.

In both cases, $\mathbf{F}$ specializes to the corresponding classical structure, showing that IndetermForestSoft Sets indeed generalize both IndetermSoft and ForestSoft concepts. □

3.3 IndetermSuperHyperSoft Set

The IndetermSuperHyperSoft Set combines the concepts of SuperHyperSoft Sets and IndetermHyperSoft Sets. Its definition is provided below.

Definition 3.10 (IndetermSuperHyperSoft Set)**.** Let U be a universal set, and let $\{a_1, a_2, \ldots, a_n\}$ be n distinct attributes with corresponding sets of values A_i, each of which may exhibit partial or total indeterminacy. Define

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \;\times\; \mathcal{P}(A_2) \;\times\; \cdots \;\times\; \mathcal{P}(A_n),$$

which forms the domain of a *SuperHyperSoft Set* (Definition [14]).

A mapping

$$F : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U)$$

is called an *IndetermSuperHyperSoft Set* if one or more of the following indeterminacy conditions hold:

1. Some $\mathcal{P}(A_i)$ is uncertain or incompletely specified (i.e., the attribute values or their subsets are not fully determined).
2. The codomain $\mathcal{P}(U)$ contains partially unknown subsets or ambiguous memberships.
3. For at least one $\gamma \in \mathcal{C}$, the set $F(\gamma) \subseteq U$ is not uniquely defined or is subject to ambiguity.

Additionally, higher-level nesting (if each A_i itself is replaced by $\tilde{\mathcal{P}}(A_i)$ for extended hyperstructures) may introduce further levels of indeterminacy, consistent with the definition of IndetermHyperSoft Sets.

Example 3.11 (Scholarship Eligibility under Uncertain Criteria)**.** Let

$$U = \{\text{Alice, Bob, Charlie}\}, \qquad H = U.$$

Define two attributes:

$$A_1 = \{\text{High, Medium, Low}\}, \quad A_2 = \{\text{Science, Arts, Sports}\}.$$

Form the SuperHyperSoft domain

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2).$$

We define an IndetermSuperHyperSoft Set

$$F : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U)$$

by specifying several values (using $\widetilde{\text{Bob}}$ to denote that Bob's eligibility is *indeterminate*):

$$\begin{aligned} F\big(\{\text{High}\}, \{\text{Science, Arts}\}\big) &= \{\text{Alice, } \widetilde{\text{Bob}}\}, \\ F\big(\{\text{Medium, Low}\}, \{\text{Sports}\}\big) &= \{\widetilde{\text{Charlie}}\}, \\ F\big(\{\text{High, Medium}\}, \{\text{Arts, Sports}\}\big) &= \{\text{Alice, Bob}\}. \end{aligned}$$

Verification of indeterminacy conditions:

(i) *Domain indeterminacy.* The set $\{\text{Science, Arts}\} \subseteq A_2$ is *uncertain*, since the distinction between Science and Arts criteria overlaps in some interdisciplinary cases.

(ii) *Codomain indeterminacy.* The images $F(\{\text{High}\}, \{\text{Science, Arts}\})$ and $F(\{\text{Medium, Low}\}, \{\text{Sports}\})$ include $\widetilde{\text{Bob}}$ and $\widetilde{\text{Charlie}}$, indicating *ambiguous* membership.

(iii) *Image indeterminacy.* For $\gamma = (\{\text{High}\}, \{\text{Science, Arts}\})$, the exact set of eligible students is not uniquely determined because Bob's interdisciplinary record is unclear.

Thus F satisfies the conditions of an IndetermSuperHyperSoft Set, modeling a real-world scholarship committee's decision process under multiple, interdependent uncertainties.

Theorem 3.12 (IndetermSuperHyperSoft Set Generalizes SuperHyperSoft and IndetermHyperSoft)**.** *Let U be a universal set, and let A_i $(1 \le i \le n)$ be attribute value sets, each possibly indeterminate. Set*

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n)$$

and let $F : \mathcal{C} \to \mathcal{P}(U)$ be an IndetermSuperHyperSoft Set as in Definition 3.10. Then:

1. *If every $\mathcal{P}(A_i)$ is fully determined and each $F(\gamma)$ is unambiguous, then F is precisely a SuperHyperSoft Set (no indeterminacy).*
2. *If $f : A_1 \times \cdots \times A_n \to \mathcal{P}(U)$ is an IndetermHyperSoft Set (with indeterminacy in the A_i, $\mathcal{P}(U)$, or in some $f(a_1, \ldots, a_n)$), then there is an embedding*

$$\widetilde{F} : \mathcal{C} \to \mathcal{P}(U), \qquad \widetilde{F}(\alpha_1, \ldots, \alpha_n) = \bigcup_{(a_1,\ldots,a_n) \in \alpha_1 \times \cdots \times \alpha_n} f(a_1, \ldots, a_n),$$

which is an IndetermSuperHyperSoft Set extending f.

Proof. **(1) Determinate case.** Suppose each $\mathcal{P}(A_i)$ is a classical power set (no ambiguity) and that for every $\gamma \in \mathcal{C}$, $F(\gamma) \subseteq U$ is uniquely specified. Then none of the indeterminacy conditions in Definition 3.10 applies, so F meets exactly the axioms of a SuperHyperSoft Set (see [14]). Hence F is a SuperHyperSoft Set.

(2) Embedding IndetermHyperSoft into IndetermSuperHyperSoft. Let

$$f : A_1 \times \cdots \times A_n \longrightarrow \mathcal{P}(U)$$

be an IndetermHyperSoft Set, so at least one of:

- some A_i or its subsets are indeterminate,
- some $f(a_1, \ldots, a_n)$ is ambiguous,
- $\mathcal{P}(U)$ contains uncertain subsets,

holds. Define

$$\widetilde{F}(\alpha_1, \ldots, \alpha_n) = \bigcup_{(a_1,\ldots,a_n) \in \alpha_1 \times \cdots \times \alpha_n} f(a_1, \ldots, a_n), \quad (\alpha_1, \ldots, \alpha_n) \in \mathcal{C}.$$

Then:

- If any $\alpha_i \subseteq A_i$ or any $f(a_1, \ldots, a_n)$ is indeterminate, the union remains indeterminate, so $\widetilde{F}$ satisfies the corresponding condition in Definition 3.10.
- $\widetilde{F}$ extends f because whenever each $\alpha_i = \{a_i\}$, we have

$$\widetilde{F}(\{a_1\}, \ldots, \{a_n\}) = f(a_1, \ldots, a_n).$$

Thus $\widetilde{F}$ is an IndetermSuperHyperSoft Set that coincides with the given IndetermHyperSoft Set on singleton inputs, demonstrating that every IndetermHyperSoft Set embeds into the IndetermSuperHyperSoft framework. □

Theorem 3.13 (Closure under Union)**.** *Let U be a universe and $\mathcal{C} = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$. Suppose*

$$F,\ G : \mathcal{C} \longrightarrow \mathcal{P}(U)$$

are two IndetermSuperHyperSoft Sets. Define

$$(F \vee G)(\gamma) \ = \ F(\gamma) \ \cup \ G(\gamma), \qquad \forall\, \gamma \in \mathcal{C}.$$

Then $F \vee G$ is again an IndetermSuperHyperSoft Set.

Proof. We must show that $F \vee G$ satisfies at least one of the indeterminacy conditions in Definition 3.10.

(i) Domain indeterminacy. If any $\mathcal{P}(A_i)$ is uncertain, this is independent of the choice of F or G, hence remains true for $F \vee G$.

(ii) Codomain indeterminacy. If $\mathcal{P}(U)$ admits ambiguous subsets, then since $F(\gamma)$ or $G(\gamma)$ may be ambiguous, their union $F(\gamma) \cup G(\gamma)$ is likewise ambiguous.

(iii) Image indeterminacy. Suppose there exists $\gamma_0 \in \mathcal{C}$ such that $F(\gamma_0)$ is ambiguous or $G(\gamma_0)$ is ambiguous. Then

$$(F \vee G)(\gamma_0) = F(\gamma_0) \cup G(\gamma_0)$$

remains ambiguous, because the union of a non-unique set with any other preserves non-uniqueness.

At least one of (i)–(iii) holds for $F \vee G$, so by Definition 3.10, $F \vee G$ is an IndetermSuperHyperSoft Set. □

Theorem 3.14 (Closure under Intersection)**.** *Under the same hypotheses as Theorem 3.13, define*

$$(F \wedge G)(\gamma) \;=\; F(\gamma) \;\cap\; G(\gamma).$$

Then $F \wedge G$ is an IndetermSuperHyperSoft Set.

Proof. Again let $H = F \wedge G$. Conditions (i) and (ii) from Definition 3.10 carry over unchanged. For (iii), if there exists γ_0 with $F(\gamma_0)$ or $G(\gamma_0)$ ambiguous, then

$$H(\gamma_0) = F(\gamma_0) \cap G(\gamma_0)$$

remains ambiguous, since intersecting a non-unique set with any other yields non-uniqueness. Hence H satisfies at least one indeterminacy condition and is an IndetermSuperHyperSoft Set. □

Theorem 3.15 (Projection onto a Subset of Attributes)**.** *Let F be an IndetermSuperHyperSoft Set on $\mathcal{C} = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$. Fix a nonempty index set $I \subseteq \{1, \ldots, n\}$ and set*

$$\mathcal{C}_I = \prod_{i \in I} \mathcal{P}(A_i).$$

Define the projection

$$F_I : \mathcal{C}_I \;\longrightarrow\; \mathcal{P}(U), \qquad F_I\big((\alpha_i)_{i \in I}\big) = \bigcup_{\substack{(\beta_1, \ldots, \beta_n) \in \mathcal{C} \\ \beta_i = \alpha_i \;(i \in I)}} F(\beta_1, \ldots, \beta_n).$$

Then F_I is an IndetermSuperHyperSoft Set on $\mathcal{C}_I$.

Proof. We verify the indeterminacy conditions for F_I:

(i) Domain indeterminacy. Any uncertainty in the original domains $\mathcal{P}(A_i)$ for $i \in I$ persists in the projected domains $\mathcal{P}(A_i)$, so F_I inherits domain-level indeterminacy.

(ii) Codomain indeterminacy. Since $F(\gamma)$ may be ambiguous for some γ, their union over any fiber of the projection remains ambiguous in $\mathcal{P}(U)$.

(iii) Image indeterminacy. If there exists $\delta \in \mathcal{C}_I$ and some extension $\gamma \in \mathcal{C}$ with $\gamma|_I = \delta$ for which $F(\gamma)$ is ambiguous, then

$$F_I(\delta) = \bigcup_{\gamma|_I = \delta} F(\gamma)$$

is ambiguous as well (the union of a non-unique set preserves non-uniqueness).

Thus F_I satisfies at least one indeterminacy condition and is an IndetermSuperHyperSoft Set on $\mathcal{C}_I$. □

3.4 Incomplete HyperSoft Set

The Incomplete HyperSoft Set is a generalized concept derived from the Incomplete Soft Set. Its formal definition is provided below.

Definition 3.16 (Incomplete HyperSoft Set)**.** Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be m (possibly indeterminate) attribute domains. Define the Cartesian product

$$\mathcal{C} \;=\; \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

An *Incomplete HyperSoft Set* over U is defined as a tuple

$$\Big(U,\ \mathcal{A}_1,\ \mathcal{A}_2,\ \ldots,\ \mathcal{A}_m,\ G\Big),$$

where $G : \mathcal{C} \to \big(\mathcal{P}(U) \cup \{*\}\big)$. Concretely,

$$G(\gamma) \;=\; \begin{cases} S_\gamma \subseteq U, & \text{if sufficient information is available,} \\ *, & \text{if the information is missing or undefined,} \end{cases}$$

for each $\gamma \in \mathcal{C}$. Here, $*$ denotes incomplete or missing data in analogy with the incomplete soft set.

In this definition:

1. U is a non-empty universe of discourse.
2. $\mathcal{A}_i$ are attribute domains, each of which may be partially incomplete or fully defined.
3. $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$ represents all possible combinations of attribute values.
4. $G(\gamma)$ lies in $\mathcal{P}(U) \cup \{*\}$. If $G(\gamma) = *$, the mapping is incomplete for the given γ.

Example 3.17 (Clothing Purchase Data with Incomplete Records)**.** Let

$$U = \{\text{Alice, Bob, Charlie}\}$$

be the set of customers. Consider two product attributes:

$$\mathcal{A}_1 = \{\text{Red, Blue}\}, \quad \mathcal{A}_2 = \{\text{Small, Medium, Large}\}.$$

The Cartesian product

$$\mathcal{C} \;=\; \mathcal{A}_1 \times \mathcal{A}_2 \;=\; \{\,(c, s) \mid c \in \{\text{Red}, \text{Blue}\},\ s \in \{\text{Small}, \text{Medium}, \text{Large}\}\}$$

indexes all color-size variants.

Define

$$G : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U) \cup \{*\}$$

by the following table:

$$\begin{aligned} G(\text{Red}, \text{Small}) &= \{\text{Alice, Bob}\}, \\ G(\text{Red}, \text{Medium}) &= *, \\ G(\text{Red}, \text{Large}) &= \{\text{Charlie}\}, \\ G(\text{Blue}, \text{Small}) &= \{\text{Bob}\}, \\ G(\text{Blue}, \text{Medium}) &= \{\text{Alice}\}, \\ G(\text{Blue}, \text{Large}) &= *. \end{aligned}$$

Here, $*$ denotes that the sales data for that specific color-size combination is missing or was not recorded.

Then

$$\big(U,\ \mathcal{A}_1,\ \mathcal{A}_2,\ G\big)$$

is an *Incomplete HyperSoft Set* over U. It models customer purchase behavior under incomplete data: for some variants (e.g. Red–Medium and Blue–Large), the mapping G returns $*$, indicating unavailable information.

Theorem 3.18. *Every Incomplete HyperSoft Set is an IndetermHyperSoft Set, and the Incomplete Soft Set arises as the single-attribute special case.*

Proof. Let

$$\big(U,\ \mathcal{A}_1,\dots,\mathcal{A}_m,\ G\big)$$

be an Incomplete HyperSoft Set as in Definition 3.16, with $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$.

(1) Embedding into IndetermHyperSoft. Recall (Definition of IndetermHyperSoft Set) that a pair $(F, \mathcal{C})$ is an *IndetermHyperSoft Set* if

$$F : \mathcal{C} \ \longrightarrow\ \mathcal{P}(U)$$

is allowed to take *indeterminate* values in any of the following ways:

- one or more domains $\mathcal{A}_i$ are themselves uncertain or incomplete,
- the codomain $\mathcal{P}(U)$ admits fuzzy or ambiguous subsets,
- there exists $\gamma \in \mathcal{C}$ with $F(\gamma)$ not uniquely determined.

Define

$$F : \mathcal{C} \ \longrightarrow\ \mathcal{P}(U) \cup \{*\}, \qquad F(\gamma) = G(\gamma).$$

Then each occurrence $G(\gamma) = *$ is exactly an *indeterminate image* in the sense of IndetermHyperSoft Sets. No further restrictions are imposed on the domains $\mathcal{A}_i$. Hence $(F, \mathcal{C})$ satisfies the definition of an IndetermHyperSoft Set.

(2) Reduction to Incomplete Soft Set. Let (U, E, f) be an Incomplete Soft Set with

$$f : E \ \longrightarrow\ \mathcal{P}(U) \cup \{*\}.$$

Take $m = 1$ and $\mathcal{A}_1 = E$. Then $\mathcal{C} = \mathcal{A}_1 = E$, and setting $G = f$ yields an Incomplete HyperSoft Set

$$\big(U, E,\ G = f\big).$$

Thus every Incomplete Soft Set is exactly the single-attribute case of an Incomplete HyperSoft Set. □

Theorem 3.19 (Closure under Union)**.** *Let*

$$\big(U,\ \mathcal{A}_1,\dots,\mathcal{A}_m,\ G_1\big) \quad \textit{and} \quad \big(U,\ \mathcal{A}_1,\dots,\mathcal{A}_m,\ G_2\big)$$

be two Incomplete HyperSoft Sets with the same attribute domains and $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$*. Define*

$$G(\gamma) = \begin{cases} G_1(\gamma) \cup G_2(\gamma), & G_1(\gamma) \neq *,\ G_2(\gamma) \neq *, \\ G_1(\gamma), & G_1(\gamma) \neq *,\ G_2(\gamma) = *, \\ G_2(\gamma), & G_1(\gamma) = *,\ G_2(\gamma) \neq *, \\ *, & G_1(\gamma) = *,\ G_2(\gamma) = *. \end{cases}$$

Then $\big(U, \mathcal{A}_1, \dots, \mathcal{A}_m, G\big)$ *is an Incomplete HyperSoft Set.*

Proof. We must show $G : \mathcal{C} \to \mathcal{P}(U) \cup \{*\}$ satisfies Definition 3.16.

(Domain indeterminacy) Any incompleteness in the domains $\mathcal{A}_i$ remains unchanged by forming unions, so condition (1) of Definition 3.16 persists.

(Codomain indeterminacy) If $\mathcal{P}(U)$ admits ambiguous or partial subsets, then whenever $G_1(\gamma)$ or $G_2(\gamma)$ is a non-unique subset, their union is also non-unique, satisfying condition (2).

(Image indeterminacy) Take any $\gamma_0 \in \mathcal{C}$. If $G_1(\gamma_0) = *$ or $G_2(\gamma_0) = *$, then by construction $G(\gamma_0) = *$. Otherwise $G_1(\gamma_0)$ or $G_2(\gamma_0)$ may be ambiguous, and so $G(\gamma_0) = G_1(\gamma_0) \cup G_2(\gamma_0)$ is ambiguous, satisfying condition (3).

Thus G meets at least one indeterminacy condition for each γ, and is an Incomplete HyperSoft Set. □

Theorem 3.20 (Closure under Intersection)**.** *With the same setup as Theorem 3.19, define*

$$G'(\gamma) = \begin{cases} G_1(\gamma) \cap G_2(\gamma), & G_1(\gamma) \neq *,\ G_2(\gamma) \neq *, \\ *, & \textit{otherwise.} \end{cases}$$

Then $\big(U, \mathcal{A}_1, \ldots, \mathcal{A}_m, G'\big)$ *is an Incomplete HyperSoft Set.*

Proof. Again, domain indeterminacy (1) and codomain indeterminacy (2) carry over from G_1, G_2. For image indeterminacy (3), if either $G_1(\gamma)$ or $G_2(\gamma)$ equals $*$, then $G'(\gamma) = *$. If both are non-$*$ but at least one is ambiguous, their intersection remains ambiguous. Hence G' fulfills Definition 3.16. □

Theorem 3.21 (Projection onto a Subset of Attributes)**.** *Let* $\big(U, \mathcal{A}_1, \ldots, \mathcal{A}_m, G\big)$ *be an Incomplete HyperSoft Set on* $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$*. Fix a nonempty index set* $I \subseteq \{1, \ldots, m\}$ *and let* $\mathcal{C}_I = \prod_{i \in I} \mathcal{A}_i$*. Define the* projection

$$G_I(\alpha) = \begin{cases} \bigcup\limits_{\substack{\gamma \in \mathcal{C} \\ \gamma|_I = \alpha}} G(\gamma), & \textit{if } G(\gamma) \neq * \textit{ for all such } \gamma, \\ *, & \textit{otherwise,} \end{cases}$$

for each $\alpha \in \mathcal{C}_I$*. Then* $\big(U, \{\mathcal{A}_i\}_{i \in I}, G_I\big)$ *is an Incomplete HyperSoft Set on* $\mathcal{C}_I$*.*

Proof. **(1) Domain indeterminacy.** Any incompleteness in the original domains $\mathcal{A}_i$ for $i \in I$ remains in $\mathcal{C}_I$.

(2) Codomain indeterminacy. If $\mathcal{P}(U)$ is ambiguous, then unions of its elements remain ambiguous.

(3) Image indeterminacy. For a given $\alpha \in \mathcal{C}_I$, if any extension γ with $\gamma|_I = \alpha$ has $G(\gamma) = *$, then $G_I(\alpha) = *$. Otherwise, the union over a family of possibly ambiguous subsets is ambiguous.

Thus G_I satisfies at least one indeterminacy condition for each α, making it an Incomplete HyperSoft Set. □

3.5 Incomplete SuperHyperSoft Set

The Incomplete SuperHyperSoft Set is a generalized concept derived from the Incomplete HyperSoft Set. Its formal definition is provided below.

Definition 3.22 (Incomplete SuperHyperSoft Set)**.** Let U be a universal set, and let $\{a_1, a_2, \ldots, a_n\}$ be $n \geq 1$ distinct attributes, each with a set of possible values A_i. Define

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \;\times\; \mathcal{P}(A_2) \;\times\; \cdots \;\times\; \mathcal{P}(A_n),$$

as in the definition of a *SuperHyperSoft Set*. An *Incomplete SuperHyperSoft Set* is a tuple

$$\Big(U,\ \{A_i\}_{i=1}^{n},\ F\Big),$$

where

$$F : \mathcal{C} \;\longrightarrow\; \big(\mathcal{P}(U) \cup \{*\}\big).$$

Specifically, for each $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \subseteq A_i$,

$$F(\gamma) \;=\; \begin{cases} S_\gamma \subseteq U, & \text{if the information is complete,} \\ *, & \text{if the information is incomplete or undefined.} \end{cases}$$

Here, $*$ indicates a missing or indeterminate value at the superhyper level (e.g., incomplete data about chosen subsets α_i or how they map into U).

Example 3.23 (Smartphone Models under Incomplete SuperHyperSoft Criteria)**.** Let

$$U = \{\text{ModelX, ModelY, ModelZ}\}$$

be the set of available smartphone models. Consider three attribute domains:

$$A_1 = \{\text{5G, NFC}\}, \quad A_2 = \{\text{Black, White}\}, \quad A_3 = \{\text{64GB, 128GB}\}.$$

Form the SuperHyperSoft domain

$$\mathcal{C} = \mathcal{P}(A_1) \;\times\; \mathcal{P}(A_2) \;\times\; \mathcal{P}(A_3).$$

Define

$$F : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U) \cup \{*\}$$

by specifying its values on several representative points (where "$*$" indicates missing or uncollected data):

$$\begin{aligned} F\big(\{\text{5G}\}, \{\text{Black}\}, \{\text{64GB}\}\big) &= \{\text{ModelX, ModelY}\}, \\ F\big(\{\text{NFC}\}, \{\text{White}\}, \{\text{128GB}\}\big) &= \{\text{ModelZ}\}, \\ F\big(\{\text{5G, NFC}\}, \{\text{White}\}, \{\text{128GB}\}\big) &= *, \\ F\big(\varnothing, \{\text{Black, White}\}, \{\text{64GB, 128GB}\}\big) &= \{\text{ModelX, ModelZ}\}. \end{aligned}$$

Explanation of indeterminacy:

- *Domain-level incompleteness:* For the combination $(\{\text{5G, NFC}\}, \{\text{White}\}, \{\text{128GB}\})$, customer preference data were not gathered, so $F = *$.
- *Image-level incompleteness:* Even when a subset of attributes is specified, the corresponding models may be under review (e.g. ModelY's 128GB sensor performance), yielding $*$.
- *Complete entries:* Other entries, such as $(\{\text{5G}\}, \{\text{Black}\}, \{\text{64GB}\})$, have full data and map to concrete subsets of U.

Thus $\big(U, \{A_i\}, F\big)$ is an *Incomplete SuperHyperSoft Set*, modeling a real-world scenario where multi-attribute purchase data may be partially missing or undefined.

Theorem 3.24. *An Incomplete SuperHyperSoft Set is a special case of an IndetermHyperSoft Set (with superhyper structure) and also extends the concept of a SuperHyperSoft Set to accommodate incomplete information.*

Proof. Let

$$\big(U, \{A_i\}_{i=1}^n, F\big)$$

be an Incomplete SuperHyperSoft Set as in Definition 3.22, with

$$\mathcal{C} = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n).$$

(1) It is an IndetermHyperSoft Set with superhyper-level indeterminacy. Recall (Definition of IndetermHyperSoft Set) that an IndetermHyperSoft Set over U with domain $\mathcal{C}$ is any mapping

$$F' : \mathcal{C} \;\longrightarrow\; \mathcal{P}(U)$$

which is allowed to exhibit indeterminacy in any of the following ways:

(a) Some attribute domain $\mathcal{P}(A_i)$ is itself uncertain or partially specified.

(b) The codomain $\mathcal{P}(U)$ admits ambiguous or fuzzy subsets.

(c) There exists $\gamma \in \mathcal{C}$ for which $F'(\gamma)$ is not uniquely determined.

In our Incomplete SuperHyperSoft Set, the map

$$F(\gamma) \ \in \ \mathcal{P}(U) \cup \{*\}$$

explicitly returns $*$ whenever the information for γ is missing or undefined. Viewing $*$ as "indeterminate output," we see that each occurrence of $F(\gamma) = *$ satisfies condition (c) above. Moreover, if any $\mathcal{P}(A_i)$ is incomplete or if $\mathcal{P}(U)$ contains ambiguous subsets, then conditions (a) or (b) are also met. Hence F meets the definition of an IndetermHyperSoft Set on the same superhyper domain $\mathcal{C}$.

(2) It extends the SuperHyperSoft Set by allowing $*$. A SuperHyperSoft Set over U with domain $\mathcal{C}$ is precisely a mapping

$$F_0 : \mathcal{C} \ \longrightarrow \ \mathcal{P}(U),$$

with no undefined values. The Incomplete SuperHyperSoft Set broadens this by permitting $* \in \mathcal{P}(U) \cup \{*\}$ whenever the data are missing. Concretely, if one restricts to those $\gamma \in \mathcal{C}$ for which $F(\gamma) \neq *$, then

$$F|_{\{\gamma : F(\gamma) \neq *\}} \ : \ \mathcal{C} \ \longrightarrow \ \mathcal{P}(U)$$

coincides with the original SuperHyperSoft mapping. Thus, in the absence of any $*$-values, F reduces exactly to a SuperHyperSoft Set.

Combining (1) and (2), we conclude that every Incomplete SuperHyperSoft Set is simultaneously

- a special case of an IndetermHyperSoft Set (with nesting up to superhyper level, and indeterminacy encoded by $*$), and
- a strict generalization of a SuperHyperSoft Set (recovering it when no information is missing).

This establishes the desired generalization. □

4 Conclusion

In this paper, we proposed three novel extensions—*IndetermSuperHyperSoft Sets*, *IndetermForestSoft Sets*, and *IndetermTreeSoft Sets*—providing rigorous definitions, illustrative examples, and a comprehensive study of how these structures extend and unify existing soft–set models.

In the future, we hope that further research will address extensions of these frameworks to Graphs [40,41], HyperGraphs [42–44], and SuperHyperGraphs [45–47], as well as to QuantumSoft Sets [48–50]. We also anticipate progress in the development of algorithms and practical case studies related to these models.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors affirm that, to the best of their knowledge, this manuscript represents their original research. It has not been previously published in any journal, nor is it currently being considered for publication elsewhere.

Disclaimer on Computational Tools

No computer-based tools—such as symbolic computation systems, automated theorem provers, or proof assistants (e.g., Mathematica, SageMath, Coq)—were employed in the development, analysis, or verification of the results contained in this paper. All derivations and proofs were conducted manually through analytical methods by the authors.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Consent to Participate

Not applicable.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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TABLE II. COMPARISON WITH STATE-OF-THE-ART METHODS

Ref	Method	Accuracy (%)	Precision (%)	Recall (%)
[4]	Hybrid (CNN-ResNet50)	92	94	95
[5]	Transfer Learning with MobileNetV2	89	90	91
[6]	VGG19-DenseNet	90	91	92
[7]	DarkCovidNet-CNN	87	89	88
[8]	Ensemble(InceptionV3, Xception, ResNet)	93	94	93
[9]	CNN-LSTM	91	92	91
[10	ResNet50-InceptionV3	91	92	91
[11]	EfficientNet	89	90	89
[12]	SqueezeNet	88	89	88
This	**CViTLNN**	**94**	**95**	**94**

IV. CONCLUSION

This work introduced CViTLNN, a hybrid architecture based on ViT and LNN for COVID-19 detection from CXR images. The ViT architecture employed 24 encoder blocks to extract deep features from chest X-ray images, using self-attention mechanisms to capture both local and global representations. Subsequently, the LNN was integrated with ViT for dynamic feature refinement for COVID-19 detection. Experimental results demonstrated that CViTLNN achieved superior performance by surpassing existing methods, achieving a competitive test accuracy of 94%, a precision of 95%, and a recall of 94%, highlighting its robustness and reliability in detecting COVID-19 from CXR images. This work is particularly significant for deployment in low-resource settings, where access to expert radiologists and advanced diagnostic tools is limited. The proposed architecture provides an efficient and effective solution for COVID-19 detection, offering the potential to assist healthcare professionals in underserved populations. Future work will involve expanding the application of the ViTLNN model to other biomedical image analysis tasks, such as lung cancer detection.

DATA AVAILABILITY STATEMENT

Dataset: https://data.mendeley.com/datasets/dvntn9yhd2/1

Code: https://github.com/BioAIMLDL/CViTLNN

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New Types of Soft Sets: HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set

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Abstract

This is a review paper, where we recall the definitions together with practical applications of the Soft Set and its extensions to HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set.

Keywords: Soft Set; HyperSoft Set; IndetermSoft Set; IndetermHyperSoft Set; MultiSoft Set; TreeSoft Set.

1. Introduction

The Soft Set was introduced by Molodtsov [1] in 1999. Further on, the HyperSoft Set (2018), IndetermSoft Set (2022), IndetermHyperSoft Set (2022), and TreeSoft Set (2022) were introduced by Smarandache [2-7].

The MultiSoft Set (2010) was introduced by Alkhazaleh et al. [8]. The soft set and its extensions have many applications in our real world. Many hybrid version of the soft set have been proposed and used, combined with fuzzy or fuzzy extension sets, such as: fuzzy soft set, intuitionistic fuzzy soft set, neutrosophic soft set, picture fuzzy soft set, spherical fuzzy soft set, plithogenic soft set, and similarly for fuzzy hypersoft set, intuitionistic fuzzy hypersoft set, neutrosophic hypersoftsoft set, picture fuzzy hypersoft set, spherical fuzzy hypersoft soft set, plithogenic sift set.

Future research may also investigate and apply the newly forms of soft sets, in combinations with fuzzy and fuzzy extension sets, that will result in fuzzy / intuitionistic fuzzy / neutrosophic / picture fuzzy / spherical fuzzy / Pythagorean fuzzy etc. / plithogenic IndetermSoft / IndetermHyperSoft / TreeSoft Set . respectively.

Let us recall their definitions together with real examples.

2. Definition of Soft Set

Let U be a universe of discourse, $P(U)$ the power set of U, and A a set of attributes. Then, the pair (F, U), where $F: A \to P(U)$ is called a Soft Set over U.

3. Real Example of Soft Set

Let U = {Helen, George, Mary, Richard} and a set M = {Helen, Mary, Richard} included in U.

Let the attribute be: a = size, and its attribute' values respectively:

$$\text{Size} = A_1 = \{\text{small, medium, tall}\}.$$

Let the function be: $F: A_1 \rightarrow P(U)$.

Then, for example:

$$F(\text{tall}) = \{\text{Helen, Mary}\},$$

which means that both Helen and Mary are tall.

4. Definition of IndetermSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ be the powerset of H. Let a be an attribute, and A be a set of this attribute-values. Then $F: A \rightarrow P(H)$ is called an IndetermSoft Set if at least one of the bellow occurs:

i) the set A has some indeterminacy;

ii) the sets H or $P(H)$ have some indeterminacy;

iii) the function F has some indeterminacy, i.e. there exist at least an attribute-value v that belongs to A, such that $F(v)$ = indeterminate (unclear, incomplete, conflicting, or not unique).

IndetermSoft Set, as an extension of the classical (determinate) Soft Set, deals with indeterminate data, because there are sources unable to provide exact or complete information on the sets A, H, or $P(H)$, nor on the function F. We did not add any indeterminacy, we found the indeterminacy in our real world. Because many sources give approximate/uncertain/incomplete/conflicting information, not exact information as in the Soft Set, as such we still need to deal with such situations.

Herein, 'Indeterm' stands for 'Indeterminate' (uncertain, conflicting, incomplete, not unique outcome).

Similarly, distinctions between determinate and indeterminate operators are taken into consideration. Afterwards, an IndetermSoft Algebra is built, using a determinate soft operator (joinAND), and three indeterminate soft operators (disjoinOR, exclussiveOR, NOT), whose properties are further on studied.

Smarandache has generalized the Soft Set to the HyperSoft set by transforming the function F into a multi-attribute function, and then he introduced the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, other fuzzy extensions, and Plithogenic HyperSoft Set.

The classical Soft Set is based on a determinate function (whose values are certain, and unique), but in our world there are many sources that, because of lack of information or ignorance, provide indeterminate (uncertain, and not unique – but hesitant or alternative) information. They can be modeled by operators having some degree of indeterminacy due to the imprecision of our world.

5. Real Example of IndetermSoft Set

Assume a town has many houses.

1) Indeterminacy with respect to the function.

1a) You ask a source:

— What houses have the red color in the town?

The source:

— I am not sure, I think the houses h_1 or h_2.

Therefore, $F(\text{red}) = h_1$ or h_2 (indeterminate / uncertain answer).

1b) You ask again:

— But, what houses are yellow?

The source:

— I do not know, the only thing I know is that the house h_5 is not yellow because I have visited it.

Therefore, F(yellow) = not h_5 (again indeterminate / uncertain answer).

1c) Another question you ask:

— Then what houses are blue?

The source:

— For sure, either h_8 or h_9.

Therefore, F(blue) = either h_8 or h_9 (again indeterminate / uncertain answer).

2) Indeterminacy with respect to the set H of houses.

You ask the source:

— How many houses are in the town?

The source:

— I never counted them, but I estimate their number to be between 100-120 houses.

3) Indeterminacy with respect to the set A of attributes.

You ask the source:

— What are all colors of the houses?

The source:

— I know for sure that there are houses of colors red, yellow, and blue, but I do not know if there are houses of other colors (?)

This is the IndetermSoft Set.

6. Definition of HyperSoft Set

The soft set was extended to the hypersoft set by transforming the function F into a multi-attribute function. Afterwards, the hybrids of HyperSoft Set with the Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, other fuzzy extensions, and Plithogenic Set were introduced.

Let U be a universe of discourse, $P(U)$ the power set of U. Let $a_1, a_2, \dots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \Phi$, for $i \neq j$, and i, j in $\{1, 2, \dots, n\}$. Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $F: A_1 \times A_2 \times \dots \times A_n \rightarrow P(U)$, is called a HyperSoft Set over U.

7. Real Example of HyperSoft Set

Let U = {Helen, George, Mary, Richard} and a set M = {Helen, Mary, Richard} included in U.

Let the attributes be: a_1 = size, a_2 = color, a_3 = gender, a_4 = nationality, and their attributes' values respectively:

$$\text{Size} = A_1 = \{\text{small, medium, tall}\},$$

$$\text{Color} = A_2 = \{\text{white, yellow, red, black}\},$$

$$\text{Gender} = A_3 = \{\text{male, female}\},$$

$$\text{Nationality} = A_4 = \{\text{American, French, Spanish, Italian, Chinese}\}.$$

Let the function be: $F: A_1 \times A_2 \times A_3 \times A_4 \rightarrow P(U)$.

Then, for example:

$F(\{\text{tall, white, female, Italian}\}) = \{\text{Helen, Mary}\}$, which means that both Helen and Mary are tall, and white, and female, and Italian.

Notice that this is an extension of the previous Real Example of Soft Set.

8. Definition of IndetermHyperSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the powerset of H. Let $a_1, a_2, \dots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute-values are respectively the sets $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \Phi$ for $i \neq j$, and i, j in $\{1, 2, \dots, n\}$. Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $F: A_1 \times A_2 \times \dots \times A_n \rightarrow P(H)$, is called an IndetermHyperSoft Set over U if at least one of the bellow occurs:

i) at least one of the sets $A_1, A_2, \dots, A_n$ has some indeterminacy;

ii) the sets H or $P(H)$ have some indeterminacy;

iii) there exist at least one n-plet $(e_1, e_2, \dots, e_n) \in A_1 \times A_2 \times \dots \times A_n$ such that $F(e_1, a_2, \dots, e_n)$ = indeterminate (unclear, uncertain, conflicting, or not unique).

Similarly, IndetermHyperSoft Set ia an extension of the HyperSoft Set, when there is indeterminate data, or indeterminate functions, or indeterminate sets.

9. Real Example of IndetermHyperSoft Set

Assume a town has many houses.

1) Indeterminacy with respect to the function.

1a) You ask a source:

— What houses are of red color and big size in the town?

The source:

— I am not sure, I think the houses h_1 or h_2.

Therefore, F(red, big) = h_1 or h_2 (indeterminate / uncertain answer).

1b) You ask again:

— But, what houses are yellow and small?

The source:

— I do not know, the only thing I know is that the house h_5 is neither yellow nor small because I have visited it.

Therefore, F(yellow, small) = not h_5 (again indeterminate / uncertain answer).

1c) Another question you ask:

— Then what houses are blue and big?

The source:

— For sure, either h_8 or h_9.

Therefore, F(blue, big) = either h_8 or h_9 (again indeterminate / uncertain answer).

2) Indeterminacy with respect to the set H of houses.

You ask the source:

— How many houses are in the town?

The source:

— I never counted them, but I estimate their number to be between 100-120 houses.

3) Indeterminacy with respect to the product set $A_1 \times A_2 \times \ldots \times A_n$ of attributes.

You ask the source:

— What are all colors and sizes of the houses?

The source:

— I know for sure that there are houses of colors of red, yellow, and blue, but I do not know if there are houses of other colors (?) About the size, I saw many houses that are small, but I do not remember to have seeing big houses.

This is the IndetermHyperSoft Set.

10. Definition of TreeSoft Set

Let U be a universe of discourse, and H a non-empty subset of U, with $P(H)$ the powerset of H.

Let A be a set of attributes (parameters, factors, etc.),

$A = \{A_1, A_2, \ldots, A_n\}$, for integer $n \geq 1$, where $A_1, A_2, \ldots, A_n$ are considered attributes of first level (since they have one-digit indexes).

Each attribute A_i, $1 \leq i \leq n$, is formed by sub-attributes:

$$A_1 = \{A_{1,1}, A_{1,2}, \ldots\}$$

$$A_2 = \{A_{2,1}, A_{2,2}, \ldots\}$$

$$\ldots\ldots\ldots\ldots$$

$$A_n = \{A_{n,1}, A_{n,2}, \ldots\}$$

where the above $A_{i,j}$ are sub-attributes (or attributes of second level) (since they have two-digit indexes).

Again, each sub-attribute $A_{i,j}$ is formed by sub-sub-attributes (or attributes of third level):

$$A_{i,j,k}$$

And so on, as much refinement as needed into each application, up to sub-sub-…-sub-attributes (or attributes of m-level (or having m digits into the indexes):

$$A_{i1,i2,\ldots,im}$$

Therefore, a graph-tree is formed, that we denote as Tree(A), whose root is A (considered of level zero), then nodes of level 1, level 2, up to level m.

We call *leaves* of the graph-tree, all terminal nodes (nodes that have no descendants).

Then the TreeSoft Set is:

$$F: P(Tree(A)) \rightarrow P(H)$$

Tree(A) is the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and P(Tree(A)) is the powerset of the Tree(A).

All node sets of the *TreeSoft Set of level m* are:

$$\text{Tree}(A) = \{A_{i1} \mid_{i1} = 1, 2, \ldots\}$$

The first set is formed by the nodes of level 1, second set by the nodes of level 2, third set by the nodes of level 3, and so on, the last set is formed by the nodes of level m. If the graph-tree has only two levels ($m = 2$), then the TreeSoft Set is reduced to a MultiSoft Set [8].

11. Practical Example of TreeSoft Set of Level 3

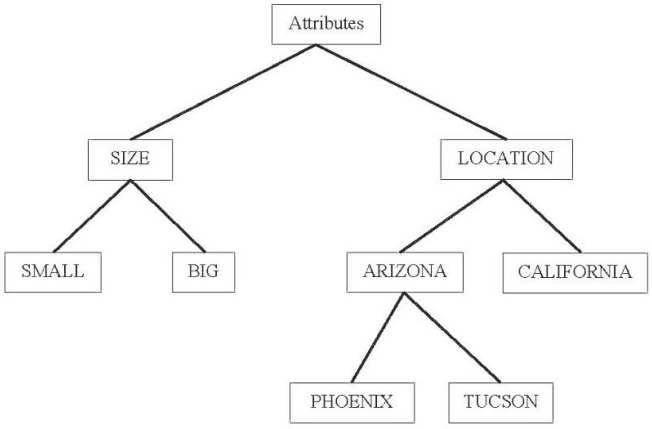


Figure 1: A TreeSoft Set of Level 3

Assume a town has many houses.

This is a classical tree as shown in Figure 1, whose:

Level 0 (the root) is the node Attributes;

Level 1 is formed by the nodes: Size, Location;

Level 2 is formed by the nodes Small, Big, Arizona, California;

Level 3 is formed by the nodes Phoenix, Tucson.

Let's consider $H = \{h_1, h_2, ..., h_{10}\}$ be a set of houses, and $P(H)$ the powerset of H.

And the set of Attributes: $A = \{A_1, A_2\}$, where A_1 = Size, A_2 = Location.

Then $A_1 = \{A_{11}, A_{12}\}$ = {Small, Big}, $A_2 = \{A_{21}, A_{22}\}$ = {Arizona, California} as American states.

Further on, $A_{22} = \{A_{211}, A_{212}\}$ = {Phoenix, Tucson} as Arizonian cities.

Let's assume that the function F gets the following values:

$$F(\text{Big, Arizona, Phoenix}\} = \{ h_9, h_{10} \}$$

$$F(\text{Big, Arizona, Tucson}) = \{ h_1, h_2, h_3, h_4 \}$$

F(Big, Arizona) = all big houses from both cities, Phoenix and Tucson

$$= F(\text{Big, Arizona, Phoenix}) \cup F(\text{Big, Arizona, Tucson}) = \{ h_1, h_2, h_3, h_4, h_9, h_{10} \}.$$

12. Conclusion

This paper reviews Soft Set and its extensions to HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set. The HyperSoft Set (2018) is a generalization of Soft Set (1999) and MultiSoft Set (2010), from a uni-variate function to a multi-variate function F; IndetermSoft Set (2022) is an extension of the Soft Set, from the determinate data to indeterminate data; IndetermHyperSoft Set (2022) is an extension of the HyperSoft Set, from the determinate data to indeterminate data; and TreeSoft Set (2022) that is a generalization of the MultiSoft Set.

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New Types of Soft Sets "HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set": An Improved Version

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Abstract: This is an improved paper of [10], where we recall the definitions together with practical applications of the Soft Set and its extensions to HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set.

Keywords: Soft Set; HyperSoft Set; IndetermSoft Set; IndetermHyperSoft Set; MultiSoft Set; TreeSoft Set.

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1. Introduction

The Soft Set was introduced by Molodtsov [1] in 1999.

Further on, the HyperSoft Set (2018), IndetermSoft Set (2022), IndetermHyperSoft Set (2022), and TreeSoft Set (2022) were introduced by Smarandache [2-7]. The definitions of IndetermSoft Set and IndetermHyperSoft Set [9] are updated now.

The MultiSoft Set (2010) was introduced by Alkhazaleh et al. [8].

The soft set and its extensions have many applications in our real world.

Many hybrid version of the soft set have been proposed and used, combined with fuzzy or fuzzy extension sets, such as: fuzzy soft set, intuitionistic fuzzy soft set, neutrosophic soft set, picture fuzzy soft set, spherical fuzzy soft set, plithogenic soft set, and similarly for fuzzy hypersoft set, intuitionistic fuzzy hypersoft set, neutrosophic hypersoftsoft set, picture fuzzy hypersoft set, spherical fuzzy hypersoft soft set, plithogenic sift set.

Future research may also investigate and apply the newly forms of soft sets, in combinations with fuzzy and fuzzy extension sets, that will result in fuzzy / intuitionistic fuzzy / neutrosophic / picture fuzzy / spherical fuzzy / Pythagorean fuzzy etc. / plithogenic IndetermSoft / IndetermHyperSoft / TreeSoft Set, respectively.

Let us recall their definitions together with real examples.

2. Definition of Soft Set

Let U be a universe of discourse, $P(U)$ the power set of U, and A a set of attributes. Then, the pair (F, U), where $F: A \to P(U)$ is called a Soft Set over U.

3. Real Example of Soft Set

Let U = {Helen, George, Mary, Richard} and a set M = {Helen, Mary, Richard} included in U.

Let the attribute be: a = size, and its attribute' values respectively:

$$\text{Size} = A_1 = \{\text{small, medium, tall}\}.$$

Let the function be: $F: A_1 \to P(U)$.

Then, for example:

$$F(\text{tall}) = \{\text{Helen, Mary}\},$$

which means that both Helen and Mary are tall.

4. Definition of IndetermSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ be the powerset of H. Let a be an attribute, and A be a set of this attribute-values. Then $F: A \rightarrow P(H)$ is called an IndetermSoft Set if at least one of the bellow occurs:

i) The set A has some indeterminacy;
ii) At least one of the sets H or $P(H)$ has some indeterminacy;
iii) The function F has some indeterminacy, i.e. there exist at least one relationship $F(a) = M$ where a or M have some indeterminacy (not unique, unclear, incomplete, unknown).

IndetermSoft Set, as an extension of the classical (determinate) Soft Set, deals with indeterminate data, because there are sources unable to provide exact or complete information on the sets A, H, or $P(H)$, nor on the function F. We did not add any indeterminacy, we found the indeterminacy in our real world. Because many sources give approximate/uncertain/incomplete/conflicting information, not exact information as in the Soft Set, as such we still need to deal with such situations.

Herein, 'Indeterm' stands for 'Indeterminate' (uncertain, conflicting, incomplete, not unique outcome).

Similarly, distinctions between determinate and indeterminate operators are taken into consideration. Afterwards, an IndetermSoft Algebra is built, using a determinate soft operator (joinAND), and three indeterminate soft operators (disjoinOR, exclussiveOR, NOT), whose properties are further on studied.

Smarandache has generalized the Soft Set to the HyperSoft set by transforming the function F into a multi-attribute function, and then he introduced the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, other fuzzy extensions, and Plithogenic HyperSoft Set.

The classical Soft Set is based on a determinate function (whose values are certain, and unique), but in our world there are many sources that, because of lack of information or ignorance, provide indeterminate (uncertain, and not unique – but hesitant or alternative) information. They can be modeled by operators having some degree of indeterminacy due to the imprecision of our world.

5. Real Example of IndetermSoft Set

The present study represents an application of modern material statistics, specifically leveraging the principles of neutrosophic statistics.

Assume a town has many houses.

1) Indeterminacy with respect to the set A of attributes.

You ask the source:

- What are all colors of the houses?

The source:

- I know for sure that there are houses of colors red, yellow, and blue, but I do not know if there are houses of other colors (?)

This is the IndetermSoft Set.

2) Indeterminacy with respect to the set H of houses.

You ask the source:

- How many houses are in the town?

The source:

- I never counted them, but I estimate their number to be between 100-120 houses.

3) Indeterminacy with respect to the function $F: A \rightarrow P(H)$.

3a) You ask a source:
- What houses have the red color in the town?
The source:
- I am not sure, I think the houses h_1 or h_2.
Therefore, F(red) = h_1 or h_2 (indeterminate / uncertain answer).
3b) You ask again:
- But, what houses are yellow?
The source:
- I do not know, the only thing I know is that the house h_5 is not yellow because I have visited it.
Therefore, F(yellow) = not h_5 (again indeterminate / uncertain answer).
3c) Another question you ask:
- Then what houses are blue?
The source:
- For sure, either h_8 or h_9.
Therefore, F(blue) = either h_8 or h_9 (again indeterminate / uncertain answer).

6. Definition of HyperSoft Set

The soft set was extended to the hypersoft set by transforming the function F into a multi-attribute function. Afterwards, the hybrids of HyperSoft Set with the Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, other fuzzy extensions, and Plithogenic Set were introduced.

Let U be a universe of discourse, $P(U)$ the power set of U. Let $a_1, a_2, \dots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \Phi$, for $i \neq j$, and i, j in $\{1, 2, \dots, n\}$. Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $F: A_1 \times A_2 \times \dots \times A_n \rightarrow P(U)$, is called a HyperSoft Set over U.

7. Real Example of HyperSoft Set

Let U = {Helen, George, Mary, Richard} and a set M = {Helen, Mary, Richard} included in U.

Let the attributes be: a_1 = size, a_2 = color, a_3 = gender, a_4 = nationality, and their attributes' values respectively:

$$\text{Size} = A_1 = \{\text{small, medium, tall}\},$$

$$\text{Color} = A_2 = \{\text{white, yellow, red, black}\},$$

$$\text{Gender} = A_3 = \{\text{male, female}\},$$

$$\text{Nationality} = A_4 = \{\text{American, French, Spanish, Italian, Chinese}\}.$$

Let the function be: $F: A_1 \times A_2 \times A_3 \times A_4 \rightarrow P(U)$.
Then, for example:
F({tall, white, female, Italian}) = {Helen, Mary}, which means that both Helen and Mary are tall, and white, and female, and Italian.
Notice that this is an extension of the previous Real Example of Soft Set.

8. Definition of IndetermHyperSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the powerset of H. Let $a_1, a_2, \dots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute-values are respectively the sets $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \Phi$ for $i \neq j$, and i, j in $\{1, 2, \dots, n\}$. Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $F: A_1 \times A_2 \times \dots \times A_n \rightarrow P(H)$, is called an IndetermHyperSoft Set over U if at least one of the bellow occurs:

i) at least one of the sets $A_1, A_2, \dots, A_n$ has some indeterminacy;
ii) at least one of the sets H or $P(H)$ has some indeterminacy;
iii) the function F has some indeterminacy, i.e. there exist at least one relationship

$F(e_1, e_2, \ldots, e_n) = M$, where some of $e_1, e_2, \ldots, e_n$, or M *have indeterminacy* (not unique, unclear, incomplete, unknown).

Similarly, IndetermHyperSoft Set ia an extension of the HyperSoft Set, when there is indeterminate data, or indeterminate functions, or indeterminate sets.

9. Real Example of IndetermHyperSoft Set

Assume a town has many houses.

1) Indeterminacy with respect to the product set $A_1 \times A_2 \times \ldots \times A_n$ of attributes.

You ask the source:

- What are all colors and sizes of the houses?

The source:

- I know for sure that there are houses of the following colors: red, yellow, and blue, but I do not know if there are houses of other colors (?)

About the size, I saw many houses that are small, but I do not remember to have seeing big houses.

2) Indeterminacy with respect to the set H of houses.

You ask the source:

- How many houses are in the town?

The source:

- I never counted them, but I estimate their number to be between 100-120 houses.

3) Indeterminacy with respect to the function $F: A_1 \times A_2 \times \ldots \times A_n \rightarrow P(H)$.

3a) You ask a source:

- What houses are of red color and big size in the town?

The source:

- I am not sure, I think the houses h_1 or h_2.

Therefore, F(red, big) = h_1 or h_2 (indeterminate / not unique / uncertain answer).

3b) You ask again:

- But, what houses are yellow and small?

The source:

- I do not know, the only thing I know is that the house h_5 is neither yellow nor small because I have visited it.

Therefore, F(yellow, small) = not h_5 (again indeterminate / uncertain answer).

3c) Another question you ask:

- Then what houses are blue and big?

The source:

- For sure, either h_8 or h_9.

Therefore, F(blue, big) = either h_8 or h_9 (again indeterminate / uncertain answer).

This is the IndetermHyperSoft Set.

10. Definition of TreeSoft Set

Let U be a universe of discourse, and H a non-empty subset of U, with $P(H)$ the powerset of H.

Let A be a set of attributes (parameters, factors, etc.),

$A = \{A_1, A_2, \ldots, A_n\}$, for integer $n \geq 1$, where $A_1, A_2, \ldots, A_n$ are considered attributes of first level (since they have one-digit indexes).

Each attribute A_i, $1 \leq i \leq n$, is formed by sub-attributes:

$$A_1 = \{A_{1,1}, A_{1,2}, \ldots\}$$

$$A_2 = \{A_{2,1}, A_{2,2}, \ldots\}$$

.....

$$A_n = \{A_{n,1}, A_{n,2}, \dots\}$$

where the above $A_{i,j}$ are sub-attributes (or attributes of second level) (since they have two-digit indexes).

Again, each sub-attribute $A_{i,j}$ is formed by sub-sub-attributes (or attributes of third level):

$$A_{i,j,k}$$

And so on, as much refinement as needed into each application, up to sub-sub-…-sub-attributes (or attributes of m-level (or having m digits into the indexes):

$$A_{i1,i2,\dots,im}$$

Therefore, a graph-tree is formed, that we denote as Tree(A), whose root is A (considered of level zero), then nodes of level 1, level 2, up to level m.

We call *leaves* of the graph-tree, all terminal nodes (nodes that have no descendants).

Then the TreeSoft Set is:

$$F: P(Tree(A)) \to P(H)$$

Tree(A) is the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and P(Tree(A)) is the powerset of the Tree(A).

All node sets of the *TreeSoft Set of level m* are:

$$\text{Tree}(A) = \{A_{i1} \mid i1 = 1, 2, \dots\}$$

The first set is formed by the nodes of level 1, second set by the nodes of level 2, third set by the nodes of level 3, and so on, the last set is formed by the nodes of level m. If the graph-tree has only two levels (m = 2), then the TreeSoft Set is reduced to a MultiSoft Set [8].

11. Practical Example of TreeSoft Set of Level 3

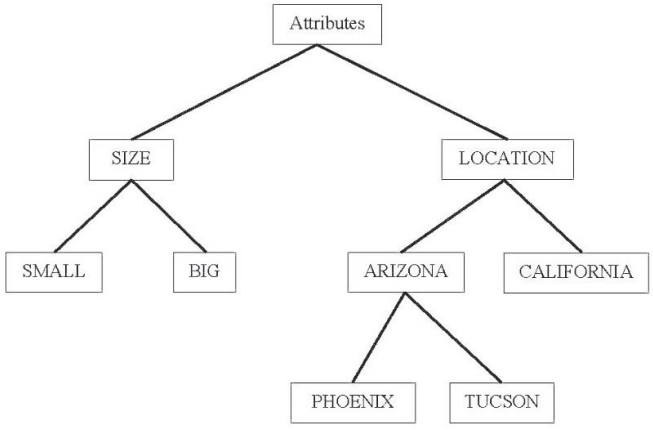


Figure 1. A TreeSoft Set of Level 3.

This is a classical tree, whose:
Level 0 (the root) is the node Attributes;
Level 1 is formed by the nodes: Size, Location;
Level 2 is formed by the nodes Small, Big, Arizona, California;
Level 3 is formed by the nodes Phoenix, Tucson.
Let's consider $H = \{h_1, h_2, \dots, h_{10}\}$ be a set of houses, and $P(H)$ the powerset of H.
And the set of Attributes: $A = \{A_1, A_2\}$, where A_1 = Size, A_2 = Location.
Then $A_1 = \{A_{11}, A_{12}\}$ = {Small, Big}, $A_2 = \{A_{21}, A_{22}\}$ = {Arizona, California} as American states.
Further on, $A_{22} = \{A_{211}, A_{212}\}$ = {Phoenix, Tucson} as Arizonian cities.

Let's assume that the function F gets the following values:

$$F(\text{Big, Arizona, Phoenix}\} = \{ h_9, h_{10} \}$$

$$F(\text{Big, Arizona, Tucson}) = \{ h_1, h_2, h_3, h_4 \}$$

$$F(\text{Big, Arizona}) = \text{all big houses from both cities, Phoenix and Tucson}$$

$$= F(\text{Big, Arizona, Phoenix}) \cup F(\text{Big, Arizona, Tucson}) = \{ h_1, h_2, h_3, h_4, h_9, h_{10} \}.$$

12. Conclusion

The present study represents an application of modern material statistics, specifically leveraging the principles of neutrosophic statistics.

The HyperSoft Set (2018) is a generalization of Soft Set (1999) and MultiSoft Set (2010), from a uni-variate function to a multi-variate function F;

IndetermSoft Set (2022) is an extension of the Soft Set, from the determinate data to indeterminate data;

IndetermHyperSoft Set (2022) is an extension of the HyperSoft Set, from the determinate data to indeterminate data;

and TreeSoft Set (2022) that is a generalization of the MultiSoft Set.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflict of interest

The authors declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Soft Set Product Extended to HyperSoft Set and IndetermSoft Set Product Extended to IndetermHyperSoft Set (updated)

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Abstract

In this paper we define the Soft Set Product as a product of many soft sets and afterwards we extend it to the HyperSoft Set. Similarly, the IndetermSoft Product is extended to the IndetermHyperSoft Set. We also present several applications of the Soft Set Product to Fuzzy (and fuzzy-extensions) Soft Set Product and to IndetermSoft Set and IndetermHyperSoft Set.

Keywords: HyperSoft set, IndetermSoft set, IndetermHyperSoft set, product.

1 | Introduction

Using the Product of the Soft Set [1] we connect it with its generalization called HyperSoft Set [2], [3]. Then, similarly, through another Product the IndetermSoft Set [4] is connected to the IndetermHyperSoft Set [4] and we present several applications of them.

2 | Definition of Soft Set Product

Let's have $n \geq 2$ Soft Sets

$$F_1: A_1 \rightarrow P(H),$$
$$F_2: A_2 \rightarrow P(H),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$F_n: A_n \rightarrow P(H),$$

where $a_1, a_2, \ldots, a_n$ are n distinct attributes, and respectively $A_1, A_2, \ldots, A_n$ their corresponding sets of attributes' values, such that

$A_i \cap A_j = \varnothing$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$.

The Soft Set Product $F_1 \times F_2 \times \ldots \times F_n$ is defined as follows:

$$F_1 \times F_2 \times \ldots \times F_n : A_1 \times A_2 \times \ldots \times A_n \rightarrow \underbrace{P(H) \times P(x) \times \ldots \times P(H)}_{n \text{ times}}.$$

3 | The Soft Set Product transformed into a HyperSoft Set

Let's denote $F = F_1 \times F_2 \times \ldots \times F_n$.

Then we define the HyperSoft Set in the following way:

$$F : A_1 \times A_2 \times \ldots A_n \rightarrow P(H).$$

such that for each $(e_1, e_2, \ldots, e_n) \in A_1 \times A_2 \times \ldots A_n$,

$$F(e_1, e_2, \ldots, e_n) = F_1(e_1) \cap F_2(e_2) \cap \ldots \cap F_n(e_n).$$

4 | Fuzzy (and Any Fuzzy-Extension) Soft Set Product

Let's denote the fuzzy (and any fuzzy-extension degree in general) by d_0.

For example, if one uses the Fuzzy Soft Set, then d_0 = T (degree of truth), $T \in [0, 1]$.

If one uses the Intuitionistic Fuzzy Set, the d_0 = (T, F) or degree of truth/ falsehood, $T, F \in [0, 1]$, $0 \leq T + F \leq 1$;

If one uses the Neutrosophic Set, then d_0 = (T, I, F), or degree of truth/ indeterminacy/ falsehood, one has:

$$0 \leq T + I + F \leq 3.$$

Let $H = \{h_1, h_2, h_3, h_4\}$, a set of houses.

Let the attributes a_1 = size, whose set of values is A_1 = {small, medium, big} and a_2 = location, whose set of values is A_2 = {central, peripherical}.

The soft sets

$$F_1: A_1 \rightarrow P(H),$$

$$F_2: A_2 \rightarrow P(H),$$

and their Product

$$F = F_1 \times F_2,$$

$$F: A_1 \times A_2 \rightarrow P(H).$$

F: {small, medium, big} $\times$ {central, peripherical} $\rightarrow P(\{h_1, h_2, h_3, h_4\})$.

F_1 (small) = $\{h_1, h_3, h_4\} \overset{def}{=} H_{11}$.

F_1 (medium) = $\{h_1, h_4\} \overset{def}{=} H_{12}$.

F_1 (big) = $\{h_2\} \overset{def}{=} H_{13}$.

F_2 (central) = $\{h_1\} \overset{def}{=} H_{21}$.

F_2 (peripheral) = $\{h_2, h_3, h_4\} \overset{def}{=} H_{22}$.

Size \ Location	Central	Peripheral
small	$\{h_1\}$	$\{h_3, h_4\}$
medium	$\{h_1\}$	$\{h_4\}$
big	∅ (no house)	$\{h_2\}$

5 | HyperSoft Set Resulted from a Product of Two Soft Sets

How to read the above table, for example the intersection between the row 1 (small) and the column 2 (peripheral) gives

F [small, peripheral] = F_1 (small) ∩ F_2 (peripheral) = $H_{11} \cap H_{22} = \{h_1, h_3, h_4\} \cap \{h_2, h_3, h_4\} = \{h_3, h_4\}$.

6 | Fuzzy and Fuzzy-Extensions Soft Set and HyperSoft Set

F_1 (small) = $H_2(d_1^0) = \{h_1, h_3, h_4\}(d_{11}^0)$.

F_2 (peripheral) = $H_2(d_2^0) = \{h_2, h_3, h_4\}(d_{22}^0)$.

F (small, peripheral) = $H_{11}(d_{11}^0) \cap H_{22}(d_{22}^0) = \{h_3, h_4\}(d_{11}^0 \wedge d_{22}^0)$.

7 | Definition of the IndetermSoft Set

Let U be a universe of discourse, H a non-empty subset of U, and P(H) the powerset of H. Let a be an attribute, and A be a set of this attribute values.

A function F: A → (H) is called an IndetermSoft Set (Function) if

I. The set A has some indeterminacy with respect to one or more attribute's values.
II. Or P(H) has some indeterminacy.
III. Or there exist at least an attribute value v ∈ A, such that F(v) = indeterminate (unclear, uncertain, or not unique).
IV. Or any two or all three of the above situations.

In other words, an IndetermSoft Set, introduced by Smarandache in [4], is a soft set that has some degree of indeterminate (unclear, uncertain, alternative, conflicting) data or procedure.

8 | Example of IndetermSoft Set and IndetermHyperSoft Set

F_1 (small) = $\{h_1, or\ h_4\}$ = {either h_1, or h_2, or h_1 and h_2} $\overset{def}{=} H_{12}$.

F_2 (peripheral) = $\{ noth_2 \}$ = {h_1, or h_3, or h_4} $\overset{def}{=} H_{134}$.

F (small, peripheral) = $H_{12} \cap H_{134}$ = {h_1, or ϕ (no house)} $\overset{def}{=} H_{10}$.

The first two equations above represent two IndetermSoft Set, only the third one gets the Product of them and obtains a IndetermHyperSoft Set.

As seeing, one has indeterminacy with respect to H_{12}, H_{123}, and the final result H_{10}.

Conclusion

We presented for the first time the connection between the Product of Soft Sets and the HyperSoft Set, and afterwards the Product of IndetermSoft Set that produced the IndetermHyperSoft Set, together with several applications.

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Practical Applications of IndetermSoft Set and IndetermHyperSoft Set and Introduction to TreeSoft Set as an extension of the MultiSoft Set

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Smarandache, Florentin (2022). "Practical Applications of IndetermSoft Set and IndetermHyperSoft Set and Introduction to TreeSoft Set as an extension of the MultiSoft Set." *Neutrosophic Sets and Systems,* Vol. 51, 2022, 939-947.

Abstract: The IndetermSoft Set is as an extension of the Soft Set, because the data, or the function, or the sets involved in the definition of the soft set have indeterminacy - as in our everyday life, and we still need to deal with such situations.

And similarly, IndetermHyperSoft Set as extension of the HyperSoft Set, when there is indeterminate data, or indeterminate functions, or indeterminate sets.

Herein, 'Indeterm' stands for 'Indeterminate' (uncertain, conflicting, incomplete, not unique outcome).

We now introduce for the first time the TreeSoft Set as extension of the MultiSoft Set.

Several applications are presented for each type of soft set.

Keywords: Soft Set, IndetermSoft Set, HyperSoft Set, IndetermHyperSoft Set, MultiSoft Set, TreeSoft Set

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1. Introduction

We have extended the Soft Set to HyperSoft Set [2, 3] in 2018, then both of them to IndetermSoft Set and IndetermHyperSoft Set [4, 8] respectively in 2022, and we have introduced Indeterminate Soft and HyperSoft operators.

The operations (complement, intersection, union) for IndetermSoft Set and IndetermHyperSoft Set respectively are to be done in the future research.

And in this paper a new type of soft set, called TreeSoft Set, is introduced for the first time as an extension of the MultiSoft Set.

Several applications are presented for each type of soft set.

2. Definition of Soft Set

Let U be a universe of discourse, H a non-empty subset of U, with $P(H)$ the powerset of H, and a an attribute (parameter, factor, etc.), with its set of attribute-values denoted by A. Then, the pair (F, A), with $F: A \to P(H)$, is called a (Classical) Soft Set over H.

Molodtsov [1] has defined in 1999 the Soft Set.

3. Real Example of (Classical) Soft Set

Let $H = \{h_1, h_2, h_3, h_4\}$ be a set of houses, and a an attribute, $a = color$, and its set of attribute-values $A = \{white, green, red\}$. The function $F : A \to P(H)$, as:

$F(white) = \{h_1, h_2, h_4\}$, $F(green) = h_3$, $F(red) = \phi$ *(no house).*

4. Definition of IndetermSoft Set

Smarandache [4, 8] introduced it in 2022.

Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the powerset of H. Let a be an attribute, and A be a set of this attribute-values.

Then $F: A \to P(H)$ is called an IndetermSoft Set if:

i) the set A has some indeterminacy;
ii) or the set $P(H)$ has some indeterminacy;
iii) or there exist at least an attribute-value $v \in A$, such that $F(v)$ = indeterminate (unclear, incomplete, conflicting, or not unique);
iv) or any two or all three of the above situations.

The IndetermSoft Set has some degree of indeterminacy, and as such it is a particular case of the NeutroFunction [5, 6], defined in 2014 – 2015, which is a function that is only partially well-defined (inner-defined), partially indeterminate, and partially outer-defined. The NeutroFunction is a generalization of the classical function, that is totally well-defined.

IndetermSoft Set, as extension of the classical (determinate) Soft Set, deals with indeterminate data, because there are sources [4, 8] unable to provide exact or complete information on the sets A, H or $P(H)$, and on the function F.

We did not add any indeterminacy, we found the indeterminacy in our real world. Because many sources give approximate/uncertain/incomplete/conflicting information, not exact information as in the Soft Set, as such we still need to deal with such situations.

For more information on IndetermSoft Set consult [4, 8].

5. Real Example of IndetermSoft Set:

Assume a town has many houses.

1) Indeterminacy with respect to the function.

1a) You ask a source:
- What houses have the red color in the town?
The source:
- I am not sure, I think the houses h_1 or h_2.
Therefore, $F(red) = h_1$ *or* h_2
(indeterminate / uncertain answer).
1b) You ask again:
- But, what houses are yellow?
The source:
- I do not know, the only thing I know is that the house h_5 is not yellow because I have visited it.
Therefore, $F(yellow) = not\ h_5$
(again indeterminate / uncertain answer).
1c) Another question you ask:
- Then what houses are blue?
The source:
- For sure, *either* h_8 *or* h_9
Therefore, $F(blue) =$ *either* h_8 *or* h_9
(again indeterminate / uncertain answer).

2) Indeterminacy with respect to the set H of houses.

You ask the source:

- How many houses are in the town?

The source:

- I never counted them, but I estimate their number to be *between 100-120 houses*.

3) Indeterminacy with respect to the set A of attributes.

You ask the source:

What are all colors of the houses?

The source:

I know for sure that there are houses of colors *red, yellow, and blue*, but I do not know if there are houses of other colors (?)

This is the IndetermSoft Set.

6. Definition of HyperSoft Set

Smarandache has extended in 2018 the Soft Set to the HyperSoft Set [3, 4, 8] by transforming the function F from a uni-attribute function into a multi-attribute function.

Let $\mathcal{U}$ be a universe of discourse, H a non-empty set included in U, and $P(H)$ the powerset of H. Let $a_1, a_2, \ldots, a_n$, where $n \geq 1$, be n distinct attributes, whose corresponding attribute-values are respectively the sets $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$.

Then the pair $(F, A_1 \times A_2 \times \ldots \times A_n)$, where $A_1 \times A_2 \times \ldots \times A_n$ represents the Cartesian product, with $F: A_1 \times A_2 \times \ldots \times A_n \to P(H)$ is called a HyperSoft Set.

In other words, for any $(e_1, e_2, ..., e_n) \in A_1 \times A_2 \times ... \times A_n$, $F(e_1, e_2, ..., e_n) \in P(H)$

7. Real Example of HyperSoft Set

Let $H = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$ a set of houses, and two attributes a_1 and a_2, where $a_1 = color$, and its set of attribute-values $A_1 = \{white, green, red\}$, and $a_2 = size$, and its attribute-values $A_2 = \{small, big\}$. The function $F : A_1 \times A_2 \to P(H)$, as :

$F(white, small) = \{h_1, h_2\}$, $F(green, big) = \{h_4, h_6, h_7\}$, $F(red, big) = \{h_3, h_5\}$.

8. Definition of IndetermHyperSoft Set

Smarandache [4, 8] introduced it in 2022.

Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the powerset of H. Let $a_1, a_2, \ldots, a_n$, where $n \geq 1$, be n distinct attributes, whose corresponding attribute-values are respectively the sets $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$.

Then the pair $(F, A_1 \times A_2 \times \ldots \times A_n)$, where $A_1 \times A_2 \times \ldots \times A_n$ represents the Cartesian product, with $F: A_1 \times A_2 \times \ldots \times A_n \to P(H)$ is called an IndetermHyperSoft Set if:

i) at least one of the sets $A_1, A_2, \ldots, A_n$ has some indeterminacy;

ii) or the set $P(H)$ has some indeterminacy;

iii) or there exist at least one n-plet $(e_1, e_2, ..., e_n) \in A_1 \times A_2 \times ... \times A_n$ such that $F(e_1, e_2, ..., e_n)$ = indeterminate (unclear, uncertain, conflicting, or not unique);

iv) or any two or all three of the above situations.

The IndetermHyperSoft Set has some degree of indeterminacy, and it is as extension of the (determinate) HyperSoft Set.

Similarly, we did not add any indeterminacy, we found the indeterminacy in our real world. Because many sources give approximate/uncertain/incomplete/conflicting information, not exact information as in the Soft Set and in the HyperSoft Set, as such we still need to deal with such situations.

9. Real Example of IndetermHyperSoft Set

Assume a town has many houses.

1) *Indeterminacy with respect to the function*.
1a) You ask a source:
2) What houses are of red color and big size in the town?
The source:
3) I am not sure, I think the houses h_1 *or* h_2.
Therefore, $F(red, big) = h_1$ *or* h_2
(indeterminate / uncertain answer).
1b) You ask again:
4) But, what houses are *yellow and small*?
The source:
5) I do not know, the only thing I know is that the house h_5 is neither yellow nor small because I have visited it.
Therefore, $F(yellow, small) = not\ h_5$
(again indeterminate / uncertain answer).
1c) Another question you ask:
- Then what houses are *blue and big*?
The source:
- For sure, *either* h_8 *or* h_9
Therefore, $F(blue, big) = either\ h_8\ or\ h_9$
(again indeterminate / uncertain answer).

2) *Indeterminacy with respect to the set H of houses*.
You ask the source:
- How many houses are in the town?
The source:
- I never counted them, but I estimate their number to be *between 100-120 houses*.

3) *Indeterminacy with respect to the set A of attributes*.
You ask the source:
What are all colors and sizes of the houses?
The source:
I know for sure that there are houses of colors of *red, yellow,* and *blue*, but I do not know if there are houses of other colors (?)
About the size, I saw many houses that are *small*, but I do not remember to have seing be *big* houses.

This is the IndetermHyperSoft Set.

10. Definition of MultiSoft Set [7]

Let U be a universe of discourse, and H a non-empty subset of U.
And $P(H)$ is the power set of H. Let $A_1, A_2, \ldots, A_n$ be $n \geq 2$ sets of attributes (parameters) whose intersection $A_1 \cap A_2 \cap \ldots \cap A_n = \phi$.
Let $A = A_1 \cup A_2 \cup \ldots \cup A_n$ and $P(A)$ be the power set of A.
Then $F : P(A) \to P(H)$ is a MultiSoft Set over H.
For $\varepsilon \in P(A)$ one considers that $F(\varepsilon)$ is the set of ε - approximate sets of the multisoft set *(F, P(A))*.

11. Extension of the MultiSoft Set to a HyperSoft Set

One introduces the empty-element ϕ to each set of attribute-values, and let denote

$A_1^{'} = A_1 \cup \{\phi\}, A_2^{'} = A_2 \cup \{\phi\}, ..., A_n^{'} = A_n \cup \{\phi\}$.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \in A_1^{'} \times A_2^{'} \times ... \times A_n^{'}$,

then $\varepsilon_1 \in A_1^{'} = A_1 \cup \{\phi\}$ means that either $\varepsilon_1 \in A_1$ or $\varepsilon_1 = \phi$ (discarded);

similarly for all $\varepsilon_i \in A_i^{'} = A_i \cup \{\phi\}$, $1 \le i \le n$.

Thus, $F : A_1^{'} \times A_2^{'} \times ... \times A_n^{'} \to P(H)$ is a hypersoft set.

12. Real Example of MultiSoft Set

We retake the previous example and adjust it to a MultiSoft Set.

Let $H = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$ a set of houses, and two attributes a_1 and a_2, where $a_1 = color$, and its set of attribute-values $A_1 = \{white, green, red\}$, and $a_2 = size$, and its attribute-values $A_2 = \{small, big\}$. Let $A = A_1 \cup A_2 = \{white, green, red ; small, big\}$, and $P(A)$ be the power set of A. Then $F : P(A) \to P(H)$ is defined as follows:

$F(white) = \{h_1\}$, $F(green, big) = \{h_4, h_6\}$, $F(big) = \{h_3, h_5\}$.

13. Real MultiSoft Set extended to a HyperSoft Set

Let's enlarge A1 and A2:

$A_1^{'}$ = {white, green, red, ϕ}, and $A_2^{'}$ = {small, big, ϕ}

Then $F^{'} : A_1^{'} \times A_2^{'} \to P(H)$

F′(white, ϕ) ≡ F(white) ={h1} (since the attribute-value ϕ was discarded).

F′(green, big) ≡ F(green, big) = {h4, h6}.

F′(ϕ, big) ≡ F(big) = {h3, h5} (since the attribute-value ϕ was discarded).

14. Generalization of MultiSoft Set to the TreeSoft Set

Let U be a universe of discourse, and H a non-empty subset of U, with $P(H)$ the powerset of H.

Let A be a set of attributes (parameters, factors, etc.),

$A = \{A_1, A_2, ..., A_n\}$, for integer $n \ge 1$,

where $A_1, A_2, ..., A_n$ are <u>*attributes of first level*</u> (since they have one-digit indexes).

Each attribute $A_i, 1 \le i \le n,$ is formed by sub-attributes:

$$A_1 = \{A_{1,1}, A_{1,2}, ...\}$$
$$A_2 = \{A_{2,1}, A_{2,2}, ...\}$$
$$.$$
$$.$$
$$.$$
$$A_n = \{A_{n,1}, A_{n,2}, ...\}$$

where $A_{i,j}$ are <u>*sub-attributes (or attributes of second level)*</u> (since they have two-digit indexes).

Again, each sub-attribute $A_{i,j}$ is formed by <u>*sub-sub-attributes (or attributes of third level)*</u>:

$$A_{i,j,k}$$

And so on, as much refinement as needed into each application, up to *sub-sub-…-sub-attributes (or attributes of m-level* (or having *m* digits into the indexes):

$$A_{i_1,i_2,...,i_m}$$

Therefore, a graph-tree is formed, that we denote as *Tree(A)*, whose root is *A* (considered of *level zero*), then nodes of *level 1, level 2,* up to *level m.*

We call *leaves* of the graph-tree, all terminal nodes (nodes that have no descendants).

Then the TreeSoft Set is:

$$F : P(Tree(A)) \to P(H).$$

Tree(A) is the set of all nodes and leaves (from *level 1* to *level m*) of the graph-tree, and $P(Tree(A))$ is the powerset of the *Tree(A).*

All node sets of the *TreeSoft Set of level m* are:

$$Tree(A) = \{A_{i_1} \mid i_1 = 1,2,...\} \cup \{A_{i_1,i_2} \mid i_1, i_2 = 1,2,...\} \cup \{A_{i_1,i_2,i_3} \mid i_1, i_2, i_3 = 1,2,...\} \cup ... \cup \{A_{i_1,i_2,...,i_m} \mid i_1, i_2, ..., i_m = 1,2,...\}$$

The first set is formed by the *nodes of level 1*, second set by the *nodes of level 2*, third set by the *nodes of level 3*, and so on, the last set is formed by the *nodes of level m*.

If the graph-tree has only two levels (*m = 2*), then the TreeSoft Set is reduced to a MultiSoft Set.

15. Example of TreeSoft Set of Level 3

Node of *level 0* (the graph-tree root): A.

Nodes of *level 1*: A_1, A_2.

Nodes of *level 2*: A_{11}, A_{12}; A_{21}, A_{22}.

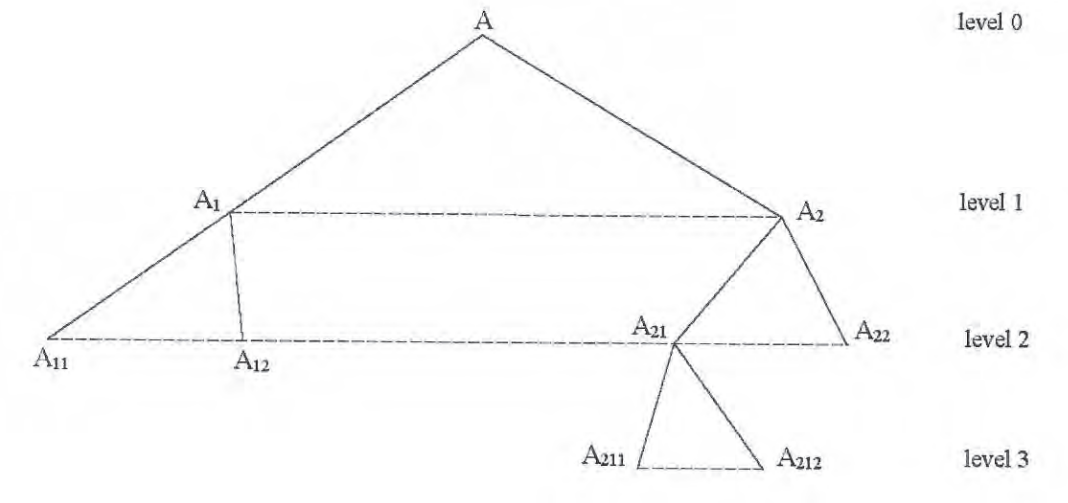
Nodes of *level 3*: A_{211}, A_{212}.

Whence *Tree(A)* = {A_1, A_2; A_{11}, A_{12}; A_{21}, A_{22}; A_{211}, A_{212}}.

The *leaves* are: A_{11}, A_{12}; A_{211}, A_{212}; A_{22}. As we see, the leaves may have various levels, in this case: *2*, or *3*.

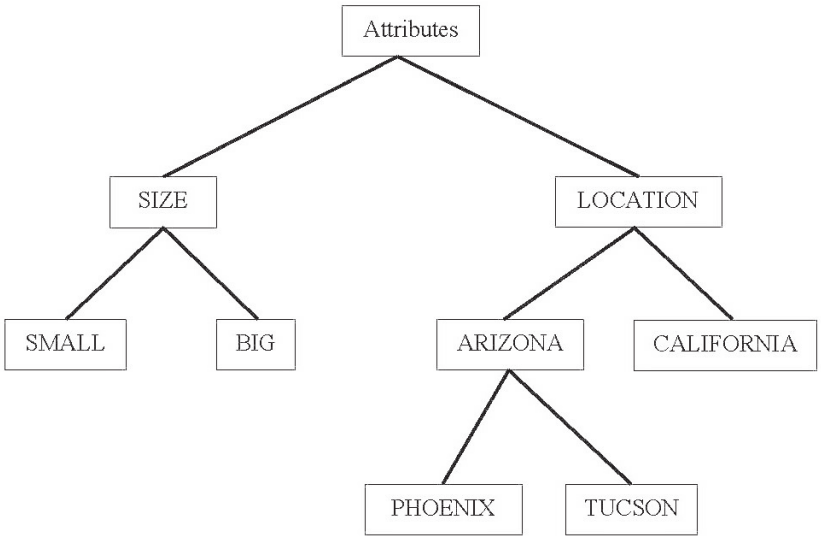
$P(Tree(A))$ is the powerset of $Tree(A)$.

F: $P(Tree(A)) \to P(H)$ is a TreeSoft Set of Level *3*.



Graph 1: TreeSoft Set of Level 3

16. Practical Example of TreeSoft Set of Level 3



Graph 2: Practical TreeSoft Set of Level 3

et's consider H = $\{h_1, h_2, \dots, h_{10}\}$ be a set of houses, and *P(H)* the powerset of *H*.
And the set of attributes: *A* = $\{A_1, A_2\}$,
where A_1 = *size,* A_2 = *location.*
Then $A_1 = \{A_{11}, A_{12}\}$ = *{small, big}.*
$A_2 = \{A_{21}, A_{22}\}$ = *{Arizona, California},* American states.
Further on, $A_{21} = \{A_{211}, A_{212}\}$ = *{Phoenix, Tucson},* Arizonian cities.
Let's assume that the function F gets the following values:
F(big, Arizona, Phoenix) = $\{h_9, h_{10}\}$
F(big, Arizona, Tucson) = $\{h_1, h_2, h_3, h_4\}$
F(big, Arizona) = all big houses from both cities, Phoenix and Tucson,
= *F(big, Arizona, Phoenix)* $\cup$ *F(big, Arizona, Tucson)*= $\{h_1, h_2, h_3, h_4, h_9, h_{10}\}$.

17. Properties of the TreeSoft Set

17.1. Theorem 1

F(node) includes all node's descendants, and sub-descendants, then sub-sub-descendants, and so on up to the corresponding leaves.

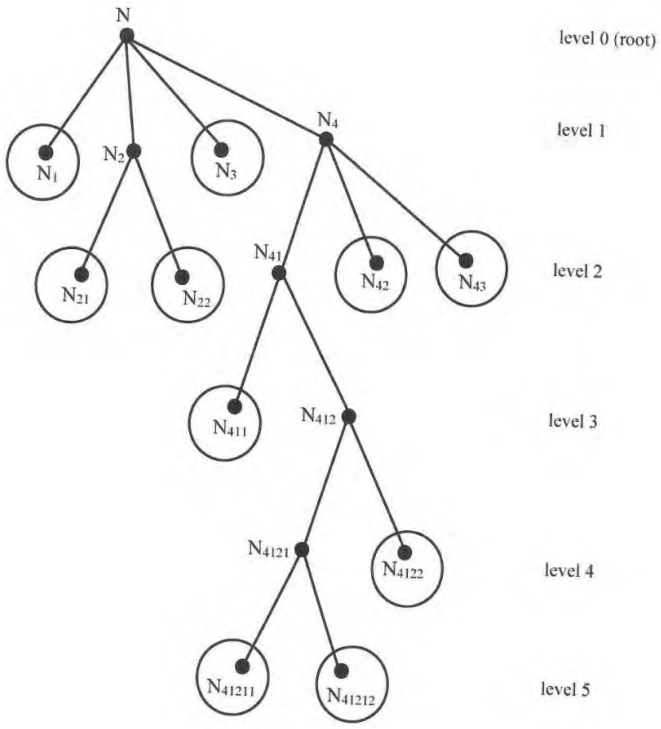
From previous Example 15, one has:
$F(A_{21}) = F(A_{211}) \cup F(A_{212})$,
and consequently
$F(A_{12}, A_{21}) = F(A_{12}, A_{211}) \cup F(A_{12}, A_{212})$.

17.2. Theorem 2

Let $N \in Tree(A)$ be a node.
N generates a *SubTree(N)* whose root is *N* itself.
Then $F(N) = \cup_{\varphi(i)} F(N_{\varphi(i)})$
where $N_{\varphi(i)}$ are all leaves of the SubTree (*N*).
From previous Example 15:

$F(A_2) = F(A_{21}) \cup F(A_{22}) = \big(F(A_{211}) \cup F(A_{212})\big) \cup F(A_{22}) = F(A_{211}) \cup F(A_{212}) \cup F(A_{22})$
where A_{211}, A_{212}, A_{22} are all leaves of the SubTree whose root is A_2 {i.e. *SubTree*(A_2)}.

The *proof* of Theorem 2 is obvious, no matter what graph-tree one has, and it is similar to the below Example:



Graph 3: Tree(N)

The circled nodes are the leaves.
$F(N) = F(N_1) \cup F(N_2) \cup F(N_3) \cup F(N_4)$
$= F(N_1) \cup [F(N_{21}) \cup F(N_{22}) \cup F(N_3) \cup F(N_{41}) \cup F(N_{42}) \cup F(N_{43})]$
$= F(N_1) \cup F(N_{21}) \cup F(N_{22}) \cup F(N_3) \cup [F(N_{411}) \cup F(N_{412})] \cup F(N_{42}) \cup F(N_{43})$
$= F(N_1) \cup F(N_{21}) \cup F(N_{22}) \cup F(N_3) \cup F(N_{411}) \cup [F(N_{4121}) \cup F(N_{4122})] \cup F(N_{42}) \cup F(N_{43})$
$= F(N_1) \cup F(N_{21}) \cup F(N_{22}) \cup F(N_3) \cup F(N_{411}) \cup F(N_{41211}) \cup F(N_{41212}) \cup F(N_{42}) \cup F(N_{43})$
which is the union of the soft-values *F(.)* of all leaves of the *SubTree(N).*
Actually Theorems 1 and 2 are equivalent.

17.3. Theorem 3

$F\left(N_{i_1}, N_{i_2}, \ldots, N_{i_p}\right) = F\left(N_{i_1}\right) \cap F\left(N_{i_2}\right) \cap \ldots \cap F\left(N_{i_p}\right)$,
where $N_{i_1}, N_{i_2}, \ldots, N_{i_p}$ are nodes of various levels into the TreeSoft Set of N.

The *proof* results from the fact that $F(N_{i_1})$ represents the subset H_1 of elements in H that have the attribute-value N_{i_1}, and $F(N_{i_2})$ represents the subset H_2 of elements in H that have the attribute-value N_{i_2}, and so on $F(N_{i_p})$ represents the subset H_p of elements in H that have the attribute-value N_{i_p}, therefore to get the elements that have all these attribute-values one needs to intersect these subsets $H_1 \cap H_2 \cap \ldots \cap H_p$.

18. Future Research

To define the operations (complement, intersection, union) for IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set respectively and to use them in real applications.

19. Conclusion

We introduced the TreeSoft Set as an extension of the MultiSoft Set. We presented simple practical applications of IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set respectively for better understanding.

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Aplicaciones prácticas de IndetermSoft Set e IndetermHyperSoft Set e Introducción a TreeSoft Set como una extensión del MultiSoft Set

Practical applications of IndetermSoft Set and IndetermHyperSoft Set and Introduction to TreeSoft Set as an extension to MultiSoft Set

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Resumen. El IndetermSoft Set es como una especie de extensión del Soft Set, porque los datos, o la función, o los conjuntos involucrados en la definición del Soft Set tienen indeterminación, como en nuestra vida cotidiana, y todavía necesitamos lidiar con tales situaciones. Del mismo modo, el IndetermHyperSoft Set es una extensión del HyperSoft Set, cuando hay datos indeterminados, o funciones indeterminadas, o conjuntos indeterminados. Aquí, 'Indeterm' significa 'Indeterminado' (resultado incierto, conflictivo, incompleto, no único). Ahora se presenta por primera vez el TreeSoft Set como extensión del MultiSoft Set. Se presentan varias aplicaciones para cada tipo de Soft Set.

Palabras clave: Soft Set, IndetermSoft Set, HyperSoft Set, IndetermHyperSoft Set, MultiSoft Set, TreeSoft Set.

Abstract. The IndetermSoft Set is kind of an extension of the Soft Set, because the data, or the function, or the sets involved in the Soft Set definition have indeterminacy, as in our daily life, and we still need to deal with such situations. Similarly, the IndetermHyperSoft Set is an extension of the HyperSoft Set, when there are indeterminate data, or indeterminate functions, or indeterminate sets. Here, 'Indeterm' means 'Indetermi-nate' (uncertain, conflicting, incomplete, non-unique result). The TreeSoft Set is now presented for the first time as an extension of the MultiSoft Set. Several applications are presented for each type of Soft Set.

Keywords: Soft Set, IndetermSoft Set, HyperSoft Set, IndetermHyperSoft Set, MultiSoft Set, TreeSoft Set.

1 Introducción

Se ha extendido el Soft Set a HyperSoft Set [2, 3] en 2018, luego ambos a IndetermSoft Set e IndetermHyperSoft Set [4, 8] respectivamente en 2022, y se han introducido los operadores Indeterminate Soft e HyperSoft.

Las operaciones (complemento, intersección, unión) para IndetermSoft Set e IndetermHyperSoft Set respectivamente se realizarán en futuras investigaciones.

Y en este documento, se presenta por primera vez un nuevo tipo de Soft Set, llamado TreeSoft Set, como una extensión del MultiSoft Set.

Se presentan varias aplicaciones para cada tipo de Soft Set.

2 Definición de Soft Set

Sea U un universo de discurso, H un subconjunto no vacío de U, con $P(H)$ el conjunto de potencias de H, y a un atributo (parámetro, factor, etc.) con su conjunto de valores de atributo denotado por A. Entonces, el par (F, A) con F: A.), con su conjunto de valores de atributo denotado por A. Entonces, el par (F, A), con $F: A \to P(H)$, se denomina Soft Set (Clásico) sobre H.

Molodtsov [1] definió el Soft Set en 1999.

3 Ejemplo real de Soft Set (clásico)

Sea H = {h1, h2, h3, h4} un conjunto de casas, y a un atributo, a = color, y su conjunto de valores de atributo A = {blanco, verde, rojo}. La función F: A ⊓ (H), como:
F(blanco) = {h1, h2, h4}, F(verde) = h3, F(rojo) = (sin casa).

4 Definición de IndetermSoft Set

Smarandache [4, 8] lo introdujo en 2022.

Sea U un universo del discurso, H un subconjunto no vacío de U, y P(H) el conjunto de potencias de HH. Sea a sea un atributo, y A un conjunto de valores de este atributo. Entonces $F: A \rightarrow P(H)$ se llama Conjunto IndetermSoft si:

i) el conjunto A tiene alguna indeterminación;
ii) o el conjunto P(H) tiene alguna indeterminación;
iii) o existe al menos un atributo-valor $v \in A$, tal que F(v) = indeterminado (poco claro, incompleto, conflictivo o no único);
iv) o dos o las tres situaciones anteriores.

El IndetermSoft Set tiene cierto grado de indeterminación, y como tal es un caso particular de la NeutroFunción [5, 6], definida en 2014 - 2015, que es una función que sólo está parcialmente bien definida (definida interiormente), parcialmente indeterminada y parcialmente definida exteriormente. La Neutrofunción es una generalización de la función clásica, que está totalmente bien definida.

El IndetermSoft Set, como extensión del Soft Set clásico (determinado), trata con datos indeterminados, porque hay fuentes [4, 8] incapaces de proporcionar información exacta o completa sobre los conjuntos A, H o P(H), y la función F.

No se ha añadido ninguna indeterminación. La indeterminación es propia de nuestro mundo real. Porque muchas fuentes dan información aproximada/incierta/incompleta/conflictiva, no información exacta como en el Soft Set, por lo que todavía tenemos que lidiar con estas situaciones.

Para más información sobre el IndetermSoft Set, consulte [4, 8].

5 Ejemplo real de IndetermSoft Set:

Supongamos que una ciudad tiene muchas casas.

1) *Indeterminación con respecto a la función*.
1a) Si se le pregunta a una fuente:
- ¿Qué casas tienen el color rojo en la ciudad?

La fuente:
- No estoy seguro, creo que las casas h1 o h2.

Por lo tanto, F(rojo) = h1 o h2 (respuesta indeterminada / incierta).
1b) Vuelve a preguntar:
- Pero ¿qué casas son amarillas?

La fuente:
- No lo sé, lo único que sé es que la casa h5 no es amarilla porque la he visitado.

Por lo tanto, F(amarillo) = no h5 (de nuevo respuesta indeterminada / incierta).
1c) Otra pregunta que se puede hacer:
- Entonces, ¿qué casas son azules?

La fuente:
- Con certeza, h8 o h9

Por lo tanto, F(azul) = h8 o h9
(de nuevo respuesta indeterminada / incierta).

2) *Indeterminación con respecto al conjunto H de casas.*
Pregunta a la fuente:
- ¿Cuántas casas hay en la ciudad?

La fuente:
- Nunca las he contado, pero estimo que su número oscila entre 100 y 120 casas.

3) *Indeterminación con respecto al conjunto A de atributos.*
Pregunta a la fuente:
¿Cuáles son los colores de las casas?
La fuente:
Sé con certeza que hay casas de colores rojo, amarillo y azul, pero no sé si hay casas de otros colores (?)

Este es el IndetermSoft Set.

6 Definición de Hypersoft Set

Smarandache extendió en 2018 el Soft Set al Hypersoft Set [3, 4, 8] transformando la función F de una función uniatributo a una función multiatributo.

Sea $\mathcal{U}$ un universo de discurso, H un conjunto no vacío incluido en U, y P(H) el conjunto de potencias de H. Sean $a1$, $a2$, ..., an, donde $n \geq 1$, n atributos distintos, cuyos correspondientes atributos-valores son respectivamente los conjuntos $A1$, $A2$, ..., An, con $Ai \cap Aj = \emptyset$, para $i \neq j$, y $i, j \in \{1, 2, ..., n\}$.

Entonces el par (F, $A1 \times A2 \times ... \times An$), donde $A1 \times A2 \times ... \times An$ representa el producto cartesiano, con F: $A1 \times A2 \times ... \times An \rightarrow P(H)$ se denomina Hypersoft Set.

En otras palabras, para cualquier (e_1, e_2, ..., e_n) A_1 A_2 ... A_n, F (e_1, e_2, ..., e_n) Π(H)

7 Ejemplo real de HyperSoft Set

Sea H = {h1, h2, h3, h4, h5, h6, h7} un conjunto de casas, y dos atributos a1 y a2, donde a1 = color, y su conjunto de valores-atributo A1 = {blanco, verde, rojo}, y a2 = tamaño, y sus valores-atributo A2 = {pequeño, grande}. La función F: A1×A2 → P(H), tal que:

F(blanco, pequeño) = {h1, h2}, F(verde, grande) = {h4, h6, h7}, F(rojo, grande) = {h3, h5}.

8 Definición de IndetermHyperSoft Set

Smarandache [4, 8] lo introdujo en 2022.

Sea U un universo del discurso, H un subconjunto no vacío de U, y P(H) el conjunto de potencias de H. Sea $a1$, $a2$, ..., an, donde $n \geq 1$, sean n atributos distintos, cuyos correspondientes atributos-valores son respectivamente los conjuntos A_1, A_2, ..., A_n, con $A_i \cap A_j = \emptyset$, para $i \neq j$, y $i, j \in \{1, 2, ..., n\}$.

Entonces el par (F, $A_1 \times A_2 \times ... \times An$), donde $A1 \times A2 \times ... \times An$ representa el producto cartesiano, con F: $A_1 \times A_2 \times ... \times An \rightarrow P(H)$ se denomina IndetermHyperSoft Set si:

i) al menos uno de los conjuntos A1, A2, ..., An tiene alguna indeterminación;

ii) o el conjunto P(H) tiene alguna indeterminación;

iii) o existe al menos un n-pleto (e1, e2, ..., en) A_1 A_2 ... A_n tal que F (e1, e2, ..., en) = indeterminado (poco claro, incierto, conflictivo o no único);

iv) o dos o las tres situaciones anteriores.

El Conjunto IndetermHyperSoft tiene cierto grado de indeterminación y es una extensión del Conjunto HyperSoft (determinado).

Del mismo modo, no se añade ninguna indeterminación, sino que la indeterminación se encuentra en nuestro mundo real. Porque muchas fuentes dan información aproximada/incierta/incompleta/conflictiva, no información exacta como en el Soft Set y en el Hypersoft Set, por lo que todavía tenemos que tratar con esas situaciones.

9 Ejemplo real de IndetermSoft Set

Supongamos que una ciudad tiene muchas casas.

1) *Indeterminación con respecto a la función.*

1a) Se pregunta a una fuente:

- ¿Qué casas son de color rojo y gran tamaño en la ciudad?

La fuente:

- No estoy seguro, creo que las casas h1 o h2.

Por lo tanto, F(rojo, grande) = h1 o h2 (respuesta indeterminada / incierta).

1b) Vuelve a preguntar:

- Pero ¿qué casas son amarillas y pequeñas?

La fuente:

- No lo sé, lo único que sé es que la casa h5 no es amarilla ni pequeña porque la he visitado.

Por lo tanto, F(amarillo, pequeño) = no h5 (de nuevo respuesta indeterminada / incierta).

1c) Otra pregunta:

- Entonces, ¿qué casas son azules y grandes?

La fuente:

- Ciertamente, h8 o h9

Por lo tanto, F (azul, grande) = h8 o h9

(de nuevo respuesta indeterminada / incierta).

2) *Indeterminación respecto al conjunto H de casas.*
Pregunta a la fuente:
- ¿Cuántas casas hay en la ciudad? La fuente:
- Nunca las he contado, pero estimo que su número oscila entre 100 y 120 casas.

3) *Indeterminación con respecto al conjunto A de atributos.*
Pregunta a la fuente:
¿De qué colores y tamaños son las casas?
La fuente:
Sé con certeza que hay casas de colores rojo, amarillo y azul, pero no sé si hay casas de otros colores (?)
Sobre el tamaño, vi muchas casas que son pequeñas, pero no recuerdo haber visto casas grandes.

Este es el IndetermHyperSoft Set.

10 Definición de MultiSoft Set [7]

Sea U un universo de discurso, y H un subconjunto no vacío de U.
Y $P(H)$ es el conjunto de potencias de H. Sean $A_1, A_2, ..., A_n$ n ≥ 2 conjuntos de atributos (parámetros) cuya intersección
$A_1 \cap A_2 \cap ... \cap A_n = \phi$.
Sea $A = A_1 \cup A_2 \cup ... \cup A_n$ y P(A) el conjunto de potencias de A.
Entonces $F: P(A) \rightarrow P(H)$ es un MultiSoft Set sobre H.
Para $\varepsilon \in P(A)$ se considera que $F(\varepsilon)$ es el conjunto de ε- conjuntos aproximados del MultiSoft Set $(F, P(A))$

11 Extensión del Multisoft Set a un Hipersoft Set

Se introduce el elemento vacío ϕ en cada conjunto de valores de atributo y se denota
$A_1' = A_1 \cup \{\phi\}, A_2' = A_2 \cup \{\phi\}, ..., A_n' = A_n \cup \{\phi\}$.
Sea $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_n) \in A_1' \times A_2' \times ... \times A_n'$,
entonces $\varepsilon_1 \in A_1' = A_1 \cup \{\phi\}$ significa que $\varepsilon_1 \in A_1$ ó $\varepsilon_1 = \phi$ (descartado);
de forma similar para todos $\varepsilon_i \in A_i' = A_i \cup \{\phi\}, 1 \leq i \geq n$
Por lo que, $F: A_1' \times A_2' \times ... \times A_n' \rightarrow P(H)$ es un Hipersoft Set.

12 Ejemplo real de MultiSoft Set

Se retoma el ejemplo anterior para adaptarlo a un MultiSoft Set.
Sea $H = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$ un conjunto de casas, y dos atributos a_1 y a_2, donde $a_1 = color$, y su conjunto de valores-atributo $A_1 = \{blanco, verde, rojo\}$, y $a_2 = tamaño$, y sus valores-atributo $A_2 = \{pequeño, grande\}$. Sea $A = A_1 \cup A_2 = \{blanco, verde, rojo; pequeño, grande\}$, y $P(A)$ el conjunto de potencias de A.
Entonces $F: P(A) \rightarrow P(H)$ se define del siguiente modo:
$F(blanco) = \{h_1\}, F(verde, grande) = \{h_4, h_6\}, F(grande) = \{h_3, h_5\}$.

13 Multisoft Set real ampliado a un Hipersoft Set

Ampliemos A1 y A2:
$A_1' = \{blanco, verde, rojo, \phi\}$ y $A_2' = \{pequeño, grande, \phi\}$
F': $A_1' \times A_2' \rightarrow P(H)$
Entonces
F'(blanco, ϕ) ≡ F(blanco) ={h1} (ya que se descartó el atributo-valor ϕ).
F'(verde, grande) ≡ F(verde, grande) = {h4, h6}.
F'(ϕ, grande) ≡ F(grande) = {h3, h5} (ya que se descartó el atributo-valor ϕ).

14 Generalización de MultiSoft Set a TreeSoft Set

Sea U un universo de discurso, y H un subconjunto no vacío de U, siendo $P(H)$ el conjunto de potencias de H. Sea A un conjunto de atributos (parámetros, factores, etc.),
$A = \{A_1, A_2, ..., A_n\}$, para $n \geq 1$,
donde $A_1, A_2, ..., A_n$ son atributos de primer nivel (ya que tienen índices de un dígito).
Cada atributo A_i, $1 \leq i \geq n$, está formado por subatributos:

$$A_1 = \{A_{1,1}, A_{1,2}, ...\}$$
$$A_2 = \{A_{2,1}, A_{2,2}, ...\}$$

$$A_n = \{A_{n,1}, A_{n,2}, ...\}$$

Donde $A_{i,j}$ son subatributos (o atributos de segundo nivel) (ya que tienen índices de dos dígitos).
De nuevo, cada subatributo $A_{i,j}$ está formado por sub-subatributos (o atributos de tercer nivel):

$A_{i,j,k}$

Y así sucesivamente, tanto refinamiento como sea necesario en cada aplicación, hasta sub-sub-...-sub-atributos o atributos de nivel *m* (o que tengan *m* dígitos en los índices):

$$A_{i_1,i_2,\ldots,i_m,}$$

Por lo tanto, se forma un grafo-árbol, que denotamos como *Tree(A)*, cuya raíz es *A* (considerado de *nivel cero*), luego nodos de *nivel 1*, *nivel 2*, hasta el *nivel m*.

Llamamos *hojas* del grafo-árbol, a todos los nodos terminales (nodos que no tienen descendientes). Entonces el TreeSoft Set es:

$$F: P(Tree(A)) \rightarrow P(H).$$

Tree(A) es el conjunto de todos los nodos y hojas (del *nivel 1* al *nivel m*) del grafo-árbol, y *P(Tree(A))* es el conjunto de potencias del *Tree(A)*.

Todos los conjuntos de nodos del *TreeSoft Set de nivel m* son:

$$Tree(A) = \{A_{i_1} | i_1 = 1,2,\ldots\} \cup \{A_{i_1,i_2} | i_1, i_2 = 1,2,\ldots\} \cup\cup \{A_{i_1,i_2,i_3} | i_1, i_2, i_3 = 1,2,\ldots\} \cup \ldots \cup \{A_{i_1,i_2,\ldots,i_m} | i_1, i_2, \ldots, i_m = 1,2,\ldots\}$$

El primer conjunto está formado por los *nodos del nivel 1*, el segundo conjunto por los *nodos del nivel 2*, el tercer conjunto por los *nodos del nivel 3*, y así sucesivamente, el último conjunto está formado por los *nodos del nivel m*.

Si el grafo-árbol sólo tiene dos niveles (m = 2), el TreeSoft Set se reduce a un MultiSoft Set.

15 Ejemplo de TreeSoft Set de nivel 3

Nodo de nivel 0 (raíz del grafo o árbol): A.
Nodos de nivel 1: A_1, A_2.
Nodos de nivel 2: A_{11}, A_{12}, A_{21}, A_{22}.
Nodos de nivel 3: A_{211}, A_{212}.
De donde *Tree(A) = {A1, A2; A11, A12; A21, A22; A211, A212}*.
Las hojas son: A_{11}, A_{12}; A_{211}, A_{212}; A_{22}. Como se puede ver, las hojas pueden tener varios niveles, en este caso: 2, o 3.
P(Tree(A)) es el conjunto de potencia de *Tree(A)*.
F: P(Tree(A)) → P(H) es un TreeSoft Set de Nivel 3.

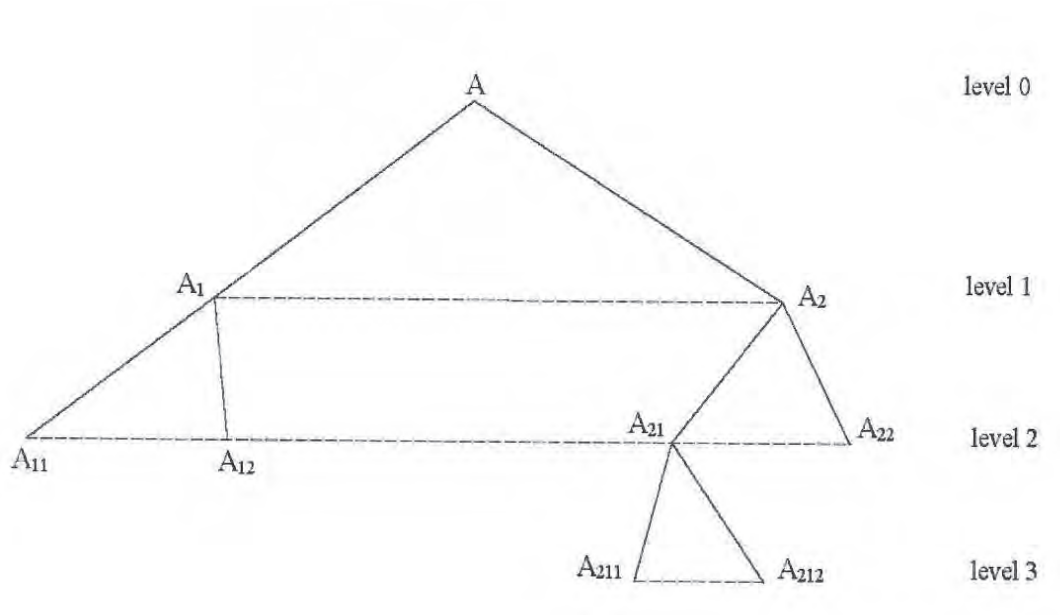
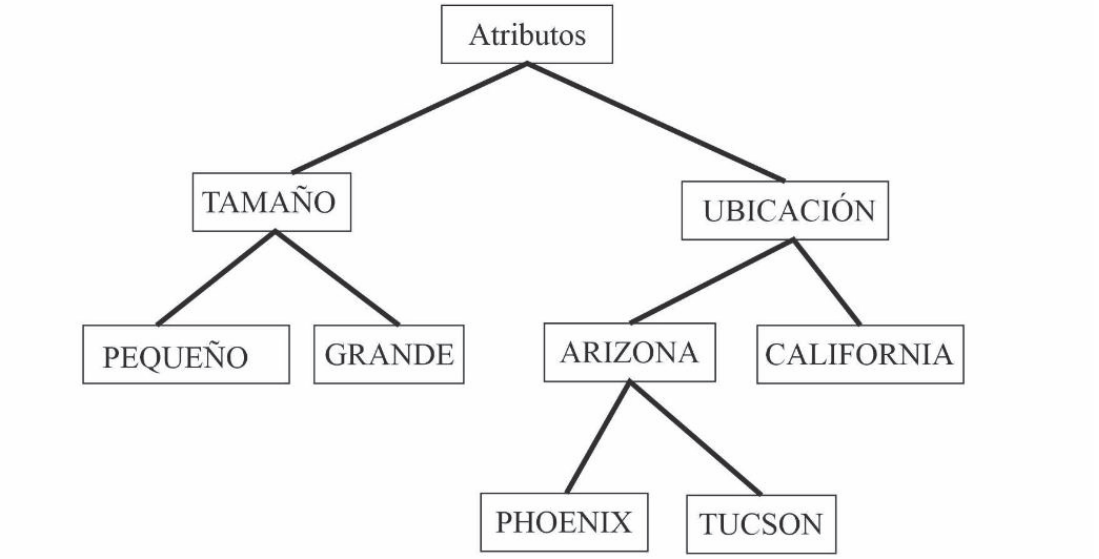


Figura 1: TreeSoft Set de Nivel 3

16 Ejemplo práctico de TreeSoft Set de nivel 3

Figura 2: TreeSoft Set de Nivel 3 práctico.



Consideremos que H = $\{h_1, h_2, ..., h_{10}\}$ es un conjunto de casas, y *P(H)* el conjunto de potencias de *H*.
Y el conjunto de atributos: $A = \{A_1, A_2\}$, donde A_1 = tamaño, A_2 = ubicación.
Entonces $A_1 = \{A_{11}, A_{12}\}$ = {pequeño, grande}.
$A_2 = \{A_{21}, A_{22}\}$ = {Arizona, California}, estados americanos.

Más adelante, $A_{21} = \{A_{211}, A_{212}\}$ = {Phoenix, Tucson}, ciudades de Arizona. Supongamos que la función F obtiene los siguientes valores:

F(grande, Arizona, Phoenix) = $\{h_9, h_{10}\}$
F(grande, Arizona, Tucson) = $\{h_1, h_2, h_3, h_4\}$
F(grande, Arizona) = todas las casas grandes de ambas ciudades, Phoenix y Tucson,
= F(grande, Arizona, Phoenix) $\cup$ F(grande, Arizona, Tucson)= $\{h_1, h_2, h_3, h_4, h_9, h_{10}\}$

17 Propiedades del TreeSoft Set

17.1 Teorema 1

F(nodo) incluye todos los descendientes del nodo, y los sub-descendientes, luego los sub-sub-descendientes, y así sucesivamente hasta las hojas correspondientes.

Del Ejemplo anterior (15), se tiene:

$F(A_{21}) = F(A_{211}) \cup F(A_{212})$, y por consiguiente
$F(A_{12}, A_{21}) = F(A_{12}, A_{211}) \cup F(A_{12}, A_{212})$.

17.2 Teorema 2

Sea $N \in \text{Tree}(A)$ un nodo.
N genera un *SubTree*(*N*) cuya raíz es el propio *N*.
Entonces $F(N) = \bigcup_{\varphi(i)} F(N_{\varphi(i)})$
donde $N_{\varphi(i)}$ son todas las hojas del SubTree(N).

Del Ejemplo anterior (15):

$F(A_2) = F(A_{21}) \cup F(A_{22}) = (F(A_{211}) \cup F(A_{212})) \cup F(A_{22}) = F(A_{211}) \cup F(A_{212}) \cup F(A_{22})$
donde A_{211}, A_{212}, A_{22} son todas las hojas del SubTree cuya raíz es A_2 {es decir, SubTree(A_2)}.

La *prueba* del Teorema 2 es obvia, independientemente del grafo-árbol que se tenga, y es similar al siguiente Ejemplo:

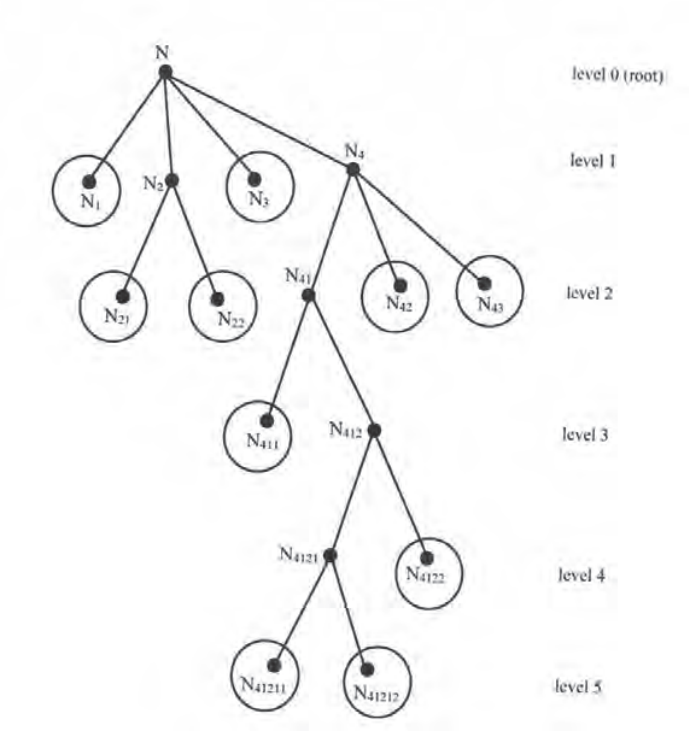


Figura 3: Tree (N).

Los nodos marcados con un círculo son las hojas.
$F(N) = F(N_1) \cup F(N_2) \cup F(N_3) \cup F(N_4)$
$= F(N1) \cup [F(N21) \cup F(N22) \cup F(N3) \cup F(N41) \cup F(N42) \cup F(N43)]$
$= F(N1) \cup F(N21) \cup F(N22) \cup F(N3) \cup [F(N411) \cup F(N412)] \cup F(N42) \cup F(N43)$
$= F(N1) \cup F(N21) \cup F(N22) \cup F(N3) \cup F(N411) \cup [F(N4121) \cup F(N4122)] \cup F(N42) \cup F(N43)$
$= F(N1) \cup F(N21) \cup F(N22) \cup F(N3) \cup F(N411) \cup F(N41211) \cup F(N41212) \cup F(N42) \cup F(N43)$
que es la unión de los soft-values F(.) de todas las hojas del SubTree(N).
En realidad, los teoremas 1 y 2 son equivalentes.

17.3 Teorema 3

$F\left(N_{i_1}, N_{i_2}, \dots, N_{i_p}\right) = F\left(N_{i_1}\right) \cap F\left(N_{i_2}\right) \dots \cap F\left(N_{i_p}\right)$,

donde $N_{i_1}, N_{i_2}, \dots, N_{i_p}$ son nodos de varios niveles en el TreeSoft Set de N.

La *prueba* resulta del hecho de que F (N_{i_1}) representa el subconjunto H_1 de elementos en H que tienen el atributo-valor N_{i_1}, y F (N_{i_2}) representa el subconjunto H_2 de elementos en H que tienen el atributo-valor N_{i_2} *y* así sucesivamente F (N_{i_p}) representa el subconjunto H_p de elementos en H que tienen el atributo-valor N_{i_p}, por lo que para obtener los elementos que tienen todos estos valores de atributo hay que intersecan estos subconjuntos

$H_1 \cap H_2 \cap \dots \cap H_p$.

18 Investigación futura

Definir las operaciones (complemento, intersección, unión) para IndetermSoft Set, IndetermHyperSoft Set y TreeSoft Set respectivamente y utilizarlas en aplicaciones reales.

Conclusión

Presentamos el TreeSoft Set como una extensión del MultiSoft Set. Presentamos aplicaciones prácticas sencillas de IndetermSoft Set, IndetermHyperSoft Set y TreeSoft Set, respectivamente, para una mejor comprensión.

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A Concise Study of Some Superhypergraph classes

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Abstract. In graph theory, the hypergraph [22] extends the traditional graph structure by allowing edges to connect multiple vertices, and this concept is further broadened by the superhypergraph [174,176]. Additionally, several types of uncertain graphs have been explored, including fuzzy graphs [136, 153], neutrosophic graphs [35, 36], and plithogenic graphs [66, 75, 185].

This study explores the SuperHyperGraph, Single-Valued Neutrosophic Quasi SuperHyperGraph, and Plithogenic Quasi SuperHyperGraph, analyzing their relationships with other graph classes. Future work will define the Semi Superhypergraph, Multi Superhypergraph, Pseudo Superhypergraph, Mixed Superhypergraph, and Bidirected Superhypergraph and examine their connections to existing classes in hypergraphs and graphs.

Keywords: Neutrosophic graph, Fuzzy graph, Hypergraph, Superhypergraph

1. **Short Introduction**

1.1. *Hypergraph and many applications*

Graphs are widely recognized as powerful tools for representing concepts and the relationships between them [57]. Graphs continue to play a significant role in computer science and societal applications, especially in areas such as graph neural networks [14, 39, 60, 76, 121, 123, 163, 211, 212, 214, 226]. In graph theory, various classes of graphs have been developed to address diverse applications and objectives [30].

In graph theory, the concept of a hypergraph [22], which extends the idea of a traditional graph, is well established and further generalized by the concept of a superhypergraph [174, 176]. The hypergraph concept has found applications in fields such as database theory [87, 100] and hypergraph neural networks [44, 94, 117, 122, 206, 213]. Examples in database theory include hypertree and hypergraph databases [13, 90], while in hypergraph neural networks, various applications have emerged, such as GroupNet, Metro, Hypergraph Neural Networks

(HyperNN), RAHGAR, and others. In graph parameters, hypertree-width is a notable metric that has been widely studied, among other research efforts [1, 61, 86, 113, 128, 218].

Directed hypergraphs, a variant where directionality is added to hypergraphs [85, 107, 108, 132, 135], have also been defined and studied in various research areas, including applications in pattern mining [149], Hypernetworks [205], traffic forecasting [124], Directed hypergraph neural network [200, 216], Personalized E-learning [191], and PageRank analysis [199]. Superhypergraphs further generalize hypergraphs by incorporating concepts from supergraph theory, expanding the framework for hypergraphs [68, 174, 176]. The author anticipates that superhypergraphs will inspire extensive research into both their mathematical structures and various applications, similar to the broad interest seen in hypergraphs.

1.2. *Various types of Uncertain Hypergraphs*

Additionally, various types of uncertain graphs have been explored, including fuzzy graphs [136, 153], neutrosophic graphs [35, 36], and plithogenic graphs [66, 75, 185], along with their corresponding hypergraph versions [23]. In these contexts, extensive research has also been conducted on applications such as decision-making [5, 10, 34, 106, 116, 164, 186] and uncertain neural networks [42, 89, 109, 120, 146, 208, 227].

Although some superhypergraph concepts, such as Fuzzy Superhypergraphs [91–93, 175], have been explored, their full mathematical properties and applications remain largely unexplored.

1.3. *Our Contribution in This Short Paper*

The discussion above highlights the mathematical value and significance of research on uncertain graphs, hypergraphs, and their generalizations to superhypergraphs. In this brief paper, we explore superhypergraphs in greater depth, specifically presenting key findings by defining the Intuitionistic Fuzzy Quasi SuperHyperGraph, the Single-Valued Neutrosophic Quasi SuperHyperGraph, and the Plithogenic Quasi SuperHyperGraph, as well as examining their relationships with other graph classes. Future work will involve defining the Semi Superhypergraph, Multi Superhypergraph, Pseudo Superhypergraph, Mixed Superhypergraph, and Bidirected Superhypergraph and investigating their connections to existing classes in hypergraphs and graphs.

2. **Preliminaries and Definitions**

This section provides an overview of the fundamental definitions and notations used throughout the paper. Basic concepts from set theory are also applied in parts of this work; for additional details, please refer to relevant references as needed [63, 95, 97, 102, 115].

2.1. *Basic Graph Concepts*

This subsection explains the basic concepts of graph theory. Below are some of the foundational ideas in this field. For more in-depth information on graph theory and its notations, please refer to [52, 55, 57, 210].

Definition 2.1 (Graph)**.** [57] A *graph* G is a mathematical structure that represents relationships between objects. It consists of a set of vertices $V(G)$ and a set of edges $E(G)$, where each edge connects a pair of vertices. Formally, a graph is represented as $G = (V, E)$, where V is the set of vertices and E is the set of edges.

Definition 2.2 (Degree)**.** [57] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is defined as the number of edges connected to v. For undirected graphs, the degree is given by:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

For directed graphs, the *in-degree* $\deg^-(v)$ refers to the number of edges directed towards v, while the *out-degree* $\deg^+(v)$ represents the number of edges directed away from v.

2.2. *Uncertain Graph*

This subsection explains the basic concepts of graph theory. Below are some of the foundational ideas in this field. For more in-depth information on graph theory and its notations, please refer to [52, 55, 57, 210]. As highlighted in the introduction, this paper centers on uncertain graph models, including Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Graphs. Addressing real-world uncertainties requires robust mathematical frameworks, such as Fuzzy Sets [219–225], Soft Sets [131], Picture Fuzzy Sets [45,77,105,159,168,188,207], Spherical Fuzzy Sets [16, 111, 112], Vague Sets [4, 38, 43], Hesitant Fuzzy Sets [197, 198], Rough Sets [143, 144], and Neutrosophic Sets [169–172, 184, 187]. These frameworks are invaluable for analyzing ambiguous or imprecise information. The concept of Fuzzy Graphs and related uncertain graph models can be viewed as extensions of these sets into graph theory, offering versatile tools for managing uncertainty in network structures.

The following sections define these graph concepts. For more on uncertain graphs, recent surveys provide comprehensive insights [67, 69–71, 192].

Definition 2.3 (Uncertain Graphs)**.** (cf. [69]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, falsity, and other nuanced measures of uncertainty.

(1) *Fuzzy Graph [153]:*

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0,1]$.
- Each edge $e = (u,v) \in E$ is assigned a membership degree $\mu(u,v) \in [0,1]$.

(2) *Intuitionistic Fuzzy Graph (IFG) [47, 134, 195]:*

- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0,1]$ (degree of membership) and $\nu_A(v) \in [0,1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
- Each edge $e = (u,v) \in E$ is assigned two values: $\mu_B(u,v) \in [0,1]$ and $\nu_B(u,v) \in [0,1]$, with $\mu_B(u,v) + \nu_B(u,v) \leq 1$.

(3) *Neutrosophic Graph [11, 104]:*

- Each vertex $v \in V$ is assigned a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where $\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0,1]$ and $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
- Each edge $e = (u,v) \in E$ is assigned a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.

2.3. *Plithogenic Graph*

This subsection provides an overview of the plithogenic graph. A plithogenic graph extends Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Graphs, representing the graphical counterpart of a plithogenic set [64, 167, 173].

A plithogenic set consists of elements with multiple attributes, where each attribute has specific values accompanied by degrees of membership and contradiction (fuzzy, intuitionistic, or neutrosophic) [173]. The formal definition of a plithogenic graph is outlined below [173,190].

Definition 2.4. [173,190] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

(1) *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:

- $M \subseteq V$ is the set of vertices.
- l is an attribute associated with the vertices.
- Ml is the range of possible attribute values.
- $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
- $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

(2) *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

- $N \subseteq E$ is the set of edges.
- m is an attribute associated with the edges.
- Nm is the range of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.

- $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

(1) *Edge Appurtenance Constraint*: For all $(x,a),(y,b) \in M \times Ml$:

$$bdf\left((xy),(a,b)\right) \leq \min\{adf(x,a), adf(y,b)\}$$

where $xy \in N$ is an edge between vertices x and y, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.

(2) *Contradiction Function Constraint*: For all $(a,b),(c,d) \in Nm \times Nm$:

$$bCf\left((a,b),(c,d)\right) \leq \min\{aCf(a,c), aCf(b,d)\}$$

(3) *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(a,a) &= 0, && \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), && \forall a,b \in Ml \\ bCf(a,a) &= 0, && \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

Example 2.5. (cf. [67]) The following examples are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic Graph*.
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic Graph*.
- When $s = 6, t = 1$, PG is called a *Plithogenic hexapartitioned Neutrosophic Graph*.
- When $s = 7, t = 1$, PG is called a *Plithogenic heptapartitioned Neutrosophic Graph*.
- When $s = 8, t = 1$, PG is called a *Plithogenic octapartitioned Neutrosophic Graph*.
- When $s = 9, t = 1$, PG is called a *Plithogenic nonapartitioned Neutrosophic Graph*.

2.4. *Hypergraph and Uncertain Hypergraph*

This subsection provides an overview of Hypergraphs and Uncertain Hypergraphs. A hypergraph is a generalized graph structure that extends traditional graph concepts by allowing hyperedges, which connect multiple vertices rather than just pairs, enabling more complex relationships between elements [19, 22, 87, 88]. Hypergraphs have a wide range of applications, notably in database systems [100]. For additional information, readers are encouraged to consult comprehensive surveys on hypergraphs, such as those in [32, 33, 204].

Definition 2.6 (Hypergraph)**.** [22] A *hypergraph* $H = (V, E)$ consists of a set V of vertices and a set E of hyperedges. Each hyperedge $e \in E$ is defined as a subset of V, thus $e \subseteq V$, and $E \subseteq \mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the power set of V.

Extensions of the hypergraph concept, including Fuzzy Hypergraphs [8,23,133], incorporate elements from Fuzzy, Intuitionistic Fuzzy [2, 28, 51, 141], and Neutrosophic theories [6, 7, 125, 126]. These graphs extend traditional hypergraphs by applying concepts from Fuzzy Graphs, Intuitionistic Fuzzy Graphs, and Neutrosophic Graphs to create more nuanced hypergraph structures. For more details on specific operations and further theoretical underpinnings, readers are encouraged to consult the cited papers.

Definitions of these extended hypergraph types are provided below. Note that the Single-Valued Neutrosophic Hypergraph generalizes the Intuitionistic Fuzzy Hypergraph, which in turn generalizes the Fuzzy Hypergraph, and the Fuzzy Hypergraph itself generalizes the Hypergraph.

Definition 2.7 (Fuzzy Hypergraph)**.** [8, 23, 133] A *fuzzy hypergraph* $G = (V, E, \psi, w)$ is a hypergraph where vertices have fuzzy membership degrees in hyperedges, and each hyperedge has an associated weight. The fuzzy hypergraph is defined as follows:

- V is the set of vertices.
- E is the set of hyperedges, where each hyperedge $e \in E$ is a subset of V.
- $\in [0,1]^{|E|\times|V|}$ is a matrix where ψ_{ei} represents the degree of membership of vertex $i \in V$ in hyperedge $e \in E$, satisfying $\sum_{i\in V} \psi_{ei} = 1$ for each $e \in E$ and $\sum_{e\in E} \psi_{ei} > 0$ for each $i \in V$.
- $w : E \to \mathbb{R}_+$ assigns a positive weight $w(e)$ to each hyperedge $e \in E$.

Here, the matrix ψ serves as the incidence matrix of the fuzzy hypergraph, where each hyperedge quantifies the participation of each vertex. The weight function w provides a quantitative measure for the importance or relevance of each hyperedge.

Definition 2.8 (Intuitionistic Fuzzy Hypergraph)**.** [140] An *intuitionistic fuzzy hypergraph* (IFHG) $H = (V, E)$ consists of:

- A finite set $V = \{x_1, x_2, \ldots, x_n\}$ of vertices.
- A family $E = \{E_1, E_2, \ldots, E_m\}$ of intuitionistic fuzzy subsets of V, where each hyperedge E_j is defined as:

$$E_j = \{(x_i, \mu_j(x_i), \nu_j(x_i)) \mid x_i \in V\},$$

with $\mu_j(x_i)$ denoting the degree of membership of vertex x_i to the hyperedge E_j, and $\nu_j(x_i)$ denoting the degree of non-membership. It is required that $0 \leq \mu_j(x_i)+\nu_j(x_i) \leq 1$ for all $x_i \in V$ and all $E_j \in E$.

Additionally, it holds that:

$$\bigcup_{j=1}^{m} \text{supp}(E_j) = V,$$

where $\text{supp}(E_j) = \{x_i \in V \mid \mu_j(x_i) > 0 \text{ and } \nu_j(x_i) > 0\}$ is the support of E_j.

Definition 2.9 (Single-Valued Neutrosophic Hypergraph)**.** [12] Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite set of vertices, and let $E = \{E_1, E_2, \ldots, E_m\}$ be a collection of single-valued neutrosophic subsets of V, such that:

$$\bigcup_{i=1}^{m} \text{supp}(E_i) = V,$$

where each edge E_i is defined as:

$$E_i = \{(v_j, T_{E_i}(v_j), I_{E_i}(v_j), F_{E_i}(v_j)) \mid v_j \in V\},$$

with $T_{E_i}(v_j), I_{E_i}(v_j), F_{E_i}(v_j)$ representing the truth-membership, indeterminacy-membership, and falsity-membership of vertex v_j to edge E_i respectively. The pair $H = (V, E)$ is called a *single-valued neutrosophic hypergraph* (SVNH).

2.5. *Superhypergraph and Quasi-SuperHyperGraph*

This subsection explains the concepts of the Superhypergraph and Quasi-SuperHyperGraph. The SuperHyperGraph is a generalized graph concept that extends the Hypergraph by introducing supervertices and superedges [174, 175, 177, 178]. This framework includes related structures such as the quasi-SuperHyperGraph [92]. As briefly mentioned in the short introduction of this paper, these structures are further extended using fuzzy relations, leading to the concept known as the Fuzzy SuperHyperGraph [92]. The definitions are provided below.

Definition 2.10 (SuperHyperGraph)**.** [174] Let V be a finite set of vertices. A *superhypergraph* is an ordered pair $H = (V, E)$, where:

- $V \subseteq P(V)$ (the power set of V), meaning that each element of V can be either a single vertex or a subset of vertices (called a *supervertex*).
- $E \subseteq P(V)$ represents the set of edges, called *superedges*, where each $e \in E$ can connect multiple supervertices.

In this framework, a superhypergraph can accommodate complex relationships among groups of vertices, including single edges, hyperedges, superedges, and multi-edges. Superhypergraphs provide a flexible structure to represent high-order and hierarchical relationships.

The following relationship between a superhypergraph and a hypergraph is well-known and clearly holds.

Proposition 2.11. *Every superhypergraph can be transformed into a hypergraph.*

Proof. Let $H = (V, E)$ be a superhypergraph, where:

- $V \subseteq P(U)$ for some underlying set U, meaning each $v \in V$ is either a single element of U or a subset of U (termed a *supervertex*).
- $E \subseteq P(V)$, with each $e \in E$ linking subsets of elements in V (known as *superedges*).

Our objective is to construct a hypergraph $H' = (V', E')$ that is equivalent to H but adheres to the structure of a hypergraph.

Construction of H': 1. Define V' as the union of all elements in the supervertices of V, specifically:

$$V' = \bigcup_{v \in V} v.$$

Here, each element in V' is a unique element of U, effectively "flattening" the supervertices of V into single elements in V'.

2. Define the edge set E' as follows: for each superedge $e = \{v_1, v_2, \ldots, v_n\} \in E$ (where $v_i \in V$), construct an edge $e' \subseteq V'$ by taking the union of the elements within each supervertex v_i, i.e.,

$$e' = \bigcup_{v \in e} v.$$

This construction ensures that each $e' \subseteq V'$, thus meeting the definition of a hyperedge in H'.

Verification: The transformed structure $H' = (V', E')$ satisfies the following properties of a hypergraph:

- V' is a set of vertices, where each element corresponds to an element in the underlying set U (not subsets of U), ensuring $V' \subseteq U$.
- Each edge $e' \in E'$ is a subset of V' and does not contain supervertices, adhering to the standard hypergraph structure where each edge is a subset of the vertex set.

Thus, the transformation from $H = (V, E)$ to $H' = (V', E')$ is valid, and H' is a hypergraph as required. □

Definition 2.12 (Quasi-SuperHyperGraph)**.** [92] A *quasi-superhypergraph* is a triple $H = (V, S, \Phi)$ where:

- V is a set of elements called *vertices*.
- $S = \{S_i\}_{i=1}^{k} \subset P(V)$ is a family of subsets of V called *supervertices*.
- $\Phi = \{\phi_{i,j} \mid i \neq j\}$ is a set of mappings $\phi_{i,j} : S_i \to S_j$, known as *superedges*, linking different supervertices.

The quasi-superhypergraph becomes a complete superhypergraph if, for each supervertex S_i, there exists at least one supervertex S_j such that S_i links to S_j.

The following relationship between a quasi-superhypergraph and a superhypergraph is well-known and clearly holds.

Proposition 2.13. *Every quasi-superhypergraph can be transformed into a superhypergraph.*

Proof. Let $H = (V, S, \Phi)$ be a quasi-superhypergraph where:

- V is a set of vertices.
- $S = \{S_i\}_{i=1}^{k} \subset P(V)$ is a collection of subsets of V (supervertices).
- $\Phi = \{\phi_{i,j} \mid i \neq j\}$ is a set of mappings $\phi_{i,j} : S_i \to S_j$, linking different supervertices S_i and S_j.

We aim to construct a superhypergraph $H' = (V', E')$ from H by defining an appropriate vertex set V' and edge set E' that satisfy the properties of a superhypergraph.

To define the vertex set V', let $V' = S$, where each supervertex S_i in S becomes a vertex in V'. Thus, each vertex in V' represents a subset of elements from V, satisfying the requirement that $V' \subseteq P(V)$.

Next, we define the edge set E' based on the mappings Φ. For each mapping $\phi_{i,j} \in \Phi$, where $\phi_{i,j} : S_i \to S_j$, we define a corresponding edge $e_{i,j}$ in E' that links S_i and S_j. Thus, we set $E' = \{e_{i,j} \mid \phi_{i,j} \in \Phi\}$, where each $e_{i,j} = \{S_i, S_j\}$. This construction ensures that each superedge in the quasi-superhypergraph H has a corresponding edge in H' connecting supervertices in V'.

Finally, we verify that $H' = (V', E')$ satisfies the properties of a superhypergraph. Each element in V' is a subset of V, fulfilling the requirement that vertices may represent groups of elements. Furthermore, each edge in E' connects subsets of V, with each edge $e_{i,j}$ linking two supervertices. Thus, the structure H' constructed from H meets the definition of a superhypergraph.

Since this transformation process can be applied to any quasi-superhypergraph $H = (V, S, \Phi)$, we conclude that every quasi-superhypergraph can indeed be transformed into a superhypergraph, as required. □

Next, the definitions for the Fuzzy Quasi-SuperHyperGraph and the q-Fuzzy Quasi-SuperHyperGraph are provided below. These are graph concepts that extend the Quasi-SuperHyperGraph through fuzzy relations.

Definition 2.14 (q-Fuzzy Quasi-SuperHyperGraph)**.** [92] Let $H^* = (V, S, \Phi)$ be a quasi-superhypergraph, and let $\sigma_i = \{(x, \sigma_i(x)) \mid x \in S_i, 0 \leq \sigma_i(x) \leq 1\}$ represent fuzzy supervertices, where $\sigma_i(x)$ denotes the degree of membership of x in S_i. Define $\mu_{i,j} : \phi_{i,j} \to [0, 1]$ as fuzzy superedges.

Then $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ is called a *q-fuzzy quasi-superhypergraph* on H^*, where:

$$\mu_{i,j}(x, \phi_{i,j}(x)) \leq \frac{\sigma_i(x)}{q\, w(\sigma_i)} \wedge \frac{\sigma_j(\phi_{i,j}(x))}{q\, w(\sigma_j)},$$

for all $x \in V$, and where $w(\sigma_t) = \sum_{x \in S_t} \sigma_t(x)$ denotes the total weight of the fuzzy supervertex σ_t. Here, σ_i and σ_j represent fuzzy supervertices, and $\mu_{i,j}$ represents fuzzy superedges or fuzzy links from σ_i to σ_j.

If $q = 1$, we denote this as a *fuzzy quasi-superhypergraph*.

3. **Result in this paper**

In this section, we present the main results of this paper by defining the Intuitionistic Fuzzy Quasi SuperHyperGraph, Single-Valued Neutrosophic Quasi SuperHyperGraph, and Plithogenic Quasi SuperHyperGraph, and examining their relationships with other graph classes. While the ideas underlying these graph concepts have been suggested in sources such as [174], this paper reinterprets them through formal mathematical definitions.

3.1. *Intuitionistic Fuzzy Quasi SuperHyperGraph*

The definition of an Intuitionistic Fuzzy Quasi SuperHyperGraph and its relationships with other graph classes are presented below.

Definition 3.1 (Intuitionistic Fuzzy Quasi SuperHyperGraph)**.** Let $H^* = (V, S, \Phi)$ be a quasi-superhypergraph, where:

- V is a non-empty set of vertices.
- $S = \{S_i\}_{i=1}^k$ is a family of non-empty subsets of V called *supervertices*, i.e., $S_i \subseteq V$.
- $\Phi = \{\phi_{i,j}\}_{i,j}$ is a set of mappings $\phi_{i,j} : S_i \to S_j$ called *superedges*, for $i \neq j$.

An *intuitionistic fuzzy quasi superhypergraph* (IFQSHG)

$$H = (\{\langle S_i, \mu_{S_i}, \nu_{S_i}\rangle\}_{i=1}^k, \{\langle \phi_{i,j}, \mu_{i,j}, \nu_{i,j}\rangle\}_{i,j})$$

on H^* consists of:

- For each supervertex S_i, membership and non-membership functions $\mu_{S_i}, \nu_{S_i} : S_i \to [0, 1]$ satisfying:

$$0 \leq \mu_{S_i}(x) + \nu_{S_i}(x) \leq 1, \quad \forall x \in S_i.$$

- For each superedge $\phi_{i,j} : S_i \to S_j$, membership and non-membership functions $\mu_{i,j}, \nu_{i,j} : \phi_{i,j} \to [0, 1]$ satisfying:

$$0 \leq \mu_{i,j}(x, \phi_{i,j}(x)) + \nu_{i,j}(x, \phi_{i,j}(x)) \leq 1, \quad \forall x \in S_i.$$

Additionally, we define the hesitation degree for vertices and edges as:

$$\pi_{S_i}(x) = 1 - \mu_{S_i}(x) - \nu_{S_i}(x), \quad \pi_{i,j}(x, \phi_{i,j}(x)) = 1 - \mu_{i,j}(x, \phi_{i,j}(x)) - \nu_{i,j}(x, \phi_{i,j}(x)),$$

representing the indeterminacy or hesitation of x in S_i and the connection from x to $\phi_{i,j}(x)$, respectively.

Theorem 3.2. *Any intuitionistic fuzzy quasi superhypergraph H can be transformed into a fuzzy quasi superhypergraph H' by neglecting the non-membership functions and adjusting the membership functions accordingly.*

Proof. Given an intuitionistic fuzzy quasi superhypergraph H, we construct a fuzzy quasi superhypergraph H' as follows:

- For each supervertex S_i, define the membership function $\mu'_{S_i} : S_i \to [0, 1]$ by:

$$\mu'_{S_i}(x) = \mu_{S_i}(x), \quad \forall x \in S_i.$$

- For each superedge $\phi_{i,j}$, define the membership function $\mu'_{i,j} : \phi_{i,j} \to [0, 1]$ by:

$$\mu'_{i,j}(x, \phi_{i,j}(x)) = \mu_{i,j}(x, \phi_{i,j}(x)), \quad \forall x \in S_i.$$

By omitting the non-membership functions ν_{S_i} and $\nu_{i,j}$, we effectively reduce the intuitionistic fuzzy structure to a fuzzy one. The resulting fuzzy membership functions μ'_{S_i} and $\mu'_{i,j}$ retain the original membership degrees, ensuring that the essential relationships in H are preserved in H'. □

Theorem 3.3. *Any intuitionistic fuzzy quasi superhypergraph H can be transformed into an intuitionistic fuzzy hypergraph H'' by representing the supervertices and superedges as standard vertices and edges in a hypergraph.*

Proof. To transform H into an intuitionistic fuzzy hypergraph H'', we proceed as follows:

- Let the vertex set V' of H'' be the union of all supervertices:

$$V' = \bigcup_{i=1}^{k} S_i.$$

- For each element $x \in V'$, define the membership and non-membership functions $\mu_{V'}(x)$ and $\nu_{V'}(x)$ as:

$$\mu_{V'}(x) = \mu_{S_i}(x), \quad \nu_{V'}(x) = \nu_{S_i}(x), \quad \text{if } x \in S_i.$$

- For each mapping $\phi_{i,j} : S_i \to S_j$, create edges in H'' connecting $x \in S_i$ to $y = \phi_{i,j}(x) \in S_j$, with membership and non-membership functions $\mu_E(x, y)$ and $\nu_E(x, y)$ defined by:

$$\mu_E(x, y) = \mu_{i,j}(x, y), \quad \nu_E(x, y) = \nu_{i,j}(x, y).$$

By treating each element of the supervertices as individual vertices and each mapping as edges, we construct an intuitionistic fuzzy hypergraph H'' that encapsulates the same relationships as H. □

3.2. *Single-Valued Neutrosophic Quasi SuperHyperGraph*

The definition of a Single-Valued Neutrosophic Quasi SuperHyperGraph and its relationship with other graph classes are outlined below.

Definition 3.4 (Single-Valued Neutrosophic Quasi SuperHyperGraph)**.** A *Single-Valued Neutrosophic Quasi SuperHyperGraph* $H = (X, S, \Phi, T, I, F)$ is a quasi superhypergraph where:

- For each supervertex S_i:
  - Truth-membership function $T_{S_i} : S_i \to [0, 1]$.
  - Indeterminacy-membership function $I_{S_i} : S_i \to [0, 1]$.
  - Falsity-membership function $F_{S_i} : S_i \to [0, 1]$.
  - For all $x \in S_i$:

$$0 \leq T_{S_i}(x) + I_{S_i}(x) + F_{S_i}(x) \leq 1.$$

- For each superedge $\phi_{i,j}$:
  - Truth-membership function $T_{\phi_{i,j}} : \phi_{i,j} \to [0, 1]$.
  - Indeterminacy-membership function $I_{\phi_{i,j}} : \phi_{i,j} \to [0, 1]$.
  - Falsity-membership function $F_{\phi_{i,j}} : \phi_{i,j} \to [0, 1]$.
  - For all $(x, y) \in \phi_{i,j}$:

$$0 \leq T_{\phi_{i,j}}(x, y) + I_{\phi_{i,j}}(x, y) + F_{\phi_{i,j}}(x, y) \leq 1.$$

Theorem 3.5. *Any single-valued neutrosophic quasi superhypergraph H can be transformed into an intuitionistic fuzzy quasi superhypergraph H' by merging the indeterminacy and falsity memberships.*

Proof. Given H, we construct an intuitionistic fuzzy quasi superhypergraph H' as follows:

- For each supervertex S_i, define the membership and non-membership functions μ_{S_i} and ν_{S_i} by:

$$\mu_{S_i}(x) = T_{S_i}(x), \quad \nu_{S_i}(x) = F_{S_i}(x) + I_{S_i}(x), \quad \forall x \in S_i.$$

- Since $0 \leq T_{S_i}(x) + I_{S_i}(x) + F_{S_i}(x) \leq 1$, it follows that $\mu_{S_i}(x) + \nu_{S_i}(x) \leq 1$, satisfying the conditions of an intuitionistic fuzzy membership.
- For each superedge $\phi_{i,j}$, define $\mu_{i,j}$ and $\nu_{i,j}$ similarly:

$$\mu_{i,j}(x, \phi_{i,j}(x)) = T_{i,j}(x, \phi_{i,j}(x)), \quad \nu_{i,j}(x, \phi_{i,j}(x)) = F_{i,j}(x, \phi_{i,j}(x)) + I_{i,j}(x, \phi_{i,j}(x)).$$

By combining the indeterminacy and falsity degrees into a single non-membership function, we effectively reduce the neutrosophic structure to an intuitionistic fuzzy structure. □

Theorem 3.6. *Any single-valued neutrosophic quasi superhypergraph H can be transformed into a single-valued neutrosophic hypergraph H'' by representing supervertices and superedges as standard vertices and edges.*

Proof. We construct H'' as follows:

- The vertex set V' of H'' is $V' = \bigcup_{i=1}^{k} S_i$.
- For each vertex $x \in V'$, define the neutrosophic membership functions:

$$T_{V'}(x) = T_{S_i}(x), \quad I_{V'}(x) = I_{S_i}(x), \quad F_{V'}(x) = F_{S_i}(x), \quad \text{if } x \in S_i.$$

- For each mapping $\phi_{i,j}$, create edges between $x \in S_i$ and $y = \phi_{i,j}(x) \in S_j$, with neutrosophic membership functions:

$$T_E(x, y) = T_{i,j}(x, y), \quad I_E(x, y) = I_{i,j}(x, y), \quad F_E(x, y) = F_{i,j}(x, y).$$

This construction transforms the quasi superhypergraph H into a single-valued neutrosophic hypergraph H'' by flattening the supervertex and superedge structures into standard vertices and edges. □

Theorem 3.7. *The intersection of two Single-Valued Neutrosophic Quasi SuperHyperGraphs with the same vertex set X results in a Single-Valued Neutrosophic Quasi SuperHyperGraph.*

Proof. Let $H_1 = (X, S^1, \Phi^1, T^1, I^1, F^1)$ and $H_2 = (X, S^2, \Phi^2, T^2, I^2, F^2)$. Define $H = (X, S, \Phi, T, I, F)$ where:

$$\begin{aligned} S &= S^1 \cap S^2, \\ \Phi &= \Phi^1 \cap \Phi^2, \\ T_{S_i}(x) &= \min\{T^1_{S_i}(x), T^2_{S_i}(x)\}, \\ I_{S_i}(x) &= \max\{I^1_{S_i}(x), I^2_{S_i}(x)\}, \\ F_{S_i}(x) &= \max\{F^1_{S_i}(x), F^2_{S_i}(x)\}. \end{aligned}$$

Since $T + I + F \leq 1$, H satisfies the conditions of a Single-Valued Neutrosophic Quasi SuperHyperGraph. □

3.3. *Plithogenic Quasi SuperHyperGraph*

The definition of a Plithogenic Quasi SuperHyperGraph is outlined below.

Definition 3.8 (Plithogenic Quasi SuperHyperGraph)**.** A *Plithogenic Quasi SuperHyperGraph* $H = (X, S, \Phi, \Theta)$ extends the quasi superhypergraph by incorporating attributes and their associated degrees. It consists of:

- A non-empty set X of elements called *vertices*.
- A set $S = \{S_i\}_{i=1}^{k}$ of non-empty subsets of X, called *supervertices*.
- A set of mappings $\Phi = \{\phi_{i,j}\}_{i \neq j}$, where $\phi_{i,j} : S_i \to S_j$ are called *superedges*.
- For each supervertex S_i:
  - An attribute l_i with a domain of possible values M_{l_i}.
  - A *Degree of Appurtenance Function* (DAF) $\mathrm{adf}_{S_i} : S_i \times M_{l_i} \to [0,1]^s$.
- For each superedge $\phi_{i,j}$:
  - An attribute $m_{i,j}$ with a domain of possible values $M_{m_{i,j}}$.
  - A DAF $\mathrm{adf}_{\phi_{i,j}} : \phi_{i,j} \times M_{m_{i,j}} \to [0,1]^s$.
- *Degrees of Contradiction Functions* (DCFs):
  - For supervertices: $\mathrm{aCf}_{S_i} : M_{l_i} \times M_{l_i} \to [0,1]^t$.
  - For superedges: $\mathrm{aCf}_{\phi_{i,j}} : M_{m_{i,j}} \times M_{m_{i,j}} \to [0,1]^t$.

These functions satisfy:

(1) Reflexivity and symmetry of DCFs:

$$\begin{aligned}
\mathrm{aCf}_{S_i}(a,a) &= 0, \quad \forall a \in M_{l_i}, \\
\mathrm{aCf}_{S_i}(a,b) &= \mathrm{aCf}_{S_i}(b,a), \quad \forall a,b \in M_{l_i}, \\
\mathrm{aCf}_{\phi_{i,j}}(c,c) &= 0, \quad \forall c \in M_{m_{i,j}}, \\
\mathrm{aCf}_{\phi_{i,j}}(c,d) &= \mathrm{aCf}_{\phi_{i,j}}(d,c), \quad \forall c,d \in M_{m_{i,j}}.
\end{aligned}$$

(2) The overall membership degree of an element $x \in S_i$ with attribute value a is adjusted by the contradiction between a and other attribute values.

Theorem 3.9. *Every Plithogenic Quasi SuperHyperGraph can be transformed into a Single-Valued Neutrosophic Quasi SuperHyperGraph.*

Proof. Given a Plithogenic Quasi SuperHyperGraph $H = (X, S, \Phi, \Theta)$, we can define the neutrosophic membership functions for each supervertex and superedge based on the Degrees of Appurtenance and Contradiction.

For each supervertex S_i and $x \in S_i$ with attribute value $a \in M_{l_i}$:

$$T_{S_i}(x) = \mathrm{adf}_{S_i}(x, a) \cdot (1 - \mathrm{aCf}_{S_i}(a, a')),$$
$$I_{S_i}(x) = \mathrm{aCf}_{S_i}(a, a'),$$
$$F_{S_i}(x) = 1 - T_{S_i}(x) - I_{S_i}(x),$$

where a' is a reference attribute value.

Similarly, for each superedge $\phi_{i,j}$ and $(x, y) \in \phi_{i,j}$ with attribute value $c \in M_{m_{i,j}}$:

$$T_{\phi_{i,j}}(x, y) = \mathrm{adf}_{\phi_{i,j}}((x, y), c) \cdot (1 - \mathrm{aCf}_{\phi_{i,j}}(c, c')),$$
$$I_{\phi_{i,j}}(x, y) = \mathrm{aCf}_{\phi_{i,j}}(c, c'),$$
$$F_{\phi_{i,j}}(x, y) = 1 - T_{\phi_{i,j}}(x, y) - I_{\phi_{i,j}}(x, y),$$

with c' as a reference attribute value for edges.

Since the sum $T + I + F \leq 1$ holds by construction, the transformed structure satisfies the conditions of a Single-Valued Neutrosophic Quasi SuperHyperGraph. □

Theorem 3.10. *In a Plithogenic Quasi SuperHyperGraph, the degree of contradiction function aCf is symmetric and reflexive.*

Proof. By the definition of a Plithogenic Quasi SuperHyperGraph, the degree of contradiction functions satisfy:

$$\mathrm{aCf}(a, a) = 0, \quad \mathrm{aCf}(a, b) = \mathrm{aCf}(b, a).$$

Reflexivity follows from $\mathrm{aCf}(a, a) = 0$, and symmetry from $\mathrm{aCf}(a, b) = \mathrm{aCf}(b, a)$. □

4. Future tasks and Discussions

The following outlines future prospects for this study.

4.1. *Semi Superhypergraph*

As noted in the introduction, numerous graph classes have been proposed within graph theory, alongside ongoing efforts to generalize these classes [30]. The semigraph is recognized as a generalized graph class extending the standard concept of a graph [46,161,165]. Recently, an even broader generalization, the semihypergraph, has been introduced, further expanding on this structure [101]. Similarly, it is reasonable to expect the existence of a semi superhypergraph, which would build on the structure of the superhypergraph. In future work, we aim to explore the mathematical structure of the semi superhypergraph, as well as potential extensions incorporating Fuzzy and Neutrosophic properties and concepts such as the semi n-superhypergraph [174].

The definitions of the already established concepts of the semigraph and semihypergraph, along with the conceptual framework for the semi superhypergraph, are presented below.

Definition 4.1 (Semigraph)**.** [161] A *semigraph* $G = (V, E)$ consists of:

- A non-empty set V of elements called *vertices*.
- A set E of ordered n-tuples of distinct vertices, called *edges*, where $n \geq 2$ for each edge. Each edge connects a specific sequence of vertices.

The semigraph G must satisfy the following conditions:

(1) *Intersection Condition*: Any two edges in E have at most one vertex in common.
(2) *Edge Equivalence*: Two edges $E_1 = (u_1, u_2, \ldots, u_m)$ and $E_2 = (v_1, v_2, \ldots, v_n)$ are considered equal if and only if:
   - $m = n$ (they have the same length).
   - Either $u_i = v_i$ for all $i = 1, \ldots, n$ (identical ordering), or $u_i = v_{n-i+1}$ for all $i = 1, \ldots, n$ (reverse ordering).

Definition 4.2 (Semihypergraph)**.** [101] A *semihypergraph* $H_s = (V, E_h)$ consists of:

- A finite set $V = \{v_1, v_2, \ldots, v_n\}$ of vertices.
- A set $E_h = \{E_h^1, E_h^2, \ldots, E_h^p\}$ of ordered hyperedges, where each hyperedge E_h^j is an ordered k_j-tuple of distinct vertices from V, with $k_j \geq 2$.

The semihypergraph H_s must satisfy the following conditions:

(1) *Intersection Condition*: Any two hyperedges in E_h have at most one vertex in common.
(2) *Hyperedge Equivalence*: Two hyperedges $E_h^m = (u_1, u_2, \ldots, u_m)$ and $E_h^n = (v_1, v_2, \ldots, v_n)$ are considered equal if and only if:
   - $m = n$ (they have the same length).
   - Either $u_i = v_i$ for all $i = 1, \ldots, n$ (identical ordering), or $u_i = v_{n-i+1}$ for all $i = 1, \ldots, n$ (reverse ordering).

Proposition 4.3. *Every semihypergraph $H_s = (V, E_h)$ can be transformed into a semigraph $G = (V, E)$.*

Proof. Let $H_s = (V, E_h)$ be a semihypergraph with a vertex set V and a set E_h of hyperedges, where each hyperedge $E_h^j \in E_h$ is an ordered k_j-tuple of distinct vertices from V with $k_j \geq 2$.

We construct a corresponding semigraph $G = (V, E)$ as follows:

(1) Set V as the vertex set of G, identical to the vertex set of H_s.
(2) For each hyperedge $E_h^j = (v_{j_1}, v_{j_2}, \ldots, v_{j_{k_j}})$ in H_s, define a corresponding edge $E_j = (v_{j_1}, v_{j_2}, \ldots, v_{j_{k_j}})$ in E of G. Add E_j to E.

This construction ensures that each hyperedge in H_s corresponds directly to an edge in G with identical ordering and vertex sequence. Next, we verify that $G = (V, E)$ satisfies the conditions of a semigraph.

The set V in G is identical to that in H_s, so V is non-empty, meeting the requirement for a semigraph. Additionally, each edge $E_j \in E$ is an ordered k_j-tuple of distinct vertices from V, with $k_j \geq 2$ by construction, which aligns with the edge structure required for semigraphs.

By the intersection condition of the semihypergraph H_s, any two hyperedges E_h^a and E_h^b share at most one vertex. Since each edge in G corresponds to a hyperedge in H_s, the intersection condition is preserved in G; any two edges in E have at most one vertex in common.

The hyperedge equivalence condition in H_s specifies that two hyperedges are equal if they have the same length and either identical or reverse ordering of vertices. Since each edge in G corresponds to a hyperedge in H_s, this equivalence criterion holds for the edges in G as well.

