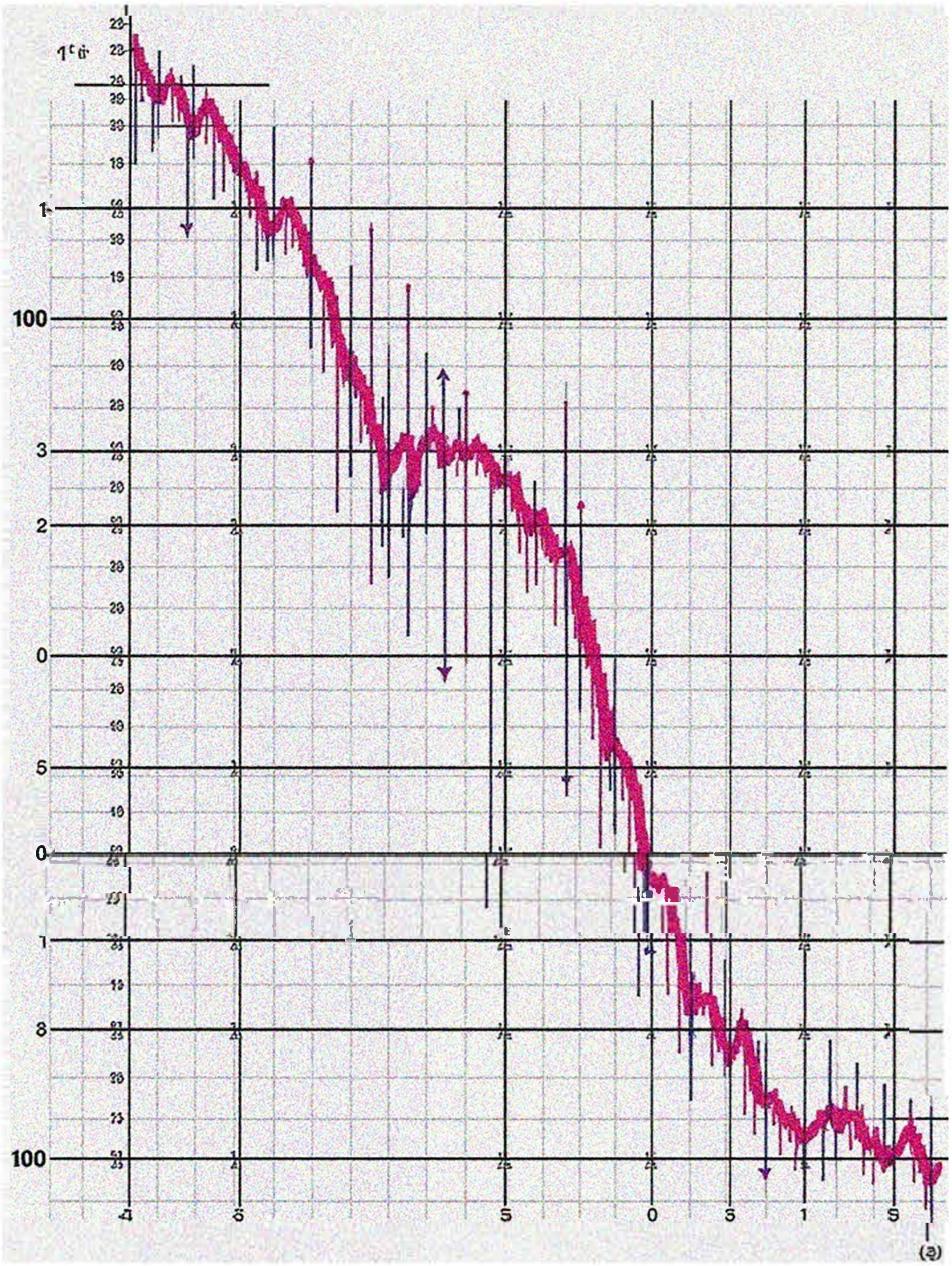
Takaaki Fujita

Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization:

Fuzzy, Neutrosophic, Soft, Rough, and Beyond



Biblio Publishing
Publishing for the 21st Century

Takaaki Fujita
Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan
Emails: t171d603@gunma-u.ac.jp

Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond

Foreword

This book explores the advancement of uncertain combinatorics through innovative methods such as graphization, hyperization, and uncertainization, incorporating concepts from fuzzy, neutrosophic, soft, and rough set theory, among others. Combinatorics and set theory are fundamental mathematical disciplines that focus on counting, arrangement, and the study of collections under specified rules. While combinatorics excels at solving problems involving uncertainty, set theory has expanded to include advanced concepts like fuzzy and neutrosophic sets, which are capable of modeling complex real-world uncertainties by accounting for truth, indeterminacy, and falsehood. These developments intersect with graph theory, leading to novel forms of uncertain sets in "graphized" structures, such as hypergraphs and superhypergraphs. Innovations like Neutrosophic Oversets, Undersets, and Offsets, as well as the Nonstandard Real Set, build upon traditional graph concepts, pushing the boundaries of theoretical and practical advancements. This synthesis of combinatorics, set theory, and graph theory provides a strong foundation for addressing the complexities and uncertainties present in mathematical and real-world systems, paving the way for future research and application.

Florentin Smarandache

Peer-reviewers:

Florentin Smarandache
University of New Mexico, Mathematics, Physics, and Natural Science Division, Gallup Campus, NM 87301, USA.
https://fs.unm.edu/FlorentinSmarandache.htm
Email: smarand@unm.edu

Maikel Leyva Vazquez
Faculty of Mathematical and Physical Sciences, University of Guayaquil, Guayaquil, Ecuador. Email: maikel.leyvav@ug.edu.ec

Xiaohong Zhang
Department of Mathematics, Shaanxi University of Science & Technology, Xian 710021, China.

Takaaki Fujita

Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond

Biblio Publishing
Publishing for the 21st Century

Grandview Heights
United States of America
2024

Biblio Publishing
1091 West 1st Ave
Grandview Heights, OH 43212
United States of America
Phone: +614.485.0721
Email: Info@BiblioPublishing.com
https://BiblioPublishing.com/

ISBN 978-1-59973-812-3

Abstract

Combinatorics is a branch of mathematics focused on counting, arranging, and combining elements within a set under specific rules and constraints. This field is particularly fascinating due to its ability to yield novel results through the integration of concepts from various mathematical domains. Its significance remains unchanged in areas that address uncertainty in the real world.

Set theory, another foundational area of mathematics, explores "sets," which are collections of objects that can be finite or infinite. Recent years have seen growing interest in "non-standard set theory" and "non-standard analysis." To better handle real-world uncertainty, concepts such as fuzzy sets, neutrosophic sets, rough sets, and soft sets have been introduced. For example, neutrosophic sets, which simultaneously represent truth, indeterminacy, and falsehood, have proven to be valuable tools for modeling uncertainty in complex systems. These set concepts are increasingly studied in graphized forms, and generalized graph concepts now encompass well-known structures such as hypergraphs and superhypergraphs. Furthermore, hyperconcepts and superhyperconcepts are being actively researched in areas beyond graph theory.

Combinatorics, uncertain sets (including fuzzy sets, neutrosophic sets, rough sets, soft sets, and plithogenic sets), uncertain graphs, and hyper and superhyper concepts are active areas of research with significant mathematical and practical implications. Recognizing their importance, this paper explores new graph and set concepts, as well as hyper and superhyper concepts, as detailed in the "Results" section of "The Structure of the Paper." Additionally, this work aims to consolidate recent findings, providing a survey-like resource to inform and engage readers.

For instance, we extend several graph concepts by introducing Neutrosophic Oversets, Neutrosophic Undersets, Neutrosophic Offsets, and the Nonstandard Real Set [885]. This paper defines a variety of concepts with the goal of inspiring new ideas and serving as a valuable resource for researchers in their academic pursuits.

Keywords: Neutrosophic Set, plithogenic set, fuzzy set, Neutrosophic Graph

1 Short Introduction

1.1 Uncertain Combinatorics

Combinatorics is a branch of mathematics that focuses on the counting, arrangement, and combination of elements within a set, following specific rules and constraints. This field is essential in diverse areas, including computer science, statistics, and probability, where it aids in solving problems involving discrete structures [75, 142, 148, 198, 212, 333, 650, 652, 965].

In combinatorial research, various approaches and perspectives may be found depending on the author or study. Typical areas of overlap include set theory [267,953,955], number theory [91,222,662,769], graph theory [279], topology [84, 126, 339, 669], matroid theory [709], partition theory [11, 816], geometry [118, 620, 743, 869], probability theory [214, 300, 324, 542, 557, 581, 746], algebra [105, 120, 304, 488, 575, 665], formal languages [7, 504], and group theory [175, 659, 708]. Research often combines these fields or applies concepts from logic [197, 231, 309, 388, 441, 541, 845], Combinatorial optimization, Computer Complexity [3, 93, 471, 720], and algorithms [107, 133, 225–227, 317, 481, 663, 846] to address complex combinatorial problems.

These combinatorial methods also play a significant role in fields dealing with uncertainty, such as fuzzy set theory and neutrosophic theory, where they help to model and analyze indeterminate structures and relationships (cf. [99, 100, 102–104, 124, 246, 294–296, 305, 459, 583, 613, 621, 641, 737, 855, 1013, 1015–1018, 1018, 1019]).

1.2 Neutrosophic Set and Related Set Theory

Set theory is a branch of mathematics focused on the study of "sets," or collections of objects [267,953,955]. It explores the properties, relationships, and operations of sets. Over time, numerous derived concepts related to sets have emerged, such as ordered sets [297, 473, 829], point sets [673, 675], convex sets [446, 606, 677, 753], alternative sets [964], internal sets [693], open sets [41,43,238,521,839], closed sets [40,42,47,644,818,932], and directed sets [116].

Sets can be either finite or infinite, and recent years have seen increased attention on "Non-standard set theory" and "Non-standard analysis." [234, 506, 565, 633, 777, 778, 792] Non-standard set theory extends classical set theory by introducing infinitesimal and infinitely large elements, allowing for detailed modeling of real-world uncertainties and complex mathematical structures. Within this framework, numerous new concepts have been proposed, including the non-standard real number system [139, 560].

To better address real-world uncertainties, various mathematical concepts related to sets have been proposed, such as Fuzzy Sets [246, 291, 294, 558, 921, 1013, 1035, 1036], vague sets [25, 182, 208, 495, 1028], soft sets [56,59,639,670,1000], rough sets [729,733], soft expert sets [44,46,70,71,745,804], hypersoft sets [4,248,646, 682, 683, 756, 820, 861, 1009], Hypersoft Expert Sets [10, 523–530], and Neutrosophic Sets [33, 168, 306, 695, 855, 856, 894, 969]. Each of these approaches is designed to handle different types of ambiguity. Neutrosophic Sets, for example, simultaneously account for truth, indeterminacy, and falsehood, making them versatile tools for modeling uncertainty in complex systems [855, 856]. Plithogenic Sets, which generalize these uncertain sets, are also an area of active research [6, 425, 773, 848, 864, 865, 889, 895, 917]. Examples of applications for uncertain sets include fields like traffic control and decision-making [32,61,64,109,123,237,425,494,546,558, 612, 613, 640, 807, 864, 877, 1004, 1022]. Uncertain sets, such as those mentioned above, have been extensively studied and discussed in numerous research papers [552, 855, 856, 865].

This paper extends several graph concepts to include Neutrosophic Overset [885], Neutrosophic Underset [885], Neutrosophic Offset [885], MultiNeutrosophic Set, Hyperfuzzy Set [411, 551, 902], and Nonstandard Real Set [885]. We then examine their relationships with other classes of graphs.

For reference, the relationships between Uncertain sets are illustrated in Figure 1. (cf. [378])

1.3 Graphization: Transforming to Classes of Uncertain Graphs

This paper addresses concepts related to uncertain graphs and other advanced graph structures. Graph theory, a fundamental branch of mathematics, models relationships within networks by connecting vertices (or nodes) through edges. Since its inception in the 1700s, graph theory has evolved significantly and continues to be a dynamic area of research [151, 161, 279, 710, 738, 1027].

In disciplines such as set theory and topology, a technique called "graphization" is frequently employed, wherein sets are mapped onto collections of vertices and edges. This approach enables the exploration of their mathematical structures, the development of novel graph algorithms, and various practical applications. A primary advantage of graphization lies in its visual clarity; relationships among vertices are effectively represented by edges, making complex connections easier to visualize. Moreover, by incorporating foundational concepts from classical graph theory, including well-established graph algorithms and structural principles, we can analyze these uncertain graph classes with greater efficiency and achieve a more comprehensive understanding of their properties.

The importance of graph theory has led to extensive research across diverse fields, including many practical applications [129, 251, 396, 465, 469, 580, 582, 602, 676, 788, 940, 989, 1010, 1026], graph neural networks [76, 183,319,391,627,628,827,987,988,992,1033], Bayesian networks [711], protein structure analysis [423,944], chemical graph theory [115,945], machine learning [433,510,697,774,988,1001], and graph databases [86,532, 660]. Additionally, research within graph theory has placed significant focus on understanding graph structures [90, 152, 161, 694] and developing numerous graph algorithms [133, 159, 334, 346, 355, 402, 573, 691, 935].

Based on the above, sets that handle uncertainty, such as Fuzzy Sets and Neutrosophic Sets, are often "graphized" for further study [786]. This paper investigates various models of uncertain graphs, which expand classical graph theory by integrating dimensions of uncertainty to better represent complex and ambiguous

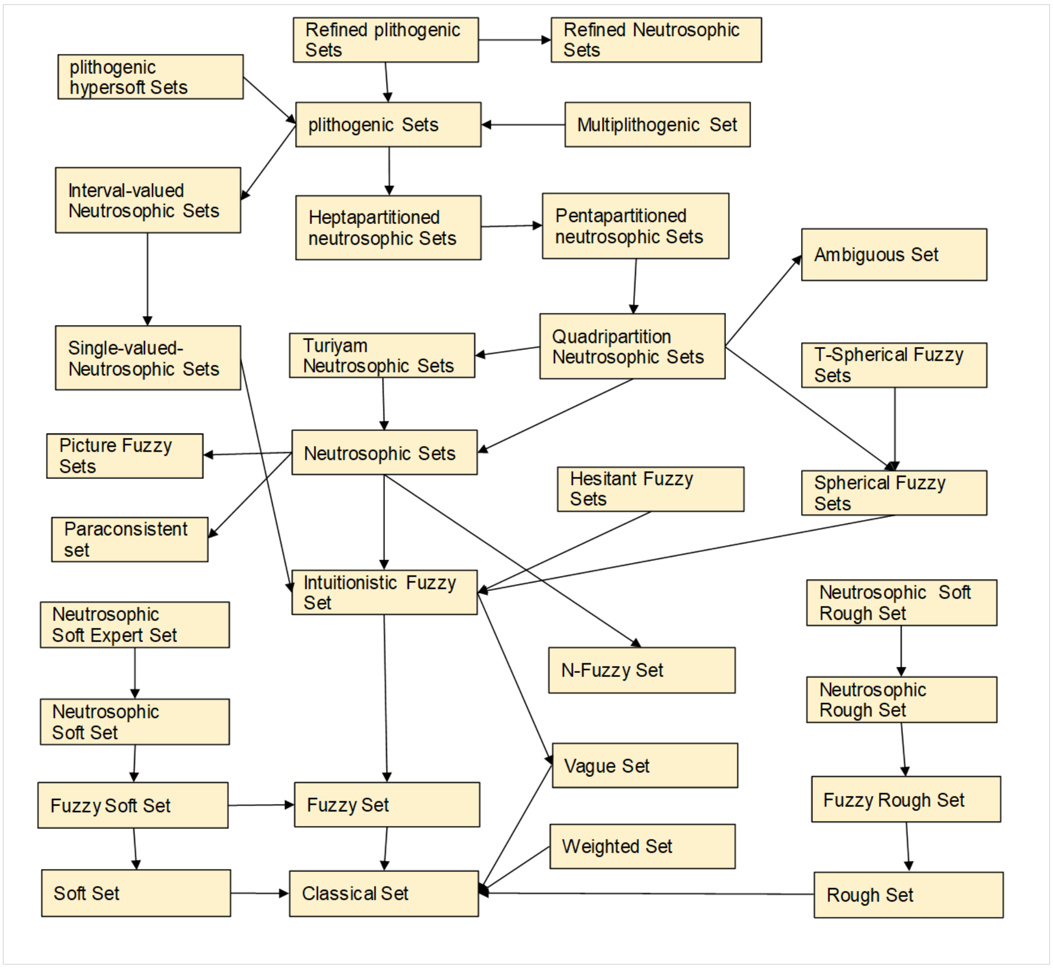


Figure 1: Some Uncertain sets Hierarchy. The set class at the origin of an arrow contains the set class at the destination of the arrow(cf. [378]).

relationships. For example, a Neutrosophic Graph models connections by incorporating degrees of truth, indeterminacy, and falsehood, allowing greater flexibility in capturing uncertainty within networks. These models of uncertain graphs have proven valuable across numerous real-world fields, resulting in the creation of many specialized graph classes [14–16,20–28,342,347,349,351–353,356,358,364–366,371,373,375–381,383,386]. Similar to uncertain sets, these uncertain graphs are particularly effective in decision-making applications [12, 32, 201–203, 578, 586, 836, 877, 1021].

Given the depth of research and wide-ranging applications in this field, uncertain graphs have become an essential area of study. For further insights and recent advancements, readers are encouraged to consult recent survey papers [368, 373, 375].

1.4 Hyperizaion and Superhyperizaion

Generalized graph concepts encompass well-known structures such as hypergraphs [164, 183, 328, 736] and superhypergraphs [867,868]. Similarly, in the field of soft sets, hypersoft sets [862] and superhypersoft sets have been introduced as generalized forms. Additionally, various other hyperized and superhyperized concepts exist. These types of hyperconcepts and superhyperconcepts are widely studied across numerous mathematical fields, with substantial attention directed toward their applications [532]. For example, hypergraphs have applications in areas such as hypergraph neural networks [184, 210, 327, 398, 486, 509, 513, 544, 931, 974] and hypergraph databases [52, 191, 406, 457]. Note that in some fields, multiple definitions exist, and the meanings of ”hyper” or ”super” may vary.

1.5 Our Contribution in This Paper

In this subsection, we summarize the contributions of this paper. Active research areas such as combinatorics, uncertain sets (including fuzzy sets, neutrosophic sets, and plithogenic sets), uncertain graphs, hyper concepts, and superhyper concepts provide significant mathematical insights and practical applications. Recognizing their importance, this paper explores new graph and set concepts, as well as hyper and superhyper constructs, as presented in the "Results" section. Additionally, this paper serves as a structured survey of recent findings, aiming to inform and inspire readers. It should be noted that some concepts listed in the "Future Tasks" section are preliminary definitions and remain under investigation.

Moreover, the Discussion section includes an example of the Procedure of Graphization, Hyperization, and Uncertainization. Due to the inherent complexity of defining a mathematically comprehensive procedure, the provided example represents just one possible approach. This paper defines a variety of concepts with the hope of sparking new ideas and serving as a valuable resource for readers in their research endeavors.

1.6 The Structure of the Paper

The structure of this paper is as follows. Section 2 provides an overview of existing definitions, Section 3 presents the main results, Section 4 examines an example of the Procedure of Graphization, Hyperization, and Uncertainization, and Section 5 outlines future directions and tasks.

List of Figures

2 Preliminaries and Definitions

Some foundational concepts from set theory are applied in parts of this work. For further details on these foundational concepts, please consult the relevant references as needed [338,475,507,543,610]. Additionally, for operations and related topics concerning each concept, please refer to the respective references as necessary.

2.1 Uncertain Set Theory

This subsection outlines fundamental set concepts and sets designed to handle uncertainty. Examples of these include Fuzzy Sets [1013,1015–1019,1036], Soft Sets, Neutrosophic Sets [855,856], Vague Sets [25,182,208, 495,1028], and Rough Sets [221,706,729,733,852,975]. For further details on each type of set, please refer to the relevant sources as needed.

2.1.1 Basic Set Theory

Below are some fundamental concepts in set theory. For more comprehensive details, please refer to the relevant references as needed [224,338,475,492,493,507,543,547,610,825,840].

Definition 2.1 (Set). [543] A *set* is a collection of distinct objects, known as elements, that are clearly defined, allowing any object to be identified as either belonging to or not belonging to the set. If A is a set and x is an element of A, this membership is denoted by $x \in A$. Sets are typically represented using curly brackets. For example, $A = \{1, 2, 3\}$ indicates that the set A contains the elements 1, 2, and 3.

Definition 2.2 (Union). [543] The *union* of two sets A and B is the set of all elements that are in either A, B, or both. The union is denoted by $A \cup B$ and is formally defined as:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

In other words, an element x is in $A \cup B$ if and only if x is in A, in B, or in both.

Definition 2.3 (Intersection). [543] The *intersection* of two sets A and B is the set of all elements that A and B have in common. The intersection is denoted by $A \cap B$ and is formally defined as:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

In other words, an element x is in $A \cap B$ if and only if x is in both A and B.

Definition 2.4 (Completement). [543] The *complement* of a set A, denoted by A^c or $\overline{A}$, consists of all elements in the set U that are not in A. The complement is formally defined as:

$$A^c = \{x \in U \mid x \notin A\}$$

Definition 2.5 (Difference). [543] The *difference* of two sets A and B, denoted $A \setminus B$, is the set of elements that belong to A but not to B. The difference of sets is formally defined as:

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Definition 2.6 (Product Set). [976] Let A and B be two sets. The *product set*, or *Cartesian product* of A and B, denoted by $A \times B$, is defined as the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Formally, we write:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Each element of $A \times B$ is an ordered pair, and the product set represents all possible combinations of elements from A and B in the specified order.

Example 2.7 (Product Set in Finite Sets). Let $A = \{1, 2\}$ and $B = \{x, y\}$. The Cartesian product $A \times B$ is:

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}.$$

This set consists of all ordered pairs where the first element is from A and the second is from B.

Definition 2.8 (Empty Set). [543] The *empty set*, denoted by $\emptyset$, is the unique set that contains no elements. Formally, the empty set is defined as:

$$\emptyset = \{x \mid x \neq x\},$$

indicating that there are no elements x for which the condition $x = x$ fails, thereby resulting in an empty collection. The empty set is a subset of every set and has a cardinality of zero.

Example 2.9 (Product of Empty Set). If $A = \{1, 2, 3\}$ and $B = \emptyset$ (the empty set), the Cartesian product $A \times B$ is:

$$A \times B = \emptyset.$$

Since B has no elements, there are no ordered pairs (a, b) where $b \in B$, making the product empty.

Definition 2.10 (Non-Empty Set). A *non-empty set* is a set that contains at least one element. Formally, a set S is non-empty if:

$$\exists x \in S.$$

In contrast to the empty set $\emptyset$, a non-empty set has a cardinality $|S| > 0$.

Definition 2.11 (Underlying Set of a Family of Sets). Let $\mathcal{F}$ be a family of subsets of a universal set U. The *underlying set* (or *whole set*) of $\mathcal{F}$, denoted by under$(\mathcal{F})$, is defined as the union of all subsets in $\mathcal{F}$. Formally:

$$\text{under}(\mathcal{F}) = \bigcup_{A \in \mathcal{F}} A.$$

Remark 2.12 (Underlying Set for General Structures)**.** For a mathematical structure S (e.g., groups, rings, graphs), the *underlying set* of S is the set of elements that form the base of S, abstracted from any additional operations or relations. Examples include:

1. *Graph:* If $G = (V, E)$ is a graph with V as the vertex set and E as the edge set, the underlying set is:

$$\text{under}(G) = V \cup \bigcup_{e \in E} e,$$

where e is the set of vertices that form the edge.

2. *Ring:* If $R = (X, +, \cdot)$ is a ring with X as the set of elements, $+$ as addition, and $\cdot$ as multiplication, the underlying set is:

$$\text{under}(R) = X.$$

Definition 2.13 (Power Set)**.** (cf. [268]) Let S be a set. The *power set* of S, denoted by $\mathcal{P}(S)$, is defined as the set of all subsets of S, including the empty set and S itself. Formally, we write:

$$\mathcal{P}(S) = \{T \mid T \subseteq S\}.$$

The power set $\mathcal{P}(S)$ contains $2^{|S|}$ elements, where $|S|$ represents the cardinality of S. This is because each element of S can either be included in or excluded from each subset.

Definition 2.14 (Partition of a Set)**.** Let S be a non-empty set. A *partition* of S is a collection of non-empty subsets $\mathcal{P} = \{A_1, A_2, \ldots, A_k\}$ such that:

- $A_i \cap A_j = \emptyset$ for all $i \neq j$ (disjoint subsets),
- $\bigcup_{i=1}^{k} A_i = S$ (complete coverage).

In other words, $\mathcal{P}$ divides S into mutually disjoint, non-empty subsets that collectively cover S.

Additionally, the definitions of related concepts, such as open [912] and closed intervals [912], are provided below. As these are fundamental concepts, please refer to lecture notes or other resources as needed.

Definition 2.15 (Real Number)**.** (cf. [145, 508, 771, 810]) The set of *real numbers*, denoted by $\mathbb{R}$, is a complete ordered field. It can be rigorously constructed as the set of equivalence classes of rational Cauchy sequences, where two sequences are equivalent if their difference converges to zero. This ensures that every Cauchy sequence in $\mathbb{R}$ converges to a limit within $\mathbb{R}$.

Definition 2.16 (open interval)**.** An *open interval* in the real numbers $\mathbb{R}$, denoted by (a, b), where $a, b \in \mathbb{R}$ and $a < b$, is defined as the set of all real numbers x such that:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$

This interval includes all values strictly between a and b but excludes the endpoints a and b.

Definition 2.17 (closed interval)**.** A *closed interval* in the real numbers $\mathbb{R}$, denoted by $[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$, is defined as the set of all real numbers x such that:

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

This interval includes all values from a to b, including the endpoints a and b.

2.1.2 Crisp Sets and Neutrosophic Sets

When dealing with Fuzzy Sets or Neutrosophic Sets, they are often discussed alongside their foundational Crisp Sets. The definition of a Crisp Set is provided below.

Definition 2.18 (Universe Set)**.** (cf. [696]) A *universe set*, often denoted by U, is a set that contains all the elements under consideration for a particular discussion or problem domain. Formally, U is defined as a set that encompasses every element within the scope of a given context or framework, so that any subset of interest can be regarded as a subset of U.

In set theory, the universe set U is typically assumed to contain all elements relevant to the discourse, meaning that for any set A, if $A \subseteq U$, then all elements of A are elements of U. Related concepts include underlying sets and whole sets.

Definition 2.19 (Non-empty Universe Set)**.** A *non-empty universe set*, denoted by U, is a set that contains all elements under consideration in a specific context and satisfies $U \neq \emptyset$. Formally:

$$U = \{x \mid x \text{ is relevant to the problem domain}\}, \quad \text{with } U \neq \emptyset.$$

Every subset of interest is considered a subset of U, ensuring that $A \subseteq U$ for any A.

Definition 2.20 (Crisp Set)**.** [706] Let X be a universe set, and let $P(X)$ denote the power set of X, which represents all subsets of X. A *crisp set* $A \subseteq X$ is defined by a characteristic function $\chi_A : X \to \{0, 1\}$, where:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

This function χ_A assigns a value of 1 to elements within the set A and 0 to those outside it, creating a clear boundary. Crisp sets are thus bivalent and follow the principle of binary classification, where each element is either a member of the set or not.

Definition 2.21 (Crisp Set Representation of the Empty Set)**.** The empty set $\emptyset$ in a universe X is represented as a Crisp Set $A \subseteq X$ by the characteristic function $\chi_\emptyset : X \to \{0, 1\}$, where:

$$\chi_\emptyset(x) = 0 \quad \text{for all } x \in X.$$

The Fuzzy Set is a well-known concept used to handle uncertainty in set theory. The definition is provided below [1013].

Definition 2.22. [1013, 1018] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 2.23. A *Fuzzy Set* A over X is defined by a membership function $\mu_A : X \to [0, 1]$, which assigns to each element $x \in X$ a membership degree $\mu_A(x)$ representing its degree of belonging to the set A.

Let us define the fuzzy set A as follows:

$$\mu_A(x_1) = 0.2,$$
$$\mu_A(x_2) = 0.5,$$
$$\mu_A(x_3) = 0.7.$$

This means:

- The element x_1 has a membership degree of 0.2 in A.
- The element x_2 has a membership degree of 0.5 in A.

- The element x_3 has a membership degree of 0.7 in A.

Proposition 2.24. *A Fuzzy Set is a generalization of a Crisp Set.*

Proof. This follows directly from the definition. □

One of the extended concepts of a fuzzy set is the Vague Set [25,182,208,495,1028]. The definition is provided below.

Definition 2.25 (Vague Set)**.** [208] Let U be a universe of discourse, defined as $U = \{u_1, u_2, \ldots, u_n\}$. A *vague set* A in U is characterized by two functions:

$$t_A : U \to [0,1] \quad \text{and} \quad f_A : U \to [0,1],$$

where:

- $t_A(u_i)$ is the *truth-membership function*, providing a lower bound on the membership degree of u_i based on supporting evidence for $u_i \in A$.
- $f_A(u_i)$ is the *false-membership function*, offering a lower bound on the negation of u_i based on evidence against $u_i \in A$.

These functions satisfy the constraint:

$$t_A(u_i) + f_A(u_i) \leq 1, \quad \text{for all } u_i \in U.$$

The degree of membership of u_i in the vague set A is thus constrained within a subinterval of $[0,1]$ defined by:

$$t_A(u_i) \leq \mu_A(u_i) \leq 1 - f_A(u_i),$$

where $\mu_A(u_i)$ represents the true membership grade of u_i in A. The interval $[t_A(u_i), 1 - f_A(u_i)]$ indicates that, although the exact membership degree may be uncertain, it is bound within this range.

If U is continuous, a vague set A can be represented as:

$$A = \int_U [t_A(u), 1 - f_A(u)]/u.$$

In the case of a discrete universe U, A is expressed as:

$$A = \sum_{i=1}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i.$$

Proposition 2.26. *A Vague Set is a generalization of a Fuzzy Set.*

Proof. This follows directly from the definition. □

The definition of Neutrosophic Sets, which is frequently referenced throughout this paper, is provided below [855–857, 859]. Neutrosophic Sets represent a generalization of Fuzzy Sets [1013, 1014, 1018] and are a significant concept due to the extensive array of derived concepts and applications they have inspired [61, 62, 258, 260, 719, 805, 853, 858, 949, 1005, 1012, 1025].

Definition 2.27. [855] Let X be a given set. A Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

In the framework of Neutrosophic Sets, the following properties hold. Moreover, it is well known that Neutrosophic Sets generalize not only crisp sets but also paraconsistent sets [189, 236, 619, 977–979], dialethist sets [856], paradoxist sets [893], intuitionistic fuzzy sets [73, 99–102, 104, 246, 305, 400, 611, 811, 849, 921], and tautological sets [681, 742, 968]. Furthermore, extensions of Neutrosophic Sets, such as quadpartitioned Neutrosophic Sets and pentapartitioned Neutrosophic Sets, have been introduced. These sets are further generalized by the concept of Plithogenic Sets, which will be discussed later.

Proposition 2.28. *A Neutrosophic Set is a generalization of a Vague Set.*

Proof. This follows directly from the definition. □

Proposition 2.29. *A Neutrosophic Set is a generalization of a Fuzzy Set.*

Proof. This follows directly from the definition. □

The Plithogenic Set is known as a type of set that can generalize Neutrosophic Sets, Fuzzy Sets, and other similar sets [864, 865]. The definition of the Plithogenic Set is provided below.

Definition 2.30. [864, 865] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)*.
- $pCF : Pv \times Pv \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a, a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a, b) = pCF(b, a)$$

The following is an already known result, but it is presented here for reference.

Proposition 2.31. *A Plithogenic Set $PS = (P, v, Pv, pdf, pCF)$ can be reduced to a Fuzzy Set, Vague Set, or Neutrosophic Set based on specific values of the parameters s and t:*

- *If $s = 1$ and $t = 1$, PS becomes a Fuzzy Set.*
- *If $s = 2$ and $t = 1$, PS becomes a Vague Set.*
- *If $s = 3$ and $t = 1$, PS becomes a Neutrosophic Set.*

Proof. Let $PS = (P, v, Pv, pdf, pCF)$ be a Plithogenic Set, where $P \subseteq S$ is a subset of the universal set, v is an attribute, Pv is the set of possible values of v, $pdf : P \times Pv \to [0, 1]^s$ is the Degree of Appurtenance Function, and $pCF : Pv \times Pv \to [0, 1]^t$ is the Degree of Contradiction Function.

- *Reduction to a Fuzzy Set:*

  When $s = 1$ and $t = 1$:

  - Let $Pv = \{1\}$, meaning that v has only one possible value.
  - Define $pdf(p, 1) = \mu_{PS}(p)$ for each $p \in P$, where $\mu_{PS} : P \to [0, 1]$ represents the degree of membership of p in P.

  Since pdf now assigns a single degree of membership for each element in P, the Plithogenic Set PS simplifies to a Fuzzy Set $F = \{(p, \mu_{PS}(p)) : p \in P\}$.

- *Reduction to a Vague Set:*

  When $s = 2$ and $t = 1$:

  - Let $Pv = \{\text{truth}, \text{falsity}\}$, so that v takes on two possible values.
  - Define $pdf(p, \text{truth}) = t_{PS}(p)$ and $pdf(p, \text{falsity}) = f_{PS}(p)$ for each $p \in P$, where $t_{PS}(p)$ and $f_{PS}(p)$ denote the degrees of truth and falsity, respectively.
  - Impose the constraint $t_{PS}(p) + f_{PS}(p) \leq 1$ to satisfy the structure of a Vague Set.

  These configurations yield a Vague Set $V = \{(p, [t_{PS}(p), 1 - f_{PS}(p)]) : p \in P\}$, where $t_{PS}(p)$ and $f_{PS}(p)$ define a bounded interval for the membership degree of each $p \in P$, consistent with the Vague Set framework.

- *Reduction to a Neutrosophic Set:*

  When $s = 3$ and $t = 1$:

  - Let $Pv = \{\text{truth}, \text{indeterminacy}, \text{falsity}\}$, so v has three possible values.
  - Define $pdf(p, \text{truth}) = T_{PS}(p)$, $pdf(p, \text{indeterminacy}) = I_{PS}(p)$, and $pdf(p, \text{falsity}) = F_{PS}(p)$ for each $p \in P$, where $T_{PS}(p)$, $I_{PS}(p)$, and $F_{PS}(p)$ represent the degrees of truth, indeterminacy, and falsity, respectively.
  - Impose the condition $0 \leq T_{PS}(p) + I_{PS}(p) + F_{PS}(p) \leq 3$, consistent with the Neutrosophic Set requirements.

  Under these configurations, PS becomes a Neutrosophic Set $N = \{(p, (T_{PS}(p), I_{PS}(p), F_{PS}(p))) : p \in P\}$, where each element in P has associated degrees of truth, indeterminacy, and falsity, following the Neutrosophic Set structure.

In conclusion, by setting the values of s and t appropriately, we can obtain a Fuzzy Set, Vague Set, or Neutrosophic Set as special cases of the Plithogenic Set. □

As demonstrated in the proofs above, the following generalization relationships are known.

Example 2.32. (cf. [368, 374]) The following examples of Plithogenic Sets are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Set*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Set*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Set*.
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic Set*.
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic Set*.

2.1.3 Neutrosophic triplet

A *Neutrosophic Triplet* is an ordered triple $\langle T, I, F\rangle$, where T, I, and F represent the degrees of truth, indeterminacy, and falsehood, respectively [63, 806, 886, 890, 891, 1029]. A related concept, the Neutrosophic Duplet, is also recognized [562, 656, 863, 1030]. This structure allows for nuanced modeling of uncertainty in various applications. The definition is provided below. For more details, please refer to sources such as [886, 890].

Definition 2.33. An *ordered triple* is a collection of three elements in a specified order, denoted by (a, b, c), where the position of each element matters. Formally, an ordered triple can be defined as follows:

Let a, b, and c be elements from any set. The ordered triple (a, b, c) is defined as the ordered pair:

$$(a, b, c) := (a, (b, c)),$$

where the element a is paired with the ordered pair (b, c).

This construction preserves the order of elements, meaning that:

$$(a, b, c) \neq (b, a, c) \quad \text{unless} \quad a = b = c.$$

Definition 2.34. [886, 890] A *neutrosophic triplet* is an ordered triple $\langle T, I, F\rangle$, where:

- T represents the degree of truth,
- I represents the degree of indeterminacy,
- F represents the degree of falsehood,

such that $T, I, F \in [0, 1]$ and

$$(T, I, F) \neq (1, 0, 0) \quad \text{and} \quad (T, I, F) \neq (0, 0, 1).$$

Here, $(1, 0, 0)$ represents a classical truth and $(0, 0, 1)$ represents a classical falsehood. This triplet provides a framework for representing truth, indeterminacy, and falsehood simultaneously in various degrees.

In applications such as NeutroTopology, we use:

- *Axiom* $\langle 1, 0, 0\rangle$: A classical axiom, fully true.
- *NeutroAxiom* $\langle T, I, F\rangle$: An axiom with degrees of truth, indeterminacy, and falsehood.
- *AntiAxiom* $\langle 0, 0, 1\rangle$: An axiom that is entirely false.

2.1.4 NonStandard Real Set

The *NonStandard Real Set* is an extension of the standard real number set R, encompassing both infinitesimally small values (infinitesimals [244, 574, 605, 914, 915]) and infinitely large values [866, 886]. Definitions of this concept, along with related concepts, are provided below.

Definition 2.35. A real number x is considered a positive real number if it is greater than zero. The set of all positive real numbers is denoted by $\mathbb{R}^+$ and is formally defined as:

$$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}.$$

In this context, each element $x \in \mathbb{R}^+$ satisfies the condition $x > 0$, indicating that positive real numbers are located on the right side of zero on the real number line.

Definition 2.36. A real number x is considered a negative real number if it is less than zero. The set of all negative real numbers is denoted by $\mathbb{R}^-$ and is formally defined as:

$$\mathbb{R}^- = \{x \in \mathbb{R} \mid x < 0\}.$$

Each element $x \in \mathbb{R}^-$ satisfies the condition $x < 0$, meaning negative real numbers are located on the left side of zero on the real number line.

Definition 2.37. (cf. [303,707]) The *real number set*, denoted by $\mathbb{R}$, is the set of all real numbers. This includes both rational numbers, such as $\mathbb{Q} \subset \mathbb{R}$, where:

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\},$$

and irrational numbers, which cannot be expressed as a ratio of integers.

The set $\mathbb{R}$ is formally defined as:

$\mathbb{R} = \{x \mid x$ is a solution to a polynomial equation with real coefficients, or $x \in$ limit points of such solutions$\}$.

The real number set $\mathbb{R}$ forms a complete ordered field, meaning it is closed under addition, subtraction, multiplication, and division (excluding division by zero), and satisfies the properties of order completeness.

Definition 2.38. (cf. [244, 574, 605, 914, 915]) Consider an element $x^* \in \mathbb{R}^*$ within the NonStandard real numbers. We define the following classifications for x^*:

1. x^* is said to be *infinitesimal* if, for every positive real number $a \in \mathbb{R}^+$,

$$-a < x^* < a.$$

2. x^* is said to be *finite* if there exists some positive real number $a \in \mathbb{R}^+$ such that

$$-a < x^* < a.$$

3. x^* is said to be *infinite* if, for every positive real number $a \in \mathbb{R}^+$,

$$x^* \leq -a \quad \text{or} \quad x^* \geq a.$$

Definition 2.39. [886] The NonStandard Real Set, denoted as $\mathbb{R}^*$, is an extension of the standard real number set $\mathbb{R}$ that includes infinitesimal and infinitely large elements. It is defined as follows:

$$\mathbb{R}^* = \mathbb{R} \cup \mathbb{R}^{+*} \cup \mathbb{R}^{-*},$$

where:

- $\mathbb{R}$ represents the standard real numbers,
- $\mathbb{R}^{+*}$ denotes the set of positive infinitesimals, i.e., elements $\epsilon \in \mathbb{R}^*$ such that $0 < \epsilon < \frac{1}{n}$ for any positive integer n,
- $\mathbb{R}^{-*}$ denotes the set of negative infinitesimals, i.e., elements $-\epsilon$ where $\epsilon \in \mathbb{R}^{+*}$.

Thus, $\mathbb{R}^*$ contains all standard real numbers along with infinitesimally small values near zero.

2.1.5 Single-Valued Neutrosophic OverSet

A Neutrosophic OverSet includes elements with a degree of membership exceeding the standard limit, expanding beyond typical bounds [854, 871, 874].

Definition 2.40 (Single-Valued Neutrosophic OverSet)**.** [874] A *Single-Valued Neutrosophic OverSet*, denoted $A_{\text{over}} \subseteq U_{\text{over}}$, is a set in a universe of discourse U_{over} where certain elements possess at least one neutrosophic value—truth, indeterminacy, or falsity—that exceeds the conventional upper limit of 1. It is formally defined as:

$$A_{\text{over}} = \{(x, \langle T(x), I(x), F(x)\rangle) \mid x \in U_{\text{over}}, \exists\, T(x) > 1 \text{ or } I(x) > 1 \text{ or } F(x) > 1\},$$

where:

- $T(x)$, $I(x)$, and $F(x)$ represent the truth-membership, indeterminacy-membership, and falsity-membership degrees for an element $x \in U_{\text{over}}$, respectively.
- $T(x), I(x), F(x) \in [0, \Omega]$ with $\Omega > 1$, termed the *OverLimit*, allowing for any or all of the values $T(x)$, $I(x)$, or $F(x)$ to exceed 1 for specific elements.

The Single-Valued Neutrosophic OverSet generalizes the classical neutrosophic set by enabling at least one of the neutrosophic degrees to surpass the conventional boundary of 1, thereby extending the representational scope of uncertainty.

Proposition 2.41. *Every classical Neutrosophic Set is contained within a Neutrosophic OverSet.*

Proof. Let $A \subseteq U$ be a classical Neutrosophic Set in the universe of discourse U, defined by membership functions:

$$T_A : U \to [0,1], \quad I_A : U \to [0,1], \quad F_A : U \to [0,1],$$

where for each $x \in U$, $T_A(x)$, $I_A(x)$, and $F_A(x)$ denote the degrees of truth, indeterminacy, and falsity, respectively, with the constraint:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Now consider an *OverLimit* $\Omega > 1$ and define a Neutrosophic OverSet $B \subseteq U$ on the same universe with membership functions:

$$T_B : U \to [0,\Omega], \quad I_B : U \to [0,\Omega], \quad F_B : U \to [0,\Omega],$$

where the OverSet condition $T_B(x) \le \Omega$, $I_B(x) \le \Omega$, and $F_B(x) \le \Omega$ applies.

Since all values $T_A(x)$, $I_A(x)$, and $F_A(x)$ for a classical Neutrosophic Set A are contained within $[0,1] \subseteq [0,\Omega]$, we can define the OverSet B such that $T_B(x) = T_A(x)$, $I_B(x) = I_A(x)$, and $F_B(x) = F_A(x)$ for all $x \in U$. Hence, $A \subseteq B$, and every classical Neutrosophic Set can indeed be embedded within a Neutrosophic OverSet. □

2.1.6 Single-Valued Neutrosophic UnderSet

A Single-Valued Neutrosophic UnderSet includes elements with non-membership degrees that can fall below zero, thus allowing negative non-membership values [874]. Neutrosophic UnderSets have been widely studied in various research contexts [645, 961]. The definition is provided below.

Definition 2.42 (Single-Valued Neutrosophic UnderSet)**.** [874] A *Single-Valued Neutrosophic UnderSet*, denoted $A_{\text{under}} \subseteq U_{\text{under}}$, is a set in a universe of discourse U_{under} where certain elements have one or more neutrosophic values—truth, indeterminacy, or falsity—that fall below the standard minimum of 0. It is formally defined as:

$$A_{\text{under}} = \{(x, \langle T(x), I(x), F(x)\rangle) \mid x \in U_{\text{under}}, \exists\, T(x) < 0 \text{ or } I(x) < 0 \text{ or } F(x) < 0\},$$

where:

- $T(x)$, $I(x)$, and $F(x)$ denote the truth-membership, indeterminacy-membership, and falsity-membership degrees of an element $x \in U_{\text{under}}$, respectively.
- Each of these values $T(x), I(x), F(x) \in [\Psi, 1]$, with $\Psi < 0$ termed the *UnderLimit*, allowing $T(x)$, $I(x)$, or $F(x)$ to be negative for certain elements.

The Single-Valued Neutrosophic UnderSet extends the classical neutrosophic set by permitting at least one neutrosophic degree to fall below zero, thereby broadening the framework for uncertain representation.

Proposition 2.43. *Every classical Neutrosophic Set is contained within a Neutrosophic UnderSet.*

Proof. Let $A \subseteq X$ be a classical Neutrosophic Set in a universe of discourse X, defined by the membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, $T_A(x)$, $I_A(x)$, and $F_A(x)$ denote the degrees of truth, indeterminacy, and falsity, respectively, with the constraint:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Now consider an *UnderLimit* $\Psi < 0$ and define a Neutrosophic UnderSet $B \subseteq X$ on the same universe with membership functions:

$$T_B : X \to [\Psi, 1], \quad I_B : X \to [\Psi, 1], \quad F_B : X \to [\Psi, 1],$$

where the UnderSet condition $T_B(x) \ge \Psi$, $I_B(x) \ge \Psi$, and $F_B(x) \ge \Psi$ applies.

Since all values $T_A(x)$, $I_A(x)$, and $F_A(x)$ for a classical Neutrosophic Set A are confined within $[0, 1] \subseteq [\Psi, 1]$, we can define the UnderSet B such that $T_B(x) = T_A(x)$, $I_B(x) = I_A(x)$, and $F_B(x) = F_A(x)$ for all $x \in X$. Consequently, $A \subseteq B$, and every classical Neutrosophic Set is indeed contained within a Neutrosophic UnderSet. □

2.1.7 Single-Valued Neutrosophic OffSet

The definition of the Single-Valued Neutrosophic OffSet is also included below [854, 871, 874].

