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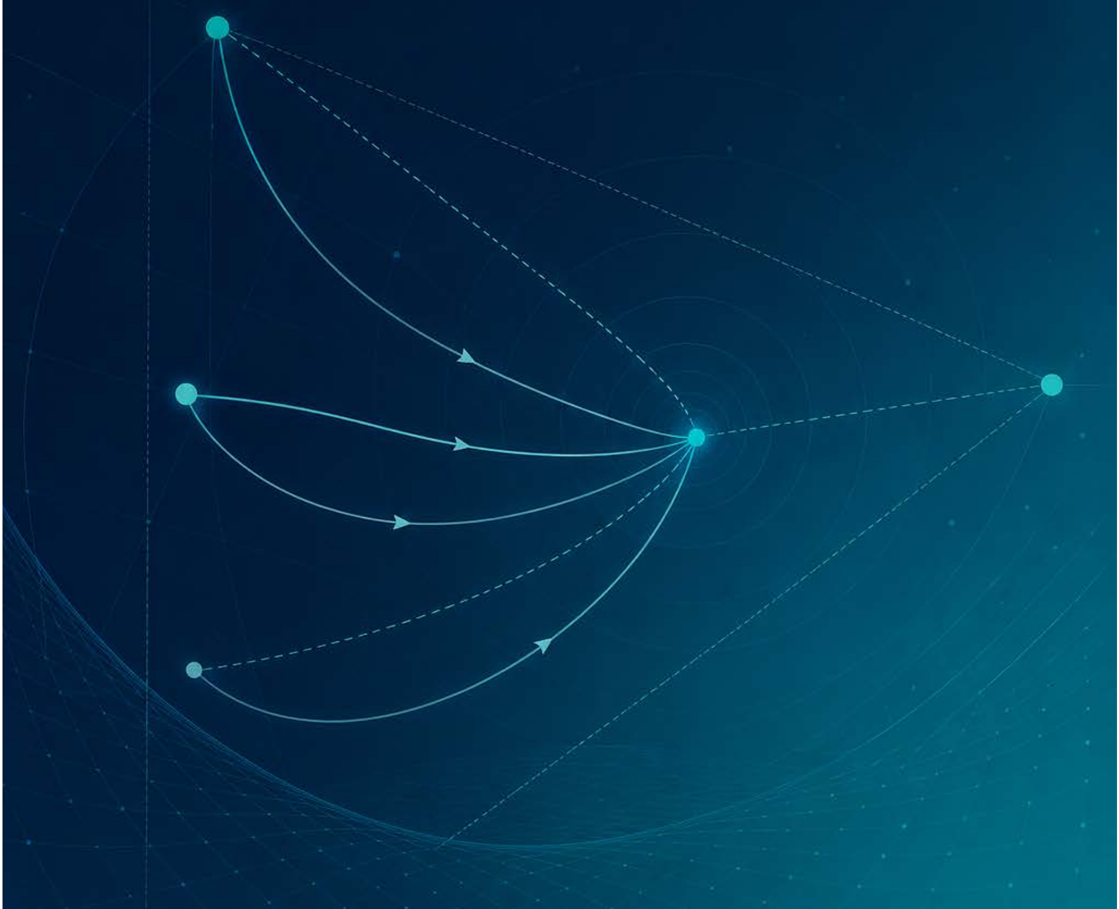
Scalarization, Quantitative Convergence, Stability, and Applications



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Neutrosophic Science International Association (NSIA)
Publishing House
2026

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Scope notice. This book is a new-direction manuscript developed from the general subject of the authors' article on integral-type common fixed points in neutrosophic metric spaces. It does not restate that article's integral CLR theorem, its corollaries, its four-map weak-compatibility argument, or its dynamic-programming theorem. The results here are organized around a different object: an exactly scalar-zable calibrated neutrosophic metric profile and the constructive algorithms that it supports. Claims of worldwide bibliographic priority should be made only after a separate database-level novelty search and independent peer review.

Neutrosophic Science International Association (NSIA)
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https://fs.unm.edu/NSIA/

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Preface

Neutrosophic metric fixed point theory is often written in a qualitative language: truth-nearness tends to one, indeterminacy and falsity tend to zero, and an appropriate contractive condition forces a fixed point. That language is useful, but it can conceal the numerical distance that controls an iteration. It can also make it difficult to answer the questions asked by a computational user. How many iterations are sufficient? How does an error in one step affect the final point? How sensitive is the solution to a perturbed model? Can a stopping rule be stated directly in the three neutrosophic components?

This monograph develops one answer. A rational calibrated profile encodes an ordinary metric exactly in a truth, indeterminacy, and falsity triple. The encoding is not decorative: it has an explicit inverse. Consequently every estimate can be read in two equivalent forms, one metric and one neutrosophic. The inverse permits proofs to be audited line by line, while the profile permits uncertainty to be reported componentwise. The central theme is therefore

$$\begin{aligned} \text{profile inequality} \quad &\Longleftrightarrow \quad \text{scalar distance inequality} \\ &\Longrightarrow \quad \text{computable certificate} \end{aligned}$$

The main new package consists of profile reconstruction, quantitative Picard and Kannan principles, block-iterate results, inexact and adaptive processes, Ulam–Hyers stability, data dependence, approximate commutation, contraction matrices, and applications to robust Bellman equations, nonlinear integral equations, fractional models, matrix equations, and network equilibria. Each principal statement has a proof. Conjectural extensions are explicitly labeled as research problems rather than presented as theorems.

The intended readers are researchers in fixed point theory, uncertainty modeling, nonlinear analysis, and computational mathematics. The text is also designed to serve as a proof-audit template: the final part isolates common logical failures and provides checklists for examples, applications, and numerical claims.

Contribution Map

The labels **N1–N28** identify the results developed in this book. They are not labels for results in the source article.

Label	Content
N1	Construction of a normalized rational neutrosophic metric from every metric.
N2	Converse recovery of a metric from a time-consistent rational profile.
N3	Exact equivalence of profile and metric topology, Cauchy property, and completeness.
N4	Multiscale reconstruction and perturbation bounds for noisy profile components.
N5	Equivalence between time-dilated profile contraction and scalar contraction.
N6	Quantitative profile Picard theorem with a priori, a posteriori, and residual bounds.
N7	Summable nonlinear comparison theorem with an explicit tail certificate.
N8	Quantitative profile Kannan theorem.
N9	Orbitwise variable-factor convergence criterion.
N10	Contractive-iterate theorem for maps whose finite power is contractive.

N11	Inexact Picard convolution bound and vanishing-error convergence.
N12	Geometric-error closed form and certified stopping rule.
N13	Shadowing bound for computed orbits.
N14	Ulam–Hyers stability in all three profile components.
N15	Data-dependence and parameter-Lipschitz estimates.
N16	Differentiability of a parameterized fixed point in Banach space.
N17	Anchor-contraction common fixed point theorem for a commuting family.
N18	Approximate-commutation displacement estimate.
N19	Alternating two-map cycle and a commutative common-fixed-point criterion.
N20	Spectral-radius theorem for coupled profile systems.
N21	Robust Bellman fixed point with value-iteration certificates.
N22	Robust Bellman model-perturbation estimate.
N23	Entropic Bellman contraction theorem.
N24	Bielecki-profile theorem for nonlinear Volterra equations.
N25	Coupled integral-system theorem via a Lipschitz matrix.
N26	Fractional initial-value profile theorem with a computable horizon condition.
N27	Nonlinear matrix-equation theorem and perturbation bound.
N28	Network-equilibrium theorem under a spectral-radius condition.

Contents

Part I

Orientation and Core Algorithmic Results

Chapter 1

Introduction

This chapter motivates an exactly calibrated neutrosophic metric profile, states the questions addressed by the monograph, and summarizes the mathematical and algorithmic contributions.

1.1 Background and motivation

Fixed point theory connects an equation with an iteration. The Banach contraction principle provides the canonical example: a contractive self-map on a complete metric space has a unique fixed point, and successive approximation converges to it [4]. Later principles weakened or reorganized the contractive hypothesis, including the Kannan, Meir–Keeler, multivalued, and generalized contractive frameworks [28, 38, 43, 8, 9]. These developments made fixed point methods useful far beyond existence proofs. When a contraction factor and a residual are available, the same argument can supply a convergence rate, a stopping rule, and a sensitivity estimate.

Uncertainty-oriented metric structures add a second line of development. Fuzzy set theory [68], statistical metric spaces [56], and fuzzy metrics [34, 10] replace a single distance value by a scale-dependent assessment. Intuitionistic fuzzy metrics add a non-membership component [47], while neutrosophic metric spaces use three components to represent truth-nearness, indeterminacy, and falsity [33]. This richer language is attractive when uncertainty is not well represented by one scalar observation.

Recently, The authors have made significant contributions to fuzzy, intuitionistic fuzzy, and neutrosophic mathematics, particularly in functional analysis, convergence theory, n-normed spaces, graph theory, and algebraic structures. Their recent works cover generalized functional analysis, advanced topics in neutrosophic mathematics, neutrosophic functional analysis, deferred statistical summability, completeness of neutrosophic rectangular n-normed spaces, deferred $\mathcal{I}$-statistical rough convergence, and rough I-γ_τ-statistical convergence [69, 70, 71, 72, 73, 74, 75]. Their related contributions further include rough I-statistical convergence, neutrosophic bidirected graphs, Fermatean intuitionistic fuzzy n-normed linear spaces, Tribonacci ideal and ideal-star convergence, and foundational topics in neutrosophic graph theory and algebra [76, 77, 78, 79, 80, 81, 82].

The extra components, however, create a quantitative interpretation problem. A typical neutrosophic fixed point theorem proves, for every $t > 0$, that

$$\mathsf{T}(x_n, p, t) \longrightarrow 1, \qquad \mathsf{I}(x_n, p, t) \longrightarrow 0, \qquad \mathsf{F}(x_n, p, t) \longrightarrow 0.$$

These limits identify convergence, but they do not automatically reveal the numerical distance from x_n to p. Nor do they by themselves answer the questions that arise in computation: How many iterations are sufficient? How should a tolerance in the three components be selected? What happens when each map evaluation is inexact? How does the fixed point change when the operator, data, or model parameters change?

This monograph addresses those questions by restricting attention to an explicitly calibrated class. Starting with an ordinary metric d and a fixed split parameter $\vartheta \in (0,1)$, it uses the rational profile

$$\begin{aligned} \mathsf{T}_d(x,y,t) &= \frac{t}{t+d(x,y)}, \qquad \mathsf{I}_d(x,y,t) = \vartheta\frac{d(x,y)}{t+d(x,y)}, \\ \mathsf{F}_d(x,y,t) &= (1-\vartheta)\frac{d(x,y)}{t+d(x,y)}. \end{aligned} \tag{1.1}$$

The central feature is not merely that equation (1.1) satisfies the required neutrosophic metric axioms. Each nondegenerate component has an explicit inverse, so the scalar distance can be reconstructed at every positive scale. The three-component description and the scalar error are therefore two exactly equivalent views of the same geometry.

1.2 Objectives and research questions

The primary objective is to build a constructive fixed point theory for calibrated neutrosophic profiles. More specifically, the monograph asks:

1. *Representation.* Under what conditions does a rational truth–indeterminacy–falsity profile encode one metric exactly, and how can that metric be reconstructed from noisy or multiscale observations?

2. *Equivalence.* Which topological and completeness properties are preserved by the profile, and when is a time-dilated profile contraction exactly a scalar contraction?

3. *Computation.* What a priori, a posteriori, and residual bounds follow for Picard, Kannan, nonlinear-comparison, variable-factor, and block iterations?

4. *Robustness.* How do summable evaluation errors, approximate commutation, model perturbations, and parameter variations prop-

agate to the fixed point?

5. *Applications.* How can one use the same calibrated machinery for robust Bellman operators, nonlinear integral equations, fractional models, matrix equations, and network equilibria?

The intended outcome is a certificate rather than a qualitative assertion alone. In the simplest contractive setting, the three basic certificate forms are

$$\begin{aligned} &\text{a priori:} && d(x_n, p) \le E_n\big(d(x_0, x_1)\big), \\ &\text{a posteriori:} && d(x_n, p) \le C\, d(x_n, x_{n-1}), \\ &\text{residual:} && d(x, p) \le C\, d(x, Ax). \end{aligned}$$

Exact calibration then converts any scalar upper bound into simultaneous lower or upper bounds for T_d, I_d, and F_d.

1.3 Principal contributions

The contributions N1–N28, listed in the front matter, form five connected groups.

1. **Exact representation and reconstruction.** Every metric generates a normalized rational neutrosophic profile, and a converse recovery theorem holds under time consistency. The metric and profile topologies, Cauchy properties, and completeness are equivalent. Multiscale estimators and perturbation bounds address noisy component observations.

2. **Quantitative fixed point principles.** Exact scalarization yields Picard and Kannan theorems with convergence rates, a priori and a posteriori bounds, and residual stopping criteria. Nonlinear comparison functions, variable orbitwise factors, and contractive finite iterates are treated separately.

3. **Inexactness and stability.** The book gives convolution bounds

for inexact Picard processes, geometric-error closed forms, shadowing estimates, Ulam–Hyers stability, data-dependence bounds, and parameter sensitivity.

4. **Coupled and family-valued operators.** Commuting families are studied through a contractive anchor. Coupled systems are controlled by a nonnegative Lipschitz matrix and a spectral-radius condition.

5. **Applications and proof audit.** The theory is applied to robust and entropic Bellman operators, integral and fractional equations, nonlinear matrix equations, and network equilibria. The audit framework records the domain, scale, comparison direction, constants, residuals, and numerical assumptions.

1.4 Boundary with the source article

The source article develops an integral-type common fixed point condition for four weakly compatible self-maps satisfying a common limit range property. The present book deliberately changes all four structural choices:

1. it studies one constructive orbit or a coupled block orbit rather than a CLR sequence;

2. it uses an invertible profile rather than an integral transform of a component;

3. it derives rates, residual bounds, and perturbation estimates rather than existence alone;

4. it treats robust and inexact computation as part of the theorem, not as an afterthought.

Thus, the common vocabulary of neutrosophic metrics is retained, but the mathematical results and proof mechanism are separate.

1.5 Three design principles

1.5.1 Calibration

The three components should have a known interpretation. If a triple is generated from a metric distance $d(x, y)$, one should be able to reconstruct that distance without guessing. This requirement rules out arbitrary component formulas when a quantitative theorem is claimed.

1.5.2 Scale covariance

Changing the positive scale t should not change the recovered distance. A profile may look different at $t = 1$ and $t = 10$, but both observations must encode the same pair (x, y). This gives a testable internal consistency condition.

1.5.3 Certificate first

An iterative theorem should end with a quantity that can be computed. The three most useful certificates are:

$$\begin{aligned} &\text{a priori:} && d(x_n, p) \le E_n(d(x_0, x_1)), \\ &\text{a posteriori:} && d(x_n, p) \le C\, d(x_n, x_{n-1}), \\ &\text{residual:} && d(x, p) \le C\, d(x, Ax). \end{aligned}$$

The book translates every one of these into lower and upper bounds for T, I, and F.

1.6 What "new" means in this manuscript

The word *new* is used in two controlled senses. First, none of N1–N28 is copied from the supplied article. Second, the definitions and theorem package are derived here as one coherent calibrated-profile framework. The classical metric ideas behind contraction arguments are acknowledged and cited; re-encoding a classical theorem without additional information is not presented as a discovery. The additional information in this book consists of exact profile equivalences, componentwise certificates, noisy-profile reconstruction, inexact-orbit bounds, matrix-coupled results, and the stated applications.

