A note on computing lower and upper bounds of subjective probability from masses of belief

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This short note shows on a very simple example the consistency of free DSm model encountered in Dezert-Smarandache Theory (DSmT) [2] with a refined model for computing lower and upper probability bounds from basic belief assignments (bba). The belief functions have been introduced in 1976 by Shafer in Dempster-Shafer Theory (DST), see [1] and [2] for definitions and examples.

Let's consider the simplest free model for the frame $\Theta = \{A, B\}$ with $A \cap B \neq \emptyset$, and let's consider the following bba defined on the hyper-power set of Θ (its Dedekind's lattice)

$$m(A \cap B) = 0.4, \qquad m(A) = 0.6, \qquad m(B) = 0, \qquad m(A \cup B) = 0$$

From the definition of belief and plausibility functions, one has for this example

$$\begin{aligned} &Bel(A) = m(A \cap B) + m(A) = 0.4 + 0.6 = 1 \\ &Pl(A) = m(A \cap B) + m(A) + m(B) + m(A \cup B) = 0.4 + 0.6 + 0 + 0 = 1 \end{aligned}$$

and

$$\begin{aligned} &Bel(B) = m(A \cap B) + m(B) = 0.4 + 0 = 0.4 \\ &Pl(B) = m(A \cap B) + m(B) + m(A) + m(A \cup B) = 0.4 + 0 + 0.6 + 0 = 1 \end{aligned}$$

Note that $m(B)$ (resp. $m(A)$) is included in the computation of $Pl(A)$ (resp. $Pl(B)$) because $A \cap B$ is assumed not empty in the free model. Therefore, one sees that the unknown probabilities $P(A)$ and $P(B)$ are bounded as follows:

$$P(A) \in [Bel(A), Pl(A)] = [1, 1] \tag{1}$$

$$P(B) \in [Bel(B), Pl(B)] = [0.4, 1] \tag{2}$$

In fact from the bba $m(.)$ above, one gets finally $P(A) + P(B) \in [1.4, 2]$ (greater than 1) which is perfectly normal because the events A and B are not exclusive elements and so there is no reason to get $P(A) + P(B) = 1$ in such case.

Let's compute the bounds of $P(A)$ and $P(B)$ from another point of view based on the refinement of the frame Θ into the refined frame Θ' defined as follows:

$$\Theta' \triangleq \{\theta_1 = \bar{B}, \theta_2 = A \cap B, \theta_3 = \bar{A}\}$$

where $\bar{B}$ is the complement of B, $\bar{A}$ is the complement of A and where elements θ_1, θ_2 and θ_3 are now three truly exclusive finer elements. Based on such refinement, one has $A = \theta_1 \cup \theta_2$ and $B = \theta_2 \cup \theta_3$. The previous bba $m(.)$ is strictly equivalent to the bba $m'(.)$ defined on the power-set of Θ' with

$$m'(\theta_2 = A \cap B) = 0.4, \qquad m'(\theta_1 \cup \theta_2 = A) = 0.6$$

and having all other masses $m'(X) = 0$ for $X \neq \theta_2$, and for $X \neq \theta_1 \cup \theta_2$.

From this bba $m'(.)$ one can also compute the beliefs and plausibilities of elements of Θ' as follows:

$$\begin{aligned} &Bel(\theta_1) = m'(\theta_1) = 0 \\ &Pl(\theta_1) = m'(\theta_1) + m'(\theta_1 \cup \theta_2) + m'(\theta_1 \cup \theta_3) + m'(\theta_1 \cup \theta_2 \cup \theta_3) = 0 + 0.6 + 0 + 0 = 0.6 \end{aligned}$$

$$\begin{aligned} &Bel(\theta_2) = m'(\theta_2) = 0.4 \\ &Pl(\theta_2) = m'(\theta_2) + m'(\theta_1 \cup \theta_2) + m'(\theta_2 \cup \theta_3) + m'(\theta_1 \cup \theta_2 \cup \theta_3) = 0.4 + 0.6 + 0 + 0 = 1 \end{aligned}$$

$$Bel(\theta_3) = m'(\theta_3) = 0$$
$$Pl(\theta_3) = m'(\theta_3) + m'(\theta_1 \cup \theta_3) + m'(\theta_2 \cup \theta_3) + m'(\theta_1 \cup \theta_2 \cup \theta_3) = 0 + 0 + 0 + 0 = 0$$

Therefore, one sees that the probabilities of θ_1, θ_2 and θ_3 are bounded as follows:

$$P(\theta_1) \in [Bel(\theta_1), Pl(\theta_1)] = [0, 0.6] \tag{3}$$
$$P(\theta_2) \in [Bel(\theta_2), Pl(\theta_2)] = [0.4, 1] \tag{4}$$
$$P(\theta_3) \in [Bel(\theta_3), Pl(\theta_3)] = [0, 0] \tag{5}$$

with the condition $P(\theta_1) + P(\theta_2) + P(\theta_3) = 1$.

Because $P(\theta_3) \in [0, 0]$, then one has $P(\theta_3) = 0$ and therefore one has $P(\theta_1) + P(\theta_2) = 1$. Since $\theta_1 \cap \theta_2 = \emptyset$ (by construction of the refined frame Θ'), one has also $P(\theta_1 \cap \theta_2 = \emptyset) = 0$ and therefore the probability of $A = \theta_1 \cup \theta_2$ can be computed from probability axioms as follows

$$\begin{aligned} P(A) &= P(\theta_1 \cup \theta_2) \\ &= P(\theta_1) + P(\theta_2) - P(\theta_1 \cap \theta_2) \\ &= P(\theta_1) + P(\theta_2) \\ &= 1 = [1, 1] \end{aligned}$$

One sees that this computation of bounds of $P(A)$ is fully consistent with (1).

Now lets' examine the computation of bounds of $P(B)$ where $B = \theta_2 \cup \theta_3$. From the probability axioms, one can write

$$\begin{aligned} P(B) &= P(\theta_2 \cup \theta_3) \\ &= P(\theta_2) + P(\theta_3) - P(\theta_2 \cap \theta_3) \\ &= P(\theta_2) + P(\theta_3) \\ &= P(\theta_2) \end{aligned}$$

This result comes from the fact that $P(\theta_2 \cap \theta_3) = P(\emptyset) = 0$, and also one has $P(\theta_3) = 0$ here due to (5). So finally, according to (4) we get

$$P(B) = P(\theta_2) \in [Bel(\theta_2), Pl(\theta_2)] = [0.4, 1] \tag{6}$$

One sees that the bounds for $P(B)$ computed directly using the free model by (2) are fully consistent with the bounds of $P(B)$ given in (6) computed from the refined frame Θ'. The two approaches provide the same lower and upper bounds of unknown subjective probabilities which is normal.

From this simplest example, one has shown that the computations of the bounds of $P(A)$ and $P(B)$ are mathematically consistent using both approaches. We have to point out that it is very easy to obtain directly when working with the free model of the frame and a bit more complex when working with the refined model of the frame.

REFERENCES

[1] Shafer, G. A Mathematical Theory of Evidence. Princeton University Press, Princeton, 1976.

[2] Smarandache F., Dezert J. Advances and applications of DSmT for information fusion, Volumes 1, 2 & 3, ARP, 2004–2009 (http://www.gallup.unm.edu/~smarandache/DSmT.htm).