(T, I, N, F) Neutrosophic Set and Logic
(Truth, Indeterminacy, Neutrality, Falsehood)

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Abstract

In this paper, we introduce for the first time:

the *(T, I, N, F) Neutrosophic Set/Logic/Measure/Probability/Statistics,*, which is a particular case of the Refined Neutrosophic Set/Logic [3] by refining Indeterminacy into: Pure Indeterminacy (with no neutrality) (I), and Neutrality (N), and as extension of the Neutrosophic Set/Logic/Measure/Probability/Statistics.

And the *Refined (T, I, N, F) Neutrosophic Set/Logic/Measure/Probability/Statistics*

as extension of the Refined Neutrosophic Set/Logic/Measure/Probability/Statistics.

Furthermore:

the *(T, I, N, F) Neutrosophic Over/Under/Off-Set/logic/Measure/Probability/Statistics*

as extension of the Neutrosophic Over/Under/Off-Set/Logic/Measure/Probability/Statistics

and

Refined (T, I, N, F) Neutrosophic Over/Under/Off-Set/Logic/Measure/Probability/Statistics

as extension of

the Refined Neutrosophic Over/Under/Off-Set/Logic/Measure/Probability/Statistics.

We designed the (T, I, N, F) Neutrosophic Operators, Also, we introduced the Score, Accuracy, Extended Certainty, and (non-extended) Certainty functions in order to determine a total order on the set of quadruples (T, I, N, F).

Keywords

Pure Indeterminacy, Neutrality,

Neutrosophic Set/Logic/Measure/Probability/Statistics,

(T, I, N, F) Neutrosophic Set/Logic/Measure/Probability/Statistics,

Refined Neutrosophic Set/Logic/Measure/Probability/Statistics,

Refined (T, I, N, F) Neutrosophic Set/Logic/Measure/Probability/Statistics,

(T, I, N, F) Neutrosophic Over/Under/Off-Set/Logic/Measure/Probability/Statistics,

Refined (T, I, N, F) Neutrosophic Over/Under/Off-Set/Logic/Measure/Probability/Statistics.

1. Introduction

Previously, we had introduced the **degree of Indeterminacy/Neutrality** (I / N) as independent, or partially independent and partially dependent, or totally dependent component in 1995 (published in 1998) and defined the neutrosophic set/logic/measure/probability/statistics on triad of components: (T, I, F) = (Truth, Indeterminacy or Neutrality, Falsehood), where in general T, I, F are subsets of the interval [0, 1], satisfying the inequalities below:

$$0 \le \inf(T) + \inf(I) + \inf(F) \le \sup(T) + \sup(I) + \sup(F) \le 3.$$

In particular case T, I, F may be single-numbers, or intervals, or hesitant sets [1].

Indeterminacy (or *Neutrality*), as independent or dependent component from the truth and the falsehood, was the main distinction between Neutrosophic Theories and other classical and fuzzy theory or fuzzy extension theories [2].

2. **(T, I, N, F) Neutrosophic Set/Logic/Measure/Probability/Statistics**.

Now, we extend for the first time the Triad (T, I, F) to the Quadruple (T, I, N, F), where T, I, N, F $\in \mathcal{P}([0, 1])$, which is the powerset of the interval [0, 1],

with T = Truth, I = Pure Indeterminacy (indeterminacy that excludes Neutrality), N = Neutrality, F = Falsehood, and we call it : *(T, I, N, F) Neutrosophic Set/Logic/Measure/Probability/Statistics.*

In the case that T, I, N, F are subsets of [0, 1] one has:

$$0 \le \inf(T) + \inf(I) + \inf(N) + \inf(F) \le \sup(T) + \sup(I) + \sup(N) + \sup(F) \le 4.$$

The components T, I, N, F may be totally independent, partially independent and partially dependent, or totally dependent.

3. **Refined Neutrosophic Components** (Set/Logic/Measure/Probability/Statistics)

In 2013 Smarandache [3] refined / split the Neutrosophic Components (T, I, F) into Neutrosophic SubComponents:

$(T_1, T_2, ...; I_1, I_2, ...; F_1, F_2, ...)$, where all subcomponents are single values, or intervals, or hesitant sets, or any subsets of [0, 1], and similarly may be totally independent, partially independent and partially dependent, or totally dependent.