An exhaustive novelty assertion would require searches in MathSciNet, zbMATH, Scopus, Web of Science, Crossref, and full-text databases. That search is outside a mathematical proof. Accordingly, the responsible publication claim at this stage is that this is a distinct, newly developed manuscript direction whose global novelty must still be independently checked.

1.7 Reading paths and standing shorthand

Part I presents the algorithmic core first. Until the formal construction in Chapter 6, a *complete calibrated space* means a complete metric space (X, d) equipped, for a fixed $\vartheta \in (0,1)$, with

$$\begin{aligned} \mathsf{T}(x,y,t) &= \frac{t}{t+d(x,y)}, \\ \mathsf{I}(x,y,t) &= \frac{\vartheta d(x,y)}{t+d(x,y)}, \\ \mathsf{F}(x,y,t) &= \frac{(1-\vartheta)d(x,y)}{t+d(x,y)}. \end{aligned}$$

Part II proves that this triple is a valid neutrosophic metric and establishes its topological and reconstruction properties. Part III develops exact contraction mechanisms, and Part IV supplies applications. Chapter 17 is the proof-audit guide.

1.8 Chapter exercises

1. Explain why convergence of only $\mathsf{T}(x_n, p, t)$ at one fixed scale need not determine a numerical error unless an inverse calibration is known.

2. Give an example of two different scalar distances that lead to the same rounded truth value at a fixed scale.

3. Identify which of the four structural choices above is most important for a computer-assisted stopping rule, and justify the choice mathematically.

Literature Review

Classical metric fixed point theory

The Banach principle established the prototype in which completeness, contraction, existence, uniqueness, and convergence of successive approximations appear in one theorem [4]. Alternative contractive mechanisms soon showed that the same fixed point conclusion does not require the Banach inequality in its original form. Kannan's condition uses the displacements $d(x, Ax)$ and $d(y, Ay)$ [28]; Meir and Keeler formulated a nonlinear uniform contractivity condition [38]; Nadler passed to multivalued contractions with the Hausdorff metric [43]; and C̀irić developed a broad generalized contraction principle [8, 9]. Altering-distance functions supplied another way to formulate comparison without relying on a single linear factor [30]. Matkowski's comparison-function approach and Jachymski's analysis of Meir–Keeler-type conditions further clarified the relation between nonlinear contractivity and convergence [36, 37, 24].

Later work continued in two directions relevant to this book: weakening the contractive hypothesis and making the iteration more explicit. Branciari introduced integral-type contractive conditions [7]. Suzuki found a generalized Banach condition that characterizes metric completeness [61]. Pata gave a weaker asymptotic contraction with an explicit convergence estimate [46], while Wardowski introduced F-contractions [63]. Jleli and Samet proposed a further generalization in a Branciari-type distance setting [25], and simulation functions were developed as a unifying language for several contractive schemes [31].

This line of research provides much of the scalar analytic engine used in the monograph. Its methodological lesson is important: a fixed point principle should be separated into the geometry of the space, the comparison law, and the information available about the orbit. The present book retains that separation when the scalar distance is encoded by a neutrosophic profile.

From probabilistic and fuzzy metrics to intuitionistic fuzzy metrics

The uncertainty-oriented lineage begins with statistical metric spaces, in which a distribution function replaces a deterministic distance [56], and with Zadeh's theory of fuzzy sets [68]. Kramosil and Michálek combined these ideas in an early axiomatization of fuzzy metric spaces [34]. Kaleva and Seikkala developed a related fuzzy metric framework [29], and Grabiec established a fuzzy counterpart of the contraction principle [14]. George and Veeramani's formulation became especially influential because it supports a natural Hausdorff topology [10]; their later work developed additional analytic properties [11]. Gregori and Sapena examined fixed point results in that setting and highlighted the role of the exact definition of fuzzy contractivity and completeness [16].

Atanassov's intuitionistic fuzzy sets distinguish membership and non-membership rather than forcing one to be the complement of the other [3]. Park incorporated this distinction into intuitionistic fuzzy metric spaces [47]. Fixed point results in the resulting two-component geometry were developed by Alaca, Türkoğlu, and Yildiz [1], by Mohamad [40], and in the modified framework of Saadati, Sedghi, and Shobe [54]. These studies demonstrate that componentwise uncertainty and fixed point arguments can coexist, but they also show that conclusions depend sensitively on the adopted notions of convergence and completeness.

More recent fuzzy metric research has refined both the contraction and the underlying topology. Wardowski formulated fuzzy contractive mappings directly in fuzzy metric spaces [64]. Shukla, Gopal, and Sintunavarat introduced a new class of fuzzy contractions [57], and extended simulation functions have been used for fixed and best-proximity results [41]. Miñana, Šostak, and Valero studied metrization and its application to fixed point theory [39]. Gregori, Miñana, Roig, and Sapena revisited the relation between completeness assumptions and fuzzy fixed point theorems [15]. The 2025 survey of Moussaoui and Radenović records the continuing development of contraction principles and simulation functions in fuzzy metric spaces [42].

For the present monograph, the most relevant conclusion from this literature is that a scale-dependent metric object can be topologically meaningful without automatically being quantitatively invertible. Metrizability identifies a compatible ordinary metric; exact calibration demands more—a specified formula that recovers the same distance from every scale and from each nondegenerate component.

Neutrosophic metric spaces and fixed point theory

Neutrosophy separates truth, indeterminacy, and falsity as three coordinates rather than reducing uncertainty to one membership grade or to a complementary pair [59]. Building on that perspective and on the fuzzy-metric tradition, Kirişci and Şimşek introduced neutrosophic metric spaces with continuous triangular norms and conorms and studied their basic topological structure [33]. Şimşek and Kirişci gave early fixed point theorems in this setting [58]; Sowndrarajan, Jeyaraman, and Smarandache treated Banach- and Edelstein-type contractions [60]; and Kirişci, Şimşek, and Akyiğit developed further fixed point results for the new metric structure [32]. These papers established the central program: translate metric convergence and contractivity into conditions on three scale-dependent functions.

The subsequent literature enlarged both the geometry and the class of admissible mappings. Orthogonality was incorporated by Ishtiaq and coauthors [22]. Uddin and coauthors introduced neutrosophic double controlled metric spaces and proved contraction results with an integral-equation application [62]. Riaz and coauthors considered chainable neutrosophic spaces and generalized neutrosophic cone metric spaces [52]. Saleem and coauthors used multivalued maps and the Hutchinson–Barnsley operator to construct neutrosophic fractals [55]. Together, these contributions show that the three-component formalism is flexible enough to support ordered, controlled, cone-valued, and set-valued variants.

Research published in 2024 and 2025 confirms that this is an active rather than a closed line of inquiry. Mani and coauthors obtained a fixed point result in an extended neutrosophic rectangular metric space and applied it to a Caputo fractional differential equation [35]. Akram and coauthors proposed several generalized neutrosophic metric spaces and applications [2], while Ghosh and coauthors investigated a neutrosophic fuzzy metric and its topology [12]. Non-Archimedean generalized neutrosophic metrics were treated by Johnsy, Jeyaraman, and Shukla [26], and generalized neutrosophic rectangular b-metrics by Hussain, Alharbi, and Basendwah [21]. Recent work has also used subsequential continuity and subcompatibility [17], and a multi-parameter ς-neutrosophic fuzzy metric has been proposed [50].

The cited neutrosophic fixed point papers primarily ask whether a generalized contractive condition guarantees existence, uniqueness, or a common fixed point in the chosen axiomatic space. That question is fundamental, but it is different from the computational question pursued here. In the literature reviewed above, the three components usually certify convergence through inequalities and limits; they are not generally designed as redundant, exactly invertible measurements of one specified scalar error. This book therefore studies a deliberately narrower calibrated subclass. Its restriction is a source of additional information: every nondegenerate component can be inverted to the same $d(x, y)$, every scale supplies an independent reconstruction, and disagreement between reconstructions becomes a diagnostic for noise or model inconsistency.

Quantitative iteration, perturbation, and certification

Existence and convergence do not by themselves specify how many iterations are needed or whether a computed iterate is already reliable. For a Banach contraction, the geometric estimate can be written both a priori, using the initial error, and a posteriori, using successive increments. Nonlinear generalizations such as Pata's contractions, F-contractions, and simulation-function contractions broaden the comparison mechanism [46, 63, 31], but a usable algorithm still requires an observable quantity that bounds the unknown error. This distinction between a convergence theorem and an error certificate is a central organizing principle of the present manuscript.

The distinction becomes sharper under inexact computation. If the actual update only satisfies $d(x_{n+1}, Ax_n) \leq \epsilon_n$, then the orbit contains both propagated initial error and a weighted history of perturbations. A qualitative fixed point statement does not reveal this convolution. Nor does a rounded profile value, by itself, reveal the metric perturbation unless the profile has a known inverse. Accordingly, the chapters that follow derive deterministic bounds for absolute, relative, adaptive, and data-driven errors; separate a posteriori from a priori stopping rules; and propagate component noise through explicit inverse formulas.

This quantitative emphasis also changes the role of the three neutrosophic coordinates. Rather than treating truth, indeterminacy, and falsity solely as three logical assessments, the calibrated model treats them as three channels that observe the same distance with different sensitivities. The equality of their exact reconstructions is a structural identity. With noisy observations it becomes a testable compatibility condition, while interval inversion and weighted projection produce certified or statistically stabilized distance estimates. Thus the profile is used not only to restate convergence but also to audit it.

Application literature and the role of contraction

Fixed point methods are useful in applications when the operator, domain, metric, and contraction constant can all be exhibited. Integral equations provide a classical example: integral-type contractive conditions were studied by Branciari [7], and neutrosophic variants have recently been used for Fredholm integral equations [52, 62]. The same pattern appears in fractional differential equations, where a fixed point argument proves existence only after the kernel and parameter range yield a genuine contraction [35]. These examples motivate the book's insistence on displaying the constant, verifying invariance of the domain, and converting the abstract bound into a computable certificate.

Dynamic programming supplies a second application family. Both ordinary and robust Bellman operators are naturally studied by contraction. The robust Markov decision process literature includes rectangular uncertainty models [23, 44], distributionally robust formulations [67], general ambiguity sets and tractability questions [66], and recent models that move beyond rectangularity [13]. These works provide the optimization setting; the calibrated profile in this book is added as an error-observation and stopping mechanism. It does not alter the Bellman operator or manufacture robustness.

Order and coupling offer a third bridge between theory and computation. Ran and Reurings used an order-sensitive fixed point theorem for matrix equations [51]. In coupled systems, a scalar contraction factor can be too coarse because each component may amplify a different combination of errors. The matrix-coupled part of this book therefore tracks a nonnegative error matrix and uses its spectral radius. Calibrated component profiles then translate the resulting vector estimates into observable, componentwise certificates.

Synthesis and research gap

The literature supports four conclusions that delimit the contribution of this monograph.

1. Classical and generalized fixed point theory offers a mature supply of contractive mechanisms, but many results are formulated at the level of existence and asymptotic convergence.

2. Fuzzy, intuitionistic fuzzy, and neutrosophic metric theories provide rich scale-dependent uncertainty geometries, yet topological compatibility alone does not provide an exact numerical inverse for each profile component.

3. Recent neutrosophic studies extend the admissible spaces and mappings and demonstrate applications, while the papers reviewed here do not assemble exact componentwise inversion, cross-scale consistency, noisy reconstruction, inexact iteration, and matrix-coupled certification in one framework.

4. Applied contraction models become computationally credible only when the invariant domain, contraction modulus, perturbation model, and stopping certificate are all explicit.

These conclusions identify the book's gap and its scope. The calibrated profile is not proposed as the most general neutrosophic metric. It is proposed as an auditable interface between an ordinary metric proof and three observable uncertainty coordinates. The resulting chain is

$$\begin{aligned} \text{operator hypothesis} &\Longrightarrow \text{metric bound}, \\ \text{metric bound} &\Longrightarrow \text{profile certificate}, \\ \text{profile certificate} &\Longrightarrow \text{stopping/diagnostic decision}. \end{aligned}$$

The chapters that follow develop each arrow, including the cases of noisy data, inexact updates, adaptive tolerances, nonlinear moduli, coupled systems, and the application classes surveyed above.

Chapter 2

Inexact, Adaptive, and Perturbed Picard Processes

Exact map evaluations are idealizations. This chapter quantifies deterministic computational errors and produces stopping rules that remain valid for inexact steps.

2.1 The inexact recurrence

Let A be a q-contraction with fixed point p. An inexact orbit satisfies

$$d(x_{n+1}, Ax_n) \leq \epsilon_n, \qquad \epsilon_n \geq 0. \tag{2.1}$$

The error may represent an approximate optimization, a truncated integral, finite precision, or a deliberately early inner solve.

Theorem 2.1 (Inexact Picard bound)**.** ***New result N11.*** *Every* *sequence satisfying (2.1) obeys*

$$d(x_n, p) \le q^n d(x_0, p) + \sum_{k=0}^{n-1} q^{n-1-k} \epsilon_k. \tag{2.2}$$

If $\epsilon_n \to 0$, then $x_n \to p$. If merely $\sup_n \epsilon_n \le \bar{\epsilon}$, then

$$\limsup_{n\to\infty} d(x_n, p) \le \frac{\bar{\epsilon}}{1-q}. \tag{2.3}$$

Both estimates translate to the three components through (9.6).

Proof. The triangle inequality and contractivity give

$$d(x_{n+1}, p) \le d(x_{n+1}, Ax_n) + d(Ax_n, Ap) \le \epsilon_n + q d(x_n, p).$$

Iteration proves (2.2). If $\epsilon_n \to 0$, split the convolution at a fixed N. The finitely many early terms vanish after multiplication by powers of q, while the remaining tail is at most $\sup_{k\ge N} \epsilon_k/(1-q)$. Let first $n \to \infty$ and then $N \to \infty$. The bounded-error assertion follows directly from the geometric sum. □

2.2 Geometric inner accuracy

Theorem 2.2 (Closed form and stopping certificate)**.** ***New result N12.*** *If $\epsilon_k \le Cr^k$ with $C \ge 0$ and $r \in [0, 1)$, then*

$$d(x_n, p) \le q^n d(x_0, p) + \begin{cases} C\dfrac{q^n - r^n}{q - r}, & q \ne r, \\ Cnq^{n-1}, & q = r. \end{cases} \tag{2.4}$$

Moreover, at any computed point x_n,

$$d(x_n, p) \le \frac{d(x_n, x_{n+1}) + \epsilon_n}{1-q}. \tag{2.5}$$

Thus the right side of (2.5) is a certified stopping test even when $x_{n+1} \neq Ax_n$.

Proof. Insert Cr^k into (2.2) and evaluate the finite geometric convolution. For the stopping estimate,

$$d(x_n, Ax_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Ax_n) \leq d(x_n, x_{n+1}) + \epsilon_n,$$

then apply (9.5). □

2.3 Shadowing an exact orbit

Let $y_0 = x_0$ and $y_{n+1} = Ay_n$ be the exact orbit from the same initial point.

Theorem 2.3 (Deterministic shadowing). ***New result N13.*** *Under (2.1),*

$$d(x_n, y_n) \leq \sum_{k=0}^{n-1} q^{n-1-k} \epsilon_k. \tag{2.6}$$

In particular, if $\epsilon_k \leq \bar{\epsilon}$, the computed orbit never departs from the exact orbit by more than $\bar{\epsilon}(1 - q^n)/(1 - q)$.

Proof. Use

$$d(x_{n+1}, y_{n+1}) \leq d(x_{n+1}, Ax_n) + d(Ax_n, Ay_n) \leq \epsilon_n + qd(x_n, y_n)$$

and iterate from $d(x_0, y_0) = 0$. □

2.4 Perturbed maps

Suppose the available map at step n is A_n and

$$\sup_{x \in X} d(A_n x, Ax) \leq \delta_n.$$

The iteration $x_{n+1} = A_n x_n$ is an inexact Picard process with $\epsilon_n = \delta_n$. N11 therefore handles changing discretizations, sampled expectations, and approximate Bellman operators without another fixed point proof.