Definition 2.44 (Single-Valued Neutrosophic OffSet)**.** [874] A *Single-Valued Neutrosophic OffSet*, denoted $A_{\text{off}} \subseteq U_{\text{off}}$, is a set within a universe of discourse U_{off} in which certain elements may possess neutrosophic degrees—truth, indeterminacy, or falsity—that extend beyond the standard limits, either above 1 or below 0. It is formally defined as:

$$A_{\text{off}} = \{(x, \langle T(x), I(x), F(x)\rangle) \mid x \in U_{\text{off}}, \exists\ (T(x) > 1 \text{ or } F(x) < 0)\},$$

where:

- $T(x)$, $I(x)$, and $F(x)$ denote the truth-membership, indeterminacy-membership, and falsity-membership degrees of each $x \in U_{\text{off}}$.
- $T(x), I(x), F(x) \in [\Psi, \Omega]$, where $\Omega > 1$ (termed the *OverLimit*) and $\Psi < 0$ (termed the *UnderLimit*), allow the possibility for $T(x)$, $I(x)$, or $F(x)$ to take values beyond the conventional bounds of $[0, 1]$.

Proposition 2.45. *A Single-Valued Neutrosophic OffSet can be transformed into a Single-Valued Neutrosophic OverSet or a Single-Valued Neutrosophic UnderSet.*

Proof. Let X be a universal set, and let $A_{\text{off}} \subseteq X$ be a Single-Valued Neutrosophic OffSet. By definition, an OffSet A_{off} permits neutrosophic values that can exceed 1 or fall below 0.

Define a Single-Valued Neutrosophic OverSet $A_{\text{over}} \subseteq X$ by allowing truth values $T(x)$ to exceed the conventional upper limit of 1:

$$A_{\text{over}} = \{(x, \langle T(x), I(x), F(x)\rangle) \mid x \in X,\ T(x) > 1\}.$$

Since $T(x) \in [0, \Omega]$ in A_{off} with $\Omega > 1$, any x for which $T(x) > 1$ in A_{off} qualifies for inclusion in A_{over}. Thus, by selecting only those elements $x \in X$ with $T(x) > 1$ from A_{off}, we construct a valid Single-Valued Neutrosophic OverSet:

$$A_{\text{over}} \subseteq A_{\text{off}}.$$

Similarly, define a Single-Valued Neutrosophic UnderSet $A_{\text{under}} \subseteq X$ by permitting falsity values $F(x)$ to fall below the conventional lower limit of 0:

$$A_{\text{under}} = \{(x, \langle T(x), I(x), F(x)\rangle) \mid x \in X,\ F(x) < 0\}.$$

Since $F(x) \in [\Psi, 1]$ in A_{off} with $\Psi < 0$, any x with $F(x) < 0$ in A_{off} meets the criteria for inclusion in A_{under}. Thus, by selecting only those elements $x \in X$ with $F(x) < 0$ from A_{off}, we form a valid Single-Valued Neutrosophic UnderSet:

$$A_{\text{under}} \subseteq A_{\text{off}}.$$

Therefore, each Single-Valued Neutrosophic OffSet A_{off} can be transformed into a Single-Valued Neutrosophic OverSet A_{over} by restricting to elements with $T(x) > 1$ or into a Single-Valued Neutrosophic UnderSet A_{under} by restricting to elements with $F(x) < 0$. □

2.1.8 Fuzzy and Intuitionistic Fuzzy OverSet/Offset/Underset

The definitions of Fuzzy and Intuitionistic Fuzzy OverSets, Offsets, and UnderSets are provided below [860].

Definition 2.46 (Fuzzy Overset)**.** (cf. [860]) Let X be a universe of discourse. A *Fuzzy Overset* $\tilde{A}$ in X is defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X,\ \mu_{\tilde{A}}(x) \in [0, \Omega]\},$$

where $\Omega > 1$ represents the *Overlimit*, allowing membership degrees greater than 1. There exists at least one $x \in X$ such that $\mu_{\tilde{A}}(x) > 1$.

Definition 2.47 (Intuitionistic Fuzzy Overset)**.** (cf. [860]) Let X be a universe of discourse. An *Intuitionistic Fuzzy Overset* A in X is defined as:

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X,\ \mu_A(x), \nu_A(x) \in [0, \Omega]\},$$

where $\Omega > 1$. There exists at least one $x \in X$ such that $\mu_A(x) > 1$ or $\nu_A(x) > 1$.

Definition 2.48 (Fuzzy Underset)**.** (cf. [860]) Let X be a universe of discourse. A *Fuzzy Underset* $\tilde{A}$ in X is defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X,\ \mu_{\tilde{A}}(x) \in [\Psi, 1]\},$$

where $\Psi < 0$ is the *Underlimit*, allowing membership degrees below 0. There exists at least one $x \in X$ such that $\mu_{\tilde{A}}(x) < 0$.

Definition 2.49 (Intuitionistic Fuzzy Underset)**.** (cf. [860]) Let X be a universe of discourse. An *Intuitionistic Fuzzy Underset* A in X is defined as:

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X,\ \mu_A(x), \nu_A(x) \in [\Psi, 1]\},$$

where $\Psi < 0$. There exists at least one $x \in X$ such that $\mu_A(x) < 0$ or $\nu_A(x) < 0$.

Definition 2.50 (Fuzzy Offset)**.** (cf. [860]) Let X be a universe of discourse. A *Fuzzy Offset* $\tilde{A}$ in X is defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X,\ \mu_{\tilde{A}}(x) \in [\Psi, \Omega]\},$$

where $\Omega > 1$ and $\Psi < 0$. There exist elements $x, y \in X$ such that $\mu_{\tilde{A}}(x) > 1$ and $\mu_{\tilde{A}}(y) < 0$.

Definition 2.51 (Intuitionistic Fuzzy Offset)**.** (cf. [860]) Let X be a universe of discourse. An *Intuitionistic Fuzzy Offset A* in X is defined as:

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X,\ \mu_A(x) \in [\Psi, \Omega],\ \nu_A(x) \in [\Psi, \Omega]\},$$

where $\Omega > 1$ and $\Psi < 0$. There exist elements $x, y \in X$ such that $\mu_A(x) > 1$ or $\nu_A(x) > 1$, and $\mu_A(y) < 0$ or $\nu_A(y) < 0$.

Corollary 2.52. *A Fuzzy OffSet can be transformed into a Fuzzy OverSet or a Fuzzy UnderSet.*

Proof. The proof follows similarly to the case of the Neutrosophic set. □

Corollary 2.53. *An Intuitionistic Fuzzy OffSet can be transformed into an Intuitionistic Fuzzy OverSet or an Intuitionistic Fuzzy UnderSet.*

Proof. The proof follows similarly to the case of the Neutrosophic set. □

Proposition 2.54. *The Neutrosophic Offset generalizes both the Fuzzy Offset and the Intuitionistic Fuzzy Offset.*

Proof. A Neutrosophic Offset permits the truth-membership degrees $T_A(x)$, indeterminacy degrees $I_A(x)$, and non-membership degrees $F_A(x)$ to exceed 1 or be less than 0. For a Fuzzy Offset, setting $I_A(x) = 0$ and $F_A(x) = 1 - \mu_{\tilde{A}}(x)$ effectively restricts the neutrosophic framework to represent fuzzy offsets. For the Intuitionistic Fuzzy Offset, assigning $T_A(x) = \mu_A(x)$, $F_A(x) = \nu_A(x)$, and $I_A(x) = 1 - \mu_A(x) - \nu_A(x)$ captures intuitionistic fuzzy offsets within the neutrosophic domain. Therefore, the Neutrosophic Offset is a generalization of both concepts. □

Proposition 2.55. *The Neutrosophic Underset generalizes both the Fuzzy Underset and the Intuitionistic Fuzzy Underset.*

Proof. In a Neutrosophic Underset, any of the degrees $T_A(x)$, $I_A(x)$, or $F_A(x)$ can be less than 0, encompassing fuzzy and intuitionistic fuzzy undersets where such degrees may be negative. Mapping $I_A(x) = 0$, $F_A(x) = 1 - \mu_{\tilde{A}}(x)$ captures the Fuzzy Underset, and defining $T_A(x) = \mu_A(x)$, $F_A(x) = \nu_A(x)$, and $I_A(x) = 1 - \mu_A(x) - \nu_A(x)$ similarly captures the Intuitionistic Fuzzy Underset. □

Proposition 2.56. *The Neutrosophic Overset generalizes both the Fuzzy Overset and the Intuitionistic Fuzzy Overset.*

Proof. Allowing degrees in the Neutrosophic Overset to exceed 1 generalizes fuzzy and intuitionistic fuzzy oversets. Assigning $I_A(x) = 0$ and $F_A(x) = 1 - \mu_{\tilde{A}}(x)$ represents the Fuzzy Overset, while defining $T_A(x) = \mu_A(x)$, $F_A(x) = \nu_A(x)$, and $I_A(x) = 1 - \mu_A(x) - \nu_A(x)$ captures the Intuitionistic Fuzzy Overset within the neutrosophic framework. □

Additionally, a Crisp Set can also be defined using Offset, Overset, and Underset concepts, resulting in definitions for Crisp Offset, Crisp Overset, and Crisp Underset. As an example, the definition of a Crisp Offset is provided below.

Definition 2.57 (Crisp Offset)**.** Let X be a universe of discourse, and let Ψ and Ω represent 0 and 1, respectively. A *Crisp Offset $A \subseteq X$* is defined by a characteristic function $\chi_A : X \to \{\Psi, \Omega\}$, where:

$$\chi_A(x) = \begin{cases} \Omega & \text{if } x \in A, \\ \Psi & \text{if } x \notin A. \end{cases}$$

In this context, the function χ_A assigns a value of Ω (1) to elements that are within the set A and Ψ (0) to elements that are outside A. This structure adheres to the principle of binary classification, as each element is either fully included in the set A or completely excluded from it.

The concept of a Crisp Offset, unlike fuzzy or neutrosophic sets, does not allow for intermediate degrees of membership. Instead, membership is strictly limited to the values Ψ and Ω, reflecting the clear-cut, deterministic nature of this classification approach. This discrete boundary is a distinguishing feature of Crisp Offsets, contrasting with the gradual membership levels typical of fuzzy sets.

Proposition 2.58. *A Fuzzy Offset is a generalization of a Crisp Offset.*

Proof. For a Fuzzy Offset $\tilde{A}$ to reduce to a Crisp Offset, we restrict $\mu_{\tilde{A}}(x)$ to take values only in $\{\Psi, \Omega\}$. By setting $\Psi = 0$ and $\Omega = 1$, the Fuzzy Offset behaves identically to a Crisp Offset, assigning membership degrees of 1 (indicating inclusion) or 0 (indicating exclusion), thus satisfying the binary classification of a Crisp Set.

Unlike the binary nature of a Crisp Offset, a Fuzzy Offset allows for continuous membership values between Ψ and Ω, providing a more flexible framework. This flexibility supports scenarios where partial, over-, or under-membership is required, thereby extending the representation of set membership.

Since the Crisp Offset can be obtained as a special case of the Fuzzy Offset by restricting membership values to $\{\Psi, \Omega\} = \{0, 1\}$, it follows that the Fuzzy Offset generalizes the concept of a Crisp Offset. □

Corollary 2.59. *A Fuzzy Overset is a generalization of a Crisp Overset.*

Proof. The proof can be conducted in the same manner as previously shown. □

Corollary 2.60. *A Fuzzy Underset is a generalization of a Crisp Underset.*

Proof. The proof can be conducted in the same manner as previously shown. □

2.1.9 Neutrosophic Triplet Set

A Neutrosophic Triplet Strong Set is a set where each element has a unique neutral and opposite element, under a specified operation [886].

Definition 2.61. [886] A Neutrosophic Triplet Strong Set is a set N equipped with a binary operation $*$, under which every element $x \in N$ has an associated neutral element $\text{neut}(x)$ and an opposite element $\text{anti}(x)$ such that:

$$x * \text{neut}(x) = \text{neut}(x) * x = x \quad \text{and} \quad x * \text{anti}(x) = \text{anti}(x) * x = \text{neut}(x).$$

In this set:

- $*$ is a well-defined associative operation,
- $\text{neut}(x) \in N$ serves as the neutral element distinct from any classical identity element if one exists,
- $\text{anti}(x) \in N$ acts as the opposite of x, ensuring the operation yields the neutral element.

2.1.10 MultiNeutrosophic Set

A MultiNeutrosophic Set is a Neutrosophic Set in which each element's degrees of truth, indeterminacy, and falsehood are evaluated by multiple sources [879]. The formal definition is provided below.

Definition 2.62 (MultiNeutrosophic Set)**.** [50, 879] Let $\mathcal{U}$ be a universe of discourse, and M a subset of it. Then, a *MultiNeutrosophic Set* is defined as:

$$M = \{x, x(T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s) \mid x \in \mathcal{U}\},$$

where:

- p, r, s are integers ≥ 0 with $p + r + s = n \geq 2$, ensuring the existence of multiplicity in at least one neutrosophic component (truth/membership, indeterminacy, or falsehood/nonmembership).
- $T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s \subseteq [0, 1]$, with at least one of $p, r, s \geq 2$.

The subsets $T_1, T_2, \ldots, T_p$ represent multiple truth values provided by different sources; $I_1, I_2, \ldots, I_r$ represent multiple indeterminacy values; and $F_1, F_2, \ldots, F_s$ represent multiple falsehood values. Additionally, the condition:

$$0 \leq \sum_{j=1}^{p} \inf T_j + \sum_{k=1}^{r} \inf I_k + \sum_{l=1}^{s} \inf F_l \leq \sum_{j=1}^{p} \sup T_j + \sum_{k=1}^{r} \sup I_k + \sum_{l=1}^{s} \sup F_l \leq n,$$

holds for these values. No other restrictions apply to these neutrosophic components.

Explanation of Components

- *MultiTruth (MultiMembership)*: Degrees $T_1, T_2, \ldots, T_p$ of an element x with respect to set M, each evaluated by a different source.
- *MultiIndeterminacy (MultiNeutrality)*: Degrees $I_1, I_2, \ldots, I_r$ representing the indeterminacy or neutrality of x with respect to M, each provided by a different source.
- *MultiFalsehood (MultiNonmembership)*: Degrees $F_1, F_2, \ldots, F_s$ of x with respect to M, each evaluated by a different source.

Remark 2.63. [879] Depending on the values and types of neutrosophic components, several particular cases of MultiNeutrosophic Sets can be identified:

- *Single-Valued MultiNeutrosophic Set (SVMNS)*: All components T_j, I_k, F_l are single values in $[0, 1]$.
- *Interval-Valued MultiNeutrosophic Set (IVMNS)*: All components are intervals within $[0, 1]$.

Proposition 2.64. *The MultiNeutrosophic Set is a generalization of the Neutrosophic Set.*

Proof. In a Neutrosophic Set, each element x in X has a single degree of truth, indeterminacy, and falsity. However, in a MultiNeutrosophic Set, x can have multiple values for each of these components,

To see that the MultiNeutrosophic Set generalizes the Neutrosophic Set, consider the specific case where $p = r = s = 1$ in the MultiNeutrosophic Set. Each element $x \in X$ would then be represented by a single value T_1 for truth, I_1 for indeterminacy, and F_1 for falsity, effectively reducing the MultiNeutrosophic Set to:

$$M = \{x, x(T_1; I_1; F_1) \mid x \in X\},$$

which matches the structure of a Neutrosophic Set.

Thus, the MultiNeutrosophic Set includes the Neutrosophic Set as a specific instance, demonstrating that it is indeed a generalization. □

As a related concept, the MultiCrisp Set is defined as follows. It can be generalized within the framework of the MultiNeutrosophic Set.

Definition 2.65. Formally, a MultiCrisp Set M over a universe $\mathcal{U}$ is defined as:

$$M = \{(x, \mu_1(x), \mu_2(x), \ldots, \mu_k(x)) \mid x \in \mathcal{U}\},$$

where:

- $k \geq 2$ is the number of crisp membership evaluations (multi-membership conditions).
- $\mu_i(x) : \mathcal{U} \to \{0, 1\}$ is a crisp membership function for the i-th condition, defined as:

$$\mu_i(x) = \begin{cases} 1 & \text{if } x \text{ satisfies the } i\text{-th membership condition,} \\ 0 & \text{otherwise.} \end{cases}$$

1. Each membership function $\mu_i(x)$ evaluates to 0 or 1, maintaining the crispness of each membership criterion.
2. The tuple $(\mu_1(x), \mu_2(x), \ldots, \mu_k(x))$ encodes the multi-membership of x, providing multiple perspectives on whether x belongs to M.

Proposition 2.66. *The MultiCrisp Set is a generalization of the Crisp Set.*

Proof. Let $\mathcal{U}$ be a universal set. A Crisp Set $C \subseteq \mathcal{U}$ is defined as:

$$C = \{x \in \mathcal{U} \mid \mu(x) = 1\}.$$

Now, consider a MultiCrisp Set M defined as:

$$M = \{(x, \mu_1(x), \mu_2(x), \ldots, \mu_k(x)) \mid x \in \mathcal{U}\},$$

where:

- $k \geq 2$, representing multiple membership conditions.
- $\mu_i(x) : \mathcal{U} \to \{0, 1\}$ is the crisp membership function for the i-th condition, satisfying:

$$\mu_i(x) = \begin{cases} 1 & \text{if } x \text{ satisfies the } i\text{-th condition,} \\ 0 & \text{otherwise.} \end{cases}$$

For $k = 1$, the MultiCrisp Set reduces to:

$$M = \{(x, \mu_1(x)) \mid x \in \mathcal{U}\}.$$

By identifying $\mu_1(x)$ with the single membership function $\mu(x)$ of a Crisp Set, it follows that:

$$M = \{x \in \mathcal{U} \mid \mu_1(x) = 1\} = C.$$

Thus, every Crisp Set C is a special case of a MultiCrisp Set M when $k = 1$. Conversely, for $k > 1$, the MultiCrisp Set provides additional structure by encoding multiple membership conditions simultaneously, which the Crisp Set cannot represent. Therefore, the MultiCrisp Set strictly generalizes the Crisp Set. □

Theorem 2.67. *The MultiNeutrosophic Set generalizes the MultiCrisp Set.*

Proof. A MultiNeutrosophic Set M_N over a universe $\mathcal{U}$ is defined as:

$$M_N = \{x, x(T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s) \mid x \in \mathcal{U}\},$$

where:

- $T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s \subseteq [0, 1]$.
- At least one of $p, r, s \geq 2$, ensuring multiplicity in truth, indeterminacy, or falsehood components.
- The values satisfy:

$$0 \leq \sum_{j=1}^{p} \inf T_j + \sum_{k=1}^{r} \inf I_k + \sum_{l=1}^{s} \inf F_l \leq \sum_{j=1}^{p} \sup T_j + \sum_{k=1}^{r} \sup I_k + \sum_{l=1}^{s} \sup F_l \leq n,$$

  where $n = p + r + s$.

To show that the MultiNeutrosophic Set generalizes the MultiCrisp Set, consider the specific case where:

$$p = k, \quad r = 0, \quad s = 0, \quad T_i = \{\mu_i(x)\} \text{ for } i = 1, 2, \ldots, k.$$

Under this setup:

- Each truth component T_i of the MultiNeutrosophic Set corresponds directly to the crisp membership function $\mu_i(x)$ of the MultiCrisp Set, with $T_i \subseteq \{0, 1\}$.
- Indeterminacy and falsehood components I_k, F_l are absent, as $r = s = 0$.
- The MultiNeutrosophic Set reduces to:

  $$M_N = \{(x, T_1, T_2, \ldots, T_k) \mid x \in \mathcal{U}\},$$

  which is structurally equivalent to:

  $$M = \{(x, \mu_1(x), \mu_2(x), \ldots, \mu_k(x)) \mid x \in \mathcal{U}\}.$$

Thus, every MultiCrisp Set is a specific case of a MultiNeutrosophic Set when $r = s = 0$. Conversely, the MultiNeutrosophic Set introduces additional flexibility by allowing multiple values for truth, indeterminacy, and falsehood components, which the MultiCrisp Set cannot represent.

Therefore, the MultiNeutrosophic Set strictly generalizes the MultiCrisp Set. □

Furthermore, the MultiNeutrosophic Offset can also be defined as follows. This is a set concept that combines the principles of the Single-Valued Neutrosophic Offset and the MultiNeutrosophic Set. Similarly, the Multi-Neutrosophic Overset and MultiNeutrosophic Underset can be defined using the same approach; however, they are omitted here.

Definition 2.68 (Single-Valued MultiNeutrosophic Offset)**.** Let $\mathcal{U}$ be a universe of discourse. A *Single-Valued MultiNeutrosophic Offset (SVMNO)* is defined as:

$$SVMNO = \{\langle x, T_i, I_j, F_k \rangle \mid x \in \mathcal{U}, i = 1, \ldots, p; j = 1, \ldots, r; k = 1, \ldots, s\},$$

where:

- $T_i, I_j, F_k \in [\Psi, \Omega]$ for all i, j, k, where $\Psi < 0$ and $\Omega > 1$.
- $p, r, s \geq 1$, representing the number of evaluations for truth, indeterminacy, and falsehood, respectively.
- The normalization condition holds:

  $$\Psi \leq \sum_{i=1}^{p} T_i + \sum_{j=1}^{r} I_j + \sum_{k=1}^{s} F_k \leq \Omega.$$

Here, T_i represents the truth-membership degree, I_j represents the indeterminacy-membership degree, and F_k represents the falsehood-membership degree of x in the SVMNO.

Theorem 2.69. *A Single-Valued MultiNeutrosophic Offset (SVMNO) generalizes a Single-Valued Neutrosophic Offset (SVNO).*

Proof. Let $SVNO = \{\langle x, T, I, F \rangle \mid x \in \mathcal{U}\}$ with membership degrees $T, I, F \in [\Psi, \Omega]$ satisfying $\Psi \leq T + I + F \leq \Omega$.

Define $SVMNO = \{\langle x, T_i, I_j, F_k \rangle \mid x \in \mathcal{U}, i = 1, \ldots, p; j = 1, \ldots, r; k = 1, \ldots, s\}$, where $T_i, I_j, F_k \in [\Psi, \Omega]$ for all i, j, k and:

$$\Psi \leq \sum_{i=1}^{p} T_i + \sum_{j=1}^{r} I_j + \sum_{k=1}^{s} F_k \leq \Omega.$$

To show that $SVNO$ is a special case of $SVMNO$, consider the case where $p = r = s = 1$. Then, $SVMNO$ reduces to:

$$SVMNO = \{\langle x, T_1, I_1, F_1 \rangle \mid x \in \mathcal{U}\},$$

where $T_1 = T, I_1 = I, F_1 = F$. The normalization condition becomes:

$$\Psi \leq T_1 + I_1 + F_1 \leq \Omega,$$

which is identical to the normalization condition for $SVNO$.

Thus, $SVNO$ is a special case of $SVMNO$ when $p = r = s = 1$. Conversely, $SVMNO$ extends $SVNO$ by allowing multiple truth, indeterminacy, and falsehood evaluations, enabling more detailed representations of uncertainty.

Hence, $SVMNO$ generalizes $SVNO$. □

2.1.11 Soft Set and Soft Expert Set

The Soft Expert Set is known as one of the generalized concepts of the Soft Set. Comprehensive studies have been conducted on Soft Expert Sets and their applications, reinforcing their value in fields where nuanced expert input is crucial [44, 46, 70, 71, 745, 804].

Definition 2.70. [639] Let U be a universal set and E a set of parameters. A *soft set* over U is defined as an ordered pair (F, E), where F is a mapping from E to the power set $\mathcal{P}(U)$:

$$F : E \to \mathcal{P}(U).$$

For each parameter $e \in E$, $F(e) \subseteq U$ represents the set of e-approximate elements in U, with (F, E) forming a parameterized family of subsets of U.

Proposition 2.71. *A Soft Set is a generalization of a Crisp Set.*

Proof. A Crisp Set $A \subseteq U$ is defined by a characteristic function:

$$\chi_A : U \to \{0, 1\}.$$

A Soft Set (F, E) over a universal set U and a parameter set E is defined as:

$$F : E \to \mathcal{P}(U).$$

To demonstrate that a Crisp Set is a special case of a Soft Set, consider the following:

- Let the parameter set E contain a single parameter e, i.e., $E = \{e\}$.
- Define the mapping $F : E \to \mathcal{P}(U)$ such that $F(e) = A$, where $A \subseteq U$ is the given Crisp Set.

Under this setup:

- The Soft Set (F, E) reduces to $(F, \{e\})$, with $F(e) = A$.
- The characteristic function $\chi_A(x)$ of the Crisp Set is implicitly represented by the membership condition $x \in F(e)$, which evaluates to 1 if $x \in A$ and 0 otherwise.

Thus, the Soft Set (F, E) captures the structure of the Crisp Set when E contains only one parameter and $F(e)$ corresponds to the characteristic function $\chi_A(x)$.

Conversely, a Soft Set allows for multiple parameters and associates subsets of U with each parameter, thereby generalizing the concept of membership to a parameterized family of subsets. This flexibility goes beyond the binary membership of Crisp Sets and enables Soft Sets to handle uncertainty and approximation.

Therefore, the Soft Set is a generalization of the Crisp Set. □

Definition 2.72. [639] Let (F, A) and (G, B) be two soft sets over the same universe U, where $A \subseteq E$ and $B \subseteq E$. We say (F, A) is a *soft subset* of (G, B), denoted $(F, A) \subseteq (G, B)$, if the following conditions hold:

1. $A \subseteq B$,
2. $F(e) \subseteq G(e)$ for every $e \in A$.

Thus, (F, A) is considered a soft subset of (G, B) if all approximations in F are contained within the approximations in G for each parameter in A.

Definition 2.73 (Null Soft Set [639])**.** Let U be a universe of discourse, and let A be a non-empty set of parameters. A soft set (F, A) over U is called a *Null Soft Set*, denoted by Φ, if for every $\varepsilon \in A$, the subset $F(\varepsilon)$ of U is empty. Formally, we have:

$$(F, A) = \Phi \quad \text{if and only if} \quad F(\varepsilon) = \emptyset \quad \forall \varepsilon \in A.$$

Definition 2.74 (Full Soft Set)**.** [326] Let $S = (F, A)$ be a soft set over a universal set U, where F is a mapping $F : A \to \mathcal{P}(U)$, assigning each parameter $a \in A$ to a subset $F(a) \subseteq U$. The soft set S is called a *Full Soft Set* if:

$$\bigcup_{a \in A} F(a) = U.$$

This condition ensures that every element of U is included in at least one subset $F(a)$ for some $a \in A$.

Definition 2.75 (Soft Expert Set)**.** [70] Let U be a universe, E a set of parameters, X a set of experts, and O a set of possible opinions. Define $Z = E \times X \times O$ as the set of all parameter-expert-opinion combinations, and let $A \subseteq Z$.

A pair (F, A) is called a *Soft Expert Set* over the universe U, where F is a mapping given by:

$$F : A \to \mathcal{P}(U),$$

where $\mathcal{P}(U)$ denotes the power set of U. The mapping F associates each element in A, representing a combination of parameter, expert, and opinion, with a subset of U. Specifically, for a parameter $e \in E$, expert $x \in X$, and opinion $o \in O$, $F(e, x, o) \subseteq U$ represents the subset of U corresponding to this particular combination.

2.1.12 Neutrosophic Axial Set and Partner Multineutrosophic Set

The Neutrosophic Axial Set [458] and Partner Multineutrosophic Set [49, 50] have been recently proposed as related concepts of the Neutrosophic Set. The definitions of the Neutrosophic Axial Set and Partner Multineutrosophic Set are provided below.

Definition 2.76 (Discrete Subset)**.** A *discrete subset* of a topological space X is a subset $D \subseteq X$ such that every point in D is an isolated point relative to D. Specifically, for every $x \in D$, there exists an open set U in X such that $U \cap D = \{x\}$.

Definition 2.77 (Neutrosophic Axial Set). [458] Let X be a set. A *Neutrosophic Axial Set* $\mathfrak{NA}A$ is defined as:

$$\mathfrak{NA}A = \{\langle A, A_1, A_2\rangle \mid A \cap A_i = \emptyset,\ i = 1, 2\}$$

where A is any subset of X, and A_1 and A_2 are called the *parts* of $\langle A, A_1, A_2\rangle$.

For example, if we let $X = \mathbb{R}$ (the set of real numbers), then

$$\mathfrak{NA}(1,2) = \{\langle(1,2), A_1, A_2\rangle \mid (1,2) \cap A_i = \emptyset,\ i = 1, 2\}$$

where A_i could be either the empty set or a discrete subset of $\mathbb{R}$, for example:

$$A_i = \begin{cases} \emptyset \text{ or discrete subsets in } \mathbb{R} \setminus (1,2) \\ [2, x) \text{ for some } x \geq 2 \\ (y, 1] \text{ for some } y \leq 1 \end{cases}.$$

Definition 2.78 (Partner Multineutrosophic Set). [49, 50] Let X be a non-empty universal set, and let M_n be a multineutrosophic set represented by:

$$M_n = \{\langle x, T_1(x), \ldots, T_r(x), I_1(x), \ldots, I_s(x), F_1(x), \ldots, F_t(x)\rangle : x \in X,\ r + s + t = n\}$$

where:

- $T_i : X \to [0,1]$ for $i = 1, \ldots, r$ are the truth-membership degrees,
- $I_j : X \to [0,1]$ for $j = 1, \ldots, s$ are the indeterminacy-membership degrees, and
- $F_k : X \to [0,1]$ for $k = 1, \ldots, t$ are the falsity-membership degrees.

The set M_n is considered a *Partner Multineutrosophic Set* if it is associated with a fuzzy set M_n^p, termed the partner set of M_n, defined as:

$$M_n^p = \left\{\langle x, f_{M_n}(x)\rangle : x \in X\right\}$$

where $f_{M_n} : X \to [0,1]$ is defined by:

$$f_{M_n}(x) = \frac{1}{n}\left(\sum_{i=1}^{r} T_i(x) + \sum_{j=1}^{s} I_j(x) + \sum_{k=1}^{t} F_k(x)\right)$$

for all $x \in X$.

2.1.13 Meta Set

Meta sets extend fuzzy set theory, allowing elements to belong to sets with varying degrees of membership within a hierarchical structure [553, 554, 904–910]. In meta sets, each element is associated with multiple membership degrees that are represented through nodes in a binary tree, enabling complex hierarchical membership modeling.

Definition 2.79 (Infinite Binary Tree). [57, 187, 188] An *infinite binary tree* $T = (V, E)$ is a tree structure defined as follows:

- The set V of *vertices* consists of all finite binary sequences. Each vertex represents a unique sequence $v = (v_1, v_2, \ldots, v_n)$ where $v_i \in \{0, 1\}$ for all i and n is finite.
- The set E of *edges* consists of ordered pairs (u, v) where v is obtained from u by appending exactly one additional binary digit (either 0 or 1) to the end of u. Formally, if $u = (u_1, u_2, \ldots, u_n)$, then $v = (u_1, u_2, \ldots, u_n, b)$ with $b \in \{0, 1\}$.

Each vertex has exactly two children, corresponding to appending 0 and 1 respectively, and exactly one parent, obtained by removing the last binary digit. The root of T is the empty sequence $()$, denoted by ϵ.

Definition 2.80. Let X be a finite universe of discourse, and let T denote a full infinite binary tree. A *meta set* ρ over X is a collection of pairs $\langle x, p\rangle$, where:

$$\rho = \{\langle x, p\rangle \mid x \in X, p \in T \text{ and } \mu_\rho(x, p) \in [0, 1]\},$$

where $\mu_\rho : X \times T \to [0, 1]$ is a *membership function* mapping each pair (x, p) to a membership grade in the interval $[0, 1]$.

Proposition 2.81. *A meta set generalizes the concept of a fuzzy set. Specifically, every fuzzy set can be represented as a special case of a meta set where membership degrees are associated with a single node in a binary tree structure.*

Proof. Let A be a fuzzy set on a universe X with membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ denotes the membership degree of x in A.

To represent A as a meta set, define ρ_A over X using a full binary tree T with root node $\mathcal{R}$. Construct ρ_A such that

$$\rho_A = \{\langle x, \mathcal{R}\rangle \mid x \in X \text{ and } \mu_{\rho_A}(x, \mathcal{R}) = \mu_A(x)\},$$

where $\mu_{\rho_A}(x, \mathcal{R}) = \mu_A(x)$ for each $x \in X$. This assignment mirrors the membership structure of A within the meta set framework.

Since meta sets allow associations with multiple nodes in T, representing multi-level membership, the case where each element x is associated only with $\mathcal{R}$ reduces ρ_A to the single-level structure of a fuzzy set. Therefore, every fuzzy set can be viewed as a meta set where all memberships are confined to the root level, establishing that meta sets generalize fuzzy sets. □

2.1.14 Binary fuzzy set

A binary fuzzy set assigns each element in a set two membership degrees, reflecting distinct perspectives or dimensions of membership, typically within the range [958]. The definition is provided below.

Definition 2.82. [957, 958] A binary set B on a universe X is a mapping $\mu_B : X \to \{0, 1\}$ where each element $x \in X$ is assigned either 0 or 1. The value $\mu_B(x) = 1$ indicates that x is a member of the binary set B, and $\mu_B(x) = 0$ indicates that x is not a member of B.

A binary fuzzy set B on a universe X is defined by a pair of membership functions $\mu_{B_\rho}, \mu_{B_\nu} : X \to [0, 1]$ where each element $x \in X$ is assigned two degrees of membership. The function $\mu_{B_\rho}(x)$ indicates the degree of membership of x under one perspective, and $\mu_{B_\nu}(x)$ indicates the degree of membership of x under an alternative perspective. These mappings allow binary fuzzy sets to capture more nuanced, two-dimensional membership information.

Additionally, the definition of a Binary Neutrosophic Set is also well-known. The definition is provided below.

Definition 2.83 (Binary Neutrosophic Set (BNCS))**.** [954] Let (X, Y) be a non-empty fixed space, where X and Y are distinct sets. A *Binary Neutrosophic Set* (BNCS) C in (X, Y) is an ordered pair represented by

$$C = ((C_{11}, C_{12}, C_{13}), (C_{21}, C_{22}, C_{23})),$$

where $C_{11}, C_{12}, C_{13} \subseteq X$ and $C_{21}, C_{22}, C_{23} \subseteq Y$.

Each subset C_{ij} represents distinct elements related to the concept of truth, indeterminacy, and falsity, respectively, within the sets X and Y.

The binary neutrosophic set C can further be categorized into three types based on the properties of these subsets:

- *Type 1:* C is a BNCS-Type1 if it satisfies the conditions:

$$(C_{11} \cap C_{12}, C_{21} \cap C_{22}) = (\emptyset, \emptyset), \quad (C_{11} \cap C_{13}, C_{21} \cap C_{23}) = (\emptyset, \emptyset), \quad (C_{12} \cap C_{13}, C_{22} \cap C_{23}) = (\emptyset, \emptyset).$$

- *Type 2:* C is a BNCS-Type2 if it satisfies:

$$(C_{11} \cap C_{12}, C_{21} \cap C_{22}) = (\emptyset, \emptyset), \quad (C_{11} \cap C_{13}, C_{21} \cap C_{23}) = (\emptyset, \emptyset), \quad (C_{12} \cap C_{13}, C_{22} \cap C_{23}) = (\emptyset, \emptyset),$$

and

$$(C_{11} \cup C_{12} \cup C_{13}, C_{21} \cup C_{22} \cup C_{23}) = (X, Y).$$

- *Type 3:* C is a BNCS-Type3 if it satisfies:

$$(C_{11} \cap C_{12} \cap C_{13}, C_{21} \cap C_{22} \cap C_{23}) = (\emptyset, \emptyset),$$

and

$$(C_{11} \cup C_{12} \cup C_{13}, C_{21} \cup C_{22} \cup C_{23}) = (X, Y).$$

2.1.15 Cohesive Fuzzy Set

The Cohesive Fuzzy Set is a generalized concept that extends both the Complex Fuzzy Set [73, 273, 758, 759, 947] and the Hesitant Fuzzy Set [55, 780, 809, 942, 943, 995]. The definition is provided below.

Definition 2.84 (Cohesive Fuzzy Set (CHFS))**.** [996] Let S be a fixed universe of discourse and $T \subset S$ a fuzzy set defined over S. A *Cohesive Fuzzy Set* on T is defined by a membership function h_T that, when applied to each $x \in S$, returns a subset of the unit circle in the complex plane, representing the possible membership degrees of x in T.

For each $x \in S$, the membership degree $h_T(x)$ is expressed as a set of complex numbers in the form:

$$h_T(x) = \{r_T(x) \exp(iw_T(x)) : r_T(x) \in [0, 1],\ w_T(x) \in \mathbb{R}\},$$

where $r_T(x)$ represents the magnitude of membership, $w_T(x)$ represents the phase in radians, and $i = \sqrt{-1}$ denotes the imaginary unit.

The cohesive fuzzy set T is therefore represented as:

$$T = \{\langle x, h_T(x)\rangle : x \in S\}.$$

2.1.16 Ranked Soft Set

The Ranked Soft Set is known as a related concept to the Soft Set. Its definition is provided below [819].

Definition 2.85. [819] A *Ranked Partition* of the set U is an ordered collection of subsets $(V_0, V_1, \ldots, V_k)$ such that:

1. The subsets $V_i \subseteq U$ (where $i = 0, 1, \ldots, k$) are pairwise disjoint: $V_i \cap V_j = \emptyset$ for $i \neq j$.
2. The union of all subsets covers U: $V_0 \cup V_1 \cup \cdots \cup V_k = U$.
3. V_0 may be an empty subset, representing elements that do not satisfy any property within the context of the parameter to which the ranked partition applies. The subsets $V_1, V_2, \ldots, V_k$ represent increasingly higher ranks or levels of satisfaction or confidence.

The set of all ranked partitions of U is denoted as $\mathcal{R}(U)$.

Definition 2.86. A *Ranked Soft Set* (R, E) over the set U is a pair where R is a mapping from the parameter set E to the set of all ranked partitions of U. Formally, this is expressed as:

$$R : E \to \mathcal{R}(U),$$

where each parameter $t \in E$ is associated with a ranked partition $(V_0, V_1, \ldots, V_k)$ of U.

In a ranked soft set (R, E), each ranked partition $(V_0, V_1, \ldots, V_k)$ associated with a parameter t has the following interpretation:

- The subsets $V_i \subseteq U$ (for $i > 0$) represent increasing levels of satisfaction, confidence, or relevance for the parameter t.
- V_0, which may be empty, consists of elements that do not satisfy the parameter t.

Thus, for any element $o_j \in U$:

- $o_j \in V_i$ (with $i > 0$) implies that o_j satisfies the parameter t to a certain degree, with higher indices i indicating higher levels of satisfaction.
- Elements in V_0, if any, do not satisfy the parameter t.

The Ranked Soft Set model provides a qualitative hierarchy of satisfaction for each parameter, differing from traditional soft set models by representing satisfaction levels non-numerically through the structure of ranked partitions.

2.1.17 Bijective Soft Set

The definition of a Bijective Soft Set is provided below [426, 427, 599, 938, 939]. A Bijective Soft Set is a set concept that extends the Soft Set by incorporating the notion of a bijective mapping.

Definition 2.87 (Bijective Function)**.** (cf. [647,705,966,1037]) Let A and B be two sets. A function $f : A \to B$ is said to be *bijective* if it satisfies the following two conditions:

1. *Injective (One-to-one):* For every pair of elements $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. Formally:

$$\forall x_1, x_2 \in A, \quad f(x_1) = f(x_2) \implies x_1 = x_2.$$

2. *Surjective (Onto):* For every element $y \in B$, there exists at least one element $x \in A$ such that $f(x) = y$. Formally:

$$\forall y \in B, \quad \exists x \in A \text{ such that } f(x) = y.$$

If f is both injective and surjective, it is called *bijective*. In this case, f establishes a one-to-one correspondence between the elements of A and B, meaning every element of A is paired with a unique element of B, and every element of B is paired with exactly one element of A.

Definition 2.88 (Bijective Soft Set)**.** [427] Let U be a universe of discourse, and let B be a non-empty set of parameters. A pair (F, B) is called a *bijective soft set* over U if F is a mapping $F : B \to \mathcal{P}(U)$ that satisfies the following conditions:

- Exhaustivity: The union of all $F(e)$ for $e \in B$ covers U:

$$\bigcup_{e \in B} F(e) = U.$$

- Disjointness: For any two distinct parameters $e_i, e_j \in B$, where $e_i \neq e_j$, the subsets $F(e_i)$ and $F(e_j)$ are disjoint:

$$F(e_i) \cap F(e_j) = \emptyset.$$

In other words, a bijective soft set is a soft set in which each element of U is uniquely associated with exactly one parameter in B, ensuring that each subset $F(e)$ is distinct and collectively exhaustive.

2.2 Uncertain Graph Theory

In this subsection, we provide the necessary definitions and terms essential for the discussion.

2.2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [274, 275, 275, 279, 443, 985].

Definition 2.89 (Graph)**.** [279] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2.90 (Subgraph)**.** Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of G is a graph where:

$$V_H \subseteq V \quad \text{and} \quad E_H \subseteq E \cap (V_H \times V_H).$$

That is, H consists of a subset of the vertices and a subset of the edges of G, such that the edges in H are only between vertices in V_H.

Definition 2.91 (Degree)**.** [279] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 2.92 (Connected Graph)**.** Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of edges. G is said to be *connected* if for every pair of vertices $u, v \in V$, there exists a path in G that connects u and v. Formally:

$$\forall u, v \in V,\ u \neq v,\ \exists \text{path } P \text{ in } G \text{ such that } u, v \in P.$$

Definition 2.93 (Tree)**.** A graph $T = (V, E)$ is called a *tree* if it satisfies the following conditions:

1. T is connected.
2. T is acyclic (contains no cycles).

Equivalently, a tree is a connected graph with $|E| = |V| - 1$, where $|V|$ and $|E|$ denote the number of vertices and edges, respectively.

Definition 2.94 (Weighted Graph)**.** (cf. [171, 194, 228, 668, 850, 930, 951, 1011, 1032]) A *weighted graph* $G = (V, E, w)$ is a graph structure consisting of a set of vertices V, a set of edges $E \subseteq V \times V$, and a weight function $w : E \to \mathbb{R}$. Each edge $e \in E$ has an associated real number $w(e)$, representing the *weight* of the edge.

Formally, a weighted graph G is defined by the following components:

- V: The set of vertices (or nodes) in the graph.
- E: The set of edges, where each edge $e = (u, v) \in E$ represents a connection between two vertices $u, v \in V$.
- $w : E \to \mathbb{R}$: The weight function, which assigns a real-valued weight $w(e)$ to each edge $e \in E$.

The weight $w(e)$ can represent various attributes, such as distance, cost, or capacity, depending on the application context. If all edge weights are equal, the weighted graph reduces to an unweighted graph.

Example 2.95. An example of a weighted graph is as follows:

$$V = \{v_1, v_2, v_3\},$$
$$E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\},$$
$$w((v_1, v_2)) = 5, \quad w((v_2, v_3)) = 3, \quad w((v_3, v_1)) = 7.$$

In this example, the weight function w assigns specific weights to each edge in E.

Proposition 2.96. *A Weighted Graph is a generalization of a Classic Graph.*

Proof. This follows directly from the definition. □

2.2.2 Uncertain Graph

Approximately half a century has passed since the introduction of the Fuzzy Set, leading to the development of various graph concepts designed to handle uncertainty. Here, we provide definitions for frameworks including Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Single-Valued Pentapartitioned Neutrosophic.

A Fuzzy Graph is often studied in relation to a Crisp Graph [368]. To begin, we present the definition of a Crisp Graph below [368].

Definition 2.97 (Crisp Graph)**.** A *Crisp Graph* $G = (V, E)$ is defined as follows:

1. V: A non-empty finite set of vertices (or nodes).
2. $E \subseteq \{\{u, v\} \mid u, v \in V \text{ and } u \neq v\}$: A set of unordered pairs of vertices, called edges. Each edge is associated with exactly two vertices, referred to as its endpoints. An edge is said to connect its endpoints.

Special Cases.

- A graph G with $E = \emptyset$ is called an *edgeless graph*.

Definition 2.98 (Unified Uncertain Graphs Framework)**.** (cf. [373]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, falsity, and other nuanced measures of uncertainty.

1. *Fuzzy Graph [137, 394, 413, 569, 674, 716, 775, 786, 916, 981]:*
   - Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$.
   - Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$.
2. *Intuitionistic Fuzzy Graph (IFG) [19, 136, 239, 539, 679, 924, 950, 1031]:*
   - Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $\nu_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
   - Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ and $\nu_B(u, v) \in [0, 1]$, with $\mu_B(u, v) + \nu_B(u, v) \leq 1$.
3. *Neutrosophic Graph [29, 35, 169, 368, 376, 449, 514, 563, 808, 868, 892]:*
   - Each vertex $v \in V$ is assigned a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where $\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0, 1]$ and $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
   - Each edge $e = (u, v) \in E$ is assigned a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.

4. *Hesitant Fuzzy Graph [113, 429, 718, 725, 995]:*

   - Each vertex $v \in V$ is assigned a hesitant fuzzy set $\sigma(v)$, represented by a finite subset of $[0, 1]$, denoted $\sigma(v) \subseteq [0, 1]$.
   - Each edge $e = (u, v) \in E$ is assigned a hesitant fuzzy set $\mu(e) \subseteq [0, 1]$.
   - Operations on hesitant fuzzy sets (e.g., intersection, union) are defined to handle the hesitation in membership degrees.