4. **Refined (T, I, N, F) Neutrosophic Components**
(Set/Logic/Measure/Probability/Statistics)

$(T_1, T_2, ...; I_1, I_2, ...; N_1, N_2, ...; F_1, F_2, ...)$, where similarly all these subcomponents are single values, or intervals, or any subsets of [0, 1].

5. Neutrosophic Over/Under/Off-Set/Logic/Measure/Probability/Statistics

By "uncertain" set/logic we mean all types of fuzzy and fuzzy-extensions (intuitionistic fuzzy, neutrosophic, refined neutrosophic, spherical fuzzy, plithogenic, etc.) set/logic.

In 2017 the Uncertain Set was extended by Smarandache [7] to Uncertain **OverSet** (when some component is > 1), since we observed that, for example, an employee working overtime deserves a degree of membership > 1, with respect to an employee that only works regular full-time and whose degree of membership = 1;
and to Uncertain **UnderSet** (when some neutrosophic component from T, I, F is < 0), since, for example, an employee making more damage than benefit to his company deserves a degree of membership < 0, with respect to an employee that does not work but produces no damage (whose membership = 0), and an employee that produces benefit to the company and has the degree of membership > 0;
and to Uncertain **OffSet** (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and other neutrosophic component < 0).
Then, similarly, the uncertain Logic/Measure/Probability/Statistics etc. were extended to respectively uncertain **Over-/Under-/Off- Logic / Measure / Probability / Statistics** etc.

By "uncertain" we mean all types of fuzzy and fuzzy-extensions (intuitionistic fuzzy, neutrosophic, spherical fuzzy, plithogenic, etc.).

6. (T, I, N, F) Neutrosophic Over/Under/Off-Set/Logic/Measure/Probability/Statistics

Similarly,

(i) when some component from T, I, N, F is > 1 (and none of the others is < 0), one has a *(T, I, N, F) Neutrosophic Over-Set/Logic/Measure/Probability/Statistics*;

(ii) and if some component is < 0 (and none of the others is > 1) one has a *(T, I, N, F) Neutrosophic Under-Set/Logic/Measure/Probability/Statistics;*

(iii) while if some component is > 1 and other component is < 0, one has a *Neutrosophic Off-Set/Logic/Measure/Probability/Statistics.*

7. (T, I, N, F) Neutrosophic Set/Logic Operators

These operators - similarly to the Neutrosophic Set and Logic, are using the classical t-norms and t-conorms on their components T, I, N, F separately.

T and N are considered of positive value (the bigger, the better), while I and F have negative value (the bigger, the worse). If on T, N one applies the t-norms, then on I, F one must apply the t-conorm (the opposite), and vice versa.

8. How to deal with *N* (Neutrality)

Examples of Neutralities:

1. Assume there is a soccer game between team A and team B (which may be considered as the fight between the opposites Truth and Falsehood). The referee team is the *N* (Neutrality) part. If the referee team arbitrates correctly all playing time, then *N = 1* (Neutrality is 100%).

   If the referee team commits intentionally or unintentionally some mistakes (favorizing or de-favorizing one or both teams), then N < 1, let's say *N = 0.7*.

2. Two countries, C_1 and C_2 are at war. Other countries are neutral. Assume some neutral country politically pledges for C_1, then N < 1.

Neutrality is assumed to be in the middle of the opposites (T and F), the further is Neutrality from the middle (to the left, or to right side) the smaller is the value of neutrality.

9. t-norms and t-conorms

Let's first recall the classical t-norms and t-conorms that are used in the construction of the (T, I, N, F) Neutrosophic Set/Logic Operators.

Triangular Norms (t-norms)

9.1. Definition & Overview

A t-norm (triangular norm) is a binary operation T : [0, 1] × [0, 1] ∈ [0, 1] (or denoted as

T : [0, 1] × [0, 1] → [0, 1]) that generalizes logical conjunction (∧) to continuous multi-valued logic and fuzzy set theory.