2.5 An adaptive accuracy schedule

Assume an outer tolerance τ is required. A simple balanced design assigns half the error budget to contraction and half to inexactness. Choose n so that

$$q^n d(x_0, p) \leq \frac{\tau}{2}$$

using an a priori upper bound for $d(x_0, p)$, and choose inner errors so that

$$\sum_{k=0}^{n-1} q^{n-1-k} \epsilon_k \leq \frac{\tau}{2}.$$

One sufficient schedule is $\epsilon_k \leq (1-q)\tau/2$ for all k, though decreasing schedules often spend computational work more efficiently near the solution. The theorem certifies any schedule satisfying the convolution inequality; optimization of computational cost requires a separate cost model.

2.6 Chapter exercises

1. Prove convergence in N11 when the errors are not summable but tend to zero.

2. Derive the smallest constant error bound that guarantees the asymptotic error floor is below a prescribed τ.

3. Translate (2.6) into a lower bound for truth-nearness.

4. Explain why a stochastic error model needs assumptions beyond N11 to produce probabilistic confidence statements.

Chapter 3

Stability, Data Dependence, and Parameter Sensitivity

A fixed point is useful only if it is stable under residual error, changes in the operator, and changes in model parameters.

3.1 Ulam–Hyers stability

An ϵ-fixed point of A is a point z satisfying $d(z, Az) \le \epsilon$.

Theorem 3.1 (Profile Ulam–Hyers stability)**.** ***New result N14.*** *If A is a q-contraction with fixed point p, every ϵ-fixed point z satisfies*

$$d(z, p) \le \frac{\epsilon}{1-q}. \tag{3.1}$$

Equivalently, for every scale t,

$$\mathsf{T}(z, p, t) \ge \frac{t(1-q)}{t(1-q)+\epsilon},$$

with the analogous upper bounds for I *and* F *obtained from* (9.6)*.*

Proof. This is the residual estimate (9.5) with $d(z, Az) \le \epsilon$; profile substitution gives the component form. □

3.2 Two operators

Let A and B be contractions with fixed points p_A and p_B. Define the uniform operator discrepancy

$$\eta = \sup_{x \in X} d(Ax, Bx),$$

assuming it is finite.

Theorem 3.2 (Data dependence)**.** ***New result N15.*** *If A and B have contraction factors $q_A, q_B < 1$, then*

$$d(p_A, p_B) \le \frac{\eta}{1 - \min\{q_A, q_B\}}. \tag{3.2}$$

If A_λ is a family with a common contraction factor $q < 1$ and

$$\sup_x d(A_\lambda x, A_\mu x) \le L|\lambda - \mu|,$$

then its fixed point satisfies

$$d(p_\lambda, p_\mu) \le \frac{L}{1 - q}|\lambda - \mu|. \tag{3.3}$$

Proof. Using A as the anchor,

$$d(p_A, p_B) = d(Ap_A, Bp_B) \le q_A d(p_A, p_B) + \eta,$$

so $d(p_A, p_B) \le \eta/(1 - q_A)$. Reversing the roles gives $\eta/(1 - q_B)$; take the smaller bound to obtain (3.2). Formula (3.3) is the special case $\eta = L|\lambda - \mu|$. □

3.3 Differentiability of a fixed point

The Lipschitz estimate describes finite changes. In a Banach space, differentiability gives a local sensitivity equation.

Theorem 3.3 (Fixed-point derivative)**.** ***New result N16.*** *Let X be a Banach space, let Λ be an open subset of another Banach space, and let $A : \Lambda \times X \to X$ be continuously Fréchet differentiable. Assume $\|D_x A(\lambda, x)\| \leq q < 1$ on a neighborhood of $(\lambda_0, p_{\lambda_0})$ and that each nearby $A_\lambda = A(\lambda, \cdot)$ maps a common closed neighborhood into itself. Then the local fixed point map is continuously differentiable and*

$$Dp_\lambda = (I - D_x A(\lambda, p_\lambda))^{-1} D_\lambda A(\lambda, p_\lambda). \tag{3.4}$$

Moreover, $\|(I - D_x A)^{-1}\| \leq (1 - q)^{-1}$.

Proof. Define $G(\lambda, x) = x - A(\lambda, x)$. Then $D_x G = I - D_x A$. The Neumann series $\sum_{n \geq 0} (D_x A)^n$ converges in operator norm, so $D_x G$ is invertible and its inverse has norm at most $(1 - q)^{-1}$. The Banach-space implicit function theorem provides a differentiable solution of $G(\lambda, p_\lambda) = 0$. Differentiating this identity gives

$$(I - D_x A) Dp_\lambda = D_\lambda A,$$

which is (3.4). Local uniqueness from contraction identifies the implicit solution with the fixed point. □

3.4 Componentwise interpretation

If a parameter perturbation yields the scalar bound $d(p_\lambda, p_\mu) \leq E$, the truth-nearness of the two solutions is at least $t/(t + E)$. The split then allocates the complement between indeterminacy and falsity. This provides a transparent uncertainty report: the analytic sensitivity is carried by E, while the semantic allocation is carried by ϑ.

3.5 Chapter exercises

1. Show that (3.2) improves either one-sided estimate.
2. Give a family A_λ for which equality holds in (3.3).
3. Derive a norm bound for Dp_λ from N16.
4. Explain why pointwise, rather than uniform, convergence of A_n to A is insufficient for the stated data-dependence estimate.

Chapter 4

Commuting Families, Approximate Commutation, and Alternation

We obtain common fixed points without CLR assumptions by using one contractive anchor, and we quantify what happens when commutation is imperfect.

4.1 A contractive anchor

Theorem 4.1 (Commuting-family anchor theorem). ***New result N17.*** *Let A be a contraction on a complete calibrated space, and let $\{S_i : i \in J\}$ be any family of self-maps satisfying $AS_i = S_iA$. Then the unique fixed point p of A is a common fixed point of every S_i. No continuity or contractivity of the maps S_i is required.*

Proof. For each i,

$$A(S_ip) = S_i(Ap) = S_ip.$$

Thus, $S_i p$ is a fixed point of A. Uniqueness of the anchor's fixed point gives $S_i p = p$. □

Remark 4.2. This mechanism is entirely different from a common limit range argument. It does not search for coincidence points of two weakly compatible pairs. One map supplies existence and uniqueness; algebraic commutation transfers the fixed point to every other map.

4.2 Approximate commutation

Theorem 4.3 (Displacement under approximate commutation)**.** ***New result N18.*** *Let A be a q-contraction with fixed point p, and let $S : X \to X$ satisfy*

$$\sup_{x \in X} d(ASx, SAx) \le \eta. \tag{4.1}$$

Then

$$d(Sp, p) \le \frac{\eta}{1-q}. \tag{4.2}$$

Hence Sp is an η-approximate fixed point of the anchor and has the profile certificate obtained from N14.

Proof. Because $Ap = p$,

$$d(ASp, Sp) = d(ASp, SAp) \le \eta.$$

Apply the residual bound for A at the point Sp. □

4.3 Alternating maps

Let $A, B : X \to X$ and set $C = B \circ A$. An alternating orbit uses $y_n = Ax_n$ and $x_{n+1} = By_n$. A fixed point of C produces a two-cycle of the alternation.

Theorem 4.4 (Alternating-cycle theorem). ***New result N19.*** *If $C = B \circ A$ is a contraction on a complete calibrated space, then there is a unique pair (p, y) satisfying*

$$y = Ap, \qquad p = By. \tag{4.3}$$

If, in addition, A and B commute, then $p = y$ and p is the unique common fixed point of A and B.

Proof. The contraction C has a unique fixed point p. Setting $y = Ap$ gives $By = BAp = Cp = p$. Any other pair satisfying (4.3) would give another fixed point of C, proving uniqueness of p and then of y.

If $AB = BA$, then $CA = BAA = ABA = AC$. Hence $C(Ap) = A(Cp) = Ap$, so Ap is a fixed point of C. Uniqueness gives $Ap = p$, hence $y = p$ and $Bp = BAp = Cp = p$. Any common fixed point is fixed by C, so it is p. □

4.4 A noncommuting example

On $\mathbb{R}$, take $A(x) = x + 1$ and $B(x) = x/2$. Then, $C(x) = (x + 1)/2$ is a 1/2-contraction with fixed point $p = 1$, while $y = Ap = 2$ and $By = 1$. The alternating process has the unique two-cycle $(1, 2)$, but neither number is a common fixed point. This demonstrates why the commutation clause in the second part of N19 is substantive.

4.5 Chapter exercises

1. Extend N17 to a semigroup of maps commuting with the anchor.
2. In N18, replace the uniform bound by the single-point condition $d(ASp, SAp) \leq \eta$ and identify what remains true.
3. Compute the alternating orbit in the noncommuting example from $x_0 = 0$.

4. Show that commutation of A and B can be weakened to commutation of A with $C = B \circ A$ for the conclusion $Ap = p$; state an analogous condition for B.

Chapter 5

Coupled Systems and Contraction Matrices

A coupled operator can fail to be contractive in the naive product norm even when its component interactions are jointly stable. A spectral-radius argument supplies the correct weights.

5.1 Lipschitz matrices

Let (X_i, d_i), $1 \le i \le m$, be complete metric spaces, let $X = \prod_i X_i$, and write $F = (F_1, \ldots, F_m) : X \to X$. Suppose a nonnegative matrix $M = (m_{ij})$ satisfies

$$d_i(F_i(x), F_i(y)) \le \sum_{j=1}^{m} m_{ij} d_j(x_j, y_j). \tag{5.1}$$

The row sums may exceed one, so the usual product max norm may not reveal a contraction.

Theorem 5.1 (Spectral-radius profile theorem). ***New result N20.*** *If the spectral radius $r(M) < 1$, then there exist weights $w_i > 0$ and a* *number $\beta < 1$ such that F is a β-contraction in the weighted metric (7.4). Therefore F has a unique fixed point p, block Picard iteration converges to p, and all estimates of N6 hold in d_w and its rational profile.*

More explicitly, choose any q with $r(M) < q < 1$, and set

$$w = \left(I - \frac{M}{q}\right)^{-1} \mathbf{1} = \sum_{k=0}^{\infty} \left(\frac{M}{q}\right)^{k} \mathbf{1}. \tag{5.2}$$

Then, one may take

$$\beta = \max_i \frac{(Mw)_i}{w_i} < q < 1. \tag{5.3}$$

Proof. Because $r(M/q) < 1$, the nonnegative Neumann series in (5.2) converges and $w_i \geq 1$. The identity $(I - M/q)w = \mathbf{1}$ gives $Mw = q(w - \mathbf{1}) < qw$, hence (5.3). If $D = d_w(x, y)$, then $d_j(x_j, y_j) \leq w_j D$. Therefore,

$$\frac{d_i(F_i(x), F_i(y))}{w_i} \leq \frac{(Mw)_i}{w_i} D \leq \beta D.$$

Taking the maximum over i proves contractivity. Completeness of the weighted product and **N6** finish the proof. □

5.2 Gauss–Seidel and approximate blocks

The theorem applies directly to simultaneous block updates. A sequential Gauss–Seidel map has a different Lipschitz matrix because updated coordinates enter later blocks. Once a valid nonnegative matrix for the complete sweep is derived, the same spectral-radius theorem applies. If block i is computed with error $\epsilon_{i,n}$, the weighted step error is

$$\epsilon_n = \max_i \frac{\epsilon_{i,n}}{w_i},$$

and N11 gives the coupled inexact bound.

5.3 A two-component illustration

Suppose

$$M = \begin{pmatrix} 0.2 & 0.7 \\ 0.1 & 0.3 \end{pmatrix}.$$

The first row sum is 0.9 and the second is 0.4, so the unweighted max norm already works. More instructively, the matrix

$$\widetilde{M} = \begin{pmatrix} 0.6 & 0.7 \\ 0.05 & 0.1 \end{pmatrix}$$

has a row sum 1.3 but spectral radius approximately $0.661 < 1$. By **N20** and therefore constructs weights for which the system contracts, even though the naive row-sum test fails.

5.4 Componentwise error recovery

If $d_w(x_n, p) \leq E_n$, then every block satisfies $d_i(x_{n,i}, p_i) \leq w_i E_n$. Thus one global profile certificate can be expanded into block-specific thresholds:

$$\mathsf{T}_i(x_{n,i}, p_i, t_i) \geq \frac{t_i}{t_i + w_i E_n}.$$

Weights have two roles: they establish contraction and convert the global error back to physical units in each block.

5.5 Chapter exercises

1. Compute the spectral radius of $\widetilde{M}$ and construct weights using $q = 0.8$.
2. Prove that scaling all weights by the same positive constant leaves β unchanged.
3. Derive the inexact convolution bound for block errors.

4. Give a matrix with every row sum greater than one but spectral radius less than one, or prove that none exists in dimension two under an additional condition of your choice.

Part II

Calibrated Profiles and Exact Scalarization

Chapter 6

Normalized Rational Neutrosophic Profiles

We introduce the calibrated object used throughout the book and prove that it is a valid neutrosophic metric with an exact inverse.

6.1 Continuous norms and the ambient axioms

A continuous t-norm is an associative, commutative, continuous, and coordinatewise nondecreasing operation $* : [0,1]^2 \to [0,1]$ with $a * 1 = a$. A continuous t-conorm is defined dually, with neutral element zero. Two standard pairs are

$$a * b = \min\{a, b\}, \quad a \diamond b = \max\{a, b\},$$

and

$$a * b = ab, \quad a \diamond b = a + b - ab.$$

For the purposes of this book, a neutrosophic metric space is a sextuple $(X, \mathsf{T}, \mathsf{I}, \mathsf{F}, *, \diamond)$ in which the three maps have domain $X \times X \times (0, \infty)$

and range $[0,1]$, are symmetric in their first two arguments, and satisfy

$$\begin{aligned} \mathsf{T}(x,y,t) = 1 &\iff x = y, \\ \mathsf{I}(x,y,t) = 0 &\iff x = y, \\ \mathsf{F}(x,y,t) = 0 &\iff x = y, \end{aligned}$$

together with

$$\begin{aligned} \mathsf{T}(x,y,t) * \mathsf{T}(y,z,s) &\le \mathsf{T}(x,z,t+s), \\ \mathsf{I}(x,z,t+s) &\le \mathsf{I}(x,y,t) \diamond \mathsf{I}(y,z,s), \\ \mathsf{F}(x,z,t+s) &\le \mathsf{F}(x,y,t) \diamond \mathsf{F}(y,z,s). \end{aligned}$$

The component functions are continuous in the scale, and as $t \to \infty$ their limits are $1, 0, 0$, respectively. These axioms are consistent with the foundational formulation of Kirişci and Şimşek [33]. Some authors add or rearrange normalization conditions. Our profile below satisfies the stronger identity $\mathsf{T} + \mathsf{I} + \mathsf{F} = 1$, so those minor variants cause no ambiguity here.

6.2 The rational calibration

Definition 6.1 (Normalized rational profile)**.** Let (X, d) be a metric space and let $\vartheta \in (0,1)$. For $x, y \in X$ and $t > 0$, define

$$\begin{aligned} \mathsf{T}_d(x,y,t) &= \frac{t}{t + d(x,y)}, \\ \mathsf{I}_d(x,y,t) &= \vartheta \frac{d(x,y)}{t + d(x,y)}, \\ \mathsf{F}_d(x,y,t) &= (1-\vartheta) \frac{d(x,y)}{t + d(x,y)}. \end{aligned} \tag{6.1}$$

The resulting triple is called the normalized rational profile of d, and ϑ is called the indeterminacy split.