5. *Quadripartitioned Neutrosophic Graph (QNG) [517–519, 823, 842]*:

   - Each vertex $v \in V$ is associated with a quadripartitioned neutrosophic membership, represented as $(T(v), C(v), U(v), F(v))$, where:
     - $T(v) \in [0, 1]$: truth-membership degree,
     - $C(v) \in [0, 1]$: contradiction-membership degree,
     - $U(v) \in [0, 1]$: ignorance-membership degree,
     - $F(v) \in [0, 1]$: false-membership degree,
     - with the condition $0 \leq T(v) + C(v) + U(v) + F(v) \leq 4$.
   - Each edge $e = (u, v) \in E$ in a QNG is also assigned a quadripartitioned membership $(T(e), C(e), U(e), F(e))$, where:

$$\begin{cases} T(e) \leq \min\{T(u), T(v)\}, \\ C(e) \leq \min\{C(u), C(v)\}, \\ U(e) \leq \max\{U(u), U(v)\}, \\ F(e) \leq \max\{F(u), F(v)\}, \end{cases}$$

   and the total membership degree of each edge satisfies $0 \leq T(e) + C(e) + U(e) + F(e) \leq 4$.

6. *Single-Valued Pentapartitioned Neutrosophic Graph [240, 516, 518, 751]:*

   - Each vertex $v \in V$ is assigned a quintuple $\sigma(v) = (T(v), C(v), R(v), U(v), F(v))$, where:
     - $T(v) \in [0, 1]$ is the truth-membership degree.
     - $C(v) \in [0, 1]$ is the contradiction-membership degree.
     - $R(v) \in [0, 1]$ is the ignorance-membership degree.
     - $U(v) \in [0, 1]$ is the unknown-membership degree.
     - $F(v) \in [0, 1]$ is the false-membership degree.
     - $T(v) + C(v) + R(v) + U(v) + F(v) \leq 5$.
   - Each edge $e = (u, v) \in E$ is assigned a quintuple $\mu(e) = (T(e), C(e), R(e), U(e), F(e))$, satisfying:

$$\begin{cases} T(e) \leq \min\{T(u), T(v)\}, \\ C(e) \leq \min\{C(u), C(v)\}, \\ R(e) \geq \max\{R(u), R(v)\}, \\ U(e) \geq \max\{U(u), U(v)\}, \\ F(e) \geq \max\{F(u), F(v)\}. \end{cases}$$

A Plithogenic Graph is a generalized graph based on the concept of a Plithogenic Set. The definition is provided below [864].

Definition 2.99. [425, 864, 865, 895, 919] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:

- $M \subseteq V$ is the set of vertices.
- l is an attribute associated with the vertices.
- Ml is the range of possible attribute values.
- $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
- $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
   - $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x,a), (y,b) \in M \times Ml$:

$$bdf\,((xy),(a,b)) \leq \min\{adf(x,a), adf(y,b)\}$$

   where $xy \in N$ is an edge between vertices x and y, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.

2. *Contradiction Function Constraint*: For all $(a,b),(c,d) \in Nm \times Nm$:

$$bCf\,((a,b),(c,d)) \leq \min\{aCf(a,c), aCf(b,d)\}$$

3. *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned}
aCf(a,a) &= 0, && \forall a \in Ml \\
aCf(a,b) &= aCf(b,a), && \forall a,b \in Ml \\
bCf(a,a) &= 0, && \forall a \in Nm \\
bCf(a,b) &= bCf(b,a), && \forall a,b \in Nm
\end{aligned}$$

Example 2.100. (cf. [368, 373]) The following examples are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic Graph*.
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic Graph*.

The plithogenic graph encompasses various graph types that have been actively studied. This graph concept is capable of handling multiple layers of uncertainty while generalizing numerous existing graph concepts. Its flexibility allows for selecting different graph types depending on the research objectives or practical applications, making it mathematically significant and versatile. Furthermore, it is anticipated that practical applications and further studies on its utility will be explored in the future.

Definition 2.101 (Plithogenic Graph Type)**.** (cf. [368, 373]) A plithogenic graph PG is a graph that satisfies one of the following types of plithogenic characteristics (referred to as PG of the i-th type) or any combination thereof:

(i) $PG_1 = \{G_1, G_2, G_3, \ldots, G_P\}$ where plithogenic characteristics exist in each graph G_i, incorporating different attributes and degrees of appurtenance and contradiction for vertices and edges.

(ii) $PG_2 = \{V, E_P\}$ where the edge set E_P is plithogenic, meaning that each edge is associated with a range of possible attributes and corresponding degrees of appurtenance and contradiction.

(iii) $PG_3 = \{V, E(t_P, h_P)\}$ where both the vertex set V and edge set E are crisp, but the edges have plithogenic heads $h(e_i)$ and plithogenic tails $t(e_i)$ with respect to certain attributes.

(iv) $PG_4 = \{V_P, E\}$ where the vertex set V_P is plithogenic, meaning each vertex has attributes with varying degrees of appurtenance and contradiction.

(v) $PG_5 = \{V, E(w_P)\}$ where both the vertex set V and edge set E are crisp, but the edges have plithogenic weights w_P, indicating the attribute-based degrees of appurtenance and contradiction.

The General Plithogenic Graph is a generalization of the Plithogenic Graph (cf. [340, 368, 700]). The General Plithogenic Graph relaxes certain conditions, such as the Edge Appurtenance Constraint.

By incorporating constraints from Pythagorean fuzzy sets [232,233], spherical fuzzy sets [68,94,329,451,452, 637,649,967], (m, n)-fuzzy sets [48,933], and q-rung orthopair fuzzy sets [60,262,624,626,740,761,971,998], we hope to explore new mathematical characteristics and applications, such as in decision-making and other domains.

Definition 2.102 (General Plithogenic Graph)**.** [368] Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges.

A General Plithogenic Graph $G^{GP} = (PM, PN)$ consists of:

1. *General Plithogenic Vertex Set PM*:

$$PM = (M, l, Ml, adf, aCf)$$

where:

- $M \subseteq V$: Set of vertices.
- l: Attribute associated with the vertices.
- Ml: Range of possible attribute values.
- $adf : M \times Ml \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for vertices.
- $aCf : Ml \times Ml \to [0, 1]^t$: Degree of Contradiction Function (DCF) for vertices.

2. *General Plithogenic Edge Set PN*:

$$PN = (N, m, Nm, bdf, bCf)$$

where:

- $N \subseteq E$: Set of edges.
- m: Attribute associated with the edges.
- Nm: Range of possible attribute values.
- $bdf : N \times Nm \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for edges.
- $bCf : Nm \times Nm \to [0, 1]^t$: Degree of Contradiction Function (DCF) for edges.

The General Plithogenic Graph G^{GP} only needs to satisfy the following *Reflexivity and Symmetry* properties of the Contradiction Functions:

- Reflexivity and Symmetry of Contradiction Functions:

$$\begin{aligned} aCf(a,a) &= 0, & \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), & \forall a,b \in Ml \\ bCf(a,a) &= 0, & \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), & \forall a,b \in Nm \end{aligned}$$

In graphs dealing with uncertainty, the following has been established [368].

Theorem 2.103. *[368] In each graph class, the following relationships hold.*

- *An empty graph and a null graph can be represented as 2-valued graphs and 3-valued graphs.*
- *Every edge-fuzzy graph can be transformed into a 2-valued graph by thresholding the edge membership values.*
- *Every fuzzy graph can be transformed into a 3-valued graph by mapping the fuzzy membership values of vertices and edges to the values {-1, 0, 1}.*
- *Every Intuitionistic Fuzzy Graph can be transformed into a Fuzzy Graph by restricting the non-membership function ν_A to 0 for all vertices.*
- *Every Neutrosophic Graph can be transformed into an Intuitionistic Fuzzy Graph by setting the indeterminacy value to zero.*
- *Every Extended Turiyam Neutrosophic Graph is a generalization of the Turiyam Neutrosophic Graph.*
- *A plithogenic Graphs generalize Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, Extended Turiyam Neutrosophic Graphs.*
- *Every general plithogenic Graphs can be transformed into General Turiyam Neutrosophic Graph, General Fuzzy Graph, General Intuitionistic Fuzzy Graph, Four-Valued Fuzzy graph, Ambiguous graph, Picture Fuzzy Graph, Hesitant Fuzzy Graph, Intuitionistic Hesitant Fuzzy Graph, Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Quadripartitioned Neutrosophic graph, Pentapartitioned Neutrosophic graph, Turiyam Neutrosophic Graphs, and Spherical Fuzzy Graphs.*

2.2.3 Soft Graph and Multisoft Graph

This subsection explains the concept of a Soft Set. A soft set over a universe U assigns subsets of U to parameters from a parameter set A, allowing for flexible representations of uncertain data. The formal definition of a Soft Set is provided below [56, 59, 639, 670, 1000]. For readers interested in details on operations within Soft Set theory, please refer to [670] as needed.

Definition 2.104. [670] Let U be a non-empty finite set, called the *universe* of discourse, and let E be a non-empty set of parameters. A *soft set* over U is defined as follows:

$$F = (F, A) \text{ over } U \text{ is an ordered pair, where } A \subseteq E \text{ and } F : A \to P(U),$$

where $F(a) \subseteq U$ for each $a \in A$ and $P(U)$ denotes the power set of U. The set of all soft sets over U is denoted by $S(U)$.

1. *Soft Subset:* Let $F = (F, A)$ and $G = (G, B)$ be two soft sets over the common universe U. We say that F is a soft subset of G, denoted $F \subseteq G$, if:
   - $A \subseteq B$
   - $F(a) \subseteq G(a)$ for all $a \in A$.

2. *Union of Soft Sets:* The union of two soft sets $F = (F, A)$ and $G = (G, B)$ over U is defined as $H = (H, C)$ where $C = A \cup B$ and

$$H(e) = \begin{cases} F(e), & e \in A - B \\ G(e), & e \in B - A \\ F(e) \cup G(e), & e \in A \cap B \end{cases}$$

3. *Intersection of Soft Sets:* The intersection of two soft sets $F = (F, A)$ and $G = (G, B)$ with disjoint parameter sets $A \cap B = \emptyset$ is defined as $H = (H, C)$, where $C = A \cap B$ and

$$H(e) = F(e) \cap G(e), \quad \forall e \in C.$$

The Soft Graph is a concept that represents the "graphization" of a Soft Set, and discussions on this concept frequently consider its relationship to related structures such as fuzzy graphs and Neutrosophic graphs [30, 31, 82, 106, 110, 476, 515, 549, 948]. The definition is provided below.

Definition 2.105 (Soft Graph)**.** [30, 476] Let $G^* = (V, E)$ be a simple graph, where V represents the vertex set and E represents the edge set of G^*. Let A be a nonempty set of parameters associated with G^*. Define two soft sets (F, A) and (K, A), where:

1. (F, A) is a soft set over V, representing the vertices associated with each parameter in A.
2. (K, A) is a soft set over E, representing the edges associated with each parameter in A.

Then, a *soft graph* $G = (G^*, F, K, A)$ is defined as the 4-tuple satisfying the following conditions:

1. $G^* = (V, E)$ is a simple graph.
2. A is a nonempty set of parameters.
3. (F, A) is a soft set over V.
4. (K, A) is a soft set over E.
5. For each $a \in A$, the pair $(F(a), K(a))$ forms a subgraph of G^*, which we denote by $H(a)$.

A soft graph can thus be represented as:

$$G = (F, K, A) = \{H(a) \mid a \in A\}.$$

The set of all soft graphs over G^* is denoted by $\mathcal{S}(G^*)$.

In recent years, the concept of the Multisoft Set has been defined, and its definition, as well as an extension to graphs, is presented below. Related concepts to the Multisoft Set include IndetermSoft Sets [880, 882], IndetermHyperSoft Sets [880, 882], and TreeSoft Sets [270, 719, 880, 882, 887].

Definition 2.106. [72, 412, 875] Let U be a universal set, and let E_i represent distinct sets of parameters such that $E_i \cap E_j = \emptyset$ for $i \neq j$. Define the union of all parameters as $E = \bigcup_i E_i$, and let $\mathcal{P}(U)$ denote the power set of U.

A Multisoft Set over a universal set U is a pair (F, A), where:

- $A \subseteq \mathcal{P}(E)$ is a set of attribute combinations, and
- $F : A \to \mathcal{P}(U)$ is a function that maps each combination of attributes in A to a subset of U.

For each $a \in A$, the set $F(a) \subseteq U$ is referred to as the a-approximate set of the multisoft set (F, A).

Definition 2.107. A *Multisoft Graph* $G = (G^*, F, K, A)$ over a graph $G^* = (V, E)$ consists of the following components:

1. $G^* = (V, E)$ is the underlying simple graph.
2. A is a set of attribute combinations where $A \subseteq \mathcal{P}(E)$.
3. (F, A) is a multisoft set over V, where $F : A \to \mathcal{P}(V)$ maps each combination of attributes in A to a subset of vertices V.
4. (K, A) is a multisoft set over E, where $K : A \to \mathcal{P}(E)$ maps each combination of attributes in A to a subset of edges E.

For each $a \in A$, the pair $(F(a), K(a))$ forms a subgraph of G^*, which we denote by $H(a)$. The multisoft graph G can thus be represented as:

$$G = (F, K, A) = \{H(a) \mid a \in A\}.$$

The set of all multisoft graphs over G^* is denoted by $\mathcal{MS}(G^*)$.

Proposition 2.108. *Every Multisoft Graph $G = (G^*, F, K, A)$ over a simple graph $G^* = (V, E)$ can be transformed into an equivalent Soft Graph $G' = (G^*, F', K', A')$.*

Proof. To demonstrate that any multisoft graph $G = (G^*, F, K, A)$ can be transformed into a soft graph, we proceed as follows:

1. *Define a Soft Graph Parameter Set A'*: Construct A' as a parameter set corresponding to the set of attribute combinations A in the multisoft graph. Specifically, define $A' = A$.

2. *Construct Soft Sets (F', A') and (K', A')*: For each $a \in A'$, define $F'(a)$ as the union of all subsets $F(a')$ for every $a' \subseteq a$ in A. Formally, we have:

$$F'(a) = \bigcup_{\substack{a' \subseteq a \\ a' \in A}} F(a').$$

Similarly, for each $a \in A'$, define $K'(a)$ as the union of all subsets $K(a')$ for every $a' \subseteq a$ in A:

$$K'(a) = \bigcup_{\substack{a' \subseteq a \\ a' \in A}} K(a').$$

3. *Verification of Conditions for a Soft Graph*: By construction, (F', A') is a soft set over V with respect to the parameter set A', and (K', A') is a soft set over E. Since $F'(a) \subseteq V$ and $K'(a) \subseteq E$ for all $a \in A'$, the conditions of a soft graph are satisfied.

Thus, we have constructed a soft graph $G' = (G^*, F', K', A')$ that is equivalent to the original multisoft graph $G = (G^*, F, K, A)$, completing the proof. □

The Soft Graph is often discussed alongside other types, such as Fuzzy Graphs and Neutrosophic Graphs. For example, the Fuzzy Soft Graph [82, 89, 106, 202, 515, 851] and the Neutrosophic Soft Graph [36, 167, 835, 948] are generalized extensions of the Soft Graph concept. This paper focuses on the Neutrosophic Soft Graph, and thus, its formal definition is provided below.

Definition 2.109. [36, 167, 835, 948] *Neutrosophic Soft Graph* is defined as a 4-tuple $G = (G^*, J, K, A)$, where:

- $G^* = (V, E)$ is a *neutrosophic graph*, where:
  - V is the set of vertices.
  - $E \subseteq V \times V$ is the set of edges.

– For each vertex $x \in V$, there exist three functions:

$$T : V \to [0, 1], \quad I : V \to [0, 1], \quad F : V \to [0, 1],$$

representing the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively.

– For each vertex $x \in V$, the constraint holds:

$$0 \leq T(x) + I(x) + F(x) \leq 3.$$

- A is a non-empty set of parameters.
- (J, A) is a *neutrosophic soft set* over the vertex set V, where $J : A \to \rho(V)$, and $\rho(V)$ denotes the set of all neutrosophic sets of V.
- (K, A) is a *neutrosophic soft set* over the edge set E, where $K : A \to \rho(E)$, and $\rho(E)$ denotes the set of all neutrosophic sets of E.

Proposition 2.110. *The Neutrosophic Soft Graph generalizes the Soft Graph.*

Proof. If the neutrosophic memberships $T(x)$, $I(x)$, and $F(x)$ are replaced with a single crisp membership value (e.g., $T(x) = 1, I(x) = 0, F(x) = 0$), the neutrosophic graph G^* reduces to a crisp graph. If (J, A) and (K, A) are defined over the power sets $\rho(V)$ and $\rho(E)$, respectively, instead of the neutrosophic sets, the neutrosophic soft sets reduce to classical soft sets. The conditions and mappings in G_N encompass those of G_S, making G_S a special case of G_N.

Thus, the Neutrosophic Soft Graph G_N generalizes the Soft Graph G_S. □

2.2.4 Neutrosophic OverGraph, UnderGraph, and OffGraph

Recently, OverGraphs [263, 264, 638], UnderGraphs, and OffGraphs [266] have been defined to graphically represent the concepts of Overset, Underset, and Offset. The following sections introduce these definitions.

Definition 2.111. (cf. [263,264,638]) A *Single-Valued Neutrosophic OverGraph* is a graph $G = (V, E)$ defined over a universe of discourse U_{over}, where:

- Each vertex $v \in V$ is assigned a truth-membership degree $T(v)$, an indeterminacy-membership degree $I(v)$, and a falsity-membership degree $F(v)$, such that $T(v) \in [0, \Omega]$ with $\Omega > 1$, allowing $T(v) > 1$.
- Each edge $e = (u, v) \in E$ is similarly assigned degrees $T(e) \in [0, \Omega]$, $I(e) \in [0, \Omega]$, and $F(e) \in [0, \Omega]$.
- For all $v \in V$, $T(v) + I(v) + F(v) \leq 3\Omega$.

Definition 2.112. A *Single-Valued Neutrosophic UnderGraph* is a graph $G = (V, E)$ defined over a universe of discourse U_{under}, where:

- Each vertex $v \in V$ is assigned degrees $T(v)$, $I(v)$, and $F(v)$, with $F(v) \in [\Psi, 1]$ where $\Psi < 0$, allowing $F(v) < 0$.
- Each edge $e = (u, v) \in E$ is assigned degrees $T(e) \in [\Psi, 1]$, $I(e) \in [\Psi, 1]$, and $F(e) \in [\Psi, 1]$.
- For all $v \in V$, $T(v) + I(v) + F(v) \leq 3$.

Definition 2.113. (cf. [266]) A *Single-Valued Neutrosophic OffGraph* is a graph $G = (V, E)$ defined over a universe U_{off}, where:

- Each vertex $v \in V$ is assigned degrees $T(v)$, $I(v)$, and $F(v)$, with $T(v) \in [0, \Omega]$ and $F(v) \in [\Psi, \Omega]$, where $\Omega > 1$ and $\Psi < 0$, allowing $T(v) > 1$ and $F(v) < 0$.

- Each edge $e = (u, v) \in E$ is assigned degrees $T(e) \in [\Psi, \Omega]$, $I(e) \in [\Psi, \Omega]$, and $F(e) \in [\Psi, \Omega]$.
- For all $v \in V$, $T(v) + I(v) + F(v) \leq 3\Omega$.

Proposition 2.114. *In a Single-Valued Neutrosophic OverGraph $G = (V, E)$, if $\Omega = 1$, then G becomes a standard Single-Valued Neutrosophic Graph.*

Proof. Setting $\Omega = 1$ restricts membership degrees to the interval $[0, 1]$. Therefore, G conforms to the standard Single-Valued Neutrosophic Graph definition. □

Proposition 2.115. *In a Single-Valued Neutrosophic UnderGraph $G = (V, E)$, if $\Psi = 0$, then G becomes a standard Single-Valued Neutrosophic Graph.*

Proof. When $\Psi = 0$, the membership degrees are constrained to $[0, 1]$, disallowing values less than 0. This removes the "under" characteristic of the graph, making it identical to a standard Single-Valued Neutrosophic Graph. □

Proposition 2.116. *In a Single-Valued Neutrosophic OffGraph $G = (V, E)$, if $\Omega = 1$ and $\Psi = 0$, then G becomes a standard Single-Valued Neutrosophic Graph.*

Proof. Setting $\Omega = 1$ and $\Psi = 0$ confines all membership degrees $T(v)$, $I(v)$, $F(v)$ within $[0, 1]$, disallowing any values outside this range. Thus, the "off" aspect, which allows degrees to be greater than 1 or less than 0, is eliminated, and G becomes a standard Single-Valued Neutrosophic Graph. □

Proposition 2.117. *In a Single-Valued Neutrosophic OverGraph $G = (V, E)$, the sum $T(v) + I(v) + F(v)$ can exceed 3 when $\Omega > 1$.*

Proof. Obviously holds. □

Definition 2.118 (Fuzzy Overgraph)**.** A *Fuzzy Overgraph* $G = (V, E, \mu_V, \mu_E)$ consists of:

- A set of vertices V.
- A set of edges $E \subseteq V \times V$.
- A vertex membership function $\mu_V : V \to [0, \Omega]$, with $\Omega > 1$ and $\exists v \in V$ such that $\mu_V(v) > 1$.
- An edge membership function $\mu_E : E \to [0, \Omega]$, where $\exists e \in E$ such that $\mu_E(e) > 1$.

Definition 2.119 (Fuzzy Undergraph)**.** A *Fuzzy Undergraph* $G = (V, E, \mu_V, \mu_E)$ consists of:

- A set of vertices V.
- A set of edges $E \subseteq V \times V$.
- A vertex membership function $\mu_V : V \to [\Psi, 1]$, with $\Psi < 0$ and $\exists v \in V$ such that $\mu_V(v) < 0$.
- An edge membership function $\mu_E : E \to [\Psi, 1]$, where $\exists e \in E$ such that $\mu_E(e) < 0$.

Definition 2.120 (Fuzzy Offgraph)**.** A *Fuzzy Offgraph* $G = (V, E, \mu_V, \mu_E)$ consists of:

- A set of vertices V.
- A set of edges $E \subseteq V \times V$.
- A vertex membership function $\mu_V : V \to [\Psi, \Omega]$, with $\Omega > 1$, $\Psi < 0$, and $\exists v \in V$ such that $\mu_V(v) > 1$ or $\mu_V(v) < 0$.
- An edge membership function $\mu_E : E \to [\Psi, \Omega]$, where $\exists e \in E$ such that $\mu_E(e) > 1$ or $\mu_E(e) < 0$.

Definition 2.121 (Intuitionistic Fuzzy Overgraph)**.** An *Intuitionistic Fuzzy Overgraph* $G = (V, E, \mu_V, \nu_V, \mu_E, \nu_E)$ consists of:

- A set of vertices V.
- A set of edges $E \subseteq V \times V$.
- Vertex membership $\mu_V : V \to [0, \Omega]$ and non-membership $\nu_V : V \to [0, \Omega]$ functions.
- Edge membership $\mu_E : E \to [0, \Omega]$ and non-membership $\nu_E : E \to [0, \Omega]$ functions.

where $\Omega > 1$, and at least one $\mu_V(v) > 1$, $\nu_V(v) > 1$, $\mu_E(e) > 1$, or $\nu_E(e) > 1$.

Definition 2.122 (Intuitionistic Fuzzy Undergraph)**.** An *Intuitionistic Fuzzy Undergraph* $G = (V, E, \mu_V, \nu_V, \mu_E, \nu_E)$ consists of:

- A set of vertices V.
- A set of edges $E \subseteq V \times V$.
- Vertex membership $\mu_V : V \to [\Psi, 1]$ and non-membership $\nu_V : V \to [\Psi, 1]$ functions.
- Edge membership $\mu_E : E \to [\Psi, 1]$ and non-membership $\nu_E : E \to [\Psi, 1]$ functions.

where $\Psi < 0$, and at least one $\mu_V(v) < 0$, $\nu_V(v) < 0$, $\mu_E(e) < 0$, or $\nu_E(e) < 0$.

Definition 2.123 (Intuitionistic Fuzzy Offgraph)**.** An *Intuitionistic Fuzzy Offgraph* $G = (V, E, \mu_V, \nu_V, \mu_E, \nu_E)$ consists of:

- A set of vertices V.
- A set of edges $E \subseteq V \times V$.
- Vertex membership $\mu_V : V \to [\Psi, \Omega]$ and non-membership $\nu_V : V \to [\Psi, \Omega]$ functions.
- Edge membership $\mu_E : E \to [\Psi, \Omega]$ and non-membership $\nu_E : E \to [\Psi, \Omega]$ functions.

where $\Omega > 1$ and $\Psi < 0$, allowing degrees that can exceed 1 or be less than 0.

Proposition 2.124. *The Neutrosophic Undergraph generalizes both the Fuzzy Undergraph and the Intuitionistic Fuzzy Undergraph.*

Proof. The proof can be constructed in a manner analogous to the set-based case. □

Proposition 2.125. *The Neutrosophic Overgraph generalizes both the Fuzzy Overgraph and the Intuitionistic Fuzzy Overgraph.*

Proof. The proof can be constructed in a manner analogous to the set-based case. □

Proposition 2.126. *The Neutrosophic Offgraph generalizes both the Fuzzy Offgraph and the Intuitionistic Fuzzy Offgraph.*

Proof. The proof can be constructed in a manner analogous to the set-based case. □

Proposition 2.127. *A Single-Valued Neutrosophic OffGraph can be transformed into a Single-Valued Neutrosophic OverGraph or a Single-Valued Neutrosophic UnderGraph.*

Proof. The proof can be constructed in a manner analogous to the set-based case. □

2.2.5 Neutrosophic Soft offgraph/overgraph/undergraph

The Neutrosophic Soft Offgraph/Overgraph/Undergraph is an extension of the Neutrosophic Offgraph/Overgraph/Undergraph. The definitions are provided below. Note that the Neutrosophic Soft Overgraph was originally defined in [92].

Definition 2.128 (Neutrosophic Soft Over Graph (NSOG))**.** [92] Let $G = (V, E)$ be a classical graph where V represents the set of vertices and $E \subseteq V \times V$ represents the set of edges. A *Neutrosophic Soft Over Graph (NSOG)*, denoted $G_{NSOG} = (F_V, F_E, T, I, F)$, is a graph where each vertex and edge is assigned neutrosophic soft oversets. These oversets capture neutrosophic information with the possibility of exceeding the traditional boundaries (i.e., values greater than 1), allowing a flexible representation of uncertain, indeterminate, and contradictory information.

The structure of G_{NSOG} is defined as follows:

1. F_V and F_E are neutrosophic soft oversets associated with the vertices V and edges E, respectively:

$$F_V(v) = \{(v, T(v), I(v), F(v)) \mid v \in V\}, \quad F_E(e) = \{(e, T(e), I(e), F(e)) \mid e \in E\},$$

   where:

   - $T(v)$, $I(v)$, and $F(v)$ represent the truth-membership, indeterminacy-membership, and falsity-membership degrees for each vertex $v \in V$.
   - $T(e)$, $I(e)$, and $F(e)$ represent the corresponding degrees for each edge $e \in E$.
   - Each degree $T(x)$, $I(x)$, and $F(x)$ (for $x = v$ or e) can take values in the extended interval $[0, \Omega]$, where $\Omega > 1$, allowing degrees to exceed the conventional bound of 1.

2. For all $v \in V$ and $e \in E$, the sum of the degrees satisfies:

$$T(v) + I(v) + F(v) \leq 3\Omega, \quad T(e) + I(e) + F(e) \leq 3\Omega,$$

   maintaining an upper bound on the combined neutrosophic components for each vertex and edge.

3. The NSOG is termed *pure* if each vertex $v \in V$ and each edge $e \in E$ has at least one degree that exceeds the standard maximum of 1:

$$T(v) > 1 \quad \text{or} \quad I(v) > 1 \quad \text{or} \quad F(v) > 1,$$

   and similarly,

$$T(e) > 1 \quad \text{or} \quad I(e) > 1 \quad \text{or} \quad F(e) > 1.$$

In this structure, the Neutrosophic Soft Over Graph G_{NSOG} extends the concept of a classical graph by incorporating neutrosophic soft oversets for both vertices and edges, where at least one of the truth, indeterminacy, or falsity values may surpass 1, thus enhancing the graph's flexibility in handling various levels of uncertainty, indeterminacy, and opposition.

Definition 2.129 (Neutrosophic Soft UnderGraph)**.** A *Neutrosophic Soft UnderGraph* is a graph $G = (G^*, J, K, A)$, where:

- $G^* = (V, E)$ is a *neutrosophic graph* with the following membership functions for each vertex $x \in V$:

$$T : V \to [\Psi, 1], \quad I : V \to [\Psi, 1], \quad F : V \to [\Psi, 1],$$

  where $\Psi < 0$ allows falsity degrees to be less than 0.

- A is a non-empty set of parameters.

- (J, A) is a neutrosophic soft set over the vertex set V, where $J : A \to \rho(V)$, and $\rho(V)$ denotes the set of all neutrosophic sets of V.

- (K, A) is a neutrosophic soft set over the edge set E, where $K : A \to \rho(E)$.

The sum of degrees for any vertex $x \in V$ satisfies:

$$T(x) + I(x) + F(x) \leq 3.$$

Definition 2.130 (Neutrosophic Soft OffGraph)**.** A *Neutrosophic Soft OffGraph* is a graph $G = (G^*, J, K, A)$, where:

- $G^* = (V, E)$ is a *neutrosophic graph* with the following membership functions for each vertex $x \in V$:

$$T : V \to [\Psi, \Omega], \quad I : V \to [\Psi, \Omega], \quad F : V \to [\Psi, \Omega],$$

  where $\Omega > 1$ allows truth degrees to exceed 1, and $\Psi < 0$ allows falsity degrees to be less than 0.
- A is a non-empty set of parameters.
- (J, A) is a neutrosophic soft set over the vertex set V, where $J : A \to \rho(V)$, and $\rho(V)$ denotes the set of all neutrosophic sets of V.
- (K, A) is a neutrosophic soft set over the edge set E, where $K : A \to \rho(E)$.

The sum of degrees for any vertex $x \in V$ satisfies:

$$T(x) + I(x) + F(x) \leq 3\Omega.$$

Proposition 2.131. *A Neutrosophic Soft OffGraph can be transformed into a standard Neutrosophic OffGraph by collapsing the soft parameters into a single, unified membership value over the vertex and edge sets.*

Proof. Let $G = (G^*, J, K, A)$ be a Neutrosophic Soft OffGraph, where:

- $G^* = (V, E)$ is a neutrosophic graph with membership functions $T(x) \in [0, \Omega]$, $I(x) \in [0, 1]$, and $F(x) \in [\Psi, 1]$ for each $x \in V \cup E$.
- (J, A) and (K, A) are neutrosophic soft sets over V and E, respectively.

Define the transformed graph $G' = (V, E)$ as a Neutrosophic OffGraph by setting each vertex $v \in V$ and each edge $e \in E$ to have the following unified membership functions:

$$T'(v) = \sup_{a \in A} T_{J(a)}(v), \quad I'(v) = \sup_{a \in A} I_{J(a)}(v), \quad F'(v) = \inf_{a \in A} F_{J(a)}(v),$$

where $T_{J(a)}(v)$, $I_{J(a)}(v)$, and $F_{J(a)}(v)$ are the membership degrees from the soft set $J(a)$ on v. A similar transformation applies for each edge $e \in E$.

This construction ensures that G' retains the core properties of G with unified membership degrees, thus transforming the soft OffGraph structure into a standard Neutrosophic OffGraph. □

Proposition 2.132. *A Neutrosophic Soft OverGraph can be transformed into a standard Neutrosophic OverGraph by consolidating the parameterized degrees.*

Proof. Given a Neutrosophic Soft OverGraph $G = (G^*, J, K, A)$, where membership degrees in (J, A) and (K, A) may exceed 1, construct $G' = (V, E)$ as follows:

$$T'(v) = \sup_{a \in A} T_{J(a)}(v), \quad I'(v) = \sup_{a \in A} I_{J(a)}(v), \quad F'(v) = \inf_{a \in A} F_{J(a)}(v).$$

This transformation results in a standard Neutrosophic OverGraph where the truth-membership can exceed the standard bound 1, satisfying the criteria for an OverGraph. □

Proposition 2.133. *A Neutrosophic Soft UnderGraph can be transformed into a standard Neutrosophic UnderGraph by reducing the parameters to single values.*

Proof. Let $G = (G^*, J, K, A)$ be a Neutrosophic Soft UnderGraph with each $J(a)$ and $K(a)$ containing non-membership degrees below zero. Define the transformed Neutrosophic UnderGraph $G' = (V, E)$ by setting:

$$T'(v) = \sup_{a \in A} T_{J(a)}(v), \quad I'(v) = \sup_{a \in A} I_{J(a)}(v), \quad F'(v) = \inf_{a \in A} F_{J(a)}(v),$$

where $F_{J(a)}(v) < 0$. This results in a single Neutrosophic UnderGraph that mirrors the original soft graph structure with combined membership degrees. □

Proposition 2.134. *A Neutrosophic Soft OffGraph, OverGraph, or UnderGraph can be transformed into a Soft Graph by disregarding the neutrosophic attributes and only using the soft set structure over parameters.*

Proof. For any Neutrosophic Soft OffGraph, OverGraph, or UnderGraph $G = (G^*, J, K, A)$, we can create a Soft Graph $G' = (V, E, A)$ by defining soft sets (J, A) and (K, A) without the neutrosophic components. Each vertex $v \in V$ and edge $e \in E$ is thus represented by soft sets without the need for neutrosophic membership degrees. This construction yields a Soft Graph G' that represents the parameterized structure without neutrosophic complexity. □

Proposition 2.135. *A Neutrosophic Soft OffGraph can be transformed into a Neutrosophic Soft OverGraph or a Neutrosophic Soft UnderGraph.*

Proof. The proof can be constructed in a manner analogous to the set-based case. □

Theorem 2.136. *Neutrosophic Soft OffGraph, OverGraph, and UnderGraph generalize the concept of a Soft Graph. Specifically:*

- *A Neutrosophic Soft OffGraph generalizes a Soft Graph by incorporating neutrosophic truth, indeterminacy, and falsity degrees over an extended range* $[\Psi, \Omega]$*, where* $\Psi < 0$ *and* $\Omega > 1$.
- *A Neutrosophic Soft OverGraph generalizes a Soft Graph by allowing truth and other membership degrees to exceed the standard maximum of 1 (*$T, I, F \in [0, \Omega]$*,* $\Omega > 1$*).*
- *A Neutrosophic Soft UnderGraph generalizes a Soft Graph by permitting falsity and other membership degrees to fall below zero (*$T, I, F \in [\Psi, 1]$*,* $\Psi < 0$*).*

Proof. Let $G = (G^*, J, K, A)$ be a Soft Graph, where $G^* = (V, E)$, J is a soft set over V, and K is a soft set over E. For each parameter $a \in A$, $J(a) \subseteq V$ and $K(a) \subseteq E$.

1. A Neutrosophic Soft OffGraph $G_{\text{NSO}} = (G^*, J, K, A, T, I, F)$ extends G by assigning:

   $$T, I, F : V \cup E \to [\Psi, \Omega], \quad T(x) + I(x) + F(x) \leq 3\Omega, \quad \forall x \in V \cup E.$$

   When $\Psi = 0$ and $\Omega = 1$, this reduces to the original Soft Graph.

2. A Neutrosophic Soft OverGraph $G_{\text{NSOver}} = (G^*, J, K, A, T, I, F)$ uses:

   $$T, I, F : V \cup E \to [0, \Omega], \quad T(x) + I(x) + F(x) \leq 3\Omega, \quad \forall x \in V \cup E.$$

   Setting $\Omega = 1$ confines the membership degrees to $[0, 1]$, recovering the Soft Graph.

3. A Neutrosophic Soft UnderGraph $G_{\text{NSUnder}} = (G^*, J, K, A, T, I, F)$ uses:

   $$T, I, F : V \cup E \to [\Psi, 1], \quad T(x) + I(x) + F(x) \leq 3, \quad \forall x \in V \cup E.$$

   When $\Psi = 0$, this becomes a Soft Graph.

In all cases, the Neutrosophic extensions allow for broader ranges of membership degrees, reducing to the Soft Graph when $\Psi = 0$ and $\Omega = 1$. This proves the generalization. □

2.2.6 Rough set and Rough Graph

This paper partially focuses on Rough Sets and Rough Graphs. A Rough Set [440, 726, 728–735] (or a Rough Graph [204, 265, 479, 703, 941]) is a mathematical model developed to approximate uncertain or imprecise data through the use of lower and upper approximations. Given the significant amount of research extending Rough Sets and Rough Graphs using concepts like Fuzzy Sets, Neutrosophic Sets, and Soft Sets, the study of these frameworks is evidently of great importance [29, 140, 204, 265, 440, 618, 648, 667, 701, 726, 760, 770].

The definitions are provided below.

Definition 2.137. [729] Let X be the universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or an indiscernibility relation) on X, partitioning X into equivalence classes. For any subset $U \subseteq X$, the lower approximation $\underline{U}$ and the upper approximation $\overline{U}$ are defined as follows:

1. *Lower Approximation* $\underline{U}$:

$$\underline{U} = \{x \in X \mid R(x) \subseteq U\}$$

This is the set of all elements in X that certainly belong to U based on the equivalence classes defined by R.

2. *Upper Approximation* $\overline{U}$:

$$\overline{U} = \{x \in X \mid R(x) \cap U \neq \emptyset\}$$

This set contains all elements in X that possibly belong to U.

The pair $(\underline{U}, \overline{U})$ constitutes a rough set representation of U, where $\underline{U} \subseteq U \subseteq \overline{U}$.

Remark 2.138. The lower approximation provides a conservative estimate of U, including only elements that definitely belong to U. The upper approximation provides a liberal estimate, including all elements that could potentially belong to U. Together, these approximations characterize the uncertainty of the set U relative to the equivalence relation R.

Proposition 2.139. *A Rough Set is a generalization of a Crisp Set.*

Proof. A Crisp Set $A \subseteq X$ is defined by a characteristic function:

$$\chi_A : X \to \{0, 1\}.$$

A Rough Set is defined based on a universe X and an equivalence relation $R \subseteq X \times X$ (or an indiscernibility relation). For any subset $U \subseteq X$, the lower approximation $\underline{U}$ and the upper approximation $\overline{U}$ are given by:

$$\underline{U} = \{x \in X \mid R(x) \subseteq U\},$$

$$\overline{U} = \{x \in X \mid R(x) \cap U \neq \emptyset\}.$$

To show that a Crisp Set is a special case of a Rough Set, consider the specific scenario where the equivalence relation R induces single-element equivalence classes, i.e., $R(x) = \{x\}$ for all $x \in X$. In this case:

- The lower approximation $\underline{U}$ becomes:

$$\underline{U} = \{x \in X \mid \{x\} \subseteq U\} = \{x \in X \mid x \in U\} = U.$$

- The upper approximation $\overline{U}$ becomes:

$$\overline{U} = \{x \in X \mid \{x\} \cap U \neq \emptyset\} = \{x \in X \mid x \in U\} = U.$$

Thus, the rough set representation $(\underline{U}, \overline{U})$ reduces to (U, U), which exactly matches the Crisp Set U represented by the characteristic function $\chi_U(x)$.

Conversely, for a general equivalence relation R, the rough set representation $(\underline{U}, \overline{U})$ captures the uncertainty in set membership caused by the equivalence classes. This flexibility allows Rough Sets to model imprecise or uncertain boundaries, extending the strict binary membership of Crisp Sets.

Therefore, a Rough Set generalizes a Crisp Set by reducing to the latter when the equivalence relation R induces singleton equivalence classes. □

Definition 2.140. [479] Let $G = (V, E)$ be a graph, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices and $E \subseteq V \times V$ is the set of edges. Additionally, let R be an equivalence relation over some attribute space associated with the vertices, creating equivalence classes of edges.

1. *Rough Vertex Set*: For each vertex $v_i \in V$, we define its lower approximation $\underline{v_i}$ and upper approximation $\overline{v_i}$, representing the subsets of V in terms of their certainty of inclusion based on relation R.

2. *Rough Edge Set*: For each edge $e = (v_i, v_j) \in E$, the lower and upper approximations are defined similarly. Specifically, we form the following:

- *Lower Approximate Edge Set* $\underline{E}$:

$$\underline{E} = \{e = (v_i, v_j) \mid R(e) \subseteq E\}$$

representing edges that certainly exist between vertices based on R.

- *Upper Approximate Edge Set* $\overline{E}$:

$$\overline{E} = \{e = (v_i, v_j) \mid R(e) \cap E \neq \emptyset\}$$

representing edges that possibly exist.

The Rough Graph $G_R = (\underline{V}, \overline{V}, \underline{E}, \overline{E})$ is thus described by its lower and upper approximations of vertices and edges, enabling the representation of uncertainty in network structures.

2.3 Hyperconcepts and Superhyperconcepts

In this subsection, we introduce several types of hyperconcepts and superhyperconcepts. While these terms may have slightly different meanings across various mathematical fields, many concepts in each field are defined from the perspective of hyper and superhyper concepts. These definitions often serve to generalize classical concepts. Here, we briefly present a few examples of hyperconcepts and superhyperconcepts. It is important to note that in some fields, multiple definitions exist, and the interpretations of "hyper" or "super" may vary. However, concepts such as Superhypergraph can be understood as structures based on the n-th PowerSet framework.

Definition 2.141 (n-th PowerSet)**.** [867] Let H be a set representing a system or structure, such as a set of items, a company, an institution, a country, or a region. The *n-th PowerSet*, denoted as $\mathcal{P}_n^*(H)$, describes a hierarchical organization of H into subsystems, sub-subsystems, and so forth. It is defined recursively as follows:

1. **Base Case:**
$$\mathcal{P}_0^*(H) := H.$$
2. **First-Level PowerSet:**
$$\mathcal{P}_1^*(H) = \mathcal{P}(H),$$
where $\mathcal{P}(H)$ is the power set of H.
3. **Higher Levels:** For $n \geq 2$, the n-th PowerSet is defined recursively as:
$$\mathcal{P}_n^*(H) = \mathcal{P}(\mathcal{P}_{n-1}^*(H)).$$

Thus, $\mathcal{P}_n^*(H)$ represents a nested hierarchy, where the power set operation $\mathcal{P}$ is applied n times. Formally:

$$\mathcal{P}_n^*(H) = \mathcal{P}(\mathcal{P}(\cdots \mathcal{P}(H) \cdots)),$$

where the power set operation $\mathcal{P}$ is repeated n times.

2.3.1 Hypergraph and Superhypergraph

A hypergraph is a generalized graph concepts that extends traditional graph concepts by allowing hyperedges, which connect multiple vertices rather than just pairs, enabling more complex relationships between elements [111, 112, 132, 435–437]. Hypergraphs have a wide range of applications, notably in database systems [532]. For additional information, readers are encouraged to consult comprehensive surveys on hypergraphs, such as those in [163, 164, 963].

Definition 2.142 (Hypergraph). [132] A *hypergraph* $H = (V, E)$ consists of a set V of vertices and a set E of hyperedges. Each hyperedge $e \in E$ is defined as a subset of V, thus $e \subseteq V$, and $E \subseteq \mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the power set of V.

Proposition 2.143. *A HyperGraph generalizes a graph.*

Proof. This is evident. □

Definition 2.144 (Supergraph). (cf. [466,467,970,994]) Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs, where V_G and E_G represent the vertex set and edge set of G, and V_H and E_H represent the vertex set and edge set of H.

The graph G is called a *supergraph* of H if:

$$V_H \subseteq V_G \quad \text{and} \quad E_H \subseteq E_G.$$

In other words, G contains all the vertices and edges of H, and possibly more. Formally, this can be written as:

$$H \subseteq G \quad \text{if and only if} \quad V_H \subseteq V_G \text{ and } E_H \subseteq E_G.$$

Remarks:

- If $V_H = V_G$ and $E_H \subset E_G$, G is called an *edge-supergraph* of H.
- If $E_H = E_G$ and $V_H \subset V_G$, G is called a *vertex-supergraph* of H.

The SuperHyperGraph is a generalized graph concept that extends the Hypergraph by incorporating supervertices and superedges [370, 410, 462–464, 867, 868, 873, 877, 884]. Definitions of this and related concepts are introduced below.

Definition 2.145 (SuperHyperGraph). [867] Let V be a finite set of vertices. A *superhypergraph* is an ordered pair $H = (V, E)$, where:

- $V \subseteq P(V)$ (the power set of V), meaning that each element of V can be either a single vertex or a subset of vertices (called a *supervertex*).
- $E \subseteq P(V)$ represents the set of edges, called *superedges*, where each $e \in E$ can connect multiple supervertices.

In this framework, a superhypergraph can accommodate complex relationships among groups of vertices, including single edges, hyperedges, superedges, and multi-edges. Superhypergraphs provide a flexible structure to represent high-order and hierarchical relationships.

The following is clearly valid.

Proposition 2.146. *A SuperHyperGraph generalizes a Hypergraph.*

Proof. To show that a SuperHyperGraph generalizes a Hypergraph, consider the special case where:

- All elements of V_S are singletons: $V_S = V$.
- All superedges $e \in E_S$ are subsets of V_S, i.e., $e \subseteq V_S$.

Under these constraints, $H_S = (V_S, E_S)$ reduces to the definition of a Hypergraph $H = (V, E)$, where $V = V_S$ and $E = E_S$.

Conversely, a SuperHyperGraph allows for:

- Supervertices, which are subsets of V_S, enabling hierarchical relationships among vertices.
- Superedges, which connect supervertices, providing a higher level of connectivity and structural complexity.

This additional flexibility makes a SuperHyperGraph a strict generalization of a Hypergraph, as it encompasses all Hypergraphs as a special case. □

Proposition 2.147. *A SuperHyperGraph generalizes the concept of a SuperGraph.*

Proof. For a given supergraph $G = (V_G, E_G)$, define a corresponding SuperHyperGraph $H_s = (V_s, E_s)$ as follows:

- Set $V_s = \{\{v\} \mid v \in V_G\}$, where each vertex in G is represented as a singleton subset in H_s.
- Set $E_s = \{\{u, v\} \mid (u, v) \in E_G\}$, where each edge in G is represented as a superedge in H_s.

This construction ensures that H_s contains all vertices and edges of G, preserving the structure of the supergraph.

