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Research Paper

Characterizations and Cartesian Products of Smarandache Semigroups (S-semigroups)

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Abstract
Let $(S, \ast)$ be a semigroup. A semigroup S is called a *Smarandache semigroup* (or S*-semigroup*) if it contains a proper subset $A \subset S$ such that $(A, \ast)$ forms a group under the same binary operation defined on S. In general, not every semigroup admits a proper subset that is a group; hence, not all semigroups are S-semigroups. In this paper, several structural conditions related to Smarandache semigroups are investigated. In particular, we study the role of idempotent and completely regular elements in the structure of S-semigroups. These conditions provide a characterization of S-semigroups. Furthermore, this study investigates whether the Cartesian product of two or more S-semigroups is again an S-semigroup.

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1. INTRODUCTION

A nonempty set equipped with at least one binary operation and satisfying certain axioms is called an algebraic structure. Well-known examples of algebraic structures include groups and semigroups. The most basic algebraic structure is a groupoid, namely a nonempty set G together with a binary operation "$\ast$". A groupoid $(G, \ast)$ becomes a semigroup if the binary operation "$\ast$" satisfies the associative property, that is,

$$(a \ast b) \ast c = a \ast (b \ast c), \quad \text{for all } a, b, c \in G.$$

Furthermore, if a semigroup $(G, \ast)$ contains an identity element e such that $a \ast e = e \ast a = a$ for all $a \in G$, then $(G, \ast)$ is called a monoid. A monoid is also known as a semigroup with identity (or a semigroup with unity when the operation is multiplication). Moreover, a monoid $(G, \ast)$ is called a group if for every element $a \in G$, there exists an element $a^{-1} \in G$ such that $a \ast a^{-1} = e$, where a^{-1} is called the inverse of a.

From these definitions, it can be concluded that semigroups are generalizations of monoids and groups. Consequently, every group or monoid is necessarily a semigroup. However, the focus of this paper is restricted to semigroups.

Early developments in semigroup theory can be traced back to the work of A.K. Suschkewitsch in 1928 and later developments in the theory of semigroups can be found in standard references such as [1]. One significant development in semigroup theory was presented by Raul Padilla in 1998 through his work on Smarandache algebraic structures entitled *Special Algebraic Structures* [2]. In this work, Padilla introduced the concept and notion of Smarandache structures in algebra [2, 3]. Further developments and variations of Smarandache semigroups have been studied by several authors [3, 4].

In classical semigroup theory, various structural properties of semigroups have been extensively studied, including regular semigroups, completely regular semigroups, and the structural analysis provided by Green's relations. Regular semigroups are characterized by the property that for every element a, there exists an element x in the semigroup such that $axa = a$, while completely regular semigroups consist of unions of groups. These notions play an important role in understanding the internal structure of semigroups and their subgroup-like components [1]. The concept of Smarandache algebraic structures provides a different perspective by focusing on the existence of a proper subset that forms a stronger algebraic structure within a weaker one. In the context of semigroups, this leads to the notion of a Smarandache semigroup, where a semigroup contains a proper subset that forms a group under the same binary operation.

Motivated by this concept, this paper investigates the structural properties of Smarandache semigroups. A semigroup S is called a Smarandache semigroup if it contains a proper subset that forms a group under the same binary operation defined on S [4]. However, such a group-forming proper subset does not necessarily exist in every semigroup. Therefore, it is important to determine structural conditions that guarantee the existence of such a subgroup within a semigroup. However, to the best of our knowledge, structural characterizations of Smarandache semigroups in terms of special elements such as idempotent and completely regular elements have not been systematically studied. Furthermore, the behavior of Smarandache semigroups under Cartesian products has received little attention in the existing literature. In particular, we prove several characterization results for Smarandache semigroups and study the preservation of the Smarandache property under Cartesian products. The main contributions of this paper are summarized as follows.

First, we establish several characterization results describing structural conditions related to Smarandache semigroups in terms of special elements such as idempotent and completely regular elements. These characterizations provide a structural viewpoint that connects Smarandache semigroups with classical concepts in semigroup theory. Furthermore, we investigate the behavior of Smarandache semigroups under Cartesian products. Although the direct product of groups is again a group, the preservation of the Smarandache property under Cartesian products of semigroups is not immediate, since the existence of a subgroup-forming proper subset may depend on the interaction between the component semigroups. Therefore, it is natural to examine conditions under which the Cartesian product of Smarandache semigroups is again a Smarandache semigroup. In particular, we study conditions under which the Cartesian product of two or more Smarandache semigroups is again a Smarandache semigroup. Our results show that certain structural properties related to the Cartesian product of groups can be extended to the setting of Smarandache semigroups.