Thus, the constructed graph $G = (V, E)$ satisfies all conditions of a semigraph. Therefore, any semihypergraph H_s can indeed be transformed into a semigraph G, as required. □

Definition 4.4 (Semi SuperHyperGraph)**.** A *semi superhypergraph* $H = (V, S, E)$ consists of:

- A finite set V of elements called *vertices*.
- A collection $S = \{S_1, S_2, \ldots, S_k\}$ of non-empty subsets of V, called *supervertices*.
- A set E of ordered n-tuples of supervertices, called *superedges*, where $n \geq 2$ for each superedge.

The semi superhypergraph H must satisfy the following conditions:

(1) *Intersection Condition*: Any two superedges in E have at most one supervertex in common.

(2) *Superedge Equivalence*: Two superedges $E_1 = (S_{i_1}, S_{i_2}, \ldots, S_{i_m})$ and $E_2 = (S_{j_1}, S_{j_2}, \ldots, S_{j_n})$ are considered equal if and only if:
   - $m = n$ (they have the same length).
   - Either $S_{i_k} = S_{j_k}$ for all $k = 1, \ldots, n$ (identical ordering), or $S_{i_k} = S_{j_{n-k+1}}$ for all $k = 1, \ldots, n$ (reverse ordering).

We can show that both superhypergraphs and semihypergraphs can be represented as semi superhypergraphs.

Theorem 4.5. *Every superhypergraph and every semihypergraph can be represented as a semi superhypergraph.*

Proof. We will prove this theorem in two parts.

Part 1: Superhypergraphs as Semi SuperHyperGraphs

Let $G = (V, E)$ be a superhypergraph, where:

- V is a set of supervertices, each being a subset of some underlying set U.
- $E \subseteq \mathcal{P}(V)$ is a set of superedges, where each superedge $e \in E$ is a subset of V.

To represent G as a semi superhypergraph $H = (V', S, E')$, we proceed as follows:

- Set $V' = V$. That is, the vertices of the semi superhypergraph are the supervertices of the superhypergraph.
- Set $S = V'$. Each supervertex in V' is a supervertex in H.
- For each superedge $e = \{S_{i_1}, S_{i_2}, \ldots, S_{i_m}\} \in E$, define an ordered superedge $E' = (S_{i_1}, S_{i_2}, \ldots, S_{i_m})$ and include it in E'.

The Intersection Condition and Superedge Equivalence in the semi superhypergraph are satisfied because they correspond directly to those in the superhypergraph, with the addition of ordering, which is accounted for in the equivalence condition.

Part 2: Semihypergraphs as Semi SuperHyperGraphs

Let $H_s = (V, E_h)$ be a semihypergraph, where:

- $V = \{v_1, v_2, \ldots, v_n\}$ is a set of vertices.
- E_h is a set of ordered hyperedges, each hyperedge being an ordered tuple of vertices from V.

To represent H_s as a semi superhypergraph $H = (V, S, E')$, we proceed as follows:

- Define the supervertex set $S = \{\{v\} \mid v \in V\}$, where each supervertex is a singleton set containing a single vertex from V.
- The vertex set V remains the same.
- For each hyperedge $E_h^j = (v_{j_1}, v_{j_2}, \ldots, v_{j_m}) \in E_h$, define a superedge $E' = (\{v_{j_1}\}, \{v_{j_2}\}, \ldots, \{v_{j_m}\})$ and include it in E'.

Again, the Intersection Condition and Superedge Equivalence in the semi superhypergraph are satisfied because they correspond to those in the semihypergraph, with vertices replaced by singleton supervertices. □

4.2. *Soft Semihypergraph*

In this paper, we also introduce the concept of a Soft Semihypergraph, a hybrid graph structure that merges elements from both Soft Hypergraphs [9, 82, 162] and Soft Semigraphs [79–81]. A soft graph, in general, is a mathematical framework that extends soft set theory to graph structures, assigning parameters to vertices and edges to effectively manage uncertainty. Below, we provide definitions of Soft Hypergraphs and Soft Semigraphs, followed by an initial definition of a Soft Semihypergraph as a concept still under development.

Definition 4.6. [131] Let U be a non-empty finite set, called the *universe* of discourse, and let E be a non-empty set of parameters. A *soft set* over U is defined as follows:

$$F = (F, A) \text{ over } U \text{ is an ordered pair, where } A \subseteq E \text{ and } F : A \to P(U),$$

where $F(a) \subseteq U$ for each $a \in A$ and $P(U)$ denotes the power set of U. The set of all soft sets over U is denoted by $S(U)$.

(1) *Soft Subset:* Let $F = (F, A)$ and $G = (G, B)$ be two soft sets over the common universe U. We say that F is a soft subset of G, denoted $F \subseteq G$, if:
- $A \subseteq B$
- $F(a) \subseteq G(a)$ for all $a \in A$.

(2) *Union of Soft Sets:* The union of two soft sets $F = (F, A)$ and $G = (G, B)$ over U is defined as $H = (H, C)$ where $C = A \cup B$ and

$$H(e) = \begin{cases} F(e), & e \in A - B \\ G(e), & e \in B - A \\ F(e) \cup G(e), & e \in A \cap B \end{cases}$$

(3) *Intersection of Soft Sets:* The intersection of two soft sets $F = (F, A)$ and $G = (G, B)$ with disjoint parameter sets $A \cap B = \emptyset$ is defined as $H = (H, C)$, where $C = A \cap B$ and

$$H(e) = F(e) \cap G(e), \quad \forall e \in C.$$

Definition 4.7. [82] Let $H^* = (V, E)$ be a hypergraph with a vertex set V and a hyperedge set E. A *soft hypergraph* is defined by a 4-tuple $H = (H^*, A, B, C)$, where:

(1) $H^* = (V, E)$ is a simple hypergraph.
(2) C is a nonempty set of parameters.
(3) (A, C) is a soft set over V, where $A : C \to \mathcal{P}(V)$ is a mapping such that $A(c) \subseteq V$ for each $c \in C$.
(4) (B, C) is a soft set over E_s, the set of all subhyperedges of H^*, where $B : C \to \mathcal{P}(E_s)$ maps each $c \in C$ to a subset of the maximal subhyperedges of H^*.
(5) For each $c \in C$, the pair $(A(c), B(c))$ forms a semisubhypergraph of H^*, meaning each element in $B(c)$ is a subhyperedge of a hyperedge in H^* and corresponds to the vertices in $A(c)$.

If we denote $(A(c), B(c))$ by $F(c)$, then the soft hypergraph H is represented by $\{F(c) : c \in C\}$, where each $F(c)$ is termed as a *hyperpart* (or simply *h-part*) of the soft hypergraph H.

Definition 4.8 (Soft Semigraph)**.** [79, 81] Let $S^* = (T, D)$ be a semigraph with vertex set T and edge set D. A *soft semigraph* is defined by a 4-tuple $S = (S^*, I, J, K)$, where:

(1) $S^* = (T, D)$ is a semigraph with a set T of vertices and a set D of edges.
(2) K is a nonempty set of parameters.
(3) (I, K) is a soft set over T, where $I : K \to \mathcal{P}(T)$ is a mapping such that $I(k) \subseteq T$ for each $k \in K$. For each $k \in K$, $I(k)$ denotes the subset of vertices associated with parameter k.
(4) (J, K) is a soft set over D_p, the set of all maximal partial edges (or *mp edges*) in S^*. Here, $J : K \to \mathcal{P}(D_p)$ maps each $k \in K$ to a subset $J(k)$ of D_p, where each element in $J(k)$ is an mp edge formed by some or all vertices in $I(k)$.

The soft semigraph $S = (S^*, I, J, K)$ represents a collection of partial semigraphs of S^* defined by different parameters in K. Specifically, for each $k \in K$, the pair $L(k) = (I(k), J(k))$ is called a *partial semigraph* (or *p-part*) of S^*.

Definition 4.9 (Soft Semihypergraph)**.** Let $H^* = (V, E_h)$ be a semihypergraph, where:

- V is a finite set of vertices.
- $E_h = \{E_h^1, E_h^2, \ldots, E_h^p\}$ is a set of hyperedges, where each hyperedge E_h^j is an ordered n_j-tuple of distinct vertices from V, with $n_j \geq 2$.

Let C be a non-empty set of parameters. A *soft semihypergraph* is a quadruple $H = (H^*, A, B, C)$, where:

(1) (A, C) is a soft set over V, i.e., $A : C \to \mathcal{P}(V)$ is a mapping such that $A(c) \subseteq V$ for each $c \in C$.
(2) (B, C) is a soft set over E_s, where E_s is the set of all subhyperedges of H^*. The mapping $B : C \to \mathcal{P}(E_s)$ assigns to each $c \in C$ a set $B(c)$ of maximal subhyperedges corresponding to $A(c)$.
(3) For each $c \in C$, the pair $F(c) = (A(c), B(c))$ forms a *partial semihypergraph* of H^*, meaning:
    - Each hyperedge in $B(c)$ is a subhyperedge of some hyperedge in E_h.
    - The vertices of each hyperedge in $B(c)$ are elements of $A(c)$.

The collection $H = \{F(c) : c \in C\}$ represents the soft semihypergraph, where each $F(c)$ is called a *partial semihypergraph* (or *p-part*) of H.

Theorem 4.10. *The concept of a soft semihypergraph generalizes both semihypergraphs and soft semigraphs. Specifically:*

(1) *Every semihypergraph can be considered as a soft semihypergraph.*
(2) *Every soft semigraph can be represented as a soft semihypergraph.*

Proof. Let $H^* = (V, E_h)$ be a semihypergraph. We can construct a soft semihypergraph $H = (H^*, A, B, C)$ by choosing:

- $C = \{c\}$ is a singleton parameter set.
- Define $A : C \to \mathcal{P}(V)$ by $A(c) = V$.
- Define $B : C \to \mathcal{P}(E_h)$ by $B(c) = E_h$.

Then, $(A(c), B(c)) = (V, E_h)$ is the original semihypergraph H^*. Since H consists of a single partial semihypergraph identical to H^*, the semihypergraph is represented as a soft semihypergraph.

Let $S = (S^*, I, J, K)$ be a soft semigraph as defined in [82], where:

- $S^* = (T, D)$ is a semigraph with vertex set T and edge set D.
- K is a non-empty set of parameters.
- $I : K \to \mathcal{P}(T)$ is a mapping representing the soft set over T.
- $J : K \to \mathcal{P}(D_p)$, where D_p is the set of partial edges of S^*.

We can construct a semihypergraph $H^* = (V, E_h)$ corresponding to S^* by setting:

- $V = T$.
- $E_h = D_p$, treating each partial edge as an ordered hyperedge.

Now, define the soft semihypergraph $H = (H^*, A, B, K)$ by:

- $A = I$.
- $B = J$.

For each $k \in K$, the partial semihypergraph $F(k) = (A(k), B(k))$ corresponds to the partial semigraph $L(k) = (I(k), J(k))$ in the soft semigraph S.

Therefore, the soft semigraph $S = (S^*, I, J, K)$ is represented as a soft semihypergraph $H = (H^*, A, B, K)$. □

Question 4.11. Is it possible to extend a Soft Semihypergraph to a Soft Semisuperhypergraph? What potential applications might such an extension have?

4.3. *Multi-Superhypergraph*

One well-known graph class is the multigraph, which allows for multiple (parallel) edges between the same pair of vertices [40,62,110,127]. This concept extends to the multi-hypergraph, a generalization to hypergraphs where parallel hyperedges are permitted. Both multigraphs and multi-hypergraphs [145, 209] have numerous applications, particularly in areas such as neural network theory [25, 114, 119, 189]. Here, we aim to explore an even broader extension with the definition of the Multi-Superhypergraph, provided below.

Definition 4.12 (Multiset)**.** [26, 27, 166] A multiset M over a set S is a collection where each element $x \in S$ can appear more than once. Formally, a multiset is defined as a pair $M = (S, m)$, where:

- S is the underlying set of distinct elements, called the *support* of M.
- $m : S \to \mathbb{Z}_{\geq 0}$ is a function called the *multiplicity function*, which assigns to each element $x \in S$ a non-negative integer $m(x)$, representing the number of occurrences of x in M.

If $m(x) > 0$ for some $x \in S$, then x is said to be an element of M with multiplicity $m(x)$. A multiset with all multiplicities equal to 1 is equivalent to a set.

Definition 4.13 (Multigraph)**.** [40] Let $G = (V, E)$ be a multigraph, where:

- V is a non-empty set of vertices.
- E is a multiset of unordered pairs of vertices, meaning that E allows multiple edges (parallel edges) between the same pair of vertices.

In a multigraph, loops (edges that connect a vertex to itself) may or may not be permitted, depending on the specific multigraph definition being used.

Definition 4.14 (Multi-Hypergraph)**.** [209] Let $H = (V, E)$ be a multi-hypergraph, where:

- V is a non-empty set of vertices.
- E is a multiset of subsets of V, called hyperedges, where each hyperedge can contain any number of vertices. The multiset property allows for multiple occurrences of the same hyperedge (parallel hyperedges).

In a multi-hypergraph, hyper-loops (hyperedges that contain repeated vertices) may be allowed, providing further generalization.

Definition 4.15 (Multi-Superhypergraph)**.** A *Multi-Superhypergraph* is a triple $H = (V, S, E)$, where:

- V is a non-empty finite set of elements called *vertices*.
- S is a multiset of non-empty subsets of V, called *supervertices*. Each supervertex $s \in S$ satisfies $s \subseteq V$, and the multiset allows for multiple occurrences of the same supervertex.
- E is a multiset of non-empty subsets of S, called *superedges*. Each superedge $e \in E$ is a subset of supervertices from S, and multiple occurrences of the same superedge are permitted.

Theorem 4.16. *Every Multigraph and every Multi-Hypergraph can be represented as a Multi-Superhypergraph. Conversely, the Multi-Superhypergraph generalizes both the Multigraph and the Multi-Hypergraph.*

Proof. Let $G = (V, E_G)$ be a multigraph, where:

- V is a non-empty set of vertices.

- E_G is a multiset of unordered pairs of vertices, allowing multiple edges between the same pair of vertices.

Construction:

Define a Multi-Superhypergraph $H = (V', S, E)$ as follows:

- Set $V' = V$.
- Let $S = \{\{v\} \mid v \in V\}$. Each vertex v becomes a singleton supervertex $\{v\}$. Since S is a multiset, multiplicities are considered if necessary.
- For each edge $e = \{u, v\} \in E_G$, include the superedge $e' = \{\{u\}, \{v\}\}$ in the multiset E. The multiplicity of e' in E matches the multiplicity of e in E_G.

Verification:

- Supervertices in S correspond to individual vertices in V.
- Superedges in E connect singleton supervertices, reflecting the edges in G.
- Multiplicities of edges are preserved due to the multiset nature of E.

And let $H_G = (V, E_H)$ be a multi-hypergraph, where:

- V is a non-empty set of vertices.
- E_H is a multiset of subsets of V, called hyperedges, allowing multiple occurrences of the same hyperedge.

Construction:

Define a Multi-Superhypergraph $H = (V', S, E)$ as follows:

- Set $V' = V$.
- Let $S = \{\{v\} \mid v \in V\}$. Each vertex becomes a singleton supervertex.
- For each hyperedge $e_H \in E_H$, include the superedge $e = \{\{v\} \mid v \in e_H\}$ in the multiset E. The multiplicity of e in E matches that of e_H in E_H.

Verification:

- Supervertices represent the original vertices as singleton sets.
- Superedges correspond to hyperedges by connecting the supervertices representing the vertices in each hyperedge.
- Multiplicities of hyperedges are preserved in E.

The Multi-Superhypergraph accommodates:

- Supervertices that are subsets of V, not limited to singletons.
- Superedges connecting any subsets of supervertices.
- Multiplicities of supervertices and superedges through the use of multisets.

Thus, the Multi-Superhypergraph generalizes both the Multigraph and the Multi-Hypergraph by allowing more complex relationships and multiplicities. □

4.4. *Pseudo-SuperHypergraph*

A pseudograph is a type of graph that allows parallel edges and self-loops (edges that return to the originating vertex) [21, 29, 110]. This structure enables the representation of more complex relationships and intricate networks compared to standard graphs [203, 217]. A pseudo-hypergraph generalizes the pseudograph concept by incorporating hypergraphs, extending its capabilities for modeling even more sophisticated connections [17, 31, 118]. Moving forward, We aim to elucidate the mathematical structure and potential applications of the pseudo-superhypergraph. Below, we present the established definitions of pseudograph and pseudo-hypergraph, followed by an overview of the pseudo-superhypergraph.

Definition 4.17 (Pseudograph)**.** [202] A *pseudograph* $G = (V, E)$ is a graph that consists of:

- A non-empty set V of elements called *vertices*.
- A set E of unordered pairs of vertices, where each edge $e \in E$ is of the form $\{u, v\}$ with $u, v \in V$. The following are permitted in E:
  - *Parallel edges*: Multiple instances of the same pair $\{u, v\}$ are allowed in E.
  - *Loops*: An edge of the form $\{v, v\}$, which connects a vertex to itself, is allowed.

Definition 4.18 (Pseudo-Hypergraph)**.** [17] A *pseudo-hypergraph* $H = (V, E)$ generalizes the concept of a hypergraph and consists of:

- A non-empty set V of elements called *vertices*.
- A set E of multisets of vertices, where each hyperedge $e \in E$ is a multiset over V. This allows:
  - *Multiple occurrences of vertices within a hyperedge*: A vertex $v \in V$ may appear more than once in a single hyperedge e, resembling loops in pseudographs.
  - *Multiple identical hyperedges*: The same multiset of vertices may appear multiple times in E, resembling parallel edges in pseudographs.

Remark 4.19 (Pseudo-Hypergraph)**.** In a pseudo-hypergraph, each hyperedge can connect any subset of vertices from V, with repetitions of vertices within hyperedges permitted. This allows a single hyperedge to act both as a multi-edge and as a "loop" by including repeated occurrences of vertices.

Definition 4.20 (Pseudo-Superhypergraph)**.** A *pseudo-superhypergraph* $H = (V, S, E)$ consists of:

- *Vertices*: A non-empty finite set V of elements called *vertices*.
- *Supervertices*: A multiset S of multisets over V, called *supervertices*. Each supervertex $s \in S$ is a multiset of vertices from V, allowing:

  - Multiple occurrences of the same vertex within a supervertex (vertices can repeat within s).
  - Multiple occurrences of the same supervertex in S (supervertices can repeat in S).
- *Superedges*: A multiset E of multisets over S, called *superedges*. Each superedge $e \in E$ is a multiset of supervertices from S, allowing:
  - Multiple occurrences of the same supervertex within a superedge (supervertices can repeat within e).
  - Multiple occurrences of the same superedge in E (superedges can repeat in E).

Remark 4.21 (Pseudo-Superhypergraph)**.** In a pseudo-superhypergraph:

- Supervertices can have repeated vertices from V.
- Superedges can have repeated supervertices from S.
- Superedges themselves can appear multiple times in E.

Theorem 4.22. *Every pseudograph and every pseudo-hypergraph can be represented as a pseudo-superhypergraph. Conversely, the pseudo-superhypergraph generalizes both the pseudograph and the pseudo-hypergraph.*

Proof. We prove this theorem in two parts.

Part 1: Pseudographs as Pseudo-Superhypergraphs. Let $G = (V, E_G)$ be a pseudograph, where:

- V is a non-empty set of vertices.
- E_G is a multiset of unordered pairs $\{u, v\}$ with $u, v \in V$, allowing for loops and multiple edges.

Construction:

Define a pseudo-superhypergraph $H = (V', S, E)$ as follows:

- Set $V' = V$.
- Let $S = \{\{v\} \mid v \in V\}$. Each vertex v becomes a supervertex $s_v = \{v\}$ in S.
- For each edge $e = \{u, v\} \in E_G$:
  - If $u \neq v$:
    - Define $e' = \{s_u, s_v\}$, a multiset containing s_u and s_v.
    - Include e' in E with multiplicity equal to the number of edges between u and v in E_G.
  - If $u = v$ (a loop at v):
    - Define $e' = \{s_v, s_v\}$, a multiset containing s_v twice.
    - Include e' in E with multiplicity equal to the number of loops at v in E_G.

Verification:

- Supervertices represent the vertices in V.
- Superedges correspond to edges in E_G, including loops and multiple edges.
- Repeated supervertices within a superedge represent loops.
- Multiplicities in E reflect the multiplicities in E_G.

Part 2: Pseudo-Hypergraphs as Pseudo-Superhypergraphs. Let $H_G = (V, E_H)$ be a pseudo-hypergraph, where:

- V is a non-empty set of vertices.
- E_H is a multiset of multisets over V, allowing for repeated vertices within hyperedges and multiple hyperedges.

Construction:

Define a pseudo-superhypergraph $H = (V', S, E)$ as follows:

- Set $V' = V$.
- Let $S = \{\{v\} \mid v \in V\}$.
- For each hyperedge $e_H \in E_H$:
  - Construct a superedge e by replacing each occurrence of v in e_H with s_v.
  - Thus, if v appears m times in e_H, s_v appears m times in e.
  - Include e in E with multiplicity equal to the multiplicity of e_H in E_H.

Verification:

- Supervertices correspond to vertices in V.
- Superedges represent hyperedges, preserving multiplicities and repeated vertices.
- Repeated supervertices within a superedge represent repeated vertices within a hyperedge.
- Multiplicities of hyperedges are preserved in E.

Since both pseudographs and pseudo-hypergraphs can be represented as pseudo-superhypergraphs, and the pseudo-superhypergraph allows for more general structures, it follows that the pseudo-superhypergraph generalizes both concepts. □

4.5. *Mixed Superhypergraph*

A mixed graph integrates undirected and directed edges, allowing both bidirectional and directional connections between vertices [64, 154]. This concept has been expanded to mixed hypergraphs [193], extending the structure to hypergraphs, whose mathematical properties have been examined in depth. In this paper, we further introduce the framework of mixed superhypergraphs. Future work will explore fuzzy mixed hypergraphs and neutrosophic mixed hypergraphs, combining elements of fuzzy and neutrosophic graphs within this extended structure. Although detailed definitions are beyond this paper's scope, readers needing background

on directed graphs [24, 37, 103], directed hypergraphs [85, 108, 132, 135], or directed superhypergraphs [65] are encouraged to consult relevant literature as needed.

Definition 4.23 (Mixed Graph)**.** [152] A *mixed graph* $G = (V, E \cup A)$ is a graph structure with:

- V, a non-empty set of vertices.
- E, a set of undirected edges, where each edge $e \in E$ is an unordered pair $\{u, v\}$ of distinct vertices $u, v \in V$.
- A, a set of directed edges (also called arcs), where each arc $a \in A$ is an ordered pair (u, v) of distinct vertices $u, v \in V$.

In a mixed graph, an edge can either connect two vertices bidirectionally (undirected) or directionally (from one vertex to another).

Remark 4.24. A mixed graph combines features of both undirected and directed graphs. While undirected edges $e = \{u, v\}$ imply symmetric relationships between vertices u and v, directed edges $a = (u, v)$ define an asymmetric connection from u to v.

Definition 4.25 (Mixed Hypergraph)**.** [193] A *mixed hypergraph* $H = (V, E \cup A)$ is a hypergraph structure with:

- V, a non-empty set of vertices.
- E, a set of undirected hyperedges, where each hyperedge $e \in E$ is a non-empty subset of V containing at least two vertices.
- A, a set of directed hyperedges (also known as *dyperedges*), where each directed hyperedge $a = (Z, z) \in A$ is an ordered pair consisting of:
  - Z, a non-empty subset of $V \setminus \{z\}$, called the *tail set* of a.
  - $z \in V$, a single vertex called the *head* of a.

In a mixed hypergraph, undirected hyperedges connect subsets of vertices without directionality, whereas directed hyperedges represent a directed relationship from the vertices in the tail set Z to the head z.

Remark 4.26. Mixed hypergraphs generalize mixed graphs by allowing hyperedges, which can involve more than two vertices, and by permitting directed connections (dyperedges) between sets of vertices and individual vertices. The structure enables modeling of more complex relationships in network theory, such as group-to-individual influence dynamics.

Theorem 4.27. *Any mixed hypergraph can be transformed into a mixed graph.*

Proof. Given a mixed hypergraph $H = (V, E \cup A)$, we construct a mixed graph $G = (V', E' \cup A')$ as follows:

(1) *Vertex Set V':* Define $V' = V \cup \{e \mid e \in E\}$. For each undirected hyperedge $e \in E$, introduce a new vertex in V' to represent it.
(2) *Edge Set E':* For each undirected hyperedge $e \in E$ and each $v \in e$, create an undirected edge $\{v, e\}$ in E'.
(3) *Arc Set A':* For each directed hyperedge $a = (Z, z) \in A$, choose a vertex $y \in Z$ and create a directed edge (y, z) in A'.

Verification:

- The structure of the undirected hyperedges E is represented in G by the connections between the original vertices in V and the new vertices representing hyperedges in V'.
- Directed hyperedges in A are represented by directed edges connecting a vertex in the tail set to the head vertex, thereby preserving directionality in the transformed graph.

The resulting G is a mixed graph that represents the original mixed hypergraph H. □

Definition 4.28 (Mixed Superhypergraph)**.** A *mixed superhypergraph* $H = (V, S, E, A)$ consists of:

- A non-empty set V of elements called *vertices*.
- A set S of non-empty subsets of V, called *supervertices*, where each supervertex $s \in S$ satisfies $s \subseteq V$.
- A set E of undirected *superedges*, where each superedge $e \in E$ is a non-empty subset of S.
- A set A of directed *superedges* (also known as *super-dyperedges*), where each directed superedge $a = (Z, z) \in A$ consists of:
  - A non-empty subset $Z \subseteq S \setminus \{z\}$, called the *tail set*.
  - A supervertex $z \in S$, called the *head*.

In a mixed superhypergraph, undirected superedges connect subsets of supervertices without directionality, while directed superedges represent a directed relationship from the tail set Z to the head z.

Theorem 4.29. *Any mixed superhypergraph can be transformed into a mixed hypergraph.*

Proof. Let $H = (V, S, E, A)$ be a mixed superhypergraph. We construct a mixed hypergraph $H' = (V', E', A')$ as follows:

(1) *Vertex Set V':* Introduce a new vertex v_s in V' for each supervertex $s \in S$. Thus,

$$V' = \{v_s \mid s \in S\}.$$

(2) *Undirected Hyperedges E':* For each undirected superedge $e \in E$, create an undirected hyperedge e' in E' defined by:

$$e' = \{v_s \mid s \in e\}.$$

(3) *Directed Hyperedges A':* For each directed superedge $a = (Z, z) \in A$, create a directed hyperedge a' in A' defined by:

$$a' = (\{v_s \mid s \in Z\}, v_z),$$

where v_z is the vertex corresponding to the head supervertex z.

Verification:

- The vertex set V' represents the supervertices of H.
- Undirected superedges in E are transformed into undirected hyperedges in E' by mapping the supervertices in e to their corresponding vertices in V'.
- Directed superedges in A are transformed into directed hyperedges in A', preserving the direction from the tail set Z to the head z.

The mixed hypergraph $H' = (V', E', A')$ represents the original mixed superhypergraph H, with supervertices replaced by vertices and superedges appropriately transformed. □

Theorem 4.30. *Any mixed superhypergraph can be transformed into an undirected superhypergraph by removing the directed superedges.*

Proof. Given a mixed superhypergraph $H = (V, S, E, A)$, we construct an undirected superhypergraph $H' = (V, S, E')$ as follows:

(1) *Supervertex Set S:* Retain the same supervertex set from H.

(2) *Undirected Superedges E':* Form E' by combining the undirected superedges E with the directed superedges A converted to undirected superedges:

$$E' = E \cup \{e_a \mid a = (Z, z) \in A\},$$

where for each $a = (Z, z) \in A$, the undirected superedge e_a is defined as:

$$e_a = Z \cup \{z\}.$$

Verification:

- The undirected superedges in E remain unchanged.
- Each directed superedge $a = (Z, z)$ is converted into an undirected superedge e_a connecting all supervertices in $Z \cup \{z\}$, thus removing directionality while preserving the connectivity.

The undirected superhypergraph $H' = (V, S, E')$ maintains the structure of H without directionality, representing all relationships among supervertices with undirected superedges.

□

4.6. *Bidirected Superhypergraph*

The concept of the bidirected graph [15, 59, 78] has been introduced in recent years. Below, we provide definitions for the bidirected hypergraph and bidirected superhypergraph, which generalize bidirected graphs.

Definition 4.31 (Bidirected Graph)**.** [15, 59, 78] A *bidirected graph* (also known as a *bigraph*) is a pair $B = (G, \tau)$, where:

- $G = (V, E)$ is a simple undirected graph, where V is a non-empty set of vertices and E is a set of edges (without parallel edges or loops).
- $\tau : V \times E \to \{-1, 0, 1\}$ is a function called the *bidirection function*, which assigns a *local orientation* to each vertex-edge pair (v, e) as follows:
  - $\tau(v, e) = 1$: Edge e is directed *towards* vertex v.
  - $\tau(v, e) = -1$: Edge e is directed *away from* vertex v.
  - $\tau(v, e) = 0$: Vertex v is not incident to edge e.

The graph G is referred to as the *underlying graph* of B, and the function τ provides the bidirected structure on G by assigning a direction at each endpoint of every edge in E.

Definition 4.32 (Bidirected Hypergraph)**.** A *bidirected hypergraph* is a triple $H = (V, E, \tau)$, where:

- V is a non-empty set of vertices.
- E is a set of hyperedges, where each hyperedge $e \in E$ is a non-empty subset of V.
- $\tau : V \times E \to \{-1, 0, 1\}$ is a function called the *bidirection function*, assigning a *local orientation* to each vertex-hyperedge pair (v, e) such that:
  - $\tau(v, e) = 1$ if hyperedge e is directed *towards* vertex v.
  - $\tau(v, e) = -1$ if hyperedge e is directed *away from* vertex v.
  - $\tau(v, e) = 0$ if $v \notin e$.

Additionally, for each hyperedge $e \in E$, we require that:

$$\sum_{v \in e} \tau(v, e) = 0.$$

This condition ensures that the hyperedge is balanced in terms of its local orientations.

Definition 4.33 (Bidirected Superhypergraph)**.** A *bidirected superhypergraph* is a quadruple $H = (V, S, E, \tau)$, where:

- V is a non-empty set of vertices.
- S is a set of non-empty subsets of V, called *supervertices*.
- E is a set of superedges, where each superedge $e \in E$ is a non-empty subset of S.
- $\tau : S \times E \to \{-1, 0, 1\}$ is a function called the *bidirection function*, assigning a *local orientation* to each supervertex-superedge pair (s, e) such that:
  - $\tau(s, e) = 1$ if superedge e is directed *towards* supervertex s.
  - $\tau(s, e) = -1$ if superedge e is directed *away from* supervertex s.
  - $\tau(s, e) = 0$ if $s \notin e$.

Similarly to bidirected hypergraphs, for each superedge $e \in E$, we require that:

$$\sum_{s \in e} \tau(s, e) = 0.$$

This condition ensures that the superedge is balanced in terms of its local orientations.

Theorem 4.34. *Any bidirected hypergraph can be transformed into a bidirected graph.*

Proof. Given a bidirected hypergraph $H = (V, E, \tau)$, we construct a bidirected graph $G = (V', E', \tau')$ as follows:

(1) *Vertex Set V':* Let $V' = V \cup \{e \mid e \in E\}$. That is, for each hyperedge $e \in E$, introduce a new vertex representing e.

(2) *Edge Set E':* For each hyperedge $e \in E$ and each vertex $v \in e$, create an edge $\{v, e\}$ connecting vertex v to the hyperedge vertex e.

(3) *Bidirection Function τ':* For each edge $\{v, e\} \in E'$, define:

$$\tau'(v, \{v, e\}) = \tau(v, e), \quad \tau'(e, \{v, e\}) = -\tau(v, e).$$

Verification:

- The sum of the bidirections at the endpoints of each edge $\{v, e\}$ is zero:

$$\tau'(v, \{v, e\}) + \tau'(e, \{v, e\}) = \tau(v, e) - \tau(v, e) = 0.$$

- This construction preserves the bidirectional relationships of the original hypergraph H in the graph G.

The resulting G is a bidirected graph that represents the original bidirected hypergraph H. Therefore, any bidirected hypergraph can be transformed into a bidirected graph. □

Theorem 4.35. *Any bidirected superhypergraph can be transformed into a bidirected hypergraph.*

Proof. Given a bidirected superhypergraph $H = (V, S, E, \tau)$, we construct a bidirected hypergraph $H' = (V', E', \tau')$ as follows:

(1) *Vertex Set V':* Let $V' = S$. Each supervertex $s \in S$ becomes a vertex in V'.

(2) *Hyperedge Set E':* For each superedge $e \in E$, define a hyperedge $e' = e$, consisting of the supervertices in e.

(3) *Bidirection Function τ':* For each $s \in V'$ and $e' \in E'$, define:

$$\tau'(s, e') = \tau(s, e).$$

Verification:

- The bidirected hypergraph H' has vertices corresponding to the supervertices of H, and hyperedges corresponding to the superedges.
- The local orientations in H' are preserved from H via τ'.

Thus, any bidirected superhypergraph can be transformed into a bidirected hypergraph by considering supervertices as vertices and superedges as hyperedges. □

Theorem 4.36. *Any bidirected superhypergraph can be transformed into an undirected superhypergraph by removing the bidirections.*

Proof. Given a bidirected superhypergraph $H = (V, S, E, \tau)$, we construct an undirected superhypergraph $H' = (V, S, E)$ by:

- Retaining the same vertex set V, supervertex set S, and superedge set E.
- Omitting the bidirection function τ.

Verification:

- The connectivity among supervertices via superedges remains unchanged.
- The removal of τ eliminates all local orientations, resulting in an undirected structure.

The resulting H' is an undirected superhypergraph that retains the original connectivity of H without bidirections. Therefore, any bidirected superhypergraph can be transformed into an undirected superhypergraph. □

We also aim to explore extensions of these concepts, such as fuzzy bidirected graphs and neutrosophic bidirected graphs, by integrating elements from fuzzy and neutrosophic graph theory. While these ideas are still in the conceptual stage, we propose the following definitions as a foundation. Further research into the mathematical structures of these graphs is expected to advance in the future.

Definition 4.37 (Bidirected Fuzzy Graph)**.** A *Bidirected Fuzzy Graph* is a tuple $BFG = (G, \tau, \sigma, \mu)$, where:

- $G = (V, E)$ is a simple undirected graph, where V is the set of vertices and E is the set of edges.

- $\tau : V \times E \to \{-1, 0, 1\}$ is the *bidirection function*, which assigns a direction to each vertex-edge pair as follows:

$$\tau(v, e) = \begin{cases} 1 & \text{if } e \text{ is directed towards } v, \\ -1 & \text{if } e \text{ is directed away from } v, \\ 0 & \text{if } v \text{ is not incident to } e. \end{cases}$$

- $\sigma : V \to [0, 1]$ is the vertex membership function, assigning a membership degree to each vertex $v \in V$.
- $\mu : E \to [0, 1]$ is the edge membership function, assigning a membership degree to each edge $e \in E$.

The membership functions must satisfy the condition:

$$\mu(e) \leq \min\{\sigma(u), \sigma(v)\}, \quad \text{for each edge } e = (u, v) \in E.$$

Definition 4.38 (Bidirected Neutrosophic Graph)**.** A *Bidirected Neutrosophic Graph* is a tuple $BNG = (G, \tau, \sigma, \mu)$, where:

- $G = (V, E)$ is a simple undirected graph.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function as defined above.
- $\sigma : V \to [0, 1]^3$ assigns a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$ to each vertex $v \in V$, representing the truth, indeterminacy, and falsity degrees, respectively.
- $\mu : E \to [0, 1]^3$ assigns a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$ to each edge $e \in E$.

The membership functions must satisfy the following conditions:

$$\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3, \quad \forall v \in V,$$
$$\mu_T(e) + \mu_I(e) + \mu_F(e) \leq 3, \quad \forall e \in E.$$

Definition 4.39 (Bidirected Plithogenic Graph)**.** A *Bidirected Plithogenic Graph* is a tuple $BPG = (G, \tau, PM, PN)$, where:

- $G = (V, E)$ is a simple undirected graph.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function as defined above.
- $PM = (M, l, Ml, adf, aCf)$ is the Plithogenic Vertex Set:
  - $M \subseteq V$ is the set of vertices.
  - l is the vertex attribute.
  - Ml is the range of possible vertex attribute values.
  - $adf : M \times Ml \to [0, 1]^s$ is the Degree of Appurtenance Function.
  - $aCf : Ml \times Ml \to [0, 1]^t$ is the Degree of Contradiction Function for vertices.
- $PN = (N, m, Nm, bdf, bCf)$ is the Plithogenic Edge Set:
  - $N \subseteq E$ is the set of edges.

- m is the edge attribute.
- Nm is the range of possible edge attribute values.
- $bdf : N \times Nm \to [0,1]^s$ is the Degree of Appurtenance Function.
- $bCf : Nm \times Nm \to [0,1]^t$ is the Degree of Contradiction Function for edges.

The functions adf and bdf must satisfy the Appurtenance Constraint, and aCf and bCf must satisfy the Contradiction Function Constraints as defined in Plithogenic Graphs.

Theorem 4.40. *Bidirected Fuzzy Graphs, Bidirected Neutrosophic Graphs, and Bidirected Plithogenic Graphs are generalizations of Bidirected Graphs.*

Proof. Let $B = (G, \tau)$ be a Bidirected Graph, where:

- $G = (V, E)$ is a simple undirected graph, with V as the set of vertices and E as the set of edges.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function, assigning a local direction to each vertex-edge pair.

A Bidirected Fuzzy Graph $BFG = (G, \tau, \sigma, \mu)$ extends B by introducing:

- $\sigma : V \to [0,1]$, the vertex membership function.
- $\mu : E \to [0,1]$, the edge membership function.

When $\sigma(v) = 1$ for all $v \in V$ and $\mu(e) = 1$ for all $e \in E$, the fuzzy graph reduces to the classical bidirected graph B. Hence, BFG generalizes B.

A Bidirected Neutrosophic Graph $BNG = (G, \tau, \sigma, \mu)$ extends B by introducing:

- $\sigma : V \to [0,1]^3$, assigning truth, indeterminacy, and falsity degrees to each vertex.
- $\mu : E \to [0,1]^3$, assigning truth, indeterminacy, and falsity degrees to each edge.

When $\sigma(v) = (1, 0, 0)$ and $\mu(e) = (1, 0, 0)$ for all $v \in V$ and $e \in E$, the neutrosophic graph reduces to B. Thus, BNG generalizes B.

A Bidirected Plithogenic Graph $BPG = (G, \tau, PM, PN)$ extends B by incorporating:

- Plithogenic Vertex Set $PM = (M, l, Ml, adf, aCf)$, which assigns attributes and appurtenance/contradiction values to vertices.
- Plithogenic Edge Set $PN = (N, m, Nm, bdf, bCf)$, which assigns attributes and appurtenance/contradiction values to edges.

When $adf(x, a) = 1$, $bdf(e, b) = 1$, and contradiction functions are zero for all $x \in M, e \in N$, and attributes a, b, the plithogenic graph reduces to B. Hence, BPG generalizes B.

Since each of these graph types reduces to the classical Bidirected Graph under specific conditions, it follows that Bidirected Fuzzy Graphs, Bidirected Neutrosophic Graphs, and Bidirected Plithogenic Graphs are generalizations of Bidirected Graphs. □

Theorem 4.41. *A Bidirected Plithogenic Graph is a generalization of a Plithogenic Graph.*

Proof. Let $PG = (G, PM, PN)$ be a Plithogenic Graph. Now, let $BPG = (G, \tau, PM, PN)$ be a Bidirected Plithogenic Graph, where:

- $G = (V, E)$ and PM, PN are as defined in PG.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function, which assigns a local orientation to each vertex-edge pair.

To prove the generalization, consider the following:

- If $\tau(v, e) = 0$ for all $v \in V$ and $e \in E$, the bidirection function does not influence the structure of the graph. In this case, BPG reduces to PG, as the bidirection function becomes irrelevant.
- All other components of BPG, including the vertex and edge attribute structures (DAF and DCF), remain identical to those in PG.

Since BPG retains all the characteristics of PG and extends it with the bidirection function τ, it follows that every Plithogenic Graph PG can be represented as a Bidirected Plithogenic Graph BPG under appropriate conditions ($\tau(v, e) = 0$ for all v and e).

Thus, Bidirected Plithogenic Graphs generalize Plithogenic Graphs. □

4.7. *Other future tasks: More Graph Classes*

Additionally, we aim to generalize above structures using Fuzzy and Neutrosophic principles [169,170], examining their mathematical frameworks and applications, particularly in decision-making contexts. Furthermore, we plan to extend the concept of the Soft Semihypergraph by integrating ideas from Rough Graphs [41, 49, 96, 137, 196], Soft Expert Graphs [18, 58, 151, 194, 201], Hypersoft Graphs [73, 74, 148, 156–158], Superhypersoft Sets [83, 130, 179, 182], TreeSoft Sets [50,139,180,181,183], Intersection Graphs [69,129], Vague Graphs [3,4,150,160], Single-Valued Pentapartitioned Neutrosophic Graphs [48,72,98,99,147], Hesitant Fuzzy Graphs [20, 84, 138, 142, 215], Infinite Graphs [53, 54, 56], Directed Graphs [24, 37, 103], Mixed Graphs [64, 154, 155], and Superhypergraphs. This approach aims to facilitate a deeper investigation and broader applicability of these advanced structures.

As an additional note, applications of the aforementioned graph classes extend beyond decision-making, which is often explored in uncertain graphs, to promising advancements in areas like superhypergraph neural networks, fuzzy hypergraph neural networks, neutrosophic graph networks, and more. The authors hopes that research in these areas will gain momentum, fostering further developments across these fields.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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A Short Note for Hypersoft Rough Graphs

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Abstract

Graph theory, a branch of mathematics, explores relationships among entities using vertices and edges. To address the uncertainties inherent in real-world networks, Uncertain Graph Theory has emerged as a significant extension within this field. Hypersoft Graphs advance traditional graphs further by incorporating multi-attribute nodes, allowing each node to represent multiple attribute values, thus capturing more complex, multi-dimensional relationships. The Soft Rough Set and Soft Rough Graph combine the principles of Soft Sets and Rough Sets to manage uncertainty in a more refined manner. In this paper, we introduce the Hypersoft Rough Graph and analyze its relationships with other graph classes.

Keywords: Hypersoft Rough Graph; Rough Graph; Soft Graph; Hypersoft Graph; Soft Set.

1 | Introduction

1.1 | Soft Graph and Rough Graph Classes

Graph theory, a fundamental branch of mathematics, models relationships within networks using vertices (nodes) and edges to represent connections and interactions among entities such as individuals, locations, or organizations. Since its inception in the 1700s, graph theory has evolved significantly, yielding profound theoretical advancements and a wide range of applications [31, 32, 47, 153, 162, 236]. In recent years, the concept of graphs has also found broad application in artificial intelligence, particularly in areas like Graph Neural Networks, further demonstrating its relevance [22, 35, 49, 83, 134, 135, 182, 216, 218, 237].

To address real-world uncertainty, mathematical frameworks such as Fuzzy Sets [224 229, 231], Soft Sets [141], Rough Sets [158], and Neutrosophic Sets [188-191, 203, 205] provide essential tools for analyzing ambiguous or imprecise information. This paper examines several models of uncertain graphs, which hold promise in a variety of real-world applications, inspiring diverse graph classes [55-59, 61-71, 73, 73-82].

A Soft Set [15, 16, 137, 219], or Soft Graph [7, 9, 110], offers a flexible framework for dealing with uncertainty by linking elements to specific parameters, thereby supporting adaptable decision-making processes. Extensive research has been conducted on soft graphs, soft sets, and their various extensions [8, 25, 26, 28, 34, 41, 48, 93, 110, 114, 123, 126, 140, 150, 184-186, 213].

Similarly, Rough Sets [101, 157-160] and Rough Graphs [37, 43, 111, 150, 212] manage uncertainty by using lower and upper boundaries to approximate imprecise data.

Hypersoft Graphs build on traditional graphs by incorporating multi-attribute nodes, enabling each node to represent multiple attribute values and thereby capture more complex, multi-dimensional relationships [171, 175-178]. Conceptually, this development aligns with Hypersoft Sets [1, 40, 138, 144, 145, 172, 181, 192, 223].

The Soft Rough Set and Soft Rough Graph [5, 6, 13, 18, 42, 52, 53, 125, 128, 151, 183, 214, 222, 235] combine principles from Soft Sets and Rough Sets, providing nuanced approaches to uncertainty. Similarly, Hypersoft Rough Sets integrates concepts from Hypersoft Sets and Rough Sets, extending their applicability to uncertain data analysis [124, 169, 208, 209].

1.2 | Contributions

As previously mentioned, while research on Soft Expert Graphs and related fields is advancing, studies on Hypersoft Expert Graphs remain scarce. In this paper, we define the Hypersoft Rough Graph and explore its relationships with other graph classes. In the Future Tasks section, we discuss concepts related to Hypersets, Superhypersets, and N-soft sets. These efforts are anticipated to further the study of Soft Graph Theory and Rough Graph Theory.

2 | Preliminaries and Definitions

This section offers an overview of the fundamental definitions and notations used throughout the paper. Additionally, some foundational concepts from set theory are applied in parts of this work. For further details, please consult relevant references as needed [54, 109, 113, 116, 127].

2.1 | Basic Graph Concepts

This section provides a concise overview of essential concepts in graph theory. For a more detailed study and additional notational conventions, see [45-47, 102, 215].

Definition 1 (Graph). [47] A graph G is a mathematical structure used to model pairwise relations between objects. It is composed of a set of vertices $V(G)$ and a set of edges $E(G)$, where each edge represents a connection between two vertices. Formally, a graph is denoted by $G = (V, E)$, with V as the set of vertices and E as the set of edges.

Definition 2 (Vertex Degree). [47] For a graph $G = (V, E)$, the degree of a vertex $v \in V$, denoted deg (v), is defined as the number of edges incident to v. For an undirected graph, this is expressed as:

$$\deg(v) = |\{e \in E \mid v \in e\}|$$

This measure reflects the connectivity of v within the graph.

2.2 | Soft Graph

A soft set over a universe U associates subsets of U to parameters from a parameter set A, capturing flexible relationships in uncertain data. A soft graph is a soft set over the vertices V of a graph G where each subset forms a connected subgraph, capturing parameter-based connections within G. The definitions are provided as follows.

Definition 3. [141] Let U be a non-empty finite set, called the universe of discourse, and let E be a non-empty set of parameters. A soft set over U is defined as follows:

$$F = (F, A) \text{ over } U \text{ is an ordered pair, where } A \subseteq E \text{ and } F: A \to P(U)$$

where $F(a) \subseteq U$ for each $a \in A$ and $P(U)$ denotes the power set of U. The set of all soft sets over U is denoted by $S(U)$.

1. Soft Subset: Let $F = (F, A)$ and $G = (G, B)$ be two soft sets over the common universe U. We say that F is a soft subset of G, denoted $F \subseteq G$, if:

- $A \subseteq B$
- $F(a) \subseteq G(a)$ for all $a \in A$.

2. Union of Soft Sets: The union of two soft sets $F = (F, A)$ and $G = (G, B)$ over U is defined as $H = (H, C)$ where $C = A \cup B$ and

$$H(e) = \begin{cases} F(e), & e \in A - B \\ G(e), & e \in B - A \\ F(e) \cup G(e), & e \in A \cap B \end{cases}$$

3. The intersection of Soft Sets: The intersection of two soft sets $F = (F, A)$ and $G = (G, B)$ with disjoint parameter sets $A \cap B = \emptyset$ is defined as $H = (H, C)$, where $C = A \cap B$ and

$$H(e) = F(e) \cap G(e), \ \forall e \in C$$

Definition 4. [7, 110] Let $G = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges. Let A be a non-empty set of parameters and $R \subseteq A \times V$ be a relation between elements of A and elements of V. Define a set-valued function $F: A \to P(V)$ by

$$F(x) = \{y \in V \mid xRy\}$$

The pair (F, A) is a soft set over V.

A Soft Graph of G is defined as follows:

A soft set (F, A) over V is said to be a soft graph of G if the subgraph $F(x)$ is a connected subgraph of G for all $x \in A$. The set of all soft graphs of G is denoted by $SG(G)$.

2.3 | Hypersoft Graph

A HyperSoft Graph represents multi-attribute nodes where each node can hold distinct attribute values, enabling complex, multi-dimensional relationships. The definition of a Hypersoft Graph is provided as follows [171, 175-178].

Definition 5 (Hypersoft Set). [192] Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n-distinct attributes with corresponding disjoint sets $J_1, J_2, \ldots, J_n$. A pair (F, J) is called a hypersoft set over the universal set X, where F is a mapping defined by

$$F: J \to \mathcal{P}(X)$$

with $J = J_1 \times J_2 \times \cdots \times J_n$.

Definition 6 (Hypersoft Graph). Let $G = (V, E)$ be a simple connected graph, where V is the set of vertices and E is the set of edges. Consider $J = J_1 \times J_2 \times \cdots \times J_n$, where each $J_i \subseteq V$ and $J_i \cap J_j = \emptyset$ for $i \neq j$. A Hypersoft Graph (HS-Graph) of G is defined as a hypersoft set (F, J) over V such that for each $x \in J, F(x)$ induces a connected subgraph of G. The set of all HS-Graphs of G is denoted by $\mathrm{HsG}(G)$.

Proposition 7. Any Hypersoft Graph can be transformed into a Soft Graph and subsequently into a Classical Graph.

Proof. Let $G = (V, E)$ be a Hypersoft Graph defined with $J = J_1 \times J_2 \times \cdots \times J_n$ and a hypersoft set $F: J \to \mathcal{P}(V)$. For each attribute tuple $x = (j_1, j_2, \ldots, j_n) \in J$, the mapping $F(x) \subseteq V$ defines subsets of V that induce connected subgraphs of G. To transform this into a Soft Graph, we redefine the parameter set $A = J_1 \cup J_2 \cup \cdots \cup J_n$ by flattening the multi-attribute parameters into a single layer of parameters. For each $a \in A$, define $F'(a)$ as the union of all $F(x)$ where $x \in J$ includes$a$ as one of its components. This mapping $F': A \to \mathcal{P}(V)$ now defines a Soft Graph, where each subset $F'(a)$ still induces connected subgraphs of G

based on parameter a. Thus, the Hypersoft Graph (F, J) has been transformed into a Soft Graph (F', A) over G.

And let $G = (V, E)$ be the Soft Graph obtained from the previous transformation with parameter set A and mapping. $F': A \to \mathcal{P}(V)$. To reduce this to a Classical Graph, we ignore the parameterization by A and consider only the underlying vertex set V and edge set E. Define a Classical Graph $G' = (V, E')$, where $E' \subseteq E$ consists of edges connecting vertices within any subset $F'(a), a \in A$, such that each $F'(a)$ maintains connectivity among its vertices. Since $F'(a)$ induces connected subgraphs in the Soft Graph, the resultant graph G' will be a connected graph without parameterization, representing the underlying structure of the Soft Graph.

By this two-step process, a Hypersoft Graph can be systematically reduced to a Soft Graph, which can then be further reduced to a Classical Graph by eliminating parameters.

2.4 | Rough Set and Rough Graph

Rough Set provides a mathematical tool to handle uncertain information by approximating the set of elements. Rough Graph extends Rough Set Theory to graphs, where uncertainty in relationships (edges) is represented through lower and upper approximations. Rough Graphs are especially useful in knowledge mining, social networks, and systems where connections among entities are partially known or inherently uncertain.

Definition 8 (Partition of Sets). Let U be a non-empty set. A partition of U is a collection of non-empty, pairwise disjoint subsets. $\{P_i\}_{i \in I}$ Such that:

- $P_i \subseteq U$ for each $i \in I$,
- $P_i \cap P_j = \emptyset$ for all $i \neq j$,
- $\bigcup_{i \in I} P_i = U$.

Each subset P_i is called a block or cell of the partition, and the elements within each block are equivalence-related under some equivalence relation R.

Definition 9 (Rough Set). [101, 157-160] Let U be a universe of discourse and R a relation on U that induces a partition of U into equivalence classes. For a subset $X \subseteq U$, the lower approximation of X nnonr eqnoc R (denoted $R(X)$) is the set of elements that are certainly in X given the information provided by R :

$$R(X) = \{x \in U \mid [x]_R \subseteq X\}$$

The upper approximation of X (denoted (X)) is the set of elements that possibly belong to X :

$$R(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}.$$

The pair $(R(X), R(X))$ is called the Rough Set approximation of X with respect to R.
Definition 10 (Rough Graph). [37, 43, 111, 150, 212] Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of edges. Let R be an attribute set on E, inducing an equivalence relation on the edges. For any edge set $X \subseteq E$, the lower approximation of X with respect to R (denoted $R(X)$) is defined as:

$$R(X) = \{e \in E \mid [e]_R \subseteq X\}$$

where $[e]_R$ Denotes the equivalence class of e under R. The upper approximation of X(denoted $R(X)$) is defined as:

$$R(X) = \{e \in E \mid [e]_R \cap X \neq \emptyset\}.$$

A graph $G = (V, E)$ is called an R-rough graph if X is not exactly definable under R, and it is characterized by the pair $(R(X), R(X))$, where $R(X)$ is the lower approximation graph and $R(X)$ is the upper approximation graph.

2.5 | Soft Rough Set and Soft Rough Graph

A soft rough set extends soft sets by using lower and upper approximations over a universe U, capturing uncertainty within the set [53]. A soft rough graph combines soft rough set concepts over a graph's vertices and edges, forming subgraphs to represent uncertain or approximate connections. The definitions are provided as follows.

Definition 11. [53] Consider a soft approximation space $S = (U, F)$, where:

- U represents the universe of discourse.
- $F: A \to P(U)$ is a set-valued mapping from a set $A \subseteq P$, where P is the set of all parameters.

A soft rough set over a subset $X \subseteq U$ has both lower and upper approximations defined as follows:

$$F_*(X) = \{u \in U \mid \exists a \in A \text{ such that } u \in F(a) \subseteq X\}$$
$$F^*(X) = \{u \in U \mid \exists a \in A \text{ such that } u \in F(a) \text{ and } F(a) \cap X \neq \emptyset\}$$

If $F_*(X) = F^*(X)$, then X is said to be softly definable; otherwise, X is a soft rough set, denoted by $(F_*(X), F^*(X), A)$.

Definition 12. [151] Let $G = (V, E)$ be a simple graph where V is the set of vertices and E is the set of edges. Suppose $S = (U, F)$ is a soft approximation space, and $A \subseteq P$ is a non-empty set of parameters. Define $(F_*(X), F^*(X), A)$ as a soft rough set over V, and $(K_*(X), K^*(X), A)$ as a soft rough set over E, with:

$$K_*(X) = \{e \in E \mid \exists a \in A \text{ such that } e \in K(a) \subseteq X\}$$
$$K^*(X) = \{e \in E \mid \exists a \in A \text{ such that } e \in K(a) \text{ and } K(a) \cap X \neq \emptyset\}$$

Here, $K: A \to P(E)$ maps each parameter $a \in A$ to subsets of E based on adjacency relations. A graph $\tilde{G} = (G, F_*, K_*, F^*, K^*, A, X)$ is called a soft rough graph if it satisfies:

1. $G = (V, E)$ is a simple graph.
2. A is a non-empty set of parameters.
3. $X \subseteq U$ is a non-empty subset.
4. $(F_*(X), F^*(X), A)$ is a soft rough set over V.
5. $(K_*(X), K^*(X), A)$ is a soft rough set over E.
6. $H_*(X) = (F_*(X), K_*(X))$ and $H^*(X) = (F^*(X), K^*(X))$ are subgraphs of G.

A soft rough graph can be represented by

$$\tilde{G} = \langle F_*, K_*, F^*, K^*, A, X\rangle = \{H_*(X), H^*(X)\}$$

The collection of all soft rough graphs of G is denoted by $\mathrm{SRG(G)}$.

Proposition 13. Any Soft Rough Graph can be transformed into a Rough Graph and a Soft Graph. **Proof.** To prove this theorem, we demonstrate the transformations of a Soft Rough Graph into a Rough Graph and then into a Soft Graph.

To transform a Soft Rough Graph $\tilde{G} = \langle F_*, K_*, F^*, K^*, A, X\rangle$ Into a Rough Graph, we ignore the parameter set A while retaining the approximation sets $F_*(X)$ and $F^*(X)$ over vertices, and $K_*(X)$ and $K^*(X)$ over edges. The rough graph thus formed retains the inherent approximations in vertex and edge connectivity

without explicit parameter dependence. Specifically, the lower approximation graph $H_*(X) = (F_*(X), K_*(X))$ and the upper approximation graph $H^*(X) = (F^*(X), K^*(X))$ Serve as the representations of uncertainty through rough approximations, and this transformed graph is a Rough Graph over G.

Next, we consider transforming the Soft Rough Graph into a Soft Graph. We redefine the parameterization by considering only one approximation level at a time. For instance, we treat either the lower or the upper approximation set $F_*(X)$ or $F^*(X)$ over vertices, and $K_*(X)$ or $K^*(X)$ over edges, as separate soft sets $F: A \to P(V)$ and $K: A \to P(E)$, while associating each approximation level with a distinct parameter. This enables the conversion of the uncertainty in connectivity into a parameterized soft graph structure. By this construction, the approximations of vertices and edges in the Soft Rough Graph are reinterpreted within a flexible parameterized structure, thus forming a Soft Graph over G without explicitly relying on rough approximations.

Consequently, we conclude that any Soft Rough Graph $\tilde{G}$ Can be effectively transformed into both a Rough Graph and a Soft Graph by reinterpreting its approximation components and parameter dependencies.

2.6 | Hypersoft Rough Set

A hypersoft rough set uses lower and upper approximations over an approximation space to represent multi-attribute uncertainty in sets. Note that an approximation space is a mathematical structure that models uncertainty, consisting of a universe of objects and an equivalence relation, enabling the definition of lower and upper approximations for subsets.

Definition 14 (Pawlak Approximation Space). [220, 221] Let U be a non-empty set, known as the universe of discourse, and let R be an equivalence relation on U. The pair (U, R) is called a Pawlak approximation space. The relation R induces a partition of U into equivalence classes, where each element in U is indiscernible from others within its equivalence class.

Given a subset $X \subseteq U$, the lower approximation $R(X)$ and the upper approximation $\bar{R}(X)$ of X with respect to R are defined as follows:

- The lower approximation of X, denoted $R(X)$, is the set of all elements $u \in U$ such that the equivalence class $[u]_R \subseteq X$:

$$R(X) = \{u \in U \mid [u]_R \subseteq X\}$$

  This set contains elements that are certainly in X based on the information provided by R.

- The upper approximation of X, denoted $\bar{R}(X)$, is the set of all elements $u \in U$ such that the intersection $[u]_R \cap X \neq \emptyset$:

$$\bar{R}(X) = \{u \in U \mid [u]_R \cap X \neq \emptyset\}$$

  This set contains elements that possibly belong to X given the information provided by R.

The pair $(R(X), \bar{R}(X))$ represents the rough set approximation of X within the Pawlak approximation space (U, R).

Definition 15 (Hypersoft Rough Set). [124] Let (X, R) be a Pawlak approximation space, where R is an equivalence relation on X. Given a Hypersoft Set (F, J) over X, the Hypersoft Lower Approximation $F_*(\mathrm{j})$ and Hypersoft Upper Approximation $F^*(\mathrm{j})$ of F with respect to R are defined for each $\mathrm{j} \in J$ as:

$$F_*(\mathrm{j}) = \{x \in X \mid [x]_R \subseteq F(\mathrm{j})\},$$
$$F^*(\mathrm{j}) = \{x \in X \mid [x]_R \cap F(\mathrm{j}) \neq \emptyset\}$$

where $[x]_R$ Denotes the equivalence class of x under R.

The Hypersoft Rough Set is then the pair. (F_*, F^*, J).

3 | Results

The results of this paper are described below.

3.1 | Hypersoft Rough Graph

A hypersoft rough graph extends rough graph theory by using multi-attribute approximations over vertices and edges to capture complex relationships.

Definition 16 (Hypersoft Rough Graph). Let $G = (V, E)$ be a simple graph, and let R be an equivalence relation on V. Consider n distinct attributes with disjoint parameter sets $J_1, J_2, \dots, J_n$, and let $J = J_1 \times J_2 \times \cdots \times J_n$.

Define a mapping $F: J \to \mathcal{P}(V)$, assigning to each parameter tuple $\mathrm{j} \in J$ a subset of vertices. The Hypersoft Lower Approximation $F_*(\mathrm{j})$ and Hypersoft Upper Approximation $F^*(\mathrm{j})$ of F with respect to R are:

$$F_*(\mathrm{j}) = \{v \in V \mid [v]_R \subseteq F(\mathrm{j})\}$$
$$F^*(\mathrm{j}) = \{v \in V \mid [v]_R \cap F(\mathrm{j}) \neq \emptyset\}$$

Similarly, define a mapping $K: J \to \mathcal{P}(E)$ for edges, and define the edge approximations. $K_*(\mathrm{j})$ and $K^*(\mathrm{j})$.

A Hypersoft Rough Graph $\tilde{G} = (G, F_*, K_*, F^*, K^*, J)$ is defined, where for each $\mathrm{j} \in J$:

- $H_*(\mathrm{j}) = (F_*(\mathrm{j}), K_*(\mathrm{j}))$ is a subgraph of G.
- $H^*(\mathrm{j}) = (F^*(\mathrm{j}), K^*(\mathrm{j}))$ is a subgraph of G.

Theorem 17. Every Hypersoft Rough Graph can be transformed into a Soft Rough Graph, a Rough Graph, and a Soft Graph.

Proof. We will demonstrate a systematic transformation from a Hypersoft Rough Graph to a Soft Rough Graph, then to a Rough Graph, and finally to a Soft Graph.

Let $\tilde{G} = (G, F_*, K_*, F^*, K^*, J)$ be a Hypersoft Rough Graph, where $J = J_1 \times J_2 \times \cdots \times J_n$.

Select one parameter set, say $J_1 = A$, and fix the other parameters. Define a mapping $F': A \to \mathcal{P}(V)$ by:

$$F'(a) = \bigcup_{\substack{\mathrm{j} \in J \\ j_1 = a}} F(\mathrm{j}),\ \forall a \in A$$

Define the lower and upper approximations:

$$F'_*(a) = \bigcup_{\substack{\mathrm{j} \in J \\ j_1 = a}} F_*(\mathrm{j}),\ F'^*(a) = \bigcup_{\substack{\mathrm{j} \in J \\ j_1 = a}} F^*(\mathrm{j})$$

Similarly, define $K': A \to \mathcal{P}(E)$ and $K'_*(a), K'^*(a)$.

Then $\tilde{G}' = (G, F'_*, K'_*, F'^*, K'^*, A)$ is a Soft Rough Graph. Justification:

- The mapping F' is from A to $\mathcal{P}(V)$, satisfying the definition of a soft set over V.
- The lower and upper approximations $F'_*(a)$ and $F'^*(a)$ are well-defined and correspond to those in Soft Rough Sets.
- The subgraphs $H'_*(a) = (F'_*(a), K'_*(a))$ and $H'^*(a) = (F'^*(a), K'^*(a))$ are subgraphs of G.

Next, consider the Soft Rough Graph $\tilde{G}' = (G, F'_*, K'_*, F'^*, K'^*, A)$.

Define:

$$V_R = \bigcup_{a \in A} F'^*(a),\ E_R = \bigcup_{a \in A} K'^*(a)$$

Then $G_R = (V_R, E_R)$ is a Rough Graph.

Justification:

- The vertex set V_R and edge set E_R are formed by the upper approximations over all parameters in A.
- This represents the rough approximation of G with respect to the equivalence relation R.

Step 3: Transforming a Rough Graph into a Soft Graph

Define a mapping $F'': A \to \mathcal{P}(V)$ by:

$$F''(a) = F'^*(a),\ \forall a \in A$$

Then (F'', A) is a Soft Graph over G.

Justification:

- The mapping F'' assigns to each parameter $a \in A$ a subset of V, forming a soft set over V.
- If the subgraphs induced by $F''(a)$ are connected, (F'', A) Satisfies the definition of a Soft Graph.

Through these transformations, we have shown that any Hypersoft Rough Graph can be systematically transformed into a Soft Rough Graph, a Rough Graph, and a Soft Graph.

4 | Future Tasks and Discussion of this Research

This section outlines future directions for this research.

4.1 | Discussion: N-Soft Graphs, N-hypersoft Graphs and N-bipolar hypersoft Graphs

The Soft Set, due to its intriguing mathematical structure and ease of application, has led to the development of various derivative concepts, such as the N -Soft Set [3, 4, 14, 50, 51], N-HyperSoft Set [148], bipolar hypersoft set [11, 27, 142, 146, 147], and N-bipolar hypersoft set [143, 149]. Visualizing these sets as graphs could provide valuable insights and further facilitate their application in future studies. **Definition 18** (N-Soft Set). [14] Let U be a universe of objects under consideration, and let E denote a set of attributes with $A \subseteq E$. Let $R = \{0,1,2,\dots,N-1\}$ represent an ordered set of grades, where $N \in \{2,3,\dots\}$.

An N-soft set on U is defined as a triple (F, A, N), where F is a mapping from A to $\mathcal{P}(U \times R)$ such that for each $a \in A$ and $u \in U$, there exists a specific pair $(u, r) \in U \times R$ with $(u, r) \in F(a)$.

The interpretation of the pair $(u, r) \in F(a)$ is that the element u belongs to the a-approximation of U with grade r. The tabular representation of an N-soft set is given as follows:

(F, A, N)	a_1	a_2	...	a_q
u_1	$\{r_{11}\}$	$\{r_{12}\}$	...	$\{r_{1q}\}$
u_2	$\{r_{21}\}$	$\{r_{22}\}$	...	$\{r_{2q}\}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_p	$\{r_{p1}\}$	$\{r_{p2}\}$	...	$\{r_{pq}\}$

Definition 19 (N-Soft Graph). Let $G = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges. Let A be a non-empty set of parameters, and let $R = \{0,1,2, \dots, N-1\}$ be an ordered set of grades, with $N \geq 2$.

An N-Soft Graph over G is defined as a triple (F, A, N), where:

- $F: A \to \mathcal{P}(V \times R)$ is a mapping such that for each $a \in A$ and each $v \in V$, there exists a unique grade $r \in R$ with $(v, r) \in F(a)$ (equivalently, $F(a)(v) = r$).

  The interpretation of $(v, r) \in F(a)$ is that the vertex v has a grade r with respect to the parameter a. The N -Soft Graph (F, A, N) represents graded relationships between parameters and vertices in G.

Theorem 20. Every N-Soft Graph can be transformed into a Soft Graph.

Proof. Let $G = (V, E)$ be a simple graph, A be a non-empty set of parameters, and (F, A, N) be an N-Soft Graph over G, where $F: A \to \mathcal{P}(V \times R)$ with $R = \{0,1, \dots, N-1\}$.

We define a Soft $\text{Graph}(F', A)$ over G as follows:

For each parameter $a \in A$, define the mapping $F': A \to \mathcal{P}(V)$ by:

$$F'(a) = \{v \in V \mid F(a)(v) \geq T\}$$

Where T is a fixed threshold in R.

Explanation:

- For each parameter $a, F(a)(v)$ is the grade assigned to vertex v.
- We select a threshold $T \in R$ to determine which vertices are included in $F'(a)$.
- The set $F'(a)$ includes all vertices v such that $F(a)(v)$ meets or exceeds the threshold T.

Verification:

- F' maps A to subsets of V, i.e., $F': A \to \mathcal{P}(V)$, satisfying the definition of a Soft Graph.
- Since F is defined for all $a \in A$ and $v \in V$, so is F'.

  By applying the threshold T, any N-Soft $\text{Graph}(F, A, N)$ can be transformed into a $\text{SoftGraph}(F', A)$.

Definition 21 (N-Hypersoft Set). [148] Let Ω be a universe of objects, and let E denote a set of parameters, partitioned into disjoint subsets $E_1, E_2, \dots, E_n$. Each E_i represents a distinct category of parameters. Define $R = \{0,1,2, \dots, N-1\}$ as an ordered set of grades, where $N \geq 2$. An N-hypersoft set over Ω is a triple (∇, J, N), where:

- $J = E_1 \times E_2 \times \cdots \times E_n$ is the set of all tuples formed by selecting one parameter from each subset E_i,
- ∇ is a mapping $\nabla: J \to \mathcal{P}(\Omega \times R)$, with the property that for each tuple $\mathrm{j} \in J$ and each object $\omega \in \Omega$, there exists a unique grade $r_{\mathrm{j}} \in R$ such that $(\omega, r_{\mathrm{j}}) \in \nabla(\mathrm{j})$.

The interpretation of $(\omega, r_{\mathrm{j}}) \in \nabla(\mathrm{j})$ is that the object ω has a grade r_{j} under the parameter tuple j . The N-hypersoft set can be represented in tabular form, where each row corresponds to an object in Ω and each column corresponds to a parameter tuple in J.

Thus, the N-hypersoft set (∇, J, N) can be expressed as:

$$(\nabla, J, N) = \{(\mathrm{j}, \{(\omega, \nabla(\mathrm{j})(\omega)) \mid \omega \in \Omega\}) \mid \mathrm{j} \in J\}$$

Definition 22 (N-Hypersoft Graph). Let $G = (V, E)$ be a simple graph. Consider n distinct attributes with corresponding disjoint parameter sets $E_1, E_2, \ldots, E_n$, and let $J = E_1 \times E_2 \times \cdots \times E_n$. Let $R = \{0,1,2, \ldots, N-1\}$ be an ordered set of grades, where $N \geq 2$.

An N-Hypersoft Graph over G is defined as a triple (∇, J, N), where:

- $\nabla: J \to \mathcal{P}(V \times R)$ is a mapping such that for each parameter tuple $\mathrm{j} \in J$ and each $v \in V$, there exists a unique grade $r_{\mathrm{j}} \in R$ with $(v, r_{\mathrm{j}}) \in \nabla(\mathrm{j})$ (equivalently, $\nabla(\mathrm{j})(v) = r_{\mathrm{j}}$). The interpretation of $(v, r_{\mathrm{j}}) \in \nabla(\mathrm{j})$ is that the vertex v has a grade r_{j} under the parameter tuple j . The N -Hypersoft Graph captures multi-parameter graded relationships within G.

Theorem 23. The following holds.

1. Every N-Hypersoft Graph can be transformed into an N-Soft Graph.
2. Every N-Hypersoft Graph can be transformed into a Hypersoft Graph.
3. Every N-Hypersoft Graph can be transformed into a Soft Graph.

Proof. Let $G = (V, E)$ be a simple graph, and (∇, J, N) be an N-Hypersoft Graph over G, where $J = E_1 \times E_2 \times \cdots \times E_n$, and $\nabla: J \to \mathcal{P}(V \times R)$ with $R = \{0,1, \ldots, N-1\}$.

Part 1: Transformation into an N-Soft Graph

We can transform the N-Hypersoft Graph into an N-Soft Graph (F, A, N) over G as follows:

- Let $A = J$ (i.e., the set of parameter tuples in).
- Define $F: A \to \mathcal{P}(V \times R)$ by $F(a) = \nabla(a)$ for all $a \in A$.

Explanation:

- Since ∇ maps parameter tuples to graded subsets of V, we can view each tuple $a \in A$ as a single parameter in the N-Soft Graph.
- The mapping F retains the grading information from ∇.

  Thus, (∇, J, N) can be viewed as an N-Soft Graph (F, A, N).

  Part 2: Transformation into a Hypersoft Graph We can transform the N-Hypersoft Graph into a Hypersoft Graph (∇', J) Over G by selecting a threshold $T \in R$ and defining:

$$\nabla'(\mathrm{j}) = \{v \in V \mid \nabla(\mathrm{j})(v) \geq T\}$$

for all $\mathrm{j} \in J$.

The mapping ∇' now assigns to each parameter tuple j a subset of V, where the grade meets or exceeds the threshold T. Thus, (∇, J, N) can be transformed into a Hypersoft Graph. (∇', J).

Part 3: Transformation into a Soft Graph To further transform the Hypersoft Graph into a Soft Graph, we can:

Fix one parameter set E_i and select a particular value $e_i \in E_i$.

Consider the reduced parameter set $J' = E_1 \times \cdots \times E_{i-1} \times E_{i+1} \times \cdots \times E_n$.

Define $F'': J' \to \mathcal{P}(V)$ by:

$$F''(a) = \nabla'(a, e_i)$$

for all $a \in J'$.

Therefore, (∇, J, N) can be transformed into a Soft Graph (F'', J').

Definition 24 (Bipolar Hypersoft Set (BHS Set)). [142] Let Ω be a universe of objects, E a set of attributes, and $Q_1 \subseteq E$ a subset of attributes. A bipolar hypersoft set over Ω is defined as a triple (Δ, r, Q_1), where:

- $\Delta: Q_1 \to \mathcal{P}(\Omega)$ is a mapping assigning each parameter $q \in Q_1$ a subset of Ω.
- $r: \neg Q_1 \to \mathcal{P}(\Omega)$ is a mapping assigning each parameter $\neg q \in \neg Q_1$ (where $\neg q$ is the complement or opposite of q) a subset of Ω.
- For each $q \in Q_1, \Delta(q) \cap r(\neg q) = \emptyset$.

The BHS set (Δ, r, Q_1) can be represented as:

$$(\Delta, r, Q_1) = \{(q, \Delta(q), r(\neg q)) \mid q \in Q_1, \Delta(q), r(\neg q) \subseteq \Omega\}$$

Definition 25 (Bipolar Hypersoft Graph). Let $G = (V, E)$ be a simple graph, and let $Q_1 \subseteq E$ be a subset of parameters. The Bipolar Hypersoft Graph over G is defined as a triple (Δ, r, Q_1), where:

- $\Delta: Q_1 \to \mathcal{P}(V)$ assigns to each parameter $q \in Q_1$ a subset $\Delta(q) \subseteq V$.
- $r: \neg Q_1 \to \mathcal{P}(V)$ assigns to each opposite parameter $\neg q \in \neg Q_1$ a subset $r(\neg q) \subseteq V$.
- For each $q \in Q_1$, it holds that $\Delta(q) \cap r(\neg q) = \emptyset$.

The Bipolar Hypersoft Graph (Δ, r, Q_1) captures both positive and negative relationships between parameters and vertices in G.

Theorem 26. The following holds.

1. Every Bipolar Hypersoft Graph can be transformed into a Hypersoft Graph.
2. Every Bipolar Hypersoft Graph can be transformed into a Soft Graph.

Proof. Let $G = (V, E)$ be a simple graph, and (Δ, r, Q_1) be a Bipolar Hypersoft Graph over G. Part 1: Transformation into a Hypersoft Graph

We can construct a Hypersoft Graph (∇, J) over G as follows:

- Let $J = Q_1$.
- Define $\nabla(q) = \Delta(q)$ for all $q \in Q_1$.

Thus, the Bipolar Hypersoft Graph (Δ, r, Q_1) can be transformed into a Hypersoft Graph (∇, J). Part 2: Transformation into a Soft Graph

To further transform the Hypersoft Graph into a Soft Graph:

- If Q_1 consists of single parameters (not tuples), then (∇, Q_1) is a Soft Graph over G.

Therefore, the Bipolar Hypersoft Graph can be transformed into a Soft Graph. **Definition 27** (N-Bipolar Hypersoft Set (N-BHS Set)). [143] Let Ω be a universe of objects, E a set of attributes, and $Q_1 \subseteq E$ a subset of attributes. Consider $R = \{0, 1, \dots, N-1\}$ as an ordered set of grades, where $N \geq 2$. An N-bipolar hypersoft set over Ω is defined as a quadruple (Δ, r, Q_1, N), where:

- $\Delta: Q_1 \to \mathcal{P}(\Omega \times R)$ maps each parameter $q \in Q_1$ to a subset of $\Omega \times R$.
- $r: \neg Q_1 \to \mathcal{P}(\Omega \times R)$ maps each parameter $\neg q \in \neg Q_1$ (opposite of q) to a subset of $\Omega \times R$.

- For each $q \in Q_1$ and $\omega \in \Omega$, there exist unique values $r_q, r_{\neg q} \in R$ such that $(\omega, r_q) \in \Delta(q)$ and $(\omega, r_{\neg q}) \in r(\neg q)$ with the condition $r_q + r_{\neg q} \leq N - 1$.

The N-BHS set (Δ, r, Q_1, N) can be written as:

$$(\Delta, r, Q_1, N) = \{(q, \{(\omega, \Delta(q)(\omega), r(\neg q)(\omega))\}) \mid q \in Q_1, \omega \in \Omega, \Delta(q)(\omega), r(\neg q)(\omega) \in R\}$$

Definition 28 (N-Bipolar Hypersoft Graph). Let $G = (V, E)$ be a simple graph, $Q_1 \subseteq E$ a subset of parameters, and $R = \{0,1,2, \dots, N-1\}$ an ordered set of grades, with $N \geq 2$.

An N-Bipolar Hypersoft Graph over G is defined as a quadruple (Δ, r, Q_1, N), where:

- $\Delta: Q_1 \to \mathcal{P}(V \times R)$ assigns to each parameter $q \in Q_1$ a set of pairs (v, r_q), where $v \in V$ and $r_q \in R$.
- $r: \neg Q_1 \to \mathcal{P}(V \times R)$ assigns to each opposite parameter $\neg q \in \neg Q_1$ a set of pairs $(v, r_{\neg q})$, where $v \in V$ and $r_{\neg q} \in R$.
- For each $q \in Q_1$ and $v \in V$, there exist unique grades $r_q, r_{\neg q} \in R$ such that:
    1. $(\Delta(q))(v) = r_q$ and $(r(\neg q))(v) = r_{\neg q}$.
    2. The grades satisfy $r_q + r_{\neg q} \leq N - 1$.

The N-Bipolar Hypersoft Graph (Δ, r, Q_1, N) captures graded positive and negative relationships between parameters and vertices in G, with the grading condition ensuring the compatibility between positive and negative assessments.

Theorem 29. The following holds.

1. Every N-Bipolar Hypersoft Graph can be transformed into a Bipolar Hypersoft Graph.
2. Every N-Bipolar Hypersoft Graph can be transformed into a Hypersoft Graph.
3. Every N-Bipolar Hypersoft Graph can be transformed into an N-Soft Graph.
4. Every N-Bipolar Hypersoft Graph can be transformed into an N-Hypersoft Graph.

Proof. Let $G = (V, E)$ be a simple graph, and (Δ, r, Q_1, N) be an N-Bipolar Hypersoft Graph over G.
Part 1: Transformation into a Bipolar Hypersoft Graph

Define a Bipolar Hypersoft Graph (Δ', r', Q_1) over G as follows:

$$\Delta'(q) = \{v \in V \mid \Delta(q)(v) \geq T\}$$
$$r'(\neg q) = \{v \in V \mid r(\neg q)(v) \geq T'\}$$

where $T, T' \in R$ are thresholds.

Thus, (Δ, r, Q_1, N) can be transformed into a Bipolar Hypersoft Graph (Δ', r', Q_1).

Part 2: Transformation into a Hypersoft Graph

By ignoring the negative mapping r', we define a Hypersoft $\mathrm{Graph}(\nabla, J)$:

Let $J = Q_1$.

Define $\nabla(q) = \Delta'(q)$ for all $q \in Q_1$.

(Δ, r, Q_1, N) can be transformed into a Hypersoft $\mathrm{Graph}(\nabla, J)$.

Part 3: Transformation into an N-Soft Graph

Define an N-Soft Graph (F, A, N) as follows:

$$A = Q_1,\ F(a) = \Delta(a) \text{ for all } a \in A$$

(Δ, r, Q_1, N) can be transformed into an N-Soft $\text{Graph}(F, A, N)$.

Part 4: Transformation into an N-Hypersoft Graph

If Q_1 is a Cartesian product, i.e., $Q_1 = E_1 \times E_2 \times \cdots \times E_n$, then we define an N-Hypersoft Graph (∇, J, N) as follows:

$$J = Q_1,\ \nabla(\mathrm{j}) = \Delta(\mathrm{j}) \text{ for all } \mathrm{j} \in J$$

(Δ, r, Q_1, N) can be transformed into an N-Hypersoft $\text{Graph}(\nabla, J, N)$. .

4.2 | Future tasks: Hypersoft Hyperset and SuperHypersoft Superhyperset

This paper outlines future directions for exploration. In the realm of soft set theory, related concepts such as hypersoft sets and super hypersoft sets [96, 139, 197, 201] are recognized for their potential to broaden the structure and applications of generalized sets. In graph theory, analogous concepts like hypergraphs [33 98-100, 161] and super hypergraphs [72, 94, 104-106, 193-196, 200] are well-known, as construct such as hyperalgebras [115, 87, 204] serve to generalize traditional graph and algebraic concepts. Additionally, in set theory, we find concepts such as hypersets [30, 97, 152] and supersets, which we propose to integrate into a unified concept termed the Superhyperset. We aim to investigate the mathematical structures and applications of sets, hypersets, and super hypersets, further enriched by the theoretical foundations of soft, hypersoft, and superhypersoft frameworks.

Although still in the conceptual stage, we outline below the definitions of Superhypersets and related concepts.

Definition 30 (Hyperset). [97] A Hyperset is a set that can contain itself as a member, either directly or indirectly. This is permitted in Non-Well-Founded Set Theory, where the Axiom of Foundation is replaced by the AntiFoundation Axiom (AFA). In this context, sets are allowed to have membership structures that include cycles or infinite descending chains.

Example 31 (Hyperset). Let A be defined as:

$$A = \{A\}$$

Here, A contains itself as its sole member, which is permissible in Hyperset Theory under the Anti-Foundation Axiom.

Definition 32 (Superset). In set theory, a Superset is a set whose elements are themselves sets. Formally, a set S is a Superset if and only if for every element $x \in S, x$ is a set. That is,

$$\forall x \in S,\ x \text{ is a set.}$$

Example 33 (Superset). Let $B = \{1,2,3\}$ and $C = \{1,2\}$. Since $C \subseteq B$, we have:

$$B \supseteq C.$$

Every element of C is contained in B, so B is a Superset of C.

Definition 34 (Superhyperset). Let $\mathcal{H}$ be the class of all Hypersets. A Superhyperset S is a set such that:

$$S \subseteq \mathcal{P}(\mathcal{H})$$

Where $\mathcal{P}(\mathcal{H})$ denotes the power set of $\mathcal{H}$. Elements of S are subsets of Hypersets, and S may include itself as an element, directly or indirectly, thus allowing for non-well-founded and higher-order membership structures.

Example 35 (Superhyperset). Let $A = \{A\}$ (a Hyperset) and $H = \{A\}$. Define:

$$S = \{\{A\}, \{A, H\}\}$$

Here, $S \subseteq \mathcal{P}(\mathcal{H})$ with $\mathcal{H} = \{A, H\}$, and S is a Superhyperset since its elements are subsets of Hypersets.

Theorem 36. Every Superhyperset can be transformed into a Hyperset and a Superset.

Proof. Let S be a Superhyperset, so $S \subseteq \mathcal{P}(\mathcal{H})$, where $\mathcal{H}$ is the class of all Hypersets. Transformation into a Hyperset:

Define a function $f: S \to \mathcal{H}$ by selecting an element from each subset $s \in S$. Since each $s \subseteq \mathcal{H}$, we have $f(s) \in \mathcal{H}$.

Consider the set:

$$H = \{f(s) \mid s \in S\}$$

Since $f(s) \in \mathcal{H}$ for all $s \in S, H$ is a set of Hypersets. If H contains itself as an element (directly or indirectly), then H is a Hyperset.

Transformation into a Superset:

Let:

$$U = \bigcup_{s \in S} s$$

Then U is a set containing all elements of the subsets in S. For any set A such that $A \in s$ for some $s \in S$, we have $A \in U$, so U is a Superset of A.

Thus, the Superhyperset S can be associated with a Hyperset H and a Superset U, demonstrating that a Superhyperset can be transformed into both a Hyperset and a Superset.

Additionally, We would like to extend the concept of the SuperHyperSoft Graph to Rough Graphs, Expert Graphs, Fuzzy Graphs, and Neutrosophic Graphs, and to explore their mathematical structures as well as applications in decision-making and related areas.

Definition 37 (SuperHyperSoft Set). [197] Let U be a universe of discourse, and let $P(U)$ denote the power set of U. Let $a_1, a_2, \dots, a_n$ be n distinct attributes, where $n \geq 1$. Each attribute a_i has a corresponding set of attribute values A_i, with the property that $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Let $P(A_i)$ denote the power set of A_i for each $i = 1,2, \dots, n$.

Then, the pair $\big(F, P(A_1) \times P(A_2) \times \cdots \times P(A_n)\big)$, where

$$F: P(A_1) \times P(A_2) \times \cdots \times P(A_n) \to P(U)$$

is called a SuperHyperSoft Set over U.

Definition 38 (SuperHyperSoft Graph). [60] Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. Let $U = V \cup E$, and let $P(U)$ denote the power set of U.

Let $a_1, a_2, \dots, a_n$ be n distinct attributes, each with a corresponding set of attribute values A_i, such that $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Let $P(A_i)$ denote the power set of A_i for each $i = 1,2, \dots, n$.

Define two functions:

1. Vertex Function:

$$F_V: P(A_1) \times P(A_2) \times \cdots \times P(A_n) \to P(V)$$

Which maps combinations of attribute value subsets to subsets of vertices.

2. Edge Function:

$$F_E: P(A_1) \times P(A_2) \times \cdots \times P(A_n) \to P(E)$$

Which maps combinations of attribute value subsets to subsets of edges.

Then, the pair (F_V, F_E) is called a SuperHyperSoft Graph over G.

Due to the versatility and ease of application of soft sets, a variety of soft set extensions, beyond the above-mentioned Superhypersoft, have been developed. The concepts of IndetermSoft Sets [198, 199], IndetermHyperSoft Sets [198, 199], Binary soft sets [2, 155, 173], Generalized soft rough sets [17, 18, 23] and TreeSoft Sets [44, 154, 198, 199, 202] are recognized as extensions of the Soft Set. In the future, additional research into the mathematical structures and applications of these sets in the contexts of Expert Sets, Rough Sets, Expert Graphs, and Rough Graphs is anticipated.

4.3 | Discussion: Hypersoft Digraphs

We intend to explore the mathematical structures of Hypersoft Digraphs, extending the concepts of Soft Directed Graphs as established by [89, 117-120]. A Directed Graph consists of vertices connected by directed edges (arcs), where each edge has a direction from one vertex to another. The definitions, along with relevant related concepts, will be outlined below.

Definition 39. A directed graph (or digraph) is a pair $D = (V, A)$ where:

- V is a non-empty finite set of elements called vertices or nodes.
- $A \subseteq V \times V$ is a set of ordered pairs of distinct vertices called arcs or directed edges.

Definition 40. [117] Let $D^* = (V, A)$ be a directed graph and P be a non-empty set of parameters. A soft-directed graph over D^* is a quadruple $D = (D^*, J, L, P)$, where:

- $D^* = (V, A)$ is a directed graph.
- P is a non-empty set of parameters.
- $J: P \to \mathcal{P}(V)$ is a mapping that assigns to each parameter $p \in P$ a subset $J(p) \subseteq V$.
- $L: P \to \mathcal{P}(A)$ is a mapping that assigns to each parameter $p \in P$ a subset $L(p) \subseteq A$.

These mappings satisfy the condition that for each $p \in P$, the pair $M(p) = (J(p), L(p))$ is a subdigraph of D^*, i.e., $L(p) \subseteq J(p) \times J(p)$.

The collection $D = \{M(p) \mid p \in P\}$ represents the soft directed graph over D^*.

Definition 41. Let $D^* = (V, A)$ be a directed graph. Consider n distinct attributes with corresponding disjoint parameter sets $E_1, E_2, \ldots, E_n$, and let $J = E_1 \times E_2 \times \cdots \times E_n$.

A hypersoft directed graph over D^* is a triple $D = (D^*, \nabla, J)$, where:

- $D^* = (V, A)$ is a directed graph.
- $J = E_1 \times E_2 \times \cdots \times E_n$ is the set of all parameter tuples formed by selecting one parameter from each E_i.
- $\nabla: J \to \mathcal{P}(V)$ is a mapping that assigns to each parameter tuple $\mathrm{j} \in J$ a subset $\nabla(\mathrm{j}) \subseteq V$.

The mapping ∇ must satisfy the condition that for each $\mathrm{j} \in J$, the pair $M(\mathrm{j}) = (\nabla(\mathrm{j}), L(\mathrm{j}))$ is a subdigraph of D^*, where:

$$L(\mathrm{j}) = \{(u, v) \in A \mid u, v \in \nabla(\mathrm{j})\}$$

The collection $D = \{M(\mathrm{j}) \mid \mathrm{j} \in J\}$ represents the hypersoft directed graph over D^*.

Theorem 42. Every hypersoft-directed graph can be transformed into a soft-directed graph and a directed graph.

Proof. Let $D = (D^*, \nabla, J)$ be a hypersoft directed graph over the directed graph $D^* = (V, A)$, where $J = E_1 \times E_2 \times \cdots \times E_n$ and $\nabla: J \to \mathcal{P}(V)$.

We can construct a soft-directed graph $D' = (D^*, J', L', P)$ as follows:

- Let $P = J$ (i.e., the set of all parameter tuples).
- Define the mapping $J': P \to \mathcal{P}(V)$ by:

$$J'(\mathrm{j}) = \nabla(\mathrm{j}),\ \forall \mathrm{j} \in P$$

- Define the mapping $L': P \to \mathcal{P}(A)$ by:

$$L'(\mathrm{j}) = \{(u, v) \in A \mid u, v \in \nabla(\mathrm{j})\},\ \forall \mathrm{j} \in P$$

Since $M(\mathrm{j}) = (J'(\mathrm{j}), L'(\mathrm{j}))$ is a subdigraph of D^* for each $\mathrm{j} \in P$, the quadruple $D' = (D^*, J', L', P)$ Satisfies the definition of a soft-directed graph.

To transform the hypersoft directed graph D into a directed graph D'', we can consider the union of all subdigraphs $M(\mathrm{j})$:

$$V'' = \bigcup_{\mathrm{j} \in J} \nabla(\mathrm{j}) \subseteq V,$$
$$A'' = \bigcup_{\mathrm{j} \in J} L(\mathrm{j}) \subseteq A.$$

Then, $D'' = (V'', A'')$ is a subdigraph of D^*, and thus a directed graph.

By constructing D' and D'' as shown, we have demonstrated that a hypersoft-directed graph D can be transformed into a soft-directed graph D' and a directed graph D''.

4.4 | Discussion: Hypersoft Semingraphs

We aim to explore the mathematical structures of Hypersoft Semigraphs, building on the concepts of Soft Semigraphs as developed in works such as [84-88, 90, 91]. A Semigraph generalizes graphs by allowing edges to connect multiple vertices with specified order but with constraints on shared vertices between edges. Below, we will provide the definitions and discuss relevant related concepts. We also plan to extend these ideas by incorporating Directed Graphs, Soft Expert Graphs, and Rough Graphs to further analyze their mathematical structures and potential applications.

Definition 43. [180] A semigraph is a pair $G^* = (V, E)$ where:

- V is a non-empty finite set whose elements are called vertices.
- E is a set of finite sequences (tuples) of distinct vertices called edges, such that:
  1. Any two edges have at most one vertex in common.
  2. Two edges $(v_1, v_2, \ldots, v_n)$ and $(u_1, u_2, \ldots, u_m)$ are considered equal if and only if:
     - $n = m$, and
     - Either $v_i = u_i$ for all $1 \le i \le n$, or $v_i = u_{n-i+1}$ for all $1 \le i \le n$ (i.e., they are reverses of each other).

An edge $e = (v_1, v_2, \ldots, v_n)$ is an ordered sequence where:

- v_1 and v_n are called the end vertices of e.

- The vertices $v_2, v_3, \dots, v_{n-1}$ are called the middle vertices or m-vertices of e.

Definition 44. [92] Let $G^* = (V, E)$ be a semigraph, and let A be a non-empty set of parameters. A soft semigraph over G^* is a quadruple $G = (G^*, Q, W, A)$, where:

- $G^* = (V, E)$ is a semigraph.
- A is a non-empty set of parameters.
- $Q: A \to \mathcal{P}(V)$ is a mapping that assigns to each parameter $a \in A$ a subset $Q(a) \subseteq V$.
- $W: A \to \mathcal{P}(E_p)$ is a mapping that assigns to each parameter $a \in A$ a set $W(a)$ of maximal partial edges induced by $Q(a)$, where E_p is the set of all partial edges of G^*.

The mappings Q and W must satisfy the condition that for each $a \in A$, the pair $H(a) = (Q(a), W(a))$ is a partial semigraph of G^*.

- Partial Edge: A partial edge of an edge $e = (v_1, v_2, \dots, v_n)$ is a subsequence $e' = (v_{i_1}, v_{i_2}, \dots, v_{i_k})$, where $1 \le i_1 < i_2 < \cdots < i_k \le n$ or $n \ge i_k > i_{k-1} > \cdots > i_1 \ge 1$.
- Maximal Partial Edge: A partial edge that is not a subsequence of any other partial edge induced by the same vertex set.

Definition 45. Let $G^* = (V, E)$ be a semigraph. Consider n distinct attributes with corresponding disjoint parameter sets $E_1, E_2, \dots, E_n$, and let $J = E_1 \times E_2 \times \cdots \times E_n$. A hypersoft semigraph over G^* is a triple $G = (G^*, F, J)$, where:

- $G^* = (V, E)$ is a semigraph.
- $J = E_1 \times E_2 \times \cdots \times E_n$ is the set of all parameter tuples formed by selecting one parameter from each E_i.
- $F: J \to \mathcal{P}(V)$ is a mapping that assigns to each parameter tuple $\mathrm{j} \in J$ a subset $F(\mathrm{j}) \subseteq V$.

For each $\mathrm{j} \in J$, define $W(\mathrm{j})$ as the set of maximal partial edges induced by $F(\mathrm{j})$. The pair $H(\mathrm{j}) = (F(\mathrm{j}), W(\mathrm{j}))$ must be a partial semigraph of G^*.

The collection $G = \{H(\mathrm{j}) \mid \mathrm{j} \in J\}$ represents the hypersoft semigraph over G^*.

Theorem 46. Every hypersoft semigraph can be transformed into a semigraph and a soft semigraph. Proof. Let $G = (G^*, F, J)$ be a hypersoft semigraph over the semigraph $G^* = (V, E)$, where $J = E_1 \times E_2 \times \cdots \times E_n$ and $F: J \to \mathcal{P}(V)$.

To transform the hypersoft semigraph G into a semigraph $G' = (V', E')$, proceed as follows:

Let $V' = \bigcup_{\mathrm{j} \in J} F(\mathrm{j}) \subseteq V$.

Let $E' = \bigcup_{\mathrm{j} \in J} W(\mathrm{j}) \subseteq E_p$, where $W(\mathrm{j})$ is the set of maximal partial edges induced by $F(\mathrm{j})$.

Since each $H(\mathrm{j}) = (F(\mathrm{j}), W(\mathrm{j}))$ is a partial semigraph of G^*, their union $G' = (V', E')$ forms a semigraph, as it satisfies the conditions of a semigraph:

Any two edges in E' have at most one vertex in common because this property holds in G^*.

The edges in E' are sequences of distinct vertices, satisfying the edge definition in a semigraph.

We can construct a soft semigraph $G'' = (G^*, Q, W', A)$ as follows:

Let $A = J$ (i.e., the set of all parameter tuples).

Define the mapping $Q: A \to \mathcal{P}(V)$ by:

$$Q(\mathrm{j}) = F(\mathrm{j}),\ \forall \mathrm{j} \in A$$

Define the mapping $W' : A \to \mathcal{P}(E_p)$ by:

$$W'(\mathrm{j}) = W(\mathrm{j}),\ \forall \mathrm{j} \in A,$$

where $W(\mathrm{j})$ is the set of maximal partial edges induced by $F(\mathrm{j})$.

Since $H(\mathrm{j}) = (Q(\mathrm{j}), W'(\mathrm{j}))$ is a partial semigraph of G^* for each $\mathrm{j} \in A$, the quadruple $G'' = (G^*, Q, W', A)$ satisfies the definition of a soft semigraph.

By constructing G' and G'' as shown, we have demonstrated that a hypersoft semigraph G can be transformed into a semigraph G' and a soft semigraph G''.

4.5 | Future Tasks: Extension of Near Set and Decision-theoretic Rough Set

Future research directions include various extensions of existing concepts. For example, We aim to define new sets, such as Soft Near Sets, Soft Expert Near Sets, Hypersoft Near Sets, Soft Decision-Theoretic Rough Sets, and Hypersoft Decision-Theoretic Rough Sets by extending Near Sets [108, 112, 165-167] and Decision Theoretic Rough Sets [129-133, 170] through Soft Sets, Soft Expert Sets(cf., 10, 12, 20, 21, 168, 179]), and Hypersoft Sets. Near sets are collections that are either spatially or descriptively close, often sharing characteristics or elements. Decision-Theoretic Rough Sets, meanwhile, utilize rough set approximations for classification, leveraging Bayesian decision theory to minimize risk based on observed evidence. Alongside these definitions, I intend to explore their graphical representations, mathematical properties, potential applications, and construction algorithms.

4.6 | Future Tasks: Various Hypersoft Set

In the future, we intend to explore the mathematical structures of Z-hypersoft sets and Z-hypersoft fuzzy rough sets as extensions of Z-soft sets and Z-soft fuzzy rough sets [232-234, 238]. This exploration includes related concepts, such as Z-numbers [19, 38, 156, 230]. Furthermore, We anticipate progress in future research on Z-hypersoft graphs and Z-hypersoft fuzzy rough graphs, which graphically represent these concepts.

Other types of sets, such as Positive Sets [107, 163], Boffa Sets [29, 36 136], Meta Sets [122, 206, 207], Hyperfuzzy Sets [95, 121], and Naive Sets [103, 211], are also recognized. We aim to consider generalized concepts of Soft Sets, Hypersoft Sets, Soft Graphs, Hypersoft Graphs, and Rough Graphs by incorporating these ideas. For overviews of well-known types of sets, please refer to survey papers as needed [79].

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Author Contribution

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Development of Some New Hybrid Structures of Hypersoft Set with Possibility-degree Settings

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Abstract

The concept of a hypersoft membership function is introduced in the extension of a soft set known as a hypersoft set, permitting it to handle complicated and uncertain information in a more powerful and flexible manner. Many academics have already become fascinated with this new area of study, leading to the development of a number of hybrid structures. This chapter develops some new hybrid hypersoft set structures by taking into account multiple fuzzy set-like settings and possibility degree-based settings collectively. Additionally, numerical examples are included to clarify the concept of these structures. Researchers can utilize this work to better understand and apply a variety of mathematical ideas.

Keywords: Fuzzy Set, Intuitionistic Fuzzy Set, Neutrosophic Set, Soft Set, Hypersoft Set, Possibility Theory, Uncertainty, Attribute-valued Sets

1. Introduction

While analyzing information-based data, the data analyst has to cope with various kinds of uncertainties. Without a doubt, it is a challenging task in all respects. The ideas [1-3] are considered trustworthy regarding the handling of such uncertainties and impreciseness. However, it is observed that these ideas are unable to manage situations where attributes/parameters have to be considered. Therefore, Molodtsov [4] put forward the idea of the soft set (SOS) that is meant to equip the fuzzy set-like structures with parameterization mode. In SOS, the classical belonging mapping is replaced with approximate mapping that considers the set of parameters as its domain and the power set of initial space of alternatives as its co-domain. In other words, the SOS provides a single argument domain to approximate the alternatives. However, the SOS itself is inadequate with the settings which demand a multi-argument domain for the approximation of alternatives. Smarandache [5] addressed this issue by putting forward the idea of the

hypersoft set (HypSOS) that employs MaaM to provide a multi-argument domain. This domain is obtained by taking the product of attribute-valued sub-classes. Saeed et al. [6] characterized various fundamental concepts and notions of HypSOS to enhance its applicability in other branches of study. Yolcu & Öztürk [7], Debnath [8], Ihsan et al. [9], Khan et al. [10], and Kamacı & Saqlain [11] developed the hybrid structures of HypSOS with fuzzy set-like environments and discussed their applications. Yolcu et al. [12] developed the hybrid structure of HypSOS with an intuitionistic fuzzy set and discussed its applications.

Zadeh [13] discussed the possibility theory as a basis for the fuzzy set. In such a theory, a possibility grade is used that is meant to assess the acceptance level of any approximation. Several authors [14-24] have already used possibility setting with soft set-like environments to develop various possibility SOS-like structures. Recently such settings have been employed with HypSOS-like environments to introduce new structures [25-31].

This chapter basically is aimed to extend the above-mentioned possibility SOS-like and HypSOS-like structures and then to introduce different types of possibility HypSOS-like structures.

Table 1: Notations and Abbreviations

Notations	Full Name	Notations	Full Name
$\mathfrak{A}^{F}$	Collection of fuzzy subsets	$\mathfrak{A}^{IVIF}$	Collection of interval-valued Intuitionistic fuzzy subsets
$\mathfrak{A}^{IF}$	Collection of intuitionistic fuzzy subsets	$\mathfrak{A}^{IVPF}$	Collection of interval-valued picture fuzzy subsets
$\mathfrak{A}^{PyF}$	Collection of Pythagorean fuzzy subsets	$\mathfrak{A}^{IVsvN}$	Collection of interval-valued sv-neutrosophic subsets
$\mathfrak{A}^{PF}$	Collection of picture fuzzy subsets	$\mathfrak{A}^{IVF}$	Collection of interval-valued fuzzy subsets
$\mathfrak{A}^{svN}$	Collection of sv-neutrosophic subsets	MaaM	Multi-argument approximate mapping

2. Preliminaries

This part presents some essential terms for proper understanding of the main results. The notation $2^{\hat{U}}$ represents the power set of $\hat{U}$ (initial space of objects).

Definition 2.1: Soft Set [1]

A SOS A is the collection of object $(\Psi_A, \hat{E})$ characterized by an approximate mapping $\Psi_A : \hat{E} \to 2^{\hat{U}}$ and defined as $A = \left\{ \left(\Psi_A(e), e \right) : e \in \hat{E} \wedge \Psi_A(e) \subseteq 2^{\hat{U}} \right\}$ where $\Psi_A(e)$ is *e*-approximate element of A corresponding to attribute *e* and $\hat{E}$ is the set of distinct attributes.

Definition 2.2: Hypersoft Set [5]

A HypSS H is the collection of object $(\Psi_H, \hat{\Theta})$ characterized by an approximate mapping $\Psi_H : \hat{\Theta} \to 2^{\hat{U}}$ and defined as $H = \left\{ \left(\Psi_H(\hat{\theta}), \hat{\theta} \right) : \hat{\theta} \in \hat{\Theta} \wedge \Psi_H(\hat{\theta}) \subseteq 2^{\hat{U}} \right\}$ where $\Psi_H(\hat{\theta})$ is $\hat{\theta}-$ approximate element of H corresponding to attribute-valued tuple $\wp$ and $\hat{\Theta} = \hat{\Theta}_1 \times \hat{\Theta}_2 \times \ldots \times \hat{\Theta}_n$. The sets $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_n$ are attribute-valued non-overlapping sets.

3. Hybrid Structures of Hypersoft Sets

In this section, some new hybrid structures of hypersoft set are discussed with illustrative examples.

3.1 Hybrid Structures of Fuzzy Hypersoft Sets with Possibility Settings

In accessible literature, Rahman et al. [25-29], Zhao et al. [30] and Al-Hijjawi & Alkhazaleh [31] developed hybrid structures of hypersoft set by considering possibility degree in terms of fuzzy membership grades. However, in this section, possibility degree is considered in terms of other fuzzy set-like membership grades, e.g., intuitionistic fuzzy membership grades, neutrosophic membership grades, etc. The set $\hat{\Theta} = \hat{\Theta}_1 \times \hat{\Theta}_2 \times \ldots \times \hat{\Theta}_n$ is the product of attribute-valued disjoint sets $\hat{\Theta}_i, i = 1, 2, 3, \ldots, n$ with respect to *n* different attributes in

Definition 3.1.1: **Possibility Fuzzy Hypersoft Set of Type-1**

A possibility fuzzy hypersoft set of type-1 $\aleph_1$ is defined as

$$\mathfrak{C}_1=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_F(\hat{\theta})(\hat{u})},\mu_F(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_F(\hat{\theta})(\hat{u}),\mu_F(\hat{\theta})\subseteq\mathfrak{A}^F$ and $\psi_F(\hat{\theta})(\hat{u}),\mu_F(\hat{\theta})\in[0,1]$.

Example 3.1.1: Let $\hat{U}=\{\hat{u}_1,\hat{u}_2,\hat{u}_3,\hat{u}_4\}$ consisting of four models of air coolers and $\hat{\Theta}=\{\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3,\hat{\theta}_4\}$ corresponding to attribute-valued sets $\hat{\Theta}_1=\{800,1000\},\hat{\Theta}_2=\{180,230\},\hat{\Theta}_3=\{55\}$, and $\hat{\Theta}_4=\{200\}$ with respect to attributes $\wp_1$ = water tank capacity in liters, $\wp_2$ = voltage, $\wp_3$ = air pressure in feet, and $\wp_4$ = weight in kilograms respectively then possibility fuzzy hypersoft set of type-1 $\mathfrak{C}_1$ can be constructed as

$$\mathfrak{C}_1=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{0.21},0.31\right\rangle,\left\langle\frac{\hat{u}_2}{0.31},0.41\right\rangle,\left\langle\frac{\hat{u}_3}{0.41},0.51\right\rangle,\left\langle\frac{\hat{u}_4}{0.51},0.61\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{0.22},0.32\right\rangle,\left\langle\frac{\hat{u}_2}{0.32},0.42\right\rangle,\left\langle\frac{\hat{u}_3}{0.42},0.52\right\rangle,\left\langle\frac{\hat{u}_4}{0.52},0.62\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{0.23},0.33\right\rangle,\left\langle\frac{\hat{u}_2}{0.33},0.43\right\rangle,\left\langle\frac{\hat{u}_3}{0.43},0.53\right\rangle,\left\langle\frac{\hat{u}_4}{0.53},0.63\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{0.24},0.34\right\rangle,\left\langle\frac{\hat{u}_2}{0.34},0.44\right\rangle,\left\langle\frac{\hat{u}_3}{0.44},0.54\right\rangle,\left\langle\frac{\hat{u}_4}{0.54},0.64\right\rangle\right\}\right),\end{array}\right\}.$$

Definition 3.1.2: **Possibility Fuzzy Hypersoft Set of Type-2**

A possibility fuzzy hypersoft set of type-2 $\mathfrak{C}_2$ is defined as

$$\mathfrak{C}_1=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_F(\hat{\theta})(\hat{u})},\mu_{IF}(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_F(\hat{\theta})(\hat{u})\subseteq\mathfrak{A}^F,\mu_{IF}(\hat{\theta})\subseteq\mathfrak{A}^{IF}$ and $\psi_F(\hat{\theta})(\hat{u})\in[0,1],\mu_{IF}(\hat{\theta})=\left\langle T_\mu(\hat{\theta}),\mathrm{F}_\mu(\hat{\theta})\right\rangle$ with $T_\mu(\hat{\theta}),\mathrm{F}_\mu(\hat{\theta})\in[0,1]$ and $0\le T_\mu(\hat{\theta})+\mathrm{F}_\mu(\hat{\theta})\le 1$.

Example 3.1.2: Considering the assumptions from Example 3.1.1, we can construct possibility fuzzy hypersoft set of type-2 $\mathfrak{C}_2$ can be constructed as

$$\mathfrak{C}_2 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{0.21}, \langle 0.31, 0.21 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{0.31}, \langle 0.41, 0.31 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{0.41}, \langle 0.51, 0.41 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{0.51}, \langle 0.61, 0.21 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{0.22}, \langle 0.32, 0.22 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{0.32}, \langle 0.42, 0.32 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{0.42}, \langle 0.52, 0.42 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{0.52}, \langle 0.62, 0.22 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{0.23}, \langle 0.33, 0.23 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{0.33}, \langle 0.43, 0.33 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{0.43}, \langle 0.53, 0.43 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{0.53}, \langle 0.63, 0.23 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{0.24}, \langle 0.34, 0.24 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{0.34}, \langle 0.44, 0.34 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{0.44}, \langle 0.54, 0.44 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{0.54}, \langle 0.64, 0.24 \rangle \right\rangle \right\} \right), \end{array} \right\}.$$

Definition 3.1.3: **Possibility Fuzzy Hypersoft Set of Type-3**

A possibility fuzzy hypersoft set of type-3 $\mathfrak{C}_3$ is defined as

$$\mathfrak{C}_3 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_F(\hat{\theta})(\hat{u})}, \mu_{PyF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_F(\hat{\theta})(\hat{u}) \subseteq \mathfrak{A}^F$, $\mu_{PyF}(\hat{\theta}) \subseteq \mathfrak{A}^{PyF}$ and $\psi_F(\hat{\theta})(\hat{u}) \in [0,1]$, $\mu_{PyF}(\hat{\theta}) = \left\langle T_\mu(\hat{\theta}), \mathrm{F}_\mu(\hat{\theta}) \right\rangle$ with $T_\mu(\hat{\theta}), \mathrm{F}_\mu(\hat{\theta}) \in [0,1]$ and $0 \le T_\mu^2(\hat{\theta}) + \mathrm{F}_\mu^2(\hat{\theta}) \le 1$.

Example 3.1.3: Considering the assumptions from Example 3.1.1, we can construct possibility fuzzy hypersoft set of type-3 $\mathfrak{C}_3$ can be constructed as

$$\mathfrak{C}_3 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{0.21}, \langle 0.5, 0.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{0.31}, \langle 0.7, 0.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{0.41}, \langle 0.5, 0.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{0.51}, \langle 0.6, 0.5 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{0.22}, \langle 0.9, 0.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{0.32}, \langle 0.4, 0.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{0.42}, \langle 0.4, 0.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{0.52}, \langle 0.5, 0.8 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{0.23}, \langle 0.7, 0.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{0.33}, \langle 0.8, 0.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{0.43}, \langle 0.6, 0.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{0.53}, \langle 0.5, 0.6 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{0.24}, \langle 0.4, 0.9 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{0.34}, \langle 0.8, 0.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{0.44}, \langle 0.7, 0.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{0.54}, \langle 0.8, 0.5 \rangle \right\rangle \right\} \right), \end{array} \right\}.$$

Definition 3.1.4: **Possibility Fuzzy Hypersoft Set of Type-4**

A possibility fuzzy hypersoft set of type-4 $\mathfrak{F}_4$ is defined as

$$\mathfrak{F}_4=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_F(\hat{\theta})(\hat{u})},\mu_{PF}(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_F(\hat{\theta})(\hat{u})\subseteq\mathfrak{A}^F,\mu_{PF}(\hat{\theta})\subseteq\mathfrak{A}^{PF}$ and $\psi_F(\hat{\theta})(\hat{u})\in[0,1],\mu_{PF}(\hat{\theta})=\left\langle T_\mu(\hat{\theta}),\mathrm{I}_\mu(\hat{\theta}),\mathrm{F}_\mu(\hat{\theta})\right\rangle$ with $T_\mu(\hat{\theta}),\mathrm{I}_\mu(\hat{\theta}),\mathrm{F}_\mu(\hat{\theta})\in[0,1]$ and $0\le T_\mu(\hat{\theta})+I_\mu(\hat{\theta})+\mathrm{F}_\mu(\hat{\theta})\le 1$.

Example 3.1.4: Considering the assumptions from Example 3.1.1, we can construct possibility fuzzy hypersoft set of type-4 $\mathfrak{F}_4$ can be constructed as

$$\mathfrak{F}_4=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{.2},\langle.5,.1,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{.3},\langle.2,.5,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{.4},\langle.4,.2,.3\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{.5},\langle.6,.1,.2\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{.3},\langle.1,.4,.4\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{.4},\langle.3,.4,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{.5},\langle.2,.3,.3\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{.6},\langle.2,.4,.2\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{.4},\langle.3,.5,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{.5},\langle.2,.5,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{.6},\langle.6,.2,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{.7},\langle.5,.2,.2\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{.5},\langle.4,.1,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{.6},\langle.2,.2,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{.7},\langle.7,.1,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{.8},\langle.4,.2,.1\rangle\right\rangle\right\}\right),\end{array}\right\}.$$

Definition 3.1.5: **Possibility Fuzzy Hypersoft Set of Type-5**

A possibility fuzzy hypersoft set of type-5 $\mathfrak{F}_5$ is defined as

$$\mathfrak{F}_5=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_F(\hat{\theta})(\hat{u})},\mu_{svN}(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_F(\hat{\theta})(\hat{u})\subseteq\mathfrak{A}^F,\mu_{svN}(\hat{\theta})\subseteq\mathfrak{A}^{svN}$ and $\psi_F(\hat{\theta})(\hat{u})\in[0,1],\mu_{svN}(\hat{\theta})=\left\langle T_\mu(\hat{\theta}),\mathrm{I}_\mu(\hat{\theta}),\mathrm{F}_\mu(\hat{\theta})\right\rangle$ with $T_\mu(\hat{\theta}),\mathrm{I}_\mu(\hat{\theta}),\mathrm{F}_\mu(\hat{\theta})\in[0,1]$ and $0\le T_\mu(\hat{\theta})+I_\mu(\hat{\theta})+\mathrm{F}_\mu(\hat{\theta})\le 3$.

Example 3.1.5: Considering the assumptions from Example 3.1.1, we can construct possibility fuzzy hypersoft set of type-5 $\mathfrak{F}_5$ can be constructed as

$$\mathfrak{E}_5=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{.2},\langle.5,.6,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{.3},\langle.6,.7,.7\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{.4},\langle.7,.8,.8\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{.5},\langle.8,.9,.9\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{.3},\langle.6,.5,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{.4},\langle.7,.6,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{.5},\langle.8,.7,.7\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{.6},\langle.9,.8,.8\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{.4},\langle.7,.6,.7\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{.5},\langle.8,.7,.8\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{.6},\langle.9,.8,.9\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{.7},\langle.6,.5,.9\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{.5},\langle.8,.7,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{.6},\langle.9,.6,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{.7},\langle.6,.7,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{.8},\langle.6,.8,.8\rangle\right\rangle\right\}\right),\end{array}\right\}.$$

Definition 3.1.6: **Possibility Fuzzy Hypersoft Set of Type-6**

A possibility fuzzy hypersoft set of type-6 $\mathfrak{E}_6$ is defined as

$$\mathfrak{E}_6=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_F(\hat{\theta})(\hat{u})},\mu_{IVF}(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_F(\hat{\theta})(\hat{u})\subseteq\mathfrak{A}^F,\mu_{IVF}(\hat{\theta})\subseteq\mathfrak{A}^{IVF}$ and $\psi_F(\hat{\theta})(\hat{u})\in[0,1],\mu_{IVF}(\hat{\theta})=\left[L_\mu(\hat{\theta}),\mathrm{U}_\mu(\hat{\theta})\right]$ with $L_\mu(\hat{\theta}),\mathrm{U}_\mu(\hat{\theta})\in[0,1]$.

Example 3.1.6: Considering the assumptions from Example 3.1.1, we can construct possibility fuzzy hypersoft set of type-6 $\mathfrak{E}_6$ can be constructed as

$$\mathfrak{E}_6=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{0.21},[0.21,0.31]\right\rangle,\left\langle\frac{\hat{u}_2}{0.31},[0.31,0.41]\right\rangle,\left\langle\frac{\hat{u}_3}{0.41},[0.41,0.51]\right\rangle,\left\langle\frac{\hat{u}_4}{0.51},[0.21,0.61]\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{0.22},[0.22,0.32]\right\rangle,\left\langle\frac{\hat{u}_2}{0.32},[0.32,0.42]\right\rangle,\left\langle\frac{\hat{u}_3}{0.42},[0.42,0.52]\right\rangle,\left\langle\frac{\hat{u}_4}{0.52},[0.22,0.62]\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{0.23},[0.23,0.33]\right\rangle,\left\langle\frac{\hat{u}_2}{0.33},[0.33,0.43]\right\rangle,\left\langle\frac{\hat{u}_3}{0.43},[0.43,0.53]\right\rangle,\left\langle\frac{\hat{u}_4}{0.53},[0.23,0.63]\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{0.24},[0.24,0.34]\right\rangle,\left\langle\frac{\hat{u}_2}{0.34},[0.34,0.44]\right\rangle,\left\langle\frac{\hat{u}_3}{0.44},[0.44,0.54]\right\rangle,\left\langle\frac{\hat{u}_4}{0.54},[0.24,0.64]\right\rangle\right\}\right),\end{array}\right\}.$$

3.2 Hybrid Structures of Intuitionistic Fuzzy Hypersoft Sets with Possibility Settings

Definition 3.2.1: **Possibility Intuitionistic Fuzzy Hypersoft Set of Type-1**

A possibility intuitionistic fuzzy hypersoft set of type-1 λ_1 is defined as

$$\lambda_1 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IF}(\hat{\theta})(\hat{u})}, \mu_F(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IF}(\hat{\theta})(\hat{u}) = \left\langle T_\psi(\hat{\theta})(\hat{u}), \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{IF}$, $\mu_F(\hat{\theta}) \subseteq \mathfrak{A}^F$ and $\mathrm{T}_\psi(\hat{\theta})(\hat{u}), \mathrm{F}_\psi(\hat{\theta})(\hat{u}), \mu_F(\hat{\theta}) \in [0,1]$ such that $0 \le \mathrm{T}_\psi(\hat{\theta})(\hat{u}) + F_\psi(\hat{\theta})(\hat{u}) \le 1$.

Example 3.2.1: Considering the assumptions from Example 3.1.1, we can construct possibility intuitionistic fuzzy hypersoft set of type-1 λ_1 can be constructed as

$$\lambda_1 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.5 \rangle}, 0.31 \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.6 \rangle}, 0.41 \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.4 \rangle}, 0.51 \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.2 \rangle}, 0.61 \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.3 \rangle}, 0.32 \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.4 \rangle}, 0.42 \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.2 \rangle}, 0.52 \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.1 \rangle}, 0.62 \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.5 \rangle}, 0.33 \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.3 \rangle}, 0.43 \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.1 \rangle}, 0.53 \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.2 \rangle}, 0.63 \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.2 \rangle}, 0.34 \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.5 \rangle}, 0.44 \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.5 \rangle}, 0.54 \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.3 \rangle}, 0.64 \right\rangle \right\} \right), \end{array} \right\}.$$

Definition 3.2.2: **Possibility Intuitionistic Fuzzy Hypersoft Set of Type-2**

A possibility intuitionistic fuzzy hypersoft set of type-2 λ_2 is defined as

$$\lambda_2 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IF}(\hat{\theta})(\hat{u})}, \mu_{IF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IF}(\hat{\theta})(\hat{u}) = \left\langle T_\psi(\hat{\theta})(\hat{u}), \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{IF}$, $\mu_{IF}(\hat{\theta}) \subseteq \mathfrak{A}^{IF}$ and $T_\psi(\hat{\theta})(\hat{u}), \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \in [0,1]$, $\mu_{IF}(\hat{\theta}) = \left\langle T_\mu(\hat{\theta}), \mathrm{F}_\mu(\hat{\theta}) \right\rangle$, $T_\mu(\hat{\theta}), \mathrm{F}_\mu(\hat{\theta}) \in [0,1]$ with $0 \le T_\psi(\hat{\theta})(\hat{u}) + \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \le 1$ and $0 \le T_\mu(\hat{\theta}) + \mathrm{F}_\mu(\hat{\theta}) \le 1$.

Example 3.2.2: Considering the assumptions from Example 3.1.1, we can construct possibility intuitionistic fuzzy hypersoft set of type-2 λ_2 can be constructed as

$$\lambda_2=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.2,.4\rangle},\langle 0.31,0.21\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.3,.4\rangle},\langle 0.41,0.31\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.4,.5\rangle},\langle 0.51,0.41\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.5,.2\rangle},\langle 0.61,0.21\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.2,.3\rangle},\langle 0.32,0.22\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.3,.2\rangle},\langle 0.42,0.32\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.4,.4\rangle},\langle 0.52,0.42\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.5,.1\rangle},\langle 0.62,0.22\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.1,.5\rangle},\langle 0.33,0.23\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.1,.4\rangle},\langle 0.43,0.33\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.5,.1\rangle},\langle 0.53,0.43\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.4,.2\rangle},\langle 0.63,0.23\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.3,.3\rangle},\langle 0.34,0.24\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.3,.5\rangle},\langle 0.44,0.34\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.4,.5\rangle},\langle 0.54,0.44\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.5,.3\rangle},\langle 0.64,0.24\rangle\right\rangle\right\}\right),\end{array}\right\}$$

.

Definition 3.2.3: **Possibility Intuitionistic Fuzzy Hypersoft Set of Type-3**

A possibility intuitionistic fuzzy hypersoft set of type-3 λ_3 is defined as

$$\lambda_3=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_{IF}(\hat{\theta})(\hat{u})},\mu_{PyF}(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_{IF}(\hat{\theta})(\hat{u})=\left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}),\mathrm{F}_{\psi}(\hat{\theta})(\hat{u})\right\rangle\subseteq\mathfrak{A}^{IF}$, $\mu_{PyF}(\hat{\theta})\subseteq\mathfrak{A}^{PyF}$ and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}),\mathrm{F}_{\psi}(\hat{\theta})(\hat{u})\in[0,1]$ with $0\le \mathrm{T}_{\psi}(\hat{\theta})(\hat{u})+F_{\psi}(\hat{\theta})(\hat{u})\le 1$ and $\mu_{PyF}(\hat{\theta})=\left\langle T_{\mu}(\hat{\theta}),\mathrm{F}_{\mu}(\hat{\theta})\right\rangle$ with $T_{\mu}(\hat{\theta}),\mathrm{F}_{\mu}(\hat{\theta})\in[0,1]$ and $0\le T_{\mu}^2(\hat{\theta})+\mathrm{F}_{\mu}^2(\hat{\theta})\le 1$.

Example 3.2.3: Considering the assumptions from Example 3.1.1, we can construct possibility intuitionistic fuzzy hypersoft set of type-3 λ_3 can be constructed as

$$\lambda_3=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.2,.1\rangle},\langle 0.5,0.6\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.2,.3\rangle},\langle 0.7,0.5\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.2,.4\rangle},\langle 0.5,0.8\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.2,.5\rangle},\langle 0.6,0.5\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.3,.1\rangle},\langle 0.9,0.4\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.3,.2\rangle},\langle 0.4,0.8\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.3,.4\rangle},\langle 0.4,0.7\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.3,.5\rangle},\langle 0.5,0.8\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.4,.1\rangle},\langle 0.7,0.5\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.4,.2\rangle},\langle 0.8,0.5\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.4,.3\rangle},\langle 0.6,0.5\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.4,.5\rangle},\langle 0.5,0.6\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.5,.1\rangle},\langle 0.4,0.9\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.5,.2\rangle},\langle 0.8,0.4\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.5,.3\rangle},\langle 0.7,0.4\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.5,.4\rangle},\langle 0.8,0.5\rangle\right\rangle\right\}\right),\end{array}\right\}$$

.

Definition 3.2.4: **Possibility Intuitionistic Fuzzy Hypersoft Set of Type-4**

A possibility intuitionistic fuzzy hypersoft set of type-4 λ_4 is defined as

$$\lambda_4 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IF}(\hat{\theta})(\hat{u})}, \mu_{PF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IF}(\hat{\theta})(\hat{u}) = \left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{IF}$, and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$ with $0 \leq \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}) + F_{\psi}(\hat{\theta})(\hat{u}) \leq 1$. Similarly $\mu_{PF}(\hat{\theta}) \subseteq \mathfrak{A}^{PF}$, $\mu_{PF}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \leq T_{\mu}(\hat{\theta}) + I_{\mu}(\hat{\theta}) + \mathrm{F}_{\mu}(\hat{\theta}) \leq 1$.

Example 3.2.4: Considering the assumptions from Example 3.1.1, we can construct possibility intuitionistic fuzzy hypersoft set of type-4 λ_4 can be constructed as

$$\lambda_4 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.1 \rangle}, \langle .5,.1,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .2,.3 \rangle}, \langle .2,.5,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .2,.4 \rangle}, \langle .4,.2,.3 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .2,.5 \rangle}, \langle .6,.1,.2 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .3,.1 \rangle}, \langle .1,.4,.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.2 \rangle}, \langle .3,.4,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .3,.4 \rangle}, \langle .2,.3,.3 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .3,.5 \rangle}, \langle .2,.4,.2 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .4,.1 \rangle}, \langle .3,.5,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .4,.2 \rangle}, \langle .2,.5,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.3 \rangle}, \langle .6,.2,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .4,.5 \rangle}, \langle .5,.2,.2 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .5,.1 \rangle}, \langle .4,.1,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .5,.2 \rangle}, \langle .2,.2,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .5,.3 \rangle}, \langle .7,.1,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.4 \rangle}, \langle .4,.2,.1 \rangle \right\rangle \right\} \right), \end{array} \right\}$$

.

Definition 3.2.5: **Possibility Intuitionistic Fuzzy Hypersoft Set of Type-5**

A possibility intuitionistic fuzzy hypersoft set of type-5 λ_5 is defined as

$$\lambda_5 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IF}(\hat{\theta})(\hat{u})}, \mu_{svN}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IF}(\hat{\theta})(\hat{u}) = \left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{IF}$, and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$ with $0 \leq \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}) + F_{\psi}(\hat{\theta})(\hat{u}) \leq 1$. Similarly $\mu_{svN}(\hat{\theta}) \subseteq \mathfrak{A}^{svN}$, $\mu_{svN}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \leq T_{\mu}(\hat{\theta}) + I_{\mu}(\hat{\theta}) + \mathrm{F}_{\mu}(\hat{\theta}) \leq 3$.

Example 3.2.5: Considering the assumptions from Example 3.1.1, we can construct possibility intuitionistic fuzzy hypersoft set of type-5 λ_5 can be constructed as

$$\lambda_5 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.1 \rangle}, \langle .5,.6,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .2,.3 \rangle}, \langle .6,.7,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .2,.4 \rangle}, \langle .7,.8,.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .2,.5 \rangle}, \langle .8,.9,.9 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .3,.1 \rangle}, \langle .6,.5,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.2 \rangle}, \langle .7,.6,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .3,.4 \rangle}, \langle .8,.7,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .3,.5 \rangle}, \langle .9,.8,.8 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .4,.1 \rangle}, \langle .7,.6,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .4,.2 \rangle}, \langle .8,.7,.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.3 \rangle}, \langle .9,.8,.9 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .4,.5 \rangle}, \langle .6,.5,.9 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .5,.1 \rangle}, \langle .8,.7,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .5,.2 \rangle}, \langle .9,.6,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .5,.3 \rangle}, \langle .6,.7,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.4 \rangle}, \langle .6,.8,.8 \rangle \right\rangle \right\} \right), \end{array} \right\}$$

.

Definition 3.2.6: **Possibility Intuitionistic Fuzzy Hypersoft Set of Type-6**

A possibility intuitionistic fuzzy hypersoft set of type-6 λ_6 is defined as

$$\lambda_6 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IF}(\hat{\theta})(\hat{u})}, \mu_{IVF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IF}(\hat{\theta})(\hat{u}) = \left\langle T_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{IF}$ and $T_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$ such that $0 \le T_{\psi}(\hat{\theta})(\hat{u}) + F_{\psi}(\hat{\theta})(\hat{u}) \le 1$. Similarly $\mu_{IVF}(\hat{\theta}) \subseteq \mathfrak{A}^{IVF}$ with $\mu_{IVF}(\hat{\theta}) = \left[L_{\mu}(\hat{\theta}), \mathrm{U}_{\mu}(\hat{\theta}) \right]$ and $L_{\mu}(\hat{\theta}), \mathrm{U}_{\mu}(\hat{\theta}) \in [0,1]$.

Example 3.2.6: Considering the assumptions from Example 3.1.1, we can construct possibility intuitionistic fuzzy hypersoft set of type-6 λ_6 can be constructed as

$$\lambda_6=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.2,.1\rangle},[0.21,0.31]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.2,.3\rangle},[0.31,0.41]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.2,.4\rangle},[0.41,0.51]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.2,.5\rangle},[0.21,0.61]\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.3,.1\rangle},[0.22,0.32]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.3,.2\rangle},[0.32,0.42]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.3,.4\rangle},[0.42,0.52]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.3,.5\rangle},[0.22,0.62]\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.4,.1\rangle},[0.23,0.33]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.4,.2\rangle},[0.33,0.43]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.4,.3\rangle},[0.43,0.53]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.4,.5\rangle},[0.23,0.63]\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.5,.1\rangle},[0.24,0.34]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.5,.2\rangle},[0.34,0.44]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.5,.3\rangle},[0.44,0.54]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.5,.5\rangle},[0.24,0.64]\right\rangle\right\}\right),\end{array}\right\}$$

3.3 Hybrid Structures of Pythagorean Fuzzy Hypersoft Sets with Possibility Settings

Definition 3.3.1: **Possibility Pythagorean Fuzzy Hypersoft Set of Type-1**

A possibility Pythagorean fuzzy hypersoft set of type-1 η_1 is defined as

$$\eta_1=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_{PyF}(\hat{\theta})(\hat{u})},\mu_F(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_{PyF}(\hat{\theta})(\hat{u})=\left\langle T_\psi(\hat{\theta})(\hat{u}),\mathrm{F}_\psi(\hat{\theta})(\hat{u})\right\rangle\subseteq\mathfrak{A}^{PyF}$ and $\mathrm{T}_\psi(\hat{\theta})(\hat{u}),\mathrm{F}_\psi(\hat{\theta})(\hat{u})\in[0,1]$ such that $0\le\mathrm{T}_\psi^2(\hat{\theta})(\hat{u})+F_\psi^2(\hat{\theta})(\hat{u})\le1$. Similarly $\mu_F(\hat{\theta})\subseteq\mathfrak{A}^F$ with $\mu_F(\hat{\theta})\in[0,1]$.

Example 3.3.1: Considering the assumptions from Example 3.1.1, we can construct possibility Pythagorean fuzzy hypersoft set of type-1 η_1 can be constructed as

$$\eta_1=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.5,.6\rangle},0.31\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.5,.7\rangle},0.41\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.5,.8\rangle},0.51\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.5,.8\rangle},0.61\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.6,.5\rangle},0.32\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.6,.6\rangle},0.42\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.6,.7\rangle},0.52\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.6,.7\rangle},0.62\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.7,.5\rangle},0.33\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.7,.6\rangle},0.43\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.7,.7\rangle},0.53\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.7,.4\rangle},0.63\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.8,.5\rangle},0.34\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.8,.5\rangle},0.44\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.8,.4\rangle},0.54\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.8,.4\rangle},0.64\right\rangle\right\}\right),\end{array}\right\}.$$

Definition 3.3.2: **Possibility Pythagorean Fuzzy Hypersoft Set of Type-2**

A possibility Pythagorean fuzzy hypersoft set of type-2 η_2 is defined as

$$\eta_2 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{PyF}(\hat{\theta})(\hat{u})}, \mu_{IF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{PyF}(\hat{\theta})(\hat{u}) = \left\langle T_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{PyF}, \mu_{IF}(\hat{\theta}) \subseteq \mathfrak{A}^{IF}$ and

$T_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1], \mu_{IF}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle, T_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ with

$0 \leq \mathrm{T}_{\psi}^2(\hat{\theta})(\hat{u}) + F_{\psi}^2(\hat{\theta})(\hat{u}) \leq 1$ and $0 \leq T_{\mu}(\hat{\theta}) + \mathrm{F}_{\mu}(\hat{\theta}) \leq 1$.

Example 3.3.2: Considering the assumptions from Example 3.1.1, we can construct possibility Pythagorean fuzzy hypersoft set of type-2 η_2 can be constructed as

$$\eta_2 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .5,.6 \rangle}, \langle 0.31, 0.21 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .5,.7 \rangle}, \langle 0.41, 0.31 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .5,.8 \rangle}, \langle 0.51, 0.41 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.8 \rangle}, \langle 0.61, 0.21 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .6,.5 \rangle}, \langle 0.32, 0.22 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .6,.6 \rangle}, \langle 0.42, 0.32 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .6,.7 \rangle}, \langle 0.52, 0.42 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .6,.7 \rangle}, \langle 0.62, 0.22 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .7,.5 \rangle}, \langle 0.33, 0.23 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .7,.6 \rangle}, \langle 0.43, 0.33 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .7,.7 \rangle}, \langle 0.53, 0.43 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .7,.4 \rangle}, \langle 0.63, 0.23 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .8,.5 \rangle}, \langle 0.34, 0.24 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .8,.5 \rangle}, \langle 0.44, 0.34 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .8,.4 \rangle}, \langle 0.54, 0.44 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .8,.4 \rangle}, \langle 0.64, 0.24 \rangle \right\rangle \right\} \right), \end{array} \right\}$$

.

Definition 3.3.3: **Possibility Pythagorean Fuzzy Hypersoft Set of Type-3**

A possibility Pythagorean fuzzy hypersoft set of type-3 η_3 is defined as

$$\eta_3 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{PyF}(\hat{\theta})(\hat{u})}, \mu_{PyF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{PyF}(\hat{\theta})(\hat{u}) = \left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{PyF}, \mu_{PyF}(\hat{\theta}) \subseteq \mathfrak{A}^{PyF}$ and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$

with $0 \leq \mathrm{T}_{\psi}^2(\hat{\theta})(\hat{u}) + F_{\psi}^2(\hat{\theta})(\hat{u}) \leq 1$ and $\mu_{PyF}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and

$0 \leq T_{\mu}^2(\hat{\theta}) + \mathrm{F}_{\mu}^2(\hat{\theta}) \leq 1$.

Example 3.3.3: Considering the assumptions from Example 3.1.1, we can construct possibility Pythagorean fuzzy hypersoft set of type-3 η_3 can be constructed as

$$\eta_3 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .5,.6 \rangle}, \langle 0.5, 0.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .5,.7 \rangle}, \langle 0.7, 0.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .5,.8 \rangle}, \langle 0.5, 0.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.8 \rangle}, \langle 0.6, 0.5 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .6,.5 \rangle}, \langle 0.9, 0.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .6,.6 \rangle}, \langle 0.4, 0.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .6,.7 \rangle}, \langle 0.4, 0.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .6,.7 \rangle}, \langle 0.5, 0.8 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .7,.4 \rangle}, \langle 0.7, 0.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .7,.5 \rangle}, \langle 0.8, 0.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .7,.6 \rangle}, \langle 0.6, 0.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .7,.7 \rangle}, \langle 0.5, 0.6 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .8,.4 \rangle}, \langle 0.4, 0.9 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .8,.5 \rangle}, \langle 0.8, 0.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .8,.5 \rangle}, \langle 0.7, 0.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .8,.4 \rangle}, \langle 0.8, 0.5 \rangle \right\rangle \right\} \right), \end{array} \right\}$$

.

Definition 3.3.4: **Possibility Pythagorean Fuzzy Hypersoft Set of Type-4**

A possibility Pythagorean fuzzy hypersoft set of type-4 η_4 is defined as

$$\eta_4 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{PyF}(\hat{\theta})(\hat{u})}, \mu_{PF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{PyF}(\hat{\theta})(\hat{u}) = \left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{PyF}$, and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$ with $0 \leq \mathrm{T}_{\psi}^2(\hat{\theta})(\hat{u}) + F_{\psi}^2(\hat{\theta})(\hat{u}) \leq 1$. Similarly $\mu_{PF}(\hat{\theta}) \subseteq \mathfrak{A}^{PF}$, $\mu_{PF}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \leq T_{\mu}(\hat{\theta}) + I_{\mu}(\hat{\theta}) + \mathrm{F}_{\mu}(\hat{\theta}) \leq 1$.

Example 3.3.4: Considering the assumptions from Example 3.1.1, we can construct possibility Pythagorean fuzzy hypersoft set of type-4 η_4 can be constructed as

$$\eta_4 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .5,.6 \rangle}, \langle .5,.1,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .5,.7 \rangle}, \langle .2,.5,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .5,.8 \rangle}, \langle .4,.2,.3 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.8 \rangle}, \langle .6,.1,.2 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .6,.5 \rangle}, \langle .1,.4,.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .6,.6 \rangle}, \langle .3,.4,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .6,.7 \rangle}, \langle .2,.3,.3 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .6,.7 \rangle}, \langle .2,.4,.2 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .7,.4 \rangle}, \langle .3,.5,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .7,.5 \rangle}, \langle .2,.5,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .7,.6 \rangle}, \langle .6,.2,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .7,.7 \rangle}, \langle .5,.2,.2 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .8,.4 \rangle}, \langle .4,.1,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .8,.5 \rangle}, \langle .2,.2,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .8,.5 \rangle}, \langle .7,.1,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .8,.4 \rangle}, \langle .4,.2,.1 \rangle \right\rangle \right\} \right), \end{array} \right\}$$

.

Definition 3.3.5: **Possibility Pythagorean Fuzzy Hypersoft Set of Type-5**

A possibility Pythagorean fuzzy hypersoft set of type-5 η_5 is defined as

$$\eta_5 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{PyF}(\hat{\theta})(\hat{u})}, \mu_{svN}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{PyF}(\hat{\theta})(\hat{u}) = \left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{PyF}$, and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$ with $0 \leq \mathrm{T}_{\psi}^2(\hat{\theta})(\hat{u}) + F_{\psi}^2(\hat{\theta})(\hat{u}) \leq 1$. Similarly $\mu_{svN}(\hat{\theta}) \subseteq \mathfrak{A}^{svN}$, $\mu_{svN}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \leq T_{\mu}(\hat{\theta}) + I_{\mu}(\hat{\theta}) + \mathrm{F}_{\mu}(\hat{\theta}) \leq 3$.

Example 3.3.5: Considering the assumptions from Example 3.1.1, we can construct possibility Pythagorean fuzzy hypersoft set of type-5 η_5 can be constructed as

$$\eta_5=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.5,.6\rangle},\langle.5,.6,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.5,.7\rangle},\langle.6,.7,.7\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.5,.8\rangle},\langle.7,.8,.8\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.5,.8\rangle},\langle.8,.9,.9\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.6,.5\rangle},\langle.6,.5,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.6,.6\rangle},\langle.7,.6,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.6,.7\rangle},\langle.8,.7,.7\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.6,.7\rangle},\langle.9,.8,.8\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.7,.4\rangle},\langle.7,.6,.7\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.7,.5\rangle},\langle.8,.7,.8\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.7,.6\rangle},\langle.9,.8,.9\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.7,.7\rangle},\langle.6,.5,.9\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.8,.4\rangle},\langle.8,.7,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.8,.4\rangle},\langle.9,.6,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.8,.5\rangle},\langle.6,.7,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.8,.5\rangle},\langle.6,.8,.8\rangle\right\rangle\right\}\right),\end{array}\right\}$$

.

Definition 3.3.6: **Possibility Pythagorean Fuzzy Hypersoft Set of Type-6**

A possibility Pythagorean fuzzy hypersoft set of type-6 η_6 is defined as

$$\eta_6=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_{PyF}(\hat{\theta})(\hat{u})},\mu_{IVF}(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_{PyF}(\hat{\theta})(\hat{u})=\left\langle T_\psi(\hat{\theta})(\hat{u}),\mathrm{F}_\psi(\hat{\theta})(\hat{u})\right\rangle\subseteq\mathfrak{A}^{PyF}$ and $T_\psi(\hat{\theta})(\hat{u}),\mathrm{F}_\psi(\hat{\theta})(\hat{u})\in[0,1]$ such that $0\le\mathrm{T}_\psi^2(\hat{\theta})(\hat{u})+F_\psi^2(\hat{\theta})(\hat{u})\le1$ .Similarly $\mu_{IVF}(\hat{\theta})\subseteq\mathfrak{A}^{IVF}$ with $\mu_{IVF}(\hat{\theta})=\left[L_\mu(\hat{\theta}),\mathrm{U}_\mu(\hat{\theta})\right]$ and $L_\mu(\hat{\theta}),\mathrm{U}_\mu(\hat{\theta})\in[0,1]$.

Example 3.3.6: Considering the assumptions from Example 3.1.1, we can construct possibility Pythagorean fuzzy hypersoft set of type-6 η_6 can be constructed as

$$\eta_6=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.5,.6\rangle},[0.21,0.31]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.5,.7\rangle},[0.31,0.41]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.5,.8\rangle},[0.41,0.51]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.5,.8\rangle},[0.21,0.61]\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.6,.5\rangle},[0.22,0.32]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.6,.6\rangle},[0.32,0.42]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.6,.7\rangle},[0.42,0.52]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.6,.7\rangle},[0.22,0.62]\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.7,.4\rangle},[0.23,0.33]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.7,.5\rangle},[0.33,0.43]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.7,.6\rangle},[0.43,0.53]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.7,.7\rangle},[0.23,0.63]\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.8,.4\rangle},[0.24,0.34]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.8,.5\rangle},[0.34,0.44]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.8,.4\rangle},[0.44,0.54]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.8,.5\rangle},[0.24,0.64]\right\rangle\right\}\right),\end{array}\right\}$$

3.4 Hybrid Structures of Picture Fuzzy Hypersoft Sets with Possibility Settings

Definition 3.4.1: **Possibility Picture Fuzzy Hypersoft Set of Type-1**

A possibility picture fuzzy hypersoft set of type-1 δ_1 is defined as

$$\delta_1 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{PF}(\hat{\theta})(\hat{u})}, \mu_F(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{PF}(\hat{\theta})(\hat{u}) = \left\langle T_\psi(\hat{\theta})(\hat{u}), \mathrm{I}_\psi(\hat{\theta})(\hat{u}), \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{PF}$ and $\mathrm{T}_\psi(\hat{\theta})(\hat{u}), \mathrm{I}_\psi(\hat{\theta})(\hat{u}), \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \in [0,1]$ such that $0 \le \mathrm{T}_\psi(\hat{\theta})(\hat{u}) + I_\psi(\hat{\theta})(\hat{u}) + F_\psi(\hat{\theta})(\hat{u}) \le 1$. Similarly $\mu_F(\hat{\theta}) \subseteq \mathfrak{A}^F$ with $\mu_F(\hat{\theta}) \in [0,1]$.

Example 3.4.1: Considering the assumptions from Example 3.1.1, we can construct possibility picture fuzzy hypersoft set of type-1 λ_1 can be constructed as

$$\lambda_1 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.5,.1 \rangle}, .3 \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.5,.1 \rangle}, .4 \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.4,.1 \rangle}, .5 \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.2,.1 \rangle}, .6 \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.3,.2 \rangle}, .2 \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.4,.2 \rangle}, .3 \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.2,.2 \rangle}, .4 \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.1,.2 \rangle}, .5 \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.5,.3 \rangle}, .1 \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.3,.3 \rangle}, .2 \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.1,.3 \rangle}, .3 \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.2,.2 \rangle}, .4 \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.2,.4 \rangle}, .5 \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.5,.1 \rangle}, .6 \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.4,.1 \rangle}, .7 \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.3,.1 \rangle}, .8 \right\rangle \right\} \right), \end{array} \right\}.$$

Definition 3.4.2: **Possibility Picture Fuzzy Hypersoft Set of Type-2**

A possibility picture fuzzy hypersoft set of type-2 δ_2 is defined as

$$\delta_2 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{PF}(\hat{\theta})(\hat{u})}, \mu_{IF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{PF}(\hat{\theta})(\hat{u}) = \left\langle T_\psi(\hat{\theta})(\hat{u}), \mathrm{I}_\psi(\hat{\theta})(\hat{u}), \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{PF}$ and $T_\psi(\hat{\theta})(\hat{u}), \mathrm{I}_\psi(\hat{\theta})(\hat{u}), \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \in [0,1]$, with $0 \le T_\psi(\hat{\theta})(\hat{u}) + I_\psi(\hat{\theta})(\hat{u}) + \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \le 1$. Similarly $\mu_{IF}(\hat{\theta}) \subseteq \mathfrak{A}^{IF}$, $\mu_{IF}(\hat{\theta}) = \left\langle T_\mu(\hat{\theta}), \mathrm{F}_\mu(\hat{\theta}) \right\rangle$ and $T_\mu(\hat{\theta}), \mathrm{F}_\mu(\hat{\theta}) \in [0,1]$ with $0 \le T_\mu(\hat{\theta}) + \mathrm{F}_\mu(\hat{\theta}) \le 1$.

Example 3.4.2: Considering the assumptions from Example 3.1.1, we can construct possibility picture fuzzy hypersoft set of type-2 δ_2 can be constructed as

$$\delta_2 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.4,.1\rangle}, \langle .3,.2\rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.4,.1\rangle}, \langle .4,.3\rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.4,.1\rangle}, \langle .5,.4\rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.2,.1\rangle}, \langle .6,.2\rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.3,.2\rangle}, \langle .3,.1\rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.2,.2\rangle}, \langle .4,.2\rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.4,.1\rangle}, \langle .5,.3\rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .5,.1,.2\rangle}, \langle .6,.1\rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .1,.5,.3\rangle}, \langle .3,.3\rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .1,.4,.3\rangle}, \langle .3,.4\rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .5,.1,.3\rangle}, \langle .3,.5\rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .4,.2,.3\rangle}, \langle .3,.6\rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .3,.3,.3\rangle}, \langle .3,.4\rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.5,.2\rangle}, \langle .3,.5\rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.4,.2\rangle}, \langle .3,.6\rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .3,.3,.2\rangle}, \langle .3,.3\rangle \right\rangle \right\} \right), \end{array} \right\}.$$

Definition 3.4.3: **Possibility Picture Fuzzy Hypersoft Set of Type-3**

A possibility picture fuzzy hypersoft set of type-3 δ_3 is defined as

$$\delta_3 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{PF}(\hat{\theta})(\hat{u})}, \mu_{PyF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{PF}(\hat{\theta})(\hat{u}) = \left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{PF}$ and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$ with $0 \le \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}) + I_{\psi}(\hat{\theta})(\hat{u}) + F_{\psi}(\hat{\theta})(\hat{u}) \le 1$. Similarly $\mu_{PyF}(\hat{\theta}) \subseteq \mathfrak{A}^{PyF}$ and $\mu_{PyF}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \le T_{\mu}^2(\hat{\theta}) + \mathrm{F}_{\mu}^2(\hat{\theta}) \le 1$.

Example 3.4.3: Considering the assumptions from Example 3.1.1, we can construct possibility picture fuzzy hypersoft set of type-3 δ_3 can be constructed as

$$\lambda_3=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.2,.1,.1\rangle},\langle.5,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.2,.3,.1\rangle},\langle.7,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.2,.4,.1\rangle},\langle.5,.8\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.2,.5,.1\rangle},\langle.6,.5\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.3,.1,.2\rangle},\langle.9,.4\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.3,.2,.2\rangle},\langle.4,.8\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.3,.4,.2\rangle},\langle.4,.7\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.3,.5,.1\rangle},\langle.5,.8\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.4,.1,.3\rangle},\langle.7,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.4,.2,.3\rangle},\langle.8,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.4,.3,.1\rangle},\langle.6,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.4,.4,.1\rangle},\langle.5,.6\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.5,.1,.2\rangle},\langle.4,.9\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.5,.2,.1\rangle},\langle.8,.4\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.5,.3,.1\rangle},\langle.7,.4\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.5,.2,.1\rangle},\langle.8,.5\rangle\right\rangle\right\}\right),\end{array}\right\}.$$

Definition 3.4.4: **Possibility Picture Fuzzy Hypersoft Set of Type-4**

A possibility picture fuzzy hypersoft set of type-4 δ_4 is defined as

$$\delta_4=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_{PF}(\hat{\theta})(\hat{u})},\mu_{PF}(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_{PF}(\hat{\theta})(\hat{u})=\left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}),\mathrm{I}_{\psi}(\hat{\theta})(\hat{u}),\mathrm{F}_{\psi}(\hat{\theta})(\hat{u})\right\rangle\subseteq\mathfrak{A}^{PF}$, and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}),\mathrm{I}_{\psi}(\hat{\theta})(\hat{u}),\mathrm{F}_{\psi}(\hat{\theta})(\hat{u})\in[0,1]$ with $0\le\mathrm{T}_{\psi}(\hat{\theta})(\hat{u})+I_{\psi}(\hat{\theta})(\hat{u})+F_{\psi}(\hat{\theta})(\hat{u})\le1$. Similarly $\mu_{PF}(\hat{\theta})\subseteq\mathfrak{A}^{PF}$, $\mu_{PF}(\hat{\theta})=\left\langle T_{\mu}(\hat{\theta}),\mathrm{I}_{\mu}(\hat{\theta}),\mathrm{F}_{\mu}(\hat{\theta})\right\rangle$ with $T_{\mu}(\hat{\theta}),\mathrm{I}_{\mu}(\hat{\theta}),\mathrm{F}_{\mu}(\hat{\theta})\in[0,1]$ and $0\le T_{\mu}(\hat{\theta})+I_{\mu}(\hat{\theta})+\mathrm{F}_{\mu}(\hat{\theta})\le1$.

Example 3.4.4: Considering the assumptions from Example 3.1.1, we can construct possibility picture fuzzy hypersoft set of type-4 δ_4 can be constructed as

$$\delta_4=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.2,.1,.1\rangle},\langle.5,.1,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.2,.3,.1\rangle},\langle.2,.5,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.2,.4,.1\rangle},\langle.4,.2,.3\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.2,.5,.1\rangle},\langle.6,.1,.2\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.3,.1,.2\rangle},\langle.1,.4,.4\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.3,.2,.2\rangle},\langle.3,.4,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.3,.4,.2\rangle},\langle.2,.3,.3\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.3,.5,.1\rangle},\langle.2,.4,.2\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.4,.1,.2\rangle},\langle.3,.5,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.4,.2,.1\rangle},\langle.2,.5,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.4,.3,.1\rangle},\langle.6,.2,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.4,.4,.1\rangle},\langle.5,.2,.2\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.5,.1,.2\rangle},\langle.4,.1,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.5,.2,.1\rangle},\langle.2,.2,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.5,.3,.1\rangle},\langle.7,.1,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.2,.4,.1\rangle},\langle.4,.2,.1\rangle\right\rangle\right\}\right),\end{array}\right\}$$

.

Definition 3.4.5: **Possibility Picture Fuzzy Hypersoft Set of Type-5**

A possibility picture fuzzy hypersoft set of type-5 δ_5 is defined as

$$\delta_5 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{PF}(\hat{\theta})(\hat{u})}, \mu_{svN}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{PF}(\hat{\theta})(\hat{u}) = \left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{PF}$, and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$ with $0 \le \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}) + I_{\psi}(\hat{\theta})(\hat{u}) + F_{\psi}(\hat{\theta})(\hat{u}) \le 1$. Similarly $\mu_{svN}(\hat{\theta}) \subseteq \mathfrak{A}^{svN}$, $\mu_{svN}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \le T_{\mu}(\hat{\theta}) + I_{\mu}(\hat{\theta}) + \mathrm{F}_{\mu}(\hat{\theta}) \le 3$.

Example 3.4.5: Considering the assumptions from Example 3.1.1, we can construct possibility picture fuzzy hypersoft set of type-5 δ_5 can be constructed as

$$\delta_5 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.1,.1 \rangle}, \langle .5,.6,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .2,.3,.1 \rangle}, \langle .6,.7,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .2,.4,.1 \rangle}, \langle .7,.8,.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .2,.5,.1 \rangle}, \langle .8,.9,.9 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .3,.1,.2 \rangle}, \langle .6,.5,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.2,.2 \rangle}, \langle .7,.6,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .3,.4,.2 \rangle}, \langle .8,.7,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .3,.5,.1 \rangle}, \langle .9,.8,.8 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .4,.1,.3 \rangle}, \langle .7,.6,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .4,.2,.3 \rangle}, \langle .8,.7,.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.3,.2 \rangle}, \langle .9,.8,.9 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .4,.4,.1 \rangle}, \langle .6,.5,.9 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .5,.1,.2 \rangle}, \langle .8,.7,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .5,.2,.1 \rangle}, \langle .9,.6,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .5,.3,.1 \rangle}, \langle .6,.7,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .3,.4,.1 \rangle}, \langle .6,.8,.8 \rangle \right\rangle \right\} \right), \end{array} \right\}$$

.

Definition 3.4.6: **Possibility Picture Fuzzy Hypersoft Set of Type-6**

A possibility picture fuzzy hypersoft set of type-6 δ_6 is defined as

$$\delta_6 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{PF}(\hat{\theta})(\hat{u})}, \mu_{IVF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{PF}(\hat{\theta})(\hat{u}) = \left\langle T_{\psi}(\hat{\theta})(\hat{u}), \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{PF}$ and $T_{\psi}(\hat{\theta})(\hat{u}), \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$ such that $0 \le T_{\psi}(\hat{\theta})(\hat{u}) + \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}) + F_{\psi}(\hat{\theta})(\hat{u}) \le 1$ .Similarly $\mu_{IVF}(\hat{\theta}) \subseteq \mathfrak{A}^{IVF}$ with $\mu_{IVF}(\hat{\theta}) = \left[L_{\mu}(\hat{\theta}), \mathrm{U}_{\mu}(\hat{\theta}) \right]$ and $L_{\mu}(\hat{\theta}), \mathrm{U}_{\mu}(\hat{\theta}) \in [0,1]$.

Example 3.4.6: Considering the assumptions from Example 3.1.1, we can construct possibility picture fuzzy hypersoft set of type-6 δ_6 can be constructed as

$$\delta_6 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .2,.1,.1 \rangle}, [.2,.3] \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .2,.3,.1 \rangle}, [.3,.4] \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .2,.4,.1 \rangle}, [.4,.5] \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .2,.5,.1 \rangle}, [.5,.6] \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .3,.1,.2 \rangle}, [.6,.7] \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .3,.2,.2 \rangle}, [.7,.8] \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .3,.4,.2 \rangle}, [.8,.9] \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .3,.5,.1 \rangle}, [.2,.4] \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .4,.1,.2 \rangle}, [.3,.5] \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .4,.2,.2 \rangle}, [.4,.6] \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .4,.3,.1 \rangle}, [.5,.7] \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .4,.4,.1 \rangle}, [.6,.8] \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .5,.1,.1 \rangle}, [.7,.9] \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .5,.2,.1 \rangle}, [.2,.5] \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .5,.3,.1 \rangle}, [.3,.6] \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .2,.4,.1 \rangle}, [.4,.7] \right\rangle \right\} \right), \end{array} \right\}$$

3.5 Hybrid Structures of sv-Neutrosophic Hypersoft Sets with Possibility Settings

Definition 3.5.1: **Possibility sv-Neutrosophic Fuzzy Hypersoft Set of Type-1**

A possibility sv-neutrosophic hypersoft set of type-1 ϖ_1 is defined as

$$\varpi_1 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{svN}(\hat{\theta})(\hat{u})}, \mu_F(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{svN}(\hat{\theta})(\hat{u}) = \left\langle T_\psi(\hat{\theta})(\hat{u}), \mathrm{I}_\psi(\hat{\theta})(\hat{u}), \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{svN}$ and $\mathrm{T}_\psi(\hat{\theta})(\hat{u}), \mathrm{I}_\psi(\hat{\theta})(\hat{u}), \mathrm{F}_\psi(\hat{\theta})(\hat{u}) \in [0,1]$ such that $0 \le \mathrm{T}_\psi(\hat{\theta})(\hat{u}) + I_\psi(\hat{\theta})(\hat{u}) + F_\psi(\hat{\theta})(\hat{u}) \le 3$. Similarly $\mu_F(\hat{\theta}) \subseteq \mathfrak{A}^F$ with $\mu_F(\hat{\theta}) \in [0,1]$.