The split does not change nearness. It assigns the complement $1 - \mathsf{T}_d$ between indeterminacy and falsity:

$$\mathsf{T}_d + \mathsf{I}_d + \mathsf{F}_d = 1, \qquad \frac{\mathsf{I}_d}{\mathsf{F}_d} = \frac{\vartheta}{1-\vartheta} \quad (x \neq y).$$

This is useful when a modeler wants to report a fixed total uncertainty while distinguishing epistemic indeterminacy from falsity.

Theorem 6.2 (Metric-to-profile construction)**.** ***New result N1.*** *For every metric space (X, d) and every $\vartheta \in (0,1)$, the functions in* (6.1) *form a neutrosophic metric for each of the following pairs:*

1. *the Gödel pair $* = \min$ and $\diamond = \max$;*
2. *the product pair $a * b = ab$ and $a \diamond b = a + b - ab$.*

Proof. All range, symmetry, identity, continuity, normalization, and limiting properties follow immediately from the metric axioms and (6.1). It remains to prove the triangle inequalities. Put $a = d(x,y)$, $b = d(y,z)$, and $c = d(x,z) \leq a + b$.

For the Gödel pair, observe that

$$\frac{c}{t+s} \leq \frac{a+b}{t+s} \leq \max\left\{\frac{a}{t}, \frac{b}{s}\right\}.$$

Since $r \mapsto (1+r)^{-1}$ is decreasing,

$$\mathsf{T}_d(x,z,t+s) \geq \min\{\mathsf{T}_d(x,y,t), \mathsf{T}_d(y,z,s)\}.$$

Taking complements gives

$$1 - \mathsf{T}_d(x,z,t+s) \leq \max\{1 - \mathsf{T}_d(x,y,t), 1 - \mathsf{T}_d(y,z,s)\}.$$

Multiplication by ϑ and $1-\vartheta$ proves the I_d and F_d inequalities.

For the product pair, direct expansion yields

$$(t+s)(t+a)(s+b) - ts(t+s+a+b) = t^2 b + as^2 + (t+s)ab \geq 0.$$

Using $c \le a + b$ therefore gives

$$\frac{t+s}{t+s+c} \ge \frac{t}{t+a}\frac{s}{s+b}.$$

Thus, the truth triangle inequality holds. If $u = 1 - \mathsf{T}_d(x,y,t)$ and $v = 1 - \mathsf{T}_d(y,z,s)$, it also follows that $1 - \mathsf{T}_d(x,z,t+s) \le u + v - uv$. Since

$$\vartheta u + \vartheta v - \vartheta^2 uv \ge \vartheta(u + v - uv),$$

the probabilistic-sum inequality holds for I_d. Replacing ϑ by $1 - \vartheta$ proves it for F_d. $\square$

Remark 6.3. The theorem permits either of two algebraic conventions without changing any scalar estimate in later chapters. This prevents a common defect in examples: declaring one t-norm in a theorem and checking the example under another.

6.3 Exact inversion and a converse

Every nontrivial component has an inverse. Define

$$\begin{aligned}
\rho_t^{\mathsf{T}}(x,y) &= t\left(\frac{1}{\mathsf{T}_d(x,y,t)} - 1\right), \\
\rho_t^{\mathsf{I}}(x,y) &= \frac{t\mathsf{I}_d(x,y,t)}{\vartheta - \mathsf{I}_d(x,y,t)}, \\
\rho_t^{\mathsf{F}}(x,y) &= \frac{t\mathsf{F}_d(x,y,t)}{(1-\vartheta) - \mathsf{F}_d(x,y,t)}.
\end{aligned} \tag{6.2}$$

Then,

$$\rho_t^{\mathsf{T}}(x,y) = \rho_t^{\mathsf{I}}(x,y) = \rho_t^{\mathsf{F}}(x,y) = d(x,y) \tag{6.3}$$

for every $t > 0$. At $x = y$ the last two expressions are interpreted by their continuous value zero.

Definition 6.4 (Time-consistent rational triple)**.** A triple $(\mathsf{T}, \mathsf{I}, \mathsf{F})$

with split ϑ is time-consistent rational if the three expressions in (6.2) are finite, equal, and independent of t. Their common value is denoted by $\rho(x, y)$.

Theorem 6.5 (Profile-to-metric recovery). ***New result N2.*** *Let* $(X, \mathsf{T}, \mathsf{I}, \mathsf{F}, \min, \max)$ *be a neutrosophic metric space whose triple is time-consistent rational. Then* ρ *is a metric and the original triple is exactly the normalized rational profile generated by* ρ.

Proof. Nonnegativity, symmetry, and separation follow from the component axioms and the inverse formulas. Let $a = \rho(x, y)$, $b = \rho(y, z)$, and $c = \rho(x, z)$. If $a, b > 0$, apply the truth triangle inequality at scales $t = a$ and $s = b$:

$$\frac{a+b}{a+b+c} \geq \min\left\{\frac{a}{a+a}, \frac{b}{b+b}\right\} = \frac{1}{2}.$$

Hence, $c \leq a + b$. If $a = 0$, then $x = y$ and $c = b$; the case $b = 0$ is analogous. Thus, ρ is a metric. Solving any equality in (6.2) for its component recovers (6.1) with $d = \rho$. □

Example 6.6 (A calibrated profile on a function space). Let $X = C([0, 1])$ and $d(u, v) = \|u - v\|_\infty$. For $\vartheta = 2/5$,

$$\mathsf{T}(u, v, t) = \frac{t}{t + \|u - v\|_\infty}, \quad \mathsf{I}(u, v, t) = \frac{2\|u - v\|_\infty}{5(t + \|u - v\|_\infty)},$$

and $\mathsf{F} = (3/5)(1 - \mathsf{T})$. If $u(s) = s$ and $v(s) = s^2$, then $d(u, v) = 1/4$. At $t = 1$, the triple is $(4/5, 1/10, 3/20)$, and each formula in (6.2) returns $1/4$.

6.4 Why all three inverses matter

In exact mathematics, one component is enough to recover d. In measured or rounded data, the three reconstructions will generally differ. Their disagreement is therefore an internal diagnostic rather than redundant information. Chapter 4 turns this observation into perturbation and consistency bounds.

6.5 Chapter exercises

1. Verify **N1** for the metric $d(x, y) = |x - y|$ at two arbitrary positive scales.
2. Show that allowing $\vartheta = 0$ destroys the separation axiom for I.
3. Determine the metric encoded by $\mathsf{T}(x, y, t) = t/(t + |x^3 - y^3|)$ on $[0, \infty)$ and decide whether it is a metric.
4. Prove directly that $\rho_t^{\mathsf{I}} = \rho_t^{\mathsf{F}}$ for the normalized profile.

Chapter 7

Topology, Completeness, and Compactness

Exact scalarization is useful only if it preserves the analytic structure needed by fixed point theory. Here we prove that it preserves the relevant structure exactly.

7.1 Profile balls

For $x \in X$, $t > 0$, and $\varepsilon \in (0, 1)$, define the calibrated profile ball

$$\begin{aligned}\mathcal{B}_{\mathfrak{P}}(x; t, \varepsilon) = \{y : \mathsf{T}(x, y, t) > 1 - \varepsilon,& \\ \mathsf{I}(x, y, t) < \vartheta\varepsilon,& \\ \mathsf{F}(x, y, t) < (1 - \vartheta)\varepsilon\}.&\end{aligned} \tag{7.1}$$

The three inequalities are equivalent for a normalized rational profile. Indeed,

$$\mathsf{T}(x, y, t) > 1 - \varepsilon \iff d(x, y) < \frac{t\varepsilon}{1 - \varepsilon}.$$

Theorem 7.1 (Analytic equivalence)**.** ***New result N3.*** *Let (X, d) carry its normalized rational profile. Then:*

1. *the profile topology generated by (7.1) equals the metric topology;*
2. *x_n converges to x in the profile if and only if $d(x_n, x) \to 0$;*
3. *(x_n) is profile Cauchy if and only if it is metric Cauchy;*
4. *the profile is complete if and only if (X, d) is complete;*
5. *a subset is profile closed, totally bounded, or compact if and only if it has the corresponding metric property.*

Proof. The ball identity

$$\mathcal{B}_{\mathfrak{P}}(x; t, \varepsilon) = B_d\left(x, \frac{t\varepsilon}{1-\varepsilon}\right)$$

proves the first assertion because every positive metric radius r is obtained, for example, by fixing $t = 1$ and taking $\varepsilon = r/(1+r)$. It also proves the convergence and Cauchy equivalences directly. Completeness is equivalence of the same Cauchy sequences and the same limits. Closedness follows from equality of topologies. Total boundedness follows because every profile-ball cover at fixed parameters is a metric-ball cover at the corresponding radius and conversely. Compactness follows from completeness plus total boundedness, or directly from the equality of topologies in metric spaces. $\square$

Corollary 7.2 (Transfer principle)**.** *Any argument whose hypotheses and conclusion depend only on distances, convergence, Cauchy sequences, closed subsets, completeness, or compactness can be translated losslessly between a metric space and its normalized rational profile.*

Remark 7.3. The word *losslessly* is important. A general neutrosophic metric need not have an exact scalar inverse. The transfer principle is asserted only for the calibrated class, where (6.3) has been proved.

7.2 Sequential characterizations

For a fixed $t > 0$, each of the following is equivalent to $x_n \to x$:

$$\mathsf{T}(x_n, x, t) \to 1, qquad \mathsf{I}(x_n, x, t) \to 0, qquad \mathsf{F}(x_n, x, t) \to 0.$$

Moreover, the recovered error is exact:

$$d(x_n, x) = t\frac{1 - \mathsf{T}(x_n, x, t)}{\mathsf{T}(x_n, x, t)}. \tag{7.2}$$

Thus, a target $d(x_n, x) \le \tau$ is equivalent to

$$\mathsf{T}(x_n, x, t) \ge \frac{t}{t+\tau}, \quad \mathsf{I}(x_n, x, t) \le \frac{\vartheta\tau}{t+\tau}, \quad \mathsf{F}(x_n, x, t) \le \frac{(1-\vartheta)\tau}{t+\tau}. \tag{7.3}$$

7.3 Changing the split

Let $\mathfrak{P}_{\vartheta_1}$ and $\mathfrak{P}_{\vartheta_2}$ be profiles generated by the same metric. Their truth components coincide, while

$$\mathsf{I}_{\vartheta_2} = \frac{\vartheta_2}{\vartheta_1}\mathsf{I}_{\vartheta_1}, \qquad \mathsf{F}_{\vartheta_2} = \frac{1-\vartheta_2}{1-\vartheta_1}\mathsf{F}_{\vartheta_1}.$$

Consequently, changing the interpretation of uncertainty does not alter topology, completeness, fixed points, rates, or scalar error certificates. This separates a modeling choice from an analytic fact.

7.4 Products and weighted metrics

Let (X_i, d_i), $1 \le i \le m$, be metric spaces and let $w_i > 0$. The weighted sup metric on $X = \prod_i X_i$ is

$$d_w(x, y) = \max_{1 \le i \le m} \frac{d_i(x_i, y_i)}{w_i}. \tag{7.4}$$

Its rational profile provides a single three-component report for an entire coupled system. Completeness of every X_i implies completeness of (X, d_w). The choice of weights is not merely cosmetic; Chapter 10 chooses them from a nonnegative Lipschitz matrix to reveal contraction that is invisible in the unweighted norm.

7.5 Chapter exercises

1. For a desired metric radius r and fixed scale t, solve for the profile tolerance ε in (7.1).
2. Prove the compactness assertion in **N3** using open covers rather than total boundedness.
3. Show that two different splits produce uniformly equivalent triples.
4. Verify completeness of the weighted product metric (7.4).

Chapter 8

Multiscale Reconstruction and Noisy Profiles

Exact profiles contain redundant scalar information. This chapter uses that redundancy to detect and bound component errors.

8.1 Componentwise distance estimators

Suppose a device or numerical routine returns a triple $(\widetilde{\mathsf{T}}, \widetilde{\mathsf{I}}, \widetilde{\mathsf{F}})$ at scale t. Define

$$\widehat{d}_{\mathsf{T}} = t\left(\frac{1}{\widetilde{\mathsf{T}}} - 1\right), \quad \widehat{d}_{\mathsf{I}} = \frac{t\widetilde{\mathsf{I}}}{\vartheta - \widetilde{\mathsf{I}}}, \quad \widehat{d}_{\mathsf{F}} = \frac{t\widetilde{\mathsf{F}}}{(1-\vartheta) - \widetilde{\mathsf{F}}}, \tag{8.1}$$

whenever the denominators are positive. An exact rational profile makes all three equal. The consistency index

$$\mathcal{C}_t = \max\{|\widehat{d}_{\mathsf{T}} - \widehat{d}_{\mathsf{I}}|, |\widehat{d}_{\mathsf{T}} - \widehat{d}_{\mathsf{F}}|, |\widehat{d}_{\mathsf{I}} - \widehat{d}_{\mathsf{F}}|\}. \tag{8.2}$$

Therefore, measures internal disagreement in distance units.

Theorem 8.1 (Noisy reconstruction)**.** ***New result N4.*** *Let* $(\mathsf{T}, \mathsf{I}, \mathsf{F})$

be the rational profile of a distance d at scale t. Assume

$$|\widetilde{\mathsf{T}} - \mathsf{T}| \le \epsilon_{\mathsf{T}}, \quad |\widetilde{\mathsf{I}} - \mathsf{I}| \le \epsilon_{\mathsf{I}}, \quad |\widetilde{\mathsf{F}} - \mathsf{F}| \le \epsilon_{\mathsf{F}}.$$

If both T and $\widetilde{\mathsf{T}}$ are at least $m_{\mathsf{T}} > 0$, both $\vartheta - \mathsf{I}$ and $\vartheta - \widetilde{\mathsf{I}}$ are at least $m_{\mathsf{I}} > 0$, and both $(1-\vartheta) - \mathsf{F}$ and $(1-\vartheta) - \widetilde{\mathsf{F}}$ are at least $m_{\mathsf{F}} > 0$, then

$$|\widehat{d}_{\mathsf{T}} - d| \le \frac{t\epsilon_{\mathsf{T}}}{m_{\mathsf{T}}^2}, \tag{8.3}$$

$$|\widehat{d}_{\mathsf{I}} - d| \le \frac{t\vartheta\epsilon_{\mathsf{I}}}{m_{\mathsf{I}}^2}, \tag{8.4}$$

$$|\widehat{d}_{\mathsf{F}} - d| \le \frac{t(1-\vartheta)\epsilon_{\mathsf{F}}}{m_{\mathsf{F}}^2}. \tag{8.5}$$

Consequently, $\mathcal{C}_t$ is no larger than the sum of the two largest bounds. Any convex combination of the three estimates has error no larger than the same convex combination of their bounds.

Proof. Apply the mean value theorem to $g(u) = t(u^{-1} - 1)$, whose derivative has absolute value t/u^2. This proves (8.3). For $h(u) = tu/(\vartheta - u)$, $h'(u) = t\vartheta/(\vartheta - u)^2$, giving (8.4). The last estimate is identical with ϑ replaced by $1-\vartheta$. Pairwise differences are bounded by the triangle inequality. Finally, if $\widehat{d} = \sum_j a_j \widehat{d}_j$, $a_j \ge 0$, and $\sum_j a_j = 1$, then $|\widehat{d} - d| \le \sum_j a_j |\widehat{d}_j - d|$. □

8.2 Multiscale estimators

At scales $t_1, \ldots, t_k$, let $\widehat{d}_{j,\mathsf{T}}$ be the truth-based estimates and choose nonnegative weights summing to one. The multiscale estimate

$$\widehat{d}_{\mathrm{ms}} = \sum_{j=1}^{k} a_j \widehat{d}_{j,\mathsf{T}}$$

is exact when the data are exact. Under deterministic component error it inherits the weighted bound from **N4**. Under independent, zero-mean random errors, the weights can instead be chosen inversely proportional to estimated variances. The latter is a statistical modeling choice, not part of the deterministic theorem.