9.2. Axiomatic Definition

A binary operation *T* is a t-norm if it satisfies the following four conditions for all x, y, z ∈ [0, 1]:

- Commutativity: T(x, y) = T(y, x)
- Associativity: T(T(x, y), z) = T(x, T(y, z))
- Monotonicity: $y \le z \implies T(x, y) \le T(x, z)$
- Boundary Condition (Identity): T(x, 1) = x

From these axioms, it directly follows that T(x, 0) = 0 for all x ∈ [0, 1].

9.3. Three Fundamental T-Norms

Łukasiewicz (T_L), $T_L(x,y) = \max(0, x + y - 1)$

Product (T_P), $T_P(x,y) = x \cdot y$

Minimum (T_m) (Zadeh), $T_M(x,y) = \min(x, y)$

9.4. Dual T-Conorms (Fuzzy Disjunction)

Every t-norm *T* has a dual t-conorm S (also referred to as an *s-norm*), which models logical disjunction (∨) using standard negation N(x) = 1 - x:

$$S(x, y) = 1 - T(1 - x, 1 - y)$$

Common dual pairs include:

- Minimum (T_M) pairs with Maximum (S_M): $S_M(x, y) = \max(x, y)$
- Product (T_P) pairs with Probabilistic Sum (S_P): $S\_P(x, y) = x + y - x{\cdot}y$
- Łukasiewicz (T_L) pairs with Bounded Sum (S_L): $S\_L(x, y) = \min(1, x + y)$

10. (T, I, N, F) Neutrosophic Set/Logic Operators

In the below formulas, we use the classical t-norms (simply denoted by ∧) and the classical t-conorm (∨).

10.1 (T, I, N, F) Neutrosophic Negation (Complement) (¬):

For any (T, I, N, F) ∈ Q, the negation

$$\neg(T, I, N, F) = (F, N, I, T)$$

Which satisfies the double negation axiom: $\neg(\neg(T, I, N, F)) = (T, I, N, F)$.

10.2 (T, I, N, F) Neutrosophic Intersection (Conjunction) (∧)

For any $(T_1, I_1, N_1, F_1), (T_2, I_2, N_2, F_2) \in Q$ one has

$$(T_1, I_1, N_1, F_1) \wedge (T_2, I_2, N_2, F_2) = (T_1 \wedge T_2, I_1 \vee I_2, N_1 \wedge N_2, F_1 \vee F_2)$$

where $T_1 \wedge T_2$ = t-norm(T_1, T_2), and $N_1 \wedge N_2$ = t-norm(N_1, N_2),

while $I_1 \vee I_2$ = t-conorm(I_1, I_2), and $F_1 \vee F_2$ = t-conorm(F_1, F_2).

10.3 (T, I, N, F) Neutrosophic Union (Disjunction) (∨)

For any $(T_1, I_1, N_1, F_1), (T_2, I_2, N_2, F_2) \in Q$ one has

$$(T_1, I_1, N_1, F_1) \vee (T_2, I_2, N_2, F_2) = (T_1 \vee T_2, I_1 \wedge I_2, N_1 \vee N_2, F_1 \wedge F_2)$$

where $T_1 \vee T_2$ = t-conorm(T_1, T_2), and $N_1 \vee N_2$ = t-conorm(N_1, N_2),

while $I_1 \wedge I_2$ = t-norm(I_1, I_2), and $F_1 \wedge F_2$ = t-norm(F_1, F_2).

10.4 (T, I, N, F) Neutrosophic Implication (→)

For any $(T_1, I_1, N_1, F_1), (T_2, I_2, N_2, F_2) \in Q$ one has

$$(T_1, I_1, N_1, F_1) \rightarrow (T_2, I_2, N_2, F_2) \equiv \neg(T_1, I_1, N_1, F_1) \vee (T_2, I_2, N_2, F_2)$$
$$\equiv (F_1, N_1, I_1, T_1) \vee (T_2, I_2, N_2, F_2)$$
$$\equiv (F_1 \vee T_2, N_1 \vee I_2, I_1 \wedge N_2, T_1 \wedge F_2).$$