Example 3.5.1: Considering the assumptions from Example 3.1.1, we can construct possibility sv-neutrosophic hypersoft set of type-1 ϖ_1 can be constructed as

$$\varpi_1=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.8,.8,.9\rangle},.3\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.8,.8,.8\rangle},.4\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.8,.8,.7\rangle},.5\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.8,.8,.6\rangle},.6\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.7,.7,.9\rangle},.2\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.7,.7,.8\rangle},.3\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.7,.7,.7\rangle},.4\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.7,.7,.6\rangle},.5\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.6,.6,.9\rangle},.1\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.6,.6,.8\rangle},.2\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.6,.6,.7\rangle},.3\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.6,.9,.6\rangle},.4\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.9,.9,.8\rangle},.5\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.9,.9,.7\rangle},.6\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.9,.9,.6\rangle},.7\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.9,.9,.5\rangle},.8\right\rangle\right\}\right),\end{array}\right\}.$$

Definition 3.5.2: **Possibility sv-Neutrosophic Hypersoft Set of Type-2**

A possibility sv-neutrosophic hypersoft set of type-2 ϖ_2 is defined as

$$\varpi_2=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_{svN}(\hat{\theta})(\hat{u})},\mu_{IF}(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_{svN}(\hat{\theta})(\hat{u})=\left\langle T_\psi(\hat{\theta})(\hat{u}),\mathrm{I}_\psi(\hat{\theta})(\hat{u}),\mathrm{F}_\psi(\hat{\theta})(\hat{u})\right\rangle\subseteq\mathfrak{A}^{svN}$ and $T_\psi(\hat{\theta})(\hat{u}),\mathrm{I}_\psi(\hat{\theta})(\hat{u}),\mathrm{F}_\psi(\hat{\theta})(\hat{u})\in[0,1]$, with $0\le T_\psi(\hat{\theta})(\hat{u})+I_\psi(\hat{\theta})(\hat{u})+\mathrm{F}_\psi(\hat{\theta})(\hat{u})\le 3$. Similarly $\mu_{IF}(\hat{\theta})\subseteq\mathfrak{A}^{IF}$, $\mu_{IF}(\hat{\theta})=\left\langle T_\mu(\hat{\theta}),\mathrm{F}_\mu(\hat{\theta})\right\rangle$ and $T_\mu(\hat{\theta}),\mathrm{F}_\mu(\hat{\theta})\in[0,1]$ with $0\le T_\mu(\hat{\theta})+\mathrm{F}_\mu(\hat{\theta})\le 1$.

Example 3.5.2: Considering the assumptions from Example 3.1.1, we can construct possibility sv-neutrosophic hypersoft set of type-2 ϖ_2 can be constructed as

$$\varpi_2=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.8,.8,.9\rangle},\langle.3,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.8,.8,.8\rangle},\langle.4,.3\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.8,.8,.7\rangle},\langle.5,.4\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.8,.8,.6\rangle},\langle.6,.2\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.7,.7,.9\rangle},\langle.3,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.7,.7,.8\rangle},\langle.4,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.7,.7,.7\rangle},\langle.5,.3\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.7,.7,.6\rangle},\langle.6,.1\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.6,.6,.9\rangle},\langle.3,.3\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.6,.6,.8\rangle},\langle.3,.4\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.6,.6,.7\rangle},\langle.3,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.6,.9,.6\rangle},\langle.3,.6\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.9,.9,.8\rangle},\langle.3,.4\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.9,.9,.7\rangle},\langle.3,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.9,.9,.6\rangle},\langle.3,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.9,.9,.5\rangle},\langle.3,.3\rangle\right\rangle\right\}\right),\end{array}\right\}$$

.

Definition 3.5.3: **Possibility sv-Neutrosophic Hypersoft Set of Type-3**

A possibility sv-neutrosophic hypersoft set of type-3 ϖ_3 is defined as

$$\varpi_3 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{svN}(\hat{\theta})(\hat{u})}, \mu_{PyF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{svN}(\hat{\theta})(\hat{u}) = \left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{svN}$ and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$ with $0 \leq \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}) + I_{\psi}(\hat{\theta})(\hat{u}) + F_{\psi}(\hat{\theta})(\hat{u}) \leq 3$. Similarly $\mu_{PyF}(\hat{\theta}) \subseteq \mathfrak{A}^{PyF}$ and $\mu_{PyF}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \leq T_{\mu}^2(\hat{\theta}) + \mathrm{F}_{\mu}^2(\hat{\theta}) \leq 1$.

Example 3.5.3: Considering the assumptions from Example 3.1.1, we can construct possibility sv-neutrosophic hypersoft set of type-3 ϖ_3 can be constructed as

$$\varpi_3 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{\langle .8,.8,.9 \rangle}, \langle .5,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .8,.8,.8 \rangle}, \langle .7,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .8,.8,.7 \rangle}, \langle .5,.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .8,.8,.6 \rangle}, \langle .6,.5 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{\langle .7,.7,.9 \rangle}, \langle .9,.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .7,.7,.8 \rangle}, \langle .4,.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .7,.7,.7 \rangle}, \langle .4,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .7,.7,.6 \rangle}, \langle .5,.8 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{\langle .6,.6,.9 \rangle}, \langle .7,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .6,.6,.8 \rangle}, \langle .8,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .6,.6,.7 \rangle}, \langle .6,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .6,.9,.6 \rangle}, \langle .5,.6 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{\langle .9,.9,.8 \rangle}, \langle .4,.9 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{\langle .9,.9,.7 \rangle}, \langle .8,.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{\langle .9,.9,.6 \rangle}, \langle .7,.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{\langle .9,.9,.5 \rangle}, \langle .8,.5 \rangle \right\rangle \right\} \right), \end{array} \right\}$$

.

Definition 3.5.4: **Possibility sv-Neutrosophic Hypersoft Set of Type-4**

A possibility sv-neutrosophic hypersoft set of type-4 ϖ_4 is defined as

$$\varpi_4 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{svN}(\hat{\theta})(\hat{u})}, \mu_{PF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{svN}(\hat{\theta})(\hat{u}) = \left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \right\rangle \subseteq \mathfrak{A}^{svN}$ and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{I}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$ with $0 \leq \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}) + I_{\psi}(\hat{\theta})(\hat{u}) + F_{\psi}(\hat{\theta})(\hat{u}) \leq 3$. Similarly $\mu_{PF}(\hat{\theta}) \subseteq \mathfrak{A}^{PF}$, $\mu_{PF}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \leq T_{\mu}(\hat{\theta}) + I_{\mu}(\hat{\theta}) + \mathrm{F}_{\mu}(\hat{\theta}) \leq 1$.

Example 3.5.4: Considering the assumptions from Example 3.1.1, we can construct possibility sv-neutrosophic hypersoft set of type-4 ϖ_4 can be constructed as

$$\varpi_4=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.8,.8,.9\rangle},\langle.5,.1,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.8,.8,.8\rangle},\langle.2,.5,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.8,.8,.7\rangle},\langle.4,.2,.3\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.8,.8,.6\rangle},\langle.6,.1,.2\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.7,.7,.9\rangle},\langle.1,.4,.4\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.7,.7,.8\rangle},\langle.3,.4,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.7,.7,.7\rangle},\langle.2,.3,.3\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.7,.7,.6\rangle},\langle.2,.4,.2\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.6,.6,.9\rangle},\langle.3,.5,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.6,.6,.8\rangle},\langle.2,.5,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.6,.6,.7\rangle},\langle.6,.2,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.6,.9,.6\rangle},\langle.5,.2,.2\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.9,.9,.8\rangle},\langle.4,.1,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.9,.9,.7\rangle},\langle.2,.2,.2\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.9,.9,.6\rangle},\langle.7,.1,.1\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.9,.9,.5\rangle},\langle.4,.2,.1\rangle\right\rangle\right\}\right),\end{array}\right\}$$

.

Definition 3.5.5: **Possibility sv-Neutrosophic Hypersoft Set of Type-5**

A possibility sv-neutrosophic hypersoft set of type-5 ϖ_5 is defined as

$$\varpi_5=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_{svN}(\hat{\theta})(\hat{u})},\mu_{svN}(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_{svN}(\hat{\theta})(\hat{u})=\left\langle \mathrm{T}_{\psi}(\hat{\theta})(\hat{u}),\mathrm{I}_{\psi}(\hat{\theta})(\hat{u}),\mathrm{F}_{\psi}(\hat{\theta})(\hat{u})\right\rangle\subseteq\mathfrak{A}^{svN}$ and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}),\mathrm{I}_{\psi}(\hat{\theta})(\hat{u}),\mathrm{F}_{\psi}(\hat{\theta})(\hat{u})\in[0,1]$ with $0\le \mathrm{T}_{\psi}(\hat{\theta})(\hat{u})+I_{\psi}(\hat{\theta})(\hat{u})+F_{\psi}(\hat{\theta})(\hat{u})\le 3$. Similarly $\mu_{svN}(\hat{\theta})\subseteq\mathfrak{A}^{svN}$, $\mu_{svN}(\hat{\theta})=\left\langle T_{\mu}(\hat{\theta}),\mathrm{I}_{\mu}(\hat{\theta}),\mathrm{F}_{\mu}(\hat{\theta})\right\rangle$ with $T_{\mu}(\hat{\theta}),\mathrm{I}_{\mu}(\hat{\theta}),\mathrm{F}_{\mu}(\hat{\theta})\in[0,1]$ and $0\le T_{\mu}(\hat{\theta})+I_{\mu}(\hat{\theta})+\mathrm{F}_{\mu}(\hat{\theta})\le 3$.

Example 3.5.5: Considering the assumptions from Example 3.1.1, we can construct possibility sv-neutrosophic hypersoft set of type-5 ϖ_5 can be constructed as

$$\varpi_5=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.8,.8,.9\rangle},\langle.5,.6,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.8,.8,.8\rangle},\langle.6,.7,.7\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.8,.8,.7\rangle},\langle.7,.8,.8\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.8,.8,.6\rangle},\langle.8,.9,.9\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.7,.7,.9\rangle},\langle.6,.5,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.7,.7,.8\rangle},\langle.7,.6,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.7,.7,.7\rangle},\langle.8,.7,.7\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.7,.7,.6\rangle},\langle.9,.8,.8\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.6,.6,.9\rangle},\langle.7,.6,.7\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.6,.6,.8\rangle},\langle.8,.7,.8\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.6,.6,.7\rangle},\langle.9,.8,.9\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.6,.9,.6\rangle},\langle.6,.5,.9\rangle\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.9,.9,.8\rangle},\langle.8,.7,.6\rangle\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.9,.9,.7\rangle},\langle.9,.6,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.9,.9,.6\rangle},\langle.6,.7,.5\rangle\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.9,.9,.5\rangle},\langle.6,.8,.8\rangle\right\rangle\right\}\right),\end{array}\right\}$$

.

Definition 3.5.6: **Possibility sv-Neutrosophic Hypersoft Set of Type-6**

A possibility sv-neutrosophic hypersoft set of type-6 ϖ_6 is defined as

$$\delta_6=\left\{\left(\hat{\theta},\left\langle\frac{\hat{u}}{\psi_{PF}(\hat{\theta})(\hat{u})},\mu_{IVF}(\hat{\theta})\right\rangle\right):\hat{u}\in\hat{U}\wedge\hat{\theta}\in\hat{\Theta}\right\}$$

where $\psi_{svN}(\hat{\theta})(\hat{u})=\left\langle T_\psi(\hat{\theta})(\hat{u}),\mathrm{I}_\psi(\hat{\theta})(\hat{u}),\mathrm{F}_\psi(\hat{\theta})(\hat{u})\right\rangle\subseteq\mathfrak{A}^{svN}$ and $T_\psi(\hat{\theta})(\hat{u}),\mathrm{I}_\psi(\hat{\theta})(\hat{u}),\mathrm{F}_\psi(\hat{\theta})(\hat{u})\in[0,1]$ such that $0\le T_\psi(\hat{\theta})(\hat{u})+\mathrm{I}_\psi(\hat{\theta})(\hat{u})+F_\psi(\hat{\theta})(\hat{u})\le 3$ .Similarly $\mu_{IVF}(\hat{\theta})\subseteq\mathfrak{A}^{IVF}$ with $\mu_{IVF}(\hat{\theta})=\left[L_\mu(\hat{\theta}),\mathrm{U}_\mu(\hat{\theta})\right]$ and $L_\mu(\hat{\theta}),\mathrm{U}_\mu(\hat{\theta})\in[0,1]$.

Example 3.5.6: Considering the assumptions from Example 3.1.1, we can construct possibility sv-neutrosophic hypersoft set of type-6 ϖ_6 can be constructed as

$$\varpi_6=\left\{\begin{array}{l}\left(\hat{\theta}_1,\left\{\left\langle\frac{\hat{u}_1}{\langle.8,.8,.9\rangle},[.2,.3]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.8,.8,.8\rangle},[.3,.4]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.8,.8,.7\rangle},[.4,.5]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.8,.8,.6\rangle},[.5,.6]\right\rangle\right\}\right),\\ \left(\hat{\theta}_2,\left\{\left\langle\frac{\hat{u}_1}{\langle.7,.7,.9\rangle},[.6,.7]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.7,.7,.8\rangle},[.7,.8]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.7,.7,.7\rangle},[.8,.9]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.7,.7,.6\rangle},[.2,.4]\right\rangle\right\}\right),\\ \left(\hat{\theta}_3,\left\{\left\langle\frac{\hat{u}_1}{\langle.6,.6,.9\rangle},[.3,.5]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.6,.6,.8\rangle},[.4,.6]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.6,.6,.7\rangle},[.5,.7]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.6,.9,.6\rangle},[.6,.8]\right\rangle\right\}\right),\\ \left(\hat{\theta}_4,\left\{\left\langle\frac{\hat{u}_1}{\langle.9,.9,.8\rangle},[.7,.9]\right\rangle,\left\langle\frac{\hat{u}_2}{\langle.9,.9,.7\rangle},[.2,.5]\right\rangle,\left\langle\frac{\hat{u}_3}{\langle.9,.9,.6\rangle},[.3,.6]\right\rangle,\left\langle\frac{\hat{u}_4}{\langle.9,.9,.5\rangle},[.4,.7]\right\rangle\right\}\right),\end{array}\right\}$$

3.6 Hybrid Structures of Interval-valued Fuzzy Hypersoft Sets with Possibility Settings

Definition 3.6.1: **Possibility Interval-valued Fuzzy Hypersoft Set of Type-1**

A possibility interval-valued fuzzy hypersoft set of type-1 π_1 is defined as

$$\pi_1 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IVF}(\hat{\theta})(\hat{u})}, \mu_F(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IVF}(\hat{\theta})(\hat{u}) = \left[L_\psi(\hat{\theta})(\hat{u}), \mathrm{U}_\psi(\hat{\theta})(\hat{u}) \right] \subseteq \mathfrak{A}^{IVF}$ and $L_\psi(\hat{\theta})(\hat{u}), \mathrm{U}_\psi(\hat{\theta})(\hat{u}) \in [0,1]$. Similarly $\mu_F(\hat{\theta}) \subseteq \mathfrak{A}^F$ with $\mu_F(\hat{\theta}) \in [0,1]$.

Example 3.6.1: Considering the assumptions from Example 3.1.1, we can construct possibility interval-valued fuzzy hypersoft set of type-1 π_1 can be constructed as

$$\pi_1 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{[.2,.3]}, 0.31 \right\rangle, \left\langle \frac{\hat{u}_2}{[.2,.4]}, 0.41 \right\rangle, \left\langle \frac{\hat{u}_3}{[.2,.5]}, 0.51 \right\rangle, \left\langle \frac{\hat{u}_4}{[.2,.6]}, 0.61 \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{[.3,.4]}, 0.32 \right\rangle, \left\langle \frac{\hat{u}_2}{[.3,.5]}, 0.42 \right\rangle, \left\langle \frac{\hat{u}_3}{[.3,.6]}, 0.52 \right\rangle, \left\langle \frac{\hat{u}_4}{[.3,.7]}, 0.62 \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{[.4,.5]}, 0.33 \right\rangle, \left\langle \frac{\hat{u}_2}{[.4,.6]}, 0.43 \right\rangle, \left\langle \frac{\hat{u}_3}{[.4,.7]}, 0.53 \right\rangle, \left\langle \frac{\hat{u}_4}{[.4,.8]}, 0.63 \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{[.5,.6]}, 0.34 \right\rangle, \left\langle \frac{\hat{u}_2}{[.5,.7]}, 0.44 \right\rangle, \left\langle \frac{\hat{u}_3}{[.5,.8]}, 0.54 \right\rangle, \left\langle \frac{\hat{u}_4}{[.5,.9]}, 0.64 \right\rangle \right\} \right), \end{array} \right\}.$$

Definition 3.6.2: **Possibility Interval-valued Fuzzy Hypersoft Set of Type-2**

A possibility interval-valued fuzzy hypersoft set of type-2 π_2 is defined as

$$\pi_2 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IVF}(\hat{\theta})(\hat{u})}, \mu_{IF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IVF}(\hat{\theta})(\hat{u}) = \left[L_\psi(\hat{\theta})(\hat{u}), \mathrm{U}_\psi(\hat{\theta})(\hat{u}) \right] \subseteq \mathfrak{A}^{IVF}$ and $L_\psi(\hat{\theta})(\hat{u}), \mathrm{U}_\psi(\hat{\theta})(\hat{u}) \in [0,1]$. Similarly $\mu_{IF}(\hat{\theta}) \subseteq \mathfrak{A}^{IF}$ with $\mu_{IF}(\hat{\theta}) = \left\langle T_\mu(\hat{\theta}), \mathrm{F}_\mu(\hat{\theta}) \right\rangle, T_\mu(\hat{\theta}), \mathrm{F}_\mu(\hat{\theta}) \in [0,1]$ such that $0 \le T_\mu(\hat{\theta}) + \mathrm{F}_\mu(\hat{\theta}) \le 1$.

Example 3.6.2: Considering the assumptions from Example 3.1.1, we can construct possibility interval-valued fuzzy hypersoft set of type-2 π_2 can be constructed as

$$\pi_2 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{[.2,.3]}, \langle .31,.21 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.2,.4]}, \langle .41,.31 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.2,.5]}, \langle .51,.41 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.2,.6]}, \langle .61,.21 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{[.3,.4]}, \langle .32,.22 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.3,.5]}, \langle .42,.32 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.3,.6]}, \langle .52,.42 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.3,.7]}, \langle .62,.22 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{[.4,.5]}, \langle .33,.23 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.4,.6]}, \langle .43,.33 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.4,.7]}, \langle .53,.43 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.4,.8]}, \langle .63,.23 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{[.5,.6]}, \langle .34,.24 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.5,.7]}, \langle .44,.34 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.5,.8]}, \langle .54,.44 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.5,.9]}, \langle .64,.24 \rangle \right\rangle \right\} \right), \end{array} \right\}.$$

Definition 3.6.3: **Possibility Interval-valued Fuzzy Hypersoft Set of Type-3**

A possibility interval-valued fuzzy hypersoft set of type-3 π_3 is defined as

$$\pi_3 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IVF}(\hat{\theta})(\hat{u})}, \mu_{PyF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IVF}(\hat{\theta})(\hat{u}) = \left[L_{\psi}(\hat{\theta})(\hat{u}), \mathrm{U}_{\psi}(\hat{\theta})(\hat{u}) \right] \subseteq \mathfrak{A}^{IVF}$ and $\mathrm{T}_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$. Similarly $\mu_{PyF}(\hat{\theta}) \subseteq \mathfrak{A}^{PyF}$ with $\mu_{PyF}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$, $T_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \le T_{\mu}^2(\hat{\theta}) + \mathrm{F}_{\mu}^2(\hat{\theta}) \le 1$.

Example 3.6.3: Considering the assumptions from Example 3.1.1, we can construct possibility interval-valued fuzzy hypersoft set of type-3 π_3 can be constructed as

$$\pi_3 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{[.2,.3]}, \langle .5,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.2,.4]}, \langle .7,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.2,.5]}, \langle .5,.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.2,.6]}, \langle .6,.5 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{[.3,.4]}, \langle .9,.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.3,.5]}, \langle .4,.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.3,.6]}, \langle .4,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.3,.7]}, \langle .5,.8 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{[.4,.5]}, \langle .7,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.4,.6]}, \langle .8,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.4,.7]}, \langle .6,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.4,.8]}, \langle .5,.6 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{[.5,.6]}, \langle .4,.9 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.5,.7]}, \langle .8,.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.5,.8]}, \langle .7,.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.5,.9]}, \langle .8,.5 \rangle \right\rangle \right\} \right), \end{array} \right\}.$$

Definition 3.6.4: **Possibility Interval-valued Fuzzy Hypersoft Set of Type-4**

A possibility interval-valued fuzzy hypersoft set of type-4 π_4 is defined as

$$\pi_4 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IVF}(\hat{\theta})(\hat{u})}, \mu_{PF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IVF}(\hat{\theta})(\hat{u}) = \left[L_{\psi}(\hat{\theta})(\hat{u}), \mathrm{U}_{\psi}(\hat{\theta})(\hat{u}) \right] \subseteq \mathfrak{A}^{IVF}$, and $L_{\psi}(\hat{\theta})(\hat{u}), \mathrm{U}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$. Similarly $\mu_{PF}(\hat{\theta}) \subseteq \mathfrak{A}^{PF}$, $\mu_{PF}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \leq T_{\mu}(\hat{\theta}) + I_{\mu}(\hat{\theta}) + \mathrm{F}_{\mu}(\hat{\theta}) \leq 1$.

Example 3.6.4: Considering the assumptions from Example 3.1.1, we can construct possibility interval-valued fuzzy hypersoft set of type-4 π_4 can be constructed as

$$\pi_4 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{[.2,.3]}, \langle .5,.1,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.2,.4]}, \langle .2,.5,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.2,.5]}, \langle .4,.2,.3 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.2,.6]}, \langle .6,.1,.2 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{[.3,.4]}, \langle .1,.4,.4 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.3,.5]}, \langle .3,.4,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.3,.6]}, \langle .2,.3,.3 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.3,.7]}, \langle .2,.4,.2 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{[.4,.5]}, \langle .3,.5,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.4,.6]}, \langle .2,.5,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.4,.7]}, \langle .6,.2,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.4,.8]}, \langle .5,.2,.2 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{[.5,.6]}, \langle .4,.1,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.5,.7]}, \langle .2,.2,.2 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.5,.8]}, \langle .7,.1,.1 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.5,.9]}, \langle .4,.2,.1 \rangle \right\rangle \right\} \right), \end{array} \right\}.$$

Definition 3.6.5: **Possibility Interval-valued Fuzzy Hypersoft Set of Type-5**

A possibility interval-valued fuzzy hypersoft set of type-5 π_5 is defined as

$$\pi_5 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IVF}(\hat{\theta})(\hat{u})}, \mu_{svN}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IVF}(\hat{\theta})(\hat{u}) = \left[L_{\psi}(\hat{\theta})(\hat{u}), \mathrm{U}_{\psi}(\hat{\theta})(\hat{u}) \right] \subseteq \mathfrak{A}^{IVF}$, and $L_{\psi}(\hat{\theta})(\hat{u}), \mathrm{U}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$. Similarly $\mu_{svN}(\hat{\theta}) \subseteq \mathfrak{A}^{svN}$, $\mu_{svN}(\hat{\theta}) = \left\langle T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \right\rangle$ with $T_{\mu}(\hat{\theta}), \mathrm{I}_{\mu}(\hat{\theta}), \mathrm{F}_{\mu}(\hat{\theta}) \in [0,1]$ and $0 \leq T_{\mu}(\hat{\theta}) + I_{\mu}(\hat{\theta}) + \mathrm{F}_{\mu}(\hat{\theta}) \leq 3$.

Example 3.6.5: Considering the assumptions from Example 3.1.1, we can construct possibility interval-valued fuzzy hypersoft set of type-5 π_5 can be constructed as

$$\pi_5 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{[.2,.3]}, \langle .5,.6,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.2,.4]}, \langle .6,.7,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.2,.5]}, \langle .7,.8,.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.2,.6]}, \langle .8,.9,.9 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{[.3,.4]}, \langle .6,.5,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.3,.5]}, \langle .7,.6,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.3,.6]}, \langle .8,.7,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.3,.7]}, \langle .9,.8,.8 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{[.4,.5]}, \langle .7,.6,.7 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.4,.6]}, \langle .8,.7,.8 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.4,.7]}, \langle .9,.8,.9 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.4,.8]}, \langle .6,.5,.9 \rangle \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{[.5,.6]}, \langle .8,.7,.6 \rangle \right\rangle, \left\langle \frac{\hat{u}_2}{[.5,.7]}, \langle .9,.6,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_3}{[.5,.8]}, \langle .6,.7,.5 \rangle \right\rangle, \left\langle \frac{\hat{u}_4}{[.5,.9]}, \langle .6,.8,.8 \rangle \right\rangle \right\} \right), \end{array} \right\}.$$

Definition 3.6.6: **Possibility Interval-valued Fuzzy Hypersoft Set of Type-6**

A possibility interval-valued fuzzy hypersoft set of type-6 π_6 is defined as

$$\pi_6 = \left\{ \left(\hat{\theta}, \left\langle \frac{\hat{u}}{\psi_{IVF}(\hat{\theta})(\hat{u})}, \mu_{IVF}(\hat{\theta}) \right\rangle \right) : \hat{u} \in \hat{U} \wedge \hat{\theta} \in \hat{\Theta} \right\}$$

where $\psi_{IVF}(\hat{\theta})(\hat{u}) = \left[L_{\psi}(\hat{\theta})(\hat{u}), \mathrm{U}_{\psi}(\hat{\theta})(\hat{u}) \right] \subseteq \mathfrak{A}^{IVF}$ and $T_{\psi}(\hat{\theta})(\hat{u}), \mathrm{F}_{\psi}(\hat{\theta})(\hat{u}) \in [0,1]$. Similarly $\mu_{IVF}(\hat{\theta}) \subseteq \mathfrak{A}^{IVF}$ with $\mu_{IVF}(\hat{\theta}) = \left[L_{\mu}(\hat{\theta}), \mathrm{U}_{\mu}(\hat{\theta}) \right]$ and $L_{\mu}(\hat{\theta}), \mathrm{U}_{\mu}(\hat{\theta}) \in [0,1]$.

Example 3.6.6: Considering the assumptions from Example 3.1.1, we can construct possibility interval-valued fuzzy hypersoft set of type-6 π_6 can be constructed as

$$\pi_6 = \left\{ \begin{array}{l} \left(\hat{\theta}_1, \left\{ \left\langle \frac{\hat{u}_1}{[.2,.3]}, [.21,.31] \right\rangle, \left\langle \frac{\hat{u}_2}{[.2,.4]}, [.31,.41] \right\rangle, \left\langle \frac{\hat{u}_3}{[.2,.5]}, [.41,.51] \right\rangle, \left\langle \frac{\hat{u}_4}{[.2,.6]}, [.21,.61] \right\rangle \right\} \right), \\ \left(\hat{\theta}_2, \left\{ \left\langle \frac{\hat{u}_1}{[.3,.4]}, [.22,.32] \right\rangle, \left\langle \frac{\hat{u}_2}{[.3,.5]}, [.32,.42] \right\rangle, \left\langle \frac{\hat{u}_3}{[.3,.6]}, [.42,.52] \right\rangle, \left\langle \frac{\hat{u}_4}{[.3,.7]}, [.22,.62] \right\rangle \right\} \right), \\ \left(\hat{\theta}_3, \left\{ \left\langle \frac{\hat{u}_1}{[.4,.5]}, [.23,.33] \right\rangle, \left\langle \frac{\hat{u}_2}{[.4,.6]}, [.33,.43] \right\rangle, \left\langle \frac{\hat{u}_3}{[.4,.7]}, [.43,.53] \right\rangle, \left\langle \frac{\hat{u}_4}{[.4,.8]}, [.23,.63] \right\rangle \right\} \right), \\ \left(\hat{\theta}_4, \left\{ \left\langle \frac{\hat{u}_1}{[.5,.6]}, [.24,.34] \right\rangle, \left\langle \frac{\hat{u}_2}{[.5,.7]}, [.34,.44] \right\rangle, \left\langle \frac{\hat{u}_3}{[.5,.8]}, [.44,.54] \right\rangle, \left\langle \frac{\hat{u}_4}{[.5,.9]}, [.24,.64] \right\rangle \right\} \right), \end{array} \right\}$$

4. Conclusion

The expansion of a soft set known as a hypersoft set introduces the idea of a hypersoft membership function, enabling it to handle complex and uncertain information in a more powerful and flexible way. In this chapter, several settings that are similar to fuzzy sets are considered, along with settings based on possibility degree. Numerical examples are also provided to help explain the principle behind these structures. This work can be used by researchers to comprehend and apply a range of mathematical concepts. The proposed structures can be used for developing various algebraic and topological structures.

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Decision Making Based on Valued Fuzzy Superhypergraphs

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ABSTRACT

This paper explores the defects in fuzzy (hyper) graphs (as complex (hyper) networks) and extends the fuzzy (hyper) graphs to fuzzy (quasi) superhypergraphs as a new concept. We have modeled the fuzzy superhypergraphs as complex superhypernetworks in order to make a relation between labeled objects in the form of details and generalities. Indeed, the structure of fuzzy (quasi) superhypergraphs collects groups of labeled objects and analyzes them in the form of the part to part of objects, the part of objects to the whole group of objects, and the whole to the whole group of objects at the same time. We have investigated the properties of fuzzy (quasi) superhypergraphs based on any positive real number as valued fuzzy (quasi) superhypergraphs, considering the complement of valued fuzzy (quasi) superhypergraphs, the notation of isomorphism of valued fuzzy (quasi) superhypergraphs based on the permutations, and we have presented the isomorphic conditions of (self complemented) valued fuzzy (quasi) superhypergraphs. The concept of impact membership value of fuzzy (quasi) superhypergraphs is introduced in this study and it is applied in designing the real problem in the real world. Finally, the problem of business superhypernetworks is presented as an application of fuzzy valued quasi superhypergraphs in the real world.

KEYWORDS

(Quasi) superhypergraph; valued fuzzy (quasi) superhypergraph; impact membership value; complex superhypernetwork

1 Introduction

The theory of hypergraph as a model of hypernetwork has been introduced by Berge as a generalization of graph theory in 1960 [1]. A graph cannot connect more than two elements, which is a defect when discussing the connection between a group of elements is considered. A hypergraph is a generalization of a graph such that its edge can relate any number of vertices, in which a graph, and its edge connects exactly two vertices. Indeed, a hypergraph is a useful tool to analyze the structure of a system that connects a group of elements (more than two elements) together, and in this sense, it can play an important role in embedding, classifying, partitioning, covering, and clustering elements in different classes. Therefore, a hypergraph as an extension and covering of the graph has attracted the attention of many researchers, and this theory has spread rapidly, especially since it has many applications in real world (its uses in model gene interactions (ER-type hypergraph [2]), machine

learning (spectral hypergraph [3]), computer networks (WIS hypergraph [4]), chemistry (molecular hypergraph [5]), visual classification (hypergraph-induced convolutional network [6]), and social media (soft hypergraph [7])). Because of the more important and updated hypergraphs, a hypergraph structure as a hypernetwork has various applications, among which it can refer to updated works such as soft hypergraph for modeling global interactions via social media networks [7], session-based recommendation with hypergraph convolutional networks and sequential information embeddings [8], hypergraph-based analysis and design of intelligent collaborative manufacturing space [9] and hypergraph-based centrality metrics for maritime container service networks: a worldwide application [10]. In classical set theory, the mathematical concepts introduce purely, and without any quality or criteria, it is not attractive to be used in the world. Zadeh introduced the concept of fuzzy set theory as a generalization of set theory to deal with uncertainties [11]. Furthermore, the Plithogenic set (as a generalization of crisp, fuzzy, Intuitionistic fuzzy, and neutrosophic sets) was introduced by Smarandache in 2017. The Plithogenic set is a set whose elements are characterized by attribute values. Recently, Edalatpanah et al. considered new and recent developments in methodologies, techniques, and applications of Neutrosophic and Plithogenic sets for various practical problems and demonstrated the challenging issues as a generalization of fuzzy sets [12]. Sometimes graphs are not able to correctly analyze many phenomena because the uncertainty of various characteristics of systems naturally exists, so fuzzy graphs can cover this defect and this is an important motivation to introduce fuzzy graph theory. After that, the fuzzy graph theory was proposed as a generalization of the graph by Rosenfeld [13], which has many applications in real world (see more details in [14]). Since a fuzzy graph as a complex graph theory gives very limited information about complex networks, so we propose that the main motivation of complex hypergraph structures is for covering fuzzy graph defects in applications. Lee-Kwang et al. [15], generalized and redefined the concept of fuzzy hypergraphs (as a useful tool for the analysis and fuzzy partition of a system) whose basic idea was given by Kaufmann [16]. Akram et al. have written a book on fuzzy hypergraph which presents the fundamental and technical concepts of fuzzy hypergraphs and explains their extensions and applications. It discusses applied generalized mathematical models of hypergraphs, including complex, intuitionistic, bipolar, m-polar fuzzy, Pythagorean, complex Pythagorean, and q-rung ortho-pair hypergraphs, as well as single-valued neutrosophic, complex neutrosophic, and bipolar neutrosophic hypergraphs. In addition, the book also sheds light on real-world applications of these hypergraphs, making it a valuable resource for students and researchers in the field of mathematics, as well as computer and social scientists [17]. There is also some research about fuzzy (hyper) graphs and their applications in complex hypernetworks, such as the implementation of single-valued neutrosophic soft hypergraphs on the human nervous system [18], decision-making methods based on fuzzy soft competition hypergraphs [19], hypergraph and network flow-based quality function deployment [20], global domination in fuzzy graphs using strong arcs [21], fuzzy hypergraph modeling, analysis and prediction of crimes [22], single-valued neutrosophic directed (hyper) graphs and applications in networks [23], achievable single-valued neutrosophic graphs in wireless sensor networks [24], fuzzy hypergraph network for recommending top-k profitable stocks [25], an algorithm to compute the strength of competing interactions in the bearing sea based on Pythagorean fuzzy hypergraphs [26] and centrality measures in fuzzy social networks [27]. Recently, Smarandache extended hypergraphs to a new concept as *n*-superhypergraph and Plithogenic *n*-superhypergraph which have several properties and are connected with the real-world [28]. Indeed, *n*-superhypergraphs are a generalization of hypergraphs, with the advantage that they can communicate between the hyperedges. Recently, Hamidi et al. introduced the concept of quasi superhypergraphs as a special concept of *n*-superhypergraphs and showed that in hypergraph theory, any hypergraph can relate a set of elements, while without any details, it makes some conflicts, defects, and shortcomings in hypergraph theory [29]. Thus, by introducing superhypergraph, they try to eliminate defects of graphs (sometimes graph structures give very limited information

about complex networks) structures and hypergraph structures (although the hypergraph structures are for covering graph defects in the applications in hypergraphs, the relation between vertices can't be described in full details). They introduced the incidence matrix of superhypergraph and computed the characteristic polynomial for the incidence matrix of superhypergraph, so they obtained the spectrum of superhypergraphs. Also, by computingthe number of superedges of any given superhypergraphs and based on superedges and partitions of an underlying set of superhypergraph, the number of all superhypergraphs on any nonempty set is obtained.

Regarding these points, we introduce the concept of valued fuzzy superhypergraphs as a generalization of fuzzy hypergraphs. Valued fuzzy superhypergraphs are dependent on the concept of fuzzy supervertices and fuzzy superedges or fuzzy links, which are defined in this study. The motivation of valued fuzzy superhypergraphs is based on a design of real problems as complex superhypernetworks. Indeed, we modify a real problem as a fuzzy superhypergraph, and with respect to the notation of the impact membership value of modeled fuzzy superhypergraph, we have made the best decision. This paper presents the fuzzy algebraic structures on valued fuzzy superhypergraphs such as strong valued fuzzy superhypergraphs, isomorphic valued fuzzy superhypergraphs, the complement of fuzzy link, and self-complemented valued fuzzy superhypergraphs. Also, the relation between fuzzy hypergraphs and strongly valued fuzzy superhypergraphs has been investigated.

Motivation and advantage: Modeling based on fuzzy (hyper) graphs is a clustering or grouping of elements based on certain properties, in which the properties of its elements are checked in each cluster, and this check has nothing to do with the properties of other clusters. We need more complete modeling in order to be able to analyze the effect of the elements in the entire modeling and with other cluster elements at the same time. Our motivation in introducing valued fuzzy superhypergraphs is the coverage of this problem in fuzzy (hyper) graphs and therefore we define it such a way that in each cluster and group the elements are related to the elements of other clusters and groups in order to analyze the effect of each element in the (super) hypernetwork on the whole (super) hypernetwork. One of the motivations for valued fuzzy superhypergraphs is that we can have the best modeling of complex (super) hypernetworks where the role of all the details that make up this complex (super) hypernetwork is important in detail and in general. In fact, in this modeling, for each complex (super) hypernetwork, we examine the relationship between all the components of the systems that make up the complex (super) hypernetwork in detail and analyze the impact of these components in the entire complex (super) hypernetwork. The highest advantage of modeling the world's issues based on valued fuzzy superhypergraph regarding the modeling of the fuzzy (hyper) graph is in the complex (super) hypernetworks based on the fuzzy superhypergraph, and the relationship between the components is checked only in the components of all clusters based on the map of clusters. This advantage leads to obtaining, the optimal results in the complex (super) hypernetwork, based on our mathematical methods and computations. In this method, we consider the values of components in clusters, the values between clusters, and the weight of clusters and closely and carefully find the suitable positive real values for optimal computations in complex (super) hypernetworks. The positive real values play the main role in the extremum in calculus and so the optimization problem plays a significant role in these calculations.

2 Preliminaries

In this section, we recall some definitions and results, which we need as follows:

Definition 2.1. [1] Let X be a finite set. A *hypergraph* on X is a pair $H = (X, \{E_i\}_{i=1}^{m})$ such that for all $1 \leq i \leq m, \emptyset \neq E_i \subseteq X$ and $\bigcup_{i=1}^{m} E_i = X$. The elements $x_1, x_2, \ldots, x_n$ of X are called *vertices*, and the sets

$E_1, E_2, \ldots, E_m$ are called the *hyperedges* of the hypergraph H. In hypergraphs, hyperedges can contain an element (*loop*) two elements (*edge*) or more than three elements. A hypergraph $H = (X, \{E_i\}_{i=1}^{m})$ is called a complete hypergraph, if for any $x, y \in X$, there is $1 \leq i \leq m$ such that $\{x, y\} \subseteq E_i$. A hypergraph $H = (X, \{E_i\}_{i=1}^{n})$ is called a joint complete hypergraph, if $|X| = n$ for all $1 \leq i \leq n$, $|E_i| = i$ and $E_i \subseteq E_{i+1}$ element (*loop*). If for all $1 \leq k \leq m$ $|E_k| = 2$, the hypergraph becomes an ordinary (undirected) graph. and n rows representing the vertices $x_1, x_2, \ldots, x_n$, where for all $1 \leq i \leq n$ and for all $1 \leq j \leq m$, $m_{ij} = 1$ if $x_i \in E_j$ and $m_{ij} = 0$ if $x_i \notin E_j$.

Definition 2.2. [30] Let $m \in \mathbb{N}$ and $V = \{v_1, v_2, \ldots, v_m\}$ be a set of vertices, that contain single vertices (the classical ones), indeterminant vertices (unclear, vague, unknown) and null vertices (unknown, empty). Consider $P(V)$ as the power set of V, $P^2(V) = P(P(V)) \ldots$, and $P^{n+1}(V) = P(P^n(V))$ be the n-power set of the set V. Then the n-superhypergraph (n-*SHG*) is an ordered pair n-$SHG = (G_n, E_n)$, where for any $n \in \mathbb{N}$, $G_n \subseteq P^n(V)$, is the set of vertices and $E_n \subseteq P^n(V)$ is the set of edges. The set G_n contains some type of vertices, such as single vertices (the classical ones), indeterminate vertices (unclear, vague, partially unknown), null vertices (totally unknown, empty), supervertices (or subset vertex), i.e., two or more (single, indeterminate, or null) vertices that together from a group (organization). An n-supervertex is a collection of many vertices such that at least one is a $(n-1)$-supervertex and all other supervertices in the collection have the order $r \leq n-1$. The set of edges E_n contains some edges such as single edges (the classical ones), indeterminant (unclear, vague, partially unknown), null-edge (empty, totally unknown), hyperedge (containing three or more single vertices), superedge (containing two vertices, at least one of them being a supervertex), n-superedge (containing two vertices, at least one being an n-supervertex and the other of order r-supervertex with $r \leq n$), superhyperedge (containing three or more vertices, at least one being a supervertex), n-superhyper edge (containing three or more vertices, at least one being an n-supervertex and the other r-supervertices with $r \leq n$), multi edges (two or more edges connecting the same two vertices), and loop (an edge that connects an element).

Definition 2.3. [29] Let X be a non-empty set. Then

(*i*) $H = (X, S = \{S_i\}_{i=1}^{k}, \Phi = \{\varphi_{i,j}\}_{i,j})$ is called a *quasi superhypergraph*, if $\Phi \neq \emptyset$ and $X = \bigcup_{i=1}^{n} S_i$, where $k \geq 2$, and for all $1 \leq i \leq k, S_i \in P^*(X)$, is called a *supervertex* and for any $i \neq j$, the map $\varphi_{i,j} : S_i \to S_j$ (say S_i links to S_j) is called a *superedge*,

(*ii*) the quasi superhypergraph $H = (X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}_{i,j})$ is called a *superhypergraph*, if for any $S_i \in P^*(X)$ there exists at least one $S_j \in P^*(X)$ such that S_i links to S_j (it is not necessary all supervertices be linked).

(*iii*) The superhypergraph $H = (X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}_{i,j})$ is called a *trivial* superhypergraph, if $k = 1$ (S_1 can't link to itself).

Let X be a nonempty set, $\mathcal{SH}(X) = \{H \mid H \text{ is a superhypergraph on } X\}$ and $\mathcal{SH}(n_1, n_2, \ldots, n_k) = \{(X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}) \in \mathcal{SH} \mid |S_i| = n_i\}$.

Theorem 2.1. [29] Let X be a nonempty set and $|X| = n$. If $\alpha = \{(n_1, n_2, \ldots, n_r) \mid \sum_{i=1}^{r} n_i = n, n_i, r \in \mathbb{N}\}$ and $m = |\{i \mid n_i = n_j\}|$, then $|\mathcal{SH}(X)| = \sum_{\alpha} \left(\frac{1}{m!} \prod_{i=1}^{r} \binom{n - \sum_{j=1}^{i-1} n_j}{n_i}\right)\left(\sum_{1 \leq i \neq j \leq r} n_i^{n_j}\right)\right)$.

3 Fuzzy Superhypergraphs

In this section, we introduce the novel concept of fuzzy supervertices, fuzzy superedges or fuzzy links, and fuzzy superhypergraph.

Example 3.1. Let $X = \{a, b, c, d, e, f\}$ and consider the fuzzy hypergraph $H = (X, \sigma_1, \sigma_2, \sigma_3)$ as shown in Fig. 1. In this fuzzy hypergraph as a complex (super)hypernetwork, we can find the relation between a, b in σ_1, c, d in σ_2 and e, f in σ_3, but we cannot find the relation between a in σ_1 with to e in σ_3, for instance. Also, we cannot find the impact of element c in cluster σ_1 and the impact of element f in cluster σ_2, for instance. In addition, in this model, we cannot find the relation or impact of cluster $\sigma_i's$ with $\sigma_j's$, where $i, j \in \{1, 2, 3\}$. Let's assume that in this real-problem, as a complex (super)hypernetwork, we want to check the influence of and the relationship all components, either component by component or component in one cluster with another cluster and cluster to cluster. By the above reason, we cannot analyze in this form, since we do not know the relation among the elements of a cluster with elements of the other cluster and a cluster to another cluster at the same time and it is the main defect. Therefore, we need this component-to-component connection and group-to-group connection in the other model as a complex (super)hypernetwork.

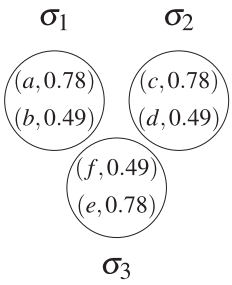


Figure 1: Hypergraph $H = (X, \sigma_1, \sigma_2, \sigma_3)$

Definition 3.1. Let $j, k \in \mathbb{N}, 1 \leq j \leq k, q \in \mathbb{R}^+$, $H^* = (X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}_{i,j})$ be a quasi superhypergraph, $\sigma_i = \{(x, \sigma_i(x)) \,|\, x \in S_i, 0 \leq \sigma_i(x) \leq 1\}$ and $\mu_{i,j} : \varphi_{i,j} \to [0, 1]$ be fuzzy subsets. Then $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ is called a q-fuzzy quasi superhypergraph on H^*, if $X = \bigcup_{i=1}^{k} supp(\sigma_i)$ and $\mu_{i,j}(x, \varphi_{i,j}(x)) \leq \frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}$, where $w(\sigma_t) = \sum_{x_t \in S_t} \sigma(x_t)$. We will call σ_i's by fuzzy supervertices and $\mu_{i,j}$'s by fuzzy superedges or fuzzy links from σ_i's to σ_j's and 1-fuzzy quasi superhypergraph is denoted by fuzzy quasi superhypergraph.

From now on, if there is not any fuzzy links from σ_i's to σ_j's, it means that $\mu_{i,j} \equiv 0$. In this case $H^* = (X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}_{i,j})$ converts to a fuzzy hypergraph.

Example 3.2. Let $X = \{x_i\}_{i=1}^7$. Then $H = (X, \{S_i\}_{i=1}^3, \{\varphi_{1,2}, \varphi_{2,3}, \varphi_{1,3}\})$ is a quasi superhypergraph in Fig. 2, where $\varphi_{1,2} = \{(x_1, x_4), (x_2, x_4), (x_3, x_5)\}$, $\varphi_{2,3} = \{(x_4, x_6), (x_5, x_7)\}$ and $\varphi_{1,3} = \{(x_1, x_6), (x_2, x_6), (x_3, x_7)\}$. Hence, $H = (X, \{\sigma_i\}_{i=1}^3, \mu_{1,2}, \mu_{2,3}, \mu_{1,3})$ is a $\frac{20}{33}$-fuzzy quasi superhypergraph as Fig. 2, while is not a fuzzy quasi superhypergraph.

Theorem 3.1. Let $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ be a q-fuzzy quasi superhypergraph on H^* and $q, q' \in \mathbb{R}^+$. If $q' \leq q$, then $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ is a q'-fuzzy quasi superhypergraph on H^*.

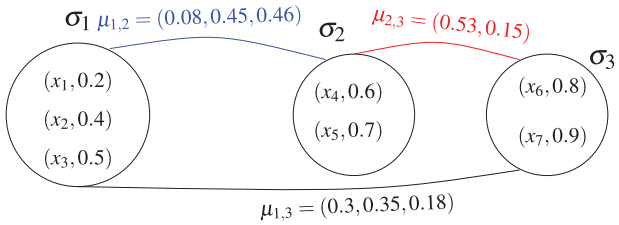


Figure 2: $\frac{20}{33}$-Fuzzy quasi superhypergraph $H = (X, \sigma_1, \sigma_2, \sigma_3, \mu_{1,2}, \mu_{2,3}, \mu_{1,3})$

Proof. Let $q < q' \in \mathbb{R}^+$ and $x \in X$. Since $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ is a q-fuzzy quasi superhypergraph on H^*, come to conclusion $\mu_{i,j}(x, \varphi_{i,j}(x)) \leq \frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))} \leq \frac{\sigma_i(x)}{q'(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q'(w(\sigma_j))}$. Thus $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ is a q'-fuzzy quasi superhypergraph on H^*.

Proposition 3.1. The following hold:

(*i*) Every fuzzy hypergraph is a fuzzy quasi superhypergraph.

(*ii*) Every fuzzy graph is a fuzzy quasi superhypergraph.

Proof. The proof is clear by definition.

Theorem 3.2. Let $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ be a q-fuzzy quasi superhypergraph on H^*. If for all $1 \leq i \leq k, |S_i| = 1$, then

(*i*) For all $x \in X$, $s = sup\{\mu_{i,j}(x, \varphi_{i,j}(x)) \mid x \in X, i, j\} \leq \frac{1}{q}$ is obtained.

(*ii*) If $q \in [t, s]$, then $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ is a fuzzy graph, where $t = \max\{\frac{1}{\sigma_i(x) \wedge \sigma_j(\varphi_{i,j}(x))} \mid x \in X, i, j \in \mathbb{N}\}$.

(*iii*) If $q \leq 1$, then for all $1 \leq i \leq k, |S_i| = 1$. Then for any fuzzy subsets σ_i and $\mu_{i,j}$, $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ is a fuzzy quasi superhypergraph on H^*.

Proof. Let $x \in X$. Then

(*i*) $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ is a q-fuzzy quasi superhypergraph on H^* and for all $1 \leq i \leq k, |S_i| = 1$, so

$$\mu_{i,j}(x, \varphi_{i,j}(x)) \leq \frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))} = \frac{\sigma_i(x)}{q\sigma_i(x)} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q\sigma_j(\varphi_{i,j}(x))} = \frac{1}{q}.$$

It follows that $q \leq s$.

(*ii*) By (*i*), $q \leq s$. Let $x \in X$. If $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ is a fuzzy graph, then it must be $\mu_{i,j}(x, \varphi_{i,j}(x)) \leq \frac{1}{q} \leq (\sigma_i(x) \wedge \sigma_j(\varphi_{i,j}(x)))$. Hence for any $x \in X$ and i, j, $q \geq \frac{1}{\sigma_i(x) \wedge \sigma_j(\varphi_{i,j}(x))}$ and so $q \in [t, s]$.

(*iii*) It is obtained from (*ii*), (*iii*) and Theorem 3.1.

Corollary 3.1. Let $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ be a q-fuzzy quasi superhypergraph on H^*. If for all $1 \leq i \leq k, q(w(\sigma_i) > 1$, then $\mu_{i,j}(x, \varphi_{i,j}(x)) \leq \sigma_i(x) \wedge \sigma_j(\varphi_{i,j}(x))$.

Definition 3.2. Let $q, q' \in \mathbb{R}^+$, $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ and $H' = (\{\sigma_i'\}_{i=1}^{k'}, \{\mu_{i,j}'\}_{i,j})$ be q-fuzzy quasi superhypergraph and q'-fuzzy quasi superhypergraph on quasi superhypergraphs $H^* = (X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}_{i,j})$ and $H'^* = (X', \{S_i'\}_{i=1}^k, \{\varphi_{i,j}'\}_{i,j})$, respectively.

(*i*) A bijective mapping $h : H \to H'$ is called an isomorphism, if for any $x \in X$, there exists a permutation π, ν on $I_k = \{1, 2, \ldots, k\}$ such that $\sigma_i(x) = \sigma'_{\pi(i)}(h(x))$ and $\mu_{i,j}(x, \varphi_{i,j}(x)) = \mu'_{\pi(i)\nu(j)}(h(x), \varphi'_{\pi(i)\nu(j)}(h(x))$. In this case will say that $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ and $H' = (\{\sigma_i'\}_{i=1}^{k'}, \{\mu_{i,j}'\}_{i,j})$ are isomorphic and will denote it by $H \cong H'$.

(*ii*) H is called a self complemented fuzzy quasi superhypergraph, if $H \cong H^c$.

Example 3.3. Let $X = \{x, y, z, w\}$ and $X' = \{x', y', z', w'\}$. Then $H = (X, \{\sigma_i\}_{i=1}^2, \mu_{1,2})$ and $H' = (X', \{\sigma_i'\}_{i=1}^2, \mu_{1,2}')$ are 2-fuzzy quasi superhypergraphs as subfigures 3a, 3b in Fig. 3, where $\varphi_{1,2} = \{(x, z), (y, w)\}$ and $\varphi_{2,1}' = \{(x', z'), (y', w')\}$. Consider $h : H \to H'$, by $h = \{(x, y'), (y, x'), (z, w'), (w, z')\}$ and permutations $\pi = \nu = (1, 2)$. Then $H \cong H'$.

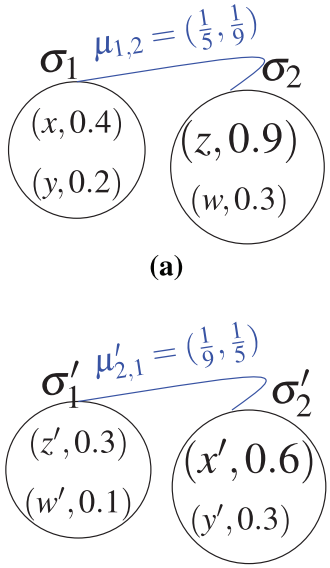


Figure 3: Isomorphic fuzzy 2-quasi superhypergraphs H, H'

Theorem 3.3. Let $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ and $H' = (\{\sigma_i'\}_{i=1}^{k'}, \{\mu_{i,j}'\}_{i,j})$ be q-fuzzy quasi superhypergraph and q'-fuzzy quasi superhypergraph on quasi superhypergraphs $H^* = (X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}_{i,j})$ and $H'^* = (X', \{S_i'\}_{i=1}^k, \{\varphi_{i,j}'\}_{i,j})$, respectively. If $H \cong H'$, then $q = q'$.

Proof. Since $H \cong H'$, there exists a bijective mapping $h : H \to H'$ is called an isomorphism, if for any $x \in X$, there exist permutations π, ν on $I_k = \{1, 2, \ldots, k\}$ such that $\sigma_i(x) = \sigma'_{\pi(i)}(h(x))$ and $\mu_{i,j}(x, \varphi_{i,j}(x)) = \mu'_{\pi(i)\nu(j)}(h(x), \varphi'_{\pi(i)\nu(j)}(h(x))$. Without loss of generality and a rearrangement, we consider an arbitrary fuzzy supervertex σ_m and suppose that $\sigma_m = \{(x_i, \sigma_m(x_i)) \mid 1 \leq i \leq r\}$ in q-fuzzy quasi superhypergraph $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$. Thus there exists a fuzzy supervertex $\sigma'_{m'} = \{(x_i', \sigma_{m'}(x_i')) \mid 1 \leq i \leq r'\}$ in q'-fuzzy quasi superhypergraph $H' = (\{\sigma_i'\}_{i=1}^{k'}, \{\mu_{i,j}'\}_{i,j})$ such that $\sigma_m(x_i) = \sigma'_{m'}(x_i')$, which $m' = \pi(m)$

and $h(x_i) = x'_i$. It follows that $r = r'$ and for any given $1 \leq i, i' \leq r$, we have $\frac{x_i}{q(\sum_{i=1}^{r} x_i)} = \frac{x'_i}{q'(\sum_{i=1}^{r} x'_i)}$.

Hence $\sum_{i=1}^{m} \frac{x_i}{q(\sum_{i=1}^{r} x_i)} = \sum_{i=1}^{m} \frac{x'_i}{q'(\sum_{i=1}^{r} x'_i)}$ and so $q = q'$.

Let $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$ and $H' = (\{\sigma'_i\}_{i=1}^{k'}, \{\mu'_{i,j}\}_{i,j})$ be q-fuzzy quasi superhypergraph and q'-fuzzy quasi superhypergraph on quasi superhypergraphs $H^* = (X, \{S_i\}_{i=1}^{k}, \{\varphi_{i,j}\}_{i,j})$ and $H'^* = (X', \{S'_i\}_{i=1}^{k}, \{\varphi'_{i,j}\}_{i,j})$, respectively. From now on, based on Theorem 3.3, when we say that $H \cong H'$, then $q = q'$.

Theorem 3.4. Let $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$ and $H' = (\{\sigma'_i\}_{i=1}^{k'}, \{\mu'_{i,j}\}_{i,j})$ be isomorphic q-fuzzy quasi superhypergraphs.

(*i*) For any given fuzzy supervertex σ_i in H, there exists a fuzzy supervertex σ'_i in H' and $\lambda \in \mathbb{R}$, such that $\frac{w(\sigma_i)}{w(\sigma_j)} = \lambda$.

(*ii*) For any $x \in X$, there exist $\lambda \in \mathbb{R}$ and $x' \in X'$ such that $\frac{x}{x'} = \lambda$.

Proof. (*i*) Let $\sigma_i = \{(x_m, \sigma_i(x_m)) \mid 1 \leq m \leq r\}$ be a fuzzy supervertex in H. Clearly

$h(\sigma_i) = \{(h(x_m), \sigma_i(h(x_m)) \mid 1 \leq m \leq r\}$ is a fuzzy supervertex in H' and using Theorem 3.3, $\frac{x_m}{w(\sigma_i)} = \frac{x'_m}{w(h(\sigma_i))}$. It follows that $\frac{w(\sigma_i)}{w(h(\sigma_i))} = \frac{x_m}{x'_m}$.

(*ii*) Let $x \in X$. Since $X = \bigcup_{i=1}^{k} supp(\sigma_i)$, there exists a fuzzy supervertex σ_i in H such that $x \in supp(\sigma_i)$. Hence by item (*i*), there exists a fuzzy supervertex σ'_i in H' and $\lambda \in \mathbb{R}$, such that $x \in supp(\sigma'_i)$ and $\frac{w(\sigma_i)}{w(\sigma'_i)} = \frac{x}{x'}$.

Example 3.4. Consider 2-fuzzy quasi superhypergraphs $H = (X, \{\sigma_i\}_{i=1}^{2}, \mu_{1,2})$ and $H' = (X', \{\sigma'_i\}_{i=1}^{2}, \mu'_{1,2})$ as shown in subfigures 3a, 3b of Fig. 3. Clearly $H \cong H'$, while for any $i \in \{1, 2\}$, $w(\sigma_i) \neq w(\sigma'_i)$. This shows that if q-fuzzy quasi superhypergraphs are isomorphic, it is not necessary that their fuzzy superverties have the same weights.

Definition 3.3. Let $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$ be a q-fuzzy quasi superhypergraph on H^*. Then $H^c = (\{\sigma_i^c\}_{i=1}^{k}, \{\mu_{i,j}^c\}_{i,j})$ is called a complement q-fuzzy quasi superhypergraph of $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$, if $\sigma_i^c = \sigma_i$ and $\mu_{i,j}^c(x, \varphi_{i,j}(x)) = (\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) - \mu_{i,j}(x, \varphi_{i,j}(x))$.

Lemma 3.1. Let $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$ be a q-fuzzy quasi superhypergraph on H^*. Then $\mu_{i,j}^c = \mu_{i,j}$ if and only if $\mu_{i,j}(x, \varphi_{i,j}(x)) = \frac{1}{2}(\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))})$, for all $x \in X$ and $q \geq \frac{1}{2}$.

Theorem 3.5. Let $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$ be a q-fuzzy quasi superhypergraph on H^*. Then $(H^c)^c \cong H$.

Proof. Let $x \in X$. Then for any $1 \leq i \leq k$, $(\sigma_i^c)^c(x) = \sigma_i^c(x) = \sigma_i(x)$ and

$$\begin{aligned}(\mu_{i,j}^c)^c(x, \varphi_{i,j}(x)) &= (\frac{\sigma_i^c(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j^c(\varphi_{i,j}(x))}{q(w(\sigma_j))}) - \mu_{i,j}^c(x, \varphi_{i,j}(x)) \\ &= (\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) - \mu_{i,j}^c(x, \varphi_{i,j}(x)) \\ &= (\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) - ((\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) - \mu_{i,j}(x, \varphi_{i,j}(x))) \\ &= \mu_{i,j}(x, \varphi_{i,j}(x))).\end{aligned}$$

It follows that for any $1 \leq i \leq k$, $(\sigma_i^c)^c = \sigma_i$ and $(\mu_{i,j}^c)^c = \mu$. Hence $(H^c)^c \cong H$.

Theorem 3.6. Let $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ and $H' = (\{\sigma_i'\}_{i=1}^{k'}, \{\mu_{i,j}'\}_{i,j})$ be isomorphic q-fuzzy quasi superhypergraphs. Then H^c and H'^c are isomorphic q-fuzzy quasi superhypergraphs and conversely.

Proof. Since $H \cong H'$, there exists a bijective mapping $h : H \to H'$ is called an isomorphism, if for any $x \in X$, there exists a permutations π, ν on $I_k = \{1, 2, \ldots, k\}$ such that $\sigma_i(x) = \sigma'_{\pi(i)}(h(x))$ and $\mu_{i,j}(x, \varphi_{i,j}(x)) = \mu'_{\pi(i)\nu(j)}(h(x), \varphi'_{\pi(i)\nu(j)}(h(x))$. Using the definition of complement, we have

$$\begin{aligned}\mu_{i,j}^c(x, \varphi_{i,j}(x)) &= (\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) - \mu_{i,j}(x, \varphi_{i,j}(x)) \\ &= (\frac{\sigma'_{\pi(i)}(h(x))}{q(w(\sigma'_{\pi(i)}))} \wedge \frac{\sigma'_{\pi(j)}(\varphi_{i,j}(h(x)))}{q(w(\sigma'_{\pi(j)}))}) - \mu'_{\pi(i)\nu(j)}(h(x), \varphi'_{\pi(i)\nu(j)}(h(x)) \\ &= \mu^c_{\pi(i),\nu(j)}(h(x), \varphi_{\pi(i),\nu(j)}(h(x)))\end{aligned}$$

Hence $H^c \cong H'^c$. Conversely, let $H^c \cong H'^c$, then by Theorem 3.5, $H \cong (H^c)^c \cong (H'^c)^c \cong H'$.

Theorem 3.7. Let $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ be a self complemented q-fuzzy quasi superhypergraph on H^*. Then $\frac{1}{2}(\sum(\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) = \sum \mu_{i,j}(x, \varphi_{i,j}(x))$.

Proof. Since $H = (\{\sigma_i\}_{i=1}^k, \{\mu_{i,j}\}_{i,j})$ is a self complemented q-fuzzy quasi superhypergraph, there exists a map $h : X \to X$ such that for any $x \in X$, $\sigma_i(x) = \sigma_i'(h(x))$ and $\mu_{i,j}^c(h(x), \varphi_{i,j}(h(x)) = \mu_{i,j}(x, \varphi_{i,j}(x))$. Then $(\frac{\sigma_i(h(x))}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(h(x))}{q(w(\sigma_j))}) - \mu_{i,j}(h(x), \varphi_{i,j}(h((x)) = \mu_{i,j}(x, \varphi_{i,j}(x))$. Since $h : X \to X$ is a bijection, we get to the result that

$$\sum \big((\frac{\sigma_i(h(x))}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(h(x))}{q(w(\sigma_j))}) - \mu_{i,j}(h(x), \varphi_{i,j}(h((x))\big) = \sum \mu_{i,j}(x, \varphi_{i,j}(x)),$$

so

$$\sum((\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j)})) - \sum(\mu_{i,j}(x, \varphi_{i,j}(x)) = \sum \mu_{i,j}(x, \varphi_{i,j}(x)).$$

It follows that $\frac{1}{2}(\sum(\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) = \sum \mu_{i,j}(x, \varphi_{i,j}(x)$.

Example 3.5. Let $X = \{x_i\}_{i=1}^{6}$. Then $H^* = (X, \{S_i\}_{i=1}^{3}, \{\varphi_{1,2}, \varphi_{2,3}\})$ is a 4-quasi superhypergraph in Fig. 4, where $\varphi_{1,2} = \{(x_1, x_3), (x_2, x_4)\}$ and $\varphi_{2,3} = \{(x_3, x_6), (x_4, x_7)\}$. If $H = (X, \{\sigma_i\}_{i=1}^{3}, \mu_{1,2}, \mu_{2,3})$ is a fuzzy quasi superhypergraph as shown in Fig. 4, where $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq 1$. The equation $\frac{1}{2}(\sum(\frac{\sigma_i(x)}{4(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{4(w(\sigma_j))})) = \sum \mu_{i,j}(x, \varphi_{i,j}(x))$, implies that

$$\frac{1}{2}(\frac{\alpha_1}{4(\alpha_1+\alpha_2)} \wedge \frac{\alpha_3}{4(\alpha_3+\alpha_4)} + \frac{\alpha_2}{4(\alpha_1+\alpha_2)} \wedge \frac{\alpha_4}{4(\alpha_3+\alpha_4)} + \frac{\alpha_3}{4(\alpha_3+\alpha_4)} \wedge \frac{\alpha_5}{4(\alpha_5+\alpha_6)}$$
$$+ \frac{\alpha_4}{4(\alpha_3+\alpha_4)} \wedge \frac{\alpha_6}{4(\alpha_5+\alpha_6)}) = \frac{1}{2}(\frac{\alpha_1}{4(\alpha_1+\alpha_2)} + \frac{\alpha_2}{4(\alpha_1+\alpha_2)} + \frac{\alpha_3}{4(\alpha_3+\alpha_4)} + \frac{\alpha_4}{4(\alpha_3+\alpha_4)})$$
$$= \frac{1}{8} = (\beta_1 + \beta_2 + \beta_3 + \beta_4),$$

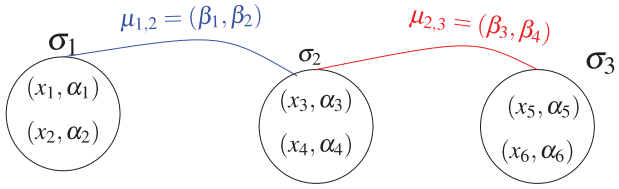


Figure 4: Fuzzy quasi superhypergraph $H = (X, \sigma_1, \sigma_2, \sigma_3, \mu_{1,2}, \mu_{2,3}, \mu_{1,3})$

where $\beta_1 \leq \frac{\alpha_1}{4(\alpha_1+\alpha_2)}, \beta_2 \leq \frac{\alpha_2}{4(\alpha_1+\alpha_2)}, \beta_3 \leq \frac{\alpha_3}{4(\alpha_3+\alpha_4)}$ and $\beta_4 \leq \frac{\alpha_4}{4(\alpha_3+\alpha_4)}$. Suppose that

$\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.3, \alpha_4 = 0.4, \alpha_5 = 0.5, \alpha_6 = 0.6, \beta_1 = 0.01, \beta_2 = 0.03, \beta_3 = 0.04$ *and* $\beta_4 = 0.17$.

It follows that $\frac{1}{2}(\sum(\frac{\sigma_i(x)}{4(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{4(w(\sigma_j))})) = \sum \mu_{i,j}(x, \varphi_{i,j}(x))$, while $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$ is not a self complemented fuzzy quasi superhypergraph on H^* as shown in Fig. 5. Thus the converse of Theorem 3.7, is not necessarily true.

Theorem 3.8. Let $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$ be a q-fuzzy quasi superhypergraph on H^*. If for all $x \in X$, $\mu^c_{i,j}(x, \varphi_{i,j}(x)) = \frac{1}{2}(\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))})$, then $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$ is a self complemented fuzzy quasi superhypergraph.

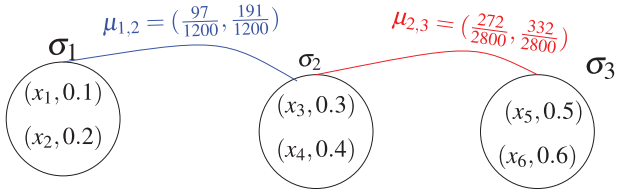


Figure 5: Fuzzy quasi superhypergraph $H^c = (X, \sigma_1, \sigma_2, \sigma_3, \mu_{1,2}, \mu_{2,3}, \mu_{1,3})$ of Fig. 4

Proof. Define $h : X \to X$ by $h(x) = x$, where $x \in X$. Clearly $\sigma^c(h(x)) = \sigma^c(x) = \sigma(x)$. In addition,

$$\begin{aligned}\mu^c_{i,j}(x, \varphi_{i,j}(x)) &= \mu^c_{i,j}(h(x), \varphi_{i,j}(h(x)) \\ &= (\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) - \mu_{i,j}(x, \varphi_{i,j}(x)) \\ &= (\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) - \frac{1}{2}(\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) \\ &= \frac{1}{2}(\frac{\sigma_i(x)}{q(w(\sigma_i)} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) = \mu_{i,j}(x, \varphi_{i,j}(x)).\end{aligned}$$

It follows that $\mu^c_{i,j}(x, \varphi_{i,j}(x)) = \mu_{i,j}(x, \varphi_{i,j}(x))$ and so $H = (\{\sigma_i\}^k_{i=1}, \{\mu_{i,j}\}_{i,j})$ is a self complemented fuzzy quasi superhypergraph.

Definition 3.4. Let $H^* = (X, \{S_i\}^k_{i=1}, \{\varphi_{i,j}\}_{i,j})$ be a quasi superhypergraph, $\sigma_i = \{(x, \sigma_i(x)) \mid x \in S_i, 0 \le \sigma_i(x) \le 1\}$ and $\mu_{i,j} : \varphi_{i,j} \to [0, 1]$ be fuzzy subsets. Then $H = (\{\sigma_i\}^k_{i=1}, \{\mu_{i,j}\}_{i,j})$ is called a strong q-fuzzy quasi superhypergraph on H^*, if $X = \bigcup_{i=1}^{k} supp(\sigma_i)$ and $\mu_{i,j}(x, \varphi_{i,j}(x)) = \frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}$, where $w(\sigma_t) = \sum_{x_t \in S_t} \sigma(x_t)$.

Theorem 3.9. Let $H = (\{\sigma_i\}^k_{i=1}, \{\mu_{i,j}\}_{i,j})$ be a strong q-fuzzy quasi superhypergraph on H^*. Then $H^c = (\{\sigma^c_i\}^k_{i=1}, \{\mu^c_{i,j}\}_{i,j})$ is a fuzzy hypergraph.

Proof. Let $x \in X$. Since $H = (\{\sigma_i\}^k_{i=1}, \{\mu_{i,j}\}_{i,j})$ is a strong q-fuzzy quasi superhypergraph on H^*, it is resulted in

$$\begin{aligned}\mu^c_{i,j}(x, \varphi_{i,j}(x)) &= (\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) - \mu_{i,j}(x, \varphi_{i,j}(x)) \\ &= (\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) - (\frac{\sigma_i(x)}{q(w(\sigma_i))} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{q(w(\sigma_j))}) = 0.\end{aligned}$$

Thus for any $x \in X$, we have $\mu^c_{i,j}(x, \varphi_{i,j}(x)) =$ or $\mu_{i,j} \equiv 0$. It follows that for any given fuzzy superverteices $\sigma_i \neq \sigma_j$, there is not any fuzzy links from σ_i's to σ_j's. Thus $H = (\{\sigma_i\}^k_{i=1}, \{\mu_{i,j}\}_{i,j}) = (\{\sigma_i\}^k_{i=1})$ is a fuzzy hypergraph, by Definition 3.1.

Definition 3.5. Let $H = (\{\sigma_i\}^k_{i=1}, \{\mu_{i,j}\}_{i,j})$ be a q-fuzzy quasi superhypergraph on H^* and $x \in X$. Then $d(x, H) = \begin{cases} \sum_{(x, \varphi_{i,j}(x)) \in \varphi_{i,j}} \mu_{i,j}(x, \varphi_{i,j}(x)) & \text{if } (x, \varphi_{i,j}(x)) \in \varphi_{i,j} \\ 0 & \text{if } (x, \varphi_{i,j}(x)) \notin \varphi_{i,j} \end{cases}$ is the impact membership value of x in $H = (\{\sigma_i\}^k_{i=1}, \{\mu_{i,j}\}_{i,j})$.

Theorem 3.10. Let $H = (\{\sigma_i\}^k_{i=1}, \{\mu_{i,j}\}_{i,j})$ be a q-fuzzy quasi superhypergraph on H^* and $x \in X$. Then

(*i*) if $H = (\{\sigma_i\}^k_{i=1}, \{\mu_{i,j}\}_{i,j})$ is strong, then $qd(x, H) = \sum(\frac{\sigma_i(x)}{w(\sigma_i)} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{w(\sigma_j)})$.

(*ii*) $d(x, H^c) + d(x, H) = \sum(\frac{\sigma_i(x)}{qw(\sigma_i)} \wedge \frac{\sigma_j(\varphi_{i,j}(x))}{qw(\sigma_j)})$.

(*iii*) if $H = (\{\sigma_i\}^k_{i=1}, \{\mu_{i,j}\}_{i,j})$ is strong, then $d(x, H^c) = 0$.

Proof. I is clear by Definition 3.3 and Theorem 3.9.

Corollary 3.2. Let $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$ and $H' = (\{\sigma_i'\}_{i=1}^{k'}, \{\mu_{i,j}'\}_{i,j})$ be q-fuzzy quasi superhypergraphs. If $H \cong H'$, then $\sum_{x \in H} d(x, H) = \sum_{x \in H'} d(x, H')$.

4 Application of Fuzzy q-quasi Superhypergraph in Real World

In this section, we apply the concept of fuzzy valued quasi superhypergraph in the real world. To simplify and better display the method of application of fuzzy q-quasi superhypergraph in a practical problem, the following algorithm can be used:

Step 1: Consider a real problem as a complex (super)hypernetwork,

Step 2: Separate the factors and components of this complex (super)hypernetwork according to the type of application,

Step 3: Put the factors and constituent factors classified according to their values in different tables,

Step 4: According to the tables of step 3, model the relationship between components in an optimal state through a fuzzy q-quasi superhypergraph (in this step, one must find the q, and compute the maximum of fuzzy superedges based on fuzzy supervertices)

Step 5: Compute the impact membership value of components of the q-fuzzy quasi superhypergraph,

Step 6: Find the extremum case based on step 5 and the conditions we need to solve the real problem.

In the following section, we consider a real problem as a business (super)hypernetwork, and in this regard, first, we analyze and describe a business network. A **business network** is a complex network of companies, working together to accomplish certain objectives. These objectives, which are strategic and operational, are adopted by business networks based on their role in the market. In follows, we define a fuzzy (super)hypernetwork as a q-fuzzy quasi superhypergraph, such that its objects are called fuzzy supervertices and their connections are called fuzzy superedges or fuzzy links. For example, when we represent the business superhyper network, it concludes a collection of people and their fund, products and their quality, and the buying and selling market and their liquidity as a fuzzy quasi superhypergraph. The fuzzy supervertices are the people under consideration, products, and their quality and the buying and selling market and their liquidity, while the fuzzy superedges list all ability level, communication level, and the service time between them.

Business superhypernetwork: Business fuzzy superhypernetworking is the process of establishing a mutually beneficial relationship with other business people and potential clients or customers. The main purpose of business fuzzy superhypernetworking is to tell others about your business and hopefully convert them into customers. The most valuable advantage of the fuzzy superhypernetwork is meeting potential customers or creating referrals that you can then follow up to hopefully add to your customer base. In Table 1, we introduce a group of people with capital percentage for the participation of a business superhypernetwork that is supposed to work. In Table 2, we introduce a carpet production factory with high production quality for the participation of business superhypernetwork and in Table 3, we introduce shopping markets with liquidity for the participation of a business superhypernetwork.

Table 1: People and amount of capital

People	Amount of capital
Benjamin	0.6
Oliver	0.68
James	0.7
William	0.84
Lucas	0.65

Table 2: Machinery and high production quality

Machinery	High production quality
M_1	0.5
M_2	0.55
M_3	0.65
M_4	0.75
M_5	0.85

Table 3: Shopping market and liquidity

Shopping market	Liquidity
SM_1	0.4
SM_2	0.35
SM_3	0.75
SM_4	0.45
SM_5	0.55

Case 1: We want to analyze the minimum ability of people on machines, the minimum consultation of people with the capital market, and the minimum distance between factories and the capital market. For instance, we compute the minimum ability of Benjamin and Machinery M_1 as follows:

$$\mu_{i,j}(Benjamin, M_1) \leq \frac{\sigma_i(Benjamin)}{w(\sigma_i)} \wedge \frac{\sigma_j(\varphi_{i,j}(Benjamin))}{w(\sigma_j)} \Rightarrow$$

$$\mu_{i,j}(Benjamin, M_1) \leq \frac{0.6}{3.47} \wedge \frac{0.5}{3.3} = \frac{0.5}{3.3} \Rightarrow \frac{0.5}{3.3} = \min\{\mu_{i,j}(Benjamin, M_k) \mid 1 \leq k \leq 5\}.$$

It follows that:

$$\varphi_{1,2} = \{(B, M_1), (L, M_2), (O, M_3), (J, M_4), (W, M_5)\},$$

$$\varphi_{1,3} = \{(B, SM_2), (L, SM_1), (O, SM_4), (J, SM_5), (W, SM_3)\} \textit{ and}$$

$$\varphi_{2,3} = \{(M_1, SM_2), (M_2, SM_1), (M_3, SM_4), (M_4, SM_5), (M_5, SM_3)\}.$$

All these qualities will be represented in the fuzzy quasi superhypergraph, where the fuzzy supervertices are the people, Machinery and the fuzzy superedges list all the ability levels, communication levels, and service time (Fig. 6). Now we see the minimum ability of people on machines, the minimum consultation of people with the capital market, and the minimum distance between factories and the capital market as shown in Tables 4–6.

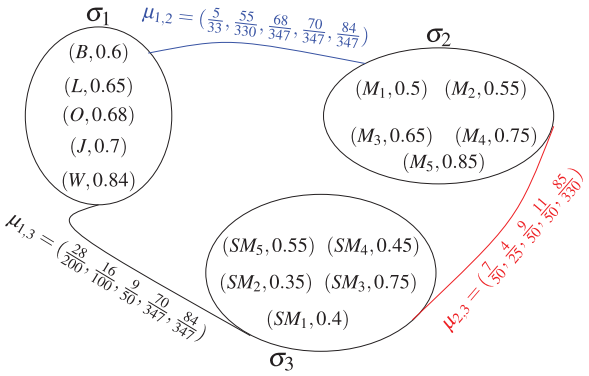


Figure 6: Fuzzy quasi superhypergraph $H = (X, \sigma_1, \sigma_2, \sigma_3, \mu_{1,2}, \mu_{2,3}, \mu_{1,3})$

Table 4: Business fuzzy superhypernetworking

(People, Machinery)	(B, M_1)	(L, M_2)	(O, M_3)	(J, M_4)	(W, M_5)
(Amount of capital, High production quality)	$\frac{5}{33}$	$\frac{55}{330}$	$\frac{68}{347}$	$\frac{70}{347}$	$\frac{84}{347}$

Table 5: Business fuzzy superhypernetworking

(People, Shopping market)	(B, SM_1)	(L, SM_2)	(O, SM_3)	(J, SM_4)	(W, SM_5)
(Amount of capital, Liquidity)	$\frac{7}{50}$	$\frac{4}{25}$	$\frac{9}{50}$	$\frac{70}{347}$	$\frac{84}{347}$

Table 6: Business fuzzy superhypernetworking

(Machinery, Shopping market)	(M_1, SM_1)	(M_2, SM_2)	(M_3, SM_4)	(M_4, SM_5)	(M_5, SM_3)
(High production quality, Liquidity)	$\frac{28}{200}$	$\frac{16}{100}$	$\frac{9}{50}$	$\frac{11}{50}$	$\frac{85}{330}$

Case 2: We want to identify the best one in terms of the impact membership value in this superhypernetwork. For instance, we compute the impact membership value of William as follows:

$$d(William, H) = \sum_{(x,\varphi_{i,j}(x))\in\varphi_{i,j}} \mu_{i,j}(x, \varphi_{i,j}(x)) = \mu_{1,2}(William, M_5) + \mu_{1,3}(William, SM_3)$$

$$= \frac{84}{347} + \frac{84}{347} = \frac{168}{347}.$$

In Table 7, we compute that

$$d(Oliver, H) \geq d(Lucas, H) \geq d(William, H) \geq d(James, H) \geq d(Benjamin, H) \geq d(M_5, H)$$
$$\geq d(M_4, H) \geq d(M_3, H) \geq d(M_2, H) \geq d(M_1, H).$$

Table 7: Impact member value in business fuzzy superhypernetworking

x:	Benjamin	Oliver	James	William	Lucas	M_1	M_2	M_3	M_4	M_5
$d(x, H)$:	$\frac{481}{1650}$	$\frac{539}{165}$	$\frac{6523}{17350}$	$\frac{140}{347}$	$\frac{164}{347}$	$\frac{7}{50}$	$\frac{4}{25}$	$\frac{9}{50}$	$\frac{11}{50}$	$\frac{85}{330}$

5 Conclusion

The current paper has defined and explored the notion of fuzzy quasi superhypergraphs. This study has tried to prove that the fuzzy links in the concept of fuzzy quasi superhypergraphs are fundamental and play a main role in the impact value of quasi superhypergraphs. The main motivation of this work is to apply fuzzy quasi superhypergraphs in the real world and to generalize the application of fuzzy hypergraphs in the real world. Indeed, all results of fuzzy hypergraphs can be extended to fuzzy quasi superhypergraphs both theoretically and practically. The merit of the proposed method is to fix the defects of fuzzy hypergraph theory. Indeed, fuzzy hypergraph theory investigates the optimal case for limited elements, while fuzzy quasi superhypergraphs consider the optimal case for the set of elements or object (object can be set). Also, we have shown that:

(*i*) The extension of a valued fuzzy quasi superhypergraph depends on its value.

(*ii*) Every fuzzy hypergraph is a fuzzy quasi superhypergraph.

(*iii*) The supremum of fuzzy superedges is less than or equal to the inverse of the real value.

(*iv*) Every fuzzy graph is a fuzzy quasi superhypergraph.

(*v*) The concept of isomorphic valued fuzzy quasi superhypergraphs is introduced and proved that the necessary condition for isomorphism of two valued fuzzy quasi superhypergraphs is equality of their values.

(*vi*) The concept of the complement of valued fuzzy quasi superhypergraphs is introduced and proved that the complement of isomorphic valued fuzzy quasi superhypergraphs is isomorphic. Also, it is proved that the complement of valued fuzzy quasi superhypergraphs satisfies involution properties.

(*vii*) The concept of self complemented valued fuzzy quasi superhypergraphs is introduced and the conditions for a valued fuzzy quasi-superhypergraph to be self-complemented have been investigated.

(*viii*) The impacts membership value of any valued fuzzy quasi superhypergraph is introduced and the properties of impact membership value of strong valued fuzzy quasi superhypergraph have been explored. We have particularly proven that the impact membership value of strong valued fuzzy quasi superhypergraph is zero.

(*ix*) We proved that the sum of all impact membership values of isomorphic valued fuzzy quasi superhypergraphs is equal.

(*x*) Based on the impact membership value of valued fuzzy quasi superhypergraph, we presented business superhypernetwork as a real-world problem and reached an optimal decision in this type of problem.

We hope that these results are helpful for further studies in complex (super)hypernetwork via algebraic structures and hyperstructures and fuzzy quasi superhypergraphs. In our future studies, we hope to obtain more results through the comparison of the suggested method with some existing methods and prove the effectiveness of the method, the fundamental relation on fuzzy quasi superhypergraphs, and their applications in other research. Also, we intend to work on the Plithogenics and new types of (hyper)soft sets and on superhypergraph and its relation to real-world problems.

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A Comprehensive Study on Decision-Making Algorithms in Retail and Project Management using Double Framed Hypersoft Sets

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Abstract

In today's dynamic business environment, effective decision-making plays a crucial role in the success of retail businesses and project management endeavors. This paper presents a comprehensive study of decision-making algorithms, focusing on the utilization of Double Framed Hypersoft Sets (DFHS) in retail and project management contexts. The study explores the theoretical foundations of DFHS and its practical applications in customer segmentation, personalized marketing, project selection, and resource allocation. Various algorithms and methodologies incorporating DFHS are discussed and analyzed, highlighting their advantages, limitations, and real-world implications. Through a series of numerical examples, case studies, and comparative analyses, this study provides valuable insights into the role of DFHS in optimizing decision processes, improving business outcomes, and fostering innovation in retail and project management domains.

Keywords: Soft Set, Hypersoft Set, Double Framed Hypersoft Set, Decision-making.

1 | Introduction

In recent years, there has been a significant expansion of soft set theory and its applications across various fields, particularly in decision-making and optimization. This introduction aims to provide an overview of the advancements in soft set theory, encompassing its various extensions and practical applications.

Soft set theory, initially introduced by Molodtsov [1], has undergone numerous modifications and extensions to address various complex decision-making problems. One notable extension is the hypersoft set, which incorporates multiple levels of parameters to enhance the decision-making process. Saeed et al. [2] extended the concept of Double Frame Soft Sets (DFSS) to Double Frame Hypersoft Sets (DFHSS), providing a more robust framework for dealing with multi-criteria decision-making problems. This extension builds on earlier work by Saeed et al. [7] on n-framed soft sets.

Fuzzy soft set theory, which integrates the fuzziness of parameters, has been particularly influential in decision-making scenarios. For instance, Riaz and Hashmi [3] explored fuzzy parameterized soft compact spaces for decision-making, demonstrating the practical applications of this theory in complex decision

contexts. Sooraj et al. [4] also highlighted the application of fuzzy soft set theory in group decision-making, emphasizing its utility in aggregating diverse opinions and preferences.

Zadeh [5] proposed a generalized theory of uncertainty, which laid the groundwork for many subsequent developments in fuzzy set and soft set theories. Building on these foundations, Gul et al. [6] conducted a comprehensive study on the (α, β)-bipolar fuzzified rough set model, which has significant implications for decision-making applications.

Further advancements include the development of soft matrix theory by Çağman and Enginoğlu [8], which provided a novel approach to decision-making by utilizing matrices to represent and manipulate soft sets. Ali et al. [9] expanded on this by introducing new operations in soft set theory, enhancing its applicability and computational efficiency. Singh and Onyeozili [10] contributed additional insights into the operations of soft matrices, further enriching this area of research.

The hypersoft set theory has also seen significant developments. Smarandache [11] extended the soft set theory to hypersoft sets and pathogenic hypersoft sets, offering new perspectives and tools for handling complex data and decision-making problems. Saeed and Harl [12] introduced the fundamental picture of fuzzy hypersoft sets, which incorporate degrees of positivity, neutrality, and negativity, enhancing the expressiveness of soft sets in decision-making scenarios.

Debnath [13] presented the fuzzy hypersoft set and its weightage operator, providing a framework for more nuanced decision-making by considering the relative importance of different parameters. Rahman et al. [14] proposed a novel decision support system based on fuzzy parameterized hypersoft sets and Riesz summability, demonstrating its application in the medical field for diagnosing heart diseases.

Saqlain et al. [15] explored distance and similarity measures for intuitionistic fuzzy hypersoft sets, applying these concepts to evaluate air pollution levels in cities based on air quality indices. This work illustrates the practical applications of hypersoft set theory in environmental monitoring and public health. Saqlain et al. [16] also investigated single and multi-valued neutrosophic hypersoft sets, providing tangent similarity measures for these sets. Jafar et al. [17] extended this research by developing trigonometric similarity measures for neutrosophic hypersoft sets and applying them to the selection of renewable energy sources.

These advancements underscore the versatility and robustness of soft set theory and its extensions, particularly in decision-making and optimization contexts. The ongoing research and development in this field continue to expand its applicability and effectiveness in addressing complex, real-world problems. In today's rapidly evolving business landscape, effective decision-making is a critical determinant of success for organizations operating in retail and project management sectors. The ability to make informed, data-driven decisions that align with organizational goals and customer preferences is paramount in achieving competitive advantage, optimizing resource utilization, and driving business growth. This paper focuses on exploring decision-making algorithms in the context of retail businesses and project management, with a specific emphasis on the utilization of Double Framed Hypersoft Sets (DFHS). DFHS is a powerful mathematical framework that enables decision-makers to model and analyze complex decision scenarios involving multiple attributes and criteria. The primary objective of this study is to conduct a comprehensive examination of DFHS-based decision-making algorithms, their theoretical foundations, practical applications, and real-world implications. By leveraging DFHS, organizations can enhance their decision accuracy, personalize customer experiences, optimize project selection, and allocate resources effectively.

The paper is structured as follows:

Section 2 serves as a foundational overview of essential concepts related to soft sets, hypersoft sets, double-framed hypersoft sets, and double-framed hypersoft subsets. Section 3 delves into specific applications of DFHS in retail, such as customer segmentation, personalized marketing, and inventory management. Additionally, the paper explores how DFHS can be applied in project management, particularly in project selection, resource allocation, and risk management. Numerical examples, case studies, and comparative

analyses are presented to illustrate the practical utility of DFHS-based algorithms and their impact on business outcomes. Finally, in Section 4, the paper concludes with key insights, implications for practice, and recommendations for future research in the field of decision-making algorithms using DFHS.

By delving into the intricacies of DFHS and its applications in retail and project management, this study aims to contribute to the growing body of knowledge on advanced decision-making techniques and their role in driving organizational success in dynamic business environments.

2 | Preliminaries

In this segment, we revisit fundamental concepts pertinent to soft sets, hypersoft sets, double-framed hypersoft sets, and double-framed hypersoft subsets.

Definition 2.1 [1]. A soft set over a universe U is a pair (F, E) where E is a set of parameters and F is a mapping given by

$$F: E \to \mathcal{P}(U)$$

where $\mathcal{P}(U)$ denotes the power set of U. In other words, for each parameter $e \in E, F(e)$ is a subset of U.

Definition 2.2 [11]. Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ be the distinct attributes whose attribute values belong to the sets $X_1, X_2, X_3, \dots, X_n$ respectively, where $X_i \cap X_j = \emptyset$ for $i \neq j$.

A pair $(\psi, W_1 \times W_2 \times W_3 \times \cdots \times W_n)$ is called a hypersoft set over the universal set U, where ψ is the mapping given by

$$\psi: W_1 \times W_2 \times W_3 \times \cdots \times W_n \to \mathcal{P}(U)$$

where $\mathcal{P}(U)$ denotes the power set of U.

Definition 2.3 [2]. Let ν be the universe of discourse. $\mathcal{P}(\nu)$ is the power set of ν. Let $a_1, a_2, a_3, \dots, a_n$ for $n \geq 1$ be n distinct attributes whose corresponding attribute values are respectively the sets $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n$ with $\varphi_i \cap \varphi_j = \emptyset$ for $i \neq j, i, j \in \{1,2, \dots, n\}$.

Then the pair $(\pi_1, \pi_2), \varphi_1 \times \varphi_2 \times \varphi_3 \times \cdots \times \varphi_n$, where π_1 and π_2 are mappings from

$$\varphi_1 \times \varphi_2 \times \varphi_3 \times \cdots \times \varphi_n \to \mathcal{P}(\nu)$$

is called a Double Framed Hypersoft Set.

Definition 2.4 [2]. Suppose that $\{(\pi_1, \pi_2), \varphi_1 \times \varphi_2 \times \varphi_3 \times \cdots \times \varphi_n\}$ is a double-framed hypersoft set. Then a collection $\{(\epsilon_1, \epsilon_2), B_1 \times B_2 \times B_3 \times \cdots \times B_n\}$ is called a double-framed hypersoft subset if:

i). $B_1 \times B_2 \times B_3 \times \cdots \times B_n \subseteq \varphi_1 \times \varphi_2 \times \varphi_3 \times \cdots \times \varphi_n$.

ii). $\epsilon_1(t) = \pi_1(t)$ and $\epsilon_2(t) = \pi_2(t)$ for all $t \in \varphi_1 \times \varphi_2 \times \varphi_3 \times \cdots \times \varphi_n$.

3 | Applications of Double Framed Hypersoft Sets

3.1 | Application of Double Framed Hypersoft Set Algorithm in Retail Customer Segmentation

In this section, we delve into the application of the Double Framed Hypersoft Set algorithm in retail customer segmentation. We explore how this algorithm helps businesses identify distinct customer segments based on preferences for product categories, brands, and price ranges.

This approach enables personalized marketing strategies, enhancing customer satisfaction and driving sales growth.

Algorithm 1 Algorithm for Double Framed Hypersoft Set in retail customer segmentation

Require: Attributes $\alpha_1, \alpha_2, ..., \alpha_n$ with corresponding attribute values $X_1, X_2, ..., X_n$.
Require: Subsets $G_1, G_2, ..., G_m$ of the set $G = \alpha_1 \times \alpha_2 \times ... \times \alpha_n$.
Require: Mappings $\pi_1, \pi_2 : G \to P(\nu)$ (mapping from combinations of attribute values to subsets of a universe ν).
Step 1: Initialize an empty set D to store Double Framed Hypersoft Subsets.
for each subset B in $G_1, G_2, ..., G_m$ **do**
  Step 2: Determine the corresponding subsets $\pi_1(B)$ and $\pi_2(B)$ using mappings π_1 and π_2.
  for each pair of subsets $\epsilon_1(B), \epsilon_2(B)$ in $G_1, G_2, ..., G_m$ **do**
    if $\epsilon_1(B) \subseteq \pi_1(B)$ and $\epsilon_2(B) \subseteq \pi_2(B)$ **then**
      Step 3: Add $(\epsilon_1, \epsilon_2), B$ to D.
    end if
  end for
end for
return Step 4: D (contains all valid Double Framed Hypersoft Subsets).

Application: Consider a shopping website that tracks customer preferences for products based on attributes like color, brand, price range, and ratings. The Double Framed Hypersoft Set can help in categorizing customer preferences and recommending products accordingly.

Example:

- Attributes: Color {Red, Blue}, Brand {Nike, Adidas}, Price Range {Low, Medium, High}.
- Subsets: $G_1 =$ {(Red, Nike, Low), (Red, Nike, Medium), (Blue, Adidas, High)}, $G_2 =$ {(Blue, Nike, Medium), (Red, Adidas, Low)}.
- Mappings: π_1((Red, Nike, Low)) ={Shoe1, Shoe2}, π_2((Red, Nike, Low)) = { Shoe3, Shoe4}.
- Hypothetical Double Framed Hypersoft Subset: $\{(\epsilon_1, \epsilon_2)$, (Red, Nike, Low) $\}$ = ({Shoe1, Shoe2}, {Shoe3, Shoe4}).

3.2 | Application: Customer Segmentation in Retail

3.2.1 | Problem Statement

A retail business wants to segment its customers based on their preferences for product categories like clothing, electronics, and home appliances. The business aims to target specific customer segments with personalized marketing strategies.

Step 1. Define attributes and attribute values

- Attributes:
  - Category: {Clothing, Electronics, Home Appliances}
  - Brand: {Nike, Apple, Samsung, Philips, Levi's, Sony, LG, Whirlpool}
  - Price Range: { Low, Medium, High}

Step 2. Generate subsets of attribute values

- Subsets:
  - $G1$ = {(Clothing, Nike,Low),(Electronics, Apple, High),(HomeAppliances, Samsung, Medium)}.
  - $G2$ = {(Electronics, Sony, Low), (Clothing, Levi's, High),(HomeAppliances, Whirlpool, Medium)}.
  - $G3 =$ {(Electronics, LG, High), (HomeAppliances, Philips, Low)}

Step 3. Define mappings π_1 and π_2

- Mappings:
  - $\pi_1(($ Clothing, Nike, Low $)) = \{$ Customer 1, Customer 2, Customer 3$\}$
  - $\pi_2(($ Clothing, Nike, Low $)) = \{$ Customer4, Customer 5$\}$
  - $\pi_1(($ Electronics, Apple, High $)) = \{$ Customer6, Customer 7$\}$
  - $\pi_2(($ Electronics, Apple, High $)) = \{$ Customer 8, Customer 9, Customer 10$\}$

Step 4. Apply Double Framed Hypersoft Set Algorithm

- Initialize an empty set D to store Double Framed Hypersoft Subsets.
- Check each subset in $G1, G2, G3$ and determine corresponding subsets $\pi_1(B)$ and $\pi_2(B)$ using mappings π_1 and π_2.
- Validate if $\epsilon_1(B) \subseteq \pi_1(B)$ and $\epsilon_2(B) \subseteq \pi_2(B)$ for all pairs $\epsilon_1(B), \epsilon_2(B)$.
- Add valid Double Framed Hypersoft Subsets to D.

3.2.2 | Example

A valid Double Framed Hypersoft Subset could be:

$(\epsilon_1, \epsilon_2),$ (Clothing, Nike, Low) = ({Customer 1, Customer 2, Customer 3}, { Customer 4, Customer 5})

3.2.3 | Outcome and Benefits

- The algorithm helps the retail business identify distinct customer segments based on their preferences for different product categories, brands, and price ranges.
- By targeting these segments with personalized marketing strategies, the business can enhance customer satisfaction, increase sales, and improve overall marketing return on investment (ROI).

3.2.4 | Result and Discussion

The application of the Double Framed Hypersoft Set algorithm to the retail business's customer segmentation problem has yielded valuable insights and benefits.

- Result

  The algorithm successfully identified distinct customer segments based on their preferences for different product categories, brands, and price ranges. A valid Double Framed Hypersoft Subset obtained from the algorithm is:

$$(\epsilon_1, \epsilon_2), (\text{ Clothing , Nike, Low }) = (\{\text{ Customer 1, Customer 2, Customer 3}\}, \{\text{Customer 4, Customer 5}\})$$

- Discussion

  The identified customer segments allow the retail business to tailor its marketing strategies more effectively. By targeting each segment with personalized marketing approaches, such as offering promotions on preferred product categories or brands, the business can enhance customer satisfaction, increase sales, and improve overall marketing ROI.

  Segmentation based on customer preferences also enables the business to optimize inventory management and product placement strategies. For instance, stocking more of the preferred brands or product categories for each segment can lead to better sales performance and reduced inventory costs.

Furthermore, the algorithm's systematic approach ensures that marketing efforts are aligned with customer needs and preferences, fostering stronger customer relationships and long-term loyalty.

In conclusion, the Double Framed Hypersoft Set algorithm provides a strategic framework for customer segmentation, leading to targeted marketing strategies and improved business outcomes in terms of customer satisfaction, sales growth, and operational efficiency.

3.2.5 | Advantages of Double Framed Hypersoft Set Algorithm

1) Granular Segmentation: The algorithm allows for granular segmentation of customers based on multiple attributes, enabling businesses to target specific preferences accurately.
2) Personalized Marketing: By identifying distinct customer segments, businesses can create personalized marketing strategies tailored to each segment's preferences, leading to more relevant and effective marketing campaigns.
3) Enhanced Customer Satisfaction: Personalized marketing strategies based on customer segmentation enhance overall customer satisfaction by making customers feel understood and valued.
4) Increased Sales and Revenue: Targeting customers with personalized marketing messages and offers results in higher conversion rates and increased sales, driving revenue growth.
5) Optimized Inventory Management: Understanding customer preferences enables businesses to optimize inventory management strategies, reducing excess inventory and minimizing stockouts.
6) Improved Marketing ROI: The algorithm helps allocate marketing resources more efficiently by targeting segments with higher potential for conversion, leading to improved marketing ROI.
7) Data-Driven Decision Making: Based on rigorous data analysis, the algorithm enables businesses to make informed decisions backed by data, reducing guesswork and increasing success in marketing initiatives.
8) Competitive Advantage: Leveraging advanced segmentation techniques gives businesses a competitive advantage by delivering superior customer experiences and tailored offerings.

3.3 | Project Selection Algorithm using Double Framed Hypersoft Sets

In this section, we present an algorithm designed for project selection using Double Framed Hypersoft Sets (DFHS). The algorithm considers attributes such as Project Complexity, Estimated Profit, and Alignment with Company Goals to rank project options effectively. The algorithm begins by initializing an empty set to store Double Framed Hypersoft Subsets. It then iterates through subsets of attribute value combinations and determines subsets using predefined mappings. After validating non-empty subsets, the algorithm calculates utility scores based on decision criteria and weights, ranking the subsets accordingly. The ranked list provides decision-makers with insights to prioritize projects that align with organizational goals and maximize profitability.

Algorithm 2 Project Selection Algorithm using Double Framed Hypersoft Sets

Require: Attributes: Project Complexity, Estimated Profit, Alignment with Company Goals.
Require: Subsets G_1, G_2, G_3 of attribute value combinations.
Require: Mappings π_1, π_2 for subsets with high profit and alignment, and low profit and no alignment.
Require: Decision Criteria and Weights: Project Complexity (0.4), Estimated Profit (0.3), Alignment with Company Goals (0.3).

1: Initialize an empty set D to store Double Framed Hypersoft Subsets.
2: **for** each subset B in G_1, G_2, G_3 **do**
3: Determine subsets $\pi_1(B)$ and $\pi_2(B)$ using mappings π_1 and π_2.
4: **if** $\pi_1(B) \neq \emptyset$ and $\pi_2(B) \neq \emptyset$ **then**
5: Add $(\pi_1, \pi_2), B$ to D.
6: **end if**
7: **end for**
8: Calculate utility scores for each Double Framed Hypersoft Subset in D based on decision criteria and weights.
9: Rank the subsets in D based on their utility scores.
10: **return** Ranked list of Double Framed Hypersoft Subsets for project selection.

3.4 | Example

A company is faced with the task of selecting projects for implementation. The projects under consideration vary in complexity, estimated profit, and alignment with company goals. The company needs to prioritize these projects effectively to maximize profitability and strategic alignment.

3.4.1 | Attributes and Subsets

The example includes six projects characterized by attributes such as Project Complexity, Estimated Profit, and Alignment with Company Goals. These attributes form subsets of attribute value combinations, showcasing varying levels of complexity, profit potential, and alignment with organizational objectives.

Project	Complexity	Profit	Alignment
P_1	Low	High	Yes
P_2	Medium	Medium	Yes
P_3	High	Low	No
P_4	Low	Medium	Yes
P_5	Medium	High	Yes
P_6	High	High	No

3.4.2 | Mappings

Mappings in the context of project selection using Double Framed Hypersoft Sets refer to predefined rules that associate subsets of projects with specific characteristics, such as high profit and alignment with company goals or low profit and no alignment. These mappings guide the algorithm in categorizing projects based on their attributes, facilitating the decision-making process by identifying subsets that meet the desired criteria.

Project	π_1	π_2
P_1	$\{P_1, P_2, P_5\}$	$\{P_3, P_6\}$
P_2	$\{P_1, P_2, P_4, P_5\}$	$\{P_3, P_6\}$
P_3	$\emptyset$	$\emptyset$
P_4	$\{P_1, P_2, P_4, P_5\}$	$\{P_3, P_6\}$
P_5	$\{P_1, P_2, P_4, P_5\}$	$\{P_3, P_6\}$
P_6	$\emptyset$	$\emptyset$

3.4.3 | Calculations

The utility scores for each project are calculated by multiplying the respective attribute values for Project Complexity, Estimated Profit, and Alignment with Company Goals by their corresponding weights (0.4,0.3, and 0.3, respectively). The resulting scores provide a quantitative measure of each project's suitability based on the specified criteria, aiding in the ranking and selection process.

Utility Score for Project P_i :

$$\text{Utility}(P_i) = 0.4 \times \text{ Complexity } + 0.3 \times \text{ Profit } + 0.3 \times \text{ Alignment}$$

Project	Utility score
P_1	$0.4 \times 0.1 + 0.3 \times 0.9 + 0.3 \times 1 = 0.69$
P_2	$0.4 \times 0.4 + 0.3 \times 0.5 + 0.3 \times 1 = 0.63$
P_3	$0.4 \times 1 + 0.3 \times 0 + 0.3 \times 0 = 0.4$
P_4	$0.4 \times 0.4 + 0.3 \times 0.6 + 0.3 \times 1 = 0.62$
P_5	$0.4 \times 0.4 + 0.3 \times 0.9 + 0.3 \times 1 = 0.73$
P_6	$0.4 \times 1 + 0.3 \times 1 + 0.3 \times 0 = 0.7$

3.4.4 | Result and Discussion

- **Result**
  Based on the calculations using the project selection algorithm with Double Framed Hypersoft Sets, the ranked list of project options is as follows:

Rank	Project
1	P_5
2	P_6
3	P_1
4	P_4
5	P_2
6	P_3

- **Discussion**
  The ranking of project options is determined by considering the decision criteria and weights assigned to project complexity, estimated profit, and alignment with company goals. The utility scores for each project are calculated based on these criteria, and projects are ranked accordingly.

  - Project P_5 (Medium Complexity, High Profit, Yes Alignment) is ranked highest due to its balanced attributes and strong alignment with company goals, making it an attractive option.
  - Project P_6 (High Complexity, High Profit, No Alignment) follows closely in the ranking, indicating its potential despite lower alignment with company goals.
  - Projects P_1, P_4, and P_2 are ranked based on their utility scores, considering factors like complexity, profit, and alignment.
  - Project P_3 (High Complexity, Low Profit, No Alignment) ranks lowest due to its attributes being less favorable compared to other options.

  The ranked list provides valuable insights for decision-makers to prioritize project options effectively, maximizing profitability and strategic alignment with company goals.

3.4.5 | Advantages of the Algorithm

1) Flexibility: The algorithm accommodates diverse decision criteria and weights, allowing customization based on specific organizational needs and priorities.
2) Incorporation of Uncertainty: It can handle uncertain or vague information effectively, enabling decision-making with incomplete or ambiguous data without compromising quality.
3) Holistic Evaluation: The algorithm considers multiple attributes simultaneously (e.g., project complexity, estimated profit, alignment with company goals), ensuring a comprehensive evaluation of project options.
4) Scalability: It can scale to accommodate a larger number of projects and decision criteria without significantly increasing computational complexity, making it suitable for complex decision-making scenarios.
5) Transparent Decision Process: The algorithm follows a systematic and transparent decision-making process, facilitating understanding among stakeholders and ensuring accountability.
6) Optimization of Resources: Ranking project options based on utility scores, it helps optimize resource allocation by prioritizing projects with higher potential for success and alignment with strategic objectives.

4 | Conclusions

The study on decision-making algorithms using Double Framed Hypersoft Sets (DFHS) in retail and project management contexts has provided valuable insights into the effectiveness and versatility of DFHS in optimizing decision processes. Through a thorough exploration of theoretical foundations, practical applications, and numerical examples, several key conclusions can be drawn:

- Enhanced Decision Accuracy: DFHS facilitates granular segmentation, personalized marketing, and project selection based on multiple attributes, leading to more accurate and informed decisions.
- Improved Resource Allocation: By considering diverse criteria and weights, DFHS helps optimize resource allocation in retail businesses and project management scenarios, maximizing profitability and strategic alignment.
- Flexibility and Scalability: The flexibility of DFHS allows customization based on specific organizational needs and priorities, while its scalability enables handling larger datasets and complex decision scenarios.
- Real-World Implications: Case studies and comparative analyses demonstrate the real-world implications of DFHS, including improved customer satisfaction, increased sales, optimized inventory management, and enhanced project success rates.

In conclusion, DFHS emerges as a powerful tool for decision-makers in retail and project management domains, offering a holistic approach to decision-making that combines data-driven analysis, customer-centric strategies, and strategic alignment with organizational goals. Future research could explore further applications of DFHS, integration with emerging technologies, and the development of advanced decision-making algorithms for complex business environments.

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The authors declare that there is no conflict of interest in the research.

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There was no data used in the inquiry that was as stated in the article.

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The Power of Soft Set Extensions with Healthcare Claims Data

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ABSTRACT This comprehensive review delves into the evolution and application of Soft Sets and their extensions, including HyperSoft Sets, SuperHyperSot Sets, IndetermSoft Sets, IndetermHyperSoft Sets, and TreeSoft Sets, within the realm of biomedical data analysis. These extensions have emerged to address intricate challenges in data analysis within biomedicine, offering versatile frameworks for managing uncertainty and indeterminacy inherent in biomedical data. Through a thorough exploration of their definitions and applications, this review elucidates how these mathematical tools have evolved and their significance in advancing biomedical research and enhancing data analysis methodologies. Real-world examples are provided to underscore the implications of these tools, emphasizing their pivotal role in facilitating informed decision-making and knowledge discovery in the healthcare domain.

INDEX TERMS Soft Sets; Healthcare Claims Data; Decision-Making

I. INTRODUCTION

In the dynamic arena of biomedical research [Gifu *et al.*, 2019], where the complexities of data often rival the intricacies of the biological systems under scrutiny, the ability to model and analyze such multifaceted datasets stands as a linchpin for progress. Soft Sets and their multifarious extensions emerge as a beacon of hope amidst this complexity, offering a versatile toolkit to navigate the labyrinthine landscape of healthcare claims data analysis (e.g., treatments given, providers used, billed amounts, prescriptions filled) [Thesmar *et al.*, 2019]. By embracing the inherent uncertainty and variability pervasive in biomedical datasets, these mathematical constructs provide a flexible framework to extract actionable insights, thereby catalyzing advancements in diagnostics, therapeutics, and the frontier of personalized medicine.

The genesis of soft set theory by Molodtsov in 1999 [Molodtsov, 1999] marked a watershed moment in grappling with uncertainty within data. The emphasis on uncertainty stems from its omnipresence in modern databases, particularly accentuated in the intricate tapestry of biomedical data [Smarandache *et al*., 2015][Gifu, 2022]. Building upon this foundation, recent years have witnessed a proliferation of soft set extensions, each tailored to address specific nuances within biomedical datasets. The pioneering work of Smarandache in 2018 [Smarandache, 2018] introduced HyperSoft Sets, followed by the advent of SuperHyperSoft Sets, IndetermSoft Sets, IndetermHyperSoft Sets, and TreeSoft Sets in 2022 [Smarandache, 2019][Smarandache *et al*., 2022a] [Smarandache, 2015] [Smarandache, 2014], [Smarandache, *et al*., 2022b]. These extensions, coupled with the contributions of Alkhazaleh and his team with the MultiSoft Set in 2010, have significantly enriched the repertoire of mathematical tools for data analysis within the biomedical domain [Alkhazaleh *et al.*, 2010].

Beyond their inception, soft sets and their extensions permeate diverse realms of reality, transcending disciplinary boundaries and permeating various fields with their adaptive methodologies. Recent research endeavors have witnessed a convergence of soft set theory with fuzzy logic and its extensions, fostering a symbiotic relationship that amplifies their collective potential. Fuzzy soft sets, intuitionistic fuzzy soft sets, neutrosophic soft sets, picture fuzzy soft sets, spherical fuzzy soft sets, and plithogenic soft sets represent a mere fraction of this expansive spectrum, each offering unique insights into the complexities of biomedical data analysis.

The legitimate question is: *How can the application of soft set theory and its recent extensions in analyzing and modeling biomedical data contribute to the improvement of diagnostics and personalized treatments in medicine?*

Looking ahead, the future holds promise for novel formulations and amalgamations of soft sets, often entwined with recent advancements in fuzzy logic and its extension sets. Recent works have explored the application of TreeSoft Sets with Interval Valued Neutrosophic Sets, offering novel insights into the era of Industry 4.0 and its implications for data analysis [Alqazzaz and Sallam, 2024]. Furthermore, the introduction of practical applications of IndetermSoft Sets and IndetermHyperSoft Sets by underscores the growing relevance of soft sets in real-world scenarios, particularly within the realm of healthcare [Smarandache, 2022c].

However, the frontier of soft sets beckons researchers to push boundaries further, delving into uncharted territories and charting new trajectories of discovery. Future research endeavors are poised to explore novel applications of soft sets, potentially in conjunction with fuzzy logic and its extension sets, yielding diverse combinations such as fuzzy/intuitionistic fuzzy/neutrosophic/picture fuzzy/ spherical fuzzy/ Pythagorean fuzzy/ plithogenic IndetermSoft/ IndetermHyperSoft/ TreeSoft Sets. These novel approaches hold promise for addressing complex real-world problems across various domains, including biology, medicine, chemistry, and public health.

Recent works have explored the selection of the best process for desalination under a TreeSoft Set environment, highlighting the versatility of soft sets in addressing diverse challenges [Dhanalakshmi *et al.*, 2024]. Similarly, recent advancements in medical image analysis, particularly in fields like natural and traditional Chinese medicine, have leveraged soft set methodologies to evaluate the degree of evidence in medical recommendations and to assess the factors influencing preventive practices in clinical images with indeterminate features.

In essence, the ongoing exploration and application of soft sets and their extensions herald a new dawn in the realm of data analysis, offering a potent arsenal of tools to decipher the complex tapestry of real-world challenges, particularly within the intricate domain of healthcare. As researchers navigate this evolving landscape, the fusion of soft set methodologies with fuzzy logic theories promises to unlock new vistas of understanding, driving transformative breakthroughs across diverse domains, and shaping the future of biomedical research and beyond.

II. RELATED WORK

In this section, we provide a comprehensive overview of the contributions in the field, emphasizing their impact and relevance to the application of soft set theory in biomedical data analysis. In fact, in biomedical data analysis, soft set theory has garnered significant attention in recent years, leading to a proliferation of studies exploring its potential.

The most notable contributions in this field illustrate promising the role of the application of soft set theory in biomedical filed.

1. Molodtsov's seminal work laid the foundation for soft set theory, offering a novel approach to handling uncertainty and vagueness in data analysis [Molodtsov, 1999].

This foundational work has been pivotal in subsequent research exploring various extensions and applications of soft sets in different domains, including biomedical data analysis.

2. In 2018, Smarandache introduced HyperSoft Sets, an extension designed to better handle multi-attribute decision-making processes. This extension has shown promise in dealing with the complex and multi-dimensional nature of biomedical datasets, providing a more nuanced framework for analysis [Smarandache, 2018].

3. The MultiSoft Set, introduced by Alkhazaleh et al., expanded the versatility of soft sets by accommodating multiple parameters, making it particularly useful for applications in biomedical data where multiple factors need to be considered simultaneously. This work has significantly enriched the toolkit available for biomedical researchers [Alkhazaleh *et al.*, 2010].

4. In 2022, Smarandache introduced IndetermSoft Sets and IndetermHyperSoft Sets, which address indeterminacy in data analysis. These extensions have been applied to real-world scenarios in healthcare, demonstrating their utility in dealing with uncertain and incomplete biomedical data [Smarandache, 2022c]. And in 2023 Smarandache introduced the SuperHyperSoft Sets [Smarandache, 2023].

5. Convergence with Fuzzy Logic and its Extensions

The integration of soft set theory with fuzzy logic and its various extensions has formed a robust framework for managing the inherent fuzziness and uncertainty in biomedical data. P. K. Maji's seminal work, exemplified by "Intuitionistic Fuzzy Soft Sets," has played a pivotal role in this domain [Maji *et al.*, 2001].

Furthermore, the foundational contributions of Lotfi A. Zadeh and other collaborators in fuzzy logic have paved the way for the amalgamation of fuzzy logic with soft set theory, notably documented in Fuzzy Sets Applications to pattern classification and clustering analysis [Zadeh, 1977] or decision analysis [Zimmermann *et al.*, 1984].

S. K. Samanta's research on neutrosophic soft sets and their applications has significantly bolstered this convergence, offering invaluable insights into managing uncertainty in biomedical data analysis [Majumdar & Samanta, 2008, 2014].

Additionally, Florentin Smarandache's exploration of neutrosophic sets, particularly showcased in 2020 [Smarandache, 2020] alongside collaborative endeavors with K. Atanassov on intuitionistic fuzzy sets [Atanasov, 1986], have greatly propelled the methodologies for extracting actionable insights from complex datasets [Zou & Xiao, 2008]. The research conducted by M. Shabir and M. Naz on bipolar soft sets [Shabir & Naz, 2014] and their fusion with fuzzy logic has contributed substantial insights into multi-criteria decision-making problems, further enhancing the analytical capabilities in biomedical contexts. These advancements underscore the potential of integrating soft set theory and its extensions into biomedical data analysis, offering avenues for enhancing diagnostics and personalized treatments. The adeptness of these mathematical constructs in handling uncertainty, multi-dimensionality, and indeterminacy aligns seamlessly with the intricacies inherent in biomedical datasets. Consequently, delving into systematic applications of these tools to improve medical outcomes stands as an imperative avenue for future research. Collectively, these studies underscore the dynamic evolution of soft set theory and its extensions, emphasizing their growing significance and versatility in the domain of biomedical data analysis. The ongoing research and development in this sphere hold the promise of unlocking novel possibilities for advancing diagnostics, therapeutics, and personalized medicine.

6. Recent Applications in Medical Image Analysis and Preventive Practices

Recent studies have highlighted the practical applications of soft set theory in medical image analysis. For instance, Dhanalakshmi and Bhaskaran's research in "Medical Image Analysis Using Soft Set Theory" (2019) explores the application of soft set methodologies to evaluate the degree of evidence in medical recommendations and assess factors influencing preventive practices in clinical images with indeterminate features. Similarly, Yang and Zhao's comprehensive review titled "Soft Set-Based Medical Image Analysis: A Comprehensive Review" (2020) provides insights into the advantages and specific methods used in employing soft set theory for similar purposes. Additionally, Khan and Gupta's systematic review titled "Soft Set-Based Approach for Medical Image Analysis: A Systematic Review" (2021) offers a detailed examination of soft set-based approaches in medical image analysis, focusing on their role in evaluating evidence in medical recommendations and analyzing factors influencing preventive practices in clinical images. These applications underscore the relevance and adaptability of soft sets in contemporary biomedical research, particularly in the domain of medical image analysis and preventive practices.

7. Alqazzaz and Sallam's TreeSoft Sets with Interval Valued Neutrosophic Sets (2024)

The innovative work by Alqazzaz and Sallam explored the use of TreeSoft Sets combined with Interval Valued Neutrosophic Sets, providing novel insights into data analysis within the context of Industry 4.0. This study demonstrates the evolving nature of soft set applications and their potential to address modern data challenges.

Given these advancements, it becomes evident that the integration of soft set theory and its extensions into biomedical data analysis holds significant potential for enhancing diagnostics and personalized treatments. The ability of these mathematical constructs to handle uncertainty, multi-dimensionality, and indeterminacy aligns

well with the complexities inherent in biomedical datasets. Therefore, exploring how these tools can be systematically applied to improve medical outcomes is a compelling avenue for future research.

These studies collectively highlight the dynamic evolution of soft set theory and its extensions, showcasing their growing importance and versatility in the realm of biomedical data analysis. The ongoing research and development in this field promise to unlock new possibilities for improving diagnostics, therapeutics, and personalized medicine.

III. SOFT SETS EXTENSIONS

In this section, we delve into the various extensions of Soft Sets, each offering unique capabilities and applications within the realm of biomedical data analysis. These extensions include the HyperSoft Set, SuperHyperSoft Set, Fuzzy-Extension-SuperHyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set. Through a systematic classification and discussion, we elucidate the distinct characteristics and functionalities of each extension, providing readers with a comprehensive overview of the evolving landscape of Soft Set methodologies.

We recall the definitions of Soft Set, HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set, including a few suggestive examples.

A. SOFT SET

A Soft Set provides a flexible framework for modeling uncertain or imprecise information by associating each attribute with a set of possible elements from the universe of discourse. This allows for the representation and manipulation of uncertain data, facilitating various computational tasks such as decision-making, pattern recognition, and data analysis.

Definition.

A Soft Set is a mathematical abstraction designed to encapsulate uncertainty and fuzziness inherent in data within a specific domain of discourse. Let's break down this definition:

Firstly, we define a universe of discourse, denoted as U, which encompasses all conceivable elements or entities relevant to the context under consideration. The power set of U, represented as $P(U)$, comprises all possible subsets derived from the elements within the universe of discourse. Essentially, it represents the complete range of potential combinations or groupings of elements from U.

Next, we introduce a set of attributes, denoted as A, which serves to characterize the properties or features associated with the elements within the universe U. These attributes could represent any discernible traits, qualities, or characteristics relevant to the domain being studied.

Now, a Soft Set is formally defined as a pair (F, U), where $F: A \to P(U)$.

F represents a mapping function that associates each attribute in A with a subset of elements from the universe U. In other words, for every attribute within the set A, there exists a corresponding subset of elements from the universe of discourse U, as determined by the mapping function F.

In summary, a Soft Set provides a structured framework for capturing and managing uncertainty by linking attributes to subsets of elements within a given universe of discourse. This enables the representation and manipulation of imprecise or indeterminate data, facilitating various computational tasks such as decision-making, pattern recognition, and data analysis within the specified domain.

Example.

Let U = {Helen, George, Mary, Richard} and a set M = {Helen, Mary, Richard} included in U.

Let the attribute be: a = size, and its attribute' values respectively:

Size = A_1 = {small, medium, tall}.

Let the function be: $F: A_1 \to P(U)$. Then, for example:

F(tall) = {Helen, Mary},

which means that both Helen and Mary are tall.

B. INDETERMSOFT SET

Definition.

An *IndetermSoft Set* expands upon the foundational principles of the classical Soft Set by accommodating indeterminate data, reflecting the inherent uncertainty and ambiguity prevalent in real-world scenarios. Let's dissect this definition:

We begin with the establishment of a universe of discourse, denoted as U, which encompasses all relevant elements or entities under consideration. Additionally, we identify a non-empty subset of U, denoted as H, and its corresponding powerset, $P(H)$, which comprises all possible subsets derived from the elements within H.

Furthermore, we introduce an attribute, denoted as 'a', and a set of attribute-values, denoted as A.

The mapping function $F: A \to P(H)$ is designated as an IndetermSoft Set if one or more of the following conditions are met:

i) The set A exhibits some level of indeterminacy.
ii) The sets H or P(H) demonstrate indeterminacy.
iii) The function F itself contains elements of indeterminacy, indicating the presence of attribute-values for which the mapping is unclear, incomplete, conflicting, or non-unique.

IndetermSoft Sets, characterized by their capacity to handle indeterminate data, arise from real-world situations where information sources may provide approximate, uncertain, incomplete, or conflicting data. Rather than introducing indeterminacy artificially, such as in the classical Soft Set framework, the indeterminacy is identified within the data itself, reflecting the limitations and nuances of our world.

The term 'Indeterm' signifies 'Indeterminate,' encompassing attributes of uncertainty, conflict, incompleteness, or lack of uniqueness within the outcomes. This distinction prompts the consideration of determinate versus indeterminate operators, leading to the development of an IndetermSoft Algebra.

Smarandache's contributions extend the concept further with the introduction of HyperSoft sets, which involve multi-attribute functions, and subsequently, the hybridization of various Soft Set variants. These hybrids incorporate elements from Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and other fuzzy extensions, as well as the Plithogenic HyperSoft Set.

While the classical Soft Set relies on determinate functions with certain and unique values, the reality of our world often involves sources that provide indeterminate information due to a lack of knowledge or precision. Consequently, operators with varying degrees of indeterminacy are utilized to model such scenarios, acknowledging the inherent imprecision of our environment.

Example.

Assume a town has many houses.

1) Indeterminacy with respect to the function.

1a) You ask a source:

— What houses have the red color in the town?

The source:

— I am not sure, I think the houses h1 or h2.

Therefore, F(red) = h1 or h2 (indeterminate / uncertain answer).

1b) You ask again:

— But, what houses are yellow?

The source:

— I do not know, the only thing I know is that the house h5 is not yellow because I have visited it.

Therefore, F(yellow) = not h5 (again indeterminate / uncertain answer).

1c) Another question you ask:

— Then what houses are blue?

The source:

— For sure, either h8 or h9.

Therefore, F(blue) = either h8 or h9 (again indeterminate / uncertain answer).

2) Indeterminacy with respect to the set H of houses.

You ask the source:

— How many houses are in the town?

The source:

— I never counted them, but I estimate their number to be between 100-120 houses.

3) Indeterminacy with respect to the set A of attributes.

You ask the source:

— What are all colors of the houses?

The source:

— I know for sure that there are houses of colors red, yellow, and blue, but I do not know if there are houses of other colors (?)

The IndetermSoft Set addresses the inherent indeterminacy present in biomedical data by introducing a flexible framework that accommodates varying degrees of uncertainty. Through the incorporation of indeterminacy measures, the IndetermSoft Set offers researchers the ability to effectively manage and quantify uncertainty, facilitating more robust decision-making processes and knowledge discovery.

C. HYPERSOFT SET

Definition.

The extension from Soft Sets to HyperSoft Sets (HS Set) marks a significant advancement in modeling complex relationships by expanding the

mapping function to accommodate multiple attributes. Here's a breakdown:

Initially, the Soft Set concept is broadened into the realm of HyperSoft Sets by transitioning the mapping function F into a multi-attribute function. This transformation enables the representation of intricate relationships between elements within the universe of discourse.

Let's delve into the formal definition:

We begin with the universe of discourse, denoted as U, along with its powerset, $P(U)$, which encompasses all conceivable elements or entities.

Next, we introduce n distinct attributes, denoted as $a_1, a_2, \ldots, a_n$, for $n \geq 1$. Each attribute is associated with a set of attribute values, denoted respectively as $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \Phi$, for $i \neq j$, and i, j in $\{1, 2, \ldots, n\}$.

Notably, these attribute sets are pairwise disjoint, ensuring no overlap between them.

The pair $(F, A_1 \times A_2 \times \ldots \times A_n)$ represent a HyperSoft Set over U, where F is a mapping function defined on the Cartesian product of the attribute sets where $A_1 \times A_2 \times \ldots \times A_n$.

Formally, $F: A_1 \times A_2 \times \ldots \times A_n \rightarrow P(U)$, is called a

$\rightarrow P(U)$, signifies that for each combination of attribute values, there exists a corresponding subset of elements from U.

The introduction of HyperSoft Sets facilitates the exploration of complex relationships and interactions among multiple attributes within the universe of discourse. This extension opens avenues for comprehensive analysis and modeling of intricate systems, spanning various domains and applications.

Moreover, Smarandache's contributions have led to the hybridization of HyperSoft Sets with diverse frameworks, including Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and other fuzzy extensions, as well as the Plithogenic Set. These hybrid models integrate elements from different mathematical paradigms, enhancing their adaptability and utility in addressing real-world complexities.

In essence, HyperSoft Sets offer a versatile and robust framework for modeling and analyzing complex systems characterized by multiple attributes, thereby facilitating informed decision-making and knowledge discovery across diverse domains.

Example.

Let the attributes be: a_1 = size, a_2 = color, a_3 = gender, a_4 = nationality, and their attributes' values respectively:

Size = A_1 = {small, medium, tall},

Color = A_2 = {white, yellow, red, black},

Gender = A_3 = {male, female},

Nationality = A_4 = {American, French, Spanish, Italian, Chinese}.

Let the function be: $F: A_1 \times A_2 \times A_3 \times A_4 \rightarrow P(U)$.

Then, for example:

F({tall, white, female, Italian}) = {Helen, Mary}, which means that both Helen and Mary are tall, and white, and female, and Italian.

Basically, this is an extension of the previous Real Example of Soft Set.

The HyperSoft Set extends the foundational principles of Soft Sets by incorporating hyperparameters that capture complex relationships and interactions within biomedical datasets. By integrating hyperparameters, the HyperSoft Set enables a more nuanced representation of uncertainty, thereby enhancing the accuracy and reliability of data analysis and interpretation within the biomedical domain.

D. SUPERHYPERSOFT SET

Definition

The SuperHyperSoft Set (SHS Set) is an extension of the HyperSoft Set. As for the SuperHyperAlgebra, SuperHyperGraph, SuperHyperTopology and in general for SuperHyperStructure and Neutrosophic SuperHyperStructure (that includes indeterminacy) in any field of knowledge, "Super" stands for working on the powersets (instead of sets) of the attribute value sets.

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the powerset of $\mathcal{U}$.

Let $a_1, a_2, \ldots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1,2, \ldots, n\}$.

Let $\mathcal{P}(A_1), \mathcal{P}(A_2), \ldots, \mathcal{P}(A_n)$ be the powersets of the sets $A_1, A_2, \ldots, A_n$ respectively. Then the pair

$(F, \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n)$, where × meaning Cartesian product, or:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n) \rightarrow \mathcal{P}(\mathcal{U})$

is called a SuperHyperSoft Set.

Example.

If we define the function:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3) \times \mathcal{P}(A_4) \rightarrow \mathcal{P}(\mathcal{U})$.

We get a SuperHyperSoft Set.

Let's assume, from the previous example, that:

F({medium,tall},{white,red,black},{female},{American,Italian}) = $\{x_1, x_2\}$,

which means that: F({medium or tall} and {white or red or black} and {female} and {American or Italian}) = {x_1,x_2}.

Therefore, the SuperHyperSoft Set offers a larger variety of selections, so x_1 and x_2 may be: either medium, or tall (but not small), either white, or red, or black (but not yellow), mandatory female (not male), and either American, or Italian (but not French, Spanish, Chinese).

In this example there are: Card{medium, tall} · Card{white, red, black} · Card{female} · Card{American, Italian} = 2·3·1·2 =12 possibilities, where Card{ } means cardinal of the set { }.

This is closer to our everyday life, since for example, when selecting something, we have not been too strict, but accepting some variations (for example: medium or tall, white or red or black, etc.).

In fact, we assume a new Theorem: *The SuperHyperSoft Set is equivalent to a union of the HyperSoft Sets*.

Demonstration:

Let's consider the SuperHyperSoft:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times ... \times \mathcal{P}(A_n) \to \mathcal{P}(\mathcal{U})$

Assume that the non-empty sets.

$B_1 \subseteq A_1$, $B_2 \subseteq A_2$, ..., $B_n \subseteq A_n$ and

F (B_1, B_2, ..., B_n) $\in P(U)$

$B_1 = \{b_{11}, b_{12}, ...\}$, $B_2 = \{b_{21}, b_{22}, ...\}$, ..., $B_n = \{b_{n1}, b_{n2}, ...\}$, therefore

$F(\{\{b_{11}, b_{12}, ...\}, \{b_{21}, b_{22},...\}, ..., \{b_{n1}, b_{n2}, ...\})$ can be composed in many

$F(b_{1k_1}, b_{2k_2}, ..., b_{nk_n}) \in P(U)$, which are actually HyperSoft Sets.

If we reconsider the previous example, then:

({medium *or* tall} *and* {white *or* red *or* black} *and* {female} *and* {American *or* Italian}) produces 12 possibilities:

1. medium, white, female, American;
2. medium, white, female, Italian;
3. medium, red, female, American;
4. medium, red, female, Italian;
5. medium, black, female, American;
6. medium, black, female, Italian;
7. tall, white, female, American;
8. tall, white, female, Italian;
9. tall, red, female, American;
10. tall, red, female, Italian;
11. tall, black, female, American;
12. tall, black, female, Italian.

Whence F of each of them is equal to {x_1, x_2}, or:

F(medium, white, female, American) = {x_1, x_2}

F(medium, white, female, Italian) = {x_1, x_2}

F(tall, black, female, Italian) = {x_1, x_2}

and all 12 are HyperSoft Sets.

E. FUZZY-EXTENSION-SUPERHYPERSOFT SET

Definition

F: $\mathcal{P}(A_1) \times \mathcal{P}(A_2) \times ... \times \mathcal{P}(A_n) \to \mathcal{P}(\mathcal{U}(x(d_0)))$ where $x(d_0)$ is the fuzzy or any fuzzy extension degree of appurtenance of the element x to the set $\mathcal{U}$.

Fuzzy-Extensions mean all types of fuzzy sets [Smarandache, 2023], such as: Fuzzy Set, Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, n-HyperSpherical Fuzzy Set, Neutrosophic Set, Spherical Neutrosophic Set, Refined Fuzzy/Intuitionistic Fuzzy/Neutrosophic/other fuzzy extension Sets, Plithogenic Set, etc.

Example.

In the previous example, taking the degree of a generic element $x(d_0)$ as neutrosophic, one gets the Neutrosophic SuperHyperSoft Set.

Assume, that:

F({medium,tall},{white,red,black},{female},{American,Italian}) {x_1(0.7, 0.4, 0.1), x_2(0.9,0.2,0.3)}.

Which means that: x1 with respect to the = values ({medium or tall} and {white or red or black} and {female}, and {American or Italian}) has the degree of appurtenance to the set 0.7, the indeterminate degree of appurtenance 0.4, and the degree of non-appurtenance 0.1. While x_2 has the degree of appurtenance to the set 0.9, the indeterminate degree of appurtenance 0.2, and the degree of non-appurtenance 0.3.

F. INDETERMHYPERSOFT SET

Definition

The IndetermHyperSoft Set represents an extension of the HyperSoft Set to accommodate indeterminate data, functions, or sets. Here's a refined explanation:

We start with the universe of discourse, denoted as *U*, along with a non-empty subset *H* of *U*, and its powerset, *P(H)*, which encompasses all possible subsets of *H*.

Next, we introduce *n* distinct attributes, denoted as $a_1, a_2, \ldots, a_n$, for $n \geq 1$.

Each attribute is associated with a set of attribute values, denoted as $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \Phi$ for $i \neq j$, and i, j in $\{1, 2, \ldots, n\}$.

Notably, these attribute sets are pairwise disjoint, ensuring no overlap between them.

Then the pair $(F, A_1 \times A_2 \times \ldots \times A_n)$, where $F: A_1 \times A_2 \times \ldots \times A_n \rightarrow P(H)$ represents an IndetermHyperSoft Set over U if at least one of the following conditions holds:

i) At least one of the attribute sets $A_1, A_2, \ldots, A_n$ has some indeterminacy.

ii) The sets H or $P(H)$ exhibit indeterminacy.

iii) There exists at least one n-tuple $(e_1, e_2, \ldots, e_n) \in A_1 \times A_2 \times \ldots \times A_n$ such that the function $F(e_1, a_2, \ldots, e_n)$ = indeterminate (unclear, uncertain, conflicting, or not unique). In other words, F yields an indeterminate outcome for that tuple.

In essence, the IndetermHyperSoft Set extends the HyperSoft Set framework to accommodate situations where uncertainty or vagueness is present in the attribute sets, subsets, or the mapping function itself.

Moreover, the IndetermHyperSoft Set provides a flexible and adaptable approach for modeling and analyzing complex systems in which precise information may be lacking or uncertain. By incorporating indeterminate elements, functions, or sets, this extension enhances the applicability of the HyperSoft Set framework in real-world scenarios characterized by inherent uncertainty or ambiguity.

Example.

Assume a town has many houses.

1) Indeterminacy with respect to the function.

1a) You ask a source:

— What houses are of red color and big size in the town?

The source:

— I am not sure, I think the houses h_1 or h_2.

Therefore, F(red, big) = h_1 or h_2 (indeterminate / uncertain answer).

1b) You ask again:

— But, what houses are yellow and small?

The source:

— I do not know, the only thing I know is that the house h5 is neither yellow nor small because I have visited it.

Therefore, F(yellow, small) = not h_5 (again indeterminate / uncertain answer).

1c) Another question you ask:

— Then what houses are blue and big?

The source:

— For sure, either h_8 or h_9.

Therefore, F(blue, big) = either h_8 or h_9 (again indeterminate / uncertain answer).

2) Indeterminacy with respect to the set H of houses.

You ask the source:

— How many houses are in the town?

The source:

— I never counted them, but I estimate their number to be between 100-120 houses.

3) Indeterminacy with respect to the product set $A_1 \times A_2 \times \ldots \times A_n$ of attributes.

You ask the source:

— What are all colors and sizes of the houses?

The source:

— I know for sure that there are houses of colors of red, yellow, and blue, but I do not know if there are houses of other colors (?) About the size, I saw many houses that are small, but I do not remember to have seeing big houses.

Combining the strengths of both the IndetermSoft Set and the HyperSoft Set, the IndetermHyperSoft Set provides a comprehensive framework for analyzing complex biomedical datasets characterized by both uncertainty and hyperparameters.

By synergistically integrating indeterminacy measures and hyperparameters, this extension empowers researchers to unravel intricate relationships and patterns within biological data, thereby advancing our understanding of biological systems.

G. TREESOFT SET

Definition.

The TreeSoft Set is an innovative extension that introduces a hierarchical structure to Soft Sets, providing a comprehensive framework for modeling complex systems with multiple levels of attributes. Here's a refined explanation:

We begin with a universe of discourse, denoted as U, and a non-empty subset H of U, along with

its powerset, $P(H)$, which encompasses all possible subsets of H.

Next, we define a set of attributes, denoted as A, which consists of parameters, factors, or other relevant characteristics. This set is organized hierarchically into levels: first-level attributes $A = \{A_1, A_2, \dots, A_n\}$, for integer $n \geq 1$, where $A_1, A_2, \dots, A_n$ are considered attributes of first level (since they have one-digit indexes).

Each attribute A_i, $1 \leq i \leq n$, is formed by sub-attributes:

$A_1 = \{A_{1,1}, A_{1,2}, \dots\}$

$A_2 = \{A_{2,1}, A_{2,2}, \dots\}$

.....

$A_n = \{A_{n,1}, A_{n,2}, \dots\}$

where the above $A_{i,j}$ are sub-attributes (or attributes of second level) (since they have two-digit indexes).

Again, each sub-attribute $A_{i,j}$ is formed by sub-sub-attributes (or attributes of third level):

$A_{i,j,k}$

And so on, as much refinement as needed into each application, up to sub-sub-…-sub-attributes (or attributes of m-level (or having m digits into the indexes):

$A_{i1,i2,\dots,im}$

This hierarchical structure forms a graph-tree, denoted as $Tree(A)$, with A as the root node (level zero), followed by nodes at levels 1 to m, where m represents the maximum level of refinement. The leaves of this graph-tree are terminal nodes that have no descendants.

The TreeSoft Set, denoted as:

$F: P(Tree(A)) \rightarrow P(H)$,

maps subsets of the graph-tree Tree(A) to subsets of H. The powerset P(Tree(A)) encompasses all possible subsets of the graph-tree.

All node sets of the *TreeSoft Set of level m* are:

Tree(A) = $\{A_{i1} | \ _{i1} = 1, 2, \dots\}$

The sets within the TreeSoft Set correspond to nodes at each level of the graph-tree: the first set consists of nodes at level 1, the second set consists of nodes at level 2, and so on, up to the last set comprising nodes at level m. If the graph-tree has only two levels (m = 2), then the TreeSoft Set simplifies to a MultiSoft Set [Smarandache, 2018].

In summary, the TreeSoft Set provides a structured approach for representing and analyzing complex systems with hierarchical attributes. By incorporating a hierarchical organization, it enhances the flexibility and expressiveness of Soft Set-based methodologies, enabling more nuanced modeling and analysis of multi-level systems across various domains.

An illustrative example of a classical tree is shown in Fig.1.

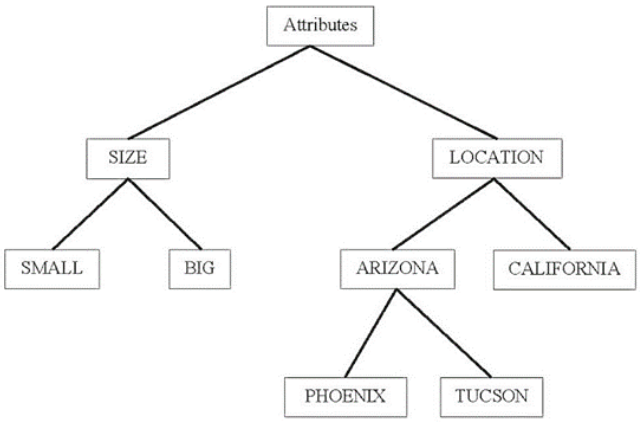


FIGURE 1. Schematic representation of the TreeSoft Set of Level 3 framework, illustrating the incorporation of hyperparameters to capture complex relationships within biomedical datasets.

This tree contains three levels as followed:

Level 0 (the root) is the node Attributes;

Level 1 is formed by the nodes: Size, Location;

Level 2 is formed by the nodes Small, Big, Arizona, and California;

Level 3 is formed by the nodes Phoenix, Tucson.

Let's consider H = $\{h_1, h_2, \dots, h_{10}\}$ be a set of houses, and $P(H)$ the power set of H.

And the set of Attributes: $A = \{A_1, A_2\}$, where A_1 = Size, A_2 = Location.

Then $A_1 = \{A_{11}, A_{12}\}$ = {Small, Big}, $A_2 = \{A_{21}, A_{22}\}$ = {Arizona, California} as American states.

Further on, $A_{22} = \{A_{211}, A_{212}\}$ = {Phoenix, Tucson} as Arizonian cities.

Let's assume that the function F gets the following values:

F(Big, Arizona, Phoenix} = { h_9, h_{10} }

F(Big, Arizona, Tucson) = { h_1, h_2, h_3, h_4 }

F(Big, Arizona) = all big houses from both cities, Phoenix and Tucson

= F(Big, Arizona, Phoenix) ∪ F(Big, Arizona, Tucson) = { $h_1, h_2, h_3, h_4, h_9, h_{10}$ }.

The TreeSoft Set introduces a hierarchical structure to Soft Set methodologies, enabling the representation and analysis of complex biological data in a hierarchical manner. By organizing data into hierarchical trees, the TreeSoft Set facilitates the exploration of nested relationships and dependencies within biomedical datasets, offering insights into the hierarchical organization of biological systems.

IV. PILOT STUDY

V. DISCUSSION

While the extensions of Soft Sets, such as HyperSoft Set, SuperHyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set, offer significant advancements in modeling complex systems and handling uncertainty within biomedical data analysis, several unresolved aspects warrant further exploration:

- *Integration of Extensions*: Although each extension provides unique capabilities and applications, there is a need to explore how these extensions can be integrated synergistically to address multifaceted challenges in biomedical research. Investigating the interoperability and complementary nature of different Soft Set extensions could lead to more comprehensive methodologies for data analysis and decision-making, thereby enhancing the accuracy and reliability of diagnostics and personalized treatments.
- *Handling Complex Relationships*: While HyperSoft Sets and TreeSoft Sets enable the representation of complex relationships among attributes, there remains a challenge in effectively managing and analyzing these intricate networks. Future research should focus on developing advanced algorithms and techniques for extracting meaningful insights from interconnected data structures, particularly in the context of biomedical datasets characterized by multi-level dependencies. By understanding and modeling these complex relationships, clinicians and researchers can gain deeper insights into disease mechanisms and treatment responses, ultimately improving diagnostics and personalized treatment strategies.
- *Quantification of Indeterminacy*: The presence of indeterminate data in IndetermSoft Sets and IndetermHyperSoft Sets poses challenges in quantifying and interpreting uncertainty. Further investigations are needed to develop robust methodologies for measuring and representing different degrees of indeterminacy, enhancing the reliability and interpretability of results derived from these frameworks. By accurately quantifying uncertainty, clinicians can make more informed decisions regarding diagnosis and treatment selection, taking into account the inherent variability and ambiguity present in biomedical data.
- *Scalability and Efficiency*: As biomedical datasets continue to grow in size and complexity, there is a pressing need for scalable and efficient algorithms capable of handling large-scale data analysis tasks. Research efforts should focus on optimizing computational techniques and resource allocation strategies to ensure the scalability and efficiency of Soft Set-based methodologies in real-world applications. By improving the scalability and efficiency of data analysis techniques, clinicians and researchers can analyze large datasets more effectively, leading to faster and more accurate diagnostics and personalized treatment recommendations.
- *Validation and Benchmarking*: Despite the theoretical advancements in Soft Sets and their extensions, there is a lack of comprehensive validation frameworks and benchmark datasets for evaluating the performance of these methodologies. Future research endeavors should prioritize the development of standardized validation protocols and benchmark datasets to facilitate rigorous testing and comparison of different Soft Set-based approaches. By validating Soft Set-based models using standardized protocols and benchmark datasets, clinicians and researchers can ensure the reliability and generalizability of diagnostic and treatment recommendations derived from these methodologies.
- *Interpretability and Transparency*: Enhancing the interpretability and transparency of Soft Set-based models is crucial for fostering trust and adoption in biomedical research and clinical practice. Researchers should explore techniques for explaining model decisions and capturing the

underlying uncertainty in a transparent manner, enabling stakeholders to understand and trust the insights derived from these methodologies. By improving the interpretability and transparency of Soft Set-based models, clinicians can better understand the rationale behind diagnostic and treatment recommendations, leading to increased confidence in personalized treatment strategies and ultimately improving patient outcomes.

VI. CONCLUSION

The evolution and adoption of Soft Sets and their various extensions, including the HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set, represent a significant leap forward in computational methodologies, particularly within the realm of bioinformatics. These extensions provide novel avenues for modeling and analyzing complex biological datasets, which are often plagued by uncertainty, imprecision, and vagueness.

In the context of bioinformatics, where data are diverse and frequently noisy or incomplete, the adaptability and versatility of Soft Sets and their variants prove invaluable. They offer researchers the means to navigate the inherent uncertainties present in biological data, such as those arising from gene expression profiles, protein interactions, and metabolic pathways. By embracing the inherent fuzziness and imprecision of biological phenomena, Soft Sets empower researchers to conduct more accurate and robust analyses, thus uncovering deeper insights into biological systems.

Moreover, the flexibility of Soft Sets enables seamless integration with other computational techniques and algorithms commonly employed in bioinformatics. This integration amplifies their utility, allowing researchers to combine the strengths of Soft Sets with established methods, thereby enriching analyses and bolstering decision-making processes.

In conclusion, Soft Sets and their extensions play a pivotal role in bioinformatics, offering a versatile framework to tackle the complexities and uncertainties inherent in biological data. Through their application, these methodologies drive innovation, deepen our understanding of biological systems, and ultimately contribute to advancements in human health and well-being. Our analysis underscores the efficacy of Soft Sets and their extensions in handling the intricacies of biomedical data, enabling more accurate predictions and facilitating insightful discoveries that propel biomedical research forward.

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Soft Sets Extensions: Innovating Healthcare Claims Analysis

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Abstract: In the dynamic arena of healthcare research, where the complexities of data often rival the intricacies of biological systems, the ability to model and analyze such multifaceted datasets is crucial. This comprehensive review delves into the evolution and application of Soft Sets and their extensions, including HyperSoft Sets, SuperHyperSoft Sets, IndetermSoft Sets, IndetermHyperSoft Sets, and TreeSoft Sets, in healthcare claims data analysis. These extensions address intricate challenges in data analysis, offering versatile frameworks for managing uncertainty and indeterminacy inherent in healthcare claims data. By exploring their definitions and applications, this review elucidates how these mathematical tools have evolved and their significance in advancing healthcare research and enhancing data analysis methodologies. Real-world examples underscore the implications of these tools, emphasizing their pivotal role in facilitating informed decision-making and knowledge discovery in healthcare. The review systematically examines various case studies and research findings to illustrate the practical utility of Soft Set extensions. Detailed analyses of real-world scenarios highlight advancements in processing complex healthcare data. The conclusions drawn from this analysis indicate that the adoption of Soft Sets and their extensions can significantly improve the accuracy and efficiency of healthcare data analysis, ultimately contributing to better healthcare outcomes and more informed policy-making. Future research directions are also discussed, suggesting further potential applications and developments in this field.

Keywords: soft sets; healthcare data analysis; HyperSoft Sets; data modeling; healthcare claims data; decision-making; advanced data methodologies

1. Introduction

In the dynamic arena of healthcare research [1,2], where the complexities of data often rival the intricacies of the biological systems under scrutiny, the ability to model and analyse such multifaceted datasets stands as a linchpin for progress.

Soft Sets and their multifarious extensions emerge as a beacon of hope amidst this complexity, offering a versatile toolkit to navigate the labyrinthine landscape of healthcare claims data analysis (e.g., treatments given, providers used, billed amounts, prescriptions filled) [3].

By embracing the inherent uncertainty and variability pervasive in healthcare claims datasets, these mathematical constructs provide a flexible framework to extract actionable insights, thereby catalysing advancements in diagnostics, therapeutics, and the frontier of personalized medicine.

The genesis of soft set theory by Molodtsov in 1999 [4] marked a watershed moment in grappling with uncertainty within data. The emphasis on uncertainty stems from its omnipresence in modern databases, particularly accentuated in the intricate tapestry of healthcare claims data [5,6]. Building upon this foundation, recent years have witnessed a proliferation of soft set extensions, each tailored to address specific nuances within healthcare claims datasets.

The pioneering work of Smarandache in 2018 [7] introduced HyperSoft Sets, followed by the advent of SuperHyperSot Sets, IndetermSoft Sets, IndetermHyperSoft Sets, and TreeSoft Sets in 2022

[8–12]. These extensions, coupled with the contributions of Alkhazaleh and his team with the MultiSoft Set in 2010, have significantly enriched the repertoire of mathematical tools for data analysis within the healthcare domain [13].

Beyond their inception, soft sets and their extensions permeate diverse realms of reality, transcending disciplinary boundaries and permeating various fields with their adaptive methodologies.

Recent research endeavours have witnessed a convergence of soft set theory with fuzzy logic and its extensions, fostering a symbiotic relationship that amplifies their collective potential.

Fuzzy soft sets, intuitionistic fuzzy soft sets, neutrosophic soft sets, picture fuzzy soft sets, spherical fuzzy soft sets, and plithogenic soft sets represent a mere fraction of this expansive spectrum, each offering unique insights into the complexities of healthcare claims data analysis.

The legitimate question is: How can the application of soft set theory and its recent extensions in analysing and modelling healthcare claims data contribute to the improvement of diagnostics and personalized treatments in medicine?

Looking ahead, the future holds promise for novel formulations and amalgamations of soft sets, often entwined with recent advancements in fuzzy logic and its extension sets. Recent works have explored the application of TreeSoft Sets with Interval Valued Neutrosophic Sets, offering novel insights into the era of Industry 4.0 and its implications for data analysis [14].

Furthermore, the introduction of practical applications of IndetermSoft Sets and IndetermHyperSoft Sets by underscores the growing relevance of soft sets in real-world scenarios, particularly within the realm of healthcare [15]. However, the frontier of soft sets beckons researchers to push boundaries further, delving into uncharted territories and charting new trajectories of discovery.

Future research endeavours are poised to explore novel applications of soft sets, potentially in conjunction with fuzzy logic and its extension sets, yielding diverse combinations such as fuzzy / intuitionistic fuzzy /neutrosophic / picture fuzzy / spherical fuzzy / Pythagorean fuzzy / plithogenic IndetermSoft / IndetermHyperSoft / TreeSoft Sets. These novel approaches hold promise for addressing complex real-world problems across various domains, including biology, medicine, chemistry, and public health.

Recent works have explored the selection of the best process for desalination under a TreeSoft Set environment, highlighting the versatility of soft sets in addressing diverse challenges [16].

Similarly, recent advancements in medical image analysis, particularly in fields like natural and traditional Chinese medicine, have leveraged soft set methodologies to evaluate the degree of evidence in medical recommendations and to assess the factors influencing preventive practices in clinical images with indeterminate features.

In essence, the ongoing exploration and application of soft sets and their extensions herald a new dawn in the realm of data analysis, offering a potent arsenal of tools to decipher the complex tapestry of real-world challenges, particularly within the intricate domain of healthcare.

As researchers navigate this evolving landscape, the fusion of soft set methodologies with fuzzy logic theories promises to unlock new vistas of understanding, driving transformative breakthroughs across diverse domains, and shaping the future of healthcare research and beyond.

Current Survey Mission

- This paper explores the evolution and application of Soft Sets and their extensions in healthcare claims data analysis, addressing the inherent complexities and uncertainties in such datasets. The main contributions of our research are as follows:
- **Comprehensive Review:** We provide a thorough examination of the evolution and application of Soft Sets, including HyperSoft Sets, SuperHyperSoft Sets, IndetermSoft Sets, IndetermHyperSoft Sets, and TreeSoft Sets, specifically within the context of healthcare claims data analysis.
- **Analysis of Real-World Applications:** We present detailed analyses and real-world examples demonstrating the practical utility of these Soft Set extensions in processing complex healthcare data, emphasizing their role in informed decision-making and knowledge discovery.

- **Advancements in Methodologies:** Our review highlights significant advancements in data analysis methodologies enabled by Soft Sets and their extensions, showcasing how these tools can enhance the accuracy and efficiency of healthcare data analysis.
- **Future Research Directions:** We discuss potential future research avenues, suggesting novel applications and combinations of Soft Sets with fuzzy logic and its extensions to further improve data analysis in healthcare and beyond.

The data discussed in this paper are made available as open-source collections (Browse by Research Unit, Center, or Department | UNM Digital Repository, accessed on 02 August 2024), with the aim of fostering further research and development in this field, along with the demonstrations of Soft Sets and their multifarious extensions (available at http://fs.unm.edu/NSS/ExtensionOfSoftSetToHypersoftSet.pdf; http://fs.unm.edu/NSS/IndetermSoftIndetermHyperSoft38.pdf, accessed on 02 August 2024).

2. Related Work

In this section, we provide a comprehensive overview of the contributions in the field, emphasizing their impact and relevance to the application of soft set theory in healthcare claims data analysis. In fact, soft set theory, applied to healthcare claims data, provides a flexible framework for analysing the uncertainty and imprecision inherent in medical records. Consider a scenario where a patient's diagnosis is uncertain due to incomplete information or conflicting test results. Traditional methods may struggle to handle such ambiguity, leading to inaccurate assessments or diagnoses.

However, by employing soft set theory, we can represent the uncertainty associated with each diagnosis or treatment option using membership functions. These membership functions assign degrees of certainty to various outcomes based on available evidence, allowing healthcare practitioners to make informed decisions despite incomplete or conflicting data.

For example, a soft set approach could be used to determine the likelihood of a patient having a particular condition based on their symptoms, medical history, and test results, even when some information is missing or contradictory. This flexibility makes soft set theory a valuable tool for analysing healthcare claims data, improving diagnostic accuracy, and ultimately enhancing patient care.

The most notable contributions in this field illustrate are mention below.

1. Molodtsov's seminal work laid the **foundation for soft set theory**, offering a novel approach to handling uncertainty and vagueness in data analysis [3]. This foundational work has been pivotal in subsequent research exploring various extensions and applications of soft sets in different domains, including healthcare claims data analysis.

2. In 2018, **Smarandache introduced HyperSoft Sets**, an extension designed to better handle multi-attribute decision-making processes. This extension has shown promise in dealing with the complex and multi-dimensional nature of healthcare claims datasets, providing a more nuanced framework for analysis [7].