The remainder of this paper is organized as follows. Section 2 presents the research methods and the theoretical framework used in this study. Section 3 discusses the main results, including the characterization of Smarandache semigroups and the properties of their Cartesian products. Finally, Section 4 provides the conclusions of this study.

2. METHODS

This study employs a theoretical approach based on the analysis of fundamental concepts and results in semigroup theory. The focus of the study is on semigroup structures and Smarandache semigroups, particularly their definitions, structural properties, and distinguishing characteristics. In addition, special elements in semigroup structures, such as idempotent elements and regular elements, which play an important role in the study of Smarandache semigroups, are also considered.

The theoretical framework of this study is developed through an in-depth examination of standard textbooks and relevant research articles in algebra and semigroup theory [1, 2, 5, 6, 7]. Based on this framework, the study investigates structural conditions related to Smarandache semigroups and analyzes the behavior of Smarandache semigroups under Cartesian products. In particular, the study examines conditions under which the Cartesian product of two or more Smarandache semigroups preserves the Smarandache property.

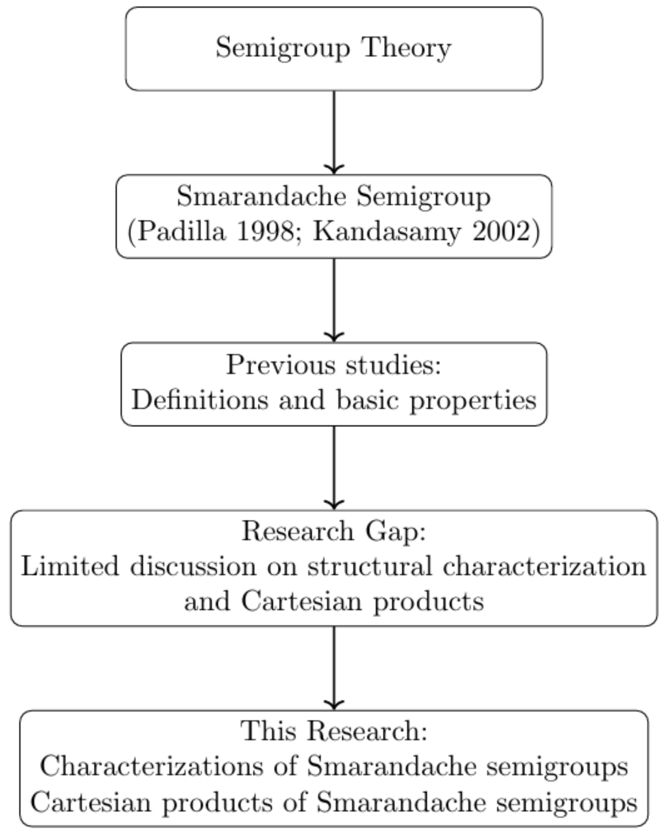

Figure 1. Research Position Within The Development of Smarandache Semigroup Theory

The position of this research within the development of Smarandache semigroup theory is illustrated in Figure 1. Previous studies mainly focus on the definitions and basic properties of Smarandache semigroups. However, the structural characterization of Smarandache semigroups and their behavior under Cartesian products have received limited attention. This study aims to contribute to this area by investigating characterization aspects of Smarandache semigroups and examining the preservation of the Smarandache property under Cartesian products.

3. RESULTS AND DISCUSSION

This section presents the main theoretical results obtained in this study, along with their discussions. We begin with the fundamental definition of a semigroup, which serves as the basis for the development of Smarandache semigroups.

Definition 1. *[1] Let $S \neq \emptyset$ be a nonempty set equipped with a binary operation $*\colon\ S \times S \to S$. The algebraic structure $(S, *)$ is*

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called a ***semigroup*** *if the operation $*$ is associative, that is,*

$$(a * b) * c = a * (b * c), \quad \text{for all } a, b, c \in S.$$

Definition 2. *[1] Let $(S, *)$ be a semigroup. If there exists an element $e \in S$ such that*

$$e * s = s * e = s \quad \text{for all } s \in S,$$

then $(S, *)$ is called a ***monoid****, and the element e is called the* ***identity element*** *of S.*

Example 1. *Let $M_{2\times2}(\mathbb{R})$ denote the set of all 2×2 real matrices equipped with the usual matrix multiplication. Since matrix multiplication is associative, $(M_{2\times2}(\mathbb{R}), \cdot)$ forms a semigroup. Moreover, there exists an identity element*

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_{2\times2}(\mathbb{R})$$

such that

$$AE = EA = A \quad \text{for all } A \in M_{2\times2}(\mathbb{R}).$$

Therefore, $(M_{2\times2}(\mathbb{R}), \cdot)$ is a monoid.

After defining the identity element, we introduce the notion of an idempotent element.

Definition 3 (Idempotent Element [8, 9])**.** *Let S be a semigroup. An element $x \in S$ is called an* ***idempotent element*** *if*

$$x \cdot x = x^2 = x.$$

If a semigroup S has an identity element, then the identity element is necessarily idempotent.