In a SuperHyperGraph $H_s = (V_s, E_s)$, V_s and E_s are subsets of the power set $P(V_s)$, allowing:

- Vertices to represent subsets of V_s (supervertices), enabling hierarchical or group-based relationships.
- Edges to connect multiple supervertices (superedges), accommodating complex and high-order connections.

This extension provides additional flexibility not available in a standard supergraph. For example:

- A supervertex can model a group of related entities.
- A superedge can represent relationships among multiple groups or individuals simultaneously.

Every supergraph G can be embedded into a SuperHyperGraph H_s as a special case where V_s and E_s contain only singletons and pairwise connections. The additional structure of supervertices and superedges in H_s extends the modeling capabilities of G, demonstrating that SuperHyperGraphs generalize SuperGraphs. □

Definition 2.148 (Quasi-SuperHyperGraph)**.** [463] A *quasi-superhypergraph* is a triple $H = (V, S, \Phi)$ where:

- V is a set of elements called *vertices*.
- $S = \{S_i\}_{i=1}^{k} \subset P(V)$ is a family of subsets of V called *supervertices*.

- $\Phi = \{\varphi_{i,j} \mid i \neq j\}$ is a set of mappings $\varphi_{i,j} : S_i \to S_j$, known as *superedges*, linking different supervertices.

Remark 2.149. The quasi-superhypergraph becomes a complete superhypergraph if, for each supervertex S_i, there exists at least one supervertex S_j such that S_i links to S_j.

The following is clearly valid.

Proposition 2.150. *A quasi-superhypergraph is a generalization of a hypergraph.*

Proof. In a quasi-superhypergraph $H = (V, S, \Phi)$:

- The set of vertices V is the same as in a hypergraph.
- The family $S = \{S_i\}$ of subsets of V, called supervertices, can be interpreted as a generalization of the set of vertices V in a hypergraph, allowing for hierarchical relationships or grouping of vertices.
- The set of mappings Φ introduces additional structure by linking supervertices, enabling relationships not possible in a standard hypergraph.

If S consists solely of single-element subsets of V (i.e., $S = \{\{v\} \mid v \in V\}$) and the mappings Φ are absent or trivial, the quasi-superhypergraph reduces to a hypergraph.

Therefore, the quasi-superhypergraph extends the concept of a hypergraph by incorporating supervertices and their mappings. This demonstrates that the quasi-superhypergraph is a generalization of the hypergraph. □

Definition 2.151 (Mixed Superhypergraph)**.** [384] A *mixed superhypergraph* $H = (V, S, E, A)$ consists of:

- A non-empty set V of elements called *vertices*.
- A set S of non-empty subsets of V, called *supervertices*, where each supervertex $s \in S$ satisfies $s \subseteq V$.
- A set E of undirected *superedges*, where each superedge $e \in E$ is a non-empty subset of S.
- A set A of directed *superedges* (also known as *super-dyperedges*), where each directed superedge $a = (Z, z) \in A$ consists of:
  - A non-empty subset $Z \subseteq S \setminus \{z\}$, called the *tail set*.
  - A supervertex $z \in S$, called the *head*.

In a mixed superhypergraph, undirected superedges connect subsets of supervertices without directionality, while directed superedges represent a directed relationship from the tail set Z to the head z.

Definition 2.152 (Pseudo-Superhypergraph)**.** [384] A *pseudo-superhypergraph* $H = (V, S, E)$ consists of:

- *Vertices*: A non-empty finite set V of elements called *vertices*.
- *Supervertices*: A multiset S of multisets over V, called *supervertices*. Each supervertex $s \in S$ is a multiset of vertices from V, allowing:
  - Multiple occurrences of the same vertex within a supervertex (vertices can repeat within s).
  - Multiple occurrences of the same supervertex in S (supervertices can repeat in S).
- *Superedges*: A multiset E of multisets over S, called *superedges*. Each superedge $e \in E$ is a multiset of supervertices from S, allowing:
  - Multiple occurrences of the same supervertex within a superedge (supervertices can repeat within e).

– Multiple occurrences of the same superedge in E (superedges can repeat in E).

Proposition 2.153. *A pseudo-superhypergraph is a superhypergraph.*

Proof. This is evident. □

A SuperHyperGraph can be generalized to an n-SuperHyperGraph. The definition is provided below. While this paper primarily deals with SuperHyperGraphs, it can be inferred that SuperHyperConcepts can be extended to n-SuperHyperConcepts.

Definition 2.154 (n-SuperHyperGraph)**.** [867] A *n-SuperHyperGraph* (SHG) is an ordered pair $H = (V, E)$, where:

1. $V \subseteq \mathcal{P}^n(V_0)$: The set of vertices, where $\mathcal{P}^n(V_0)$ represents the n-power set of a base vertex set V_0.
2. $E \subseteq \mathcal{P}^n(V_0)$: The set of edges, called *superedges*.
3. Each vertex in V can be:
   - A single vertex ($v \in V_0$),
   - A subset of vertices ($v \subseteq V_0$),
   - An indeterminate vertex (partially defined membership),
   - A null vertex ($\emptyset$).
4. Each edge in E can connect:
   - A set of single vertices,
   - A set of supervertices,
   - Or higher-order subsets (depending on n).

Superhypergraphs provide a generalization of hypergraphs by introducing hierarchical and high-order relationships between vertices and edges.

Theorem 2.155. *A SuperHyperGraph can be generalized to an n-SuperHyperGraph.*

Proof. By definition, a SuperHyperGraph $H = (V, E)$ is a pair where:

$$V \subseteq \mathcal{P}(V_0), \quad E \subseteq \mathcal{P}(V_0),$$

and $\mathcal{P}(V_0)$ denotes the power set of the base vertex set V_0.

For $n = 1$, the vertex and edge sets are subsets of $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$, which is equivalent to the definition of a SuperHyperGraph.

For $n > 1$, we recursively define $\mathcal{P}^n(V_0)$ as:

$$\mathcal{P}^n(V_0) = \mathcal{P}(\mathcal{P}^{n-1}(V_0)).$$

Thus, the vertices and edges of an n-SuperHyperGraph are subsets of $\mathcal{P}^n(V_0)$.

Since $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$, the SuperHyperGraph corresponds to the special case where $n = 1$. Hence, every SuperHyperGraph is an n-SuperHyperGraph for $n = 1$, proving the generalization. □

Theorem 2.156. *An n-SuperHyperGraph has a structure based on the n-th PowerSet.*

Proof. Let $H = (V, E)$ be an n-SuperHyperGraph. By definition:

$$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}^n(V_0),$$

where $\mathcal{P}^n(V_0)$ represents the n-th PowerSet of the base set V_0.

The n-th PowerSet $\mathcal{P}^n(V_0)$ is constructed recursively as:

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^1(V_0) = \mathcal{P}(V_0), \quad \mathcal{P}^n(V_0) = \mathcal{P}(\mathcal{P}^{n-1}(V_0)).$$

This recursive definition implies that $\mathcal{P}^n(V_0)$ contains subsets of subsets up to n levels of hierarchy.

Since V and E are subsets of $\mathcal{P}^n(V_0)$, the structure of an n-SuperHyperGraph inherently relies on the n-th PowerSet. Each vertex or edge can represent:

1. Single vertices (base elements of V_0),
2. Subsets of vertices,
3. Higher-order subsets depending on n.

This confirms that an n-SuperHyperGraph has a structure based on the n-th PowerSet. □

The following diagram illustrates their relationship with these graph classes. The same holds true for directed graphs as well [368].

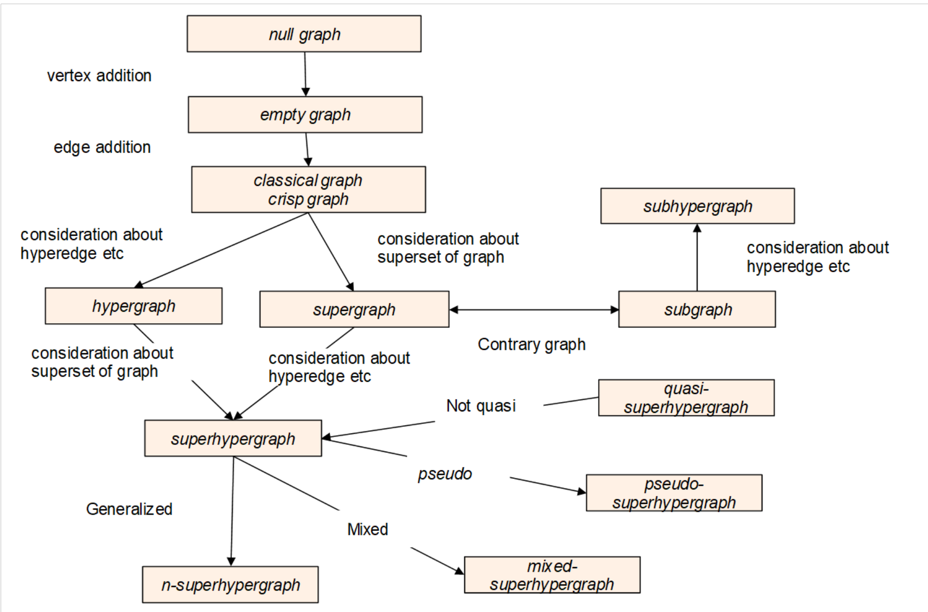


Figure 2: Graph Hierarchy for Hypergraph. This figure is referenced from [368].

2.3.2 Hypersoft Graph

A HyperSoft Graph represents nodes with multiple attributes, where each node can hold unique attribute values, facilitating complex, multi-dimensional relationships. This type of graph is derived by applying the concept of a HyperSoft Set [2, 5, 249, 484, 535, 657, 755, 795, 837, 861, 1008] to graph theory. The formal definitions of a HyperSoft Set and a HyperSoft Graph are presented below [754, 797–800].

Definition 2.157 (Hypersoft Set)**.** [861] Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n-distinct attributes with corresponding disjoint sets $J_1, J_2, \ldots, J_n$. A pair (F, J) is called a *hypersoft set* over the universal set X, where F is a mapping defined by

$$F : J \to \mathcal{P}(X),$$

with $J = J_1 \times J_2 \times \cdots \times J_n$.

Proposition 2.158. *A hypersoft Set is a generalization of a soft Set.*

Proof. This is evident. □

Definition 2.159 (Null Hypersoft Set)**.** Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n-distinct attributes with corresponding disjoint sets $J_1, J_2, \ldots, J_n$. A hypersoft set (F, J) over X, where $J = J_1 \times J_2 \times \cdots \times J_n$ and $F : J \to \mathcal{P}(X)$, is called a *Null Hypersoft Set*, denoted by Φ, if for every $j \in J$, the subset $F(j)$ of X is empty. Formally:

$$(F, J) = \Phi \quad \text{if and only if} \quad F(j) = \emptyset \quad \forall j \in J.$$

Proposition 2.160. *A null hypersoft Set is a hypersoft Set.*

Proof. This is evident. □

Definition 2.161 (Full Hypersoft Set)**.** Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n-distinct attributes with corresponding disjoint sets $J_1, J_2, \ldots, J_n$. A hypersoft set (F, J) over X, where $J = J_1 \times J_2 \times \cdots \times J_n$ and $F : J \to \mathcal{P}(X)$, is called a *Full Hypersoft Set* if:

$$\bigcup_{j \in J} F(j) = X.$$

This condition ensures that every element of the universe X is included in at least one subset $F(j)$ for some $j \in J$.

Proposition 2.162. *A Full hypersoft Set is a hypersoft Set.*

Proof. This is evident. □

Theorem 2.163. *A Null Hypersoft Set generalizes a Null Soft Set.*

Proof. To show generalization, consider $n = 1$ and $J_1 = A$. In this case, $J = J_1 = A$, and $F : J_1 \to \mathcal{P}(X)$ reduces to $F : A \to \mathcal{P}(U)$. Since $F(j) = \emptyset$ for all $j \in J$ is equivalent to $F(\varepsilon) = \emptyset$ for all $\varepsilon \in A$, the Null Hypersoft Set becomes a Null Soft Set when $n = 1$. Hence, the Null Hypersoft Set generalizes the Null Soft Set. □

Theorem 2.164. *A Full Hypersoft Set generalizes a Full Soft Set.*

Proof. To show generalization, consider $n = 1$ and $J_1 = A$. In this case, $J = J_1 = A$, and $F : J_1 \to \mathcal{P}(X)$ reduces to $F : A \to \mathcal{P}(U)$. Since $\bigcup_{j \in J} F(j) = X$ is equivalent to $\bigcup_{\varepsilon \in A} F(\varepsilon) = U$, the Full Hypersoft Set becomes a Full Soft Set when $n = 1$. Hence, the Full Hypersoft Set generalizes the Full Soft Set. □

Definition 2.165 (Hypersoft Graph)**.** (cf. [797–800]) Let $G = (V, E)$ be a simple connected graph, where V is the set of vertices and E is the set of edges. Consider $J = J_1 \times J_2 \times \cdots \times J_n$, where each $J_i \subseteq V$ and $J_i \cap J_j = \emptyset$ for $i \neq j$. A *Hypersoft Graph* (HS-Graph) of G is defined as a hypersoft set (F, J) over V such that for each $x \in J$, $F(x)$ induces a connected subgraph of G. The set of all HS-Graphs of G is denoted by $\mathrm{HsG}(G)$.

Proposition 2.166. *A hypersoft graph is a generalization of a soft graph.*

Proof. This is evident. □

The related concepts of SuperHyperSoft Sets [350, 666, 878], IndetermSoft Sets [880, 882], and IndetermHyperSoft Sets [880, 882] are well-recognized in this area. The definitions are provided below.

Definition 2.167 (SuperHyperSoft Set)**.** [878] Let U be a universe of discourse, and let $\mathcal{P}(U)$ denote the power set of U. Let $a_1, a_2, \ldots, a_n$, where $n \geq 1$, represent n distinct attributes, with corresponding attribute value sets $A_1, A_2, \ldots, A_n$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$.

For each attribute a_i, let $\mathcal{P}(A_i)$ denote the power set of A_i. Then, the pair

$$(F, \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n)),$$

where $F : \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n) \to \mathcal{P}(U)$, is called a *SuperHyperSoft Set* over U.

In this structure:

- Each attribute a_i is associated with its power set $\mathcal{P}(A_i)$, allowing for subsets of attribute values to be used as inputs in the mapping.
- The function F assigns each n-tuple $(S_1, S_2, \ldots, S_n) \in \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n)$ to a subset $F(S_1, S_2, \ldots, S_n) \subseteq U$, effectively mapping combinations of attribute subsets to elements in U.

The SuperHyperSoft Set generalizes the HyperSoft Set by utilizing the power sets of attribute values, thereby enabling a wider range of input combinations and facilitating complex relationships between attributes and the universe U.

Proposition 2.168. *A superhypersoft Set is a generalization of a hypersoft Set.*

Proof. This is evident. □

Definition 2.169 (Null SuperHyperSoft Set)**.** Let U be a universe of discourse, and let $\mathcal{P}(U)$ denote the power set of U. Let $a_1, a_2, \ldots, a_n$ represent n-distinct attributes with corresponding attribute value sets $A_1, A_2, \ldots, A_n$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$.

A *SuperHyperSoft Set* (F, J) over U, where $J = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n)$ and $F : J \to \mathcal{P}(U)$, is called a *Null SuperHyperSoft Set*, denoted by Φ, if for every $j \in J$, the subset $F(j)$ of U is empty. Formally:

$$(F, J) = \Phi \quad \text{if and only if} \quad F(j) = \emptyset \quad \forall j \in J.$$

Proposition 2.170. *A Null superhypersoft Set is a superhypersoft Set.*

Proof. This is evident. □

Definition 2.171 (Full SuperHyperSoft Set)**.** Let U be a universe of discourse, and let $\mathcal{P}(U)$ denote the power set of U. Let $a_1, a_2, \ldots, a_n$ represent n-distinct attributes with corresponding attribute value sets $A_1, A_2, \ldots, A_n$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$.

A *SuperHyperSoft Set* (F, J) over U, where $J = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n)$ and $F : J \to \mathcal{P}(U)$, is called a *Full SuperHyperSoft Set* if:

$$\bigcup_{j \in J} F(j) = U.$$

This condition ensures that every element of the universe U is included in at least one subset $F(j)$ for some $j \in J$.

Proposition 2.172. *A Full superhypersoft Set is a superhypersoft Set.*

Proof. This is evident. □

Theorem 2.173. *A Null SuperHyperSoft Set generalizes a Null HyperSoft Set.*

Proof. To show generalization, consider the case where $A_1, A_2, \ldots, A_n$ are singletons such that $\mathcal{P}(A_i) = \{A_i\}$ for all i. Then, $J = A_1 \times A_2 \times \cdots \times A_n$, and $F : J \to \mathcal{P}(U)$ reduces to $F : J_1 \times J_2 \times \cdots \times J_n \to \mathcal{P}(X)$. Since $F(j) = \emptyset$ for all $j \in J$ remains consistent in both cases, the Null SuperHyperSoft Set reduces to a Null HyperSoft Set under these constraints. Hence, the Null SuperHyperSoft Set generalizes the Null HyperSoft Set. □

Theorem 2.174. *A Full SuperHyperSoft Set generalizes a Full HyperSoft Set.*

Proof. To show generalization, consider the case where $A_1, A_2, \ldots, A_n$ are singletons such that $\mathcal{P}(A_i) = \{A_i\}$ for all i. Then, $J = A_1 \times A_2 \times \cdots \times A_n$, and $F : J \to \mathcal{P}(U)$ reduces to $F : J_1 \times J_2 \times \cdots \times J_n \to \mathcal{P}(X)$. Since $\bigcup_{j \in J} F(j) = U$ becomes $\bigcup_{j \in J} F(j) = X$ under these constraints, the Full SuperHyperSoft Set reduces to a Full HyperSoft Set. Hence, the Full SuperHyperSoft Set generalizes the Full HyperSoft Set. □

For reference, the relationships between the Soft set are illustrated in Figure 3. (cf. [378])

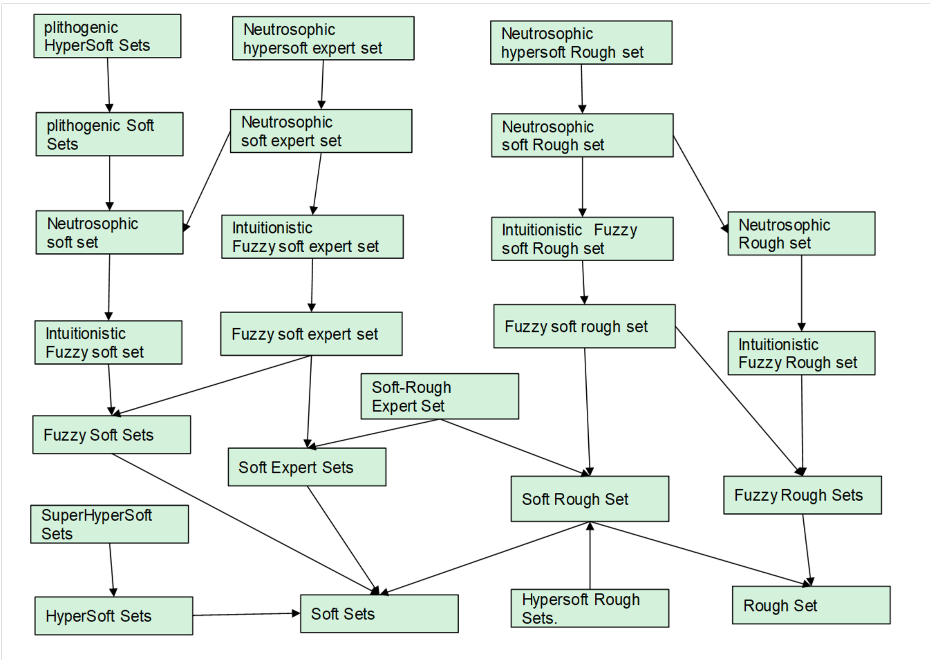


Figure 3: Some Soft sets Hierarchy. The set class at the origin of an arrow contains the set class at the destination of the arrow(cf. [378]).

Definition 2.175 (IndetermSoft Set)**.** [880] Let U be a universe of discourse, H a non-empty subset of U, and $\mathcal{P}(H)$ the power set of H. Let a represent an attribute, and let A be a set of attribute values associated with a. Then, a function $F : A \to \mathcal{P}(H)$ is called an *IndetermSoft Set* if at least one of the following conditions holds:

1. The set A has some indeterminacy.
2. At least one of the sets H or $\mathcal{P}(H)$ has some indeterminacy.
3. The function F itself has some indeterminacy, meaning there exists at least one pair $F(a) = M$, where either a or M is not well-defined (e.g., non-unique, unclear, incomplete, or unknown).

An IndetermSoft Set, as an extension of the classical Soft Set, is used to model data that may contain indeterminate elements. This type of set arises naturally due to sources that cannot provide exact or complete information about A, H, $\mathcal{P}(H)$, or F. The term Indeterm here signifies Indeterminate and reflects uncertainty or incompleteness in the set.

Proposition 2.176. *The IndetermSoft Set generalizes the Soft Set.*

Proof. To prove that the IndetermSoft Set generalizes the Soft Set, consider the definition of a Soft Set. A Soft Set is defined as a pair (F, A), where A is a set of parameters and $F : A \to \mathcal{P}(H)$ is a mapping from A to the power set of a non-empty subset $H \subseteq U$.

The IndetermSoft Set introduces the possibility of indeterminacy in three components:

- The parameter set A,
- The universe H or its power set $\mathcal{P}(H)$,
- The mapping F.

If all indeterminacies are removed (i.e., A, H, $\mathcal{P}(H)$, and F are completely well-defined), an IndetermSoft Set reduces to a classical Soft Set. Specifically, the condition of indeterminacy in the IndetermSoft Set does not restrict it from satisfying the definition of a Soft Set. Therefore, the IndetermSoft Set includes the Soft Set as a special case where there is no indeterminacy in any of its components.

Thus, the IndetermSoft Set generalizes the Soft Set. □

Definition 2.177 (IndetermHyperSoft Set)**.** [880] Let U be a universe of discourse, H a non-empty subset of U, and $\mathcal{P}(H)$ the power set of H. Let $a_1, a_2, \ldots, a_n$, where $n \geq 1$, be distinct attributes, and let $A_1, A_2, \ldots, A_n$ be their corresponding sets of attribute values, with $A_i \cap A_j = \emptyset$ for $i \neq j$. Then the pair $(F, A_1 \times A_2 \times \cdots \times A_n)$, where $F : A_1 \times A_2 \times \cdots \times A_n \to \mathcal{P}(H)$, is called an *IndetermHyperSoft Set* over U if at least one of the following conditions holds:

1. At least one of the sets $A_1, A_2, \ldots, A_n$ has some indeterminacy.
2. At least one of the sets H or $\mathcal{P}(H)$ has some indeterminacy.
3. The function F has some indeterminacy, meaning there exists at least one relationship $F(e_1, e_2, \ldots, e_n) = M$ where some of $e_1, e_2, \ldots, e_n$, or M are not well-defined (e.g., non-unique, unclear, incomplete, or unknown).

The IndetermHyperSoft Set generalizes the HyperSoft Set by allowing indeterminate elements in the attribute sets, universe, or function, capturing more realistic cases where data may be uncertain or incomplete.

Proposition 2.178. *The IndetermHyperSoft Set generalizes both the IndetermSoft Set and the HyperSoft Set.*

Proof. To demonstrate that the IndetermHyperSoft Set generalizes both the IndetermSoft Set and the HyperSoft Set, consider their definitions:

1. **Generalization of the HyperSoft Set:** The HyperSoft Set is defined as (F, J), where $J = J_1 \times J_2 \times \cdots \times J_n$, and $F : J \to \mathcal{P}(X)$. In the IndetermHyperSoft Set, the mappings F and the attribute value sets $J_1, J_2, \ldots, J_n$ can exhibit indeterminacy, such as non-unique or incomplete information. When all indeterminacies are removed, an IndetermHyperSoft Set reduces to a classical HyperSoft Set. Hence, the IndetermHyperSoft Set includes the HyperSoft Set as a special case.

2. **Generalization of the IndetermSoft Set:** An IndetermSoft Set (F, A) maps an attribute value set A to $\mathcal{P}(H)$, allowing for indeterminacy in A, H, or F. The IndetermHyperSoft Set extends this concept by introducing multiple attributes $A_1, A_2, \ldots, A_n$ and their corresponding mappings $F : A_1 \times A_2 \times \cdots \times A_n \to \mathcal{P}(H)$. When $n = 1$, the IndetermHyperSoft Set reduces to an IndetermSoft Set. Therefore, the IndetermHyperSoft Set includes the IndetermSoft Set as a special case.

Since the IndetermHyperSoft Set encompasses the conditions and structures of both the IndetermSoft Set and the HyperSoft Set, it is a generalization of both. □

Below, we provide the definition of the TreeSoft Set, one of the related extensions of the soft set. The TreeSoft Set serves as a generalization of the MultiSoft Set concept.

Definition 2.179. [880] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1,i_2,\ldots,i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as $\text{Tree}(A)$, with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set* F is defined as a function:

$$F : P(\text{Tree}(A)) \to P(H),$$

where $\text{Tree}(A)$ represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\text{Tree}(A))$ denotes its power set.

Proposition 2.180. *A treesoft Set is a generalization of a soft Set.*

Proof. This is evident. □

Definition 2.181 (Null TreeSoft Set)**.** Let U be a universe of discourse, and let $H \subseteq U$ be a non-empty subset with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes, where each attribute A_i consists of sub-attributes forming a hierarchical tree-like structure $\text{Tree}(A)$.

A *TreeSoft Set* F is defined as a function:

$$F : P(\text{Tree}(A)) \to P(H).$$

The TreeSoft Set F is called a *Null TreeSoft Set*, denoted by Φ, if:

$$F(t) = \emptyset \quad \forall t \in P(\text{Tree}(A)).$$

This condition ensures that every subset t of the tree structure maps to an empty subset of H.

Definition 2.182 (Full TreeSoft Set)**.** Let U be a universe of discourse, and let $H \subseteq U$ be a non-empty subset with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes, where each attribute A_i consists of sub-attributes forming a hierarchical tree-like structure $\text{Tree}(A)$.

A *TreeSoft Set* F is defined as a function:

$$F : P(\text{Tree}(A)) \to P(H).$$

The TreeSoft Set F is called a *Full TreeSoft Set* if:

$$\bigcup_{t \in P(\text{Tree}(A))} F(t) = H.$$

This condition ensures that every element of H is included in at least one subset $F(t)$ for some $t \in P(\text{Tree}(A))$.

Theorem 2.183. *A Null TreeSoft Set generalizes a Null Soft Set.*

Proof. To show generalization, consider the case where Tree(A) has only one level, i.e., Tree$(A) = A$. Then:

$$P(\text{Tree}(A)) = P(A), \quad F : P(A) \to \mathcal{P}(U).$$

Since $F(t) = \emptyset \, \forall t \in P(A)$ reduces to $F(a) = \emptyset \, \forall a \in A$ when $t = \{a\}$, the Null TreeSoft Set becomes equivalent to the Null Soft Set in this case. Thus, the Null TreeSoft Set generalizes the Null Soft Set. □

Theorem 2.184. *A Full TreeSoft Set generalizes a Full Soft Set.*

Proof. To show generalization, consider the case where Tree(A) has only one level, i.e., Tree$(A) = A$. Then:

$$P(\text{Tree}(A)) = P(A), \quad F : P(A) \to \mathcal{P}(U).$$

Since $\bigcup_{t \in P(A)} F(t) = U$ reduces to $\bigcup_{a \in A} F(a) = U$ when $t = \{a\}$, the Full TreeSoft Set becomes equivalent to the Full Soft Set in this case. Thus, the Full TreeSoft Set generalizes the Full Soft Set. □

Proposition 2.185. *(cf. [880]) Let U be a universal set, and let $H \subseteq U$ be a non-empty subset with $P(H)$ denoting its power set. Suppose $A = \{A_1, A_2, \ldots, A_n\}$ is a set of attributes, where each attribute A_i (for $1 \leq i \leq n$) consists of sub-attributes of the second level $\{A_{i,1}, A_{i,2}, \ldots\}$, forming a tree structure Tree(A) of depth $m = 2$.*

Then, a TreeSoft Set $F : P(Tree(A)) \to P(H)$ at level $m = 2$ is equivalent to a MultiSoft Set (G, B), where:

- *$B \subseteq \mathcal{P}(A)$ represents the attribute combinations at level 2, and*
- *$G : B \to P(H)$ maps each attribute combination to a subset of H.*

Proof. To show that a TreeSoft Set F with $m = 2$ is equivalent to a MultiSoft Set, we proceed as follows:

At level $m = 2$, each attribute A_i in A has a set of sub-attributes $\{A_{i,1}, A_{i,2}, \ldots\}$. Therefore, we can define a set of attribute combinations B as the power set of the union of all second-level sub-attributes:

$$B = \mathcal{P}\left(\bigcup_{i=1}^{n} \{A_{i,1}, A_{i,2}, \ldots\}\right).$$

For each combination $b \in B$ of second-level attributes, we define a corresponding subset of H under the TreeSoft Set mapping F as follows:

$$G(b) = F(b),$$

where $G : B \to P(H)$ maps each combination $b \in B$ to the same subset $F(b)$ in H.

Since F at level $m = 2$ only maps combinations of second-level sub-attributes to subsets of H, it follows that each mapping $F(b)$ is equivalent to the MultiSoft Set mapping $G(b)$. Therefore, the TreeSoft Set $(F, P(\text{Tree}(A)))$ at $m = 2$ effectively reduces to the MultiSoft Set (G, B), where B is the set of all possible combinations of second-level attributes, and G provides the corresponding mapping.

Thus, we conclude that a TreeSoft Set of level $m = 2$ is equivalent to a MultiSoft Set. □

Definition 2.186. Let U be a universe of discourse, and let $H \subseteq U$, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes, where $n \geq 1$. Each attribute A_i is hierarchically structured into sub-attributes as follows:

1. **First-level attributes:** $A_i = \{A_{i,1}, A_{i,2}, \ldots\}$,
2. **Second-level attributes:** $A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\}$,
3. **Higher levels:** Continuing recursively to m-th level attributes, indexed as $A_{i_1, i_2, \ldots, i_m}$, where m is the depth of the tree.

The hierarchical structure of A forms a tree-like graph denoted as $\text{Tree}(A)$, with:

- A as the root (level 0),
- Nodes at successive levels representing sub-attributes,
- Leaves as terminal nodes without descendants.

A **TreeCrisp Set** F is defined as a mapping:

$$F : \text{Tree}(A) \to \{0, 1\},$$

where F assigns to each node or leaf in $\text{Tree}(A)$ a crisp membership value:

$$F(x) = \begin{cases} 1, & \text{if } x \text{ belongs to the TreeCrisp Set,} \\ 0, & \text{otherwise.} \end{cases}$$

This setup provides a crisp classification for all nodes and leaves in $\text{Tree}(A)$, preserving the hierarchical structure.

Theorem 2.187. *A TreeCrisp Set generalizes a Crisp Set. In particular, when the hierarchical structure of attributes in a TreeCrisp Set is reduced to a single level, it becomes equivalent to a Crisp Set.*

Proof. A TreeCrisp Set F over a hierarchical structure $\text{Tree}(A)$ is defined as a mapping:

$$F : \text{Tree}(A) \to \{0, 1\}$$

.

If the hierarchical structure of $\text{Tree}(A)$ is reduced to a single level, then $\text{Tree}(A)$ becomes equivalent to the universe X. The mapping F reduces to:

$$F : X \to \{0, 1\}.$$

This is identical to the characteristic function $\chi_A(x)$ of a Crisp Set.

A Crisp Set is a specific instance of a TreeCrisp Set where the hierarchical structure contains only one level. Therefore, the TreeCrisp Set is a generalization of the Crisp Set, as it allows for hierarchical membership assignments, whereas the Crisp Set is limited to a flat structure. □

Proposition 2.188. *A TreeSoft Set is a generalization of a TreeCrisp Set. Specifically, when the mapping of a TreeSoft Set is restricted to binary membership values (either the subset $t \in P(Tree(A))$ maps to H or it does not), it reduces to a TreeCrisp Set.*

Proof. A TreeCrisp Set F over a hierarchical tree structure $\text{Tree}(A)$ is defined as:

$$F : P(\text{Tree}(A)) \to \{0, 1\}.$$

For each subset $t \in P(\text{Tree}(A))$, $F(t) = 1$ if t is part of the TreeCrisp Set, and $F(t) = 0$ otherwise. Thus, a TreeCrisp Set assigns binary membership values to subsets of $\text{Tree}(A)$.

A TreeSoft Set F over the same hierarchical structure is defined as:

$$F : P(\text{Tree}(A)) \to P(H),$$

where $P(H)$ denotes the power set of a non-empty subset $H \subseteq U$. Each subset $t \in P(\text{Tree}(A))$ is mapped to a subset of H, allowing for more complex relationships and memberships compared to binary values.

If the mapping F in the TreeSoft Set is constrained such that:

$$F(t) = \begin{cases} H, & \text{if } t \text{ belongs to the TreeCrisp Set,} \\ \emptyset, & \text{otherwise,} \end{cases}$$

then the TreeSoft Set reduces to the definition of a TreeCrisp Set. Here, $F(t)$ either maps to H (indicating membership) or $\emptyset$ (indicating non-membership), which corresponds to binary values $\{0, 1\}$.

Since a TreeSoft Set allows for more flexible mappings (any subset of H) compared to the binary mapping of a TreeCrisp Set, the TreeSoft Set generalizes the TreeCrisp Set. □

2.3.3 HyperFuzzy Set

The HyperFuzzy Set concept generalizes both fuzzy sets and interval-valued fuzzy sets by allowing each element in X to have an associated subset of membership degrees within $[0, 1]$, rather than a single degree or interval [154, 411, 551, 692, 902].

Definition 2.189 (HyperFuzzy Set)**.** [154, 411, 551, 692, 902] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0, 1])$ is called a *hyperfuzzy set* over X, where $\tilde{P}([0, 1])$ denotes the family of all non-empty subsets of the interval $[0, 1]$.

Example 2.190. A *Hyperfuzzy Set* $\tilde{A}$ over X is defined by a mapping $\tilde{\mu}_{\tilde{A}} : X \to \tilde{\mathcal{P}}([0, 1])$, where $\tilde{\mathcal{P}}([0, 1])$ denotes the family of all non-empty subsets of the interval $[0, 1]$.

We extend the fuzzy set A to a hyperfuzzy set $\tilde{A}$ by allowing each element to have a set of possible membership degrees, representing uncertainty or variability.

Define $\tilde{\mu}_{\tilde{A}}$ as:

$$\begin{aligned}
\tilde{\mu}_{\tilde{A}}(x_1) &= \{0.1, 0.2, 0.3\}, \\
\tilde{\mu}_{\tilde{A}}(x_2) &= \{0.4, 0.5, 0.6\}, \\
\tilde{\mu}_{\tilde{A}}(x_3) &= \{0.6, 0.7, 0.8\}.
\end{aligned}$$

This means:

- The element x_1 may have membership degrees 0.1, 0.2, or 0.3 in $\tilde{A}$.
- The element x_2 may have membership degrees 0.4, 0.5, or 0.6 in $\tilde{A}$.
- The element x_3 may have membership degrees 0.6, 0.7, or 0.8 in $\tilde{A}$.

Proposition 2.191. *A HyperFuzzy Set is a generalization of a Fuzzy Set.*

Proof. This follows directly from the definition. □

We introduce the notion of a *HyperFuzzy Graph*. This structure builds upon the fuzzy graph by incorporating hyperfuzzy sets as both vertex and edge membership functions. The formal definition is provided below [354].

Definition 2.192 (HyperFuzzy Graph)**.** [354] A *HyperFuzzy Graph* G_H is defined as a structure

$$G_H = (V, E, \tilde{\sigma}, \tilde{\mu}),$$

where:

- V is a non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges.
- $\tilde{\sigma} : V \to \tilde{P}([0, 1])$ is a hyperfuzzy set that assigns to each vertex a set of membership degrees within $[0, 1]$.
- $\tilde{\mu} : E \to \tilde{P}([0, 1])$ is a hyperfuzzy set that assigns to each edge a set of membership degrees within $[0, 1]$.

Proposition 2.193. *A HyperFuzzy Graph is a generalization of a Fuzzy Graph.*

Proof. This follows directly from the definition. □

2.3.4 SuperHyperFunction

The concept of SuperHyperFunction has been studied in recent years. It is a generalized notion of a traditional function. The definition is provided below [876, 884].

Definition 2.194 (SuperHyperFunction)**.** [876] A *SuperHyperFunction* generalizes classical functions, hyperfunctions, and superfunctions by allowing both the domain and codomain to be elements of the n-th PowerSet hierarchy of a given set S. Formally:

$$\text{SHF} : \mathcal{P}_r^*(S) \to \mathcal{P}_n^*(S),$$

where:

- S is a non-empty set,
- $\mathcal{P}_r^*(S)$ denotes the r-th PowerSet of S (the domain of the function),
- $\mathcal{P}_n^*(S)$ denotes the n-th PowerSet of S (the codomain of the function),
- $r, n \geq 0$ are integers.

A SuperHyperFunction assigns to each element $A \in \mathcal{P}_r^*(S)$ a subset of $\mathcal{P}_n^*(S)$. For $r = 0$ and $n = 0$, this reduces to a classical function.

2.3.5 Hypercube and Hypersphere

A hypercube is an n-dimensional generalization of a cube, consisting of 2^n vertices and edges connecting each vertex to n others in $\mathbb{R}^n$ [138, 284, 302, 903, 911, 962, 999]. A hypersphere is a generalization of a sphere to higher dimensions, representing points equidistant from a center in n-dimensional space [114]. The definitions are provided below.

Definition 2.195 (Hypercube)**.** [138, 284] An n-dimensional hypercube, often referred to as an n-cube, is a generalization of a three-dimensional cube (or regular polyhedron) to n dimensions. Formally, an n-dimensional hypercube C_n is defined as the set of points in $\mathbb{R}^n$ where each coordinate x_i of a point $(x_1, x_2, \ldots, x_n) \in C_n$ satisfies:

$$x_i \in \{0, 1\} \quad \text{for } i = 1, 2, \ldots, n.$$

The properties of the n-dimensional hypercube include:

- The hypercube C_n has 2^n vertices, corresponding to the 2^n possible combinations of the coordinates.
- Each vertex in C_n is connected to n other vertices by edges, resulting in a total of $\frac{n \cdot 2^{n-1}}{2} = n \cdot 2^{n-1}$ edges.
- The n-dimensional hypercube can be constructed recursively: an n-dimensional hypercube is formed by taking two copies of an $(n-1)$-dimensional hypercube and connecting each vertex in one copy to the corresponding vertex in the other copy.

The 0-dimensional hypercube C_0 is a single point, the 1-dimensional hypercube C_1 is a line segment, the 2-dimensional hypercube C_2 is a square, the 3-dimensional hypercube C_3 is a cube, and so forth.

Definition 2.196 (Hypersphere)**.** [114] An n-dimensional hypersphere, often called an n-sphere and denoted S^n, is the set of points in $(n+1)$-dimensional Euclidean space that lie at a fixed distance, or radius r, from a central point $c = (c_1, c_2, \ldots, c_{n+1})$. Mathematically, the hypersphere S^n is defined as:

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x - c\| = r\},$$

where $\|x - c\|$ represents the Euclidean distance between x and c, calculated as:

$$\|x - c\| = \sqrt{(x_1 - c_1)^2 + (x_2 - c_2)^2 + \cdots + (x_{n+1} - c_{n+1})^2}.$$

In the case of the standard n-sphere, the center c is located at the origin (i.e., $c = 0$) and the radius $r = 1$, yielding:

$$S^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

The properties of S^n include:

- The 0-sphere S^0 consists of two discrete points, forming the ends of an interval.
- The 1-sphere S^1 is a circle, while the 2-sphere S^2 is the surface of a standard 3-dimensional ball.
- Generally, an n-sphere S^n has an n-dimensional surface but exists within $(n+1)$-dimensional space.

2.3.6 Real Hypernumbers

An extended concept of numbers, known as hypernumbers [684, 685], is also well-established. As an example, the definition of a Real Hypernumber is provided below.

Definition 2.197. (cf. [176, 178–181]) 1. *Sequence of Real Numbers:*

Let $\mathbb{R}^\omega$ denote the set of all sequences of real numbers, where each sequence $a = (a_i)_{i\in\omega}$ is an ordered collection of real numbers:

$$a = (a_i) = (a_1, a_2, \dots), \quad \text{where } a_i \in \mathbb{R}.$$

2. *Equivalence Relation on Sequences:*

For sequences $a = (a_i)$ and $b = (b_i)$ in $\mathbb{R}^\omega$, we define an equivalence relation $\sim$ by:

$$a \sim b \quad \text{if and only if} \quad \lim_{i\to\infty} |a_i - b_i| = 0.$$

This equivalence relation identifies sequences that converge towards each other in terms of the limit of their absolute differences.

3. *Definition of Real Hypernumbers:*

A *real hypernumber* associated with a sequence $a = (a_i)$ is the equivalence class of all sequences $b \in \mathbb{R}^\omega$ such that $b \sim a$. We denote this equivalence class by $\alpha = H(a_i)$, where $H(a_i)$ represents all sequences equivalent to a. The set of all such equivalence classes of sequences of real numbers forms the space of real hypernumbers, denoted as $\mathbb{R}^\omega$.

Thus, a real hypernumber $\alpha \in \mathbb{R}^\omega$ can be represented as:

$$\alpha = H(a_i) = \{b \in \mathbb{R}^\omega \mid b \sim a\},$$

where each a_i is a term in the sequence.

4. *The Space of Real Hypernumbers:*

We define the space of real hypernumbers as:

$$\mathbb{R}^\omega = \{H(a_i) \mid a = (a_i) \in \mathbb{R}^\omega\}.$$

In this framework, *real numbers* correspond to stable hypernumbers, which are represented by constant sequences within their equivalence classes.

2.3.7 Hypersets and SuperHypersets in set theory

In the literature [385], hypersets and superhypersets have been studied in depth. In fields such as database systems, various concepts of hypersets have been defined to suit different objectives. The definitions draw inspiration from structures like hypergraphs and superhypergraphs in [385]. We hope that future research will further explore the applications, validity, and potential generalizations of these concepts. Definitions of related concepts are provided below. Note that a Hyperset is very similar in structure to a Power Set.

Definition 2.198 (Hyperset)**.** [385] A *hyperset* is a set that extends the concept of traditional sets by allowing its elements to be sets themselves, possibly including the hyperset itself, forming nested or recursive structures analogous to hyperedges in a hypergraph. Formally, a hyperset H is defined over a universal set U as:

$$H \subseteq \mathcal{P}(U),$$

where elements of H can be subsets of U or H itself. This allows for self-referential membership, where a set can contain itself directly or indirectly.

Definition 2.199 (Superset)**.** (cf. [654]) Let A and B be sets. We say that B is a *superset* of A if every element of A is also an element of B. This relationship is denoted by:

$$A \subseteq B \quad \text{or equivalently} \quad B \supseteq A.$$

Definition 2.200 (SuperHyperset)**.** [385] A *superhyperset* is an extension of a hyperset, inspired by the superhypergraph, where elements can be subsets of the power set of the universal set, allowing for higher-order nesting and relationships among sets. Formally, a superhyperset SH is defined as:

$$SH \subseteq \mathcal{P}(\mathcal{P}(U)),$$

where U is a universal set, and $\mathcal{P}(\mathcal{P}(U))$ is the power set of the power set of U. Elements of SH are subsets of $\mathcal{P}(U)$, enabling the construction of sets of sets of subsets of U, and so on, allowing for complex hierarchical and recursive structures.

Theorem 2.201. *The superhyperset generalizes both the hyperset and the superset.*

Proof. To prove that the *superhyperset* generalizes the *hyperset* and the *superset*, we proceed as follows:

Generalization of Hyperset

1. By definition, a hyperset H is a subset of the power set of a universal set U, i.e.,

$$H \subseteq \mathcal{P}(U).$$

2. A superhyperset SH is a subset of the power set of the power set of U, i.e.,

$$SH \subseteq \mathcal{P}(\mathcal{P}(U)).$$

3. Since $\mathcal{P}(U)$ is itself a set, any hyperset $H \subseteq \mathcal{P}(U)$ is an element of $\mathcal{P}(\mathcal{P}(U))$. Therefore, $H \in SH$ is allowed under the structure of a superhyperset.

Thus, every hyperset H can be interpreted as a special case of a superhyperset SH, where the elements of SH are restricted to subsets of $\mathcal{P}(U)$.

Generalization of Superset

1. By definition, a superset B is a set that contains every element of another set A, denoted by $A \subseteq B$ or $B \supseteq A$.
2. A superhyperset SH contains elements that are subsets of $\mathcal{P}(U)$. Let $A, B \subseteq U$, then $\{A, B\} \in \mathcal{P}(U)$.
3. The inclusion property $A \subseteq B$ can be embedded in the structure of a superhyperset by interpreting B as a collection of subsets $\{A_1, A_2, \ldots, A_k\} \subseteq \mathcal{P}(U)$, where $A_i \subseteq B$.
4. Hence, the superhyperset allows for relationships and nesting among subsets, extending the concept of a traditional superset.

The hyperset $H \subseteq \mathcal{P}(U)$ and the superset $B \supseteq A$ are both contained within the structural framework of a superhyperset $SH \subseteq \mathcal{P}(\mathcal{P}(U))$. Therefore, the superhyperset is a generalization of both the hyperset and the superset. □

Theorem 2.202. *The sets of supervertices and superedges in a superhypergraph exhibit the structure of a superhyperset.*

Proof. Let $H = (V, E)$ be a superhypergraph, where V is the set of supervertices and E is the set of superedges, as defined in [867]:

- Each supervertex $v \in V$ is a subset of the power set $\mathcal{P}(U)$ of a universal set U.
- Each superedge $e \in E$ is also a subset of $\mathcal{P}(U)$, possibly connecting multiple supervertices.