8.3 Selecting a stable scale

The derivative t/T^2 in (8.3) explains why a very small truth value is numerically unstable. For an anticipated distance range $d \in [0, D]$, choosing t comparable to D keeps $\mathsf{T} = t/(t+d)$ away from zero. On the other hand, a very large t compresses all truth values near one and may lose resolution after rounding. A practical design interval is $t \in [D/2, 2D]$ when a reliable upper scale D is known.

8.4 A profile audit protocol

Given observed triples at several scales:

1. verify $0 < \widetilde{\mathsf{T}} \le 1$, $0 \le \widetilde{\mathsf{I}} < \vartheta$, and $0 \le \widetilde{\mathsf{F}} < 1 - \vartheta$;
2. compute the three estimators in (8.1);
3. compute $\mathcal{C}_t$ at every scale;
4. compare recovered distances across scales;
5. reject exact-calibration claims if discrepancies exceed justified error bounds.

This protocol distinguishes a mathematical profile from an arbitrary triple of scores.

8.5 Chapter exercises

1. Derive an interval containing d from a truth observation $\widetilde{\mathsf{T}}$ with absolute error at most ϵ without using the mean value theorem.
2. Find the truth scale t that makes $\mathsf{T} = 1/2$ for a known calibration distance D.
3. Show that the median of the three component estimates has error at most the second-largest of their individual error bounds.
4. Construct a rounded triple whose components sum to one but whose consistency index is nonzero.

Part III

Constructive Contractions and Iteration Mechanisms

Chapter 9

Time-Dilated Contractions and Picard Certificates

The exact inverse converts a componentwise time-dilation condition into an ordinary contraction and then returns sharp error estimates to all three components.

9.1 The profile contraction

Let $A : X \to X$ and $q \in [0, 1)$. The natural rational-profile form of a contraction is

$$\begin{aligned} \mathsf{T}(Ax, Ay, t) &\geq \mathsf{T}(x, y, t/q), \\ \mathsf{I}(Ax, Ay, t) &\leq \mathsf{I}(x, y, t/q), \\ \mathsf{F}(Ax, Ay, t) &\leq \mathsf{F}(x, y, t/q), \end{aligned} \tag{9.1}$$

where the right side is interpreted as $(1, 0, 0)$ when $q = 0$. The appearance of t/q is forced by calibration:

$$\mathsf{T}(x, y, t/q) = \frac{t}{t + qd(x, y)}.$$

Theorem 9.1 (Exact contraction equivalence). ***New result N5.*** *On a normalized rational profile, each one of the three inequalities in* (9.1) *is equivalent to*

$$d(Ax, Ay) \leq q d(x, y). \tag{9.2}$$

Consequently, imposing all three component inequalities is logically redundant but useful for reporting.

Proof. The truth inequality is

$$\frac{t}{t + d(Ax, Ay)} \geq \frac{t}{t + q d(x, y)},$$

which is equivalent to (9.2). The maps $r \mapsto \vartheta r/(t + r)$ and $r \mapsto (1 - \vartheta) r/(t + r)$ are strictly increasing, so the same equivalence holds for the other components. □

9.2 Existence, uniqueness, and rates

Starting from $x_0 \in X$, the Picard orbit is $x_{n+1} = A x_n$. Write $s_0 = d(x_1, x_0)$.

Theorem 9.2 (Quantitative profile Picard theorem). ***New result N6.*** *Let (X, d) be complete, let its normalized rational profile have split ϑ, and assume* (9.1) *for some $q < 1$. Then, A has a unique fixed point p, every Picard orbit converges to p, and for $n \geq 0$,*

$$d(x_n, p) \leq \frac{q^n}{1 - q} s_0, \tag{9.3}$$

$$d(x_n, p) \leq \frac{q}{1 - q} d(x_n, x_{n-1}) \quad (n \geq 1), \tag{9.4}$$

$$d(x, p) \leq \frac{d(x, Ax)}{1 - q} \quad (x \in X). \tag{9.5}$$

If E denotes any right side in these inequalities, then for every $t > 0$,

$$\mathsf{T}(x,p,t) \geq \frac{t}{t+E}, \quad \mathsf{I}(x,p,t) \leq \frac{\vartheta E}{t+E}, \quad \mathsf{F}(x,p,t) \leq \frac{(1-\vartheta)E}{t+E}. \tag{9.6}$$

Proof. By **N5**, A is a q-contraction in d. Therefore,

$$d(x_{n+1}, x_n) \leq q^n s_0.$$

For $m > n$, the triangle inequality gives

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} q^k s_0 \leq \frac{q^n}{1-q} s_0.$$

Completeness provides a limit p. The Lipschitz property makes A continuous, so $Ap = p$. If p' is another fixed point, then $d(p,p') \leq qd(p,p')$, hence $p = p'$. Letting $m \to \infty$ gives (9.3). Repeating the tail estimate from the step $d(x_{n+1}, x_n)$ gives (9.4). Finally,

$$d(x,p) \leq d(x,Ax) + d(Ax,Ap) \leq d(x,Ax) + qd(x,p),$$

which proves (9.5). Formula (9.6) follows by substituting $d(x,p) \leq E$ into the monotone profile formulas. □

Corollary 9.3 (Iteration budget)**.** *If $s_0 > 0$, $0 < q < 1$, and a scalar tolerance $\tau > 0$ is prescribed, the a priori condition*

$$n \geq \left\lceil \frac{\log(\tau(1-q)/s_0)}{\log q} \right\rceil$$

is sufficient whenever the expression inside the ceiling is positive; otherwise $n = 0$ is already sufficient. For $q = 0$, one exact map evaluation reaches the fixed point. The equivalent component thresholds are given by (7.3).

9.3 A certified implementation

Algorithm 1. Profile-certified Picard iteration

Input: initial point x_0, contraction factor $q < 1$, and tolerance $\tau > 0$.

1. For $n = 0, 1, 2, \ldots$, compute $x_{n+1} = Ax_n$.
2. Compute $r_{n+1} = d(x_{n+1}, Ax_{n+1})$.
3. If $r_{n+1}/(1-q) \le \tau$, return x_{n+1} together with the profile certificate (9.6); otherwise continue.

The residual version costs one additional evaluation of A per test. If that cost is undesirable, use the a posteriori step bound $qd(x_{n+1}, x_n)/(1-q)$. The residual certificate is preferable when q is a conservative global bound but the local residual becomes small quickly.

9.4 Example: a scalar nonlinear map

Let $X = [0, 1]$ with $d(x, y) = |x - y|$ and

$$A(x) = \frac{1}{4} + \frac{1}{2}\sin x.$$

Since $|A'(x)| \le 1/2$, the map is a contraction with $q = 1/2$ and maps X into $[1/4, 1/4 + \sin(1)/2] \subset X$. It has a unique fixed point. Starting from $x_0 = 0$, **N6** yields the bound

$$|x_n - p| \le 2^{-n-1},$$

because $s_0 = 1/4$ and $q^n s_0/(1-q) = 2^{-n-1}$. At scale $t = 1$ and split $\vartheta = 1/3$, this becomes

$$\mathsf{T}(x_n, p, 1) \ge \frac{1}{1 + 2^{-n-1}}, \quad \mathsf{I}(x_n, p, 1) \le \frac{2^{-n-1}}{3(1 + 2^{-n-1})}.$$

9.5 Chapter exercises

1. Derive the falsity certificate in the scalar example for $n = 5$.
2. Prove **N5** using the indeterminacy component only.
3. Compare the a priori and a posteriori bounds when the actual local contraction ratio is much smaller than q.
4. Modify Algorithm 1 so that it never performs an extra map evaluation.

Chapter 10

Nonlinear Comparison and Kannan Profiles

We develop summable comparison and Kannan profile theorems with explicit tail bounds.

10.1 Summable comparison functions

Definition 10.1. A function $\phi : [0, \infty) \to [0, \infty)$ is a summable comparison function if it is nondecreasing and continuous, $\phi(r) < r$ for $r > 0$, and

$$\sum_{n=0}^{\infty} \phi^n(r) < \infty \quad \text{for every } r > 0,$$

where $\phi^0(r) = r$.

For examples, include $\phi(r) = qr$ with $q < 1$ and $\phi(r) = r/(1 + cr)$ on a bounded interval only after an additional summability condition is imposed; the latter's iterates behave like $1/n$ and hence are not summable globally. This distinction matters because our direct Cauchy proof uses a tail series.

Theorem 10.2 (Summable profile comparison)**.** ***New result N7.*** *Let (X, d) be complete and let $A : X \to X$ satisfy*

$$\rho_t(Ax, Ay) \leq \phi(\rho_t(x, y)) \tag{10.1}$$

for a summable comparison function ϕ. Then, A has a unique fixed point p, and the Picard orbit satisfies

$$d(x_n, p) \leq \sum_{k=n}^{\infty} \phi^k(s_0). \tag{10.2}$$

Substitution of the right side into (9.6) gives the truth, indeterminacy, and falsity certificates.

Proof. The exact inverse gives $d(Ax, Ay) \leq \phi(d(x, y))$. Induction yields $d(x_{n+1}, x_n) \leq \phi^n(s_0)$. Hence, for $m > n$,

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} \phi^k(s_0),$$

whose tail tends to zero. Completeness gives $x_n \to p$. Continuity of A follows from the comparison inequality and continuity of ϕ at zero, so $Ap = p$. For two fixed points, $d(p, p') \leq \phi(d(p, p'))$, impossible if the distance is positive. Letting $m \to \infty$ proves (10.2). □

10.2 A Kannan profile

Kannan's classical condition replaces the mutual distance $d(x, y)$ by the two displacements $d(x, Ax)$ and $d(y, Ay)$. In the calibrated setting it is most clearly written after exact inversion.

Theorem 10.3 (Quantitative profile Kannan theorem)**.** ***New result N8.*** *Let (X, d) be complete and suppose $A : X \to X$ satisfies, for some $\alpha \in [0, 1/2)$,*

$$\rho_t(Ax, Ay) \leq \alpha\big(\rho_t(x, Ax) + \rho_t(y, Ay)\big). \tag{10.3}$$

Put $r = \alpha/(1-\alpha) < 1$. Then, A has a unique fixed point p, and

$$d(x_n, p) \leq \frac{r^n}{1-r} d(x_1, x_0). \tag{10.4}$$

The componentwise certificate follows from (9.6) with the right side of (10.4).

Proof. Exact inversion turns (10.3) into the metric Kannan inequality. With $x = x_n$ and $y = x_{n-1}$,

$$d(x_{n+1}, x_n) \leq \alpha\{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)\},$$

so $d(x_{n+1}, x_n) \leq r d(x_n, x_{n-1})$. The geometric tail proves that (x_n) is Cauchy and gives (10.4). Let $x_n \to p$. Applying the Kannan inequality to x_n and p gives

$$d(x_{n+1}, Ap) \leq \alpha\{d(x_n, x_{n+1}) + d(p, Ap)\}.$$

Taking limits yields $d(p, Ap) \leq \alpha d(p, Ap)$, hence $Ap = p$; no continuity of A is needed. If p, p' are fixed, then $d(p, p') \leq \alpha(0 + 0) = 0$. □

10.3 Comparison of the two mechanisms

The Picard contraction compares images of two arbitrary inputs. The Kannan condition compares images using only input-to-image displacements. Neither condition contains the other in general. Calibration does not erase this distinction; it makes the scalar mechanism visible and permits the same output language for both.

10.4 Chapter exercises

1. Show that $\phi(r) = qr/(1+r)$ is a summable comparison function for $q < 1$.
2. Locate the exact point in the proof of **N7** where summability is used.
3. Prove uniqueness in **N8** without referring to continuity.
4. Translate (10.3) entirely into T using $\rho_t = t(1/\mathsf{T} - 1)$.

Chapter 11

Variable Factors and Contractive Iterates

Some algorithms contract only along their realized orbit or only after several steps. We give separate theorems for these two situations.

11.1 Orbitwise factors

Let $x_{n+1} = Ax_n$ and suppose there are factors $q_n \geq 0$ such that

$$d(x_{n+1}, x_n) \leq q_n d(x_n, x_{n-1}) \qquad (n \geq 1). \tag{11.1}$$

Define $Q_0 = 1$ and $Q_n = \prod_{k=1}^{n} q_k$.

Theorem 11.1 (Variable-factor orbit theorem)**.** ***New result N9.*** *Assume (X, d) is complete, A is continuous, and the Picard orbit satisfies (11.1) with $\sum_{n=0}^{\infty} Q_n < \infty$. Then the orbit converges to a fixed point p and*

$$d(x_n, p) \leq d(x_1, x_0) \sum_{k=n}^{\infty} Q_k. \tag{11.2}$$

If, in addition, $d(Ax, Ay) < d(x, y)$ *whenever* $x \neq y$*, the fixed point is unique.*

Proof. Induction in (11.1) gives $d(x_{n+1}, x_n) \leq Q_n d(x_1, x_0)$. Summing the tail shows that the orbit is Cauchy and proves (11.2) after passage to the limit. Completeness gives $x_n \to p$ and continuity yields $Ap = p$. Two distinct fixed points would satisfy $d(p, p') = d(Ap, Ap') < d(p, p')$, a contradiction. $\square$

Remark 11.2. **N9** is intentionally an orbit theorem. A step-ratio estimate on one orbit does not by itself guarantee that every orbit converges or that a fixed point is unique. The continuity and strictness clauses state exactly what is additionally used.

11.2 Finite powers

An operator may be expansive in one step but contractive over a complete cycle. This occurs in alternating algorithms, seasonal models, and block updates.

Theorem 11.3 (Contractive iterate)**.** ***New result N10.*** *Let* (X, d) *be complete and suppose that, for some integer* $m \geq 2$*, the iterate* A^m *is a* q*-contraction with* $q < 1$*. Then* A *has a unique fixed point* p*. For* $n = km + r$*,* $0 \leq r < m$*, if* A^r *is* L_r*-Lipschitz, then*

$$d(A^n x, p) \leq L_r q^k d(x, p). \tag{11.3}$$

The corresponding profile bounds follow by substituting the right side into (9.6)*.*

Proof. By N6 applied to A^m, there is a unique p with $A^m p = p$. Since $A^m(Ap) = A(A^m p) = Ap$, the point Ap is also fixed by A^m. Uniqueness gives $Ap = p$. Any fixed point of A is a fixed point of A^m, so it equals p. Finally,

$$d(A^{km+r}x, p) = d(A^r(A^{km}x), A^r p) \leq L_r d((A^m)^k x, p) \leq L_r q^k d(x, p).$$

$\square$

11.3 Example: a two-stage update

Let $X = \mathbb{R}$ and define $A(x) = 2x + 1$ for $x \geq 0$ and $A(x) = x/8 - 1/2$ for $x < 0$. A global analysis must first verify whether A^2 maps branches as expected; a superficial multiplication of slopes is not enough. This example illustrates a general audit rule: when a map is piecewise defined, the finite-iterate contraction must be checked on every possible itinerary. A clean example is instead obtained on $X = \mathbb{R}^2$ with

$$A(x, y) = (2y, x/8).$$

Then, $A^2(x, y) = (x/4, y/4)$ is a $1/4$-contraction in the max norm, although A has one-step Lipschitz constant 2. From **N10** gives the unique fixed point $(0, 0)$ and a block rate 4^{-k}.