3. The MultiSoft Set, **introduced by Alkhazaleh** and his team, expanded the versatility of soft sets by accommodating multiple parameters, making it particularly useful for applications in healthcare claims data where multiple factors need to be considered simultaneously. This work has significantly enriched the toolkit available for researchers and specialists in healthcare [11].

4. In 2022, Smarandache introduced **IndetermSoft Sets and IndetermHyperSoft Sets**, which address indeterminacy in data analysis. These extensions have been applied to real-world scenarios in healthcare, demonstrating their utility in dealing with uncertain and incomplete healthcare claims data [6,15]. Next year, Smarandache proposed SuperHyperSoft Sets [17].

5. Convergence with Fuzzy Logic and its Extensions

The integration of soft set theory with fuzzy logic and its various extensions has formed a robust framework for managing the inherent fuzziness and uncertainty in healthcare claims data. P. K. Maji's seminal work, exemplified by "Intuitionistic Fuzzy Soft Sets," has played a pivotal role in this domain [18].

Furthermore, the foundational contributions of Lotfi A. Zadeh and other collaborators in fuzzy logic have paved the way for the amalgamation of fuzzy logic with soft set theory, notably documented in Fuzzy Sets Applications to pattern classification and clustering analysis [19] or decision analysis [20].

S. K. Samanta's research on neutrosophic soft sets and their applications has significantly bolstered this convergence, offering invaluable insights into managing uncertainty in biomedical data analysis [21,22].

Additionally, Florentin Smarandache's exploration of neutrosophic sets, particularly showcased in 2020 [23] alongside collaborative endeavours with K. Atanassov on intuitionistic fuzzy sets [24], have greatly propelled the methodologies for extracting actionable insights from complex datasets [25]. The research conducted by M. Shabir and M. Naz on bipolar soft sets [26] and their fusion with fuzzy logic has contributed substantial insights into multi-criteria decision-making problems, further enhancing the analytical capabilities in healthcare contexts.

These advancements underscore the potential of integrating soft set theory and its extensions into healthcare data analysis, offering avenues for enhancing diagnostics and personalized treatments.

The adeptness of these mathematical constructs in handling uncertainty, multi-dimensionality, and indeterminacy aligns seamlessly with the intricacies inherent in healthcare claims dataset.

Consequently, delving into systematic applications of these tools to improve medical outcomes stands as an imperative avenue for future research.

Collectively, these studies underscore the dynamic evolution of soft set theory and its extensions, emphasizing their growing significance and versatility in the domain of healthcare claims data analysis. The ongoing research and development in this sphere hold the promise of unlocking novel possibilities for advancing diagnostics, therapeutics, and personalized medicine.

6. Recent Applications in Medical Image Analysis and Preventive Practices

Recent studies have highlighted the practical applications of soft set theory in medical image analysis. For instance, Dhanalakshmi and Bhaskaran explore the application of soft set methodologies to evaluate the degree of evidence in medical recommendations and assess factors influencing preventive practices in clinical images with indeterminate features [27].

Similarly, Yang and Zhao provide insights into the advantages and specific methods used in employing soft set theory for similar purposes [28]. Additionally, Khan and Gupta offer a detailed examination of soft set-based approaches in medical image analysis, focusing on their role in evaluating evidence in medical recommendations and analysing factors influencing preventive practices in clinical images [29].

These applications underscore the relevance and adaptability of soft sets in contemporary healthcare research, particularly in the domain of medical image analysis and preventive practices.

7. The innovative work by **Alqazzaz and Sallam explored the use of TreeSoft Sets** combined with Interval Valued Neutrosophic Sets, providing novel insights into data analysis within the context of Industry 4.0. [14]. This study demonstrates the evolving nature of soft set applications and their potential to address modern data challenges.

Given these advancements, it becomes evident that the integration of soft set theory and its extensions into healthcare claims data analysis holds significant potential for enhancing diagnostics and personalized treatments. The ability of these mathematical constructs to handle uncertainty, multi-dimensionality, and indeterminacy aligns well with the complexities inherent in healthcare claims datasets. Therefore, exploring how these tools can be systematically applied to improve medical outcomes is a compelling avenue for future research.

These studies collectively highlight the dynamic evolution of soft set theory and its extensions, showcasing their growing importance and versatility in the realm of healthcare claims data analysis.

The ongoing research and development in this field promise to unlock new possibilities for improving diagnostics, therapeutics, and personalized medicine.

2. Soft Sets Extensions

In this section, we delve into the various extensions of Soft Sets, each offering unique capabilities and applications within the realm of healthcare claims data analysis.

These extensions include the HyperSoft Set, SuperHyperSoft Set, Fuzzy-Extension-SuperHyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set.

Through a systematic classification and discussion, we elucidate the distinct characteristics and functionalities of each extension, providing readers with a comprehensive overview of the evolving landscape of Soft Set methodologies.

We recall the definitions of Soft Set, HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set, including a few suggestive examples applied to healthcare claims data.

2.1. Soft Set

A Soft Set provides a flexible framework for modelling uncertain or imprecise information by associating each attribute with a set of possible elements from the universe of discourse. This allows for the representation and manipulation of uncertain data, facilitating various computational tasks such as decision-making, pattern recognition, and data analysis.

Definition

A Soft Set is a mathematical abstraction designed to encapsulate uncertainty and fuzziness inherent in data within a specific domain of discourse. Let's break down this definition:

Firstly, we define a universe of discourse, denoted as U, which encompasses all conceivable elements or entities relevant to the context under consideration. The power set of U, represented as $P(U)$, comprises all possible subsets derived from the elements within the universe of discourse. Essentially, it represents the complete range of potential combinations or groupings of elements from U.

Next, we introduce a set of attributes, denoted as A, which serves to characterize the properties or features associated with the elements within the universe U. These attributes could represent any discernible traits, qualities, or characteristics relevant to the domain being studied.

Now, a Soft Set is formally defined as a pair (F, U), where $F: A \to P(U)$.
F represents a mapping function that associates each attribute in A with a subset of elements from the universe U. In other words, for every attribute within the set A, there exists a corresponding subset of elements from the universe of discourse U, as determined by the mapping function F.

In summary, a Soft Set provides a structured framework for capturing and managing uncertainty by linking attributes to subsets of elements within a given universe of discourse. This enables the representation and manipulation of imprecise or indeterminate data, facilitating various computational tasks such as decision-making, pattern recognition, and data analysis within the specified domain.

Example

Let's define the universe of discourse U as a set of patients.

U = {Patient1, Patient2, Patient3, Patient4}

and a subset included in U representing patients with specific conditions:

M = {Patient1, Patient3, Patient4}.

Now, let's consider an attribute related to medical conditions:

a = condition,

with attribute values representing different medical conditions:

Condition = A_1 = {diabetes, hypertension, asthma}.

We define a function: $F: A_1 \to P(U)$,
where $P(U)$ represents the power set of U.
Then, for example:
$F(asthma)$ = {Patient2, Patient3},
This means that both Patient2 and Patient3 have been diagnosed with asthma.

2.2. Indetermsoft Set

An IndetermSoft Set provides a flexible framework for modelling uncertain or imprecise information by associating each attribute with a set of possible elements from the universe of discourse. This enables the representation and manipulation of uncertain data, facilitating various computational tasks such as decision-making, pattern recognition, and data analysis.

Definition

An IndetermSoft Set expands upon the foundational principles of the classical Soft Set by accommodating indeterminate data, reflecting the inherent uncertainty and ambiguity prevalent in real-world scenarios. Let's dissect this definition:

We begin with the establishment of a universe of discourse, denoted as U, which encompasses all relevant elements or entities under consideration. Additionally, we identify a non-empty subset of U, denoted as H, and its corresponding powerset, $P(H)$, which comprises all possible subsets derived from the elements within H.

Furthermore, we introduce an attribute, denoted as 'a', and a set of attribute-values, denoted as A.

The mapping function $F: A \rightarrow P(H)$ is designated as an IndetermSoft Set if one or more of the following conditions are met:

i). The set A exhibits some level of indeterminacy.
ii). The sets H or $P(H)$ demonstrate indeterminacy.
iii). The function F itself contains elements of indeterminacy, indicating the presence of attribute-values for which the mapping is unclear, incomplete, conflicting, or non-unique.

IndetermSoft Sets, characterized by their capacity to handle indeterminate data, arise from real-world situations where information sources may provide approximate, uncertain, incomplete, or conflicting data. Rather than introducing indeterminacy artificially, such as in the classical Soft Set framework, the indeterminacy is identified within the data itself, reflecting the limitations and nuances of our world.

The term "Indeterm" signifies "Indeterminate", encompassing attributes of uncertainty, conflict, incompleteness, or lack of uniqueness within the outcomes. This distinction prompts the consideration of determinate versus indeterminate operators, leading to the development of an IndetermSoft Algebra.

Smarandache's contributions extend the concept further with the introduction of HyperSoft sets, which involve multi-attribute functions, and subsequently, the hybridization of various Soft Set variants. These hybrids incorporate elements from Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and other fuzzy extensions, as well as the Plithogenic HyperSoft Set.

While the classical Soft Set relies on determinate functions with certain and unique values, the reality of our world often involves sources that provide indeterminate information due to a lack of knowledge or precision. Consequently, operators with varying degrees of indeterminacy are utilized to model such scenarios, acknowledging the inherent imprecision of our environment.

Example

Consider a dataset comprising healthcare claims from various patients.

I. Indeterminacy with respect to the function:

1a) You inquire from a source:

— "Which patients have been diagnosed with diabetes?"

The source responds:

— "I'm uncertain; it could be patients Patient1 or Patient2."

Thus, F(diabetes) = Patient1 or Patient2 (an indeterminate/uncertain response).

1b) Another query:

— "And which patients have undergone surgery?"

The source replies:

— "I'm not certain; all I can confirm is that Patient5 has not had surgery because I have their records."

Thus, F(surgery) = not Patient5 (again, an indeterminate/uncertain response).

1c) Further inquiry:

— "Then, which patients have high blood pressure?"

The source asserts:

— "It's either Patient8 or Patient9 for sure."

Thus, F(high blood pressure) = either Patient8 or Patient9 (yet another indeterminate/uncertain response).

II. Indeterminacy with respect to the set P of patients:

You ask the source:

— "How many patients are included in the dataset?"

The source replies:

— "I haven't counted them, but I estimate the number to be between 100-120 patients."

III. Indeterminacy with respect to the set C of medical conditions:

You inquire:

— "What are all the medical conditions diagnosed in the patients?"

The source states:

— "I'm certain there are patients diagnosed with diabetes, high blood pressure, and heart disease, but I'm unsure if there are patients with other conditions."

The IndetermSoft Set addresses the inherent indeterminacy present in healthcare claims data by introducing a flexible framework that accommodates varying degrees of uncertainty. Through the incorporation of indeterminacy measures, the IndetermSoft Set offers researchers the ability to effectively manage and quantify uncertainty, facilitating more robust decision-making processes and knowledge discovery.

2.3. Hypersoft Set

A HyperSoft set presents a dynamic framework for modelling uncertain or imprecise information, where each attribute is linked to a set of potential elements from the universe of discourse. This framework enables the comprehensive representation and manipulation of uncertain data, empowering various computational tasks such as decision-making, pattern recognition, and data analysis.

Definition

The extension from Soft Sets to HyperSoft Sets (HS Set) marks a significant advancement in modelling complex relationships by expanding the mapping function to accommodate multiple attributes.

Here's a breakdown:

Initially, the Soft Set concept is broadened into the realm of HyperSoft Sets by transitioning the mapping function F into a multi-attribute function. This transformation enables the representation of intricate relationships between elements within the universe of discourse.

Let's delve into the formal definition:

We begin with the universe of discourse, denoted as U, along with its powerset, $P(U)$, which encompasses all conceivable elements or entities.

Next, we introduce n distinct attributes, denoted as $a_1, a_2, \ldots, a_n$, for $n \geq 1$. Each attribute is associated with a set of attribute values, denoted respectively as $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \Phi$, for $i \neq j$, and i, j in $\{1, 2, \ldots, n\}$.

Notably, these attribute sets are pairwise disjoint, ensuring no overlap between them.

The pair $(F, A_1 \times A_2 \times \ldots \times A_n)$ represent a HyperSoft Set over U, where F is a mapping function defined on the Cartesian product of the attribute sets where $A_1 \times A_2 \times \ldots \times A_n$.

Formally,

$F: A_1 \times A_2 \times \ldots \times A_n \rightarrow P(U)$, is called a $\rightarrow P(U)$,

signifies that for each combination of attribute values, there exists a corresponding subset of elements from U.

The introduction of HyperSoft Sets facilitates the exploration of complex relationships and interactions among multiple attributes within the universe of discourse. This extension opens avenues for comprehensive analysis and modelling of intricate systems, spanning various domains and applications.

Moreover, Smarandache's contributions have led to the hybridization of HyperSoft Sets with diverse frameworks, including Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and other fuzzy extensions, as well as the Plithogenic Set. These hybrid models integrate elements from different mathematical paradigms, enhancing their adaptability and utility in addressing real-world complexities.

In essence, HyperSoft Sets offer a versatile and robust framework for modelling and analysing complex systems characterized by multiple attributes, thereby facilitating informed decision-making and knowledge discovery across diverse domains.

Exemple

Let the attributes be:

a_1 = diagnosis,

a_2 = treatment,

a_3 = cost,

a_4 = duration,

and their attributes' values respectively:

Diagnosis = A_1 = {diabetes, heart condition, respiratory issue},

Treatment = A_2 = {medication, surgery, therapy},

Cost = A_3 = {low, medium, high},

Duration = A_4 = {short-term, medium-term, long-term}.

Let the function be: $F: A_1 \times A_2 \times A_3 \times A_4 \rightarrow P(U)$.

Then, for example:

F({diabetes, medication, low, short-term}) = {Claim1, Claim2}, which means that both Claim1 and Claim2 involve a diagnosis of diabetes, medication as treatment, low cost, and short-term duration.

Basically, this is an extension of the previous Real Example of Soft Set.

The HyperSoft Set extends the foundational principles of Soft Sets by incorporating hyperparameters that capture complex relationships and interactions within healthcare claims datasets.

By integrating hyperparameters, the HyperSoft Set enables a more nuanced representation of uncertainty, thereby enhancing the accuracy and reliability of data analysis and interpretation within the healthcare domain.

2.4. SuperHypersoft Set

A SuperHyperSoft Set introduces an innovative framework for modelling complex and uncertain information, where each attribute is associated with an expansive set of potential elements from the universe of discourse. This advanced approach enables the comprehensive representation and manipulation of intricate data, facilitating advanced computational tasks including decision-making, pattern recognition, and data analysis at a highly refined level.

Definition

The SuperHyperSoft Set (SHS Set) is an extension of the HyperSoft Set. As for the SuperHyperAlgebra, SuperHyperGraph, SuperHyperTopology and in general for SuperHyperStructure and Neutrosophic SuperHyperStructure (that includes indeterminacy) in any field of knowledge, "Super" stands for working on the powersets (instead of sets) of the attribute value sets.

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the powerset of $\mathcal{U}$.

Let $a_1, a_2, \ldots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1,2, \ldots, n\}$.

Let $\mathcal{P}(A_1), \mathcal{P}(A_2), \ldots, \mathcal{P}(A_n)$ be the powersets of the sets $A_1, A_2, \ldots, A_n$ respectively. Then the pair

$(F, \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n)$, where × meaning Cartesian product, or:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n) \to \mathcal{P}(\mathcal{U})$

is called a SuperHyperSoft Set.

Example

If we define the function:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3) \times \mathcal{P}(A_4) \to \mathcal{P}(\mathcal{U})$.

We get a SuperHyperSoft Set.

Let's consider a scenario involving healthcare claim data, extending the previous examples. Assume we have a dataset comprising healthcare claims, and we want to categorize them based on various attributes.

Let's define the attributes and their possible values as follows:

Attribute A_1: Type of Treatment (e.g., Surgery, Medication, Therapy)

A_1: {Surgery, Medication, Therapy}

Attribute A_2: Diagnosis Code (e.g., Injury, Illness, Chronic Condition)

A_2: {Injury, Illness, Chronic Condition}

Attribute A_3: Patient Age Group (e.g., Child, Adult, Senior)

A_3: {Child, Adult, Senior}

Attribute A_4: Insurance Provider (e.g., Company A, Company B, Company C)

A_4: {Company A, Company B, Company C}

Let the function $F: A_1 \times A_2 \times A_3 \times A_4 \to P(U)$ map combinations of these attributes to subsets of the set of healthcare claims U.

$F(\{Surgery, Medication\}, \{Injury, Illness\}, \{Adult\}, \{CompanyA, CompanyB\}) = \{claim_1, claim_2\}$,

this means that claims $claim_1$ and $claim_2$ involve either surgery or medication, are related to either injury or illness, are for adult patients, and are covered by either CompanyA or CompanyB insurance providers.

This SuperHyperSoft Set approach allows for a flexible categorization of healthcare claims, accommodating various combinations of treatment types, diagnoses, patient age groups, and insurance providers, reflecting the complexity and diversity of real-world healthcare scenarios.

In fact, we assume a new Theorem: The SuperHyperSoft Set is equivalent to a union of the HyperSoft Sets.

Demonstration

Let's consider the SuperHyperSoft:

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n) \to \mathcal{P}(\mathcal{U})$

Assume that the non-empty sets.

$B_1 \subseteq A_1, B_2 \subseteq A_2, \ldots, B_n \subseteq A_n$ and

$F(B_1, B_2, \ldots, B_n) \in P(U)$

$B_1 = \{b_{11}, b_{12}, \ldots\}, B_2 = \{b_{21}, b_{22}, \ldots\}, \ldots, B_n = \{b_{n1}, b_{n2}, \ldots\}$, therefore

$F(\{b_{11}, b_{12}, \ldots\}, \{b_{21}, b_{22}, \ldots\}, \ldots, \{b_{n1}, b_{n2}, \ldots\})$ can be composed in many

$F\left(b_{1k_1}, b_{2k_2}, \ldots, b_{nk_n}\right) \in P(U)$, which are actually HS Sets.

Considering the attributes diagnosis, treatment, cost, and duration, we can derive the following 12 possibilities:

1. Diagnosis: diabetes, Treatment: medication, Cost: low, Duration: short-term
2. Diagnosis: diabetes, Treatment: medication, Cost: low, Duration: medium-term
3. Diagnosis: diabetes, Treatment: medication, Cost: low, Duration: long-term
4. Diagnosis: diabetes, Treatment: medication, Cost: medium, Duration: short-term
5. Diagnosis: diabetes, Treatment: medication, Cost: medium, Duration: medium-term
6. Diagnosis: diabetes, Treatment: medication, Cost: medium, Duration: long-term
7. Diagnosis: diabetes, Treatment: medication, Cost: high, Duration: short-term
8. Diagnosis: diabetes, Treatment: medication, Cost: high, Duration: medium-term
9. Diagnosis: diabetes, Treatment: medication, Cost: high, Duration: long-term
10. Diagnosis: diabetes, Treatment: surgery, Cost: low, Duration: short-term
11. Diagnosis: diabetes, Treatment: surgery, Cost: low, Duration: medium-term
12. Diagnosis: diabetes, Treatment: surgery, Cost: low, Duration: long-term.

For each of these combinations, the function F yields the set of patients who meet these criteria, represented by $\{x_1, x_2\}$. Totally: 12 are HyperSoft Sets.

2.5. Fuzzy-Extension-SuperhyperSoft Set

A Fuzzy-Extension-SuperhyperSoft Set introduces an advanced framework that combines fuzzy logic with Hypersoft Set theory, providing a robust approach for modelling highly complex and uncertain information. Each attribute is associated with an expansive set of potential elements from the universe of discourse, allowing for nuanced representation and manipulation of uncertain data. This innovative approach empowers advanced computational tasks such as decision-making, pattern recognition, and data analysis with enhanced adaptability, precision, and the ability to handle fuzzy boundaries effectively.

Definition

$F: \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \ldots \times \mathcal{P}(A_n) \rightarrow \mathcal{P}(\mathcal{U}(x(d_0)))$ where $x(d_0)$ is the fuzzy or any fuzzy extension degree of appurtenance of the element x to the set $\mathcal{U}$.

Fuzzy-Extensions mean all types of fuzzy sets [16], such as: Fuzzy Set, Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), Fermatean Fuzzy Set, q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, n-HyperSpherical Fuzzy Set, Neutrosophic Set, Spherical Neutrosophic Set, Refined Fuzzy/Intuitionistic Fuzzy/Neutrosophic/other fuzzy extension Sets, Plithogenic Set, etc.

Example

In the previous example, considering the attributes diagnosis, treatment, cost, and duration, we can envision a Neutrosophic SuperHyperSoft Set.

Let's assume:

$(\{\text{diabetes}\},\{\text{medication}\},\{\text{low}\},\{\text{short-term}\}) = x_1(0.7, 0.4, 0.1)$

$F(\{\text{diabetes}\},\{\text{medication}\},\{\text{low}\},\{\text{medium-term}\}) = x_2(0.9,0.2,0.3)$.

This would mean that x_1, corresponding to the values ({diabetes}, {medication}, {low}, {short-term}), holds an appurtenance degree of 0.7, an indeterminate degree of 0.4, and a non-appurtenance degree of 0.1.

Similarly, x_2, associated with the values ({diabetes}, {medication}, {low}, {medium-term}), exhibits an appurtenance degree of 0.9, an indeterminate degree of 0.2, and a non-appurtenance degree of 0.3.

2.6. IndetermHyperSoft Set

An IndetermHyperSoft Set offers a sophisticated framework for modelling uncertain or imprecise information, where each attribute is linked to a set of potential elements from the universe

of discourse. This advanced approach allows for the comprehensive representation and manipulation of uncertain data, facilitating complex computational tasks such as decision-making, pattern recognition, and data analysis with enhanced adaptability and granularity.

Definition

The IndetermHyperSoft Set represents an extension of the HyperSoft Set to accommodate indeterminate data, functions, or sets. Here's a refined explanation:

We start with the universe of discourse, denoted as U, along with a non-empty subset H of U, and its powerset, $P(H)$, which encompasses all possible subsets of H.

Next, we introduce n distinct attributes, denoted as $a_1, a_2, \dots, a_n$, for $n \geq 1$.

Each attribute is associated with a set of attribute values, denoted respectively as $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \Phi$ for $i \neq j$, and i, j in $\{1, 2, \dots, n\}$.

Notably, these attribute sets are pairwise disjoint, ensuring no overlap between them.

Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where $F: A_1 \times A_2 \times \dots \times A_n \rightarrow P(H)$ represents an IndetermHyperSoft Set over U if at least one of the following conditions holds:

i). At least one of the attribute sets $A_1, A_2, \dots, A_n$ has some indeterminacy.

ii). The sets H or $P(H)$ exhibit indeterminacy.

iii). There exists at least one n-tuple $(e_1, e_2, \dots, e_n) \in A_1 \times A_2 \times \dots \times A_n$ such that the function $F(e_1, a_2, \dots, e_n)$ = indeterminate (unclear, uncertain, conflicting, or not unique). In other words, F yields an indeterminate outcome for that tuple.

In essence, the IndetermHyperSoft Set extends the HyperSoft Set framework to accommodate situations where uncertainty or vagueness is present in the attribute sets, subsets, or the mapping function itself.

Moreover, the IndetermHyperSoft Set provides a flexible and adaptable approach for modeling and analysing complex systems in which precise information may be lacking or uncertain. By incorporating indeterminate elements, functions, or sets, this extension enhances the applicability of the HyperSoft Set framework in real-world scenarios characterized by inherent uncertainty or ambiguity.

Example

Assume there are many patients in a hospital database.

1. Indeterminacy with respect to the function.

1a) You ask a source:

— What patients have been diagnosed with diabetes and prescribed medication?

The source:

— I am not sure, I think it's either Patient1 or Patient2. Therefore, F(diabetes, medication) = Patient1 or Patient2 (indeterminate / uncertain answer).

1b) You ask again:

— But, what patients have hypertension and are undergoing surgery?

The source:

— I do not know, the only thing I know is that Patient5 does not have hypertension or undergo surgery because I have checked their records.

Therefore, F(hypertension, surgery) = not Patient5 (again indeterminate / uncertain answer).

1c) Another question you ask:

— Then what patients have asthma and are being treated with therapy?

The source:

— For sure, either Patient8 or Patient9.

Therefore, F(asthma, therapy) = either Patient8 or Patient9 (again indeterminate / uncertain answer).

2. Indeterminacy with respect to the set P of patients.

You ask the source:
— How many patients are in the database?
The source:
— I never counted them, but I estimate their number to be between 100-120 patients.

3. Indeterminacy with respect to the product set $A_1 \times A_2 \times \dots \times A_n$ of attributes.

You ask the source:
— What are all diagnoses and treatments of the patients?
The source:
— I know for sure that there are patients diagnosed with diabetes, hypertension, and asthma, but I do not know if there are patients with other diagnoses (?) About the treatments, I recall seeing many patients receiving medication, but I do not remember seeing patients undergoing surgery or therapy.

Combining the strengths of both the IndetermSoft Set and the HyperSoft Set, the IndetermHyperSoft Set provides a comprehensive framework for analysing complex healthcare claims datasets characterized by both uncertainty and hyperparameters.

By synergistically integrating indeterminacy measures and hyperparameters, this extension empowers researchers to unravel intricate relationships and patterns within biological data, thereby advancing our understanding of biological systems.

2.7. TreeSoft Set

A TreeSoft Set introduces a structured framework for modelling uncertain or imprecise information, where each attribute is organized in a hierarchical tree-like structure, associating each node with a set of potential elements from the universe of discourse. This hierarchical approach enables the systematic representation and manipulation of uncertain data, facilitating various computational tasks such as decision-making, pattern recognition, and data analysis with a focus on hierarchical relationships and dependencies.

Definition

The TreeSoft Set is an innovative extension that introduces a hierarchical structure to Soft Sets, providing a comprehensive framework for modelling complex systems with multiple levels of attributes. Here's a refined explanation:

We begin with a universe of discourse, denoted as U, and a non-empty subset H of U, along with its powerset, $P(H)$, which encompasses all possible subsets of H.

Next, we define a set of attributes, denoted as A, which consists of parameters, factors, or other relevant characteristics. This set is organized hierarchically into levels: first-level attributes $A = \{A_1, A_2, \dots, A_n\}$, for integer $n \geq 1$, where $A_1, A_2, \dots, A_n$ are considered attributes of first level (since they have one-digit indexes).

Each attribute A_i, $1 \leq i \leq n$, is formed by sub-attributes:

$A_1 = \{A_{1,1}, A_{1,2}, \dots\}$

$A_2 = \{A_{2,1}, A_{2,2}, \dots\}$

.....

$A_n = \{A_{n,1}, A_{n,2}, \dots\}$

where the above $A_{i,j}$ are sub-attributes (or attributes of second level) (since they have two-digit indexes).

Again, each sub-attribute $A_{i,j}$ is formed by sub-sub-attributes (or attributes of third level):

$A_{i,j,k}$

And so on, as much refinement as needed into each application, up to sub-sub-…-sub-attributes (or attributes of *m*-level (or having *m* digits into the indexes):

$A_{i1,i2,...,im}$

This hierarchical structure forms a graph-tree, denoted as *Tree(A)*, with *A* as the root node (level zero), followed by nodes at levels 1 to *m*, where *m* represents the maximum level of refinement. The leaves of this graph-tree are terminal nodes that have no descendants.

The TreeSoft Set, denoted as:

F: P(Tree(A)) → P(H),

maps subsets of the graph-tree Tree(*A*) to subsets of H. The powerset P(Tree(A)) encompasses all possible subsets of the graph-tree.

All node sets of the *TreeSoft Set of level m* are:

$Tree(A) = \{A_{i1} \mid i_1 = 1, 2, ... \}$

The sets within the TreeSoft Set correspond to nodes at each level of the graph-tree: the first set consists of nodes at level 1, the second set consists of nodes at level 2, and so on, up to the last set comprising nodes at level *m*. If the graph-tree has only two levels (*m* = 2), then the TreeSoft Set simplifies to a MultiSoft Set [7].

In summary, the TreeSoft Set provides a structured approach for representing and analysing complex systems with hierarchical attributes.

By incorporating a hierarchical organization, it enhances the flexibility and expressiveness of Soft Set-based methodologies, enabling more nuanced modelling and analysis of multi-level systems across various domains.

An illustrative example of a classical tree is shown in Figure 1.

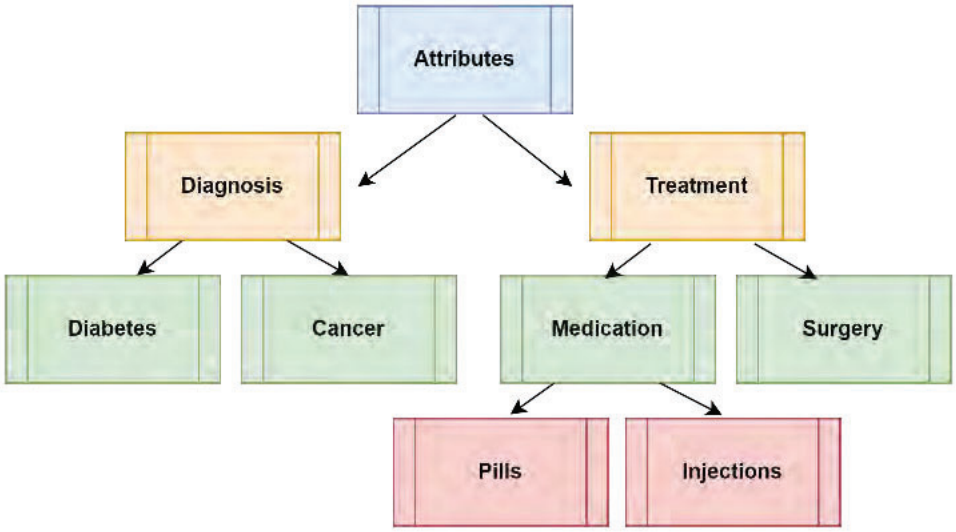


Figure 1. Schematic representation of the TreeSoft Set of Level 3 framework, illustrating the incorporation of hyperparameters to capture complex relationships within healthcare claims datasets.

This tree contains three levels as followed:
Level 0 (the root) is the node Attributes;
Level 1 is formed by the nodes: Diagnosis, Treatment;
Level 2 is formed by the nodes Diabetes, Cancer, Medication, and Surgery;
Level 3 is formed by the nodes Pills, Injections.
Let's consider $p = \{p_1, p_2, ..., p_{10}\}$ be a set of patients, and $P(p)$ the power set of p.
The attributes are defined as follows: $A=\{A_1,A_2\}$
were
A_1=Diagnosis
and

A_2=Treatment

Then,

A_1={A_{11},A_{12}}={Diabetes,Cancer}

and A_2={A_{21},A_{22}}={Medication,Surgery}.

Let's further break down A_{22} into A_{221} and A_{222}, representing specific treatments:

A_{221}={Pills,Injections} for medication and A_{222}={Chemotherapy,Radiation}for surgery.

Now, let's assume the function F has the following values:

1. F(Diabetes,Medication,Pills) = {$p1$,$p2$,$p3$,$p4$}
2. F(Diabetes,Medication,Injections) = {$p5$,$p6$}
3. F(Diabetes,Surgery,Chemotherapy) = {$p7$,$p8$}
4. F(Cancer,Surgery,Radiation) = {$p9$,$p10$}.

The TreeSoft Set introduces a hierarchical structure to Soft Set methodologies, enabling the representation and analysis of complex biological data in a hierarchical manner.

By organizing data into hierarchical trees, the TreeSoft Set facilitates the exploration of nested relationships and dependencies within healthcare claims datasets, offering insights into the hierarchical organization of biological systems.

3. Discussion

While the extensions of Soft Sets, such as HyperSoft Set, SuperHyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set, offer significant advancements in modeling complex systems and handling uncertainty within healthcare claims data analysis, several unresolved aspects warrant further exploration:

Integration of Extensions: Although each extension provides unique capabilities and applications, there is a need to explore how these extensions can be integrated synergistically to address multifaceted challenges in healthcare research. Investigating the interoperability and complementary nature of different Soft Set extensions could lead to more comprehensive methodologies for data analysis and decision-making, thereby enhancing the accuracy and reliability of diagnostics and personalized treatments.

- *Handling Complex Relationships*: While HyperSoft Sets and TreeSoft Sets enable the representation of complex relationships among attributes, there remains a challenge in effectively managing and analysing these intricate networks. Future research should focus on developing advanced algorithms and techniques for extracting meaningful insights from interconnected data structures, particularly in the context of healthcare claims datasets characterized by multi-level dependencies. By understanding and modelling these complex relationships, clinicians and researchers can gain deeper insights into disease mechanisms and treatment responses, ultimately improving diagnostics and personalized treatment strategies.
- *Quantification of Indeterminacy*: The presence of indeterminate data in IndetermSoft Sets and IndetermHyperSoft Sets poses challenges in quantifying and interpreting uncertainty. Further investigations are needed to develop robust methodologies for measuring and representing different degrees of indeterminacy, enhancing the reliability and interpretability of results derived from these frameworks. By accurately quantifying uncertainty, clinicians can make more informed decisions regarding diagnosis and treatment selection, taking into account the inherent variability and ambiguity present in healthcare claims data.
- *Scalability and Efficiency*: As healthcare claims datasets continue to grow in size and complexity, there is a pressing need for scalable and efficient algorithms capable of handling large-scale data analysis tasks. Research efforts should focus on optimizing computational techniques and resource allocation strategies to ensure the scalability and efficiency of Soft Set-based methodologies in real-world applications. By improving the scalability and efficiency of data analysis techniques, clinicians and researchers can analyse large datasets more effectively, leading to faster and more accurate diagnostics and personalized treatment recommendations.
- *Validation and Benchmarking*: Despite the theoretical advancements in Soft Sets and their extensions, there is a lack of comprehensive validation frameworks and benchmark datasets for evaluating the performance of these methodologies. Future research endeavors should prioritize

the development of standardized validation protocols and benchmark datasets to facilitate rigorous testing and comparison of different Soft Set-based approaches. By validating Soft Set-based models using standardized protocols and benchmark datasets, clinicians and researchers can ensure the reliability and generalizability of diagnostic and treatment recommendations derived from these methodologies.

- *Interpretability and Transparency*: Enhancing the interpretability and transparency of Soft Set-based models is crucial for fostering trust and adoption in healthcare research and clinical practice. Researchers should explore techniques for explaining model decisions and capturing the underlying uncertainty in a transparent manner, enabling stakeholders to understand and trust the insights derived from these methodologies. By improving the interpretability and transparency of Soft Set-based models, clinicians can better understand the rationale behind diagnostic and treatment recommendations, leading to increased confidence in personalized treatment strategies and ultimately improving patient outcomes.

4. Conclusions

The evolution and adoption of Soft Sets and their various extensions, including the HyperSoft Set, IndetermSoft Set, IndetermHyperSoft Set, and TreeSoft Set, represent a significant leap forward in computational methodologies, particularly within the realm of bioinformatics. These extensions provide novel avenues for modelling and analysing complex biological datasets, which are often plagued by uncertainty, imprecision, and vagueness.

In the context of bioinformatics [30,31], where data are diverse and frequently noisy or incomplete [32,33], the adaptability and versatility of Soft Sets and their variants prove invaluable. They offer researchers the means to navigate the inherent uncertainties present in biological data, such as those arising from gene expression profiles, protein interactions, and metabolic pathways. By embracing the inherent fuzziness and imprecision of biological phenomena, Soft Sets empower researchers to conduct more accurate and robust analyses, thus uncovering deeper insights into biological systems.

Moreover, the flexibility of Soft Sets enables seamless integration with other computational techniques and algorithms commonly employed in bioinformatics. This integration amplifies their utility, allowing researchers to combine the strengths of Soft Sets with established methods, thereby enriching analyses and bolstering decision-making processes.

In conclusion, Soft Sets and their extensions play a pivotal role in bioinformatics, offering a versatile framework to tackle the complexities and uncertainties inherent in biological data. Through their application, these methodologies drive innovation, deepen our understanding of biological systems, and ultimately contribute to advancements in human health and well-being. Our analysis underscores the efficacy of Soft Sets and their extensions in handling the intricacies of healthcare claims data, enabling more accurate predictions and facilitating insightful discoveries that propel healthcare research forward.

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Some Similarity Measures on Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets and Their Applications in Medical Diagnosis

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Abstract: In recent years, the application of fuzzy sets has gained significant attraction in various fields, including medical diagnosis, due to their ability to manage uncertainties and imprecise information. This paper focuses on the comparative analysis of similarity measures within the realm of Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets (GIVIFSESs) and explores their application in the domain of medical diagnosis. Most of the important topics in fuzzy set theory are the similarity measures between the generalizations of fuzzy set theory. Similarity measures are a crucial tool which was used in data science. In this process, we measure how much the data sets are related and comparable. Measures of similarity give a numerical value that reveals the strength of associations between sets or sets of variables. In this paper, we initiate a new concept of generalized interval-valued intuitionistic fuzzy soft expert sets and their fundamental operations. This new concept is more flexible than existing concepts based on their algebraic definition. Unlike fuzzy sets, the concept of generalized interval-valued intuitionistic fuzzy soft expert sets is characterized by a degree of membership and degree of non-membership along with fuzzy set theory. The proposed methodology is validated through an empirical application in medical diagnosis, where (GIV-IFSESs) are employed to model the uncertainty and imprecision inherent in expert assessments. The selected similarity measures are then applied to quantify the degree of resemblance between different medical cases, facilitating a more informed decision-making process. We introduce several types of similarity measures on generalized interval-valued intuitionistic fuzzy soft expert sets. We also discuss a similarity measure of Type-I, Type-II, and Type-III for two (GIVIFSESs) and its application in medical diagnosis problems.

Keywords: Fuzzy Sets; Soft Sets; Interval-valued Intuitionistic Fuzzy Soft Sets; Similarity Measures.

1. Introduction

The majority of real-world problems include data that is highly unclear and imprecise. To cope with uncertainties and fuzziness, classical mathematical theories such as fuzzy mathematics, probability theory, and interval mathematics have traditionally been used. However, as Molodtsov [1] pointed out, all of these theories have inherent challenges and shortcomings. He pioneered the idea of soft sets as a new mathematical technique for dealing with uncertainties that traditional mathematical tools cannot rectify. He has demonstrated multiple applications of this theory in the solution of several significant issues in economics, engineering, social science, medicine, and other

domains. The soft setting is also a fascinating and well-liked subject, where various forms of decision-making issues are frequently resolved. The ideas of soft subset, soft superset, soft equality, null soft set, and absolute soft set were first developed by Maji and Biswas [2]. They also performed some soft-set operations and confirmed De Morgan's laws. Ali et al. [3] addressed some previous studies' mistakes and defined some new soft-set procedures. Following that, Ali et al. [4] examined certain algebraic structures of soft sets and studied several essential features linked with the new operations. Maji et al. and Roy [5-7] have investigated soft set theory further and applied it to solve several decision-making difficulties. They also created and researched the concept of fuzzy soft set, a more generalized concept that is a blend of fuzzy set and soft set. Maji et al. defined the soft set operations union and intersection. Soft sets and fuzzy soft sets were introduced into the incomplete environment by Zou and Xiao [8], respectively. Wajid et al. [9] proposed an enhanced tool for fuzzy sets, intuitionistic fuzzy sets, and image fuzzy sets. They also presented spherical fuzzy sets and their representations of spherical fuzzy T-norm and T-conorm. As a generalization of the soft set, Alkhazaleh et al. [10] developed the notion of soft multiset. They also defined fuzzy parameterized interval-valued fuzzy soft sets [11] and the possibility of fuzzy soft sets [12] and proposed their applications for decision-making and medical diagnosis. Alkhazaleh and Salleh [13] established the concept of a soft expert set, in which the user can obtain the opinions of all experts in a single model without performing any activities. Even after a procedure, the user can obtain the opinions of all professionals. Salleh [14] provided a quick survey from the soft set to the intuitionistic fuzzy soft set in 2011. By integrating the interval-valued fuzzy set [16, 17] and soft set models, Yang et al. [15] proposed the notion of interval-valued fuzzy soft set. Yang et al. [18] discussed the application of soft sets with interval-valued intuitionistic fuzzy soft sets. Interval-valued intuitionistic fuzzy sets are extensively used to select the best alternative in multi-attribute decision-making issues, however, how to evaluate uncertainty is a big open problem. Majumdar and Samanta [19] proposed and investigated a generalized fuzzy soft set in which the degree is associated with the parameterization of fuzzy sets for defining a fuzzy soft set. Babitha and John [20] presented generalized intuitionistic fuzzy soft sets and solved a multi-criteria decision-making problem using generalized intuitionistic fuzzy soft sets.

1.1 Motivation and Purpose of the Paper

In this paper, we simplify the concept of generalized interval-valued fuzzy soft sets as introduced by Majumdar and Samanta to generalized interval-valued intuitionistic fuzzy soft expert sets. In our generalization of interval-valued intuitionistic fuzzy soft expert sets, a degree is attached to the parameterization of fuzzy sets while defining an interval-valued intuitionistic fuzzy soft expert set. Also, we give some applications of generalized interval-valued intuitionistic fuzzy soft expert set-in decision-making problems. We have further studied the similarity between two generalized interval-valued intuitionistic fuzzy soft expert sets, and it has been applied in medical diagnosis.

In the realm of decision-making and information processing, the ability to effectively manage uncertainty and imprecision is paramount. Fuzzy set theory has emerged as a powerful tool for capturing and modeling vagueness inherent in real-world data. The integration of fuzzy sets with expert opinions further enhances decision-making processes, providing a comprehensive framework for tackling complex problems [20-29]. This research endeavors to explore and compare various similarity measures within the context of Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets (GIVIFSESs) and subsequently apply these measures in the domain of medical diagnosis.

i). As medical diagnosis involves subjective expert opinions, imprecise data, and uncertainty, traditional approaches may fall short of providing accurate and reliable results. Fuzzy set theory, coupled with soft expert information, offers a promising avenue to address these challenges. The motivation for this study lies in the need for a robust methodology that not

only represents the complexity of medical diagnostic scenarios but also enables a rigorous comparison of similarity measures within the (GIVIFSESs) framework.

ii). The (GIVIFSESs) framework is introduced as a comprehensive model that extends the capabilities of traditional fuzzy sets by incorporating interval-valued intuitionistic fuzzy sets and soft expert information. This hybrid approach aims to capture not only the uncertainty and imprecision associated with medical diagnoses but also the inherent hesitancy present in expert opinions. The integration of these elements enhances the fidelity of the representation and provides a more realistic model for decision-making in medical contexts.

iii). Similarity measures play a crucial role in assessing the degree of resemblance between different entities represented by (GIVIFSESs). The selection of an appropriate similarity measure is pivotal for ensuring the accuracy and reliability of decision support systems. This study seeks to compare and evaluate various similarity measures to identify the most suitable measure for capturing the subtleties of expert opinions in the context of medical diagnosis.

iv). The primary objectives of this research are twofold:

a. To conduct a comprehensive comparative analysis of similarity measures within the (GIVIFSESs) framework.

b. To demonstrate the practical application of the selected similarity measures in medical diagnosis scenarios, highlighting their efficacy in handling uncertainty and imprecision.

The organization of this paper is as follows: In section 2, some preliminary definitions and results are given which will be used in the rest of the paper. In section 3 a definition of generalized interval-valued intuitionistic fuzzy soft expert set is given and fundamental operations of GIVIFSESs namely Sum, Product, Subtraction, Max- Product, Min-Product, Complement, Power, and Scaler product. In section 4, the similarity between two generalized interval-valued intuitionistic fuzzy soft expert sets has been discussed. In this section, we defined several types of similarity measures and also discussed some of the results related to those measures. An application of these similarity measures in medical diagnosis has been shown in section 5. In this section, we also compared the results of the similarity measures which were discussed in the previous section. Section 6 concludes the paper.

2. Preliminaries

In this section, we recall some basic definitions such as soft set, soft expert set, fuzzy soft expert set, interval-valued fuzzy set, interval-valued intuitionistic fuzzy soft expert set, and generalized interval-valued fuzzy soft set which are required in the construction of new concepts.

Definition 2.1. [1] *Let* $\widehat{F}: \mathcal{A} \to P(U)$ Is a mapping, whose domain is a set of parameters, and codomain is a power set of universal setU. **Soft set** over the universal set U is represented as a pair $(\widehat{\mathrm{F}}: \mathcal{A})$. For $\propto \in \mathcal{A}$, $\widehat{\mathrm{F}}(\propto)$ gives the set of $\propto$-approximate elements of the soft set $(\widehat{\mathrm{F}}: \mathcal{A})$. It is considered a parameterized family of subsets of the set U.

Definition 2.2. [13] Let $\widehat{F}: \mathcal{A} \to P(U)$ Is a mapping, whose domain is the cartesian product of the set of parameters, the panel of experts, and the set of opinions i-e $\mathrm{Ą} = \hat{\mathrm{E}} \times \mathrm{Ķ} \times \mathrm{Ǫ}$ and codomain is a power set of universal set U. **Soft expert set** over the universal set U is represented as a pair $(\widehat{\mathrm{F}}: \mathcal{A})$.

Definition 2.3. [21] Let $\widehat{F}: \mathcal{A} \to I^U$ is a mapping, whose domain is the cartesian product of a set of parameters, set of experts, and set of opinions i-e $\mathrm{Ą} = \hat{\mathrm{E}} \times \mathrm{Ķ} \times \mathrm{Ǫ}$ and codomain I^Udenoted all fuzzy subsets of U is a power set of universal set U. **Fuzzy soft expert set** over the universal set U is represented as a pair $(\widehat{\mathrm{F}}: \mathcal{A})$.

Definition 2.4. [15] Let U be a non-empty set. A function $\mathrm{Ą}: \mathrm{U} \to \mathrm{Int}[0,1]$ is called an **interval-valued fuzzy set**, whose domain is a universal set and codomain is the set of all closed sub-intervals of

[0,1], the set of all interval-valued fuzzy sets on U is denoted by I^U. For every $Ą \in I^U$ and u$\in U$, $Ą(u) = [Ą^-(u), Ą^+(u)]$ is called the degree of membership of an element u to Ą, lower fuzzy set, and upper fuzzy set in U represented by the following maps $Ą^-: U \to I$ and $Ą^+: U \to I$ respectively.

Definition 2.5. [22] Let $\xi: Ą \times Ç \to Ķ(U)$ is a mapping, whose domain is a cartesian product of the set of attributes Ą and the set of experts Ç and codomain in the set of the interval-valued intuitionistic fuzzy sets on the universe set U which is denoted as Ķ(U). **Interval-valued intuitionistic fuzzy soft expert set** (IVIFSE set) is a triplet$(\xi: Ą, Ç)$. For b$\in$ Ą and p$\in$ Ç we define

$$\xi_p^b = \left\{< u, \left[\gamma^{-b}_p(u), \gamma^{+b}_p(u)\right], \left[\zeta^{-b}_p(u), \zeta^{+b}_p(u)\right] >: u \in U\right\}.$$

Definition 2.6. [23] Let $Ŭ = \{u_1, u_2, u_3 \ldots\ldots u_n\}$ be the universal set of elements and $Ê = \{v_1, v_2, v_3 \ldots\ldots v_n\}$ be the universal set of parameters. The pair $(Ŭ, Ê)$ will be called a soft universe. Let $Ƒ: Ê \to Int(Ŭ)$ and η be a fuzzy set of Ê i.e η: $Ê \to I = [0,1]$, where $Int(Ŭ)$ is the set of all interval-valued fuzzy subsets on Ŭ. Let $Ƒ_\eta: Ê \to Int(Ŭ) \times I$ be a function defined as follows:

$$Ƒ_\eta(v) = \big(Ƒ(v)(u), \eta(v)\big), \forall u \in Ŭ.$$

Then $Ƒ_\eta$ is called a **generalized interval-valued fuzzy soft set** (GIVFS set) over the soft set $(Ŭ, Ê)$.

Definition 2.7 [24] If m(ƒ,ɡ) is the similarity measure between two interval-valued intuitionistic fuzzy soft sets (F,A) and (G,B) then,

1. $m(ƒ, ɡ) = m(ɡ, ƒ,)$
2. $0 \le m(ƒ, ɡ) \le 1$
3. $m(ƒ, ɡ) = 1 \Leftrightarrow (F, A) = (G, B)$

3. The Proposed concept of Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets and Their Fundamental Operations.

The terms Generalized Interval-Valued Fuzzy Soft set and Interval-Valued Intuitionistic Fuzzy Soft Expert set are combined in this section to create the new concept known as Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets. We define the idea of Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets and define some related concepts pertaining to this notion as well as their basic operations on this concept.

Definition 3.1. Let $Û = \{\xi_1, \xi_2, \xi_3, \ldots\ldots \xi_n\}$ be the initial universal set of elements, $Â = \{\sigma_1, \sigma_2, \sigma_3 \ldots\ldots \sigma_n\}$ be the set of attributes and $Ĝ = \{\rho_1, \rho_2, \rho_3 \ldots\ldots \rho_n\}$ be the set of experts. The Interval-Valued Intuitionistic Fuzzy Soft Expert set is a triplet $(Û, Â, Ĝ)$ that is described by a mapping $£: Â \times Ĝ \to Ķ(Û)$ where $Ķ(Û)$ is the set of all Interval-Valued Intuitionistic Fuzzy subset on Û. Let $£: Â \times Ĝ \to Ķ(Û)$ and η be a fuzzy soft expert set of $Â \times Ĝ$, i.e., $\eta: Â \times Ĝ \to I = [0,1]$. Let $£_\eta: Â \times Ĝ \to Ķ(Û) \times$ I be a function defined as follows:

$$£_{\eta^\rho_\sigma} = \left(\left\{< \xi, \left[\alpha^{-\rho}_\sigma, \alpha^{+\rho}_\sigma(u)\right], \left[\beta^{-\rho}_\sigma, \beta^{+\rho}_\sigma\right] >\right\}, \eta^{\,\rho}_\sigma(\xi)\right),$$

Then $£_{\eta^\rho_\sigma}$ is called a Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert set. GIVIFSEs over the soft expert set $(Û, Â, Ĝ)$.

Example 3.1. Let $Û = \{\xi_1, \xi_2, \xi_3\}$ be the set of universe, $Â = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of attributes and $Ĝ = \{\rho_1, \rho_2\}$ be the set of expert and $\eta: Â \times Ĝ \to I$ define a function $£_\eta: Â \times Ĝ \to Ķ(Û) \times$ I as follows:

$$£_{\eta^{\rho 1}_{\sigma 1}} = \left(\left\{\frac{\xi_1}{[0.2,0,4],[0.1,0.3]}, \frac{\xi_2}{[0.1,0.5],[0.4,0.5]}, \frac{\xi_3}{[0.3,0.4],[0.4,0.5]}\right\}, 0.5\right),$$

$$\pounds_{\eta_{\sigma 2}}^{\rho 1} = \left(\left\{\frac{\xi_1}{[0.3,0,4],[0.1,0.2]}, \frac{\xi_2}{[0.5,0.6],[0.4,0.6]}, \frac{\xi_3}{[0.2,0.4],[0.3,0.5]}\right\}, 0.6\right),$$

$$\pounds_{\eta_{\sigma 3}}^{\rho 1} = \left(\left\{\frac{\xi_1}{[0.2,0,5],[0.1,0.3]}, \frac{\xi_2}{[0.4,0.6],[0.5,0.7]}, \frac{\xi_3}{[0.1,0.2],[0.3,0.4]}\right\}, 0.3\right),$$

$$\pounds_{\eta_{\sigma 1}}^{\rho 2} = \left(\left\{\frac{\xi_1}{[0.1,0,3],[0.4,0.6]}, \frac{\xi_2}{[0.5,0.6],[0.1,0.3]}, \frac{\xi_3}{[0.7,0.8],[0.1,0.2]}\right\}, 0.4\right),$$

$$\pounds_{\eta_{\sigma 2}}^{\rho 2} = \left(\left\{\frac{\xi_1}{[0.0,0.4],[0.0,0.3]}, \frac{\xi_2}{[0.1,0.2],[0.6,0.7]}, \frac{\xi_3}{[0.1,0.5],[0.0,0.3]}\right\}, 0.1\right),$$

$$\pounds_{\eta_{\sigma 3}}^{\rho 2} = \left(\left\{\frac{\xi_1}{[0.1,0.4],[0.1,0.2]}, \frac{\xi_2}{[0.2,0.5],[0.0,0.5]}, \frac{\xi_3}{[0.3,0.5],[0.0,0.1]}\right\}, 0.7\right),$$

Then $\pounds_\eta$ is called a GIVIFSEs over $(\hat{U}, \hat{A}, \hat{G})$. In matrix notation we write in Table 1 and 2,

Table 1. Opinion of expert $\boldsymbol{\rho_1}$.

$\boldsymbol{\rho_1}$	$\boldsymbol{\xi_1}$	$\boldsymbol{\xi_2}$	$\boldsymbol{\xi_3}$	$\eta_\sigma^{\rho}(\xi)$
$\boldsymbol{\sigma_1}$	[0.2,0,4],[0.1,0.3]	[0.1,0.5],[0.4,0.5]	[0.3,0.4],[0.4,0.5]	0.5
$\boldsymbol{\sigma_2}$	[0.3,0,4],[0.1,0.2]	[0.5,0.6],[0.4,0.6]	[0.2,0.4],[0.3,0.5]	0.6
$\boldsymbol{\sigma_3}$	[0.2,0,5],[0.1,0.3]	[0.4,0.6],[0.5,0.7]	[0.1,0.2],[0.3,0.4]	0.3

Table 2. Opinion of expert $\boldsymbol{\rho_2}$.

$\boldsymbol{\rho_2}$	$\boldsymbol{\xi_1}$	$\boldsymbol{\xi_2}$	$\boldsymbol{\xi_3}$	$\eta_\sigma^{\rho}(\xi)$
$\boldsymbol{\sigma_1}$	[0.1,0,3],[0.4,0.6]	[0.5,0.6],[0.1,0.3]	[0.7,0.8],[0.1,0.2]	0.5
$\boldsymbol{\sigma_2}$	[0.0,0.4],[0.0,0.3]	[0.1,0.2],[0.6,0.7]	[0.1,0.5],[0.0,0.3]	0.6
$\boldsymbol{\sigma_3}$	[0.1,0.4],[0.1,0.2]	[0.2,0.5],[0.0,0.5]	[0.3,0.5],[0.0,0.1]	0.3

Definition 3.2. Let $\pounds_{\eta_1}$ and Γ_{η_2} be two GIVIFSEs over $(\hat{U}, \hat{A}, \hat{G})$. Then $\pounds_{\eta_1}$ is called a *GIVIFSE* subset of Γ_{η_2} and we write $\pounds_{\eta_1} \subseteq \Gamma_{\eta_2}$ if.

a) ${\eta_1}_\sigma^{\rho}(\xi)$ is a fuzzy subset of ${\eta_2}_\sigma^{\rho}(\xi)$ $\forall\, \xi \in \hat{U}$,

b) $\pounds_{\eta_\sigma}^{\rho}$ is an IVIF subset of $\Gamma_{\eta_\sigma}^{\rho}$, $\forall\ \sigma \in \hat{A}$ *and* $\rho \in \hat{G}$.

Definition 3.3. The complement of GIVIFSEs over $(\hat{U}, \hat{A}, \hat{G})$ is denoted by $(\hat{U}, \hat{A}, \hat{G})^{\acute{c}}$ and is defined for all $\sigma \in \hat{A}$ *and* $\rho \in \hat{G}$ as follows:

$$(\hat{U}, \hat{A}, \hat{G})^{\acute{c}} = \left(\left\{< \xi, \left[\beta_\sigma^{-\rho}, \beta_\sigma^{+\rho}\right], \left[\alpha_\sigma^{-\rho}, \alpha_\sigma^{+\rho}(u)\right] >\right\}, 1 - \eta_\sigma^{\rho}(\xi)\right),$$

Where $\pounds_{\eta_\sigma}^{\rho} = \left(\left\{< \xi, \left[\alpha_\sigma^{-\rho}, \alpha_\sigma^{+\rho}(u)\right], \left[\beta_\sigma^{-\rho}, \beta_\sigma^{+\rho}\right] >\right\}, \eta_\sigma^{\rho}(\xi)\right)$.

Definition 3.4. Let $\pounds_{\eta_1}$ be a GIVIFSEs over $(\hat{U}, \hat{A}, \hat{G})$. Then the complement of $\pounds_{\eta_1}$ denoted by $\pounds_{\eta_1}^{\acute{c}}$ and is defined by $\pounds_{\eta_1}^{\acute{c}} = \Gamma_\eta$ such that ${\eta_2}_\sigma^{\rho}(\xi) = \acute{c}\,({\eta_1}_\sigma^{\rho}(\xi))$ and $\Gamma_\sigma^{\rho} = \acute{c}\,(\pounds_\sigma^{\rho})$ $\forall\ \sigma \in \hat{A}$ *and* $\rho \in \hat{G}$ where $\acute{c}$ is a fuzzy complement and $\hat{c}$ is an Interval-Valued Intuitionistic Fuzzy complement.

Definition 3.5. The sum of two GIVIFSEs $(\pounds_1, \hat{A}_1, \hat{G}_1)$ and $(\pounds_2, \hat{A}_2, \hat{G}_2)$ over ξ is denoted as $(\pounds_1, \hat{A}_1, \hat{G}_1) + (\pounds_2, \hat{A}_2, \hat{G}_2)$ and defined as:

$$\pounds_{1\sigma}^{\rho} + \pounds_{2\sigma}^{\rho} = \left(\left\{< \xi, \begin{bmatrix}\alpha_1{}_\sigma^{-\rho}(\xi) + \alpha_2{}_\sigma^{-\rho}(\xi) - \alpha_1{}_\sigma^{-\rho}(\xi)\alpha_2{}_\sigma^{-\rho}(\xi), \\ \alpha_1{}_\sigma^{+\rho}(\xi) + \alpha_2{}_\sigma^{+\rho}(\xi) - \alpha_1{}_\sigma^{+\rho}(\xi)\alpha_2{}_\sigma^{+\rho}(\xi),\end{bmatrix}, \begin{bmatrix}\beta_1{}_\sigma^{-\rho}(\xi)\beta_2{}_\sigma^{-\rho}(\xi), \beta_1{}_\sigma^{+\rho}(\xi)\beta_2{}_\sigma^{+\rho}(\xi)\end{bmatrix}, > \right\}, \begin{matrix}\\ \\ {\eta_1}_\sigma^{\rho}(\xi) + {\eta_2}_\sigma^{\rho}(\xi) - {\eta_1}_\sigma^{\rho}(\xi){\eta_2}_\sigma^{\rho}(\xi)\end{matrix}\right),$$

Where $\mathcal{E}_{\eta_{1\sigma}}^{\rho} = \left(\{< \xi, \left[{\alpha_1}_{\sigma}^{-\rho}, {\alpha_1}_{\sigma}^{+\rho}(u)\right], \left[{\beta_1}_{\sigma}^{-\rho}, {\beta_1}_{\sigma}^{+\rho}\right] >\}, {\eta_1}_{\sigma}^{\rho}(\xi)\right)$,

and $\mathcal{E}_{\eta_{2\sigma}}^{\rho} = \left(\{< \xi, \left[{\alpha_2}_{\sigma}^{-\rho}, {\alpha_2}_{\sigma}^{+\rho}(u)\right], \left[{\beta_2}_{\sigma}^{-\rho}, {\beta_2}_{\sigma}^{+\rho}\right] >\}, {\eta_2}_{\sigma}^{\rho}(\xi)\right)$.

Definition 3.6. The difference between two GIVIFSEs $(\mathcal{E}_1, \hat{A}_1, \hat{G}_1)$ and $(\mathcal{E}_2, \hat{A}_2, \hat{G}_2)$ over ξ is denoted as $(\mathcal{E}_1, \hat{A}_1, \hat{G}_1) - (\mathcal{E}_2, \hat{A}_2, \hat{G}_2)$ and defined as:

$$\mathcal{E}_{1\sigma}^{\rho} - \mathcal{E}_{2\sigma}^{\rho} = \left(\begin{matrix}\left\{\begin{matrix}< \xi, \left[{\alpha_1}_{\sigma}^{-\rho}(\xi) \wedge {\beta_2}_{\sigma}^{-\rho}(\xi), {\alpha_1}_{\sigma}^{+\rho}(\xi) \wedge {\beta_2}_{\sigma}^{+\rho}(\xi)\right], \\ \left[{\beta_1}_{\sigma}^{-\rho}(\xi) \vee {\alpha_2}_{\sigma}^{-\rho}(\xi), {\beta_1}_{\sigma}^{+\rho}(\xi) \vee {\alpha_1}_{\sigma}^{+\rho}(\xi)\right] >\end{matrix}\right\}, \\ {\eta_1}_{\sigma}^{\rho}(\xi) \wedge 1 - {\eta_2}_{\sigma}^{\rho}(\xi)\end{matrix}\right),$$

Where $\mathcal{E}_{\eta_{1\sigma}}^{\rho} = \left(\{< \xi, \left[{\alpha_1}_{\sigma}^{-\rho}, {\alpha_1}_{\sigma}^{+\rho}(u)\right], \left[{\beta_1}_{\sigma}^{-\rho}, {\beta_1}_{\sigma}^{+\rho}\right] >\}, {\eta_1}_{\sigma}^{\rho}(\xi)\right)$,

and $\mathcal{E}_{\eta_{2\sigma}}^{\rho} = \left(\{< \xi, \left[{\alpha_2}_{\sigma}^{-\rho}, {\alpha_2}_{\sigma}^{+\rho}(u)\right], \left[{\beta_2}_{\sigma}^{-\rho}, {\beta_2}_{\sigma}^{+\rho}\right] >\}, {\eta_2}_{\sigma}^{\rho}(\xi)\right)$.

Definition 3.7. The product of two GIVIFSEs $(\mathcal{E}_1, \hat{A}_1, \hat{G}_1)$ and $(\mathcal{E}_2, \hat{A}_2, \hat{G}_2)$ over ξ is denoted as $(\mathcal{E}_1, \hat{A}_1, \hat{G}_1) \times (\mathcal{E}_2, \hat{A}_2, \hat{G}_2)$ and defined as:

$$\mathcal{E}_{1\sigma}^{\rho}\mathcal{E}_{2\sigma}^{\rho} = \left(\begin{matrix}\left\{\begin{matrix}< \xi, \left[{\alpha_1}_{\sigma}^{-\rho}(\xi){\alpha_2}_{\sigma}^{-\rho}(\xi), {\alpha_1}_{\sigma}^{+\rho}(\xi){\alpha_2}_{\sigma}^{+\rho}(\xi)\right], \\ \left[{\beta_1}_{\sigma}^{-\rho}(\xi) + {\beta_2}_{\sigma}^{-\rho}(\xi) - {\beta_1}_{\sigma}^{-\rho}(\xi){\beta_2}_{\sigma}^{-\rho}(\xi), \right. \\ \left. {\beta_1}_{\sigma}^{+\rho}(\xi) + {\beta_2}_{\sigma}^{+\rho}(\xi) - {\beta_1}_{\sigma}^{+\rho}(\xi){\beta_2}_{\sigma}^{+\rho}(\xi)\right] >\end{matrix}\right\}, \\ {\eta_1}_{\sigma}^{\rho}(\xi){\eta_2}_{\sigma}^{\rho}(\xi)\end{matrix}\right),$$

Where $\mathcal{E}_{\eta_{1\sigma}}^{\rho} = \left(\{< \xi, \left[{\alpha_1}_{\sigma}^{-\rho}, {\alpha_1}_{\sigma}^{+\rho}(u)\right], \left[{\beta_1}_{\sigma}^{-\rho}, {\beta_1}_{\sigma}^{+\rho}\right] >\}, {\eta_1}_{\sigma}^{\rho}(\xi)\right)$

and $\mathcal{E}_{\eta_{2\sigma}}^{\rho} = \left(\{< \xi, \left[{\alpha_2}_{\sigma}^{-\rho}, {\alpha_2}_{\sigma}^{+\rho}(u)\right], \left[{\beta_2}_{\sigma}^{-\rho}, {\beta_2}_{\sigma}^{+\rho}\right] >\}, {\eta_2}_{\sigma}^{\rho}(\xi)\right)$

Definition 3.8. The scalar product of GIVIFSEs $(\xi, \hat{A}, \hat{G})$ with an arbitrary real number $r > 0$ is denoted by $r(\xi, \hat{A}, \hat{G})$ and defined as follows:

$$r\mathcal{E}_{\sigma}^{\rho} = \left(\begin{matrix}\left\{< \xi, \left[1 - \left(1 - \alpha^{-\rho}_{\sigma}\right)^{r}, 1 - \left(1 - \alpha^{+\rho}_{\sigma}\right)^{r}\right], \left[\left(\beta^{-\rho}_{\sigma}\right)^{r}, \left(\beta^{+\rho}_{\sigma}\right)^{r}\right] >, \right\} \\ 1 - \left(1 - \eta_{\sigma}^{\rho}\right)^{r}\end{matrix}\right)$$

Definition 3.9 The power of GIVIFSEs$(\xi, \hat{A}, \hat{G})$ with an arbitrary real number $r > 0$ is denoted by $(\xi, \hat{A}, \hat{G})^{r}$ over ξ and defined as follow:

$$\left(\mathcal{E}_{\sigma}^{\rho}\right)^{r} = \left(\begin{matrix}\left\{< \xi, \left[\left(\alpha^{-\rho}_{\sigma}\right)^{r}, \left(\alpha^{+\rho}_{\sigma}\right)^{r}\right]\left[1 - \left(1 - \beta^{-\rho}_{\sigma}\right)^{r}, 1 - \left(1 - \beta^{+\rho}_{\sigma}\right)^{r}\right] >, \right\} \\ \left(\eta_{\sigma}^{\rho}\right)^{r}\end{matrix}\right).$$

4. Some Novel Similarity Measures for Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets

We are frequently interested in determining if two patterns, photos or designs are similar, or at least to what degree they are similar. There are different methods for calculating the similarity measures between two fuzzy sets, hesitant fuzzy sets, fuzzy soft sets, intuitionistic fuzzy sets, Interval-Valued Intuitionistic Fuzzy Soft sets, and Generalized Interval-Valued Fuzzy Soft set. In this section, we explain three distinct types of similarity measures for GIVIFSESs. Recently Majumdar and Samanta [25, 26] have studied the similarity measure of soft sets and fuzzy soft sets. Rafiq, Muhammad et al. [27] investigated the family of novel similarity measures between spherical fuzzy sets based on cosine function by considering the positive, neutral, negative and refusal grades in SFSs. We have further studied the similarity between two GIVIFSESs and it has been applied in medical

diagnosis. Similarity measures have extensive application in several areas such as pattern recognition, image processing, region extraction, and coding theory. We are using the set theoretic approach because it is more straightforward to calculate and is also a commonly used technique. Let us suppose that.

$$\pounds_{\eta 1_{1\sigma_1}}^{\rho_1} = \left(\left\{< \xi, \left[{\alpha_1}^{-}{}_{\sigma}^{\rho}(\xi), {\alpha_1}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[{\beta_1}^{-}{}_{\sigma}^{\rho}(\xi), {\beta_1}^{+}{}_{\sigma}^{\rho}(\xi)\right] >\right\}, \eta_1{}_{\sigma}^{\rho}(\xi)\right): \xi \in \hat{U}, {}_{\sigma_1}^{\rho_1}\in \hat{A}_1 \times \hat{G}_1\Big),$$

$$\Gamma_{\eta_{2\sigma_2}}^{\rho_2} = \left(\left\{< \xi, \left[{\alpha_2}^{-}{}_{\sigma}^{\rho}(\xi), {\alpha_2}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[{\beta_2}^{-}{}_{\sigma}^{\rho}(\xi), {\beta_2}^{+}{}_{\sigma}^{\rho}(\xi)\right] >\right\}, \eta_2{}_{\sigma}^{\rho}(\xi)\right): \xi \in \hat{U}, {}_{\sigma_2}^{\rho_2}\in \hat{A}_2 \times \hat{G}_2\Big),$$

be two GIVIFSESs over ξ.

4.1 Type-I Similarity Measures for Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets

Definition 4.1. Let $\hat{U} = \{\xi_x : x = 1,2,3, \ldots\ldots, m\}$ be the finite universal set of elements ξ_x , $\hat{A} = \{\sigma_{\hat{y}} : \hat{y} = 1,2,3, \ldots\ldots, n\}$ be the set of attributes and$\hat{G} = \{\rho_{\check{z}} : \check{z} = 1.2.3, \ldots\ldots\ldots, t\}$ be the set of experts (or decision makers). Let $\pounds_{\eta 1}$ and $\Gamma_{\eta 2}$ be two GIVIFSESs over ξ. Let SM($\pounds_{\eta 1}, \Gamma_{\eta 2}$) specify the similarity measures between GIVIFSESs $\pounds_{\eta 1}$ and $\Gamma_{\eta 2}$. We must first find similarities between their $\rho_{\check{z}}$th approximations to see similar measurements $\pounds_{\eta 1}$ and $\Gamma_{\eta 2}$. Let $S_i(\pounds_{\eta 1}, \Gamma_{\eta 2})$ indicate the similarity measure between the two $\rho_{\check{z}}$th approximation of :

$$\pounds_{\eta 1_{1\sigma}}^{\rho} = \left(\left\{< \xi, \left[{\alpha_1}^{-}{}_{\sigma}^{\rho}(\xi), {\alpha_1}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[{\beta_1}^{-}{}_{\sigma}^{\rho}(\xi), {\beta_1}^{+}{}_{\sigma}^{\rho}(\xi)\right] >\right\}, \eta_1{}_{\sigma}^{\rho}(\xi)\right),$$

$$\Gamma_{\eta_{2\sigma}}^{\rho} = \left(\left\{< \xi, \left[{\alpha_2}^{-}{}_{\sigma}^{\rho}(\xi), {\alpha_2}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[{\beta_2}^{-}{}_{\sigma}^{\rho}(\xi), {\beta_2}^{+}{}_{\sigma}^{\rho}(\xi)\right] >\right\}, \eta_2{}_{\sigma}^{\rho}(\xi)\right),$$

Then $S_i(\pounds_{\eta 1}, \Gamma_{\eta 2})$ is defined as follows:

$$S_i\left(\pounds_{\eta 1}, \Gamma_{\eta 2}\right) = \max_{\xi \in \hat{U}} \min_{\sigma \in \hat{A}, \rho \in \hat{G}} \left\lfloor \begin{matrix} \left[{\alpha_1}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}})}(\xi_x), {\alpha_1}^{+}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}})}(\xi_x)\right], \\ \left[{\beta_1}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}})}(\xi_x), {\beta_1}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}})}(\xi_x)\right], \\ \left[{\alpha_2}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}+1})}(\xi_x), {\alpha_2}^{+}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}+1})}(\xi_x)\right], \\ \left\langle \begin{matrix} \left[{\beta_2}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}+1})}(\xi_x), {\beta_2}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}+1})}(\xi_x)\right], \\ \eta_1{}_{(\sigma_{\hat{y}}, \rho_{\check{z}})}(\xi_x), \eta_2{}_{(\sigma_{\hat{y}}, \rho_{\check{z}+1})}(\xi_x) \end{matrix} \right\rangle \end{matrix} \right\rfloor \quad (4.1)$$

where i=1,2,3....,.

when i=1, $\hat{y}$ =1, $\check{z}$=1 and x=1,2,3....,m, then by using (4.1) we get:

$$S_1\left(\pounds_{\eta 1}, \Gamma_{\eta 2}\right) = \begin{matrix} \max \\ \xi \in \hat{U} \end{matrix} \left\{ \begin{matrix} \min \\ \sigma_1 \in \hat{A}, \rho \in \hat{G} \end{matrix} \left\{ \begin{matrix} \left[{\alpha_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_1), {\alpha_1}^{+}{}_{(\sigma_1,\rho_1)}(\xi_1)\right], \\ \left[{\beta_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_1), {\beta_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_1)\right], \\ \left[{\alpha_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_1), {\alpha_2}^{+}{}_{(\sigma_1,\rho_2)}(\xi_1)\right], \\ \left\langle \begin{matrix} \left[{\beta_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_1), {\beta_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_1)\right], \\ \eta_{1(\sigma_1,\rho_1)}(\xi_1), \eta_{2(\sigma_1,\rho_2)}(\xi_1) \end{matrix} \right\rangle \end{matrix} \right\}, \right.$$

$$\begin{matrix} \max \\ \xi \in \hat{U} \end{matrix} \left\{ \begin{matrix} \min \\ \sigma_1 \in \hat{A}, \rho \in \hat{G} \end{matrix} \left\{ \begin{matrix} \left[{\alpha_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_2), {\alpha_1}^{+}{}_{(\sigma_1,\rho_1)}(\xi_2)\right], \\ \left[{\beta_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_2), {\beta_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_2)\right], \\ \left[{\alpha_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_2), {\alpha_2}^{+}{}_{(\sigma_1,\rho_2)}(\xi_2)\right], \\ \left\langle \begin{matrix} \left[{\beta_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_2), {\beta_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_2)\right], \\ \eta_{1(\sigma_1,\rho_1)}(\xi_2), \eta_{2(\sigma_1,\rho_2)}(\xi_2) \end{matrix} \right\rangle \end{matrix} \right\}, \right.$$

$$\begin{matrix} \max \\ \xi \in \hat{U} \end{matrix} \left\{ \begin{matrix} \min \\ \sigma_1 \in \hat{A}, \rho \in \hat{G} \end{matrix} \left\{ \begin{matrix} \left[{\alpha_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_3), {\alpha_1}^{+}{}_{(\sigma_1,\rho_1)}(\xi_3)\right], \\ \left[{\beta_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_3), {\beta_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_3)\right], \\ \left[{\alpha_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_3), {\alpha_2}^{+}{}_{(\sigma_1,\rho_2)}(\xi_3)\right], \\ \left\langle \begin{matrix} \left[{\beta_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_3), {\beta_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_3)\right], \\ \eta_{1(\sigma_1,\rho_1)}(\xi_3), \eta_{2(\sigma_1,\rho_2)}(\xi_3) \end{matrix} \right\rangle \end{matrix} \right\}, \quad \ldots\ldots\ldots, \right.$$

$$\begin{matrix} \max \\ \xi \in \hat{U} \end{matrix} \left\{ \begin{matrix} \min \\ \sigma_1 \in \hat{A}, \rho \in \hat{G} \end{matrix} \left\{ \begin{matrix} \left[{\alpha_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_m), {\alpha_1}^{+}{}_{(\sigma_1,\rho_1)}(\xi_m)\right], \\ \left[{\beta_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_m), {\beta_1}^{-}{}_{(\sigma_1,\rho_1)}(\xi_m)\right], \\ \left[{\alpha_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_m), {\alpha_2}^{+}{}_{(\sigma_1,\rho_2)}(\xi_m)\right], \\ \left\langle \begin{matrix} \left[{\beta_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_m), {\beta_2}^{-}{}_{(\sigma_1,\rho_2)}(\xi_m)\right], \\ \eta_{1(\sigma_1,\rho_1)}(\xi_m), \eta_{2(\sigma_1,\rho_2)}(\xi_m) \end{matrix} \right\rangle \end{matrix} \right\}. \right. \quad (4.2)$$

and when i=2, ŷ=2, ž=1 and x=1,2,3....,m, then (4.1):

$$S_2(\pounds_{\eta 1}, \Gamma_{\eta 2}) = \begin{matrix} \max \\ \xi \in \hat{U} \end{matrix} \left\lfloor \begin{matrix} \min \\ \sigma_2 \in \hat{A}, \rho \in \hat{G} \end{matrix} \left\langle \begin{matrix} \left[{\alpha_1}^-_{(\sigma_2,\rho_1)}(\xi_1), {\alpha_1}^+_{(\sigma_2,\rho_1)}(\xi_1)\right], \\ \left[{\beta_1}^-_{(\sigma_2,\rho_1)}(\xi_1), {\beta_1}^-_{(\sigma_2,\rho_2)}(\xi_1)\right], \\ \left[{\alpha_2}^-_{(\sigma_2,\rho_2)}(\xi_1), {\alpha_2}^+_{(\sigma_2,\rho_2)}(\xi_1)\right], \\ \left\langle \begin{matrix} \left[{\beta_2}^-_{(\sigma_2,\rho_2)}(\xi_1), {\beta_2}^-_{(\sigma_2,\rho_2)}(\xi_1)\right], \\ \eta_{1(\sigma_2,\rho_1)}(\xi_1), \eta_{2(\sigma_2,\rho_2)}(\xi_1) \end{matrix} \right\rangle \end{matrix} \right\rfloor , \right.$$

$$\begin{matrix} \max \\ \xi \in \hat{U} \end{matrix} \left\lfloor \begin{matrix} \min \\ \sigma_2 \in \hat{A}, \rho \in \hat{G} \end{matrix} \left\langle \begin{matrix} \left[{\alpha_1}^-_{(\sigma_2,\rho_1)}(\xi_2), {\alpha_1}^+_{(\sigma_2,\rho_1)}(\xi_2)\right], \\ \left[{\beta_1}^-_{(\sigma_2,\rho_1)}(\xi_2), {\beta_1}^-_{(\sigma_2,\rho_1)}(\xi_2)\right], \\ \left[{\alpha_2}^-_{(\sigma_2,\rho_2)}(\xi_2), {\alpha_2}^+_{(\sigma_2,\rho_2)}(\xi_2)\right], \\ \left\langle \begin{matrix} \left[{\beta_2}^-_{(\sigma_2,\rho_2)}(\xi_2), {\beta_2}^-_{(\sigma_2,\rho_2)}(\xi_2)\right], \\ \eta_{1(\sigma_2,\rho_1)}(\xi_2), \eta_{2(\sigma_2,\rho_2)}(\xi_2) \end{matrix} \right\rangle \end{matrix} \right\rfloor , \quad ,$$

$$\begin{matrix} \max \\ \xi \in \hat{U} \end{matrix} \left\lfloor \begin{matrix} \min \\ \sigma_2 \in \hat{A}, \rho \in \hat{G} \end{matrix} \left\langle \begin{matrix} \left[{\alpha_1}^-_{(\sigma_2,\rho_1)}(\xi_3), {\alpha_1}^+_{(\sigma_2,\rho_1)}(\xi_3)\right], \\ \left[{\beta_1}^-_{(\sigma_2,\rho_1)}(\xi_3), {\beta_1}^-_{(\sigma_2,\rho_1)}(\xi_3)\right], \\ \left[{\alpha_2}^-_{(\sigma_2,\rho_2)}(\xi_3), {\alpha_2}^+_{(\sigma_2,\rho_2)}(\xi_3)\right], \\ \left\langle \begin{matrix} \left[{\beta_2}^-_{(\sigma_2,\rho_2)}(\xi_3), {\beta_2}^-_{(\sigma_2,\rho_2)}(\xi_3)\right], \\ \eta_{1(\sigma_2,\rho_1)}(\xi_3), \eta_{2(\sigma_2,\rho_2)}(\xi_3) \end{matrix} \right\rangle \end{matrix} \right\rfloor , \quad \dots\dots\dots ,$$

$$\begin{matrix} \max \\ \xi \in \hat{U} \end{matrix} \left\lfloor \begin{matrix} \min \\ \sigma_2 \in \hat{A}, \rho \in \hat{G} \end{matrix} \left\langle \begin{matrix} \left[{\alpha_1}^-_{(\sigma_2,\rho_1)}(\xi_m), {\alpha_1}^+_{(\sigma_2,\rho_1)}(\xi_m)\right], \\ \left[{\beta_1}^-_{(\sigma_2,\rho_1)}(\xi_m), {\beta_1}^-_{(\sigma_2,\rho_1)}(\xi_m)\right], \\ \left[{\alpha_2}^-_{(\sigma_2,\rho_2)}(\xi_m), {\alpha_2}^+_{(\sigma_2,\rho_2)}(\xi_m)\right], \\ \left\langle \begin{matrix} \left[{\beta_2}^-_{(\sigma_2,\rho_2)}(\xi_m), {\beta_2}^-_{(\sigma_2,\rho_2)}(\xi_m)\right], \\ \eta_{1(\sigma_2,\rho_1)}(\xi_m), \eta_{2(\sigma_2,\rho_2)}(\xi_m) \end{matrix} \right\rangle \end{matrix} \right\rfloor \quad (4.3)$$

when i =l, ŷ=n, ž=1 and x=1,2,3....,m, then by using (4.1):

$$S_l(\pounds_{\eta 1}, \Gamma_{\eta 2}) = \begin{pmatrix} \max\limits_{\xi \in \hat{U}} \boxed{\min\limits_{\sigma_n \in \hat{A}, \rho \in \hat{G}}} \begin{pmatrix} \left[\alpha_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_1), \alpha_1{}^{+}{}_{(\sigma_n,\rho_1)}(\xi_1)\right], \\ \left[\beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_1), \beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_1)\right], \\ \left[\alpha_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_1), \alpha_2{}^{+}{}_{(\sigma_n,\rho_2)}(\xi_1)\right], \\ \left[\beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_1), \beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_1)\right], \\ \eta_{1(\sigma_n,\rho_1)}(\xi_1), \eta_{2(\sigma_n,\rho_2)}(\xi_1) \end{pmatrix}, \\ \max\limits_{\xi \in \hat{U}} \boxed{\min\limits_{\sigma_n \in \hat{A}, \rho \in \hat{G}}} \begin{pmatrix} \left[\alpha_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_2), \alpha_1{}^{+}{}_{(\sigma_n,\rho_1)}(\xi_2)\right], \\ \left[\beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_2), \beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_2)\right], \\ \left[\alpha_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_2), \alpha_2{}^{+}{}_{(\sigma_n,\rho_2)}(\xi_2)\right], \\ \left[\beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_2), \beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_2)\right], \\ \eta_{1(\sigma_n,\rho_1)}(\xi_2), \eta_{2(\sigma_n,\rho_2)}(\xi_2) \end{pmatrix}, \\ \max\limits_{\xi \in \hat{U}} \boxed{\min\limits_{\sigma_n \in \hat{A}, \rho \in \hat{G}}} \begin{pmatrix} \left[\alpha_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_3), \alpha_1{}^{+}{}_{(\sigma_n,\rho_1)}(\xi_3)\right], \\ \left[\beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_3), \beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_3)\right], \\ \left[\alpha_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_3), \alpha_2{}^{+}{}_{(\sigma_n,\rho_2)}(\xi_3)\right], \\ \left[\beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_3), \beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_3)\right], \\ \eta_{1(\sigma_n,\rho_1)}(\xi_3), \eta_{2(\sigma_n,\rho_2)}(\xi_3) \end{pmatrix}, \quad \dots\dots\dots, \\ \max\limits_{\xi \in \hat{U}} \boxed{\min\limits_{\sigma_n \in \hat{A}, \rho \in \hat{G}}} \begin{pmatrix} \left[\alpha_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_m), \alpha_1{}^{+}{}_{(\sigma_n,\rho_1)}(\xi_m)\right], \\ \left[\beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_m), \beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_m)\right], \\ \left[\alpha_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_m), \alpha_2{}^{+}{}_{(\sigma_n,\rho_2)}(\xi_m)\right], \\ \left[\beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_m), \beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_m)\right], \\ \eta_{1(\sigma_n,\rho_1)}(\xi_m), \eta_{2(\sigma_n,\rho_2)}(\xi_m) \end{pmatrix} \end{pmatrix} \quad (4.4)$$

And so on. where

$$\left[\alpha_1{}^{-\rho}_{\sigma}(\xi), \alpha_1{}^{+\rho}_{\sigma}(\xi)\right], \left[\beta_1{}^{-\rho}_{\sigma}(\xi), \beta_1{}^{+\rho}_{\sigma}(\xi)\right] = \frac{1}{3}\begin{pmatrix} \alpha_1{}^{-\rho}_{\sigma}(\xi) + \alpha_1{}^{+\rho}_{\sigma}(\xi) - \\ \alpha_1{}^{-\rho}_{\sigma}(\xi)\alpha_1{}^{+\rho}_{\sigma}(\xi) + \\ \beta_1{}^{-\rho}_{\sigma}(\xi) + \beta_1{}^{+\rho}_{\sigma}(\xi) - \\ \beta_1{}^{-\rho}_{\sigma}(\xi)\beta_1{}^{+\rho}_{\sigma}(\xi) \end{pmatrix}, \quad (4.5)$$

$$\left[\alpha_2{}^{-\rho}_{\sigma}(\xi), \alpha_2{}^{+\rho}_{\sigma}(\xi)\right], \left[\beta_2{}^{-\rho}_{\sigma}(\xi), \beta_2{}^{+\rho}_{\sigma}(\xi)\right] = \frac{1}{3}\begin{pmatrix} \alpha_2{}^{-\rho}_{\sigma}(\xi) + \alpha_2{}^{+\rho}_{\sigma}(\xi) - \\ \alpha_2{}^{-\rho}_{\sigma}(\xi)\alpha_2{}^{+\rho}_{\sigma}(\xi) + \\ \beta_2{}^{-\rho}_{\sigma}(\xi) + \beta_2{}^{+\rho}_{\sigma}(\xi) - \\ \beta_2{}^{-\rho}_{\sigma}(\xi)\beta_2{}^{+\rho}_{\sigma}(\xi) \end{pmatrix}, \quad (4.6)$$

Then the similarity measures between the GIVIFSESs $\pounds_{\eta_1}$ and Γ_{η_2} is defined as,

$$SM(\pounds_{\eta_1}, \Gamma_{\eta_2}) = \frac{1}{n}\Sigma S_i(\pounds_{\eta_1}, \Gamma_{\eta_2}) \quad (4.7)$$

Definition 4.2. Let us suppose that.

$$\pounds_{\eta_1}{}^{\rho_1}_{\sigma_1} = \left(\left(\{< \xi, \left[\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}\right], \left[\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}\right] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right): \xi \in \hat{U}, {}^{\rho_1}_{\sigma_1} \in \hat{A}_1 \times \hat{G}_1\right),$$

$$\Gamma_{\eta_2}{}^{\rho_2}_{\sigma_2} = \left(\left(\{< \xi, \left[\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}\right], \left[\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}\right] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right): \xi \in \hat{U}, {}^{\rho_2}_{\sigma_2} \in \hat{A}_2 \times \hat{G}_2\right),$$

be two GIVIFSESs over ξ. Then $\mathcal{E}_{\eta_1}$ and Γ_{η_2} are said to be $\lambda-$similar, denoted by $\mathcal{E}_{\eta_1} \overset{\lambda}{\cong} \Gamma_{\eta_2}$ if and only if $S(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2})$ for $\lambda \in (0,1)$. We call the two GIVIFSESs significantly similar if $S(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2}) > \frac{1}{2}$.

Theorem 4.1. If $SM(\mathcal{E}_{\eta 1}, \Gamma_{\eta_2})$ be the similarity measures between two GIVIFSESs $\mathcal{E}_{\eta_1}$ and Γ_{η_2} then.

i. $SM(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2}) = SM(\Gamma_{\eta_2}, \mathcal{E}_{\eta_1})$,
ii. $0 \leq SM(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2}) \leq 1$,
iii. $SM(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2}) = 1 \Leftrightarrow \mathcal{E}_{\eta_1} = \Gamma_{\eta_2}$.

4.2 Type-II Similarity Measures for Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets

Definition 4.3. Let $\xi = \{\xi_x : x = 1,2,3, \ldots\ldots, m\}$ be the finite universal set of elements ξ_x: , $\hat{A} = \{\sigma_{\hat{y}} : \hat{y} = 1,2,3, \ldots\ldots, n\}$ be the set of attributes and $\hat{G} = \{\rho_{\check{z}} : \check{z} = 1.2.3, \ldots\ldots, t\}$ be the set of experts (or decision makers).

Let $\mathcal{E}_{\eta_1}$ and Γ_{η_2} be two GIVIFSESs over ξ. Let $\widetilde{SM}(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2})$ specify the similarity measures between GIVIFSESs $\mathcal{E}_{\eta_1}$ andΓ_{η_2}. We must first find similarity between their $\rho_{\check{z}}$th approximations in order to see the similarity measurements $\mathcal{E}_{\eta_1}$ and Γ_{η_2}. Let $S_i(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2})$ indicate the similarity measure between the two $\rho_{\check{z}}$th approximation of

$$\mathcal{E}_{\eta_1}{}_{\sigma}^{\rho} = \left(\left\{< \xi, \left[{\alpha_1}^{-}{}_{\sigma}^{\rho}(\xi), {\alpha_1}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[{\beta_1}^{-}{}_{\sigma}^{\rho}(\xi), {\beta_1}^{+}{}_{\sigma}^{\rho}(\xi)\right] >\right\}, \eta_1{}_{\sigma}^{\rho}(\xi)\right),$$

$$\Gamma_{\eta_2}{}_{\sigma}^{\rho} = \left(\left\{< \xi, \left[{\alpha_2}^{-}{}_{\sigma}^{\rho}(\xi), {\alpha_2}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[{\beta_2}^{-}{}_{\sigma}^{\rho}(\xi), {\beta_2}^{+}{}_{\sigma}^{\rho}(\xi)\right] >\right\}, \eta_2{}_{\sigma}^{\rho}(\xi)\right),$$

Then $S_i(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2})$ is defined as follow:

$$S_i(\mathcal{E}_{\eta 1}, \Gamma_{\eta 2}) = \underset{\xi \in \hat{U}}{min} \left[\underset{\sigma \in \hat{A}, \rho \in \hat{G}}{max}\right] \begin{pmatrix} \left[{\alpha_1}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}})}(\xi_x), {\alpha_1}^{+}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}})}(\xi_x)\right], \\ \left[{\beta_1}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}})}(\xi_x), {\beta_1}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}})}(\xi_x)\right], \\ \left[{\alpha_2}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}+1})}(\xi_x), {\alpha_2}^{+}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}+1})}(\xi_x)\right], \\ \left[{\beta_2}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}+1})}(\xi_x), {\beta_2}^{-}{}_{(\sigma_{\hat{y}}, \rho_{\check{z}+1})}(\xi_x)\right], \\ \eta_1{}_{(\sigma_{\hat{y}}, \rho_{\check{z}})}(\xi_x), \eta_2{}_{(\sigma_{\hat{y}}, \rho_{\check{z}+1})}(\xi_x) \end{pmatrix} \quad (4.8)$$

Where i= 1, 2, 3...., l

when i=1, $\hat{y}$=1, $\check{z}$=1 and x=1,2,3...,m, then by using equation (4.8) we get:

$$S_1(\pounds_{\eta 1}, \Gamma_{\eta 2}) = \begin{matrix} \min \\ \xi \in \hat{U} \end{matrix} \boxed{\begin{matrix} \max \\ \sigma_1 \in \hat{A}, \rho \in \hat{G} \end{matrix}} \left(\begin{matrix} \left[\alpha_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_1), \alpha_1^{+}{}_{(\sigma_1,\rho_1)}(\xi_1)\right], \\ \left[\beta_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_1), \beta_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_1)\right], \\ \left[\alpha_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_1), \alpha_2^{+}{}_{(\sigma_1,\rho_2)}(\xi_1)\right], \\ \left[\beta_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_1), \beta_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_1)\right], \\ \eta_{1(\sigma_1,\rho_1)}(\xi_1), \eta_{2(\sigma_1,\rho_2)}(\xi_1) \end{matrix} , \right.$$

$$\left. \begin{matrix} \left[\alpha_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_2), \alpha_1^{+}{}_{(\sigma_1,\rho_1)}(\xi_2)\right], \\ \left[\beta_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_2), \beta_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_2)\right], \\ \left[\alpha_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_2), \alpha_2^{+}{}_{(\sigma_1,\rho_2)}(\xi_2)\right], \\ \left[\beta_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_2), \beta_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_2)\right], \\ \eta_{1(\sigma_1,\rho_1)}(\xi_2), \eta_{2(\sigma_1,\rho_2)}(\xi_2) \end{matrix} , \right.$$

$$\begin{matrix} \min \\ \xi \in \hat{U} \end{matrix} \boxed{\begin{matrix} \max \\ \sigma_1 \in \hat{A}, \rho \in \hat{G} \end{matrix}} \left(\begin{matrix} \left[\alpha_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_3), \alpha_1^{+}{}_{(\sigma_1,\rho_1)}(\xi_3)\right], \\ \left[\beta_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_3), \beta_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_3)\right], \\ \left[\alpha_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_3), \alpha_2^{+}{}_{(\sigma_1,\rho_2)}(\xi_3)\right], \\ \left[\beta_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_3), \beta_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_3)\right], \\ \eta_{1(\sigma_1,\rho_1)}(\xi_3), \eta_{2(\sigma_1,\rho_2)}(\xi_3) \end{matrix} \right) \quad \dots\dots\dots,$$

$$\begin{matrix} \min \\ \xi \in \hat{U} \end{matrix} \boxed{\begin{matrix} \max \\ \sigma_1 \in \hat{A}, \rho \in \hat{G} \end{matrix}} \left(\begin{matrix} \left[\alpha_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_m), \alpha_1^{+}{}_{(\sigma_1,\rho_1)}(\xi_m)\right], \\ \left[\beta_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_m), \beta_1^{-}{}_{(\sigma_1,\rho_1)}(\xi_m)\right], \\ \left[\alpha_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_m), \alpha_2^{+}{}_{(\sigma_1,\rho_2)}(\xi_m)\right], \\ \left[\beta_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_m), \beta_2^{-}{}_{(\sigma_1,\rho_2)}(\xi_m)\right], \\ \eta_{1(\sigma_1,\rho_1)}(\xi_m), \eta_{2(\sigma_1,\rho_2)}(\xi_m) \end{matrix} \right) \quad (4.9)$$

and when i=2, ŷ=2, ž=1 and x=1,2,3....,m, then equation (4.8):

$$S_2\left(\pounds_{\eta 1}, \Gamma_{\eta 2}\right) = \left\{ \begin{array}{l} \underset{\xi \in \hat{U}}{min} \left[\underset{\sigma_2 \in \hat{A}, \rho \in \hat{G}}{max} \left(\begin{array}{c} \left[{\alpha_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_1), {\alpha_1}^{+}{}_{(\sigma_2,\rho_1)}(\xi_1)\right], \\ \left[{\beta_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_1), {\beta_1}^{-}{}_{(\sigma_2,\rho_2)}(\xi_1)\right], \\ \left[{\alpha_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_1), {\alpha_2}^{+}{}_{(\sigma_2,\rho_2)}(\xi_1)\right], \\ \left(\begin{array}{c} \left[{\beta_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_1), {\beta_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_1)\right], \\ \eta_{1(\sigma_2,\rho_1)}(\xi_1), \eta_{2(\sigma_2,\rho_2)}(\xi_1) \end{array} \right) \end{array} \right) \right], \\ \underset{\xi \in \hat{U}}{min} \left[\underset{\sigma_2 \in \hat{A}, \rho \in \hat{G}}{max} \left(\begin{array}{c} \left[{\alpha_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_2), {\alpha_1}^{+}{}_{(\sigma_2,\rho_1)}(\xi_2)\right], \\ \left[{\beta_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_2), {\beta_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_2)\right], \\ \left[{\alpha_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_2), {\alpha_2}^{+}{}_{(\sigma_2,\rho_2)}(\xi_2)\right], \\ \left(\begin{array}{c} \left[{\beta_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_2), {\beta_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_2)\right], \\ \eta_{1(\sigma_2,\rho_1)}(\xi_2), \eta_{2(\sigma_2,\rho_2)}(\xi_2) \end{array} \right) \end{array} \right) \right], \\ \underset{\xi \in \hat{U}}{min} \left[\underset{\sigma_2 \in \hat{A}, \rho \in \hat{G}}{max} \left(\begin{array}{c} \left[{\alpha_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_3), {\alpha_1}^{+}{}_{(\sigma_2,\rho_1)}(\xi_3)\right], \\ \left[{\beta_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_3), {\beta_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_3)\right], \\ \left[{\alpha_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_3), {\alpha_2}^{+}{}_{(\sigma_2,\rho_2)}(\xi_3)\right], \\ \left(\begin{array}{c} \left[{\beta_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_3), {\beta_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_3)\right], \\ \eta_{1(\sigma_2,\rho_1)}(\xi_3), \eta_{2(\sigma_2,\rho_2)}(\xi_3) \end{array} \right) \end{array} \right) \right], \\ \underset{\xi \in \hat{U}}{min} \left[\underset{\sigma_2 \in \hat{A}, \rho \in \hat{G}}{max} \left(\begin{array}{c} \left[{\alpha_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_m), {\alpha_1}^{+}{}_{(\sigma_2,\rho_1)}(\xi_m)\right], \\ \left[{\beta_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_m), {\beta_1}^{-}{}_{(\sigma_2,\rho_1)}(\xi_m)\right], \\ \left[{\alpha_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_m), {\alpha_2}^{+}{}_{(\sigma_2,\rho_2)}(\xi_m)\right], \\ \left(\begin{array}{c} \left[{\beta_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_m), {\beta_2}^{-}{}_{(\sigma_2,\rho_2)}(\xi_m)\right], \\ \eta_{1(\sigma_2,\rho_1)}(\xi_m), \eta_{2(\sigma_2,\rho_2)}(\xi_m) \end{array} \right) \end{array} \right) \right] \end{array} \right\} \quad (4.10)$$

when i=l, ŷ=n, ž=1 and x =1,2,3....,m, then by using equation (4.8):

$$S_l(\pounds_{\eta 1}, \Gamma_{\eta 2}) = \begin{pmatrix} \min\limits_{\xi \in \hat{U}} \boxed{\max\limits_{\sigma_n \in \hat{A}, \rho \in \hat{G}}} \begin{pmatrix} \left[\alpha_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_1), \alpha_1{}^{+}{}_{(\sigma_n,\rho_1)}(\xi_1)\right], \\ \left[\beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_1), \beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_1)\right], \\ \left[\alpha_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_1), \alpha_2{}^{+}{}_{(\sigma_n,\rho_2)}(\xi_1)\right], \\ \left[\beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_1), \beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_1)\right], \\ \eta_{1(\sigma_n,\rho_1)}(\xi_1), \eta_{2(\sigma_n,\rho_2)}(\xi_1) \end{pmatrix}, \\ \min\limits_{\xi \in \hat{U}} \boxed{\max\limits_{\sigma_n \in \hat{A}, \rho \in \hat{G}}} \begin{pmatrix} \left[\alpha_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_2), \alpha_1{}^{+}{}_{(\sigma_n,\rho_1)}(\xi_2)\right], \\ \left[\beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_2), \beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_2)\right], \\ \left[\alpha_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_2), \alpha_2{}^{+}{}_{(\sigma_n,\rho_2)}(\xi_2)\right], \\ \left[\beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_2), \beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_2)\right], \\ \eta_{1(\sigma_n,\rho_1)}(\xi_2), \eta_{2(\sigma_n,\rho_2)}(\xi_2) \end{pmatrix}, \\ \min\limits_{\xi \in \hat{U}} \boxed{\max\limits_{\sigma_n \in \hat{A}, \rho \in \hat{G}}} \begin{pmatrix} \left[\alpha_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_3), \alpha_1{}^{+}{}_{(\sigma_n,\rho_1)}(\xi_3)\right], \\ \left[\beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_3), \beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_3)\right], \\ \left[\alpha_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_3), \alpha_2{}^{+}{}_{(\sigma_n,\rho_2)}(\xi_3)\right], \\ \left[\beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_3), \beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_3)\right], \\ \eta_{1(\sigma_n,\rho_1)}(\xi_3), \eta_{2(\sigma_n,\rho_2)}(\xi_3) \end{pmatrix}, \\ \min\limits_{\xi \in \hat{U}} \boxed{\max\limits_{\sigma_n \in \hat{A}, \rho \in \hat{G}}} \begin{pmatrix} \left[\alpha_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_m), \alpha_1{}^{+}{}_{(\sigma_n,\rho_1)}(\xi_m)\right], \\ \left[\beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_m), \beta_1{}^{-}{}_{(\sigma_n,\rho_1)}(\xi_m)\right], \\ \left[\alpha_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_m), \alpha_2{}^{+}{}_{(\sigma_n,\rho_2)}(\xi_m)\right], \\ \left[\beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_m), \beta_2{}^{-}{}_{(\sigma_n,\rho_2)}(\xi_m)\right], \\ \eta_{1(\sigma_n,\rho_1)}(\xi_m), \eta_{2(\sigma_n,\rho_2)}(\xi_m) \end{pmatrix} \end{pmatrix} \quad (4.11)$$

And so on. Where

$$\left(\left[\alpha_1{}^{-}{}_{\sigma}^{\rho}(\xi), \alpha_1{}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[\beta_1{}^{-}{}_{\sigma}^{\rho}(\xi), \beta_1{}^{+}{}_{\sigma}^{\rho}(\xi)\right]\right) = \frac{1}{3}\begin{pmatrix} \alpha_1{}^{-}{}_{\sigma}^{\rho}(\xi) + \alpha_1{}^{+}{}_{\sigma}^{\rho}(\xi) - \\ \alpha_1{}^{-}{}_{\sigma}^{\rho}(\xi)\alpha_1{}^{+}{}_{\sigma}^{\rho}(\xi) + \\ \beta_1{}^{-}{}_{\sigma}^{\rho}(\xi) + \beta_1{}^{+}{}_{\sigma}^{\rho}(\xi) - \\ \beta_1{}^{-}{}_{\sigma}^{\rho}(\xi)\beta_1{}^{+}{}_{\sigma}^{\rho}(\xi) \end{pmatrix}, \quad (4.12)$$

$$\left(\left[\alpha_2{}^{-}{}_{\sigma}^{\rho}(\xi), \alpha_2{}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[\beta_2{}^{-}{}_{\sigma}^{\rho}(\xi), \beta_2{}^{+}{}_{\sigma}^{\rho}(\xi)\right]\right) = \frac{1}{3}\begin{pmatrix} \alpha_2{}^{-}{}_{\sigma}^{\rho}(\xi) + \alpha_2{}^{+}{}_{\sigma}^{\rho}(\xi) - \\ \alpha_2{}^{-}{}_{\sigma}^{\rho}(\xi)\alpha_2{}^{+}{}_{\sigma}^{\rho}(\xi) + \\ \beta_2{}^{-}{}_{\sigma}^{\rho}(\xi) + \beta_2{}^{+}{}_{\sigma}^{\rho}(\xi) - \\ \beta_2{}^{-}{}_{\sigma}^{\rho}(\xi)\beta_2{}^{+}{}_{\sigma}^{\rho}(\xi) \end{pmatrix}, \quad (4.13)$$

Then the similarity measures between the GIVIFSESs $\pounds_{\eta_1}$ and Γ_{η_2} is defined as,

$$\widetilde{SM}\left(\pounds_{\eta_1}, \Gamma_{\eta_2}\right) = \frac{1}{n}\Sigma S_i\left(\pounds_{\eta_1}, \Gamma_{\eta_2}\right) \quad (4.14)$$

Theorem 4.3 If $\widetilde{SM}\left(\pounds_{\eta 1}, \Gamma_{\eta_2}\right)$ be the similarity measures between two GIVIFSESs $\pounds_{\eta_1}$ and Γ_{η_2} Then.

i. $\widetilde{SM}\left(\pounds_{\eta_1}, \Gamma_{\eta_2}\right) = \widetilde{SM}\left(\Gamma_{\eta_2}, \pounds_{\eta_1}\right)$,
ii. $0 \leq \widetilde{SM}\left(\pounds_{\eta_1}, \Gamma_{\eta_2}\right) \leq 1$,
iii. $\widetilde{SM}\left(\pounds_{\eta_1}, \Gamma_{\eta_2}\right) = 1 \Leftrightarrow \pounds_{\eta_1} = \Gamma_{\eta_2}$.

4.3 Type-III Similarity Measures for Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets

Definition 4.4 Let $\xi = \{\xi_x : x = 1,2,3,\dots\dots,m\}$ be the finite universal set of elements ξ_x, $\hat{A} = \{\sigma_{\hat{y}} : \hat{y} = 1,2,3,\dots\dots,n\}$ be the set of attributes and $\hat{G} = \{\rho_{\check{z}} : \check{z} = 1.2.3,\dots\dots\dots,t\}$ be the set of experts (or decision makers). Let $\mathcal{E}_{\eta_1}$ and Γ_{η_2} be two GIVIFSESs over ξ. Let $\widehat{SM}(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2})$specify the similarity measures between GIVIFSESs $\mathcal{E}_{\eta_1}$and Γ_{η_2}. We must first find similarities between their $\rho_{\check{z}}$th approximations to see similar measurements $\mathcal{E}_{\eta_1}$and Γ_{η_2}. Let $S_i(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2})$ indicate the similarity measure between the two $\rho_{\check{z}}$th approximation of

$$\mathcal{E}_{\eta_1}{}_{\sigma}^{\rho} = \left(\{< \xi, \left[{\alpha_1}^{-}{}_{\sigma}^{\rho}(\xi), {\alpha_1}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[{\beta_1}^{-}{}_{\sigma}^{\rho}(\xi), {\beta_1}^{+}{}_{\sigma}^{\rho}(\xi)\right] >\}, {\eta_1}_{\sigma}^{\rho}(\xi)\right),$$

$$\Gamma_{\eta_2}{}_{\sigma}^{\rho} = \left(\{< \xi, \left[{\alpha_2}^{-}{}_{\sigma}^{\rho}(\xi), {\alpha_2}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[{\beta_2}^{-}{}_{\sigma}^{\rho}(\xi), {\beta_2}^{+}{}_{\sigma}^{\rho}(\xi)\right] >\}, {\eta_2}_{\sigma}^{\rho}(\xi)\right),$$

Then $S_i(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2})$ is defined as follow:

$S_i(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2}) =$

$$\sum_{\check{z}=1}^{t}\sum_{\hat{y}=1}^{n}\sum_{x=1}^{m}\left\lfloor \frac{\left\lfloor \frac{\left|\left[{\alpha_2}^{-}{}_{(\sigma_{\hat{y}},\rho_{\check{z}+1})}(\xi_x), {\alpha_2}^{+}{}_{(\sigma_{\hat{y}},\rho_{\check{z}+1})}(\xi_x)\right], \left[{\beta_2}^{-}{}_{(\sigma_{\hat{y}},\rho_{\check{z}+1})}(\xi_x), {\beta_2}^{+}{}_{(\sigma_{\hat{y}},\rho_{\check{z}+1})}(\xi_x)\right]\right|_{-}}{{\eta_2}_{(\sigma_{\hat{y}},\rho_{\check{z}+1})}(\xi_x)}\right\rfloor}{\left|\frac{\left|\left[{\alpha_1}^{-}{}_{(\sigma_{\hat{y}},\rho_{\check{z}})}(\xi_x), {\alpha_1}^{+}{}_{(\sigma_{\hat{y}},\rho_{\check{z}})}(\xi_x)\right], \left[{\beta_1}^{-}{}_{(\sigma_{\hat{y}},\rho_{\check{z}})}(\xi_x), {\beta_1}^{+}{}_{(\sigma_{\hat{y}},\rho_{\check{z}})}(\xi_x)\right]\right|_{-}}{{\eta_1}_{(\sigma_{\hat{y}},\rho_{\check{z}})}(\xi_x)}\right|_{\vee}}\right\rfloor \quad (4.15)$$

Where i=1, 2, 3,……., l and

$$\left(\left[{\alpha_1}^{-}{}_{\sigma}^{\rho}(\xi), {\alpha_1}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[{\beta_1}^{-}{}_{\sigma}^{\rho}(\xi), {\beta_1}^{+}{}_{\sigma}^{\rho}(\xi)\right]\right) = \frac{1}{2}\begin{pmatrix} {\alpha_1}^{-}{}_{\sigma}^{\rho}(\xi) + {\alpha_1}^{+}{}_{\sigma}^{\rho}(\xi) \\ + \\ {\beta_1}^{-}{}_{\sigma}^{\rho}(\xi) + {\beta_1}^{+}{}_{\sigma}^{\rho}(\xi) \end{pmatrix}, \quad (4.16)$$

$$\left(\left[{\alpha_2}^{-}{}_{\sigma}^{\rho}(\xi), {\alpha_2}^{+}{}_{\sigma}^{\rho}(\xi)\right], \left[{\beta_2}^{-}{}_{\sigma}^{\rho}(\xi), {\beta_2}^{+}{}_{\sigma}^{\rho}(\xi)\right]\right) = \frac{1}{2}\begin{pmatrix} {\alpha_2}^{-}{}_{\sigma}^{\rho}(\xi) + {\alpha_2}^{+}{}_{\sigma}^{\rho}(\xi) \\ + \\ {\beta_2}^{-}{}_{\sigma}^{\rho}(\xi) + {\beta_2}^{+}{}_{\sigma}^{\rho}(\xi) \end{pmatrix}, \quad (4.17)$$

then the similarity measures between the GIVIFSESs $\mathcal{E}_{\eta_1} and\ \Gamma_{\eta_2}$ is described as,

$$\widehat{SM}(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2}) = \frac{1}{n}\Sigma S_i(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2}) \qquad (4.18)$$

Theorem 4.4 If $\widehat{SM}(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2})$ be the similarity measures between two GIVIFSESs $\mathcal{E}_{\eta_1}$and Γ_{η_2}then.

i. $\widehat{SM}(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2}) = \widehat{SM}(\Gamma_{\eta_2}, \mathcal{E}_{\eta_1})$
ii. $0 \le \widehat{SM}(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2}) \le 1$
iii. $\widehat{SM}(\mathcal{E}_{\eta_1}, \Gamma_{\eta_2}) = 1 \Leftrightarrow \mathcal{E}_{\eta_1} = \Gamma_{\eta_2}$

5. Application in Decision-Making Problems

In this section, we will apply the proposed similarity measure for the decision-making problem in the healthcare field. The step-by-step algorithm is defined below.

5.1 Algorithm for Decision-making based on the Proposed Model

Here, we introduce an algorithm depending on the similarity measures of two GIVIFSESs to evaluate the possibility that the sick person having specific symptoms is suffering from typhoid fever. For this purpose, we first established a model of GIVIFSESs by using the opinion of a specialist over the criteria of the symptoms for a sick person and another GIVIFSESs using a different opinion of the next specialist using the same criteria for sickness. Then we calculate the similarity measures of these two GIVIFSESs. We also suppose that if the similarity measures between these two GIVIFSESs is greater than or equal to 0.5 then we decide that the person is possibly affected by typhoid fever.

The algorithm of the similarity measures for GIVIFSESs is as follows:

Step 1: Construct a model GIVIFSES over the universe ξ based on the opinion of a specialist for a sick person.

Step 2**:** Construct GIVIFSESs over the universe ξ based on another specialist opinion for sickness.

Step 3**:** Calculate Type-I, Type-II, and Type-III similarity measures between the two GIVIFSESs.

Step 4**:** Estimate the result by using similarity measures, where if the similarity measures between these two GIVIFSESs is greater than or equal to 0.5 then the person is possibly affected by typhoid fever, and if the similarity measures between these two GIVIFSESs is less than 0.5 then the person is not possibly affected by typhoid fever.

5.2 Mathematical Applications of Proposed Model

Let $\xi = \{\xi_1, \xi_2\}$ be the universal set, where ξ holds only two elements which are ξ_1=typhoid fever and ξ_2= Not typhoid fever. Here the set of attributes σ is the set of certain visible symptoms of typhoid fever. Let $\hat{A} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, where σ_1 =body temperature, σ_2 =diarrhea, σ_3 =weakness, σ_4 =muscle aches and $\hat{G} = \{\rho_1, \rho_2\}$Be the set of specialists.

Step 1: Construct a model for GIVIFSES $\pounds_{\eta_1}$ over the universe ξ based on the opinion of a specialist ρ_1 for a sick person as shown in Table 3:

Table 3. Opinion of specialist $\boldsymbol{\rho_1}$.

$\pounds_{\eta_1}$	$(\boldsymbol{\sigma_1}, \boldsymbol{\rho_1}), \eta_{\sigma}^{\rho}(\boldsymbol{\xi})$	$(\boldsymbol{\sigma_2}, \boldsymbol{\rho_1}), \eta_{\sigma}^{\rho}(\boldsymbol{\xi})$	$(\boldsymbol{\sigma_3}, \boldsymbol{\rho_1}), \eta_{\sigma}^{\rho}(\boldsymbol{\xi})$	$(\boldsymbol{\sigma_4}, \boldsymbol{\rho_1}), \eta_{\sigma}^{\rho}(\boldsymbol{\xi})$
$\boldsymbol{\xi_1}$	$\left(\begin{matrix}[0.1,0.3],[0.2,0.4],\\ 0.1\end{matrix}\right)$	$\left(\begin{matrix}[0.1,0.2],[0.3,0.4],\\ 0.2\end{matrix}\right)$	$\left(\begin{matrix}[0.3,0.4],[0.4,0.5],\\ 0.4\end{matrix}\right)$	$\left(\begin{matrix}[0.4,0.6],[0.1,0.5],\\ 0.6\end{matrix}\right)$
$\boldsymbol{\xi_2}$	$\left(\begin{matrix}[0.3,0.4],[0.3,0.5],\\ 0.5\end{matrix}\right)$	$\left(\begin{matrix}[0.2,0.3],[0.3,0.4],\\ 0.4\end{matrix}\right)$	$\left(\begin{matrix}[0.7,0.8],[0.1,0.2],\\ 0.3\end{matrix}\right)$	$\left(\begin{matrix}[0.3,0.7],[0.2,0.4],\\ 0.7\end{matrix}\right)$

Step 2: Construct a model for GIVIFSES Γ_{η_2}over the universe ξ for typhoid fever based on another specialist ρ_2 opinion for a sickness as shown in Table 4:

Table 4. Opinion of specialist $\boldsymbol{\rho_2}$.

Γ_{η_2}	$(\boldsymbol{\sigma_1}, \boldsymbol{\rho_2}), \eta_{\sigma}^{\rho}(\boldsymbol{\xi})$	$(\boldsymbol{\sigma_2}, \boldsymbol{\rho_2}), \eta_{\sigma}^{\rho}(\boldsymbol{\xi})$	$(\boldsymbol{\sigma_3}, \boldsymbol{\rho_2}), \eta_{\sigma}^{\rho}(\boldsymbol{\xi})$	$(\boldsymbol{\sigma_4}, \boldsymbol{\rho_2}), \eta_{\sigma}^{\rho}(\boldsymbol{\xi})$
$\boldsymbol{\xi_1}$	$\left(\begin{matrix}[0.4,0.5],[0.1,0.2],\\ 0.4\end{matrix}\right)$	$\left(\begin{matrix}[0.3,0.4],[0.2,0.3],\\ 0.7\end{matrix}\right)$	$\left(\begin{matrix}[0.4,0.5],[0.3,0.4],\\ 0.0\end{matrix}\right)$	$\left(\begin{matrix}[0.4,0.7],[0.1,0.3],\\ 0.5\end{matrix}\right)$
$\boldsymbol{\xi_2}$	$\left(\begin{matrix}[0.6,0.7],[0.2,0.4],\\ 0.32\end{matrix}\right)$	$\left(\begin{matrix}[0.4,0.5],[0.1,0.4],\\ 0.9\end{matrix}\right)$	$\left(\begin{matrix}[0.0,0.3],[0.1,0.5],\\ 0.8\end{matrix}\right)$	$\left(\begin{matrix}[0.6,0.7],[0.1,0.4],\\ 0.1\end{matrix}\right)$

Step 3: Calculate Type-I, Type-II, and Type-III similarity measures between the two GIVIFSESs $\pounds_{\eta_1}$and Γ_{η_2} Respectively, we have.

Type-I similarity measure for Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets:
Now we use equations 4.5 and 4.6 respectively, we get.

$$\pounds_{\eta_1} = \begin{Bmatrix} ((\sigma_1,\rho_1),\eta_1(\xi)) & ((\sigma_2,\rho_1),\eta_1(\xi)) & ((\sigma_3,\rho_1),\eta_1(\xi)) & ((\sigma_4,\rho_1),\eta_1(\xi)) \\ \xi_1 = (0.29,0.1) & (0.28,0.2) & (0.42,0.4) & (0.43,0.6) \\ \xi_2 = (0.41,0.5) & (0.34,0.4) & (0.40,0.3) & (0.43,0.7) \end{Bmatrix},$$

$$\Gamma_{\eta_2} = \begin{Bmatrix} ((\sigma_1,\rho_2),\eta_1(\xi)) & ((\sigma_2,\rho_2),\eta_1(\xi)) & ((\sigma_3,\rho_2),\eta_1(\xi)) & ((\sigma_4,\rho_2),\eta_1(\xi)) \\ \xi_1 = (0.32,0.4) & (0.34,0.7) & (0.42,0.0) & (0.39,0.5) \\ \xi_2 = (0.46,0.32) & (0.38,0.9) & (0.28,0.8) & (0.44,0.1) \end{Bmatrix},$$

Then by using Eq. (4.1) when i=1

$$S_1(\pounds_{\eta_1},\Gamma_{\eta_2}) = max_{\xi_1}\left\{\begin{matrix} min \\ \rho_1,\rho_2(0.29,0.1,0.32,0.4) \\ \sigma_1 \end{matrix}\right\}, max_{\xi_2}\left\{\begin{matrix} min \\ \rho_1,\rho_2(0.41,0.5,0.46,0.32) \\ \sigma_1 \end{matrix}\right\}, max\{(0.1,0.32)\}, 0.32$$

Similarly, by using Eq. (4.1) when i=2

$$S_2(\pounds_{\eta_1},\Gamma_{\eta_2}) = max_{\xi_1}\left\{\begin{matrix} min \\ \rho_1,\rho_2(0.28,0.2,0.34,0.7) \\ \sigma_2 \end{matrix}\right\}, max_{\xi_2}\left\{\begin{matrix} min \\ \rho_1,\rho_2(0.34,0.4,0.38,0.9) \\ \sigma_2 \end{matrix}\right\}, max\{(0.2,0.34)\}, 0.34$$

Similarly, by using Eq. (4.1) when i=3

$$S_3(\pounds_{\eta_1},\Gamma_{\eta_2}) = max_{\xi_1}\left\{\begin{matrix} min \\ \rho_1,\rho_2(0.42,0.4,0.42,0.0) \\ \sigma_3 \end{matrix}\right\}, max_{\xi_2}\left\{\begin{matrix} min \\ \rho_1,\rho_2(0.40,0.3,0.28,0.8) \\ \sigma_3 \end{matrix}\right\}, max\{(0.0,0.28)\}, 0.28$$

Similarly, by using Eq. (4.1) when i=4

$$S_4(\pounds_{\eta_1},\Gamma_{\eta_2}) = max_{\xi_1}\left\{\begin{matrix} min \\ \rho_1,\rho_2(0.43,0.6,0.39,0.5) \\ \sigma_4 \end{matrix}\right\}, max_{\xi_2}\left\{\begin{matrix} min \\ \rho_1,\rho_2(0.43,0.7,0.44,0.1) \\ \sigma_4 \end{matrix}\right\}, max\{(0.39,0.1)\}, 0.39$$

Now by using Eq. (4.7) the similarity measure of Type-I between the GIVIFSESs $\pounds_{\eta_1}$ and Γ_{η_2} is,

$$SM(\pounds_{\eta_1},\Gamma_{\eta_2}) = \frac{1}{4}(0.32 + 0.34 + 0.28 + 0.39)$$

$$= 0.33$$

Type-II similarity measure for Generalized Interval Valued Intuitionistic Fuzzy Soft Expert sets:
Now by using Eq. (4.8) we have, when i = 1

$$S_1(\pounds_{\eta_1}, \Gamma_{\eta_2}) = min_{\xi_1}\left\{\begin{matrix}max\\ \rho_1,\rho_2(0.29,0.1,0.32,0.4)\\ \sigma_1\end{matrix}\right\},$$
$$min_{\xi_2}\left\{\begin{matrix}max\\ \rho_1,\rho_2(0.41,0.5,0.46,0.32)\\ \sigma_1\end{matrix}\right\},$$
$$min\{(0.4,0.5)\},$$
$$0.4$$

Similarly, by using Eq. (4.8) when i=2

$$S_2(\pounds_{\eta_1}, \Gamma_{\eta_2}) = min_{\xi_1}\left\{\begin{matrix}max\\ \rho_1,\rho_2(0.28,0.2,0.34,0.7)\\ \sigma_2\end{matrix}\right\},$$
$$min_{\xi_2}\left\{\begin{matrix}max\\ \rho_1,\rho_2(0.34,0.4,0.38,0.9)\\ \sigma_2\end{matrix}\right\},$$
$$min\{(0.7,0.9)\},$$
$$0.7$$

Similarly, by using Eq. (4.8) when i=3

$$S_3(\pounds_{\eta_1}, \Gamma_{\eta_2}) = min_{\xi_1}\left\{\begin{matrix}max\\ \rho_1,\rho_2(0.42,0.4,0.42,0.0)\\ \sigma_3\end{matrix}\right\},$$
$$min_{\xi_2}\left\{\begin{matrix}max\\ \rho_1,\rho_2(0.40,0.3,0.28,0.8)\\ \sigma_3\end{matrix}\right\},$$
$$min\{(0.42,0.8)\},$$
$$0.42$$

Similarly, by using Eq. (4.8) when i=4

$$S_4(\pounds_{\eta_1}, \Gamma_{\eta_2}) = min_{\xi_1}\left\{\begin{matrix}max\\ \rho_1,\rho_2(0.43,0.6,0.39,0.5)\\ \sigma_4\end{matrix}\right\},$$
$$min_{\xi_2}\left\{\begin{matrix}max\\ \rho_1,\rho_2(0.43,0.7,0.44,0.1)\\ \sigma_4\end{matrix}\right\},$$
$$min\{(0.6,0.7)\},$$
$$0.6$$

Now by using Eq. (4.14) the similarity measure of Type-II between the GIVIFSESs $\pounds_{\eta_1}$ and Γ_{η_2} is,

$$\widetilde{SM}\left(\pounds_{\eta_1}, \Gamma_{\eta_2}\right) = \frac{1}{4}(0.4 + 0.7 + 0.42 + 0.6)$$

$$= 0.53$$

Type-III similarity measure for Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets:
Now using Eq. (4.16) and Eq. (4.17)

$$\pounds_{\eta 1} = \begin{Bmatrix} ((\sigma_1,\rho_1) - \eta_1(\xi)) & ((\sigma_2,\rho_1) - \eta_1(\xi)) & ((\sigma_3,\rho_1) - \eta_1(\xi)) & ((\sigma_4,\rho_1) - \eta_1(\xi)) \\ \xi_1 = |0.5 - 0.1| = 0.4 & |0.5 - 0.2| = 0.3 & |0.8 - 0.4| = 0.4 & |0.8 - 0.6| = 0.2 \\ \xi_2 = |0.75 - 0.5| = 0.25 & |0.6 - 0.4| = 0.2 & |0.9 - 0.3| = 0.6 & |0.8 - 0.7| = 0.1 \end{Bmatrix},$$

$$\Gamma_{\eta_2} = \begin{Bmatrix} ((\sigma_1,\rho_2) - \eta_2(\xi)) & ((\sigma_2,\rho_2) - \eta_2(\xi)) & ((\sigma_3,\rho_2) - \eta_2(\xi)) & ((\sigma_4,\rho_2) - \eta_2(\xi)) \\ \xi_1 = |0.6 - 0.4| = 0.2 & |0.6 - 0.7| = 0.1 & |0.8,0.0| = 0.8 & |1.75,0.5| = 0.25 \\ \xi_2 = |0.95 - 0.32| = 0.63 & |0.7 - 0.9| = 0.2 & |0.45 - 0.8| = 0.35 & |0.9,0.1| = 0.8 \end{Bmatrix},$$

Now by using Eq. (4.15), we have, when i= 1

$$S_1(\pounds_{\eta_1}, \Gamma_{\eta_2}) = \frac{0.2 + 0.25}{0.4 + 0.63} = 0.43,$$

Now by using Eq. (4.15), we have, when i= 2

$$S_2(\pounds_{\eta_1}, \Gamma_{\eta_2}) = \frac{0.1 + 0.2}{0.3 + 0.2} = 0.6,$$

Now by using Eq. (4.15), we have, when i= 3

$$S_3(\pounds_{\eta_1}, \Gamma_{\eta_2}) = \frac{0.4 + 0.35}{0.8 + 0.6} = 0.53,$$

Now by using Eq. (4.15), we have, when i= 4

$$S_4(\pounds_{\eta_1}, \Gamma_{\eta_2}) = \frac{0.2 + 0.1}{0.25 + 0.8} = 0.28,$$

Therefore, by using Eq. (4.18) the similarity measure between GIVIFSESs $\pounds_{\eta_1}$ and Γ_{η_2} is specified by.

$$\widehat{SM}(\pounds_{\eta_1}, \Gamma_{\eta_2}) = \frac{1}{4}(0.43 + 0.6 + 0.53 + 0.28),$$

$$= 0.46$$

Step 4: Now we can easily see that from Type-I and Type-III there are less than 0.5 so we can say the person is not possibly affected by typhoid fever. But on another hand the result from Type-II similarity measures is $\widetilde{SM}(\pounds_{\eta_1}, \Gamma_{\eta_2}) = 0.53 > 0.5$ then we conclude that the person is possibly affected by typhoid fever.

6. Conclusion

In conclusion, this paper has introduced a novel framework of Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets (GIVIFSESs) and a suite of similarity measures—Type-I, Type-II, and Type-III—demonstrating their practical application in the medical diagnosis domain. The flexibility of GIVIFSESs in handling uncertainty and imprecision, along with the ability to quantify resemblance between medical cases, enhances decision-making accuracy in diagnostic processes. The empirical validation confirms the potential of these measures to support more informed and reliable medical decisions.

For future research, several promising directions can be explored. First, the extension of the proposed similarity measures to other domains, such as risk assessment, financial forecasting, and environmental monitoring, could further validate the adaptability of GIVIFSESs in handling complex, uncertain data. Second, integrating these similarity measures with machine learning algorithms may enhance automated decision-support systems, combining the strengths of data-driven models with expert knowledge. Additionally, more sophisticated types of similarity measures could be developed to capture higher-order relationships between variables. Lastly, conducting large-scale real-world experiments across different medical fields will further solidify the practical benefits of GIVIFSESs in improving diagnostic outcomes in the given fuzzy data [30-33].

Declarations

Ethics Approval and Consent to Participate

The results/data/figures in this manuscript have not been published elsewhere, nor are they under consideration by another publisher. All the material is owned by the authors, and/or no permissions are required.

Consent for Publication

This article does not contain any studies with human participants or animals performed by any of the authors.

Availability of Data and Materials

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Competing Interests

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Time-Leveled Hypersoft Matrix, Level Cuts, Operators, and COVID-19 Collective Patient Health State Ranking Model

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This article is the first step to formulate such higher dimensional mathematical structures in the extended fuzzy set theory that includes time as a fundamental source of variation. To deal with such higher dimensional information, some modern data processing structures had to be built. Classical matrices (connecting equations and variables through rows and columns) are a limited approach to organizing higher dimensional data, composed of scattered information in numerous forms and vague appearances that differ on time levels. To extend the approach of organizing and classifying the higher dimensional information in terms of specific time levels, this unique plithogenic crisp time-leveled hypersoft-matrix (PCTLHS matrix) model is introduced. This hypersoft matrix has multiple parallel layers that describe parallel universes/realities/information on some specific time levels as a combined view of events. Furthermore, a specific kind of view of the matrix is described as a top view. According to this view, *i*-level cuts, sublevel cuts, and sub-sublevel cuts are introduced. These level cuts sort the clusters of information initially, subject-wise then attribute-wise, and finally time-wise. These level cuts are such matrix layers that focus on one required piece of information while allowing the variation of others, which is like viewing higher dimensional images in lower dimensions as a single layer of the PCTLHS matrix. In addition, some local aggregation operators are designed to unify i-level cuts. These local operators serve the purpose of unifying the material bodies of the universe. This means that all elements of the universe are fused and represented as a single body of matter, reflecting multiple attributes on different time planes. This is how the concept of a unified global matter (something like dark matter) is visualized. Finally, to describe the model in detail, a numerical example is constructed to organize and classify the states of patients with COVID-19.

1. Introduction

The discipline of modeling and decision-making in an uncertain and ambiguous environment is an incredible endeavor for the human mind. To optimize the field of modeling and decision-making in an uncertain and ambiguous universe, Zadeh [1] developed the fuzzy set. The author extended the crisp state (either yes or no) of the human mind by introducing fuzziness into mathematical structures, i.e., partial yes and partial no represented by some membership and nonmembership in the area of decision-making. Later, in 1986, Atanassov [2] extended this state of vagueness further by introducing intuition, or hesitation, into decision-making structures, which were called intuitionistic fuzzy set theory. This means adding some layers of hesitation with the layers of partial yes and partial no. These three states of the human mind were represented by the degree of membership, the degree of nonmembership, and the degree of hesitation. In addition, Atanassov [3] introduced an interval-valued fuzzy set (IVFS) in 1999, which is

somewhat extended form of IFS (hesitating partial yes and partial no values packed in interval units). However, these extended set theories were unable to deal with uncertainty. To deal with uncertainty, Smarandache [4] introduced neutrosophy by generalizing hesitation as an independent indeterminate neutral factor and mentioning that the neutrosophic set is a generalization of the intuitionistic fuzzy set, the inconsistent intuitionistic fuzzy set, and the Pythagorean fuzzy set, and some other applications of the neutrosophic set were discussed by Smarandache and coauthors [5, 6]. Later, Molodtsov formulated a soft set in 1999 [7–10], in which the author further extended this extended fuzzy theory by considering multiple attributes parameterized by multiple subjects. Later, in 2018 [11], Smarandache further expanded this soft set to a hypersoft set and a plithogenic hypersoft set by splitting the attributes into various levels of attributes called subattributes. The author presented the hypersoft set as a set containing many subjects (matter bodies) parametrized by a combination of attributes/subattributes. In the case of the hypersoft set, the observer experiences an outside perception of the information that can be displayed in any extended fuzzy environment such as fuzzy, intuitionistic, and neutrosophic. The plithogenic hypersoft set is a set whose elements are structured by one or more attributes, and each attribute can have many values such that each attribute value has a corresponding degree of membership of an element x (subject) to the set P, with respect to each separately given criterion. It is an extended version of the hypersoft set and more widely applicable since the observer can intrinsically see the state of the element x (subject) by looking at each attribute separately. Smarandache introduced plithogeny [12–14] and expanded the vision of an ambiguous, uncertain universe. The author raised some open problems in this extended fuzzy set theory such as designing multicriterion decision-making techniques and construction of operators. Rana et al. [15] addressed these open problems and expanded the dimension of the plithogenic hypersoft set [11] by introducing and representing these PFWHSS in a matrix form called the plithogenic fuzzy whole hypersoft matrix. In addition, some local aggregation operators have been developed for the plithogenic fuzzy hypersoft set (PFHSS). This matrix was developed for a specific combination of attributes and subattributes, which was a limited structure designed for a single combination of attributes and subattributes, and a general model was required to be formulated. Therefore, later, Rana et al. [16] generalized the plithogenic whole hypersoft matrix to an extended form of the matrix called the plithogenic subjective hyper-supersoft matrix. It is a generalized form and a matrix superior to the previously developed matrix. It has a greater capacity to express the variations of connected attributive levels as well. These attributive levels are represented in the form of matrix layers. The application of this matrix is provided in the form of a new ranking model called the plithogenic subjective local-global universal ranking model. Some other researchers also addressed this extended fuzzy set theory. Saeed et al. [17] discussed the prognosis of allergy-based diseases by using Pythagorean fuzzy hypersoft and recommended medication. Muhammad et al. and Zulqarna et al. [18–21] described a correlation coefficient-based decision-making approach and discussed some of its properties for interval-valued neutrosophic hypersoft sets. The authors also outlined some basic operations and discussed their properties for interval-valued neutrosophic hypersoft sets. Furthermore, generalized aggregation operators have been established for neutrosophic hypersoft sets, and robust aggregation operators have been established for intuitionistic fuzzy hypersoft sets. Yolcu and Ozturk [22] described fuzzy hypersoft sets and discussed their application as a decision-making model. Siddique et al. [23] established a multicriterion decision-making approach for aggregation operators by using Pythagorean fuzzy hypersoft sets. Abdel-Basset et al. [24] designed an integrated plithogenic MCDM approach for the financial performance evaluation of manufacturing industries. Grida et al. [25] evaluated the impact of COVID-19 prevention policies on aspects of the supply chain under uncertainty. Akram et al. stated that after all these developments, there is still a vacuum in this extended fuzzy set theory. As we know, the source of the variation of the universe as information, data, and realities is time. It is a grim need to discuss and include this source of variation in the extended fuzzy set theory, and it is essential to design such mathematical higher dimensional structures that include time as a variation unit.

This recent paper is a first step toward designing such higher dimensional mathematical structures that include time as a fundamental source of variation. It further provides an upgraded and broadened plithogenic universal model by introducing a time-leveled variation in a hypersoft matrix, which deals with data and information as a magnified angle of vision. As we know, most of the variations in this universe are time-dependent like weather graphs, stock exchange, and website ratings, therefore, it is of tremendous help if we use a plithogenic hypersoft matrix to cope with scattered time-varying pieces of information of the plithogenic universe in crisp and other environments (i.e., fuzzy and neutrosophic environments). Initially, a three-dimensional broadened view of the PCHS-Matrix is portrayed to represent the plithogenic crisp time-leveled hypersoft set. This PCTLHS matrix is a rank-three tensor that shows three types of variations. It contains several matrix layers, whereas each layer is a tensor of rank two (i.e., a case of the ordinary matrix) expressed in the crisp environment. Furthermore, this PCTLHS matrix represents time-dependent multiple parallel universes/parallel realities/information. For example, if we are organizing the information of COVID-19 patients (subjects) admitted to a certain hospital, the record of their symptoms (attributes) at some time levels can be organized as a PCTLHS matrix. By using this connected matrix expression, we can see and classify all information immediately; i.e., the information from a group of patients is assigned to a combination of attributes and observed at different time levels. The three types of PCTLHS matrix variation indices would describe patients (subjects), their symptoms (attributes), and the time-leveled states of their symptoms as time-based subattributes, whereas one kind of level cuts (i-level cuts) related to the top view of the

PCTLHS matrix focuses on the subject (patient) separately and would display their attributes (symptoms) at several time levels as matrix layers.

In this way, these level cuts can focus on required information while displaying the variation of other information as a single matrix layer of the hypersoft matrix. Furthermore, these level cuts further cleave into sublevel cuts by splitting the matrix layer at one of the two remaining variation indices, whereas these sublevel cuts do offer the display of the previous lower dimension in the further lower dimension and enable us to sneak in an inside view of the expanded universe; i.e., after focusing on a subject explicitly through an i-level cut (single layer of the layered matrix), our next focus is on the attribute (a specific symptom) of that subject (patient) through the sublevel cut (row or a column of the one layer of the multiple layered matrix). One may call the expanded view of the matrix an implicit view of the universe. In the next stage of these sub-level cuts, sub-sublevel cuts are constructed by splitting the sublevel cuts (row or column of a specific layer of the PCTLHS matrix) at the third variation index of the matrix, which means after focusing on the subject (patient) and attribute (symptom) through the level cut and sublevel cut, our next focus is on the specific time level of information. After applying all splits to all indices, outcomes would be reflected as singletons. These level cuts, sublevel cuts, and sub-sublevel cuts are displaying their traits as zoom-in and zoom-out functions to provide the interior and exterior view of these time-bounded events, which can be argued as a contraction of the expanded higher dimensional universe. The sub-sublevel cuts provide a contract picture of the smallest part of a single or multiple universe. In this way, the expanded universe of matrix layers could be contracted to a single point. Similarly, by reversing the process, one can expand the same singleton into higher dimensions of rows, columns, matrices, matrix layers, and clusters of matrix layers. The matrix expression is the more appropriate expression to represent multidimensional data compared to the classic set expression. The question now arises as to why a collective state ranking would be preferable to an individual state ranking. The answer is precise and obvious since the collective state ranking sorts attributes of a group of people (subjects) as compared to an individual state ranking in which attributes of a person (subject) are categorized. By using the collective state ranking, one can categorize attributes on a broader spectrum such as attributes (health states) of a group of patients that can be distinguished, and hence, remedy for the most dominant attribute can be identified. Then, the cure for that attribute, which would be cough syrup, would be introduced to the market and manufactured on a large scale. Therefore, by considering a collective state ranking, any product associated with the dominant attribute may be imported or manufactured on a larger scale, which is not possible when considering an individual state ranking.

In the final stage, plithogenic local aggregation operators are developed and utilized to elaborate the activity of these several types of level cuts based on variation indices. These local operators serve the purpose of the unification of matter bodies in the universe, which means all their attributes are observed as they are reflected from a single entity. In this way, attributes are focused and subjects (matter bodies) are merged. This means that all elements of the universe are fused and represented as a single body of matter reflecting multiple attributes at different time levels. These i-level cuts unify the subjects using the aggregation operators, which help introduce the concept of a unified global matter that appears like dark matter that can have attributes but not individual bodies of matter. These operators are named plithogenic disjunction, plithogenic conjunction, and plithogenic averaging operators. For further precise applications of the model, a numerical example is derived for the organization and classification of COVID-19 data or information at two distinct time levels.

This article is organized into seven basic sections. After the introduction (Section 1), Section 2 summarizes some related preliminaries. Section 3 introduces some fundamental new concepts and definitions of the PTCHS-Matrix. Section 4 gives a mathematical description of the top-to-bottom view of the PLCHS-Matrix, and its splits major structures (connected matrix layers) into smaller structures such as i-level cuts, sublevel cuts, and sub-sublevel cuts. Section 5 describes the application of these split structures as the health state model for patients withCOVID-19 by using PCTLHS-Matrix. Section 6 introduces the set-based operations for the unification of i-level cuts, sublevel cuts, and sub-sublevel cuts of the PCTLHS matrix. Section 7 summarizes some conclusions and discussions on future research.

2. Preliminaries

This section summarizes some basic definitions of soft sets, hypersoft sets, crisp hypersoft sets, plithogenic hypersoft sets, and plithogenic crisp hypersoft Sets. These definitions would help expand the theory of plithogenesis.

Definition 1 (see [7]). (Soft set) Let U be the initial Universe of discourse and E be a set of parameters or attributes with respect to U, let $P(U)$ denote the power set of U, and $A \subseteq E$ is a set of attributes. Then, the pair (F, A) where F: $A \longrightarrow P(U)$ is called soft set over U. In other words, a soft set (F, A) over U is a parameterized family of subsets of U. For $e \in A$, $F(e)$ may be considered as a set of e elements or e approximate elements:

$$(F, A) = \{(F(e) \in P(U)): e \in E, F(e) = \varphi \quad \text{if } e \notin A\}. \quad (1)$$

Definition 2 (see [11]). (Hypersoft set)Let U be the initial universe of discourse and $P(U)$ be the power set of U.

Let $a_1, a_2, \ldots, a_n$ for $n \geq 1$ be n distinct attributes, whose corresponding attributes values are, respectively, the sets $A_1, A_2, \ldots, A_n$ with $A_i \cap A_j = \varphi$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$.

Then, the pair $(F, A_1 \times A \times \cdots \times A_n)$ where

$$F\colon A_1 \times A \times \cdots \times A_n \longrightarrow P(U), \tag{2}$$

is called a hypersoft set over$U$.

Definition 3 (see [11]). (Plithogenic crisp hypersoft set) Let U_c be the initial crisp universe of discourse and $P(U_c)$ be the power set of U. Let $a_1, a_2, \ldots, a_n$ for $n \geq 1$ be n distinct attributes, whose corresponding attributes values are, respectively, the sets $A_1, A_2, \ldots, A_n$ with $A_i \cap A_j = \varphi$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$. Then $\{F_c, A_1 \times A_2 \times \cdots \times A_n\}$ is called plithogenic crisp hypersoft set over U_c where $F_c\colon A_1 \times A \times \cdots \times A_n \longrightarrow P(U_c)$.

Definition 4 (see [25, 26]). (Supermatrices) Square or rectangular arrangements of numbers in rows and columns are matrices, and we call them simple matrices, while the supermatrix is the one whose elements are themselves matrices with elements that can be either scalars or other matrices.

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{3}$$

where

$$\begin{aligned} a_{11} &= \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}, \\ a_{12} &= \begin{bmatrix} 0 & 40 \\ 21 & -12 \end{bmatrix}, \\ a_{21} &= \begin{bmatrix} 3 & -1 \\ 5 & 7 \\ -2 & 9 \end{bmatrix}, \\ a_{22} &= \begin{bmatrix} 4 & 12 \\ -17 & 6 \\ 3 & 7 \end{bmatrix}, \end{aligned} \tag{4}$$

where a is a supermatrix.

Note: the elements of supermatrices are called submatrices; i.e., $a_{11}, a_{12}, a_{21}, a_{22}$ are submatrices of the supermatrix a. In this example, the order of the supermatrix a is 2×2, and the order of submatrices a_{11} is 2×2,a_{12} is 2×2, anda_{21} is 3×2, and the order of the submatrix a_{22} is 3×2, and we can see that the order of the supermatrix does not tell us about the order of its submatrices.

Definition 5 (see [27]). (Hypermatrices) For $n_1, \ldots, n_d \in N$, a function $f\colon (n_1) \times \cdots \times (n_d) \longrightarrow F$ is a hypermatrix, also called an order-d hypermatrix or d-hypermatrix. We often just write $a_{k_1 \cdots k_d}$ to denote the value $f(k_1 \cdots k_d)$ of f at $(k_1 \cdots k_d)$ and think of f (renamed as A) as specified by a d-dimensional table of values, writing $A = [a_{k_1 \cdots k_d}]_{k_1 \cdots k_d}^{n_1, \ldots, n_d}$.

A 3-hypermatrix may be conveniently written down on a (2-dimensional) piece of paper as a list of usual matrices, called slices. For example, we obtain

$$A = \left[a_{ijk}\right] = \begin{bmatrix} a_{111} & a_{121} & a_{131} & \cdot & a_{112} & a_{122} & a_{132} \\ a_{211} & a_{221} & a_{231} & \cdot & a_{212} & a_{222} & a_{232} \\ a_{311} & a_{321} & a_{331} & \cdot & a_{312} & a_{322} & a_{332} \end{bmatrix}. \tag{5}$$

3. Plithogenic Crisp Time-Leveled Hypersoft Matrix

3.1. (PCTLHS Matrix). Let $U_C(X)$ be the crisp universe of discourse and $P(U_C)$ be the power set ofU_C. A_j^k is a combination of attributes and subattributes (time-leveled attributes) for some $j = 1, 2, 3, \ldots, N$ represents N number of attributes, $k = 1, 2, 3, \ldots, L$ represents L number of time levels and $x_i\, i = 1, 2, 3, \ldots,$ and M represents the M number of subjects under consideration. Then, plithogenic crisp time-leveled hypersoft matrix (PCTLHS matrix) is a mapping from the cross product of attributes/time-leveled attributes on the power set of the universe $P(U_C)$, represented in the matrix form. This mapping with its matrix form is described in equations as follows:

$$F\colon A_1^k \times A_2^k \times A_3^k \times \cdots \times A_N^k \longrightarrow P(U_C), \tag{6}$$

$$F = \left[\mu_{A_j^k}(x_i)\right], \tag{7}$$

where $\mu_{A_j^k}(x_i) \in \{0.1\}$ are crisp memberships for a given x_i subject regarding each given A_j^k attribute/time-leveled attributes.

In simple words, a plithogenic crisp hypersoft set, represented in the matrix form, is called plithogenic crisp time-lined hypersoft matrix (PCTLHS matrix).

This matrix has three possible expansions associated with its three-dimensional views, which are described in crisp environments.

3.2. Three-Dimensional Views and Level Cuts of the PCTLHS Matrix. As we know, all ordinary matrices in the real vector space are rank 2 tensors. Similarly, as an extended matrix version, the PCTLHS matrix with its three variation indices is a rank 3 tensor. The PCTLHS matrix contains layers of ordinary matrices called matrix layers or level cuts.

For a detailed description, we consider the example of the PCTLHS matrix $A = [A_{ijk}]$. The index i refers to variations of rows used to represent the subjects under consideration, j specifies a variation of columns used to represent attributes of subjects, and k provides variations of layers of rows and columns that would be used to represent the attributes on specific time levels (varying matrix layers as clusters of rows and columns). Similarly, $[A_{jki}]$ is interpreted as the index, and j offers a variation of rows, k gives a variation of columns, and i offers variation of clusters of rows and columns.

The PCTLHS matrix contains layers of ordinary matrices, termed matrix layers or level cuts of the PCTLHS matrix

The level cuts, sublevel cuts, and sub-sublevel cuts would be defined by specifying variation indices i, j, k for their positive integer values.

3.2.1. Level Cuts. Level cuts are submatrices (first-level splits) of the PCTLHS matrix that can be further described as parallel matrix layers. The PCTLHS matrix is generated by uniting these matrix layers. These level cuts of the PCTLHS matrix are obtained by assigning a specific integer value to the first variation index at a time. According to three types of view of the PCTLHS matrix, level cuts are distributed in three categories, i.e.,

i-level cuts

j-level cuts

k-level cuts

Note: in this article, only the first type of level cuts are formulated and explored, whereas the other two types would be discussed in the upcoming version.

3.2.2. Sublevel Cuts. Sublevel cuts are perceived as level Cuts of level cuts (second splits applied over first splits) of the PCTLHS matrix. These sublevel cuts are columns or rows of the submatrix (the matrix obtained after the first split). The sublevel cuts are obtained by assigning a specific integer value to one of the two variation indices of a parallel layer (submatrix) of the PCHS matrix.

3.2.3. Sub-Sublevel Cuts. Sub-sublevel cuts are obtained by assigning a specific integer value to the variation index of sublevel cuts (the third-level split over the second split). The sub-sublevel cut is one specific element (point) of the sublevel cut (column or row). These level cuts, sublevel cuts, and sub-sublevel cuts are images of the higher dimensional universe in lower dimensions and can be used as tools for getting images and transformations. The detailed classification of these level cuts, sublevel cuts, and sub-sublevel cuts is described below.

The utilization of these cuts is that one can contract the expanded dimension of the PCTLHS matrix to a matrix, then to a row or column matrix, and then further to a single point. Similarly, the reverse procedure would provide an expansion of the universe.

3.2.4. Variation Indices of the PCTLHS Matrix. Three types of variation indices are used to represent a general element $(\mu_{A_j^k}(x_i))$ of the PCHS matrix. The first variation index i (associated with subjects) represents M rows of an $M \times N$ submatrix which is a single layer of the $M \times N \times L$ PCHS matrix. The second variation j (used to specify attributes) represents N columns of this submatrix. A third variation index k (represent attributive levels) represents L layers or L level cuts of the $M \times N \times L$ PCHS matrix. The index-based views, level cuts, sublevel cuts, and Sub-sublevel cuts are categorized into the following types.

4. *i*-Level Cuts, Sublevel Cuts, and Sub-Sublevel Cuts of the PCTLHS Matrix

The top-to-bottom view of the PCTLHS matrix consists of top to the bottom layer of the matrix. These layers are formulated by specifying the variation index i. Equation (8a) describes the top-to-bottom view of the PCTLHS matrix:

$$A = \begin{bmatrix} \mu_{A_1^1}(x_i) & \mu_{A_1^2}(x_i) & \cdots & \mu_{A_1^L}(x_i) \\ \mu_{A_2^1}(x_i) & \mu_{A_2^2}(x_i) & \cdots & \mu_{A_2^L}(x_i) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_{A_N^1}(x_i) & \mu_{A_N^2}(x_i) & \cdots & \mu_{A_N^L}(x_i) \end{bmatrix}. \tag{8a}$$

The top-down view of the PCTLHS matrix in an expanded form is described as follows:

$$A = \begin{bmatrix} \begin{bmatrix} \mu_{A_1^1}(x_1) & \mu_{A_1^2}(x_1) & \cdots & \mu_{A_1^L}(x_1) \\ \mu_{A_2^1}(x_1) & \mu_{A_2^2}(x_1) & \cdots & \mu_{A_2^L}(x_1) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_{A_N^1}(x_1) & \mu_{A_N^2}(x_1) & \cdots & \mu_{A_N^L}(x_1) \end{bmatrix} \\ \begin{bmatrix} \mu_{A_1^1}(x_2) & \mu_{A_1^2}(x_2) & \cdots & \mu_{A_1^L}(x_2) \\ \mu_{A_2^1}(x_2) & \mu_{A_2^2}(x_2) & \cdots & \mu_{A_2^L}(x_2) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_{A_N^1}(x_2) & \mu_{A_N^2}(x_2) & \cdots & \mu_{A_N^L}(x_2) \end{bmatrix} \\ \cdot \\ \cdot \\ \begin{bmatrix} \mu_{A_1^1}(x_M) & \mu_{A_1^2}(x_M) & \cdots & \mu_{A_1^L}(x_M) \\ \mu_{A_2^1}(x_M) & \mu_{A_2^2}(x_M) & \cdots & \mu_{A_2^L}(x_M) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_{A_N^1}(x_M) & \mu_{A_N^2}(x_M) & \cdots & \mu_{A_N^L}(x_M) \end{bmatrix} \end{bmatrix}. \tag{8b}$$

4.1. i-Level Cuts $A^{[i]}$. i-level cuts expressed as $A^{[i]}$ are the top-to-bottom split layers of the PCTLHS matrix. Figure 1 depicts i-level cuts of the PCHTLS matrix. The observer can sort the clusters of information subject-wise, using this top-down view of the PCTLHS matrix. For each specific value of i, an $N \times L$ submatrix is obtained termed as i-level cut. These i-level cuts are M numbers of top-to-bottom matrix layers. Each layer is an $N \times L$ submatrix of the $M \times N \times L$ PCTLH matrix. The top-to-bottom layers of equation (9) represent i-level cuts.

Furthermore, these i-level cuts focus on each subject (patient) separately and display variations of their attributes (symptoms) at several time levels.

$A = [\mu_{A_j^k}(x_i)]$ is a PCTLHS matrix.

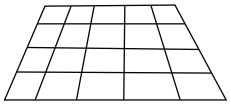


Figure 1: Level cut.

For some given $i = 1, 2, \ldots, M$, $j = 1, 2, \ldots, N$ and $k = 1, 2, 3, \ldots, L$.

The i-level cuts of the PCTLHS matrix, constructed by splitting the index i, are called i-leveled splits or i-level cuts.

i-level cuts $A^{[i]}$ of the PCTLHS matrix (equations (8a) and (8b)) for fixed $i = 1, 2, \ldots, M$ are given as follows:

$$A^{[1]} = \begin{bmatrix} \mu_{A_1^1}(x_1) & \mu_{A_2^1}(x_1) & \cdots & \mu_{A_N^1}(x_1) \\ \mu_{A_1^2}(x_1) & \mu_{A_2^2}(x_1) & \cdots & \mu_{A_N^2}(x_1) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_{A_1^L}(x_1) & \mu_{A_2^L}(x_1) & \cdots & \mu_{A_N^L}(x_1) \end{bmatrix}, \tag{9}$$

$$A^{[2]} = \begin{bmatrix} \mu_{A_1^1}(x_2) & \mu_{A_2^1}(x_2) & \cdots & \mu_{A_N^1}(x_2) \\ \mu_{A_1^2}(x_2) & \mu_{A_2^2}(x_2) & \cdots & \mu_{A_N^2}(x_2) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_{A_1^L}(x_2) & \mu_{A_2^L}(x_2) & \cdots & \mu_{A_N^L}(x_2) \end{bmatrix}, \tag{10}$$

$$A^{[M]} = \begin{bmatrix} \mu_{A_1^1}(x_M) & \mu_{A_2^1}(x_M) & \cdots & \mu_{A_N^1}(x_M) \\ \mu_{A_1^2}(x_M) & \mu_{A_2^2}(x_M) & \cdots & \mu_{A_N^2}(x_M) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mu_{A_1^L}(x_M) & \mu_{A_2^L}(x_M) & \cdots & \mu_{A_N^L}(x_M) \end{bmatrix}. \tag{11}$$

These i-level slices are submatrices of the PCTLHS matrix, which can be further described as parallel layers of the matrix. The PCTLHS matrix is created by uniting these matrix layers. It is observed that i-level cuts can focus and classify subjects (matter bodies) in scattered clusters of information. For example, the first i-level cut $A^{[1]}$ focuses on the first subject x_1 and displays the N-number of attributes at L-number of time levels. Similarly, the second i-level cut $A^{[2]}$ focuses on the second subject x_2 and $A^{[M]}$ focuses on Mth subject x_M.

4.2. i_j-Sublevel Cuts. i_j-sublevel cuts are constructed by first specifying $i = m$ where m is any positive integer from 1 to M, then further specifying $j = n$ such that n is a positive integer from 1 to N, and then varying $k = 1, 2, \ldots, L$ as given in equation (12). The observer can sort the clusters of information first subject-wise and then attribute-wise, respectively, by using this top-down view of the PCTLHS matrix These i_j-sublevel cuts are the column-wise split of each top-to-bottom layer of the matrix:

$$A^{[m_n]} = \begin{bmatrix} \mu_{A_n^1}(x_m) \\ \mu_{A_n^2}(x_m) \\ \cdot \\ \cdot \\ \cdot \\ \mu_{A_n^L}(x_m) \end{bmatrix}. \tag{12}$$

It is observed that after classifying information subject-wise through i-level cuts, one would further distinguish information attribute-wise by using i_j-sublevel cuts. For example, for a fixed $i = 1$ and $j = 1$, a sublevel cut is a column matrix, displaying the first subject's information for its first attribute at k time levels where $k = 1, 2, \ldots, L$.

4.3. i_k-Sublevel Cuts. i_k-sublevel cuts of i-level cut $A^{[i]}$ are obtained by first specifying the index $i = m$, then further specifying the index k, and varying only j. These sub-level cuts are rows of an $N \times L$ submatrix as shown in equation:

$$A^{[m_l]} = \left[\mu_{A_1^l}(x_m) \;\; \mu_{A_2^l}(x_m) \;\; \cdots \;\; \mu_{A_N^l}(x_m) \right]. \tag{13}$$

4.4. i_{k_j}-Sub-Sublevel Cuts. i_{k_j}-sub-sublevel cuts $A^{[i_{k_j}]}$ are constructed by first specifying i and then further specifying both j and k. The mathematical expression of these sub-sublevel cuts is given as follows:

$$A^{\left[i_{k_j}\right]} = A^{\left[m_{l_n}\right]} = \left[\mu_{A_n^l}(x_m)\right]. \tag{14}$$

Note: it is obvious that $A^{[i_{j_k}]} = [\mu_{A_n^l}(x_m)]$ represents a single element of the expanded matrix and serves as a zoom-in view of the matrix. For example, $A^{[1_{1_1}]}$ is an i_{k_j}-sub-sublevel cut that focuses on the information of the first subject for its first attribute at the first time level.

i_{k_j} sub-sublevel cut: for a given i-level cut by specifying j, one can obtain the i_{k_j}-sub-sublevel cut.