To clarify the above definition, the following example concerning idempotent elements is given.

Example 2. *Consider the semigroup $\mathbb{Z}_{10} = \{0, 1, 2, \ldots, 9\}$ under multiplication modulo 10. There exist idempotent elements in $\mathbb{Z}_{10}$, namely $\bar{5}$ and $\bar{6}$, since*

$$\bar{5}^2 \equiv \bar{5} \pmod{10} \quad \text{and} \quad \bar{6}^2 \equiv \bar{6} \pmod{10}.$$

Furthermore, in the study of semigroups, there are several elements that possess special properties. The definitions of such special elements are adopted from [8] and will be presented in the following definitions.

Definition 4 (Special Elements in a Semigroup [8])**.** *Let S be a semigroup and let $a \in S$. The following notions are defined.*

1. *An element $b \in S$ is called a* ***left divisor*** *of a if there exists $x \in S$ such that $bx = a$. Similarly, b is called a* ***right divisor*** *of a if there exists $y \in S$ such that $yb = a$.*
2. *An element $b \in S$ is called a* ***left unit*** *of a if $ba = a$, and a* ***right unit*** *of a if $ab = a$. If b is both a left and a right unit of a, then b is called a* ***unit*** *of a.*
3. *An element $a \in S$ is called* ***regular*** *if there exists $x \in S$ such that*

   $$axa = a.$$

4. *An element $a \in S$ is called* ***completely regular*** *if there exists $x \in S$ such that*

   $$axa = a \quad \text{and} \quad ax = xa.$$

5. *An element $e \in S$ is called a* ***regular left unit*** *of a if e is a left unit of a and a is a left divisor of e. Similarly, e is called a* ***regular right unit*** *of a if e is a right unit of a and a is a right divisor of e. An element $e \in S$ is called a* ***regular unit*** *of a if it is a unit of a and a is both a left and a right divisor of e.*

After reviewing the fundamental theory of semigroups and the special elements contained in them, including their definitions and illustrative examples, these concepts are then employed to study the notion of Smarandache semigroups.

3.1 Smarandache Semigroups

This subsection is devoted to the discussion of Smarandache semigroups. Before introducing the definition of a Smarandache semigroup, it is necessary to first recall the definition of a subset, proper subset, and subsemigroup along with a relevant example, which will serve as a foundational concept for the subsequent discussion.

Definition 5 (Proper Subset [10])**.** *A set A is called a* proper subset *of a set B if every element of A is an element of B and at least one element of B is not an element of A. A proper subset is denoted by the symbol $\subset$.*

After introducing the notion of a proper subset, we now discuss the concept of a non-proper subset. In general, a non-proper subset is simply referred to as a subset. The definition is given as follows.

Definition 6 (Subset [11])**.** *A set S is said to be a* subset *of a set T if every element of S is an element of T, denoted by $S \subseteq T$. If $S \subseteq T$ and $T \subseteq S$, then $S = T$.*

From Definitions 5 and 6, it can be concluded that every proper subset is a subset, but the converse does not hold. The examples are given as follows.

Example 3. *Let A and B be non-empty sets defined by*

$$A = \{4, 3, 2, 1\}$$

and

$$B = \{x \mid x \text{ is a solution of } x^4 - 10x^3 + 35x^2 - 50x + 12 = 0\}.$$

Since $(x - 1)(x - 2)(x - 3)(x - 4) = 0$, we obtain $B = \{4, 3, 2, 1\}$. Thus, $A \subseteq B$ and $B \subseteq A$, which implies $A = B$.

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Example 4. *Let*

$$R = \{11, 12, 13, 14, 18\}, \quad P = \{11, 13\}.$$

Since every element of P is contained in R, it follows that $P \subseteq R$.

Example 5. *Let*

$$S = \{9, 8, 7, 6, 5\}, \quad T = \{9, 8, 7, 6, 5, 4\},$$

$$U = \{9, 8, 7, 6, 5, 4\}.$$

Then $S \subset T$ and $T \subseteq U$. However, T is not a proper subset of U.

Definition 7. *[8] Let $(S, *)$ be a semigroup. A nonempty subset $P \subseteq S$ is called a **subsemigroup** of S if $(P, *)$ is itself a semigroup.*

To clarify the above definition, the following simple example of a subsemigroup is provided.

Example 6. *Let $\mathbb{Z}_{12} = \{0, 1, 2, \dots, 11\}$ be a semigroup under multiplication modulo 12. The subset $P = \{0, 2, 4, 8\}$ is a subsemigroup of $\mathbb{Z}_{12}$.*

The theory of Smarandache algebraic structures was introduced by the Romanian mathematician; Florentin Smarandache and Raul Padilla in 1998, one of which was developed within the framework of semigroup theory [2, 5]. In his work, Raul Padilla defined and studied Smarandache semigroups, which are denoted by S-semigroups.