10.5 (T, I, N, F) Neutrosophic Equivalence (↔)

$$(T_1, I_1, N_1, F_1) \leftrightarrow (T_2, I_2, N_2, F_2) \equiv (T_1, I_1, N_1, F_1) \rightarrow (T_2, I_2, N_2, F_2)$$
$$\text{and } (T_2, I_2, N_2, F_2) \rightarrow (T_1, I_1, N_1, F_1)$$

From the above implication formula, we have:

$$(T_1, I_1, N_1, F_1) \rightarrow (T_2, I_2, N_2, F_2) \equiv (F_1 \vee T_2, N_1 \vee I_2, I_1 \wedge N_2, T_1 \wedge F_2)$$

while

$$(T_2, I_2, N_2, F_2) \rightarrow (T_1, I_1, N_1, F_1) \equiv \neg(T_2, I_2, N_2, F_2) \vee (T_1, I_1, N_1, F_1)$$
$$\equiv (F_2, N_2, I_2, T_2) \vee (T_1, I_1, N_1, F_1)$$
$$\equiv (F_2 \vee T_1, N_2 \vee I_1, I_2 \wedge N_1, T_2 \wedge F_1)$$

And finally, the (T, I, N, F) neutrosophic equivalence becomes:

$$(T_1, I_1, N_1, F_1) \leftrightarrow (T_2, I_2, N_2, F_2) \equiv$$
$$\equiv (F_1 \vee T_2, N_1 \vee I_2, I_1 \wedge N_2, T_1 \wedge F_2) \quad \wedge \quad (F_2 \vee T_1, N_2 \vee I_1, I_2 \wedge N_1, T_2 \wedge F_1)$$
$$\equiv \big((F_1 \vee T_2) \wedge (F_2 \vee T_1),\ (N_1 \vee I_2) \wedge (N_2 \vee I_1),\ (I_1 \wedge N_2) \vee (I_2 \wedge N_1),\ (T_1 \wedge F_2) \vee (T_2 \wedge F_1) \big).$$

11. Single-Valued (T, I, N, F) Neutrosophic Score, Accuracy, Extended Certainty, and (Non-Extended) Certainty Functions

Borrowing and extending from the classical Neutrosophic Set/Logic field [6], we defined the single-valued score, accuracy, extended-certainty, and (non-extended) certainty functions.

Let Q be the set of single-valued (T, I, N, F) neutrosophic numbers,

$Q = \{(T, I, N, F)$, with $T, I, N, F \in [0, 1],\ 0 \leq T + I + N + F \leq 4\}$,

where *T = Truth, I = Indeterminacy, N = Neutrality, F = Falsehood.*

T, N are considered of positive quality; while *I, F* of negative quality – therefore their opposites *1- I* and *1- F* respectively are of positive quality.

(i) Single-Valued (T, I, N, F) Neutrosophic Score Function:

s : Q → [0, 1]

$s(T, I, N, F) = \frac{T+N+(1-I)+(1-F)}{4} = \frac{2+T+N-I-F}{4}$

This is the average of positive qualities.

(ii) Single-Valued (T, I, N, F) Neutrosophic Accuracy Function:

a : Q → [-1, 2]

$a(T, I, N, F) = T + N - F$

This is the difference between positive quality and negative quality.

(iii) Single-Valued (T, I, N, F) Neutrosophic Extended Certainty Function:

ec : Q → [0, 2]

$ec(T, I, N, F) = T + N$

This is the sum of the positive extended qualities.

(iv) Single-Valued (T, I, N, F) Neutrosophic (Non-Extended) Certainty Function:

c : Q → [0, 1]

$c(T, N, I, F) = T$

This is the non-extended positive quality.

12. Algorithm for Ranking the Single-Valued (T, I, N, F) Neutrosophic Quadruples

Let (T_1, I_1, N_1, F_1) *and* (T_2, I_2, N_2, F_2) be two single-valued neutrosophic quadruples from Q, i.e. $T_1, I_1, N_1, F_1, T_2, I_2, N_2, F_2 \in [0, 1]$.