To demonstrate that V and E conform to the structure of a superhyperset, we consider the following:

By definition, a superhyperset $SH \subseteq \mathcal{P}(\mathcal{P}(U))$ contains elements that are subsets of the power set $\mathcal{P}(U)$. In the case of a superhypergraph, each supervertex $v \in V$ satisfies:

$$v \in \mathcal{P}(\mathcal{P}(U)).$$

Thus, the set of supervertices $V \subseteq \mathcal{P}(\mathcal{P}(U))$ qualifies as a superhyperset.

Similarly, each superedge $e \in E$ in a superhypergraph is a subset of $\mathcal{P}(U)$, and the set of all superedges satisfies:

$$E \subseteq \mathcal{P}(\mathcal{P}(U)).$$

Therefore, the set of superedges E also conforms to the structure of a superhyperset.

Both V and E allow for hierarchical and recursive relationships among subsets of $\mathcal{P}(U)$, fulfilling the criteria for superhypersets. For instance, a superedge $e \in E$ may connect multiple supervertices $v_1, v_2, \ldots, v_k \in V$, where each v_i itself is a subset of $\mathcal{P}(U)$.

Thus, the sets V (supervertices) and E (superedges) inherently exhibit the properties of a superhyperset. □

Theorem 2.203. *The n-th PowerSet, $\mathcal{P}_n^*(H)$, generalizes the concepts of supersets, hypersets, and superhypersets.*

Proof. A superset $B \supseteq A$ is defined as a set that contains all elements of another set A. The n-th PowerSet $\mathcal{P}_n^*(H)$ inherently generalizes the superset relationship because:

- For $n = 0$, $\mathcal{P}_0^*(H) = H$, representing the original set H.
- For $n = 1$, $\mathcal{P}_1^*(H) = \mathcal{P}(H)$, which is the set of all subsets of H, making $\mathcal{P}_1^*(H)$ the largest possible superset containing all subsets of H.

Thus, $\mathcal{P}_n^*(H)$ trivially includes all supersets of H for $n \geq 1$.

A hyperset $H \subseteq \mathcal{P}(U)$ extends traditional sets by allowing self-referential or recursive membership. The n-th PowerSet generalizes this structure because:

- For $n = 1$, $\mathcal{P}_1^*(H) = \mathcal{P}(H)$, which includes all subsets of H, allowing for a structure analogous to hypersets where elements can be subsets of the base set H.
- For $n \geq 2$, $\mathcal{P}_n^*(H)$ recursively applies the power set operation, allowing for nested subsets of subsets, which align with the recursive nature of hypersets.

Therefore, $\mathcal{P}_n^*(H)$ captures the hierarchical and recursive relationships of hypersets.

A superhyperset $SH \subseteq \mathcal{P}(\mathcal{P}(U))$ introduces higher-order nesting by considering subsets of the power set of the power set. The n-th PowerSet naturally generalizes this concept:

- For $n = 2$, $\mathcal{P}_2^*(H) = \mathcal{P}(\mathcal{P}(H))$, which corresponds to the definition of a superhyperset.
- For $n > 2$, $\mathcal{P}_n^*(H) = \mathcal{P}(\mathcal{P}_{n-1}^*(H))$, allowing for even higher-order nesting, surpassing the complexity of superhypersets.

Thus, $\mathcal{P}_n^*(H)$ extends the hierarchical structure of superhypersets to arbitrary levels.

The n-th PowerSet $\mathcal{P}_n^*(H)$ provides a unified framework that generalizes supersets ($n = 1$), hypersets ($n \geq 1$), and superhypersets ($n \geq 2$) by recursively applying the power set operation, enabling hierarchical and recursive structures at arbitrary levels. □

The concept of a partially ordered set (poset) is well-known as a fundamental structure related to sets and has been extensively studied in numerous research papers [241, 242, 261]. Based on this foundation, the superhyperposet is defined as follows. The validity and mathematical structure of these definitions are areas for potential exploration in future research. These definitions are inspired by structures such as hypergraphs and superhypergraphs, as discussed in [385].

Definition 2.204 (Poset)**.** A *partially ordered set (poset)* is a pair $(P, \leq)$, where:

- P is a set.
- $\leq$ is a binary relation on P that satisfies the following conditions for all $x, y, z \in P$:
  1. *Reflexivity*: $x \leq x$.
  2. *Antisymmetry*: If $x \leq y$ and $y \leq x$, then $x = y$.
  3. *Transitivity*: If $x \leq y$ and $y \leq z$, then $x \leq z$.

Definition 2.205 (Hyperposet)**.** A *hyperposet* generalizes a poset by allowing the elements of the set P to be subsets of a universal set U, rather than individual elements. Formally, a hyperposet is a pair $(H, \leq)$, where:

- $H \subseteq \mathcal{P}(U)$, the power set of a universal set U.
- $\leq$ is a binary relation on H that satisfies reflexivity, antisymmetry, and transitivity as defined for posets.

Definition 2.206 (n-SuperHyperPoset)**.** An *n-SuperHyperPoset* is a hierarchical generalization of posets, hyperposets, superposets, and superhyperposets, allowing elements to be subsets of the n-th PowerSet $\mathcal{P}_n^*(H)$. Formally, an n-SuperHyperPoset is defined as a pair $(SHP_n, \leq)$, where:

- $SHP_n \subseteq \mathcal{P}_n^*(H)$, the n-th PowerSet of a universal set H.

- $\leq$ is a binary relation on SHP_n that satisfies:
  1. **Reflexivity:** $x \leq x$ for all $x \in SHP_n$.
  2. **Antisymmetry:** If $x \leq y$ and $y \leq x$, then $x = y$.
  3. **Transitivity:** If $x \leq y$ and $y \leq z$, then $x \leq z$.

Example 2.207 (Examples for Specific n). 1. For $n = 0$: $SHP_0 = H$, reducing to the classical poset.

2. For $n = 1$: $SHP_1 \subseteq \mathcal{P}(H)$, representing a hyperposet.

3. For $n = 2$: $SHP_2 \subseteq \mathcal{P}(\mathcal{P}(H))$, aligning with the superhyperset.

4. For $n > 2$: $SHP_n = \mathcal{P}_n^*(H)$, extending hierarchical structures recursively.

Theorem 2.208. *A superhyperposet generalizes both the hyperposet.*

Proof. This is evident. □

Theorem 2.209. *A superhyperposet is a superhyperset.*

Proof. This is evident. □

As discussed above, it is inferred that the superhyperset and superhyperposet are not entirely new concepts but can be generalized from existing ones. Additionally, it should be noted that general superhyperconcepts are already considered superhyperconcepts starting from $n = 1$.

2.3.8 Other Hyperconcepts and Superhyperconcepts

Numerous hyper concepts are well-known across various fields, including hyperrandom variables [431, 432], hypertangles [9], hypertrees [8, 9, 408, 434–436], superhypertrees [409], hypercomplete graphs [219], hyperfunctions [531, 564, 589, 828], hyperlattice [79, 477, 478], superhyperfunctions [876, 884], hyperprobabilities [177], hyperconvex spaces [172, 316, 576, 722], hypergraph neural network [184, 210, 327, 398, 486, 509, 513, 544, 931, 974], hyperlogic [587],decision hypertrees [460], hyperalgebras [424, 741, 785, 868], superhyperalgebras [537, 870, 872, 897, 898], superhypergroups [567], semihypergroups [158, 320, 928, 1003], hypervector spaces [80, 81, 522, 925], hypertopologies [272, 632, 689], hypergraph grammars [229, 230, 311–313, 559], and superhypertopologies [885].

The following question reflects the author's curiosity about these concepts, demonstrating an interest in how mathematical extensions using such ideas might evolve and become more prevalent in the future. Note that in some fields, multiple definitions exist, and the meanings of "hyper" or "super" may vary.

Question 2.210. Is it possible to define a Hyper-random Graph? Additionally, can we define structures such as Hyper-random fuzzy sets and Hyper-random neutrosophic sets? What mathematical properties and potential applications could these constructs have?

Question 2.211. Is it possible to define concepts such as Fuzzy Hyperlogic and Neutrosophic Hyperlogic? Additionally, what would be their properties and potential applications?

Question 2.212. Is it possible to define a superhypergraph grammar as an extension of hypergraph grammar? Additionally, what would be its properties and potential applications? Furthermore, could these concepts be further generalized using fuzzy or neutrosophic frameworks?

Question 2.213. Is it possible to define a superhypergraph neural network as an extension of a hypergraph neural network? Additionally, what would be its properties and potential applications? Furthermore, could these concepts be further generalized using fuzzy or neutrosophic frameworks?

Question 2.214. Can the concept of hypergraphic matroids [318, 337] be extended to the framework of superhypergraphs to define superhypergraphic matroids?

Question 2.215. Can the concept of hypergraphic sequences [801–803] be extended to the framework of superhypergraphs to define superhypergraphic sequences?

Question 2.216. Define submodular hyperfunctions and submodular superhyperfunctions mathematically, and determine their potential applications. Note that a submodular function is a set function that exhibits diminishing returns, meaning adding an element to a smaller set yields a larger gain than adding it to a larger set [83,335].

Question 2.217. For concepts within Superhyper theory that have not yet been defined, is it possible to establish mathematically precise definitions? Additionally, are there potential applications? Furthermore, can these concepts be generalized using approaches that handle uncertainty, such as Fuzzy and Neutrosophic theories, to enable practical applications?

Question 2.218. In fields or concepts where semihyper concepts have not been examined, can semihyper conceptualization be applied? (cf. [384, 536])

3 Result in this paper

This section outlines the results presented in this paper. The concepts visualized as graphs, those designed to handle more complex uncertainties through specific methods, and concepts extended using hyper concepts or superhyper concepts are detailed below. The division of subsections reflects the author's personal perspective, and your understanding in this regard is appreciated.

3.1 Graph Concepts

We will examine several graph concepts.

3.1.1 NonStandard Real Graph

We present the definition of the NonStandard Real Graph as an extension of the NonStandard Real Set within the framework of graph theory.

Definition 3.1. A *NonStandard Real Graph* is a graph $G = (V, E)$ where:

- V is a non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges.
- Each vertex $v \in V$ is associated with a value $\sigma(v) \in \mathbb{R}^*$, where $\mathbb{R}^*$ is the NonStandard Real Set, including infinitesimals and infinite numbers.
- Each edge $e = (u, v) \in E$ is associated with a value $\mu(e) \in \mathbb{R}^*$.

Theorem 3.2. *In a NonStandard Real Graph $G = (V, E)$, if all vertex values $\sigma(v)$ are standard real numbers, then G reduces to a standard weighted graph.*

Proof. By definition, if all $\sigma(v) \in \mathbb{R}$ (standard real numbers), then the NonStandard Real Set $\mathbb{R}^*$ reduces to $\mathbb{R}$ for this graph. Thus, the graph G is a standard weighted graph where weights are real numbers. The inclusion of infinitesimal or infinite elements is nullified, and standard graph theory applies. □

Question 3.3. Is it possible to extend the above concepts using Superhypergraph frameworks? Furthermore, what are the practical applications, construction algorithms, and mathematical properties associated with these extended concepts?

3.1.2 pentapartitioned neutrosophic offgraph/overgraph/undergraph

The definitions of quadripartitioned neutrosophic offgraph/overgraph/undergraph and pentapartitioned neutrosophic offgraph/overgraph/undergraph are outlined as follows.

Definition 3.4 (Quadripartitioned Neutrosophic OverGraph (QNOvG))**.** A *Quadripartitioned Neutrosophic OverGraph* is a quadripartitioned neutrosophic graph in which membership degrees may exceed 1. For each vertex $v \in V$ and edge $e \in E$, the membership values are given by:

$$T(v) \in [0, \Omega_T], \quad C(v) \in [0, \Omega_C], \quad U(v) \in [0, \Omega_U], \quad F(v) \in [0, \Omega_F],$$

where $\Omega_T, \Omega_C, \Omega_U, \Omega_F > 1$ are the respective overlimits. The sum of the membership degrees satisfies:

$$0 \leq T(v) + C(v) + U(v) + F(v) \leq \Omega_T + \Omega_C + \Omega_U + \Omega_F.$$

Definition 3.5 (Quadripartitioned Neutrosophic UnderGraph (QNUdG))**.** A *Quadripartitioned Neutrosophic UnderGraph* is a quadripartitioned neutrosophic graph where membership degrees may be less than 0. Specifically, for each vertex $v \in V$ and edge $e \in E$, we have:

$$T(v) \in [\Psi_T, 1], \quad C(v) \in [\Psi_C, 1], \quad U(v) \in [\Psi_U, 1], \quad F(v) \in [\Psi_F, 1],$$

where $\Psi_T, \Psi_C, \Psi_U, \Psi_F < 0$ are underlimits. The sum of the membership degrees satisfies:

$$\Psi_T + \Psi_C + \Psi_U + \Psi_F \leq T(v) + C(v) + U(v) + F(v) \leq 4.$$

Definition 3.6 (Quadripartitioned Neutrosophic OffGraph (QNOfG))**.** A *Quadripartitioned Neutrosophic OffGraph* is a quadripartitioned neutrosophic graph where membership degrees may exceed 1 or fall below 0. For each vertex $v \in V$ and edge $e \in E$:

$$T(v) \in [\Psi_T, \Omega_T], \quad C(v) \in [\Psi_C, \Omega_C], \quad U(v) \in [\Psi_U, \Omega_U], \quad F(v) \in [\Psi_F, \Omega_F],$$

where $\Psi_i < 0$ and $\Omega_i > 1$ for $i \in \{T, C, U, F\}$. The sum satisfies:

$$\Psi_T + \Psi_C + \Psi_U + \Psi_F \leq T(v) + C(v) + U(v) + F(v) \leq \Omega_T + \Omega_C + \Omega_U + \Omega_F.$$

Definition 3.7 (Pentapartitioned Neutrosophic OverGraph (PNOvG))**.** A *Pentapartitioned Neutrosophic OverGraph* allows membership degrees to exceed 1. For each vertex $v \in V$ and edge $e \in E$:

$$T(v) \in [0, \Omega_T], \quad C(v) \in [0, \Omega_C], \quad R(v) \in [0, \Omega_R], \quad U(v) \in [0, \Omega_U], \quad F(v) \in [0, \Omega_F].$$

The total satisfies:

$$0 \leq T(v) + C(v) + R(v) + U(v) + F(v) \leq \sum_i \Omega_i.$$

Definition 3.8 (Pentapartitioned Neutrosophic UnderGraph (PNUdG))**.** A *Pentapartitioned Neutrosophic UnderGraph* allows membership degrees to be less than 0. For each vertex $v \in V$ and edge $e \in E$:

$$T(v) \in [\Psi_T, 1], \quad C(v) \in [\Psi_C, 1], \quad R(v) \in [\Psi_R, 1], \quad U(v) \in [\Psi_U, 1], \quad F(v) \in [\Psi_F, 1],$$

where $\Psi_T, \Psi_C, \Psi_R, \Psi_U, \Psi_F < 0$. The total satisfies:

$$\sum_i \Psi_i \leq T(v) + C(v) + R(v) + U(v) + F(v) \leq 5.$$

Definition 3.9 (Pentapartitioned Neutrosophic OffGraph (PNOfG))**.** A *Pentapartitioned Neutrosophic OffGraph* allows for both overlimits and underlimits. For each vertex $v \in V$ and edge $e \in E$:

$$T(v) \in [\Psi_T, \Omega_T], \quad C(v) \in [\Psi_C, \Omega_C], \quad R(v) \in [\Psi_R, \Omega_R], \quad U(v) \in [\Psi_U, \Omega_U], \quad F(v) \in [\Psi_F, \Omega_F].$$

The sum satisfies:

$$\sum_i \Psi_i \leq T(v) + C(v) + R(v) + U(v) + F(v) \leq \sum_i \Omega_i.$$

Theorem 3.10. *A quadripartitioned neutrosophic OffGraph can generalize both the quadripartitioned neutrosophic UnderGraph and the quadripartitioned neutrosophic OverGraph.*

Proof. The proof follows similarly to the case of Neutrosophic structures. □

Theorem 3.11. *Any quadripartitioned neutrosophic overgraph (undergraph, offgraph) can be transformed into a standard neutrosophic overgraph (undergraph, offgraph), preserving its structural properties.*

Proof. Let $G = (V, E)$ be a quadripartitioned neutrosophic overgraph. Define the standard graph $G' = (V, E)$ with transformed membership degrees for each vertex v and edge e:

$$\sigma'(v) = (T'(v), I'(v), F'(v)),$$

where:

$$T'(v) = \frac{T(v) + C(v)}{2}, \quad I'(v) = U(v), \quad F'(v) = F(v).$$

This transformation maintains the overlimit, underlimit, or both, as required, ensuring G' retains the essential properties of G. □

Theorem 3.12. *A pentapartitioned neutrosophic OffGraph can generalize both the pentapartitioned neutrosophic UnderGraph and the pentapartitioned neutrosophic OverGraph.*

Proof. The proof follows similarly to the case of Neutrosophic structures. □

Theorem 3.13. *Any pentapartitioned neutrosophic overgraph (undergraph, offgraph) can be transformed into a quadripartitioned neutrosophic overgraph (undergraph, offgraph), preserving its structural properties.*

Proof. Let $G = (V, E)$ be a pentapartitioned neutrosophic overgraph. Define the quadripartitioned graph $G' = (V, E)$ with membership values:

$$\sigma'(v) = (T'(v), C'(v), U'(v), F'(v)),$$

where:

$$T'(v) = T(v), \quad C'(v) = C(v), \quad U'(v) = U(v) + R(v), \quad F'(v) = F(v).$$

This preserves both the structure and properties, effectively reducing the pentapartition to a quadripartition while maintaining overlimits and underlimits. □

Question 3.14. Is it possible to extend the above concepts using Hypergraph and Superhypergraph frameworks? Furthermore, what are the practical applications, construction algorithms, and mathematical properties associated with these extended concepts?

3.1.3 Plithogenic offgraph/overgraph/undergraph

In this subsection, we define the Plithogenic OffGraph, OverGraph, and UnderGraph. These concepts extend the Plithogenic Graph by incorporating the notions of offset, overset, and underset into the graph structure.

First, we provide the definitions for Plithogenic offset, overset, and underset below. We will later extend these concepts to graph structures for further analysis.

Definition 3.15 (Plithogenic Overset)**.** Let S be a universal set, and $P \subseteq S$. A *Plithogenic Overset* PS_{over} is defined as:

$$PS_{\text{over}} = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.

- Pv is the set of possible values for the attribute v.
- $pdf : P \times Pv \to [0, \Omega_v]^s$ is the *Degree of Appurtenance Function (DAF)*, where $\Omega_v > 1$.
- $pCF : Pv \times Pv \to [0, \Omega_v]^t$ is the *Degree of Contradiction Function (DCF)*.

In this definition, the DAF allows the membership degrees $pdf(x, a)$ to exceed the standard maximum value of 1, up to an *overlimit* Ω_v.

Definition 3.16 (Plithogenic Underset)**.** Let S be a universal set, and $P \subseteq S$. A *Plithogenic Underset* PS_{under} is defined as:

$$PS_{\text{under}} = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the set of possible values for the attribute v.
- $pdf : P \times Pv \to [\Psi_v, 1]^s$ is the *Degree of Appurtenance Function (DAF)*, where $\Psi_v < 0$.
- $pCF : Pv \times Pv \to [\Psi_v, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

In this definition, the DAF allows the membership degrees $pdf(x, a)$ to be less than the standard minimum value of 0, down to an *underlimit* Ψ_v.

Definition 3.17 (Plithogenic Offset)**.** Let S be a universal set, and $P \subseteq S$. A *Plithogenic Offset* PS_{off} is defined as:

$$PS_{\text{off}} = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the set of possible values for the attribute v.
- $pdf : P \times Pv \to [\Psi_v, \Omega_v]^s$ is the *Degree of Appurtenance Function (DAF)*, where $\Psi_v < 0$ and $\Omega_v > 1$.
- $pCF : Pv \times Pv \to [\Psi, \Omega]^t$ is the *Degree of Contradiction Function (DCF)*.

In this definition, the DAF allows the membership degrees $pdf(x, a)$ to range from below 0 to above 1, between the underlimit Ψ_v and the overlimit Ω_v.

Definition 3.18 (Plithogenic OverGraph)**.** A *Plithogenic OverGraph* is a Plithogenic Graph where the membership degrees μ_{A_i} can exceed the standard maximum of 1. That is, $\mu_{A_i}(x) \in [0, \Omega_i]$ with $\Omega_i > 1$ for all attributes A_i and elements $x \in V \cup E$.

Definition 3.19 (Plithogenic UnderGraph)**.** A *Plithogenic UnderGraph* is a Plithogenic Graph where the membership degrees μ_{A_i} can be less than the standard minimum of 0. That is, $\mu_{A_i}(x) \in [\Psi_i, 1]$ with $\Psi_i < 0$ for all attributes A_i and elements $x \in V \cup E$.

Definition 3.20 (Plithogenic OffGraph)**.** A *Plithogenic OffGraph* is a Plithogenic Graph where the membership degrees μ_{A_i} can both exceed 1 and be less than 0. That is, $\mu_{A_i}(x) \in [\Psi_i, \Omega_i]$ with $\Psi_i < 0$ and $\Omega_i > 1$ for all attributes A_i and elements $x \in V \cup E$.

Theorem 3.21. *A Plithogenic OffGraph can generalize both the Plithogenic UnderGraph and the Plithogenic OverGraph.*

Proof. The proof follows similarly to the case of Neutrosophic structures. □

Theorem 3.22. *Any Plithogenic OffGraph (OverGraph, UnderGraph) can be transformed into a Pentapartitioned Neutrosophic OffGraph (OverGraph, UnderGraph), preserving its structural properties.*

Proof. To prove this theorem, we will demonstrate that for a given Plithogenic OffGraph $G_P = (V, E, \{A_i\}_{i=1}^s, \{\mu_{A_i}\}_{i=1}^s, pCF)$ with $s = 5$ attributes and $t = 1$ contradiction function, there exists a corresponding Pentapartitioned Neutrosophic OffGraph $G_N = (V, E, T, C, R, U, F)$ that preserves the structural properties of G_P.

In a Plithogenic OffGraph, each vertex $v \in V$ and edge $e \in E$ is characterized by:

- A set of $s = 5$ attributes $\{A_i\}_{i=1}^5$.
- Degree of Appurtenance Functions (DAFs) $\mu_{A_i} : V \cup E \to [\Psi_i, \Omega_i]$, where $\Psi_i < 0$ (UnderLimit) and $\Omega_i > 1$ (OverLimit), allowing membership degrees to be less than 0 or greater than 1.
- A Degree of Contradiction Function (DCF) $pCF : Pv \times Pv \to [0, 1]^t$ with $t = 1$.

In a Pentapartitioned Neutrosophic OffGraph, each vertex $v \in V$ and edge $e \in E$ has five membership degrees:

$$T(v), \quad C(v), \quad R(v), \quad U(v), \quad F(v),$$

where each degree can range between its UnderLimit Ψ_i and OverLimit Ω_i:

$$T(v) \in [\Psi_T, \Omega_T], \quad C(v) \in [\Psi_C, \Omega_C], \quad R(v) \in [\Psi_R, \Omega_R], \quad U(v) \in [\Psi_U, \Omega_U], \quad F(v) \in [\Psi_F, \Omega_F].$$

The sum of these degrees satisfies:

$$\sum_i \Psi_i \leq T(v) + C(v) + R(v) + U(v) + F(v) \leq \sum_i \Omega_i.$$

We will map the five attributes $\{A_i\}_{i=1}^5$ of the Plithogenic OffGraph to the five membership degrees of the Pentapartitioned Neutrosophic OffGraph. Specifically, we define:

$$\begin{aligned} T(v) &= \mu_{A_1}(v), \\ C(v) &= \mu_{A_2}(v), \\ R(v) &= \mu_{A_3}(v), \\ U(v) &= \mu_{A_4}(v), \\ F(v) &= \mu_{A_5}(v). \end{aligned}$$

Similarly, for edges $e \in E$:

$$T(e) = \mu_{A_1}(e),$$
$$C(e) = \mu_{A_2}(e),$$
$$R(e) = \mu_{A_3}(e),$$
$$U(e) = \mu_{A_4}(e),$$
$$F(e) = \mu_{A_5}(e).$$

Since $t = 1$, the DCF pCF in the Plithogenic OffGraph maps to a single contradiction degree between attribute values. In the Pentapartitioned Neutrosophic OffGraph, contradictions are inherently represented through the indeterminacy components $C(v)$ and $U(v)$, which capture conflicting and unknown information.

We need to ensure that:

- The membership degrees in G_N are within the appropriate ranges, matching the OverLimits and UnderLimits from G_P:

  $$T(v) \in [\Psi_{A_1}, \Omega_{A_1}], \quad C(v) \in [\Psi_{A_2}, \Omega_{A_2}], \quad R(v) \in [\Psi_{A_3}, \Omega_{A_3}], \quad U(v) \in [\Psi_{A_4}, \Omega_{A_4}], \quad F(v) \in [\Psi_{A_5}, \Omega_{A_5}].$$

- The sum of the membership degrees in G_N satisfies:

  $$\sum_{i=1}^{5} \Psi_{A_i} \leq T(v) + C(v) + R(v) + U(v) + F(v) \leq \sum_{i=1}^{5} \Omega_{A_i}.$$

- The structural properties, such as adjacency and incidence relations, are preserved since the vertex and edge sets remain unchanged, and the mapping does not alter the underlying graph structure.

By establishing a one-to-one correspondence between the attributes of the Plithogenic OffGraph and the membership degrees of the Pentapartitioned Neutrosophic OffGraph, and ensuring that the ranges and sums of these degrees are preserved, we have demonstrated that any Plithogenic OffGraph can be transformed into a Pentapartitioned Neutrosophic OffGraph without loss of structural properties. Therefore, the theorem is proved. □

For reference, the relationships between the Uncertain Offset are illustrated in Figure 4. Offset is a concept that increases the degree of freedom for the values of intervals.

Question 3.23. Is it possible to extend the above concepts using Hypergraph and Superhypergraph frameworks? Furthermore, what are the practical applications, construction algorithms, and mathematical properties associated with these extended concepts?

Definition 3.24 (General Plithogenic OverGraph)**.** A *General Plithogenic OverGraph* is a General Plithogenic Graph $G^{GP} = (PM, PN)$ where the membership degrees μ_{A_i} of vertices and edges can exceed the standard maximum of 1. Specifically:

- For each vertex $v \in M$ and edge $e \in N$:

  $$\mu_{A_i}(v) \in [0, \Omega_i], \quad \mu_{A_i}(e) \in [0, \Omega_i],$$

  where $\Omega_i > 1$ for all attributes A_i.

- The Degree of Appurtenance Function (DAF) is extended as:

  $$adf : M \times Ml \to [0, \Omega_i]^s, \quad bdf : N \times Nm \to [0, \Omega_i]^s,$$

  allowing the membership degrees to exceed the standard maximum.

Theorem 3.25. *A Plithogenic OverGraph is General Plithogenic OverGraph.*

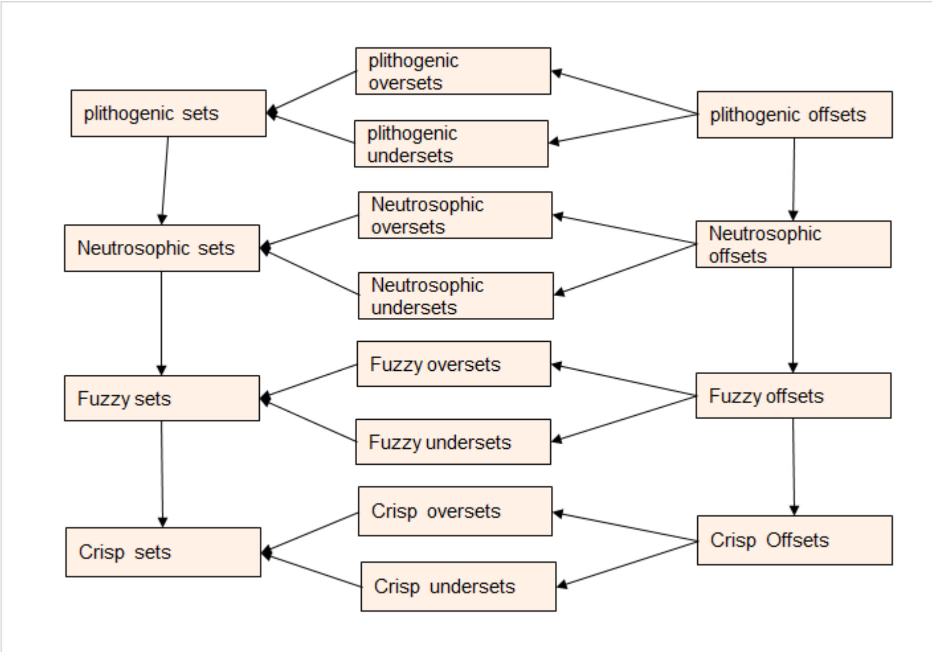


Figure 4: Some the Uncertain offsets Hierarchy. The set class at the origin of an arrow contains the set class at the destination of the arrow.

Proof. This is evident. □

Definition 3.26 (General Plithogenic UnderGraph)**.** A *General Plithogenic UnderGraph* is a General Plithogenic Graph $G^{GP} = (PM, PN)$ where the membership degrees μ_{A_i} of vertices and edges can be less than the standard minimum of 0. Specifically:

- For each vertex $v \in M$ and edge $e \in N$:

$$\mu_{A_i}(v) \in [\Psi_i, 1], \quad \mu_{A_i}(e) \in [\Psi_i, 1],$$

  where $\Psi_i < 0$ for all attributes A_i.

- The Degree of Appurtenance Function (DAF) is extended as:

$$adf : M \times Ml \to [\Psi_i, 1]^s, \quad bdf : N \times Nm \to [\Psi_i, 1]^s,$$

  allowing the membership degrees to be below the standard minimum.

Theorem 3.27. *A Plithogenic UnderGraph is General Plithogenic UnderGraph.*

Proof. This is evident. □

Definition 3.28 (General Plithogenic OffGraph)**.** A *General Plithogenic OffGraph* is a General Plithogenic Graph $G^{GP} = (PM, PN)$ where the membership degrees μ_{A_i} of vertices and edges can both exceed 1 and be less than 0. Specifically:

- For each vertex $v \in M$ and edge $e \in N$:

$$\mu_{A_i}(v) \in [\Psi_i, \Omega_i], \quad \mu_{A_i}(e) \in [\Psi_i, \Omega_i],$$

  where $\Psi_i < 0$ and $\Omega_i > 1$ for all attributes A_i.

- The Degree of Appurtenance Function (DAF) is extended as:

$$adf : M \times Ml \to [\Psi_i, \Omega_i]^s, \quad bdf : N \times Nm \to [\Psi_i, \Omega_i]^s,$$

  allowing the membership degrees to exceed 1 and go below 0.

Theorem 3.29. *A Plithogenic OffGraph is General Plithogenic OffGraph.*

Proof. This is evident. □

3.1.4 MultiNeutrosophic Graph

We define the MultiNeutrosophic Graph as an extension of the MultiNeutrosophic Set to the structure of a graph.

Definition 3.30. Let $G = (V, E)$ be a classical graph, where V is the set of vertices and E is the set of edges. In this context, we define a *MultiNeutrosophic Graph* G_{MN}, which is a Neutrosophic Graph extended to allow multiple evaluations for the truth, indeterminacy, and falsehood degrees associated with each vertex and edge. This structure provides a more nuanced representation of uncertainty by incorporating multiple sources of evaluation.

A *MultiNeutrosophic Graph* G_{MN} is defined as follows:

- For each vertex $v \in V$:
  - A set of truth degrees:
  $$T_v = \{T_{v,1}, T_{v,2}, \dots, T_{v,p_v}\}, \quad T_{v,j} \in [0,1], \quad j = 1, 2, \dots, p_v,$$
  where $p_v \geq 1$ is the number of truth evaluations for vertex v.
  - A set of indeterminacy degrees:
  $$I_v = \{I_{v,1}, I_{v,2}, \dots, I_{v,r_v}\}, \quad I_{v,k} \in [0,1], \quad k = 1, 2, \dots, r_v,$$
  where $r_v \geq 1$ is the number of indeterminacy evaluations for vertex v.
  - A set of falsehood degrees:
  $$F_v = \{F_{v,1}, F_{v,2}, \dots, F_{v,s_v}\}, \quad F_{v,l} \in [0,1], \quad l = 1, 2, \dots, s_v,$$
  where $s_v \geq 1$ is the number of falsehood evaluations for vertex v.
  - At least one of $p_v \geq 2$, $r_v \geq 2$, or $s_v \geq 2$ to ensure multiplicity in at least one component.
- Similarly, for each edge $e \in E$:
  - A set of truth degrees:
  $$T_e = \{T_{e,1}, T_{e,2}, \dots, T_{e,p_e}\}, \quad T_{e,j} \in [0,1], \quad j = 1, 2, \dots, p_e,$$
  where $p_e \geq 1$ is the number of truth evaluations for edge e.
  - A set of indeterminacy degrees:
  $$I_e = \{I_{e,1}, I_{e,2}, \dots, I_{e,r_e}\}, \quad I_{e,k} \in [0,1], \quad k = 1, 2, \dots, r_e,$$
  where $r_e \geq 1$ is the number of indeterminacy evaluations for edge e.
  - A set of falsehood degrees:
  $$F_e = \{F_{e,1}, F_{e,2}, \dots, F_{e,s_e}\}, \quad F_{e,l} \in [0,1], \quad l = 1, 2, \dots, s_e,$$
  where $s_e \geq 1$ is the number of falsehood evaluations for edge e.

In addition, for each vertex $v \in V$, the following condition holds to ensure the neutrosophic bounds are respected:

$$0 \leq \sum_{j=1}^{p_v} \inf T_{v,j} + \sum_{k=1}^{r_v} \inf I_{v,k} + \sum_{l=1}^{s_v} \inf F_{v,l} \leq \sum_{j=1}^{p_v} \sup T_{v,j} + \sum_{k=1}^{r_v} \sup I_{v,k} + \sum_{l=1}^{s_v} \sup F_{v,l} \leq n_v,$$

where $n_v = p_v + r_v + s_v \geq 2$. A similar condition applies to each edge $e \in E$.

Explanation of Components Each component provides a specific representation of uncertainty, as described below:

- *MultiTruth Degrees (T_v, T_e)*: A set of evaluations for the truth-membership degree for each vertex and edge.
- *MultiIndeterminacy Degrees (I_v, I_e)*: A set of evaluations for the indeterminacy-membership degree for each vertex and edge.
- *MultiFalsehood Degrees (F_v, F_e)*: A set of evaluations for the falsehood-membership degree for each vertex and edge.

Theorem 3.31. *Every Neutrosophic Graph is a specific case of a MultiNeutrosophic Graph.*

Proof. Let $G_N = (V, E)$ be a Neutrosophic Graph, where each vertex $v \in V$ is associated with single truth, indeterminacy, and falsehood degrees $T(v), I(v), F(v) \in [0, 1]$, and each edge $e \in E$ is similarly assigned single truth, indeterminacy, and falsehood degrees.

To construct a MultiNeutrosophic Graph G_{MN} that represents G_N, we define the degrees for each vertex $v \in V$ as follows:

- $T_v = \{T(v)\}$, making $p_v = 1$.
- $I_v = \{I(v)\}$, making $r_v = 1$.
- $F_v = \{F(v)\}$, making $s_v = 1$.

Similarly, for each edge $e \in E$:

- $T_e = \{T(e)\}$, making $p_e = 1$.
- $I_e = \{I(e)\}$, making $r_e = 1$.
- $F_e = \{F(e)\}$, making $s_e = 1$.

Since a MultiNeutrosophic Graph permits singleton sets for the degrees of truth, indeterminacy, and falsehood, this setup implies that a Neutrosophic Graph G_N can indeed be represented as a MultiNeutrosophic Graph G_{MN} with $p_v = r_v = s_v = 1$ for all vertices $v \in V$ and $p_e = r_e = s_e = 1$ for all edges $e \in E$.

Thus, the MultiNeutrosophic Graph G_{MN} generalizes the concept of a Neutrosophic Graph. □

3.1.5 Subset-Valued Neutrosophic Graph and NonStandard Neutrosophic Graph

In the literature [888], the Subset-Valued Neutrosophic Set and Single-Valued NonStandard Neutrosophic Set were defined. This work extends these concepts to graphs. Moving forward, we aim to investigate the mathematical structures and validity of these graph definitions.

Definition 3.32 (Infimum)**.** Let $S \subseteq \mathbb{R}$ be a subset of the real numbers. The *infimum* (or greatest lower bound) of S, denoted by $\inf(S)$, is defined as:

$$\inf(S) = \sup\{x \in \mathbb{R} \mid x \leq s, \forall s \in S\}.$$

Equivalently, $\inf(S) = m$ if and only if:

1. $m \leq s$ for all $s \in S$ (i.e., m is a lower bound of S).

2. For any $\epsilon > 0$, there exists an $s \in S$ such that $m + \epsilon > s$ (i.e., m is the greatest such lower bound).

If S is empty, we define $\inf(S) = +\infty$.

Definition 3.33 (Supremum)**.** Let $S \subseteq \mathbb{R}$ be a subset of the real numbers. The *supremum* (or least upper bound) of S, denoted by $\sup(S)$, is defined as:

$$\sup(S) = \inf\{x \in \mathbb{R} \mid x \geq s, \forall s \in S\}.$$

Equivalently, $\sup(S) = M$ if and only if:

1. $M \geq s$ for all $s \in S$ (i.e., M is an upper bound of S).
2. For any $\epsilon > 0$, there exists an $s \in S$ such that $M - \epsilon < s$ (i.e., M is the least such upper bound).

If S is empty, we define $\sup(S) = -\infty$.

Definition 3.34 (Subset-Valued Neutrosophic Set)**.** [888] Let X be a given set. A *Subset-Valued Neutrosophic Set (SVNS) A* on X is defined by three membership functions:

$$T_A : X \to \mathcal{P}([0,1]), \quad I_A : X \to \mathcal{P}([0,1]), \quad F_A : X \to \mathcal{P}([0,1]),$$

where $\mathcal{P}([0,1])$ denotes the power set of the interval $[0,1]$. For each $x \in X$, $T_A(x), I_A(x), F_A(x)$ are subsets of $[0,1]$ representing the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq \inf(T_A(x)) + \inf(I_A(x)) + \inf(F_A(x)) \leq \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \leq 3,$$

where inf and sup represent the infimum and supremum, respectively. The values $T_A(x), I_A(x), F_A(x)$ provide a subset-based representation of uncertainty.

Definition 3.35 (Left Monad)**.** [888] The *Left Monad*, denoted by $\mu(-a)$, is defined as:

$$\mu(-a) = \{-a + \epsilon \mid \epsilon \in \mathbb{R}_+^*, \epsilon \text{ is infinitesimal}\},$$

where $\mathbb{R}_+^*$ is the set of positive hyperreal numbers and ϵ is an infinitesimal value.

Definition 3.36 (Right Monad)**.** [888] The *Right Monad*, denoted by $\mu(a^+)$, is defined as:

$$\mu(a^+) = \{a + \epsilon \mid \epsilon \in \mathbb{R}_+^*, \epsilon \text{ is infinitesimal}\}.$$

This represents the set of hyperreal numbers infinitesimally greater than a.

Definition 3.37 (Left Monad Closed to the Right)**.** [888] The *Left Monad Closed to the Right*, denoted by $\mu(-0a)$, is defined as:

$$\mu(-0a) = \{-a + \epsilon \mid \epsilon = 0 \text{ or } \epsilon \in \mathbb{R}_+^*, \epsilon \text{ is infinitesimal}\} \cup \{-a\}.$$

It includes all elements of $\mu(-a)$ along with $-a$ itself.

Definition 3.38 (Right Monad Closed to the Left)**.** [888] The *Right Monad Closed to the Left*, denoted by $\mu(0a^+)$, is defined as:

$$\mu(0a^+) = \{a + \epsilon \mid \epsilon = 0 \text{ or } \epsilon \in \mathbb{R}_+^*, \epsilon \text{ is infinitesimal}\} \cup \{a\}.$$

It includes all elements of $\mu(a^+)$ along with a itself.

Definition 3.39 (Pierced Binad)**.** [888] The *Pierced Binad*, denoted by $\mu(-a^+)$, is defined as:

$$\mu(-a^+) = \{-a + \epsilon \mid \epsilon \in \mathbb{R}_+^*, \epsilon \text{ is infinitesimal}\} \cup \{a + \epsilon \mid \epsilon \in \mathbb{R}_+^*, \epsilon \text{ is infinitesimal}\}.$$

This represents the union of the left monad and the right monad without closure at $-a$ or a.

Definition 3.40 (Unpierced Binad)**.** [888] The *Unpierced Binad*, denoted by $\mu(-a^+ \cup a)$, is defined as:

$\mu(-a^+ \cup a) = \{-a+\epsilon \mid \epsilon = 0 \text{ or } \epsilon \in \mathbb{R}_+^*, \epsilon \text{ is infinitesimal}\} \cup \{a+\epsilon \mid \epsilon = 0 \text{ or } \epsilon \in \mathbb{R}_+^*, \epsilon \text{ is infinitesimal}\} \cup \{-a, a\}.$

This includes both left and right monads closed at $-a$ and a, creating a symmetric and fully closed structure.

Definition 3.41 (Single-Valued NonStandard Neutrosophic Set)**.** [888] Let X be a given set. A *Single-Valued NonStandard Neutrosophic Set (NoS SVNS) A* on X is defined by three membership functions:

$$T_A : X \to]-0, 1+[, \quad I_A : X \to]-0, 1+[, \quad F_A : X \to]-0, 1+[,$$

where $]-0, 1+[$ denotes the non-standard interval that extends $[0, 1]$ by including additional elements:

- Left Monads ($\mu(-a)$),
- Right Monads ($\mu(a^+)$),
- Left Monads Closed to the Right ($\mu(-0a)$),
- Right Monads Closed to the Left ($\mu(0a^+)$),
- Pierced Binads ($\mu(-a^+)$),
- Unpierced Binads ($\mu(-a^+ \cup a)$).

For each $x \in X$, the values $T_A(x), I_A(x), F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively, in the extended framework. These values satisfy the condition:

$$0 \le \inf(T_A(x)) + \inf(I_A(x)) + \inf(F_A(x)) \le \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \le 3^+,$$

where inf and sup denote the infimum and supremum over the extended interval $]-0, 1+[$. This definition enables modeling of nuanced or non-standard uncertainties using hyperreal extensions.

Definition 3.42 (Subset-Valued Neutrosophic Graph)**.** Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of edges. A *Subset-Valued Neutrosophic Graph (SVNG)* is defined by three membership functions for both vertices and edges:

$$T_V : V \to \mathcal{P}([0, 1]), \quad I_V : V \to \mathcal{P}([0, 1]), \quad F_V : V \to \mathcal{P}([0, 1]),$$

$$T_E : E \to \mathcal{P}([0, 1]), \quad I_E : E \to \mathcal{P}([0, 1]), \quad F_E : E \to \mathcal{P}([0, 1]),$$

where $\mathcal{P}([0, 1])$ is the power set of the interval $[0, 1]$. For each vertex $v \in V$ and edge $e \in E$, the subsets $T_V(v), I_V(v), F_V(v)$ and $T_E(e), I_E(e), F_E(e)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy:

$$0 \le \inf(T_V(v)) + \inf(I_V(v)) + \inf(F_V(v)) \le \sup(T_V(v)) + \sup(I_V(v)) + \sup(F_V(v)) \le 3,$$

$$0 \le \inf(T_E(e)) + \inf(I_E(e)) + \inf(F_E(e)) \le \sup(T_E(e)) + \sup(I_E(e)) + \sup(F_E(e)) \le 3,$$

where inf and sup denote the infimum and supremum, respectively. The SVNG provides a subset-based representation of uncertainty for both vertices and edges.

Definition 3.43 (Single-Valued NonStandard Neutrosophic Graph)**.** Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of edges. A *Single-Valued NonStandard Neutrosophic Graph (NoS SVNG)* is defined by three membership functions for both vertices and edges:

$$T_V : V \to]-0, 1+[, \quad I_V : V \to]-0, 1+[, \quad F_V : V \to]-0, 1+[,$$

$$T_E : E \to]-0, 1+[, \quad I_E : E \to]-0, 1+[, \quad F_E : E \to]-0, 1+[,$$

where $]-0, 1+[$ is the non-standard interval extending $[0, 1]$ to include elements such as:

- Left Monads ($\mu(-a)$),

- Right Monads ($\mu(a^+)$),
- Left Monads Closed to the Right ($\mu(-0a)$),
- Right Monads Closed to the Left ($\mu(0a^+)$),
- Pierced Binads ($\mu(-a^+)$),
- Unpierced Binads ($\mu(-a^+ \cup a)$).

For each vertex $v \in V$ and edge $e \in E$, the values $T_V(v), I_V(v), F_V(v)$ and $T_E(e), I_E(e), F_E(e)$ represent the degrees of truth, indeterminacy, and falsity, respectively, in the extended framework. These values satisfy:

$$0 \leq \inf(T_V(v)) + \inf(I_V(v)) + \inf(F_V(v)) \leq \sup(T_V(v)) + \sup(I_V(v)) + \sup(F_V(v)) \leq 3^+,$$

$$0 \leq \inf(T_E(e)) + \inf(I_E(e)) + \inf(F_E(e)) \leq \sup(T_E(e)) + \sup(I_E(e)) + \sup(F_E(e)) \leq 3^+,$$

where inf and sup denote the infimum and supremum over the extended interval $]-0, 1+[$. This framework allows modeling of nuanced and non-standard uncertainties for vertices and edges.

The related concept of a subset-valued fuzzy set is defined as follows.