Definition 8. *[5] A **Smarandache semigroup** (denoted by an **S-semigroup**) is a semigroup S such that there exists a* proper subset *of S which forms a group under the same binary operation defined on S.*

Example 7. *Let $(\mathbb{Z}_{12}, \cdot_{12})$ be a semigroup under multiplication modulo 12. There exists a proper subset $B = \{\bar{3}, \bar{9}\} \subset \mathbb{Z}_{12}$ that satisfies the group axioms under the operation $\cdot_{12}$. The element $\bar{9} \in B$ acts as the identity element in $(B, \cdot_{12})$, since*

$$\bar{3} \cdot_{12} \bar{9} = \bar{9} \cdot_{12} \bar{3} = \bar{9}, \quad \bar{9} \cdot_{12} \bar{9} = \bar{9}.$$

Moreover, every element in B is its own inverse under $\cdot_{12}$. Hence, $(B, \cdot_{12})$ is a group. Therefore, $(\mathbb{Z}_{12}, \cdot_{12})$ is an S-semigroup.

Example 8. *Let $(\mathbb{Z}_4, \cdot_4)$ and $(\mathbb{Z}_5, \cdot_5)$ be semigroups under multiplication modulo 4 and 5, respectively. There exist proper subsets $A = \{\bar{1}, \bar{3}\} \subset \mathbb{Z}_4$ and $B = \{\bar{1}, \bar{4}\} \subset \mathbb{Z}_5$ such that $(A, \cdot_4)$ and $(B, \cdot_5)$ are groups.Thus, both $(\mathbb{Z}_4, \cdot_4)$ and $(\mathbb{Z}_5, \cdot_5)$ are Smarandache semigroups.*

Proposition 3.1. *Let $(\mathbb{Z}_n, \cdot_n)$ be a semigroup, where $n \in \mathbb{N}$. In $\mathbb{Z}_n$, there exists a proper subset $A = \{\bar{1}, \overline{n-1}\} \subset \mathbb{Z}_n$ such that $(A, \cdot_n)$ forms a group. Therefore, $(\mathbb{Z}_n, \cdot_n)$ is an **S-semigroup**.*

Proof. The operation $\cdot_n$ defined by

$$\bar{a} \cdot_n \bar{b} = \overline{ab}, \quad \forall\, \bar{a}, \bar{b} \in \mathbb{Z}_n,$$

is well-defined, closed, and associative. Hence, $(\mathbb{Z}_n, \cdot_n)$ is a semigroup. Consider the set $\mathbb{Z}_n = \{\bar{1}, \bar{2}, \bar{3}, \dots, \overline{n-1}\}$, there always exists a proper subset subset $A = \{\bar{1}, \overline{n-1}\}$. The element $\bar{1}$ acts as the identity in A, and each element is self-inverse since

$$\bar{1}^2 = \bar{1}, \qquad (\overline{n-1})^2 = \bar{1} \pmod{n}.$$

Thus, $(A, \cdot_n)$ is a group. Therefore, $(\mathbb{Z}_n, \cdot_n)$ is a Smarandache semigroup. □

Example 9. *Let $(\mathbb{Z}, \max)$ be a semigroup. The proper subset $A = \{1\} \in \mathbb{Z}$ forms a trivial group with maximum operation. Hence, $(\mathbb{Z}, \max)$ is a Smarandache semigroup. This result extends to $(\mathbb{Q}_{\geq 0}, \max)$, which is also a Smarandache semigroup.*

Example 10. *Let $M_n(\mathbb{R})$ denote the semigroup of all $n \times n$ real matrices under matrix multiplication. The set of invertible matrices is a proper subset that forms a group. Hence, $M_n(\mathbb{R})$ is a Smarandache semigroup.*

Every Smarandache semigroup is a semigroup, but the converse does not hold. For example, $(\mathbb{Z}_{\geq 0}, +)$ and $(\mathbb{N}, +)$ are semigroups that do not contain any proper subset forming a group. Hence, they are not Smarandache semigroups.

3.2 The Characterization of $S-$semigroups

After discussing several properties of Smarandache semigroups, it is natural to ask under what conditions a semigroup can be classified as a Smarandache semigroup. In other words, we seek conditions that guarantee the existence of a proper subset forming a group under the same binary operation. The following theorem provides a necessary condition for a semigroup to be a Smarandache semigroup.