It is obvious that by following the abovementioned split procedure, one would zoom into the given PCTLHS matrix. The level cutbehaves like a zoom into the first layer (matrix) of the hypermatrix, then the sub level cut reaches the the column or row of matrix, and the sub sub level cut would be an element of the column or row. Similarly, the reverse process can serve as a zoom-out function. In this way, one can approach the smallest unit of the extended universe, which is an explicit view of the event as an element of the PCHS matrix. Similarly, the matrix itself serves as an implicit view of the event.

5. Application

5.1. Example 5.1. Let $U_{PC} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ is a group of six patients who visited the hospital with symptoms of COVID-19. They were examined by a doctor, who examined three of them under four basic symptoms. These

symptoms were observed in their two different visits to the doctor. These two visits at two distinct times are considered two-time levels. In both these times, the symptoms (time-lined subattributes) of these patients were observed and recorded during the first and second meetings with the doctor. The three patients under observation are considered subjects. The information/data of their health condition associated with both visits are organized in a PCTLHS matrix. The clinical observations of the visits are expressed in the crisp environment and analyzed by using the plithogenic crisp hypersoft matrix.

Consider the set of the first three patients who are under observation.

Let the four attributes be $A_j^k; j = 1, 2, 3, 4$ observed at two-time levels $k = 1, 2$ which are described as follows.

Fever with numeric values, $k = 1, 2$ representing first and second-time levels

A_1^1 = State of fever at the first visit
A_1^2 = State of fever at the second visit
A_2^k = Dry cough, with numeric values, $k = 1, 2$
A_2^1 = Condition of cough at the first visit
A_2^2 = Condition of cough at the second visit
A_3^k = Breathing difficulty with numeric values, $k = 1, 2$
A_3^1 = Breathing difficulty level at the first visit
A_3^2 = Breathing difficulty level at the first visit
A_4^k = Sickness record, with numeric values $k = 1, 2$
A_4^1 = Sickness state at the first visit
A_4^2 = Sickness state at the second visit

Now, in the next two subsections, this information consisted of symptoms (attributes) of patients (subjects) observed at two levels of time. Two forms of expressions are formulated to organize information. One way of representation is the set expression, i.e., the PCTLHS set, and the other way of representation is the matrix expression, which is a connected matrix or a hypermatrix termed the PCTLHS matrix.

5.1.1. PCTLHS Set Representation. Let the function A reflects given attributes/time-leveled attributes described as follows:

$$\begin{aligned} \mathbf{A}: \quad & A_1^k \times A_2^k \times A_3^k \times A_4^k \longrightarrow P(U_C), \\ \text{S.t.} \quad & \mathbf{A}\left(A_1^1, A_2^1, A_3^1, A_4^1\right) = \{x_1, x_2, x_3\}, \\ & \mathbf{A}\left(A_1^2, A_2^2, A_3^2, A_4^2\right) = \{x_1, x_2, x_3\}. \end{aligned} \tag{15}$$

$A_1^1, A_2^1, A_3^1, A_4^1$ is a combination of attributes at the first visit level (α-combination). $A_1^2, A_2^2, A_3^2, A_4^2$ is a combination of attributes at the second visit level (β-combination).

Individual crisp memberships are assigned to $A = \{x_1, x_2, x_3\}$ according to the opinion of the doctor, and then, information is represented as the PCTLHS set by using crisp memberships; that is, if a given symptom is present in the patient, the assigned membership is one, and if it is not present, the membership is zero. The formal notation of the PCTLHS set A is described as follows:

$$\mathbf{A} = \left\{x_1\left(\mu_{A_j^1}(x_1)\right), x_2\left(\mu_{A_j^1}(x_2)\right), x_3\left(\mu_{A_j^1}(x_3)\right)\right\}, \tag{16}$$

where $\mu_{A_j^k}(x_i)$ represents crisp memberships assigned to three subjects (patients), $i = 1, 2, 3$ according to four different attributes (symptoms) $j = 1, 2, 3, 4$ that are observed at two distinct time levels $k = 1, 2$, (these plithogenic crisp memberships reflect whether the A_j^k attribute is present $(\mu_{A_j^k}(x_i) = 1)$ in the x_i subject or not present $(\mu_{A_j^k}(x_i) = 0)$.

The information of the first visit of patients x_i associated with four symptoms as α-combination of attributes is organized as a PCTLHS set:

$$A(\alpha) = A\left(A_1^1, A_2^1, A_3^1, A_4^1\right) = \left\{\begin{array}{l} x_1(1, 0, 1, 1), \\ x_2(1, 1, 1, 1), \\ x_3(1, 0, 0, 1). \end{array}\right\} \tag{17}$$

It is clear that the first-level layer of the PCTLHS matrix is generated by this $A(\alpha)$.

Now, regarding the second observation of patients as β -combination of attributes, information is portrayed as a PCTLHS set:

$$A(\beta) = A\left(A_1^1, A_2^1, A_3^1, A_4^1\right) = \left\{\begin{array}{l} x_1(0, 0, 0, 0), \\ x_2(0, 1, 0, 0), \\ x_3(1, 0, 1, 0). \end{array}\right\} \tag{18}$$

$A(\beta)$ generates the second-level layer of the matrix.

5.1.2. PCTLHS Matrix Representation. Let A is the matrix of representation for both PCTLHS Sets. Here, rows of this matrix represent x_1, x_2, x_3 (physical bodies or subjects) and columns represent (the nonphysical aspect of subjects as symptoms) attributes $A_1^k, A_2^k, A_3^k, A_4^k$.

This whole information of two sets $A(\alpha)$ and $A(\beta)$ is organized as two connected layers of PCTLHS matrix A described as follows:

$$F = \left[\begin{array}{c} \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{array}\right] \\ \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \end{array}\right]. \tag{19}$$

This PCTLHS matrix consists of two layers. The first layer is interpreted as the first observation, representing the state of health of three patients. By observing the matrix given in equation (19), one can clearly see that, on the first visit, the patient x_1 is suffering from fever with no dry cough but feeling suffocation and nausea. The patient x_2 has a fever with dry cough, suffocation, and nausea. The patient x_3 is suffering from fever with dry cough, no breathing difficulty, and feeling nausea. However, we can observe from matrix equation (19) that, on the second visit, all symptoms are cleared in the x_1 patient, while the x_2 patient is having breathing difficulty, and hence, the x_2 patient is suffering from fever and breathing

difficulty both. In this way, one can see and classify all information at a glance. Therefore, it is obvious that the matrix expression is the most appropriate expression to represent multidimensional data compared to the classical set expression.

$$A = \left[\mu_{A_j^k}(x_i)\right], \quad i = 1, 2, 3, j = 1, 2, 3, 4, \text{and } k = 1, 2. \tag{20}$$

The given PCTLHS matrix **A** is a rank-three tensor. The order of **A** is $(i \times j \times k) = 3 \times 4 \times 2$.

The top-to-bottom view of this PCTLHS Matrix consists of three parallel layers of ordinary matrices, and the order of each matrix is 2×4. These top-to-bottom layers in the separated form are i-level cuts.

5.2. i-Level Cuts $A^{[i]}$ of the PCTLHS-Matrix (Subjectwise Level Cuts)

$$A = \left[\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}\right]. \tag{21}$$

Equation (21) represents a top-to-bottom view of the PCTLHS matrix, which is used to formulate i-level cuts $A^{[i]}$ (subjectwise level cuts) of the PCTLHS matrix.

We split the matrix at $i = 1, 2, 3$. These three integer levels (i splits) are three i-level cuts known as subjectwise level splits. These level cuts focus on subjects initially as follows:

$$A^{[1]} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

$A^{[1]}$ reflects the attributive state of the first patient x_1 observed in the two given time levels.

$$A^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{23}$$

$A^{[2]}$ reflects the attributive state of the second patientx_2 observed in the two given time levels.

$$A^{[3]} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}. \tag{24}$$

$A^{[3]}$ reflects the attributive state of the third patient x_3 observed in the two given time levels.

This matrix form represents three top-to-bottom layers, and each layer is a matrix of order 2×4. These layers are obtained by splitting the given matrix on the index $i = 1, i = 2, i = 3$. Each i-level cut represents states of four attributes (symptoms) associated with a specific subject (patient) for the first and second examination levels. These three i-level cuts focus on the patient first and then describe their health state at time levels. These levels are utilized to organize the information for each patient separately. For example, the first layer (i-level cut) of the above matrix reflects the state of the patient x_1. From the first layer, it can be seen that they have all three symptoms except for cough on the first visit. During the second visit, all symptoms disappeared. Similarly, the second patient x_2 has all symptoms in the initial stage, and only one (cough) is left in the next stage. A description for the third patient can be given similarly by observing the third-level layer.

5.2.1. i_j-Sublevel Cuts. For specified $i = 1$ and then further specify each $j = 1, 2, 3, 4$, respectively, i_j-sublevel cuts $A^{[i_j]}$ of the i-level cut $A^{[1]}$ are constructed as follows:

$$\begin{aligned} A^{[1_1]} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ A^{[1_2]} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ A^{[1_3]} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ A^{[1_4]} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{25}$$

$A^{[1_1]}$represents the first symptom of the first patient in two given time levels, which means after focusing on the patient, the next focus is on the attribute. Similarly, one can describe other two level cuts $A^{[1_2]}$, $A^{[1_3]}$ and i_j-sublevel cuts.

For fixed $i = 2$ and then further specify each $j = 1, 2, 3, 4$, i_j-sublevel cuts of i-level cut $A^{[2]}$ are constructed as follows:

$$\begin{aligned} A^{[2_1]} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ A^{[2_2]} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ A^{[2_3]} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ A^{[2_4]} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{26}$$

For fixed $i = 3$ and $j = 1, 2, 3, 4$, i_j-sublevel cuts of i-level cut $A^{[3]}$ are as follows:

$$\begin{aligned} A^{[3_1]} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ A^{[3_2]} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ A^{[3_3]} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ A^{[3_4]} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{27}$$

5.2.2. i_{j_k}-Sub-Sublevel Cut. i_{j_k}-sub-sublevel cuts are constructed as described. By using a level cut, one would observe the patient first $i = m$, $j = n$ and $k = l$, then at the

next level by using a sublevel cut, one would observe his symptom (attribute) at several given time levels. And by using sub-sub-level cut, one would observe its symptom at a certain time level. The sub-sublevel cut is the smallest unit of the matrix, which is the single element. The general form is described as follows:

$$A^{\left[i_{j_k}\right]} = A^{\left[m_{n_l}\right]} = \left[\mu_{A_n^l}(x_m)\right]. \tag{28}$$

i_{j_k} *sub*-sublevel cuts of A are as follows:

$$\begin{aligned}
&A^{\left[1_{1_1}\right]} = [1],\\
&A^{\left[1_{1_2}\right]} = [0],\\
&A^{\left[1_{2_1}\right]} = [0],\\
&A^{\left[1_{2_2}\right]} = [0],\\
&A^{\left[1_{3_1}\right]} = [1],\\
&A^{\left[1_{3_2}\right]} = [0],\\
&A^{\left[1_{4_1}\right]} = [1],\\
&A^{\left[1_{4_2}\right]} = [0],\\
&A^{\left[2_{1_1}\right]} = [1],\\
&A^{\left[2_{1_2}\right]} = [0],\\
&A^{\left[2_{2_1}\right]} = [1],\\
&A^{\left[2_{2_2}\right]} = [1],\\
&A^{\left[2_{3_1}\right]} = [1],\\
&A^{\left[2_{3_2}\right]} = [0]\\
&A^{\left[2_{4_1}\right]} = [1],\\
&A^{\left[2_{4_2}\right]} = [0],\\
&A^{\left[3_{1_1}\right]} = [1],\\
&A^{\left[3_{1_2}\right]} = [1],\\
&A^{\left[3_{2_1}\right]} = [1],\\
&A^{\left[3_{2_2}\right]} = [0],\\
&A^{\left[3_{3_1}\right]} = [0],\\
&A^{\left[3_{3_2}\right]} = [1],\\
&A^{\left[3_{4_1}\right]} = [1],\\
&A^{\left[3_{4_2}\right]} = [0].
\end{aligned} \tag{29}$$

5.2.3. i_k-Sublevel Cuts. For given i-level cuts, further specifying k and varying only j, one can obtain i_k-sublevel cuts. These level cuts are rows of an $N \times L$ submatrix.

Specifying $i = mk = l$ and varying $j = 1, 2, \ldots, Ni_k$-sublevel cuts for a fixed i-level cut are constructed as follows:

$$A^{[i_k]} = A^{[m_l]} = \left[\mu_{A_1^l}(x_m)\ \ \mu_{A_2^l}(x_m)\ \cdot\ \cdot\ \cdot\ \ \mu_{A_N^l}(x_m)\right]. \tag{30}$$

i_k-sublevel cuts of i-level cut $A^{[i]}$ are as follows:

$$\begin{aligned}
A^{\left[1_1\right]} &= [\,1\ \ 0\ \ 1\ \ 1\,],\\
A^{\left[1_2\right]} &= [\,0\ \ 0\ \ 0\ \ 0\,].
\end{aligned} \tag{31}$$

For fixed $i = 2$ and $k = 1, 2, i_k$-sublevel cuts of i-level cut $A^{[2]}$ are as follows:

$$\begin{aligned}
A^{\left[2_1\right]} &= [\,1\ \ 1\ \ 1\ \ 1\,],\\
A^{\left[2_2\right]} &= [\,0\ \ 1\ \ 0\ \ 0\,].
\end{aligned} \tag{32}$$

For fixed $i = 3$ and $k = 1, 2, i_k$-sublevel cuts of i-level cut $A^{[3]}$ are as follows:

$$\begin{aligned}
A^{\left[3_1\right]} &= [\,1\ \ 1\ \ 0\ \ 1\,],\\
A^{\left[3_2\right]} &= [\,0\ \ 1\ \ 1\ \ 0\,].
\end{aligned} \tag{33}$$

5.2.4. i_{k_j}-Sub-Sublevel Cut. i_{k_j} sub-sublevel cuts are constructed by specifying $i = m$. Then, by further specifying $j = n$ and $k = l$, the general form is described as follows:

$$A^{\left[i_{k_j}\right]} = A^{\left[m_{l_n}\right]} = \left[\mu_{A_n^l}(x_m)\right]. \tag{34}$$

i_{j_k} sub-sublevel cuts of A are as follows:

$$\begin{aligned}
&A^{\left[1_{1_1}\right]} = [1],\ A^{\left[1_{1_2}\right]} = [0],\ A^{\left[1_{1_3}\right]} = [1],\ A^{\left[1_{1_4}\right]} = [1],\\
&A^{\left[1_{2_1}\right]} = [0], A^{\left[1_{2_2}\right]} = [0], A^{\left[1_{2_3}\right]} = [0], A^{\left[1_{2_4}\right]} = [0],\\
&A^{\left[2_{1_1}\right]} = [1], A^{\left[2_{1_2}\right]} = [1], A^{\left[2_{1_3}\right]} = [1], A^{\left[2_{1_4}\right]} = [1],\\
&A^{\left[2_{2_1}\right]} = [0], A^{\left[2_{2_2}\right]} = [1], A^{\left[2_{2_3}\right]} = [0], A^{\left[2_{2_4}\right]} = [0],\\
&A^{\left[3_{1_1}\right]} = [1], A^{\left[3_{1_2}\right]} = [1], A^{\left[3_{1_3}\right]} = [0], A^{\left[3_{1_4}\right]} = [1],\\
&A^{\left[3_{2_1}\right]} = [0], A^{\left[3_{2_2}\right]} = [1], A^{\left[3_{2_3}\right]} = [1], A^{\left[3_{2_4}\right]} = [0].
\end{aligned} \tag{35}$$

Example 1

$$B = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{bmatrix}. \tag{36}$$

The matrix in equation (36) is a $3 \times 3 \times 2$ PCTLHS matrix with three subjects, three attributes, and two time levels.

$i = 1, 2, 3$

$j = 1, 2, 3$

$k = 1, 2$

A top-to-bottom view of the PCTLHS matrix with three i -level cuts is given in equation (37). Every level cut is a 2×3 matrix.

$$B = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right]. \tag{37}$$

i – level cuts of B are given as follows:

$$B^{[1]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$B^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \tag{38}$$
$$B^{[3]} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The local aggregation operators for i-level cuts are the union, intersection, average, and compliment operators described as follows.

5.2.5. Union of i – Level Cuts. The union of $A^{[i]}$ is defined as follows:

$$\bigcup\left(A^{[i]}\right) = \underset{i}{\text{Max}}\left(\mu_{A_j^k}(x_i)\right) = \left[\Omega_{A_j^k}(X)\right], \tag{39}$$

$[\Omega_{A_j^k}(X)]$ is an accumulated layer (submatrix) of the highest memberships considered as the top-level layer.

Example 2. For matrix **A** given in Ex-1, $\bigcup(A^{[i]})$ is given as follows:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \bigcup \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \bigcup \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \tag{40}$$

The cumulative attributive state of all three subjects reflects the highest level of each symptom among the given group of patients.

5.2.6. Intersection of i – Level Cuts. The intersection of $A^{[i]}$ defined as follows:

$$\cap\left(A^{[i]}\right) = \underset{i}{\text{Min}}\left(\mu_{A_j^k}(x_i)\right) = \left[\Omega_{A_j^k}(x)\right]. \tag{41}$$

$[\mu_{A_j^k}(x)]$ is the accumulated lowest membership as a bottom layer that represents the lowest state of each symptom of the group of three patients.

Example 3. For Matrix A in Ex-1, $\cap(A^{[i]})$ is given as follows:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cap \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cap \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{42}$$

This is the accumulated lowest attributive state of all three subjects as a group.

5.2.7. Average of i – Level Cuts. The average memberships as an interior layer are defined as follows:

$$\Gamma\left(A^{[i]}\right) = \sum_{i=1}^{M} \frac{\left(\mu_{A_j^k}(x_i)\right)}{M} = \left[\Omega_{A_j^k}(x)\right] = \begin{cases} 1, & \text{if } \sum_{i=1}^{M} \frac{\left(\mu_{A_j^k}(x_i)\right)}{M} \geq 0.5, \\ 0, & \text{if } \sum_{i=1}^{M} \frac{\left(\mu_{A_j^k}(x_i)\right)}{M} < 0.5, \end{cases} \tag{43}$$

$[\Omega_{A_j^k}(x)]$ is the accumulated layer of average memberships considered as the interior-level layer. This average operator presents the neutral or average state of the universe.

Example 4. For matrix A given in Ex-1, $\Gamma(A^{[i]})$ is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Gamma \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Gamma \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

This is the accumulative attributive state of all three subjects as they are viewed as a single entity, reflecting the average level of each symptom.

5 .2.8.Complement of i – Level Cuts. The complement of each membership of the *i*-level cut is defined as follows:

$$C\left(A^{[i]}\right) = \left[1 - \mu_{A_j^k}(x_i)\right]. \tag{45}$$

Example 5. Complement of *i*-level cuts of *A* is given as follows:

$$C\left(A^{[1]}\right) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$
$$C\left(A^{[2]}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \tag{46}$$
$$C\left(A^{[3]}\right) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

These complements can be further accumulated through previously defined aggregation operators. These complement operators reflect the inverted state of the merged universe.

6. Conclusion

6.1. Conclusion and Discussion. This novice model would help improve and expand the field of decision-making and artificial intelligence. Using the PCTLHS matrix, one would be able to represent an extensive, indeterminate plithogenic universe. Some important facts of this model are discussed as follows:

(1) The PCTLHS matrix provides multidimensional views of the universe (subjects versus attributes and time-lined attributes), whereas in this first version, only one of the three possible views is described as a model. In this view, the subjects are initially focused by the observer, and the next focus would be attributes or time levels depending on the interest of the observer or decision-maker.

(2) One can classify and analyze the universe explicitly and implicitly through level cuts, sublevel cuts, and sub-sublevel cuts. In this way, one could be able to enhance his approach to knowing the universe (all information) in a more dynamic and modern form.

(3) By choosing specific *i*, *j*, or *k* level cuts, one can make his priority choices by subjects, attributes, or time levels, and by specifying further sublevel cuts, one can further specify his next level of selection by subject, attribute, or time levels. Through the next level of selection, i.e., using sub-sublevel cuts, one can approach a final selection of the subject, attribute, or time level as per requirements.

(4) The PCTLHS matrix provides the broader exterior and interior views of the matrix by displaying all possible events (realities) together. Therefore, we expect that this PCTLHS Matrix will deal with multiple information networks under one command and one control system.

(5) The level cuts of the PCTLHS matrix present an explicit event or reality at a single instance of time. Therefore, one would be able to observe the state of an event (health state of patient) from moment to moment. This will improve his understanding of each event and every state according to its time level during that state.

(6) Variation index-based level cuts provide the view of reality or events from multiple angles of vision. This will improve the mathematical visualization of facts/events/information.

(7) One can analyze the universe by choosing the best possible reality from several possible realities using level cuts and operators. This fact would be helpful in the development of artificial intelligence programs.

(8) This PCTLHS matrix is very close to the functions of the human mind. Therefore, it acts as a pioneer in the modeling of the human mind. If this application can be opted by some computer specialists/software developers, it has every potential not only to improve artificial intelligence (AI) systems but also has great promises in the field of real intelligence (RI).

(9) The disjunction operator, i.e., the max operator, provides the optimist view of reality.

(10) The conjunction operator, i.e., the min operator, offers the pessimist perception of the event or reality

(11) The averaging operator offers the neutral states of the unified universe, which means that all the elements of the universe are considered as a single entity and only its neutral states are focused

(12) The complement operator presents the inverted reflection of the event or reality.

(13) The local operators designed for i-level cuts serve the purpose of unifying the matter bodies of the universe. This means that all the elements of the universe are merged and presented as a single matter body reflecting several attributes in distinct time levels. In this way, the concept of the unified global matter (something like dark matter) is visualized.

(14) These local operators serve the purpose of unification of matter bodies of the universe. This means that all the elements of the universe are merged and presented as a single matter body reflecting several attributes in distinct time levels. This is how the concept of the unified global matter (something like dark matter) arises. In this way, the concept of the unified global matter (something like dark matter) emerges.

6.2. Comparisons of Former Fuzzy Extensions and Models. This section contains a brief comparison of previous and current fuzzy extensions and models.

The soft set is an advanced and extended version of the fuzzy set because it manages many attributes simultaneously regardless of the fuzzy set, which only handles one attribute at a time.

The hypersoft set is an enhanced extension of a soft set because it can accommodate multidimensional information by managing many attributes and their various values simultaneously.

The plithogenic hypersoft set is a more advanced and applicable version than the hypersoft set, soft set, and fuzzy set. It is a higher dimensional version that manages detailed information. The observer can intrinsically see the state of the element x (subject) by looking at each attribute separately. In simple terms, the plithogenic hypersoft set manages multiple attributes and their values (subattributes) simultaneously and beyond by observing each attribute separately.

The plithogenic fuzzy whole hypersoft set/matrix (PFWHS set/matrix) is a more appropriate choice than the previously mentioned extensions as it indicates the states of subjects (attributes/subattributes) at the individual level for each attribute/subattribute (as in the case of the plithogenic hypersoft set) and also at the unified or combined level for all attributes together as a whole (the case of the hypersoft set). Therefore, it is an extended hybrid version of the hypersoft set and the plithogenic hypersoft set. By using the PFWHS set/matrix, one can observe a more transparent inner perception (case of a single state representation) or outer view (case of a combined state representation) of information/facts/events.

The plithogenic subjective hyper-supersoft matrix (PSHSS matrix) is a generalized advanced form of the PFWHS matrix and is more applicable as than previously developed models, as it has a greater capacity to express and manage various connected attributes/subattributes separately and as a whole by considering connected attribute/subattribute levels.

The plithogenic time-leveled hypersoft matrix (PTLHS matrix) is a special case of the previous generalized form (PSHSS matrix) that manages time-based connected attributes. It is more suitable than other extended fuzzy sets mentioned (soft set, hypersoft set, plithogenic hypersoft set, PFWHS set/matrix, and PSHSS matrix) for the following valid reasons:

(1) The plithogenic time-leveled hypersoft matrix includes time as the fundamental source of variation. As most of the variations in this universe are time-dependent like weather graphs, stock exchange, and website ratings, therefore, it is of tremendous help if this PCTLHS matrix is used to cope with the scattered time-varying piece of information.

(2) It deals with several attributes interiorly such that each attribute has many values (subattributes). These subattributes are varying with the flow of time. By using the PCTLHS matrix, one can organize and classify multidimensional information into the shape of connected matrix layers as hypermatrices.

(3) The matrix expression is the most appropriate expression to represent multidimensional data compared to the classic set expression.

(4) This PTLHS Matrix allows the viewer to see information down to its innermost level through level cuts, sublevel cuts, and sub-sublevel cuts.

(5) In addition, it offers a broader view of multidimensional information by viewing the entire universe as a hypersoft time-leveled matrix. Therefore, the observer can see and analyze the whole universe externally at a single glance.

(6) Level cuts can focus on one required piece of information that is displayed in the form of a single matrix layer of the PTLHS matrix, while the other information can vary with the flow of time being displayed as other matrix layers.

(7) Sublevel cuts can focus on required information that is displayed as a single column or row of the given layer (submatrix) of the PTLHS matrix.

(8) Sub-sublevel cuts can focus on required information that is displayed as a single element of the submatrix of the PTLHS matrix.

(9) Sublevel cuts offer the representation of the previous lower dimension in the further lower dimension and enable us to sneak in an inside view of the expanded universe; i.e., after explicitly focusing on a subject through an i-level cut (single level of the layered matrix), our next focus is on that subject's (patient's) attribute (a particular symptom) through the sublevel cut (row or column of one layer of the multilayered matrix).

(10) It also offers the unification of the information by applying aggregation operators, and in this way, all the extended information of the universe that is represented as a matrix having multiple layers can be transformed into a single layer of the matrix.

6.3. Future Research. Now, let us list some of the open problems that might be addressed in future research.

(i) In this article, we have portrayed the plithognic hypersoft matrix in a crisp environment. One can extend this model in other environments such as fuzzy, intuitionistic, and neutrosophic, or any mixed or combined environment; i.e., containing several environments would provide the variation of fuzziness levels of reflected events.

(ii) Moreover, some other kinds of local operators can be provided for the unification purpose of littered data according to the requirement of the concerned bodies.

(iii) The operations and properties of these hypersoft matrices need to be explored.

Data Availability

No data were used to support the findings of this study as it is a theoretical model.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors reviewed the results and approved the final version of the manuscript for publication. Shazia Rana is responsible for the study, conception, drafting, design, interpretation of the work, and review of the results for the final version of the manuscript. Mohammad Saeed is responsible for the analysis of the research work and its critical revision for key intellectuals' content agreement to be responsible for all aspects of the work to ensure that issues related to the accuracy or integrity of any part of the work are properly investigated and resolved. Badria Almaz Ali Yousif is responsible for analyzing the work and critically reviewing it for important intellectual content, reviewing the results, and approving the final version of the manuscript. Florentin Smarandache is responsible for analyzing and critically revising the manuscript for the acquisition of important intellectual content and the agreement to all aspects of the work, to ensure that issues related to the accuracy or integrity of any part of the work are adequately investigated and resolved. Hamiden Abd-ul-Wahid Khalifa is responsible for critically revising the manuscript for important in-tellectual content, reviewing the results, and approving the final version of the manuscript.

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Single Valued Neutrosophic HyperSoft Set based on VIKOR Method for 5G Architecture Selection

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Abstract

This work introduces the framework for selecting architecture in 5G networks, considering various technological, performance, economic, and operational factors. With the emergence of 5G technology, the architecture selection process has become pivotal in meeting diverse requirements for ultra-high-speed connectivity, low latency, scalability, and diverse service demands. The evaluation comprehensively analyses different architecture options, including centralized, distributed, cloud-based, and virtualized architectures. Factors such as network performance, scalability, cost-effectiveness, security, and compatibility are considered within a multi-criteria decision-making framework. Findings reveal each architecture option's diverse strengths and limitations, emphasizing the importance of aligning architectural choices with specific use cases, deployment scenarios, and business objectives. We used the concept of multi-criteria decision-making (MCDM) methodology to control the various criteria. We used the single-valued neutrosophic set (SVNS) to deal with vague and inconsistent data. The SVNS is integrated with the hyperSoft set. The proposed methodology uses a single-valued neutrosophic hyperSoft set (SVNHSS) to select the best 5G architecture. We used the VIKOR method as an MCDM method to rank the alternatives. The proposed framework provides stakeholders with a structured methodology to evaluate and prioritize architecture options, facilitating informed decision-making in the complex landscape of 5G network deployments.

Keywords: HyperSoft Set; Single Valued Neutrosophic Set; VIKOR method; Multi-Criteria Decision Making; VIKOR Method.

1. Introduction

The fifth generation (5G) architecture represents a revolutionary leap in wireless communication technology, promising to transform how we connect, communicate, and interact with the digital world. Building upon the foundations of previous generations, 5G architecture introduces several vital advancements that enable faster data speeds, lower latency, higher capacity, and enhanced reliability. With its transformative potential, 5G is poised to fuel innovation and drive the development of new applications and services across various industries[1], [2].

At its core, 5G architecture is designed to meet the evolving demands of an increasingly connected society. It aims to support a massive increase in data traffic and accommodate the exponential growth of connected devices, ranging from smartphones and tablets to Internet of Things (IoT) devices, autonomous vehicles, and intelligent infrastructure. By providing an ultra-fast and robust network infrastructure, 5G architecture enables seamless connectivity and empowers many applications that rely on real-time data transfer and responsiveness[3], [4].

One of the defining features of 5G architecture is its utilization of advanced radio technologies. It incorporates new frequency bands, including traditional cellular frequencies and higher-frequency bands, such as millimetre waves

(mmWave), to deliver unparalleled data rates. These higher frequencies offer wider bandwidths, enabling the transmission of large amounts of data at ultra-fast speeds. Additionally, 5G supports massive multiple-input multiple-output (MIMO) technology, which utilizes multiple antennas to enhance the capacity and efficiency of wireless communication[5], [6].

Another critical aspect of 5G architecture is its emphasis on network virtualization and software-defined networking (SDN). By virtualizing network functions and separating control and data planes, 5G enables greater flexibility, scalability, and agility in network management. This approach allows for dynamic allocation of network resources, efficient traffic routing, and on-demand service provisioning, leading to optimized network performance and improved user experience[7], [8].

Furthermore, 5G architecture introduces edge computing capabilities, bringing computing power closer to the network edge. 5G enables low-latency processing and real-time response for latency-sensitive applications such as autonomous vehicles, industrial automation, and augmented reality. By reducing the need for data to travel long distances to centralized data centres, edge computing enhances efficiency and responsiveness, supporting time-critical and bandwidth-intensive applications[9], [10].

Security and privacy are also integral aspects of 5G architecture. Robust security measures are crucial with the proliferation of connected devices and the increasing volume of sensitive data being transmitted. 5G incorporates advanced encryption, authentication protocols, and network-slicing techniques to ensure secure and private communication. Network slicing allows the creation of virtual networks dedicated to specific applications or user groups, providing customized security measures and isolation [11], [12].

5G architecture represents a significant leap forward in wireless communication technology, bringing unprecedented speed, capacity, and responsiveness to our interconnected world. By leveraging advanced radio technologies, virtualization, edge computing, and robust security measures, 5G architecture paves the way for transformative applications and services across industries such as healthcare, transportation, manufacturing, and entertainment. In addition, with the evolution of the beyond 5G (B5G) technologies, our digital world is expecting to encounter a great shift connectivity to reach degree that even exceed the level of capabilities posed by 5G networks. Though 5G is a noteworthy leap forward with its aforementioned services, the introduction of B5G has a great promise to revolutionize different phases of our daily lives in unimaginable manners. This advancement is likely to become the game-changer in sectors whereby data trust and responsibilities are paramount, including but not limited to finance, healthcare, and national security [13], [14].

This study used the MCDM method for ranking the 5G architecture. However, the evaluation data contains uncertainty, so the concept of the fuzzy set is introduced. The notion of an interval-valued fuzzy set, in which the degree of membership is an interval value in [0,1], was introduced due to the ambiguity around the membership degree. Later, Atanassov introduced the notion of the intuitionistic fuzzy set (IFS) in 1986, which includes the non-membership degree. IFS was used for a variety of purposes[15].

Nevertheless, ambiguous information often present in belief systems is impossible for IFS. The notion of a neutrosophic set was presented by Smarandache in 2000 as a solution to this insufficiency. In this set, membership (T), indeterminacy (I), and non-membership (F) degrees were quantified separately, meaning that T, I, and F $\in$[0,1], and the sum T+I+F need not be contained in [0,1]. These mathematical methods have been extensively studied and effectively used to manage uncertainty; however, since they lack a parametrization tool and need all concepts to be precise, they often fall short when dealing with ambiguity in various real-world scenarios[16]–[18], [18].

As a result, Molodstsov proposed the soft set in 1999, a broad mathematical tool for handling uncertainty. A parameterized family of subsets of a discourse universe is what Molodstsov characterized as a soft set. Maji et al. first presented aggregation procedures on soft sets in 2003. Many applications have seen the effective use of soft sets and their hybrids[19], [20].

The characteristics must be further split into attribute values to improve decision-making in a range of real-world MCDM scenarios. In 2018, Smarandache created the notion of a hypersoft set[21], [22], a generalization of the soft set concept, therefore meeting this demand[23]–[25].

The main contributions of this study are:

- Build the decision matrix between criteria and alternatives of 5G architecture to rank and select the best option.
- The VIKOR method is used as an MCDM method to evaluate and rank the alternatives.
- The proposed framework (SVNHSS) deals with uncertainty and vague data.
- HyperSoft is used to show the relation between criteria and attribute values.
- There are 18 criteria, and 49 attribute values are used in this study.

2. HyperSoft Set and VIKOR Method

The aim of this study, rank and select best 5G architecture based on various criteria and alternatives. These problem will have solved by using the single valued neutrosophic hypersoft set VIKOR method (SVNHSS). The VIKOR method is used to rank the alternatives, while the single valued neutrosophic is used to deal with vague data with the hypersoft set. The weight values are computed by the average method. The steps of the SVNHSS is introduced. The proposed method has the three steps includes, identify a set of criteria and alternatives, compute the weights of criteria by the average method, and rank the alternatives using the VIKOR method.

2.1 Identify a set of criteria and alternatives.

The experts and decision makers are defined a set of criteria based on a set of alternatives by the Questioners and literature review.

2.2 Compute the criteria weights.

We compute the criteria weights by the average method. We let the experts used the single valued neutrosophic scale to evaluate the criteria and alternatives, then compute the criteria weights. Table 1 shows the single valued neutrosophic numbers (SVNNs)

Table 1: The single valued neutrosophic numbers.

Linguistic Variables	**SVNNs**
Absolutely High	(0.9,0.1,0.2)
Very High	(0.8,0.2,0.3)
High	(0.7,0.3,0.4)
Moderate High Equal	(0.6,0.4,0.5)
Equal	(0.5,0.5,0.5)
Moderate Low Equal	(0.4,0.5,0.6)
Low	(0.3,0.6,0.7)
Very Low	(0.2,0.7,0.8)
Absolutely Low	(0.1,0.8,0.9)

2.3 Rank the alternatives by the SVNHSS-VIKOR method.

The soft set can be defined as:

Let μ be a universe of discourse, $p(\mu)$ the power set of μ, and A a set of attributes. Then, the pair $(F, \mu), F: A \to P(\mu)$ is called a Soft Set over μ

The HyperSoft can be defined as[21], [26]–[33]:

Let μ be a universe of discourse, $p(\mu)$ the power set of μ. $Let\ a_1, a_2, \dots a_n\ for\ n \geq 1$, be n distinct attributes, whose corresponding attributes are respectively the set $A_1, A_2, \dots A_n\ with\ A_i \cap A_j = \emptyset, for\ i \neq j, and\ i, j \in \{1,2, \dots n\}$. Then the pair $\left(F: A_1 \times A_2 \times \dots A_n \to p(\mu)\right)$ is called a HyperSoft over μ.

Step 1. Build the decision matrix

The decision matrix is built based on a set of criteria $A5GA_1, A5GA_2, \dots A5GA_n$ and a set of alternatives $5GAL_1, 5GAL_2, \dots 5GAL_m$. The set of criteria are evaluated based on a their values $A5GF_1, A5GF_2$

$$X = \begin{matrix} & A5GA_1 & \cdots & A5GA_n \\ 5GAL_1 & A5GF_1 & \cdots & A5GF_n \\ \vdots & \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix} & & \\ 5GAL_m & & & \end{matrix} \quad (1)$$

Where $i = 1,2, \ldots m; j = 1,2, \ldots . n$

Step 2. Applied the HSS to choose the ordered tuple

The HSS is applied to choose the best attribute value for all criteria based on alternatives.

Step 3. Apply the score function

The score function is applied to show the crisp values for all SVNNs.

$$S(x_{ij}) = \frac{2 + T - I - F}{3} \quad (2)$$

Where T, I, F refer to the truth, indeterminacy and falsity memberships degree for the SVNNs.

Step 4. Compute the benefit and cots values.

$$b_i = \max[x_{ij}] \quad (3)$$

$$c_i = \min[x_{ij}] \quad (4)$$

Step 5. Compute the utility measure value and regret measure value [34-35].

$$u_i = \frac{w_j[b_i - x_{ij}]}{b_i - c_i} \quad (5)$$

$$u_i = \frac{w_j[x_{ij} - b_i]}{b_i - c_i} \quad (6)$$

$$R_i = \max of \left\{ \frac{w_j[b_j - x_{ij}]}{b_j - c_j} \right\} \quad (7)$$

Step 6. Compute the VIKOR index.

$$V_i = \varphi \left[\frac{u_i - u_i^-}{u_i^+ - u_i^-} \right] + (1 - \varphi) \left[\frac{R_i - R_i^-}{R_i^+ - R_i^-} \right] \quad (8)$$

Where $u_i^+ = \max u_i , u_i^- = \min u_i , R_i^+ = \max R_i \ and \ R_i^- = \min R_i$. The value φ between 0 and 1, we used it in this paper as 0.5.

3. Results and Discussion

This section introduces the results of the SVNHSS-VIKOR method for selecting best 5G architecture.

3.1 Identify a set of criteria and alternatives.

We identified 18 criteria and 5 alternatives in this study based on opinions of experts and decision makers. Then we used 49 attributes values for all criteria.

The criteria of the 5G selection Architecture and their values are:

$A5GA_1$= Throughput

$A5GA_2$=Latency

$A5GA_3$=Reliability

$A5GA_4$=Coverage

$A5GA_5$=Mobility Support

$A5GA_6$=Capacity

$A5GA_7$=Flexibility

$A5GA_8$=Expandability

$A5GA_9$=Initial Investment (INV)

$A5GA_{10}$=Total Cost of Ownership (TCO)

$A5GA_{11}$=Security Measures

$A5GA_{12}$=Redundancy and Failover

$A5GA_{13}$=Quality of Service (QoS):

$A5GA_{14}$=Interoperability (INT)

$A5GA_{15}$=Standards Compliance (STN)

$A5GA_{16}$=Power Consumption (PCN)

$A5GA_{17}$=Ease of Deployment

$A5GA_{18}$=Network Management (NM)

The attributes values are:

$A5GF_1 = \{\text{Throughput} < 1\ Gbps, \text{Throughput} > 10\ Gbps\}$

$A5GF_2 = \{\text{Latency} < 1ms, \text{Latency} = 10ms\}$

$A5GF_3 = \{high, medium, low\}$

$A5GF_4 = \{urban, suburban, rural\}$

$A5GF_5 = \{Low\ speed, medium\ speed,\ high\ speed\}$

$A5GF_6 = \{Capacity < 1000\ devices\ per\ km^2, capacity > 10{,}000\ devices\ per\ km^2\}$

$A5GF_7 = \{low\ flexible, moderate\ flexible, high\ flexible\}$

$A5GF_8 = \{limited, moderate, or\ extensive\}$.

$A5GF_9 = \{low\ \text{INV}, moderate\ \text{INV}, high\ \text{INV}\}$.

$A5GF_{10} = \{low\ \text{TCO}, moderate\ \text{TCO}, high\ \text{TCO}\}$

$A5GF_{11} = \{basic\ security, standard\ security,\ advanced\ security\}$

$A5GF_{12} = \{basic\ Redundancy, moderate\ Redundancy, comprehensive\ Redundancy\}$

$A5GF_{13} = \{basic\ QoS, acceptable\ QoS, superior\ QoS\}$

$A5GF_{14} = \{compatible\ INT, partially\ compatible\ INT, incompatible\ INT\}$

$A5GF_{15} = \{fully\ compliant\ STN, partially\ STN\ Compliance, non\ compliant\ STN\}$

$A5GF_{16} = \{low\ \text{PCN}, high\ \text{PCN}\}$

$A5GF_{17}= \{complex\ development, easy\ development\}$

$A5GF_{18}= \{manual\ NMS, semi\ automated\ NM, fully\ automated\ NM\ \}$

3.2 Compute the criteria weights. The criteria weighted are computed based on the average method as shown in Figure 1. The latency and reliability, and coverage has the highest weight but the ease of development has the lowest weight.

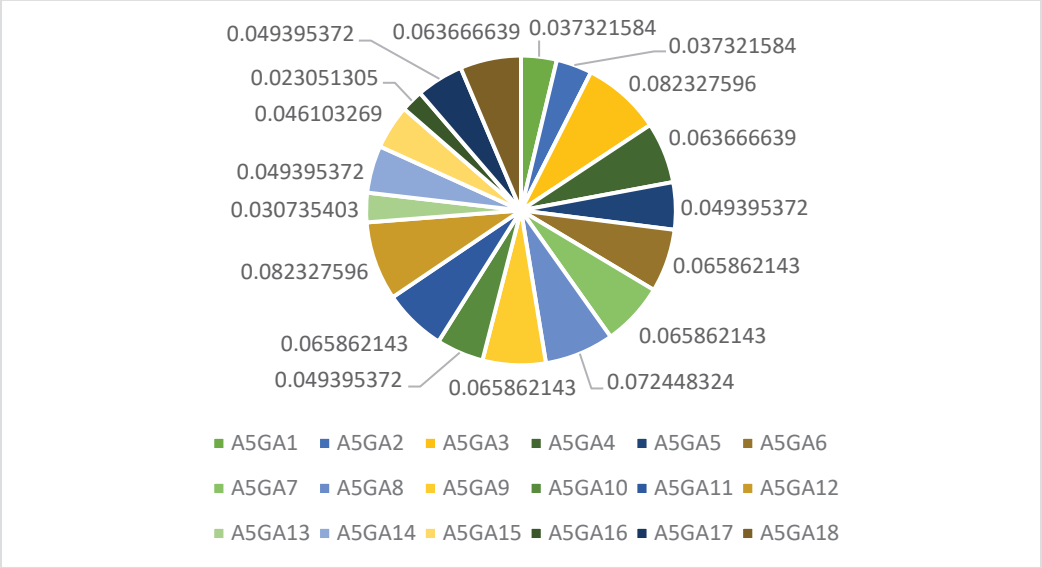


Figure 1: The criteria of 5G architecture weights.

3.3 Rank the alternatives by the SVNHSS-VIKOR method.

Step 1. Build the decision matrix between 18 criteria, 49 attributes values and 5 alternatives in this study as shown in Table 2 by Eq. (1).

Table 2: The values of criteria, attributes values and alternatives for decision matrix.

		$5GAL_1$	**$5GAL_2$**	**$5GAL_3$**	**$5GAL_4$**	**$5GAL_5$**
$A5GA_1$	$A5GF_1$	(0.9,0.1,0.2)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.9,0.1,0.2)
	$A5GF_2$	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
$A5GA_2$	$A5GF_3$	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.9,0.1,0.2)
	$A5GF_4$	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
$A5GA_3$	$A5GF_5$	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
	$A5GF_6$	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
	$A5GF_7$	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.2,0.7,0.8)	(0.1,0.8,0.9)
$A5GA_4$	$A5GF_8$	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
	$A5GF_9$	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.9,0.1,0.2)
	$A5GF_{10}$	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.6,0.4,0.5)
$A5GA_5$	$A5GF_{11}$	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.9,0.1,0.2)
	$A5GF_{12}$	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
	$A5GF_{13}$	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.6,0.4,0.5)

A5GA6	A5GF14	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF15	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GA7	A5GF16	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF17	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF18	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GA8	A5GF19	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
	A5GF20	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF21	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GA9	A5GF22	(0.9,0.1,0.2)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF23	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF24	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
A5GA10	A5GF25	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF26	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
	A5GF27	(0.8,0.2,0.3)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.9,0.1,0.2)
A5GA11	A5GF28	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF29	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF30	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GA12	A5GF31	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
	A5GF32	(0.3,0.6,0.7)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.9,0.1,0.2)
	A5GF33	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
A5GA13	A5GF34	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.9,0.1,0.2)
	A5GF35	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF36	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GA14	A5GF37	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.6,0.4,0.5)	(0.9,0.1,0.2)
	A5GF38	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.2,0.7,0.8)	(0.1,0.8,0.9)
	A5GF39	(0.9,0.1,0.2)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GA15	A5GF40	(0.3,0.6,0.7)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.9,0.1,0.2)
	A5GF41	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.2,0.7,0.8)	(0.1,0.8,0.9)
	A5GF42	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GA16	A5GF43	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.9,0.1,0.2)
	A5GF44	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GA17	A5GF45	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF46	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GA18	A5GF47	(0.3,0.6,0.7)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.2,0.7,0.8)	(0.1,0.8,0.9)
	A5GF48	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
	A5GF49	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.9,0.1,0.2)

Step 2. Applied the HSS to choose the ordered tuple.

Let $C = A5GA_1 \times A5GA_2 \times A5GA_3 \times A5GA_4 \times A5GA_5 \times A5GA_6 \times A5GA_7 \times A5GA_8 \times A5GA_9 \times A5GA_{10} \times A5GA_{11} \times A5GA_{12} \times A5GA_{13} \times A5GA_{14} \times A5GA_{15} \times A5GA_{16} \times A5GA_{17} \times A5GA_{18}$ and the attribute values are $(A5GF_1, A5GF_2, \dots A5GF_{49}) \in$. We choose the 18 attributes values as $A5GF_2 \times A5GF_4 \times A5GF_5 \times A5GF_{10} \times A5GF_{13} \times A5GF_{15} \times A5GF_{18} \times A5GF_{20} \times A5GF_{23} \times A5GF_{25} \times A5GF_{30} \times A5GF_{31} \times A5GF_{36} \times A5GF_{37} \times A5GF_{40} \times A5GF_{44} \times A5GF_{46} \times A5GF_{48}$. The decision matrix of chosen ordered tuple is shown in Table 3.

Table 3: The ordered tuple of attributes values based on alternatives.

	5GAL1	**5GAL2**	**5GAL3**	**5GAL4**	**5GAL5**
A5GF2	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
A5GF4	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
A5GF5	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
A5GF10	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.6,0.4,0.5)
A5GF13	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
A5GF15	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)

A5GF18	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GF20	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GF23	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GF25	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GF30	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GF31	(0.9,0.1,0.2)	(0.9,0.1,0.2)	(0.8,0.2,0.3)	(0.7,0.3,0.4)	(0.6,0.4,0.5)
A5GF36	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GF37	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.6,0.4,0.5)	(0.9,0.1,0.2)
A5GF40	(0.3,0.6,0.7)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.9,0.1,0.2)
A5GF44	(0.3,0.6,0.7)	(0.2,0.7,0.8)	(0.1,0.8,0.9)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GF46	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.5,0.5,0.5)	(0.4,0.5,0.6)
A5GF48	(0.9,0.1,0.2)	(0.6,0.4,0.5)	(0.5,0.5,0.5)	(0.4,0.5,0.6)	(0.9,0.1,0.2)

Step 3. Apply the score function by Eq. (2) to compute the crisp values.
Step 4. Compute the benefit and cots values by Eqs. (3 and 4). All criteria are benefit except initial investment and total cost are cost criteria.
Step 5. Compute the utility measure value and regret measure value by Eqs. (5-6) a shown in Table 4.

Table 4. The measure and utility degree.

	5GAL1	**5GAL2**	**5GAL3**	**5GAL4**	**5GAL5**
A5GF2	0.031429	0.037322	0	0.005893	0.011786
A5GF4	0.031429	0.037322	0	0.005893	0.011786
A5GF5	0	0	0.027443	0.054885	0.082328
A5GF10	0	0.044073	0.053868	0.063667	0.044073
A5GF13	0.021166	0.035278	0.049395	0	0.021166
A5GF15	0	0.019756	0.039513	0.052685	0.065862
A5GF18	0	0.019756	0.039513	0.052685	0.065862
A5GF20	0	0.016717	0.050152	0.061298	0.072448
A5GF23	0	0.019756	0.039513	0.052685	0.065862
A5GF25	0	0.024692	0.049395	0.024692	0.049395
A5GF30	0	0.019756	0.039513	0.052685	0.065862
A5GF31	0	0	0.027443	0.054885	0.082328
A5GF36	0.023643	0.030735	0	0.004729	0.00946
A5GF37	0.034194	0.041793	0.049395	0.034194	0
A5GF40	0.046103	0.025933	0.031696	0.037462	0
A5GF44	0.010478	0.016765	0.023051	0	0.004193
A5GF46	0	0.024692	0.049395	0.024692	0.049395
A5GF48	0	0.044073	0.053868	0.063667	0

Step 6. Compute the VIKOR index by Eq. (8) as shown in Figure 2. We show alternative 1 is the best and alternative 5 is the worst.

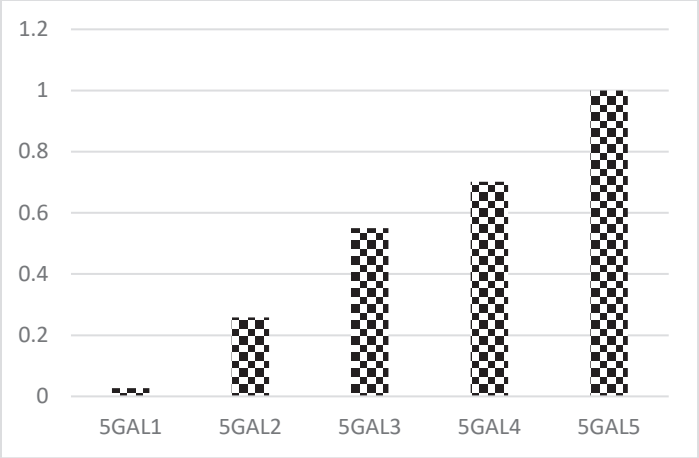


Figure 2: The values of VIKOR index.

4. Conclusions

The selection of architecture for 5G networks is a critical decision that necessitates a holistic evaluation encompassing diverse factors. The assessment of architectural options revealed a spectrum of strengths and trade-offs. Centralized architectures offer streamlined management but may encounter scalability challenges. Distributed architectures provide enhanced reliability but may face complexities in synchronization. Cloud-based architectures offer flexibility but pose concerns regarding latency and data privacy. Virtualized architectures showcase agility but demand robust hardware resources. The selection process highlighted the importance of aligning architectural choices with specific use cases and operational requirements. Moreover, ensuring interoperability, security, and compliance with 5G standards emerged as crucial considerations. The proposed MCDM framework under SVNHSS is a valuable tool for stakeholders to navigate the complexities and trade-offs associated with selecting the optimal architecture for 5G networks. Collaboration among industry stakeholders, policymakers, and technology providers will be essential in adapting architectures to meet evolving demands and ensuring the successful deployment of 5G networks. We used 18 criteria and 49 attributes values based on 5 alternatives. We used the VIKOR method to rank the alternatives. In future work, deep learning

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Hypersoft Sets in a Game Theory-Based Decision Making Model

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Abstract

In this study, we offer a hypersoft set theory-based game model for handling uncertainties. The term ”hypersoft game” refers to this newly suggested game. Four techniques of game solution are identified: hypersoft saddle points, hypersoft upper and lower values, hypersoft dominated strategy and hypersoft nash equilibrium. We build a two-person hypersoft game first. Additionally, we present real-world problems that the hypersoft saddle point approaches and hypersoft dominating strategy are used to tackle. In conclusion, we expand the hypersoft games from two players to n players.

Keywords: hypersoft set, two person hypersoft games, hypersoft payoff functions, hypersoft dominated strategies, hypersoft lower and hypersoft upper values, hypersoft nash equilibrium.

1 Introduction

Soft set theory was developed by Molodtsov[18] as a generic Mathematical instrument for handling uncertainties. Soft sets have been widely employed to solve decision-making issues. A soft set application for decision-making problems was proposed by Maji et al.,[17] Cagman and Enginoglu.[10]

By replacing the function with a multi argument function specified in the cartesian product with a new set of parameters, Smarandache[21] extended the idea of a soft set to a hypersoft set. Compared to soft sets, this idea is more flexible and more suited to problems requiring decision-making. The hypersoft sets were effectively employed in the COVID-19 decision-making approach by Inthumathi et al.[11] In addition, they presented hypersoft matrices[12] and their uses.[13] They have introduced the concept of hypersoft semi-open sets.[14]

The study of strategic decision-making is known as game theory. The two-person zero-sum game and its proof served as the foundation for contemporary game theory, which was first presented by Neumann and Morgenstern[19] in 1944. Aliprantis and Chakrabarti[2] provided decision-making games. The concepts of fuzzy sets have been embedded into several intriguing game theory applications. Numerous writers have examined the two-person zero sum games with fuzzy objectives and payoffs in game theory.[4,5,7–9,16,22] Additionally, the max-min solution for the degree of attainment of a fuzzy objective has been examined. Soft games, as described by Irfan Deli and Naim Cagman,[15] are applicable to challenges that involve ambiguity and uncertainty.

In this research, we develop a novel uncertainty-handling game model for games dubbed hypersoft games. This game is highly practical and easy to apply in practise since the hypersoft payoff functions of the hypersoft game are set valued functions and the solution of the hypersoft games acquired by applying the operations of sets.

The organisation of our article is as follows: The next part includes some fundamental definitions of hypersoft sets that are necessary for our work. The two-person hypersoft games that we construct in section 3 have four different ways to solve them: hypersoft saddle points, hypersoft upper and lower values, hypersoft dominating strategy, and hypersoft nash equilibrium. We distribute applications for two-person hypersoft games in area 4. We provide n-person hypersoft games in section 5, which are an expansion of the two-person hypersoft games. The last portion contains the conclusion.

2 Hypersoft sets

Here, we review some fundamental terms and findings related to hypersoft sets.

Definition 2.1. [21] "Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the power set $\mathcal{U}$ and $E_1, E_2, ..., E_n$ the pairwise disjoint sets of parameters. Let A_i be the non-empty subset of E_i for each $i = 1, 2, ..., n$. A hypersoft set can be identified by the pair$(\mathcal{F}, A_1 \times A_2 \times ... \times A_n)$, $\mathcal{F} : A_1 \times A_2 \times ... \times A_n \longrightarrow \mathcal{P}(\mathcal{U})$.
For sake of simplicity, we write the symbols $\boldsymbol{\mathcal{E}}$ for $E_1 \times E_2 \times ... \times E_n$, $\boldsymbol{\mathcal{A}}$ for $A_1 \times A_2 \times ... \times A_n$ and $\tilde{\boldsymbol{a}}$ for an element of the set $\boldsymbol{\mathcal{A}}$. We also suppose that none of the set A_i is empty.

Example 2.2. Let $\mathcal{U} = \{x_1, x_2, x_3, x_4\}$.
Let the attributes be $a_1 = size, a_2 = colour, a_3 = gender, a_4 = nationality$, and their attributes values respectively:

size = $A_1 = \{small, medium, tall\}$
colour =$A_2 = \{white, yellow, red, black\}$
gender =$A_3 = \{male, female\}$
nationality = $A_4 = \{American, French, Italian\}$

Let the function be $\mathcal{F} : A_1 \times A_2 \times A_3 \times A_4 \longrightarrow \mathcal{P}(\mathcal{U})$.
Then the hypersoft set is
$\mathcal{F}(\{tall, white, female, italian\}) = \{x_1, x_3\}$.

Definition 2.3. [1] Let $\mathcal{U}$ be a universe of discourse and $\boldsymbol{\mathcal{A}}$ a subset of $\boldsymbol{\mathcal{E}}$. Then $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ is called

1. a null hypersoft set if for each parameter $\tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}}$, $\mathcal{F}(\tilde{\boldsymbol{a}})$ is an 0. We will denote it by $\Phi_{\boldsymbol{\mathcal{A}}}$.
2. an absolute hypersoft set if for each parameter $\tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}}$, $\mathcal{F}(\tilde{\boldsymbol{a}}) = \mathcal{U}$. We will denote it by $\mathcal{U}_{\boldsymbol{\mathcal{A}}}$.

Definition 2.4. [1] Let $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ and $(\mathcal{G}, \boldsymbol{\mathcal{B}})$ be two hypersoft sets over $\mathcal{U}$. Then $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ is called a hypersoft subset of $(\mathcal{G}, \boldsymbol{\mathcal{B}})$ if $\boldsymbol{\mathcal{A}} \subset \boldsymbol{\mathcal{B}}$ and $\mathcal{F}(\tilde{\boldsymbol{a}}) \subseteq \mathcal{G}(\tilde{\boldsymbol{a}})$ for all $\tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}}$. We denote this by $(\mathcal{F}, \boldsymbol{\mathcal{A}}) \subseteq (\mathcal{G}, \boldsymbol{\mathcal{B}})$.

Definition 2.5. [1] The complement of a hypersoft set $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ is denoted by $(\mathcal{F}, \boldsymbol{\mathcal{A}})^c$ and is defined as $(\mathcal{F}, \boldsymbol{\mathcal{A}})^c = (\mathcal{F}^c, \boldsymbol{\mathcal{A}})$ where $\mathcal{F}^c(\tilde{\boldsymbol{a}})$ is the complement of $\mathcal{F}(\tilde{\boldsymbol{a}})$ for each $\tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}}$.

Definition 2.6. [1] Let $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ and $(\mathcal{G}, \boldsymbol{\mathcal{B}})$ be two hypersoft sets over $\mathcal{U}$.
Then

1. Union of $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ and $(\mathcal{G}, \boldsymbol{\mathcal{B}})$ is denoted by $(\mathcal{H}, \boldsymbol{\mathcal{C}}) = (\mathcal{F}, \boldsymbol{\mathcal{A}}) \cup (\mathcal{G}, \boldsymbol{\mathcal{B}})$ with $\boldsymbol{\mathcal{C}} = C_1 \times C_2 \times ... \times C_n$, where $C_i = A_i \cup B_i$ for $i = 1, 2, ..., n$, and $\mathcal{H}$ is defined by

$$\mathcal{H}(\tilde{\boldsymbol{a}}) = \begin{pmatrix} \mathcal{F}(\tilde{\boldsymbol{a}}), & if\ \tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{B}} \\ \mathcal{G}(\tilde{\boldsymbol{a}}), & if\ \tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{A}} \\ \mathcal{F}(\tilde{\boldsymbol{a}}) \cup \mathcal{G}(\tilde{\boldsymbol{a}}), & if\ \tilde{\boldsymbol{a}} \in \boldsymbol{\mathcal{A}} \cap \boldsymbol{\mathcal{B}} \\ 0, & else \end{pmatrix}$$

   where $\tilde{\boldsymbol{a}} = (c_1, c_2, ...c_n) \in \boldsymbol{\mathcal{C}}$.
2. Intersection of $(\mathcal{F}, \boldsymbol{\mathcal{A}})$ and $(\mathcal{G}, \boldsymbol{\mathcal{B}})$ is denoted by $(\mathcal{H}, \boldsymbol{\mathcal{C}}) = (\mathcal{F}, \boldsymbol{\mathcal{A}}) \cap (\mathcal{G}, \boldsymbol{\mathcal{B}})$, where $\boldsymbol{\mathcal{C}} = C_1 \times C_2 \times ... \times C_n$, where $C_i = A_i \cap B_i$ for $i = 1, 2, ..., n$, and $\mathcal{H}$ is defined as
   $\mathcal{H}(\tilde{\boldsymbol{a}}) = \mathcal{F}(\tilde{\boldsymbol{a}}) \cap \mathcal{G}(\tilde{\boldsymbol{a}})$. where $\tilde{\boldsymbol{a}} = (c_1, c_2, ...c_n) \in \boldsymbol{\mathcal{C}}$.

3 Two Person Hypersoft Games

We create two-person hypersoft games with hypersoft payoffs in this section. We then provide four ways to solve the games. For further information and fundamental terminology, see.[3,6,20]

Throughout the work, we write the symbols $\mathfrak{P}$ for $P_1 \times P_2 \times ... \times P_n$, $\mathfrak{Q}$ for $Q_1 \times Q_2 \times .... \times Q_n$, $\mathfrak{p}$ for an element of $\mathfrak{P}$ and $\mathfrak{q}$ for an element of $\mathfrak{Q}$

Definition 3.1. Let $\mathfrak{P}, \mathfrak{Q}$ are a sets of strategies. A choice of behavior is called an action. The elements of $\mathfrak{P} \times \mathfrak{Q}$ are called action pairs. That is, $\mathfrak{P} \times \mathfrak{Q}$ is the set of available actions.

Definition 3.2. Let $\mathcal{U}$ be a set of alternatives, $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathfrak{P}, \mathfrak{Q}$ are set of strategies. Then a set valued function $\Im_{\mathcal{HS}} : \mathfrak{P} \times \mathfrak{Q} \longrightarrow \mathcal{P}(\mathcal{U})$ is called a hypersoft payoff function. For each $(\mathfrak{p}, \mathfrak{q}) \in \mathfrak{P} \times \mathfrak{Q}$, the value $\Im_{\mathcal{HS}}(\mathfrak{p}, \mathfrak{q})$ is called a hypersoft payoff.

Definition 3.3. Let $\mathfrak{P}$ and $\mathfrak{Q}$ be a set of strategies of player 1 and 2 respectively, $\mathcal{U}$ be a set of alternatives and $\Im_{(\mathcal{HS})_k} : \mathfrak{P} \times \mathfrak{Q} \longrightarrow \mathcal{P}(\mathcal{U})$ be a hypersoft payoff function for player k, $(k = 1, 2)$. Then for each player k, a two person hypersoft game is defined by a hypersoft set over $\mathcal{U}$ as

$(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q}) \in \mathfrak{P} \times \mathfrak{Q}\}$

The two person hypersoft game is played as follows:

At a certain time player 1 chooses a stratagy $\mathfrak{p}_i \in \mathfrak{P}$, simultaneously player 2 chooses a strategy $\mathfrak{q}_i \in \mathfrak{Q}$ and once this is done each player $k(k = 1, 2)$ receives the hypersoft payoff $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j)$

If $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_m\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, ..., \mathfrak{q}_n\}$ then the hypersoft payoffs of $(\mathcal{HS})_k$ can be organized in the form of $m \times n$ matrix shown in table 1.

Table 1: The two person hypersoft game

$(\mathcal{HS})_k$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\cdots$	$\mathfrak{q}_n$
$\mathfrak{p}_1$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_1, \mathfrak{q}_1)$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_1, \mathfrak{q}_2)$	$\cdots$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_1, \mathfrak{q}_n)$
$\mathfrak{p}_2$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_2, \mathfrak{q}_1)$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_2, \mathfrak{q}_2)$	$\cdots$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_2, \mathfrak{q}_n)$
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\mathfrak{p}_m$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_m, \mathfrak{q}_1)$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_m, \mathfrak{q}_2)$	$\cdots$	$\Im_{(\mathcal{HS})_k}(\mathfrak{p}_m, \mathfrak{q}_n)$

We can now provide an example of a two-player hypersoft game.

Example 3.4. Let $\mathcal{R} = \{r_1, r_2, r_3, ..., r_8\}$ be a set of alternatives, $\mathcal{P}(\mathcal{R})$ be the power set of $\mathcal{R}$, $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_3, \mathfrak{p}_5\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_4\}$ be a set of the strategies of player 1 and player 2 respectively. If player 1 constructs a two person hypersoft game as follows,

$$(\mathcal{HS})_1 = \left\{ \begin{array}{c} ((\mathfrak{p}_1, \mathfrak{q}_1), \{r_1, r_2, r_5, r_8\}), ((\mathfrak{p}_1, \mathfrak{q}_2), \{r_1, r_2, r_3, r_4, r_5, r_8\}), \\ ((\mathfrak{p}_1, \mathfrak{q}_4), \{r_3, r_8\}), ((\mathfrak{p}_3, \mathfrak{q}_1), \{r_1, r_3, r_7\}), \\ ((\mathfrak{p}_3, \mathfrak{q}_2), \{r_1, r_2, r_3, r_5, r_6, r_7\}), ((\mathfrak{p}_3, \mathfrak{q}_4), \{r_1, r_2, r_3\}), \\ ((\mathfrak{p}_5, \mathfrak{q}_1), \{r_3, r_4, r_5, r_8\}), \\ ((\mathfrak{p}_5, \mathfrak{q}_2), \{r_1, r_2, r_3, r_4, r_5, r_6, r_8\}), ((\mathfrak{p}_5, \mathfrak{q}_4), \{r_1, r_2, r_3, r_8\}). \end{array} \right\}$$

The game's hypersoft payoffs may then be set up as shown in table 2.

Table 2

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_4$
$\mathfrak{p}_1$	$\{r_1, r_2, r_5, r_8\}$	$\{r_1, r_2, r_3, r_4, r_5, r_8\}$	$\{r_3, r_8\}$
$\mathfrak{p}_3$	$\{r_1, r_3, r_7\}$	$\{r_1, r_2, r_3, r_5, r_6, r_7\}$	$\{r_1, r_2, r_3\}$
$\mathfrak{p}_5$	$\{r_3, r_4, r_5, r_8\}$	$\{r_1, r_2, r_3, r_4, r_5, r_6, r_8\}$	$\{r_1, r_2, r_3, r_8\}$

Let's go over a few of this game's components:

The value of the game will be a set $\{r_1, r_2, r_3, r_8\}$ if player 1 selects $\mathfrak{p}_5$ and player 2 selects $\mathfrak{q}_4$. The set of choices $\{r_1, r_2, r_3, r_8\}$ is won by player 1 in this scenario, whereas player 2 loses the identical set of alternatives. Likewise, if player 2 builds the following two-player hypersoft game,

$$(\mathcal{HS})_2 = \left\{ \begin{array}{c} ((\mathfrak{p}_1, \mathfrak{q}_1), \{r_3, r_4, r_6, r_7\}), ((\mathfrak{p}_1, \mathfrak{q}_2), \{r_6, r_7\}), \\ ((\mathfrak{p}_1, \mathfrak{q}_4), \{r_1, r_2, r_4, r_5, r_6, r_7\}), ((\mathfrak{p}_3, \mathfrak{q}_1), \{r_2, r_4, r_5, r_6, r_8\}), \\ ((\mathfrak{p}_3, \mathfrak{q}_2), \{r_4, r_8\}), ((\mathfrak{p}_3, \mathfrak{q}_4), \{r_4, r_5, r_6, r_7, r_8\}), \\ ((\mathfrak{p}_5, \mathfrak{q}_1), \{r_1, r_2, r_6, r_7\}), ((\mathfrak{p}_5, \mathfrak{q}_2), \{r_7\}), \\ ((\mathfrak{p}_5, \mathfrak{q}_4), \{r_4, r_5, r_6, r_7\}). \end{array} \right\}$$

Then the hypersoft payoffs of the game can be arranged as in table 3.

Table 3

$(\mathcal{HS})_2$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_4$
$\mathfrak{p}_1$	$\{r_3, r_4, r_6, r_7\}$	$\{r_6, r_7, \}$	$\{r_1, r_2, r_4, r_5, r_6, r_7\}$
$\mathfrak{p}_3$	$\{r_2, r_4, r_5, r_6, r_8\}$	$\{r_4, r_8\}$	$\{r_4, r_5, r_6, r_7, r_8\}$
$\mathfrak{p}_5$	$\{r_1, r_2, r_6, r_7\}$	$\{r_7, \}$	$\{r_4, r_5, r_6, r_7\}$

Let us explain some element of this two person hypersoft game. If player 1 select $\mathfrak{p}_1$ and player 2 select $\mathfrak{q}_2$, then the value of game will be a set $\{r_6, r_7\}$.

$$\Im_{(\mathcal{HS})_2}(\mathfrak{p}_1, \mathfrak{q}_2) = \{r_6, r_7\}.$$

In this case, player 1 wins the set of alternatives $\{r_6, r_7\}$ and player 2 lost $\{r_6, r_7\}$.

Definition 3.5. Let $(\mathcal{HS})_k = \{(\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}((\mathfrak{p}, \mathfrak{q})) : ((\mathfrak{p}, \mathfrak{q}))\in\mathfrak{P} \times \mathfrak{Q}\}$ be a two person hypersoft game and $(\mathfrak{p}_i, \mathfrak{q}_j), (\mathfrak{p}_r, \mathfrak{q}_s)\in\mathfrak{P}\times\mathfrak{Q}$. Then player k is rational if the player's hypersoft payoff satisfies the following conditions.

1. Either $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j) \supseteq \Im^k_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}_r, \mathfrak{q}_s)$ or $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_r, \mathfrak{q}_s) \supseteq \Im^k_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}_i, \mathfrak{q}_j)$

2. When $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j) \supseteq \Im^k_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}_r, \mathfrak{q}_s)$ and $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_r, \mathfrak{q}_s) \supseteq \Im^k_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}_i, \mathfrak{q}_j)$, then $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j) = \Im^k_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}_r, \mathfrak{q}_s)$.

Definition 3.6. Let $(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}\}$ be a two person hypersoft game.Then an action $(\mathfrak{p}^*, \mathfrak{q}^*)\in\mathfrak{P} \times \mathfrak{Q}$ is called an optimal action if

$$\Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*) \supseteq \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q}) \text{ for all } (\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}.$$

Definition 3.7. Let $(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}\}$ be a two person hypersoft game.Then,

1. If $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j \supset \Im_{(\mathcal{HS})_k}(\mathfrak{p}_r, \mathfrak{q}_s)$, we says that a player strictly prefers action pair $(\mathfrak{p}_i, \mathfrak{q}_j)$ over action $(\mathfrak{p}_r, \mathfrak{q}_s)$,
2. If $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j) = \Im_{(\mathcal{HS})_k}(\mathfrak{p}_r, \mathfrak{q}_s)$, we says that a player is indifferent between the two actions,
3. $\Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j) \supseteq \Im_{(\mathcal{HS})_k}(\mathfrak{p}_r, \mathfrak{q}_s)$, we says that a player either prefers $(\mathfrak{p}_i, \mathfrak{q}_j)$ to $(\mathfrak{p}_r, \mathfrak{q}_s)$ or is indifferent between the two actions.

Definition 3.8. Let $(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}\}$ be a two person hypersoft game for $k = 1, 2$. Then,

1. If $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q}) = \Phi$ for all $(\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}$, then $(\mathcal{HS})_k$ is called a empty hypersoft game denoted by $(\mathcal{HS})_\Phi$.
2. $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q}) = \mathcal{U}$ for all $(\mathfrak{p}, \mathfrak{q})\in\mathfrak{P} \times \mathfrak{Q}$, then $(\mathcal{HS})_k$ is called a full hypersoft game, denoted by $(\mathcal{HS})_\mathcal{U}$

In hypersoft game theory, the two-person zero sum game from classical game theory will now be a two-person disjoint game. The definition that follows provides it.

Definition 3.9. If there is an empty set at the intersection of the players' hypersoft payoffs for every action pair, the game is referred to as a two-person disjoint hypersoft game.
For instance, example 3.4 is a two person disjoint hypersoft game.

Proposition 3.10. *Let* $(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q}) \in \mathfrak{P} \times \mathfrak{Q}\}$ *be a two person disjoint hypersoft game for* $k = 1, 2$. *Then,*

1. $((\mathcal{HS})_k^c)^c = (\mathcal{HS})_k, k = 1, 2$.
2. $(\mathcal{HS})_1 \setminus (\mathcal{HS})_2 = (\mathcal{HS})_1$
3. $(\mathcal{HS})_2 \setminus (\mathcal{HS})_1 = (\mathcal{HS})_2$
4. $(\mathcal{HS})_1 \cap (\mathcal{HS})_2 = (\mathcal{HS})_\Phi$

Proof. Proof is trivial. □

Definition 3.11. If each action pair's union of the players' hypersoft payoffs is universally established, the game is referred to as a two-person universal hypersoft game.
As an illustration, example 3.4 is a universal hypersoft game for two players.

Proposition 3.12. *Let* $(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q}) \in \mathfrak{P} \times \mathfrak{Q}\}$ *be a two person universal hypersoft game for* $k = 1, 2$. *Then,*

1. $((\mathcal{HS})_k^c)^c = (\mathcal{HS})_k, k = 1, 2$
2. $(\mathcal{HS})_1^c = (\mathcal{HS})_2$
3. $(\mathcal{HS})_2^c = (\mathcal{HS})_1$
4. $(\mathcal{HS})_1 \cup (\mathcal{HS})_2 = (\mathcal{HS})_\mathcal{U}$.

Proof. Proof is straightforward. □

Proposition 3.13. *Let* $(\mathcal{HS})_k = \{((\mathfrak{p}, \mathfrak{q}), \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})) : (\mathfrak{p}, \mathfrak{q}) \in \mathfrak{P} \times \mathfrak{Q}\}$ *be a two person both universal and disjoint hypersoft game for* $k = 1, 2$. *Then,*

1. $(\mathcal{HS})_1 \setminus (\mathcal{HS})_2 = (\mathcal{HS})_1$
2. $(\mathcal{HS})_2 \setminus (\mathcal{HS})_1 = (\mathcal{HS})_2$
3. $(\mathcal{HS})_1 \cap (\mathcal{HS})_2 = (\mathcal{HS})_\Phi$
4. $(\mathcal{HS})_1 \cup (\mathcal{HS})_2 = (\mathcal{HS})_\mathcal{U}$

Proof. Proof is straightforward. □

3.1 Method 1 : Hypersoft Saddle Point Method

Definition 3.14. Let $\Im_{(\mathcal{HS})_k}$ be a hypersoft payoff funtion of a two person hypersoft game $(\mathcal{HS})_k$. If the following propertices hold.

1. $\bigcup_{i=1}^{m} \Im_{(\mathcal{HS})_k}(\mathfrak{p}_\mathfrak{i}, \mathfrak{q}_\mathfrak{j}) = \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$

2. $\bigcap_{j=1}^{n} \Im_{(\mathcal{HS})_k}(\mathfrak{p}_i, \mathfrak{q}_j) = \Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$

Then $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$ is called a hypersoft saddle point value and $(\mathfrak{p}, \mathfrak{q})$ is called a hypersoft saddle point of player k's in the two person hypersoft game.
Note that if $\Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q})$ is a hypersoft saddle point of a two person hypersoft game $(\mathcal{HS})_1$, then player 1 can win atleast by choosing the strategy $\mathfrak{p} \in \mathfrak{P}$ and player 2 can keep loss to atmost $\Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q})$ by choosing the strategy $\mathfrak{q} \in \mathfrak{Q}$. Hence the hypersoft saddle point is a value of the two person hypersoft game.

Example 3.15. Let $\mathcal{R} = \{r_1, r_2, r_3, ..., r_{10}\}$ be a set of alternatives, $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$ be the strategies for player 1 and 2 respectively. Then two person hypersoft game of player 1 is given as in Table 4.

Table 4

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\{r_1, r_4, r_7\}$	$\{r_4,\}$	$\{r_4\}$
$\mathfrak{p}_2$	$\{r_5\}$	$\{r_7\}$	$\{r_4, r_7\}$
$\mathfrak{p}_3$	$\{r_2, r_4, r_5, r_7, r_8, r_{10}\}$	$\{r_4, r_8\}$	$\{r_7, r_8\}$
$\mathfrak{p}_4$	$\{r_2, r_4, r_5, r_7, r_8\}$	$\{r_1, r_4, r_7, r_8\}$	$\{r_4, r_7, r_8\}$

Clearly,

$\bigcup_{i=1}^{4} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_i, \mathfrak{q}_1) = \{r_1, r_2, r_4, r_5, r_7, r_8, r_{10}\}$,
$\bigcup_{i=1}^{4} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_i, \mathfrak{q}_2) = \{r_1, r_4, r_7, r_8\}$,
$\bigcup_{i=1}^{4} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_i, \mathfrak{q}_3) = \{r_4, r_7, r_8\}$,
and
$\bigcap_{j=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_1, \mathfrak{q}_j) = \{r_4\}$,
$\bigcap_{j=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_2, \mathfrak{q}_j) = \Phi$,
$\bigcap_{j=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_3, \mathfrak{q}_j) = \{r_8\}$,
$\bigcap_{j=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_4, \mathfrak{q}_j) = \{r_4, r_7, r_8\}$.

The intersection of the fourth row and the union of the third column make $\{r_4, r_7, r_8\}$ a hypersoft saddle point in the two-person hypersoft game. The two-player hypersoft game value is therefore $\{r_4, r_7, r_8\}$.

Note:
Every two person hypersoft game (tphs-game) has not a hypersoft saddle point. In the above example, if $\{r_4, r_7, r_8\}$ is replaced with $\{r_4, r_7, r_8, r_9\}$ in hypersoft payoff $\Im_{(\mathcal{HS})_1}(\mathfrak{p}_4, \mathfrak{q}_3)$, then a hypersoft saddle point of the game can not be found.

3.2 Method 2 : Hypersoft Upper and Hypersoft Lower Value Method

A tphs-game cannot employ a saddle point; instead, the hypersoft higher and hypersoft lower values of the tphs-game may be used, as stated in the description below.

Definition 3.16. Let $\Im_{(\mathcal{HS})_k}$ be a hypersoft payoff function of a tphs-game $(\mathcal{HS})_k$. Then,

1. Hypersoft upper value of the tphs-game, denoted $\overline{v}$ is defined by
$$\overline{v} = \bigcap_{\mathfrak{q}\in\mathfrak{Q}} \left(\bigcup_{\mathfrak{p}\in\mathfrak{P}} \left(\Im_{(\mathcal{HS})_k}(\mathfrak{p},\mathfrak{q}) \right) \right)$$
2. Hypersoft lower value of the tphs-game, denoted $\underline{v}$ is defined by
$$\underline{v} = \bigcup_{\mathfrak{p}\in\mathfrak{P}} \left(\bigcap_{\mathfrak{q}\in\mathfrak{Q}} \left(\Im_{(\mathcal{HS})_k}(\mathfrak{p},\mathfrak{q}) \right) \right)$$
3. If hypersoft upper and hypersoft lower value of the tphs-game are equal, they are called value of the tphs-game, denoted by v. (ie) $v = \underline{v} = \overline{v}$.

Example 3.17. Let us consider Table 4 in Example 3.15 . It is clear that hypersoft upper value $\overline{v} = \{r_4, r_7, r_8\}$ and hypersoft lower value $\underline{v} = \{r_4, r_7, r_8\}$, hence $\underline{v} = \overline{v}$. It means that value of the tphs-game is $\{r_4, r_7, r_8\}$.

Theorem 3.18. *If $\underline{v}$ and $\overline{v}$ be a hypersoft lower and hypersoft upper value of a tphs-game, respectively. Then the hypersoft lower value is subset or equal to the hypersoft upper value, that is $\underline{v} \subseteq \overline{v}$.*

Proof. Assume that $\underline{v}$ be a hypersoft lower value, $\overline{v}$ be a hypersoft upper value of a tphs-game and $\mathfrak{P} = (\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_m)$ and $\mathfrak{Q} = (\mathfrak{q}_1, \mathfrak{q}_2, ..., \mathfrak{q}_n)$ are sets of the strategies for player 1 and 2 respectively, we choose $\mathfrak{p}_i^* \in \mathfrak{P}$ and $\mathfrak{q}_j^* \in \mathfrak{Q}$. Then

$$\begin{aligned}
\underline{v} &= \bigcup_{\mathfrak{p}\in\mathfrak{P}} \left(\bigcap_{\mathfrak{q}\in\mathfrak{Q}} (\Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p},\mathfrak{q})) \right) \\
&\subseteq \bigcap_{\mathfrak{q}\in\mathfrak{Q}} (\Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}^*,\mathfrak{q})) \\
&\subseteq \Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p}^*,\mathfrak{q}^*) \\
&\subseteq \bigcup_{\mathfrak{q}\in\mathfrak{P}} (\Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p},\mathfrak{q}^*)) \\
&\subseteq \bigcap_{\mathfrak{q}\in\mathfrak{Q}} \left(\bigcup_{\mathfrak{p}\in\mathfrak{P}} (\Im_{\mathfrak{P}\times\mathfrak{Q}}(\mathfrak{p},\mathfrak{q})) \right) \\
&= \overline{v}
\end{aligned}$$

hence $\underline{v} \subseteq \overline{v}$

□

Example 3.19. Let us consider hypersoft upper value $\overline{v}$ and hypersoft lower $\underline{v}$ in example 3.17. It is clear that $\underline{v} = \{r_4, r_7, r_8\} \subseteq \overline{v} = \{r_4, r_7, r_8\}$, hence $\underline{v} \subseteq \overline{v}$.

Remark 3.20. Let $\Im_{(\mathcal{HS})_k}(\mathfrak{p},\mathfrak{q})$ be a hypersoft saddle point, $\underline{v}$ be a hypersoft lower value, $\overline{v}$ be a hypersoft upper value of a tphs-game. Then, $\underline{v} \subseteq \Im_{(\mathcal{HS})_k}(\mathfrak{p}^*,\mathfrak{q}^*) \subseteq \overline{v}$.

Proof. Assume that $\Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*)$ be a hypersoft saddle point, $\underline{v}$ be a hypersoft lower value, $\overline{v}$ be a hypersoft upper value of a tphs-game and $\mathfrak{P} = \{\mathfrak{p_1}, \mathfrak{p_2}, ..., \mathfrak{p}_m\}$ and $\mathfrak{Q} = \{\mathfrak{q_1}, \mathfrak{q_2}, ..., \mathfrak{q}_n\}$ are sets of the strategies for player 1 and 2 respectively.
We choose $\mathfrak{p}_i^* \in \mathfrak{P}$ and $\mathfrak{q}_j^* \in \mathfrak{Q}$.
Since $\Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*)$ is a hypersoft saddle point, we have

$$\bigcup_{i=1}^{m} \Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j}) = \bigcap_{j=1}^{n} \Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j}) = \Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*)$$

$$\underline{v} = \bigcup_{\mathfrak{p} \in \mathfrak{P}} \left(\bigcap_{\mathfrak{q} \in \mathfrak{Q}} (\Im_{\mathfrak{P} \times \mathfrak{Q}}(\mathfrak{p}, \mathfrak{q})) \right)$$

$$\subseteq \bigcup_{i=1}^{m} \Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j})$$

$$= \Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*)$$

$$= \bigcap_{j=1}^{n} \Im_{(\mathcal{HS})_k}(\mathfrak{p_i}, \mathfrak{q_j})$$

$$\subseteq \bigcap_{\mathfrak{q} \in \mathfrak{Q}} \left(\bigcup_{\mathfrak{p} \in \mathfrak{P}} (\Im_{\mathfrak{P} \times \mathfrak{Q}}(\mathfrak{p}, \mathfrak{q})) \right)$$

$$= \overline{v}$$

hence $\underline{v} \subseteq \Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*) \subseteq \overline{v}$.

□

Corollary 3.21. *Let $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$ be a hypersoft saddle point, $\underline{v}$ be a hypersoft lower value and $\overline{v}$ be a hypersoft upper value of a tphs-game. If $v = \underline{v} = \overline{v}$, then $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$ is exactly v.*

Example 3.22. Let us consider hypersoft upper value $\overline{v}$ and hypersoft lower value $\underline{v}$ in example 3.17. It is clear that hypersoft saddle point $\Im_{(\mathcal{HS})_k}(\mathfrak{p}, \mathfrak{q})$ is exactly $v = \underline{v} = \overline{v} = \{r_4, r_7, r_8\}$.

Note: In every tphs-game, the hypersoft lower value $\underline{v}$ can not be equals to the hypersoft upper value $\overline{v}$.
In the above example if $\{r_4, r_7, r_8\}$ is replaced with $\{r_4, r_7, r_8, r_9\}$ in hypersoft payoff $\Im_{(\mathcal{HS})_k}(\mathfrak{p_4}, \mathfrak{q_3})$, then the hypersoft lower value $\underline{v}$ can not be equals to the hypersoft upper value $\overline{v}$.

3.3 Method 3 : Hypersoft Dominated Strategy Method

If in a tphs-game $\underline{v} \neq \overline{v}$, then to get the solution of the game hypersoft dominated strategy may be used. We define hypersoft dominated strategy for tphs-game as follows.

Definition 3.23. Let $(\mathcal{HS})_1$ be a tphs- game with its hypersoft payoff function $\Im_{(\mathcal{HS})_1}$. Then

1. a strategy $\mathfrak{p_i} \in \mathfrak{P}$ is called a hypersoft dominated to another strategy $\mathfrak{p_r} \in \mathfrak{P}$, if $\Im_{(\mathcal{HS})_1}(\mathfrak{p_i}, \mathfrak{q}) \supseteq \Im_{(\mathcal{HS})_1}(\mathfrak{p_r}, \mathfrak{q}))$ for all $\mathfrak{q} \in \mathfrak{Q}$.
2. a strategy $\mathfrak{q_j} \in \mathfrak{Q}$ is called a hypersoft dominated to another strategy $\mathfrak{q_s} \in \mathfrak{Q}$, if $\Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q_j}) \subseteq \Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q_s}))$ for all $\mathfrak{p} \in \mathfrak{P}$.

Rows and columns can be removed from tphs-games in order to shrink them utilising the hypersoft dominating method. Sometimes, this process of getting rid of hypersoft-dominated tactics finds us the answer to a tphs-game. The hypersoft elimination method is the name given to this type of tphs-game solving technique. This approach may be used to solve the following tphs-game.

Example 3.24. Let $\mathcal{R} = \{r_1, r_2, r_3, ..., r_{10}\}$ be a set of alternatives, $\mathfrak{P} = \{\mathfrak{p_1}, \mathfrak{p_2}, \mathfrak{p_3}\}$ and $\mathfrak{Q} = \{\mathfrak{q_1}, \mathfrak{q_2}, \mathfrak{q_3}\}$ be the strategies for player 1 and player 2 respectively.
Then tphs-game of player 1 is given as in Table 5."

Table 5

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\{r_2, r_4, r_7\}$	$\{r_4\}$	$\{r_4\}$
$\mathfrak{p}_2$	$\{r_5\}$	$\{r_7\}$	$\{r_4, r_7\}$
$\mathfrak{p}_3$	$\{r_2, r_4, r_5, r_7, r_8, r_{10}\}$	$\{r_4, r_7, r_8\}$	$\{r_4, r_7, r_8\}$

The last column is dominated by the middle column. After removing the final column, Table 6 looks like this:

Table 6

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_2$
$\mathfrak{p}_1$	$\{r_2, r_4, r_7\}$	$\{r_4\}$
$\mathfrak{p}_2$	$\{r_5\}$	$\{r_7\}$
$\mathfrak{p}_3$	$\{r_2, r_4, r_5, r_7, r_8, r_{10}\}$	$\{r_4, r_7, r_8\}$

Now, as you can see from Table 6 (Note that this is not the case in Table 5), the bottom row dominates the top row. After removing the top row, Table 7 looks like this:

Table 7

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_2$
$\mathfrak{p}_2$	$\{r_5\}$	$\{r_7\}$
$\mathfrak{p}_3$	$\{r_2, r_4, r_5, r_7, r_8, r_{10}\}$	$\{r_4, r_7, r_8\}$

In Table 7, player 1 has a hypersoft dominate strategy $\mathfrak{p}_3$ so that $\mathfrak{p}_2$ is now eliminated. player 2 can now choose between $\mathfrak{q}_1$ and $\mathfrak{q}_2$. Player 2 will clearly choose $\mathfrak{q}_2$.
The solution using the method is ($\mathfrak{p}_3$, $\mathfrak{q}_2$), value of the tphs-game is $\{r_4, r_7, r_8\}$.

3.4 Method 4 : Hypersoft Nash Equilibrium Method

Certain tphs-games without hypersoft dominating tactics are ineligible for using the hypersoft elimination procedure. The hypersoft nash equilibrium, which is described as follows, may be applied in this situation.

Definition 3.25. Let $(\mathcal{HS})_k$ be a tphs-game with its hypersoft payoff function $\Im_{(\mathcal{HS})_k}$ for $k = 1, 2$. If the following properties hold:

1. $\Im_{(\mathcal{HS})_1}(\mathfrak{p}^*, \mathfrak{q}^*) \supseteq \Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q}^*)$ for each $\mathfrak{p} \in \mathfrak{P}$.
2. $\Im_{(\mathcal{HS})_2}(\mathfrak{p}^*, \mathfrak{q}^*) \supseteq \Im_{(\mathcal{HS})_2}(\mathfrak{p}^*, \mathfrak{q})$ for each $\mathfrak{q} \in \mathfrak{Q}$.

Then $(\mathfrak{p}^*, \mathfrak{q}^*) \in \mathfrak{P} \times \mathfrak{Q}$ is a hypersoft nash equilibrium of a tphs-game.

Note that if $(\mathfrak{p}^*, \mathfrak{q}^*) \in \mathfrak{P} \times \mathfrak{Q}$ is a hypersoft nash equilibrium of a tphs-game, then player 1 can win atleast $\Im_{(\mathcal{HS})_1}(\mathfrak{p}^*, \mathfrak{q}^*)$ by choosing strategy $\mathfrak{p}^* \in \mathfrak{Q}$ and player 2 can win atleast $\Im_{(\mathcal{HS})_2}(\mathfrak{p}^*, \mathfrak{q}^*)$ by choosing strategy $\mathfrak{q}^* \in \mathfrak{Q}$. Hence the hypersoft nash equilibrium is an optimal action for tphs-game, therefore $\Im_{(\mathcal{HS})_k}(\mathfrak{p}^*, \mathfrak{q}^*)$ is the solution of the tphs-game for player k, $k = 1, 2$.
The following tphs-game can be solved by using this method.

Example 3.26. Let $\mathcal{R} = \{r_1, r_2, r_3, ..., r_{10}\}$ be a set of alternatives, $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$ be the strategies for player 1 and player 2 respectively.
Then tphs-game of player 1 is given as in Table 8.

Table 8

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\{r_1, r_2, r_4, r_7, r_8, r_9\}$	$\{r_1, r_2, r_4, r_7, r_8\}$	$\{r_1, r_2, r_3, r_4, r_7, r_8\}$
$\mathfrak{p}_2$	$\{r_1, r_2, r_3, r_5\}$	$\{r_1, r_4, r_7, r_8\}$	$\{r_1, r_2, r_3, r_4, r_5, r_9\}$
$\mathfrak{p}_3$	$\{r_2, r_5, r_7, r_8, r_{1o}\}$	$\{r_2, r_4, r_7, r_8\}$	$\{r_4, r_5, r_7, r_8, r_{10}\}$

and tphs-game of player 2 is given as in Table 9.

Table 9

$(\mathcal{HS})_2$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\{r_3, r_5, r_6, r_{10}\}$	$\{r_3, r_5, r_6, r_9, r_{10}\}$	$\{r_5, r_6, r_9, r_{10}\}$
$\mathfrak{p}_2$	$\{r_4, r_6, r_7, r_8, r_9, r_{10}\}$	$\{r_2, r_3, r_5, r_6, r_9, r_{10}\}$	$\{r_6, r_8, r_9, r_{10}\}$
$\mathfrak{p}_3$	$\{r_1, r_3, r_4, r_6, r_9\}$	$\{r_1, r_3, r_5, r_6, r_9, r_{10}\}$	$\{r_1, r_2, r_3, r_6, r_9\}$

From the tables we have

1. $\Im_{(\mathcal{HS})_1}(\mathfrak{p}_1, \mathfrak{q}_2) \supseteq \Im_{(\mathcal{HS})_1}(\mathfrak{p}, \mathfrak{q}_2)$ for each $\mathfrak{p} \in \mathfrak{P}$.
2. $\Im_{(\mathcal{HS})_2}(\mathfrak{p}_1, \mathfrak{q}_2) \supseteq \Im_{(\mathcal{HS})_2}(\mathfrak{p}_1, \mathfrak{q})$ for each $\mathfrak{q} \in \mathfrak{Q}$.

Thus, $(\mathfrak{p}_1, \mathfrak{q}_2) \in \mathfrak{P} \times \mathfrak{Q}$ is a hypersoft nash equilibrium.
so, $\Im_{(\mathcal{HS})_1}(\mathfrak{p}_1, \mathfrak{q}_2) = \{r_1, r_2, r_4, r_7, r_8\}$ and
$\Im_{(\mathcal{HS})_2}(\mathfrak{p}_1, \mathfrak{q}_2) = \{r_3, r_5, r_6, r_9, r_{10}\}$
is the solution of the tphs-game for player 1 and player 2 respectively.

4 An Application

This section presents a real-world problem that is handled by combining the hypersoft saddle point approach and hypersoft dominating strategy. There are two businesses, let's call player 1 and player 2, that are in competition with one another to boost product sales in the nation. Therefore, they give advertisements. Assume that two companies have a set of different products $\mathcal{R} = \{r_1, r_2, r_3, ..., r_8\}$. For $i = 1, 2, 3, ...8$, the product r_i stands for Scanner, Printer, Monitor, Mouse, Keyboard, Hard Disc, Computer and Laptop respectively.
The products can be characterized by a set of strategy $\mathfrak{P} = \mathfrak{Q} = \{\mathfrak{p}_i, i = 1, 2, 3\}$ which contains styles of advertisement. For $i = 1, 2, 3$, the strategies $\mathfrak{p}_i$ stands for Social Media, Newspaper and Television respectively. Suppose that $\mathfrak{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1 = \mathfrak{p}_1, \mathfrak{q}_2 = \mathfrak{p}_2, \mathfrak{q}_3 = \mathfrak{p}_3\}$ are strategies of players 1 and 2 respectively. Then tphs-game of player 1 is given as in Table 10.

Table 10

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\{r_1, r_2, r_3, r_5, r_8\}$	$\{r_1, r_2, r_3, r_4, r_5, r_8\}$	$\{r_3\}$
$\mathfrak{p}_2$	$\{r_1, r_3, r_7\}$	$\{r_1, r_2, r_3, r_5, r_6, r_7\}$	$\{r_2, r_3\}$
$\mathfrak{p}_3$	$\{r_1, r_2, r_3, r_4, r_5\}$	$\{r_1, r_2, r_3, r_4, r_5, r_6, r_8\}$	$\{r_1, r_2, r_3\}$

In Table 10, let us explain action pair $(\mathfrak{p}_2, \mathfrak{q}_3)$:

If player 1 select $\mathfrak{p}_2$ = "Newspaper" and
player 2 select $\mathfrak{q}_3$ = "Television"

then the hypersoft payoff of player 1 is a set $\{r_2, r_3\}$. ie. $\Im_{(\mathcal{HS})_1}(\mathfrak{p}_2, \mathfrak{q}_3) = \{r_2, r_3\}$.
In this case , player 1 increase the sale of $\{r_2, r_3\}$ and player 2 decrease sale of $\{r_2, r_3\}$.
Now we solve the game. It is seen in Table 10.

$\{r_1, r_2, r_3, r_5, r_8\} \subseteq \{r_1, r_2, r_3, r_4, r_5, r_8\}$
$\{r_1, r_3, r_7\} \subseteq \{r_1, r_2, r_3, r_5, r_6, r_7\}$
$\{r_1, r_2, r_3, r_4, r_5\} \subseteq \{r_1, r_2, r_3, r_4, r_5, r_6, r_8\}$

the middle column is dominated by the right column. We then deleting the middle column we obtain Table 11.

Table 11

$(\mathcal{HS})_1$	$\mathfrak{q}_1$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\{r_1, r_2, r_3, r_5, r_8\}$	$\{r_3\}$
$\mathfrak{p}_2$	$\{r_1, r_3, r_7\}$	$\{r_2, r_3\}$
$\mathfrak{p}_3$	$\{r_1, r_2, r_3, r_4, r_5\}$	$\{r_1, r_2, r_3\}$

In Table 11, there is no another hypersoft dominated strategy, we can use hypersoft saddle point method.

$$\bigcup_{i=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_i, \mathfrak{q}_1) = \{r_1, r_2, r_3, r_4, r_5, r_7, r_8\}$$
$$\bigcup_{i=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_i, \mathfrak{q}_3) = \{r_1, r_2, r_3\}$$
$$\bigcap_{j=1,3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_1, \mathfrak{q}_j) = \{r_3\}$$
$$\bigcap_{j=1,3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_2, \mathfrak{q}_j) = \{r_3\}$$
$$\bigcap_{j=1,3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_3, \mathfrak{q}_j) = \{r_1, r_2, r_3\}.$$

Here optimal strategy for the game is $(\mathfrak{p}_3, \mathfrak{q}_3)$.
Since $\bigcup_{i=1}^{3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_i, \mathfrak{q}_3) = \bigcap_{j=1,3} \Im_{(\mathcal{HS})_1}(\mathfrak{p}_3, \mathfrak{q}_j)$.
Therefore, value of the tphs-game is $\{r_1, r_2, r_3\}$.

5 n-Person Hypersoft Games

The hypersoft games may frequently be played by more than two players in numerous applications. Consequently, it is possible to expand two-person hypersoft games (tphs-game) to n-person hypersoft games (nphs-game).

Definition 5.1. Let $\mathcal{U}$ be a set of alternatives, $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathfrak{P}_k$ is the set of strategies of player k, $(k = 1, 2, ..., n)$. Then for each player k, an n-person hypersoft game (nphs-game) is defined by a hypersoft set over $\mathcal{U}$ as:
$(\mathcal{HS})_k^n = \{((\mathfrak{p}_1, \mathfrak{p}_2, , .., \mathfrak{p}_n), \Im_{(\mathcal{HS})_k^n}(\mathfrak{p}_1, \mathfrak{p}_2, , .., \mathfrak{p}_n)) : (\mathfrak{p}_1, \mathfrak{p}_2, , .., \mathfrak{p}_n) \in (\mathfrak{P}_1 \times \mathfrak{P}_2 \times ... \times \mathfrak{P}_n)\}$
where $\Im_{(\mathcal{HS})_k^n}$ is a hypersoft payoff function of player k.
The nphs-game is played as follows:
At a certain time player 1 choose a strategy $\mathfrak{p}_1 \in \mathfrak{P}_1$ and simultaneously each player k, $(k = 2, 3...n)$ choose a strategy $\mathfrak{p}_k \in \mathfrak{P}_k$ and once this is done each player k receives the hypersoft payoff $\Im_{(\mathcal{HS})_k^n}(\mathfrak{p}_1, \mathfrak{p}_2, , .., \mathfrak{p}_n)$.

6 Conclusion

We define two-person hypersoft games (tphs-game) with hypersoft payoffs in this work. In addition, we provided four approaches to solving the tphs-games and an example demonstrating how the approaches may be effectively used to solve real-world issues. Lastly, we expanded the hypersoft games from two players (tphs-games) to n players (nphs-games). In the future, we may use nphs-games to solve decision-making issues related to daily life.

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An Efficient SuperHyperSoft Framework for Evaluating LLMs-based Secure Blockchain Platforms

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Abstract: In the age of modern technology, the use of the internet has become imperative. However, this widespread access presents a double-edged sword and opens doors for hackers and scammers to exploit vulnerabilities and engage in illegal activities. Accordingly, scholars and stakeholders are attempting to solve this matter. Large Language Models (LLMs) have provided highly effective methodologies and solutions in various cybersecurity sectors. Hence, we exhibited the efficacy of LLMs in several information and communication technologies (ICT) such as the Internet of Things (IoT), cloud computing, blockchain technology (BCT)…etc which are attacked and threatened. Accordingly, the objective of our study is to illustrate how LLMs are supporting ICT, especially BC to be secure against attacks. Another study's objective is to aid the stakeholders and enterprises that seek resilience and sustainability by recommending the most secure BC platform to adopt in critical sectors. Wherein, LLMs support BC in many directions as developing secure smart contracts and scanning the smart contract to protect it from any subversive acts by identifying anomalous activities. Hence, we suggested a soft opting model to rank the alternatives of BC platforms and recommend optimal BC. Also, the process of constructing this model requires leveraging several techniques. We applied for the first time SuperHyperSoft (SHS) as an extension of Hypersoft to treat various attributes and sub-attributes for BC based on LLMs. Multi-criteria decision-making (MCDM)techniques are utilized for their ability to treat conflicting sub-attributes. Hence, entropy and multi-objective optimization based on simple ratio analysis (MOOSRA) are utilized as techniques of MCDM. These techniques are working under the authority of the Single Value Neutrosophic (SVN) technique to support MCDM techniques in ambiguous situations.

Keywords: Large Language Models (LLMs); SuperHyperSoft; Blockchain; Cybersecurity; Multi-Criteria Decision Making (MCDM); Single Value Neutrosophic (SVN)

1. Introduction

The desire for sustainability and development on a global scale is seen as the driving force behind change and the growing rate of technological advancement. Indeed, in the recent interval, a new phenomenon dubbed Industry 4.0 [1] and eventually, Industry 5.0 has evolved.

These digital transformations permit devices and humans to communicate by utilizing smart devices as sensors in the Industrial Internet of Things (IIoT)[2] to collect data and process it through big data analytical (BDA) and store it in blockchain technology (BCT). Despite the importance of the technologies of Industry 4.0 in transferring and exchanging data, it is vulnerable to attacks. Accordingly, [3] indicated that the usage of cutting-edge technologies has expanded over the past several decades, leading to a significant increase in cyber-attacks and threats. This issue affects all stakeholders who utilize IoT systems, either directly or indirectly. Malicious assaults may cause enormous financial difficulties and incalculable losses, including data corruption, system breakdowns, privacy breaches, reputational damage, lost customers, dependability, and market share, especially for large enterprises. Wherein [4] claimed that only 16% of experts believe that their business is well-prepared to handle cybersecurity threats, even though 75% see cybersecurity as a priority.

Hence, [5] stated that it becomes imperative to have sophisticated and efficient detecting techniques. Similarly[6], as cyber threats evolve, the cybersecurity field can also benefit from state-of-the-art tools. These tools can support cybersecurity practitioners who are always looking for ways to apply new regulations or fortify technological defenses against the leakage of private data, illegal access, and other types of data abuse. According to the perspective of [7], it is imperative to develop proactiv5[5[5e technology for cyber defense. Wherein the defense is represented in harnessing various cutting-edge technologies such as Machine Learning (ML) techniques [5] and deep learning (DL) techniques [8] which are capable of automatically identifying, thwarting, also responding to various cyberattacks[9]. As well the new developments in transformers and large language models (LLMs)[10], which have demonstrated remarkable capabilities in natural language understanding, generation, and reasoning, have made significant strides in several security tasks, including threat detection, automated vulnerability analysis, intelligent defense mechanisms [11]. LLMs have a number of advantages, from customization and transparency to cutting-edge performance. LLMs are classified into open-source models like Llama [12] and Mixtral [13] or closed-source models like ChatGPT and Gemini[14]. They do, however, all have their limitations. Code-based LLMs, such as CodeLlama [15] and StarCoder [16, 17], are especially useful for tasks including automated code review, secure code development, and bug discovery. Their use in threat intelligence, binary analysis, IT operations, vulnerability detection, program repair, and anomaly detection highlights their revolutionary influence on cybersecurity.

1.1 Journey of Language Models Toward Large Language Models: Provenance

Contemporary techniques of artificial intelligence and machines have amalgamated to learn how to comprehend and communicate through human language [18] where a term of language modeling (LM) is the consequence of this amalgamation. This term has appeared in many

studies as [19] LM predominantly strives to forecast the probability of future or omitted tokens by modeling the generative probability of word sequences. As well [20] described the language model's evolution phases. The first phase formed in **statistical language models (SLMs)** [21]this model was initially introduced in the 1990s, and as it was constructed using the Markov assumption, its goal was to anticipate the next word based on the most recent context. The other side of it is bigram and trigram language models [19] where SLMs with a predetermined context length n that's why it's called n-gram language models. Generally, this is not a sufficient model, but newer versions have appeared due to some problems that this model suffered from. Worthwhile[22], neural networks (NNs) serve to predict the probability distribution of the following word in a sequence given the words that came before it as manner in phase two of **Neural Language Models (NLMs).** Wherein, the models of NLs are employed in this phase as multi-layer perceptron (MLP), Recurrent neural networks (RNNs), and (LSTM) [23]to provide a cohesive, comprehensive solution for different NLP. Phase three entails building what are known as **pre-trained language models (PLMs)** [24], which are contextualized word embeddings that seek to capture the meaning and context of words in a phrase or text. Nonetheless, several investigations have [25] examined the performance limitations PLMs encounter when training progressively broader PLMs. Thereby, Generative Pre-Trained Transformer -3 (GPT-3) and Generative Pre-Trained Transformer -4 (GPT-4) were developed to address the deficiencies in phase three. Due to[26], these models can be refined for certain downstream tasks, such as question answering or language translation, after being trained on vast volumes of text data. Accordingly, such models belong to **transformer language models [19]**, in another formula LLMs are considered motivators for phase four. Due to its ability to comprehend natural language and complete challenging tasks.

1.2 Motivations and incentives of study

Each phase finds a way to address the faults of the one before it, based on the discussion from the preceding subsection. In the final phase of LMs, GPT-3 as the model belongs to LLMs can solve a series of complex tasks that GPT-2 can't solve[27]. One of PLM's drawbacks[28], is it cannot generalize to unseen activities without task-specific training, although LLMs can do so without the need for task-specific training. As a result, LLMs serve as the study's focal point and foundation; further motivations will be covered.

A. **First incentive:** we conducted bibliometric analysis for various LMs. This analysis has been conducted on the Web of Science (WoS) database for various published studies from 2020 until 2024. Hence, we conducted a set of queries. Wherein these queries entailed the various techniques for different phases of LMs that were utilized in published studies.

B. These queries were conducted as in Fig 1

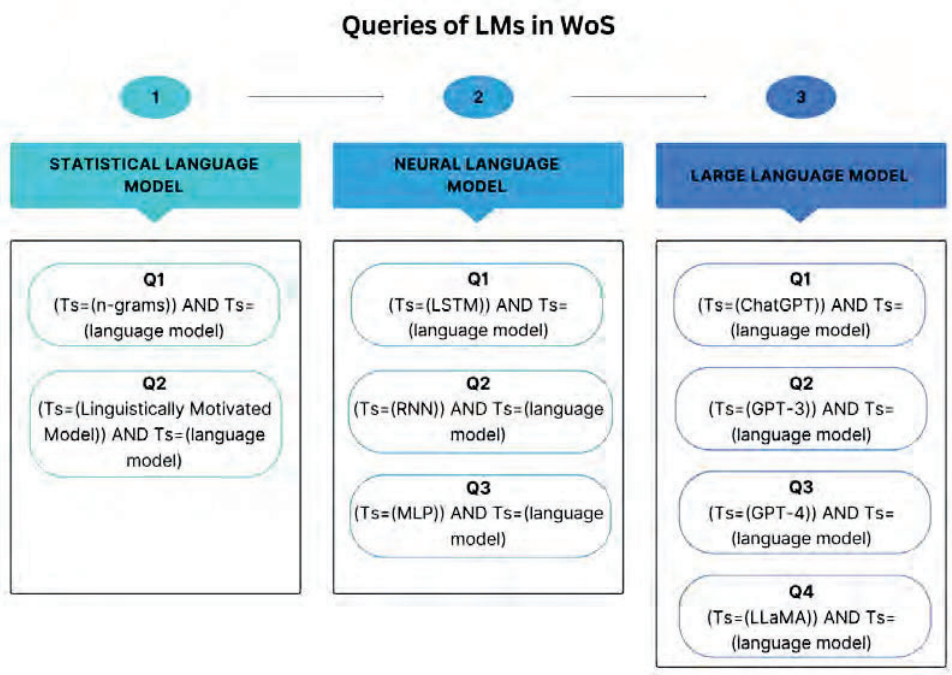
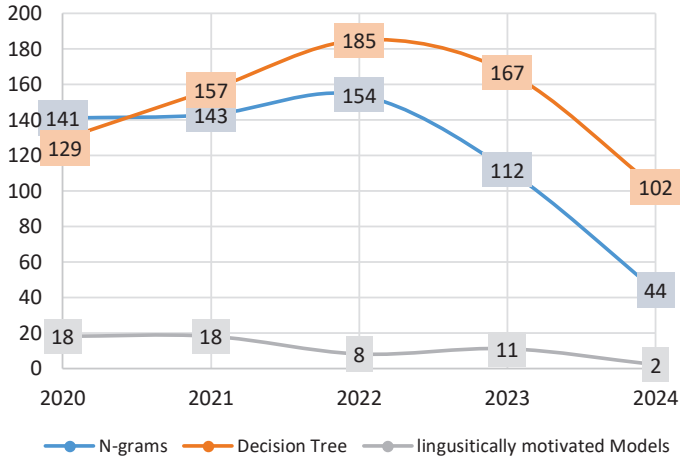


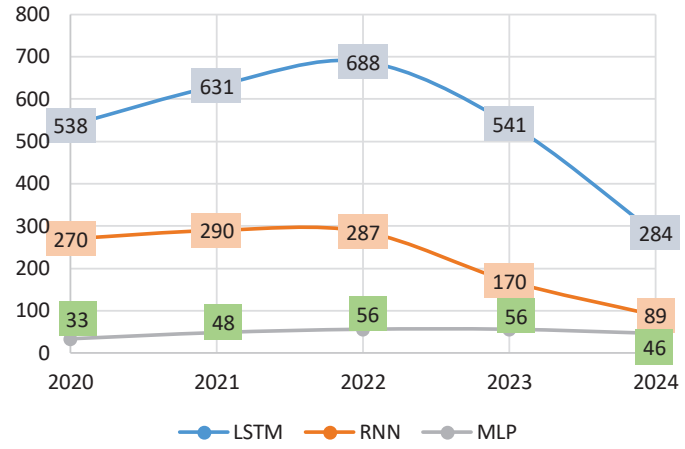
Fig 1. Methodologies of Techniques of Language Models Queries

The findings of conducted queries are illustrated in Fig 2 which comprises the queries' conclusions. This Fig indicated that the techniques of LLMs are constantly increasing over the years. In confirmation of this, in 2024 these techniques achieved the highest use in publishing studies compared to the other models of LMs.

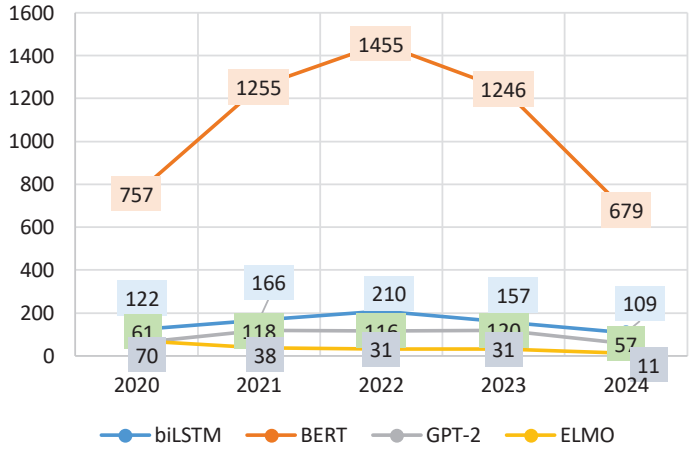
A. Second incentive: business environment and its chain, other domains such as healthcare, transportation, education…etc. are transforming their traditional strategies, planning, and operations into digital. In this light, [29] demonstrated that the urgency for effective security precautions has increased as individuals and enterprises depend on digital technology for essential infrastructure, interactions, and business. Thereby [30] stated that it is challenging for security practitioners to successfully identify, locate, and protect against them due to the magnitude and diversity of cyber threats. To address these issues, [31] suggested LLMs and [5] confirmed that LLMs help to swiftly find relevant threat intelligence, allowing consultants to decide on actions. Accordingly, [32]discussed various roles and contributions of LLMs in cybersecurity. Also [33] suggested Bidirectional Encoder Representations from Transformers (BERT) as an intrusion detection model and BERT outperforms other ML techniques in attack detection on a reputable dataset.



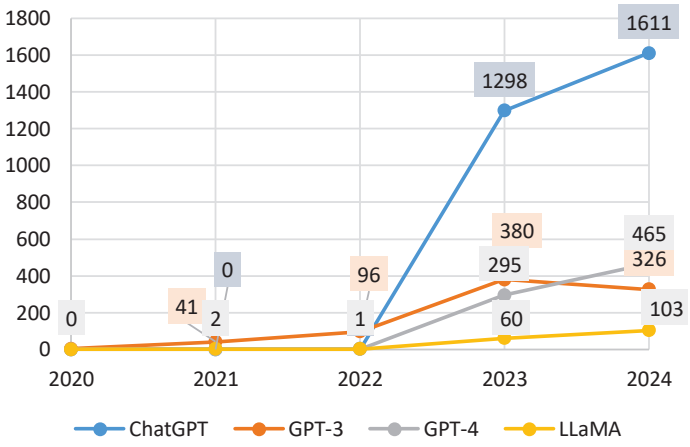
(a) Techniques of Statistical Language Models



(b) Techniques of Neural Language Model



(c) Techniques of Pretrained Language Model



(d) Techniques of Large Language Model

Fig 2. Queries' Findings for Language Models since 2020-2024

1.3 Scientific Contributions and Novelty:

This study attempts to solve the issue of cybersecurity by covering certain aspects as the following:

Theoretically: we conducted surveys in previous studies that related to our scope. We are trying to discuss and determine the main issues of security that threaten the enterprises economically and personally. Accordingly, we are studying the extent to which methods can solve this matter. The findings of our study are deployed in the next aspect (i.e. scientifically).

Scientifically: the findings of surveys are discussed by exhibiting the importance of generative artificial intelligence (AI)-LLMs for securing ICT technologies, especially BC. Also, evaluating the BC that adopting the models of LL is vital. Therefore, we proposed a soft opting model to evaluate these BC.

Practically: we applied the constructed model for two objectives. The first objective is aiding decision makers (DMs) and stakeholders at a loss to choose the secure BC by recommending secure BC based on LLMs. The second objective is to validate the accuracy of our model.

2. Empowering ICT technologies in the realm of LLMs

This section exhibits the role of LLMs in ICT technologies as security tools in various forms.

2.1 Cloud Computing Paradigm and IoT

Many scholars as[34, 35] described cloud computing (CC) as the multi-service paradigm for storing and retrieving data, computational capacity, and applications, such as internet-based on-demand services. CC has been harnessed successfully in several contexts Given that it offers a wide range of services such as control and management through various methodologies such as Infrastructure as a Service (IaaS), Platform as a Service (PaaS), and Software as a Service (SaaS).

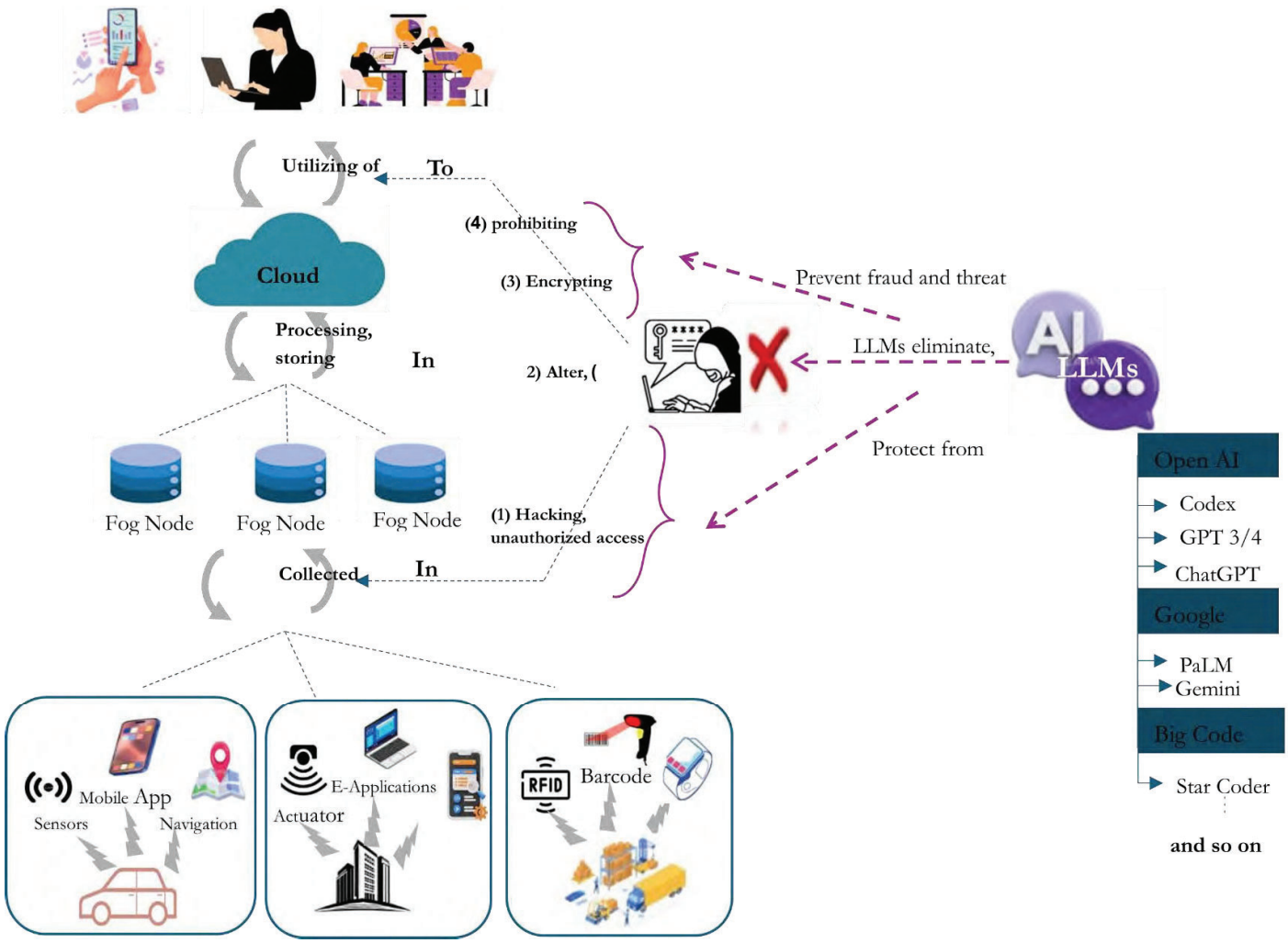


Fig 3. Role of Large Language Model in Securing Cloud-IoT against Threat

This paradigm integrated with other Contemporary technologies such as machine learning (ML) [36]to enhance the privacy, trust, and security of the cloud paradigm. But Cao et al., [37]had an alternative viewpoint and demonstrated that ML techniques are restricted to vulnerability analysis and categorization. Accordingly, the authors in [37] proposed LLMs as solvers for vulnerabilities of ML for securing the cloud. In the same vein [38] harnessed the CC paradigm in the internet of Vehicles (IoV) to store and process data collected from smart equipment as sensors in the Cloud paradigm and vice versa for retrieving data from the cloud. Wherein, LLMs play an important role as cybersecurity tools to countermeasure any threats and through [39] LLMs preserve resilience by facing hostile assaults, illegal access attempts, and data leaks. Fig 3 exhibits the role of LLMs which is discussed in [11] in securing data collected from IoT equipment and preventing hackers and spoofers from intrusion of information through unauthorized access. Also, LLMs are protecting information from encrypting, losing, and fraudulent vital information.

2.2 *Blockchain Technology*

Latterly, BCT is growing in popularity as a technology for promoting different domains[40]. As well, [41] indicated that Financial institutions brought in BCT to bolster their cybersecurity and investments. Distributed ledger (DL) technology allows BC to store transactions immutably. As stated by [42], BC security is an elaborate process that aims to protect the availability, integrity, and confidentiality of data processed and stored inside a blockchain network. Nevertheless, [43]admitted that the integrity and responsibilities of BC are in danger from several security flaws and threats. These threats formed in various forms as in [44] (see Fig 4) that exhibit vulnerabilities that jeopardize BC security.

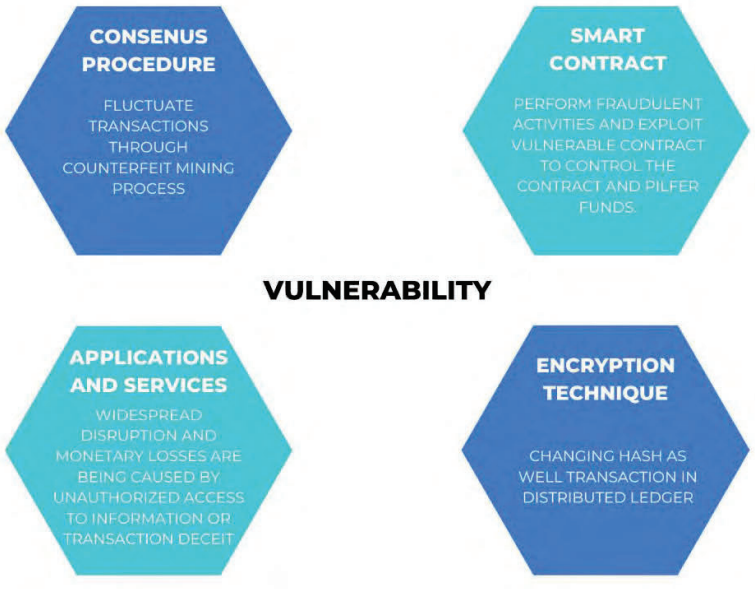


Fig 4. Blockchain Vulnerabilities

To eliminate and avoid the issues of vulnerability that jeopardize BC security, LLMs such as GPTScan [45], and AuditGPT [46] are introduced as auditors in smart contracts to detect and recognize vulnerabilities. As well, LLMs were introduced as detectors for unusual transactions and deceitful actions via leveraging BLOCKGPT[47].

3. Methodology

According to the influential role of LLMs as protectors against threats and risks, the evaluation of their roles in contemporary technology to be a secure technology is an important process. Accordingly, choosing the most secure technology amongst the set of alternatives to other technologies to serve the enterprises' objectives is crucial. This process requires prioritizing between the alternatives based on a set of conflicting criteria. Hence, MCDM techniques have been harnessed in this process due to their ability to treat such problems [48]. Yet experts may run into the issue of being unable to cope with uncertainties when using MCDM techniques, which prevents experts from being able to make an appropriate decision. To avert this issue, the SuperHyperSoft (SHs) technique is integrated with SVNSs to support experts when using MCDM techniques during the evaluation process amongst alternatives. Another impulse for this integration is to serve the study's objectives by constructing a soft opting model. These objectives entailed selecting and recommending the most secure BC platform that deploys LLMs as ChatGPT for developing smart contracts. The accuracy and security of smart contracts, which implement agreements without the need for middlemen, are crucial in BC.
Overall, the procedures of recommending a secure BC platform based on a soft opting model are conducted through the following subsections.
laying forth the assessment process's aspects

3.1 Laying forth the assessment process's main aspects

1. The alternatives of BC platforms are determined to be nominated in the evaluation process.
2. The main attributes and sub-attributes are determined to evaluate BC platforms based on them.
3. The expert panel is forming to contribute to the evaluation process for BC platforms based on attributes and sub-attributes.

3.2 Generative of attributes and sub-attributes weights: SVNSs-Entropy

The objective of this procedure is to generate weights for BC platforms' attributes and sub-attributes. Hence, the entropy of MCDM techniques is combined with SVNSs to generate weights. The generated weights are harassing in the next procedure of alternatives ranking. We implement a series of phases to accomplish the objective.

4. Transforming the utilized linguistic terms of DMs into Neutrosophic values based on the SVN scale which is mentioned in [49]. Thereby, Neutrosophic decision matrices are constructed based on the rating of DMs.
5. The score function in Eq.(1) is embraced for converting the constructed matrices into crisp matrices.

$$s(\sigma_{ij}) = \frac{(2+ \alpha - \beta - \theta)}{3} \quad (1)$$

Where α, β, θ refer to truth, false, and indeterminacy respectively.

6. Aggregate the crisp matrices into a compiled matrix based on Eq.(2).

$$\wp_{ij} = \frac{(\sum_{j=1}^{N} \sigma_{ij})}{Z} \quad (2)$$

Where σ_{ij} refers to the value of the criterion in the matrix, and Z refers to the number of decision-makers.

7. Eq.(3) is utilized in the compiled matrix to normalize it to construct a normalized matrix.

$$\mathbb{N}or_{ij} = \frac{\wp_{ij}}{\sum_{j=1}^{m} \wp_{ij}} \quad (3)$$

Where $\sum_{j=1}^{m} \wp_{ij}$ indicates the sum of each criterion in the compiled matrix per column.

8. Eq.s(4),(5) are contributed to compute entropy.

$$En_{j=-h\sum_{i=1}^{m} \mathbb{N}or_{ij}} \ln \mathbb{N}or_{ij} \quad (4)$$

where,

$$h = \frac{1}{\ln(N)} \quad (5)$$

N refers to utilized alternatives

9. Finally, the weights of attributes are generated by employing Eq. (6)

$$\omega_j = \frac{1 - En_j}{\sum_{j=1}^{n}(1 - En_j)} \quad (6)$$

3.3 Recommending the most secure BC platform: SuperHyperSoft and SVNSs-MOOSRA

Herein, three techniques are integrated to rank BC platforms. Each technique plays a vital role in solving the problem of selection. SHS is utilized to determine and employ a power set of attributes to obtain the most secure and appropriate BC platform. While SVNSs are harnessed for supporting MOOSRA in uncertain environments and ambiguity of information during the ranking process.

3.3.1 SuperHyperSoft (SHS)

SHs introduced by Smarandache [50] this technique is considered an extension of HyperSoft and consists of several HyperSoft Sets. SHS was utilized in this study as the methodology for representing the determined attributes and sub-attributes for selecting the secure BC platform based on the selected attributes and sub-attributes.

- Suppose the universe set $\mathfrak{R}$= { BC_1, BC_2,… BC_n} which was determined in the previous procedure. Moreover, P ($\mathfrak{R}$) is the powerset of $\mathfrak{R}$. As well, A_1, A_2, A_3 are utilized attributes where BC platforms have been evaluated over these attributes. Hence, $P(A_1)$, $P(A_2)$, and $P(A_3)$ are powersets of A_1, A_2, A_3.
- Let F: $P(A_1) \times P(A_2) \times P(A_3) \rightarrow P(\mathfrak{R})$, where × indicates to Cartesian product. Hence, this is called SHs over $\mathfrak{R}$.
- In this study Cartesian product for attributes and sub−attributes formed as

$$P(A_1) \times P(A_2) \times P(A_3) = \Big\{\{\{A_{11}\},\{A_{12}\},\{A_{11},A_{12}\}\} \times \{\{A_{21}\},\{A_{22}\},\{A_{21},A_{22}\},\} \times \{\{A_{31}\},\{A_{32}\},\{A_{33}\},\{A_{31},A_{32}\},\{A_{31},A_{33}\},\{A_{32},A_{33}\},\{A_{31},A_{32},A_{33}\}\}\Big\}.$$

- According to Eq. (7), $P(A_1) \times P(A_2) \times P(A_3) =$

$$\begin{pmatrix} \widetilde{s_1}\ (A_{11},A_{21},A_{31}); \widetilde{s_2}\ (A_{11},A_{21},A_{32}); \widetilde{s_3}\ (A_{11},A_{21},A_{33}); \\ \widetilde{s_4}\ (A_{11},A_{21},\{A_{31},A_{32}\}); \widetilde{s_5}\ (A_{11},A_{21},\{A_{31},A_{33}\}); \widetilde{s_6}\ (A_{11},A_{21},\{A_{32},A_{33}\}); \\ \widetilde{s_7}\ (A_{11},A_{21},\{A_{31},A_{32},A_{33}\}); \widetilde{s_8}\ (A_{11},A_{22},A_{31}); \dots \widetilde{s_{14}}\ (A_{11},A_{22},\{A_{31},A_{32},A_{33}\}); \\ \widetilde{s_{15}}\ (A_{11},\{A_{21},A_{22}\},A_{31}); \dots \widetilde{s_{21}}\ (A_{11},\{A_{21},A_{22}\},\{A_{31},A_{32},A_{33}\}); \widetilde{s_{22}}\ (A_{12},A_{21},A_{31}); \cdots \\ \widetilde{s_{28}}\ (A_{12},A_{21},\{A_{31},A_{32},A_{33}\}); \widetilde{s_{29}}\ (A_{12},A_{22},A_{31}); \cdots \widetilde{s_{35}}\ (A_{12},A_{22},\{A_{31},A_{32},A_{33}\}); \\ \widetilde{s_{36}}\ (A_{12},\{A_{21},A_{22}\},A_{31}); \ \cdots \widetilde{s_{42}}\ (A_{12},\{A_{21},A_{22}\},\{A_{31},A_{32},A_{33}\}); \widetilde{s_{43}}\ (\{A_{11},A_{12}\},A_{21},A_{31}); \cdots \\ \widetilde{s_{49}}\ (\{A_{11},A_{12}\},A_{21},\{A_{31},A_{32},A_{33}\}); \widetilde{s_{50}}\ (\{A_{11},A_{12}\},A_{22},A_{31}); \cdots \widetilde{s_{56}}\ (\{A_{11},A_{12}\},A_{22},\{A_{31},A_{32},A_{33}\}); \\ \widetilde{s_{57}}\ (\{A_{11},A_{12}\},\{A_{21},A_{22}\},A_{31}); \cdots \widetilde{s_{63}}\ (\{A_{11},A_{12}\},\{A_{21},A_{22}\},\{A_{31},A_{32},A_{33}\}). \end{pmatrix} \quad (7)$$

3.3.2 SVNSs-MOOSRA

Multi-objective optimization based on simple ratio analysis (MOOSRA)[51] is implemented under the authority of SVNSs based on SHS for ranking BC platforms that adopt ChatGPT-LLMs for developing secured smart contracts. After that recommend the most secure BC platform which implements the secured smart contract. Overall, the ranking procedures have been conducted as follows:

10. Constructing Neutrosophic decision matrices for each DM to evaluate BC platforms based on sub-attributes determined by SHS.
11. Transforming Neutrosophic decision matrices into de-neutrosophic decision matrices through using Eq.(1).
12. Eq.(2) is utilized for the second time to aggregate these matrices into an aggregated matrix.
13. Normalizing the aggregated matrix based on Eq.(8).

$$\mathrm{Nor}_{ij} = \frac{\wp_{ij}}{\left[\sum_{j}^{m} \wp_{ij}\right]^{1/2}} \quad (8$$

14. Compute a weighted decision matrix based on Eq.(9).

$$\mathrm{weighted} - \mathrm{matrix}_{ij=}\mathrm{Nor}_{ij} * \omega_j \quad (9)$$

15. Calculating ratio as in Eq.(10) to obtain final rank for alternatives.

$$\mathrm{Ratio} = \frac{\sum_{j=1}^{B} \mathrm{Nor}_{ij}}{\sum_{j=1}^{NB} \mathrm{Nor}_{ij}} \quad (10$$

4. Implementation of the constructed soft opting model: Case-study

Herein we implemented our constructed soft opting model in realistic to validate the efficacy of the constructed model.

4.1 Problem Description

We exhibit the problem that our constructed model is supposed to be applied through the following scenario.

Scenario 1
One of the most significant technologies being utilized right now across a variety of sectors in BC, particularly in sectors with a long history and significant consumer data. This is because of the utilization of robust encryption techniques such as hash. Yet, because of the extensive usage of the Internet and its applications, hackers may now easily forge hash mining to alter and steal data, as well as smart contracts that expose the money and data of consumers to jeopardy. Hence, the business sectors exploit generative AI capabilities as LLMs to bolster the efficiency of BC as ChatGPT, PaLM…etc. these models of LLM support enterprises to be proactive and become resilient. Therefore, any business aiming for sustainability and resilience must

implement the strongest BC practices involving LLMs. Accordingly, businesses may face the problem of applying the strongest and most secure BC platform. We exhibit the problem of four enterprises that need to make accurate decisions for selecting the most secured BC platform based on LLMs amongst five alternatives of BC platforms.

Action: the businesses should follow the methodology for selecting the optimal and most secured BC which deploys the robust security models based on LLM. Herein, this methodology entails implementing the constructed soft opting model to aid stakeholders in deciding the optimal decision with various propositions toward securing businesses based on robust BC.

4.2 Toward Most Secure BC platform-based LLMs: Procedures

Herein, the process of recommending optimal BC has been conducted into two subsections for generating attributes' weights. These weights are used in another subsection to recommend optimal BC platforms.

4.1.1 Generating attributes' weights

Entropy based on SVNSs is implemented to serve the target of this subsection.

1. Three Neutrosophic decision matrices are constructed based on members of the panel to evaluate five BC platforms based on four attributes (see Fig 4). These matrices are transformed into de-neutrosophic matrices based on Eq.(1).
2. These matrices aggregated into the single matrix is an aggregated matrix as listed in Table 1 based on Eq.(2).
3. Table 2 exhibits the normalized matrix which is calculated based on Eq.(3).
4. Eq.(4) is implemented to compute entropy.
5. The final attributes' weights are obtained in Fig5 through executing Eq.(6) which indicates that A2 has the highest weight otherwise, A1 has the lowest value.

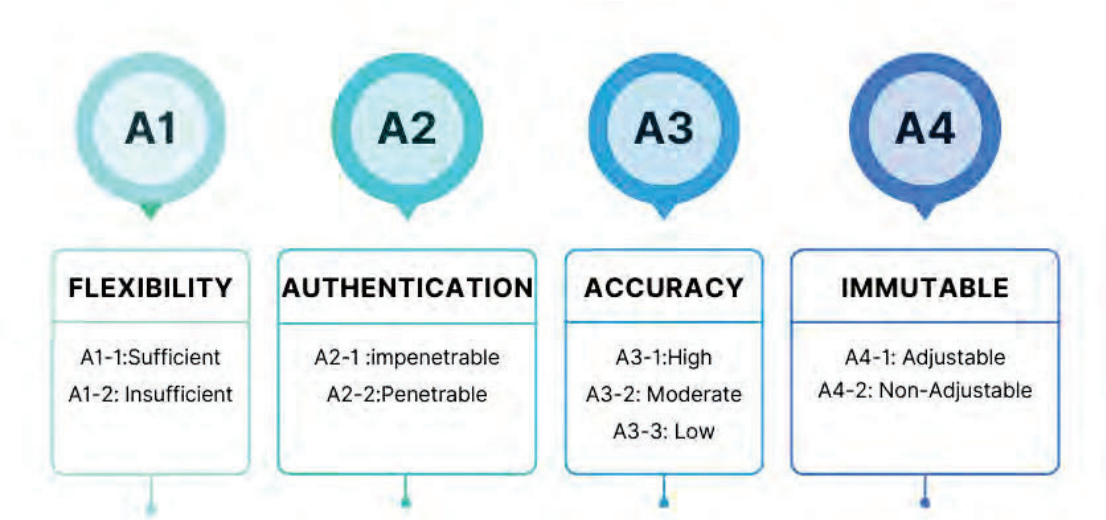


Fig 4. Attributes and Sub-attributes of BC based on LLMs

Table 1. Aggregated decision matrix

	A1	A2	A3	A4
BC1	0.604444444	0.5722222	0.42666667	0.672222222
BC2	0.816666667	0.5777778	0.71666667	0.9
BC3	0.538888889	0.38	0.5	0.537777778
BC4	0.705555556	0.8055556	0.65	0.816666667
BC5	0.644444444	0.7111111	0.81666667	0.838888889

Table 2. Normalized matrix

	A1	A2	A3	A4
BC1	0.182611615	0.18781911	0.137191854	0.178518737
BC2	0.24672709	0.189642597	0.230439443	0.239008557
BC3	0.162806311	0.124726477	0.160771704	0.14281499
BC4	0.213158778	0.264405543	0.209003215	0.216878135
BC5	0.194696207	0.233406273	0.262593783	0.222779581

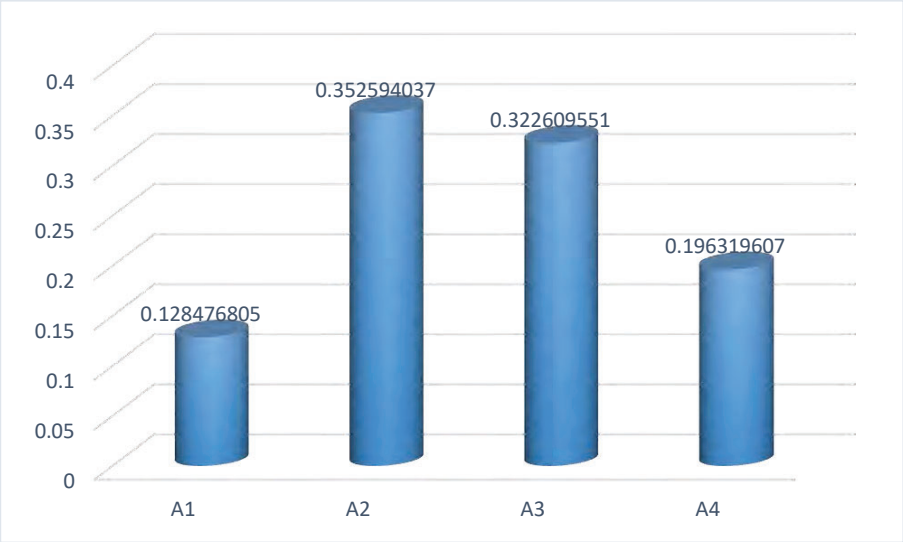


Fig 5. Attributes weights

4.2.2 Ranking BC Platforms

Based on the methodology of SHS we can suppose a set of propositions for sub-attributes to rank BC based on LLMs :

- let F({Sufficient, Non-sufficient}, {Impenetrable, Penetrable},{Low},{Non-Adjustable}).
- According to Eq.(7), there are four propositions for sub-attributes. in other words, there are four hypersofts:

proposition 1: Sufficient, Impenetrable, Low, Non-Adjustable.
proposition 2: Sufficient, Penetrable, Low, Non-Adjustable
proposition 3: Non-sufficient, Impenetrable, Low, Non-Adjustable
proposition 4: Non-sufficient, Penetrable, Low, Non-Adjustable

After that, we can implement SVNSs-MOOSRA to rank BC platforms and obtain the most secure BC.

According to Proposition 1:

- Three neutrosophic decision matrices are constructed for evaluating BC platforms based on sub-attributes according to the proposition.
- deneutrosophic these matrices based on Eq.(1).
- Aggregated these matrices into an aggregated matrix based on Eq.(2) as in Table 3.
- Table 4 represents the normalized matrix according to Eq.(8).
- Table 5 for the weighted normalized matrix is generated based on Eq.(9).

The final ranking for BC platforms is exhibited in Fig 6 which indicates that BC4 is the most secure platform.

Table 3. Aggregated matrix

	A11	A21	A31	A42
BC1	0.644444	0.56	0.353333	0.605556
BC2	0.498889	0.76	0.386667	0.777778
BC3	0.42	0.498889	0.538889	0.666667
BC4	0.598889	0.877778	0.732222	0.458889
BC5	0.565556	0.665556	0.632222	0.671111

Table 4. Normalized matrix.

	A11	A21	A31	A41
BC1	0.522865	0.365028	0.288437	0.420175
BC2	0.40477	0.495395	0.315648	0.539675
BC3	0.340764	0.325194	0.439911	0.462578
BC4	0.485904	0.572167	0.597735	0.318408
BC5	0.458859	0.433833	0.516102	0.465662

Table 5. Weighted Normalized matrix

	A11	A21	A31	A42
BC1	0.040783461	0.095272375	0.04384238	0.050421043
BC2	0.031572024	0.129298223	0.047978453	0.064760973
BC3	0.026579566	0.084875588	0.066866523	0.055509405
BC4	0.037900492	0.149335667	0.090855749	0.038208974
BC5	0.035791003	0.113230461	0.078447529	0.055879468

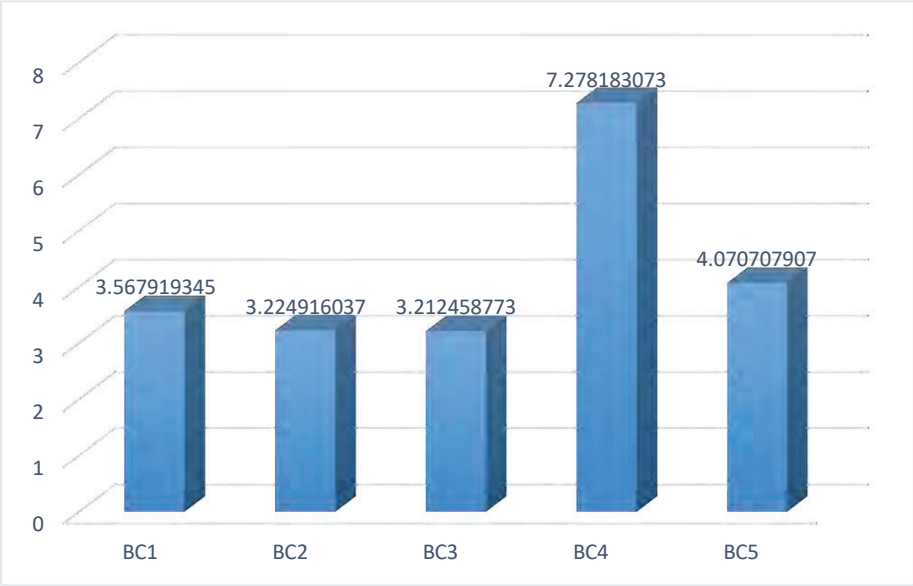


Fig 6. Final ranking for BC platforms

According to Proposition 2:

- Table 6 represents the normalized matrix according to Eq.(8).
- Table 7 for the weighted normalized matrix is generated based on Eq.(9).

Table 6. Normalized matrix

	A11	A22	A31	A42
BC1	0.404239135	0.452016567	0.288436709	0.420175361
BC2	0.546168684	0.518707536	0.315647719	0.539674775
BC3	0.360397023	0.44460646	0.439911333	0.462578379
BC4	0.471860019	0.358649211	0.597735192	0.318408118
BC5	0.430990254	0.447570503	0.516102162	0.465662235

Table 7. Weighted Normalized matrix

	A11	A22	A31	A42
BC1	0.031530653	0.041585524	0.04384238	0.050421043
BC2	0.042601157	0.047721093	0.047978453	0.064760973
BC3	0.028110968	0.040903794	0.066866523	0.055509405
BC4	0.036805082	0.032995727	0.090855749	0.038208974
BC5	0.03361724	0.041176486	0.078447529	0.055879468

According to Proposition 3:

- Table 8 represents the normalized matrix according to Eq.(8).

- Table 9 for the weighted normalized matrix is generated based on Eq.(9).

Table 8. Normalized matrix

	A12	A21	A31	A42
BC1	0.159831631	0.365028256	0.288436709	0.420175361
BC2	0.409092866	0.49539549	0.315647719	0.539674775
BC3	0.513744529	0.325193823	0.439911333	0.462578379
BC4	0.570827254	0.572167306	0.597735192	0.318408118
BC5	0.466175591	0.433833185	0.516102162	0.465662235

Table 9. Weighted Normalized matrix

	A12	A21	A31	A42
BC1	0.008151413	0.0335826	0.04384238	0.050421043
BC2	0.020863736	0.045576385	0.047978453	0.064760973
BC3	0.026200971	0.029917832	0.066866523	0.055509405
BC4	0.02911219	0.052639392	0.090855749	0.038208974
BC5	0.023774955	0.039912653	0.078447529	0.055879468

According to Proposition 4:

- Table 10 represents the normalized matrix according to Eq.(8).

- Table 11 for the weighted normalized matrix is generated based on Eq.(9).

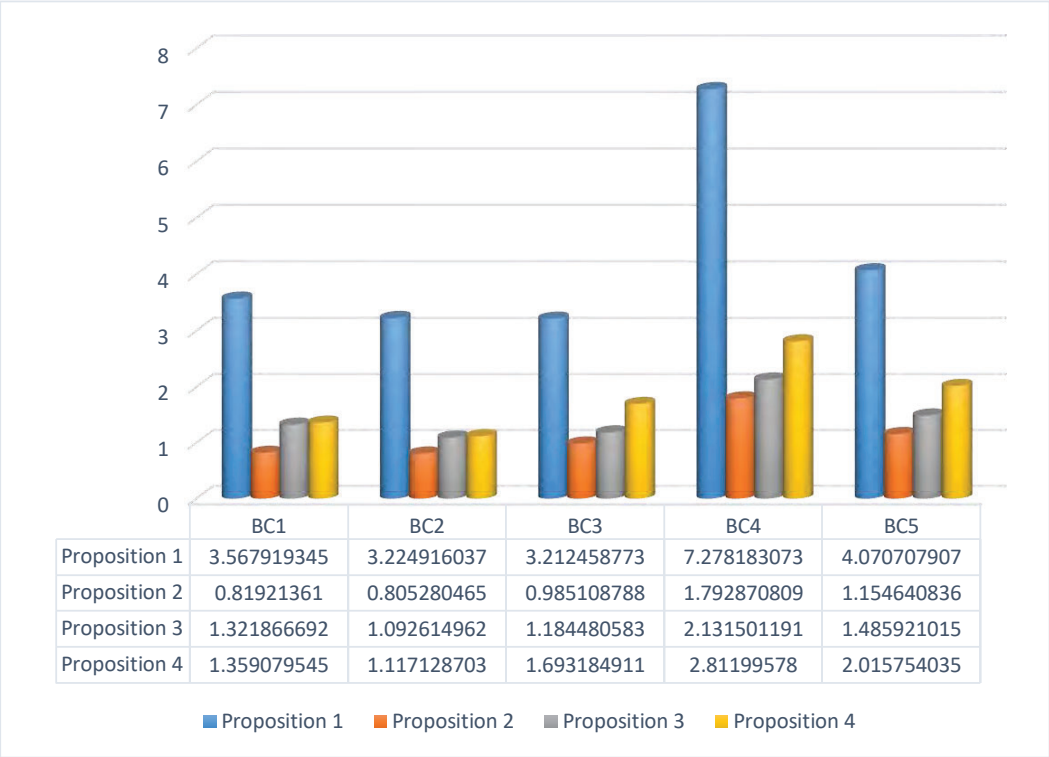
Table 10. Normalized matrix

	A12	A22	A31	A42
BC1	0.159831631	0.452016567	0.288436709	0.420175361
BC2	0.409092866	0.518707536	0.315647719	0.539674775
BC3	0.513744529	0.44460646	0.439911333	0.462578379
BC4	0.570827254	0.358649211	0.597735192	0.318408118
BC5	0.466175591	0.447570503	0.516102162	0.465662235

Table 11. Weighted Normalized matrix

	A12	A22	A31	A42
BC1	0.008151413	0.041585524	0.04384238	0.050421043
BC2	0.020863736	0.047721093	0.047978453	0.064760973
BC3	0.026200971	0.040903794	0.066866523	0.055509405
BC4	0.02911219	0.032995727	0.090855749	0.038208974
BC5	0.023774955	0.041176486	0.078447529	0.055879468

- After we applied the four propositions or four hypersoftsets, we concluded that BC5 is the most secure and appropriate for utilizing against any attack as shown in Fig 7.



	BC1	BC2	BC3	BC4	BC5
Proposition 1	3.567919345	3.224916037	3.212458773	7.278183073	4.070707907
Proposition 2	0.81921361	0.805280465	0.985108788	1.792870809	1.154640836
Proposition 3	1.321866692	1.092614962	1.184480583	2.131501191	1.485921015
Proposition 4	1.359079545	1.117128703	1.693184911	2.81199578	2.015754035

Fig 7. Final ranking for BC platforms based on various propositions

5 Conclusion

In this study, we attempt to solve the problem of attacks and exploiting vulnerabilities, especially in digital technologies. Hence, we focused on BC as one of the important technologies for many sectors Especially financial sectors and transactions that require customer information. Thus, fraudulent and dubious actions endanger financial establishments as well as people by corrupting and leaking data through encryption. Accordingly, LLMs can be harnessed to protect transactions and DL from any manipulation. through applying PaLM, BLOCKGPT, GPTu-tor….etc to scan smart contracts, detecting and recognizing unusual and abnormal activities as well, these models can develop secured smart contracts. Hence, we exhibit the importance of adopting LLMs in BC to make any business proactive and resilient against any crisis that is facing any sector.

Yet, stakeholders may face the problem of which BC platform can use. Hence, we constructed a soft opting model to aid them in recommending the most appropriate and secure BC. In these models, we applied for the first time SHS for supporting a set of propositions of sub-attributes.

Entropy is applied to generate weights for attributes and sub-attributes which are proposed by SHS to utilize in MOOSRA for recommending optimal BC based on LLMs.

The findings of a soft opting model indicated that BC 4 is the most secure platform to adopt in enterprises.

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A Decision-Making Approach Incorporating TODIM Method and Sine Entropy in q-Rung Picture Fuzzy Set Setting

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In this study, we propose a new approach based on fuzzy TODIM (Portuguese acronym for interactive and multicriteria decision-making) for decision-making problems in uncertain environments. Our method incorporates group utility and individual regret, which are often ignored in traditional multicriteria decision-making (MCDM) methods. To enhance the analysis and application of fuzzy sets in decision-making processes, we introduce novel entropy and distance measures for q-rung picture fuzzy sets. These measures include an entropy measure based on the sine function and a distance measure derived from the Jensen-Shannon divergence. In our methodology, incorporating the sine function into the entropy measure stands out as a distinctive decision, grounded in a profound understanding of the inherent characteristics of fuzzy sets. Utilizing the sine function proves especially advantageous when handling fuzzy sets that exhibit cyclical variations or fluctuations in their membership degrees. We effectively weight the criteria for an improved evaluation by using this new entropy measure. The introduced distance measure finds application in the TODIM approach, allowing the execution of TODIM method steps within a fuzzy environment until the determination of one alternative's dominance over another—an advancement beyond traditional approaches. We apply our enhanced fuzzy TODIM method to a real-life construction project management problem from the literature and compare the results with those in the literature and obtained from other MCDM methods. Our proposed measures are robust, as demonstrated by the sensitivity analysis that varied the weights of group utility and individual regret, with the results visualized in a 3D sensitivity plot. The findings demonstrate the superiority of our method in providing a more comprehensive evaluation of alternatives, making it a useful tool for decision-makers facing complex and uncertain decision-making problems.

1. Introduction

Fuzzy set theory, introduced by Zadeh [1], expands upon classical set theory's characteristic function with a membership function taking values in the closed interval [0,1]. Atanassov [2] later extended this concept to intuitionistic fuzzy sets (IFSs), which incorporate both membership and nonmembership functions. However, fuzzy sets and IFSs are sometimes insufficient for addressing real-life problems. Yager [3] introduced Pythagorean fuzzy sets (PyFSs) to overcome these limitations, characterized by membership and nonmembership functions with the sum of their squares between 0 and 1. Yager [4] also proposed q-rung orthopair fuzzy sets (q-ROFs) for better modeling real-life applications, characterized by membership and nonmembership functions with the q^{th} power between 0 and 1. q-ROFs are generalizations of IFSs and PyFSs, and some main topological properties of q-ROFs were investigated by Türkarslan et al. in [5].

Cường [6] introduced the notion of picture fuzzy sets (PFSs), characterized by membership, neutral, and nonmembership functions with their sum between 0 and 1. PFSs expand upon fuzzy sets and IFSs. Kutlu Gündoğdu and Kahraman [7] and Ashraf et al. [8] later proposed spherical fuzzy sets (SFSs) by generalizing PFSs and PyFSs. Li et al. [9] proposed the notion of q-rung picture fuzzy set (q-RPFS) in 2018, expanding upon PFs, q-ROFSs, and SFSs. All these fuzzy sets provide suitable tools for addressing unexpected

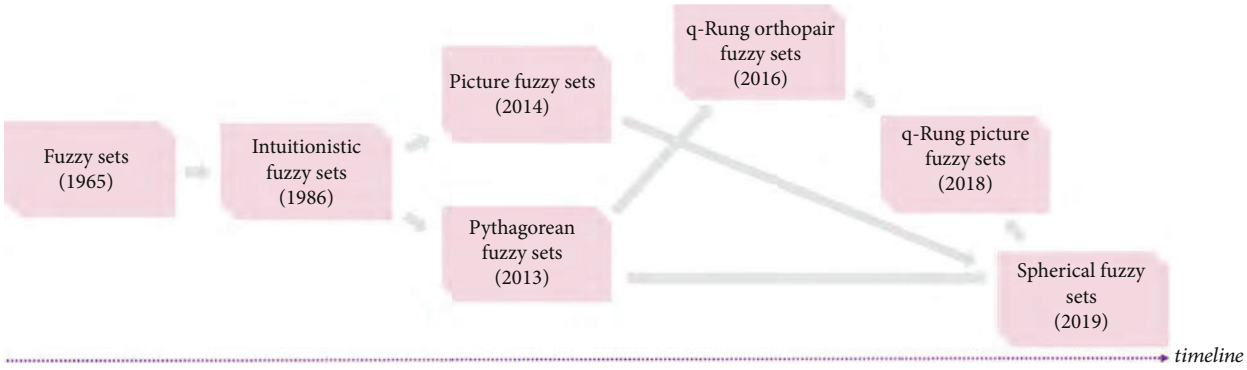


Figure 1: Development network of fuzzy set theory.

situations and uncertainties in the input data required for decision-making problems. Several researchers employed these sets in multicriteria decision-making (MCDM). For instance, Seikh and Mandal [10] introduced innovative operational laws and corresponding aggregation operators for PFSs. They also studied Archimedean aggregation operators for q-ROFSs, exploring their applications in the site selection for software operating units [11]. Ünver et al. [12] employed SFSs in addressing pattern recognition problems. Garg [13] introduced certain aggregation operators for PFSs and applied them in the context of MCDM. Özçelik [14] conducted an examination of the performances of MCDM methods and an optimization model in solving multiattribute shortest path problems under a fuzzy environment. Beg et al. [15] used q-RPFS in MCDM problems by defining some aggregation operators. Akram et al. [16] focused on MCDM with q-rung picture fuzzy information. Pinar and Boran [17] proposed a novel distance measure on q-RPFSs and its application to decision-making and classification problems. Akram et al. [18] conducted a hybrid decision-making analysis using complex q-rung picture fuzzy Einstein averaging operators. In Figure 1, the evolutionary progression of the fuzzy set theory up to q-RPFSs is depicted.