Definition 3.44 (Subset-Valued Fuzzy Set)**.** Let X be a given set. A *Subset-Valued Fuzzy Set (SVFS) A* on X is defined by a membership function:

$$\mu_A : X \to \mathcal{P}([0, 1]),$$

where $\mathcal{P}([0, 1])$ denotes the power set of the interval $[0, 1]$. For each $x \in X$, $\mu_A(x)$ is a subset of $[0, 1]$ that represents the degrees of membership of x to the set A.

The degrees of membership satisfy the following condition for all $x \in X$:

$$0 \leq \inf(\mu_A(x)) \leq \sup(\mu_A(x)) \leq 1,$$

where inf and sup denote the infimum and supremum of the subset $\mu_A(x)$, respectively.

This definition provides a flexible framework for modeling uncertainty, where an element $x \in X$ can have multiple degrees of membership represented by a subset of $[0, 1]$.

Theorem 3.45. *A Subset-Valued Fuzzy Set (SVFS) is a special case of a Subset-Valued Neutrosophic Set (SVNS).*

Proof. Let X be a given set, and consider a Subset-Valued Fuzzy Set (SVFS) A on X, characterized by the membership function:

$$\mu_A : X \to \mathcal{P}([0, 1]),$$

where $\mathcal{P}([0, 1])$ denotes the power set of the interval $[0, 1]$. For each $x \in X$, the degrees of membership satisfy:

$$0 \leq \inf(\mu_A(x)) \leq \sup(\mu_A(x)) \leq 1.$$

Now consider a Subset-Valued Neutrosophic Set (SVNS) B on the same set X, characterized by three membership functions:

$$T_B : X \to \mathcal{P}([0, 1]), \quad I_B : X \to \mathcal{P}([0, 1]), \quad F_B : X \to \mathcal{P}([0, 1]),$$

where $T_B(x)$, $I_B(x)$, and $F_B(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively, and satisfy the condition:

$$0 \leq \inf(T_B(x)) + \inf(I_B(x)) + \inf(F_B(x)) \leq \sup(T_B(x)) + \sup(I_B(x)) + \sup(F_B(x)) \leq 3.$$

To show that A is a special case of B, we define B such that:

$$T_B(x) = \mu_A(x), \quad I_B(x) = \{0\}, \quad F_B(x) = \{0\}, \quad \forall x \in X.$$

Under this construction, $T_B(x)$ retains the subset-valued membership function of $\mu_A(x)$, satisfying:

$$0 \leq \inf(T_B(x)) \leq \sup(T_B(x)) \leq 1.$$

Since $I_B(x) = \{0\}$ and $F_B(x) = \{0\}$, we have:

$$\inf(I_B(x)) = \sup(I_B(x)) = 0, \quad \inf(F_B(x)) = \sup(F_B(x)) = 0.$$

Therefore, the condition for the SVNS is satisfied:

$$0 \leq \inf(T_B(x)) + \inf(I_B(x)) + \inf(F_B(x)) \leq \sup(T_B(x)) + \sup(I_B(x)) + \sup(F_B(x)) \leq 1.$$

This demonstrates that A, the Subset-Valued Fuzzy Set, is a special case of B, the Subset-Valued Neutrosophic Set, where indeterminacy and falsity are fixed to $\{0\}$, and only the truth membership $T_B(x)$ is active. □

3.1.6 Heptapartitioned Neutrosophic Graph

The Heptapartitioned Neutrosophic Set, which generalizes the Pentapartitioned Neutrosophic Set, is a known concept. It handles seven uncertainty membership parameters. Here, we present the definition of the Heptapartitioned Neutrosophic Graph, which represents the graphization of the Heptapartitioned Neutrosophic Set [165, 688, 986].

Definition 3.46 (Heptapartitioned Neutrosophic Set (HNS))**.** [165] Let U be a non-empty universe. A *Heptapartitioned Neutrosophic Set (HNS) A* on U is characterized by seven membership functions:

$$A = \{\langle x, T_A(x), M_A(x), C_A(x), U_A(x), I_A(x), K_A(x), F_A(x)\rangle : x \in U\},$$

where:

$$T_A(x), M_A(x), C_A(x), U_A(x), I_A(x), K_A(x), F_A(x) : U \to [0, 1],$$

and the membership values satisfy the condition:

$$T_A(x) + M_A(x) + C_A(x) + U_A(x) + I_A(x) + K_A(x) + F_A(x) \leq 7, \quad \forall x \in U.$$

The interpretation of each membership function is as follows:

- $T_A(x)$: Truth membership
- $M_A(x)$: Relative truth
- $C_A(x)$: Contradiction
- $U_A(x)$: Unknown membership
- $I_A(x)$: Ignorance
- $K_A(x)$: Relative falsity
- $F_A(x)$: Absolute falsity

Theorem 3.47. *A Plithogenic Set with $s = 7$ and $t = 1$ generalizes the Heptapartitioned Neutrosophic Set (HNS).*

Proof. Let A be a Heptapartitioned Neutrosophic Set (HNS) defined on a universe U:

$$A = \{\langle x, T_A(x), M_A(x), C_A(x), U_A(x), I_A(x), K_A(x), F_A(x)\rangle : x \in U\},$$

where:

$$T_A(x) + M_A(x) + C_A(x) + U_A(x) + I_A(x) + K_A(x) + F_A(x) \le 7.$$

A Plithogenic Set $PS = (P, v, Pv, pdf, pCF)$ with $s = 7$ is characterized by:

$$pdf(x, Pv) = \big(T_A(x), M_A(x), C_A(x), U_A(x), I_A(x), K_A(x), F_A(x)\big).$$

Additionally, the single contradiction degree $t = 1$ aligns with $C_A(x)$, ensuring consistency.

As the Plithogenic Set allows for $s > 7$ and $t > 1$, it generalizes the HNS structure. When restricted to $s = 7, t = 1$, the Plithogenic Set simplifies to the HNS.

Thus, the theorem is proved. □

Definition 3.48 (Heptapartitioned Neutrosophic Graph)**.** Let $G = (V, E)$ be a graph. A *Heptapartitioned Neutrosophic Graph (HNG)* G_H is defined as:

$$G_H = \{\langle v, T_G(v), M_G(v), C_G(v), U_G(v), I_G(v), K_G(v), F_G(v)\rangle : v \in V\}$$
$$\cup\{\langle e, T_G(e), M_G(e), C_G(e), U_G(e), I_G(e), K_G(e), F_G(e)\rangle : e \in E\},$$

where:

- $T_G, M_G, C_G, U_G, I_G, K_G, F_G : V \cup E \to [0, 1]$,
- For all $x \in V \cup E$, the membership values satisfy:

$$T_G(x) + M_G(x) + C_G(x) + U_G(x) + I_G(x) + K_G(x) + F_G(x) \le 7.$$

- Each edge $e = (u, v) \in E$ satisfies:

$$\begin{aligned} &T_G(e) \le \min(T_G(u), T_G(v)), \quad M_G(e) \le \min(M_G(u), M_G(v)),\\ &C_G(e) \le \min(C_G(u), C_G(v)), \quad U_G(e) \ge \max(U_G(u), U_G(v)),\\ &I_G(e) \ge \max(I_G(u), I_G(v)), \quad K_G(e) \ge \max(K_G(u), K_G(v)),\\ &F_G(e) \ge \max(F_G(u), F_G(v)). \end{aligned}$$

Theorem 3.49. *The Heptapartitioned Neutrosophic Graph (HNG) generalizes the Pentapartitioned Neutrosophic Graph (PNG).*

Proof. Let $G_P = (V, E)$ be a Pentapartitioned Neutrosophic Graph characterized by five membership functions:

$$G_P = \{\langle x, T_G(x), C_G(x), R_G(x), U_G(x), F_G(x)\rangle : x \in V \cup E\},$$

where:

$$T_G(x) + C_G(x) + R_G(x) + U_G(x) + F_G(x) \le 5, \quad \forall x \in V \cup E.$$

Let $G_H = (V, E)$ be a Heptapartitioned Neutrosophic Graph characterized by seven membership functions:

$$G_H = \{\langle x, T_G(x), M_G(x), C_G(x), U_G(x), I_G(x), K_G(x), F_G(x)\rangle : x \in V \cup E\},$$

where:

$$T_G(x) + M_G(x) + C_G(x) + U_G(x) + I_G(x) + K_G(x) + F_G(x) \le 7, \quad \forall x \in V \cup E.$$

The five membership functions of G_P can be mapped to the seven membership functions of G_H as follows:

$$T_G^{\text{HNG}}(x) = T_G^{\text{PNG}}(x), \quad C_G^{\text{HNG}}(x) = C_G^{\text{PNG}}(x), \quad U_G^{\text{HNG}}(x) = U_G^{\text{PNG}}(x),$$

$$F_G^{\text{HNG}}(x) = F_G^{\text{PNG}}(x),$$

with:

$$M_G^{\text{HNG}}(x) = 0, \quad I_G^{\text{HNG}}(x) = R_G^{\text{PNG}}(x), \quad K_G^{\text{HNG}}(x) = 0.$$

This mapping ensures that the sum of the membership values in G_H satisfies:

$$T_G(x) + M_G(x) + C_G(x) + U_G(x) + I_G(x) + K_G(x) + F_G(x) \leq 7.$$

Furthermore, setting $M_G(x) = 0$ and $K_G(x) = 0$ reduces G_H to G_P, demonstrating that G_H generalizes G_P.

Hence, the Heptapartitioned Neutrosophic Graph generalizes the Pentapartitioned Neutrosophic Graph. □

Theorem 3.50. *A Heptapartitioned Neutrosophic Graph (HNG) can be constructed as a specific instance of a General Plithogenic Graph G^{GP} with $s = 7$ and $t = 1$.*

Proof. Now, consider a General Plithogenic Graph $G^{GP} = (PM, PN)$, where:

- $PM = (M, l, Ml, adf, aCf)$ represents the vertex set.
- $PN = (N, m, Nm, bdf, bCf)$ represents the edge set.
- $adf : M \times Ml \to [0, 1]^s$ assigns $s = 7$ membership values to each vertex.
- $bdf : N \times Nm \to [0, 1]^s$ assigns $s = 7$ membership values to each edge.

Map the vertex and edge attributes as follows:

$$adf(x, v) \mapsto \langle T_G(x), M_G(x), C_G(x), U_G(x), I_G(x), K_G(x), F_G(x) \rangle, \quad \forall x \in V,$$

$$bdf(e, m) \mapsto \langle T_G(e), M_G(e), C_G(e), U_G(e), I_G(e), K_G(e), F_G(e) \rangle, \quad \forall e \in E.$$

This mapping ensures the membership values satisfy the conditions of G_H, and the additional constraints for edges in G_H are inherited from the Plithogenic edge definitions.

Hence, a Heptapartitioned Neutrosophic Graph is a specific instance of a General Plithogenic Graph with $s = 7$ and $t = 1$. □

For reference, the relationships between the graphs are illustrated in Figure 5. (cf. [368, 373])

3.1.7 Double-valued neutrosophic Graphs

Double-Valued Neutrosophic Graphs are the graphical extension of Double-Valued Neutrosophic Sets. Like other types of neutrosophic sets, Double-Valued Neutrosophic Sets have been the subject of extensive research [286, 561, 634, 980]. The definition is provided below.

Definition 3.51 (Double-Valued Neutrosophic Set (DVNS))**.** [561] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite non-empty set ($n > 1$). A *Double-Valued Neutrosophic Set (DVNS) A* defined on X is characterized by the following membership functions:

$$T_A(x), \quad I_A^T(x), \quad I_A^F(x), \quad F_A(x),$$

where:

- $T_A(x) : X \to [0, 1]$ represents the truth membership function.
- $I_A^T(x) : X \to [0, 1]$ represents the indeterminacy leaning towards truth membership function.

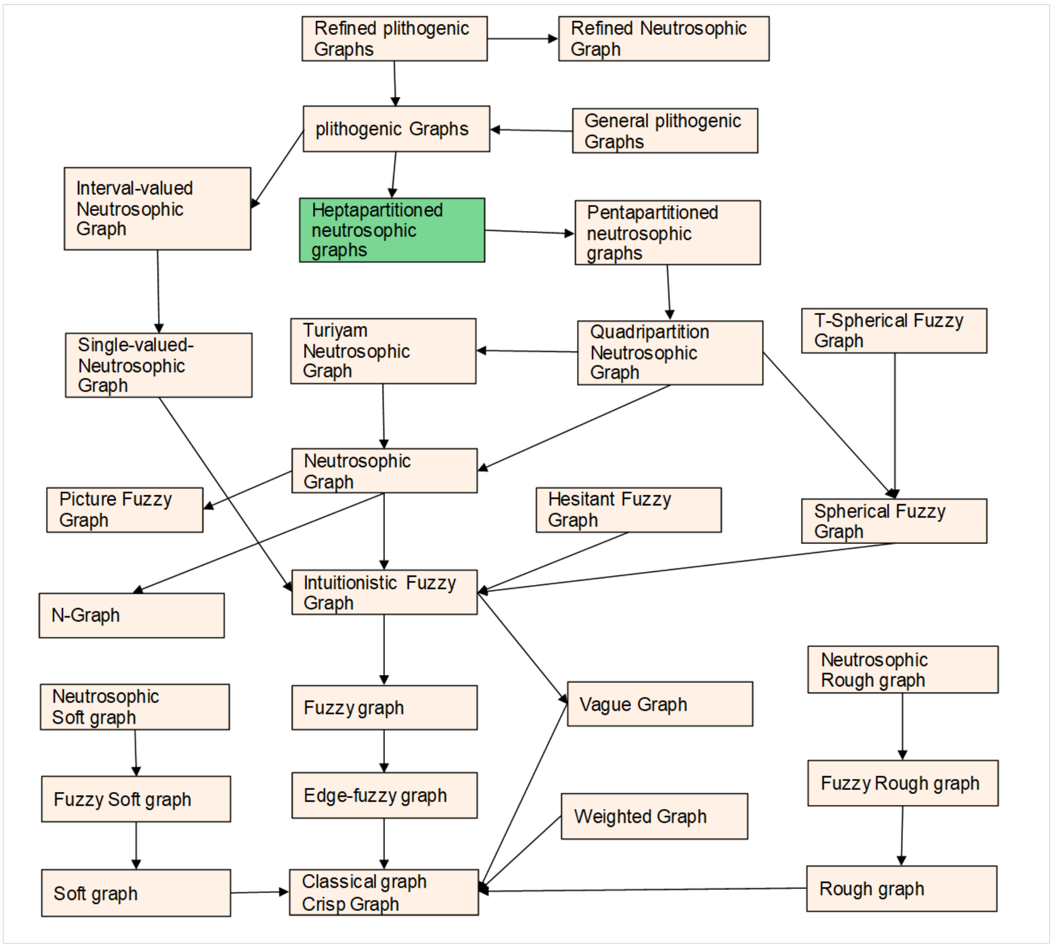


Figure 5: Some Uncertain graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

- $I_A^F(x) : X \to [0,1]$ represents the indeterminacy leaning towards falsity membership function.
- $F_A(x) : X \to [0,1]$ represents the falsity membership function.

These membership functions satisfy the following conditions:

1. For each $x \in X$:
$$T_A(x),\ I_A^T(x),\ I_A^F(x),\ F_A(x) \in [0,1].$$
2. Normalization condition:
$$0 \leq T_A(x) + I_A^T(x) + I_A^F(x) + F_A(x) \leq 4.$$
3. If X is continuous:
$$A = \left\{ \langle x, T_A(x), I_A^T(x), I_A^F(x), F_A(x) \rangle \mid x \in X \right\}.$$
If X is discrete:
$$A = \left\{ \sum_{i=1}^{n} \langle T_A(x_i), I_A^T(x_i), I_A^F(x_i), F_A(x_i) \rangle \mid x_i \in X \right\}.$$

Definition 3.52 (Double-Valued Neutrosophic Graph (DVNG))**.** A *Double-Valued Neutrosophic Graph (DVNG)* is an extension of a classical graph $G = (V, E)$, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. In a DVNG, each vertex $v \in V$ and edge $e \in E$ is associated with four membership functions that define the degrees of truth, indeterminacy (leaning towards truth and falsity), and falsity. These are denoted as:

$$T(v),\ I^T(v),\ I^F(v),\ F(v) : V \to [0,1],$$
$$T(e),\ I^T(e),\ I^F(e),\ F(e) : E \to [0,1].$$

Conditions: For each vertex $v \in V$ and edge $e \in E$, the following conditions hold:

1. Membership values are bounded:

$$T(v),\ I^T(v),\ I^F(v),\ F(v) \in [0,1], \quad T(e),\ I^T(e),\ I^F(e),\ F(e) \in [0,1].$$

2. Normalization constraint:

$$0 \leq T(v) + I^T(v) + I^F(v) + F(v) \leq 4, \quad 0 \leq T(e) + I^T(e) + I^F(e) + F(e) \leq 4.$$

Representation: A DVNG is represented as:

$$G = \left(V, E, T, I^T, I^F, F\right),$$

where T, I^T, I^F, F represent the mappings for vertices and edges.

Graphical Interpretation: Each vertex $v \in V$ and edge $e \in E$ can be graphically represented with the following membership components:

$$\langle T(v), I^T(v), I^F(v), F(v)\rangle, \quad \langle T(e), I^T(e), I^F(e), F(e)\rangle.$$

These represent the truth, indeterminacy leaning towards truth and falsity, and falsity degrees, respectively.

Theorem 3.53. *A Double-Valued Neutrosophic Graph (DVNG) is a specific case of a General Plithogenic Graph G^{GP} with $s = 4$ and $t = 1$.*

Proof. Let $G = (V, E)$ be a Double-Valued Neutrosophic Graph (DVNG), where each vertex $v \in V$ and edge $e \in E$ is associated with four membership values:

$$T(v),\ I^T(v),\ I^F(v),\ F(v), \quad T(e),\ I^T(e),\ I^F(e),\ F(e),$$

representing truth, indeterminacy leaning towards truth, indeterminacy leaning towards falsity, and falsity, respectively. These values satisfy:

$$0 \leq T(x) + I^T(x) + I^F(x) + F(x) \leq 4, \quad \forall x \in V \cup E.$$

A General Plithogenic Graph $G^{GP} = (PM, PN)$ is defined as:

$$PM = (M, l, Ml, adf, aCf), \quad PN = (N, m, Nm, bdf, bCf),$$

where s-dimensional Degree of Appurtenance Functions (DAF) and t-dimensional Degree of Contradiction Functions (DCF) are used to assign membership and contradiction values to vertices and edges.

To map G to G^{GP}, consider the following assignments:

- Vertex Set Mapping (PM):

$$M = V, \quad l = \text{vertex attributes}, \quad Ml = \{T, I^T, I^F, F\},$$

$$adf(v,a) = \begin{cases} T(v), & \text{if } a = T, \\ I^T(v), & \text{if } a = I^T, \\ I^F(v), & \text{if } a = I^F, \\ F(v), & \text{if } a = F, \end{cases}$$

  where $adf : M \times Ml \to [0,1]^4$, and aCf is trivial (e.g., $aCf(a,b) = 0$, as there is no vertex contradiction in DVNG).

- Edge Set Mapping (PN):

$$N = E, \quad m = \text{edge attributes}, \quad Nm = \{T, I^T, I^F, F\},$$

$$bdf(e,a) = \begin{cases} T(e), & \text{if } a = T, \\ I^T(e), & \text{if } a = I^T, \\ I^F(e), & \text{if } a = I^F, \\ F(e), & \text{if } a = F, \end{cases}$$

where $bdf : N \times Nm \to [0,1]^4$, and bCf is trivial (e.g., $bCf(a,b) = 0$, as there is no edge contradiction in DVNG).

The condition $s = 4$ matches the four membership functions T, I^T, I^F, F of the DVNG. The normalization condition:

$$0 \leq T(x) + I^T(x) + I^F(x) + F(x) \leq 4,$$

ensures consistency with the definition of a Plithogenic Graph, where the sum of s-dimensional membership values must lie within $[0, s]$.

Since $t = 1$, the DCF aCf and bCf are trivial, meaning there are no inter-attribute contradictions, aligning with the structure of DVNG.

Thus, a Double-Valued Neutrosophic Graph is a specific case of a General Plithogenic Graph with $s = 4$ and $t = 1$. □

3.2 Uncertain Concepts

We introduce several extended uncertain concepts.

3.2.1 MultiPlithogenic Set

The MultiPlithogenic Set is an extension of the MultiNeutrosophic Set. The definition is provided below.

Definition 3.54 (MultiPlithogenic Set). Let S be a universal set, and $P \subseteq S$. A *MultiPlithogenic Set* MPS is defined as:

$$MPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{pdf_i\}_{i=1}^n, pCF)$$

where:

- $\{v_i\}_{i=1}^n$ is a set of attributes.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $pdf_i : P \times Pv_i \to [0,1]$ is the *Degree of Appurtenance Function (DAF)*.
- $pCF : \left(\bigcup_{i=1}^n Pv_i\right) \times \left(\bigcup_{i=1}^n Pv_i\right) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

In this definition, each element $x \in P$ can be associated with multiple attributes v_i and their corresponding attribute values $a_i \in Pv_i$, each with a degree of appurtenance $pdf_i(x, a_i)$.

Theorem 3.55. *Every MultiPlithogenic Set with $s = 3$ and $t = 1$ can be transformed into an equivalent MultiNeutrosophic Set.*

Proof. Let $MPS = (P, v, Pv, pdf, pCF)$ be a MultiPlithogenic Set. Our goal is to construct a MultiNeutrosophic Set N where each element $x \in P$ has multiple degrees of truth, indeterminacy, and falsehood corresponding to the degrees in MPS.

Construction: For each element $x \in P$ and each attribute value $a \in Pv$, the DAF $pdf(x, a)$ yields a 3-tuple of values in $[0, 1]$, which we denote as:

$$pdf(x, a) = (d_1(x, a),\ d_2(x, a),\ d_3(x, a)),$$

where $d_i(x, a) \in [0, 1]$ for $i = 1, 2, 3$.

We interpret these three degrees as representing the truth, indeterminacy, and falsehood degrees of x with respect to the attribute value a.

Define the MultiNeutrosophic Set N as follows:

$$N = \{ x,\ x(T_x, I_x, F_x) \mid x \in P \},$$

where:

- The set of truth degrees for x is:
$$T_x = \{ d_1(x, a) \mid a \in Pv \}.$$
- The set of indeterminacy degrees for x is:
$$I_x = \{ d_2(x, a) \mid a \in Pv \}.$$
- The set of falsehood degrees for x is:
$$F_x = \{ d_3(x, a) \mid a \in Pv \}.$$

Verification:

1. *Multiplicity of Degrees:* Since $s = 3$, each $pdf(x, a)$ provides three degrees per attribute value a. By collecting these degrees over all $a \in Pv$, we obtain multiple truth, indeterminacy, and falsehood degrees for each x, satisfying the definition of a MultiNeutrosophic Set.
2. *Values in* $[0, 1]$*:* Each $d_i(x, a) \in [0, 1]$, so the degrees in T_x, I_x, and F_x are within the required range.
3. *Neutrosophic Conditions:* In the definition of a MultiNeutrosophic Set, the sum of the minimal degrees should satisfy:
$$0 \leq \sum_{j=1}^{p_x} \inf T_{x,j} + \sum_{k=1}^{r_x} \inf I_{x,k} + \sum_{l=1}^{s_x} \inf F_{x,l} \leq \sum_{j=1}^{p_x} \sup T_{x,j} + \sum_{k=1}^{r_x} \sup I_{x,k} + \sum_{l=1}^{s_x} \sup F_{x,l} \leq n_x,$$
where $n_x = p_x + r_x + s_x$ and p_x, r_x, s_x are the numbers of truth, indeterminacy, and falsehood degrees for x, respectively.
Since all degrees $d_i(x, a) \in [0, 1]$, the sums satisfy the above inequality naturally.
4. *Preservation of Information:* All the information from MPS is preserved in N through the mapping of degrees. Each element x retains the degrees associated with each attribute value a in Pv.

Thus, every MultiPlithogenic Set with $s = 3$ and $t = 1$ can be transformed into an equivalent MultiNeutrosophic Set by interpreting the three degrees in the DAF as the truth, indeterminacy, and falsehood degrees in the MultiNeutrosophic Set. This transformation preserves the structure and information of the original set. □

Theorem 3.56. *A MultiPlithogenic Set MPS is a generalization of a Plithogenic Set PS. Specifically, when the number of attributes in a MultiPlithogenic Set is restricted to one ($n = 1$), it reduces to a Plithogenic Set.*

Proof. A Plithogenic Set is defined as $PS = (P, v, Pv, pdf, pCF)$, where v is a single attribute, Pv is the set of possible values for v, pdf is the Degree of Appurtenance Function, and pCF is the Degree of Contradiction Function. On the other hand, a MultiPlithogenic Set is defined as $MPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{pdf_i\}_{i=1}^n, pCF)$, where $\{v_i\}_{i=1}^n$ represents multiple attributes, $\{Pv_i\}_{i=1}^n$ represents their respective value sets, $\{pdf_i\}_{i=1}^n$ represents the Degree of Appurtenance Functions, and pCF is a generalized Degree of Contradiction Function over the union of all attribute values.

When $n = 1$, the MultiPlithogenic Set reduces to $MPS = (P, v_1, Pv_1, pdf_1, pCF)$. This matches the definition of a Plithogenic Set, where $v = v_1$, $Pv = Pv_1$, and $pdf = pdf_1$. Thus, a Plithogenic Set is a special case of a MultiPlithogenic Set, proving that the latter generalizes the former. □

Theorem 3.57. *A MultiPlithogenic Set with $s = 1, t = 1$, and binary restriction generalizes a MultiCrisp Set.*

Proof. A MultiCrisp Set is defined as

$$M = \{(x, \mu_1(x), \mu_2(x), \ldots, \mu_k(x)) \mid x \in \mathcal{U}\}$$

, where each $\mu_i(x)$ is a crisp membership function taking values in $\{0, 1\}$. A MultiPlithogenic Set is defined as

$$MPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{pdf_i\}_{i=1}^n, pCF)$$

, where $pdf_i(x, a_i)$ is the Degree of Appurtenance Function.

Under the restriction $s = 1, t = 1$, and binary-valued $pdf_i(x, a_i) \in \{0, 1\}$, each $pdf_i(x, a_i)$ can be interpreted as a crisp membership function analogous to $\mu_i(x)$. Setting $n = k$, the attributes $\{v_i\}_{i=1}^n$ in the MultiPlithogenic Set correspond to the k membership conditions in the MultiCrisp Set, and the attribute values $\{Pv_i\}_{i=1}^n$ are binary ($\{0, 1\}$).

Thus, the MultiPlithogenic Set structure directly subsumes the MultiCrisp Set structure when these restrictions are applied. This proves that a MultiPlithogenic Set generalizes a MultiCrisp Set. □

3.2.2 MultiPlithogenic Graphs

The definition of MultiPlithogenic Graphs is provided below. This concept represents the graphization of the previously described MultiPlithogenic Sets.

Definition 3.58. Let $G^* = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. Let A be a set of attribute combinations where $A \subseteq \mathcal{P}(E)$.

A *MultiPlithogenic Graph* $G = (G^*, PM, PN, A)$ over the graph $G^* = (V, E)$ consists of:

1. PM is a MultiPlithogenic Vertex Set:

$$PM = (M, l, Ml, adf, aCf),$$

   where:

   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the set of possible attribute values.
   - $adf : M \times Ml \to [0, 1]^s$ is the Degree of Appurtenance Function for vertices, allowing each vertex to have multiple degrees corresponding to multiple attribute values.
   - $aCf : Ml \times Ml \to [0, 1]^t$ is the Degree of Contradiction Function for vertices.

2. PN is a MultiPlithogenic Edge Set:

$$PN = (N, m, Nm, bdf, bCf),$$

   where:

- $N \subseteq E$ is the set of edges.
- m is an attribute associated with the edges.
- Nm is the set of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$ is the Degree of Appurtenance Function for edges.
- $bCf : Nm \times Nm \to [0,1]^t$ is the Degree of Contradiction Function for edges.

In the MultiPlithogenic Graph, each vertex and edge can be associated with multiple attribute values simultaneously, each with corresponding degrees of appurtenance and contradiction.

Theorem 3.59. *Every MultiPlithogenic Graph $G = (G^*, PM, PN, A)$ over a crisp graph $G^* = (V, E)$ can be transformed into an equivalent Plithogenic Graph $G' = (G^*, PM', PN')$.*

Proof. To transform a MultiPlithogenic Graph into a Plithogenic Graph, we proceed as follows:

1. *Aggregation of Vertex Attributes*:
   - For each vertex $v \in M$, which may have multiple attribute values in the MultiPlithogenic Graph, we combine these into a single attribute value in the Plithogenic Graph.
   - Define a combined attribute value $l'(v)$ for vertex v using an appropriate method, such as selecting the most significant attribute or aggregating attributes into a composite one.
   - Aggregate the degrees of appurtenance $adf(v,l)$ into a single degree $adf'(v,l')$ using an aggregation function, for example:
   $$adf'(v,l') = \max_{l \in Ml} adf(v,l).$$
2. *Aggregation of Edge Attributes*:
   - For each edge $e \in N$, combine multiple attribute values into a single attribute value $m'(e)$ similarly.
   - Aggregate the degrees of appurtenance $bdf(e,m)$ into a single degree $bdf'(e,m')$:
   $$bdf'(e,m') = \max_{m \in Nm} bdf(e,m).$$
3. *Retaining Contradiction Functions*:
   - The Degrees of Contradiction Functions aCf and bCf remain applicable between the aggregated attribute values.
4. *Construction of the Plithogenic Graph*:
   - The resulting Plithogenic Graph is $G' = (G^*, PM', PN')$, where:
   $$PM' = (M, l', Ml', adf', aCf), \quad PN' = (N, m', Nm', bdf', bCf).$$
   - Ml' and Nm' are the sets of aggregated attribute values for vertices and edges, respectively.

By consolidating multiple attribute values and their corresponding degrees into single attribute values with aggregated degrees, we obtain a Plithogenic Graph that preserves the essential characteristics of the original MultiPlithogenic Graph. □

Question 3.60. The set known as Multineutrosophic Soft Rough Sets has been established. Is it possible to define Multineutrosophic Soft Rough OverSets, UnderSets, and OffSets? Furthermore, if these are represented graphically, what would their mathematical structures and applications entail?

For reference, the relationships between the MultiUncertain graphs are illustrated in Figure 6.

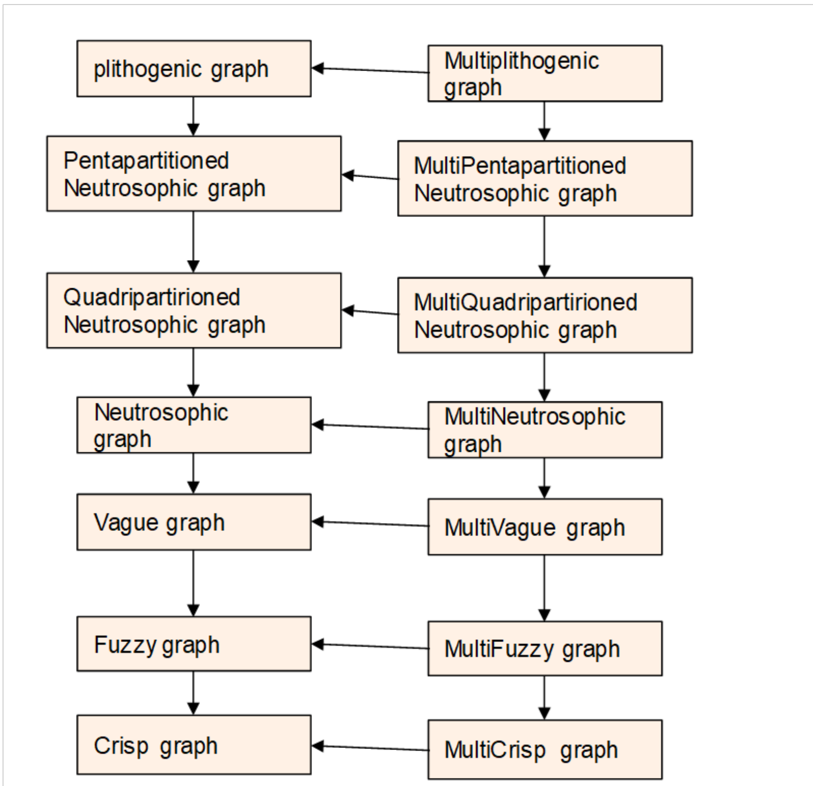


Figure 6: Some MultiUncertain graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

3.2.3 TreeFuzzy Set and TreeNeutrosophic Set

We define the TreeFuzzy Set and TreeNeutrosophic Set as follows.

Definition 3.61. A *TreeFuzzy Set* F is a mapping:

$$F : P(\text{Tree}(A)) \to [0,1]^U,$$

where $P(\text{Tree}(A))$ denotes the power set of the set of all nodes and leaves in $\text{Tree}(A)$, and $[0,1]^U$ denotes the set of all fuzzy subsets of U.

For each attribute combination $S \in P(\text{Tree}(A))$, $F(S)$ is a membership function $\mu_S : U \to [0,1]$, assigning to each element $x \in U$ a degree of membership with respect to the attribute combination S.

Definition 3.62. A *TreeNeutrosophic Set* F is a mapping:

$$F : P(\text{Tree}(A)) \to ([0,1] \times [0,1] \times [0,1])^U,$$

where for each attribute combination $S \in P(\text{Tree}(A))$, $F(S)$ assigns to each element $x \in U$ a neutrosophic membership triple:

$$F(S)(x) = (T_S(x), I_S(x), F_S(x)),$$

where $T_S(x), I_S(x), F_S(x) \in [0,1]$ represent the degrees of truth-membership, indeterminacy-membership, and falsity-membership of x with respect to the attribute combination S.

These values satisfy the condition:

$$0 \leq T_S(x) + I_S(x) + F_S(x) \leq 3,$$

for all $x \in U$ and $S \in P(\text{Tree}(A))$.

Theorem 3.63. *A TreeFuzzy Set is a generalization of a TreeCrisp Set.*

Proof. A TreeCrisp Set is defined as a mapping:

$$F : \text{Tree}(A) \to \{0, 1\},$$

where Tree(A) is the hierarchical structure of attributes, and $F(x) \in \{0, 1\}$ for each $x \in \text{Tree}(A)$. This means that each node or leaf in Tree(A) is either fully included ($F(x) = 1$) or excluded ($F(x) = 0$) from the set.

On the other hand, a TreeFuzzy Set is defined as:

$$F : P(\text{Tree}(A)) \to [0, 1]^U,$$

where $P(\text{Tree}(A))$ is the power set of the set of all nodes and leaves in Tree(A), and $[0, 1]^U$ represents the set of fuzzy subsets of U. For each subset $S \in P(\text{Tree}(A))$, $F(S)$ assigns a membership function $\mu_S : U \to [0, 1]$, where $\mu_S(x)$ denotes the degree of membership of $x \in U$ with respect to S.

To show that a TreeFuzzy Set generalizes a TreeCrisp Set, consider a TreeCrisp Set F. For each $x \in \text{Tree}(A)$, we can represent $F(x) \in \{0, 1\}$ as a membership function $\mu_x : U \to \{0, 1\}$, where:

$$\mu_x(u) = \begin{cases} 1, & \text{if } u = x \text{ and } F(x) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $F(x)$ in a TreeCrisp Set corresponds to a degenerate case of $F(S)$ in a TreeFuzzy Set, where μ_S is restricted to binary values $\{0, 1\}$.

Therefore, every TreeCrisp Set can be expressed as a special case of a TreeFuzzy Set by restricting the membership function μ_S to take only binary values. □

Corollary 3.64. *A TreeNeutrosophic Set is a generalization of a TreeCrisp Set.*

Proof. This is evident. □

Theorem 3.65. *A TreeFuzzy Set F reduces to a traditional fuzzy set when the attribute tree Tree(A) has only one level (i.e., no sub-attributes).*

Proof. When Tree(A) has only one level, it consists of a set of attributes $A = \{A_1, A_2, \dots, A_n\}$ without further sub-attributes. The power set $P(\text{Tree}(A))$ contains all possible combinations of these attributes, including the empty set.

In this case, each attribute combination $S \in P(\text{Tree}(A))$ corresponds to a subset of A. The TreeFuzzy Set mapping F assigns to each S a fuzzy subset $F(S) \in [0, 1]^U$.

If we consider the special case where Tree(A) consists of only a single attribute A_1, then $P(\text{Tree}(A))$ contains only two sets: $\emptyset$ and $\{A_1\}$.

For $S = \{A_1\}$, $F(\{A_1\})$ is a membership function $\mu_{A_1} : U \to [0, 1]$, which is exactly a traditional fuzzy set over U.

Therefore, the TreeFuzzy Set F reduces to a collection of traditional fuzzy sets when the tree has only one level, demonstrating that the TreeFuzzy Set generalizes the concept of a fuzzy set. □

Theorem 3.66. *A TreeNeutrosophic Set F generalizes both the TreeFuzzy Set and the Neutrosophic Set.*

Proof. First, we show that the TreeNeutrosophic Set generalizes the TreeFuzzy Set.

Consider a TreeNeutrosophic Set F where, for all $S \in P(\text{Tree}(A))$ and $x \in U$, the indeterminacy-membership degree $I_S(x) = 0$ and the falsity-membership degree $F_S(x) = 1 - T_S(x)$. In this case, the neutrosophic membership triple simplifies to:

$$(T_S(x), 0, 1 - T_S(x)).$$

This effectively reduces the TreeNeutrosophic Set to a TreeFuzzy Set, where the membership function is $\mu_S(x) = T_S(x)$.

Second, we show that the TreeNeutrosophic Set generalizes the traditional Neutrosophic Set.

When the attribute tree $\text{Tree}(A)$ has only a single node (no attributes or only one attribute), the power set $P(\text{Tree}(A))$ contains only the empty set or a singleton set. In this case, the TreeNeutrosophic Set F assigns to each element $x \in U$ a neutrosophic membership triple $(T(x), I(x), F(x))$, which is exactly the definition of a Neutrosophic Set over U.

Therefore, the TreeNeutrosophic Set generalizes both the TreeFuzzy Set (by allowing indeterminacy and falsity degrees) and the Neutrosophic Set (by incorporating the hierarchical attribute structure). □

Theorem 3.67. *A TreeNeutrosophic Set F reduces to a MultiNeutrosophic Set when the attribute tree $\text{Tree}(A)$ has exactly two levels: a primary level of attributes, each of which may contain sub-attributes that represent the multiple neutrosophic values (truth, indeterminacy, and falsity).*

Proof. Consider $\text{Tree}(A)$ with exactly two levels: the root level consists of primary attributes $A = \{a_1, a_2, \dots, a_n\}$, and each attribute a_i at the primary level can have multiple sub-attributes that correspond to neutrosophic components (truth, indeterminacy, and falsity) associated with different sources or perspectives.

For each combination $S \in P(\text{Tree}(A))$ of primary attributes and their sub-attributes, the mapping F assigns to each element $x \in U$ a neutrosophic membership triple $(T_S(x), I_S(x), F_S(x))$. This setup allows for multiple values for each neutrosophic component, effectively capturing the structure of a MultiNeutrosophic Set where:

- T_j values represent multiple truth-membership degrees across different sources,
- I_k values represent multiple indeterminacy-membership degrees, and
- F_l values represent multiple falsity-membership degrees.

Thus, when $\text{Tree}(A)$ has exactly two levels, the TreeNeutrosophic Set F can encapsulate the structure of a MultiNeutrosophic Set by allowing the second level of sub-attributes to capture multiple neutrosophic values. This demonstrates that a two-level attribute tree in a TreeNeutrosophic Set generalizes the MultiNeutrosophic Set structure. □

Definition 3.68. Let S be a universal set, and let $P \subseteq S$. Consider a hierarchical attribute tree $\text{Tree}(A)$, where attributes and sub-attributes are organized in levels from 1 up to m. Each node in the tree represents an attribute a_i, and for each attribute a_i, there is an associated set of possible values Pv_i.

A *TreePlithogenic Set TPS* is defined as:

$$TPS = (P, \text{Tree}(A), \{Pv_i\}, \{pdf_i\}, pCF),$$

where:

- P is a subset of the universal set S.
- $\text{Tree}(A)$ is a hierarchical tree of attributes.
- For each attribute $a_i \in \text{Tree}(A)$, Pv_i is the set of possible values of a_i.
- For each attribute a_i, $pdf_i : P \times Pv_i \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for a_i.
- $pCF : (\bigcup_i Pv_i) \times (\bigcup_i Pv_i) \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)*.

Corollary 3.69. *A TreePlithogenic Set is a generalization of a TreeCrisp Set.*

Proof. This is evident. □

Theorem 3.70. *A TreePlithogenic Set TPS reduces to a MultiPlithogenic Set when the attribute tree Tree(A) has exactly two levels: a primary level of attributes, each with a set of possible sub-attributes that represent different values for the associated plithogenic components.*

Proof. Consider Tree(A) with exactly two levels: the root level consists of primary attributes $A = \{v_1, v_2, \ldots, v_n\}$, and each attribute v_i at the primary level can have multiple sub-attributes representing possible values of the plithogenic components associated with different perspectives or conditions.

In this case, the TreePlithogenic Set TPS is defined as:

$$TPS = (P, A, \{Pv_i\}, \{pdf_i\}, pCF),$$

where:

- A is the set of primary attributes,
- Pv_i denotes the set of possible values for each primary attribute v_i,
- $pdf_i : P \times Pv_i \to [0, 1]$ represents the Degree of Appurtenance Function (DAF) for each attribute v_i, and
- $pCF : \left(\bigcup_{i=1}^{n} Pv_i\right) \times \left(\bigcup_{i=1}^{n} Pv_i\right) \to [0, 1]^t$ is the Degree of Contradiction Function (DCF).

This structure matches the definition of a MultiPlithogenic Set MPS as follows:

$$MPS = (P, \{v_i\}_{i=1}^{n}, \{Pv_i\}_{i=1}^{n}, \{pdf_i\}_{i=1}^{n}, pCF),$$

where each primary attribute v_i in MPS aligns with the attributes and sub-attributes in the two-level structure of TPS.

Thus, with a two-level attribute tree, the TreePlithogenic Set effectively reduces to the structure of a Multi-Plithogenic Set, capturing the relationships among different values and attributes, and demonstrating that the TreePlithogenic Set generalizes the MultiPlithogenic Set. □

Theorem 3.71. *When $s = 3$ and $t = 1$, the TreePlithogenic Set TPS reduces to a TreeNeutrosophic Set. Moreover, when $s = 1$ and $t = 1$, TPS reduces to a TreeFuzzy Set.*

Proof. First, consider $s = 3$ and $t = 1$. For each attribute a_i in Tree(A), let $Pv_i = \{\text{truth}, \text{indeterminacy}, \text{falsity}\}$. Define the Degree of Appurtenance Function pdf_i such that:

$$pdf_i(x, \text{truth}) = T_{a_i}(x), \quad pdf_i(x, \text{indeterminacy}) = I_{a_i}(x), \quad pdf_i(x, \text{falsity}) = F_{a_i}(x),$$

where $T_{a_i}(x), I_{a_i}(x), F_{a_i}(x) \in [0, 1]$ represent the degrees of truth-membership, indeterminacy-membership, and falsity-membership of x with respect to attribute a_i. The DCF pCF handles the contradiction between these attribute values.

Under these configurations, TPS becomes a mapping:

$$F : P(\text{Tree}(A)) \to ([0, 1] \times [0, 1] \times [0, 1])^P,$$

which matches the definition of a TreeNeutrosophic Set.

Second, consider $s = 1$ and $t = 1$. For each attribute a_i, let $Pv_i = \{\text{membership}\}$ with a single value. The Degree of Appurtenance Function simplifies to:

$$pdf_i(x, \text{membership}) = \mu_{a_i}(x),$$

where $\mu_{a_i}(x) \in [0, 1]$ is the membership degree of x with respect to attribute a_i. The TreePlithogenic Set then becomes:

$$F : P(\text{Tree}(A)) \to [0, 1]^P,$$

which is the definition of a TreeFuzzy Set.

Therefore, the TreePlithogenic Set generalizes both the TreeNeutrosophic Set and the TreeFuzzy Set by appropriately choosing the parameters s and t. □

For reference, the relationships between the TreeUncertain sets are illustrated in Figure 7.

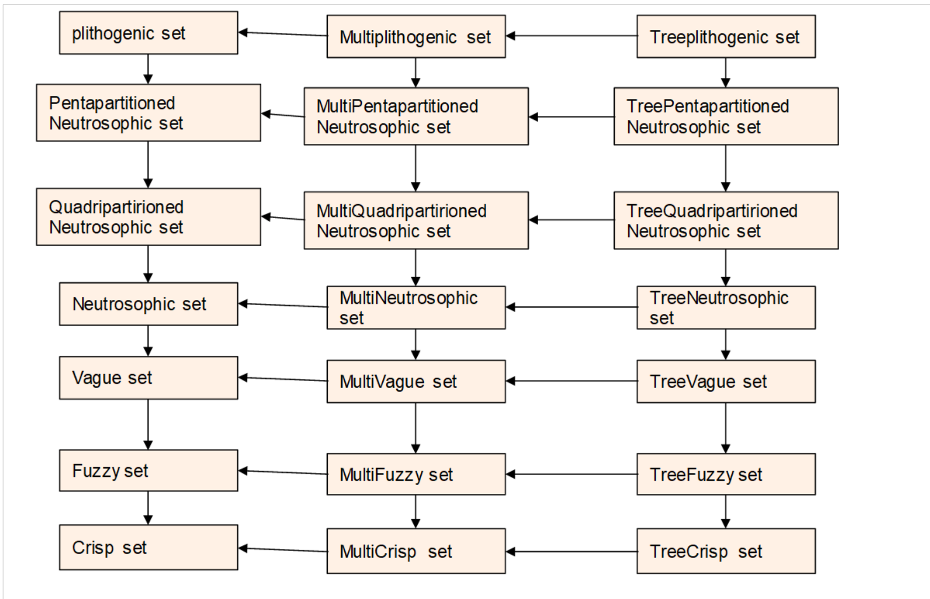


Figure 7: Some TreeUncertain sets Hierarchy. The set class at the origin of an arrow contains the set class at the destination of the arrow.