Theorem 3.1. *Let S be a semigroup. If S is a Smarandache semigroup, then S contains an idempotent element.*

Proof. Assume that S is a Smarandache semigroup. Then there exists a proper subset $G \subset S$ such that $(G, *)$ is a group under the same operation as S. Since G is a group, it contains an identity element $e \in G$ satisfying

$$e * g = g * e = g$$

for all $g \in G$. In particular,

$$e * e = e.$$

Hence, e is an idempotent element of S. Since $e \in G \subset S$, it follows that S contains an idempotent element. □

Theorem 1 shows that the existence of an idempotent element is a necessary condition for a semigroup to be a Smarandache semigroup.

The presence of an idempotent element in a semigroup often leads to the consideration of certain subsets associated with that element. In particular, for an idempotent element $e \in S$, we may consider the set of elements for which e behaves as a regular unit. This construction is described in the following remark.

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Remark 3.1. *Let S be a semigroup and let $e \in S$ be an idempotent element. Consider the set*

$$G_e = \{a \in S \mid e \text{ is a regular unit of } a\}.$$

Under suitable conditions, the set G_e may form a group with identity element e. This observation provides a useful way to investigate the relationship between idempotent elements and subgroup structures in semigroups.

The following lemma shows that the existence of a completely regular element in a semigroup guarantees the existence of an idempotent element.

Lemma 3.1. *Let S be a semigroup. If S contains a completely regular element, then S contains an idempotent element.*

Proof. Let $a \in S$ be a completely regular element. Then there exists $x \in S$ such that

$$axa = a \quad \text{and} \quad ax = xa.$$

Let $e = ax = xa$. Then

$$e^2 = ee = (ax)(ax) = (axa)x = ax = e.$$

Thus, e is an idempotent element of S. □

The presence of idempotent elements in Smarandache semigroups suggests a further structural property related to regularity. In the following theorem, we show that every Smarandache semigroup contains a completely regular element.

Theorem 3.2. *Let S be a semigroup. If S is a Smarandache semigroup, then S contains a completely regular element.*

Proof. Since S is a Smarandache semigroup, there exists a proper subset $G \subset S$ such that $(G, *)$ is a group under the same operation as S. Let $e \in G$ be the identity element of G. Then for every $x \in G$,

$$ex = xe = x.$$

In particular,

$$e = exe.$$

Thus, e is a completely regular element of S. Hence S contains a completely regular element. □

The following theorem describes the internal structure of a Smarandache semigroup. In particular, it shows that the set of all completely regular elements can be decomposed into a union of pairwise disjoint groups contained in the semigroup.

Theorem 3.3. *Let S be a Smarandache semigroup. Let C denote the set of all completely regular elements of S, and let H denote the set of all idempotent elements of S. Then*

$$C = \bigcup_{e \in H} G_e,$$

where each G_e is a group, and the family $\{G_e \mid e \in H\}$ consists of pairwise disjoint groups.

Proof. Since S is a Smarandache semigroup, there exists a subgroup $G \subset S$. Let e be the identity element of G. Then e is an idempotent element of S. Hence $H \neq \varnothing$.
Moreover, the identity element e satisfies $e = exe$, so e is a completely regular element. Thus $C \neq \varnothing$. Take an arbitrary element $c \in C$. Since c is completely regular, there exists $x \in S$ such that

$$cxc = c \quad \text{and} \quad cx = xc.$$

Let $e = cx = xc$. Then

$$ec = (cx)c = c, \quad ce = c(xc) = c,$$

and

$$e^2 = ee = (cx)(cx) = (cxc)x = cx = e.$$

Thus, e is an idempotent element and also the unit of c. Hence $c \in G_e$ for some $e \in H$, which implies

$$C \subseteq \bigcup_{e \in H} G_e. \tag{1}$$

Conversely, let $t \in \bigcup_{e \in H} G_e$. Then $t \in G_e$ for some $e \in H$. Since G_e forms a group with identity element e, we have

$$te = et = t,$$

which shows that t is a completely regular element. Therefore,

$$\bigcup_{e \in H} G_e \subseteq C. \tag{2}$$

From (1) and (2), we conclude that

$$C = \bigcup_{e \in H} G_e.$$

Finally, we show that the groups G_e are pairwise disjoint. Let $e, f \in H$ with $e \neq f$. Suppose that

$$G_e \cap G_f \neq \varnothing.$$

Then there exists an element $y \in G_e \cap G_f$. Since $y \in G_e$, the element e acts as the identity of y, and thus

$$ey = ye = y.$$

Similarly, since $y \in G_f$, the element f also acts as the identity of y, so

$$fy = yf = y.$$

Hence both e and f serve as identity elements for y. However, the identity element in a group is unique. Therefore $e = f$, which contradicts the assumption that $e \neq f$. Therefore,

$$G_e \cap G_f = \varnothing \quad \text{for all } e \neq f.$$

Thus, C is a union of pairwise disjoint groups contained in S. □

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Example 11. *Let $Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ be the semigroup under multiplication modulo 6. The set of idempotent elements is*

$$H = \{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}.$$

Since the subset $\{\bar{1}, \bar{5}\}$ forms a group under multiplication modulo 6, it follows that Z_6 is a Smarandache semigroup. For each $e \in H$, define

$$G_e = \{x \in Z_6 \mid e \text{ is the regular unit of } x\}.$$

Then we obtain

$$G_{\bar{0}} = \{\bar{0}\}, \quad G_{\bar{1}} = \{\bar{1}, \bar{5}\}, \quad G_{\bar{3}} = \{\bar{3}\}, \quad G_{\bar{4}} = \{\bar{2}, \bar{4}\}.$$