Entropy, an important measurement method for uncertainty or information, was introduced by Shannon [19]. Zadeh [20] transformed Shannon's entropy into fuzzy entropy, measuring fuzziness in a fuzzy set. De Luca and Termini [21] developed a new entropy based on Shannon's entropy, which was later extended to IFSs by Hung and Yang [22]. Arya and Kumar [23] proposed a picture fuzzy entropy. Mahnaz et al. [24] also introduced a distance-based entropy measure for q-ROPFSs, also known as t -spherical fuzzy sets [25], where the monotonicity condition of the entropy is not solely dependent on the fuzziness of the fuzzy sets. Furthermore, a distance measure, a widely used measurement form, specifies the distance between two entities and is utilized in numerous decision-making problems such as TODIM (Portuguese acronym for interactive and multicriteria decision-making) method (see, e.g., [26–28]). Distance measures are also applied in medical diagnosis, pattern recognition, classification, and MCDM. For example, Rani and Garg [29] studied distance measures for complex IFSs and their applications in MCDM. Ünver and Aydoğan [30] proposed distance measures for application in MCDM problems. Son [31] explored a generalized picture distance measure and its applications in clustering. Khan et al. [32] presented biparametric distance and similarity measures for PFSs and their applications in medical diagnosis. Jiang et al. [33] introduced a novel distance measure between IFSs based on transformed isosceles triangles, studying its applications in pattern recognition.

The TODIM method, introduced by Gomes and Lima [34] and incorporating group utility and individual regret—factors frequently overlooked in traditional MCDM methods—stands out as a widely employed discrete MCDM approach applicable to both quantitative and qualitative criteria. In this paper, we present an extended TODIM method that addresses the limitations of traditional MCDM methods by incorporating these crucial factors within the q-rung picture fuzzy environment. The distinctive choice of this method over others is grounded in its ability to comprehensively address the complexities of decision-making scenarios, considering not only the quantitative aspects but also the qualitative dimensions often crucial in real-world decision problems. By explicitly integrating group utility and individual regret, TODIM offers a more nuanced and realistic representation of decision scenarios, providing a robust framework for decision-makers to navigate uncertainties and varied criteria effectively. This adaptability and inclusiveness make TODIM a pragmatic choice, aligning with the diverse nature of decision environments encountered in practice. TODIM method determines loss and gain situations according to all criteria and uses pairwise comparisons to eliminate discrepancies [34]. It has been extensively applied to solve MCDM problems in various domains, including engineering, business, and environmental management. Recently, a number of fuzzy TODIM method have emerged in scholarly publications, which utilize fuzzy sets to address the challenges of uncertainty and vagueness in decision-making. For instance, Ju et al. [35] presented an enhanced version of TODIM method within the context of q-rung picture fuzzy framework, employing the Minkowski distance. Herrera-Viedma and Cabrerizo [36] developed a new fuzzy TODIM method to address MCDM problems with unknown attribute weights. Konwar and Debnath [37] explored continuity and the Banach contraction principle in intuitionistic fuzzy n-normed linear spaces. Lourenzutti and

Krohling [38] studied TODIM in an intuitionistic fuzzy and random environment. Ren et al. [39] proposed the Pythagorean fuzzy TODIM approach for MCDM. Wei [40] applied the TODIM method to picture fuzzy MCDM.

The present paper introduces an extended TODIM method that addresses the limitations of traditional MCDM methods within the q-rung picture fuzzy framework. The choice of q-RPFSs as the foundation for our methodology is a strategic one, driven by several compelling reasons that underscore their suitability for addressing the complexities of decision-making under uncertainty. Firstly, q-RPFSs generalize existing fuzzy set models, offering a versatile and unified framework that can seamlessly integrate various fuzzy set paradigms. Their incorporation allows us to effectively represent uncertainties and vagueness in decision-making scenarios, making them well suited for real-world applications where input data may be inherently uncertain. Moreover, the inclusion of a neutral degree for an element in a q-RPFS enhances the model's capability to handle uncertainty, providing a more nuanced representation. This is particularly advantageous in decision-making scenarios where a balanced and neutral perspective is essential. By choosing q-RPFSs, we are aligning our methodology with a proven and effective paradigm that extends beyond traditional fuzzy set models.

In the present work, we propose new entropy and distance measures for q-RPFSs, utilizing the sine function and Jensen-Shannon divergence, respectively. The choice of using the sine function instead of the absolute value function to define an entropy measure is motivated by the desire to capture the specific characteristics and properties of the fuzzy sets. While both functions can be used to quantify the spread or dispersion of values within a set, the sine function offers certain advantages. Cui and Ye [41] employed the sine function to define entropy for simplified neutrosophic sets (SNSs), applying it in the context of MCDM. Garg [42] utilized the sine function to establish operational laws for PyFSs. Ashraf et al. [43] explored fuzzy decision support modeling for Internet finance soft power evaluation, employing sine trigonometric Pythagorean fuzzy information. Türkarslan et al. [44] applied the sine function to define cross-entropy for consistency fuzzy sets (CFSs). In this paper, the sine function exhibits a periodic nature, oscillating between zero and one, which allows it to capture the cyclical patterns or fluctuations that may exist in the membership degrees. This can be particularly useful in scenarios where the data exhibits periodic or repetitive variations [42]. Furthermore, the absolute value function only considers the magnitude of the values and disregards any directional information or patterns. It treats positive and negative deviations from a central point equally, without taking into account any potential asymmetry or specific distribution characteristics. Taking into account these advanced measures, our proposed method offers a resilient approach to weighing criteria and delivers improved evaluation capabilities.

In this manuscript, we also perform sensitivity analysis by altering the weights assigned to group utility and individual regret, along with adjusting the parameters employed in the extended TODIM method. We demonstrate the practical utility of the proposed method by applying it to a real-life problem from the literature [15] in construction project management and compare the results with those obtained using other MCDM methods. The findings show that the proposed method outperforms the other methods in terms of sensitivity analysis and providing a more comprehensive evaluation of alternatives, highlighting the importance of considering group utility and individual regret in MCDM problems.

The following advantages stem from our research and the development of these pioneering information measures:

(1) This paper introduces new entropy and distance measures for q-RPFSs, expanding the existing repertoire of measurement techniques and providing alternative methods for assessing uncertainty and relationships in complex decision-making problems

(2) The proposed information measures are expected to improve the effectiveness of MCDM processes, offering better decision support by accounting for a wider range of uncertainty and vagueness inherent in real-world problems

(3) The introduced entropy measure plays a crucial role in the enhanced TODIM method by effectively assigning weights to criteria. Notably, in contrast to the classical TODIM approach, this weighting process takes place within the fuzzy environment

(4) The distance measure, derived from the Jensen-Shannon divergence, possesses the capability to gauge the similarity between objects while overcoming certain limitations of other divergence measures. In contrast to the Kullback-Leibler divergence, which lacks symmetry, the Jensen-Shannon divergence symmetrically quantifies the dissimilarity between two objects. Consequently, incorporating the proposed distance measure in the TODIM method enhances the robustness of the approach

(5) This research demonstrates the practical application of the proposed entropy and distance measures by integrating them into the well-established extended TODIM method. This integration showcases how the new measures can be used in real-life decision-making scenarios

(6) q-RPFSs extend the scope of several established fuzzy set models, facilitating the smooth integration of diverse fuzzy set paradigms. Notably, the inclusion of the neutral degree for an element in a fuzzy set enhances the treatment of uncertainty, contributing to the improved handling of fuzzy set uncertainties

(7) By comparing the results obtained using the new measures in an extended TODIM method to those from existing literature, the paper highlights the effectiveness and potential advantages of using the proposed entropy and distance measures in MCDM problems

The rest of the paper is organized as follows: In Section 2, we give a comprehensive exploration of the foundational concepts underpinning our research. We revisit key theoretical underpinnings of fuzzy set theory, emphasizing its evolution from classical set theory to more advanced forms such as q-ROFSs, PFSs, SFSs, and q-RPFSs. The section is aimed at establishing a robust theoretical foundation for the subsequent discussions on decision-making methodologies. We also give a short information about fuzzy entropy measures. Section 3 is dedicated to introducing novel information measures tailored specifically for q-RPFSs. Here, we define and elaborate on innovative entropy and distance measures, incorporating the sine function and Jensen-Shannon divergence, respectively. These measures are designed to enhance the evaluation capabilities of decision-making processes, offering a unique perspective on the weighing of criteria in complex decision scenarios. In Section 4, we present our extended TODIM method, addressing the limitations observed in traditional MCDM methods by incorporating group utility and individual regret. This section provides a step-by-step elucidation of the methodological enhancements, emphasizing the rationale behind the choice of the sine function in defining entropy measures. Furthermore, we apply the extended TODIM method to a practical, real-life construction project management problem sourced from the literature. We provide a comparative analysis of the solution using another distance-based MCDM method and conduct a sensitivity analysis. Finally, in Section 5, we conclude the paper.

2. Preliminaries

In this section, we provide a brief overview of some fundamental concepts in fuzzy set theory. Throughout this paper, we operate under the assumption that $X = \{x_1, \cdots, x_n\}$ represents a finite universal set and assume $q \geq 1$.

Definition 1 (see [4]). A q -ROFS A within X can be represented as

$$A = \{(x, \gamma_A(x), \zeta_A(x)) \colon x \in X\}, \tag{1}$$

where $\gamma_A : X \longrightarrow [0, 1]$ and $\zeta_A : X \longrightarrow [0, 1]$ are functions satisfying the condition $\gamma_A^q(x) + \zeta_A^q(x) \leq 1$ for any $x \in X$.

Definition 2 (see [6]). A PFS A within X can be represented as

$$A = \{(x, \gamma_A(x), \eta_A(x), \zeta_A(x)) \colon x \in X\}, \tag{2}$$

where $\gamma_A : X \longrightarrow [0, 1]$, $\eta_A : X \longrightarrow [0, 1]$, and $\zeta_A : X \longrightarrow [0, 1]$ are functions satisfying the condition $\gamma_A(x) + \eta_A(x) + \zeta_A(x) \leq 1$ for any $x \in X$. The functions γ_A, η_A, and ζ_A are referred to as the membership function, neutral function, and nonmembership function of A, respectively.

Definition 3 (see [7, 8]). A SFS A within X can be represented as

$$A = \{(x, \gamma_A(x), \eta_A(x), \zeta_A(x)) \colon x \in X\}, \tag{3}$$

where $\gamma_A : X \longrightarrow [0, 1]$, $\eta_A : X \longrightarrow [0, 1]$, and $\zeta_A : X \longrightarrow [0, 1]$ are functions satisfying the condition $\gamma_A^2(x) + \eta_A^2(x) + \zeta_A^2(x) \leq 1$ for any $x \in X$. The functions γ_A, η_A, and ζ_A are referred to as the membership function, neutral function, and nonmembership function of A, respectively.

To briefly revisit the concept, the Hessian matrix of a function $f(x_1, \cdots, x_n)$ with multiple variables is given by

$$\mathbb{H}(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}. \tag{4}$$

If $\mathbb{H}(f)$ is positive definite, then the function f is strictly convex. If $\mathbb{H}(f)$ is negative definite, then the function f is strictly concave at a point in its domain.

2.1. q-Rung Picture Fuzzy Sets. We revisit the definition of q-RPFS, which will be utilized to establish a knowledge measure for q-RPFSs.

Definition 4 (see [15]). A q-RPFS A in X is represented as follows:

$$A = \{(x, \gamma_A(x), \eta_A(x), \zeta_A(x)) \colon x \in X\}. \tag{5}$$

Here, $\gamma_A : X \longrightarrow [0, 1]$, $\eta_A : X \longrightarrow [0, 1]$, and $\zeta_A : X \longrightarrow [0, 1]$ are functions satisfying $\gamma_A^q(x) + \eta_A^q(x) + \zeta_A^q(x) \leq 1$ for all $x \in X$. The functions γ_A, η_A, and ζ_A are referred to as the membership function, neutral function, and nonmembership function of A, respectively. We denote the triplet of nonnegative numbers $\alpha = \langle \gamma, \eta, \zeta \rangle$ as a q-rung picture fuzzy value (q-RPFV) if $\gamma^q + \eta^q + \zeta^q \leq 1$.

It is important to note that if $\eta_A(x) = 0$ for every $x \in X$, then set A becomes a q-ROFS. For $q = 1$ and $q = 2$, set A transforms into a PFS and a SFS, respectively.

We now review some set operations for q-RPFSs.

Definition 5 (see [15]). Let $A = \{\gamma_A(x), \eta_A(x), \zeta_A(x) | x \in X\}$ and $B = \{\gamma_B(x), \eta_B(x), \zeta_B(x) | x \in X\}$ be two PFNs.

(i) $A \cup B = \{x, \max(\gamma_A(x), \gamma_B(x)), \max(\eta_A(x), \eta_B(x)), \min(\zeta_A(x), \zeta_B(x))x, \max(\gamma_A(x), \gamma_B(x)), \max(\eta_A(x), \eta_B(x)), \min(\zeta_A(x), \zeta_B(x)) | x \in X\}$

(ii) $A \cap B = \{(x, \min(\gamma_A(x), \gamma_B(x)), \min(\eta_A(x), \eta_B(x)), \max(\zeta_A(x), \zeta_B(x)) | x \in X\}$

(iii) The complement A^c of A is given with $A^c = (\zeta_A(x), \eta_A(x), \gamma_A(x)) | x \in X$

Now, we revisit a score function associated with the q-rung picture score function.

Definition 6 (see [15]). Let $A = \{(\gamma_A(x), \eta_A(x), \zeta_A(x)) | x \in X\}$ and $B = \{(\gamma_B(x), \eta_B(x), \zeta_B(x)) | x \in X\}$ be two PFNs. The score function $s(A)$ is defined as

$$s(A) = \frac{1}{3}\left[1 + \gamma^q + \eta^q - \zeta^q\right]. \tag{6}$$

The accuracy function $h(A)$ is defined as

$$h(A) = \frac{1}{2}\left(1 + \max\{\gamma^q, \eta^q\} - \zeta^q\right). \tag{7}$$

3. q-Rung Picture Fuzzy Information Measures

This section introduces new information measures for q-RPFSs, which include measures of entropy and distance. In the literature, several measures have been proposed to capture the uncertainty and vagueness inherent in fuzzy sets, with the entropy measure being particularly significant [19, 21–23, 45]. Entropy measures the uncertainty or randomness of a fuzzy set and has found various applications in the decision-making, pattern recognition, and classification. Therefore, it is essential to define a proper entropy measure for q-RPFSs to understand the degree of uncertainty or vagueness in the data.

3.1. A New q-Rung Picture Fuzzy Entropy Measure. Arya and Kumar [23] introduced an entropy measure for PFSs. Here, we represent the conditions for the entropy measures applicable to q-RPFSs.

Definition 7. Let q -RPFS(X) be the set of all q-RPFSs on X. A q-rung picture fuzzy entropy measure is a fuction $\mathbb{E} : q - RPFS(X) \longrightarrow [0, 1]$ satisfying the following properties:

E1. If A is a crisp set, then $\mathbb{E}(A) = 0$.

E2. $\mathbb{E}(A) = 1$ if and only if $A = \{(x, (1/2)^{1/q}, (1/2)^{1/q}, (1/2)^{1/q}) : x \in X\}$.

E3. $\mathbb{E}(A^c) = \mathbb{E}(A)$.

E4. $\mathbb{E}(A) \leq \mathbb{E}(B)$ if A is less fuzzier than B; that is, $\gamma_A^q(x) \leq \gamma_B^q(x)$, $\eta_A^q(x) \leq \eta_B^q(x)$, and $\zeta_A^q(x) \leq \zeta_B^q(x)$ if $\max\{\gamma_B^q(x), \eta_B^q(x), \zeta_B^q(x)\} \leq 1/2$ or $\gamma_A^q(x) \geq \gamma_B^q(x)$, $\eta_A^q(x) \geq \eta_B^q(x)$, and $\zeta_A^q(x) \geq \zeta_B^q(x)$ if $\min\{\gamma_B(x), \eta_B(x), \zeta_B(x)\} \geq 1/2$.

Inspired by [41], we introduce a novel fuzzy entropy measure for q-RPFSs. Let A be a q-RPFS in X. We define the function $\mathbb{E} : q\text{-RPFS}(X) \longrightarrow [0, 1]$ as follows:

$$\mathbb{E}(A) = \frac{1}{3n}\sum_{i=1}^{n}\left[\sin\left(\gamma_A^q(x_i)\pi\right) + \sin\left(\eta_A^q(x_i)\pi\right) + \sin\left(\zeta_A^q(x_i)\pi\right)\right], \tag{8}$$

where q-RPFS(X) is the set of all q-RPFS on X.

Theorem 8. *The measure $\mathbb{E}(A)$ is a q-rung picture fuzzy entropy measure.*

Proof. E1. If A is a crisp set, then we get

$$\gamma_A(x_i) = 1, \eta_A(x_i) = 0, \zeta_A(x_i) = 0, \tag{9}$$

or

$$\gamma_A(x_i) = 0, \eta_A(x_i) = 0, \zeta_A(x_i) = 1. \tag{10}$$

Thus, we obtain

$$\mathbb{E}(A) = \frac{1}{3n}\sum_{i=1}^{n}\left[\sin\left(\gamma_A^q(x_i)\pi\right) + \sin\left(\eta_A^q(x_i)\pi\right) + \sin\left(\zeta_A^q(x_i)\pi\right)\right] = 0. \tag{11}$$

E2. If $A = \{(x, (1/2)^{1/q}, (1/2)^{1/q}, (1/2)^{1/q}) : x \in X\}$, then we obtain

$$\begin{aligned}\mathbb{E}(A)(x_i) &= \frac{1}{3n}\sum_{i=1}^{n}\left[\sin\left(\gamma_A^q(x_i).\pi\right) + \sin\left(\eta_A^q(x_i).\pi\right) + \sin\left(\zeta_A^q(x_i).\pi\right)\right] \\ &= \frac{1}{3n}\sum_{i=1}^{n} 3\sin\frac{\pi}{2} = 1.\end{aligned} \tag{12}$$

To prove the sufficiency, consider the function f defined by $f(\gamma, \eta, \zeta) = \sin(\gamma^q\pi) + \sin(\eta^q\pi) + \sin(\zeta^q\pi)$ for $\gamma, \eta, \zeta \in [0, 1]$. Thus, we have

$$\frac{\partial f(\gamma)}{\partial \gamma} = \pi q\gamma^{q-1}\cos(\gamma^q\pi). \tag{13}$$

To find the critical points, we need to solve

$$\pi q\gamma^{q-1}\cos(\gamma^q\pi) = 0. \tag{14}$$

Then, we have $\gamma^q\pi = (\pi/2) + k\pi$ for any integer k. So, we obtain $\gamma^q = (1/2) + k$. Since $\gamma \in (0, 1]$, we get $k = 0$ and so $\gamma^q = 1/2$. Similarly, we have $\eta^q, \zeta^q = 1/2$. So, the point $P = ((1/2)^{1/q}, (1/2)^{1/q}, (1/2)^{1/q})$ is a critical point. Zero is not considered as it makes the function zero so it cannot make the function maximum. On the other hand, we have

$$K(\gamma) = \frac{\partial^2 f(\gamma)}{\partial \gamma^2} = -\pi q\gamma^{q-2}(\cos\pi\gamma^q - q\cos\pi\gamma^q + \pi q\gamma^q\sin\pi\gamma^q) \tag{15}$$

and get the following Hessian matrix:

$$\mathbb{H}(\gamma,\eta,\zeta)=\begin{bmatrix}K(\gamma) & 0 & 0\\ 0 & K(\eta) & 0\\ 0 & 0 & K(\zeta)\end{bmatrix}. \tag{16}$$

Therefore, we have

$$\mathbb{H}(P)=\begin{bmatrix}-\frac{1}{2^{(2q-2)/q}}\pi^2q^2 & 0 & 0\\ 0 & -\frac{1}{2^{(2q-2)/q}}\pi^2q^2 & 0\\ 0 & 0 & -\frac{1}{2^{(2q-2)/q}}\pi^2q^2\end{bmatrix}=\frac{1}{2^{(2q-2)/q}}\pi^2q^2\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}. \tag{17}$$

Since the Hessian matrix $\mathbb{H}(P)$ is negatively defined, then f takes its maximum value at point P.

E3. The proof is trivial.

E4. Let $\max\{\gamma_B^q(x_i),\eta_B^q(x_i),\zeta_B^q(x_i)\}\le 1/2$. In this case, if A is less fuzzier than B, we have $\gamma_A^q(x)\le\gamma_B^q(x)\le 1/2$, $\eta_A^q(x)\le\eta_B^q(x)\le 1/2$, and $\zeta_A^q(x)\le\zeta_B^q(x)\le 1/2$. Thus, we get $\pi\gamma_A^q(x)\le\pi\gamma_B^q(x)\le\pi/2$, $\pi\eta_A^q(x)\le\pi\eta_B^q(x)\le\pi/2$, and $\pi\zeta_A^q(x)\le\pi\zeta_B^q(x)\le\pi/2$. Since the sine function is increasing over $[0,\pi/2]$, the proof is completed. The proof for the other case can be established similarly. □

Remark 9. Cui and Ye [41] provided a comparable entropy for SNSs, while Türkarslan et al. [44] applied a similar concept to define a cross-entropy for CFSs. The sine function introduces a distinctive feature to the entropy measure by exhibiting a periodic nature. Unlike the absolute value function, which remains constant in its behavior, the sine function oscillates between zero and one in a periodic fashion. This periodicity enables the entropy measure to capture cyclical patterns and fluctuations that may exist within the membership degrees of fuzzy sets. In decision-making scenarios, where data might exhibit periodic or repetitive variations, the sine-based entropy provides a more nuanced and accurate representation of the underlying dynamics [42]. Moreover, the sine function introduces a directional component that is absent in the absolute value function. While the absolute value function treats positive and negative deviations from a central point equally, the sine function captures both magnitude and direction. This is particularly important in scenarios where asymmetry or specific distribution characteristics play a role in defining the fuzziness of the fuzzy sets. The sine-based entropy, therefore, offers a more nuanced and context-specific assessment of the spread or dispersion of values within a set.

3.2. A New q-Rung Picture Distance Measure. In this subsection, we discuss the concept of divergence measure and its importance in measuring the difference between two q-RPFSs. We also introduce a distance measure for q-RPFSs based on the Jensen-Shannon divergence. The reason for defining a distance measure for q-RPFSs based on the Jensen-Shannon divergence is that the Jensen-Shannon divergence has several desirable properties, such as being symmetric, nonnegative, and satisfying the triangle inequality. These properties make it a suitable choice for defining a distance measure between q-RPFSs.

Definition 10 (see [46]). Let X be a discrete random variable, and let P_1 and P_2 be two probability distributions for X. The KL divergence is defined as follows:

$$\mathrm{KL}(P_1,P_2)=\sum_{x\in X}P_1(x)\log\frac{P_1(x)}{P_2(x)}. \tag{18}$$

It is worth noting that $\mathrm{KL}(P_1,P_2)$ is nonnegative, additive, and nonsymmetric.

Now, we define a new distance measure for q-RPFSs. This new measure can provide more accurate and meaningful results in various applications where q-RPFSs are used. This measure allows for a more accurate and meaningful comparison between q-RPFSs, especially in cases where there are overlaps or uncertainties in the membership degrees of the elements. In addition, the proposed distance measure considers the underlying uncertainty and vagueness in the data, which is a common feature in real-world decision-making problems.

Definition 11. Let α and β be two q-RPFVs where

$$\begin{aligned}\alpha&=(\gamma_\alpha,\eta_\alpha,\zeta_\alpha).\\ B&=\left(\gamma_\beta,\eta_\beta,\zeta_\beta\right).\end{aligned} \tag{19}$$

A new q-rung picture fuzzy distance measure is defined as

$$\mathbb{D}(\alpha,\beta)=\frac{1}{8\log 2}\left(\sum_{J\in\mathcal{S}}J_\alpha^q\log\frac{2J_\alpha^q}{J_\alpha^q+J_\beta^q}+\sum_{J\in\mathcal{S}}J_\beta^q\log\frac{2J_\beta^q}{J_\alpha^q+J_\beta^q}\right), \tag{20}$$

where $\mathcal{S}=\{\gamma,\eta,\zeta,\pi\}$ and $\pi_A(x)=\sqrt[q]{1-\gamma_A^q(x)-\eta_A^q(x)-\zeta_A^q(x)}$.

Theorem 12. *Let α, β, and τ be three q-RPFSs in X; then we have the following:*

D1. $\mathbb{D}(\alpha,\beta)=0$ *if* $\alpha=\beta$.

D2. $\mathbb{D}(\alpha,\beta)=\mathbb{D}(\beta,\alpha)$.

D3. $\mathbb{D}(\alpha,\tau)\le\mathbb{D}(\alpha,\beta)+\mathbb{D}(\beta,\tau)$.

D4. $0\le\mathbb{D}(\alpha,\beta)\le 1$.

Proof. D1-D2. The proofs are trivial.

D3. We consider three cases.

Case 1. Let $\gamma_\alpha \le \gamma_\beta \le \gamma_\tau$ or $\gamma_\tau \le \gamma_\beta \le \gamma_\alpha$. Then, we have

$$|\gamma_\alpha^q - \gamma_\tau^q| = \left|\gamma_\alpha^q - \gamma_\beta^q\right| + \left|\gamma_\beta^q - \gamma_\tau^q\right|. \tag{21}$$

Case 2. Let $\gamma_\beta \le \min\{\gamma_\alpha, \gamma_\tau\}$. Then, we have $\gamma_\alpha^q - \gamma_\beta^q \ge 0$ and $\gamma_\tau^q - \gamma_\beta^q \ge 0$. Therefore, we have

$$\begin{aligned}
&\left|\gamma_\alpha^q - \gamma_\beta^q\right| + \left|\gamma_\beta^q - \gamma_\tau^q\right| - |\gamma_\alpha^q - \gamma_\tau^q| \\
&= \begin{cases} \gamma_\alpha^q - \gamma_\beta^q + \gamma_\tau^q - \gamma_\beta^q - \gamma_\alpha^q + \gamma_\tau^q, \text{if } \gamma_\alpha \ge \gamma_\tau \\ \gamma_\alpha^q - \gamma_\beta^q + \gamma_t^q - \gamma_\beta^q + \gamma_\alpha^q - \gamma_t^q, \text{if } \gamma_\alpha \le \gamma_\tau \end{cases} \\
&= \begin{cases} 2\left(\gamma_\tau^q - \gamma_\beta^q\right), \text{if } \gamma_\alpha \ge \gamma_\tau \\ 2\left(\gamma_\alpha^q - \gamma_\beta^q\right), \text{if } \gamma_\alpha \le \gamma_\tau \end{cases} = 2\left(\min\{\gamma_\alpha^q, \gamma_\tau^q\} - \gamma_\beta^q\right) \ge 0.
\end{aligned} \tag{22}$$

Case 3. Let $\gamma_\beta \ge \max\{\gamma_\alpha, \gamma_\tau\}$. Then, we have $\gamma_\beta^q - \gamma_\alpha^q \ge 0$ and $\gamma_\beta^q - \gamma_\tau^q \ge 0$. Thus, we obtain

$$\begin{aligned}
&\left|\gamma_\alpha^q - \gamma_\beta^q\right| + \left|\gamma_\beta^q - \gamma_\tau^q\right| - |\gamma_\alpha^q - \gamma_\tau^q| \\
&= \begin{cases} -\gamma_\alpha^q + \gamma_\beta^q - \gamma_\tau^q + \gamma_\beta^q - \gamma_\alpha^q + \gamma_\tau^q, \text{if } \gamma_\alpha \ge \gamma_\tau \\ -\gamma_\alpha^q + \gamma_\beta^q - \gamma_\tau^q + \gamma_\beta^q + \gamma_\alpha^q - \gamma_t^q, \text{if } \gamma_\alpha \le \gamma_\tau \end{cases} \\
&= \begin{cases} 2\left(\gamma_\beta^q - \gamma_\alpha^q\right), \text{if } \gamma_\alpha \ge \gamma_\tau \\ 2\left(\gamma_\beta^q - \gamma_\tau^q\right), \text{if } \gamma_\alpha \le \gamma_\tau \end{cases} = 2\left(\gamma_\beta^q - \max\{\gamma_\alpha^q, \gamma_\tau^q\}\right) \ge 0.
\end{aligned} \tag{23}$$

So far any case, we have $|\gamma_\alpha^q - \gamma_\beta^q| + |\gamma_\beta^q - \gamma_\tau^q| \ge |\gamma_\alpha^q - \gamma_\tau^q|$. On the other hand with a similar conclusion, we can get $|\eta_\alpha^q - \eta_\beta^q| + |\eta_\beta^q - \eta_\tau^q| \ge |\eta_\alpha^q - \eta_\tau^q|$, $|\zeta_\alpha^q - \zeta_\beta^q| + |\zeta_\beta^q - \zeta_\tau^q| \ge |\zeta_\alpha^q - \zeta_\tau^q|$, and $|\pi_\alpha^q - \pi_\beta^q| + |\pi_\beta^q - \pi_\tau^q| \ge |\pi_\alpha^q - \pi_\tau^q|$.

D4. Let $p_1 = \gamma_\alpha^q$, $p_2 = \eta_\alpha^q$, $p_3 = \zeta_\alpha^q$, and $p_4 = \pi_\alpha^q$ and $q_1 = \gamma_\beta^q$, $q_2 = \eta_\beta^q$, $q_3 = \zeta_\beta^q$, and $q_4 = \pi_\beta^q$. It is clear that

$$\sum_{i=1}^{4} p_i = \sum_{i=1}^{4} q_i = 1 \tag{24}$$

that yields that

$$\begin{aligned}
\sum_{J \in \mathcal{S}} J_\alpha^q \log \frac{2J_\alpha^q}{J_\alpha^q + J_\beta^q} &= \sum_{i=1}^{4} p_i \log \frac{2p_i}{p_i + q_i} = \mathrm{KL}(P, Q) \ge 0, \\
\sum_{J \in \mathcal{S}} J_\beta^q \log \frac{2J_\beta^q}{J_\alpha^q + J_\beta^q} &= \sum_{i=1}^{4} q_i \log \frac{2q_i}{p_i + q_i} = \mathrm{KL}(Q, P) \ge 0,
\end{aligned} \tag{25}$$

where $P = \{p_1, \cdots, p_4\}$ and $Q = \{q_1, \cdots, q_4\}$. Hence, $\mathbb{D}(\alpha, \beta) = (1/8 \log 2)(\mathrm{KL}(P, Q) + \mathrm{KL}(Q, P)) \ge 0$. It is clear that $\mathbb{D}(\alpha, \beta) \le 1$. □

4. An Application on MCDM

In this section, we propose an extended TODIM method in the context of q-rung picture fuzzy sets and apply it to a previously published MCDM problem. Following the application, we perform a sensitivity analysis of the extended TODIM method.

We now discuss the steps taken in implementing an extended TODIM method, using a similar approach proposed by Arya and Kumar [23].

4.1. An Extended TODIM Method. In this subsection, we present the steps of the promised extended TODIM method, which is a widely used MCDM approach and is applicable to both quantitative and qualitative criteria. The method is based on the distance measure $\mathbb{D}(A, B)$, defined in Definition 11, and the entropy measure $\mathbb{E}(A)$, defined in Definition 7, is used to weigh the criteria. The usage of the distance measure and entropy in the extended TODIM method is crucial for the proper evaluation of the MCDM problem. The distance measure $\mathbb{D}$ enables the calculation of the dissimilarity between two q-RPFSs. On the other hand, the entropy measure $\mathbb{E}$ allows for the weighting of the criteria in a more accurate manner. By utilizing both measures in the proposed extended TODIM method approach, a more comprehensive evaluation of the criteria can be achieved, resulting in more effective decision-making.

(i) Step 1: consider $B = \{B_1, \cdots, B_m\}$ and $C = \{C_1, \cdots, C_n\}$ as the sets of alternatives and criteria, respectively

(ii) Step 2: let $D = (d_{ij}) = (\gamma_{ij}, \eta_{ij}, \zeta_{ij})$ be a q-rung picture fuzzy decision matrix with alternatives $B_1, \cdots, B_m$ and criteria $C_1, \cdots, C_n$. The matrix D is given by

$$D = \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{m1} & \cdots & d_{mn} \end{bmatrix}. \tag{26}$$

The process of determining these values has been established in [15] that introduces q-RPFSs. The selection of these values depends on the specific application domain, the expertise of the decision-makers, and the nature of the problem being addressed. The values for the membership function, nonmembership function, and neutral function are typically determined through expert knowledge, domain-specific expertise, and consultation with stakeholders or decision-makers.

(iii) Step 3: to obtain a normalized q-rung picture fuzzy decision matrix $R = (r_{ij})$ from the decision matrix $D = (d_{ij})$, we use the following transformation:

$$r_{ij} = \begin{cases} d_{ij}^{c}, \text{for cost criteria,} \\ d_{ij}, \text{for benefit criteria.} \end{cases} \tag{27}$$

Here, $(d_{ij})^{c} = (\zeta_{ij}, \eta_{ij}, \gamma_{ij})$ represents the complement of d_{ij}.

(iv) Step 4: to determine the weights of the criteria, use the equation

$$w_j = \frac{\sum_{i=1}^{m} \mathbb{E}(r_{ij})}{\sum_{j=1}^{n}\sum_{i=1}^{m} \mathbb{E}(r_{ij})}, \tag{28}$$

where w_j is the weight of the criterion C_j. Next, obtain the relative weight of criterion C_j with respect to the most important criterion, denoted by rw_j, as follows:

$$rw_j = \frac{w_j}{w_r}, j, r = 1, \cdots, n, \tag{29}$$

where w_r is the maximum weight among all the criteria

(v) Step 5: calculate the dominance of alternative B_i over alternative B_t under criterion C_j using the following equation:

$$\Phi_j(B_i, B_t) = \begin{cases} \sqrt{\dfrac{rw_j \mathbb{D}(r_{ij}, r_{tj})}{\sum_{j=1}^{n} rw_j}}, & \text{if } r_{ij} < r_{tj}, \\ 0, & \text{if } r_{ij} = r_{tj}, \\ -\dfrac{1}{\theta}\sqrt{\dfrac{\mathbb{D}(r_{ij}, r_{tj})\sum_{j=1}^{n} rw_j}{rw_j}}, & \text{if } r_{ij} > r_{tj}. \end{cases} \tag{30}$$

Here, θ is the attenuation factor of the losses. The gain or loss nature of the dominance relationship is defined based on the values of r_{ij} and r_{tj}.

(vi) Step 6: construct the dominance matrix Φ_j for each criterion C_j, where each entry $\Phi_j(B_i, B_t)$ denotes the dominance of alternative B_i over B_t under criterion C_j. The matrix Φ_j has dimensions $m \times m$ and is given by

$$\Phi_j = \begin{bmatrix} 0 & \Phi_j(B_1, B_2) & \cdots & \Phi_j(B_1, B_m) \\ \Phi_j(B_2, B_1) & 0 & \cdots & \Phi_j(B_2, B_m) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_j(B_m, B_1) & \Phi_j(B_m, B_2) & \cdots & 0 \end{bmatrix}. \tag{31}$$

(vii) Step 7: compute the overall dominance degree of each alternative B_i under the criterion C_j as the summation of its dominance over all other alternatives B_t, given by

$$\delta_j(B_i) = \sum_{t=1}^{m} \Phi_j(B_i, B_t), i = 1, \cdots, m. \tag{32}$$

We can represent the total dominance consequences for each alternative as a column vector, given by

$$\Psi_j = \begin{bmatrix} \sum_{t=1}^{m} \Phi_j(B_1, B_t) \\ \vdots \\ \sum_{t=1}^{m} \Phi_j(B_m, B_t) \end{bmatrix}. \tag{33}$$

By Equation (32), the overall dominance matrix δ can be obtained as

$$(\delta_{ij}) = \begin{bmatrix} \sum_{t=1}^{m} \Phi_1(B_1, B_t) & \sum_{t=1}^{m} \Phi_2(B_1, B_t) & \cdots & \sum_{t=1}^{m} \Phi_n(B_1, B_t) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{t=1}^{m} \Phi_1(B_m, B_t) & \sum_{t=1}^{m} \Phi_2(B_m, B_t) & \cdots & \sum_{t=1}^{m} \Phi_n(B_m, B_t) \end{bmatrix}. \tag{34}$$

(viii) Step 8: compute the positive and negative ideal solutions. The positive ideal solution is denoted as

$$\delta^{+} = \left(\delta_j^{+} : j = 1, \cdots, n\right), \tag{35}$$

and the negative ideal solution is denoted as

$$\delta^{-} = \left(\delta_j^{-} : j = 1, \cdots, n\right), \tag{36}$$

where

$$\delta^{+} = \left(\max_{1\le i\le m} \sum_{t=1}^{m} \Phi_1(B_i, B_t), \cdots, \max_{1\le i\le m} \sum_{t=1}^{m} \Phi_n(B_i, B_t)\right),$$
$$\delta^{-} = \left(\min_{1\le i\le m} \sum_{t=1}^{m} \Phi_1(B_i, B_t), \cdots, \min_{1\le i\le m} \sum_{t=1}^{m} \Phi_n(B_i, B_t)\right). \tag{37}$$

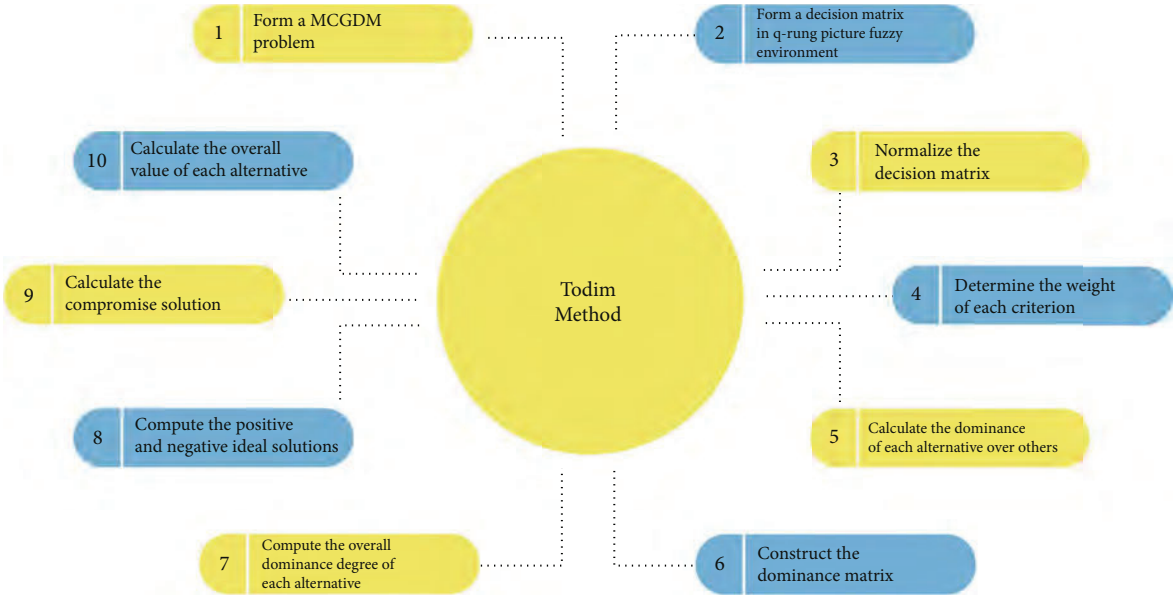


Figure 2: Flowchart of the extended TODIM method.

(ix) Step 9: use the maximum group utility M_i and the minimum individual regret value S_i to calculate the compromise solution as follows:

$$M_i = \sum_{j=1}^{n} w_j \frac{d\left(\delta_j^+, \delta_{ij}\right)}{d\left(\delta_j^+, \delta_j^-\right)},$$
$$S_i = \max_{1\le j\le n} w_j \frac{d\left(\delta_j^+, \delta_{ij}\right)}{d\left(\delta_j^+, \delta_j^-\right)}. \tag{38}$$

Here, $d(\delta_j^+, \delta_{ij})$ measures the distance between the positive ideal solution and the current solution δ_{ij}, while $d(\delta_j^+, \delta_j^-)$ measures the distance between the positive ideal solution and the negative ideal solution where

$$d\left(\delta_j^+, \delta_{ij}\right) = \max_{1\le j\le n} \sum_{t=1}^{m} \Phi_j(B_i, B_t) - \sum_{t=1}^{m} \Phi_j(B_i, B_t),$$
$$d\left(\delta_j^+, \delta_j^-\right) = \max_{1\le j\le n} \sum_{t=1}^{m} \Phi_j(B_i, B_t) - \min_{1\le j\le n} \sum_{t=1}^{m} \Phi_j(B_i, B_t). \tag{39}$$

(x) Step 10: calculate the overall value of Q_i as

$$Q_i = \tau\left[\frac{M_i - M^-}{M^\perp - M^-}\right] + (1-\tau)\left[\frac{S_i - S^-}{S^\perp - S^-}\right], \tag{40}$$

where $M^\perp = \max_i(M_i)$, $M^- = \min_i(M_i)$, $S^\perp = \max_i(S_i)$, and $S^- = \min_i(S_i)$. Here, the coefficient τ and $1-\tau$ represent the weights assigned to the maximum group utility (M_i) and individual regret (S_i), respectively

Steps of the extended TODIM method are visualized in Figure 2.

4.2. Numerical Example

(i) Step 1: in a recent study, Beg et al. [15] have investigated a MCDM problem using 3-RPFSs and aggregation operators. A construction supervisor plays a crucial role in overseeing construction activities at a worksite, involving tasks such as project planning, monitoring, and ensuring a safe working environment. They are responsible for managing contractors, staff, budgets, and policies and adhering to schedules. Hiring a competent construction supervisor is vital for a company's efficiency and productivity. In this scenario, a construction company is in the process of recruiting a supervisor for its ongoing projects. Three alternatives (B_1, B_2, B_3) have appeared for an interview as alternatives. The evaluation of these candidates is based on five attributes: the ability to read and understand blueprints, schematics, and construction documents (C_1); monitoring the project budget (C_2); being highly qualified (C_3); possessing communication skills (C_4); and having a reasonable salary demand (C_5)

(ii) Steps 2 and 3: we use the normalized decision matrix of [15] recalled in Table 1

(iii) Step 4: Table 2 displays the values of criteria weights calculated using (28)

Table 1: Normalized decision matrix.

	C_1	C_2	C_3	C_4	C_5
B_1	(0.5, 0.3, 0.5)	(0.7, 0.5, 0.6)	(0.4, 0.6, 0.2)	(0.6, 0.1, 0.7)	(0.7, 0.5, 0.3)
B_2	(0.6, 0.2, 0.4)	(0.4, 0.7, 0.2)	(0.8, 0.5, 0.2)	(0.6, 0.4, 0.4)	(0.7, 0.5, 0.6)
B_3	(0.7, 0.5, 0.4)	(0.3, 0.6, 0.8)	(0.7, 0.3, 0.4)	(0.9, 0.3, 0.4)	(0.6, 0.2, 0.4)

Table 2: Weights of the criteria.

$w_1 = 0.166916$	$w_2 = 0.248255$	$w_3 = 0.180575$	$w_4 = 0.188509$	$w_5 = 0.215745$

Table 3: Relative weights of the criteria.

$rw_1 = 0.672358$	$rw_2 = 1.000000$	$rw_3 = 0.479902$	$rw_4 = 0.759338$	$rw_5 = 0.869045$

As a result, Table 3 shows the relative weights of each criterion calculated using (29).

(iv) Steps 5 and 6: to determine the dominance of alternative B_i over B_t under criterion C_j, assuming $\theta = 1$, we calculate five dominance matrices Φ_1 to Φ_5 as follows:

$$\Phi_1 = \begin{bmatrix} 0.00000 & -0.1679 & -0.35542 \\ 0.02986 & 0.00000 & -0.32147 \\ 0.06321 & 0.05717 & 0.00000 \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} 0.0000 & -0.46662 & 0.106671 \\ 0.123422 & 0.0000 & 0.140379 \\ -0.40329 & -0.53072 & 0.0000 \end{bmatrix},$$

$$\Phi_3 = \begin{bmatrix} 0.0000 & -0.62107 & -0.56141 \\ 0.078836 & 0.0000 & 0.047903 \\ 0.071263 & -0.37738 & 0.0000 \end{bmatrix}, \quad (41)$$

$$\Phi_4 = \begin{bmatrix} 0.0000 & -0.38647 & -0.53887 \\ 0.077622 & 0.0000 & -0.51794 \\ 0.108232 & 0.104029 & 0.0000 \end{bmatrix},$$

$$\Phi_5 = \begin{bmatrix} 0.0000 & 0.066169 & 0.65004 \\ -0.28786 & 0.0000 & 0.090677 \\ -0.28279 & -0.39448 & 0.0000 \end{bmatrix}.$$

(v) Step 7: the resulting overall matrix δ is obtained as follows:

$$\delta = \begin{bmatrix} -0.52333 & -0.35994 & -1.18248 & -0.92534 & 0.131173 \\ -0.2916 & 0.263802 & 0.12674 & -0.44032 & -0.19718 \\ 0.12038 & -0.93401 & -0.30612 & 0.21226 & -0.67727 \end{bmatrix}. \quad (42)$$

(vi) Step 8: the positive and negative ideal solutions are computed as follows:

$$\delta^+ = (0.12038, 0.263802, 0.12674, 0.21226, 0.131173),$$
$$\delta^- = (-0.52333, -0.93401, -1.18248, -0.92534, -0.67727). \quad (43)$$

(vii) Step 9: for $i = 1, 2, 3$, we compute M_i and S_i as follows:

$$\begin{aligned} M_1 &= 0.60348, \\ M_2 &= 0.153399, \\ M_3 &= 0.503389, \\ S_1 &= 0.188509, \\ S_2 &= 0.108138, \\ S_3 &= 0.248255. \end{aligned} \quad (44)$$

(viii) Step 10: the total value of Q_i is computed, resulting in the ranking B_3 being superior to B_1, which in turn is superior to B_2

Table 4: Q_i influence indices.

		Ranking	Ranking of [15]
B_1	0.786801	2	3
B_2	0.286801	3	2
B_3	0.888498	1	1

4.3. Comparative Analysis

4.3.1. Comparison with the Existing Literature. The outcomes of the current study, along with the ranking of [15], are displayed in Table 4.

The results presented in Table 4 provide insights into the influence of Q_i based on the calculated Q_i values and rankings. According to the table, B_3 holds the highest influence, as it possesses the highest Q_i value and is ranked at the top. This indicates that B_3 has the most significant impact among the evaluated factors. On the other hand, B_1 and B_2 exhibit lower influence compared to B_3, with B_1 being ranked second and B_2 ranked third. The Q_i values for B_1 and B_2 are lower in magnitude, suggesting a relatively weaker impact compared to B_3. Additionally, the rankings of [15] are provided for comparison purposes. It is noteworthy that the rankings of B_1 and B_2 differ between this study and the referenced work. This discrepancy highlights the potential variability in assessing the influence of factors, which can arise from different methodologies, contexts, or perspectives employed in the studies.

The ranking order heavily depends on the specific criteria and their relative weights assigned in each evaluation. Even a slight variation in the weights can lead to different rankings. We have demonstrated the superiority of our approach over the previous one through several key aspects. The proposed approach incorporates group utility and individual regret, which are often overlooked in traditional methods. By considering both aspects, our approach provides a more comprehensive evaluation of alternatives, capturing the collective preferences of decision-makers while addressing individual concerns.

We have introduced novel entropy and distance measures for q-RPFSs, enhancing the analysis and application of fuzzy sets in decision-making processes. These measures, based on trigonometric functions and the Jensen-Shannon divergence, offer more accurate and meaningful results in various applications. The use of trigonometric functions allows for a more flexible and nuanced representation of uncertainty in q-RPFSs. The Jensen-Shannon divergence takes into account both the overlapping and differing aspects of fuzzy sets, resulting in a more comprehensive evaluation of similarity or dissimilarity. Furthermore, our approach incorporates the TODIM method, a well-established and widely used technique for multicriteria decision-making. By integrating the strengths of TODIM method with the novel measures and the consideration of group utility and individual regret, our approach combines robustness, accuracy, and practicality.

4.3.2. Comparison with Another MCDM Method. In this subsection, we address the same problem using the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) [47], another distance-based MCDM technique, and proceed to compare the outcomes. To facilitate this comparison, we perform defuzzification on the normalized decision matrix presented in Table 1 by using the score function defined in [15], as recalled in Definition 6. Thus, we obtain the decision matrix presented in Table 5.

Table 5: Defuzzified decision matrix.

	C_1	C_2	C_3	C_4	C_5
B_1	0.3423	0.4173	0.4240	0.2913	0.4803
B_2	0.3867	0.4663	0.5430	0.4053	0.4173
B_3	0.4680	0.2437	0.4353	0.5640	0.3867

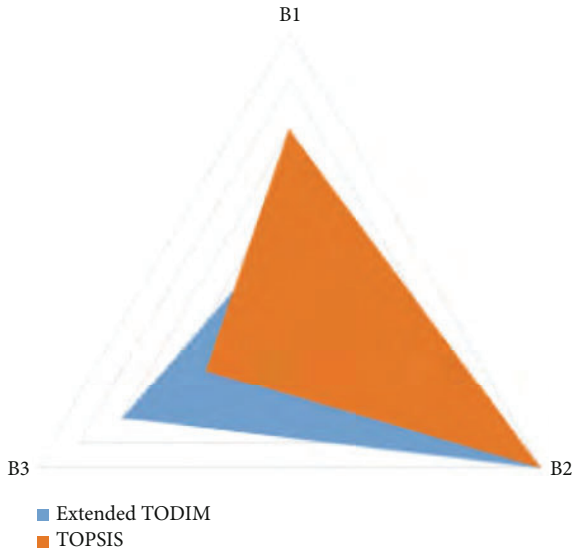


Figure 3: Comparison visualization between the extended TODIM and TOPSIS.

Upon solving the MCDM problem using TOPSIS, we identify the optimal alternative as B_2, aligning precisely with the best alternative identified by the proposed TODIM method. In this solution, the second-best alternative is B_1, while the least favorable option is B_3. The closeness coefficients provide a quantitative measure of the proximity of each alternative to the ideal solution. Higher closeness coefficients indicate greater proximity to the ideal solution. The associated closeness coefficients are 0.66, 0.46, and 0.45 for the respective alternatives.

The consistency in selecting the best alternative, B_2, by both the proposed TODIM method and TOPSIS, lends credibility to the robustness of the decision-making process. This convergence suggests that, despite the distinct methodologies employed, both approaches align in identifying B_2 as the most favorable option. The comparison between the extended TODIM and TOPSIS is visualized in Figure 3.

4.4. Sensitivity Analysis. Analyzing the sensitivity of the results to the parameters τ and θ can provide a better understanding of the robustness of the method and the relative

Table 6: Rankings for different τ values and $\theta = 1$.

τ	Ranking $(\theta = 1)$
0.5	$B_3 \succ B_1 \succ B_2$
1	$B_1 \succ B_3 \succ B_2$
1.5	$B_1 \succ B_3 \succ B_2$
2	$B_1 \succ B_3 \succ B_2$
2.5	$B_1 \succ B_3 \succ B_2$
3	$B_1 \succ B_3 \succ B_2$

importance of group utility and individual regret in the decision-making process. Varying these parameters over a range of values allows for a comparison of the rankings obtained, which can help decision-makers choose a compromise solution that is stable and satisfactory across different parameter settings. In addition, the sensitivity analysis can provide insights into the trade-off between group utility and individual regret, which can help make informed decisions. In this subsection, we present a comparison and sensitivity analysis of the results obtained with different values of τ and θ. We investigate the robustness of our proposed method to variations in these parameters and compare our results to those obtained by other methods. Specifically, we vary the values of τ and θ between 0.5 and 3 with an increment of 0.5 for each parameter.

(i) The value $\tau = 0.5$ is taken, and we get $B_3 \succ B_1 \succ B_2$. Furthermore, the ranking for different τ values is given in Table 6. Just using $\tau = 0.5$ resulted in a different ranking. The different rankings obtained for different values of τ indicate that the overall ranking of alternatives is sensitive to the weight assigned to group utility and individual regret. It is important to note that the choice of τ depends on the decision-maker's preferences and priorities. By considering different τ values, the decision-maker can have a better understanding of the impact of the weight assigned to each criterion on the overall ranking of alternatives. This sensitivity analysis can provide insights into the robustness and reliability of the decision-making process

(ii) For $\theta = 0.5$, we obtain the overall matrix δ_1, as well as the positive and negative ideal solutions, as follows:

$$\delta_1 = \begin{bmatrix} -1.04665 & -0.82656 & -2.36495 & -1.85068 & 0.131173 \\ -0.61307 & 0.263802 & 0.12674 & -0.95827 & -0.48504 \\ 0.12038 & -1.86802 & -0.18032 & 0.21226 & -1.35453 \end{bmatrix},$$

$$\delta_1^+ = (0.12038, 0.263802, 0.12674, 0.21226, 0.131173),$$

$$\delta_1^- = (-1.04665, -1.86802, -2.36495, -1.85068, -1.35453). \quad (45)$$

Table 7: Q_i influence indices for $\theta = 0.5$.

τ	Q_i values $(\theta = 0.5)$		
0.5	$Q_1 = 0.788576$	$Q_2 = 0.00000$	$Q_3 = 0.835442$
1	$Q_1 = 1.000000$	$Q_2 = 0.000000$	$Q_3 = 0.670884$
1.5	$Q_1 = 1.211424$	$Q_2 = 0.00000$	$Q_3 = 0.506326$
2	$Q_1 = 1.422848$	$Q_2 = 0.0000$	$Q_3 = 0.341768$
2.5	$Q_1 = 1.634271$	$Q_2 = 0.0000$	$Q_3 = 0.177209$
3	$Q_1 = 1.845695$	$Q_2 = 0.0000$	$Q_3 = 0.012651$

The values of M_i and S_i are calculated as follows:

$$\begin{aligned} M_1 &= 0.601539, \\ M_2 &= 0.301346, \\ M_3 &= 0.478682, \\ S_1 &= 0.188509, \\ S_2 &= 0.106962, \\ S_3 &= 0.248255, \end{aligned} \quad (46)$$

and the overall values of Q_i are computed for promised τ values, as shown in Table 7.

As a result, the rankings for various τ values are presented in Table 8.

(iii) For $\theta = 1.5$, the overall matrix δ_2 and positive and negative ideal solutions are obtained as follows:

$$\delta_2 = \begin{bmatrix} -0.34888 & -0.20441 & -0.78832 & -0.61689 & 0.131173 \\ -0.18445 & 0.263802 & 0.12674 & -0.26767 & -0.10123 \\ 0.12038 & -0.62267 & -0.18032 & 0.21226 & -0.45151 \end{bmatrix},$$

$$\delta_2^+ = (0.12038, 0.263802, 0.12674, 0.21226, 0.131173),$$

$$\delta_2^- = (-0.34888, -0.62267, -0.78832, -0.61689, -0.45151). \quad (47)$$

The values of M_i and S_i are calculated as follows:

$$\begin{aligned} M_1 &= 0.605684, \\ M_2 &= 0.30359, \\ M_3 &= 0.503978, \\ S_1 &= 0.188509, \\ S_2 &= 0.109114, \\ S_3 &= 0.248255, \end{aligned} \quad (48)$$

and thus, the overall value of Q_i are computed for different τ values and presented in Table 9. Accordingly, the rankings for different τ values are presented in Table 10.

Table 8: Rankings for different τ values and $\theta = 0.5$.

τ	Ranking $(\theta = 0.5)$
0.5	$B_3 \succ B_1 \succ B_2$
1	$B_1 \succ B_3 \succ Q_2$
1.5	$B_1 \succ B_3 \succ B_2$
2	$B_1 \succ B_3 \succ B_2$
2.5	$B_1 \succ B_3) \succ B_2$
3	$B_1 \succ B_3 \succ B_2$

Table 9: Q_i influence indices for $\theta = 1.5$.

τ	Q_i values $(\theta = 1.5)$		
0.5	$Q_1 = 0.785306$	$Q_2 = 0.00000$	$Q_3 = 0.831666$
1	$Q_1 = 1.000000$	$Q_2 = 0.00000$	$Q_3 = 0.663331$
1.5	$Q_1 = 1.214694$	$Q_2 = 0.00000$	$Q_3 = 0.494997$
2	$Q_1 = 1.429388$	$Q_2 = 0.00000$	$Q_3 = 0.326663$
2.5	$Q_1 = 1.644083$	$Q_2 = 0.00000$	$Q_3 = 0.158329$
3	$Q_1 = 1.858777$	$Q_2 = 0.00000$	$Q_3 = -0.01001$

Table 10: Rankings for different τ values and $\theta = 1.5$.

τ	Ranking $(\theta = 1.5)$
0.5	$B_3 \succ B_1 \succ B_2$
1	$B_1 \succ B_3 \succ B_2$
1.5	$B_1 \succ B_3 \succ B_2$
2	$B_1 \succ B_3 \succ B_2$
2.5	$B_1 \succ B_3 \succ B_2$
3	$B_1 \succ B_2 \succ B_3$

(iv) For $\theta = 2$, the overall matrix δ_3 and positive and negative ideal solution are obtained as follows:

$$\delta_3 = \begin{bmatrix} -0.26166 & -0.12664 & -0.59124 & -0.46267 & 0.131173 \\ -0.13087 & 0.263802 & 0.12674 & -0.18135 & -0.05325 \\ 0.12038 & -0.467 & -0.11743 & 0.21226 & -0.33863 \end{bmatrix},$$

$$\delta_3^+ = (0.12038, 0.263802, 0.12674, 0.21226, 0.131173),$$

$$\delta_3^- = (-0.26166, -0.467, -0.59124, -0.46267, -0.33863). \quad (49)$$

The values of M_i and S_i are calculated as follows:

$$\begin{aligned} M_1 &= 0.607196, \\ M_2 &= 0.304401, \\ M_3 &= 0.504516, \\ S_1 &= 0.188509, \\ S_2 &= 0.109936, \\ S_3 &= 0.248255, \end{aligned} \quad (50)$$

Table 11: Q_i influence indices for $\theta = 2$.

τ	Q_i values $(\theta = 2)$		
0.5	$Q_1 = 0.784029$	$Q_2 = 0.00000$	$Q_3 = 0.830446$
1	$Q_1 = 1.000000$	$Q_2 = 0.00000$	$Q_3 = 0.660892$
1.5	$Q_1 = 1.215971$	$Q_2 = 0.00000$	$Q_3 = 0.491338$
2	$Q_1 = 1.431941$	$Q_2 = 0.00000$	$Q_3 = 0.321784$
2.5	$Q_1 = 1.647912$	$Q_2 = 0.00000$	$Q_3 = 0.15223$
3	$Q_1 = 1.863882$	$Q_2 = 0.00000$	$Q_3 = -0.01732$

Table 12: Rankings for different τ values and $\theta = 2$.

τ	Ranking $(\theta = 2)$
0.5	$B_3 \succ B_1 \succ B_2$
1	$B_1 \succ B_3 \succ B_2$
1.5	$B_1 \succ B_3 \succ B_2$
2	$B_1 \succ B_3 \succ B_2$
2.5	$B_1 \succ B_3 \succ B_2$
3	$B_1 \succ B_2 \succ B_3$

and thus, the overall value of Q_i are computed for different τ values in Table 11.

The rankings for different values of τ are presented in Table 12.

(v) For $\theta = 2.5$, the overall matrix δ_4 and positive and negative ideal solution are obtained as follows:

$$\delta_4 = \begin{bmatrix} -0.20933 & -0.07998 & -0.47299 & -0.37014 & 0.131173 \\ -0.09873 & 0.263802 & 0.12674 & -0.12956 & -0.02447 \\ 0.12038 & -0.3736 & -0.07969 & 0.21226 & -0.27091 \end{bmatrix},$$

$$\delta_4^+ = (0.12038, 0.263802, 0.12674, 0.21226, 0.131173),$$

$$\delta_4^- = (-0.20933, -0.3736, -0.47299, -0.37014, -0.27091). \quad (51)$$

The values of M_i and S_i are calculated as follows:

$$\begin{aligned} M_1 &= 0.608457, \\ M_2 &= 0.305073, \\ M_3 &= 0.505007, \\ S_1 &= 0.188509, \\ S_2 &= 0.110922, \\ S_3 &= 0.248255, \end{aligned} \quad (52)$$

and thus, the overall value of Q_i are computed for different τ values in Table 13.

The ranking for different τ values can be seen in Table 14.

Table 13: Q_i influence indices for $\theta = 2.5$.

τ	Q_i values ($\theta = 2.5$)		
0.5	$Q_1 = 0.782479$	$Q_2 = 0.00000$	$Q_3 = 0.829507$
1	$Q_1 = 1.000000$	$Q_2 = 0.00000$	$Q_3 = 0.659014$
1.5	$Q_1 = 1.217521$	$Q_2 = 0.00000$	$Q_3 = 0.488521$
2	$Q_1 = 1.435043$	$Q_2 = 0.00000$	$Q_3 = 0.318028$
2.5	$Q_1 = 1.652564$	$Q_2 = 0.00000$	$Q_3 = 0.147535$
3	$Q_1 = 1.870085$	$Q_2 = 0.00000$	$Q_3 = -0.02296$

Table 14: Rankings for different τ values and $\theta = 2.5$.

τ	Ranking ($\theta = 2.5$)
0.5	$B_3 \succ B_1 \succ B_2$
1	$B_1 \succ B_3 \succ B_2$
1.5	$B_1 \succ B_3 \succ B_2$
2	$B_1 \succ B_3 \succ B_2$
2.5	$B_1 \succ B_3 \succ B_2$
3	$B_1 \succ B_2 \succ B_3$

(vi) For $\theta = 3$, the overall matrix δ_5 and positive and negative ideal solution are obtained as follows:

$$\delta_5 = \begin{bmatrix} -0.17444 & -0.04887 & -0.39416 & -0.30845 & 0.131173 \\ -0.07729 & 0.263802 & 0.12674 & -0.09503 & -0.000528 \\ 0.12038 & -0.31134 & -0.05453 & 0.21226 & -0.22576 \end{bmatrix},$$

$$\delta_5^+ = (0.12038, 0.263802, 0.12674, 0.21226, 0.131173),$$

$$\delta_5^- = (-0.17444, -0.31134, -0.39416, -0.30845, -0.22576). \quad (53)$$

The values of M_i and S_i are calculated as follows:

$$\begin{aligned} M_1 &= 0.609525, \\ M_2 &= 0.305637, \\ M_3 &= 0.505459, \\ S_1 &= 0.188509, \\ S_2 &= 0.111915, \\ S_3 &= 0.248255, \end{aligned} \quad (54)$$

and thus, the overall value of Q_i are computed for different τ values in Table 15.

The rankings for different τ values can be found in Table 16.

A comprehensive examination has been conducted for each τ and θ, revealing valuable insights. The results indicate that for $\tau = 0.5$, B_3 emerges as the best alternative, while B_2 is the worst alternative. However, for $\tau = 1, 1.5, 2, 2.5, 3$, the best alternatives are consistently identified as B_1, with the worst alternatives varying between B_2 and B_3. Comparing these results with the example in the literature [15], we observe that for $\tau = 0.5$, the best alternative remains as B_3, and the worst alternative is still B_2. However, for other values of τ, the best alternative changes to B_1. This discrepancy stems from our unique perspective and the utilization of a different method in examining the example.

To visually illustrate the outcomes of the sensitivity analysis for varying τ and θ values, we present a 3D scatter plot in Figure 4. The plot depicts the overall values of Q_i for each alternative B_i across different combinations of τ and θ. Through this plot, we can observe the impact of changes in τ and θ values on the overall ranking of the alternatives.

The findings of this study affirm the effectiveness of the proposed extended TODIM method in addressing MCDM problems in the q-RPFS environment. Additionally, the sensitivity analysis conducted by manipulating the τ and θ values reveals the potential variation in alternative rankings based on the assigned weights. Hence, careful consideration should be given to the selection of weights for group utility and individual regret to obtain the most appropriate ranking. Overall, this study underscores the significance of adopting different methods and perspectives when approaching MCDM problems in the q-RPFS environment.

Table 15: Q_i influence indices for $\theta = 3$.

τ	Q_i values ($\theta = 3$)		
0.5	$Q_1 = 0.780895$	$Q_2 = 0.00000$	$Q_3 = 0.828775$
1	$Q_1 = 1.000000$	$Q_2 = 0.00000$	$Q_3 = 0.65755$
1.5	$Q_1 = 1.219105$	$Q_2 = 0.00000$	$Q_3 = 0.486326$
2	$Q_1 = 1.438211$	$Q_2 = 0.00000$	$Q_3 = 0.315101$
2.5	$Q_1 = 1.657316$	$Q_2 = 0.00000$	$Q_3 = 0.143876$
3	$Q_1 = 1.876422$	$Q_2 = 0.00000$	$Q_3 = -0.02735$

Table 16: Rankings for different τ values and $\theta = 3$.

τ	Ranking ($\theta = 3$)
0.5	$B_3 \succ B_1 \succ B_2$
1	$B_1 \succ B_3 \succ B_2$
1.5	$B_1 \succ B_3 \succ B_2$
2	$B_1 \succ B_3 \succ B_2$
2.5	$B_1 \succ B_3 \succ B_2$
3	$B_1 \succ B_2 \succ B_3$

4.5. Evaluation of the Application. In this section, we assess the effectiveness of applying the extended TODIM method to the recruitment process for a construction supervisor within the context of a MCDM problem, as previously delineated by Beg et al. [15].

(1) Using the proposed entropy, our approach assigns appropriate weights to the criteria based on their relative importance in the construction supervisor selection process. This weighting reflects the

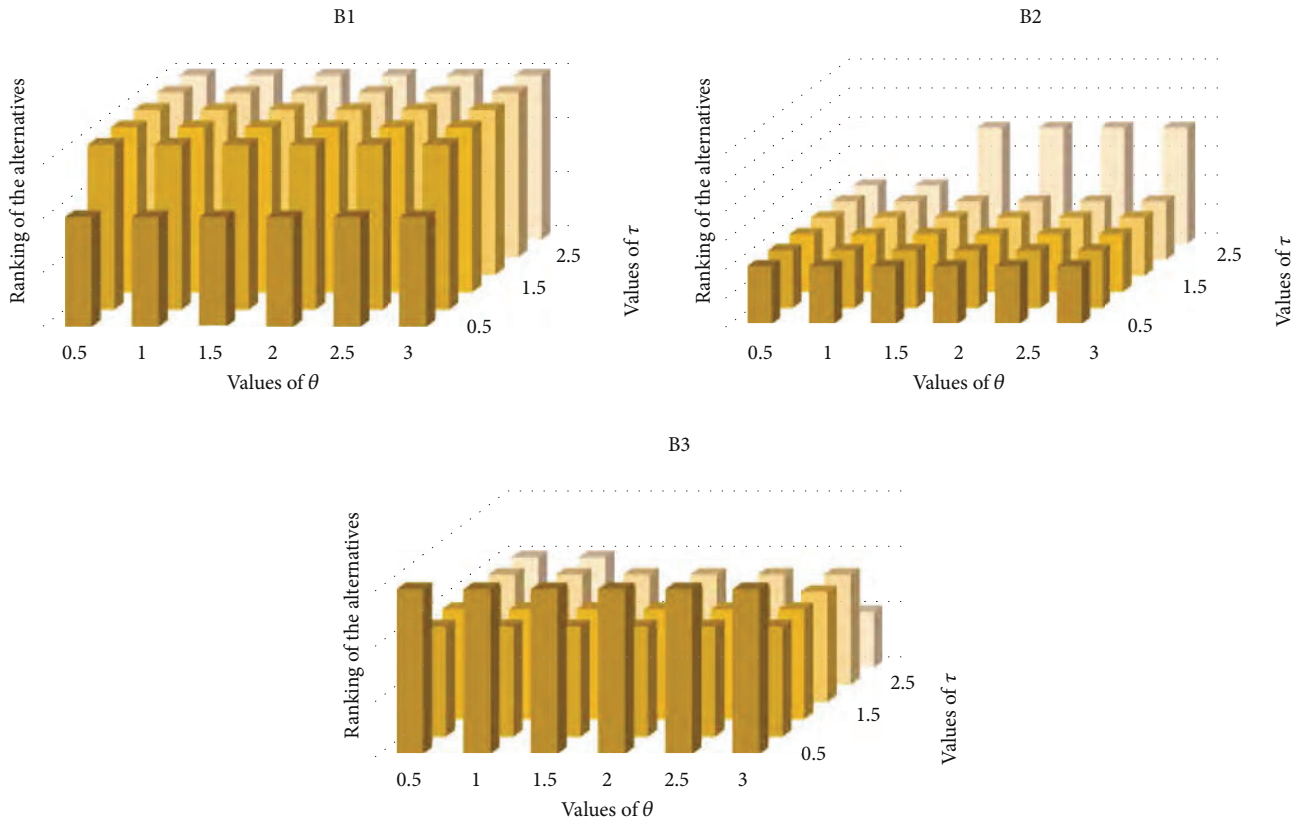


FIGURE 4: 3D sensitivity analysis plot for the extended TODIM method with varying τ and θ values.

significance of each criterion in meeting the overall objectives of the construction project management

(2) The method evaluates each candidate (B1, B2, and B3) against the established criteria (C1 to C5). The innovative entropy measures, particularly the sine function, play a crucial role in quantifying the fuzziness and uncertainties associated with each candidate's qualifications and performance in the specified criteria

(3) Unlike traditional MCDM methods, our proposed approach integrates group utility and individual regret into the decision-making process

(4) Sensitivity analysis is conducted by varying the weights assigned to group utility and individual regret, along with the parameters τ and θ used in the extended TODIM method. This analysis provides insights into the robustness of the proposed method under different scenarios, offering decision-makers a comprehensive understanding of its performance

5. Conclusion

In this paper, we proposed an extended TODIM method for MCDM problems in the q-RPFS setting. The proposed method was applied to a construction project manager selection problem from the literature, and its effectiveness was demonstrated by comparing the results with previous studies.

Through the analysis of the results, we found that the proposed method provided a flexible and efficient framework for handling MCDM problems in a q-RPFS environment. By considering the weight assigned to group utility and individual regret, the proposed method could generate different rankings for different τ values, allowing for sensitivity analysis and a more nuanced understanding of the decision problem. Furthermore, the results showed that the proposed method could yield different rankings from previous studies, indicating the importance of examining a problem from multiple viewpoints and with different methods.

In conclusion, the proposed extended TODIM method offers a promising approach for MCDM problems in q-RPFS environments and could be applied to a range of decision-making scenarios in various fields. Future research can focus on exploring additional applications of the proposed method in various domains and evaluating its effectiveness in different contexts. While the method exhibits promise in addressing the construction project management problem scrutinized in this study, its utility can be investigated in realms beyond, including finance, healthcare, and environmental management. A valuable avenue for research involves scrutinizing the sensitivity of the method to diverse parameter values, shedding light on its robustness and stability. Additionally, extending the method to tackle more intricate decision problems holds the potential to augment its practical applicability, furnishing decision-makers with

a formidable and versatile tool. Moreover, our method is adaptable for application to another extension of fuzzy sets, such as quasirung fuzzy sets (see, e.g., [48, 49]). Ultimately, the method presented here can be seamlessly incorporated into fuzzy soft sets, as demonstrated in previous works such as [50–53].

Data Availability

The data used in this paper can be found in [15].

Ethical Approval

The study was conducted in accordance with ethical guidelines and does not infringe upon any existing publication ethics policies.

Conflicts of Interest

We confirm that there are no conflicts of interest regarding this manuscript.

Authors' Contributions

We hereby confirm the final authorship for this manuscript, including BA, MO, FS, and MÜ, and ensure that all authors meet the ICMJE criteria. The contributions of each author are as follows: MO and MÜ conceptualized the study and provided critical insights throughout the research process. BA conducted the data analysis and interpretation under the supervision of MO, FS, and MÜ who also contributed to the methodological design and oversight.

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Approach to the personnel selection process in a group decision-making environment based on the PSI method

Pristup procesu selekcije kadrova u grupnom odlučivanju zasnovan na PSI metodi

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Abstract: *The selection of adequate personnel is significant for the successful operation of companies. Therefore, this article observed the application of the Preference Selection Index (PSI) method for evaluating personnel in the conditions of group decision-making. The PSI method was chosen because it does not require determining the weights of the criteria, which is why it can be suitable for application by HR managers unfamiliar with the application of multiple criteria decision-making methods.*

Keywords: *human resources management, personnel selection, recruitment, PSI method, MCDM*

Apstrakt: *Izbor adekvatnih kadrova je veoma značajan za uspešno poslovanje kompanija. Zbog toga je u ovom radu pazmatrana primena Preference Selection Index (PSI) metoda za evaluaciju kadrova u uslovima grupnog odlučivanja. PSI metoda je izabrana jer ne zahteva određivanje težina kriterijuma zbog čega može biti pogodna za primenu od strane menadžera ljudskih resursa kojima nije bliska primena metoda višekriterijumskog odlučivanja.*

Ključne reči: *upavljanje ljudskim resursima, izbor kadrova, PSI metoda, MCDM*

Introduction

Recruitment and selection of adequate personnel are essential for the efficient functioning and advancement of the company in a competitive environment. That is why almost every contemporary company has a specialized part, the Human Resources (HR) management department, which deals with recruitment and the selection of personnel needed for the efficient functioning of the company.

In addition to numerous other approaches, using multiple criteria decision-making (MCDM) methods in the recruitment and selection process can be identified as one of the actual approaches. As a result, numerous articles were published in the scientific and professional literature, such as Liang and Wang (1994), Dursun and Karsak (2010), Karabašević et al. (2015), Ulutaş et al. (2020), Popović (2021), Uslu et al. (2021), and so on.

Numerous MCDM methods were applied in these researches, such as TOPSIS (Kelemenis & Askounis, 2010; Matin et al., 2011; Samanlioglu et al., 2018; Nabeehet al. 2019), VIKOR (Liu et al., 2015; Krishankumar et al., 2020), PROMETHEE (Luo & Xing, 2019), EDAS (Karabasevic et al., 2018), COPRAS (Zolfani et al., 2012; Ighravwe & Oke, 2019), and MULTIMOORA (Baležentis et al., 2012a; Baležentis et al., 2012b; Uslu et al., 2021).

Inspired by the previously mentioned research, the article presents the application of one rarely used MCDM method for personnel evaluation in a group decision-making environment.

Methodology

The Preference Selection Index method

The PSI method was proposed by Maniya and Bhatt (2010). This method is interesting because it does not require criteria weights, considering that calculating criteria weights is an integral part of this method. In addition, this method has a relatively simple and easy-to-understand calculation procedure that can be presented as follows:

Step 1. Evaluate the alternatives and construct initial decision-making matrix *D*, as follows:

$$D = [x_{ij}]_{m\times n} \quad (1)$$

where x_{ij} denotes ratings of the alternative *i* concerning criterion *j*.

Step 2. Construct the normalized decision matrix in which the elements of the matrix are calculated as follows:

$$r_{ij} = \begin{cases} \frac{x_{ij}}{max_i x_{ij}}; & for\ beneficial\ criteria \\ \frac{max_i x_{ij}}{x_{ij}}; & for\ nonbeneficial\ criteria \end{cases}, \quad (2)$$

where r_{ij} denotes normalized ratings of the alternative *i* concerning criterion *j*.

Step 3. Calculate preference variation χ_j value to each criterion as follows:

$$\chi_j = \sum_{i=1}^{m}(r_{ij} - \bar{r}_j)^2, \quad (3)$$

where $\bar{r}_j$ denotes the mean value of normalized ratings of criterion *j*, and it is determined as follows:

$$\bar{r}_j = \frac{1}{m}\sum_{i=1}^{m} r_{ij}. \quad (4)$$

Step 4. Calculate deviation in the preference variation value Ω_j as follows:

$$\Omega_j = 1 - \frac{\chi_j}{m-1}. \quad (5)$$

Step 5. Determine the criteria weights w_j as follows:

$$w_j = \frac{\Omega_j}{\sum_{j=1}^{n}\Omega_j}. \quad (6)$$

Step 6. Calculate the preference selection index of alternatives S_i as follows:

$$S_i = \sum_{j=1}^{n} r_j w_j. \quad (7)$$

Step 7. Based on the alternatives' preference selection index values, determine the alternatives' complete ranking order. The alternative with the most extensive preference selection index represents the best-ranked alternative.

The Preference Selection Index method in group decision-making

Many complex decision-making problems, such as the ranking and selection of candidates in the recruitment process, require the participation of several or more decision-makers (DM). Numerous procedures proposed for adapting MCDM methods in a group environment are discussed in the literature. In this article, two straightforward approaches for applying the PCI method in a group environment are presented and discussed.

The first approach is based on the complete calculations using the PCI method for each DM, determining the ranking orders of alternatives based on the attitudes of each DM and finally selecting the most appropriate alternative, i.e., candidate using the Dominance Theory (DT). It should be

emphasized here that the basis of the DT is that the best alternative is the alternative that most often appears in the first position.

The second approach is based on calculating a group decision matrix based on the ratings obtained from all DMs involved in the evaluation and calculation using the PSI method based on this matrix. In this approach, the elements of the group decision matrix can be determined as follows:

$$DG = \left[x_{ij}^{G}\right]_{m\times n}, \tag{8}$$

$$x_{ij}^{G} = \frac{1}{k}\sum_{l=1}^{k} x_{ij}^{l} \tag{9}$$

where x_{ij}^{l} denotes ratings of the alternative i in relation to criterion j obtained from decision maker l, and k denotes the number of decision-makers involved in the evaluation.

Illustrative example

In order to show the usability of the PSI method for ranking and selecting candidates, a numerical example adopted from Karabasevic et al. (2018) is discussed in this section. In the mentioned example, an evaluation of a candidate for the position of human resource manager in a telecommunications company was carried out based on the following criteria: C_1 – Relevant work experience, C_2 – Education, C_3 – Communication and presentation skills, C_4 – People management skills, C_5 – Organizational and planning skills and C_6 – Foreign languages.

The decision matrices obtained from the three DMs involved in the evaluation are shown in Tables 1 to 3, while the group decision matrix is shown in Table 4.

Table 2. The decision matrix obtained from the first of three DMs

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	4	4	3	4	4	3
A_2	4	3	4	4	3	3
A_3	5	4	3	5	4	4

Table 2. The decision matrix obtained from the second of three DMs

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	3	4	3	4	4	3
A_2	5	4	3	3	3	3
A_3	3	3	3	3	4	2

Table 3. The decision matrix obtained from the third of three DMs

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	4	4	3	3	4	3
A_2	3	4	3	4	3	3
A_3	3	4	3	3	2	3

Table 4. Group decision matrix obtained from three DMs

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	3.67	4.00	3.00	3.67	4.00	3.00
A_2	4.00	3.67	3.33	3.67	3.00	3.00
A_3	3.67	3.67	3.00	3.67	3.33	3.00

The first approach

The normalized decision matrix formed based on the ratings obtained from the first DM, using Eq. (2), is shown in Table 5, while the values of χ_j, Ω_j and criteria weights, calculated using Eqs. (3), (5), and (6) are shown in Table 6.

Table 5. The normalized decision matrix formed based on the ratings obtained from the first DM

	C1	C2	C3	C4	C5	C6
A1	0.80	1.00	0.75	0.80	1.00	0.75
A2	0.80	0.75	1.00	0.80	0.75	0.75
A3	1.00	1.00	0.75	1.00	1.00	1.00

Table 6. Calculation details obtained using the PSI method

	C1	C2	C3	C4	C5	C6
χ_j	0.027	0.042	0.042	0.027	0.042	0.042
Ω_j	0.987	0.979	0.979	0.987	0.979	0.979
w_j	0.168	0.166	0.166	0.168	0.166	0.166

The weighted normalized decision matrix, calculated by multiplying the values from the normalized decision matrix, shown in Table 5, and the criteria weights, shown in Table 6, are presented in Table 7. Table 7 also shows the values of S_i, calculated using Eq. (7), as well as the ranking order of the alternatives.

Table 7. The weighted normalized decision matrix and ranking orders of alternatives

	C1	C2	C3	C4	C5	C6	S_i	Rank
A1	0.13	0.17	0.12	0.13	0.17	0.12	0.85	2
A2	0.13	0.12	0.17	0.13	0.12	0.12	0.81	3
A3	0.17	0.17	0.12	0.17	0.17	0.17	0.96	1

Using the previously presented procedure, the ranking orders of alternatives, the candidates, were determined for the second and third DMs, and the obtained results are summarized in Table 8.

Table 8. Summarized ranking orders based on the ratings obtained from three DMs

	DM I		DM II		DM III		Overall
	S_i	Rank	S_i	Rank	S_i	Rank	Rank
A1	0.85	2	0.94	1	0.96	1	1
A2	0.81	3	0.92	2	0.92	2	2
A3	0.96	1	0.80	3	0.84	3	2

As can be observed from Table 8, the alternative, that is, the candidate denoted as A1, is the best placed or the most suitable candidate selected based on the ratings obtained from three DMs involved in the evaluation using the PSI method.

The second approach

The second calculation form performed using the second proposed approach is briefly summarized in Tables 9 to 11.

Table 9. The normalized decision matrix formed based on three DMs

	C1	C2	C3	C4	C5	C6
A1	0.92	1.00	0.90	1.00	1.00	1.00
A2	1.00	0.92	1.00	1.00	0.75	1.00
A3	0.92	0.92	0.90	1.00	0.83	1.00

Table10. Calculation details were obtained using the PSI method based on the ratings obtained from three DMs

	C_1	C_2	C_3	C_4	C_5	C_6
χ_j	0.005	0.005	0.007	0.000	0.032	0.000
Ω_j	0.998	0.998	0.997	1.000	0.984	1.000
w_j	0.167	0.167	0.167	0.167	0.165	0.167

Table 11. The weighted normalized decision matrix and ranking orders of alternatives based on the ratings of three DMs

	C_1	C_2	C_3	C_4	C_5	C_6	S_i	Rank
A_1	0.15	0.17	0.15	0.17	0.16	0.17	0.97	1
A_2	0.17	0.15	0.17	0.17	0.12	0.17	0.94	2
A_3	0.15	0.15	0.15	0.17	0.14	0.17	0.93	3

As can be concluded from Table 11, the best alternative, the candidate selected by applying the second approach, is the candidate designated as A_1.

Conclusion

This article presents the application of the PSI method for evaluating candidates in the recruitment and selection process. The mentioned method does not require the determination of criteria weights because the calculation procedure of this method includes the determination of the significance of the criteria. For this reason, this method can be applicable in cases of evaluation when the criteria weights are not known. In addition, the calculation procedure of this method is still relatively simple and understandable, which is why its application can be interesting for use by DMs who need to become more familiar with the application of MCDM methods.

The article also discusses two possible approaches for using the PSI method in group decision-making. The first approach requires significant calculations but is more suitable for analysis and forming a compromise solution that satisfies the attitudes of all DMs involved in the evaluation. The second approach is more compact and involves fewer calculations because it is based on applying a group decision matrix, but it is also less informative. That is, it does not provide significant opportunities for conducting additional analyses. In addition, this approach can be suitable when several DMs are involved in the decision-making process. In addition, this approach can be easily adapted for decision-making in cases where different DMs have different weights, that is, when their attitudes can have a different impact on the final evaluation of the candidate. The main limitation of the paper is reflected through its inability to reflect the decision environment ambiguity. This shortcoming could be resolved by introducing the fuzzy or grey extensions in the evaluation process, which is also a direction for future research.

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Evaluating Blockchain Cybersecurity Based on Tree Soft and Opinion Weight Criteria Method under Uncertainty Climate

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Abstract

In the era of digital transformation (DT), many digital technologies have emerged and have had a positive impact on society. Nevertheless, because of certain issues with existing technologies, innovative technology has developed to eradicate them. Fog computing (FC) plays a vital role as an intermediate between edge layer and cloud computing (CC) to resolve limited resources and capabilities. In the same vein, blockchain technology (BCT) is responsible for resolving privacy and security issues that IoT suffers from. Due to using cryptography rules and hashing which is utilized in BCT to prevent any trickery. Hence, BC shows promise as a possible remedy for the cybersecurity issue, especially in financial services and transactions. In the context of cybersecurity, it's critical to comprehend how financial sector businesses might handle these issues by investigating the use of BCT for financial transactions. Thereby, we constructed Tree Soft Evaluator (TrSEV) framework to evaluate security for enterprises (ENs) that embrace BCT as a cyber security method. The initial and important step in the evaluation process is determining the influenced criteria. In our problem, we modeled the determined criteria of BCT in Tree soft (TrSo) model. After that these criteria are analyzed and obtaining weights for it. Hence, we harnessed Opinion Weight Criteria Method (OWCM) as a novel multi-criteria decision-making (MCDM) technique to obtain criteria weights, and weighted sum model (WSM) is utilized for ranking enterprises and recommending optimal and worst enterprise. Finally, the techniques harnessed in our constructed framework are employed under the authority of Neutrosophic theory, precisely Single value Neutrosophic sets (SVNSs).

Keywords: Blockchain Technology, Digital Transformation, Opinion Weight Criteria Method, Tree Soft, Single Value Neutrosophic sets.

1 | Introduction

Lately, Information communication technology (ICT) offered digital solutions which are represented in Industry 4.0 (Ind 4.0) which concentrates on utilizing AI-driven technology and digitization to boost efficacy and versatility. Hence, Crises such as the COVID-19 epidemic undoubtedly provided a catalyst [1] for a swift transition to new digital transformation and digitalization modes of operation, which increased the flexibility

of sectors and organizations. Confirming for that this tendency has been leveraged by [2] in a variety of organizations and industries, including finance, data management, supply chain management, medical care management, and government supremacy. Despite that [3] stated that the fundamental values of social justice and sustainability are given less attention in Ind 4.0. Thereby [4] addressed the deficiencies with Ind 4.0 by deploying Ind 5.0 where humans and machines coexist and collaborate in harmony. Accordingly, the concept of society 5.0 has emerged where humans are positioned at the center of innovation, maximizing the benefits of technology results in increased sustainability, social responsibility, and quality of life [1].

With time and the development of unpredictable circumstances and catastrophes, technology has permeated every aspect of our existence. The idea of the Internet of Things (IoT) according to [5] emerged globally in intricate ecosystems, and where employed for collecting data from different resources based on smart equipment. Whilst [6] described IoT as a vast network of physical objects connected through the Internet. Smart objects, or things, can communicate and share information because ultra-cheap sensors, actuators, and chips are becoming more and more common, and wireless networks are becoming more and more widespread. After that, the technology of cloud computing (CC) is leveraged for storing, handling, and processing the collected data from smart equipment. Afterward, fog computing (FC) emerged as in [7] to tackle cloud limitations. From the perspectives of scholars [8] for IoT systems to be deemed secure, they must address several issues, including client privacy, secure data transfer and bootstrapping, physical security design, key management, authentication, and access control methods.

In tandem with these technological advancements [5], Blockchain technology (BCT) and distributed ledgers gained prominence in technology. Due to the ability of BCT to improve security, privacy, capacity, and peer-to-peer capabilities. Additional cybersecurity-related benefits including data ownership, data openness and audibility, and fine-grained access restrictions may be obtained with BC-based systems [9]. Considering these advantages, financial institutions in [10] have been exploring ways to apply BCT to strengthen their position both financially and in terms of cybersecurity.

Scholars provided a list of arguments that corroborate the reasons for deploying BCT in various services, especially financial. (i) BCT employs a decentralized system to safeguard data while guaranteeing that its integrity is maintained[11]. (ii) this technology makes it possible to execute transactions in a decentralized way at a cheaper cost and with outstanding performance[12].(iii) BCT has largest impact on financial services, due to [13] utilized Bitcoin and other cryptocurrency alternatives have been affecting financial services.

Generally speaking, BCT is becoming more and more popular as an avenue for boosting commerce, transparency, and financial transactions. Hence, there are many perspectives that embraced the notion of evaluating the importance and influence of BCT in various industries. For instance, Alsaqer et al. [14] employed analytical hierarchy process (AHP) and technique for order of preference by similarity to the ideal solution(TOPSIS) as techniques of MCDM for evaluating BC model and select optimal one. In the same vein [15] constructed framework for selecting BC consensus protocol through using Simple Additive Weighting (SAW), TOPSIS, and IseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR).

Consequently, we are exploiting these techniques as an approach to healing our problem which represents in implementing these techniques. Hence, these techniques are motivator for this study for constructing framework for evaluating the influence of BCT as cybersecurity method for enterprises which embracing this technology in their financial services. To guarantee that this framework is used in all circumstances where there is ambiguous or perplexing information for partners or individuals who contribute to the evaluation process. we are implementing the utilized MCDM techniques under authority of Neutrosophic theory which introduced by Smarandache[16] as an advocate or supporter in these circumstances of uncertainty.

Finally, this study follows set of steps through leveraging modern techniques in constructing decision making framework.

Firstly, through surveys conducted for prior studies, we determine the latest effective techniques. These techniques are considered the main catalyst for the second step. Hence, Secondly, tree soft technique (TrSoT) which introduced by Smarandache [17] is exploited for modelling the evaluation and selection process based on influenced criteria and sub-criteria. Thirdly, MCDM techniques under authority of Neutrosophic to obtaining weights for criteria and sub-criteria which modeling by TrSoT after that these weights exploited to evaluate these criteria and sub-criteria to obtain the optimal enterprise which deploying BCT as cybersecurity method. Finally, optimal enterprise is recommending as optimal one.

2 | Background of Volunteered Techniques Toward cybersecurity.

In this study, we illustrate the previous studies which embraced the utilized techniques in decision making process in BCT through this section. Also, this section illustrates the essential concepts which contribute to construct Tree Soft evaluator (TrSEv) for evaluating enterprises based on cybersecurity BCT.

2.1 | MCDM Techniques in evaluation BCT

The BCT validated its efficiency in healthcare environment. Hence authors in [18] evaluated BC platforms through rough Analytic Hierarchy Process (RAHP) and rough Compromise Programming (RCP). Entropy and CRITIC are techniques of MCDM which exploited in [19] to obtain indicators' weights and WSM, TOPSIS, and VIKOR are utilized for ranking process. BCT implemented in supply chain (SC) in [20] for gaining competitive advantage. By enhancing chain and logistics operations in the areas of trust, transparency and accountability, collaboration, information sharing, financial exchanges, and SC integration, BC may help sustainable energy SCs. Thus, the authors applied various techniques for analyzing challenges of implementing BC in renewable energy SCs as gray evaluation based on distance from average solution (EDAS-Gray) and the gray stepwise weight assessment ratio analysis (SWARA-Gray). The findings are validated using an additional set of hybrid techniques, which include the gray weighted sum method (WSM-Gray), the gray complex proportional assessment (COPRAS-Gray), and the gray technique for order of preference by similarity to ideal solution (TOPSIS-Gray). In the same vein [21] examined the obstacles to BCT adoption in the Construction Supply Chain Management (CSCM) domain by dissecting its interconnections.

2.2 | Key concepts of Tree Soft Technique (TrSoT)

The concept of TrSoT introduced by Smarandache [17] also, he is founder for Neutrosophic theory. Whilst Smarandache described and defined this concept as:

Let $\mathfrak{H}$ be a universe of discourse, and $\mathcal{H}$ a non-empty and subset of $\mathfrak{H}$, whilst the powerset of $\mathcal{H}$ denoted as P($\mathcal{H}$).

- Main nodes encompass main attributes/criteria/factors and symbolled as $\mathfrak{R}$. Accordingly, $\mathfrak{R}$ has set of $\mathfrak{R}_s$ with (one-digit indexes) = $\{\mathfrak{R}_1, \mathfrak{R}_2, \ldots \mathfrak{R}_n\}$.
- Sub-nodes which have two-digit indexes and symbolled as:

$\{\mathfrak{R}_{11}, \ldots \mathfrak{R}_{1n}\}$ are sub-nodes of $\mathfrak{R}_1$, $\{\mathfrak{R}_{21}, \ldots \mathfrak{R}_{2n}\}$ are sub-nodes of $\mathfrak{R}_2$, and $\{\mathfrak{R}_{31}, \ldots \mathfrak{R}_{3n}\}$ are sub-nodes of $\mathfrak{R}_3$

- Generally, a graph-tree is formed, that we denote as Tree($\aleph$), whose root is considered of level zero,
- then nodes of level 1, level 2, up to level n.
- We call leaves of the graph-tree, all terminal nodes (nodes that have no descendants).

Then the TreeSoft Set is: F: P(Tree(ℵ)) → P(ℋ).

-All node sets of the TreeSoft Set of level m are: Tree(ℵ) = {ℵi1 | i1= 1, 2, ... }

3 | Methodology of Tree Soft evaluator (TrSEv)

Here in, we are employing MCDM techniques under authority the scale of Neutrosophic theory to encourage decision makers (DMs) to treat incomplete information and uncertainty during evaluation process. Thereby, our constructed TrSEV implement in our problem through following steps.

Step 1: Modelling criteria and sub-criteria as nodes into TrSo.

-Determining set of alternatives of enterprises which contribute to evaluation process based on TrSEV.
-Determining the influenced criteria and sub-criteria which contribute to evaluating security of enterprises which embracing BCT as cybersecurity method.
-DMs panel is formed for rating enterprises based on modelled criteria and sub-criteria into TrSo.

Step 2: Valuation modelled criteria and sub criteria based on OWCM.

-OWCM is applying for the first time in such a problem. This technique of MCDM is introduced by MANDIL et al.[22].

2.1Rating main criteria in level 1 of TrSo.

-Decision matrices are constructed based on number of DMs for evaluating alternatives based on main criteria through utilizing single value Neutrosophic scale (SVNS) as in [16].
-The constructed Neutrosophic matrices are convert to deneutrosophic matrices through Eq. (1)

$$\mathrm{De}_{ij} = \frac{2 + \hbar - \mathcal{F} - \ell}{3} \quad (1)$$

Where:
$\hbar, \mathcal{F}, \ell$ refer to truth, false, and indeterminacy respectively.
-Deneutrosophic matrices are aggregated into single matrix called aggregated matrix by Eq. (2).

$$\wp_{ij} = \frac{\sum_{ij}^{n} \mathrm{De}_{ij}}{\mathbb{N}} \quad (2)$$

Where:

De_{ij} refers to value of criterion in deneutrosophic matrices, $\mathbb{N}$ refers to number of decision makers

-Eq. (3) is exploited for normalizing an aggregated matrix.

$$\partial_{ij} = \frac{\wp_{ij}}{\wp_{j}^{max}} \quad (3)$$

Where:

$\wp_{j}^{max}$ refers to max number in $\wp_{ij}$ matrix per column, $\wp_{ij}$ refers to each element in $\wp_{ij}$ matrix .

-The average score of the decision matrix is calculated as following.

$$\mathcal{B} = \frac{1}{N} \sum_{i=1}^{m} \partial_{ij} \quad (4)$$

Where:
N refers to number of alternatives.
-Preference variation for each criterion is calculate as in Eq. (5)

$$\mho_{j=\sum_{j=1}^{m}[\partial_{ij-\mathcal{B}}]^2} \quad (5)$$

Where:

-The deviation in preference values is calculate based on Eq. (6).

$$\mathcal{E}_{j=1-\mho_j} \quad (6)$$

-Finally, weights for criteria is calculate as in Eq. (7)

$$\omega_{j=\frac{\varepsilon_j}{\sum_{j=1}^{m}\varepsilon_j}} \quad (7)$$

2.2 Rating sub- criteria in node 2 of TrSo.

-Decision matrices for each sub-criteria which inherent and belongs to main criterion at level 1.
-Applying SVNs in [16] to rate alternatives based on sub-criteria by DMs.
-We are following the previous steps in steps of 2.1 until sub-criteria's weights are obtaining.

Step 3: Ranking alternatives of enterprises and recommend optimal one.

Final step in our problem entails in ranking the determined alternatives through exploiting weights of criteria and sub-criteria in TrSo. Herein, we are leveraging WSM technique as ranker for alternatives based on criteria and sub-criteria.

-An aggregated matrix which obtained from previous step is leveraging for normalizing this matrix based o Eq. (8) until Eq. (10).

$$\Omega_{ij=\frac{\wp_{ij}}{\sum_{j=1}^{n}\wp_j}}, \text{For Benficial criteria} \quad (8)$$

$$\mathbb{C}_{ij} = \frac{1}{\wp_{ij}} \quad (9)$$

$$\Omega_{ij=\frac{\mathbb{C}_{ij}}{\sum_{j=1}^{n}\mathbb{C}_j}} \quad \text{, For Benficial criteria} \quad (10)$$

-Weights of main criteria in level 1 are multiply by normalized matrix for obtaining weighted decision matrix as in Eq. (11).

$$\text{weighted}_{ij=}\ \omega_j * \Omega_{ij} \quad (11)$$

-Global score computes through Eq. (11).

$$V(\text{weighted_matrix}_{ij}) = \sum_{j=1}^{n}\text{weightd}_{ij} \quad (12)$$

4 | Case Study

We are validating the ability and accuracy of our TrSEV in a real case study. Hence, we communicated with four enterprises (EN(n)) which represented as candidates of alternatives in his study. Herein, we determined two main criteria and four sub-criteria which modelled in TrSo as in Figure 1. ENs are evaluating based on main and sub criteria in Figure 1 by three DMs who related to our scope.

4.1 The influenced criteria and sub-criteria of BCT cybersecurity are modelled into TrSo as in Figure1.

4.2 Valuation main criteria weights based on OWCM and SVNs

4.2.1 Three Neutrosophic decision matrices for main criteria in level 1 are constructed through rating it by three DMs based on scales of SVN in [16].

4.2.2 Eq. (1) deneutrosophic these matrices and aggregated into an aggregated matrix by Eq. (2) and the matrix illustrated in Table 1.

4.2.3 Table 2 represents normalized matrix based on Eq.(3).

4.2.4 After that, Eq. (5) contributes to generate Preference variation for each criterion through leveraging average scores are results of Eq. (4) as listed in Table 3.

4.2.5 Eq.(6) responsible for calculating the deviation in preference values and results are leveraging in Eq.(7) to obtain final criteria's weights as recorded in Figure 2.

4.3 Valuation sub- criteria weights based on OWCM and SVNs

4.3.1 Three Neutrosophic decision matrices are constructed for each sub-criterion which is inherent from main criterion in level 1.

4.3.2 The steps in 4.2 are followed for generating weights of sub-criteria in Level 2.

4.3.3 Finally, Figure 3 involved the final sub-criteria's weights.

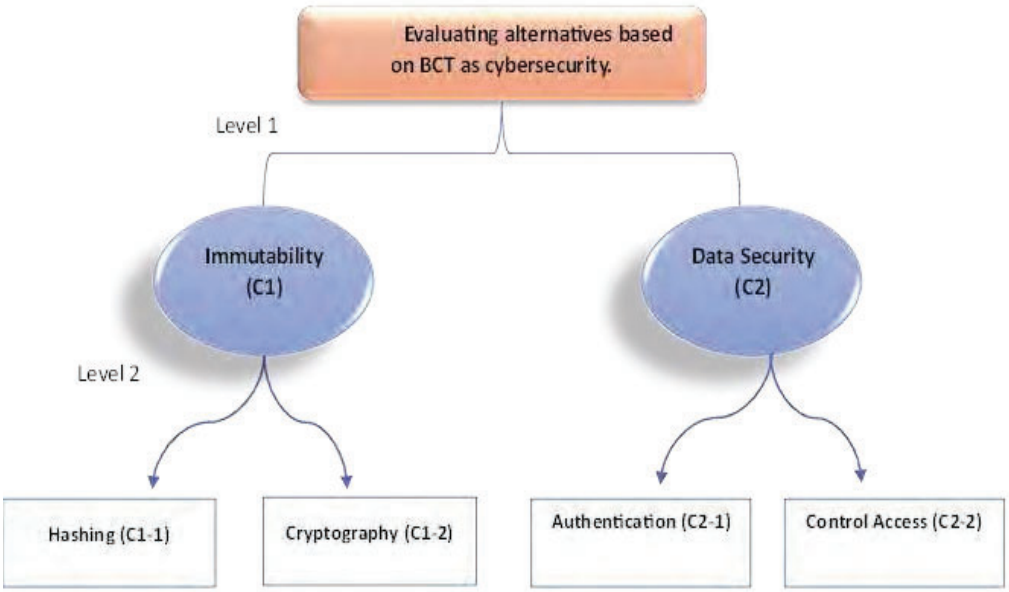


Figure 1. Modelling criteria and sub-criteria in Tree Soft.

Table 1. An aggregated decision matrix for Level 1.

	C1	C2
EN1	0.683333333	0.455555556
EN2	0.555555556	0.666666667
EN3	0.427777778	0.672222222
EN4	0.366666667	0.633333333

Table 2. Normalized matrix

	C1	C2
EN1	1	0.67768595
EN2	0.81300813	0.991735537
EN3	0.62601626	1
EN4	0.536585366	0.94214876

Table 3. Preference variation matrix.

	C1	C2
EN1	0.065585961	0.050718018
EN2	0.004775597	0.007893074
EN3	0.013897151	0.009429855
EN4	0.042980369	0.001541049

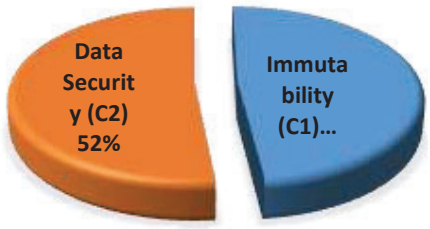
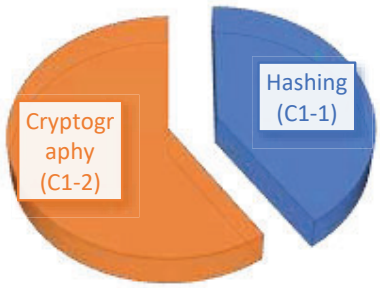
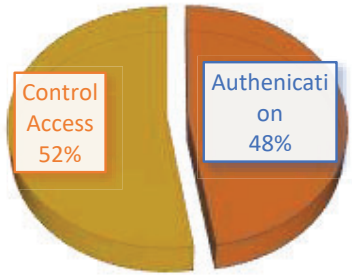


Figure 2. Final criteria weights at Level 1.



Final weights for sub-criteria at Level 2 of C1



Final weights for sub-criteria at Level 2 of C2.

Figure 3. Final weights for sub-criteria at Level.

4.4 Recommending the optimal EN based on value of WSM based on SVNs score.

4.4.1 The aggregated matrix for criteria which generated from step 4.2 is leveraging in Eq. (8) and considering the determined criteria are beneficial to generate normalized matrix and listed in Table 4.

4.4.2 According to Eq.(11) weighted decision matrix is generated and obtained in Table 5.

Eq.(12) is implemented to generate global score for each EN. Final, ranking for ENs is obtained in Figure 4. Which indicated that EN2 is the optimal otherwise, EN4 is the worst

Table 4. Normalized Matrix

	C1	C2
EN1	0.1626599	0.096821476
EN2	0.13224382	0.141689964
EN3	0.10182774	0.142870714
EN4	0.08728092	0.134605466

Table 5. Weighted Decision Matrix.

	C1	C2
EN1	0.33606557	0.187643021
EN2	0.27322404	0.274599542
EN3	0.21038251	0.276887872
EN4	0.18032787	0.260869565

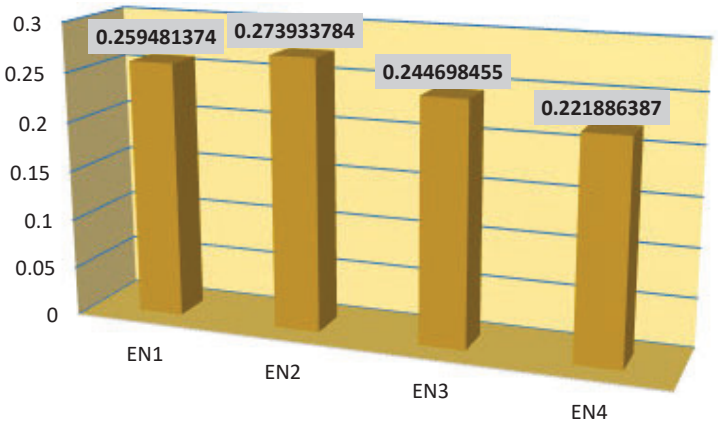


Figure 4. Final Ranking for enterprises.

5 | Conclusion

This study exploited digital transformation (DT) and discussed its ability to transform human life Positively. For instance, IoT makes people's daily life much easier by facilitating data exchange and comprehensive decision-making. Despite the importance and vital role of IoT in various industries, it suffers from some issues and concerns. One of the important concerns is security and privacy. BCT might be an effective solution to security and privacy issues in IoT. Hence, this study discussed and highlighted the role of BCT to guarantee security and protect privacy information in financial services for enterprises.

Accordingly, we analyzed its effect as cybersecurity method in finical services through analyzing the influenced criteria and its sub-criteria which deployed in evaluating the enterprises which embracing BCT cybersecurity in its services. Thereby, the influenced criteria and its sub-criteria are modelled through leveraging TrSoT which modelled it into levels. Each level includes a set of nodes, and each node represents criterion. Whilst TrSoT considered the first step in our study. After that criteria in TrSo are analyzed through utilizing constructed TrSEV framework where OWCM is a novel technique of MCDM which employed for first time in this problem to obtain weights for each criterion in each node. The finding of this technique demonstrated that data security (C2) is optimal with the highest weight in level 1. After that, WSM is exploited in our problem of evaluation as ranker technique of MCDM to rank four enterprises. The findings of this technique indicated that EN2 is optimal in contrast to EN4 is worst one. To bolster our constructed TrSEV in uncertainty circumstances, Neutrosophic theory has been harnessed in utilized techniques for evaluation security of enterprises which embracing BCT in its financial services and transactions.

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All authors contributed equally to this work.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Bv2TrS Appraiser Model: Enforcing BHARAT Version2 in Tree Soft Modelling for Appraising E-Mobility Hurdles

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Abstract: Electric vehicles (EVs) are being introduced to lessen greenhouse gas (GHG) emissions, air pollution, and reliance on fossil fuels. As a result of the government's aggressive promotion of EVs and rising environmental consciousness, EVs are quickly rising to the top of the low-carbon transportation market. Several viewpoints suggested that shifting to electric vehicles has been seen as a potential way to achieve sustainable mobility. Nevertheless, many studies discussed the obstacles and hurdles that obstruct the embracing of various electric-mobility (E-mobility) as EVs and electric-scooters (E-scooters) as eco-friendly means. Herein, we discussed these hurdles and determined them through surveys for prior studies. Therefore, appraising these hurdles is the objective of our study. Best Holistic Adaptable Ranking of Attributes Technique – version 2 (BHARAT -v2) as a novel Multi-Criteria Decision Making (MCDM) technique is leveraged as an appraiser technique for these hurdles. Tree Soft (TrS) methodology is utilized for modeling these hurdles. Hence, we hybridized and integrated two methodologies for constructing BHARAT v2 Tree Soft (Bv2TrS) as an appraiser model. Subsequently, we discussed the findings of the Bv2TrS appraiser model.

Keywords: Electric Vehicles (EVs); Electric-Mobility (E-mobility); Tree Soft; Multi-Criteria Decision Making (MCDM); Best Holistic Adaptable Ranking of Attributes Technique - version2 (BHARAT-v2).

1. Introduction

A global effort is underway to employ renewable energy sources to cut greenhouse gas emissions ;hence achieving a green environment. There are several motivations why nations embrace the concept of a green environment. Whilst [1] highlighted two well-known important environmental issues that are facing the world now are climate change and global warming. Also, the scholars demonstrated that the transport sector is considered the main contributor to these issues. This has been confirmed by [2] where more than 20% of the world's CO_2 emissions come from the transportation sector. [3] stated that fossil fuels like petroleum and diesel, whose burning produces carbon dioxide and other greenhouse gases, are largely used in transportation. As well [4] indicated that India's swift industrialization and commercialization will increase the country's need for transportation, which would increase the demand for vehicles. One growing concern among these forecasts of affluence in the future is global warming. Thereby [5] fossil fuel-powered vehicles are starting to be banned by several territories. An impending shift in the transportation system [6] toward a low-carbon, ecologically sustainable regime is necessary to combat the danger posed by greenhouse gas (GHG) emissions, air pollution, and reliance on finite fossil fuels.

Given a variety of technologies, approaches, and innovations in [7] to help decarbonization, the market for electric vehicles (EVs) is expanding quickly. Moreover, EVs were introduced by Kumar et

al. [8] as one of the best ways to address the current climate crisis. This alteration provides safe, emission-free transportation, improves health, lowers energy prices and consumption, preserves land, and creates jobs [9]. According to [10] As environmentally beneficial and sustainable substitutes for traditional internal combustion engine (ICE)-based automobiles, EVs have drawn a lot of interest.

Even while EVs are better for the environment, there are still significant barriers to their widespread adoption. prior perspectives as [11] categorized the obstacles and hurdles to market penetration from the viewpoints of investors, manufacturers, and governmental organizations. Others as [12] highlighted three categories have been identified as the elements influencing consumers' inclinations to embrace electric vehicles: A user's psychological factors include driving experience, attitudes, emotions, perceived behavioral control, societal influence, and symbolic value. (i) Situational factors include technical specifications, price, environmental factors, and subsidy policy. (ii) Individual variables include gender, age, occupation, income, and education. (iii) Household variables include vehicle ownership, accessibility to plug-in vehicles at home, and household size.

Accordingly, various scholars as [13] Prior research have attempted to determine the hurdles to EV adoption from the viewpoints of experts or users. A pilot study of the clients and expert opinion have been utilized to direct the research to narrow down the identified hurdles. Thereby [14] emphasized the existence of interrelatedness between the examined EV hurdles using factor analysis and recommended, simplifying the problem, concentrating on the reduced set of barriers according to the degree of linkage between those.

Herein, we are analyzing electric mobility in other words e-mobility's hurdles in the form of categories and sub-categories. Hence, we are structuring this problem into a hierarchical shape by using a novel methodology of tree soft sets (TSSs) introduced by Smarandache [15]. Also, this methodology has proven its efficiency and implemented in other studies and applications as evaluating Cloud Services [16] also, in bioinformatics problems [17]. Nevertheless, analyzing mobility's hurdles is based on experts who are related to this field. The appraising process is conducted through leveraging the ability of Multi-Criteria Decision Making (MCDM) techniques which are able to treat conflicting criteria[18–22]. Accordingly, this study contributed to harnessing a novel technique of MDM is Best Holistic Adaptable Ranking of Attributes Technique (BHARAT). this technique is proposed by Rao [23] which harnessed BHARAT version-1 for the first time in selecting optimal electronic commerce (E-commerce) websites in [24].

Generally speaking, the study's objectives are illustrated in a set of points:

1. Conducting surveys related to our scope based on prior studies.
2. Highlighting the importance of e-mobility and EVs toward bolstering the green environment as a result of the previous point. Also, e-mobility's hurdles impede the embracing and implementation of e-mobility and EVs.
3. Determining the hurdles of e-mobility and classifying it into a set of levels through leveraging TSSs.
4. Appraising the determined hierarchical hurdles through BHARAT -v2 of MCDM techniques based on a soft scale used by an expert for appraising. Resulting in, constructing a BHARAT-v2 tree soft model ($B_{v\text{-}2}$TrS) to appraise e-mobility's hurdles.
5. The most and least influential hurdle is the result of the $B_{v\text{-}2}$TrS model.

2. Survey of the Scientific Studies and Methodologies

The intent of conducting this section is to survey and examine the related studies that serve and cover the study's objectives which are stated in previous points.

2.1 Obstacles to Embracing Electric Mobility

For analyzing the e-mobility's hurdles, the crucial step is determining contraindications that retard the implementation of e-mobility for achieving a green environment. The result of conducted surveys of [4, 25] is aggregated in Figure 1.

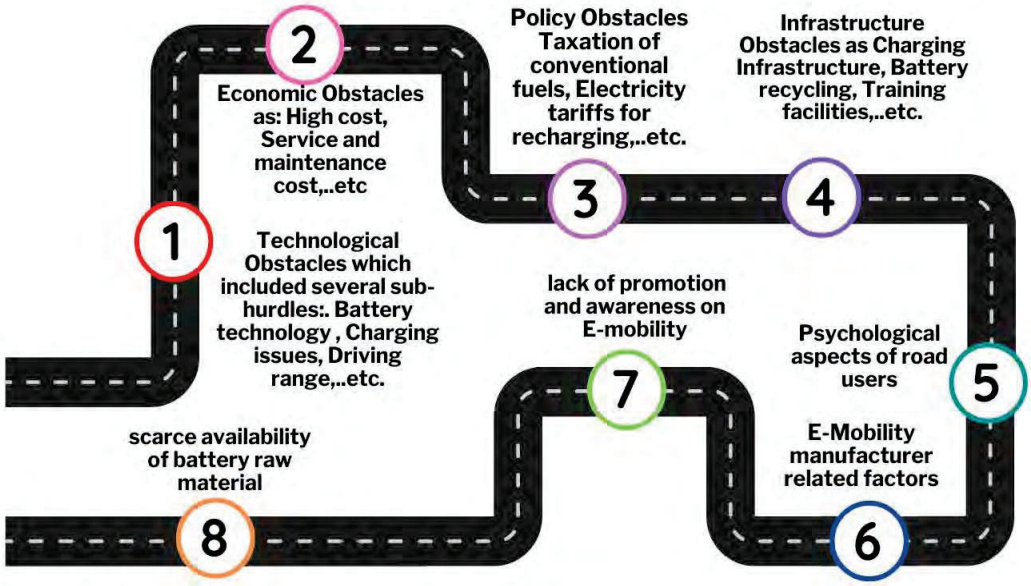


Figure 1. Electric-mobility hurdles.

2.2 *Methodologies for Evaluation E-Mobility*

Herein, we highlight the utilized methodologies that volunteered for appraising e-mobility with various purposes.

The appraising process for e-mobility is conducted by constructing pair-wise matrices based on the Analytical Hierarchical Process (AHP) in [26]. the best and worst criterion that contributes to appraising e-mobility is determined through [27] by employing the best-worst method (BWM). Others rank a set of alternatives of e-mobility and select the optimal one amongst the determined alternatives with support of the Technique for Order of Preference by Similarity to the Ideal Solution (TOPSIS) [28]. Another ranker of MCDM is exploited for ranking alternatives which is represented in VIsekriterijumska optimizacija KOmpromisno Resenje (VIKOR) [29] with support from objective MCDM method of entropy for obtaining criteria's weights. Ecer et al. [30] Used weight-multiplied comparable and weight-similar sequences for performance analysis, the COmbined COmpromise Solution (CoCoSo) technique generates a ranking list. MCDM techniques and the Delphi study are used in [31] to identify and assess the drivers for addressing LIB recycling and explore its key drivers.

2.3 *Core Concepts: New Methodologies in Decision-Making*

This sub-section intends to illustrate the utilized key conceptions in this study and their role in the appraising process.

2.3.1 **Tree Soft Methodology: Structuring Appraising Problem into Hierarchical Form**

We are using a novel expansion of the soft set involved in TSSs in this investigation, which was first introduced by Smarandache and emphasized in [15]. After that scholars of [16] embraced this new expansion and described it as:

Let $\aleph$ be a universe of discourse that includes τ a non-empty as a subset of $\aleph$, thus the powerset of τ expressed as p (τ).

Assume that our tree soft set encompasses a set of levels, each one has a multitude of nodes:

Level 1: consists of a multitude of nodes where each node represents the main criteria or hurdles, then expressed as Hu = {Hu_1, Hu_2,..Hu_n} for integer n ≥ 1.

Level 2: includes several sub-nodes of {Hu_1, Hu_2, ..Hu_n} and stated as { $Hu_{1\text{-}1,}$ …$Hu_{1\text{-}n}$ } branched of Hu_1 , { $Hu_{2\text{-}1,}$ …$Hu_{2\text{-}n}$ } branched of Hu_2 , finally { $Hu_{n\text{-}m,}$ …$Hu_{n\text{-}m}$ } branched of Hu_n .

We call the leaves of the graph-tree, all terminal nodes (nodes that have no descendants).

Then, Tree Soft Set: F: P (Tree (Hu)) → p (τ).

Tree (Hu) is the set of all nodes and leaves (from level 1 to level n) of the graph tree, and P (Tree(δ)) is the powerset of the Tree (Ind). All node sets of TSSs of level n as Tree (Hu) = {Hu_{nm} | nm= 1, 2, ...}.

2.3.2 **Novel BHARAT: Analyzing Hurdles and Sub-hurdles in Constructed Tree Soft**

Generally, BHARAT is considered a novel MCDM technique proposed by Rao [23]. Moreover, it is illustrated and applied in [24]. Herein, we applied BHARAT -v2 in our appraising problem through pair-wise matrices for hurdles and sub-hurdles in tree soft as follows:

1. For generating nodes' weights in level 1:
   1.1 Constructing pairwise matrices for main hurdles based on a number of decision-makers (DMs) who contribute to ranking hurdles. MS utilized soft scales in [32] for ranking main hurdles with each other.
   1.2 Aggregate the constructed matrices into a single decision matrix. Then, computational operations are conducted to obtain the main hurdles' weights based on Eqs. (1) and (2).

$$\wp = \sum_{i=1}^{i} \frac{1}{x_j} \quad (1)$$

$$w_i = \frac{1/\wp}{\sum_{i=1}^{m} 1/\wp} \quad (2)$$

   where x_j indicates to rank of the criterion
2. For generating sub-node weights in level 2:
   2.1 Constructing pairwise matrices for each sub-hurdle by soft scale used by DMs.
   2.2 Aggregate the constructed matrices into a single decision matrix for each sub-hurdle. After that, local weights are generated based on Eqs (1) and (2), and global weights are obtained by multiplying the main hurdles' weights by local sub-hurdles' weights.
3. Ranking main hurdles and sub-hurdles based on weight values.
   - Rank 1: largest main and sub-hurdle weight.
   - least rank: smallest main and sub-hurdle weight.

3. Description of BHARAT-v2 Tree Soft Model: Bv_2TrS

The methodologies in the previous two sub-sections are leveraged in appraising the problem of this study. The following steps illustrate procedures for implementing these novel methodologies in solving our problem of appraising.

3.1 Gathering Information and Prioritizes

- Determining the influenced hurdles and sub-hurdles.
- Forming a panel of experts or DMs who related to our appraising problem. These DMs are contributing to appraising and ranking main and sub-hurdles.
- Soft scale in [32] we utilized a 7-point scale for non-beneficial attributes, where hurdles and sub-hurdles were considered non-beneficial attributes.

3.2 Modeling Influenced Hurdles and Sub-Hurdles into Tree Soft Structure

- Forming the influenced main and sub hurdles into a multitude of levels.
- At level 1: Main hurdles resident in tree soft as nodes in this level.
- At level 2: Sub-hurdles resident in tree soft as nodes which inherent from nodes in level 1.

3.3 Implementing BHARAT v2 in Tree Soft to Obtain Main and Sub-Hurdles' Weights

BHARAT v2 works in nodes of TreeSoft to solve the problem of appraising.

3.3.1 **N of Pairwise Matrices are Constructed Based on N of DMs**

- Forming the influenced main and sub hurdles into a multitude of levels.
- At level 1: for each node or main hurdle, N of pairwise matrices is constructed.
- For each node, Eq. (3) is implemented for generating an aggregated matrix.
- Average or Aggregated of DMs'rating $= \frac{DM_1 + DM_2 + \cdots\ldots\ldots + DM_n}{n}$ (3)
  Where n is the number of DMs.
- In the aggregated matrix for each node, the average for each hurdle per column is calculated. Hence vector of values is generated.
- The generated vector is utilized to produce a new matrix. Calculating the average for each hurdle per raw $(\mathcal{g})$ and calculating the sum of the average column $(\hbar)$.
- Weights of main hurdles are computing as in Eq. (4).
  $w_{i=} \frac{\mathcal{g}}{\hbar}$ (4)
- At level 2: for each Sub-node or sub-hurdle, N of pairwise matrices is constructed.
- The aggregated matrix for each sub-hurdle is computing. The average for each sub-hurdle per column is calculated. Hence vector of values is generated.
- Eq. (4) is utilized for obtaining local weight for each sub-hurdle $(\mho_i)$.
- Finally, the global weight for each sub-hurdle $\Re_i$ based on Eq. (5).
  $\Re_{i=} w_i * \mho_i$ (5)
- Ranking sub-hurdles based on values of $\Re_i$.

4. An Empirical Case Study: Validating B_{v2}TrS

The purpose of this section is to verify the authenticity of our constructed B_{v2}TrS. Thus, we applied this B_{v2}TrS model in a case study. Accordingly, the previous procedures of B_{v2}TrS are implemented in this case study as the following.

4.1 We are communicating with a group of DMs to form a DMs panel. Thereby, three DMs are contributing to appraising e-mobility's hurdles

4.2 The appraising is conducted for three main hurdles and ten sub-hurdles as mentioned in Figure 2

4.3 Tree soft illustrates three main hurdles and ten sub-hurdles in a hierarchical structure.

4.4 BHARAT-v2 is leveraging in constructed tree soft to appraise the main hurdles of e-mobility and obtaining weights for hurdles

4.4.1 4BHARAT-v2 starts with level one to obtain weights for three nodes by appraising these nodes according to three DMs (see Appendix A, Table A1).

4.4.2 An aggregated matrix is generated by calculating the mean of three pair-wise matrices through applying Eq. (3) as listed in Table 1.

4.4.3 Vector for mean for main hurdles (H_n) through calculating the average for each main hurdle per column. Hence, {1.65666667, 1.342211, 0.59256667} are mean for H_1, H_2, H_3.

4.4.4 This vector is utilized in Table 2 to obtain hurdles weights.

4.4.5 Through applying Eq. (4), the hurdles' weights.

Table 1. Aggregated matrix

Aggregated	H_1	H_2	H_3
H_1	0.5	0.666633	0.7222
H_2	2.73	0.5	0.5555
H_3	1.74	2.86	0.5

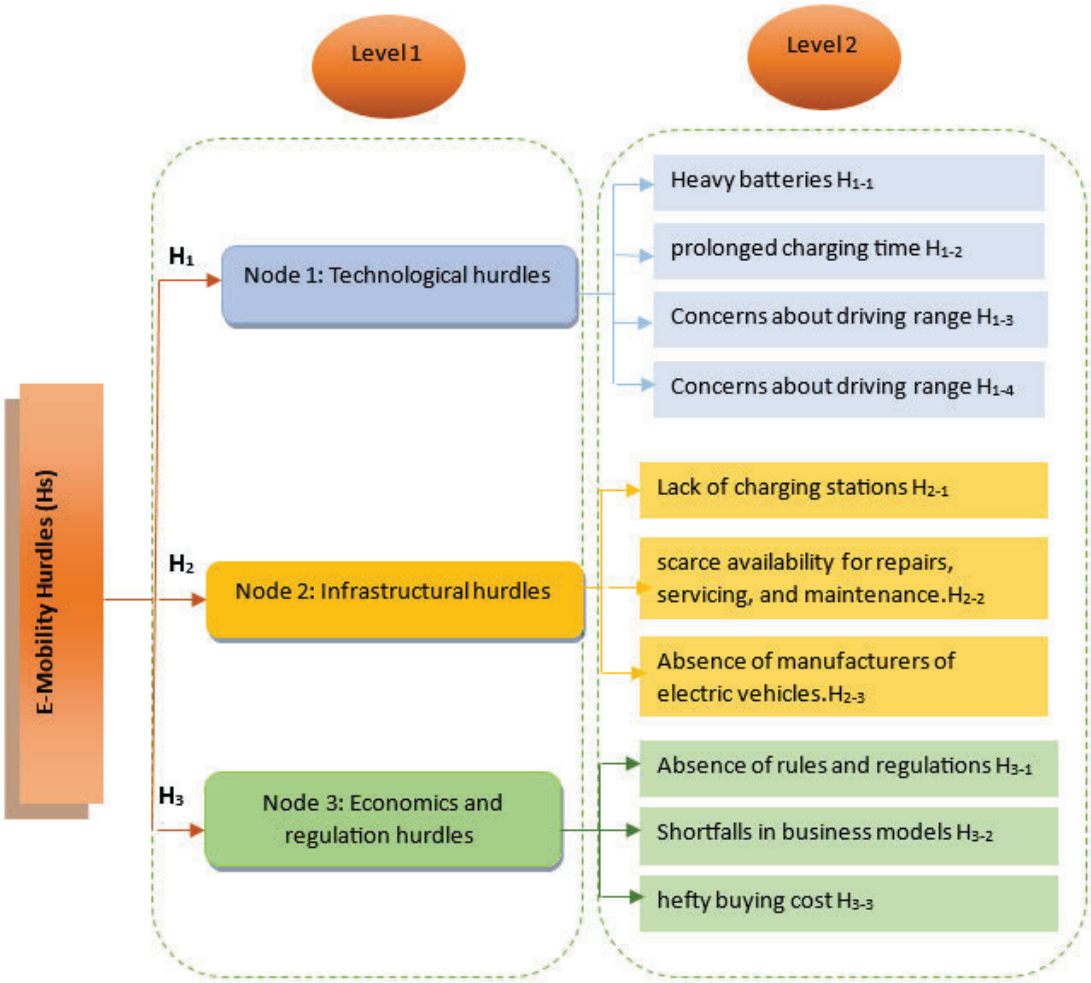


Figure 2. Tree soft model for main and sub-hurdles for e-mobility.

4.5 BHARAT-v2 is leveraging in level 2 of Tree Soft to appraise sub-hurdles of Technology and obtain weights for it

4.5.1 For each sub-hurdle, three DMs are generating pair-wise matrices (see Appendix A, in Tables A2).

4.5.2 The Aggregated matrix for each sub-hurdle is generated as listed in Table 3.

4.5.3 Vector for mean for sub-hurdles ($H_{1\text{-}n}$) through calculating the average for each sub-hurdle per column. Hence, {2.900333, 1.493688889, 0.555533} are mean for $H_{1\text{-}1}$, $H_{1\text{-}2}$, $H_{1\text{-}3}$.

4.5.4 Eq. (4) is utilized for obtaining local weights for sub-hurdles of Technology as in Table 4 and global weight is computed through using Eq. (5) and listed in Table 4.

4.6 BHARAT-v2 is leveraging in level 2 of Tree soft to appraise sub-hurdles of Infrastructure and obtain weights for it

4.6.1 For each sub-hurdle, three DMs are generating pair-wise matrices (see Appendix A, in Tables A3).

4.6.2 The Aggregated matrix for each sub-hurdle is generated as listed in Table 5.

4.6.3 Vector for mean for sub-hurdles ($H_{2\text{-}n}$) through calculating the average for each sub-hurdle per column. Hence, {2.203333, 0.972211111, 0.555533} are mean for $H_{2\text{-}1}$, $H_{2\text{-}2}$, $H_{2\text{-}3}$.

4.6.4 Eq. (4) is utilized for obtaining local weights for sub-hurdles of Technology as in Table 6 and global weight is computed through using Eq. (5) and listed in Table 6.

4.7 BHARAT-v2 is leveraging in level 2 of Tree Soft to appraise sub-hurdles of Economics and regulation and obtain weights for it

4.7.1 For each sub-hurdle, three DMs are generating pair-wise matrices (see Appendix A, in Tables A4).

4.7.2 The aggregated matrix for each sub-hurdle is generated as listed in Table 7.

4.7.3 Vector for mean for sub-hurdles (H_{3-n}) through calculating the average for each sub-hurdle per column. Hence, {0.78, 0.798877778, 0.648022} are mean for H_{3-1}, H_{3-2}, H_{2-3}.

4.7.4 Eq. (4) is utilized for obtaining local weights for sub-hurdles of Technology as in Table 6 and global weight is computed through using Eq. (5) and listed in Table 8.

4.8 Ranking sub-hurdles based on global weight values in Tables 4, 6, 8

Table 2. The main hurdles weigh in level 1.

	H_1	H_2	H_3	Average	weights
H_1	1.65666667/1.65666667	1.34/1.65666667	0.59256667/1.65666667	0.721	0.721/3.461=0.208321
H_2	1.65666667/1.34	1.34/1.34	0.59256667/1.34	0.89	0.89/3.461=0.257151
H_3	1.65666667/59256667	1.34/.59256667	59256667/59256667	1.85	1.85/3.461=0.534528
Total Sum				3.461	1

Table 3. Aggregated matrix for sub-hurdles of technology.

Aggregated	H_{1-1}	H_{1-2}	H_{1-3}
H_{1-1}	0.5	0.611066667	0.7222
H_{1-2}	2.851	0.5	0.4444
H_{1-3}	5.35	3.37	0.5

Table 4. Global weights of sub-hurdles of technology in level 2.

	H_{1-1}	H_{1-2}	H_{1-3}	Average	Local weight	Global weight
H_{1-1}	2.900333/2.900333	1.493688889/2.900333	0.555533/2.900333	0.57	0.57/4.63=0.12311	0.025646
H_{1-2}	2.900333/1.493688889	1.493688889/1.493688889	0.555533/1.493688889	1.11	1.11/4.63=0.239741	0.049943
H_{1-3}	2.900333/0.555533	1.493688889/0.555533	0.555533/0.555533	2.95	2.95/4.63=0.637149	0.132732
Total Sum				4.63	1	0.208321

Table 5. Aggregated matrix for sub-hurdles of infrastructure.

Aggregated	H_{2-1}	H_{2-2}	H_{2-3}
H_{2-1}	0.5	0.666633333	0.4444
H_{2-2}	1.81	0.5	0.7222
H_{2-3}	4.3	1.75	0.5

Table 6. Global weights of sub-hurdles of infrastructure in level 2.

	H_{2-1}	H_{2-2}	H_{2-3}	Average	Local weight	Global weight
H_{2-1}	2.203333/2.203333	0.9722111/2.203333	0.555533/2.203333	0.57	0.57/4.07=0.14004914	0.036014
H_{2-2}	2.203333/.97	0.9722111/0.9722111	0.555533/.0.9722111	1.28	1.28/4.07=0.314496314	0.080873
H_{2-3}	2.203333/.56	0.9722111/.0.555533	0.555533/0.555533	2.22	2.22/4.07=0.545454545	0.140264
Total Sum				4.07	1	0.257151

Table 7. Aggregated matrix for sub-hurdles of economics and regulation.

Aggregated	H 3-1	H 3-2	H 3-3
H 3-1	0.5	0.666633333	0.610867
H 3-2	1.84	0.5	0.8332
H 3-3	4.2.84	1.23	0.5

Table 8. Global weights of sub-hurdles of economics and regulations in level 2.

	H_{3-1}	**H_{3-2}**	**H_{3-3}**	**Average**	**Local weight**	**Global weight**
H_{3-1}	0.78/0.78	.799/.78	0.65/.78	0.953	0.953/3.033=0.314210353	0.167954104
H_{3-2}	.78/.799	.799/.799	.65/.799	0.94	0.94/3.033=0.309924167	0.165663019
H_{3-3}	.78/.65	.799/.65	.65//.65	1.14	1.14/3.033=0.37586548	0.20091047
Total Sum				3.033	1	0.534527593

5. Discussion

To validate the authenticity of the $B_{v2}TrS$ model, we applied it to a case study. Also, determining technological, infrastructure, and economics and regulations as the main hurdles for e-mobility. Accordingly, its sub-hurdles are determined.

In this study tree soft is utilized for modeling these influenced hurdles into a multitude of nodes in various levels as shown in Figure 2. After that BHARAT-v2 was implemented in tree soft modeling to obtain weights for each node contained in each level. Subsequently, the influenced hurdles of e-mobility are ranked according to global weights for each sub-hurdle. Figure 3 represents the ranking of each sub-hurdle for every main hurdle. For instance, H_{1-3} is the most influential hurdle in technology, otherwise H_{1-1} is the least. Also, H_{2-3} is the most influenced hurdle in infrastructure, otherwise H_{2-1} is the least. Finally, H_{3-3} is the most influenced hurdle in infrastructure, otherwise H_{3-2} is the least influenced.

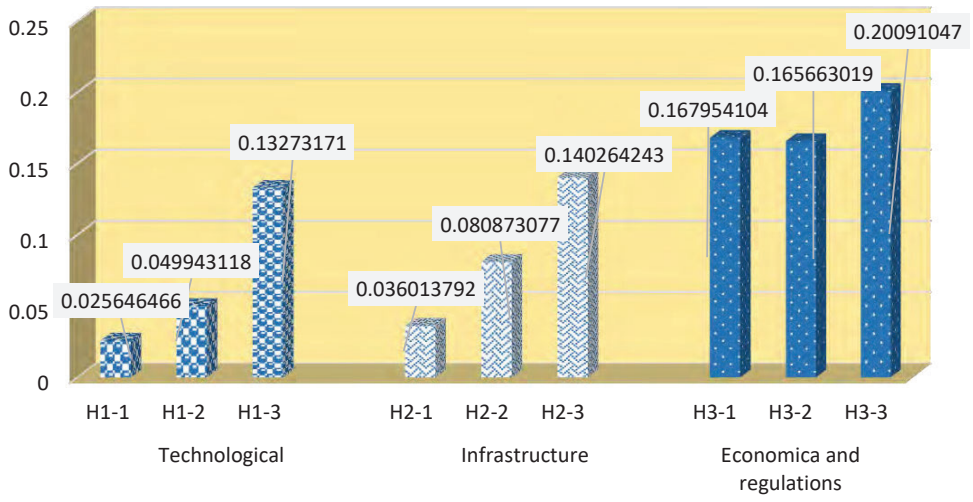


Figure 3. Ranking of sub-hurdles of e-mobility based on global weights.

6. Conclusions

E-mobility as EVs and e-scooters is an effective inducement to adopt the notion of a green environment. Consequently, the market for EVs is quickly expanding thanks to a variety of decarbonization-supporting technologies, methods, and inventions. This is due to a multitude of factors. For instance, Per the International Energy Agency's (IEA) transportation CO_2 emission report for 2022, automobiles and vans accounted for 48% of the total CO_2 emissions. All over the globe economies are moving toward the adoption of alternate fuel technologies because of the rising worldwide concern about climate change brought on by greenhouse gas emissions from vehicles and the depletion of natural resources. Hence, Electric vehicles (EVs) are positioned as a clean, green alternative technology that may make it possible to preserve natural resources and move to a low-carbon transportation system with efficiency. Although the importance of embracing EVs in our daily lives, it suffers from a set of hurdles.

Herein, the objective of this study is to analyze these hurdles and prioritize them. Therefore, to achieve this objective, we are modeling the determined hurdles through tree soft methodology. For appraising these hurdles which are modeled in tree soft, BHARAT -v2 is applied as a novel MCDM technique to appraise the influence of hurdles on e-mobility. Hence, we constructed B_{v2}TrS as an appraiser model for ranking these hurdles.

The findings of this model and according to Figure 3 indicated that $H_{1\text{-}3}$ is the highest-influenced hurdle in technology with a global weight is 0.13, otherwise, $H_{1\text{-}1}$ is the least with a global weight is 0.025. In the same vein, $H_{2\text{-}3}$ is the most influenced hurdle in infrastructure with a global weight is 0.14 and $H_{2\text{-}1}$ is the least with a global weight is 0.036. Finally, $H_{3\text{-}3}$ is the most influenced hurdle in infrastructure where its global weight is 0.20, Unlike $H_{3\text{-}2}$ is the least influenced.

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Author Contributions

All authors contributed equally to this research.

Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

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Conflict of interest

The authors declare that there is no conflict of interest in the research.

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Appendix A

Table A1. Pair-wise prioritize matrices for main hurdles in level 1.

DM_1	H_1	H_2	H_3	DM_2	H_1	H_2	H_3	DM_3	H_1	H_2	H_3
H_1	0.5	0.8333	0.3333	H_1	0.5	0.1666	1	H_1	0.5	1	0.8333
H_2	1/0.8333	0.5	0.6666	H_2	1/0.1666	0.5	0.8333	H_2	1	0.5	0.1666
H_3	1/0.3333	1/0.6666	0.5	H_3	1.0000	1/0.8333	0.5	H_3	1/0.8333	1/0.1666	0.5

Table A2. Pair-wise prioritize matrices for sub-hurdles of technological in level 2.

DM_1	$H_{1\text{-}1}$	$H_{1\text{-}2}$	$H_{1\text{-}3}$	DM_2	$H_{1\text{-}1}$	$H_{1\text{-}2}$	$H_{1\text{-}3}$	DM_3	$H_{1\text{-}1}$	$H_{1\text{-}2}$	$H_{1\text{-}3}$
$H_{1\text{-}1}$	0.5	0.1666	0.8333	$H_{1\text{-}1}$	0.5	1	0.3333	$H_{1\text{-}1}$	0.5	0.6666	1
$H_{1\text{-}2}$	1/0.1666	0.5	0.3333	$H_{1\text{-}2}$	1	0.5	0.1666	$H_{1\text{-}2}$	1/0.6666	0.5	0.8333
$H_{1\text{-}3}$	1/0.8333	1/0.3333	0.5	$H_{1\text{-}3}$	1/0.3333	1/0.1666	0.5	$H_{1\text{-}3}$	1	1/0.8333	0.5

Table A3. Pair-wise prioritize matrices for sub-hurdles of infrastructural in level 2.

DM1	**H2-1**	**H2-2**	**H2-3**	**DM2**	**H2-1**	**H2-2**	**H2-3**	**DM3**	**H2-1**	**H2-2**	**H2-3**
H2-1	0.5	0.3333	0.1666	H2-1	0.5	0.8333	0.1666	H2-1	0.5	0.8333	1
H2-2	1/0.3333	0.5	0.8333	H2-2	1/0.8333	0.5	1	H2-2	1/0.8333	0.5	0.3333
H2-3	1/0.1666	1/0.8333	0.5	H2-3	1/0.1666	1	0.5	H2-3	1	1/0.3333	0.5

Table A4. Pair-wise prioritize matrices for sub-hurdles of economics and regulation in level 2.

DM1	H3-1	H3-2	H3-3	DM2	H3-1	H3-2	H3-3	DM3	H3-1	H3-2	H3-3
H3-1	0.5	0.3333	0.1666	H3-1	0.5	0.8333	0.1666	H3-1	0.5	0.8333	1
H3-2	1/0.3333	0.5	0.8333	H3-2	1/0.8333	0.5	1	H3-2	1/0.8333	0.5	0.3333
H3-3	1/0.1666	1/0.8333	0.5	H3-3	1/0.1666	1	0.5	H3-3	1	1/0.3333	0.5

Addressing Uncertainties in Planning Sustainable Systems Through Multi-criteria Decision Analysis (MCDA)

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Sustainable development involves the creation of systems and technologies that not only improve human lives but also enable natural resources to be consumed responsibly. These systems include integration of technologies for efficient resource consumption, better production, and lower negative impacts on the environment. Over the years, humanity has developed these technologies and commercialized through a series of advancements: from prototype creation to commercialization. Even with the promise of sustainable development, these technologies took a long time before being deployed on a large scale. Decisions are needed to be made to create policies, regulations, and implementation strategies for these technologies. These decisions will be based on different economic, environmental, social, and technical factors, among others. In doing these decisions, one of the major factors needed to be addressed is uncertainty.

Statistical techniques can be used to manage uncertainties that involve randomness. For instance, the prediction of the leaching behavior of complex copper-cobalt-bearing ores is performed using an artificial neural network with a back-forward algorithm (Mbuya and Mulaba-Bafubiandi 2023). However, the existence of uncertainties in new and emerging technologies is due to the lack of information about how technologies can be deployed for use by the society. Policymakers rely on expert judgment to be able to determine which technologies can be scaled up for testing, or which technologies are better than the others. Expert judgments are based on available information where most of the uncertainties arise. Incomplete and vague information about different systems leads to inconclusive insights. Inconsistency with different expert judgments also hinders the deployment of emerging technologies. Ambiguity also adds to the challenge of uncertainty, as information can be excessive and is based on different assumptions. Several decision-making tools adopt these kinds of uncertainties into their framework. In addition, these tools make explicit the preferences and values of the stakeholders in the decision-making process. Such decision-making tools and their application, which are studied and developed in the field of *multi-criteria decision analysis* (MCDA), are as follows.

Attaining a large share of renewable energy in the energy mix is one of the major goals of sustainable development. The development of a Technique for Order Preference by Similarity to Ideal Solution (TOPSIS)–based linear programming (LP) approach enables planning of integrating renewable energy sources into the energy mix considering technical, environmental, and social factors (Oluklu and İç 2022). An MCDA tool for evaluating the transportation sustainability of different countries was also developed based on different versions of data envelopment analysis (DEA; Babaei et al. 2022). Analytic Hierarchy Process (AHP), a well-known MCDA tool, is also applied for assessing the surface irrigation potential of the Ethiopian River Basin (Gemechu 2022). The decision framework developed in these papers makes use of the available data in a deterministic or uncertain form.

Some of the widely used representations of uncertainties are fuzzy sets (Zadeh 1965), Pythagorean fuzzy sets (Yager 2014), and neutrosophic sets (Smarandache 2006) where an element's belongingness to a set is characterized in different ways. Recent work on addressing uncertainty using fuzzy set is determining the effects of green supply chain (GSC) practices on sustainable performance indicators (Panpatil et al.

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2022b) and the development of a novel decision tool for identifying factors affecting climate change and the impact of urbanization on hydropower plants (Shil et al. 2023). In this work, the subjective perspective of human judgment has been considered in developing the decision tool. A similar study where GSC enablers are now involved is done by Panpatil et al. (2022a) where the uncertainty is modelled as a Pythagorean fuzzy set. In this work, the complexity in terms of the multi-facet and multi-dimensional nature of the decision problem is considered. Neutrosophic set–based tools are also developed for the diagnosis of COVID-19 and other lung diseases (Karataş and Ozturk 2022) and for optimal resource allocation for achieving sustainable development goals in Egypt (Badr et al. 2022). Quadripartitioned single-valued neutrosophic properties are also introduced and applied to assess the influential factors affecting energy prices (Granados 2022). These works proposed decision aids that consider the degrees of membership, non-membership, and indeterminacy in which a different set of these elements provides a representation of inconsistent, vague, ambiguous, incomplete, and uncertain data.

The recent developments on addressing uncertainties in planning sustainable systems have led to different decision-making tools to enable the deployment of technologies associated with it. These tools incorporate mathematical concepts of fuzzy sets and their extension to address uncertainty in sustainable systems. The research gap that needs to be addressed for now is on how these extensions represent uncertainty specific to different systems. In the case of neutrosophic sets, the degrees of non-membership and indeterminacy should be defined in the context of the decision problem, i.e., answering the question "What do non-membership and indeterminacy mean in the system?" Such mathematical concept will be a powerful tool for improving MCDA tools for easier decision-making and policymaking. This special issue in *Process Integration and Optimization for Sustainability* provides different ways to address uncertainty with the aim of giving the readers ideas about the capabilities of different mathematical techniques.

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On Characterizing Efficient and Properly Efficient Solutions for Multi-Objective Programming Problems in a Complex Space

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Abstract

In this paper, a complex non- linear programming problem with the two parts (real and imaginary) is considered. The efficient and proper efficient solutions in terms of optimal solutions of related appropriate scalar optimization problems are characterized. Also, the Kuhn-Tuckers' conditions for efficiency and proper efficiency are derived. This paper is divided into two independently parts: The first provides the relationships between the optimal solutions of a complex single-objective optimization problem and solutions of two related real programming problems. The second part is concerned with the theory of a multi-objective optimization in complex space.

Keywords: Optimization; Complex multi- objective programming; Efficient solution; Kuhn-Tuckers' conditions; Optimal solution; Decision making

1. Introduction

Multi-objective optimization problems arise when more than one objective function is to be minimized over a given feasible region. Unlike the traditional mathematical programming with a single-objective function, an optimal solution in the sense of one that minimizes all the objective functions simultaneously does not necessarily exist in multi-objective optimization problems, and whence, we are troubled with conflicts among objectives in decision-making problems with multiple objectives. The review of the most important fundamentals in multiobjective optimization (MOO) is introduced by (Emmerich and Deutz, 2018; Gunantora and Ai, 2018) (2018). Jiang and Fan (2020) proposed a method to design thin- film stacks consisting of multiple material types. Roussel et al. (2021) investigate multiobjective Bayesian optimization to find the full pareto front of an accelerator programming problem efficiently.

In the last three decades, many authors have extended a numerous aspects of mathematical programming to complex space, which lead to the importance of this field in many applications as Resistive Network with Sinusoidal Sources, Impedance Matching of Circuits in the Sinusoidal Steady State and Jury- Lee Criterion of Absolute Stability (Abrams and Ben Israel, 1971 and Hanna and Simaan, 1985). Applications of complex programming may be found in Mathematics, engineering, and in many other areas. In earlier works in the field of complex programming problem, all the researchers have considered only the real part of the objective function of the problem as the objective function of the problem neglecting the imaginary part of the objective function, and the constraints of the problem have considered as a cone in the complex space $\mathbb{C}^n$. Abrams (1972) established sufficient conditions for optimal points of the real part of the objective function neglecting the imaginary part. Duca (1978) formulated the vectorial optimization problem in complex space and obtained some necessary and sufficient conditions for a point to be the efficient solution of a problem. Smart and Mond (1991) have shown that the necessary conditions for optimality in polyhedral- cone constrained nonlinear programming problems are sufficient under the assumption of particular form of invexity. Ferrero (1992) considered the finite dimensional spaces making use of separation arguments and the one- to- one correspondence between $\mathbb{C}^n$, and $\mathbb{R}^{2n}$. Malakooti (2010) developed complex method with interior search directions to solve linear and nonlinear programming problems. Youness and Elborolosy (2004) formulated the problem in complex space taking into account the two parts of complex objective function (real and imaginary together) and introduced an extension to necessary optimality conditions in complex programming. Zhang and Xia (2018) proposed two efficient complex- valued optimization methods for solving constrained nonlinear programming problems of real functions in complex variables. Khalifa et al. (2020) characterized the solution of complex nonlinear programming with interval- valued neutrosophic trapezoidal fuzzy parameters. A neuromas authors are studied complex multi- objective programming problem where they are established the optimality conditions, introduced several properties, constructed and provide the weak, strong and strictly converse duality theorems and introduced being a necessary conditions of local weak efficient solution for optimistic optimization problems (see, Huang and Ho, 2021; Huang and Tanaka, 2022; and Hsu and Huang, 2022; Lv et al., 2022).

This paper aims to study the necessary and sufficient conditions for a point to be an efficient or a properly efficient solution of a multi-objective complex programming problem, when the ordering is taken with respect to a pointed closed convex cone.
The outlay of the paper is constructed as follows: In the next sections, some preliminaries needed in this paper is introduced. Section 3 formulates the complex multi-objective programming problem. Section 4 introduces some of theorem, which described and derived the necessary and sufficient conditions for the complex programming involving Section 5 introduces the Kuhn-Tucker' conditions for existing the efficient and proper efficient solutions. Finally, in section 6 some concluding remarks are reported.

2. Preliminaries

In this section, some definitions, and theorems needed in this paper are recalled.

Definition 1. A non- empty set U *in* $\mathbb{C}^m$ (complex $\times\, n$) is a convex if it is closed in a convex combination, i,e., $\theta U + (1-\theta)U \subseteq U$, for $0 \le \theta \le 1$.

Definition 2. (Berman, 1973). A non- empty U in $\mathbb{C}^m$ is a cone if it is closed in a non- negative scalar multiplication, i.e., $\theta U \subseteq U$ for $\theta \ge 1$.

Remark 1. Let the cone $\{(w_1, w_2) \in \mathbb{C}^{2m}: w_2 = \overline{w_1}\}$ is denoted by P.

Definition 3.(Berman, 1973)**.** A hyperplane H in $\mathbb{C}^m$ is a set of the form

$H = \{x \in \mathbb{C}^m : \mathrm{Re}\, v^H x = \alpha\}$, where v non- zero vector in $\mathbb{C}^m$ and $\alpha \in \Re$.

Definition 4. (Berman, 1973). A polyhedral cone U in $\mathbb{C}^m$ is a convex cone generated by many vectors finitely, i.e., a set of the form $U = A\Re_+^k = \{Ax: x \in \Re_+^k\}$, for some $k \in \mathbb{Z}$ and $A \in \mathbb{C}^{n\times k}$.

Definition 5. (Berman, 1973). The dual U° of a non-empty set $U \subset \mathbb{C}^m$ is defined as

$$U^\circ = \{y \in \mathbb{C}^m: x \in U \Rightarrow \mathrm{Re}\, y^H x \ge 0\}.$$

Proposition 1. Let $U \subset \mathbb{C}^m$ be a closed convex cone, then $\widetilde{U} = \{0\}$ if and only if U is a real linear space of $\mathbb{C}^m$,i.e., U is a closed under real linear combination.

Theorem 1. (Berman, 1973). Let U be a closed convex cone in $\mathbb{C}^m$,then U is a pointed cone if and only if U° is solid. In this case $int\ U^\circ$ (interior of U°) is given by

$$\mathrm{int}\, U^\circ = \{y \in U^\circ: 0 \ne x \in U \Rightarrow \mathrm{Re}\, y^H x > 0\}.$$

It is clear that, for a closed convex cones U, U is solid iff U° is pointed. In this case,

$$\mathrm{int}\, U = \{x \in U: 0 \ne y \in U^\circ \Rightarrow \mathrm{Re}\, y^H x > 0\}.$$

Definition 6. Let M be a closed convex cone in $\mathbb{C}^m$, the largest subspace which is contained in M is $M \cap (-M)$, while the smallest subspace contains M is $M - M$. A convex cone is pointed if $M \cap (-M) = \{0\}$.

Corollary 1. (Berman, 1973). Let M and U be closed convex cones in $\mathbb{C}^n$, then

$$cl(M^\circ + U^0) = (M \cap U)^\circ.$$

Theorem 2. (Mond and Craven, 1977). Let M be a closed convex cone in $\mathbb{C}^m$ and $a \in M$. Then $y \in (M(a))^\circ$ iff $y \in M^\circ$ and $Re\ y^H = 0$.

Theorem 3. (Mond and Greenblatt, 1975; Rockafellar, 1970).

1. A set M is a polyhedron cone in $\mathbb{C}^m$ if and only if it is the intersection of finite number of closed half spaces in $\mathbb{C}^m$, each containing the origin on its boundary, i.e., $M = \bigcap_{k=1}^{p} H_{u_k}$, where $H_{u_k} = \{z \in \mathbb{C}^m: Re\ z^H u_k \ge 0\}$, for some vectors $u_1, u_2, \dots, u_p \in \mathbb{C}^m$ and integer $p > 0$. This is equivalent to M is a polyhedron cone if there is a positive integer p and a matrix $B \in \mathbb{C}^{p\times m}$ such that
$M = \{z \in \mathbb{C}^m: \mathrm{Re}\, z \ge 0\}$ and in this case $M^\circ = \{w \in \mathbb{C}^m: \exists\, u \in \mathbb{C}_+^m, w = B^H u\}$,
2. A polyhedral cone is a closed convex cone,
3. M is a polyhedral cone if and only if M° is a polyhedral cone,
4. The sum of polyhedral cones is polyhedral, and
5. The Cartesian product of polyhedral cones is a polyhedral cone.

Definition 7. (Craven and Mond, 1973).The functions $f: P \to \mathbb{C}$ and $g: P \to \mathbb{C}^n$ are said to be differentiable at the point $(z^*, \overline{z}^*) \in P$, if for every $(z, \overline{z}) \in P$,
$f(z,\overline{z}) - f(z^*,\overline{z}^*) = \nabla_z f(z^*,\overline{z}^*)(z - z^*) + \nabla_{\overline{z}} f(z^*,\overline{z}^*)(\overline{z} - \overline{z}^*) + o(\|z - z^*\|)$, and

$g(z,\overline{z}) - g(z^*,\overline{z}^*) = \nabla_z g(z^*,\overline{z}^*)(z - z^*) + \nabla_{\overline{z}} g(z^*,\overline{z}^*)(\overline{z} - \overline{z}^*) + o(\|z - z^*\|)$, where $\nabla_z f(z^*,\overline{z}^*)$ and $\nabla_{\overline{z}} f(z^*,\overline{z}^*)$ are the lower vectors of partial derivatives $\frac{\partial f(z,\overline{z}^*)}{\partial w_1^i}$ and $\frac{\partial f(z,\overline{z}^*)}{\partial w_2^i}$ respectively, While $D_z g(z^*,\overline{z}^*)$ and $D_{\overline{z}} g(z^*,\overline{z}^*)$ are the matrices $m \times n$ whose elements are $\frac{\partial g(z,\overline{z}^*)}{\partial w_1^i}$ and $\frac{\partial g(z,\overline{z}^*)}{\partial w_2^i}$ respectively, and $\frac{o(\|z-z^*\|)}{\|z-z^*\|} \to 0 as\ \ z \to z^*$.

Defintion 8. (Craven and Mond, 1973). Let f: $\mathbb{C}^n \to \mathbb{C}$ and $E \subset \mathbb{C}^n$, then $f(z)$ is called analytic in E, if in some neighborhood of every point E, it may be represented by an absolutely convergent power series about that point in the n complex variable.

Theorem 4. (Craven and Mond, 1973). Let M be a polyhedron cone in $\mathbb{C}^m$, f: $\mathbb{C}^{2n} \to \mathbb{C}$, *and* g: $\mathbb{C}^{2n} \to \mathbb{C}^m$ be two analytic functions in a neighborhood of a qualified point $(z^*, \overline{z}^*)$. Then a necessary condition for $(z^*, \overline{z}^*)$ to be a local minimum of problem(2) is that there is a vector $u \in (M(z^*))^\circ \subset M^\circ$ such that

$$\nabla_z f(z^*, \overline{z}^*) + \overline{\nabla_{\overline{z}} f}(z^*, \overline{z}^*) - u^H D_z g(z^*, \overline{z}^*) - u^T \overline{D_{\overline{z}} g}(z^*, \overline{z}^*) = 0, \quad \text{and}$$

$$Re\ u^H g(z^*, \overline{z}^*) = 0.$$

3. Complex programming Problem statement

Consider the following multiobjective complex programming problem

$$\min f(z, \overline{z}) = (f_1(z, \overline{z}), f_2(z, \overline{z}), \dots, f_p(z, \overline{z}))$$
$$\text{Subject to} \quad (z, \overline{z}) \in G = \{(z, \overline{z}) \in Q: g(z, \overline{z}) \in M\}. \qquad (1)$$

Where, $f_i: \mathbb{C}^{2n} \to \mathbb{C}, i = 1, 2, \dots, p, p \geq 2$, as well as g: $\mathbb{C}^{2n} \to \mathbb{C}^m$ are differentiable functions, and M is a closed convex cone on $\mathbb{C}^m$.