3.2.4 TreeSoft Expert Set and IndetermSoft Expert Set

Define the TreeSoft Expert Set and the IndetermSoft Expert Set as follows.

Definition 3.72 (TreeSoft Expert Set)**.** Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the power set of H. Define $A = \{A_1, A_2, \ldots, A_n\}$ as a set of attributes, where each attribute A_i (for $1 \leq i \leq n$) represents a first-level attribute. Each A_i consists of sub-attributes:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes, and each sub-attribute $A_{i,j}$ may further contain deeper levels of sub-attributes as needed. This hierarchical structure forms a tree structure, denoted Tree(A), with root A at level 0 and successive levels up to m, the depth of the tree.

Let X be a set of experts and O a set of opinions. Then, the *TreeSoft Expert Set* F is defined as:

$$F : P(\text{Tree}(A) \times X \times O) \to P(H),$$

where $P(\text{Tree}(A) \times X \times O)$ represents the power set of combinations of all attribute nodes (from level 1 to m) with expert-opinion pairs, and F maps each such combination to a subset of H.

Theorem 3.73. *A TreeSoft Expert Set F generalizes the Soft Expert Set when the attribute tree Tree(A) has only two levels.*

Proof. When Tree(A) has only two levels, the set A includes attributes $\{A_1, A_2, \ldots, A_n\}$, each having a set of first-level sub-attributes $\{A_{i,1}, A_{i,2}, \ldots\}$ without further sub-levels. Therefore, $P(\text{Tree}(A) \times X \times O)$ becomes the power set of all combinations of these sub-attributes with experts and opinions.

In this case, the TreeSoft Expert Set F is defined by:

$$F : P(\text{Tree}(A) \times X \times O) \to P(H),$$

which maps each combination of sub-attributes (first-level attributes of A), experts, and opinions to a subset of H. This matches the structure of the Soft Expert Set (F', A), where $A \subseteq E \times X \times O$ represents attribute combinations at a single expert-opinion level, and F' maps these combinations to subsets of H.

Thus, the TreeSoft Expert Set F with two levels of attributes effectively captures the Soft Expert Set structure, generalizing it to allow more intricate tree-structured combinations in higher levels. □

Theorem 3.74. *A TreeSoft Expert Set is a generalization of the TreeSoft Set.*

Proof. We begin by recalling the definition of a TreeSoft Set, where we have a hierarchical attribute structure $\text{Tree}(A)$ representing multiple levels of attributes, sub-attributes, and so forth, up to a depth m. A TreeSoft Set F is defined as a mapping:

$$F : P(\text{Tree}(A)) \to P(H),$$

where $P(\text{Tree}(A))$ is the power set of all nodes and leaves in the hierarchical tree structure of A, and $P(H)$ denotes the power set of a non-empty subset $H \subseteq U$ in the universe U.

In the TreeSoft Expert Set, we introduce the concept of expert opinions into this structure. Let X denote a set of experts, and let O denote a set of possible opinions. For each node (or attribute) in $\text{Tree}(A)$, we associate a combination of parameters, experts, and opinions, forming an enriched attribute structure $Z = \text{Tree}(A) \times X \times O$.

The TreeSoft Expert Set F_{expert} is then defined as:

$$F_{\text{expert}} : P(Z) \to P(H),$$

where $P(Z)$ is the power set of all possible combinations of attributes (from $\text{Tree}(A)$), experts, and opinions.

To show that the TreeSoft Expert Set generalizes the TreeSoft Set, observe that when we set $X = \{\text{single expert}\}$ and $O = \{\text{single opinion}\}$, the set Z reduces to $\text{Tree}(A)$ without additional layers for experts and opinions. Thus, in this special case, the TreeSoft Expert Set F_{expert} reduces to:

$$F_{\text{expert}} : P(\text{Tree}(A)) \to P(H),$$

which is precisely the definition of the TreeSoft Set.

Therefore, by introducing expert and opinion dimensions, the TreeSoft Expert Set encompasses the TreeSoft Set as a special case, demonstrating that it is a generalization of the TreeSoft Set. □

Definition 3.75 (IndetermSoft Expert Set)**.** Let U be a universe of discourse, H a non-empty subset of U, and $P(H)$ the power set of H. Let E denote a set of parameters, X a set of experts, and O a set of possible opinions, where $Z = E \times X \times O$ represents all parameter-expert-opinion combinations. Define $A \subseteq Z$.

The *IndetermSoft Expert Set* is defined as a pair (F, A) over U, where F is a mapping given by:

$$F : A \to P(H),$$

with the following indeterminate conditions:

1. There exists indeterminacy in the set A, meaning at least one element in A may not be well-defined.
2. There exists indeterminacy in the set H or $P(H)$.
3. The mapping F has indeterminacy, meaning there exists at least one pair $F(e, x, o) = M$, where e, x, o, or M may be partially defined, non-unique, or incomplete.

Theorem 3.76. *The IndetermSoft Expert Set generalizes the Soft Expert Set and the IndetermSoft Set.*

Proof. To demonstrate that the IndetermSoft Expert Set generalizes the Soft Expert Set and the IndetermSoft Set, we consider each set as a special case of the IndetermSoft Expert Set by appropriately setting the indeterminate conditions.

For an IndetermSoft Expert Set (F, A), if we assume that:

- The set A is completely well-defined, with no indeterminacy in any parameter-expert-opinion combination.
- The set H and the power set $P(H)$ are both fully determined with no elements lacking definition.

- The mapping F is deterministic and assigns a precise subset of H to each combination in A.

Then, the IndetermSoft Expert Set (F, A) becomes a standard *Soft Expert Set*. This is because each element in A is mapped by F to an exact subset of H, with no ambiguity or partial information involved. Hence, by removing all indeterminate conditions, the IndetermSoft Expert Set reduces to the Soft Expert Set.

If we consider an IndetermSoft Expert Set (F, A) with:

- A single expert and a single opinion, effectively reducing Z to the parameter set E.
- Indeterminate elements allowed in A, H, or in the mapping F, reflecting incomplete or unclear information, as in the IndetermSoft Set definition.

Then, the IndetermSoft Expert Set reduces to the *IndetermSoft Set* where we only have parameters and their possible attribute values, along with the indeterminacy in any of these elements. Therefore, the IndetermSoft Expert Set with a single expert and opinion generalizes the IndetermSoft Set.

Since the IndetermSoft Expert Set contains both the conditions and structure to encapsulate the properties of both Soft Expert Sets and IndetermSoft Sets as specific instances, it indeed generalizes both of these concepts. □

3.2.5 Multirough Set and Treerough set

The definition of a Multirough Set is provided below.

Definition 3.77. Let U be a universal set, and let $R_1, R_2, \ldots, R_n$ be equivalence relations (indiscernibility relations) on U. For any subset $X \subseteq U$, the *Multirough Set* of X is defined by the collection of lower and upper approximations with respect to each equivalence relation R_i.

For each $i = 1, 2, \ldots, n$, we define:

- The *Lower Approximation* of X with respect to R_i:

$$\underline{X}_i = \{x \in U \mid [x]_{R_i} \subseteq X\},$$

  where $[x]_{R_i}$ denotes the equivalence class of x under R_i.

- The *Upper Approximation* of X with respect to R_i:

$$\overline{X}_i = \{x \in U \mid [x]_{R_i} \cap X \neq \emptyset\}.$$

The *Multirough Set* of X is then the collection:

$$\mathcal{MR}(X) = \left\{ \left(\underline{X}_i, \overline{X}_i \right) \mid i = 1, 2, \ldots, n \right\}.$$

Theorem 3.78. *A Multirough Set is a generalization of a MultiCrisp Set.*

Proof. A MultiCrisp Set is defined as:

$$M = \{(x, \mu_1(x), \mu_2(x), \ldots, \mu_k(x)) \mid x \in \mathcal{U}\},$$

where $\mu_i(x) : \mathcal{U} \to \{0, 1\}$ is a crisp membership function for the i-th condition, and $\mu_i(x)$ evaluates whether x satisfies the i-th condition. Thus, a MultiCrisp Set provides a binary evaluation for membership across multiple criteria.

On the other hand, a Multirough Set is defined as:

$$\mathcal{MR}(X) = \left\{ \left(\underline{X}_i, \overline{X}_i \right) \mid i = 1, 2, \ldots, n \right\},$$

where $\underline{X}_i$ and $\overline{X}_i$ are the lower and upper approximations of X with respect to the equivalence relation R_i. The lower approximation $\underline{X}_i$ contains elements that certainly belong to X, while the upper approximation $\overline{X}_i$ includes elements that possibly belong to X.

To show that a Multirough Set generalizes a MultiCrisp Set, consider the special case where the universe U is partitioned into equivalence classes $[x]_{R_i}$ such that each equivalence class corresponds to a single element. In this case:

$$[x]_{R_i} = \{x\} \quad \forall x \in U, \forall i.$$

Under these conditions:

$$\underline{X}_i = \{x \in U \mid [x]_{R_i} \subseteq X\} = \{x \in U \mid x \in X\} = X,$$

and:

$$\overline{X}_i = \{x \in U \mid [x]_{R_i} \cap X \neq \emptyset\} = \{x \in U \mid x \in X\} = X.$$

Thus, the lower and upper approximations become identical and reduce to the binary membership evaluation used in a MultiCrisp Set. Specifically, for each i, the lower approximation $\underline{X}_i$ acts as the crisp membership function $\mu_i(x)$, taking values in $\{0, 1\}$.

Therefore, every MultiCrisp Set can be represented as a special case of a Multirough Set by restricting the equivalence classes $[x]_{R_i}$ to singletons and the approximations $\underline{X}_i$ and $\overline{X}_i$ to binary evaluations. □

Theorem 3.79. *If all equivalence relations $R_1, R_2, \ldots, R_n$ are identical, that is, $R_1 = R_2 = \cdots = R_n = R$, then the Multirough Set $\mathcal{MR}(X)$ reduces to the Rough Set $(\underline{X}, \overline{X})$ with respect to R.*

Proof. When $R_1 = R_2 = \cdots = R_n = R$, the lower and upper approximations for each R_i become the same. Specifically, for each i:

$$\underline{X}_i = \{x \in U \mid [x]_R \subseteq X\} = \underline{X},$$

and

$$\overline{X}_i = \{x \in U \mid [x]_R \cap X \neq \emptyset\} = \overline{X}.$$

Therefore, the collection of approximations in the Multirough Set becomes a single pair:

$$\mathcal{MR}(X) = \left\{ \left(\underline{X}, \overline{X} \right), \ldots, \left(\underline{X}, \overline{X} \right) \right\},$$

which effectively is the standard Rough Set $(\underline{X}, \overline{X})$ with respect to the equivalence relation R. Hence, the Multirough Set reduces to a Rough Set when all equivalence relations are identical. □

Building on the concept of the Treesoft Set, we propose a new definition for the Treerough Set. The definition is provided below.

Definition 3.80. Let U be a universe of discourse, and let $\mathrm{Tree}(A)$ be a hierarchical tree of attributes, where each node represents an attribute a_i. The tree has levels from 1 up to m, where $m \geq 1$. Each attribute a_i in the tree is associated with an equivalence relation R_{a_i} on U.

For any subset $X \subseteq U$, we define the *Treerough Set* $\mathcal{TR}(X)$ as the collection of lower and upper approximations of X with respect to the equivalence relations R_{a_i} associated with all attributes a_i in $\mathrm{Tree}(A)$.

For each attribute a_i in $\mathrm{Tree}(A)$, the lower and upper approximations of X are defined as:

- The *Lower Approximation* of X with respect to R_{a_i}:

$$\underline{X}_{a_i} = \{x \in U \mid [x]_{R_{a_i}} \subseteq X\},$$

  where $[x]_{R_{a_i}}$ denotes the equivalence class of x under R_{a_i}.

- The *Upper Approximation* of X with respect to R_{a_i}:

$$\overline{X}_{a_i} = \{x \in U \mid [x]_{R_{a_i}} \cap X \neq \emptyset\}.$$

The *Treerough Set* of X is then the collection:

$$\mathcal{TR}(X) = \left\{ \left(\underline{X}_{a_i}, \overline{X}_{a_i} \right) \mid a_i \in \text{Tree}(A) \right\}.$$

Theorem 3.81. *A Treerough Set is a generalization of a Treecrisp Set.*

Proof. A Treecrisp Set is defined as a mapping:

$$F : \text{Tree}(A) \to \{0, 1\},$$

where $\text{Tree}(A)$ represents a hierarchical tree of attributes, and F assigns a crisp membership value (either 0 or 1) to each node or leaf in the tree. In this structure, every attribute $a_i \in \text{Tree}(A)$ has a well-defined binary classification of membership.

On the other hand, a Treerough Set $\mathcal{TR}(X)$ is defined as the collection of lower and upper approximations of a subset $X \subseteq U$ with respect to equivalence relations R_{a_i} associated with attributes a_i in $\text{Tree}(A)$. For each a_i, the lower and upper approximations are:

$$\underline{X}_{a_i} = \{x \in U \mid [x]_{R_{a_i}} \subseteq X\},$$

$$\overline{X}_{a_i} = \{x \in U \mid [x]_{R_{a_i}} \cap X \neq \emptyset\}.$$

To show that a Treerough Set generalizes a Treecrisp Set, consider the special case where each equivalence class $[x]_{R_{a_i}}$ under R_{a_i} contains exactly one element. In this scenario:

$$[x]_{R_{a_i}} = \{x\}, \quad \forall x \in U, \forall a_i \in \text{Tree}(A).$$

Under these conditions, the lower and upper approximations simplify as follows:

$$\underline{X}_{a_i} = \{x \in U \mid \{x\} \subseteq X\} = X,$$

$$\overline{X}_{a_i} = \{x \in U \mid \{x\} \cap X \neq \emptyset\} = X.$$

Thus, the lower and upper approximations are identical and reduce to a crisp classification. In this case, the Treerough Set $\mathcal{TR}(X)$ becomes a Treecrisp Set where each node $a_i \in \text{Tree}(A)$ has a binary membership evaluation:

$$F(a_i) = \begin{cases} 1, & \text{if } a_i \text{ belongs to the Treecrisp Set,} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, every Treecrisp Set can be represented as a special case of a Treerough Set by restricting the equivalence classes $[x]_{R_{a_i}}$ to singletons and making the lower and upper approximations identical. □

Theorem 3.82. *If the attribute tree $\text{Tree}(A)$ has exactly two levels (i.e., primary attributes and their sub-attributes), then the Treerough Set $\mathcal{TR}(X)$ generalizes the Multirough Set $\mathcal{MR}(X)$.*

Proof. Consider the attribute tree $\text{Tree}(A)$ with exactly two levels:

- *Level 1 (Primary Attributes)*: $A = \{A_1, A_2, \ldots, A_n\}$.
- *Level 2 (Sub-Attributes)*: Each primary attribute A_i has a set of sub-attributes $\{A_{i1}, A_{i2}, \ldots\}$.

Associate each sub-attribute A_{ij} with an equivalence relation R_{ij} on U.

In the Treerough Set $\mathcal{TR}(X)$, we compute the lower and upper approximations of X with respect to each equivalence relation R_{ij} corresponding to sub-attributes at level 2.

The collection of all these approximations is:

$$\mathcal{TR}(X) = \left\{ \left(\underline{X}_{R_{ij}}, \overline{X}_{R_{ij}} \right) \mid i = 1, 2, \ldots, n;\ j = 1, 2, \ldots \right\}.$$

This structure is equivalent to the Multirough Set $\mathcal{MR}(X)$ defined as:

$$\mathcal{MR}(X) = \left\{ \left(\underline{X}_{R_k}, \overline{X}_{R_k} \right) \mid k = 1, 2, \ldots, N \right\},$$

where N is the total number of equivalence relations R_k (i.e., all R_{ij}).

Therefore, when the attribute tree has exactly two levels, the Treerough Set $\mathcal{TR}(X)$ captures the same information as the Multirough Set $\mathcal{MR}(X)$, but organized according to the hierarchical structure of attributes.

Hence, the Treerough Set generalizes the Multirough Set at level 2. □

For reference, the relationships between the Soft set and the rough set are illustrated in Figure 8.

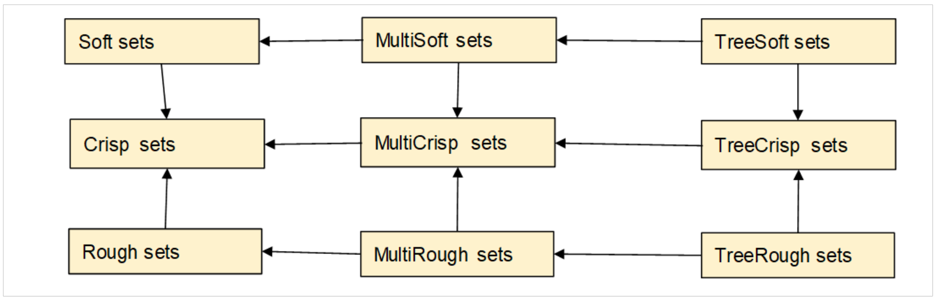


Figure 8: Some the Soft sets and the rough sets Hierarchy. The set class at the origin of an arrow contains the set class at the destination of the arrow.

3.2.6 MultiPentapartitioned Neutrosophic Graph

Define the MultiQuadripartitioned Neutrosophic Graph and MultiPentapartitioned Neutrosophic Graph, and analyze their relationships with other graph classes. The definitions and theorems are provided below.

Definition 3.83 (MultiQuadripartitioned Neutrosophic Graph)**.** Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges.

A *MultiQuadripartitioned Neutrosophic Graph* is defined by associating with each vertex $v \in V$ and each edge $e \in E$ multiple quadripartitioned neutrosophic numbers.

For each vertex $v \in V$, there exists a set of quadripartitioned neutrosophic numbers:

$$Q_v = \left\{ (T_{v,i}, C_{v,i}, U_{v,i}, F_{v,i}) \;\middle|\; i = 1, 2, \ldots, n_v \right\},$$

where:

- $T_{v,i}, C_{v,i}, U_{v,i}, F_{v,i} \in [0,1]$,
- $0 \leq T_{v,i} + C_{v,i} + U_{v,i} + F_{v,i} \leq 4$ for each i.

Similarly, for each edge $e \in E$, there exists a set of quadripartitioned neutrosophic numbers:

$$Q_e = \left\{ \left(T_{e,j}, C_{e,j}, U_{e,j}, F_{e,j}\right) \,\middle|\, j = 1, 2, \ldots, n_e \right\},$$

with analogous conditions.

Definition 3.84 (MultiPentapartitioned Neutrosophic Graph)**.** Let $G = (V, E)$ be a graph.

A *MultiPentapartitioned Neutrosophic Graph* is defined by associating with each vertex $v \in V$ and each edge $e \in E$ multiple pentapartitioned neutrosophic numbers.

For each vertex $v \in V$, there exists a set of pentapartitioned neutrosophic numbers:

$$P_v = \left\{ \left(T_{v,i}, C_{v,i}, R_{v,i}, U_{v,i}, F_{v,i}\right) \,\middle|\, i = 1, 2, \ldots, n_v \right\},$$

where:

- $T_{v,i}, C_{v,i}, R_{v,i}, U_{v,i}, F_{v,i} \in [0,1]$,
- $0 \leq T_{v,i} + C_{v,i} + R_{v,i} + U_{v,i} + F_{v,i} \leq 5$ for each i.

Similarly, for each edge $e \in E$, there exists a set of pentapartitioned neutrosophic numbers:

$$P_e = \left\{ \left(T_{e,j}, C_{e,j}, R_{e,j}, U_{e,j}, F_{e,j}\right) \,\middle|\, j = 1, 2, \ldots, n_e \right\}.$$

Theorem 3.85. *Every MultiQuadripartitioned Neutrosophic Graph can be transformed into an equivalent Quadripartitioned Neutrosophic Graph and into an equivalent MultiNeutrosophic Graph.*

Proof. To transform a MultiQuadripartitioned Neutrosophic Graph into a Quadripartitioned Neutrosophic Graph, we aggregate the multiple quadripartitioned neutrosophic numbers associated with each vertex and edge into single quadripartitioned neutrosophic numbers.

For each vertex $v \in V$, define:

$$\overline{T}_v = \frac{1}{n_v} \sum_{i=1}^{n_v} T_{v,i},$$
$$\overline{C}_v = \frac{1}{n_v} \sum_{i=1}^{n_v} C_{v,i},$$
$$\overline{U}_v = \frac{1}{n_v} \sum_{i=1}^{n_v} U_{v,i},$$
$$\overline{F}_v = \frac{1}{n_v} \sum_{i=1}^{n_v} F_{v,i}.$$

Similarly for edges.

The resulting aggregated values satisfy:

$$0 \leq \overline{T}_v + \overline{C}_v + \overline{U}_v + \overline{F}_v \leq 4.$$

Thus, we obtain a Quadripartitioned Neutrosophic Graph.

To transform into a MultiNeutrosophic Graph, for each vertex $v \in V$, we consider the following sets:

- Truth-membership degrees: $\{T_{v,i} \mid i = 1, \ldots, n_v\}$,
- Indeterminacy-membership degrees: $\{C_{v,i} + U_{v,i} \mid i = 1, \ldots, n_v\}$,
- Falsity-membership degrees: $\{F_{v,i} \mid i = 1, \ldots, n_v\}$.

Similarly for edges.

By combining the contradiction and indeterminacy components into a single indeterminacy degree, we can represent each vertex and edge in a MultiNeutrosophic Graph. □

Theorem 3.86. *Every MultiPentapartitioned Neutrosophic Graph can be transformed into an equivalent Pentapartitioned Neutrosophic Graph and into an equivalent MultiQuadripartitioned Neutrosophic Graph.*

Proof. The transformation into a Pentapartitioned Neutrosophic Graph is achieved by aggregating the multiple pentapartitioned neutrosophic numbers associated with each vertex and edge.

For each vertex $v \in V$, define:

$$\overline{T}_v = \frac{1}{n_v}\sum_{i=1}^{n_v} T_{v,i},$$
$$\overline{C}_v = \frac{1}{n_v}\sum_{i=1}^{n_v} C_{v,i},$$
$$\overline{R}_v = \frac{1}{n_v}\sum_{i=1}^{n_v} R_{v,i},$$
$$\overline{U}_v = \frac{1}{n_v}\sum_{i=1}^{n_v} U_{v,i},$$
$$\overline{F}_v = \frac{1}{n_v}\sum_{i=1}^{n_v} F_{v,i}.$$

Similarly for edges.

To transform into a MultiQuadripartitioned Neutrosophic Graph, we can merge certain components. For example, we may consider the ”R” (refusal or hesitation) component together with the ”U” (unknown) component.

Define new quadripartitioned neutrosophic numbers:

$$\left(T_{v,i}, C_{v,i}, U_{v,i} + R_{v,i}, F_{v,i}\right).$$

Thus, we obtain a MultiQuadripartitioned Neutrosophic Graph by interpreting $U'_{v,i} = U_{v,i} + R_{v,i}$. □

Theorem 3.87. *A MultiPlithogenic Graph with $s = 5$ and $t = 1$ can be transformed into a MultiPentapartitioned Neutrosophic Graph.*

Proof. In a MultiPlithogenic Graph with $s = 5$ and $t = 1$, the Degree of Appurtenance Function (DAF) outputs vectors in $[0, 1]^5$. We can interpret these five degrees as corresponding to the components of a pentapartitioned neutrosophic number.

Let the DAF for an element x be:

$$pdf(x, a) = (d_1(x, a), d_2(x, a), d_3(x, a), d_4(x, a), d_5(x, a)),$$

where each $d_i(x, a) \in [0, 1]$.

We map these degrees to the components of a pentapartitioned neutrosophic number as follows:

$$T(x) = d_1(x, a),$$
$$C(x) = d_2(x, a),$$
$$R(x) = d_3(x, a),$$
$$U(x) = d_4(x, a),$$
$$F(x) = d_5(x, a).$$

Thus, each element in the MultiPlithogenic Graph corresponds to a pentapartitioned neutrosophic number, and we obtain a MultiPentapartitioned Neutrosophic Graph. □

Theorem 3.88. *A MultiPlithogenic Graph with $s = 4$ and $t = 1$ can be transformed into a MultiQuadripartitioned Neutrosophic Graph.*

Proof. Similarly, in a MultiPlithogenic Graph with $s = 4$ and $t = 1$, the DAF outputs vectors in $[0, 1]^4$.

Let the DAF for an element x be:

$$pdf(x, a) = (d_1(x, a), d_2(x, a), d_3(x, a), d_4(x, a)).$$

We map these degrees to the components of a quadripartitioned neutrosophic number:

$$T(x) = d_1(x, a),$$
$$C(x) = d_2(x, a),$$
$$U(x) = d_3(x, a),$$
$$F(x) = d_4(x, a).$$

Thus, we obtain a MultiQuadripartitioned Neutrosophic Graph. □

3.2.7 Neutrosophic Meta Set

We define the Neutrosophic Meta Set, which extends the Meta Set, as follows.

Definition 3.89. Let X be a non-empty set, and let T denote a full infinite binary tree. A *Neutrosophic Meta Set* $\mathfrak{M}$ over X is defined as a collection of pairs $\langle x, p \rangle$ such that:

$$\mathfrak{M} = \left\{ \langle x, p \rangle \mid x \in X, p \in T, \mu_{\mathfrak{M}}(x, p) \in [0, 1]^3 \right\},$$

where $\mu_{\mathfrak{M}} : X \times T \to [0, 1]^3$ is a *neutrosophic membership function* mapping each pair (x, p) to a triplet $\langle T_{\mathfrak{M}}(x, p), I_{\mathfrak{M}}(x, p), F_{\mathfrak{M}}(x, p) \rangle$ in $[0, 1]^3$. Here:

- $T_{\mathfrak{M}}(x, p)$ represents the degree of truth of x with respect to node p,
- $I_{\mathfrak{M}}(x, p)$ represents the degree of indeterminacy of x with respect to node p,
- $F_{\mathfrak{M}}(x, p)$ represents the degree of falsity of x with respect to node p.

These values satisfy the condition:

$$0 \leq T_{\mathfrak{M}}(x, p) + I_{\mathfrak{M}}(x, p) + F_{\mathfrak{M}}(x, p) \leq 3.$$

Theorem 3.90. *A Neutrosophic Meta Set generalizes both Meta Sets and Neutrosophic Sets.*

Proof. A meta set ρ on X is defined with membership function $\mu_\rho : X \times T \to [0,1]$, which associates each pair (x, p) with a single membership degree $\mu_\rho(x, p) \in [0,1]$. To represent ρ as a Neutrosophic Meta Set $\mathfrak{M}$, let:

$$T_{\mathfrak{M}}(x,p) = \mu_\rho(x,p), \quad I_{\mathfrak{M}}(x,p) = 0, \quad F_{\mathfrak{M}}(x,p) = 1 - \mu_\rho(x,p).$$

This assignment satisfies the constraint:

$$0 \leq T_{\mathfrak{M}}(x,p) + I_{\mathfrak{M}}(x,p) + F_{\mathfrak{M}}(x,p) \leq 3,$$

preserving the structure of the meta set as a neutrosophic meta set. Thus, every meta set is a Neutrosophic Meta Set with indeterminate degree set to zero.

A neutrosophic set A on X is characterized by three membership functions $T_A, I_A, F_A : X \to [0,1]$, where $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent truth, indeterminacy, and falsity degrees, respectively. To represent A as a Neutrosophic Meta Set $\mathfrak{M}$, use a tree with a single node p_0 and define:

$$T_{\mathfrak{M}}(x,p_0) = T_A(x), \quad I_{\mathfrak{M}}(x,p_0) = I_A(x), \quad F_{\mathfrak{M}}(x,p_0) = F_A(x).$$

This representation confines the neutrosophic degrees to a single node in the tree, preserving the structure of the neutrosophic set.

Therefore, both meta sets and neutrosophic sets can be represented within the framework of Neutrosophic Meta Sets, proving that Neutrosophic Meta Sets generalize both concepts. □

3.2.8 Cohesive Neutrosophic Set

The definition of the Cohesive Neutrosophic Set is provided below.

Definition 3.91 (Cohesive Neutrosophic Set (CHNS))**.** Let S be a fixed universe of discourse, and let $T \subset S$ be a set over S. A *Cohesive Neutrosophic Set* (CHNS) on T is defined by three membership functions, $h_T^T(x)$, $h_T^I(x)$, and $h_T^F(x)$, which, when applied to each $x \in S$, return subsets of the unit circle in the complex plane. These represent the degrees of *truth*, *indeterminacy*, and *falsity* of x in T, respectively.

For each $x \in S$, the degrees are given by:

$$h_T^T(x) = \{r_T^T(x)\exp(iw_T^T(x)) : r_T^T(x) \in [0,1],\ w_T^T(x) \in \mathbb{R}\},$$

$$h_T^I(x) = \{r_T^I(x)\exp(iw_T^I(x)) : r_T^I(x) \in [0,1],\ w_T^I(x) \in \mathbb{R}\},$$

$$h_T^F(x) = \{r_T^F(x)\exp(iw_T^F(x)) : r_T^F(x) \in [0,1],\ w_T^F(x) \in \mathbb{R}\},$$

where $r_T^T(x), r_T^I(x), r_T^F(x)$ represent the magnitudes of the truth, indeterminacy, and falsity membership degrees, respectively, and $w_T^T(x), w_T^I(x), w_T^F(x)$ represent the phases in radians.

Thus, the cohesive neutrosophic set T is expressed as:

$$T = \left\{\langle x, h_T^T(x), h_T^I(x), h_T^F(x)\rangle : x \in S\right\}.$$

Theorem 3.92. *The Cohesive Neutrosophic Set (CHNS) generalizes both the Cohesive Fuzzy Set (CHFS) and the Neutrosophic Set.*

Proof. To demonstrate that the CHNS generalizes both the CHFS and the Neutrosophic Set, we consider each case:

A CHFS on T is defined by a single membership function $h_T(x)$, which gives a complex-valued degree for each element $x \in S$. To retrieve this structure from a CHNS, we set:

$$h_T^T(x) = h_T(x), \quad h_T^I(x) = \{0\}, \quad h_T^F(x) = \{0\} \quad \text{for all } x \in S.$$

With $h_T^I(x)$ and $h_T^F(x)$ fixed at zero, the CHNS reduces to a structure where only $h_T^T(x)$ is active, matching the definition of a CHFS. Hence, every CHFS can be seen as a special case of a CHNS.

A Neutrosophic Set on T is defined by three membership functions $T(x)$, $I(x)$, and $F(x)$ with real values in the range $[0,1]$. To match this structure in a CHNS, we restrict each phase component to zero radians:

$$h_T^T(x) = \{T(x)\}, \quad h_T^I(x) = \{I(x)\}, \quad h_T^F(x) = \{F(x)\} \quad \text{for all } x \in S,$$

where $T(x), I(x), F(x) \in [0,1]$. Thus, each membership function $h_T^T(x)$, $h_T^I(x)$, and $h_T^F(x)$ behaves as a real-valued Neutrosophic component. This configuration satisfies the definition of a Neutrosophic Set, showing that every Neutrosophic Set is a specific instance of a CHNS.

Since both the Cohesive Fuzzy Set and Neutrosophic Set structures can be derived as special cases of the CHNS by appropriate choices of parameters, the CHNS generalizes both concepts. □

3.2.9 Neutrosophic Multisoft Set

The definition of the Neutrosophic Multisoft Set is presented below. This set combines the concepts of the Multisoft Set and the Neutrosophic Soft Set.

Definition 3.93 (Neutrosophic Multisoft Set)**.** Let U be a universal set, and let $E_1, E_2, \ldots, E_n$ represent distinct sets of parameters such that $E_i \cap E_j = \emptyset$ for $i \neq j$. Define $E = \bigcup_{i=1}^n E_i$ as the union of all parameter sets. A *Neutrosophic Multisoft Set* NMS over U is a pair (F, A), where:

- $A \subseteq \mathcal{P}(E)$ is a set of attribute combinations.
- $F : A \to \mathcal{P}(U) \times [0,1] \times [0,1] \times [0,1]$ is a function that maps each combination of attributes in A to a subset of U along with its associated neutrosophic membership values.

For each $a \in A$, the value $F(a) = (X_a, T_a, I_a, F_a)$ consists of:

- $X_a \subseteq U$, representing the set of elements associated with a.
- $T_a : X_a \to [0,1]$, $I_a : X_a \to [0,1]$, and $F_a : X_a \to [0,1]$, representing the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively, for each $x \in X_a$.

Each membership value satisfies the constraint:

$$0 \leq T_a(x) + I_a(x) + F_a(x) \leq 3, \quad \text{for all } x \in X_a.$$

Theorem 3.94. *A Neutrosophic Multisoft Set generalizes both the Multisoft Set and the Neutrosophic Soft Set.*

Proof. To show that a Neutrosophic Multisoft Set $NMS = (F, A)$ generalizes the Multisoft Set and the Neutrosophic Soft Set, we demonstrate that each of these sets is a specific case of the Neutrosophic Multisoft Set.

A Multisoft Set is defined as (F, A), where $F : A \to \mathcal{P}(U)$ maps attribute combinations in A to subsets of U. In the Neutrosophic Multisoft Set, the function F maps each attribute combination $a \in A$ to a quadruple (X_a, T_a, I_a, F_a) where $X_a \subseteq U$. By restricting the Neutrosophic components $T_a(x) = 1$, $I_a(x) = 0$, and $F_a(x) = 0$ for all $x \in X_a$, the Neutrosophic Multisoft Set reduces to a classical Multisoft Set with $F(a) = X_a$. Thus, the Neutrosophic Multisoft Set generalizes the Multisoft Set.

A Neutrosophic Soft Set is defined as (F, A), where $F : A \to \mathcal{P}(U) \times [0,1] \times [0,1] \times [0,1]$ maps each attribute to a subset of U along with neutrosophic membership values. In the Neutrosophic Multisoft Set, each attribute combination $a \in A$ is mapped to (X_a, T_a, I_a, F_a), where T_a, I_a, and F_a are defined on $X_a \subseteq U$ and provide neutrosophic memberships. By setting A to contain single-parameter combinations (i.e., attributes without combination) and restricting $E = E_1$, the Neutrosophic Multisoft Set reduces to the structure of a Neutrosophic Soft Set. Therefore, the Neutrosophic Multisoft Set also generalizes the Neutrosophic Soft Set.

Since the Neutrosophic Multisoft Set can reduce to both the Multisoft Set and the Neutrosophic Soft Set, it follows that the Neutrosophic Multisoft Set is a generalized form of both concepts. □

As a related concept, the Neutrosophic TreeSoft Set has already been introduced. The definition is provided below [887].

Definition 3.95 (Neutrosophic TreeSoft Set)**.** Let U be a universe of discourse, and let $H \subseteq U$ be a non-empty subset with $\mathcal{P}(H)$ denoting its power set. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes, where each attribute A_i (for $1 \leq i \leq n$) is a first-level attribute.

Each first-level attribute A_i consists of sub-attributes forming a hierarchical tree structure. Specifically, each A_i has sub-attributes:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

and each $A_{i,j}$ may have its own sub-attributes:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, up to level m. This hierarchical structure forms a tree denoted as $\text{Tree}(A)$, with root A at level 0 and successive levels up to m.

A *Neutrosophic TreeSoft Set* F is a mapping:

$$F : P(\text{Tree}(A)) \to \mathcal{P}(H) \times [0,1] \times [0,1] \times [0,1],$$

where $P(\text{Tree}(A))$ denotes the power set of the nodes in the attribute tree $\text{Tree}(A)$.

For each subset $S \subseteq \text{Tree}(A)$, $F(S)$ is a quadruple (X_S, T_S, I_S, F_S), where:

- $X_S \subseteq H$ is a subset of H.
- $T_S : X_S \to [0,1]$ is the truth-membership function.
- $I_S : X_S \to [0,1]$ is the indeterminacy-membership function.
- $F_S : X_S \to [0,1]$ is the falsity-membership function.

These functions satisfy the condition:

$$0 \leq T_S(h) + I_S(h) + F_S(h) \leq 3, \quad \text{for all } h \in X_S.$$

Theorem 3.96. *The Neutrosophic TreeSoft Set F generalizes both the TreeSoft Set and the Neutrosophic MultiSoft Set. Specifically:*

1. *When the neutrosophic components $T_S(h)$, $I_S(h)$, and $F_S(h)$ are assigned specific values, the Neutrosophic TreeSoft Set reduces to a TreeSoft Set.*
2. *When the attribute tree $\text{Tree}(A)$ has exactly two levels, the Neutrosophic TreeSoft Set reduces to the Neutrosophic MultiSoft Set.*

Proof. Assume that for all $S \subseteq \text{Tree}(A)$ and $h \in X_S$, the neutrosophic components are defined as:

$$T_S(h) = 1, \quad I_S(h) = 0, \quad F_S(h) = 0.$$

Under these conditions, the mapping F simplifies to:

$$F : P(\text{Tree}(A)) \to \mathcal{P}(H),$$

where $F(S) = X_S$ for each $S \subseteq \text{Tree}(A)$.

This is exactly the definition of a TreeSoft Set, where subsets of the attribute tree are mapped to subsets of H. Therefore, the Neutrosophic TreeSoft Set reduces to the TreeSoft Set when the neutrosophic components are fixed as above.

Consider the case where the attribute tree $\text{Tree}(A)$ has exactly two levels:

- *Level 1 (Primary Attributes)*: $A = \{A_1, A_2, \ldots, A_n\}$.
- *Level 2 (Sub-Attributes)*: Each primary attribute A_i has sub-attributes $\{A_{i,1}, A_{i,2}, \ldots\}$.

Let $S \subseteq \text{Tree}(A)$ be a subset consisting of combinations of sub-attributes at level 2. The mapping F assigns to each S a quadruple (X_S, T_S, I_S, F_S), where $X_S \subseteq H$ and T_S, I_S, F_S are functions from X_S to $[0, 1]$.

This structure aligns with the definition of a Neutrosophic MultiSoft Set, where combinations of attributes are mapped to subsets of H with associated neutrosophic membership functions.

Therefore, when the attribute tree has exactly two levels, the Neutrosophic TreeSoft Set reduces to the Neutrosophic MultiSoft Set. □

3.2.10 Bijective TreeSoft Set

The definition of the Bijective TreeSoft Set is provided below.

Definition 3.97 (Bijective TreeSoft Set)**.** Let U be a universe of discourse, and let $H \subseteq U$ be a non-empty subset with $\mathcal{P}(H)$ denoting its power set. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes, where each A_i (for $1 \le i \le n$) is a first-level attribute.

Each attribute A_i may consist of sub-attributes, forming a hierarchical tree structure. Specifically, each A_i has sub-attributes:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

and each sub-attribute $A_{i,j}$ may further have its own sub-attributes:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, up to level m. This hierarchical structure forms a tree denoted as $\text{Tree}(A)$, with root A at level 0 and successive levels up to m.

A *Bijective TreeSoft Set* over H is a pair $(F, \text{Tree}(A))$ where F is a mapping:

$$F : \text{Tree}(A) \to \mathcal{P}(H),$$

satisfying the following conditions:

1. *Exhaustivity*:
$$\bigcup_{a \in \text{Tree}(A)} F(a) = H.$$
2. *Disjointness*:
$$\forall\, a, b \in \text{Tree}(A),\ a \neq b \implies F(a) \cap F(b) = \emptyset.$$

Theorem 3.98. *The Bijective TreeSoft Set generalizes both the Bijective Soft Set and the TreeSoft Set. Specifically:*

1. *When the attribute tree Tree(A) has only one level (i.e., no sub-attributes), the Bijective TreeSoft Set reduces to a Bijective Soft Set.*
2. *When the disjointness condition is relaxed (i.e., we allow $F(a) \cap F(b) \neq \emptyset$ for some $a \neq b$), the Bijective TreeSoft Set reduces to a TreeSoft Set.*

Proof. We prove each part separately.

Part 1: Reduction to Bijective Soft Set

Assume that the attribute tree $\text{Tree}(A)$ has only one level, meaning there are no sub-attributes beyond the first-level attributes $A = \{A_1, A_2, \dots, A_n\}$. In this case, $\text{Tree}(A) = A$.

The mapping F becomes:

$$F : A \to \mathcal{P}(H),$$

satisfying:

1. *Exhaustivity*:
$$\bigcup_{a \in A} F(a) = H.$$
2. *Disjointness*:
$$\forall a_i, a_j \in A,\ a_i \neq a_j \implies F(a_i) \cap F(a_j) = \emptyset.$$

This is precisely the definition of a *Bijective Soft Set* over H, where the parameters are the attributes A and each element of H is uniquely associated with exactly one parameter.

Therefore, when $\text{Tree}(A) = A$, the Bijective TreeSoft Set reduces to a Bijective Soft Set.

Part 2: Reduction to TreeSoft Set

Consider the Bijective TreeSoft Set $(F, \text{Tree}(A))$ and relax the disjointness condition. That is, we no longer require that $F(a) \cap F(b) = \emptyset$ for $a \neq b$. The mapping F still satisfies:

$$F : \text{Tree}(A) \to \mathcal{P}(H),$$

with:

$$\bigcup_{a \in \text{Tree}(A)} F(a) = H.$$

Without the disjointness condition, the mapping F assigns subsets of H to nodes in the tree, but elements of H may be associated with multiple nodes. This aligns with the definition of a *TreeSoft Set*, where:

$$F : \mathcal{P}(\text{Tree}(A)) \to \mathcal{P}(H).$$

In the original definition of the TreeSoft Set, F maps subsets of $\text{Tree}(A)$ to subsets of H without any requirement of disjointness.

Therefore, when we relax the disjointness condition, the Bijective TreeSoft Set reduces to a TreeSoft Set. □

Question 3.99. Is it possible to define Soft Sets using concepts like Quasi-Bijective [414,415], Partial Bijection [534], or Surjective? Could such definitions have any practical applications?

3.2.11 Treesoft Rough Set

The definition of a Treesoft Rough Set is provided as follows. The Soft Rough Set is an extension of the Rough Set using soft set theory [18, 69, 199, 307, 325, 448, 833, 973, 1023, 1024], and it has been studied in various research papers. Here, we extend this concept further by utilizing the Treesoft Set.

Definition 3.100 (Soft Approximation Space)**.** [326] Let U be a non-empty finite universe, and let $S = (F, A)$ be a soft set over U, where A is a subset of a parameter set E and $F : A \to \mathcal{P}(U)$ is a set-valued mapping that associates each parameter $a \in A$ with a subset $F(a) \subseteq U$. The pair $P = (U, S)$ is called a *soft approximation space*.

Definition 3.101 (Soft Rough Approximations)**.** [326] Given a soft approximation space $P = (U, S)$ and a subset $X \subseteq U$, we define:

- The *soft P-lower approximation* of X:

$$\text{apr}_P(X) = \{u \in U \mid \exists a \in A, u \in F(a) \Rightarrow F(a) \subseteq X\}.$$

- The *soft P-upper approximation* of X:

$$\text{apr}^P(X) = \{u \in U \mid \exists a \in A, u \in F(a) \text{ and } F(a) \cap X \neq \emptyset\}.$$

These approximations provide boundaries for X within the soft set context.

Definition 3.102 (Soft Rough Set)**.** [326] Let $X \subseteq U$ be any subset of the universe. The pair $(\text{apr}_P(X), \text{apr}^P(X))$ is referred to as the *soft rough set* of X with respect to the soft approximation space P.

Definition 3.103 (Treesoft Approximation Space)**.** The pair $P = (H, F)$ is called a *Treesoft Approximation Space*.

Definition 3.104. Given a subset $X \subseteq H$, we define the *Treesoft Rough Approximations*:

- The *Treesoft Lower Approximation* of X:

$$\text{apr}_P(X) = \{h \in H \mid \exists a \in \text{Tree}(A),\ h \in F(a) \text{ and } F(a) \subseteq X\}.$$

- The *Treesoft Upper Approximation* of X:

$$\text{apr}^P(X) = \{h \in H \mid \exists a \in \text{Tree}(A),\ h \in F(a) \text{ and } F(a) \cap X \neq \emptyset\}.$$

Definition 3.105. The *Treesoft Rough Set* of X is the pair $(\text{apr}_P(X), \text{apr}^P(X))$.

Definition 3.106. The *Treesoft Positive Region*, *Negative Region*, and *Boundary Region* of X are defined as:

$$\begin{aligned}
\text{Pos}_P(X) &= \text{apr}_P(X),\\
\text{Neg}_P(X) &= H \setminus \text{apr}^P(X),\\
\text{Bnd}_P(X) &= \text{apr}^P(X) \setminus \text{apr}_P(X).
\end{aligned}$$

If $\text{Bnd}_P(X) = \emptyset$, then X is called *Treesoft Definable*; otherwise, X is called a *Treesoft Rough Set*.

Theorem 3.107. *The Treesoft Rough Set generalizes both the Soft Rough Set and the TreeSoft Set. Specifically:*

1. *When the attribute tree $Tree(A)$ has only one level (i.e., no sub-attributes), the Treesoft Rough Set reduces to the Soft Rough Set.*
2. *When the Treesoft Rough Set is precise (i.e., $apr_P(X) = apr^P(X)$), it reduces to the TreeSoft Set.*

Proof. Assume that the attribute tree $\text{Tree}(A)$ has only one level, meaning there are no sub-attributes beyond the primary attributes $A = \{A_1, A_2, \ldots, A_n\}$. In this case, $\text{Tree}(A) = A$. The TreeSoft Set F becomes a mapping:

$$F : A \to \mathcal{P}(H).$$

The Treesoft Approximation Space $P = (H, F)$ is then equivalent to a Soft Approximation Space where $S = (F, A)$.

For any $X \subseteq H$, the Treesoft Lower and Upper Approximations become:

$$\text{apr}_P(X) = \{h \in H \mid \exists a \in A,\ h \in F(a) \text{ and } F(a) \subseteq X\},$$
$$\text{apr}^P(X) = \{h \in H \mid \exists a \in A,\ h \in F(a) \text{ and } F(a) \cap X \neq \emptyset\}.$$

These are exactly the definitions of the Soft Lower and Upper Approximations in the Soft Rough Set context. Therefore, when the attribute tree has only one level, the Treesoft Rough Set reduces to the Soft Rough Set.