By Theorem 3, each G_e is a group under multiplication modulo 6. Moreover,

$$G_{\bar{0}} \cup G_{\bar{1}} \cup G_{\bar{3}} \cup G_{\bar{4}} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} = Z_6,$$

and the groups are pairwise disjoint.

3.3 The Cartesian Product of S-semigroups

Before discussing the Cartesian product of Smarandache semigroups, it is necessary to recall the notion of an ordered pair. An ordered pair of elements a and b, denoted by (a, b), is different from (b, a) [12]. The order of the first and second positions is essential, such that

$$(a_1, b_1) = (a_2, b_2)$$

if and only if $a_1 = a_2$ and $b_1 = b_2$.

Definition 9 (Cartesian Product [13, 14])**.** *Let A and B be sets. The Cartesian product of A and B is the set denoted by $A \times B$, whose elements are all ordered pairs (a, b), where $a \in A$ and $b \in B$. That is,*

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

The following examples illustrate the concept of the Cartesian product.

Example 12. *Let $A = \{1, 3\}$ and $B = \{2, 4, 6, 8\}$. Then, we have the Cartesian product of A and B is $A \times B$, i.e:*

$$\{(1, 2), (1, 4), (1, 6), (1, 8), (3, 2), (3, 4), (3, 6), (3, 8)\}.$$

Example 13. *Let $J = \{x \in \mathbb{N} \mid 1 \le x \le 2\}$ and $K = \{5, 6\}$. Then $J = \{1, 2\}$. The Cartesian products are*

$$J \times K = \{(1, 5), (1, 6), (2, 5), (2, 6)\},$$

$$K \times J = \{(5, 1), (5, 2), (6, 1), (6, 2)\}.$$

Hence, $J \times K \ne K \times J$.

In classical group theory, it is well known that the direct product of groups is again a group. Since an S-semigroup is defined by the existence of a proper subset that forms a group under the same binary operation, it is natural to ask whether this property is preserved under Cartesian products. The following results show that the Smarandache property is stable under several Cartesian product constructions of semigroups. Let $(S, *)$ be an S-semigroup. Define

$$S^2 = S \times S = \{(a, b) \mid a, b \in S\}.$$

Define a binary operation Δ on S^2 by

$$(a, b)\Delta(c, d) = (a * c, b * d), \quad \text{for all } (a, b), (c, d) \in S^2.$$

Lemma 3.2. *(S^2, Δ) is an S-semigroup.*

Proof. We first verify that $(S \times S, \Delta)$ is a semigroup. Closure and well-definedness follow directly from those of $(S, *)$. Associativity holds since

$$((x_1, x_2)\Delta(y_1, y_2))\Delta(z_1, z_2) = (x_1 * y_1 * z_1, x_2 * y_2 * z_2)$$
$$= (x_1, x_2)\Delta((y_1, y_2)\Delta(z_1, z_2)),$$

by the associativity of $*$ in S. So, we have that $(S \times S, \Delta)$ is a semigroup.
Since $(S, *)$ is an S-semigroup, there exists a proper subset $T \subset S$ such that $(T, *)$ is a group. Hence, $T^2 = T \times T$ is a proper subset of S^2 and forms a group under Δ. For every $(t_1, t_2), (t_3, t_4) \in T \times T$, we have

$$(t_1, t_2)\Delta(t_3, t_4) = (t_1 * t_3, t_2 * t_4) \in T \times T,$$

so Δ is closed and well defined. Associativity is inherited from $(S \times S, \Delta)$. Then, the identity element of $(T, *)$, say $e \in T$, gives an identity $(e, e) \in T \times T$. Each $(t_1, t_2) \in T \times T$ has an inverse $(t_1^{-1}, t_2^{-1}) \in T \times T$. Hence $(T \times T, \Delta)$ is a group.
Therefore, $S \times S$ contains a proper subset $T \times T$ that forms a group under Δ, and thus (S^2, Δ) is an S-semigroup. □

This result follows naturally from the classical property that the direct product of groups forms a group. In this context, it shows that the Smarandache property is preserved when taking the Cartesian square of a semigroup.