The concept of the efficient solution, for the above complex problem, with respect to a complex domination structure of the decision maker is introduced, and a characterization of those solutions is given. Also, Kuhn-Tucker conditions for efficiency and proper efficiency of problem (1) are derived.

4. Efficient Solutions Concept in Complex Space

As we have mentioned, in a single-objective optimization problem, the meaning of optimality is clear. Whereas, in contrast, an optimal solution that minimizes all the objective functions simultaneously, can rarely be expected to exist in multi-objective optimization problems, since the objectives usually conflict with one another. Instead of optimality, the notion of efficiency.

Definition 9. A feasible point $(z^*, \overline{z}^*)$ is said to be an efficient solution of problem (1) with respect to a pointed closed convex cone (domination structure) $U \subset \mathbb{C}^p$ if there is no other feasible $(z, \overline{z})$ such that

$$f(z^*, \overline{z}^*) - f(z, \overline{z}) \in U \backslash \{0\}.$$

In other words, $(z^*, \overline{z}^*)$ is an efficient solution if and only if

$$(f(G) - f(z^*, \overline{z}^*)) \cap (-U) = \{0\}.$$

In addition, the efficient set of a non- empty set $Q \subset \mathbb{C}^p$ with respect to a cone $E \subset \mathbb{C}^p$ is given as in the following definition.

Definition 10. Let $E \subset \mathbb{C}^p$ be a cone, then the set of efficient elements of a set $Q \subset \mathbb{C}^p$ with respect to E is given by

$$D(Q, E) = \{\check{q} \in Q: there\ is\ no\ q \neq \check{q} \in Q\ such\ that\ \check{q} \in q + E\}.$$

From this definition, it follows that a point $(z^*, \overline{z}^*)$ is an efficient solution of problem (1) with respect to U if and only if $(z^*, \overline{z}^*) \in D(f(G), U)$.

Proposition 2. Let $E \subset \mathbb{C}^p$ be a non- empty pointed convex cone, then

$$D(Q, E) = E(Q + E, E).$$

Proof (Smart and Mond, 1991).

4.1. Characterization of efficient solutions

In this subsection, we characterize the efficient solutions in terms of optimal solutions of related appropriate scalar optimization problems. A method of characterizing efficient solutions is via secularization by vectors in the polar cone U° of a domination structure cone U.

Consider the following scalar optimization problem

$$\min Re\ \zeta^H f(z, \overline{z})$$
$$\text{Subject to} \quad (z, \overline{z}) \in G, \ \text{for some } \zeta \in U^\circ. \qquad (2)$$

Now, let us introduce the state with proof for the two theorems in order to study the relationships between the efficient solutions of the complex multi- objective problem (1) with respect to U and the optimal solutions of the related scalar optimization problem (2).

Theorem 5. Let $U \subset \mathbb{C}^p$ be a pointed closed convex cone and $f(G) + U$ be a convec set. If $(z^*, \overline{z}^*)$ is an efficient solution of problem (1) with respect to U, then there is a $0 \neq \zeta \in U^\circ$ such that $(z^*, \overline{z}^*)$ is an optimal solution of problem (2).

Proof. Assume that $(z^*, \overline{z}^*)$ be an efficient solution of problem (1) with respect to U, i.e., $f(z^*, \overline{z}^*) \in D(f(G), U)$, it follows that

$f(z^*, \overline{z}^*) \in D(f(G) + U, U)$ (**Proposition 2**). So, $(f(G) + U - f(z^*, \overline{z}^*)) \cap (-U) = \{0\}$. Since from the convexity of the two sets $f(G) + U$ and U, there is a hyperplane separating between them, i.e., there is a non- zero vector $\zeta \in \mathbb{C}^p$ such that:

$$Re\ \zeta^H (f(z, \overline{z}) + s - f(z^*, \overline{z}^*)) \geq 0;\ \forall\ (z, \overline{z}) \in G, s \in U, \text{ and} \qquad (3)$$

$$Re\ \zeta^H (-s') \leq 0;\ \forall s' \in U. \qquad (4)$$

From (4), it follows that $\zeta \in U^\circ$, and by setting $s = 0$ in inequality (3), we have

$Re\ \zeta^H f(z, \overline{z}) \geq Re\ \zeta^H f(z^*, \overline{z}^*);\ \forall\ (z, \overline{z}) \in G.$

We conclude that, $(z^*, \overline{z}^*)$ is an optimal solution of problem (2).

Theorem 6. A point $(z^*, \overline{z}^*)$ is an efficient solution of problem (1) with respect to a pointed closed convex cone $U \subset \mathbb{C}^p$ if there is a $\zeta \in U^\circ$ such that $(z^*, \overline{z}^*)$ is an optimal

solution of problem (2) and one of the following two conditions holds:

1. $\zeta \in int\ U^{\circ}$, or
2. $(z^*, \overline{z}^*)$ is a unique optimal solution of problem (2).

Proof. Since $(z^*, \overline{z}^*)$ solves problem (2) for some $0 \neq \zeta \in U^{\circ}$, then

$$Re\ \zeta^H \left(f(z,\overline{z}) - f(z^*,\overline{z}^*)\right) \geq 0;\ \forall\ (z,\overline{z}) \in G. \quad (5)$$

Suppose that $(z^*, \overline{z}^*)$ is not an efficient solution for problem (1) with respect to U, then there is $(\check{z}, \check{\overline{z}}) \in G$ such that:

$$f(z^*,\overline{z}^*) - f(\check{z},\check{\overline{z}}) \in U\backslash\{0\}.$$

If the **condition 1** satisfied, then

$\mathrm{Re}\ \zeta^{\mathrm{H}} \left(\mathrm{f}(z,\overline{z}) - \mathrm{f}(z^*,\overline{z}^*)\right) \geq 0$, which contradicts (5).

If the condition2 holds, then inequality (5) becomes

$Re\ \zeta^H \left(f(z,\overline{z}) - f(z^*,\overline{z}^*)\right) \geq 0;\ \forall\ (z,\overline{z}) \in G.$ While $(z^*, \overline{z}^*)$ is not efficient, which implies to the existence of $(\check{z}, \check{\overline{z}}) \in G$ such that:

$Re\ \zeta^H \left(f(z^*,\overline{z}^*) - f(\check{z},\check{\overline{z}})\right) \geq 0$, contradiction.

4. 2. Kuhn- Tucker's conditions for efficiency

As we observe, the objective function of problem (2) is the real part of the objective function of problem (1) and the optimality conditions of them have discussed by many authors. Therefore, necessary and sufficient conditions for efficiency due to the Kuhn- Tucker are as in the same analogous to those for optimality of a single objective real objective functions.

Theorem 9. Assume that M is a polyhedron cone in $\mathbb{C}^n$; f: $\mathbb{C}^{2n} \to \mathbb{C}$, g: $\mathbb{C}^{2n} \to \mathbb{C}$ are analytic functions at a feasible point $(z^*, \overline{z}^*)$ at which the Kuhn- Tucker constraint qualification holds, and $f(G) + U$ is convex set. Then, a necessary condition for $(z^*, \overline{z}^*)$ to be efficient solution for problem (1) with respect to a pointed closed convex cone $U \subset \mathbb{C}^p$ is there is $0 \neq \zeta \in U^{\circ}$ and $u \in \left(M(z^*, \overline{z}^*)\right)^{\circ} \subset M^{\circ}$ such that

$$\left(\zeta^{\mathrm{H}} D_z f(z^*,\overline{z}^*) + \zeta^{\mathrm{H}}\overline{D_{\overline{z}} f}(z^*,\overline{z}^*) - u^{\mathrm{H}} D_z g(z^*,\overline{z}^*) - u^{\mathrm{H}}\overline{D_{\overline{z}} g}(z^*,\overline{z}^*)\right) \geq 0, \text{ and} \quad (6)$$

$$\mathrm{Re}\ u^{\mathrm{H}} g(z^*,\overline{z}^*) = 0. \quad (7)$$

Proof. Since $(z^*, \overline{z}^*)$ is an efficient solution for problem (1) with respect to U, then from **Theorem 3,** there is a $0 \neq \zeta \in U^{\circ}$ such that $(z^*, \overline{z}^*)$ solves problem (1). Consequently from Theorem4, there is $u \in \left(M(z^*, \overline{z}^*)\right)^{\circ}$ such that

$$\nabla_z \zeta^{\mathrm{H}} f(z^*,\overline{z}^*) + \overline{\nabla_{\overline{z}} \zeta^{\mathrm{H}}} f(z^*,\overline{z}^*) - u^{\mathrm{H}} D_z g(z^*,\overline{z}^*) - u^{\mathrm{H}}\overline{D_{\overline{z}} g}(z^*,\overline{z}^*) = 0, \text{ and}$$

$$(\mathrm{Re}\ u^{\mathrm{H}} g(z^*,\overline{z}^*) = 0.$$

Since ζ is a constant vector, then equations (6) and (7) yield.

Theorem 10. Assume that f: $\mathbb{C}^{2n} \to \mathbb{C}$, g: $\mathbb{C}^{2n} \to \mathbb{C}$ are analytic functions, f is convex with respect to a point closed convex cone $U \subset \mathbb{C}^p$ and g is concave with respect to a closed convex cone $\subset \mathbb{C}^m$. Then a sufficient condition for $(z^*, \overline{z}^*)$ to be an efficient solution for problem (1) with respect to U is the existence of $\zeta^{*H} \in int\ U^{\circ}$ and $u^* \in \left(M(z^*, \overline{z}^*)\right)^{\circ} \subset M^{\circ}$ such that

$$\boldsymbol{\zeta^{*\mathrm{H}} D_z f(z^*,\overline{z}^*) + \zeta^{*\mathrm{H}}\overline{D_{\overline{z}} f}(z^*,\overline{z}^*) - u^{\mathrm{H}} D_z g(z^*,\overline{z}^*) - u^{\mathrm{T}}\overline{D_{\overline{z}} g}(z^*,\overline{z}^*) = 0}, \text{ and} \quad (8)$$

$$\mathrm{Re}\ u^{\mathrm{H}} g(z^*,\overline{z}^*) = 0. \quad (9)$$

Proof. At first, since $f(z,\overline{z})$ is convex with respect to U, that is for any $(z,\overline{z}) \in P$,

$$f(z,\overline{z}) - f(z^*,\overline{z}^*) - D_z f(z^*,\overline{z}^*)(z - z^*) - D_{\overline{z}} f(z^*,\overline{z}^*)(\overline{z} - \overline{z}^*) \in U. \quad (10)$$

Since $\zeta^* \in int\ U^{\circ}$, then from (9) and for any $(z,\overline{z}) \in P$, we have

$$\mathrm{Re}\ \left(\zeta^{*\mathrm{H}} f(z,\overline{z}) - \zeta^{*\mathrm{H}} f(z^*,\overline{z}^*) - \nabla_z \zeta^{*\mathrm{H}} f(z^*,\overline{z}^*)(z - z^*) - \nabla_{\overline{z}} \zeta^{*\mathrm{H}} f(z^*,\overline{z}^*)(\overline{z} - \overline{z}^*)\right) \geq 0. \quad (11)$$

This means that

$\mathrm{Re}\ \zeta^{*H} f(z,\overline{z}) = 0$ is convex with respect to $\Re_+$.

At second, equation (8) is equivalent to

$$\nabla_z \zeta^{\mathrm{H}} f(z^*,\overline{z}^*) + \overline{\nabla_{\overline{z}} \zeta^{\mathrm{H}}} f(z^*,\overline{z}^*) - u^{\mathrm{H}} D_z g(z^*,\overline{z}^*) - u^{\mathrm{H}}\overline{D_{\overline{z}} g}(z^*,\overline{z}^*) = 0. \quad (12)$$

Hence, from (9) and (12) , $(z^*, \overline{z}^*)$ is an optimal solution for the problem

$$\min \mathrm{Re}\ \zeta^{*\mathrm{H}} f(z,\overline{z})$$
$$\text{Subject to}$$
$$(z,\overline{z}) \in G.$$

Consequently, Theorem 5 implies to $(z^*, \overline{z}^*)$ is an efficient solution for problem (1) with respect to U.

5. Proper Efficiency

In this section, strengthened efficient solutions; called properly efficient solutions is introduced. In real multi-objective optimization problems, a number of various definitions to proper efficiency with respect to a domination cone(For instance, Geoffrion's concept (Chankong and Haimes, 1983) and Geoffrion, J. Borwein's notion of propernes (Borwein, 1977) . Henig (1982) introduced the concepts of local and global properness; the first is equivalent to Benson's definition and the second to Borwein's. D. Duca (1980) formlated a vectorial programming problem in complex space as

$$\min \mathrm{Re}\ f(z,\overline{z}) = (\mathrm{Re}\ f_1(z,\overline{z}), \mathrm{Re}\ f_2(z,\overline{z}), \dots, \mathrm{Re}\ f_l(z,\overline{z}))$$
$$\text{Subject to} \quad (13)$$
$$(z,\overline{z}) \in G = \{(z,\overline{z}) \in Q: g(z,\overline{z}) \in M\}.$$

Definition 11. (Proper efficient). A point $(z^*, \overline{z}^*)$is said to be a properly efficient solution of problem (13) if it is efficient and there is a real $N > 0$ such that for each $i \in \{1, 2, \dots, l\}$ and each $(z, \overline{z}) \in G$ satisfying

$\text{Re}\left(f_i(z^*, \overline{z}^*) - f_i(z, \overline{z})\right) > 0$, there exists at least $j \neq i$ such that

$\text{Re}\left(f_i(z, \overline{z}) - f_i(z^*, \overline{z}^*)\right) \leq N\text{Re}\left(f_j(z, \overline{z}) - f_j(z^*, \overline{z}^*)\right)$

Remark 2. An efficient solution which is not proper is said to be improperly efficient.

Remark 3. The proper efficiency of the problem (1) in the case that U is a complex polyhedral cone. In fact, if U is polyhedral, i.e.,

$$U = \{z \in \mathbb{C}^p : Re\ b_i z \geq 0, i = 1, 2, \dots, l; b_i \in \mathbb{C}^p\}, \quad (14)$$

Then

$f(z, \overline{z}) - f(z^*, \overline{z}^*) \in U$, for $(z, \overline{z})$ and $(z^*, \overline{z}^*) \in G$ is equivalent to

$\text{Re}b_i f(z, \overline{z}) \geq \text{Re}b_i f(z^*, \overline{z}^*)$, for i=1, 2,…, l

Therefore, by putting $f_i(z, \overline{z}) = b_i f_i(z, \overline{z}), i = 1, 2, \dots, l$, the problem (1) can be identified with the p -real objective optimization problem (13)

Definition 12. Let $U = \{z \in \mathbb{C}^p : \text{Re}\, b_i z \geq 0, i = 1, 2, \dots, l; b_i \in \mathbb{C}^p\}$. A point $(z^*, \overline{z}^*)$ is said to be a properly efficient solution of problem (1) with respect to U if it is efficient with respect to U if it is efficient with respect to U and there is a real $N > 0$ such that for each $i \in \{1, 2, \dots, l\}$ and each $(z, \overline{z}) \in G$ satisfying

$f(z, \overline{z}) - f(z^*, \overline{z}^*) \in U_i = \{z \in \mathbb{C}^p : \text{Re}\, b_i z \geq 0, i = 1, 2, \dots, l; b_i \in \mathbb{C}^p\}$ there exists at least $j \neq i$ such that

$f(z, \overline{z}) - f(z^*, \overline{z}^*) \in U_{ij} = \{z \in \mathbb{C}^p : \text{Re}\,(b_i + Nb_j)z \geq 0, i = 1, 2, \dots, l; b_i \in \mathbb{C}^p\}$.

Proposition 3. Let U defined in (14) , then the point $(z^*, \overline{z}^*)$is a properly efficient solution of problem (13) with

$$f(z, \overline{z}) = (f_i(z, \overline{z})), f_i(z, \overline{z}) = b_i f(z, \overline{z}), i = 1, 2, \dots, l \quad (15)$$

If and only if $(z^*, \overline{z}^*)$ is properly efficient solution of problem (1) with respect to U.

5.1. Characterization of Properly efficient solutions

Now, in the case of U is polyhedron which is defined as in (14), and from theorem3 it follows that for any $\zeta \in U^\circ$, there exists $\vartheta \in \Re^l, \vartheta \geq 0$ such that $\zeta = b^H \vartheta$, $b = [b_1\ b_2 \dots b_l]^T$. Therefore, problem (14) can be rewritten ad follows

$$\min Re\ \vartheta^T b f(z, \overline{z})$$
$$\text{Subject to} \quad (z, \overline{z}) \in G, for\ some\ \vartheta \in \Re^l, \vartheta \geq 0. \quad (16)$$

Theorem 7. Let U defined in (14), f be a convex function with respect to U on P and g be concave function with respect to a closed convex cone M on P. If $(z^*, \overline{z}^*)$ is a properly efficient solution for problem (1) with respect to U , then there is $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$, with $\sum_{r=1}^{l} \vartheta_r = 1$ such that $(z^*, \overline{z}^*)$ is an optimal solution for problem (16).

Proof. If $(z^*, \overline{z}^*)$ is a properly efficient for problem (1) with respect to U , then it is properly efficient for the problem (13) accompanied with (8).

On the other hand, since $f(z, \overline{z})$ is a convex with respect to U, one can easily find that $Re\ bf(z, \overline{z})$ is convex with respect to $\Re_+^l$ on P, this is leads to the existence of $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$ with $\sum_{r=1}^{l} \vartheta_r = 1$ such that $(z^*, \overline{z}^*)$ is an optimal solution for problem (9).

Theorem 8. Let U defined in (14) and $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$ with $\sum_{r=1}^{l} \vartheta_r = 1$ such that $(z^*, \overline{z}^*)$ is an optimal solution for problem (16), then $(z^*, \overline{z}^*)$ is a properly efficient for problem (13) with respect to U.

Proof. If $(z^*, \overline{z}^*)$ is a solution for problem (16), then $(z^*, \overline{z}^*)$ is a properly efficient for problem (6) satisfying equation (8), and hence $(z^*, \overline{z}^*)$ is properly efficient for problem (1) with respect to U.

5.2. Kuhn- Tucker's conditions for proper efficiency

In this section, the conditions for proper efficiency are established in the case of U is a polyhedral cone.

Theorem 11. Assume that U defined in (14), M is a polyhedral cone in $\mathbb{C}^m$; f: $\mathbb{C}^{2n} \to \mathbb{C}$, g: $\mathbb{C}^{2n} \to \mathbb{C}$ are analytic functions at a feasible point $(z^*, \overline{z}^*)$ at which the Kuhn- Tucker's constraint qualification holds, f is convex on P with respect to U, g is concave on P with respect to M. Then a necessary condition for $(z^*, \overline{z}^*)$ to be a properly efficient solution for problem (1) with respect to U is that there is $u \in \left(M\left(g(z^*, \overline{z}^*)\right)\right)^\circ$ and $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$ with $\sum_{r=1}^{l} \vartheta_r = 1$ such that

$$\vartheta^T b \nabla_z f(z^*, \overline{z}^*) + \overline{\vartheta^T b \nabla_{\overline{z}}}(z^*, \overline{z}^*) - u^H D_z g(z^*, \overline{z}^*) - \overline{u^T D_{\overline{z}} g}(z^*, \overline{z}^*) = 0, \text{ and} \quad (17)$$

$$\text{Re}\ u^H g(z^*, \overline{z}^*) = 0 \quad (18)$$

Proof. Since $(z^*, \overline{z}^*)$ is a properly efficient for problem (1) with respect to U , then from theorem7, there is a $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$ with $\sum_{r=1}^{l} \vartheta_r = 1$ such that $(z^*, \overline{z}^*)$ is an optimal solution for problem (9). From theorem4, there exist a $u \in \left(M\left(g(z^*, \overline{z}^*)\right)\right)^\circ$ such that

$$\nabla_z \vartheta^T b f(z^*, \overline{z}^*) + \overline{\nabla_{\overline{z}} \vartheta^T b}(z^*, \overline{z}^*) - u^H D_z g(z^*, \overline{z}^*) - \overline{u^T D_{\overline{z}} g}(z^*, \overline{z}^*) = 0, \text{ and}$$

$$\text{Re}\ u^H g(z^*, \overline{z}^*) = 0.$$

Theorem 12. Assume that U defined in (7), M is a closed convex cone in $\mathbb{C}^m$; f is an analytic function and convex on P with respect to U, g is an analytic function and concave on P with respect to M. Then a sufficient condition for $(z^*, \overline{z}^*) \in G$ to be properly efficient solution for the problem (1) with respect to U is that there is $u \in M^\circ$ and $\vartheta \in \Re^l, \vartheta = (\vartheta_1\ \vartheta_2 \dots \vartheta_3)^T > 0$ with $\sum_{r=1}^{l} \vartheta_r = 1$ such that the equalities (17) and (18hold.

Proof. Since f is a convex function with respect to $U, Re\ bf(z, \overline{z})$ is convex with respect to $\Re_+^l$on P. It follows from $\vartheta > 0$ that $Re\vartheta^T\ bf(z, \overline{z})$ is convex with respect to $\Re_+$ on P. Therefore, theorem4 implies to $(z^*, \overline{z}^*)$ is a

solution of problem (9), that $(z^*, \overline{z}^*)$ is a properly efficient for problem (1) with respect to U.

Table 1
List of symbols

Symbols	Meaning
$\mathbb{C}^{m\times n}$	Complex $m \times n$ matrix
A	Matrix $\mathbb{C}^{m\times n}$
A^T	Transpose of A
$\overline{A}$	Conjugate of A
A^H	Conjugate transpose
Re x	Real part
Int U	The interior of U
$\Re_+^k$	Non- negative orthant of $\Re^n$, $\Re_+^k = \{x \in \Re^n : x_j \geq 0, j = \overline{1,n}\}$.

6. Conclusions and Future Works

In this paper, the necessary and sufficient conditions for a point to be an efficient or a properly efficient solution of a multi-objective complex programming problem have studied, when the ordering has taken with respect to a pointed closed convex cone**.** The future of this work summarized as in the following points:

A. Developing the duality theory in both single-objective and multi-objective optimization problems.
B. Deriving the optimality conditions under other types of generalized convexity such as E-convexity and invexity.
C. Deriving the saddle point optimality criteria without differentiability requirements.
D. Study of fractional programming problems in both single-objective and multi-objective programming problems.
E. Study of multiplicative programming problems.
F. Study of stability in parametric programming problems over complex space.
G. Extending the various concepts of proper efficiency to complex space and establishing the relationships between them.

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IM4.0EF: Tele-Medical Realization via Integrating Vague T2NSs with OWCM-RAM Toward Intelligent Medical 4.0 Evaluator Framework

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Abstract

In the modern world, when nearly everyone considers the Internet to be indispensable, there is a constant desire to discover new applications for the technologies that are already in place. As well, recent research has found several synonyms for the adoption of information and communication technology. Industry 4.0, Industry 5.0, and modern technologies are some of these synonyms. Harnessing these technologies in important domains such as medical services and healthcare is vital. Wherein, the medical field has undergone changes, going from version 1.0 to version 4.0 currently. Medical 4.0 represents a significant advancement in medical practices and systems, leveraging cutting-edge digital technology, data analytics, automation, and artificial intelligence (AI) within the healthcare sector. Therefore, it is essential to give priority to important enabling aspects (EAs) that influence these cutting-edge technologies adoption into the healthcare industry in a methodical and successful manner. Inspired by these facts, the prioritizing of the determined EAs is robust motivator in this study. Hence, we constructed intelligent Medical 4.0 evaluator framework (IM4.0EF) which responsible for prioritizing EAs based in a set of benchmarks (Bs) through leveraging Opinion Weight Criteria Method (OWCM) as a novel multi-criteria decision-making (MCDM) technique to obtain benchmarks' weights. Wherein these weights are utilized in Root Assessment Method (RAM) as a novel MCDM to rank EAs and recommending the most influence and optimal EA as well as the worst EA.

Keywords: Medical 4.0; Cutting-edge Technologies; Opinion Weight Criteria Method; Root Assessment Method; Type-2 Neutrosophic Theory.

1 | Introduction

1.1 | Thorough View

In light of contemporary technology, handling a wide range of application sectors has become more precise and easier. In the realm of medicine and health services, human beings frequently disregard their health because they are so focused on getting by in their busy everyday lives. Therefore, health declines and then

human effectiveness is impacted. Hence, this matter directly affects overall growth, a nation's aspirations for progress cannot ignore the rights of its citizens.

Providing individuals with cutting-edge healthcare facilities is an urgent need to improve their quality of life. To enhance people's lives, many technologies have developed and are now available to the public via the internet or other networks. These technologies may enable people to receive a proper diagnosis quickly and affordably [1]. In the modern world, when nearly everyone considers the Internet to be an indispensable aspect of their existence, there is a constant desire to discover new applications for the technologies that are already in place. The medical field It has undergone changes, going from version 1.0 to version 4.0 at this time [2]. Medical 4.0 represents a significant advancement in medical practices and systems, leveraging cutting-edge digital technology, data analytics, automation, and artificial intelligence (AI) within the healthcare sector [3]. It covers patient interaction, digital health records, data management, telemedicine, remote patient monitoring, data privacy [4]. Medical 4.0 has the potential to revolutionize the way healthcare is delivered by raising overall satisfaction, cutting costs, and improving patient outcomes. But to maximize the advantages of these new technologies while addressing ethical, legal, and technical concerns, lawmakers, regulatory agencies, and healthcare providers must work together [4]. To meet these objectives, sensor-based patient monitoring systems are being employed, and they have the capacity to provide enormous amounts of data regarding the health of the patients daily. In this instance, the cloud computing concept was used for data analysis and storage [5]. Healthcare systems encounter significant obstacles including security breaches, latency, and the need to cut expenses without sacrificing the provision of high-quality patient care. These days, technological developments and breakthroughs in the healthcare industry are becoming more and more well-known [6]. However, cloud-based systems may result in misdiagnosis, unhappiness with therapeutic frameworks, and delays in application communication. Fogging, which anticipates delays by bringing capacities and abilities closer to applications and information sources, is therefore currently considered an alternative to cloud computing. Using fog, increasingly potent therapeutic innovation configurations can be created [5]. Fog computing, which is carefully considered as the addition of cloud computing to the edge of the network, is a highly virtualized perspective of the resource group that provides nearby end users with networking, processing, and data storage options [7].

1.2 | Study Motives

According to the previous perspectives as [4], there are various enabling aspects should take into consideration for embracing edge computing paradigms in healthcare domain. It is highly significant to provide stakeholders with a road map outlining how to implement this technology in the field of healthcare. As well, identifying obligations and enabling aspects that must be considered to avoid obstacles to achieving medical 4.0 is imperative.

Accordingly, this study was keen to adopt this matter of obligations and enabling aspects (EAs) and deal with it through a set of aspects. Theoretically, through conducted surveys for earlier studies [8] where human responsibilities is the main in this aspect. These responsibilities involved strategic and source planning, culture, and the ability to eliminate complexity through flexibility and proactivity for next events. Practically, taken innovations, practical deployment and leveraging resources toward optimization and achieving medical 4.0 is important EAs. Scientifically, determining the most influence EA for embracing edge computing and industry 4.0 is crucial matter. Hence, we developed IM4.0EF for evaluating these enabling aspects and its benchmarks were enabling aspects evaluating based on. Thus, MCDM techniques are utilized in prioritizing for enabling aspects through a set of benchmarks. To ensure the ability to deal with uncertain and ambiguous circumstances during judgements, T2NSs collaborated by utilizing MCDM as branch of neutrosophic vague theory.

2 | Related Work

Various researchers use different technology and methodologies to give distinct studies about healthcare issues and how they are monitored. This part considers the studies in order to provide a deeper knowledge

of the problems and difficulties that the healthcare industry and people confront [1]. Medical 4.0 is the integration of digital solutions, data analytics, and state-of-the-art technologies into medical and healthcare. The core of a Medical 4.0 application idea is to operationalize these technological developments in the healthcare and medical fields [4]. The studies presented by Silva et al. [9] demonstrate the rationale behind the fog model's implementation in healthcare through a state-of-the-art study. It was observed in the study that there is still more work to be done on the fog layer's distribution of restorative data [10]. Security and protection are key components of this approach and are frequently mentioned in [5]. Apart from the cloud, another barrier is storage capacity due to its limited space [11]. This constraint must thus be met by the fog-based storage systems. Furthermore, the review [12] validates the findings linked to the lack of healthcare knowledge by the board method in fog, asserting that no appropriate information is available. In the meantime, the review [13] views the presentation related to the control of data in applications as the main driver for the use of fog computing in the healthcare industry. Dealing with Fog's managing max is a common task. Depending on the scope of the project, timeframe for completion, and capacity limit, a management assignment and fog resource allocation strategy are recommended [14]. George et al. talked about using a smartphone as a sensor to continuously monitor a patient's health in [15]. They talked about the benefits of FC over CC in identifying a patient's health issues and employed the FC technique to analyze patient health data more quickly. They considered three different patient categories, including those who were seriously hurt, were typically hospitalized, and required future guidance for routine checkups.

Kraemer et al. [16] talked about many healthcare informatics use cases. To manage the inventory of applications utilized for a certain job, they employed the FC approach. The author also covered the various tasks that FC performs at various network levels to provide tradeoffs regarding the intended needs necessary for healthcare.

3 | Evaluation Based on MCDM-T2NSs: Intelligent Medical 4.0 Evaluator Framework

Herein, we are leveraging the ability of MCDM techniques to handle the conflict criteria that our problem is characterized by. Accordingly, OWCM [16] is utilized for obtaining benchmarks' weights where these weights have been leveraged in RAM for ranking alternatives of enabling aspects (EAs). The mentioned methods are collaborated by T2NSs and this collaboration is illustrated in the following procedures of the intelligent Medical 4.0 evaluator framework (IM4.0EF).

Procedure 1 **Consider the primary ingredients for the evaluation**

The benchmarks (Bs) that influence EAs for implementing Medical 4.0 are realized.

Decision makers (DMs) who volunteering for evaluating EAs based on Bs are opted.

T2N matrices are erected based on evaluations of volunteered DMs though using scales in Table 1[17].

Converting erected T2N matrices into crisp matrices using Eq. (1).

$$S(\tilde{Z_i}) = \frac{1}{12}(8+\left(T_{T_{Z_i}}(x)+2\left(T_{I_{Z_i}}(x)\right)+T_{F_{Z_i}}(x)\right)-\left(I_{T_{Z_i}}(x)+2\left(I_{I_{Z_i}}(x)\right)+I_{F_{Z_i}}(x)\right)$$

$$-\left(F_{T_{Z_i}}(x)+2\left(F_{I_{Z_i}}(x)\right)+F_{F_{Z_i}}(x)\right) \quad (1)$$

Procedure 2 **Figuring out the weights of benchmarks: OWCM-T2NSs [18]**

Merging the crisp metrices into an aggregated matrix based on Eq. (2):

$$x_{ij} = \frac{\sum_{j=1}^{N} s(z_i^{\sim})}{N} \quad (2)$$

Generate normalized matrix by employing Eq. (3) in an aggregated matrix based on Eq.(3):

$$R_{ij} = \frac{x_{ij}}{\underset{x_j}{max}} \quad (3)$$

x_i^{max} refers to maxnumber in x_{ij} matrix per column.

The average score of the decision matrix is calculated as following.

$$\mathcal{B} = \frac{1}{S}\sum_{i=1}^{m} R_{ij} \quad (4)$$

S refers to number of EAs.

Preference variation for each benchmark is calculate as in Eq. (5).

$$\mho_{j=\sum_{j=1}^{m}[R_{ij}-\mathcal{B}]^2}$$

The deviation in preference values is calculate based on Eq. (6)

$$\mathcal{E}_{j=1-\mho_j} \quad (6)$$

weights for benchmarks is calculate as in Eq. (7).

$$\omega_{j=\frac{\mathcal{E}_j}{\sum_{j=1}^{m}\mathcal{E}_j}} \quad (7)$$

Table 1. Linguistic variables for classification.

Linguistic Variables	T2N scale $< (T_T, T_I, T_F), (I_T, I_I, I_F), (F_T, F_I, F_F) >$
Very Bad (VB)	⟨(0.20, 0.20, 0.10),(0.65, 0.80, 0.85),(0.45, 0.80, 0.70)⟩
Bad (B)	⟨(0.35, 0.35, 0.10),(0.50, 0.75, 0.80),(0.50, 0.75, 0.65)⟩
Medium Bad (MB)	⟨(0.50, 0.30, 0.50),(0.50, 0.35, 0.45),(0.45, 0.30, 0.60)⟩
Medium (M)	⟨(0.40, 0.45, 0.50),(0.40, 0.45, 0.50),(0.35, 0.40, 0.45)⟩
Medium Good (MG)	⟨(0.60, 0.45, 0.50),(0.20, 0.15, 0.25),(0.10, 0.25, 0.15)⟩
Good (G)	⟨(0.70, 0.75, 0.80),(0.15, 0.20, 0.25),(0.10, 0.15, 0.20)⟩
Very Good (VG)	⟨(0.95, 0.90, 0.95),(0.10, 0.10, 0.05),(0.05, 0.05, 0.05)⟩

Procedure 3

Ranking enabling aspects: RAM- T2NSs [19]

Normalize the aggregated matrix in previous step using Eq. (7):

$$Nor_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}} \ (j = 1,2,\ldots,n) \quad (7)$$

Calculate the weighted decision matrix using Eq. (8):

$$y_{ij} = Nor_{ij} w_{ij} \quad (i = 1,2,..m : j = 1,2 \ldots n) \quad (8)$$

Calculate the sums of weighted normalized scores by Eqs. (9) and (10):

$$s_{+t} = \sum_{j=1}^{n} y_{+tj}, \text{ beneficial} \quad (9)$$

$$s_{-t} = \sum_{i=1}^{n} y_{-tj} \ , \text{non-beneficial} \quad (10)$$

Determine the overall score of each alternative by Eq. (11):

$$RI_i = \sqrt[2+s_{-1}]{2+s_{+t}} \quad (i = 1, 2, \ldots, m) \quad (11)$$

Rank the alternatives using the value of RI_i

4 | Illustrative Case Study

We implemented the constructed IM4.0EF on real medical field to validate the efficiency of this framework. Herein, the main ingredients as Bs and EAs are determined in this study through surveying for earlier studies as [3, 4] which resulted ten benchmarks which utilized in evaluating five enabling aspects based on five DMs as in Table 2.

4.1 | Valuating Benchmarks: OWCM-T2NSs

- Five T2NSs matrices are constructed and converted to crisp matrices using Eq. (1).
- Using Eq. (2) to blend these matrices into an aggregated matrix as in Table 3.
- Normalizing the aggregated matrix using Eq. (3) to generate normalized matrix as in Table 4.
- Final weights for Bs are generated after computing $\mathcal{B}, \mho_j \ , \mathcal{E}_j$ through deploying Eqs. (4) , (5) and (6) as shown in table 5. Figure 1 represents the valuation weights for Bs where B6 and B10 have highest weight but B2 has lowest weight.

Table 2. Main enabling aspects and its benchmarks.

Role in evaluation	Enabling Aspects (EAs)
EA1	Controlling complexity
EA2	Practical readiness
EA3	Resource Optimizing
EA4	Regulation and polices
EA5	Partner Compatibility
Benchmarks (Bs)	
B1	Environmental sustainability
B2	Compliance and deployment
B3	Cost
B4	Communication efficiency
B5	Flexibility
B6	Eliminating Uncertainty and hazard
B7	Complications
B8	Adaptation
B9	Proactivity
B10	Security and privacy

Table 3. Aggregated decision matrix

	B1	**B2**	**B3**	**B4**	**B5**	**B6**	**B7**	**B8**	**B9**	**B10**
EA1	0.4375	0.5025	0.5158	0.6742	0.5300	0.4725	0.5017	0.7283	0.5017	0.4725
EA2	0.5283	0.6008	0.5408	0.6400	0.4992	0.5067	0.6350	0.6667	0.6350	0.5067
EA3	0.5917	0.4375	0.7092	0.6817	0.5850	0.5967	0.6725	0.6100	0.6725	0.5967
EA4	0.3825	0.8333	0.5317	0.3375	0.5300	0.5400	0.5317	0.4483	0.5317	0.5400
EA5	0.7892	0.3825	0.4808	0.5708	0.7008	0.6183	0.5067	0.6442	0.5067	0.6183

Table 4. Normalizing the aggregated decision matrix.

	B1	**B2**	**B3**	**B4**	**B5**	**B6**	**B7**	**B8**	**B9**	**B10**
EA1	0.5544	0.6030	0.7274	0.9890	0.7562	0.7642	0.7460	1.0000	0.7460	0.7642
EA2	0.6695	0.7210	0.7626	0.9389	0.7122	0.8194	0.9442	0.9153	0.9442	0.8194
EA3	0.7497	0.5250	1.0000	1.0000	0.8347	0.9650	1.0000	0.8375	1.0000	0.9650
EA4	0.4847	1.0000	0.7497	0.4951	0.7562	0.8733	0.7906	0.6156	0.7906	0.8733
EA5	1.0000	0.4590	0.6780	0.8374	1.0000	1.0000	0.7534	0.8844	0.7534	1.0000

Table 5. Average score and Preference variation.

	B1	**B2**	**B3**	**B4**	**B5**	**B6**	**B7**	**B8**	**B9**	**B10**
$\mathcal{B}$	0.6917	0.6616	0.7835	0.8521	0.8119	0.8844	0.8468	0.8506	0.8468	0.8844
$\mho_j$	0.1606	0.1812	0.0627	0.1758	0.0520	0.0387	0.0550	0.0831	0.0550	0.0387
ε_j	0.8394	0.8188	0.9373	0.8242	0.9480	0.9613	0.9450	0.9169	0.9450	0.9613

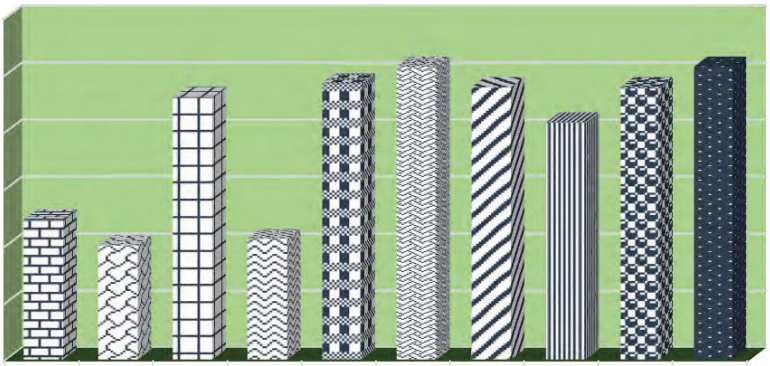


Figure 1. Final Benchmarks' weights.

4.2 | Ranking Alternatives of Enabling Aspects using RAM-T2NSs

Digital twin technology is considered one of the vital tools for smart cities, although it can also be applied in other fields.

- Eq. (7) implemented in the previous aggregated matrix to obtain normalized matrix as in Table 6.
- Table 7 represents weighted decision matrix based on Eq. (8).
- Sums of weighted normalized scores are calculated based on Eqs. (9) and (10) as in Table 8.
- The overall score of each alternative is calculated based on Eq. (11). Figure 2 show cases that partner compatibility is optimal otherwise regulations and compliance is the worst.

Table 6. Normalizing the aggregated decision matrix.

	B1 (+)	B2 (+)	B3 (-)	B4 (+)	B5 (+)	B6 (+)	B7 (-)	B8 (+)	B9 (+)	B10 (+)
EA1	0.1603	0.1823	0.1857	0.2321	0.1863	0.1728	0.1762	0.2351	0.1762	0.1728
EA2	0.1936	0.2180	0.1947	0.2204	0.1755	0.1853	0.2230	0.2152	0.2230	0.1853
EA3	0.2168	0.1587	0.2552	0.2347	0.2056	0.2182	0.2362	0.1969	0.2362	0.2182
EA4	0.1402	0.3023	0.1914	0.1162	0.1863	0.1975	0.1867	0.1447	0.1867	0.1975
EA5	0.2892	0.1388	0.1731	0.1966	0.2463	0.2262	0.1779	0.2080	0.1779	0.2262

Table 7. Weighted decision matrix.

	B1 (+)	B2 (+)	B3 (-)	B4 (+)	B5 (+)	B6 (+)	B7 (-)	B8 (+)	B9 (+)	B10 (+)
EA1	0.0148	0.0164	0.0191	0.0210	0.0194	0.0183	0.0183	0.0237	0.0183	0.0183
EA2	0.0179	0.0196	0.0201	0.0200	0.0183	0.0196	0.0232	0.0217	0.0232	0.0196
EA3	0.0200	0.0143	0.0263	0.0213	0.0214	0.0231	0.0245	0.0198	0.0245	0.0231
EA4	0.0129	0.0272	0.0197	0.0105	0.0194	0.0209	0.0194	0.0146	0.0194	0.0209
EA5	0.0267	0.0125	0.0178	0.0178	0.0257	0.0239	0.0185	0.0210	0.0185	0.0239

Table 8. Sums of weighted normalized scores and overall scores.

	s+i	s-i	RI	N-RI	Rank
EA1	0.1502	0.0374	1.4561	0.21721	4
EA2	0.1598	0.0432	1.4577	0.40248	3
EA3	0.1675	0.0508	1.4582	0.45989	2
EA4	0.1458	0.0391	1.4542	0	5
EA5	0.1699	0.0363	1.4629	1	1

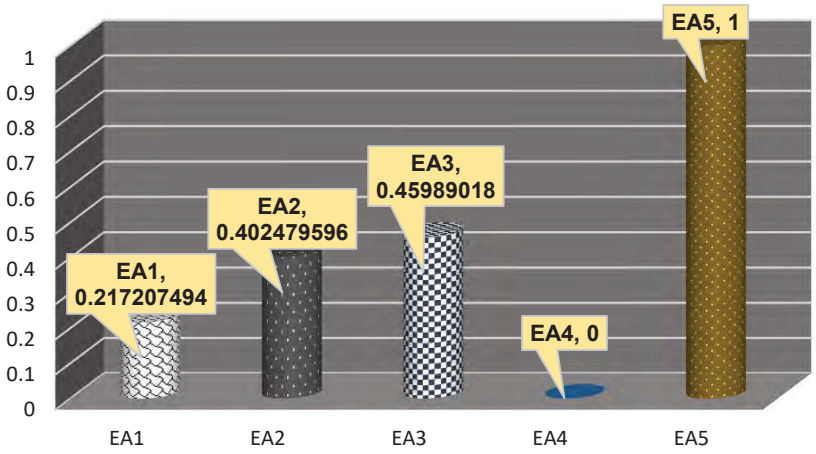


Figure 2. Ranking of enabling aspects.

5 | Conclusion

The present study enhances our comprehension of the factors propelling the adoption of digital technologies in Medical toward Medical 4.0. As well. This study provides valuable insights to facilitate the realization of the complete potential of technologically enabled advancements in healthcare. Because technologies as big data, cloud computing, 5G, digital twin...etc. are offering proximity to data sources, data offloading and processing, resilience to network outages, enhanced security and privacy, scalability, interoperability, cost optimization, Real-Time Decision Making and Regulatory Compliance. It plays a significant role in influencing the selection of Medical 4.0 enablers. Enablers can provide effective, dependable, and secure healthcare solutions that satisfy the changing requirements of contemporary healthcare systems by utilizing fog computing capabilities. Moreover, we are constructing an evaluation model for evaluating enabling aspects in Medical 4.0 based on a set of benchmarks through Multi-Criteria Decision Making (MCDM) techniques where each technique is responsible for a certain function. For instance, Opinion Weight Criteria Method (OWCM) is utilized for obtaining the benchmarks' weights and Root Assessment Method (RAM) leverages the obtained weights from OWCM for ranking the alternatives and recommends the most sustainable and worst alternative. The evaluation for alternatives is performed based on rating ten benchmarks. Finally, the utilized MCDM techniques are working under T2NSs.

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Author Contributions

All authors contributed equally to this work.

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Data Availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The author declares that there is no conflict of interest in the research.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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New Similarity and Entropy Measures of Temporal Complex Neutrosophic Set and Its application in Multi-Criteria Decision Making

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Abstract

The Temporal complex neutrosophic set (TCNS) is an effective structure for resolving information that has uncertain, indeterminate, and time-related factors in decision-making problems. In the last decade, many researchers focused on estimating the vagueness and ambiguity in knowledge using the theory of neutrosophic sets or extensions of neutrosophic sets. Similarity and entropy measures are valuable tools to measure information with the aim of dealing with real-life multi-criteria decision-making (MCDM) problems. However, the existing method did not care

or interest in the information measures of TCNS. This paper proposes several new similarity and entropy measures under the TCNS environment. The suggested measurements have been confirmed and proven to concede with the manifest definition of the similarity and entropy measure for the TCNS. Finally, a numerical example related to deciding on a tourist terminus in Vietnam is given to confirm the practical applicability of the proposed measures. The numerical example proves that the proposed similarity and entropy measures of TCNS can produce accurate and reasonable results for decision-making problems in the real world.

Keywords: MCDM; Complex Neutrosophic Set; Temporal complex neutrosophic set; Similarity measure; Entropy measure

1 Introduction

In 1999, Smarandache [1] introduced the notion of a neutrosophic set (NS) that each part of this set has a degree of truth, indeterminacy, and falsity, respectively. It's also an essential and powerful tool to deal with incomplete, indeterminate, and inconsistent information in some real-life problems. Due to its ability to deal with the uncertainty factor, it has many applications in the areas (theoretical as well as practical) related to engineering, arts, humanities, computer science, health sciences, life sciences, etc. Although NS theory is very successful in handling problems arising from vagueness and uncertainty, it cannot address the periodicity occurring in some uncertain information. Because of the indeterminate information and the complexity of the periodic form of decision-making problems, it is challenging to explain attributes in terms of crisp sets, fuzzy sets, and neutrosophic sets.

To better design and model real-life applications, the "complex" feature deals simultaneously with uncertainty, inconsistency, and periodicity data. To solve this issue, Ramot et al. [2] proposed the theory of the complex fuzzy set (CFS) that supplements a phase component to gain information regarding a particular higher dimensional periodic problem. Also, with the aim of adding the "complex" part to make NS more flexible and adaptable to occasional vague information, Ali and Smarandache [3] proposed the definition of the complex neutrosophic set (CNS) that considers a combined version of the CFS and NS. Three membership degree functions characterize a CNS tackle uncertainty, inconsistency, and indeterminacy. These functions have complex-valued and the range in the unit circle in the complex space. The constraint of the CNS is that the sum of positive, abstinence, and negative grades is less than or equal to three. After its introduction, CNS proved its significance in describing data with indeterminate, uncertain, inconsistent, and periodic factors.

The notion of similarity plays a fundamental role in solving assorted complex, indeterminate matters in human life. The similarity and entropy measures

exhibited the degree of similarity between two objects in an uncertain, indeterminate environment. Hence, many researchers have concentrated on measuring uncertainty and fuzziness in information using neutrosophic sets or extensions. Various similarity and entropy measures have been introduced. They have successfully applied in solving problems in real-life applications, such as medical diagnosis [4–6], pattern recognition [7, 8], decision making [9–11], and clustering analysis [12, 13].

In the neutrosophic environment, Broumi and Smarandache [14] introduced some similarities based on Hausdorff distance for NS. Majumdar et al. [15] proposed some measures of similarity and entropy of a single-valued NS. Jun Ye [16] presented the similarity measures that construct based on the Hamming and Euclidean distances for interval neutrosophic sets. Following that, Mukherjee and Sarkar [17] defined some similarity measures based on Harmming and Euclidean distances between two interval neutrosophic soft sets with an application in pattern recognition. Abu Qamar and Hassan [18]introduced similarity and entropy tools to the Q-Neutrosophic Soft set and examined their effectiveness for medical diagnosis problems. Rıdvan Sahin [19] proposed two novel distance measures for SVNHFNs based on single-valued neutrosophic hesitant fuzzy numbers. Besides, similarity and entropy are widely used as critical measures for solving multi-criteria decision-making problems in real life [7–10, 17, 20].

As two critical topics in the CNS, entropy and similarity measures have been extensively studied by many studies from different perspectives. Researchers made significant additions to the literature on similarity measures domains by using hybridized models to manage the uncertainty of periodic data, where time plays a vital role in depicting it. Al-Qudah and Hassan [21] introduced axiomatic definitions of entropy and similarity measure for Complex Multi Fuzzy Soft Sets. Ganeshsree Selvachandran et al. [22] established the measures of distance and similarity based on a complex vague soft set with the aim of solving pattern recognition problems involving digital images. Mondal et al. [23] proposed three complex neutrosophic similarity measures, such as cosine, Jaccard, and Dice. A numerical example of stream selection of students after secondary examination to select an appropriate educational stream for higher secondary education is given to demonstrate the proposed strategies for multi-attribute decision-making problems. Ali et al. [24] proposed novel dice similarity measures using CNS. Then, the authors applied the presented criteria to the pattern recognition model to examine the reliability and superiority of the established approaches. Faisal et al. [20] introduced some similarity measures of interval complex neutrosophic soft sets based on the distance measures: Hamming distance and Euclidean distance for solving decision-making problems in medical diagnosis.

For the purpose of modeling data with the time factor and dynamic factor of some real-world decision-making problems, recently, Lan et al. [25] introduced the definition of a Temporal Complex neutrosophic set (TCNS). TCNS is an extension of CNS. TCNS concerns decision-making problems that have

uncertain, temporal, and periodical factors. An application for tourist destination choice problems illustrated the effectiveness and flexibility of TCNS in multi-criteria decision-making. It is clear that TCNS is a new opening with many real-life applications when it considers the time cycle and the time factor. However, previous studies were still not yet taken an interest in similarity and entropy measures in the temporal complex neutrosophic environments. Therefore, the paper presents new similarity and entropy measures of the TCNS.

Specifically, the contributions of this paper are highlighted as follows: (1) Define new similarity and entropy measures of the Temporal complex neutrosophic set. (2) Construct and develop the decision-making model based on new similarity and entropy measures of TCNS. (3) Demonstrate the feasibility and rationality of solving the decision-making problem of the proposed model via an actual world-life case study of choosing a tourist destination in Vietnam and comparative analysis.

The rest of this paper is organized as follows. In Section 2, we first review some fundamental concepts of the TCNS. Section 3 defines new similarity and entropy measures based on the TCNS theory. Section 4 presents the developing multi-criteria decision-making model that uses proposed similarity and entropy measures. Section 5 describes a practical application related to a chosen tourist destination and demonstrates our method's effectiveness via a comparison experiment. Finally, the conclusions are presented in Section 6.

2 PRELIMINARIES

In this section, we recapitulate the basic notion of TCNS and their operations, namely complement, union, and intersection, which will be used in our paper.

Definition 1 ([25]) Suppose there are a universal set X and time periods $\tilde{\tau} = \{\tau_1, \tau_2, ..., \tau_{n_\tau}\}$. A Temporal complex neutrosophic set TCNS on X is denoted by

$$\text{TCNS}(x, \tilde{\tau}) = \{x, \langle T(x, \tilde{\tau}), I(x, \tilde{\tau}), F(x, \tilde{\tau}) \mid x \in X\} \tag{1}$$

Where $T(x, \tilde{\tau}) = p(x, \tau_l).e^{j\mu(\tau_l)}$; $I(x, \tilde{\tau}_x) = q(\tau_l).e^{j\nu(x,\tau_l)}$;

$F(x, \tilde{\tau}) = r(x, \tau_l).e^{j\eta(\tau_l)}$; $l = 1, 2, ..., n_\tau$; for ease of use, we'll call $\tilde{n}_E$ is a Temporal Complex Neutrosophic Element (TCNE) of element $x \in X$ to the set E and $\tilde{n}_E$ is given by the Equation (2)

$$\tilde{n}_E = \left\langle p_x(\tau_l).e^{j\mu_x(\tau_l)}, q_x(\tau_l).e^{j\nu_x(\tau_l)}, r_x(\tau_l).e^{j\eta_x(\tau_l)} \right\rangle \tag{2}$$

Definition 2 ([25]) Suppose there are two TCNSs described by their temporal complex neutrosophic membership functions $\Psi_1(x, \tilde{\tau})$ and $\Psi_2(x, \tilde{\tau})$ respectively.

$$\Psi_1(x, \tilde{\tau}) = \left\{x, \left\langle p_{\Psi_1}(x, \tilde{\tau}) e^{j\mu_{\Psi_1}(x,\tilde{\tau})}, q_{\Psi_1}(x, \tilde{\tau}) e^{j\nu_{\Psi_1}(x,\tilde{\tau})}, r_{\Psi_1}(x, \tilde{\tau}) e^{j\eta_{\Psi_1}(x,\tilde{\tau})} \right\rangle\right\}$$
$$\Psi_2(x, \tilde{\tau}) = \left\{x, \left\langle p_{\Psi_2}(x, \tilde{\tau}) e^{j\mu_{\Psi_2}(x,\tilde{\tau})}, q_{\Psi_2}(x, \tilde{\tau}) e^{j\nu_{\Psi_2}(x,\tilde{\tau})}, r_{\Psi_2}(x, \tilde{\tau}) e^{j\eta_{\Psi_2}(x,\tilde{\tau})} \right\rangle\right\}$$

The basic operations of TCNS are given as follows:
Complement:

$$C\left(\Psi_1\left(x,\tilde{\tau}\right)\right)=\left\{x,\left\langle \begin{array}{c} r_{\Psi_1}\left(x,\tilde{\tau}\right)e^{j\left(2\pi-\eta_{\Psi_1}\left(x,\tilde{\tau}\right)\right)}, \\ \left(1-q_{\Psi_1}\left(x,\tilde{\tau}\right)\right)e^{j\left(2\pi-\upsilon_{\Psi_1}\left(x,\tilde{\tau}\right)\right)}, \\ p_{\Psi_1}\left(x,\tilde{\tau}\right)e^{j\left(2\pi-\mu_{\Psi_1}\left(x,\tilde{\tau}\right)\right)} \end{array}\right\rangle \mid x\in X\right\} \quad (3)$$

Union:

$$\Psi_1\left(x,\tilde{\tau}\right)\cup\Psi_2\left(x,\tilde{\tau}\right)=\left\{x,\left\langle \begin{array}{c} \left(p_{\Psi_1}\left(x,\tilde{\tau}\right)\vee p_{\Psi_2}\left(x,\tilde{\tau}\right)\right)e^{j\mu_{\Psi_1\cup\Psi_2}\left(x,\tilde{\tau}\right)}, \\ \left(q_{\Psi_1}\left(x,\tilde{\tau}\right)\wedge q_{\Psi_2}\left(x,\tilde{\tau}\right)\right)e^{j\nu_{\Psi_1\cup\Psi_2}\left(x,\tilde{\tau}\right)}, \\ \left(r_{\Psi_1}\left(x,\tilde{\tau}\right)\wedge r_{\Psi_2}\left(x,\tilde{\tau}\right)\right)e^{j\eta_{\Psi_1\cup\Psi_2}\left(x,\tilde{\tau}\right)} \end{array}\right\rangle \mid x\in X\right\} \quad (4)$$

Intersect:

$$\Psi_1\left(x,\tilde{\tau}\right)\cap\Psi_2\left(x,\tilde{\tau}\right)=\left\{x,\left\langle \begin{array}{c} \left(p_{\Psi_1}\left(x,\tilde{\tau}\right)\wedge p_{\Psi_2}\left(x,\tilde{\tau}\right)\right)e^{j\mu_{\Psi_1\cap\Psi_2}\left(x,\tilde{\tau}\right)}, \\ \left(q_{\Psi_1}\left(x,\tilde{\tau}\right)\vee q_{\Psi_2}\left(x,\tilde{\tau}\right)\right)e^{j\nu_{\Psi_1\cap\Psi_2}\left(x,\tilde{\tau}\right)}, \\ \left(r_{\Psi_1}\left(x,\tilde{\tau}\right)\vee r_{\Psi_2}\left(x,\tilde{\tau}\right)\right)e^{j\eta_{\Psi_1\cap\Psi_2}\left(x,\tilde{\tau}\right)} \end{array}\right\rangle \mid x\in X\right\} \quad (5)$$

3 New Similarity and Entropy of Temporal complex neutrosophic set

In this section, the paper introduces some similarity measures of TCNS, such as Dice, Jaccard, Cosine, and Cotangent similarity measures. Furthermore, an entropy measurement of TCNS is also proposed as the baseline for decision-making models in temporal complex neutrosophic environments.

3.1 Dice similarity measure of TCNS

Definition 3 Assume that
$\Psi_1 = \left\langle p_{\Psi_1}\left(x_i,\tau_l\right)e^{j\mu_{\Psi_1}\left(x_i,\tau_l\right)}, q_{\Psi_1}\left(x_i,\tau_l\right)e^{j\nu_{\Psi_1}\left(x_i,\tau_l\right)}, r_{\Psi_1}\left(x_i,\tau_l\right)e^{j\eta_{\Psi_1}\left(x_i,\tau_l\right)}\right\rangle$
and $\Psi_2 = \left\langle p_{\Psi_2}\left(x_i,\tau_l\right)e^{j\mu_{\Psi_2}\left(x_i,\tau_l\right)}, q_{\Psi_2}\left(x_i,\tau_l\right)e^{j\nu_{\Psi_2}\left(x_i,\tau_l\right)}, r_{\Psi_2}\left(x_i,\tau_l\right)e^{j\eta_{\Psi_2}\left(x_i,\tau_l\right)}\right\rangle$
be are an TCNSs on universe of discourse X for all $x_i\left(1,2,3,...,n_X\right)$ and $\tau_l\left(1,2,3,...,n_\tau\right)$.
A temporal complex Dice similarity measure between TCNSs Ψ_1 and Ψ_2 (denoted as $S_{Dice}\left(\Psi_1,\Psi_2\right)$) is defined as follows:

$$S_{Dice}\left(\Psi_1,\Psi_2\right)$$
$$=\frac{1}{n_X*n_\tau}\sum_{i=1}^{n_X}\sum_{l=1}^{n_\tau}\frac{2*\left[\begin{array}{l}\left(p_{\Psi_1}\left(x_i,\tau_l\right)+\mu_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(p_{\Psi_2}\left(x_i,\tau_l\right)+\mu_{\Psi_2}\left(x_i,\tau_l\right)\right)\\+\left(q_{\Psi_1}\left(x_i,\tau_l\right)+\nu_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(q_{\Psi_2}\left(\tau_l\right)+\nu_{\Psi_2}\left(x_i,\tau_l\right)\right)\\+\left(r_{\Psi_1}\left(x_i,\tau_l\right)+\eta_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(r_{\Psi_2}\left(x_i,\tau_l\right)+\eta_{\Psi_2}\left(x_i,\tau_l\right)\right)\end{array}\right]}{\left(\begin{array}{l}\left(p_{\Psi_1}\left(x_i,\tau_l\right)+\mu_{\Psi_1}\left(x_i,\tau_l\right)\right)^2+\left(q_{\Psi_1}\left(x_i,\tau_l\right)+\nu_{\Psi_1}\left(x_i,\tau_l\right)\right)^2\\+\left(r_{\Psi_1}\left(x_i,\tau_l\right)+\eta_{\Psi_1}\left(x_i,\tau_l\right)\right)^2+\left(p_{\Psi_2}\left(x_i,\tau_l\right)+\mu_{\Psi_2}\left(x_i,\tau_l\right)\right)^2\\+\left(q_{\Psi_2}\left(x_i,\tau_l\right)+\nu_{\Psi_2}\left(x_i,\tau_l\right)\right)^2+\left(r_{\Psi_2}\left(x_i,\tau_l\right)+\eta_{\Psi_2}\left(x_i,\tau_l\right)\right)^2\end{array}\right)} \quad (6)$$

Example 1 Let $X=\{x_1,x_2\}$ be a universe of discourse; $\tilde{\tau}=\{\tau_1,\tau_2,\tau_3\}$ be time period; Ψ_1 and Ψ_2 be two TCNS in X

$$\Psi_1\left(x_1\right)=\left\{\begin{array}{l}\left\langle 0.1e^{j0.2},0.6e^{j0.45},0.3e^{j0.56}\right\rangle,\left\langle 0.8e^{j0.77},0.2e^{j0.86},0.4e^{j0.65}\right\rangle,\\\left\langle 0.6e^{j0.52},0.2e^{j0.2},0.36e^{j0.2}\right\rangle\end{array}\right\}$$
$$\Psi_1\left(x_2\right)=\left\{\begin{array}{l}\left\langle 0.5e^{j0.12},0.2e^{j0.65},0.5e^{j0.85}\right\rangle,\left\langle 0.3e^{j0.8},0.7e^{j0.23},0.1e^{j0.32}\right\rangle,\\\left\langle 0.74e^{j0.6},0.5e^{j0.15},0.66e^{j0.41}\right\rangle\end{array}\right\}$$
$$\Psi_2\left(x_1\right)=\left\{\begin{array}{l}\left\langle 0.6e^{j0.8},0.9e^{j0.25},0.4e^{j0.52}\right\rangle,\left\langle 0.6e^{j0.62},0.5e^{j0.24},0.3e^{j0.18}\right\rangle,\\\left\langle 0.3e^{j0.82},0.3e^{j0.6},0.4e^{j0.71}\right\rangle\end{array}\right\}$$
$$\Psi_2\left(x_2\right)=\left\{\begin{array}{l}\left\langle 0.3e^{j0.22},0.4e^{j0.56},0.6e^{j0.26}\right\rangle,\left\langle 0.1e^{j0.38},0.5e^{j0.43},0.5e^{j0.24}\right\rangle,\\\left\langle 0.2e^{j0.29},0.4e^{j0.72},0.5e^{j0.65}\right\rangle\end{array}\right\}$$

Then, by using Equations (6), we get Temporal complex Dice similarity measure between Ψ_1 and Ψ_2, denoted as $S_{Dice}\left(\Psi_1,\Psi_2\right)$ as follows:

$S_{Dice}\left(\Psi_1,\Psi_2\right)=0.879416$

Theorem 1 *Let Ψ_1 and Ψ_2 be two TCNSs then,*

(1) $0\leq S_{Dice}\left(\Psi_1,\Psi_2\right)\leq 1$

(2) $S_{Dice}\left(\Psi_1,\Psi_2\right)=S_{Dice}\left(\Psi_2,\Psi_1\right)$

(3) $S_{Dice}\left(\Psi_1,\Psi_2\right)=1$ *, if and only if* $\Psi_1=\Psi_2$

(4) if Ψ_3 is a TCNS in x and $\Psi_1\subseteq\Psi_2\subseteq\Psi_3$ then $S_{Dice}\left(\Psi_1,\Psi_3\right)\leq S_{Dice}\left(\Psi_1,\Psi_2\right)$ and $S_{Dice}\left(\Psi_1,\Psi_3\right)\leq S_{Dice}\left(\Psi_1,\Psi_2\right)$

Proof

(1) We can have,

$$2 * \left[\begin{array}{l} \left(p_{\Psi_1}(x_i,\tau_l) + \mu_{\Psi_1}(x_i,\tau_l)\right)\left(p_{\Psi_2}(x_i,\tau_l) + \mu_{\Psi_2}(x_i,\tau_l)\right) \\ + \left(q_{\Psi_1}(x_i,\tau_l) + \nu_{\Psi_1}(x_i,\tau_l)\right)\left(q_{\Psi_2}(x_i,\tau_l) + \nu_{\Psi_2}(x_i,\tau_l)\right) \\ + \left(r_{\Psi_1}(x_i,\tau_l) + \eta_{\Psi_1}(x_i,\tau_l)\right)\left(r_{\Psi_2}(x_i,\tau_l) + \eta_{\Psi_2}(x_i,\tau_l)\right) \end{array} \right]$$
$$\leq \left(\begin{array}{l} \left(p_{\Psi_1}(x_i,\tau_l) + \mu_{\Psi_1}(x_i,\tau_l)\right)^2 + \left(q_{\Psi_1}(x_i,\tau_l) + \nu_{\Psi_1}(x_i,\tau_l)\right)^2 \\ + \left(r_{\Psi_1}(x_i,\tau_l) + \eta_{\Psi_1}(x_i,\tau_l)\right)^2 + \left(p_{\Psi_2}(x_i,\tau_l) + \mu_{\Psi_2}(x_i,\tau_l)\right)^2 \\ + \left(q_{\Psi_2}(x_i,\tau_l) + \nu_{\Psi_2}(x_i,\tau_l)\right)^2 + \left(r_{\Psi_2}(x_i,\tau_l) + \eta_{\Psi_2}(x_i,\tau_l)\right)^2 \end{array} \right)$$

Hence, $0 \leq S_{Dice}(\Psi_1,\Psi_2) \leq 1$. So, it proves the first inequality.

(2) It is obvious that the Theorem 1 is true. This proves the second inequality.

(3) When $\Psi_1 = \Psi_2$, then it implies that $S_{Dice}(\Psi_1,\Psi_2) = 1$. On the other hand, if $S_{Dice}(\Psi_1,\Psi_2) = 1$ then,

$p_{\Psi_1}(x_i,\tau_l) = p_{\Psi_2}(x_i,\tau_l)$; $\mu_{\Psi_1}(x_i,\tau_l) = \mu_{\Psi_2}(x_i,\tau_l)$; $q_{\Psi_1}(x_i,\tau_l) = q_{\Psi_2}(x_i,\tau_l)$; $\nu_{\Psi_1}(x_i,\tau_l) = \nu_{\Psi_2}(x_i,\tau_l)$; $r_{\Psi_1}(x_i,\tau_l) = r_{\Psi_2}(x_i,\tau_l)$; $\eta_{\Psi_1}(x_i,\tau_l) = \eta_{\Psi_2}(x_i,\tau_l)$;

This implies that $\Psi_1 = \Psi_2$. So, the third inequality is proved.

(4) When $\Psi_1 \subseteq \Psi_2 \subseteq \Psi_3$, we can have

$p_{\Psi_1}(x_i,\tau_l) + \mu_{\Psi_1}(x_i,\tau_l) \leq p_{\Psi_2}(x_i,\tau_l) + \mu_{\Psi_2}(x_i,\tau_l) \leq p_{\Psi_3}(x_i,\tau_l) + \mu_{\Psi_3}(x_i,\tau_l)$;

$q_{\Psi_1}(x_i,\tau_l) + \nu_{\Psi_1}(x_i,\tau_l) \geq q_{\Psi_2}(x_i,\tau_l) + \nu_{\Psi_2}(x_i,\tau_l) \geq q_{\Psi_3}(x_i,\tau_l) + \nu_{\Psi_3}(x_i,\tau_l)$;

$r_{\Psi_1}(x_i,\tau_l) + \eta_{\Psi_1}(x_i,\tau_l) \geq r_{\Psi_2}(x_i,\tau_l) + \eta_{\Psi_2}(x_i,\tau_l) \geq r_{\Psi_3}(x_i,\tau_l) + \eta_{\Psi_3}(x_i,\tau_l)$;

So, $S_{Dice}(\Psi_1,\Psi_3) \leq S_{Dice}(\Psi_1,\Psi_2)$ and $S_{Dice}(\Psi_1,\Psi_3) \leq S_{Dice}(\Psi_2,\Psi_3)$. This proves the fourth inequality.

Finally, all the equalities in Theorem 1 are proved.

3.2 Jaccard similarity measure of TCNE

Definition 4 Let two TCNSs

$\Psi_1 = \left\langle p_{\Psi_1}(x_i,\tau_l)\, e^{j\mu_{\Psi_1}(x_i,\tau_l)}, q_{\Psi_1}(x_i,\tau_l)\, e^{j\nu_{\Psi_1}(x_i,\tau_l)}, r_{\Psi_1}(x_i,\tau_l)\, e^{j\eta_{\Psi_1}(x_i,\tau_l)} \right\rangle$ and $\Psi_2 = \left\langle p_{\Psi_2}(x_i,\tau_l)\, e^{j\mu_{\Psi_2}(x_i,\tau_l)}, q_{\Psi_2}(x_i,\tau_l)\, e^{j\nu_{\Psi_2}(x_i,\tau_l)}, r_{\Psi_2}(x_i,\tau_l)\, e^{j\eta_{\Psi_2}(x_i,\tau_l)} \right\rangle$ for all $x_i\,(1,2,3,...,n_X)$ belong to X and time period $\tau_l\,(1,2,3,...,n_\tau)$.

A temporal complex Jaccard similarity measure between TCNSs Ψ_1 and Ψ_2 (denoted

as $S_J(\Psi_1, \Psi_2)$) is defined as follows:

$$S_J(\Psi_1, \Psi_2) = \frac{1}{n_X * n_\tau} \sum_{i=1}^{n_X} \sum_{l=1}^{n_\tau} \frac{\begin{array}{l}(p_{\Psi_1}(x_i,\tau_l) + \mu_{\Psi_1}(x_i,\tau_l))(p_{\Psi_2}(x_i,\tau_l) + \mu_{\Psi_2}(x_i,\tau_l)) \\ + (q_{\Psi_1}(x_i,\tau_l) + \nu_{\Psi_1}(x_i,\tau_l))(q_{\Psi_2}(x_i,\tau_l) + q_{\Psi_2}(x_i,\tau_l)) \\ + (r_{\Psi_1}(x_i,\tau_l) + r_{\Psi_1}(x_i,\tau_l))(r_{\Psi_2}(x_i,\tau_l) + r_{\Psi_2}(x_i,\tau_l))\end{array}}{\begin{array}{l}(p_{\Psi_1}(x_i,\tau_l) + \mu_{\Psi_1}(x_i,\tau_l))^2 + (q_{\Psi_1}(x_i,\tau_l) + \nu_{\Psi_1}(x_i,\tau_l))^2 \\ + (r_{\Psi_1}(x_i,\tau_l) + r_{\Psi_1}(x_i,\tau_l))^2 + (p_{\Psi_2}(x_i,\tau_l) + \mu_{\Psi_2}(x_i,\tau_l))^2 \\ + (q_{\Psi_2}(x_i,\tau_l) + q_{\Psi_2}(x_i,\tau_l))^2 + (r_{\Psi_2}(x_i,\tau_l) + r_{\Psi_2}(x_i,\tau_l))^2 \\ - \left(\begin{array}{l}(p_{\Psi_1}(x_i,\tau_l) + \mu_{\Psi_1}(x_i,\tau_l))(p_{\Psi_2}(x_i,\tau_l) + \mu_{\Psi_2}(x_i,\tau_l)) \\ + (q_{\Psi_1}(x_i,\tau_l) + \nu_{\Psi_1}(x_i,\tau_l))(q_{\Psi_2}(x_i,\tau_l) + q_{\Psi_2}(x_i,\tau_l)) \\ + (r_{\Psi_1}(x_i,\tau_l) + r_{\Psi_1}(x_i,\tau_l))(r_{\Psi_2}(x_i,\tau_l) + r_{\Psi_2}(x_i,\tau_l))\end{array}\right)\end{array}} \tag{7}$$

Example 2 Let $X = \{x_1, x_2\}$;$\tilde{\tau} = \{\tau_1, \tau_2, \tau_3\}$; and two TCNS Ψ_1, Ψ_2 in X.

$$\Psi_1(x_1) = \left\{ \begin{array}{l} \left\langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56} \right\rangle, \left\langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65} \right\rangle, \\ \left\langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2} \right\rangle \end{array} \right\}$$

$$\Psi_1(x_2) = \left\{ \begin{array}{l} \left\langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85} \right\rangle, \left\langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32} \right\rangle, \\ \left\langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41} \right\rangle \end{array} \right\}$$

$$\Psi_2(x_1) = \left\{ \begin{array}{l} \left\langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52} \right\rangle, \left\langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18} \right\rangle, \\ \left\langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71} \right\rangle \end{array} \right\}$$

$$\Psi_2(x_2) = \left\{ \begin{array}{l} \left\langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26} \right\rangle, \left\langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24} \right\rangle, \\ \left\langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65} \right\rangle \end{array} \right\}$$

The temporal complex Jaccard similarity measure between two TCNS Ψ_1, Ψ_2 can be obtained by applying Definitions 4 as follows: $S_J(\Psi_1, \Psi_2) = 0.788072$

Theorem 2 *Let Ψ_1 and Ψ_2 be two TCNSs then,*
(1) $0 \leq S_J(\Psi_1, \Psi_2) \leq 1$
(2) $S_J(\Psi_1, \Psi_2) = S_J(\Psi_2, \Psi_1)$
(3) $S_J(\Psi_1, \Psi_2) = 1$*, if and only if* $\Psi_1 = \Psi_2$
(4) if Ψ_3 is a TCNS in X and $\Psi_1 \subseteq \Psi_2 \subseteq \Psi_3$ then $S_J(\Psi_1, \Psi_3) \leq S_J(\Psi_1, \Psi_2)$ and $S_J(\Psi_1, \Psi_3) \leq S_J(\Psi_2, \Psi_3)$

Proof

(1) Since:

$$\left(\begin{array}{l}\left(p_{\Psi_1}\left(x_i,\tau_l\right)+\mu_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(p_{\Psi_2}\left(x_i,\tau_l\right)+\mu_{\Psi_2}\left(x_i,\tau_l\right)\right)\\+\left(q_{\Psi_1}\left(x_i,\tau_l\right)+\nu_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(q_{\Psi_2}\left(x_i,\tau_l\right)+\nu_{\Psi_2}\left(x_i,\tau_l\right)\right)\\+\left(r_{\Psi_1}\left(x_i,\tau_l\right)+\eta_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(r_{\Psi_2}\left(x_i,\tau_l\right)+\eta_{\Psi_2}\left(x_i,\tau_l\right)\right)\end{array}\right)$$

$$\leq\left(\begin{array}{l}\left(p_{\Psi_1}\left(x_i,\tau_l\right)+\mu_{\Psi_1}\left(x_i,\tau_l\right)\right)^2+\left(q_{\Psi_1}\left(x_i,\tau_l\right)+\nu_{\Psi_1}\left(x_i,\tau_l\right)\right)^2\\+\left(r_{\Psi_1}\left(x_i,\tau_l\right)+\eta_{\Psi_1}\left(x_i,\tau_l\right)\right)^2+\left(p_{\Psi_2}\left(x_i,\tau_l\right)+\mu_{\Psi_2}\left(x_i,\tau_l\right)\right)^2\\+\left(q_{\Psi_2}\left(x_i,\tau_l\right)+\nu_{\Psi_2}\left(x_i,\tau_l\right)\right)^2+\left(r_{\Psi_2}\left(x_i,\tau_l\right)+\eta_{\Psi_2}\left(x_i,\tau_l\right)\right)^2\\-\left[\begin{array}{l}\left(p_{\Psi_1}\left(x_i,\tau_l\right)+\mu_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(p_{\Psi_2}\left(x_i,\tau_l\right)+\mu_{\Psi_2}\left(x_i,\tau_l\right)\right)\\+\left(q_{\Psi_1}\left(x_i,\tau_l\right)+\nu_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(q_{\Psi_2}\left(x_i,\tau_l\right)+\nu_{\Psi_2}\left(x_i,\tau_l\right)\right)\\+\left(r_{\Psi_1}\left(x_i,\tau_l\right)+\eta_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(r_{\Psi_2}\left(x_i,\tau_l\right)+\eta_{\Psi_2}\left(x_i,\tau_l\right)\right)\end{array}\right]\end{array}\right)$$

Hence, we can write $0\leq S_J\left(\Psi_1,\Psi_2\right)\leq 1$.

(2) It is obvious that the Theorem 2 is true

(3) When $\Psi_1=\Psi_2$, then obviously $S_J\left(\Psi_1,\Psi_2\right)=1$. On the other hand, if $S_J\left(\Psi_1,\Psi_2\right)=1$ then,

$p_{\Psi_1}\left(x_i,\tau_l\right)=p_{\Psi_2}\left(x_i,\tau_l\right)$; $\mu_{\Psi_1}\left(x_i,\tau_l\right)=\mu_{\Psi_2}\left(x_i,\tau_l\right)$; $q_{\Psi_1}\left(x_i,\tau_l\right)=q_{\Psi_2}\left(x_i,\tau_l\right)$; $\nu_{\Psi_1}\left(x_i,\tau_l\right)=\nu_{\Psi_2}\left(x_i,\tau_l\right)$; $r_{\Psi_1}\left(x_i,\tau_l\right)=r_{\Psi_2}\left(x_i,\tau_l\right)$; $\eta_{\Psi_1}\left(x_i,\tau_l\right)=\eta_{\Psi_2}\left(x_i,\tau_l\right)$

This implies that $\Psi_1=\Psi_2$.

(4) When $\Psi_1\subseteq\Psi_2\subseteq\Psi_3$, we can write $p_{\Psi_1}\left(x_i,\tau_l\right)+\mu_{\Psi_1}\left(x_i,\tau_l\right)\leq p_{\Psi_2}\left(x_i,\tau_l\right)+\mu_{\Psi_2}\left(x_i,\tau_l\right)\leq p_{\Psi_3}\left(x_i,\tau_l\right)+\mu_{\Psi_3}\left(x_i,\tau_l\right)$;

$q_{\Psi_1}\left(x_i,\tau_l\right)+\nu_{\Psi_1}\left(x_i,\tau_l\right)\geq q_{\Psi_2}\left(x_i,\tau_l\right)+\nu_{\Psi_2}\left(x_i,\tau_l\right)\geq q_{\Psi_3}\left(x_i,\tau_l\right)+\nu_{\Psi_3}\left(x_i,\tau_l\right)$;

$r_{\Psi_1}\left(x_i,\tau_l\right)+\eta_{\Psi_1}\left(x_i,\tau_l\right)\geq r_{\Psi_2}\left(x_i,\tau_l\right)+\eta_{\Psi_2}\left(x_i,\tau_l\right)\geq r_{\Psi_3}\left(x_i,\tau_l\right)+\eta_{\Psi_3}\left(x_i,\tau_l\right)$;

So, $S_J\left(\Psi_1,\Psi_3\right)\leq S_J\left(\Psi_1,\Psi_2\right)$ and $S_J\left(\Psi_1,\Psi_3\right)\leq S_J\left(\Psi_2,\Psi_3\right)$.

All the equalities in Theorem 2 are proved.

3.3 Cosine Similarity of TCNS

The cosine similarity measure for temporal complex neutrosophic sets is defined as follows:

Definition 5 Let

$\Psi_1=\left\langle p_{\Psi_1}\left(x_i,\tau_l\right)e^{j\mu_{\Psi_1}\left(x_i,\tau_l\right)},q_{\Psi_1}\left(x_i,\tau_l\right)e^{j\nu_{\Psi_1}\left(x_i,\tau_l\right)},r_{\Psi_1}\left(x_i,\tau_l\right)e^{j\eta_{\Psi_1}\left(x_i,\tau_l\right)}\right\rangle$ and $\Psi_2=\left\langle p_{\Psi_2}\left(x_i,\tau_l\right)e^{j\mu_{\Psi_2}\left(x_i,\tau_l\right)},q_{\Psi_2}\left(x_i,\tau_l\right)e^{j\nu_{\Psi_2}\left(x_i,\tau_l\right)},r_{\Psi_2}\left(x_i,\tau_l\right)e^{j\eta_{\Psi_2}\left(x_i,\tau_l\right)}\right\rangle$ be two TCNSs for all $x_i\left(1,2,3,...,n_X\right)$ belong to X and $\tau_l\left(1,2,3,...,n_\tau\right)$.

A temporal complex Cosine similarity measure between TCNSs Ψ_1 and Ψ_2 (denoted as $S_{\text{Cos}}(\Psi_1, \Psi_2)$ is proposed as follows:

$$S_{\text{Cos}}(\Psi_1, \Psi_2) = \frac{1}{n_X * n_\tau} \sum_{i=1}^{n_X} \sum_{l=1}^{n_\tau} \frac{\begin{array}{l}(p_{\Psi_1}(x_i,\tau_l) + \mu_{\Psi_1}(x_i,\tau_l))(p_{\Psi_2}(x_i,\tau_l) + \mu_{\Psi_2}(x_i,\tau_l)) \\ + (q_{\Psi_1}(x_i,\tau_l) + \nu_{\Psi_1}(x_i,\tau_l))(q_{\Psi_2}(x_i,\tau_l) + q_{\Psi_2}(x_i,\tau_l)) \\ + (r_{\Psi_1}(x_i,\tau_l) + r_{\Psi_1}(x_i,\tau_l))(r_{\Psi_2}(x_i,\tau_l) + r_{\Psi_2}(x_i,\tau_l))\end{array}}{\begin{array}{l}\sqrt{(p_{\Psi_1}(x_i,\tau_l) + \mu_{\Psi_1}(x_i,\tau_l))^2 + (q_{\Psi_1}(x_i,\tau_l) + \nu_{\Psi_1}(x_i,\tau_l))^2} \\ + (r_{\Psi_1}(x_i,\tau_l) + r_{\Psi_1}(x_i,\tau_l))^2 \\ \times \sqrt{(p_{\Psi_2}(x_i,\tau_l) + \mu_{\Psi_2}(x_i,\tau_l))^2 + (q_{\Psi_2}(x_i,\tau_l) + q_{\Psi_2}(x_i,\tau_l))^2} \\ + (r_{\Psi_2}(x_i,\tau_l) + r_{\Psi_2}(x_i,\tau_l))^2\end{array}} \tag{8}$$

Example 3 Let $X = \{x_1, x_2\}; \tilde{\tau} = \{\tau_1, \tau_2, \tau_3\}$ and Ψ_1 and Ψ_2 are two TCNS in X

$$\Psi_1(x_1) = \left\{ \begin{array}{l} \left\langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56} \right\rangle, \left\langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65} \right\rangle, \\ \left\langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2} \right\rangle \end{array} \right\}$$

$$\Psi_1(x_2) = \left\{ \begin{array}{l} \left\langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85} \right\rangle, \left\langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32} \right\rangle, \\ \left\langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41} \right\rangle \end{array} \right\}$$

$$\Psi_2(x_1) = \left\{ \begin{array}{l} \left\langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52} \right\rangle, \left\langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18} \right\rangle, \\ \left\langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71} \right\rangle \end{array} \right\}$$

$$\Psi_2(x_2) = \left\{ \begin{array}{l} \left\langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26} \right\rangle, \left\langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24} \right\rangle, \\ \left\langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65} \right\rangle \end{array} \right\}$$

Now, by Definition 5, we have the following results:
$S_{\text{Cos}}(\Psi_1, \Psi_2) = 0.732025$

Theorem 3 *Let Ψ_1 and Ψ_2 be two TCNSs then,*
(1) $0 \leq S_{\text{Cos}}(\Psi_1, \Psi_2) \leq 1$
(2) $S_{\text{Cos}}(\Psi_1, \Psi_2) = S_{\text{Cos}}(\Psi_2, \Psi_1)$
(3) $S_{\text{Cos}}(\Psi_1, \Psi_2) = 1$, *if and only if* $\Psi_1 = \Psi_2$
(4) if Ψ_3 is a TCNS in X and $\Psi_1 \subseteq \Psi_2 \subseteq \Psi_3$ then $S_{\text{Cos}}(\Psi_1, \Psi_3) \leq S_{\text{Cos}}(\Psi_1, \Psi_2)$ *and* $S_{\text{Cos}}(\Psi_1, \Psi_3) \leq S_{\text{Cos}}(\Psi_2, \Psi_3)$

Proof

(1) We can have,

$$\left(\begin{array}{l}\left(p_{\Psi_1}\left(x_i,\tau_l\right)+\mu_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(p_{\Psi_2}\left(x_i,\tau_l\right)+\mu_{\Psi_2}\left(x_i,\tau_l\right)\right)\\+\left(q_{\Psi_1}\left(x_i,\tau_l\right)+\nu_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(q_{\Psi_2}\left(x_i,\tau_l\right)+\nu_{\Psi_2}\left(x_i,\tau_l\right)\right)\\+\left(r_{\Psi_1}\left(x_i,\tau_l\right)+\eta_{\Psi_1}\left(x_i,\tau_l\right)\right)\left(r_{\Psi_2}\left(x_i,\tau_l\right)+\eta_{\Psi_2}\left(x_i,\tau_l\right)\right)\end{array}\right)^2$$
$$\leq\left(\begin{array}{l}\left(p_{\Psi_1}\left(x_i,\tau_l\right)+\mu_{\Psi_1}\left(x_i,\tau_l\right)\right)^2+\left(q_{\Psi_1}\left(x_i,\tau_l\right)+\nu_{\Psi_1}\left(x_i,\tau_l\right)\right)^2\\+\left(r_{\Psi_1}\left(x_i,\tau_l\right)+\eta_{\Psi_1}\left(x_i,\tau_l\right)\right)^2+\left(p_{\Psi_2}\left(x_i,\tau_l\right)+\mu_{\Psi_2}\left(x_i,\tau_l\right)\right)^2\\+\left(q_{\Psi_2}\left(x_i,\tau_l\right)+\nu_{\Psi_2}\left(x_i,\tau_l\right)\right)^2+\left(r_{\Psi_2}\left(x_i,\tau_l\right)+\eta_{\Psi_2}\left(x_i,\tau_l\right)\right)^2\end{array}\right)$$

Hence, $0\leq S_{\mathrm{Cos}}\left(\Psi_1,\Psi_2\right)\leq 1$

(2) It is obvious that the Theorem 3 is true

(3) When $\Psi_1=\Psi_2$, then obviously $S_{\mathrm{Cos}}\left(\Psi_1,\Psi_2\right)=1$. On the other hand, if $S_{\mathrm{Cos}}\left(\Psi_1,\Psi_2\right)=1$ then,

$p_{\Psi_1}\left(x_i,\tau_l\right)=p_{\Psi_2}\left(x_i,\tau_l\right)$;$\mu_{\Psi_1}\left(x_i,\tau_l\right)=\mu_{\Psi_2}\left(x_i,\tau_l\right)$; $q_{\Psi_1}\left(x_i,\tau_l\right)=q_{\Psi_2}\left(x_i,\tau_l\right)$.; $\nu_{\Psi_1}\left(x_i,\tau_l\right)=\nu_{\Psi_2}\left(x_i,\tau_l\right)$; $r_{\Psi_1}\left(x_i,\tau_l\right)=r_{\Psi_2}\left(x_i,\tau_l\right)$; $\eta_{\Psi_1}\left(x_i,\tau_l\right)=\eta_{\Psi_2}\left(x_i,\tau_l\right)$;

This implies that $\Psi_1=\Psi_2$

(4) When $\Psi_1\subseteq\Psi_2\subseteq\Psi_3$, we can have

$p_{\Psi_1}\left(x_i,\tau_l\right)+\mu_{\Psi_1}\left(x_i,\tau_l\right)\leq p_{\Psi_2}\left(x_i,\tau_l\right)+\mu_{\Psi_2}\left(x_i,\tau_l\right)\leq p_{\Psi_3}\left(x_i,\tau_l\right)+\mu_{\Psi_3}\left(x_i,\tau_l\right)$;

$q_{\Psi_1}\left(x_i,\tau_l\right)+\nu_{\Psi_1}\left(x_i,\tau_l\right)\geq q_{\Psi_2}\left(x_i,\tau_l\right)+\nu_{\Psi_2}\left(x_i,\tau_l\right)\geq q_{\Psi_3}\left(x_i,\tau_l\right)+\nu_{\Psi_3}\left(x_i,\tau_l\right)$;

$r_{\Psi_1}\left(x_i,\tau_l\right)+\eta_{\Psi_1}\left(x_i,\tau_l\right)\geq r_{\Psi_2}\left(x_i,\tau_l\right)+\eta_{\Psi_2}\left(x_i,\tau_l\right)\geq r_{\Psi_3}\left(x_i,\tau_l\right)+\eta_{\Psi_3}\left(x_i,\tau_l\right)$;

So, $S_J\left(\Psi_1,\Psi_3\right)\leq S_J\left(\Psi_1,\Psi_2\right)$ and $S_J\left(\Psi_1,\Psi_3\right)\leq S_J\left(\Psi_2,\Psi_3\right)$

Hence, all the conditions are proved.

3.4 Cotangent Similarity of TCNS

In this section, we define a Cotangent function to define the similarity measure on the TCNS environment.

Definition 6 Let two TCNSs

$\Psi_1=\left\langle p_{\Psi_1}\left(x_i,\tau_l\right)e^{j\mu_{\Psi_1}\left(x_i,\tau_l\right)},q_{\Psi_1}\left(x_i,\tau_l\right)e^{j\nu_{\Psi_1}\left(x_i,\tau_l\right)},r_{\Psi_1}\left(x_i,\tau_l\right)e^{j\eta_{\Psi_1}\left(x_i,\tau_l\right)}\right\rangle$ and $\Psi_2=\left\langle p_{\Psi_2}\left(x_i,\tau_l\right)e^{j\mu_{\Psi_2}\left(x_i,\tau_l\right)},q_{\Psi_2}\left(x_i,\tau_l\right)e^{j\nu_{\Psi_2}\left(x_i,\tau_l\right)},r_{\Psi_2}\left(x_i,\tau_l\right)e^{j\eta_{\Psi_2}\left(x_i,\tau_l\right)}\right\rangle$ for all $x_i\left(1,2,3,...,n_X\right)$ belong to X and $\tau_l\left(1,2,3,...,n_\tau\right)$.

A temporal complex Cotangent similarity measure between TCNSs Ψ_1 and Ψ_2 (denoted as $S_{\text{Cot_1}}(\Psi_1, \Psi_2)$) is defined as follows:

$$S_{\text{Cot_1}}(\Psi_1, \Psi_2) = \cot\left(\frac{\pi}{4} + \frac{\pi}{8} \times \frac{1}{n_X * n_\tau} \sum_{i=1}^{n_X} \sum_{l=1}^{n_\tau} \left[\begin{array}{l} max\left(\begin{array}{l} |\, p_{\Psi_1}(x_i, \tau_l) - p_{\Psi_2}(x_i, \tau_l)\ |, \\ |\, q_{\Psi_1}(x_i, \tau_l) - q_{\Psi_2}(x_i, \tau_l)\ |, \\ |\, r_{\Psi_1}(x_i, \tau_l) - r_{\Psi_2}(x_i, \tau_l)\ | \end{array} \right) \\ + \frac{1}{2\pi} max\left(\begin{array}{l} |\, \mu_{\Psi_1}(x_i, \tau_l) - \mu_{\Psi_2}(x_i, \tau_l)\ |, \\ |\, \nu_{\Psi_1}(x_i, \tau_l) - \nu_{\Psi_2}(x_i, \tau_l)\ |, \\ |\, \eta_{\Psi_1}(x_i, \tau_l) - \eta_{\Psi_2}(x_i, \tau_l)\ | \end{array} \right) \end{array} \right] \right) \tag{9}$$

$$S_{\text{Cot_2}}(\Psi_1, \Psi_2) = \cot\left(\frac{\pi}{4} + \frac{\pi}{24} \times \frac{1}{n_X * n_\tau} \sum_{i=1}^{n_X} \sum_{l=1}^{n_\tau} \left[\begin{array}{l} \left(\begin{array}{l} |\, p_{\Psi_1}(x_i, \tau_l) - p_{\Psi_2}(x_i, \tau_l)\ | \\ +\, |\, q_{\Psi_1}(x_i, \tau_l) - q_{\Psi_2}(x_i, \tau_l)\ | \\ +\, |\, r_{\Psi_1}(x_i, \tau_l) - r_{\Psi_2}(x_i, \tau_l)\ | \end{array} \right) \\ + \frac{1}{2\pi} max\left(\begin{array}{l} |\, \mu_{\Psi_1}(x_i, \tau_l) - \mu_{\Psi_2}(x_i, \tau_l)\ | \\ +\, |\, \nu_{\Psi_1}(x_i, \tau_l) - \nu_{\Psi_2}(x_i, \tau_l)\ | \\ +\, |\, \eta_{\Psi_1}(x_i, \tau_l) - \eta_{\Psi_2}(x_i, \tau_l)\ | \end{array} \right) \end{array} \right] \right) \tag{10}$$

Example 4 Let $X = \{x_1, x_2\}$; $\tilde{\tau} = \{\tau_1, \tau_2, \tau_3\}$; Ψ_1 and Ψ_2 are two TCNS in X

$$\Psi_1(x_1) = \left\{ \begin{array}{l} \left\langle 0.1e^{j0.2}, 0.6e^{j0.45}, 0.3e^{j0.56} \right\rangle, \left\langle 0.8e^{j0.77}, 0.2e^{j0.86}, 0.4e^{j0.65} \right\rangle, \\ \left\langle 0.6e^{j0.52}, 0.2e^{j0.2}, 0.36e^{j0.2} \right\rangle \end{array} \right\}$$

$$\Psi_1(x_2) = \left\{ \begin{array}{l} \left\langle 0.5e^{j0.12}, 0.2e^{j0.65}, 0.5e^{j0.85} \right\rangle, \left\langle 0.3e^{j0.8}, 0.7e^{j0.23}, 0.1e^{j0.32} \right\rangle, \\ \left\langle 0.74e^{j0.6}, 0.5e^{j0.15}, 0.66e^{j0.41} \right\rangle \end{array} \right\}$$

$$\Psi_2(x_1) = \left\{ \begin{array}{l} \left\langle 0.6e^{j0.8}, 0.9e^{j0.25}, 0.4e^{j0.52} \right\rangle, \left\langle 0.6e^{j0.62}, 0.5e^{j0.24}, 0.3e^{j0.18} \right\rangle, \\ \left\langle 0.3e^{j0.82}, 0.3e^{j0.6}, 0.4e^{j0.71} \right\rangle \end{array} \right\}$$

$$\Psi_2(x_2) = \left\{ \begin{array}{l} \left\langle 0.3e^{j0.22}, 0.4e^{j0.56}, 0.6e^{j0.26} \right\rangle, \left\langle 0.1e^{j0.38}, 0.5e^{j0.43}, 0.5e^{j0.24} \right\rangle, \\ \left\langle 0.2e^{j0.29}, 0.4e^{j0.72}, 0.5e^{j0.65} \right\rangle \end{array} \right\}$$

The following results can be obtained by applying Definitions 6
$S_{Cot_1}(\Psi_1, \Psi_2) = 0.690485$; $S_{Cot_2}(\Psi_1, \Psi_2) = 0.822429$

Theorem 4 *Let Ψ_1 and Ψ_2 be two TCNSs then,*
(1) $0 \leq S_{Cot}(\Psi_1, \Psi_2) \leq 1$
(2) $S_{Cot}(\Psi_1, \Psi_2) = S_{Cot}(\Psi_2, \Psi_1)$

(3) $S_{Cot}(\Psi_1, \Psi_2) = 1$*, if and only if* $\Psi_1 = \Psi_2$

(4) if Ψ_3 *is a TCNS in* X *and* $\Psi_1 \subseteq \Psi_2 \subseteq \Psi_3$ *then* $S_{Cot}(\Psi_1, \Psi_3) \leq S_{Cot}(\Psi_1, \Psi_2)$ *and* $S_{Cot}(\Psi_1, \Psi_3) \leq S_{Cot}(\Psi_2, \Psi_3)$

Proof

The proof is similar to the proof of Equation (6) and Equation (7).

(1) We have,

Hence, $0 \leq S_{Cot}(\Psi_1, \Psi_2) \leq 1$

(2) It is obvious that the Theorem 4 is true

(3) When $\Psi_1 = \Psi_2$, then obviously$S_{Cot_2}(\Psi_1, \Psi_2) = 1$. On the other hand, if $S_{Cot_2}(\Psi_1, \Psi_2) = 1$ then,

$p_{\Psi_1}(x_i, \tau_l) = p_{\Psi_2}(x_i, \tau_l); \mu_{\Psi_1}(x_i, \tau_l) = \mu_{\Psi_2}(x_i, \tau_l); q_{\Psi_1}(x_i, \tau_l) = q_{\Psi_2}(x_i, \tau_l);\ \nu_{\Psi_1}(x_i, \tau_l) = \nu_{\Psi_2}(x_i, \tau_l);\ r_{\Psi_1}(x_i, \tau_l) = r_{\Psi_2}(x_i, \tau_l);\ \eta_{\Psi_1}(x_i, \tau_l) = \eta_{\Psi_2}(x_i, \tau_l);$

This implies that $\Psi_1 = \Psi_2$

(4) When $\Psi_1 \subseteq \Psi_2 \subseteq \Psi_3$, wen can have

$p_{\Psi_1}(x_i, \tau_l) + \mu_{\Psi_1}(x_i, \tau_l) \leq p_{\Psi_2}(x_i, \tau_l) + \mu_{\Psi_2}(x_i, \tau_l) \leq p_{\Psi_3}(x_i, \tau_l) + \mu_{\Psi_3}(x_i, \tau_l);$

$q_{\Psi_1}(x_i, \tau_l) + \nu_{\Psi_1}(x_i, \tau_l) \geq q_{\Psi_2}(x_i, \tau_l) + \nu_{\Psi_2}(x_i, \tau_l) \geq q_{\Psi_3}(x_i, \tau_l) + \nu_{\Psi_3}(x_i, \tau_l);$

$r_{\Psi_1}(x_i, \tau_l) + \eta_{\Psi_1}(x_i, \tau_l) \geq r_{\Psi_2}(x_i, \tau_l) + \eta_{\Psi_2}(x_i, \tau_l) \geq r_{\Psi_3}(x_i, \tau_l) + \eta_{\Psi_3}(x_i, \tau_l);$

So, $S_{Cot_2}(\Psi_1, \Psi_3) \leq S_{Cot_2}(\Psi_1, \Psi_2)$ and $S_{Cot_2}(\Psi_1, \Psi_3) \leq S_{Cot_2}(\Psi_2, \Psi_3)$.

The proof is completed.

3.5 Entropy measures of TCNE

The entropy measure was first introduced by Zadeh [26] in 1965 with the aim of measuring fuzziness in information. Entropy measure can be considered as a measure of uncertainty and fuzziness involved in a set, whether fuzzy set, complex fuzzy set, neutrosophic set, etc. In this work, the TCNS are also capable of handling uncertain and vague data; therefore, we also care about discovering the entropy of a TCNS.

Definition 7 Let X be a finite set of objects. A function *E: TCNS(X)* $\rightarrow$ [0, 1] is called an entropy measure for a temporal complex neutrosophic set if it satisfies the following properties:

(1) $E(\Psi) = 0$ if Ψ is a crisp set

(2) $E(\Psi) = 1$ if $p_\Psi(x, \tau) = 0.5; q_\Psi(x, \tau) = 0.5; r_\Psi(x, \tau) = 0.5$ and $\mu_\Psi(x, \tau) = \pi; \nu_\Psi(x, \tau) = \pi; \eta_\Psi(x, \tau) = \pi$ for $\forall \tau \in \tilde{\tau};\ \forall x \in X$.

(3) $E(\Psi_1) \leq E(\Psi_2)$ if Ψ_2 is more uncertain that Ψ_1, that is $\mid p_{\Psi_1}(x,\tau) - 0.5 \mid \geq \mid p_{\Psi_2}(x,\tau) - 0.5 \mid$; $\mid q_{\Psi_1}(x,\tau) - 0.5 \mid \geq \mid q_{\Psi_2}(x,\tau) - 0.5 \mid$; $\mid r_{\Psi_1}(x,\tau) - 0.5 \mid \geq \mid r_{\Psi_2}(x,\tau) - 0.5 \mid$ and

$\mid \mu_{\Psi_1}(x,\tau) - \pi \mid \geq \mid \mu_{\Psi_2}(x,\tau) - \pi \mid$; $\mid \nu_{\Psi_1}(x,\tau) - \pi \mid \geq \mid \nu_{\Psi_2}(x,\tau) - \pi \mid$; $\mid \eta_{\Psi_1}(x,\tau) - \pi \mid \geq \mid \eta_{\Psi_2}(x,\tau) - \pi \mid$

(4) $E(\Psi) = E(\Psi^c)$, Ψ^c is a temporal complex neutrosophic complement of Ψ

Theorem 5 *Suppose that Ψ is a TCNS on X. A Temporal complex neutrosophic entropy measure of Ψ is proposed as follows:*

$$E(\Psi) = \tan\left(\frac{\pi}{6 * n_X * n_\tau} \sum_{i=1}^{n_X} \sum_{l=1}^{n_\tau} e_{il}(\Psi)\right) \tag{11}$$

Where

$$e_{il}(\Psi) = \begin{pmatrix} p_\Psi(x_i,\tau_l)(1 - p_\Psi(x_i,\tau_l)) + q_\Psi(x_i,\tau_l)(1 - q_\Psi(x_i,\tau_l)) \\ + r_\Psi(x_i,\tau_l)(1 - r_\Psi(x_i,\tau_l)) + \left(\frac{\mu_\Psi(x_i,\tau_l)}{2\pi}\right)\left(1 - \frac{\mu_\Psi(x_i,\tau_l)}{2\pi}\right) \\ + \left(\frac{\nu_\Psi(x_i,\tau_l)}{2\pi}\right)\left(1 - \frac{\nu_\Psi(x_i,\tau_l)}{2\pi}\right) + \left(\frac{\eta_\Psi(x_i,\tau_l)}{2\pi}\right)\left(1 - \frac{\eta_\Psi(x_i,\tau_l)}{2\pi}\right) \end{pmatrix}$$

Proof.

Consider $E(\Psi)$ be can entropy measure for TCNSs. Then, this measure must satisfy all the conditions in Definition 7.

(1) Suppose $E(\Psi) = 0$, it follows that $p(x_i,\tau_l) = 0$; $q(x_i,\tau_l) = 0$; $r(x_i,\tau_l) = 0$; $\mu(x_i,\tau_l) = 0$; $v(x_i,\tau_l) = 0$; $r(x_i,\tau_l) = 0$or $p(x_i,\tau_l) = 1$; $q(x_i,\tau_l) = 1$; $r(x_i,\tau_l) = 1$; $\mu(x_i,\tau_l) = 2\pi$; $v(x_i,\tau_l) = 2\pi$; $r(x_i,\tau_l) = 2\pi$;

(2) $E(\Psi) = 1$, if and only if $p(x_i,\tau_l) = 0.5$; $q(x_i,\tau_l) = 0.5$; $r(x_i,\tau_l) = 0.5$; $\mu(x_i,\tau_l) = \pi$; $v(x_i,\tau_l) = \pi$; $r(x_i,\tau_l) = \pi$

(3) Assum Ψ_1, Ψ_2 and Ψ_2 more uncertain than that Ψ_2. It follows that, if $p_{\Psi_1}(x,\tau) \leq p_{\Psi_2}(x,\tau) \leq 0.5$; $q_{\Psi_1}(x,\tau) \leq q_{\Psi_2}(x,\tau) \leq 0.5$; $r_{\Psi_1}(x,\tau) \leq r_{\Psi_2}(x,\tau) \leq 0.5$ or $p_{\Psi_1}(x,\tau) \geq p_{\Psi_2}(x,\tau) \geq 0.5$; $q_{\Psi_1}(x,\tau) \geq q_{\Psi_2}(x,\tau) \geq 0.5$; $r_{\Psi_1}(x,\tau) \geq r_{\Psi_2}(x,\tau) \geq 0.5$ for each $x_i \in X$.

Hence,

$p_{\Psi 1}(x_i,\tau_l)(1 - p_{\Psi_1}(x_i,\tau_l)) \leq p_{\Psi 2}(x_i,\tau_l)(1 - p_{\Psi_2}(x_i,\tau_l))$

$q_{\Psi 1}(x_i,\tau_l)(1 - q_{\Psi_1}(x_i,\tau_l)) \leq q_{\Psi 2}(x_i,\tau_l)(1 - q_{\Psi_2}(x_i,\tau_l))$;

$r_{\Psi 1}(x_i,\tau_l)(1 - r_{\Psi_1}(x_i,\tau_l)) \leq r_{\Psi 2}(x_i,\tau_l)(1 - r_{\Psi_2}(x_i,\tau_l))$;

and $\frac{\mu_{\Psi_1}(x_i,\tau_l)}{2\pi}\left(1 - \frac{\mu_{\Psi_1}(x_i,\tau_l)}{2\pi}\right) \leq \frac{\mu_{\Psi_2}(x_i,\tau_l)}{2\pi}\left(1 - \frac{\mu_{\Psi_2}(x_i,\tau_l)}{2\pi}\right)$;

$\frac{\nu_{\Psi 1}(x_i,\tau_l)}{2\pi}\left(1 - \frac{\nu_{\Psi_1}(x_i,\tau_l)}{2\pi}\right) \leq \frac{\nu_{\Psi 2}(x_i,\tau_l)}{2\pi}\left(1 - \frac{\nu_{\Psi_2}(x_i,\tau_l)}{2\pi}\right)$;

$\frac{\eta_{\Psi 1}(x_i,\tau_l)}{2\pi}\left(1-\frac{\eta_{\Psi 1}(x_i,\tau_l)}{2\pi}\right)\leq\frac{\eta_{\Psi_2}(x_i,\tau_l)}{2\pi}\left(1-\frac{\eta_{\Psi 2}(x_i,\tau_l)}{2\pi}\right)$;
which implies $E(\Psi_1)\leq E(\Psi_2)$

(4) $E(\Psi)=E(\Psi^c)$ is trivial

Example 5 Let $X=\{x_1,x_2\};\tilde{\tau}=\{\tau_1,\tau_2,\tau_3\}$ and Ψ_1 and Ψ_2 are two TCNS in X

$$\Psi_1(x_1)=\left\{\begin{array}{l}\left\langle 0.1e^{j0.2},0.6e^{j0.45},0.3e^{j0.56}\right\rangle,\left\langle 0.8e^{j0.77},0.2e^{j0.86},0.4e^{j0.65}\right\rangle,\\ \left\langle 0.6e^{j0.52},0.2e^{j0.2},0.36e^{j0.2}\right\rangle\end{array}\right\}$$

$$\Psi_1(x_2)=\left\{\begin{array}{l}\left\langle 0.5e^{j0.12},0.2e^{j0.65},0.5e^{j0.85}\right\rangle,\left\langle 0.3e^{j0.8},0.7e^{j0.23},0.1e^{j0.32}\right\rangle,\\ \left\langle 0.74e^{j0.6},0.5e^{j0.15},0.66e^{j0.41}\right\rangle\end{array}\right\}$$

$$\Psi_2(x_1)=\left\{\begin{array}{l}\left\langle 0.6e^{j0.8},0.9e^{j0.25},0.4e^{j0.52}\right\rangle,\left\langle 0.6e^{j0.62},0.5e^{j0.24},0.3e^{j0.18}\right\rangle,\\ \left\langle 0.3e^{j0.82},0.3e^{j0.6},0.4e^{j0.71}\right\rangle\end{array}\right\}$$

$$\Psi_2(x_2)=\left\{\begin{array}{l}\left\langle 0.3e^{j0.22},0.4e^{j0.56},0.6e^{j0.26}\right\rangle,\left\langle 0.1e^{j0.38},0.5e^{j0.43},0.5e^{j0.24}\right\rangle,\\ \left\langle 0.2e^{j0.29},0.4e^{j0.72},0.5e^{j0.65}\right\rangle\end{array}\right\}$$

The following results can be obtained by applying Definitions 7
$E_{\Psi_1}=0.444759$; $E_{\Psi_2}=0.475798$;

4 Multi-Criteria Decision Making

Assume $\tilde{A}=\{A_1,A_2,...,A_{n_A}\}$ and $\tilde{C}=\{C_1,C_2,...,C_{n_C}\}$ and $\tilde{D}=\{D_1,D_2,...,D_{n_D}\}$ are sets of alternatives, criteria and decision makers. For decision maker D_{i_d}; $i_d=1,2,...,n_D$, the evaluation characteristic of an alternatives A_{i_a}; $i_a=1,2,...,n_A$ on an attribute C_{i_c}; $i_c=1,2,...,n_C$ in time period τ_l; $l=1,2,...,n_\tau$ is represented by matrix $X^{i_d}(\tau_l)=\left(\theta^{i_d}_{i_ai_c}(\tau_l)\right)_{m\times n}$. where $\theta^{i_d}_{i_ai_c}(\tau_l)$ taken language label of complex neutrosophic set by time period τ_l.

Let
$x_{i_ai_ci_d}(\tau_l)=\left\langle p_{i_ai_ci_d}(\tau_l)e^{j\mu_{i_ai_ci_d}(\tau_l)},q_{i_ai_ci_d}(\tau_l)e^{j\nu_{i_ai_ci_d}(\tau_l)},r_{i_ai_ci_d}(\tau_l)e^{j\eta_{i_ai_ci_d}(\tau_l)}\right\rangle$

Step 1. Assuming that the weight information of the criterion is completely unknown. Hence, the weight w_{i_c} of criterion C_{i_c} is calculated as follows:
The averaged rating of decision makers can be evaluated by Equation (12)

$$x_{i_ai_c}=\frac{1}{n_D*n_\tau}\overset{n_\tau}{\underset{l=1}{\oplus}}\overset{n_D}{\underset{i_a=1}{\oplus}}x_{i_ai_ci_d}(\tau_l)=\left\langle\tilde{T}_{i_ai_c},\tilde{I}_{i_ai_c},\tilde{F}_{i_ai_c}\right\rangle \tag{12}$$

The weight w_{i_c} of criterion C_{i_c} is calculated by Equation (13)

$$w_{i_c} = \frac{\left(1 - E_{C_{i_c}}\right)}{n_c - \sum_{i=1}^{n_c} E_{C_{i_c}}} \tag{13}$$

Where $E_{C_{i_c}} = \frac{1}{n_A} \sum_{i_a=1}^{n_A} E\left(x_{i_a i_c}\right)$; each $E\left(x_{i_a i_c}\right)$ is computed using Equation (11)

Step 2. The temporal complex neutrosophic positive ideal solution (TCN-PIS) and negative ideal solution (TCN-NIS) are identified as follows:

$$A^+ = \left\{x, \left\langle \max\left(p_{i_a i_c}\left(x, \tilde{\tau}\right)\right) e^{j \max(\mu_{i_a i_c}(x,\tilde{\tau}))}, 0, 0 \right\rangle\right\} \tag{14}$$

$$A^- = \left\{x, \left\langle 0, \min\left(q_{i_a i_c}\left(x, \tilde{\tau}\right)\right) e^{j \min(\nu_{i_a i_c}(x,\tilde{\tau}))}, \min\left(r_{i_a i_c}\left(x, \tilde{\tau}\right)\right) e^{j \min(\eta_{i_a i_c}(x,\tilde{\tau}))} \right\rangle\right\} \tag{15}$$

Step 3. Theorem 1, 2, 3, or 4 is used to compute the weighted similarity measure for alternatives to TCN-PIS and TCN-NIS. The weighted similarity measure $S_{i_a}^+$ and $S_{i_a}^-$ of alternative A_{i_a} $(i_a = 1, 2, ..., n_A)$ is calculated by Equation (16)-(17)

$$S_{i_a}^+ = \bigoplus_{i_a=1}^{n_A} w_{i_c} S\left(x_{i_a i_c}, A^+\right) \tag{16}$$

$$S_{i_a}^- = \bigoplus_{i_a=1}^{n_A} w_{i_c} S\left(x_{i_a i_c}, A^-\right) \tag{17}$$

Step 4. The relative closeness of alternative A_{i_a} to the ideal solution is calculated as follows

$$RC_{i_a} = \frac{S_{i_a}^-}{S_{i_a}^+ + S_{i_a}^-} \tag{18}$$

Step 5. The alternatives are ranked according to their relative closeness and in descending order.

5 Numberical Example

In this section, by considering the real-life case, we present an application of proposed measures to a decision-making problem.

5.1 *Problem*

An example related to choosing a tourist destination in VN given by Lan et al. [25] demonstrates the feasibility and effectiveness of the proposed decision method. In this case, suppose a company wants to find a tourist destination in Vietnam. The company suggested a list of five tourist destinations to choose from, such as A_i; $i = 1, 2, 3, 4, 5$. For a more accurate evaluation of the different

locations, there are three group criteria and 20 sub-criteria corresponding to each location to choose the best tourist destination.

The first group of criteria includes 07 sub-criteria related to the quality of services and tourism. These sub-criteria are given below:

C_1 : the service staff is well-trained, helpful, and friendly;
C_2 : This is a value-for-money destination;
C_3 : This is a safe destination for travelers;
C_4 : It has a variety of entertainment/ nightlife activities for travelers;
C_5 : It offers many opportunities for sports and adventurous activities;
C_6 : It offers a variety of souvenirs and duty-free goods for travelers;
C_7 : Easy to move around the place

The second group includes criteria related to the Quality of life in the tourist destination. It includes the following criteria:

C_8 : This place is a country with comfortable living conditions;
C_9 : Easy to get there from home;
C_{10} : This place has good social welfare;
C_{11} : This place has good foods;
C_{12} : Local people are friendly and kindly.

The last group includes criteria related to the natural environment of the tourist destination. It includes 8 sub-criteria as follows:

C_{13} : This place has a good climate;
C_{14} : This is a good place for relaxation;
C_{15} : This place has good tourism infrastructure facilities (e.g., restaurants, accommodations, etc.);
C_{16} : This place is a country with many well-known tourist sites;
C_{17} : This place has magnificent sunny beaches;
C_{18} : The environment in this place is very clean;
C_{19} : This place has fascinating native animals and vegetation;
C_{20} : This place is a country with a vast land area and a relatively small population

To comprehensively evaluate the different locations over four periods, three experts with different experiences and knowledge have been invited to perform the evaluation. The expert's estimates are expressed by temporal complex neutrosophic numbers.

5.2 *An MCDM example using the proposed measures*

TCNS is a tool for describing two-dimensional uncertain information of a periodic nature in our day-to-day lives. In this section, we present a practical example in the TCNS environment to demonstrate that the proposed similarity measures play a significant role in solving real-life problems, such as tourist destinations chosen in multi-criteria decision-making problems.

Table 1 Language variables

Label	Value	Descriptiom
VL	$\langle 0.15e^{j0.55}, 0.65e^{j0.45}, 0.65e^{j0.35}\rangle$	Very low
L	$\langle 0.25e^{j0.65}, 0.55e^{j0.55}, 0.65e^{j0.45}\rangle$	Low
M	$\langle 0.40e^{j0.75}, 0.50e^{j0.65}, 0.45e^{j0.55}\rangle$	Medium
H	$\langle 0.55e^{j0.85}, 0.45e^{j0.75}, 0.35e^{j0.65}\rangle$	High
VH	$\langle 0.65e^{j0.95}, 0.25e^{j0.85}, 0.25e^{j0.75}\rangle$	Very high

Step 1. Determine the weight of the twenty criteria using Equations 12 and 13.

$w_1 = 0.049199;\ w_2 = 0.057871;\ w_3 = 0.052736;\ w_4 = 0.049812;\ w_5 = 0.038422;$
$w_6 = 0.042157;\ w_7 = 0.048045;\ w_8 = 0.055468;\ w_9 = 0.053227;\ w_{10} = 0.04191;$
$w_{11} = 0.049674;\ w_{12} = 0.04392;\ w_{13} = 0.044175;\ w_{14} = 0.049675;\ w_{15} = 0.058608;$
$w_{16} = 0.049995;\ w_{17} = 0.053227;\ w_{18} = 0.059684;\ w_{19} = 0.053973;\ w_{20} = 0.057452;$

Step 2. Estimate the temporal complex neutrosophic positive ideal solution (TCN-PIS) and negative ideal solution (TCN-NIS). The values are obtained as follows:

$$A^{+} = \left\{ \begin{array}{l} \left\langle 0.696238.\ \mathrm{e}^{\mathrm{j}0.916667}, 0, 0\right\rangle, \left\langle 0.696238.\mathrm{e}^{\mathrm{j}0.916667}, 0, 0\right\rangle, \\ \left\langle 0.696238.\mathrm{e}^{\mathrm{j}0.916667}, 0, 0\right\rangle, \left\langle 0.696238.\mathrm{e}^{\mathrm{j}0.916667}, 0, 0\right\rangle \end{array} \right\}$$

$$A^{-} = \left\{ \begin{array}{l} \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333}\right\rangle, \\ \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333}\right\rangle, \\ \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333}\right\rangle, \\ \left\langle 0, 0.30411.\mathrm{e}^{\mathrm{j}0.583333}, 0.279672.\mathrm{e}^{0.483333}\right\rangle \end{array} \right\}$$

Step 3. Calculate the similarity measure of five locations for TCN-PIS and TCN-NIS. They are shown in Table 2

Step 4. Determine the relative closeness of five locations to the ideal solution. They are shown in 3

Step 5. The five locations are ranked based on their relative closeness are shown in Table 3

Table 2 Similarity measures of five locations to TCN-PIS and TCN-NIS

Mesuares	Ideal solution	Location				
		A_1	A_2	A_3	A_4	A_5
Dice	S^+	0.63893	0.639313	0.637272	0.63702	0.634593
	S^-	0.656714	0.65388	0.656766	0.660245	0.660896
Jaccard	S^+	0.469471	0.469888	0.46768	0.467401	0.464794
	S^-	0.488921	0.485795	0.488989	0.492825	0.493544
Cosine	S^+	0.586304	0.588898	0.58585	0.582696	0.58158
	S^-	0.559327	0.559667	0.560673	0.560043	0.562002
Cot_1	S^+	0.641325	0.62623	0.63643	0.662388	0.652045
	S^-	0.595166	0.577427	0.594766	0.621144	0.616897
Cot_2	S^+	0.718926	0.714373	0.717399	0.724657	0.720939
	S^-	0.7867	0.767714	0.784763	0.813698	0.806133

Table 3 The Relative closeness of locations based on similarity measures

Mesuares		Location				
		A_1	A_2	A_3	A_4	A_5
Dice	RC	0.506863	0.505632	0.507532	0.508951	**0.510152**
	Ranking	4	5	3	2	1
Jaccard	RC	0.510147	0.508323	0.511137	0.513239	**0.515**
	Ranking	4	5	3	2	1
Cosine	RC	0.488226	0.487275	0.48902	0.490088	**0.49144**
	Ranking	4	5	3	2	1
Cot_1	RC	0.481335	0.479727	0.48308	0.483933	**0.486151**
	Ranking	4	5	3	2	1
Cot_2	RC	0.522507	0.517995	0.522422	**0.52894**	0.527894
	Ranking	3	5	4	1	2

5.3 A Comparison Analysis

In this section, to demonstrate the effectiveness of the proposed model, we compare the proposed model with those in Ali et al. [24] on temporal complex neutrosophic environments.

The ranking results of five locations at four time periods, $\tau_1, \tau_4, \tau_3, \tau_4$ are appropriately described in Table 4 for choosing a tourist destination problem. Table 4 shows that the rankings offive locations at four times τ_1, τ_4, τ_3 and τ_4 respectively are $A_5 \succ A_4 \succ A_1 \succ A_3 \succ A_2$; $A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$; $A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$ and $A_5 \succ A_4 \succ A_1 \succ A_3 \succ A_2$ and all 4 time periods are given A_5 as the best location. Meanwhile, the result of the proposed method gives the overall result at four time periods based on Dice, Jaccard, Cosine, and Cotangent similarity measures, respectively, are $A_5 \succ$

Table 4 Generalised-Dice-Similarity-measures of Ali et al for locations at four periods

Mesuares		Location				
		A_1	A_2	A_3	A_4	A_5
τ_1	RC	0.814267	0.808659	0.812204	0.816266	**0.819484**
	Ranking	3	5	4	2	1
τ_2	RC	0.8133	0.810594	0.813347	0.81762	**0.822208**
	Ranking	4	5	3	2	1
τ_3	RC	0.814128	0.808873	0.813478	0.817074	**0.821965**
	Ranking	4	5	3	2	1
τ_4	RC	0.813158	0.808602	0.813087	0.816648	**0.81934**
	Ranking	3	5	4	2	1

$A_4 \succ A_3 \succ A_1 \succ A_2$; $A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$; $A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$; $A_4 \succ A_5 \succ A_1 \succ A_3 \succ A_2$. The overall result of Dice, Jaccard, and Cosine similarity measures are given A_5 as the best location, and the Cotangent similarity measure is given A_4 as the best location. This result shows that the advantage and the practical applicability of the proposed measures can process the decision-making problems with time-related factors in a temporal complex neutrosophic environment. Moreover, it is more generalized and flexible than Ali et al. [24]'s method in temporal complex neutrosophic environments

6 CONCLUSIONS

In this paper, we have proposed four similarity measures, namely Dice, Jaccard, cosine, and Cotangent, in the TCNS environment. We have also proved some of their basic properties. In addition, we have defined an entropy measure of TCNS for determining unknown attribute weights in MCDM. Next, a new MCDM strategy has also been developed based on proposed similarity and entropy measures in the TCNS environment. Finally, a numerical example of decision-making problems regarding choosing a tourist destination in Vietnam is described under the temporal complex neutrosophic environment and is given to demonstrate the advantage and the practical applicability of the proposed measures. In the numerical example, it is proven that the proposed similarity and entropy measures of TCNS are able to produce reasonable results for decision-making problems in the real world. In future research, it will be interesting to develop new aggregation operators of TCNS and their applications in MADM problems such as logistics center selection, medical diagnosis, personnel selection, etc.

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Compliance with ethical standards

Conflict of interest: The authors declare no conflict of interest.
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A novel decision-making model for the selection of tobacco control strategies based on neutrosophic hypersoft set with neutrosophic possibility setting

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ABSTRACT

This study introduces the Type-NS Possibility Neutrosophic Hypersoft Set (Type-NS PNHSS), a novel framework that integrates possibility theory with neutrosophic hypersoft sets through neutrosophic possibility numbers to manage uncertain, indeterminate, and often inconsistent information across multiple attributes simultaneously in multi-criteria decision-making (MCDM). We establish the foundational theory of the Type-NS PNHSS, including its fundamental operations, measures, and aggrega-tion operators. Building upon this theoretical framework, we propose a multi-criteria decision-making model tailored to address decision-making problems within the Type-NS PNHSS environment. Finally, a real-world case study, the selection of tobacco control strategies, is conducted to demonstrate the efficacy and advantages of the proposed theory and approach. The proposed model evaluates tobacco control strategies based on 08 policies, 05 attribute groups, and 20 sub-attributes. The results indicate that the 'mass media communications' strategy is the best choice, and this finding is also further supported by comparative analyses with other existing methods, confirming the model's utility as a robust tool for advanced decision analysis.

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1. Introduction

Multi-Criteria Decision-Making (MCDM) is a subfield of operations research and decision science that deals with evaluating and selecting alternatives when multiple criteria are involved. In numerous real-world scenarios, decision-making often relies on the meticulous analysis of data derived from information; consequently, the data analyst must navigate a multitude of uncertain, indeterminate, inconsistent, and complex relationships among attributes. This endeavor undoubtedly constitutes a formidable challenge in every aspect, and the issue of addressing informational uncertainties has persistently posed significant difficulties for researchers. Various theoretical frameworks and methodologies have been formulated to address these challenges. Various theoretical frameworks and methodologies have been proposed; their capacity to fully overcome these challenges remains constrained. For example, Zadeh, (1965) introduced fuzzy sets to handle situations involving ambiguous information. Certain instances can not be addressed by fuzzy sets; thus, Atanassov, (1999) expanded the concept of fuzzy sets to encompass intuitionistic fuzzy sets. Torra (2010) proposed hesitant fuzzy sets, while Ramot et al., (2002) introduced the idea of complex fuzzy sets. Neutrosophy, introduced by Smarandache (1998) represents a more general concept than fuzzy set theory, serving as an innovative instrument for addressing issues that involve imprecise, indeterminate, and inconsistent data. Building upon these foundational theories, various decision-making methods have been implemented in published studies (Garai & Garg, 2022, 2023; Garai, Garg, & Biswas, 2023; Lan et al., 2023; Trang et al., 2024).

However, it has been found that these concepts are incapable of handling scenarios in which qualities and parameters must be addressed simultaneously. As a result, Molodtsov (Molodtsov, 1999) proposed the concept of the soft set, which is intended to provide fuzzy set-like structures with a parameterization mode. In the Soft set, the traditional belonging mapping is replaced by an approximation mapping that takes the set of parameters as its domain and the power set of the starting space of alternatives as its co-domain. In other words, the soft set offers a single argument domain for estimating the alternatives. However, the soft set itself is insufficient for scenarios that require a multi-argument domain for the approximation of alternatives. In 2018, Smarandache (Smarandache, 2018) addressed this issue by proposing the hypersoft set (HSS), which is used to create a multi-argument domain. This domain is created by multiplying attribute-valued sub-classes together. Hypersoft set points have been established in a variety of contexts, including fuzzy hypersoft sets (Afzal et al., 2024; Rahman, Saeed, Mohammed, et al., 2022; Surya et al., 2024). intuitionistic fuzzy hypersoft sets (Arshad, Saeed, Rahman, Bajri, et al., 2024; Yolcu et al., 2021; Zulqarnain et al., 2022) and Neutrosophic hypersoft set (Dhumras & Bajaj, 2023; Huong, Thong, Nhi, et al., 2025; Huong, Thong, Son, et al., 2025; Subha & Selvakumar, 2023; Valderrama Barragán et al., 2024; Zhang et al., 2022). Numerous research endeavors have subsequently been undertaken based on these frameworks.

Khan et al., (2022) developed q-rung orthopair fuzzy hypersoft sets, a concept involving fundamental operations like subset, null-set, absolute-set, union, intersection, complement, AND, and OR operators, and used numerical examples to validate their effectiveness. The research group of Zulqarnain, Siddique, et al., (2021) explored the multi-criteria decision-making issues using intuitionistic fuzzy hypersoft set (IFHSSs) data. They introduced operational laws for IFHSNs and presented two aggregation operators (AOs): IFHSWA and IFHSWG. The authors Zulqarnain, Xin, et al., (2021) proposed an extension of the neutrosophic hypersoft set (NHSS) concept, extending its basic properties and introducing basic operations like union, intersection, addition, difference, Truth-favorite, and False-favorite, as well as necessity and possibility operations. Bansal et al., (2022) introduced a new T-spherical fuzzy hypersoft set (TSFHSS) for systematic evaluation of incomplete and imprecise information in decision sciences. They discussed basic operations and aggregation operators. Arshad, Saeed, Rahman, Mohammed, et al., (2024) suggested a technique for dealing with uncertainty and imprecision that employs interval-valued multi-fuzzy hypersoft sets. This approach improves the accuracy and reliability of the decision-making process, balancing trade-offs between different criteria. Computational and structural comparisons are used to measure adaptability and dependability, resulting in the required standard. Furthermore, Zadeh, (1999) introduced the possibility theory as a foundation for the fuzzy set. In this theory, a possibility grade is employed to determine the acceptability level of any approximation. Several authors Ahmed et al., 2023; Karaaslan, 2017; Khalil et al., 2019; Palanikumar & Arulmozhi, 2021) have utilized possibility theory with soft set-like environments to create a variety of possibility soft set-like structures. Recently similar settings have been combined with Hypersoft set-like environments to provide novel structures (Musa & Al-Hagery, 2025; Palanikumar & Arulmozhi, 2023; Rahman, Saeed, & Garg, 2022; Rahman, Saeed, Khalifa, et al., 2022; Saeed, Ali, et al., 2023).

In accordance with the principles of NHSS and possibility theory, numerous scholarly groups have suggested enhancing NHSS theory through the parameter setting of possibility theory. For instance, Saeed, Wahab, et al., (2023) proposed a single-valued neutrosophic hypersoft set (psv-NHSS) for evaluating investment projects utilizing its aggregation procedures and decision-support system. Zhao et al., (2023) proposed a new method called the Possibility Neutrosophic Hypersoft Set Approach (EPDD-pNSHSS). This method employs a deep learning method for disease recognition and classification, using a possibility neutrosophic hypersoft set classifier for diseased and healthy leaf images. The method's accuracy is optimized using the whale optimization algorithm (WOA). Al-Hijjawi & Alkhazaleh, (2023) defined the Possibility Neutrosophic Hypersoft Set (PNHSS) using fuzzy numbers for possibility degree-based settings, a combination of Hypersoft Set and Neutrosophic Soft Set, to handle multiargument approximate functions. They discussed basic characteristics like subset, equality, complement, operations like union and intersection, properties like commutative, associative, distributive law, and De Morgan's law, and introduced AND and OR operations. This theory presents in detail the relationships between sub-attribute values and the possibility of degree-based settings represented by a fuzzy number. Alballa et al., (2024) introduced a framework – the fuzzy parameterized possibility single-valued neutrosophic hypersoft expert set, combined with a modified Sanchez's method to manage uncertainties in waste site selection. Rahman, Saeed, et al., (2023) addressed these uncertainties using an innovative methodology known as the fuzzy parameterized possibility single-valued neutrosophic hypersoft set (Fppsv-NHSS). The algorithm integrates Sanchez's method to resolve the SOWSSP, addressing uncertainties in decision-makers' choices of parameters and approximations. It manages uncertainties by utilizing fuzzy parameterization and possibility grades, allowing decision-makers to evaluate alternatives independently. The results are consistent and reliable due to their preferential features. Next, the authors (Rahman et al., 2024) pointed out that the possibility of a neutrosophic hypersoft set enables degrees of indeterminacy and multi-argument approximate functions, which improve decision-making flexibility and reliability. The study created a framework that hybridizes these sets, using theoretical, axiomatic, graphical, and algorithmic approaches. Furthermore, the authors (Rahman, Smarandache, et al., 2023) introduced the concept of a hypersoft membership function, extending the soft set to handle complex and uncertain information more effectively. In this research, they developed hybrid hypersoft set structures, including fuzzy sets, neutrosophic and possibility degree-based settings, and provided numerical examples for further understanding. Nevertheless, the approach of the authors focuses only on expressing the value of the relationships among sub-attributes and pays little attention to the operations, measurements, and decision-making methods.

1.1. Research gaps and motivation

The aforementioned surveys indicate that the Neutrosophic HyperSoft Set (NHSS) is a theoretical framework gaining considerable interest from researchers due to its significant applicability. The possibility degree-based setting refers to the framework in which sub-attributes are evaluated based on their possibility degrees, and has demonstrated useful results in tracking real-world decision-making environments. Moreover, extensions of NHSS that are predicated on possibility degree-based frameworks are also capturing the attention of researchers owing to their enhanced utility in practical contexts. However, the existing investigations concerning the PNHSS with possibility degree-based frameworks often employ fuzzy numbers for the possibility degree-based setting, and they have largely overlooked the comprehensive representation that utilizes neutrosophic numbers for both the interrelations among these sub-attributes and the possibility degree-based setting. Therefore, within the scope of this research paper, we extend the Neutrosophic Hypersoft Set with possibility degree-based settings, employing purely neutrosophic numbers for richer uncertainty representation in real-world decision-making environments.

1.2. Contributions

In this study, we aim to extend the Neutrosophic Hypersoft set with possibility degree-based settings, utilizing purely neutrosophic numbers, and to develop the operators and

measures, along with the decision-making model. The contributions are delineated as follows:

(a) Introducing the concept of Type-NS PHNSS based on the Neutrosophic Hypersoft set with Neutrosophic Possible degree-based settings
(b) Extending fundamental operations, measures, and aggregation operators for the Type-NS PNHSS environment to establish a comprehensive mathematical framework for computational applications in decision-making processes
(c) Developing a decision-making model based on the Type-NS PNHSS that can adapt to real-world environments
(d) In order to effectively illustrate the considerable efficacy and potential advantages of the proposed model, the article applies the model to systematically address and analyze the various selection strategies employed in tobacco control.

The subsequent research is systematically structured as follows: Section 2 encompasses essential concepts that support the enhancement of the organizational framework of the forthcoming research. Section 3 introduces the concept of Type-NS Possibility Neutrosophic Hypersoft Sets, along with operations, measures, and aggregation operations. Secton 4 presents a new decision-making model based on proposed theories in the Type-NS Possibility Neutrosophic Hypersoft environment. Section 5 applies the model to systematically address and analyze the various selection strategies that are employed in tobacco control. The conclusion and discussion are presented in Section 6.

2. Preliminaries

This part presents some fundamental principles related to the development of proposals in the next section.

Definition 1 [36]. Let Φ be the universal set and $N(\Phi)$ be set if all neutrosophic subset of Φ. For $n \geq 1$, let $l_1, l_2, \ldots, l_n$ be n well-defined attributes, whose corresponding attributes are respectively the set $L_1, L_2, \ldots, L_n$ with $L_i \cap L_j = \emptyset$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$ and their relation $L_1 \times L_2 \times \ldots \times L_n = \Omega$. The pair (Γ^{μ}, Ω) is said to be possibility neutrosophic hypersoft set over Φ where

$$\Gamma^{\mu}_{\Omega}(\theta) = \left\{ \begin{array}{l} \langle x, \Gamma_{\Omega}(\theta_i)(x), \mu(\theta)\rangle | x \in \Phi, \Gamma_{\Omega}(\theta_i) \\ (x) \in \mathrm{N}(\Phi), \mu(\theta) \in [0,1] \end{array} \right\} \quad (1)$$

Where Γ_{Ω} is a mapping given by $\Gamma_{\Omega} : L_1 \times L_2 \times \ldots \times L_n \to N(\Phi)$ and μ is fuzzy set such that $\mu : \Omega \to [0,1]$. Here $\Gamma^{\mu}_{\Omega}(\theta)$ is a mapping defined $\Gamma^{\mu}_{\Omega} : L_1 \times L_2 \times \ldots \times L_n \to N(\Phi) \times [0,1]$

Definition 2 [36]. Basic Operations of PNHSS

Let two PNHSS $\Gamma^{\mu}_{\Omega_1}$ and $\Gamma^{\eta}_{\Omega_2}$ over Φ.

(1) The complement of a PNHSS $\Gamma^{\mu}_{\Omega_1}$ is denoted $\left(\Gamma^{\mu}_{\Omega_1}\right)^c$ and define , where $\mu^c(\theta) = 1 - \mu(\theta)$ and Γ^c_{Ω} is neutrosophic soft complement with $T^c_{\Omega}(\theta) = F_{\Omega}(\theta)$; $I^c_{\Omega}(\theta) = 1 - I_{\Omega}(\theta)$; $F^c_{\Omega}(\theta) = T_{\Omega}(\theta)$;
(2) The union of two PNHSS is a PNHSS Γ^{ρ}_{Ω} defined as $\Gamma(\Omega, \rho)$ where $\Omega = \Omega_1 \cup \Omega_2$ and $\rho(\theta) = \max(\mu(\theta), \eta(\theta))$ and $\forall \theta \in \Omega$ we have follow: $\Gamma^{\rho}_{\Omega} = \Gamma^{\mu}_{\Omega_1} \hat{\cup} \Gamma^{\eta}_{\Omega_2}$. Where $\hat{\cup}$ is a NHSS union
(3) The intersection of two PNHSS is a PNHSS Γ^{ρ}_{Ω} defined as $\Gamma(\Omega, \rho)$ where $\Omega = \Omega_1 \cap \Omega_2$ and $\rho(\theta) = \min(\mu(\theta), \eta(\theta))$ and $\forall \theta \in \Omega$ we have follow: $\Gamma^{\rho}_{\Omega} = \Gamma^{\mu}_{\Omega_1} \hat{\cap} \Gamma^{\eta}_{\Omega_2}$. Where $\hat{\cap}$ is a NHSS intersection
(4) AND operation of $\Gamma^{\mu}_{\Omega_1}$ and $\Gamma^{\eta}_{\Omega_2}$ over Φ denoted by $\Gamma^{\mu}_{\Omega_1} \hat{\wedge} \Gamma^{\eta}_{\Omega_2}$ is given as $\Gamma^{\mu}_{\Omega_1} \hat{\wedge} \Gamma^{\eta}_{\Omega_2} = \Gamma^{\rho}_{\Omega_1 \times \Omega_2}$. Such that $\Gamma^{\rho}_{\Omega_1 \times \Omega_2}(\alpha, \beta) = \Gamma^{\mu}_{\Omega_1}(\alpha) \hat{\cap} \Gamma^{\eta}_{\Omega_2}(\beta)$, $\forall (\alpha, \beta) \in \Omega_1 \times \Omega_2$. Where $\hat{\cap}$ is a NHSS intersection and $\rho(\theta) = \min(\mu(\theta), \eta(\theta))$
(5) OR operation $\Gamma^{\mu}_{\Omega_1}$ and $\Gamma^{\eta}_{\Omega_2}$ over Φ denoted by $\Gamma^{\mu}_{\Omega_1} \hat{\vee} \Gamma^{\eta}_{\Omega_2}$ is given as $\Gamma^{\mu}_{\Omega_1} \hat{\wedge} \Gamma^{\eta}_{\Omega_2} = \Gamma^{\rho}_{\Omega_1 \times \Omega_2}$. Such that $\Gamma^{\rho}_{\Omega_1 \times \Omega_2}(\alpha, \beta) = \Gamma^{\mu}_{\Omega_1}(\alpha) \hat{\cup} \Gamma^{\eta}_{\Omega_2}(\beta)$, $\forall (\alpha, \beta) \in \Omega_1 \times \Omega_2$. Where $\hat{\cup}$ is a NHSS union and $\rho(\theta) = \max(\mu(\theta), \eta(\theta))$

3. Type-NS possibility neutrosophic hypersoft sets

In this section, building on the conceptual framework established by Musa & Al-Hagery, (2025), we present the concept of Type-NS PNHSS based on Neutrosophic Hypersoft set with possibility degree-based settings using neutrosophic numbers

Definition 3. Let Φ be the universal set and $N(\Phi)$ be set if all neutrosophic subset of Φ. For $n \geq 1$, let $l_1, l_2, \ldots, l_n$ be n well-defined attributes, whose corresponding attributes are respectively the set $L_1, L_2, \ldots, L_n$ with $L_i \cap L_j = \emptyset$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$ and their relation $L_1 \times L_2 \times \ldots \times L_n = \Omega$. The pair $\left(\Gamma^{\mathrm{NS}}, \Omega\right)$ is said to be Type-NS possibility neutrosophic hypersoft set over Φ where

$$\Gamma^{NS}_{\Omega}(\theta) = \{\langle \mathrm{x}, \Gamma_{\Omega}(\theta_{\mathrm{i}})(x), \mathrm{NS}(\theta)\rangle | \mathrm{x} \in \Phi, \Gamma_{\Omega}(\theta_{\mathrm{i}})(x) \in N(\Phi), NS(\theta) = \langle T(\theta), I(\theta), F(\theta)\rangle\} \quad (2)$$

Where Γ_{Ω} is a mapping given by $\Gamma_{\Omega} : L_1 \times L_2 \times \ldots \times L_n \to N(\Phi)$ and NS is a neutrosophic set such as $T, I, F : \Omega \to [0,1]$; $\theta \in \Omega$ is a n-tuple of $\theta_i \in L_i$; $i = 1, \ldots, n$; representing the corresponding relationship between the attributes.

Here Γ^{NS}_{Ω} a mapping defined $\Gamma^{NS}_{\Omega} : \mathrm{L}_1 \times \mathrm{L}_2 \times \ldots \times L_n \to N(\Phi) \times [0,1]^3$

Simple a Type-NS possibility neutrosophic hypersoft element $\Gamma^{NS}_{\Omega}(\theta) = \{\langle \theta, \Gamma_{\Omega}(\theta_{\mathrm{i}}), \mathrm{NS}(\theta) | \theta \in \Omega\rangle\}$ of $x \in \Phi$;

Example 1. Let X be the set of decision makers/alternatives $X = \{x_1, x_2, x_3\}$ and set $M = \{x_1, x_3\} \subset X$; also consider the set attributes as $\{L_1, L_2, L_3\}$. Let $\Gamma^{NS}_{\Omega} : \mathrm{L}_1 \times \mathrm{L}_2 \times \mathrm{L}_3 \to \mathrm{N}(\Phi) \times [0,1]^3$ and $T, I, F \to [0,1]$; $\theta = \{\theta_1, \theta_2, \theta_3\}$ is the set of 3-tuple values representing the relationship between the attributes

$$\Gamma_{\Omega}^{NS}(\theta)=\left\{\begin{array}{l}\langle x_1,(\{\langle 0.5,0.6,0.8\rangle,\langle 0.7,0.2,0.1\rangle,\langle 0.6,0.1,0.9\rangle\},\langle 0.4,0.8,0.6\rangle)\rangle,\\ \langle x_3,(\{\langle 0.1,0.5,0.4\rangle,\langle 0.8,0.1,0.6\rangle,\langle 0.9,0.6,0.4\rangle\},\langle 0.7,0.3,0.1\rangle)\rangle\end{array}\right\}$$

3.1. Basic operations of type-NS PHSS

We are following Operational laws of Neutrosophic Hypersoft Sets and Neutrosophic sets.

Definition 4. The Complement of Type-NS PNHSS Γ_{Ω}^{NS} over Φ defined as $\Gamma_{\Omega}^{NS}(\theta)^{c}=\{\langle \mathrm{x},(\Gamma_{\Omega}(\theta)(x))^{c},NS(\theta)^{c}\rangle|x\in\Phi\}$, where $NS(\theta)^{c}$ is complement of neutrosophic set with $T^{NS}(\theta)^{c}=F^{NS}(\theta)$; $I^{NS}(\theta)^{c}=1-I^{NS}(\theta)$; $F^{NS}(\theta)^{c}=T^{NS}(\theta)$ and $(\Gamma_{\Omega}(\theta)(x))^{c}$ is a complement of a neutrosophic hypersoft set.

Example 2. Let $\Gamma_{\Omega}^{NS}(\theta)=\left\{\begin{array}{l}\langle x_1,(\{\langle 0.5,0.6,0.8\rangle,\langle 0.7,0.2,0.1\rangle,\langle 0.6,0.1,0.9\rangle\},\langle 0.4,0.8,0.6\rangle)\rangle,\\ \langle x_3,(\{\langle 0.1,0.5,0.4\rangle,\langle 0.8,0.1,0.6\rangle,\langle 0.9,0.6,0.4\rangle\},\langle 0.7,0.3,0.1\rangle)\rangle\end{array}\right\}$

By using the Type-NS PNHSS complement, we obtain the complement given by

$$\left(\Gamma_{\Omega}^{NS}(\theta)\right)^{c}=\left\{\begin{array}{l}\langle x_1,(\{\langle 0.8,0.6,0.5\rangle,\langle 0.1,0.2,0.7\rangle,\langle 0.9,0.1,0.6\rangle\},\langle 0.6,0.8,0.4\rangle)\rangle,\\ \langle x_3,(\{\langle 0.4,0.5,0.1\rangle,\langle 0.6,0.1,0.8\rangle,\langle 0.4,0.6,0.9\rangle\},\langle 0.1,0.3,0.7\rangle)\rangle\end{array}\right\}$$

Proposition 1. Let Γ_{Ω}^{NS} be Type-NS PNHSS, then $\left(\left(\Gamma_{\Omega}^{NS}(\theta)\right)^{c}\right)^{c}=\Gamma_{\Omega}^{NS}$

Proof. Let $\Gamma_{\Omega}^{NS}(\theta)^{\mathrm{c}}=\{\langle \mathrm{x},(\Gamma_{\Omega}(\theta)(x))^{\mathrm{c}},\mathrm{NS}(\theta)^{\mathrm{c}}\rangle|\mathrm{x}\in\Phi\}$

$$=\left\{\langle x,\cup_{i=1}^{n}\left\{\langle F^{\Gamma_{\Omega}}(\theta_i),1-I^{\Gamma_{\Omega}}(\theta_i),T^{\Gamma_{\Omega}}(\theta_i)\rangle\right\},\langle F^{NS}(\theta),1-I^{NS}(\theta),T^{NS}(\theta)\rangle\rangle|x\in\Phi\right\}$$

$$\left(\Gamma_{\Omega}^{NS}(\theta)^{\mathrm{c}}\right)^{c}=\left\{\langle x,\cup_{i=1}^{n}\left\{\langle T^{\Gamma_{\Omega}}(\theta_i),1-\left(1-I^{\Gamma_{\Omega}}(\theta_i)\right),F^{\Gamma_{\Omega}}(\theta_i)\rangle\right\},\langle T^{NS}(\theta),1-\left(1-I^{NS}(\theta)\right),F^{NS}(\theta)\rangle\rangle|x\in\Phi\right\}$$

$$=\left\{\langle x,\cup_{i=1}^{n}\left\{\langle T^{\Gamma_{\Omega}}(\theta_i),I^{\Gamma_{\Omega}}(\theta_i),F^{\Gamma_{\Omega}}(\theta_i)\rangle\right\},\langle T^{NS}(\theta),I^{NS}(\theta),F^{NS}(\theta)\rangle\rangle|x\in\Phi\right\}=\Gamma_{\Omega}^{NS}(\theta)$$

Definition 5. The union of two Type-NS PNHSS $\Gamma_{\Omega_1}^{NS_1}$ and $\Gamma_{\Omega_2}^{NS_2}$ over Φ is a Type-NS PNHSS Γ_{Ω}^{NS} defined as $\Gamma(\Omega,\mathrm{NS})=\Gamma_{\Omega_1}^{NS_1}\cup\Gamma_{\Omega_2}^{NS_2}$ with $\Gamma_{\Omega}=\Gamma_{\Omega_1}\cup_{\Gamma}\Gamma_{\Omega_2}$ and $NS(r)=NS_1(r)\cup_{NS}NS_2(r)$.

Where $\cup\Gamma$ is a NHSS union and $\underset{NS}{\cup}$ is a NS union

Example 3. Assume that two Type-NS PNHSS

$$\Gamma_{\Omega_1}^{NS_1}(\theta)=\left\{\begin{array}{l}\langle x_1,(\{\langle 0.5,0.6,0.8\rangle,\langle 0.7,0.2,0.1\rangle,\langle 0.6,0.1,0.9\rangle\},\langle 0.4,0.8,0.6\rangle)\rangle,\\ \langle x_2,(\{\langle 0.1,0.5,0.4\rangle,\langle 0.8,0.1,0.6\rangle,\langle 0.9,0.6,0.4\rangle\},\langle 0.7,0.3,0.1\rangle)\rangle\end{array}\right\}$$

and $\Gamma_{\Omega_2}^{NS_2}(\theta)=\left\{\begin{array}{l}\langle x_1,(\{\langle 0.3,0.5,0.1\rangle,\langle 0.8,0.4,0.3\rangle,\langle 0.7,0.1,0.1\rangle\},\langle 0.5,0.3,0.7\rangle)\rangle,\\ \langle x_3,(\{\langle 0.2,0.6,0.7\rangle,\langle 0.4,0.2,0.9\rangle,\langle 0.1,0.5,0.2\rangle\},\langle 0.2,0.1,0.3\rangle)\rangle\end{array}\right\}$

$$\Gamma_{\Omega}^{NS}(\theta)=\left\{\begin{array}{l}\langle x_1,(\{\langle 0.5,0.5,0.1\rangle,\langle 0.8,0.2,0.1\rangle,\langle 0.7,0.1,0.1\rangle\},\langle 0.5,0.3,0.6\rangle)\rangle,\\ \langle x_2,(\{\langle 0.1,0.5,0.4\rangle,\langle 0.8,0.1,0.6\rangle,\langle 0.9,0.6,0.4\rangle\},\langle 0.7,0.3,0.1\rangle)\rangle\\ \langle x_3,(\{\langle 0.2,0.6,0.7\rangle,\langle 0.4,0.2,0.9\rangle,\langle 0.1,0.5,0.2\rangle\},\langle 0.2,0.1,0.3\rangle)\rangle\end{array}\right\}$$

Proposition 2. Let $\Gamma_{\Omega_1}^{NS_1}$, $\Gamma_{\Omega_2}^{NS_2}$ and $\Gamma_{\Omega_3}^{NS_3}$ be three Type-NS PNHSS, then,

$$\Gamma_{\Omega_1}^{NS_1}\cup\Gamma_{\Omega_2}^{NS_2}=\Gamma_{\Omega_2}^{NS_2}\cup\Gamma_{\Omega_1}^{NS_1} \tag{1}$$

$$\left(\Gamma_{\Omega_1}^{NS_1}\cup\Gamma_{\Omega_2}^{NS_2}\right)\cup\Gamma_{\Omega_3}^{NS_3}=\Gamma_{\Omega_1}^{NS_1}\cup\left(\Gamma_{\Omega_2}^{NS_2}\cup\Gamma_{\Omega_3}^{NS_3}\right) \tag{2}$$

Proof.

$$\Gamma_{\Omega_1}^{NS_1}\cup\Gamma_{\Omega_2}^{NS_2}=\left\{\begin{array}{l}\mathrm{x},\cup_{\mathrm{i}}^{\mathrm{n}}\{\langle\max(\mathrm{T}^{\Gamma_{\Omega_1}}(\theta_{\mathrm{i}},\mathrm{x}),\mathrm{T}^{\Gamma_{\Omega_2}}(\theta_{\mathrm{i}},\mathrm{x})),\min(\mathrm{I}^{\Gamma_{\Omega_1}}(\theta_{\mathrm{i}},\mathrm{x}),\mathrm{I}^{\Gamma_{\Omega_2}}(\theta_{\mathrm{i}},\mathrm{x})),\min(\mathrm{F}^{\Gamma_{\Omega_1}}(\theta_{\mathrm{i}},\mathrm{x}),\mathrm{F}^{\Gamma_{\Omega_2}}(\theta_{\mathrm{i}},\mathrm{x}))\rangle\},\\ \left\langle\max(\mathrm{T}^{NS_1}(\theta,\mathrm{x}),\mathrm{T}^{NS_2}(\theta,\mathrm{x})),\min(\mathrm{I}^{NS_1}(\theta,\mathrm{x}),\mathrm{I}^{NS_2}(\theta,\mathrm{x})),\min(\mathrm{F}^{NS_1}(\theta,\mathrm{x}),\mathrm{F}^{NS_2}(\theta,\mathrm{x}))\right\rangle\end{array}\right\}$$

$$\left(\Gamma_{\Omega_1}^{NS_1}\cup\Gamma_{\Omega_2}^{NS_2}\right)\cup\Gamma_{\Omega_3}^{NS_3}$$

$$=\left\{\begin{array}{l}x,\cup_{i}^{n}\left\{\left\langle\begin{array}{l}\max(\max(T^{\Gamma_{\Omega_1}}(\theta_i,x),T^{\Gamma_{\Omega_2}}(\theta_i,x)),T^{\Gamma_{\Omega_3}}(\theta_i,x)),\min(\min(I^{\Gamma_{\Omega_1}}(\theta_i,x),I^{\Gamma_{\Omega_2}}(\theta_i,x)),I^{\Gamma_{\Omega_3}}(\theta_i,x)),\\ \min(\min(F^{\Gamma_{\Omega_1}}(\theta_i,x),F^{\Gamma_{\Omega_2}}(\theta_i,x)),F^{\Gamma_{\Omega_3}}(\theta_i,x))\end{array}\right\rangle\right\},\\ \left\langle\max(\max(T^{NS_1}(\theta,x),T^{NS_2}(\theta,x)),T^{NS_3}(\theta,x)),\min(\min(I^{NS_1}(\theta,x),I^{NS_2}(\theta,x)),I^{NS_3}(\theta,x)),\min(\min(F^{NS_1}(\theta,x),F^{NS_2}(\theta,x)),F^{NS_3}(\theta,x))\right\rangle\end{array}\right\}$$

$$= \left\{ \begin{array}{l} x, \cup_i^n \left\{ \left\langle \begin{array}{l} \max(T^{\Gamma_{\Omega_1}}(\theta_i,x), \max(, T^{\Gamma_{\Omega_2}}(\theta_i,x), T^{\Gamma_{\Omega_3}}(\theta_i,x))), \min(I^{\Gamma_{\Omega_1}}(\theta_i,x), \min(I^{\Gamma_{\Omega_2}}(\theta_i,x), I^{\Gamma_{\Omega_3}}(\theta_i,x))), \\ \min(F^{\Gamma_{\Omega_1}}(\theta_i,x), \min(F^{\Gamma_{\Omega_2}}(\theta_i,x), F^{\Gamma_{\Omega_3}}(\theta_i,x))) \end{array} \right\rangle \right\}, \\ \left\langle \max(T^{NS_1}(\theta,x), \max(T^{NS_2}(\theta,x), T^{NS_3}(\theta,x))), \min(I^{NS_1}(\theta,x), \min(I^{NS_2}(\theta,x), I^{NS_3}(\theta,x))), \min(F^{NS_1}(\theta,x), \min(F^{NS_2}(\theta,x), F^{NS_3}(\theta,x))) \right\rangle \end{array} \right\}$$

$$= \Gamma_{\Omega_1}^{NS_1} \cup \left(\Gamma_{\Omega_2}^{NS_2} \cup \Gamma_{\Omega_3}^{NS_3} \right)$$

Definition 6. The intersection of two Type-NS PNHSS $\Gamma_{\Omega_1}^{NS_1}$ and $\Gamma_{\Omega_2}^{NS_2}$ over Φ is a Type-NS PNHSS Γ_{Ω}^{NS} defined as $\Gamma(\Omega, \mathrm{NS}) = \Gamma_{\Omega_1}^{NS_1} \cap \Gamma_{\Omega_2}^{NS_2}$ with $\Gamma_\Omega = \Gamma_{\Omega_1} \cap I \Gamma_{\Omega_2}$ and $NS(r) = NS_1(r) \underset{NS}{\cap} NS_2(r)$.