11.4 Chapter exercises

1. Give factors $q_n > 1$ infinitely often for which $\sum Q_n < \infty$.
2. Show that continuity in **N9** can be replaced by closedness of the graph of A along the orbit limit.
3. Verify the Lipschitz constants $L_0 = 1$ and $L_1 = 2$ in the two-stage example.
4. Construct a map for which A^2 is contractive and A is not.

Part IV

Applications of the New Framework

Chapter 12

Robust Bellman Operators

The source article treats a general functional equation through a common fixed point theorem. Here we take a separate robust-control direction and obtain constructive value-iteration, residual, and model-perturbation bounds.

12.1 A robust discounted model

Let S be a nonempty state set. At state s, let $A(s)$ be a nonempty action set, and let $\mathcal{U}(s,a)$ be a nonempty collection of reward–transition pairs (r,P), where $|r| \leq R$ and P is a probability measure on S. Let $\mathcal{B}(S)$ be the Banach space of bounded real functions with the sup norm. For a discount $\gamma \in (0,1)$ define

$$(\mathcal{T}V)(s) = \sup_{a\in A(s)} \inf_{(r,P)\in\mathcal{U}(s,a)} \left\{ r + \gamma \int_S V(s')\, P(ds') \right\}. \tag{12.1}$$

Rectangular uncertainty of this form is standard in robust Markov decision models [23, 44]. Our focus is the calibrated profile and the certificates it adds.

Lemma 12.1 (Sup–inf nonexpansiveness)**.** *For bounded families* $(f_{a,u})$ *and* $(g_{a,u})$*,*

$$\left|\sup_a \inf_u f_{a,u} - \sup_a \inf_u g_{a,u}\right| \le \sup_{a,u} |f_{a,u} - g_{a,u}|.$$

Proof. If $\Delta = \sup_{a,u} |f_{a,u} - g_{a,u}|$, then $f_{a,u} \le g_{a,u} + \Delta$. Taking first the infimum and then the supremum gives $\sup_a \inf_u f_{a,u} \le \sup_a \inf_u g_{a,u} + \Delta$. Interchange f and g for the reverse inequality. □

Theorem 12.2 (Robust Bellman profile theorem)**.** ***New result N21.*** *The operator* (12.1) *maps* $\mathcal{B}(S)$ *into itself and is a* γ*-contraction. Hence, it has a unique robust value* V^**, and value iteration* $V_{n+1} = \mathcal{T}V_n$ *satisfies*

$$\|V_n - V^*\|_\infty \le \frac{\gamma^n}{1-\gamma}\|V_1 - V_0\|_\infty, \tag{12.2}$$

$$\|V - V^*\|_\infty \le \frac{\|V - \mathcal{T}V\|_\infty}{1-\gamma}. \tag{12.3}$$

For the rational profile on $\mathcal{B}(S)$*, these bounds yield all three component certificates by* ***N6****. If the inner robust optimization is computed with uniform error* ϵ_n*,* ***N11*** *applies without change.*

Proof. Bounded rewards and values make $\mathcal{T}V$ bounded. By Lemma 12.1,

$$|(\mathcal{T}V)(s) - (\mathcal{T}W)(s)| \le \gamma \sup_{a,(r,P)} \left|\int (V - W)\, dP\right| \le \gamma\|V - W\|_\infty.$$

Take the supremum over s and apply **N6** and **N11**. □

12.2 A two-state numerical case study

Consider states $0, 1$, actions $0, 1$, discount $\gamma = 0.85$, and rewards

	$a = 0$	$a = 1$
$s = 0$	1.0	0.2
$s = 1$	0.5	1.4

The uncertain probability of moving to state 1 lies respectively in $[0.1, 0.3]$, $[0.55, 0.75]$, $[0.2, 0.4]$, and $[0.7, 0.9]$ in row-major order. Starting from $V_0 = (0, 0)$, robust value iteration gives:

n	$V_n(0)$	$V_n(1)$
0	0.000000	0.000000
1	1.000000	1.400000
2	1.884000	2.488000
5	3.882454	4.670615
10	5.685749	6.501104
20	6.845015	7.661341
40	7.118235	7.934561
80	7.129235	7.945562

The limiting value is approximately $V^* = (7.129251701, 7.945578231)$. The robust maximizing actions are 0 in state 0 and 1 in state 1; because $V^*(1) > V^*(0)$, nature selects the lower transition endpoints 0.1 and 0.7 for those actions. The table is illustrative; the certificate (12.3), not stabilization of printed digits, is the rigorous stopping criterion.

12.3 Model dependence

Use the total-variation convention

$$\|P - Q\|_{\mathrm{TV}} = \sup_{\|f\|_\infty \le 1} \left| \int f\, dP - \int f\, dQ \right|.$$

Theorem 12.3 (Robust model perturbation)**.** ***New result N22.*** *Let $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ be robust Bellman operators with the same state and action sets, discount γ, and reward bound R. For every (s,a), assume two-sided matching of their uncertainty sets: every (r,P) in one set has a partner $(\widetilde{r},\widetilde{P})$ in the other, and conversely, with $|r-\widetilde{r}| \le \eta_r$ and $\|P-\widetilde{P}\|_{\mathrm{TV}} \le \eta_P$. Then, their fixed points satisfy*

$$\|V^* - \widetilde{V}^*\|_\infty \le \frac{1}{1-\gamma}\left(\eta_r + \frac{\gamma R \eta_P}{1-\gamma}\right). \tag{12.4}$$

Proof. Both fixed points have sup norm at most $R/(1-\gamma)$. At any V in this ball, Lemma 12.1 and the pairing assumption give

$$\|\mathcal{T}V - \widetilde{\mathcal{T}}V\|_\infty \le \eta_r + \gamma\eta_P\|V\|_\infty \le \eta_r + \frac{\gamma R\eta_P}{1-\gamma}.$$

Apply **N15** to the two γ-contractions. □

12.4 Entropic evaluation

For $\theta \ne 0$, define the entropic Bellman operator on a finite or measurable state space by

$$(\mathcal{T}_\theta V)(s) = \sup_{a\in A(s)} \left\{ r(s,a) + \frac{1}{\theta}\log \int \exp(\theta\gamma V(s'))P(ds' \mid s,a) \right\}. \tag{12.5}$$

Theorem 12.4 (Entropic profile contraction)**.** ***New result N23.*** *If rewards are bounded and $\gamma < 1$, then $\mathcal{T}_\theta$ is a γ-contraction in the sup* *norm. It therefore has a unique bounded fixed point, with the Picard, residual, inexact, and profile certificates of **N6** and **N11**.*

Proof. Let $\delta = \|V - W\|_\infty$. Then $|\theta\gamma(V - W)| \le |\theta|\gamma\delta$. Consequently,

$$e^{-|\theta|\gamma\delta} \int e^{\theta\gamma W}\, dP \le \int e^{\theta\gamma V}\, dP \le e^{|\theta|\gamma\delta} \int e^{\theta\gamma W}\, dP.$$

Taking logarithms and dividing the absolute log difference by $|\theta|$ gives a bound $\gamma\delta$. The supremum over actions is nonexpansive, proving the claim. □

12.5 Chapter exercises

1. Prove the inf–sup variant of Lemma 12.1.
2. Compute the residual certificate at iteration 20 in the two-state example.
3. Derive an inexact value-iteration bound when every inner minimization is solved to accuracy 10^{-4}.
4. Modify **N22** when the two reward bounds are different.

Chapter 13

Nonlinear Volterra and Fredholm Equations

Integral equations provide a transparent setting in which calibrated profiles add error certificates and allow discretization error to be included in the same theorem.

13.1 A Volterra operator in a Bielecki norm

Consider

$$u(t) = g(t) + \int_0^t K(t,s) f(s, u(s))\, ds, \qquad 0 \le t \le T. \tag{13.1}$$

Assume g is continuous, K is continuous with $|K(t,s)| \le K_0$, and

$$|f(s,r) - f(s,z)| \le L|r - z|.$$

For $\beta > 0$, define the Bielecki norm

$$\|u\|_\beta = \sup_{0 \le t \le T} e^{-\beta t} |u(t)|.$$

Theorem 13.1 (Bielecki-profile Volterra theorem)**.** ***New result N24.*** *If $\beta > LK_0$, the integral operator in (13.1) is a contraction on $C([0,T])$ in the Bielecki norm, with factor*

$$q_\beta = \frac{LK_0}{\beta} < 1. \tag{13.2}$$

*Consequently, (13.1) has a unique continuous solution on any finite horizon T, and successive approximation has all metric and profile certificates of **N6**. If a quadrature approximation has step error ϵ_n in the Bielecki norm, **N11** gives a certified total error.*

Proof. Let $(Au)(t)$ denote the right side. Since $|u(s)-v(s)| \le e^{\beta s}\|u-v\|_\beta$,

$$\begin{aligned} e^{-\beta t}|(Au)(t) - (Av)(t)| &\le LK_0 e^{-\beta t}\int_0^t e^{\beta s}\,ds\,\|u-v\|_\beta \\ &= \frac{LK_0}{\beta}(1 - e^{-\beta t})\|u-v\|_\beta \\ &\le q_\beta\|u-v\|_\beta. \end{aligned}$$

The space is complete under the equivalent Bielecki norm. Apply **N6** and **N11**. □

Remark 13.2. In the ordinary sup norm the sufficient factor is LK_0T, which restricts the horizon. The Bielecki norm replaces that factor by LK_0/β and therefore provides a global finite-horizon result. The calibrated profile is built from the chosen norm; changing the norm changes the numerical interpretation but not the solution.

13.2 Fredholm equations

For

$$u(t) = g(t) + \int_a^b K(t,s)f(s,u(s))\,ds, \tag{13.3}$$

put

$$\kappa = \sup_{t\in[a,b]} \int_a^b |K(t,s)|\, ds.$$

If $L\kappa < 1$, the operator is an $L\kappa$-contraction in the sup norm. Hence, the solution and every profile certificate follow from **N6**. Unlike the Volterra case, an exponential weight does not automatically remove the smallness condition because the integration covers the entire interval in both directions.

13.3 Coupled integral systems

Let $u = (u_1, \dots, u_m)$ and consider

$$u_i(t) = g_i(t) + \sum_{j=1}^{m} \int_0^t K_{ij}(t,s) f_{ij}(s, u_j(s))\, ds. \tag{13.4}$$

Assume $|K_{ij}| \le K_{ij}^0$ and f_{ij} is Lipschitz in its final argument with constant L_{ij}.

Theorem 13.3 (Coupled integral profile theorem)**.** ***New result N25.*** *Fix $\beta > 0$ and form the nonnegative matrix*

$$M_{ij} = \frac{K_{ij}^0 L_{ij}}{\beta}. \tag{13.5}$$

If $r(M) < 1$, system (13.4) has a unique continuous solution. Weights constructed by (5.2) give a calibrated product profile, a contraction factor $\beta_M < 1$, and explicit Picard and inexact-quadrature certificates.

Proof. The calculation in **N24** gives

$$\|(Au)_i - (Av)_i\|_\beta \le \sum_j M_{ij} \|u_j - v_j\|_\beta.$$

Apply the contraction-matrix theorem **N20** to the product of the Bielecki spaces. ☐

13.4 Discretization as an inexact orbit

Let A_h be a quadrature approximation to an integral operator A and suppose

$$\sup_{u \in \mathcal{D}} \|A_h u - Au\| \leq Ch^r$$

on an invariant domain $\mathcal{D}$. The fixed points p_h and p satisfy

$$\|p_h - p\| \leq \frac{Ch^r}{1-q},$$

by **N15**, while an iteration using a changing mesh h_n satisfies **N11** with $\epsilon_n = Ch_n^r$. This cleanly separates iteration error from discretization error.

13.5 Chapter exercises

1. Show that the Bielecki and sup norms are equivalent on $C([0,T])$.
2. Derive the exact factor $(LK_0/\beta)(1 - e^{-\beta T})$ and compare it with (13.2).
3. Apply **N25** to a two-equation system and construct a Lipschitz matrix with one row sum greater than one but spectral radius below one.
4. Translate a quadrature error Ch^2 into a truth-nearness guarantee.

Chapter 14

Fractional Initial-Value Problems

A Caputo problem becomes a weakly singular Volterra equation. The calibrated theorem supplies an explicit horizon condition, convergence rate, and perturbation certificate.

14.1 Integral representation

Let $0 < \alpha \le 1$ and consider

$$^{C}D^{\alpha}u(t) = f(t, u(t)), \qquad u(0) = u_0, \quad 0 \le t \le T. \tag{14.1}$$

Under the standard continuity assumptions, a solution satisfies

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, ds. \tag{14.2}$$

Theorem 14.1 (Fractional profile theorem)**.** ***New result N26.***

Assume f is continuous and $|f(t,r) - f(t,z)| \leq L|r-z|$. If

$$q_\alpha = \frac{LT^\alpha}{\Gamma(\alpha+1)} < 1, \tag{14.3}$$

*then (14.1) has a unique solution in $C([0,T])$. Successive approximation satisfies **N6** with factor q_α. If f is replaced by $\widetilde{f}$ with* $\sup_{t,r} |f(t,r) - \widetilde{f}(t,r)| \leq \eta$, *then the two solutions satisfy*

$$\|u - \widetilde{u}\|_\infty \leq \frac{\eta T^\alpha}{\Gamma(\alpha+1)(1-q_\alpha)}, \tag{14.4}$$

provided both operators have contraction factor at most q_α.

Proof. For the integral operator A in (14.2),

$$\begin{aligned} |(Au)(t) - (Av)(t)| &\leq \frac{L\|u-v\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, ds \\ &= \frac{Lt^\alpha}{\Gamma(\alpha+1)} \|u-v\|_\infty. \end{aligned}$$

Taking the supremum gives q_α. **N6** proves existence, uniqueness, and rates. The operator discrepancy caused by $f - \widetilde{f}$ is at most $\eta T^\alpha / \Gamma(\alpha+1)$. Apply **N15**. □

14.2 Continuation on Longer Horizons

Condition (14.3) is sufficient, not necessary. When it fails on a long interval, divide the horizon into subintervals on which the corresponding local factor is below one, while accounting for the history term inherited from earlier intervals. The history is known once the previous segment has been computed and therefore acts as part of the inhomogeneous term. A rigorous continuation proof must preserve compatibility of the segment solutions; it is not valid simply to restart the Caputo derivative with zero memory.

14.3 Residual Interpretation

Let $v \in C([0,T])$ be an approximate solution of (14.2). Define its residual function $r_v : [0,T] \to \mathbb{R}$ by

$$r_v(t) = v(t) - u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, v(s))\, ds, \qquad t \in [0,T]. \tag{14.5}$$

The corresponding integral residual is

$$\mathcal{R}(v) = \|r_v\|_\infty = \sup_{0 \le t \le T} \left| v(t) - u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, v(s))\, ds \right|. \tag{14.6}$$

Equivalently, if $A : C([0,T]) \to C([0,T])$ denotes the integral operator

$$(Av)(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, v(s))\, ds,$$

then

$$\mathcal{R}(v) = \|v - Av\|_\infty.$$

Let u be the unique fixed point of A. Since A is a contraction with contraction factor

$$q_\alpha = \frac{LT^\alpha}{\Gamma(\alpha+1)} < 1,$$

we have

$$\begin{aligned} \|v - u\|_\infty &= \|v - Au\|_\infty \\ &\le \|v - Av\|_\infty + \|Av - Au\|_\infty \\ &\le \mathcal{R}(v) + q_\alpha \|v - u\|_\infty. \end{aligned}$$

Therefore,

$$\|v - u\|_\infty \le \frac{\mathcal{R}(v)}{1 - q_\alpha}. \tag{14.7}$$

Thus, $\mathcal{R}(v)$ provides an a posteriori error certificate for the approximation v. In particular, if a verified quadrature method produces a rigorous enclosure

$$\mathcal{R}(v) \leq \varepsilon_{\mathrm{res}},$$

then

$$\|v - u\|_\infty \leq \frac{\varepsilon_{\mathrm{res}}}{1 - q_\alpha}.$$

Hence, a rigorous enclosure of the residual integral yields a directly computable and mathematically certified bound on the error of the approximate solution.