Then, by extending the Cartesian product of the Smarandache semigroup $(S, *)$ to n-times, where $n \in \mathbb{N}$, that is,

$$S^n = \underbrace{S \times S \times \cdots \times S}_{n\text{-tuple}},$$

it follows that S^n is also a Smarandache semigroup.

Proposition 3.2. *Let $(S, *)$ be an S-semigroup and $n \in \mathbb{N}$. Define*

$$S^n = \underbrace{S \times S \times \cdots \times S}_{n\text{-times}},$$

and define the operation Δ on S^n by

$$(a_1, \ldots, a_n)\,\Delta\,(b_1, \ldots, b_n) = (a_1 * b_1, \ldots, a_n * b_n).$$

Then (S^n, Δ) is an S-semigroup.

Proof. Since $(S, *)$ is a semigroup, the operation Δ is closed, well-defined, and associative, hence (S^n, Δ) is a semigroup.
Because $(S, *)$ is an S-semigroup, there exists a proper subset $T \subset S$ such that $(T, *)$ is a group. Then

$$T^n = \underbrace{T \times T \times \cdots \times T}_{n\text{-times}}$$

is a proper subset of S^n and forms a group under Δ. Therefore, (S^n, Δ) is an S-semigroup. $\square$

Example 14. *Let $(\mathbb{Z}_4, \cdot_4)$ be an S-semigroup. As shown in Example 8, the proper subset*

$$A = \{\bar{1}, \bar{3}\} \subset \mathbb{Z}_4$$

is a group under $\cdot_4$. By Proposition 2, the Cartesian product

$$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$$

is an S-semigroup since it contains the proper subset $A \times A \times A$ that forms a group.

Proposition 3.3. *Let $(S, *_S)$ and $(T, *_T)$ be two distinct S-semigroups. Define a binary operation $*'$ on $S \times T$ by*

$$(s_1, t_1) *' (s_2, t_2) = (s_1 *_S s_2, t_1 *_T t_2),$$

where $(s_1, t_1), (s_2, t_2) \in S \times T$. Then $(S \times T, *')$ is an S-semigroup.

Proof. Since $(S, *_S)$ and $(T, *_T)$ are S-semigroups, closure, well-definedness, and associativity of operation $*'$ follow directly from the associativity of $*_S$ and $*_T$. Hence $(S \times T, *')$ is a semigroup.
Since S and T are S-semigroups, there exist proper subsets $S_1 \subset S$ and $T_1 \subset T$ such that $(S_1, *_S)$ and $(T_1, *_T)$ are groups. Then $S_1 \times T_1$ is a proper subset of $S \times T$ and forms a group under $*'$. Thus, $(S \times T, *')$ is an S-semigroup. $\square$

Example 15. *Let $(\mathbb{Z}_2, \cdot_2)$ and $(\mathbb{Z}_4, \cdot_4)$ be S-semigroups. By Proposition 1, each S-semigroup $(\mathbb{Z}_n, \cdot_n)$ contains a proper subset $\{\bar{1}, \overline{n-1}\} \subset \mathbb{Z}_n$ that forms a group. Hence, define $A = \{\bar{1}\} \subset \mathbb{Z}_2$ and $A' = \{\bar{1}, \bar{3}\} \subset \mathbb{Z}_4$ such that $(A, \cdot_2)$ and $(A', \cdot_4)$ are groups.*
Then, by Proposition 3, the Cartesian product $\mathbb{Z}_2 \times \mathbb{Z}_4$ with operation $*'$ defined by:

$$(x_1, y_1) *' (x_2, y_2) = (x_1 \cdot_2 x_2, y_1 \cdot_4 y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}_2 \times \mathbb{Z}_4$, is an S-semigroup. This is because the subset $A \times A' = \{(\bar{1}, \bar{1}), (\bar{1}, \bar{3})\} \subset \mathbb{Z}_2 \times \mathbb{Z}_4$ is a group under $*'$.

The previous propositions show that the Smarandache property is preserved under several Cartesian product constructions. In particular, the results extend from the Cartesian square $S \times S$ to higher products S^n, and further to mixed products such as $S^n \times T^n$. Furthermore, the S-semigroups $(S \times S, \Delta)$ and (S^n, Δ) can be extended to the set $S^n \times T^n$ as follows:

$$S^n \times T^n = \left(\underbrace{S \times S \times \cdots \times S}_{n\text{-tuple}}\right) \times \left(\underbrace{T \times T \times \cdots \times T}_{n\text{-tuple}}\right).$$

The detailed explanation of this construction is given in the following proposition.