Where $\cap I$ is a NHSS intersection and $\underset{NS}{\cap}$ is a NS intersection

Example 4. Consider an example

$$\Gamma_{\Omega_1}^{NS_1}(\theta) = \left\{ \begin{array}{l} \langle x_1, (\{\langle 0.5, 0.6, 0.8\rangle, \langle 0.7, 0.2, 0.1\rangle, \langle 0.6, 0.1, 0.9\rangle\}, \langle 0.4, 0.8, 0.6\rangle)\rangle, \\ \langle x_2, (\{\langle 0.1, 0.5, 0.4\rangle, \langle 0.8, 0.1, 0.6\rangle, \langle 0.9, 0.6, 0.4\rangle\}, \langle 0.7, 0.3, 0.1\rangle)\rangle \end{array} \right\}$$

$$\text{and } \Gamma_{\Omega_2}^{NS_2}(\theta) = \left\{ \begin{array}{l} \langle x_1, (\{\langle 0.3, 0.5, 0.1\rangle, \langle 0.8, 0.4, 0.3\rangle, \langle 0.7, 0.1, 0.1\rangle\}, \langle 0.5, 0.3, 0.7\rangle)\rangle, \\ \langle x_3, (\{\langle 0.2, 0.6, 0.7\rangle, \langle 0.4, 0.2, 0.9\rangle, \langle 0.1, 0.5, 0.2\rangle\}, \langle 0.2, 0.1, 0.3\rangle)\rangle \end{array} \right\}$$

$$\Gamma_{\Omega}^{NS}(\theta) = \left\{ \begin{array}{l} \langle x_1, (\{\langle 0.3, 0.6, 0.8\rangle, \langle 0.7, 0.4, 0.3\rangle, \langle 0.6, 0.1, 0.9\rangle\}, \langle 0.4, 0.8, 0.7\rangle)\rangle, \\ \langle x_2, (\{\langle 0.1, 0.5, 0.4\rangle, \langle 0.8, 0.1, 0.6\rangle, \langle 0.9, 0.6, 0.4\rangle\}, \langle 0.7, 0.3, 0.1\rangle)\rangle \\ \langle x_3, (\{\langle 0.2, 0.6, 0.7\rangle, \langle 0.4, 0.2, 0.9\rangle, \langle 0.1, 0.5, 0.2\rangle\}, \langle 0.2, 0.1, 0.3\rangle)\rangle \end{array} \right\}$$

Proposition 3. Let $\Gamma_{\Omega_1}^{NS_1}$, $\Gamma_{\Omega_2}^{NS_2}$ and $\Gamma_{\Omega_3}^{NS_3}$ be three Type-NS PNHSS, then,

$$\Gamma_{\Omega_1}^{NS_1} \cap \Gamma_{\Omega_2}^{NS_2} = \Gamma_{\Omega_2}^{NS_2} \cap \Gamma_{\Omega_1}^{NS_1} \tag{1}$$

$$\left(\Gamma_{\Omega_1}^{NS_1} \cap \Gamma_{\Omega_2}^{NS_2} \right) \cap \Gamma_{\Omega_3}^{NS_3} = \Gamma_{\Omega_1}^{NS_1} \cap \left(\Gamma_{\Omega_2}^{NS_2} \cap \Gamma_{\Omega_3}^{NS_3} \right) \tag{2}$$

Proof.

$$\Gamma_{\Omega_1}^{NS_1} \cap \Gamma_{\Omega_2}^{NS_2} = \left\{ \begin{array}{l} \mathrm{x}, \cup_{\mathrm{i}}^{\mathrm{n}} \{\langle \min(\mathrm{T}^{\Gamma_{\Omega_1}}(\theta_{\mathrm{i}}, \mathrm{x}), \mathrm{T}^{\Gamma_{\Omega_2}}(\theta_{\mathrm{i}}, \mathrm{x})), \max(\mathrm{I}^{\Gamma_{\Omega_1}}(\theta_{\mathrm{i}}, \mathrm{x}), \mathrm{I}^{\Gamma_{\Omega_2}}(\theta_{\mathrm{i}}, \mathrm{x})), \max(\mathrm{F}^{\Gamma_{\Omega_1}}(\theta_{\mathrm{i}}, \mathrm{x}), \mathrm{F}^{\Gamma_{\Omega_2}}(\theta_{\mathrm{i}}, \mathrm{x}))\rangle\}, \\ \left\langle \min(\mathrm{T}^{NS_1}(\theta, \mathrm{x}), \mathrm{T}^{NS_2}(\theta, \mathrm{x})), \max(\mathrm{I}^{NS_1}(\theta, \mathrm{x}), \mathrm{I}^{NS_2}(\theta, \mathrm{x})), \max(\mathrm{F}^{NS_1}(\theta, \mathrm{x}), \mathrm{F}^{NS_2}(\theta, \mathrm{x})) \right\rangle \end{array} \right\} \tag{1}$$

$$= \left\{ \begin{array}{l} x, \cup_i^n \{\langle \min(T^{\Gamma_{\Omega_2}}(\theta_i, x), T^{\Gamma_{\Omega_1}}(\theta_i, x)), \max(I^{\Gamma_{\Omega_2}}(\theta_i, x), I^{\Gamma_{\Omega_1}}(\theta_i, x)), \max(F^{\Gamma_{\Omega_2}}(\theta_i, x), F^{\Gamma_{\Omega_1}}(\theta_i, x))\rangle\}, \\ \left\langle \min(T^{NS_2}(\theta, x), T^{NS_1}(\theta, x)), \max(I^{NS_2}(\theta, x), I^{NS_1}(\theta, x)), \max(F^{NS_2}(\theta, x), F^{NS_1}(\theta, x)) \right\rangle \end{array} \right\}$$

$$= \Gamma_{\Omega_2}^{NS_2} \cap \Gamma_{\Omega_1}^{NS_1}$$

$$\Gamma_{\Omega_1}^{NS_1} \cap \Gamma_{\Omega_2}^{NS_2} = \left\{ \begin{array}{l} \mathrm{x}, \cup_{\mathrm{i}}^{\mathrm{n}} \{\langle \min(\mathrm{T}^{\Gamma_{\Omega_1}}(\theta_{\mathrm{i}}, \mathrm{x}), \mathrm{T}^{\Gamma_{\Omega_2}}(\theta_{\mathrm{i}}, \mathrm{x})), \max(\mathrm{I}^{\Gamma_{\Omega_1}}(\theta_{\mathrm{i}}, \mathrm{x}), \mathrm{I}^{\Gamma_{\Omega_2}}(\theta_{\mathrm{i}}, \mathrm{x})), \max(\mathrm{F}^{\Gamma_{\Omega_1}}(\theta_{\mathrm{i}}, \mathrm{x}), \mathrm{F}^{\Gamma_{\Omega_2}}(\theta_{\mathrm{i}}, \mathrm{x}))\rangle\}, \\ \left\langle \min(\mathrm{T}^{NS_1}(\theta, \mathrm{x}), \mathrm{T}^{NS_2}(\theta, \mathrm{x})), \max(\mathrm{I}^{NS_1}(\theta, \mathrm{x}), \mathrm{I}^{NS_2}(\theta, \mathrm{x})), \max(\mathrm{F}^{NS_1}(\theta, \mathrm{x}), \mathrm{F}^{NS_2}(\theta, \mathrm{x})) \right\rangle \end{array} \right\} \tag{2}$$

$$\left(\Gamma_{\Omega_1}^{NS_1} \cap \Gamma_{\Omega_2}^{NS_2} \right) \cap \Gamma_{\Omega_3}^{NS_3}$$

$$= \left\{ \begin{array}{l} x, \cup_i^n \left\{ \left\langle \begin{array}{l} \min(\min(T^{\Gamma_{\Omega_1}}(\theta_i,x), T^{\Gamma_{\Omega_2}}(\theta_i,x)), T^{\Gamma_{\Omega_3}}(\theta_i,x)), \max(\max(I^{\Gamma_{\Omega_1}}(\theta_i,x), I^{\Gamma_{\Omega_2}}(\theta_i,x)), I^{\Gamma_{\Omega_3}}(\theta_i,x)), \\ \max(\max(F^{\Gamma_{\Omega_1}}(\theta_i,x), F^{\Gamma_{\Omega_2}}(\theta_i,x)), F^{\Gamma_{\Omega_3}}(\theta_i,x)) \end{array} \right\rangle \right\}, \\ \left\langle \min(\min(T^{NS_1}(\theta,x), T^{NS_2}(\theta,x)), T^{NS_3}(\theta,x)), \max(\max(I^{NS_1}(\theta,x), I^{NS_2}(\theta,x)), I^{NS_3}(\theta,x)), \max(\max(F^{NS_1}(\theta,x), F^{NS_2}(\theta,x)), F^{NS_3}(\theta,x)) \right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{l} x, \cup_i^n \left\{ \left\langle \begin{array}{l} \min(T^{\Gamma_{\Omega_1}}(\theta_i,x), \min(, T^{\Gamma_{\Omega_2}}(\theta_i,x), T^{\Gamma_{\Omega_3}}(\theta_i,x))), \max(I^{\Gamma_{\Omega_1}}(\theta_i,x), \max(I^{\Gamma_{\Omega_2}}(\theta_i,x), I^{\Gamma_{\Omega_3}}(\theta_i,x))), \\ \max(F^{\Gamma_{\Omega_1}}(\theta_i,x), \max(F^{\Gamma_{\Omega_2}}(\theta_i,x), F^{\Gamma_{\Omega_3}}(\theta_i,x))) \end{array} \right\rangle \right\}, \\ \left\langle \min(T^{NS_1}(\theta,x), \min(T^{NS_2}(\theta,x), T^{NS_3}(\theta,x))), \max(I^{NS_1}(\theta,x), \max(I^{NS_2}(\theta,x), I^{NS_3}(\theta,x))), \max(F^{NS_1}(\theta,x), \max(F^{NS_2}(\theta,x), F^{NS_3}(\theta,x))) \right\rangle \end{array} \right\}$$

$$= \Gamma_{\Omega_1}^{NS_1} \cap \left(\Gamma_{\Omega_2}^{NS_2} \cap \Gamma_{\Omega_3}^{NS_3} \right)$$

Definition 7. Score, accuracy, certainty function of Type-NS PHSS Γ_{Ω}^{NS}

$\Gamma_{\Omega}^{NS}(\theta) = \left\{ \langle \Gamma_\Omega(\theta_{\mathrm{ij}}), \mathrm{NS}(\theta_{\mathrm{j}}) \rangle \right\}$, n-tupe $\hat{\theta}_j \in \Omega; \theta_{ij} \in \hat{\theta}_j; j = 1, \ldots, z; i = 1, 2, \ldots, n$

(1) Score function

$$S(\Gamma_{\Omega}^{NS}) = \frac{1}{z} \sum_{j=1}^{z} \left(\left(\frac{1}{n} \sum_{i=1}^{n} \frac{2 + T_{ij}^{\Gamma_\Omega} - I_{ij}^{\Gamma_\Omega} - F_{ij}^{\Gamma_\Omega}}{6} \right) + \left(\frac{2 + T_j^{NS} - I_j^{NS} - F_j^{NS}}{6} \right) \right) \tag{3}$$

(2) Accuracy function

$$A\left(\Gamma_{\Omega}^{NS}\right)=\frac{1}{z}\sum_{j=1}^{z}\left(\left(\frac{1}{n}\sum_{i=1}^{n}\frac{T_{ij}^{\Gamma_{\Omega}}-F_{ij}^{\Gamma_{\Omega}}}{2}\right)+\frac{\left(T_{j}^{NS}-F_{j}^{NS}\right)}{2}\right) \tag{4}$$

(3) Certainty function

$$C\left(\Gamma_{\Omega}^{NS}\right)=\frac{1}{z}\sum_{j=1}^{z}\left(\left(\frac{1}{n}\sum_{i=1}^{n}\frac{T_{ij}^{\Gamma_{\Omega}}}{2}\right)+\left(\frac{T_{j}^{NS}}{2}\right)\right) \tag{5}$$

Example 5. Consider Example 1

$$\Gamma_{\Omega}^{NS}(\theta)=\left\{\begin{array}{l}\langle x_1,(\{\langle 0.5,0.6,0.8\rangle,\langle 0.7,0.2,0.1\rangle,\langle 0.6,0.1,0.9\rangle\},\langle 0.4,0.8,0.6\rangle)\rangle,\\ \langle x_2,(\{\langle 0.1,0.5,0.4\rangle,\langle 0.8,0.1,0.6\rangle,\langle 0.9,0.6,0.4\rangle\},\langle 0.7,0.3,0.1\rangle)\rangle\end{array}\right\}$$

$S\left(\Gamma_{\Omega}^{NS}(\theta)\right)=0.57;A\left(\Gamma_{\Omega}^{NS}(\theta)\right)=0.15;C\left(\Gamma_{\Omega}^{NS}(\theta)\right)=0.58$

In MCDM methods, averaging operators play a crucial role by aggregating evaluations from multiple decision-makers and criteria. Subsequently, we introduce the averaging operators for the Type-NS PNHSS.

Definition 8. For a set of the Type-NS PNHSEs $\bar{\bar{\alpha}}\{\alpha_1,\alpha_2,\ldots,\alpha_h\}$, $\alpha_l=\left\{\langle\Gamma_{\Omega}\left(\theta_{ij}^{l}\right),NS\left(\theta_{j}^{l}\right)\rangle\right\}$; w_l and ξ_j is the respective weights of Type-NS PNHSEs and relationships, where $\sum_{l=1}^{h}w_l=1$ and $\sum_{j=1}^{z}\xi_j=1$; the Type-NS PNHSE weighted averaging and geometric operators, denoted by Type-NS PNHSSWA and Type-NS PNHSSWG respectively, are defined as:

$$NS-PNHSSWA(\alpha_1,\alpha_2,\ldots,\alpha_h)=\overset{l=1}{\underset{h}{\oplus}}w_l\alpha_l$$

$$=\left\{\begin{array}{l}\bigcup_{i=1}^{n}\left\{\left\langle 1-\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(1-T_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l},\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(I_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l},\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(F_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l}\right\rangle\right\}\\ ,\left\langle 1-\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(1-T_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l},\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(I_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l},\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(F_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l}\right\rangle\end{array}\right\} \tag{6}$$

And

$$NS-PNHSSWG(\alpha_1,\alpha_2,\ldots,\alpha_h)=\overset{l=1}{\underset{h}{\otimes}}w_l\alpha_l$$

$$=\left\{\begin{array}{l}\bigcup_{i=1}^{n}\left\{\left\langle\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(T_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l},1-\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(1-I_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l},1-\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(1-F_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l}\right\rangle\right\}\\ ,\left\langle 1-\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(1-T_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l},\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(I_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l},\prod_{l=1}^{h}\left(\prod_{j=1}^{z}\left(F_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_l}\right\rangle\end{array}\right\} \tag{7}$$

Theorem 1. *Let α be a set of Type-NS PNHSEs, then, aggregated values using Type-NS PNHSSWA and Type-NS PNHSSWG is an Type-NS PNHSE.*

Proof. Using the principle of mathematical induction, the Type-NS PNHSSWA operator can be proved by utilizing the following steps

For $h=1$, we have

$$NS-PNHSSWA(\alpha_1)=$$

$$=\left\{\overset{n}{\underset{i=1}{\cup}}\left\{\langle 1-\prod_{j=1}^{z}\left(1-T_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j},\prod_{j=1}^{z}\left(I_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j},\prod_{j=1}^{z}\left(F_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j}\rangle\right\},\langle 1-\prod_{j=1}^{z}\left(1-T_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j},\prod_{j=1}^{z}\left(I_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j},\prod_{j=1}^{z}\left(F_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j}\rangle\right\}$$

$$=\left\{\begin{array}{l}\bigcup_{i=1}^{n}\left\{\left\langle 1-\prod_{l=1}^{1}\left(\prod_{j=1}^{z}\left(1-T_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_1},\prod_{l=1}^{1}\left(\prod_{j=1}^{z}\left(I_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_1},\prod_{l=1}^{1}\left(\prod_{j=1}^{z}\left(F_{ijl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_1}\right\rangle\right\}\\ ,\left\langle 1-\prod_{l=1}^{1}\left(\prod_{j=1}^{z}\left(1-T_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_1},\prod_{l=1}^{1}\left(\prod_{j=1}^{z}\left(I_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_1},\prod_{l=1}^{1}\left(\prod_{j=1}^{z}\left(F_{jl}^{\Gamma_{\Omega}}\right)^{\xi_j}\right)^{w_1}\right\rangle\end{array}\right\}$$

Considering Equation (6) holds for $h=\beta$, such as

$$NS-PNHSSWA\left(\alpha_1,\alpha_2,\ldots,\alpha_{\beta}\right)=\overset{l=1}{\underset{\beta}{\oplus}}w_l\alpha_l$$

$$= \left\{ \begin{array}{l} \bigcup_{i=1}^{n} \left\{ \left\langle 1 - \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(1 - T_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(I_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(F_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l} \right\rangle \right\} \\ , \left\langle 1 - \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(1 - T_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(I_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(F_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l} \right\rangle \end{array} \right\}$$

Let $h = \beta + 1$; $NS - PNHSSWA\left(\alpha_1, \alpha_2, \ldots, \alpha_\beta, \alpha_{\beta+1}\right) = \bigoplus_{\beta+1}^{l=1} w_l \alpha_l = \bigoplus_{\beta}^{l=1} w_l \alpha_l \oplus w_{\beta+1} \alpha_{\beta+1}$

$$= \left\{ \begin{array}{l} \bigcup_{i=1}^{n} \left\{ \left\langle 1 - \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(1 - T_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(I_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(F_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l} \right\rangle \right\} \\ , \left\langle 1 - \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(1 - T_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(I_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta} \left(\prod_{j=1}^{z} \left(F_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l} \right\rangle \end{array} \right\}$$

$$\oplus \left\{ \begin{array}{l} \bigcup_{i=1}^{n} \left\{ \left\langle 1 - \prod_{l=\beta+1}^{\beta+1} \left(\prod_{j=1}^{z} \left(1 - T_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=\beta+1}^{\beta+1} \left(\prod_{j=1}^{z} \left(I_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=\beta+1}^{\beta+1} \left(\prod_{j=1}^{z} \left(F_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l} \right\rangle \right\} \\ , \left\langle 1 - \prod_{l=\beta+1}^{\beta+1} \left(\prod_{j=1}^{z} \left(1 - T_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=\beta+1}^{\beta+1} \left(\prod_{j=1}^{z} \left(I_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=\beta+1}^{\beta+1} \left(\prod_{j=1}^{z} \left(F_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l} \right\rangle \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \bigcup_{i=1}^{n} \left\{ \left\langle 1 - \prod_{l=1}^{\beta+1} \left(\prod_{j=1}^{z} \left(1 - T_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta+1} \left(\prod_{j=1}^{z} \left(I_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta+1} \left(\prod_{j=1}^{z} \left(F_{ijl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l} \right\rangle \right\} \\ , \left\langle 1 - \prod_{l=1}^{\beta+1} \left(\prod_{j=1}^{z} \left(1 - T_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta+1} \left(\prod_{j=1}^{z} \left(I_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{\beta+1} \left(\prod_{j=1}^{z} \left(F_{jl}^{\Gamma_\Omega}\right)^{\xi_j} \right)^{w_l} \right\rangle \end{array} \right\}$$

Theorem 1 is correct in Type-NS PNHSSWA. And Type-NS PNHSSWG is similar proof. Therefore, things proved

4. A MCDM method using possibility neutrosophic hypersoft set

Consider a multi-criteria decision-making situation using Type-NS possibility neutrosophic hypersoft sets (PNHSSs).

Suppose that $\overline{\overline{D}}\{D_1, D_2, \ldots, D_{N_D}\}$; $l = 1, 2, \ldots, N_D$ be the set of N_D decision makers. $\overline{\overline{X}}\{x_p, \ldots, x_{N_X}\}$; $x_p \in X$; $p = 1, 2, \ldots, N_X$ be the set of alternatives and $\overline{\overline{A}}\{A_1, A_2 \ldots, A_{N_A}\}$; $A_q \in \dot{A}\, q = 1, 2, \ldots, N_A$ be the sets of attributes and corresponding attributes values are respectively the set $AV = \{AV_1, \ldots, AV_{N_A}\}$; and can be specified as a set $\Omega = \mathrm{AV}_1 \times \mathrm{AV}_2 \times \mathrm{V}_{\mathrm{N_A}}$; w_l represents weight of decision makers and $\sum_{l=1}^{h} w_l = 1$; ξ_j represents weight of relationships, $\sum_{j=1}^{z} \xi_j = 1$

Figure 1 illustrates the MCDM method based on the theories in the Type-NS Possibility Neutrosophic Hypersoft environment. The detailed model is presented step by step as follows:

Step 1. Construction decision making matrix based on possibility neutrosophic hypersoft numbers (PNHNs), let $\mathrm{N_A}$-tupe-

$\theta_j(AV_1, AV_2, \ldots, AV_{N_A}) \in \Omega; \mathrm{j} = 1, \ldots, \mathrm{N_R}.\left[\Psi^{\mathrm{l}}_{\mathrm{r_{jq}}}\left(\mathrm{x_p}\right)\right]_{\mathrm{N_R} \times \mathrm{N_A}} = \left\{ \left\{ \left\langle \mathrm{T}^{\Gamma_\Omega}_{\mathrm{jql}}\left(\mathrm{x_p}\right), \mathrm{I}^{\Gamma_\Omega}_{\mathrm{jql}}\left(\mathrm{x_p}\right), \mathrm{F}^{\Gamma_\Omega}_{\mathrm{jql}}\left(\mathrm{x_p}\right) \right\rangle \right\}, \left\langle \mathrm{T}^{\mathrm{NS}}_{\mathrm{jl}}\left(\mathrm{x_p}\right), \mathrm{I}^{\mathrm{NS}}_{\mathrm{jl}}\left(\mathrm{x_p}\right), \mathrm{F}^{\mathrm{NS}}_{\mathrm{jl}}\left(\mathrm{x_p}\right) \right\rangle \right\}$

,

Step 2. Determine the aggregated values of alternatives X_p.

(a) Using aggregation operation

$$\overline{X_p} = \left\{ \begin{array}{l} \bigcup_{q=1}^{N_A} \left\{ \left\langle 1 - \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(1 - T^{\Gamma_\Omega}_{qjl}\left(x_p\right)\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(I^{\Gamma_\Omega}_{qjl}\left(x_p\right)\right)^{\xi_j} \right)^{w_l}, \right. \right. \\ \left. \left. \prod_{l=1}^{h} \left(\prod_{j=1}^{z} \left(F^{\Gamma_\Omega}_{qjl}\left(x_p\right)\right)^{\xi_j} \right)^{w_l} \right\rangle \right\}, \\ \left\langle 1 - \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(1 - T^{\Gamma_\Omega}_{jl}\left(x_p\right)\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(I^{\Gamma_\Omega}_{jl}\left(x_p\right)\right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(F^{\Gamma_\Omega}_{jl}\left(x_p\right)\right)^{\xi_j} \right)^{w_l} \right\rangle \end{array} \right\} \quad (8)$$

(b) Using geometric operation

$$\overline{X_p} = \left\{ \begin{array}{l} \bigcup_{q=1}^{N_A} \left\{ \left\langle \begin{array}{l} \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(T_{ijl}^{\Gamma_\Omega}(x_p) \right)^{\xi_j} \right)^{w_l}, 1 - \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(1 - I_{ijl}^{\Gamma_\Omega}(x_p) \right)^{\xi_j} \right)^{w_l}, \\ 1 - \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(1 - F_{ijl}^{\Gamma_\Omega}(x_p) \right)^{\xi_j} \right)^{w_l} \end{array} \right\rangle \right\}, \\ \left\langle 1 - \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(1 - T_{jl}^{\Gamma_\Omega}(x_p) \right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(I_{jl}^{\Gamma_\Omega}(x_p) \right)^{\xi_j} \right)^{w_l}, \prod_{l=1}^{N_D} \left(\prod_{j=1}^{N_R} \left(F_{jl}^{\Gamma_\Omega}(x_p) \right)^{\xi_j} \right)^{w_l} \right\rangle \end{array} \right\} \quad (9)$$

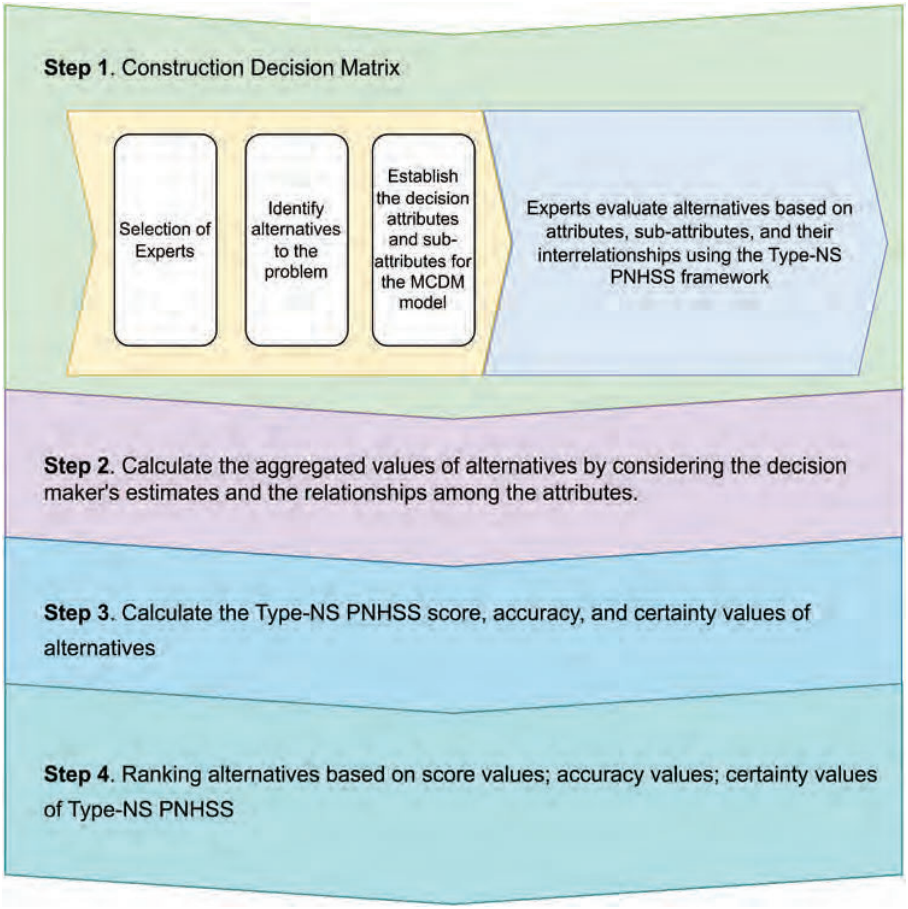


Figure 1. A MCDM method using possibility neutrosophic hypersoft set.

Step 3. Calculate the Type-NS PNHSS score, accuracy, and certainty values using Equations 3-5.

Step 4. Ranking alternatives based on score values; accuracy values; certainty values of Type-NS PNHSS

Computational complexity analysis: from Equations 8-9, we observe that the computational complexity of the averaging operation is $O(3\times N_A\times N_D\times N_R)$. The computational complexities of score, accuracy, and certainty according to Equations 3-5 is $O(N_A)$. Assume that the ranking of alternatives in Step 4 has a complexity of $O(N_X \log N_X)$. From Steps 1–4 of the proposed method, we conclude that the overall computational complexity of the proposed algorithm is $O(3\times N_A\times N_D\times N_R+3^*N_A+ N_X\log N_X)$. In MCDM problems, the number of decision makers, alternatives, and criteria is typically small; thus, the computational complexity can often be approximated as O(1).

5. A case study for the tobacco controls strategy selection

5.1. Problem Description

Smoking stands as the single most significant contributor to the onset of preventable illnesses and the premature demise of individuals worldwide, resulting in countless deaths annually. Furthermore, it serves as one of the foremost factors that exacerbate health disparities, with statistical analyses revealing that smoking rates are disproportionately higher among socially and economically disadvantaged regions. Therefore, strategic approaches to tobacco control are needed because they bring significant benefits to public health, economic stability, and social welfare. This has also prompted the development of innovative decision-making methods for the selection of tobacco-control strategies.

These methods address the intricacies of decision-making in tobacco control strategies and enable a nuanced examination of various factors affecting smoking behavior. This methodology integrates diverse criteria and uncertainties in the assessment and implementation of smoking reduction policies, making it a useful tool for policymakers and health organizations seeking to mitigate smoking-related harms. Details of the attributes and sub-attributes for the tobacco control strategy selection problem, as defined by experts in the tobacco field, are presented in Table 1.

Table 1. The tobacco controls strategy selection problem.

Attributes	Sub-attributes values
Attractiveness - $A1$	Brand exposure – $AV11$
	Brand perception – $AV12$
	Product designs – $AV13$
Affordability - $A2$	Price, current – $AV21$
	Price, change – $AV22$
	Distance to point of sale – $AV23$
Accessibility – $A3$	Easiness to locate point of sale – $AV31$
	Distance to point of sale – $AV32$
	Restrictions at point of sale – $AV33$
Awareness – $A4$	Understand about harms – $AV41$
	Understand about responsibility – $AV42$
	Understand about alternatives – $AV43$
Environment – $A5$	Social stigma – $AV51$
	Peer pressure – $AV52$
	Family influence – $AV53$

5.2. Experiment results

During the experimental phase, we used Python 3.8 and the NumPy 2.2.0 library to build the computational program for the proposed decision-making process for the tobacco control strategies problem.

Step 1. Based on the opinions of experts D_1 and D_2 in the field of tobacco control, determine 10 relationships R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10} through the sub-attributes values in Table 1 to evaluate for 8 strategies S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8 in tobacco control problems, corresponding, tax and price, smoke free air, mass media communications, marketing restrictions, health warnings, cessation treatment, youth access restrictions, retail management. Then, a decision-making matrix based on Type-NS PNHSS numbers was estimated by experts. The two experts evaluate 08 strategies based on attributes, sub-attributes, and their interrelationships using the Type-NS PNHSS framework. Table 2-3 shows the estimated values of D_1 and D_2 for strategies through 10 relationships.

Table 2. Evaluate D_1 by type-NS PNHSS number.

	S_1	S_2	S_3	S_4
R_1	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$
R_2	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle, \\ \langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$
R_3	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$
R_4	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$
R_5	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle, \\ \langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$
R_6	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$
R_7	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle\},\langle .3,.6,.7\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$
R_8	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$
R_9	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle, \\ \langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$
R_{10}	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle\},\langle .3,.6,.7\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .3,.6,.7\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle, \\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .3,.6,.7\rangle \end{matrix} \right\rangle$
	S_5	S_6	S_7	S_8
R_1	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$
R_2	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .3,.6,.7\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$
R_3	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$
R_4	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .3,.6,.7\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$
R_5	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$
R_6	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$
R_7	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .3.6,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$
R_8	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$
R_9	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle\},\langle .4,.5,.6\rangle \end{matrix} \right\rangle$
R_{10}	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle, \\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .3,.6,.7\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .3,.6,.7\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,0.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle \end{matrix} \right\rangle$	$\left\langle \begin{matrix} \{\langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle, \\ \langle .4,.5,.6\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle \end{matrix} \right\rangle$

Table 3. Evaluate D_2 by NS-PNHSS number.

	S_1	S_2	S_3	S_4
R_1	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .3,.6,.7\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$
R_2	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\\ \langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .6,.5,.4\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$
R_3	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\\ \langle .3,.6,.7\rangle,\langle .2,.5,.7\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$
R_4	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\\ \langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$
R_5	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$
R_6	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle\},\langle .3,.6,.7\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .3,.6,.7\rangle\end{matrix} \right\rangle\}$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .3,.6,.7\rangle\},\langle .3,.6,.7\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .6,.5,.4\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .3,.6,.7\rangle\end{matrix} \right\rangle$
R_7	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .6,.5,.4\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$
R_8	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\\ \langle .3,.6,.7\rangle,\langle .2,.5,.7\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$
R_9	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\\ \langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .6,.5,.4\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$
R_{10}	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .8,.3,.3\rangle,\langle .2,.5,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .3,.6,.7\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$
	S_5	S_6	S_7	S_8
R_1	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$
R_2	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,07\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\\ \langle .3,.6,.7\rangle,\langle .2,.5,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$
R_3	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .6,.5,.4\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\langle .4,.5,.6\rangle,\\ \langle .3,.6,.7\rangle,\langle .2,.5,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$
R_4	$\left\langle \begin{matrix}\{\langle .6,.5,.4\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .3,.6,.7\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\\ \langle .3,.6,.7\rangle,\langle .2,.5,.7\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$
R_5	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .2,.5,.7\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle,\\ \langle .3,.6,.7\rangle,\langle .2,.5,.7\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$
R_6	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle\},\langle .3,.6,.7\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle\},\langle .3,.6,.7\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .3,.6,.7\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .4,.5,.6\rangle,\langle .3,.6,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$
R_7	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .8,.3,.3\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .6,.5,.4\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\langle .4,.5,.6\rangle,\\ \langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$
R_8	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .8,.3,.3\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .6,.5,.4\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\langle .4,.5,.6\rangle,\\ \langle .3,.6,.7\rangle,\langle .2,.5,.7\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$
R_9	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .4,.5,.6\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle,\\ \langle .3,.6,.7\rangle,\langle .2,.5,.7\rangle\},\langle .8,.3,.3\rangle\end{matrix} \right\rangle$
R_{10}	$\left\langle \begin{matrix}\{\langle .6,.5,.4\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .3,.6,.7\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\\ \langle .8,.3,.3\rangle,\langle .6,.5,.4\rangle\},\langle .3,.6,.7\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\\ \langle .6,.5,.4\rangle,\langle .6,.5,.4\rangle\},\langle .4,.5,.6\rangle\end{matrix} \right\rangle$	$\left\langle \begin{matrix}\{\langle .2,.5,.7\rangle,\langle .3,.6,.7\rangle,\langle .3,.6,.7\rangle,\\ \langle .4,.5,.6\rangle,\langle .2,.5,.7\rangle\},\langle .6,.5,.4\rangle\end{matrix} \right\rangle$

Step 2. Within the case-study framework, the expert's weights and inter-attribute relationships were assumed equal. Using Equation (8) to calculate the average among strategies, Table 4 shows the aggregation values of 8 tobacco control strategies based on the Type-NS PNHSS.

Step 3. Calculate the aggregation of strategies based on Equations 9–10. Score, Accuracy and Certainty values of strategies for the solving of selection strategies in tobacco control is shown in Table 5.

Step 4. From Table 5 and Figure 2, the ranking of strategies is determined by their score, accuracy, and certainty values, yielding the following order: $S_3 \succ S_8 \succ S_7 \succ S_4 \succ S_6 \succ S_5 \succ S_2 \succ S_1$ and S_3 is the best strategy.

Table 4. Aggregated values of strategies.

Strategies	Aggregated values
S_1	$\left\langle \begin{matrix}\{\langle 0.5,0.5,0.5\rangle,\langle 0.8,0.3,0.3\rangle,\langle 0.4,0.5,0.56\rangle,\\ \langle 0.38,0.52,0.62\rangle,\langle 0.46,0.53,0.53\rangle\},\langle 0.52,0.5,0.49\rangle\end{matrix} \right\rangle$
S_2	$\left\langle \begin{matrix}\{\langle 0.31,0.5,0.65\rangle,\langle 0.25,0.55,0.7\rangle,\langle 0.27,0.53,0.68\rangle,\\ \langle 0.59,0.5,0.41\rangle,\langle 0.7,0.39,0.37\rangle\},\langle 0.63,0.44,0.42\rangle\end{matrix} \right\rangle$
S_3	$\left\langle \begin{matrix}\{\langle 0.77,0.32,0.33\rangle,\langle 0.43,0.5,0.53\rangle,\langle 0.31,0.5,0.65\rangle,\\ \langle 0.65,0.45,0.38\rangle,\langle 0.47,0.53,0.53\rangle\},\langle 0.63,0.44,0.41\rangle\end{matrix} \right\rangle$
S_4	$\left\langle \begin{matrix}\{\langle 0.64,0.45,0.39\rangle,\langle 0.31,0.5,0.65\rangle,\langle 0.35,0.5,0.61\rangle,\\ \langle 0.59,0.5,0.41\rangle,\langle 0.44,0.53,0.56\rangle\},\langle 0.63,0.44,0.42\rangle\end{matrix} \right\rangle$
S_5	$\left\langle \begin{matrix}\{\langle 0.56,0.48,0.46\rangle,\langle 0.31,0.5,0.65\rangle,\langle 0.22,0.51,0.7\rangle,\\ \langle 0.64,0.45,0.39\rangle,\langle 0.48,0.48,0.55\rangle\},\langle 0.63,0.44,0.42\rangle\end{matrix} \right\rangle$
S_6	$\left\langle \begin{matrix}\{\langle 0.64,0.39,0.43\rangle,\langle 0.43,0.5,0.53\rangle,\langle 0.5,0.52,0.49\rangle,\\ \langle 0.71,0.39,0.36\rangle,\langle 0.69,0.39,0.38\rangle\},\langle 0.54,0.49,0.48\rangle\end{matrix} \right\rangle$
S_7	$\left\langle \begin{matrix}\{\langle 0.64,0.39,0.43\rangle,\langle 0.43,0.5,0.53\rangle,\langle 0.5,0.52,0.49\rangle,\\ \langle 0.71,0.39,0.36\rangle,\langle 0.69,0.39,0.38\rangle\},\langle 0.54,0.49,0.48\rangle\end{matrix} \right\rangle$
S_8	$\left\langle \begin{matrix}\{\langle 0.6,0.39,0.46\rangle,\langle 0.48,0.53,0.52\rangle,\langle 0.48,0.53,0.52\rangle,\\ \langle 0.37,0.53,0.63\rangle,\langle 0.31,0.5,0.65\rangle\},\langle 0.69,0.4,0.37\rangle\end{matrix} \right\rangle$

5.3. Comparison analysis

This section will show the influence of the proposed model by comparing its performance with that of Zhao et al (Zhao et al., 2023). Type-NS PNHSS's datasets are transformed to

Table 5. Score, accuracy, certainty values.

Strategies	Score	Accuracy	Certainty	Ranking
S_1	0.509	0.013	0.511	8
S_2	0.524	0.035	0.526	7
S_3	0.562	0.134	0.581	1
S_4	0.536	0.074	0.546	4
S_5	0.530	0.050	0.534	6
S_6	0.535	0.063	0.543	5
S_7	0.550	0.112	0.569	3
S_8	0.553	0.105	0.569	2

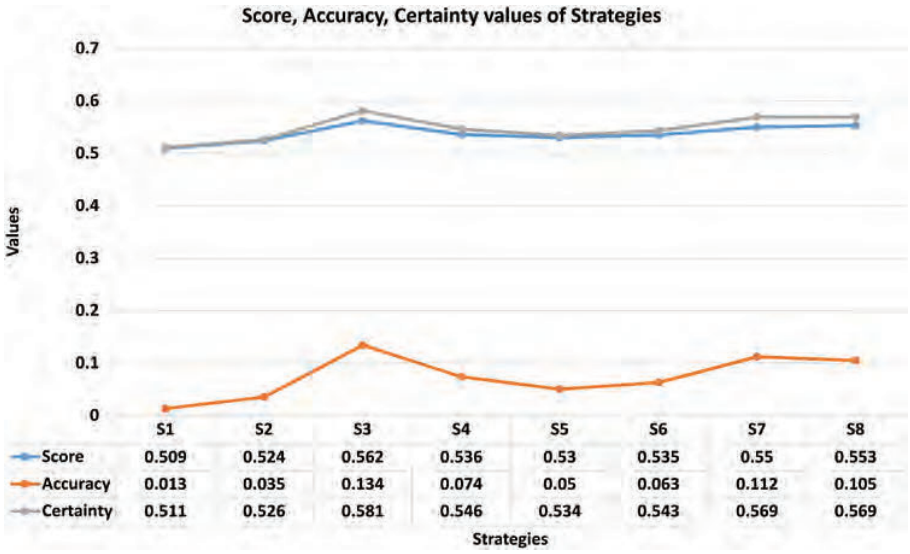


Figure 2. Score, accuracy, Certainy values of strategies.

Table 6. Values of Zhao et al.'s method.

Strategies	Min.	Max.	Scoring	Ranking
S_1	0.00695	1.40029	1.40724	3
S_2	0.00387	1.29465	1.29852	7
S_3	0.00740	1.40029	1.40769	2
S_4	0.00100	1.44000	1.44100	1
S_5	0.00663	1.38706	1.39369	4
S_6	0.00077	1.35835	1.35912	5
S_7	0.00166	1.30119	1.30285	6
S_8	0.00740	1.17947	1.18687	8

the PNHSS environment following the approach of Zhao et al. Then the measures are calculated and shown in Table 6. Results ranking of strategies the corresponding ranking is $S_4 \succ S_3 \succ S_1 \succ S_5 \succ S_6 \succ S_7 \succ S_2 \succ S_8$ and S_4 is the best strategy for solving the selection of tobacco control strategies.

The result of proposed method gives the ranking $S_3 \succ S_8 \succ S_7 \succ S_4 \succ S_6 \succ S_5 \succ S_2 \succ S_1$ and S_3 is the best strategy. This particular outcome demonstrates the substantial influence that the proposed theoretical framework has a strong impact on the ranking of strategies within the relevant environments. Furthermore, it is evident that the proposed theory presents a more relevant and appropriate set of information that aligns with the actual conditions and dynamics observed in the Type-NS PNHSS environment, thereby enhancing its applicability and effectiveness.

To underscore the advances introduced by our framework, we systematically evaluated it against state-of-the-art methods in hypersoft set theory, focusing on three aspects (1) possibility settings, (2) sub-attribute handling, and (3) neutrosophic components (truthiness/indeterminacy/falsity). Table 7 shows that, whereas existing approaches rely on fuzzy numbers, our model employs neutrosophic numbers for possibility degrees, thereby providing a superior uncertainty representation in real-world contexts. Additionally, it addresses complex information structures inherent to neutrosophic hypersoft environments, a capability not reported in prior works.

6. Conclusion

In this study, we proposed the concept of Type-NS Possibility Neutrosophic Hypersoft Set (Type-NS PNHSS) to effectively manage uncertain, indeterminate, and often inconsistent information across multiple attributes simultaneously, which synthesizes Possibility theory with the Neutrosophic hypersoft set. Subsequently, we established a crucial foundational theory within this domain, encompassing fundamental operations, measures, and aggregation operators for the Type-NS PNHSS. Following this, we developed a multi-criteria decision-making model based on the suggested theory to address the decision-making challenges inherent in the Type-NS PNHSS environment. Finally, a practical case, specifically the strategy selection dilemma concerning tobacco control strategies, was employed to demonstrate the efficacy and advantages of the theory and methodology proposed in this research. However, this study has certain limitations. The research focuses primarily on refining the mathematical framework for the Type-NS PNHSS, developing a decision-making model based on the Type-NS PNHSS theory, and applying it to a real-life case study. Several critical aspects were not addressed, including sampling techniques, statistical analyses, control mechanisms, statistical reporting, and the model's evaluation

Table 7. Comparison with other methods.

Methods	Possibility setting (mathematical space)	Sub attributes	Truthiness	Indeterminacy	Falsity
Palanikumar & Arulmozhi, (2021)	Fuzzy number (Neutrosophic soft set)	x	✓	✓	✓
Rahman, Saeed, & Garg, (2022)	Fuzzy number (fuzzy hypersoft set)	✓	✓	x	x
Saeed, Wahab, et al., (2023)	Fuzzy number (q-rung ortho-pair fuzzy hypersoft set)	✓	✓	x	x
Al-Hijjawi & Alkhazaleh, (2023)	Fuzzy number (single-valued neutrosophic hypersoft set)	✓	✓	✓	✓
Alballa et al., (2024)	Fuzzy number (single-valued neutrosophic hypersoft set)	✓	✓	✓	✓
Proposed method	Neutrosophic number (Neutrosophic hypersoft set)	✓	✓	✓	✓

metrics. Additionally, the effects of key parameter settings, such as possibility weights or neutrosophic parameters, were not examined or analyzed in this study. Furthermore, extensions of the framework to established decision-making methods (such as TOPSIS, VIKOR, etc.) in this context remain to be explored. In future work, we aim to address these gaps by integrating statistical validation, conducting sensitivity analyses, and extending the model to incorporate advanced multi-criteria decision-making techniques for broader applicability. Additionally, we plan to explore the model's potential in diverse real-world contexts, such as urban planning, healthcare, and sustainable resource management.

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Data availability statement

The data used to support the findings of this study are included in this article. However, the reader may contact the corresponding author for more details on the data.

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A Hybrid IF-FUCOM-GRA Approach and its Application to Determine Optimal Bacterial Concentrations on Mortar at Optimal Curing Day

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Abstract. In this study, Intuitionistic Fuzzy Consistency Method (IF-FUCOM) and Grey Relation Analysis (GRA) were combined to assess the effects of Bacillus subtilis bacteria on concrete properties, as well as to determine the optimal bacteria concentration and curing day. Three different concentrations of bacteria were added to the mortar mixes, like 103, 105, and 107 cells/ml of water. Mortar samples were left to cure for 7 days, 14 days, and 28 days to evaluate compressive strength, water absorption, crack healing. According to the proposed algorithm, 105 bacteria are the optimal concentration, while 28 days is the ideal curing time.

Key words: IF-FUCOM, GRA, IF-FUCOM-GRA, bacterial concentrations,, curing day.

1. Introduction

Structures must become stronger, faster, and more versatile, as well as more durable, with a huge increase in the amount of cement used in the process. Most construction projects today use Portland cement concrete, which is the predominant type of concrete. Because of the low cost of construction materials and the ease of maintenance, concrete structures can be built and maintained.

Recent research found that a biomaterial can be used to treat concrete cracks (Van Tittelboom *et al.*, 2010; Ramachandran *et al.*, 2001). Scientists have discovered that inorganic substances that are deposited by microorganisms inside cement-sand mortar or the pores of concrete can be used for filling cracks (Ghosh *et al.*, 2005; Ramakrishnan *et al.*, 1999). A concrete structure's inherent weakness is its vulnerability to cracks that allow water to penetrate, causing corrosion and reducing its durability (Chahal *et al.*, 2012).

Ramachandran *et al.* (2001) pioneered microbial concrete, and since then there has been a considerable volume of research on the topic. As ureolytic bacteria are alkali-resistant and nutrition is not necessary for survival for hundreds of years, the researchers examined Bacillus sphaericus, Sporosarcina pasteurii, and Bacillus megaterium (Arunachalam *et al.*, 2010; Dhami *et al.*, 2013; Achal *et al.*, 2011). In some studies, researchers examined the effects of adding bacteria to concrete on its compressive strength and crack healing. The findings reflect that most of them considered bacterial concentrations between 103 to 107 cells/ml when considering strength enhancement. Contrary to crack healing, researchers use higher concentrations of bacterial cells (107–109 cells/ml) (Majumdar *et al.*, 2012; Mondal and Ghosh, 2018; De Muynck *et al.*, 2008).

The versatility of concrete makes it a popular choice for building materials. Locally available, strong and durable, it is versatile. Despite its capability to resist compression loads to a limit, if the load applied on the concrete exceeds their limit of load resistance, it results in cracks in the concrete, which lowers its strength. Concrete's serviceability limit is affected by cracks. Concrete may become weaker and less durable as moisture and other chemicals get into it. In addition to that, water absorption is another major issue that reduces the life of concrete. Researchers are currently using bacteria to treat concrete mortar to overcome the problems. The selection of an optimal bacteria concentration and curing day can also pose a problem. Grey Relational Analysis (GRA) can be used in this field to find an optimal solution, since various researchers use it in different fields as an optimization technique (Dagdevir and Ozceyhan, 2021; Güler *et al.*, 2021; Roy *et al.*, 2016; Si *et al.*, 2021). A major drawback of GRA is that it assigns a similar weight to all output characteristics, even though in practice not all output characteristics are equally important (Fangfang, 2021). To overcome the problem, some researchers are using Analytical Hierarchy Process (AHP) along with GRA (Erdoğan and Sayin, 2018; Erdoğan *et al.*, 2020). An advantage, as well as a reason for using the AHP method, is that results can be validated by determining the consistency of the model with actual data. A study suggests that comparing pairwise across nine criteria by the AHP method is extremely difficult since it requires a great deal of comparisons $n(n-1)/2$ (Milićević *et al.*, 2007).

The BWM has been shown to be able to resolve certain of the previously listed constraints associated with AHP models (Rezaei, 2015). Compared to AHP's many pairwise comparisons, BWM does only a small number, such as $2n-3$. There is a direct correlation between the number of pairwise criteria comparisons and the consistency of the method. Moreover, the BWM does not require a comparison of nine criteria, merely a smaller number of criteria. The AHP model is improved by forming the Best-to-Others (BO) as well as Other-to-Worst (OW) vectors, resulting in fewer pairwise comparisons, and the resulting data are more consistent. The BWM, however, has a problem in determining the optimum weight coefficients when there is a large degree of variation inconsistency. The weight coefficient can be determined by using the average of the intervals as final values in such cases, as proposed by Rezaei (2015). Despite this, the central part of the interval is not guaranteed to be representative of the optimal weight coefficient value. A better value might lie closer to the right or left end of the interval. The interval weight values do not even cover the optimum values of priority coefficients in the cases of the greater inconsistency of results (Pamučar *et al.*, 2018a).

FUCOM uses pairwise comparisons of criteria to determine criteria priority, and it validates results across a wide range of deviations from maximum consistency in order to determine criteria priority (Pamučar *et al.*, 2018b). As compared to BWM and AHP tools, FUCOM eliminates some of their weaknesses. When using FUCOM, criteria can be compared in pairs ($n - 1$ comparisons), DMC (Deviation from the Maximum Consistency) can be calculated when comparing comparisons, and transitivity can be recognized throughout pairwise comparisons. There is a subjective effect of DMs on the weighting of criteria in FUCOM, as there is in other subjective models. As such, this refers specifically to the first, as well as second steps of the FUCOM. The FUCOM, in contrast to subjective models, shows minor deviations from the optimum value in the priority value of the criteria. In addition, the FUCOM methodological procedure removes the redundancy caused by comparing criteria pairwise, which is a problem with some subjective methods for priority value determination (Božanić *et al.*, 2019; Bozanic *et al.*, 2020; Durmić *et al.*, 2020). In recent years, FOCUM is being combined with other methods by many researchers to solve problems. For the purpose of selecting the appropriate combination of construction machines to enable mobility, Darko Boana *et al.* used a hybrid model of FUCOM and fuzzified RAFSI (Božanić *et al.*, 2021). Nunić (2018) applies a hybrid model of FOCUM-MABAC for evaluating and selecting PVC carpentry manufacturers. Real-world decision-makers often use linguistic variables instead of crisp values to evaluate attributes when they have partial knowledge or little information. Decision-makers are often left with ambiguous, imprecise, or incomplete attribute information as a result of such situations. Inaccuracies such as these can be mathematically represented by fuzzy set theory, introduced by Zadeh (1975). Since they were created, fuzzy sets have been successfully used to model MCDM problems with imprecise information. A fuzzy full consistency MCDM method was presented by Pamucar and Ecer (2020). In a hybrid model used by Baig *et al.* (2022) to enhance the resilience of oil supply chains, FOCUM prioritizes vulnerabilities while Fuzzy Quality Function Deployment identifies those capabilities that can ensure their protection. As part of the sustainability plan for urban mobility, Demir *et al.* (2022) used Fuzzy-FOCUM. A hybrid fuzzy FUCOM and neutrosophic fuzzy MARCOS methodology was used to assess alternative fuel vehicles for sustainable road transportation in the United States by Pamucar *et al.* (2021). A fuzzy-focus approach was used by Tang *et al.* (2021) for prioritizing sustainability scenarios for sewage sludge. To determine the drivers for investing in cryptocurrencies, Böyükaslan and Ecer used Fuzzy FUCOM in 2021.

The drawback of fuzzy sets is that, in some circumstances, it can be quite challenging to determine a precise membership mapping for a fuzzy set (Chiao, 2016). An intuitionistic fuzzy set (IFS), which Krassimir and Parvathi proposed in 1986, is a generalized fuzzy set that considers membership and non-membership degrees, as well as hesitation degrees. IFS can handle ambiguous information in a flexible manner (Gong *et al.*, 2014). As a result, specialists have been paying more and more attention to the IFS, which is now being used in many other domains, including decision-making (Gong *et al.*, 2014). This study addresses a vacuum in the literature, since, as far as the authors are aware, the FUCOM has not yet been used to intuitionistic contexts. In fact, extending FUCOM's research to the intuitionistic fuzzy environment is motivated in part by this.

In this study, Section 2 discusses IF-FUCOM-GRA, while Section 3 discusses step-by-step methodology with experimental details and results. Results of each method listed in Section 3 are presented in Section 4 in a step-by-step manner. Sections 5 and 6 represent the discussion and conclusion sections, respectively.

1.1 Motivation of the work:

I. However, despite the fact that many researchers have studied the effects of Bacillus subtilis bacteria on concrete properties, no studies have evaluated the optimal concentration of bacteria as per the above discussed literature. Therefore, the purpose of this study is to determine the optimal bacteria concentration as well as the effect of bacteria on concrete mortar.

II. Intuitionistic Fuzzy FUCOM Grey Relations Analysis has never been used to determine the optimal value in such an environment.

1.2 Novelty of the work:

I. In this paper, IF-FUCOM is developed that can be used to better define the weight coefficients of criteria.

II. A detailed algorithm is used in this study to calculate the weights of criteria in the intuitionistic fuzzy environment.

III. A new model for dealing with uncertainty bridges the gap between criteria weight coefficients and intuitionistic fuzzy numbers.

IV. In order to improve the methodology, a hybrid IF-FUCOM-GRA method has been proposed. It combines novel IF-FUCOM and existing GRA techniques.

V. In this study, the optimal bacteria concentration and curing day for concrete is determined based on its compressive strength, crack healing, and water absorption. The novel IF-FUCOM-GRA method is used to select the perfect bacteria and cure day.

2. Intuitionistic Fuzzy Full Consistency Method Grey Relational Analysis (IF-FUCOM-GRA)

IF-FUCOM-AHP has two phases, IF-FUCOM and IF-AHP, which are discussed respectively in Phases I and II. Phase I and Phase II discussed how to analyse criteria and alternatives to determine the priority value of criteria and alternatives. Figure 1 depicts the proposed method's computational procedure. Figure 1 illustrates how the method is computed.

Phase-1: Intuitionistic Fuzzy Full Consistency Method (IF-FUCOM):

In order to determine the priority value of criteria, FUCOM is used. It is proposed that a modified fuzzy FUCOM approach is used in the current study called Intuitionistic Fuzzy Full Consistency Method (IF-FUCOM) to find the priority values of each criterion. Five steps make up IF-FUCOM. Following are the steps:

Step-I: Identify the assessment criteria: This consists of n $(r = 1(1)q)$ decision criteria, which are represented by $\Omega = \{\xi_r : r = 1(1)q\}$.

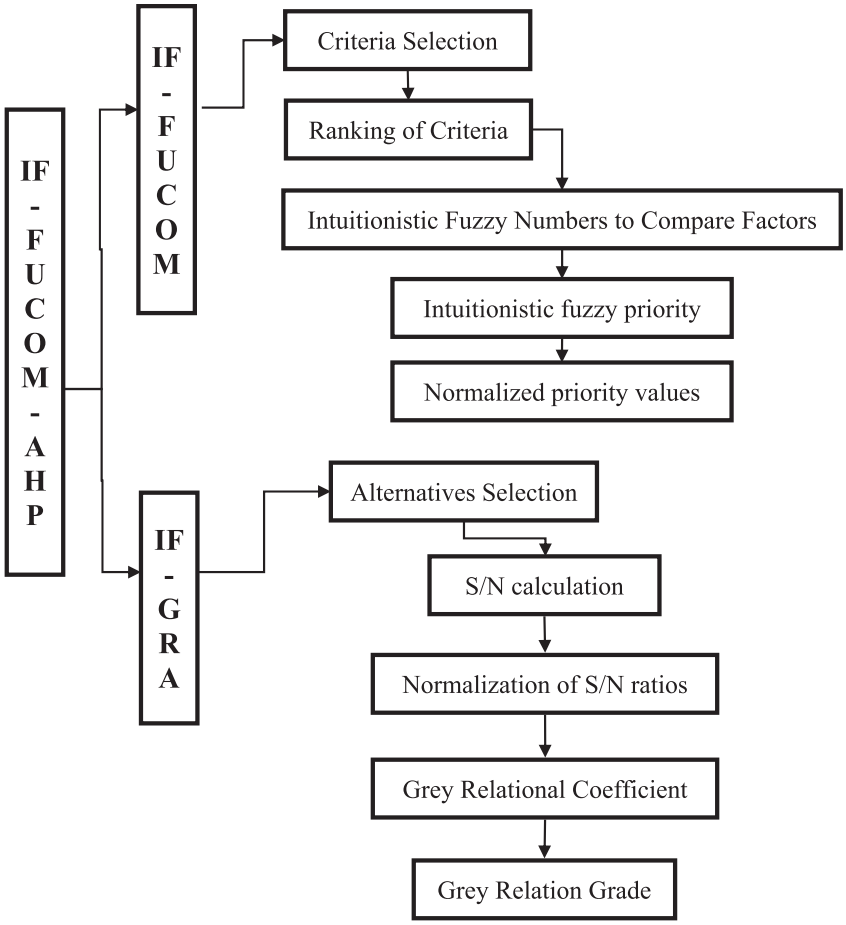


Fig. 1. Total scenario of proposed method.

Step-II: Determine the ranking of factors: The DMs determine the order of importance of factors based on their opinions. A factor is ranked in ascending order by the weight coefficient that will be assigned to it first, and so on, down to the least significant factor in the equation. The factor whose weight coefficient is expected to be the lowest is ranked last. In the resulting ranking system, $\xi_{r(1)} \succ \xi_{r(2)} \succ \cdots \succ \xi_{r(l)}$ represents the factor rank, where l represents the criterion ranking. A sign of equality replaces "$\succ$" between two or more factors that have the same ranking.

Step-III: Use intuitionistic fuzzy numbers to compare factors: Table 1 is used to compare factors. The factors are compared according to the first ranking factor. The Intuitionistic fuzzy criterion meaning ($\tilde{p}_{\xi_{r(l)}}$) is then determined for all the factors. In order to compare the remaining factors with the most important factor, a $(q-1)$ comparison is a necessity. A fuzzy Intuitionistic significance $\tilde{\wp}_{l/(l+1)}$ is derived from equation (1) by applying the defined significance of factors.

$$\tilde{\wp}_{l/(l+1)} = \frac{\tilde{p}_{\xi_{r(l+1)}}}{\tilde{p}_{\xi_{r(l)}}} = \frac{(p^{l}_{\xi_{r(l+1)}}, p^{m}_{\xi_{r(l+1)}}, p^{u}_{\xi_{r(l+1)}}; p'^{l}_{\xi_{r(l+1)}}, p'^{m}_{\xi_{r(l+1)}}, p'^{u}_{\xi_{r(l+1)}})}{(p^{l}_{\xi_{r(l)}}, p^{m}_{\xi_{r(l)}}, p^{u}_{\xi_{r(l)}}; p'^{l}_{\xi_{r(l)}}, p'^{m}_{\xi_{r(l)}}, p'^{u}_{\xi_{r(l)}})}. \tag{1}$$

Table 1
Nine-point triangular intuitionistic fuzzy scale (Otay *et al.*, 2017).

Definition	Intensity of importance $(\delta^l, \delta^m, \delta^u; \delta'^l, \delta'^m, \delta'^u)$	Reverse of intensity importance $(\frac{1}{\delta^u}, \frac{1}{\delta^m}, \frac{1}{\delta^l}; \frac{1}{\delta'^u}, \frac{1}{\delta'^m}, \frac{1}{\delta'^l})$	S.I.
EI	(1, 1, 1; 1, 1, 1)		1
AI	(8, 9, 9; 7, 9, 9)		9
MI	$(\mu-1, \mu, \mu+1; \mu-2, \mu, \mu+2)$	$(\frac{1}{\mu+1}, \frac{1}{\mu}, \frac{1}{\mu-1}; \frac{1}{\mu+2}, \frac{1}{\mu}, \frac{1}{\mu-2})$	$\mu = 3$
STI			$\mu = 5$
VSI			$\mu = 7$
Intermediate scale			$\mu = 2, 4, 6, 8$

Equation (2) provides an Intuitionistic fuzzy vector of the relative importance of the decision factors:

$$\wp = (\tilde{\wp}_{1/2}, \tilde{\wp}_{2/3}, \ldots, \tilde{\wp}_{l/(l+1)}). \tag{2}$$

Based on the factor of $\xi_{r(l+1)}$ rank, $\tilde{\wp}_{l/(l+1)}$ represents the importance that the factor of $\xi_{r(l)}$ rank possesses.

Step-IV: Calculate intuitionistic fuzzy priority: here, the Intuitionistic fuzzy priority value coefficients are calculated for factor $(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_q)^T$. As a final priority coefficient value, the following conditions must be met:

Condition 1: The weight coefficient ratio between the observed factors ($\xi_{r(l)}$ and $\xi_{r(l+1)}$) should equal the significance ratio between them ($\tilde{\wp}_{l/(l+1)}$) defined in Step II; in other words, it should satisfy:

$$\frac{\tilde{p}_l}{\tilde{p}_{l+1}} = \tilde{\wp}_{l/(l+1)}. \tag{3}$$

Condition 2: Besides satisfying the condition in expression (3), the coefficients of weights should also qualify as transitive, i.e.

$$\tilde{\wp}_{l/(l+1)} \otimes \tilde{\wp}_{(l+1)/(l+2)} = \tilde{\wp}_{l/(l+2)}, \quad \text{i.e. } \frac{\tilde{p}_l}{\tilde{p}_{l+1}} \otimes \frac{\tilde{p}_{l+1}}{\tilde{p}_{l+2}} = \frac{\tilde{p}_l}{\tilde{p}_{l+2}}.$$

It is also necessary for the final weight coefficient values to satisfy the following condition:

$$\frac{\tilde{p}_l}{\tilde{p}_{l+2}} = \tilde{\wp}_{l/(l+1)} \otimes \tilde{\wp}_{(l+1)/(l+2)}. \tag{4}$$

DMC minimum, i.e. $\nu = 0$, can only be satisfied if there is complete transitivity among priority coefficients. Then, it can be said that $\frac{\tilde{p}_l}{\tilde{p}_{l+1}} - \tilde{\wp}_{l/(l+1)} = 0$ and $\frac{\tilde{p}_l}{\tilde{p}_{l+2}} - \tilde{\wp}_{l/(l+1)} \otimes \tilde{\wp}_{(l+1)/(l+2)} = 0$. Accordingly, DMC is $\nu = 0$, when such coefficients are obtained. To satisfy these conditions, the weight coefficients for each criterion $(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_q)^T$ must satisfy the condition that $\left|\frac{\tilde{p}_l}{\tilde{p}_{l+1}} - \tilde{\wp}_{l/(l+1)}\right| \leqslant \nu$ and $\left|\frac{\tilde{p}_l}{\tilde{p}_{l+2}} - \tilde{\wp}_{l/(l+1)} \otimes \tilde{\wp}_{(l+1)/(l+2)}\right| \leqslant \nu$ minimize the value ν.

The final nonlinear model for computing the ideal Intuitionistic fuzzy values of the relative weights of each factor can then be set to $(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_q)^T$.

$$\begin{aligned}
&\text{Min } \nu \\
&\text{s.t.} \\
&\begin{cases}
\left|\dfrac{\tilde{p}_l}{\tilde{p}_{l+1}} - \tilde{\wp}_{l/(l+1)}\right| \leqslant \nu, & \text{for all } r = 1(1)q, \\
\left|\dfrac{\tilde{p}_l}{\tilde{p}_{l+2}} - \tilde{\wp}_{l/(l+1)} \otimes \tilde{\wp}_{(l+1)/(l+2)}\right| \leqslant \nu, & \text{for all } r = 1(1)q, \\
\displaystyle\sum_{r=1}^{q} \tilde{p}_r = 1, & \\
0 \leqslant p_r'^l \leqslant p_r^l \leqslant p_r^m = p_r'^m \leqslant p_r^u \leqslant p_r'^u, & \text{for all } r = 1(1)q,
\end{cases}
\end{aligned} \tag{5}$$

where $\tilde{p}_r = (p_r^l, p_r^m, p_r^u; p_r'^l, p_r'^m, p_r'^u)$ and $\tilde{\wp}_{l/(l+1)} = (\wp_{l/(l+1)}^l, \wp_{l/(l+1)}^m, \wp_{l/(l+1)}^u; \wp_{l/(l+1)}'^l, \wp_{l/(l+1)}'^m, \wp_{l/(l+1)}'^u)$.

The highest consistency can only be obtained by following the condition that $\frac{\tilde{p}_l}{\tilde{p}_{l+1}} - \tilde{\wp}_{l/(l+1)} = 0$ and $\frac{\tilde{p}_l}{\tilde{p}_{l+2}} - \tilde{\wp}_{l/(l+1)} \otimes \tilde{\wp}_{(l+1)/(l+2)} = 0$ are both met. In this way, the model (5) can be re-formulated into an Intuitionistic fuzzy nonlinear model (6). Intuitionistic fuzzy priority value coefficients are obtained $(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_q)^T$, if this problem is solved.

$$\begin{aligned}
&\text{Min } \nu \\
&\text{s.t.} \\
&\begin{cases}
|\tilde{p}_l - \tilde{p}_{l+1} \otimes \tilde{\wp}_{l/(l+1)}| \leqslant \nu, & \text{for all } r = 1(1)q, \\
|\tilde{p}_l - \tilde{p}_{l+2} \otimes \tilde{\wp}_{l/(l+1)} \otimes \tilde{\wp}_{(l+1)/(l+2)}| \leqslant \nu, & \text{for all } r = 1(1)q, \\
\displaystyle\sum_{r=1}^{q} \tilde{p}_r = 1, & \\
0 \leqslant p_r'^l \leqslant p_r^l \leqslant p_r^m = p_r'^m \leqslant p_r^u \leqslant p_r'^u, & \text{for all } r = 1(1)q,
\end{cases}
\end{aligned} \tag{6}$$

where $\tilde{p}_j = (p_j^l, p_j^m, p_j^u; p_j'^l, p_j'^m, p_j'^u)$ and $\tilde{\wp}_{l/(l+1)} = (\wp_{l/(l+1)}^l, \wp_{l/(l+1)}^m, \wp_{l/(l+1)}^u; \wp_{l/(l+1)}'^l, \wp_{l/(l+1)}'^m, \wp_{l/(l+1)}'^u)$.

Convert optimal Intuitionistic fuzzy priority value $(\tilde{p}_1^*, \tilde{p}_2^*, \ldots, \tilde{p}_q^*)$, where $\tilde{p}_r^* = (p_r^{l*}, p_r^{m*}, p_r^{u*}; p_r'^{l*}, p_r'^{m*}, p_r'^{u*})$, for all $r = 1(1)q$ into crisp value using the formula (7):

$$R\left(\tilde{p}_r^*\right) = \left[\frac{(p_r^{l*} + 2p_r^{m*} + p_r^{u*}) + (p_r'^{l*} + 2p_r'^{m*} + p_r'^{u*})}{8}\right], \quad \text{for all } r = 1(1)q. \tag{7}$$

Step-V: Normalized priority values: equation (8) is used to calculate the normalized priority values of criteria.

$$\tilde{w}_r^* = \frac{R(\tilde{p}_r^*)}{\sum_{r=1}^{q} R(\tilde{p}_r^*)}, \quad \forall r = 1(1)q. \tag{8}$$

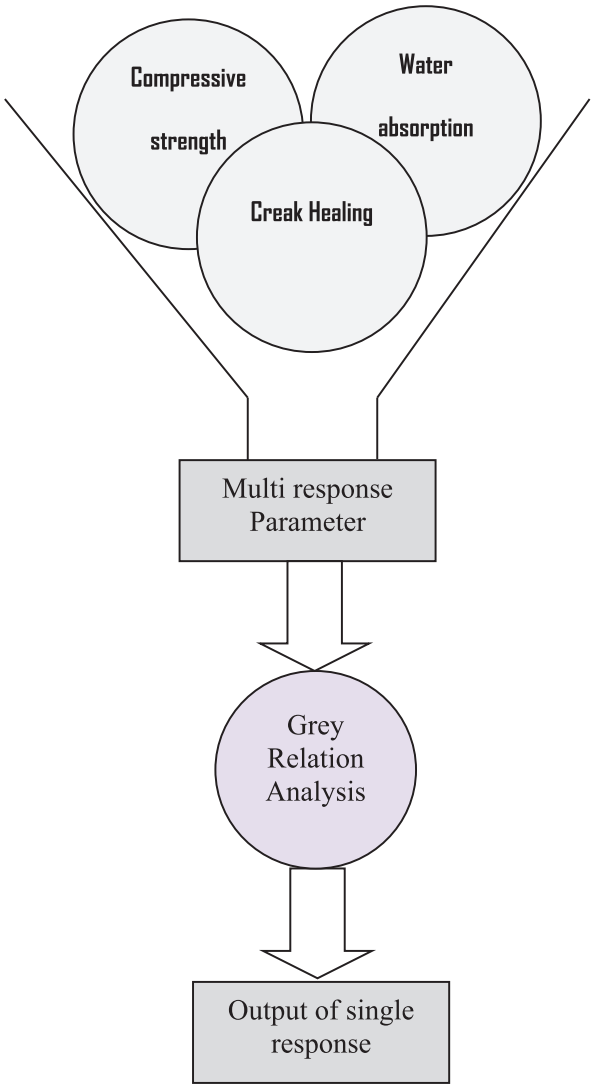


Fig. 2. Graph of a straightforward grey relational analysis.

Phase-2: Grey Relational Analysis:

The grey theory is an immense concept used to explore uncertainty, multi-input, and discrete data. Decision analysis is used to estimate the degree of relation according to the grey relational grade. A multi-objective optimization makes it more complex to analyse the effects and relationships between design factors in experiments at their various levels that result uncertain and insignificant information. In this paper, GRA is proposed for investigating and optimizing the complexity of multi-variable problems by exploiting the concept of information. As shown in Fig. 2, GRA reduces a multi-objective question to a single objective answer (referred to as single relational grade).

The present study is conducted based upon Taguchi's orthogonal array, which corresponds to nine trails, where every trail is known as a comparison sequence. The GRA places these trails into nine subsystems. The effect of these factors on the outcome variable is assessed through regression analysis. Using GRA, the multi-objective problem is

transformed into a single-objective problem by using the parameters corresponding to the greatest weighted grey relational grade.

Step-I: S/N calculation:

Greater, nominal, and lower signal-to-noise ratio analyses are the three possible approaches. For water absorption in this study, smaller-is-better, however, higher-is-better for compressive strength and creak healing. The S/N ratios of water absorption are calculated by equation (9), and compressive strength and creak healing by equation (10).

$$(S/N)_{\text{Smller-the-better}} = -10 \times \log\left(\frac{\sum_{k=1}^{m} \lambda_k^2}{m}\right), \tag{9}$$

$$(S/N)_{\text{Higher-the-better}} = -10 \times \log\left(\frac{\sum_{k=1}^{m} \frac{1}{\lambda_k^2}}{m}\right), \tag{10}$$

where, λ_k is the kth experiment's observed data and m is representing the observations' number.

Step-II: S/N ratio normalization:

To lessen unpredictability, the S/N ratio of attribute data is modified. Data preparation is the term for this. Pre-processing of the data is needed for grey analysis (Grzenda *et al.*, 2012; Kao *et al.*, 2008). The following equation (11) normalizes the original sequence:

$$\zeta_k^*(t) = \frac{\zeta_k(t) - \min_{1 \leqslant t \leqslant n} \zeta_k(t)}{\max_{1 \leqslant t \leqslant n} \zeta_k(t) - \min_{1 \leqslant t \leqslant n} \zeta_k(t)}. \tag{11}$$

However, the data is normalized using equation (12); the smaller the characteristic, the better.

$$\zeta_k^*(t) = \frac{\max_{1 \leqslant t \leqslant n} \zeta_k(t) - \zeta_k(t)}{\max_{1 \leqslant t \leqslant n} \zeta_k(t) - \min_{1 \leqslant t \leqslant n} \zeta_k(t)}, \tag{12}$$

ζ represents desired value, $\zeta_k^*(t)$ indicates normalized value, where n stands for the number of experiments, and m for the number of answers, and $t = 1(1)n$; $k = 1(1)n$.

Step-III: Grey Relational Coefficient:

The GRC, a series of information, is used by GRA to assess the relevance of two systems. Equation (13) can be used to calculate GRC (μ_k).

$$\mu_k(t) = \frac{\delta_{\min} + \tau \times \delta_{\max}}{\delta_k(t) + \tau \times \delta_{\max}}, \tag{13}$$

where $\delta_k(t) = \|\zeta_k(0) - \zeta_k(t)\|$

$$\delta_{\min}(t) = \min\left\{\left\|\zeta_k(0) - \zeta_k(t)\right\| : t = 1(1)n\right\},$$
$$\delta_{\max}(t) = \max\left\{\left\|\zeta_k(0) - \zeta_k(t)\right\| : t = 1(1)n\right\},$$

$\zeta_k(0) =$ reference value $(= 1)$; $\zeta_k(t) =$ specific comparison value, where τ is the distinguishing coefficient $0 \leqslant \tau \leqslant 1$.

Step-IV: Grey Relation Grade:

In real engineering systems, different parts have different weights based on the circumstances. Then, equation (14) grades the grey relational coefficient (GRC) (Saaty, 1980).

$$\eta(t) = \frac{\sum_{k=1}^{m} p_k \times \mu_k(t)}{\sum_{k=1}^{m} p_k}, \tag{14}$$

where p_k stands for factor k's normalized weight. The proposed IF-FUCOM approach yields the weight of each attribute. The higher the value of grey relational grade is, the greater is the desirability.

3. Methodology

The objective of this study is to find the optimal bacterial concentrations and curing days for concrete simultaneously while considering compressive strength (CS), crack healing (CH) and water absorption (WA) as outputs using a novel MCDM technique. During the present investigation, there are six phases. A schematic representation of the detailed methodology is shown in Fig. 3.

Phase-I: The Experimental details include details on the materials, bacteria culture, mixing procedure, compression strength (ξ_1), crack healing (ξ_2) and water absorption (ξ_3) test on mortar surfaces.

Phase-II: The criteria and alternatives were discussed in this phase.

Phase-III: Determine the weights of all consideration criteria using the proposed MCDM method.

Phase-IV: Evaluate alternative weights using another existing MCDM technique.

Phase-V: Comparison of the results determined by the proposed MCDM with the existing model.

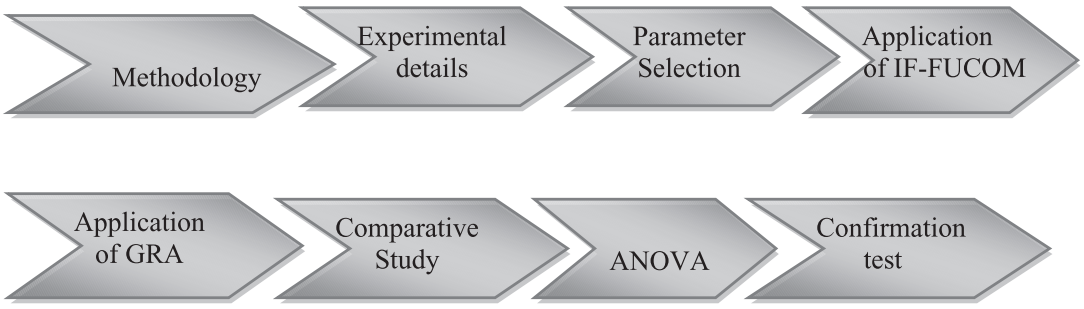


Fig. 3. A diagrammatic representation of the methodology.

Table 2
Composition and physical properties of cement.

Physical properties	
Colour	Grey
Specific gravity	3
Chemical constituents (%)	
Al_2O_3	3.78
SiO_2	21.5
MgO	1.79
Fe_2O_3	3.78
CaO	63.69
SO_3	3
Na_2O	–
K_2O	–

Phase-VI: An analysis is made of the degree to which each of the chosen parameter values contributes to the output responses.

Phase-VII: Validation tests are run to confirm forecasts and results.

Phase-VIII: Sensitivity analysis investigates how the indicators, which were calculated using the MCDM method, affect the anticipated result.

Phase-I. Experimental details:

This section explains the material choice, the bacteria mixing process, and several tests like compressive strength and water absorption.

Step-I. Materials:

Ordinary Portland Cement (OPC) 43 Grade conforms to IS 8112 : 2013, locally available Fine Aggregate, Bacillus Subtilis and potable water is used in this study. Here cement to sand ratio and water to cement ratio were 1 : 3 and 0.4 (by weight) respectively. For preparing mortar water, distilled water is used. Mortar cubes of dimension 70.6 mm^3 are prepared for both control and bacterial mortar specimens. In fresh water, curing can be conducted at room temperature 27 °C. According to information provided by the manufacturer, OPC cement's chemical composition and physical properties are presented in Table 2.

Step-II. Bacteria culture:

For this experimental work, selected bacterial sample Bacillus Subtilisis is used in this study. For bacterial culture nutrient broth was made (1.0 gm/lBeef Extract, 5.0 gm/lPeptone, 2.0 gm/lYeast Extract, 5.0 gm/NaCl). Growth conditions of Bacteria are maintained at 37 °C temperature. After 6–7 days, about 10 µl of the nutrient broth is obtained and haemocytometer counting is done. Here, the bacterial concentrations in solution used are 10^3 cells/ml, 10^5 cells/ml, 10^7 cells/ml.

Step-III. The mixing procedure:

Cement and sand is well mixed in 1:3 proportions and a mixture of water and the needed cell concentration is then prepared. After casting and compacting in a vibration

Table 3
Levels and values of the input parameters.

Parameters	Level-1	Level-2	Level-3
Concentration	0	5	7
Days	3	7	28

machine, specimens are removed and compression tests are performed after 3, 7, 14 and 28 days in air at room temperature (30 °C).

Step-IV. Compressive strength and water absorption test:

Compressive strength and water absorption of control and bacterial mortar cubes are measured in 3, 7 and 28 days after curing. The compressive strength test was done under compression testing machine.

Step-V. Crack healing on mortar surfaces:

A 28-day crack healing test is performed on microbial concrete to determine its self-healing ability at different bacteria concentrations. The crack-measuring instrument measured the crack widths. In this study, crack widths range from 0.11 mm to 1.5 mm; water is used to immerse the cracked specimens and their crack dimensions are recorded after 3, 7 and 28 days.

Phase-II. Parameter Selection:

The PV of each criterion and alternative will be calculated in the section that follows. In the present study, compressive strength (ξ_1), crack healing (ξ_2) and water absorption (ξ_3) are considered as a set of criteria. Also, nine considering trials, namely, 10^7 Concentration with 28 days, 10^5 Concentration with 28 days, 10^3 Concentration with 28 days, 10^7 Concentration with 7 days, 10^5 Concentration with 7 days, 10^3 Concentration with 7 days, 10^7 Concentration with 3 days, 10^5 Concentration with 3 days and 10^3 Concentration with 3 days as a set of alternatives.

For the present study, the design factors chosen are bacteria concentration and curing day so as to determine their influence on the outcome parameters of compressive strength, crack healing, and water absorption. Table 3 represents the levels of input parameters (bacteria concentration and curing day) that are considered as control factors for the experiment. In Table 4, based on the number of tests, a Taguchi L9 (32) orthogonal array comprising 9 rows has been calculated.

Phase-III. Application of IF-FUCOM:

Collect all factors based on the literature review, and then send them to three experts, and expert responds. Following the determination of the first-level criteria, the ranking is determined on a second level. Dimensions are ranked in this order: $\xi_1 > \xi_2 > \xi_3$. In Table 5, the linguistic variables represent the relative importance of the criteria ranked according to decision-makers preferences.

The fuzzy linguistic scale was used to transform linguistic variables into Intuitionistic fuzzy numbers (IFNs), as shown in Table 6.

Table 4
Results of an experiment using L_9 orthogonal arrays.

Trial No.	Concentration	Days	CS (Map)	CH (%)	WA (%)
1	10^3	3	30.1206	30.3167	5.5
2	10^3	7	37.6342	50.2262	4.66667
3	10^3	28	48.1967	60.6335	4.25
4	10^5	3	32.7521	70.1357	5.08333
5	10^5	7	41.2611	84.6154	4.375
6	10^5	28	52.462	90.0452	3.79167
7	10^7	3	31.4065	94.5701	4.91667
8	10^7	7	39.9128	98.5	4.04167
9	10^7	28	49.4737	99.6	3.58333

Table 5
A linguistic assessments of the main dimensions.

Dimensions	ξ_1	ξ_2	ξ_3
Linguistic variables	EI	MI	STI

Table 6
Evaluations transformed by IFNs.

Dimensions	ξ_1	ξ_2	ξ_3
IFNs	(1, 1, 1; 1, 1, 1)	(2, 3, 4; 1, 3, 5)	(4, 5, 6; 3, 5, 7)

According to expression (1), the relative importance of the criteria is as follows.

$$\tilde{\wp}_{\xi_1/\xi_2} = \frac{\tilde{p}_{\xi_1}}{\tilde{p}_{\xi_2}} = \frac{(2,3,4;1,3,5)}{(1,1,1;1,1,1)} = (2,3,4;1,3,5),$$
$$\tilde{\wp}_{\xi_2/\xi_3} = \frac{\tilde{p}_{\xi_2}}{\tilde{p}_{\xi_3}} = \frac{(4,5,6;3,5,7)}{(2,3,4;1,3,5)} = (1,1.667,3;0.6,1.667,7).$$

A vector comparative significance is therefore defined as follows:

$$\tilde{\wp} = \big\{(2,3,4;1,3,5),(1,1.667,3;0.6,1.667,7)\big\}.$$

Three constraints are imposed by equation (4) based on the conditions of relation transitivity as follows:

$$\frac{\tilde{p}_{\xi_1}}{\tilde{p}_{\xi_3}} = (2,3,4;1,3,5).(1,1.667,3;0.6,1.667,7) = (2,5.001,12;0.6,5.001,35).$$

Optimization Problem:

$$
\begin{aligned}
&\text{Min } \nu \\
&\text{s.t.} \\
&\left.\begin{aligned}
&\left|p_1^l - 2p_2^u\right| \leqslant \nu, \\
&\left|p_1^u - 4p_2^l\right| \leqslant \nu, \\
&\left|p_1'^l - p_2'^u\right| \leqslant \nu, \\
&\left|p_1'^u - 5p_2'^l\right| \leqslant \nu, \\
&\left|p_1^m - 3p_2^m\right| \leqslant \nu, \\
&\left|p_2^l - p_3^u\right| \leqslant \nu, \\
&\left|p_2^u - 3p_3^l\right| \leqslant \nu, \\
&\left|p_2'^l - 0.6p_3'^u\right| \leqslant \nu, \\
&\left|p_2'^u - 7p_3'^l\right| \leqslant \nu, \\
&\left|p_2^m - 1.667p_3^m\right| \leqslant \nu, \\
&\left|p_1^l - 2p_3^u\right| \leqslant \nu, \\
&\left|p_1^u - 12p_3^l\right| \leqslant \nu, \\
&\left|p_1'^l - 0.6p_3'^u\right| \leqslant \nu, \\
&\left|p_1'^u - 35p_3'^l\right| \leqslant \nu, \\
&\left|p_1^m - 5.001p_3^m\right| \leqslant \nu, \\
&p_1^m = p_1'^m, \\
&p_2^m = p_2'^m, \\
&p_3^m = p_3'^m, \\
&\sum_{j=1}^{3} \frac{(p_j^l + 2p_j^m + p_j^u) + (p_j'^l + 2p_j'^m + p_j'^u)}{8} = 1, \\
&0 \leqslant p_j'^l \leqslant p_j^l \leqslant p_j^m = p_j'^m \leqslant p_j^u \leqslant p_j'^u \quad \text{for all } j = 1(1)3, \\
&\nu \geqslant 0.
\end{aligned}\right\} \qquad (15)
\end{aligned}
$$

Phase-IV. Application of GRA:

To calculate the score, gray relation grading is used after determining the relative weights of each criterion. GRCs are used to calculate gray reasoning grades using equation (14). GRCs are weighted from 0 to 1. IF-FUCOM decides the weight of each characteristic. A gray relational grade of the higher value indicates greater desirability.

Phase-V. Study of Comparisons:

A comparative research can identify and quantify the relationships between at least two factors by studying different groups that have been exposed to diverse treatments either by choice or circumstance. A relative study is made possible by contrasting two sets of individuals, entities, or circumstances. In the current work, the proposed technique has been contrasted with three sophisticated models.

Model-I: The local weight must be calculated using AHP (Saaty, 1980), and the global weight must be calculated using GRA (Julong, 1989).

Table 7
Best to others criteria.

Best to others	ξ_1	ξ_2	ξ_3
ξ_1	1	3	7

Table 8
Others to worst criteria.

Others to worst	ξ_3
ξ_1	7
ξ_2	5
ξ_3	1

Model-II: Identify the local weights of the BWM (Rezaei, 2015) and the GRA (Julong, 1989) alternative, as well as identifying the global weights of the BWM-GRA alternative.

Model-III: Determine the local weight of an alternative GRA (Julong, 1989), the local weight of the FUCOM (Pamučar *et al.*, 2018b), then calculate the global weight using a hybrid strategy known as FUCOM GRA.

Here are three mathematical formulations of the models discussed below.

Model-I. AHP result:

The weights of the different criteria are determined by experts within related fields who collaborate in a pairwise comparison between each criterion. This comparison matrix is shown below.

$$\begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{array} \begin{array}{c} \begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \end{array} \\ \begin{pmatrix} 1 & 3 & 4 \\ & 1 & 3 \\ & & 1 \end{pmatrix} \end{array}.$$

Model-II. BWM result:

For the purpose of weighing the criteria in BWM, relevant experts are asked to identify the most and least significant factors in the case study, along with the best-to-others and other-to-worst vectors. According to expert consensus, ξ_1 and ξ_3 are the best and worst criteria, respectively. The best-to-others and worst-to-others vectors are shown in Tables 7 and 8.

The weight of each criterion can be calculated, as well as the consistency rate, using the non-linear mathematical model.

Table 9
Comparative significance levels for the evaluation criteria.

ξ_3	ξ_3	ξ_3
1	1.08	1.25

$$\begin{aligned}
&\max \chi \\
&\text{s.t.} \left|\frac{p_1}{p_2} - 3\right| < \chi, \\
&\left|\frac{p_1}{p_3} - 7\right| < \chi, \\
&\left|\frac{p_2}{p_3} - 5\right| < \chi, \\
&\sum_{j=1}^{3} p_j = 1, \\
&p_j \geqslant 0, \quad \text{for all } j = 1(1)3.
\end{aligned} \tag{16}$$

Model-III. FUCOM result:

The criteria are ranked in order of importance. The ranking is determined by consensus among experts. According to experts, the relation (17) criteria are ranked. Comparisons are based on a scale of Van Tittelboom *et al.* (2010), Achal *et al.* (2011), which is shown in Table 9.

$$\xi_1 > \xi_2 > \xi_3. \tag{17}$$

The relative importance of each criterion can be gauged by calculating the comparison importance values based on the obtained importance values $\theta_{C_1/C_2} = \frac{1.08}{1} = 1.08$, $\theta_{C_2/C_3} = \frac{1.25}{1.08} = 1.15740741$ and $\theta_{C_1/C_3} = 1.08 \times 1.15740741 = 1.25$.

The final weight coefficients can be determined by applying expression (18)

$$\begin{aligned}
&\max \chi \\
&\text{s.t.} \left|\frac{p_1}{p_2} - 1.08\right| < \chi, \\
&\left|\frac{p_2}{p_3} - 1.15740741\right| < \chi, \\
&\left|\frac{p_1}{p_3} - 1.25\right| < \chi, \\
&\sum_{j=1}^{3} p_j = 1, \\
&p_j \geqslant 0, \quad \text{for all } j = 1(1)3.
\end{aligned} \tag{18}$$

Phase-V. ANOVA:

Statistically, the difference among available scores can be evaluated through Analysis of Variance (ANOVA). In ANOVA, the level of contribution of each of the chosen parameter values over the output responses is analysed (Pattnaik *et al.*, 2013). ANOVA results can be used to determine which variables are responsible for the performance of a selected process and to control these variables to obtain a better result. ANOVA cannot provide data analysis, but this statistical method can assess variance of the data.

Phase-VI. Confirmation test:

A confirmation test is done to verify the forecast and the outcome after the S/N ratio plot is used to estimate the optimal output. The IF-FUCUM-GRG values delivered at the optimal output are predicted by equation (19).

$$\vartheta_{predicted} = \vartheta_{mean} + \sum_{i=1}^{n} (\vartheta_i - \vartheta_{mean}). \tag{19}$$

The group's reasoning grade ϑ_{mean} stands for the overall grade mean, ϑ_i for the grade at the best level, and n is the output regulating parameter.

4. Result

The six parts of the proposed model are described in the results section.

Part-I: Calculate the gray relation coefficient.

Phase-II: In this phase, IF-FUCOM is used to determine weights for the criteria. The weights and rankings of alternatives are determined using IF-FUCOM and GRC.

Phase-III: A comparison of the results offered by novel IF-FUCOM-AHP and some existing methods.

Phase-IV: Next, the ANOVA result is used to determine the influential input parameter.

Phase-V: The percentage significance of input factors can be analysed with ANOVA.

In the following, all phases are discussed in detail.

Phase-I: Result of GRC:

To assess the impact of each parameter, the SN ratio of every trail is computed based on equation (9) for compressive strength and equation (10) for crack healing and water absorption. Equations (11) and (12) are used to normalize the acquired value of the SN ratio while taking the higher-the-better and smaller-the-better qualities into consideration, respectively. Equation (13) is used to calculate the GRC after determining the normalized SN ratios for each investigation. Table 10 displays the SN ratios of the output parameters together with the corresponding GRCs.

Phase-II. Result from IF-FUCOM-GRG:

The IF-FUCOM-GRG result is divided into two parts, namely the result of IF-FUCOM and the result of GRG. All the parts are discussed below.

Step-I. Result from IF-FUCOM:

Table 10
SN ratio and GRC associated with output parameter.

SN RATIO			GRC		
CS (Mpa)	CH (%)	WA (%)	CS (Mpa)	CH (%)	WA (%)
29.57727237	29.63363852	−14.8072538	0.333333333	0.333333333	1
31.51165377	34.01860643	−13.3801418	0.455106149	0.464867652	0.565946459
33.66034607	35.65425276	−12.5677786	0.765904066	0.545102165	0.453817715
30.30478303	36.91878272	−14.1229661	0.37063052	0.629038242	0.731129591
32.31081604	38.54898824	−12.8195611	0.536003016	0.784836765	0.483508457
34.39689686	39.08921134	−11.5766106	1	0.855012953	0.365470817
29.94039081	39.51507697	−13.8334212	0.350961336	0.919850586	0.656446929
32.02224392	39.86872461	−12.131217	0.503674319	0.98166898	0.410148263
33.88748783	39.96518677	−11.0857361	0.825498313	1	0.333333333

The best values of the criteria can be found by solving the fuzzy linear model in equation (11), which is shown.

The weight coefficients for the criteria compressive strength (ξ_1), crack healing (ξ_2) and water absorption (ξ_3) are (0.443, 0.654, 0.866; 0.231, 0.654, 0.943), (0.212, 0.212, 0.212; 0.192, 0.212, 0.212) and (0.0770, 0.135, 0.231; 0.231, 0.135, 0.353), respectively, with a deviation from maximum consistency $\nu = 0.0192$. Lingo 17.0 is used to solve the model (11).

Next, use equations (7) to calculate the crisp weights for the criteria compressive strength (ξ_1), crack healing (ξ_2) and water absorption (ξ_3), which are, respectively, 0.637, 0.209, and 0.153.

Use equation (8) to calculate normalized weights for these three criteria, which are 0.499, 0.245, and 0.256.

The weights of the compressive strength (ξ_1), crack healing (ξ_2) and water absorption (ξ_3) using FUCOM-F (Pamucar and Ecer, 2020) are 0.400, 0.388, and 0.212, respectively, with $\nu = 0.001$. Despite significant discrepancies in the weights of those criteria, all FUCOM-F, as well as IF-FUCOM, algorithms rank each criterion in the same order.

Step-II: Result of IF-FUCOM-GRA:

In GRA, the relative weights of the criteria are obtained by IF-FUCOM. After determining the relative weights of the criteria, the score is calculated using grey relation grading. Using equation (16), GRCs are used to calculate the different grey reasoning grades. GRCs are weighted from 0 to 1, with p_k equal to 1. p_1, p_2, and p_3 are used as weighting factors in this study for compressive strength, crack healing, and water absorption, respectively. Gray relational grade (GRG) determined by different MCDM methods are presented in Table 11 for each trial using the L9 orthogonal array data.

Phase-III. Result of Comparative study:

This study validates the result of the proposed model by comparing it to three existing MCDM techniques. There are four steps in this phase. Determine the PV for each criterion using AHP, BWM, and FUCOM methods in the first three steps. As a last step, determine the weights of the alternatives using GRA.

Step-I: Result from AHP:

Table 11
Gray relational grade determined by IF-FUCOM.

Trail No.	IF-FUCOM-GRG	Rank
1	0.435	9
2	0.47365	8
3	0.671241	3
4	0.479423	7
5	0.579442	5
6	0.871615	1
7	0.516248	6
8	0.588762	4
9	0.785842	2

Calculate the priority value of each criterion using the AHP algorithm as described in Section. The priority value of criteria are $p_1 = 0.614$, $p_2 = 0.268$, $p_3 = 0.117$ and maximum eigen value $\lambda_{\max} = 3.074$. These values indicate that the most important criterion is CS (ξ_1), while the least important criterion is WA (ξ_3). To determine CI and CR:

$$\text{CI} = \frac{3.074 - 3}{3 - 1} = 0.037,$$
$$\text{CR} = \frac{0.037}{0.58} = 0.06379.$$

Step-II: Result from BWM:

Based on the solution to the above BWM-model (16), the following criteria weights are optimal: $p_1^* = 0.66154$, $p_2^* = 0.26154$, $p_3^* = 0.07692$ and $\chi* = 0.12308$. According to these values, the outputs CS (ξ_1) and WA (ξ_3) are the most important and the least important criteria, respectively. The degree of consistency is as follows:

$$\text{CR} = \frac{0.12308}{3.73} = 0.032996.$$

As suggested by the obtained CR value (0.032996), the obtained criteria weights have a satisfactory degree of consistency.

Step-III: Result from FUCOM:

Based on the solution to the above model (18), the following criteria weights are optimal: $p_1^* = 0.3668478$, $p_2^* = 0.3396739$, $p_3^* = 0.2934783$ and $\chi* = 0.2471142 \times E^{-08}$. According to these values, the outputs CS (ξ_1) and WA (ξ_3) are the most important and the least important criteria, respectively.

Step-IV: Different Gray relation grade:

In GRA, the relative weights of the criteria are obtained by AHP, BWM, and FUCOM. After determining the relative weights of the criteria, the score is calculated using grey relation grading. Using equation (16), GRCs are used to calculate the different grey reasoning grades. GRCs are weighted from 0 to 1, with p_k equal to 1. p_1, p_2, and p_3 are

Table 12
Gray relational grade determined by different MCDM techniques.

Trail No.	GRG	Rank	AHP-GRG	Rank	BWM-GRG	Rank	FUCOM-GRG	Rank
1	0.5555556	8	0.411	9	0.3846153	9	0.528986	8
2	0.4953068	9	0.470235	8	0.4661853	7	0.490951	9
3	0.5882746	6	0.669449	3	0.6841493	3	0.599313	6
4	0.5769328	7	0.481692	7	0.4659447	8	0.564204	7
5	0.6014494	5	0.596013	5	0.5970446	5	0.605119	5
6	0.7401613	1	0.885904	1	0.9132704	1	0.764531	1
7	0.6424196	3	0.538815	6	0.5232466	6	0.633852	4
8	0.6318305	4	0.620331	4	0.621494	4	0.638589	3
9	0.7196105	2	0.813856	2	0.8332784	2	0.740332	2

Table 13
IF-FUCOM-GRG response table.

Parameters	Level-1	Level-2	Level-3
Concentration	0.4932	0.6458	0.6391
Days	0.4632	0.5516	0.7633

used as weighting factors in this study for compressive strength, crack healing, and water absorption, respectively. Gray relational grade determined by different MCDM methods are presented in Table 12 for each trial using the L9 orthogonal array data.

Phase-IV. ANOVA result:

The S/N ratio is used to analyse the IF-FUCOM-GRG data to find the best combination of factors. According to the higher, the better criterion, the optimal combination should correspond to the highest S/N value on each factor. Results are analysed using Minitab software. Table 13 summarizes the main influences of control factors on mean grey relational grades. As a result of each level of the input control parameter, the S/N ratio plots of IF-FUCOM-GRG are shown in Fig. 4. Combining the highest factor levels calculated from bacteria concentration at 105 and 28 curing days yields the best factor level combination.

ANOVA is employed to determine whether design elements have a substantial impact on response (Haq *et al.*, 2008). ANOVA may examine the percentage importance of input factors. When $F > 4$ (Yang and Tarng, 1998) and a parameter is significant, Fisher's F-test is employed to evaluate the effect of the parameter on output quality. The ANOVA findings for the IF-FUCOM-GRG are displayed in Table 14. According to the ANOVA results, both input parameters are important for the study, but curing day is more important than bacteria concentration.

Phase-V. Confirmation test result:

In Table 15, the optimal output parameter is tested for actual and predicted IF-FUCOM-GRA values. Equation (19) predicts the IF-FUCOM-GRA values provided at the optimal output. IF-FUCOM-GRA, as predicted and experimentally determined as an optimum level, are 0.818256 and 0.88929, respectively.

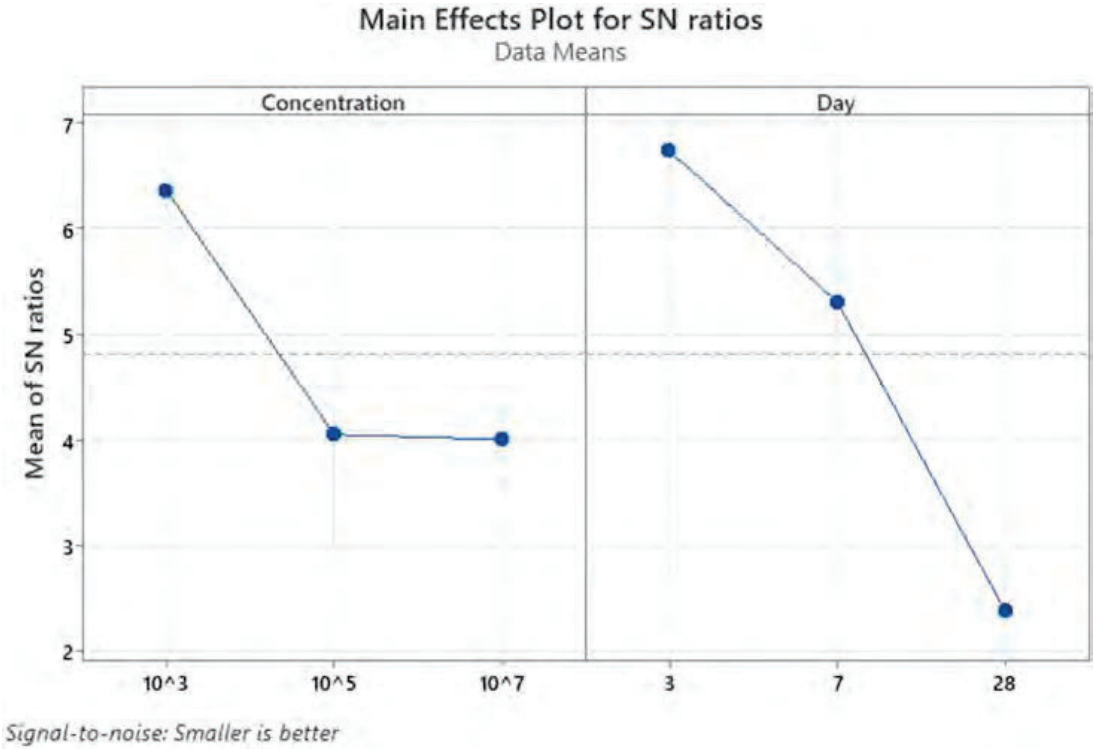


Fig. 4. Response of the IF-FUCOM-GRG SN ratio.

Table 14
Results of the ANOVA for the IF-FUCOM-GRG.

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Concentration	2	0.044646	0.022323	11.88	0.021
Days	2	0.142686	0.071343	37.96	0.003
Error	4	0.007519	0.001880		
Total	8	0.194851			

Table 15
Confirmation test table.

	Optimal input parameter	
	Predicted	Experimental
Level	10^5 bacteria concentration, 28 curing day	10^5 bacteria concentration, 28 curing day
IF-FUCOM-GRA grade	0.816385	0.88929
S/N Ratio	1.64405	1.71291

5. Discussion

In this study, three criteria, crack healing, water absorption, and compressive strength, are used to determine the optimal bacteria concentration and curing day. The values that corresponded to all three criteria are obtained through experimentation. Using the experimental data, equation (16) calculates the gray relation coefficient. A variety of MCDM

techniques are used for weighting criteria, including BWM, AHP, FUCOM, and intuitive fuzzy FUCOM. After that, a comparative study has been conducted among AHP-GRG, FUCOM-GRG, BWM-GRG, and IF-FUCOM-GRG. The comparison shows that IF-FUCOM-GRG produces similar rankings to other methods in most cases. Based on the proposed method, the optimal bacteria concentration is 105, and the optimal curing time is 28 days. The confirmation test result shown in Table 15 predicts IF-FUCOM-GRA grade pretty well, and is almost in agreement with the experimental results.

6. Conclusion

MCDM problems are solved by considering different levels of importance of the criteria. A number of weighting methods have been used in the literature to determine the importance levels of expert opinions, including SAW, AHP/ANP, SWARA, BWM, and FUCOM. The fuzzy set theory can be used to solve ambiguous and vague problems. It is possible to improve the reliability of these weighting methods by incorporating fuzzy set theory, which reflects the way humans think and reason. The intuitionistic fuzzy set solves this problem by defining the non-membership degree and the two membership levels for each element. In this study, intuitionistic fuzzy sets are combined with FUCOM to come up with the Intuitionistic Fuzzy FUCOM (IF-FUCOM). Moreover, linguistic variables are used instead of crisp values in pairwise comparisons for criteria in the decision-making process.

In comparison to the IF-BWM and IF-AHP models, the IF-FUCOM model has the advantage of offering similar results by using only $(n - 1)$ pairwise comparisons. By eliminating the inconsistency of expert preferences with respect to the final weights of criteria, the influence of inconsistency is reduced. The IF-FUCOM is considered the best method for determining criteria weights because it requires a minimum number of expert comparisons. In addition, the mathematical apparatus provides easy-to-understand weight coefficients to facilitate rational decision-making (Fazlollahtabar *et al.*, 2019). Hence, the IF-FUCOM tool allows decision-makers to ignore their own preferences in order to deal with subjectivity in prioritizing criteria.

In the present study, Intuitionistic Fuzzy FUCOM Grey Relations Analysis is used to select optimal bacteria concentrations and optimal curing time in days. The proposed algorithm obtains a grey reasoning grade according to the grey relational coefficients of each test run in order to convert multi-response optimization to single objective optimization. The intuitionist fuzzy-FUCOM-grey reasoning grades (IF-FUCOM-GRG) are compared with different grades like AHP-grey reasoning grade (AHP-GRG), BWM-grey reasoning grade (BWM-GRG), FUCOM-grey reasoning grade (FUCOM-GRG). All the algorithms have been employed to obtain the optimal input factor corresponding to the estimated values of output response. IF-FUCOM-GRG produces similar rankings with other methods in most cases, but in some cases it produces better results. According to the proposed method, the optimal bacteria concentration is 105, and the optimal curing time is 28 days. Using a confirmation experiment, the computed factor combination based on the highest ranking of IF-FUCOM-GRG is validated.

Limitations of the study:

i. There are three criteria used in this study to select the best bacteria concentration and curing time. However, the results may vary if other criteria are added.
ii. The ranking order may change as the number of alternatives increases, which is the shortcoming of this model.

Future scope:

i. In the future, the proposed method can be applied to all fields of science, engineering, and social sciences. Additionally, this method can be used in conjunction with other ranking methods (COPRAS, CODAS, ARAS, TOPSIS, EDAS, MAIRCA, etc.) to select the most appropriate alternative to solve MCDM problems.
ii. This study found optimal bacteria concentrations in concrete mortar, but optimal bacteria concentrations can also be found when cement is partially replaced by other additives like rice husk, fly ash, etc.

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The Rank Order of the Neutrosophic Triplets (T, I, F) May Change after Normalization

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Abstract

This paper examines the effects of normalization on the ranking order of single-valued neutrosophic triplets. Neutrosophic triplets are often normalized so that their sum equals one. However, we demonstrate that the ranking order of these triplets, when determined by score, accuracy, and certainty functions, can change upon normalization. Through formal proofs and illustrative counter-examples, we show that inequalities between unnormalized triplets do not always hold after normalization, affecting ranking stability. These findings highlight the importance of considering normalization's impact when using neutrosophic triplets for decision-making and uncertainty modeling.

Keywords: Neutrosophic set, Ranking functions, Score function, Accuracy function, Decision-making, Rank order consistency.

1|Introduction

Neutrosophic theory [1], [2] extends the framework of fuzzy and intuitionistic fuzzy sets by incorporating an additional degree of indeterminacy, thereby representing truth, indeterminacy, and falsity as independent values within a triplet (T, I, F). Neutrosophic sets allow each component of this triplet to vary independently in the range [0, 1], providing a more nuanced representation of uncertainty and indeterminacy in data [3], which has proven valuable in fields such as decision-making [4–7], optimization [8–11], medical diagnosis [12], risk management [13], [14], graphs [15], Statistics [16], building contexts [17], supply chain [18], topology

[19], [20], numerical analysis [21], performance evaluations [22–24], tourism strategies [25], and artificial intelligence [26–28]; see also [29–32].

One key aspect of neutrosophic triplets is their normalization. Normalization can simplify comparisons across triplets by standardizing values; however, the implications of this transformation for the ranking of neutrosophic triplets have not been fully examined. Ranking triplets is crucial in applications that require ordering based on degrees of truth, indeterminacy, or falsity, yet it is unclear whether normalization preserves ranking consistency.

This paper addresses this gap by exploring how normalization impacts the relative ordering of neutrosophic triplets according to three commonly used functions: the score function S(T, I, F), which reflects an aggregate measure of positiveness; the accuracy function A(T, I, F), which indicates the difference between truth and falsity components; and the certainty function C(T, I, F), which measures the degree of truth alone. We demonstrate that normalization can alter the ranking order of neutrosophic triplets as established by these functions, and we provide proofs and counter-examples illustrating conditions under which inconsistencies arise.

The findings of this study highlight the necessity of caution in applying normalized neutrosophic triplets in applications where ranking accuracy is essential. By showing that ranking reversals may occur post-normalization, we contribute to a more robust understanding of the neutrosophic framework, enhancing its applicability and reliability in uncertainty modeling. This work thus builds on prior studies of ranking functions for neutrosophic sets while offering new insights into the effects of normalization on rank stability.

2 | Ranking of Neutrosophic Triplets Preliminaries

In this section, we outline the essential definitions and functions related to neutrosophic triplets and discuss their properties in the context of ranking. These preliminaries are the foundation for analyzing how normalization influences the rank order of neutrosophic triplets; see [3].

2.1 | Neutrosophic Triplets and their Representation

A neutrosophic triplet (T, I, F) consists of three components: T, representing truth; I, representing indeterminacy; and F, representing falsity. Each component is defined on the interval [0,1], collectively capturing the uncertainty and indeterminacy associated with a given concept or decision. Formally, we define the set of un-normalized (non-normalized) neutrosophic triplets as

$$\mathrm{UN} = \{(\mathrm{T}, \mathrm{I}, \mathrm{F}), \mathrm{T}, \mathrm{I}, \mathrm{F} \in [0, 1], 0 \leq \mathrm{T}+\mathrm{I}+\mathrm{F} \leq 3\}.$$

For consistency, we exclude the trivial case where (T, I, F) = (0, 0, 0), as it cannot be normalized.

In many applications, it is desirable to normalize the triplet so that the sum T+I+F=1. We obtain the normalized triplet T', I', F') by dividing each component by the total sum

$$\mathrm{T}' = \frac{\mathrm{T}}{\mathrm{T}+\mathrm{I}+\mathrm{F}}, \mathrm{I}' = \frac{\mathrm{I}}{\mathrm{T}+\mathrm{I}+\mathrm{F}}, \mathrm{F}' = \frac{\mathrm{F}}{\mathrm{T}+\mathrm{I}+\mathrm{F}'}$$

provided that T+I+F>0. This normalization ensures that all values are scaled relative to the total degree of truth, indeterminacy, and falsity, effectively mapping the triplet to a bounded simplex within [0,1].

2.2 | Ranking Functions for Neutrosophic Triplets

Several ranking functions have been proposed in the literature to evaluate and compare neutrosophic triplets. These functions aim to assign a quantitative measure to each triplet based on its components. In this paper, we focus on three key functions commonly used in ranking neutrosophic sets:

I. Score function S(T, I, F): the score function aggregates the values of T, I, and F into a single measure of positiveness or overall degree. It is defined as

$$S: UN \to [0, 1],$$

$$S(T, I, F) = \frac{T+(1-I)+(1-F)}{3}.$$

Higher scores correspond to triplets with higher truth values and lower indeterminacy and falsity values, indicating greater positiveness.

II. Accuracy function A(T, I, F): the accuracy function represents the difference between the truth and falsity components, indicating certainty in the triplet. It is given by

$$A: UN \to [-1, 1],$$

$$A(T, I, F) = T - F.$$

This function takes values in [−1,1], with higher values suggesting greater accuracy due to a larger truth component relative to falsity.

III. Certainty function C(T, I, F): the certainty function is defined as the truth component alone

$$C: UN \to [0, 1]$$

$$C(T, I, F) = T.$$

This function independently reflects the degree of truth without considering indeterminacy or falsity.

2.3 | Problem Statement

The primary goal of this paper is to investigate whether the rank order of neutrosophic triplets, as determined by the score, accuracy, and certainty functions, is preserved after normalization. Given two neutrosophic triplets $N_1=(T_1, I_1, F_1)$ and $N_2=(T_2, I_2, F_2)$, we examine how applying these ranking functions before and after normalization can lead to inconsistent ordering. Specifically, we seek to identify conditions under which $S(N_1)>S(N_2)$ in the unnormalized case may yield $S(N_1')\leq S(N_2')$ after normalization and similarly for the accuracy and certainty functions.

This theoretical analysis, supported by counter-examples, demonstrates that normalization can alter the relative order of neutrosophic triplets, which has implications for applications requiring reliable ranking of uncertain information.

3 | Ranking Algorithm and Theoretical Analysis

This section presents the algorithm used for ranking neutrosophic triplets based on the score, accuracy, and certainty functions. We then analyze the effects of normalization on these rankings, providing proofs and examples to illustrate cases where the rank order changes after normalization.

3.1 | Ranking Algorithm for Neutrosophic Triplets

Given two neutrosophic triplets $N_1 = (T_1, I_1, F_1)$ and $N_2 = (T_2, I_2, F_2)$, we propose a hierarchical ranking algorithm that sequentially applies the score, accuracy and certainty functions to determine the ordering of the triplets (no matter if they are un-normalized or normalized). The steps of the algorithm are as follows:

I. Apply the score function: if $S(N_1) > S(N_2)$, then $N_1 > N_2$; or if $S(N_1) < S(N_2)$, then $N_1 < N_2$.

II. Apply the accuracy function (if necessary): if $S(N_1) = S(N_2)$, then apply the accuracy function. If $A(N_1) > A(N_2)$, then $N_1 > N_2$; or if $A(N_1) < A(N_2)$, then $N_1 < N_2$.

III. Apply the certainty function (if necessary): if $A(N_1) = A(N_2)$, then apply the certainty function. If $C(N_1) > C(N_2)$, then $N_1 > N_2$; or if $C(N_1) < C(N_2)$, then $N_1 < N_2$. But, if $C(N_1) = C(N_2)$, then $N_1 = N_2$.

This algorithm ensures a total ordering for neutrosophic triplets based on their respective truth, indeterminacy, and falsity components. However, as we will show, normalization can affect these comparisons, leading to situations where the ranking order established by the algorithm changes post-normalization.

3.2 | Theoretical Analysis: Effects of Normalization on Ranking Order

We now analyze how normalization impacts the ranking of neutrosophic triplets by examining each ranking function. The following theorem formalizes conditions under which normalization can lead to rank reversals.

Theorem 1. Ranking order reversal upon normalization.

Let the un-normalized triplets be (T_1, I_1, F_1) and $(T_2, I_2, F_2) \in UN$. Also, let their corresponding normalized forms (obtained by dividing each neutrosophic component T, I, F by the sum of all neutrosophic components $T + I + F \neq 0$) are $\left(\frac{T_1}{D_1}, \frac{I_1}{D_1}, \frac{F_1}{D_1}\right)$, where $T_1 + I_1 + F_1 = D_1$, and $\left(\frac{T_2}{D_2}, \frac{I_2}{D_2}, \frac{F_2}{D_2}\right)$, where $T_2 + I_2 + F_2 = D_2$.

I. If $S(T_1, I_1, F_1) > S(T_2, I_2, F_2)$, then $S\left(\frac{T_1}{D_1}, \frac{I_1}{D_1}, \frac{F_1}{D_1}\right)$ is not necessarily $> S\left(\frac{T_2}{D_2}, \frac{I_2}{D_2}, \frac{F_2}{D_2}\right)$.

II. If $A(T_1, I_1, F_1) > A(T_2, I_2, F_2)$, then $A\left(\frac{T_1}{D_1}, \frac{I_1}{D_1}, \frac{F_1}{D_1}\right)$ is not necessarily $> A\left(\frac{T_2}{D_2}, \frac{I_2}{D_2}, \frac{F_2}{D_2}\right)$.

III. If $C(T_1, I_1, F_1) > C(T_2, I_2, F_2)$, then $C\left(\frac{T_1}{D_1}, \frac{I_1}{D_1}, \frac{F_1}{D_1}\right)$ is not necessarily $> C\left(\frac{T_2}{D_2}, \frac{I_2}{D_2}, \frac{F_2}{D_2}\right)$.

Similarly, in the case of opposite inequalities ($<$).

Proof:

I. The score function:

Un-Normalized Triplets

Let $S(T_1, I_1, F_1) > S(T_2, I_2, F_2)$, then $\frac{T_1+(1-I_1)+(1-F_1)}{3} > \frac{T_2+(1-I_2)+(1-F_2)}{3}$

or $2 + T_1 - I_1 - F_1 > 2 + T_2 - I_2 - F_2$,

or $T_1 - I_1 - F_1 > T_2 - I_2 - F_2$.

Normalized Triplets

To normalize, we divide each component by the sum of all three components:

$(T, I, F) \to \left(\frac{T}{T+I+F}, \frac{I}{T+I+F}, \frac{F}{T+I+F}\right)$.

From $S\left(\frac{T_1}{D_1}, \frac{I_1}{D_1}, \frac{F_1}{D_1}\right) > S\left(\frac{T_2}{D_2}, \frac{I_2}{D_2}, \frac{F_2}{D_2}\right)$, one gets

$$\frac{\frac{T_1}{D_1} + \left(1 - \frac{I_1}{D_1}\right) + \left(1 - \frac{F_1}{D_1}\right)}{3} > \frac{\frac{T_2}{D_2} + \left(1 - \frac{I_2}{D_2}\right) + \left(1 - \frac{F_2}{D_2}\right)}{3},$$

or $\frac{T_1}{D_1} + 1 - \frac{I_1}{D_1} + 1 - \frac{F_1}{D_1} > \frac{T_2}{D_2} + 1 - \frac{I_2}{D_2} + 1 - \frac{F_2}{D_2}$,

or $\frac{T_1-I_1-F_1}{D_1} > \frac{T_2-I_2-F_2}{D_2}$,

or

$$T_1 - I_1 - F_1 > (T_2 - I_2 - F_2) \cdot \frac{D_1}{D_2}. \quad (1)$$

But, if for the un-normalized triplets (T_1, I_1, F_1) and (T_2, I_2, F_2) we have

$$T_1 - I_1 - F_1 > T_2 - I_2 - F_2.$$

The *Previous Inequality (1)* may not hold for some $D_2 < D_1$.

II. The accuracy function:

Un-Normalized Triplets

Let $A(T_1, I_1, F_1) > A(T_2, I_2, F_2)$, then $T_1 - F_1 > T_2 - F_2$.

Normalized Triplets

$$A(\frac{T_1}{D_1}, \frac{I_1}{D_1}, \frac{F_1}{D_1}) > A(\frac{T_2}{D_2}, \frac{I_2}{D_2}, \frac{F_2}{D_2}),$$

or $\frac{T_1}{D_1} - \frac{F_1}{D_1} > \frac{T_2}{D_2} - \frac{F_2}{D_2}$,

or $\frac{T_1 - F_1}{D_1} > \frac{T_2 - F_2}{D_2}$,

or $T_1 - F_1 > (T_2 - F_2) \cdot \frac{D_1}{D_2}$.

Similarly, this inequality may differ for some $D_2 < D_1$.

III. The certainty function:

Un-Normalized Triplets

Let $C(T_1, I_1, F_1) > C(T_2, I_2, F_2)$,

or $T_1 > T_2$.

Normalized Triplets

$$C(\frac{T_1}{D_1}, \frac{I_1}{D_1}, \frac{F_1}{D_1}) > C(\frac{T_2}{D_2}, \frac{I_2}{D_2}, \frac{F_2}{D_2}),$$

or $\frac{T_1}{D_1} > \frac{T_2}{D_2}$,

or $T_1 > T_2 \cdot \frac{D_1}{D_2}$, which may not hold for some $D_2 < D_1$.

This analysis shows that the normalized ranking order of neutrosophic triplets can differ from their unnormalized ranking order under certain conditions. In the next section, we illustrate these scenarios with specific counter-examples.

4 | Counter-Examples Demonstrating Rank order Reversal

To illustrate the theoretical results, we present counter-examples showing cases where the ranking order of neutrosophic triplets changes after normalization. These examples demonstrate the impact of normalization on the score, accuracy, and certainty functions, highlighting instances where inequalities between triplets are reversed.

4.1 | Counter-Example for the Score Function

Un-Normalized Neutrosophic Triplets

Let $N_1 = (0.2, 0, 0)$ and $N_2 = (0.1, 0. 0)$. Using the score function, we calculate

$S(N_1) = 0.2 + (1 - 0) + (1 - 0) = 2.2.$

$S(N_2) = 0.1 + (1 - 0) + (1 - 0) = 2.1.$

Hence, $S(N_1) > S(N_2)$ indicates that N_1 ranks higher than N_2 in the unnormalized form.

Normalized Neutrosophic Triplets:

I. Normalized $N_1 = (\frac{0.2}{0.2+0+0}, \frac{0}{0.2+0+0}, \frac{0}{0.2+0+0}) = (1, 0, 0).$

II. Normalized $N_2 = (\frac{0.1}{0.1+0+0}, \frac{0}{0.1+0+0}, \frac{0}{0.1+0+0}) = (1, 0, 0).$

We see that normalizedN_1 = normalizedN_2, and of course:

$S(\text{normalized}N_1) = S(\text{normalized}N_2) = 1 + (1 - 0) + (1 - 0) = 3.$

Therefore, $S(N_1) > S(N_2)$ but, after normalization,

$S(\text{normalized}N_1) = S(\text{normalized}N_2).$

The inequality was transformed into equality.

Second Counter-Example: Consider the triplets $N_1 = (0, 0.8, 0.8)$ and $N_2 = (0.1, 0.9. 0.9)$. So we have:

$S(N_1) = 0 + (1 - 0.8) + (1 - 0.8) = 0.4,$

$S(N_2) = 0.1 + (1 - 0.9) + (1 - 0.9) = 0.3.$

Whence $S(N_1) > S(N_2)$.

Normalized Triplets:

I. Normalized $N_1 = (\frac{0}{0+0.8+0.8}, \frac{0.8}{0+0.8+0.8}, \frac{0.8}{0+0.8+0.8}) = (0, 0.5, 0.5).$

II. Normalized $N_2 = (\frac{0.1}{0.1+0.9+0.9}, \frac{0.9}{0.1+0.9+0.9}, \frac{0.9}{0.1+0.9+0.9}) = (\frac{1}{19}, \frac{9}{19}, \frac{9}{19}).$

$S(\text{normalized}N_1) = 0 + (1 - 0.5) + (1 - 0.5) = 1$

$S(\text{normalized}N_2) = \frac{1}{19} + (1 - \frac{9}{19}) + (1 - \frac{9}{19}) = \frac{1}{19} + \frac{18}{19} + \frac{18}{19} = \frac{37}{19} = 1.94\ldots$

whence $S(\text{normalized}N_1) < S(\text{normalized}N_2)$.

Therefore, $S(N_1) > S(N_2)$, but $S(\text{normalized}N_1) < S(\text{normalized}N_2)$.

4.2 | Counter-Example for the Accuracy Function

Unnormalized Neutrosophic Triplets: Let $N_1 = (0.9, 0.8, 0.6)$ and $N_2 = (0.8, 0.1, 0.6)$. Using the accuracy function, we calculate $A(N_1) = 0.9 - 0.6 = 0.3$, $A(N_2) = 0.8 - 0.6 = 0.2$.

Thus $A(N_1) > A(N_2)$, so N_1 ranks higher than N_2 in terms of accuracy in the unnormalized form.

Normalized Neutrosophic Triplets:

I. Normalized $N_1 = (\frac{9}{23}, \frac{8}{23}, \frac{6}{23}).$

II. normalized $N_2 = (\frac{8}{15}, \frac{1}{15}, \frac{6}{15})$.

$A(\text{normalized}N_1) = \frac{9}{23} - \frac{6}{23} = \frac{3}{23} = 0.130434...$

$A(\text{normalized}N_2) = \frac{8}{15} - \frac{6}{15} = \frac{2}{15} = 0.333333...$

Thus, $A(\text{normalized}N_1) < A(\text{normalized}N_2)$.

Hence, from $A(N_1) > A(N_2)$ one arrives by normalization to the opposite inequality: $A(\text{normalized}N_1) < A(\text{normalized}N_2)$. This reversal illustrates how normalization can impact accuracy-based ranking.

4.3 | Counter-Example for the Certainty Function

Un-Normalized Neutrosophic Triplets:

Let $N_1 = (0.9, 0.8, 0.6)$ and $N_2 = (0.8, 0.3, 0.4)$. Using the certainty function $C(T,I,F)=T$, we compute:

$C(N_1) = 0.9$, $C(N_2) = 0.8$.

Thus, $C(N_1) > C(N_2)$, so N_1 ranks higher than N_2 in terms of certainty in the unnormalized form.

Normalized Neutrosophic Triplets:

I. Normalized $N_1 = (\frac{9}{23}, \frac{8}{23}, \frac{6}{23})$.

II. Normalized $N_2 = (\frac{8}{15}, \frac{3}{15}, \frac{4}{15})$.

III. $C(\text{normalized}N_1) = \frac{9}{23} = 0.130434...$

IV. And $C(\text{normalized}N_2) = \frac{8}{15} = 0.533333...$

As such, from $C(N_1) > C(N_2)$, one gets, after normalization, the opposite inequality: $C(\text{normalized}N_1) < C(\text{normalized}N_2)$.

This result shows that even certainty-based ranking can be affected by normalization.

These counter-examples confirm that normalization can lead to reversals in the ranking order of neutrosophic triplets across all three ranking functions. This inconsistency suggests that normalization may introduce significant biases in applications requiring stable rankings, which could impact decision-making processes.

Remark 1. There are numerous cases when the inequalities of score, accuracy, and certainty functions, respectively, for un-normalized and normalized neutrosophic triplets, are the same, but also numerous other cases when the inequalities are respectively opposite to each other, or they become equalities as shown in the previous counter-examples.

Remark 2. This proves, once more, that the ranking of the neutrosophic triplets (T, I, F) is, in general, different from the ranking of the intuitionistic fuzzy set duplets (T, F) or triplets (T, H, F), where H is hesitancy and $T + H + F = 1$.

5 | Conclusion

In this study, we explored how normalization affects the ranking order of neutrosophic triplets (T, I, F), showing that it can lead to inconsistencies across score, accuracy, and certainty functions. Through theoretical

analysis and counter-examples, we demonstrated that normalization, while often used to facilitate comparison, can unintentionally alter ranking orders, potentially impacting decision-making and uncertainty modeling applications. Our findings underscore the need for caution when normalizing neutrosophic data, suggesting practitioners consider using normalized and unnormalized rankings or alternative methods to preserve consistency. This work lays the foundation for further research into ranking methods that balance standardization with reliability, fostering more robust applications of neutrosophic theory in fields that rely on accurately handling indeterminate information.

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Data Availability

Data is not available due to privacy.

Conflicts of Interest

The authors declare no conflict of interest.

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La neutrosophie dans l'intelligence artificielle

Avancées et Applications issues des Conférences Internationales Conjointes de la BARNA Management School (République Dominicaine) et de l'Université du Travail de l'Uruguay (6–8 août 2025)

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Abstract: Ce numéro spécial de la revue scientifique *Neutrosophic Sets and Systems* est consacré aux avancées et applications de la neutrosophie dans le domaine de l'intelligence artificielle (IA). Fruit d'une collaboration entre la BARNA Management School (République Dominicaine), l'Université du Travail de l'Uruguay (UTU) et l'Association Latino-Américaine des Sciences Neutrosophiques, ce volume regroupe des contributions issues de conférences tenues du 6 au 8 août 2025. L'article aborde l'insuffisance des systèmes d'IA traditionnels face à l'incertitude, à l'ambiguïté et à la contradiction, en proposant la neutrosophie comme une solution. La neutrosophie, en introduisant l'indétermination comme un troisième élément aux côtés du vrai et du faux, offre un cadre robuste pour modéliser la complexité de la connaissance humaine et surmonter les limitations des grands modèles de langage (LLM). Les articles couvrent un large éventail d'applications pratiques, démontrant le caractère interdisciplinaire de la neutrosophie et son potentiel.

Keywords: Neutrosophie, Intelligence Artificielle, Incertitude, Pensée Critique, Indétermination, Systèmes Neutrosophiques, Modèles de Langage, Logique, Plithogénie, Applications IA.

Nous avons le plaisir de vous présenter ce numéro spécial de la revue scientifique *Neutrosophic Sets and Systems*, consacré aux avancées et applications de la neutrosophie dans l'Intelligence Artificielle. Ce volume est le fruit d'un effort conjoint issu des conférences organisées entre la BARNA Management School de la République Dominicaine, l'Université du Travail de l'Uruguay (UTU) et l'Association Latino-Américaine des Sciences Neutrosophiques. Les conférences, qui se sont déroulées du 6 au 8 août 2025, portaient le titre : « Penser dans l'incertitude : Neutrosophie et Pensée Critique pour dépasser les limites de l'Intelligence Artificielle ».

L'événement en Uruguay, qui s'est tenu le 6 août dans la Salle d'Assemblée du siège de l'UTU à Montevideo, a constitué une journée mémorable de réflexion et un atelier interactif. Cette activité, organisée par la Direction Générale de l'Enseignement Technico-Professionnel – UTU, avait pour objectif de diffuser ce domaine novateur auprès des enseignants, étudiants, chercheurs et du grand public. La venue des conférenciers ne visait pas seulement à enrichir le débat académique, mais également à ouvrir une fenêtre stratégique pour forger des liens de coopération durables.

La conférence a réuni des chercheurs de renommée internationale dont les contributions ont enrichi les discussions et approfondi la compréhension collective de la neutrosophie. Leurs présentations ont offert aux participants une occasion unique d'explorer les fondements théoriques

de cette discipline émergente, une fusion innovante de la logique, des mathématiques, de la philosophie et de la pensée computationnelle, tout en mettant en lumière ses implications pratiques dans divers domaines.

La théorie philosophique et mathématique de la neutrosophie, conçue par le Dr Florentin Smarandache dans les années 1990, a transformé notre approche de la logique et du raisonnement. À la logique classique, fondée sur des principes binaires tels que vrai/faux ou appartenance/non-appartenance, la neutrosophie ajoute un troisième composant fondamental : l'indétermination. Cette approche nouvelle n'est pas seulement académique; elle constitue un outil indispensable pour naviguer dans la complexité du monde réel, caractérisé par l'incertitude, l'ambiguïté et la contradiction.

La pertinence de la neutrosophie est particulièrement notable dans le contexte de l'intelligence artificielle moderne. Même les systèmes d'IA les plus avancés rencontrent aujourd'hui des difficultés majeures face à des informations incomplètes, contradictoires ou imprécises. Les Grands Modèles de Langage (LLM), par exemple, fonctionnent en générant du texte probable à partir de motifs statistiques extraits d'immenses ensembles de données. Leur nature probabiliste peut les amener à produire des réponses qui, bien que statistiquement plausibles, sont parfois factuellement ou contextuellement incorrectes.

Comme l'écrivain et homme politique dominicain Juan Bosch l'a exprimé :

> *« C'est une loi de la nature : il n'existe rien de si bon qu'il ne laisse un résidu de quelque chose de mauvais, ni rien de si mauvais qu'il ne produise un certain résultat positif. »*

Cette observation illustre parfaitement la dualité des technologies aussi puissantes que l'IA.

L'application des principes neutrosophiques en intelligence artificielle offre un cadre robuste pour surmonter ces limitations. En incluant explicitement l'indétermination comme un composant distinct, aux côtés de la vérité et de la fausseté, les systèmes neutrosophiques peuvent modéliser plus fidèlement la complexité des connaissances humaines et de la prise de décision dans des environnements incertains.

Les articles de ce numéro spécial de la revue scientifique *Neutrosophic Sets and Systems* couvrent un large éventail d'applications pratiques. La rencontre entre la BARNA Management School et l'Université du Travail de l'Uruguay dépasse la simple collaboration académique; elle représente la convergence de différentes perspectives sur l'avenir de l'intelligence artificielle. L'UTU, avec son accent sur l'enseignement technico-professionnel, fournit une vision pratique et appliquée. La BARNA, pour sa part, apporte son expertise dans la gestion des technologies émergentes. Cette synergie institutionnelle démontre le caractère intrinsèquement interdisciplinaire de la neutrosophie et son potentiel dans le champ de l'IA.

L'importance de ce travail dépasse la dimension purement technique. À une époque où l'IA redéfinit nos façons de travailler et d'interagir, la capacité à gérer l'incertitude devient une compétence cruciale. Le « paradoxe du déchargement cognitif » illustre comment une dépendance excessive à la technologie peut affaiblir nos facultés cognitives naturelles, rendant d'autant plus impératif de préserver et de renforcer nos compétences en pensée critique à mesure que le monde devient plus complexe.

Notre avantage compétitif unique sur les systèmes d'IA réside dans la pensée critique : la capacité d'analyser et de juger la cohérence du raisonnement. Comme l'a magistralement exprimé José Martí :

> *« Être un grand logicien est plus important que d'être un grand gymnaste, car la logique nous enseigne à raisonner, le raisonnement conduit à la sagesse, et la sagesse – disait Lord Bacon – est puissance. »*

L'IA actuelle fournit souvent des réponses unidimensionnelles basées sur les modèles les plus fréquents de ses données d'entraînement. Cependant, le perspectivisme humain, la capacité naturelle à considérer simultanément plusieurs points de vue, nous permet d'intégrer des perspectives culturelles, historiques et personnelles d'une manière que les systèmes d'IA actuels ne peuvent pas reproduire sans orientation explicite. La compréhension contextuelle est un autre domaine où la pensée humaine reste largement supérieure.

Ce numéro spécial de la revue scientifique *Neutrosophic Sets and Systems* aborde également les implications plus larges de la fusion entre neutrosophie et IA. La croissance rapide des publications sur la neutrosophie, en particulier dans les applications à l'IA, montre que la communauté scientifique reconnaît l'importance de disposer de cadres conceptuels capables de traiter l'indétermination de manière rigoureuse et systématique.

La plithogénie, une généralisation traitant des degrés d'appartenance à de multiples attributs, élargit encore la complexité et l'utilité de ces idées. La combinaison de technologies émergentes comme la blockchain, l'informatique quantique et les systèmes autonomes avec les technologies existantes ouvre de nouvelles voies pour gérer l'indétermination dans des environnements de haute technologie.

La perspective latino-américaine enrichit encore cette réflexion. Les cultures Quechuas, Aymaras, Mayas et guaranies ont développé des concepts tels que la relationalité (principe *d'ayni*), la dualité complémentaire (*yanantin*) et l'indétermination ontologique, qui présentent de fortes analogies avec les idées neutrosophiques. Le principe *In Lak'ech – Hala Ken* (« Je suis un autre toi, tu es un autre moi ») constitue une éthique relationnelle qui transcende la pensée binaire occidentale et offre des pistes précieuses pour concevoir des systèmes d'IA plus ouverts et sensibles au contexte.

Les articles de ce numéro spécial de la revue scientifique *Neutrosophic Sets and Systems* sont des travaux novateurs et originaux qui font progresser la théorie et la pratique de la neutrosophie appliquée à l'intelligence artificielle. Chaque contribution a été rigoureusement évaluée par des pairs experts. Nous espérons que ce numéro spécial inspirera de futures recherches et collaborations entre la neutrosophie et l'IA. Les défis contemporains, tels que le changement climatique et la transformation numérique, sont trop complexes pour être abordés par des idées simplistes. Nous avons besoin de cadres conceptuels capables de traiter l'incertitude de manière intelligente et nuancée.

Enfin, nous sommes conscients que l'avenir de l'IA ne réside pas dans le remplacement de l'intelligence humaine, mais dans la symbiose entre systèmes artificiels et capacités humaines, permettant un enrichissement mutuel.

Comme le rappelait Mario Benedetti dans son poème "No te rindas" (N'abandonne pas) :

« N'abandonne pas, s'il te plaît, ne cède pas,
même si le froid brûle, même si la peur morde,
même si le soleil se couche, et le vent se tue,
il y a encore du feu dans ton âme
il y a encore de la vie dans tes rêves. »

Ces mots résonnent ici comme un rappel : malgré les avancées technologiques, la créativité humaine, l'intuition et la capacité à donner du sens à l'incertitude demeurent essentielles dans notre quête de connaissance et de compréhension.

Nous invitons nos lecteurs à explorer les articles de ce volume de la revue scientifique *Neutrosophic Sets and Systems* avec un esprit ouvert et critique. La véritable sagesse ne consiste pas à choisir entre technologie et humanité, mais à trouver l'équilibre juste où les deux peuvent s'enrichir mutuellement.

4. Applications

Authors should discuss the results and how they can be interpreted in perspective of previous studies and of the working hypotheses. The findings and their implications should be discussed in the broadest context possible. Future research directions may also be highlighted.

Conclusion

Ce numéro spécial de *Neutrosophic Sets and Systems* met en évidence le rôle crucial de la neutrosophie dans l'avenir de l'intelligence artificielle. Les articles publiés dans ce numéro spécial démontrent comment la prise en compte explicite de l'indétermination peut surmonter les limitations des systèmes d'IA actuels, en leur permettant de gérer de manière plus efficace et nuancée les

informations complexes, contradictoires ou incomplètes. Au-delà des avancées techniques, cette collaboration souligne l'importance d'une perspective holistique, intégrant les savoirs académiques, pratiques et même culturels pour développer une IA plus robuste et éthique. En fin de compte, la véritable sagesse ne réside pas dans le remplacement de l'intelligence humaine par la technologie, mais dans une symbiose mutuellement enrichissante. L'IA peut servir d'outil puissant, mais c'est la pensée critique, la créativité et la capacité à donner du sens à l'incertitude qui demeureront les atouts essentiels de l'humanité dans un monde de plus en plus piloté par les données.



Session d'ouverture de la conférence à la BARNA Management School, à Saint-Domingue, République Dominicaine.



Conférence dans la Salle d'Assemblée de l'UTU à Montevideo, Uruguay, avec la participation des autorités, du corps enseignant, des étudiants et du grand public.



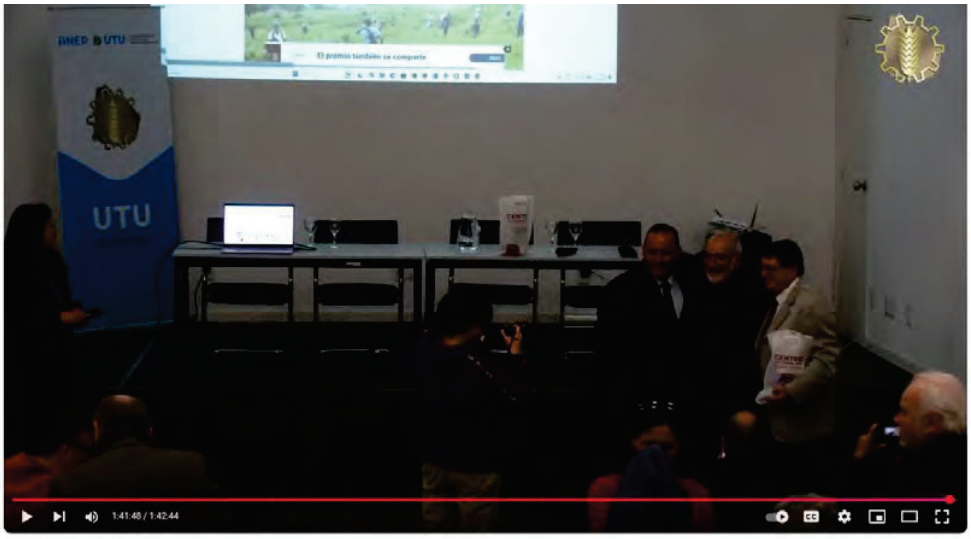
Participants lors de la séance d'ouverture à la BARNA Management School, Saint-Domingue, République Dominicaine.



Public assistant à la conférence dans la Salle d'Assemblée de l'UTU à Montevideo, Uruguay.



L'un des lieux de l'événement, l'Instituto de Alta Especialización José F. Arias (IAE Montevideo – UTU), où se sont tenues des sessions académiques et des rencontres. Tout à gauche, le Dr Manuel Enrique Puebla Martínez, chercheur éminent et principal organisateur des sessions en Uruguay.



Clôture de la conférence « Penser dans l'incertitude » à l'UTU, Montevideo.



L'événement a été couvert par la presse et largement relayé sur les réseaux sociaux.



Karina Pérez Teruel, PhD, professeure à la BARNA Management School.



Dr. Florentin Smarandache, creator of neutrosophy, delivering his presentation on the theoretical foundations and interdisciplinary applications of the discipline during the conference at the Assembly Hall of the Universidad del Trabajo del Uruguay (UTU).

Remerciements : Nous remercions Mademoiselle Jeanne Serry pour la révision en langue française du texte de cette présentation.

Liens vers les actualités et publications consultées

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Neutrosofia în Inteligența Artificială

Progrese și Aplicații prezentate la Conferințele Internaționale comune
ale BARNA Management School din Republica Dominicană
și Universidad del Trabajo del Uruguay
(6–8 august 2025)

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Suntem încântați să prezentăm un număr special al revistei „Neutrosophic Sets and Systems", dedicat progreselor și aplicațiilor neutrosofiei în domeniul Inteligenței Artificiale. Acest volum este rezultatul unui efort comun care își are originea în conferințele organizate între BARNA Management School din Republica Dominicană, Universidad del Trabajo del Uruguay (UTU) și Asociația Latino-Americană a Științelor Neutrosofice. Conferințele, desfășurate în perioada 6–8 august 2025, au purtat titlul „Gândirea în incertitudine: Neutrosofia și gândirea critică pentru depășirea limitelor Inteligenței Artificiale."

Evenimentul din Uruguay, desfășurat pe 6 august în Sala de Adunări a sediului central al UTU din Montevideo, a reprezentat o zi remarcabilă de reflecție și un atelier interactiv. Această activitate, organizată de Direcția Generală de Educație Tehnico-Profesională – UTU, a fost concepută pentru a disemina acest domeniu inovator de cunoaștere în rândul profesorilor, studenților, cercetătorilor și publicului larg. Vizita conferențiarilor a avut ca scop nu doar îmbogățirea dezbaterii academice, ci și deschiderea unei ferestre strategice pentru consolidarea unor legături de cooperare durabile.



Sesiunea de deschidere a conferinței a avut loc la BARNA Management School din Santo Domingo, Republica Dominicană.

Conferința a reunit cercetători de renume internațional, ale căror contribuții au îmbogățit discuțiile și au aprofundat înțelegerea colectivă a neutrosofiei. Prezentările lor au oferit participanților o oportunitate unică de a explora fundamentele teoretice ale acestei discipline emergente — o fuziune inovatoare între logică, matematică, filosofie și gândire computațională — evidențiind totodată implicațiile sale practice în domenii diverse.

Teoria filosofică și matematică a neutrosofiei, concepută de dr. Florentin Smarandache în anii 1990, a transformat modul nostru de a aborda logica și raționamentul. La logica clasică, bazată pe principii binare precum adevărat/fals sau apartenență/neapartenență, neutrosofia adaugă o a treia componentă fundamentală: indeterminarea. Această nouă abordare nu este doar una academică; ea constituie un instrument indispensabil pentru navigarea în complexitatea lumii reale, caracterizată de incertitudine, ambiguitate și contradicție.



Conferință în Sala de Adunări a UTU din Montevideo, Uruguay, cu participarea autorităților, cadrelor didactice, studenților și publicului larg.

Relevanța neutrosofiei este deosebit de semnificativă în contextul inteligenței artificiale moderne. Chiar și cele mai avansate sisteme de IA din prezent se confruntă cu dificultăți majore atunci când trebuie să proceseze informații incomplete, contradictorii sau neclare. Modelele de limbaj de mari dimensiuni (LLM), de exemplu, operează prin generarea de text probabilistic pe baza tiparelor statistice extrase din seturi masive de date. Natura lor probabilistică le poate conduce la producerea unor răspunsuri care, deși statistic posibile, pot fi factual sau contextual eronate.

Scriitorul și politicianul dominican Juan Bosch a afirmat cu elocvență: „Este o lege a naturii că nu există nimic atât de bun încât să nu lase un sediment de ceva rău, și nimic atât de rău încât să nu producă un rezultat bun.” Această observație surprinde perfect dualitatea tehnologiilor la fel de puternice precum inteligența artificială.



Participanți în timpul sesiunii de deschidere la BARNA Management School, Santo Domingo, Republica Dominicană, implicați în activitățile conferinței.

Aplicarea principiilor neutrosofice în inteligența artificială oferă un cadru solid pentru depășirea acestor limitări. Prin includerea explicită a indeterminării ca o componentă distinctă, alături de adevăr și falsitate, sistemele neutrosofice pot modela mai fidel complexitatea cunoașterii umane și a procesului decizional în medii incerte.

Articolele din acest număr special al „Neutrosophic Sets and Systems" acoperă o gamă largă de aplicații practice. Întâlnirea de la BARNA Management School și Universidad del Trabajo del Uruguay reprezintă mai mult decât o simplă colaborare academică; ea simbolizează convergența diferitelor perspective asupra viitorului inteligenței artificiale. UTU, prin orientarea sa către educația tehnico-vocațională, oferă o viziune practică și aplicată. BARNA, la rândul său, contribuie cu expertiza sa în managementul tehnologiilor emergente. Această sinergie instituțională demonstrează natura intrinsec interdisciplinară a neutrosofiei și potențialul său în domeniul inteligenței artificiale.



Publicul prezent la conferința desfășurată în Sala de Adunări a Universidad del Trabajo del Uruguay (UTU) din Montevideo, participând activ la prezentări.

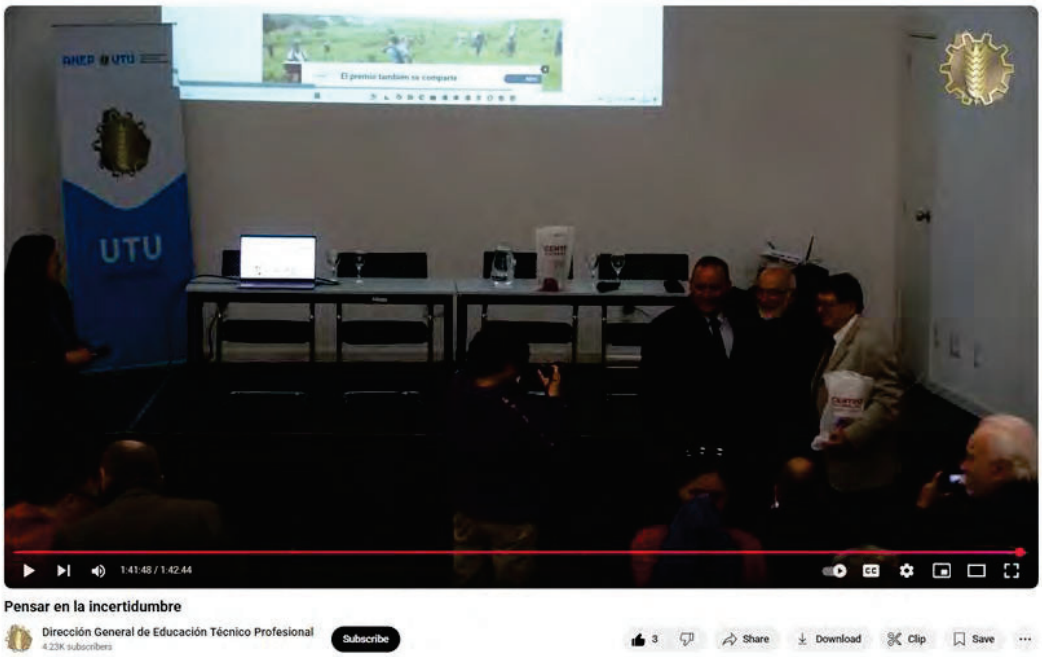
Importanța acestui demers depășește dimensiunea pur tehnică. Într-o eră în care inteligența artificială redefinește modul nostru de a lucra și de a interacționa, capacitatea de a gestiona incertitudinea devine o competență esențială. „Paradoxul descărcării cognitive" evidențiază modul în care o dependență excesivă de tehnologie poate slăbi facultățile noastre cognitive naturale, făcând cu atât mai imperativă păstrarea și cultivarea gândirii critice, pe măsură ce lumea devine tot mai complexă.

Avantajul nostru competitiv unic în raport cu sistemele de inteligență artificială constă în gândirea critică: capacitatea de a analiza și de a evalua coerența raționamentului. José Martí a exprimat magistral această idee: „A fi un mare logician este mai important decât a fi un mare gimnast, deoarece logica ne învață să raționăm, raționamentul conduce la înțelepciune, iar înțelepciunea — spunea Lord Bacon — este putere."



Una dintre locațiile evenimentului, Instituto de Alta Especialización José F. Arias (IAE Montevideo – UTU), unde s-au desfășurat sesiuni academice.
În stânga, dr. Manuel Enrique Puebla Martínez, cercetător distins și principalul organizator al sesiunilor din Uruguay.

Această afirmație a gânditorului cubanez are o relevanță deosebită astăzi, când abilitatea de a gândi logic și de a gestiona incertitudinea devine o competență fundamentală pentru a trăi într-o lume în care inteligența umană și cea artificială coexistă.



Momentele de închidere ale conferinței „Pensar en la incertidumbre”, desfășurată în Sala de Adunări a Universidad del Trabajo del Uruguay (UTU).
O parte a evenimentului a fost transmisă online și rămâne disponibilă pe *YouTube*.

Inteligența artificială actuală oferă adesea răspunsuri unidimensionale, bazate pe cele mai frecvente tipare din datele sale de antrenament. Totuși, perspectivismul uman — capacitatea naturală de a lua simultan în considerare multiple puncte de vedere — ne permite să integrăm perspective culturale, istorice și personale diverse, într-un mod pe care sistemele actuale de IA nu îl pot reproduce fără ghidare explicită.

Înțelegerea contextuală reprezintă un alt domeniu în care gândirea umană rămâne net superioară. A înțelege modul în care sensul se modifică în funcție de factori culturali, istorici, sociali sau situaționali necesită o abilitate interpretativă profundă, pe care sistemele de inteligență artificială nu o pot încă procesa adecvat doar prin informații contextuale implicite.



Dr. Karina Pérez Teruel, profesoară la BARNA Management School, una dintre organizatoare și o cercetătoare proeminentă în domeniul neutrosofiei.

Acest număr special al publicației “Neutrosophic Sets and Systems” abordează, de asemenea, implicațiile mai ample ale fuziunii dintre neutrosofie și inteligența artificială. Creșterea rapidă a numărului de

publicații dedicate neutrosofiei, în special în aplicațiile legate de IA, demonstrează că comunitatea științifică recunoaște importanța existenței unor cadre conceptuale capabile să gestioneze indeterminarea într-un mod riguros și sistematic.



Dr. Florentin Smarandache, creatorul neutrosofiei, participând la conferința desfășurată în Sala de Adunări a Universidad del Trabajo del Uruguay (UTU). Participarea sa a oferit o perspectivă neprețuită asupra originii, fundamentelor teoretice și aplicațiilor neutrosofiei.

Plitogenia, o generalizare care abordează gradele de apartenență pentru atribute multiple, extinde și mai mult complexitatea și utilitatea acestor concepte. Combinarea tehnologiilor emergente precum *blockchain*, calculul cuantic și sistemele autonome cu tehnologiile existente ar putea deschide noi direcții pentru gestionarea indeterminării în medii tehnologice avansate.

Perspectiva latino-americană adaugă o dimensiune unică acestei discuții. Culturile indigene Quechua, Aymara, Maya și Guaraní au dezvoltat concepte precum relaționalitatea (principiul *ayni*), dualitatea complementară (*yanantin*) și indeterminarea ontologică, care prezintă o asemănare remarcabilă cu ideile neutrosofice. Principiul *In Lak'ech – Hala Ken* („Eu

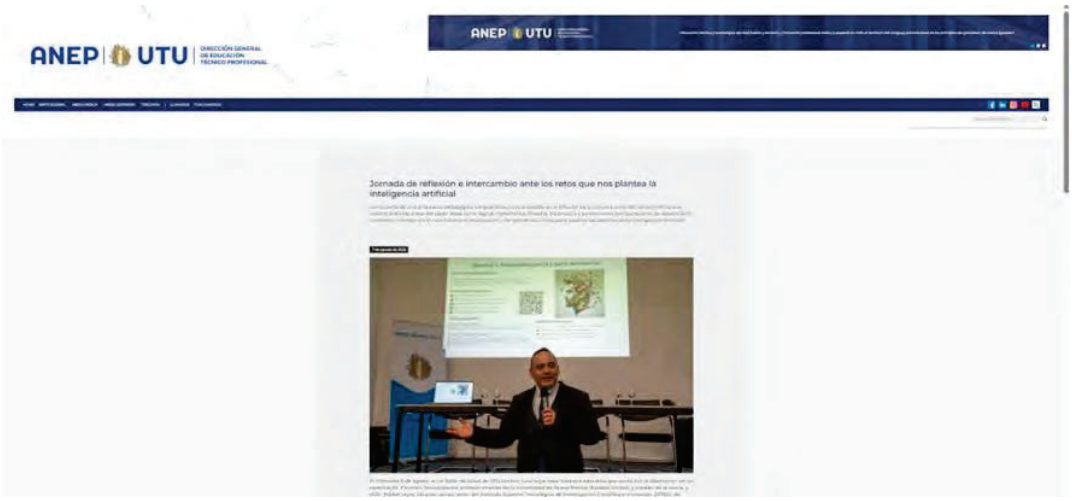
sunt un alt tu, tu ești un alt eu") reprezintă o etică relațională ce transcende gândirea binară occidentală și oferă perspective valoroase pentru dezvoltarea unor sisteme de inteligență artificială mai deschise și mai sensibile la context.



Dr. Florentin Smarandache, creatorul neutrosofiei, susținând prezentarea sa privind fundamentele teoretice și aplicațiile interdisciplinare ale disciplinei în cadrul conferinței desfășurate în Sala de Adunări a Universidad del Trabajo del Uruguay (UTU).

Articolele din acest număr special al „Neutrosophic Sets and Systems" reprezintă lucrări noi și originale care avansează teoria și practica neutrosofiei aplicate în inteligența artificială. Fiecare contribuție a fost revizuită riguros de experți în domeniu. Sperăm că această ediție specială a revistei va inspira cercetări viitoare și colaborări între neutrosofie și inteligența artificială. Provocările actuale, precum schimbările climatice și transformarea digitală, sunt prea complexe pentru idei simpliste. Avem nevoie de cadre conceptuale care să abordeze incertitudinea într-un mod inteligent și nuanțat. Neutrosofia oferă un drum util pentru reflecția asupra acestor probleme, integrând adevărul, falsitatea și incertitudinea într-un singur cadru.

În final, suntem conștienți că viitorul inteligenței artificiale nu constă în înlocuirea inteligenței umane, ci în simbioza dintre sistemele artificiale și capacitățile umane, pentru a se completa reciproc. Neutrosofia ne oferă un cadru conceptual pentru a reflecta asupra acestei colaborări, clarificând punctele forte și slăbiciunile fiecărui tip de inteligență.



Evenimentul a fost mediatizat de presă și distribuit pe scară largă pe rețelele sociale.

Mario Benedetti, în poezia sa „No te rindas” („Nu te da bătut”), ne reamintește că „încă există foc în sufletul tău, încă există viață în visele tale.” În contextul acestui număr special, aceste cuvinte ne sugerează că, în ciuda progreselor tehnologice, creativitatea umană, intuiția și capacitatea de a găsi sens în incertitudine rămân componente esențiale ale căutării noastre de cunoaștere și înțelegere.

Îi invităm pe cititorii noștri să exploreze articolele din acest volum cu mintea deschisă și spirit critic. Adevărata înțelepciune nu constă în a alege între tehnologie și umanitate, ci în a găsi echilibrul potrivit, în care ambele să se îmbogățească reciproc.

Linkuri către știrile și publicațiile consultate

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Greece	1
India	10
Indonesia	1
Iran	2
Japan	1
Pakistan	12
Philippines	2
Romania	4
Saudi Arabia	3
Serbia	3
Sudan	1
Syria	1
Turkey	3
USA	3
Vietnam	10

Collected Papers, Vol. XVII – A Multi-Disciplinary Approach to Uncertainty, Indeterminacy, and Complex Systems is a substantial and wide-ranging volume, bringing together 74 articles across nearly 1,000 pages. The contributions, originally published in a variety of international scientific journals, reflect the breadth and maturity of contemporary research in neutrosophy and its related domains.

The volume explores a rich spectrum of interdisciplinary fields, including logic, philosophy, physics, mathematics, statistics, information fusion, artificial intelligence, decision theory, complex systems, and robotics, offering both foundational developments and applied methodologies for modeling uncertainty, indeterminacy, and contradiction.

All papers are authored or co-authored by Florentin Smarandache, either individually or in collaboration with 69 distinguished co-authors from 18 countries, underscoring the global reach and collaborative nature of neutrosophic research, and revealing a coherent yet diverse intellectual landscape, ranging from theoretical frameworks—such as neutrosophic logic, hypersoft sets, superhyperstructures, and revolutionary topologies—to practical applications in healthcare analytics, artificial intelligence, sustainability, engineering, and decision-making under uncertainty.

Together, the works collected in this seventeenth volume provide both a comprehensive reference and a forward-looking perspective on the evolving role of neutrosophic and multi-valued approaches in addressing complex real-world problems.

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