For an approximate function v, define the integral residual

$$\mathcal{R}(v) = \left\| v - u_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, v(s))\, ds \right\|_\infty .$$

Then, **N14** gives $\|v-u\|_\infty \leq \mathcal{R}(v)/(1-q_\alpha)$. This statement is directly computable if the residual integral is enclosed with verified quadrature.

14.4 Chapter exercises

1. Recover the ordinary first-order Picard factor when $\alpha = 1$.
2. Determine the largest sufficient horizon for given L and α.
3. Translate (14.4) into the three profile components.
4. Explain why pointwise quadrature residuals without a uniform bound do not establish the stated sup-norm certificate.

Chapter 15

Nonlinear Matrix Equations

Matrix problems show how operator norms, nonlinearities, and calibrated perturbation bounds combine.

15.1 A structured equation

On $\mathbb{R}^{n\times n}$ with the Frobenius norm, consider

$$X = Q + \sum_{k=1}^{m} A_k^\top \Phi_k(X) A_k, \tag{15.1}$$

where each Φ_k is Lipschitz in the Frobenius norm:

$$\|\Phi_k(X) - \Phi_k(Y)\|_F \le L_k \|X - Y\|_F.$$

Theorem 15.1 (Matrix profile theorem)**.** ***New result N27.*** *If*

$$q_M = \sum_{k=1}^{m} L_k \|A_k\|_2^2 < 1, \tag{15.2}$$

then (15.1) *has a unique solution. The iteration*

$$X_{j+1} = Q + \sum_k A_k^\top \Phi_k(X_j) A_k$$

*converges with the certificates of **N6**. If Q is replaced by $\widetilde{Q}$ and Φ_k by $\widetilde{\Phi}_k$ with uniform discrepancies η_k on an invariant set, then*

$$\|X^* - \widetilde{X}^*\|_F \le \frac{\|Q - \widetilde{Q}\|_F + \sum_k \|A_k\|_2^2 \eta_k}{1 - q_M}, \tag{15.3}$$

assuming the perturbed map has contraction factor no larger than q_M.

Proof. Use $\|A^\top Y A\|_F \le \|A\|_2^2 \|Y\|_F$ to obtain

$$\|\mathcal{A}(X) - \mathcal{A}(Y)\|_F \le q_M \|X - Y\|_F.$$

The matrix space is complete, so **N6** applies. The numerator in (15.3) bounds the uniform discrepancy of the two operators, and **N15** gives the result. □

15.2 A reproducible two-by-two example

Let

$$A = \begin{pmatrix} 0.3 & 0.1 \\ 0 & 0.25 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0.2 \\ 0.2 & 0.8 \end{pmatrix},$$

and let $\Phi(X)$ apply tanh entrywise. Entrywise tanh is 1-Lipschitz in the Frobenius norm, and $\|A\|_2^2 = 0.1125$. Hence, **N27** applies with $q_M = 0.1125$. Starting from the zero matrix gives

$$X^* \approx \begin{pmatrix} 1.07108838 & 0.24146183 \\ 0.24146183 & 0.86336678 \end{pmatrix}.$$

The displayed digits are supported by a residual check in the same Frobenius norm; they are not justified merely by successive iterates appearing equal after rounding.

15.3 Invariant cones

If an application requires X to remain symmetric or positive semidefinite, the operator must preserve the corresponding closed cone. Symmetry follows when Q is symmetric and every Φ_k maps symmetric matrices to symmetric matrices. Positive semidefiniteness additionally requires $Q \succeq 0$ and $\Phi_k(X) \succeq 0$ for $X \succeq 0$. Once invariance is proved, the closed cone is complete and **N27** applies there. One must not infer positivity from scalar entrywise monotonicity alone.

15.4 Chapter exercises

1. Prove $\|A^\top Y A\|_F \le \|A\|_2^2 \|Y\|_F$.
2. Compute an a priori iteration budget for the two-by-two example.
3. State sufficient conditions under which the matrix map preserves symmetry.
4. Derive a profile truth bound from (15.3).

Chapter 16

Network Equilibria and Data Fusion

A nonnegative influence matrix provides a natural bridge from coupled Lipschitz estimates to stable network equilibria and componentwise uncertainty reports.

16.1 A nonlinear network map

Let $x \in \mathbb{R}^m$ and define

$$F_i(x) = \sigma_i \left(b_i + \sum_{j=1}^{m} w_{ij} x_j \right), \tag{16.1}$$

where σ_i is Lipschitz with constant ℓ_i. Set

$$M_{ij} = \ell_i |w_{ij}|. \tag{16.2}$$

Theorem 16.1 (Network profile equilibrium)**.** ***New result N28.*** *If $r(M) < 1$, the network map (16.1) has a unique equilibrium x^*. Weights from **N20** produce a contraction factor $\beta < 1$ and a calibrated* *weighted profile. Synchronous iteration converges with **N6**, nodewise evaluation errors converge according to **N11** with* $\epsilon_n = \max_i \epsilon_{i,n}/w_i$*; and a bias perturbation* $|b_i - \widetilde{b}_i| \le \delta_i$ *displaces the equilibrium by at most*

$$d_w(x^*, \widetilde{x}^*) \le \frac{1}{1-\beta} \max_i \frac{\ell_i \delta_i}{w_i}. \tag{16.3}$$

Proof. For each component,

$$|F_i(x) - F_i(y)| \le \sum_j M_{ij} |x_j - y_j|.$$

N20 supplies the weighted contraction and unique equilibrium. **N11** gives the node-error result. The bias perturbation changes component i by at most $\ell_i \delta_i$, its weighted uniform discrepancy is the maximum in (16.3). Apply N15. □

16.2 Interpretation for data fusion

Suppose each node fuses a local datum b_i with neighbor states. The scalar equilibrium bound records worst-case displacement in weighted physical units. At a node-specific reporting scale t_i, it becomes

$$\mathsf{T}_i(x_i^*, \widetilde{x}_i^*, t_i) \ge \frac{t_i}{t_i + w_i E},$$

where E is the right side of (16.3). The split ϑ_i may differ among nodes if they classify uncertainty differently; truth-nearness and the recovered scalar distance remain unchanged.

16.3 Damping a marginal network

If $r(M) \ge 1$, uniqueness cannot be concluded from **N28**. A damped update

$$G(x) = (1-\lambda)x + \lambda F(x)$$

does not automatically cure an expansive F: a Lipschitz upper bound is $1 - \lambda + \lambda\beta$, which is below one only when an underlying bound $\beta < 1$ already exists. Damping can improve oscillatory numerical behavior, but it is not a substitute for a valid contraction proof.

16.4 Asynchronous computation

Delayed updates require assumptions on delay bounds and update frequency. **N11** can absorb an asynchronous step only after its difference from the synchronous map has been bounded as an error ϵ_n. Claiming convergence from $r(M) < 1$ while allowing arbitrary unbounded delays would exceed the theorem.

16.5 Chapter exercises

1. Apply **N28** to a three-node network with logistic activations, whose Lipschitz constants are at most $1/4$.
2. Construct weights for a nonnegative influence matrix with spectral radius below one but a row sum above one.
3. Derive the falsity bound corresponding to (16.3).
4. Give an example showing that damping an expansive scalar map need not make it contractive.

Part V

Proof Audit, Research Design, and Extensions

Chapter 17

From Theorem to Certified Computation

A convergence theorem becomes computationally useful only after norms, constants, residuals, and arithmetic error have been connected in a reproducible workflow.

17.1 Four layers of a certificate

A reliable fixed point computation has four logically distinct layers.

1. **Invariant domain.** Prove that $A(D) \subseteq D$ for a complete closed set D. A Lipschitz estimate on a domain is irrelevant if the iteration immediately leaves that domain.

2. **Contraction.** Derive a justified factor $q < 1$ in a named metric or norm. Numerical samples of a derivative do not establish a global bound.

3. **Residual enclosure.** Bound $d(x, Ax)$ using arithmetic whose error is itself controlled.

4. **Profile report.** Convert the scalar bound through (9.6), do not treat the split parameter as an additional source of analytic accuracy.

17.2 Interval residuals

Suppose a computed vector $\widehat{x}$ is stored in floating-point arithmetic and an interval evaluation proves

$$A(\widehat{x}) \in [\underline{y}, \overline{y}]$$

componentwise. In a weighted max norm, a valid residual enclosure is

$$R = \max_i \frac{\max\{|\widehat{x}_i - \underline{y}_i|, |\widehat{x}_i - \overline{y}_i|\}}{w_i}.$$

Then, $d_w(\widehat{x}, p) \le R/(1-q)$. The interval step matters when the printed residual is close to the requested tolerance. Without it, rounding may make an uncertified residual appear to vanish.

17.3 Choosing between certificates

Certificate	Best use
A priori	Planning an iteration budget before computation.
A posteriori	Cheap monitoring when consecutive iterates are available.
Residual	Final verification, especially when local convergence is faster than the global factor.
Inexact convolution	Nested algorithms, quadrature, sampled expectations, and early inner stopping.
Data dependence	Comparing models, discretizations, or parameter values.

No certificate is uniformly strongest. A conservative q can make residual bounds large even when the observed orbit stabilizes rapidly; conversely, a small observed step is not reliable if an inexact solver has an unaccounted error.

17.4 Scale selection for reporting

Once a scalar error bound E is known, the reporting scale t changes the visual size of each component but not the error. The truth lower bound is $t/(t+E)$. Choosing $t = E$ makes the bound $1/2$; choosing $t = 9E$ makes it 0.9. Thus a large truth value can be manufactured by increasing t. Meaningful comparisons must use the same scale or report the recovered scalar gauge alongside the triple.

17.5 A reproducibility template

For each numerical table, record:

1. the space, invariant domain, metric or norm, and profile split;
2. the exact operator and all parameter values;
3. the derivation of q or of the Lipschitz matrix;
4. the initial point, stopping rule, and inexactness schedule;
5. the final residual and its arithmetic enclosure;
6. the scalar error bound and the three component bounds at a declared scale;
7. enough digits or code to reproduce the displayed values.

17.6 Complexity accounting

If a map evaluation costs C_A and the a priori budget is n, the basic cost is nC_A. Residual testing at every iteration can nearly double the number of map evaluations unless Ax_n is reused. For nested solves, a useful report separates outer iterations, total inner iterations, and the convolution-weighted error. A claim that one method is faster should use wall-clock time or operation counts under the same tolerance certificate, not merely fewer outer iterations.

17.7 Chapter exercises

1. Show how to reuse Ax_n in a residual test without an extra evaluation.
2. Explain why reporting $\mathsf{T} \geq 0.99$ without a scale is incomplete.
3. Design a table that separates iteration error from discretization error for a Fredholm equation.
4. Give an example in which the a posteriori bound is much sharper than the a priori bound.

Chapter 18

Proof Audit and Counterexamples

This chapter isolates logical checks that are especially important in three-component fixed point arguments. Each warning is accompanied by a correction strategy.

18.1 Direction of comparison

Audit Warning 18.1 (An impossible self-comparison). Let $J \geq 0$ and let $\Omega(J) > J$ for $J > 0$. A hypothesis that reduces at some substitution to

$$J \geq \Omega(J)$$

is impossible unless $J = 0$. If an example computes $J = 1.5$ and $\Omega(J) = 3$, it disproves, rather than verifies, the displayed inequality.

An expansive comparison function can be used with the opposite argument placement, but the proof must be rederived. The safe audit is to substitute a proposed fixed point or coincidence point into every contractive inequality and simplify it to a one-variable statement.

18.2 The comparison at zero

If an integral transform is used,

$$J(a) = \int_0^a \mu(r)\,dr,$$

then $J(a) = 0 \Rightarrow a = 0$ requires more than $\mu \geq 0$. A sufficient condition is $\int_0^\varepsilon \mu(r)\,dr > 0$ for every $\varepsilon > 0$. Without such a condition, the transform may vanish on a nontrivial interval and cannot separate points. The calibrated gauge avoids this ambiguity because its inverse is explicit.

18.3 Dummy variables and scale variables

In an expression such as

$$\int_0^{\mathsf{T}(Ax,Ay,t)} \mu(r)\,dr,$$

t is the neutrosophic scale and r is the integration variable. Writing s in both roles can produce an expression whose meaning changes within the same line. Every proof should use distinct symbols for scale, integration, sequence index, and state.

18.4 Consistency of algebraic operations

Audit Warning 18.2 (Mismatched t-norm)*.* An example checked with the product t-norm does not verify a theorem stated only for the minimum t-norm. The operations are part of the space, not optional notation. Either prove the theorem for both operations, as in **N1**, or use the stated operation throughout.

The same warning applies to changing $\oplus$ from maximum to probabilistic sum, or changing the component definitions while retaining a theorem whose proof uses specific identities.

18.5 Vacuous hypotheses

A theorem can be formally true but useless if its hypotheses are inconsistent for all nontrivial points. Three quick tests are:

1. set $x = y$;
2. substitute a claimed fixed point;
3. set all maps equal to the identity.

If a condition fails in all three basic regimes, one must determine whether that is intended or a sign error.

18.6 A false contraction inference

From

$$\mathsf{T}(Ax, Ay, t) \geq \mathsf{T}(x, y, t)$$

one obtains only $d(Ax, Ay) \leq d(x, y)$ in the rational profile: nonexpansiveness, not contraction. The identity map satisfies this inequality and can have every point fixed. A strict time dilation t/q with a uniform $q < 1$, or another valid strict mechanism, is necessary for **N6**.

18.7 Limits through semicontinuous functions

If $a_n \to a$ and Ω is lower semicontinuous, then

$$\Omega(a) \leq \liminf_n \Omega(a_n).$$

If ϕ is upper semicontinuous, then

$$\limsup_n \phi(a_n) \leq \phi(a).$$

Reversing either inequality invalidates a contradiction argument. Continuity is often assumed simply because it removes this bookkeeping, but if only semicontinuity is claimed, the correct liminf or limsup must appear.

18.8 Continuity of the metric components

Convergence $x_n \to x$ does not automatically justify $\mathsf{T}(x_n, y, t) \to \mathsf{T}(x, y, t)$ in every abstract definition unless joint continuity or a suitable triangle argument has been established. In the calibrated rational profile, the conclusion follows from continuity of d:

$$|d(x_n, y) - d(x, y)| \leq d(x_n, x) \to 0.$$

18.9 Examples must check all cases

For piecewise maps, verifying a contractive condition only on the region containing the proposed fixed point is not a global verification. Every pair of branches must be checked unless the invariant domain has first been restricted to one branch. Similarly, a robust Bellman example must check every state and action; a matrix example must specify the norm used to compute the constant.

18.10 Numerical equality is not mathematical equality

Two iterates printed with six decimals may agree while their residual is too large for the required tolerance. Conversely, a residual can be small even when some digits still change. The theorem certifies the residual, not visual stabilization.

18.11 Audit checklist for a theorem

No.	Question
1	Are the domain, range, t-norm, and t-conorm fixed consistently?
2	Does every symbol have one role and every denominator stay nonzero?
3	Can the contractive hypotheses be satisfied by a non-trivial example?
4	Is the topology used in the limit passage defined and compatible with the components?
5	Is completeness invoked for a sequence actually proved to be Cauchy?
6	Is existence separated from uniqueness?
7	Is weak compatibility used only after a coincidence point exists?
8	Are liminf and limsup directions compatible with semi-continuity?
9	Does the example verify all hypotheses on the entire stated domain?
10	Does the application operator map the claimed function space into itself?
11	Is every numerical constant derived in the declared norm?
12	Are novelty and generalization claims supported by an explicit comparison table?