Proposition 3.4. *Let $(S, *)$ and $(T, *)$ be S-semigroups. From the Smarandache semigroups S and T, consider the set*

$$S^n \times T^n = \left(\underbrace{S \times S \times \cdots \times S}_{n\text{-tuple}}\right) \times \left(\underbrace{T \times T \times \cdots \times T}_{n\text{-tuple}}\right)$$

$$= \{(a_i, b_i) \mid a_i \in S^n,\ b_i \in T^n\},$$

where

$$a_i = (a_1, a_2, \ldots, a_n), \quad b_i = (b_1, b_2, \ldots, b_n), \quad i \in \mathbb{N}.$$

Define a binary operation $\star$ on $S^n \times T^n$ by

$$(a_i, b_i) \star (c_j, d_j) = \big((a_i \Delta c_j), (b_i \Delta d_j)\big)$$
$$= \big((a_1 * c_1, \ldots, a_n * c_n), (b_1 * d_1, \ldots, b_n * d_n)\big),$$

for all $(a_i, b_i), (c_j, d_j) \in S^n \times T^n$ and $i, j = 1, \ldots, n \in \mathbb{N}$. Then $(S^n \times T^n, \star)$ is a Smarandache semigroup.

Proof. Before proving that $(S^n \times T^n, *)$ forms a Smarandache semigroup, note that

$$S^2 = S \times S, \quad S^3 = S \times S \times S, \quad S^n = \underbrace{S \times S \times \cdots \times S}_{n\text{-tuple}}.$$

Thus,

$$S^n = \underbrace{S \times S \times \cdots \times S}_{n\text{-tuple}} = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in S\}.$$

Similarly,

$$T^n = \underbrace{T \times T \times \cdots \times T}_{n\text{-tuple}} = \{(b_1, \ldots, b_n) \mid b_1, \ldots, b_n \in T\}.$$

Hence,

$$S^n \times T^n = \{(a_i, b_i) \mid a_i \in S^n,\ b_i \in T^n\},$$

where

$$a_i = (a_1, a_2, \ldots, a_n), \quad b_i = (b_1, b_2, \ldots, b_n)$$

From Proposition 2, it has been shown that $S^n \neq \varnothing$ and (S^n, Δ) is a S-semigroup. Analogously, $T^n \neq \varnothing$ and (T^n, Δ) is also a S-semigroup. Therefore, there exists at least one element

$$\big((e, e, \ldots, e), (e, e, \ldots, e)\big) \in S^n \times T^n,$$

which implies that $S^n \times T^n \neq \varnothing$.
Next, it will be shown that $(S^n \times T^n, \star)$ is a Smarandache semigroup. Let $(a_i, b_i), (c_j, d_j) \in S^n \times T^n$. Then

$$(a_i, b_i) \star (c_j, d_j) = \big((a_i \Delta c_j), (b_i \Delta d_j)\big) \in S^n \times T^n.$$

Thus, the operation $\star$ is closed on $S^n \times T^n$.
Then, let

$$X = ((a_1, \ldots, a_n), (b_1, \ldots, b_n)),$$

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$$Y = ((c_1, \dots, c_n), (d_1, \dots, d_n)),$$

and

$$Z = ((e_1, \dots, e_n), (f_1, \dots, f_n))$$

be arbitrary elements of $S^n \times T^n$. Then we have that

$$(X \star Y) \star Z = ((a_i \Delta c_i)\Delta e_i, (b_i \Delta d_i)\Delta f_i)$$

$$= (a_i \Delta (c_i \Delta e_i), b_i \Delta (d_i \Delta f_i)) = X \star (Y \star Z),$$

since Δ is associative in both S and T. Hence, $\star$ is associative. Therefore, $(S^n \times T^n, \star)$ is a semigroup.
Since $(S, *)$ and $(T, *)$ are Smarandache semigroups, there exists proper subsets $G \subset S$ and $H \subset T$. Define

$$G^n = \underbrace{G \times G \times \cdots \times G}_{n\text{-tuple}}, \quad H^n = \underbrace{H \times H \times \cdots \times H}_{n\text{-tuple}}.$$

Then $G^n \subset S^n$ and $H^n \subset T^n$ are proper subsets that form groups under the operation Δ on S^n and T^n, respectively. Consequently, $G^n \times H^n$ is a group under $\star$ and is a proper subset of $S^n \times T^n$.
Therefore, $(S^n \times T^n, \star)$ contains a proper subset forming a group under the same operation, and hence $(S^n \times T^n, \star)$ is a Smarandache semigroup. □

4. CONCLUSIONS

In this paper, we have shown that Smarandache semigroups (or S-semigroups) are special types of semigroups containing at least one proper subset which forms a group under the same binary operation. Every S-semigroup is a semigroup, but not every semigroup is an S-semigroup. One structural property of S-semigroups is the existence of idempotent element, completely regular elements, and the set C of all completely regular elements of S can be expressed as the union of non-intersecting groups. Moreover, the Cartesian product of two or more S-semigroups is again an S-semigroup. This property also extends to higher Cartesian powers and mixed products of different S-semigroups.

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