Apply the *(T, I, N, F) Neutrosophic Score Function (s)*:

1. If $s(T_1, I_1, N_1, F_1) > s(T_2, I_2, N_2, F_2)$, then $(T_1, I_1, N_1, F_1) > (T_2, I_2, N_2, F_2)$.
2. If $s(T_1, I_1, N_1, F_1) < s(T_2, I_2, N_2, F_2)$, then $(T_1, I_1, N_1, F_1) < (T_2, I_2, N_2, F_2)$.
3. If $s(T_1, I_1, N_1, F_1) = s(T_2, I_2, N_2, F_2)$,
4. then apply the *(T, I, N, F) Neutrosophic Accuracy Function (a)*:

4.1 If $a(T_1, I_1, N_1, F_1) > a(T_2, I_2, N_2, F_2)$, then $(T_1, I_1, N_1, F_1) > (T_2, I_2, N_2, F_2)$.

4.2 If $a(T_1, I_1, N_1, F_1) < a(T_2, I_2, N_2, F_2)$, then $(T_1, I_1, N_1, F_1) < (T_2, I_2, N_2, F_2)$.

4.3 If $a(T_1, I_1, N_1, F_1) = a(T_2, I_2, N_2, F_2)$,

5. then apply the *(T, I, N, F) Neutrosophic Extended Certainty Function (ec)*.

5.1.1 If $ec(T_1, I_1, N_1, F_1) > ec(T_2, I_2, N_2, F_2)$, then $(T_1, I_1, N_1, F_1) > (T_2, I_2, N_2, F_2)$.

5.1.2 If $ec(T_1, I_1, N_1, F_1) < ec(T_2, I_2, N_2, F_2)$, then $(T_1, I_1, N_1, F_1) < (T_2, I_2, N_2, F_2)$.

3.3.1 If $ec(T_1, I_1, N_1, F_1) = ec(T_2, I_2, N_2, F_2)$,

6. then apply the (T, I, N, F) Neutrosophic (Non-extended) Certainty

Function (c)

6.1.1 If $c(T_1, I_1, N_1, F_1) > c(T_2, I_2, N_2, F_2)$, then $(T_1, I_1, N_1, F_1) > (T_2, I_2, N_2, F_2)$.

6.1.2 If $c(T_1, I_1, N_1, F_1) < c(T_2, I_2, N_2, F_2)$, then $(T_1, I_1, N_1, F_1) < (T_2, I_2, N_2, F_2)$.

3.3.2 If $c(T_1, I_1, N_1, F_1) = c(T_2, I_2, N_2, F_2)$,

7. then $(T_1, I_1, N_1, F_1) \equiv (T_2, I_2, N_2, F_2)$, i.e. $T_1 = T_2$, $I_1 = I_2$, $N_1, = N_2$, $F_1 = F_2$.

Proof:

We have four variables T, I, N, F and four equations involving them:

The first equation $(2 + T_1 + N_1 - I_1 - F_1)/4 = (2 + T_2 + N_2 - I_2 - F_2)/4$

is equivalent to $T_1 + N_1 - I_1 - F_1 = T_2 + N_2 - I_2 - F_2$

The system is:

(1) $T_1 + N_1 - I_1 - F_1 = T_2 + N_2 - I_2 - F_2$

(2) $T_1 + N_1 - F_1 = T_2 + N_2 - F_2$

(3) $T_1 + N_1 = T_2 + N_2$

(4) $T_1 = T_2$

Solution:

Replacing eq, (4) into eq. (3), from $T_1 = T_2$ we get $N_1 = N_2$.

From $T_1 = T_2$ and $N_1 = N_2$ replaced into eq. (2), we get $F_1 = F_2$.

Finally, replacing all we got so far ($T_1 = T_2$, $N_1 = N_2$, $F_1 = F_2$) into eq. (1), we get $I_1 = I_2$.

QED.

13. Conclusion

We have refined the neutrosophic Indeterminacy into Pure Indeterminacy (meaning indeterminacy without neutrality), denoted by I, and Neutrality (denoted by N). We try to give importance to neutrality as well. And we extended the Neutrosophic Set and Logic (based on three components T, I, F) to (T, I, N, F) Neutrosophic Set and Logic,

We provided the (T, I, N, F) Neutrosophic Operators and gave a total order the the set of quadruples (T, I, N, F) using the Score, Accuracy, Extended Certainty, and (non-extended) Certainty.

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