18.12 Chapter exercises

1. Construct a nonnegative integrand μ for which $\int_0^a \mu = 0$ for some $a > 0$.
2. Give a nonexpansive map with more than one fixed point.
3. Write the correct liminf statement for a lower semicontinuous comparison function and use it in a one-line contradiction argument.
4. Audit one theorem in this book using the twelve questions above.

Chapter 19

Research Extensions and Open Problems

Theorems end where their proofs end. This chapter identifies plausible extensions but labels them as problems or conjectural programs until complete proofs and novelty searches are available.

19.1 Beyond exact rational profiles

The rational profile has the advantage of a simple exact inverse. A wider class can be built from a decreasing calibration kernel $k : [0, \infty) \to (0, 1]$ with $k(0) = 1$ by setting

$$\mathsf{T}_k(x, y, t) = k(d(x, y)/t), \quad \mathsf{I}_k = \vartheta(1 - \mathsf{T}_k), \quad \mathsf{F}_k = (1 - \vartheta)(1 - \mathsf{T}_k).$$

If k is strictly decreasing, the scalar inverse is $d = tk^{-1}(\mathsf{T}_k)$. The triangle axioms impose additional algebraic conditions on k.

Research Problem 19.1 (Kernel characterization)**.** Characterize all continuous decreasing kernels k for which every metric generates a neutrosophic metric under the Gödel pair, and separately under the product–probabilistic-sum pair. Determine which kernels give the best conditioning for noisy inversion on a prescribed distance interval.

19.2 Profiles that are only approximately calibrated

N4 treats component noise around an exact profile. A more structural problem is to start from a general neutrosophic metric and ask whether it lies near any calibrated metric profile.

Research Problem 19.2 (Nearest calibrated profile)**.** Given triples observed on a finite set at several scales, minimize a discrepancy to a rational profile over all metrics d and splits ϑ, subject to triangle inequalities. Study existence, uniqueness, statistical consistency, and algorithms for this projection problem.

The finite problem resembles metric nearness and can be formulated with triangle constraints, but the nonlinear component inverses and scale coupling require a new analysis.

19.3 Set-valued contractions

Nadler's Hausdorff contraction theorem [43] suggests defining a calibrated profile on nonempty closed bounded subsets via the Hausdorff distance.

Research Problem 19.3 (Set-valued profile algorithms)**.** Develop inexact selection algorithms for a Hausdorff-contractive multifunction, including a posteriori componentwise bounds that account for both selection error and set-approximation error. Separate fixed points $x \in F(x)$ from invariant sets $F(K) = K$.

19.4 Random and stochastic errors

The deterministic convolution **N11** remains valid pathwise whenever an almost-sure error bound is available. Mean-square or high-probability conclusions need martingale or concentration assumptions.

Research Problem 19.4 (Probabilistic profile certificate)**.** Assume the inexact errors form a martingale-difference sequence with conditional sub-Gaussian tails. Derive a time-uniform high-probability bound for $d(x_n, p)$ and translate it into simultaneous truth, indeterminacy, and falsity confidence bands without treating the three components as independent observations.

19.5 Noncontractive operators

Many important maps are nonexpansive rather than contractive. Krasnosel'skiĭ–Mann, Halpern, and proximal iterations require geometric structure such as uniform convexity, not just completeness of a metric space.

Research Problem 19.5 (Calibrated nonexpansive iteration)**.** On a uniformly convex Banach space, determine which classical weak and strong convergence theorems admit quantitatively meaningful profile rates. Distinguish an exact scalar reformulation from a genuinely new componentwise stability estimate.

19.6 Operator learning

If an unknown contraction is learned from data, both the approximation error and the estimated Lipschitz factor are uncertain. **N15** controls the former only after a uniform bound is known.

Research Problem 19.6 (Learned contractive operator)**.** Construct statistically valid upper confidence bounds for the operator discrepancy and contraction factor on an invariant domain. Combine them with **N15** to obtain a finite-sample profile certificate for the learned fixed point.

19.7 Robust control without rectangularity

The contraction proof of **N21** uses only boundedness and the discount, but dynamic programming representation of a nonrectangular robust problem can fail because nature's choices are coupled across states or time.

Research Problem 19.7 (Coupled ambiguity)**.** Identify state augmentation or dual formulations that restore a contractive Bellman operator for selected nonrectangular ambiguity sets. Quantify the price of the augmentation through the calibrated data-dependence estimate.

19.8 A publication program

A defensible sequence of papers can be narrower than the present book:

1. foundational paper: **N1**–**N7** and **N14**–**N15** with counterexamples;
2. algorithms paper: **N11**–**N13**, stopping rules, and verified experiments;
3. coupled systems paper: **N20**, **N25**, and **N28**;
4. robust control paper: **N21**–**N23** with larger computational studies;
5. applications paper: one of the integral, fractional, or matrix directions.

Each paper should contain a comparison matrix against the closest literature and should avoid claiming that all metric consequences are automatically novel merely because they are displayed in three components.

19.9 Chapter exercises

1. Find conditions on k ensuring the Gödel truth triangle inequality.
2. Formulate the finite nearest-profile problem as a constrained optimization problem.
3. Explain why the three profile components do not provide three independent confidence levels in the normalized model.
4. Choose one research problem and list the minimal lemmas needed before a main theorem could be stated responsibly.

Appendix A

Scalar Lemmas Used Throughout

A.1 Geometric convolutions

For $q \in [0, 1)$ and nonnegative e_k, the recurrence $a_{n+1} \le qa_n + e_n$ gives

$$a_n \le q^n a_0 + \sum_{k=0}^{n-1} q^{n-1-k} e_k.$$

This follows by induction. If $e_k \le Cr^k$, the convolution is

$$\sum_{k=0}^{n-1} q^{n-1-k} Cr^k = \begin{cases} C(q^n - r^n)/(q - r), & q \ne r, \\ Cnq^{n-1}, & q = r. \end{cases}$$

If $e_n \to 0$, the convolution tends to zero: for any $\delta > 0$, choose N with $e_k \le (1 - q)\delta/2$ for $k \ge N$; the late terms sum to at most $\delta/2$, and the finite early part tends to zero.

A.2 Supremum and infimum inequalities

For bounded functions f, g on a common set,

$$|\sup f - \sup g| \leq \sup |f - g|, \qquad |\inf f - \inf g| \leq \sup |f - g|.$$

These inequalities remain true when suprema or infima are not attained. They are the correct replacement for selecting an optimizer in an existence proof.

A.3 Neumann series

If L is a bounded linear operator with $\|L\| < 1$, then

$$(I - L)^{-1} = \sum_{n=0}^{\infty} L^n, \qquad \|(I - L)^{-1}\| \leq \frac{1}{1 - \|L\|}.$$

For a nonnegative matrix M with $r(M) < q$, the same series applied to M/q has nonnegative entries. This is the constructive source of the weights in N20.

A.4 Basic matrix norm inequalities

For compatible matrices,

$$\|ABC\|_F \leq \|A\|_2 \|B\|_F \|C\|_2.$$

It follows from unitary invariance of the Frobenius norm and the submultiplicative operator bound. In particular, $\|A^\top BA\|_F \leq \|A\|_2^2 \|B\|_F$.

A.5 The rational inverse calculus

For $t > 0$,

$$R_t(r) = \frac{t}{t+r}, \qquad R_t^{-1}(u) = t\left(\frac{1}{u} - 1\right).$$

Both R_t and $1 - R_t$ are monotone, in opposite directions. Thus, a scalar upper bound $r \le E$ is equivalent to

$$R_t(r) \ge R_t(E), \qquad \vartheta(1 - R_t(r)) \le \vartheta(1 - R_t(E)).$$

Every component certificate in the book is an application of this elementary fact.

Appendix B

Selected Exercise Solutions

B.1 Foundational solutions

Chapter 2, Exercise 2. If $\vartheta = 0$, then $\mathsf{I}_d(x, y, t) = 0$ for every x, y, t, so the assertion $\mathsf{I}_d(x, y, t) = 0$ if and only if $x = y$ fails whenever the space has two distinct points. The same issue occurs for F when $\vartheta = 1$.

Chapter 2, Exercise 3. The truth formula encodes $d(x, y) = |x^3 - y^3|$. This is a metric on $[0, \infty)$ because it is the pullback of the Euclidean metric under the injective map $x \mapsto x^3$; the triangle inequality follows from $|x^3 - z^3| \leq |x^3 - y^3| + |y^3 - z^3|$.

Chapter 3, Exercise 1. Solving $r = t\varepsilon/(1 - \varepsilon)$ gives $\varepsilon = r/(t + r)$. This is also $1 - \mathsf{T}$ for a pair at distance r.

Chapter 4, Exercise 1. If $|\widetilde{\mathsf{T}} - \mathsf{T}| \leq \epsilon$ and $0 < \epsilon < \widetilde{\mathsf{T}}$, then

$$\mathsf{T} \in [\widetilde{\mathsf{T}} - \epsilon, \min\{1, \widetilde{\mathsf{T}} + \epsilon\}].$$

Because the inverse is decreasing,

$$t\left(\frac{1}{\min\{1,\widetilde{\mathsf{T}}+\epsilon\}}-1\right)\le d\le t\left(\frac{1}{\widetilde{\mathsf{T}}-\epsilon}-1\right).$$

B.2 Iteration solutions

Chapter 5, Exercise 2. Since $r\mapsto \vartheta r/(t+r)$ is strictly increasing,

$$\mathsf{I}(Ax,Ay,t)\le \mathsf{I}(x,y,t/q)$$

is equivalent to $d(Ax,Ay)\le qd(x,y)$. No truth inequality is needed for this inference.

Chapter 6, Exercise 1. For $\phi(r)=qr/(1+r)$, one has $\phi(r)\le qr$, hence by induction $\phi^n(r)\le q^n r$. Therefore $\sum_n \phi^n(r)\le r/(1-q)<\infty$.

Chapter 7, Exercise 1. Take $q_{2k-1}=2$ and $q_{2k}=1/8$. Then, the two-step product is $1/4$, so $Q_{2k}=4^{-k}$ and $Q_{2k-1}=2\cdot 4^{-(k-1)}$. The series of products converges although infinitely many factors equal 2.

Chapter 8, Exercise 2. From (2.3), it suffices to impose $\bar{\epsilon}/(1-q)\le \tau$, or $\bar{\epsilon}\le(1-q)\tau$. This bound is sharp for the scalar recurrence $x_{n+1}=qx_n+\bar{\epsilon}$ with fixed point of the unperturbed map equal to zero.

Chapter 9, Exercise 2. Let $X=\mathbb{R}$ and $A_\lambda(x)=qx+L\lambda$. Then, $p_\lambda=L\lambda/(1-q)$ and $|p_\lambda-p_\mu|=L|\lambda-\mu|/(1-q)$, attaining (3.3).

B.3 Application solutions

Chapter 11, Exercise 1. For any bounded double family,

$$\left|\inf_a \sup_u f_{a,u} - \inf_a \sup_u g_{a,u}\right| \leq \sup_{a,u} |f_{a,u} - g_{a,u}|.$$

The proof repeats Lemma 12.1, reversing the outer order operations.

Chapter 12, Exercise 1. For $u \in C([0,T])$, $e^{-\beta T}\|u\|_\infty \leq \|u\|_\beta \leq \|u\|_\infty$. Thus, the norms induce the same completeness and topology.

Chapter 13, Exercise 1. When $\alpha = 1$, $\Gamma(\alpha+1) = \Gamma(2) = 1$ and $q_\alpha = LT$, the standard sup-norm Picard factor for an ordinary integral form of an initial-value problem.

Chapter 14, Exercise 1. Write a singular value decomposition $A = U\Sigma V^\top$. Unitary invariance and two applications of $\|BC\|_F \leq \|B\|_2\|C\|_F$ give

$$\|A^\top Y A\|_F \leq \|A^\top\|_2\|YA\|_F \leq \|A\|_2^2\|Y\|_F.$$

Chapter 15, Exercise 4. For $F(x) = 2x$ and $0 < \lambda \leq 1$, $G(x) = (1-\lambda)x + \lambda F(x) = (1+\lambda)x$, whose Lipschitz factor exceeds one. Thus, damping does not turn this expansive map into a contraction.

Appendix C

Notation and Reproducible Algorithms

C.1 Notation table

Symbol	Meaning
d	Underlying calibrated metric.
t	Positive profile reporting scale.
ϑ	Indeterminacy split in $(0,1)$.
$\mathsf{T}, \mathsf{I}, \mathsf{F}$	Truth-nearness, indeterminacy, and falsity components.
ρ_t	Exact scalar inverse of a calibrated component.
$A, F, \mathcal{T}$	Generic, coupled, and Bellman operators.
q	Scalar contraction factor.
ϵ_n	Error in the nth inexact map evaluation.
M	Nonnegative Lipschitz matrix.
w	Positive vector defining a weighted sup metric.

$r(M)$	Spectral radius of M.
$\|\cdot\|_\beta$	Bielecki norm.
$\|\cdot\|_F$	Frobenius matrix norm.

C.2 Reference pseudocode for an inexact process

Algorithm 2. Inexact contraction with a rigorous certificate

Input: x_0, $q < 1$, tolerance τ, scale t, and split ϑ.

1. For $n = 0, 1, 2, \ldots$, compute x_{n+1} and a verified $\epsilon_n \geq d(x_{n+1}, Ax_n)$.
2. Set $E_n = [d(x_n, x_{n+1}) + \epsilon_n]/(1-q)$.
3. If $E_n \leq \tau$, set $T_{\min} = t/(t + E_n)$, $I_{\max} = \vartheta E_n/(t + E_n)$, and $F_{\max} = (1-\vartheta)E_n/(t + E_n)$, then return these values with x_n.

C.3 Reference pseudocode for coupled weights

Algorithm 3. Weights from a Lipschitz matrix

Input: a nonnegative matrix M with a verified bound $r(M) < 1$.

1. Choose q with $r(M) < q < 1$ and solve $(I - M/q)w = \mathbf{1}$.
2. Verify every $w_i > 0$.
3. Set $\beta = \max_i (Mw)_i/w_i$ and verify $\beta < 1$.
4. Return w and β.

In floating-point arithmetic, the two "verify" steps should use interval or backward-error bounds when the spectral radius is close to one. A computed $\beta = 0.999999$ without an enclosure may not prove strict contractivity.

Conclusion

The calibrated rational profile gives neutrosophic metric fixed point theory an exact numerical spine. It preserves the three-component language while exposing a distance that can be reconstructed, bounded, and audited. The book's new direction therefore differs from integral-type CLR common fixed point theory both in hypotheses and in output: it emphasizes constructive orbits, explicit rates, residual stopping, inexactness, stability, coupled systems, and applications with reproducible certificates.

The strongest practical lesson is simple. A profile value is meaningful only with its scale and calibration; a convergence claim is meaningful only with the metric, domain, and constant that prove it; and a numerical solution is meaningful only with a residual or error bound. These requirements do not weaken neutrosophic analysis. They make its uncertainty statements quantitatively testable.

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This monograph develops a constructive framework for fixed-point analysis in calibrated neutrosophic metric spaces. Through exact scalarization, it transforms truth, indeterminacy, and falsity profiles into computable distance estimates and convergence certificates. The theory encompasses quantitative iterations, stability, perturbation analysis, and contraction matrices, with applications to Bellman operators, integral and fractional equations, matrix problems, and network equilibria. Designed for researchers in fixed point theory, nonlinear analysis, uncertainty modeling, and computational mathematics, the book connects rigorous proofs with certified computation.

ISBN 978-197250251-8

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