Next, Consider the case where the Treesoft Rough Set is precise, i.e., $\text{apr}_P(X) = \text{apr}^P(X)$ for all $X \subseteq H$.

In this scenario, the boundary region $\text{Bnd}_P(X) = \emptyset$, which means that each subset $X \subseteq H$ is Treesoft Definable.

Since there is no uncertainty (no boundary region), the Treesoft Rough Set simplifies to the original TreeSoft Set mapping F, where elements are precisely classified according to the attributes in $\text{Tree}(A)$.

Therefore, when the Treesoft Rough Set is precise, it reduces to the TreeSoft Set. □

3.2.12 n-Dimensional Neutrosophic set

We mathematically define the n-Dimensional Fuzzy OffSet and n-Dimensional Neutrosophic Set as extensions of the n-Dimensional Fuzzy Set.

Definition 3.108 (n-Dimensional Fuzzy Set)**.** [658] Let U be a non-empty universe of discourse, and $n \in \mathbb{N}^+$ (the set of positive integers). An n-dimensional fuzzy set A over U is defined as:

$$A = \left\{\left(x, \mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x)\right) \mid x \in U\right\},$$

where:

1. For each $i = 1, \ldots, n$, $\mu_{A_i} : U \to [0, 1]$ is called the i-th membership function of A.
2. The membership degrees satisfy the condition:

$$\mu_{A_1}(x) \leq \mu_{A_2}(x) \leq \cdots \leq \mu_{A_n}(x), \quad \forall x \in U.$$

Definition 3.109 (n-Dimensional Fuzzy Offset)**.** Let U be a non-empty universe of discourse, and let $n \in \mathbb{N}^+$. An *n-Dimensional Fuzzy Offset* A over U is defined as:

$$A = \left\{\left(x, \mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x)\right) \mid x \in U\right\},$$

where:

1. For each $i = 1, \ldots, n$, $\mu_{A_i} : U \to [\Psi_i, \Omega_i]$ is called the i-th membership function of A, where $\Omega_i > 1$ and $\Psi_i < 0$ are termed the *Overlimit* and *Underlimit*, respectively, allowing for membership degrees beyond the conventional range of $[0, 1]$.
2. The membership degrees satisfy the ordering condition:

$$\mu_{A_1}(x) \leq \mu_{A_2}(x) \leq \cdots \leq \mu_{A_n}(x), \quad \forall x \in U.$$

In this context:

- There exist elements $x, y \in U$ such that $\mu_{A_i}(x) > 1$ for some i, and $\mu_{A_j}(y) < 0$ for some j.
- This definition allows the membership values $\mu_{A_i}(x)$ to take on values within an extended interval, accounting for scenarios where the membership or non-membership degree may exceed the traditional bounds.

Definition 3.110 (n-Dimensional Neutrosophic Set)**.** Let U be a non-empty universe of discourse, and let $n \in \mathbb{N}^+$. An *n-Dimensional Neutrosophic Set A* over U is defined as:

$$A = \left\{ \left(x, T_{A_1}(x), T_{A_2}(x), \ldots, T_{A_n}(x), I_{A_1}(x), \ldots, I_{A_n}(x), F_{A_1}(x), \ldots, F_{A_n}(x) \right) \mid x \in U \right\},$$

where:

1. For each $i = 1, \ldots, n$:

$$T_{A_i} : U \to [0,1], \quad I_{A_i} : U \to [0,1], \quad F_{A_i} : U \to [0,1],$$

   represent the i-th degrees of truth, indeterminacy, and falsity, respectively.

2. The degrees satisfy the ordering conditions:

$$T_{A_1}(x) \le T_{A_2}(x) \le \cdots \le T_{A_n}(x), \quad I_{A_1}(x) \le I_{A_2}(x) \le \cdots \le I_{A_n}(x),$$
$$F_{A_1}(x) \le F_{A_2}(x) \le \cdots \le F_{A_n}(x), \quad \forall x \in U.$$

3. Additionally, the constraints:

$$0 \le T_{A_i}(x) + I_{A_i}(x) + F_{A_i}(x) \le 3, \quad \forall x \in U, \ i = 1, \ldots, n,$$

   must be satisfied.

Theorem 3.111. *An n-Dimensional Neutrosophic Set is a generalization of an n-Dimensional Fuzzy Set.*

Proof. To prove this, we show that an n-Dimensional Fuzzy Set is a special case of an n-Dimensional Neutrosophic Set when the indeterminacy and falsity components are set to zero. Let U be a non-empty universe of discourse, and let A be an n-Dimensional Neutrosophic Set over U, defined as follows:

$$A = \left\{ \left(x, T_{A_1}(x), T_{A_2}(x), \ldots, T_{A_n}(x), I_{A_1}(x), \ldots, I_{A_n}(x), F_{A_1}(x), \ldots, F_{A_n}(x) \right) \mid x \in U \right\},$$

where for each $i = 1, \ldots, n$,

$$T_{A_i}, I_{A_i}, F_{A_i} : U \to [0,1]$$

represent the i-th truth, indeterminacy, and falsity membership degrees of A, respectively.

Now, let us consider an n-Dimensional Fuzzy Set B over U, defined by:

$$B = \left\{ \left(x, \mu_{B_1}(x), \mu_{B_2}(x), \ldots, \mu_{B_n}(x) \right) \mid x \in U \right\},$$

where $\mu_{B_i} : U \to [0,1]$ is the i-th membership function of B, and the membership degrees satisfy

$$\mu_{B_1}(x) \le \mu_{B_2}(x) \le \cdots \le \mu_{B_n}(x), \quad \forall x \in U.$$

To embed B within A, we set:

$$T_{A_i}(x) = \mu_{B_i}(x), \quad I_{A_i}(x) = 0, \quad F_{A_i}(x) = 0, \quad \forall x \in U, \ i = 1, \ldots, n.$$

Under these settings, A reduces to:

$$A = \left\{ \left(x, \mu_{B_1}(x), \mu_{B_2}(x), \ldots, \mu_{B_n}(x), 0, \ldots, 0, 0, \ldots, 0 \right) \mid x \in U \right\},$$

which matches the structure of the n-Dimensional Fuzzy Set B with $T_{A_i}(x) = \mu_{B_i}(x)$ and indeterminacy and falsity components equal to zero.

Thus, every n-Dimensional Fuzzy Set can be viewed as a specific case of an n-Dimensional Neutrosophic Set by choosing indeterminacy and falsity components to be zero. Therefore, the n-Dimensional Neutrosophic Set generalizes the n-Dimensional Fuzzy Set. □

Definition 3.112 (n-Dimensional Neutrosophic Offset)**.** Let U be a non-empty universe of discourse, and let $n \in \mathbb{N}^+$. An *n-Dimensional Neutrosophic Offset* A over U is defined as:

$$A = \left\{ \left(x, T_{A_1}(x), T_{A_2}(x), \ldots, T_{A_n}(x), I_{A_1}(x), \ldots, I_{A_n}(x), F_{A_1}(x), \ldots, F_{A_n}(x)\right) \mid x \in U \right\},$$

where:

1. For each $i = 1, \ldots, n$:

$$T_{A_i} : U \to [\Psi_T, \Omega_T], \quad I_{A_i} : U \to [\Psi_I, \Omega_I], \quad F_{A_i} : U \to [\Psi_F, \Omega_F],$$

   with $\Psi_T, \Psi_I, \Psi_F < 0$ and $\Omega_T, \Omega_I, \Omega_F > 1$, allowing degrees of truth, indeterminacy, and falsity to exceed conventional bounds.

2. The degrees satisfy the ordering conditions:

$$T_{A_1}(x) \leq T_{A_2}(x) \leq \cdots \leq T_{A_n}(x), \quad I_{A_1}(x) \leq I_{A_2}(x) \leq \cdots \leq I_{A_n}(x),$$

$$F_{A_1}(x) \leq F_{A_2}(x) \leq \cdots \leq F_{A_n}(x), \quad \forall x \in U.$$

3. Additionally, the constraints:

$$\Psi_T \leq T_{A_i}(x) + I_{A_i}(x) + F_{A_i}(x) \leq \Omega_T + \Omega_I + \Omega_F, \quad \forall x \in U, \ i = 1, \ldots, n,$$

   must be satisfied.

Corollary 3.113. *An n-Dimensional Neutrosophic OffSet is a generalization of an n-Dimensional Fuzzy OffSet.*

Proof. This can be proven in the same manner as for the n-Dimensional Neutrosophic Set. □

3.2.13 Strait Neutrosophic Set

A set known as the Strait Fuzzy Set is already established. This concept will now be extended to the Strait Neutrosophic Set.

Definition 3.114 (Strait Fuzzy Set)**.** [98, 749] Let A be a universal set, and let $\mathcal{P}[0,1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$ denote the family of all finite partitions of the interval $[0,1]$. A *strait fuzzy set* S generated from a fuzzy set $\tilde{A}$ on A using a partition $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\}$ is defined as:

$$S(\alpha_k) = \{Y_a(x) \mid a \in \{1, 2, \ldots, r\}, x \in A, \psi(x) \in Y_a\},$$

where:

- $\psi : A \to [0,1]$ is the membership function of the fuzzy set $\tilde{A}$.
- Y_a represents the intervals in the partition α_k, and $Y_a \cap Y_b = \emptyset$ for all $a \neq b$, with $\bigcup_{a=1}^{r} Y_a = [0,1]$.
- For each $Y_a \in \alpha_k$, there exists at least one $x \in A$ such that $\psi(x) \in Y_a$.

If it causes no confusion, the strait fuzzy set $S(\alpha_k)$ is denoted by S.

Definition 3.115 (Strait Neutrosophic Set)**.** Let X be a universal set, and let $\mathcal{P}[0,1] = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots\}$ denote the family of all finite partitions of the interval $[0,1]$. A *strait neutrosophic set* S generated from a neutrosophic set A on X using a partition $\alpha_k = \{Y_1, Y_2, \ldots, Y_r\}$ is defined as:

$$S(\alpha_k) = \left\{ \left(Y_{T_a}(x), Y_{I_a}(x), Y_{F_a}(x)\right) \mid a \in \{1, 2, \ldots, r\}, x \in X, T_A(x) \in Y_{T_a}, I_A(x) \in Y_{I_a}, F_A(x) \in Y_{F_a} \right\},$$

where:

- $T_A : X \to [0,1]$, $I_A : X \to [0,1]$, and $F_A : X \to [0,1]$ are the truth, indeterminacy, and falsity membership functions of the neutrosophic set A.
- $Y_{T_a}, Y_{I_a}, Y_{F_a}$ represent the intervals in the partition α_k, and $Y_{T_a} \cap Y_{T_b} = \emptyset$, $Y_{I_a} \cap Y_{I_b} = \emptyset$, $Y_{F_a} \cap Y_{F_b} = \emptyset$ for all $a \neq b$, with:
$$\bigcup_{a=1}^{r} Y_{T_a} = [0,1], \quad \bigcup_{a=1}^{r} Y_{I_a} = [0,1], \quad \bigcup_{a=1}^{r} Y_{F_a} = [0,1].$$
- For each $Y_{T_a}, Y_{I_a}, Y_{F_a} \in \alpha_k$, there exists at least one $x \in X$ such that:
$$T_A(x) \in Y_{T_a}, \quad I_A(x) \in Y_{I_a}, \quad F_A(x) \in Y_{F_a}.$$

If it causes no confusion, the strait neutrosophic set $S(\alpha_k)$ is denoted by S.

Theorem 3.116. *A Strait Neutrosophic Set is a generalization of a Strait Fuzzy Set.*

Proof. In a Strait Neutrosophic Set, the three membership functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ map elements $x \in X$ to intervals in $[0,1]$. To obtain a Strait Fuzzy Set, we set:

$$T_A(x) = \psi(x), \quad I_A(x) = 0, \quad F_A(x) = 1 - \psi(x),$$

for all $x \in X$. Here, the indeterminacy membership $I_A(x)$ is fixed at 0, and the falsity membership $F_A(x)$ is fully determined by the complement of the truth membership $\psi(x)$. This configuration reduces the Strait Neutrosophic Set to a Strait Fuzzy Set:

$$S(\alpha_k) = \{Y_a(x) \mid a \in \{1,2,\ldots,r\}, x \in X, \psi(x) \in Y_a\}.$$

Since the Strait Neutrosophic Set can reproduce the definition of a Strait Fuzzy Set under the above specialization, the Strait Neutrosophic Set is a generalization of the Strait Fuzzy Set. □

Theorem 3.117. *A Strait Neutrosophic Set is a generalization of a Neutrosophic Set.*

Proof. In a Strait Neutrosophic Set, the membership functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ map each $x \in X$ to intervals in $[0,1]$, rather than single values as in a traditional Neutrosophic Set. To recover a Neutrosophic Set from a Strait Neutrosophic Set, we can impose the condition that each partition in the interval $[0,1]$ is a singleton. Specifically:
$$Y_{T_a} = \{T_A(x)\}, \quad Y_{I_a} = \{I_A(x)\}, \quad Y_{F_a} = \{F_A(x)\},$$
for all $x \in X$, where $T_A(x)$, $I_A(x)$, and $F_A(x)$ are single values within $[0,1]$.

Under this constraint, the Strait Neutrosophic Set $S(\alpha_k)$ reduces to:

$$S(\alpha_k) = \{(T_A(x), I_A(x), F_A(x)) \mid x \in X\}.$$

This is equivalent to the definition of a Neutrosophic Set:

$$A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\},$$

where $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the truth, indeterminacy, and falsity membership functions of the Neutrosophic Set.

Thus, by setting the partitions in a Strait Neutrosophic Set to singletons, we retrieve the classical definition of a Neutrosophic Set. Therefore, the Strait Neutrosophic Set is a generalization of the Neutrosophic Set. □

3.2.14 Neutrosophic Distribution Sets

The definition of the Neutrosophic Distribution Set, which generalizes the Fuzzy Distribution Set [121, 122, 122], is provided below.

Definition 3.118 (Fuzzy Distribution Set)**.** [121] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite non-empty set ($n > 1$). A *Fuzzy Distribution Set (FDS) d* defined on X is a function:

$$d : X \to [0, 1],$$

satisfying the normalization condition:

$$\sum_{i=1}^{n} d(x_i) = 1.$$

The set of membership values $D = \{d_1, d_2, \ldots, d_n\}$, where $d_i = d(x_i)$ for $i = 1, \ldots, n$, is called a *distribution*. For convenience, the distribution D can be represented as an n-tuple:

$$D = (d_1, d_2, \ldots, d_n).$$

A degenerate or point distribution D is defined as:

$$D(i) = (d_1, \ldots, d_n), \quad \text{where } d_i = 1 \text{ for some } i, \text{ and } d_j = 0 \text{ for all } j \neq i.$$

Definition 3.119 (Neutrosophic Distribution Set)**.** Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite non-empty set ($n > 1$). A *Neutrosophic Distribution Set (NDS) N* defined on X is characterized by three membership functions:

$$T_N : X \to [0, 1], \quad I_N : X \to [0, 1], \quad F_N : X \to [0, 1],$$

where for each $x \in X$, the values $T_N(x)$, $I_N(x)$, and $F_N(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These membership functions satisfy the following conditions:

1. Normalization condition:

$$\sum_{i=1}^{n} T_N(x_i) = 1, \quad \sum_{i=1}^{n} I_N(x_i) = 1, \quad \sum_{i=1}^{n} F_N(x_i) = 1.$$

2. For each $x \in X$:

$$0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3.$$

The set of membership values $T = \{T_N(x_1), T_N(x_2), \ldots, T_N(x_n)\}$, $I = \{I_N(x_1), I_N(x_2), \ldots, I_N(x_n)\}$, and $F = \{F_N(x_1), F_N(x_2), \ldots, F_N(x_n)\}$ are called the distributions of truth, indeterminacy, and falsity, respectively. These can be conveniently represented as n-tuples:

$$T = (T_N(x_1), T_N(x_2), \ldots, T_N(x_n)), \quad I = (I_N(x_1), I_N(x_2), \ldots, I_N(x_n)), \quad F = (F_N(x_1), F_N(x_2), \ldots, F_N(x_n)).$$

A degenerate or point distribution for truth T is defined as:

$$T(i) = (T_N(x_1), \ldots, T_N(x_n)), \quad \text{where } T_N(x_i) = 1 \text{ for some } i, \text{ and } T_N(x_j) = 0 \text{ for all } j \neq i.$$

Similarly, degenerate distributions for indeterminacy I and falsity F are defined analogously.

Theorem 3.120. *A Neutrosophic Distribution Set is a generalization of a Fuzzy Distribution Set.*

Proof. To show that an NDS(Neutrosophic Distribution Set) generalizes an FDS (Fuzzy Distribution Set), we specialize the NDS by setting the indeterminacy and falsity membership values to fixed values:

$$I_N(x) = 0, \quad F_N(x) = 1 - T_N(x), \quad \forall x \in X.$$

Under this configuration:

$$T_N(x) = d(x), \quad \forall x \in X,$$

and the normalization condition for truth membership becomes:

$$\sum_{i=1}^{n} T_N(x_i) = \sum_{i=1}^{n} d(x_i) = 1.$$

This reduces the NDS to the definition of an FDS:

$$N = \{(x, T_N(x), I_N(x), F_N(x)) \mid x \in X, T_N(x) \in [0,1], I_N(x) = 0, F_N(x) = 1 - T_N(x)\}.$$

Since an NDS can replicate the structure of an FDS under the above specialization, it follows that the Neutrosophic Distribution Set is a generalization of the Fuzzy Distribution Set. □

Theorem 3.121. *A Neutrosophic Distribution Set (NDS) is a generalization of a Neutrosophic Set.*

Proof. A Neutrosophic Set N is defined as:

$$N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle : x \in X\},$$

where $T_N(x), I_N(x), F_N(x) \in [0,1]$, representing the degrees of truth, indeterminacy, and falsity of each element $x \in X$. The condition:

$$0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3, \quad \forall x \in X,$$

must be satisfied. The Neutrosophic Set does not impose additional constraints on the aggregate values of $T_N(x)$, $I_N(x)$, and $F_N(x)$ across all elements $x \in X$.

On the other hand, a Neutrosophic Distribution Set (NDS) introduces an additional structure to the membership functions by requiring normalization conditions:

$$\sum_{i=1}^{n} T_N(x_i) = 1, \quad \sum_{i=1}^{n} I_N(x_i) = 1, \quad \sum_{i=1}^{n} F_N(x_i) = 1,$$

where $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set. This ensures that the distributions of truth, indeterminacy, and falsity are normalized over the set X, providing a probabilistic-like interpretation of the membership degrees.

To see that an NDS generalizes a Neutrosophic Set, consider the case where $T_N(x_i), I_N(x_i), F_N(x_i) \in [0,1]$ satisfy:

$$T_N(x_i) + I_N(x_i) + F_N(x_i) \leq 3, \quad \forall x_i \in X,$$

without imposing the normalization conditions. In this case, the NDS reduces to a standard Neutrosophic Set. Thus, every Neutrosophic Set is a special case of an NDS, where the normalization conditions are relaxed.

Furthermore, the additional normalization constraints in the NDS provide a richer structure that allows for distributions of membership values across the elements of X, which are not present in the standard Neutrosophic Set. This establishes that the NDS extends the Neutrosophic Set by introducing a more generalized framework for handling membership distributions. □

3.2.15 Neutrosophic Multiple Set

A Neutrosophic Multiple Set is an extension of a Multiple Set [843, 844, 952].

Definition 3.122. [843] Let X be a non-empty universal set. A *Multiple Set A* drawn from X is a mathematical structure that associates each element $x \in X$ with an $n \times k_x$ membership matrix:

$$A(x) = \begin{bmatrix} A_1^1(x) & A_2^1(x) & \cdots & A_{k_x}^1(x) \\ A_1^2(x) & A_2^2(x) & \cdots & A_{k_x}^2(x) \\ \vdots & \vdots & \ddots & \vdots \\ A_1^n(x) & A_2^n(x) & \cdots & A_{k_x}^n(x) \end{bmatrix},$$

where:

- $A_i^j(x)$ is the j-th membership value of the i-th membership function for the element $x \in X$.
- The values $A_i^j(x)$ are ordered in decreasing order, satisfying $A_i^1(x) \geq A_i^2(x) \geq \cdots \geq A_i^{k_x}(x)$.

To standardize the dimensions across all $x \in X$, the maximum $k = \sup\{k_x \mid x \in X\}$ is taken, and zero-padding is used to construct a uniform $n \times k$ matrix for all $x \in X$.

Thus, $A(x)$ becomes an $n \times k$ membership matrix:

$$A(x) = \begin{bmatrix} A_1^1(x) & A_2^1(x) & \cdots & A_k^1(x) \\ A_1^2(x) & A_2^2(x) & \cdots & A_k^2(x) \\ \vdots & \vdots & \ddots & \vdots \\ A_1^n(x) & A_2^n(x) & \cdots & A_k^n(x) \end{bmatrix}.$$

The *Multiple Set A* is then represented as:

$$A = \{(x, A(x)) \mid x \in X\},$$

and is called a *Multiple Set of order* (n, k).

Example 3.123. [843] Let A be a multiple set of order $n \times k$.

1. If $k = 1$ and $n = 1$, then A is a fuzzy set. This is a special case of a crisp set.
2. If $n = 1$, then A is a fuzzy multi-set.
3. If $k = 1$, then A is a multi-fuzzy set (finite case).
4. If $A_i^j(x) = 0$ or 1 for every $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, then A is a multi-set.

Definition 3.124 (Neutrosophic Multiple Set)**.** Let X be a non-empty universal set. A *Neutrosophic Multiple Set N* drawn from X associates each element $x \in X$ with three $n \times k_x$ membership matrices:

$$T_N(x), \quad I_N(x), \quad F_N(x),$$

where:

- $T_N(x)$ is the truth membership matrix, $I_N(x)$ is the indeterminacy membership matrix, and $F_N(x)$ is the falsity membership matrix.
- Each matrix has the form:

$$M(x) = \begin{bmatrix} M_1^1(x) & M_2^1(x) & \cdots & M_{k_x}^1(x) \\ M_1^2(x) & M_2^2(x) & \cdots & M_{k_x}^2(x) \\ \vdots & \vdots & \ddots & \vdots \\ M_1^n(x) & M_2^n(x) & \cdots & M_{k_x}^n(x) \end{bmatrix},$$

  where $M \in \{T_N, I_N, F_N\}$.

- The entries $M_i^j(x)$ satisfy:

$$0 \leq T_N^i(x) + I_N^i(x) + F_N^i(x) \leq 3 \quad \text{for all } i, j.$$

To standardize dimensions, let $k = \sup\{k_x \mid x \in X\}$, and construct $n \times k$ matrices for all $x \in X$ using zero-padding if necessary. Thus, the membership matrices are represented as:

$$T_N(x), \quad I_N(x), \quad F_N(x) \in \mathbb{R}^{n\times k}.$$

The Neutrosophic Multiple Set N is then expressed as:

$$N = \{(x, T_N(x), I_N(x), F_N(x)) \mid x \in X\},$$

and is called a *Neutrosophic Multiple Set of order* (n, k).

Conjecture 3.125. *Let N be a Neutrosophic Multiple Set of order $n \times k$. The following special cases are derived:*

1. *If $T_N(x) + I_N(x) + F_N(x) = 1$ for all x, then N reduces to a Multiple Set.*
2. *If $k = 1$ and $n = 1$, N becomes a Neutrosophic Set.*
3. *If $n = 1$, N reduces to a Neutrosophic Multi-Set.*
4. *If $k = 1$, N reduces to a Multi Neutrosophic Set.*
5. *If $T_N^i(x), I_N^i(x), F_N^i(x) \in \{0, 1\}$, N becomes a classical Multi-Set.*

Theorem 3.126. *A Neutrosophic Multiple Set is a generalization of a Multiple Set.*

Proof. Let X be a non-empty universal set, and let N be a Neutrosophic Multiple Set defined on X. By definition, N associates each $x \in X$ with three $n \times k$ membership matrices $T_N(x)$, $I_N(x)$, and $F_N(x)$, which represent the degrees of truth, indeterminacy, and falsity, respectively, satisfying the following conditions:

$$0 \leq T_N^i(x) + I_N^i(x) + F_N^i(x) \leq 3 \quad \text{for all } x \in X,\ i = 1, \dots, n,\ j = 1, \dots, k.$$

To obtain a Multiple Set M from N, we set:

$$T_N(x) = M(x), \quad I_N(x) = 0, \quad F_N(x) = 1 - M(x),$$

where $M(x)$ is an $n \times k$ membership matrix associated with the Multiple Set M.

The indeterminacy membership $I_N(x)$ is fixed at 0, eliminating any uncertainty. The falsity membership $F_N(x)$ is determined as $1 - M(x)$, fully complementing the truth membership. Thus, the Neutrosophic Multiple Set N reduces to:

$$N = \{(x, T_N(x)) \mid x \in X\}, \quad \text{where } T_N(x) = M(x).$$

This is precisely the definition of a Multiple Set M, where M associates each $x \in X$ with an $n \times k$ membership matrix.

Since the configuration above demonstrates that a Multiple Set can be derived from a Neutrosophic Multiple Set by restricting $I_N(x)$ to 0 and $F_N(x)$ to $1-T_N(x)$, it follows that the Neutrosophic Multiple Set is a generalization of the Multiple Set. □

3.2.16 Granular Neutrosophic Set

A Granular Neutrosophic Set is a generalized form of a Granular Fuzzy Set [622,631,787,1002]. The definition is provided below.

Definition 3.127 (Granular Fuzzy Set)**.** [622] Let apr $= (U, U/E)$ denote an approximation space, where:

- U is a finite set called the *ground set (underlying set)*.
- U/E is the quotient set induced by an equivalence relation E on U, where each equivalence class is denoted as $[x] \subseteq U$ for $x \in U$.

A *Granular Fuzzy Set* G on the quotient space U/E is a mapping:

$$G : U/E \to [0,1],$$

such that $G([x])$ assigns a membership grade to the equivalence class $[x] \in U/E$, representing the degree to which the granule $[x]$ belongs to the granular fuzzy set.

For each equivalence class $[x] \in U/E$, the granular fuzzy set $G([x])$ can be constructed from the fuzzy membership grades of the individual elements in $[x]$ in the ground space. Common constructions include:

1. *Minimum membership value:*
$$G([x]) = \min\{F(y) \mid y \in [x]\},$$
where $F : U \to [0,1]$ is a fuzzy set on U.
2. *Maximum membership value:*
$$G([x]) = \max\{F(y) \mid y \in [x]\}.$$
3. *Average membership value:*
$$G([x]) = \frac{1}{|[x]|} \sum_{y \in [x]} F(y),$$
where $|[x]|$ denotes the cardinality of the equivalence class $[x]$.

Theorem 3.128. *A Granular Fuzzy Set (GFS) can be transformed into a classical Fuzzy Set (FS) by assigning the granular membership grade of each equivalence class to all its elements in the underlying ground set.*

Proof. Let apr $= (U, U/E)$ be an approximation space, where:

- U is the ground set.
- U/E is the quotient set induced by an equivalence relation E on U, with equivalence classes $[x] \subseteq U$ for $x \in U$.

A Granular Fuzzy Set G on U/E is defined as:

$$G : U/E \to [0,1].$$

To construct a classical Fuzzy Set $F : U \to [0,1]$, we define the membership grade of each $y \in U$ as:

$$F(y) = G([y]),$$

where $[y]$ is the equivalence class containing y.

Since $G([x]) \in [0,1]$ for all $[x] \in U/E$, it follows that $F(y) \in [0,1]$ for all $y \in U$. By assigning the granular membership $G([x])$ to all elements in $[x]$, the granular structure is consistently mapped to a fuzzy set on U.

The mapping $F(y) = G([y])$ transforms the Granular Fuzzy Set G into a classical Fuzzy Set F, completing the proof. □

Definition 3.129 (Granular Neutrosophic Set)**.** Let $\text{apr} = (U, U/E)$ denote an approximation space, where:

- U is a finite set called the *ground set*.
- U/E is the quotient set induced by an equivalence relation E on U, where each equivalence class is denoted as $[x] \subseteq U$ for $x \in U$.

A *Granular Neutrosophic Set* G on the quotient space U/E is defined as a triple of mappings:

$$G = (T, I, F),$$

where:

- $T : U/E \to [0, 1]$ is the *truth-membership function*, assigning a degree of truth to each equivalence class $[x] \in U/E$.
- $I : U/E \to [0, 1]$ is the *indeterminacy-membership function*, assigning a degree of indeterminacy to each equivalence class $[x] \in U/E$.
- $F : U/E \to [0, 1]$ is the *falsity-membership function*, assigning a degree of falsity to each equivalence class $[x] \in U/E$.

For each equivalence class $[x] \in U/E$, the values of $T([x])$, $I([x])$, and $F([x])$ can be derived from the neutrosophic membership functions of the individual elements in $[x]$ in the ground space U. Common constructions include:

1. *Minimum values:*

   $$T([x]) = \min\{T_U(y) \mid y \in [x]\}, \quad I([x]) = \min\{I_U(y) \mid y \in [x]\}, \quad F([x]) = \min\{F_U(y) \mid y \in [x]\},$$

   where T_U, I_U, and F_U are the truth, indeterminacy, and falsity membership functions on U, respectively.
2. *Maximum values:*

   $$T([x]) = \max\{T_U(y) \mid y \in [x]\}, \quad I([x]) = \max\{I_U(y) \mid y \in [x]\}, \quad F([x]) = \max\{F_U(y) \mid y \in [x]\}.$$

3. *Average values:*

   $$T([x]) = \frac{1}{|[x]|} \sum_{y \in [x]} T_U(y), \quad I([x]) = \frac{1}{|[x]|} \sum_{y \in [x]} I_U(y), \quad F([x]) = \frac{1}{|[x]|} \sum_{y \in [x]} F_U(y),$$

   where $|[x]|$ is the cardinality of the equivalence class $[x]$.

Theorem 3.130. *A Granular Neutrosophic Set generalizes a Neutrosophic Set.*

Proof. To show that a Granular Neutrosophic Set generalizes a Neutrosophic Set, consider the following construction:

- For each $x \in U$, the equivalence class $[x] \in U/E$ contains all elements equivalent to x under E.
- The membership values for each granule $[x] \in U/E$ in the granular neutrosophic set G can be defined as:

  $$T([x]) = T_U(x), \quad I([x]) = I_U(x), \quad F([x]) = F_U(x),$$

  assuming each granule $[x]$ contains exactly one element (i.e., $[x] = \{x\}$).

Under this construction, the granular neutrosophic set G reduces to the neutrosophic set N when the granulation is trivial, meaning each equivalence class contains only one element.

Conversely, for non-trivial granulation, the membership values for $[x]$ in G can be defined using aggregations over the membership values of elements $y \in [x]$ in the neutrosophic set N. For example:

- *Minimum values:*

$$T([x]) = \min_{y\in[x]} T_U(y), \quad I([x]) = \min_{y\in[x]} I_U(y), \quad F([x]) = \min_{y\in[x]} F_U(y).$$

- *Maximum values:*

$$T([x]) = \max_{y\in[x]} T_U(y), \quad I([x]) = \max_{y\in[x]} I_U(y), \quad F([x]) = \max_{y\in[x]} F_U(y).$$

- *Average values:*

$$T([x]) = \frac{1}{|[x]|}\sum_{y\in[x]} T_U(y), \quad I([x]) = \frac{1}{|[x]|}\sum_{y\in[x]} I_U(y), \quad F([x]) = \frac{1}{|[x]|}\sum_{y\in[x]} F_U(y),$$

  where $|[x]|$ is the cardinality of the equivalence class $[x]$.

Thus, a Granular Neutrosophic Set G reduces to a Neutrosophic Set N under trivial granulation, and generalizes N by allowing aggregation over equivalence classes in non-trivial granulations. □

Theorem 3.131. *A Granular Neutrosophic Set generalizes a Granular Fuzzy Set.*

Proof. To show that a Granular Neutrosophic Set generalizes a Granular Fuzzy Set, consider the following construction:

- For each $[x] \in U/E$, let the granular fuzzy set G be represented as:

$$G([x]) = T([x]),$$

  where T is the truth-membership function of the granular neutrosophic set G_N, and:

$$I([x]) = 0, \quad F([x]) = 1 - T([x]).$$

Under this construction:

- The indeterminacy-membership $I([x])$ is fixed at 0, representing no uncertainty in the membership grade.
- The falsity-membership $F([x])$ is fully determined by $T([x])$, ensuring that the neutrosophic membership model reduces to the fuzzy membership model.

Conversely, a Granular Neutrosophic Set G_N extends a Granular Fuzzy Set by introducing additional degrees of freedom:

- The indeterminacy-membership function $I([x])$ allows for the representation of uncertainty or hesitation.
- The falsity-membership function $F([x])$ decouples from $T([x])$, enabling independent falsity representation.

Thus, when $I([x]) = 0$ and $F([x]) = 1 - T([x])$ for all $[x] \in U/E$, the granular neutrosophic set G_N reduces to the granular fuzzy set G. Therefore, a Granular Neutrosophic Set generalizes a Granular Fuzzy Set. □

3.2.17 Hereditary Neutrosophic Set System

We define the Hereditary Neutrosophic Set System as an extension of the Hereditary Fuzzy Set System below. It is hoped that further exploration of its mathematical characteristics and operations will continue in the future.

Definition 3.132 (Hereditary Fuzzy Set System)**.** [615, 616] Let U be a non-empty finite set, and let $\mathcal{F} \subseteq [0,1]^U$, where each $\mu \in \mathcal{F}$ is a fuzzy set on U. The pair $(U, \mathcal{F})$ is called a *fuzzy set system*, and $\mathcal{F}$ is called a *hereditary fuzzy set system* if the following condition holds:

$$\forall \mu \in \mathcal{F}, \forall \nu \leq \mu \implies \nu \in \mathcal{F},$$

where $\nu \leq \mu$ means $\nu(x) \leq \mu(x)$ for all $x \in U$.

Definition 3.133 (Fundamental Sequence of a Hereditary Fuzzy Set System)**.** [615, 616] Let $(U, \mathcal{F})$ be a hereditary fuzzy set system. Define $\mathcal{F}[a] = \{\mu[a] \mid \mu \in \mathcal{F}\}$ for $a \in (0,1]$, where $\mu[a] = \{x \in U \mid \mu(x) \geq a\}$. There exists a finite sequence $\{r_0, r_1, \ldots, r_n\}$ such that:

1. $0 = r_0 < r_1 < \cdots < r_n \leq 1$,
2. If $r_i < a < b < r_{i+1}$, then $\mathcal{F}[a] = \mathcal{F}[b]$,
3. If $r_i < a < r_{i+1} < b < r_{i+2}$, then $\mathcal{F}[a] \supset \mathcal{F}[b]$,
4. If $0 < a < b \leq 1$, then $\mathcal{F}[a] \supset \mathcal{F}[b]$,
5. $\mathcal{F}[a] = \{\emptyset\}$ for all $a \in [0, r_n]$; if $r_n < 1$, then $\mathcal{F}[a] = \{\emptyset\}$ for $a \in (r_n, 1]$.

The sequence $\{r_0, r_1, \ldots, r_n\}$ is called the *fundamental sequence* of the hereditary fuzzy set system $(U, \mathcal{F})$.

Definition 3.134 (Hereditary Neutrosophic Set System)**.** Let U be a non-empty finite set, and let $\mathcal{N} \subseteq ([0,1] \times [0,1] \times [0,1])^U$, where each $N \in \mathcal{N}$ is a neutrosophic set on U. Each N is defined as:

$$N(x) = (T_N(x), I_N(x), F_N(x)) \quad \text{for all } x \in U,$$

where $T_N(x), I_N(x), F_N(x) \in [0,1]$ represent the truth, indeterminacy, and falsity membership degrees, respectively. The pair $(U, \mathcal{N})$ is called a *neutrosophic set system*, and $\mathcal{N}$ is called a *hereditary neutrosophic set system* if the following condition holds:

$$\forall N \in \mathcal{N}, \forall M \leq N \implies M \in \mathcal{N},$$

where $M \leq N$ means:

$$T_M(x) \leq T_N(x), \quad I_M(x) \geq I_N(x), \quad F_M(x) \geq F_N(x) \quad \text{for all } x \in U.$$

Theorem 3.135. *The Hereditary Neutrosophic Set System generalizes the Hereditary Fuzzy Set System.*

Proof. Let U be a non-empty finite set.

If we set the indeterminacy and falsity membership functions to constant values:

$$I_N(x) = 0, \quad F_N(x) = 0 \quad \text{for all } x \in U,$$

the neutrosophic set N reduces to a fuzzy set μ with:

$$\mu(x) = T_N(x) \quad \text{for all } x \in U.$$

Under this simplification, $M \leq N$ becomes:

$$T_M(x) \leq T_N(x) \quad \text{for all } x \in U,$$

which matches the condition for a fuzzy set system:

$$\forall \mu \in \mathcal{F}, \forall \nu \leq \mu \implies \nu \in \mathcal{F}.$$

The heredity condition of a neutrosophic set system:

$$\forall N \in \mathcal{N}, \forall M \leq N \implies M \in \mathcal{N},$$

encompasses the heredity condition of a fuzzy set system:

$$\forall \mu \in \mathcal{F}, \forall \nu \leq \mu \implies \nu \in \mathcal{F}.$$

By setting $I_N(x) = 0$ and $F_N(x) = 0$, the Hereditary Neutrosophic Set System $(U, \mathcal{N})$ reduces to the Hereditary Fuzzy Set System $(U, \mathcal{F})$. Therefore, the Hereditary Neutrosophic Set System generalizes the Hereditary Fuzzy Set System. □

3.2.18 Contextual Neutrosophic Set

The Contextual Neutrosophic Set, which extends the Contextual Fuzzy Set, is defined as follows. In the future, we aim to explore the applications of the Contextual Fuzzy Set and its related concepts.

Definition 3.136 (Contextual Fuzzy Set)**.** [533] Let X be a reference universe, and let C represent a set of contexts. A *Contextual Fuzzy Set (CFS)* $A \subseteq X$ is defined as:

$$A = \{((x, c), \mu(x), \varphi(x, c)) \mid x \in X, \ c \in C\},$$

where:

- $\mu : X \to [0, 1]$ is the membership function that provides the degree of membership of an element $x \in X$,
- $\varphi : X \times C \to [0, 1]$ is the reliability function that represents the confidence level of the membership evaluation in the context $c \in C$.

The final membership degree of $x \in X$ in the context $c \in C$ is determined jointly by $\mu(x)$ and $\varphi(x, c)$, accounting for the specific characteristics of the context.

Theorem 3.137. *A Contextual Fuzzy Set (CFS) can be transformed into a classical Fuzzy Set (FS) by integrating the reliability function into the membership function across all contexts.*

Proof. Let $A \subseteq X$ be a Contextual Fuzzy Set, defined as:

$$A = \{((x, c), \mu(x), \varphi(x, c)) \mid x \in X, \ c \in C\},$$

where:

- $\mu : X \to [0, 1]$ is the membership function of $x \in X$,
- $\varphi : X \times C \to [0, 1]$ is the reliability function for the membership of x in context $c \in C$.

To transform A into a classical Fuzzy Set $F \subseteq X$, we define the membership function $\mu_F : X \to [0, 1]$ as:

$$\mu_F(x) = \frac{1}{|C|} \sum_{c \in C} \mu(x) \cdot \varphi(x, c),$$

where $|C|$ is the cardinality of the context set C.

For each $x \in X$, $\mu_F(x)$ aggregates the context-dependent memberships $\mu(x)\cdot\varphi(x,c)$, weighted by the reliability $\varphi(x,c)$. Since $\mu(x), \varphi(x,c) \in [0,1]$, it follows that $\mu(x)\cdot\varphi(x,c) \in [0,1]$ and:

$$\frac{1}{|C|}\sum_{c\in C}\mu(x)\cdot\varphi(x,c) \in [0,1].$$

Thus, $\mu_F(x)$ is a valid membership function for a classical Fuzzy Set.

By defining $\mu_F(x) = \frac{1}{|C|}\sum_{c\in C}\mu(x)\cdot\varphi(x,c)$, the Contextual Fuzzy Set A is transformed into a classical Fuzzy Set F. This preserves the essential membership information while eliminating the contextual dependency. □

Definition 3.138 (Contextual Neutrosophic Set)**.** Let X be a reference universe, and let C represent a set of contexts. A *Contextual Neutrosophic Set (CNS)* $A \subseteq X$ is defined as:

$$A = \{((x,c), T(x,c), I(x,c), F(x,c)) \mid x \in X,\ c \in C\},$$

where:

- $T : X \times C \to [0,1]$ is the truth membership function, representing the degree to which an element $x \in X$ belongs to the set A in the context $c \in C$,
- $I : X \times C \to [0,1]$ is the indeterminacy membership function, representing the degree of uncertainty or hesitation about the membership of x in the context c,
- $F : X \times C \to [0,1]$ is the falsity membership function, representing the degree to which $x \in X$ does not belong to the set A in the context c.

These membership functions satisfy the following condition for all $x \in X$ and $c \in C$:

$$0 \le T(x,c) + I(x,c) + F(x,c) \le 3.$$

The final membership of $x \in X$ in the context $c \in C$ is determined by the combined effect of $T(x,c)$, $I(x,c)$, and $F(x,c)$, capturing the nuanced interplay of truth, indeterminacy, and falsity in a context-dependent manner.

Theorem 3.139. *A Contextual Neutrosophic Set (CNS) can be transformed into a Contextual Fuzzy Set (CFS) by defining the reliability function $\varphi(x,c)$ based on the neutrosophic components $T(x,c), I(x,c), F(x,c)$.*

Proof. Let $A = \{((x,c), T(x,c), I(x,c), F(x,c)) \mid x \in X, c \in C\}$ be a Contextual Neutrosophic Set. Define the membership and reliability functions for a Contextual Fuzzy Set as follows:

$$\mu(x) = T(x,c), \quad \varphi(x,c) = 1 - I(x,c) - F(x,c),$$

where $\mu(x)$ represents the degree of membership and $\varphi(x,c)$ represents the reliability of the membership evaluation in the context c.

To ensure that $\varphi(x,c)$ is a valid reliability function, we verify its range:

$$\varphi(x,c) = 1 - I(x,c) - F(x,c).$$

Since $0 \le T(x,c) + I(x,c) + F(x,c) \le 3$, it follows that:

$$0 \le I(x,c) + F(x,c) \le 2.$$

Thus:

$$0 \le 1 - I(x,c) - F(x,c) \le 1,$$

which implies $\varphi(x,c) \in [0,1]$, satisfying the requirements for a reliability function.

Now, construct the corresponding Contextual Fuzzy Set B as:

$$B = \{((x,c), \mu(x), \varphi(x,c)) \mid x \in X, c \in C\}.$$

Here:

$$\mu(x) = T(x, c), \quad \varphi(x, c) = 1 - I(x, c) - F(x, c).$$

This transformation preserves the information from the Contextual Neutrosophic Set while mapping it into the framework of a Contextual Fuzzy Set. Thus, every Contextual Neutrosophic Set can be expressed as a Contextual Fuzzy Set. □

Theorem 3.140. *A Contextual Neutrosophic Set (CNS) generalizes a Neutrosophic Set (NS). Specifically, a Neutrosophic Set is a special case of a Contextual Neutrosophic Set where the context $c \in C$ is constant or irrelevant.*

Proof. Let X be a reference universe, and let C represent the set of contexts. A Contextual Neutrosophic Set is defined as:

$$A = \{((x, c), T(x, c), I(x, c), F(x, c)) \mid x \in X,\ c \in C\},$$

where $T(x, c), I(x, c), F(x, c)$ are the truth, indeterminacy, and falsity membership functions, respectively, satisfying:

$$0 \leq T(x, c) + I(x, c) + F(x, c) \leq 3.$$

A Neutrosophic Set is defined as:

$$B = \{(x, T(x), I(x), F(x)) \mid x \in X\},$$

where $T(x), I(x), F(x)$ satisfy the same condition:

$$0 \leq T(x) + I(x) + F(x) \leq 3.$$

To show that B is a special case of A, define a constant context $c_0 \in C$ such that:

$$T(x, c_0) = T(x), \quad I(x, c_0) = I(x), \quad F(x, c_0) = F(x), \quad \forall x \in X.$$

This reduces A to:

$$A = \{((x, c_0), T(x), I(x), F(x)) \mid x \in X\}.$$

Since the context c_0 is fixed and does not influence the membership values, the contextual dependency vanishes, and A becomes identical to B.

Conversely, any Neutrosophic Set B can be embedded into a Contextual Neutrosophic Set A by assigning a constant context c_0 for all elements in X:

$$T(x, c) = T(x), \quad I(x, c) = I(x), \quad F(x, c) = F(x), \quad \forall c \in C.$$

Thus, every Neutrosophic Set is a Contextual Neutrosophic Set with a trivial context. Therefore, the Contextual Neutrosophic Set generalizes the Neutrosophic Set. □

3.2.19 Non-Stationary Neutrosophic Set

The concept of a Non-Stationary Neutrosophic Set [78, 511, 512, 1006], which generalizes the Non-Stationary Fuzzy Set into a broader framework, is presented below. It is hoped that further exploration and development of its mathematical structure will be accelerated in the future.

Definition 3.141 (Non-Stationary Fuzzy Set)**.** [401] Let X be a universe of discourse and let A be a fuzzy set on X, characterized by a membership function $\mu_A : X \to [0,1]$. Let T be a set of time points t_i (potentially infinite) and let $f : T \to \mathbb{R}$ denote a perturbation function.

A *Non-Stationary Fuzzy Set* $\dot{A}$ is characterized by a non-stationary membership function:

$$\mu_{\dot{A}} : T \times X \to [0,1],$$

which associates each element $(t,x) \in T \times X$ with a time-specific variation of $\mu_A(x)$. The membership function $\mu_{\dot{A}}(t,x)$ is defined as:

$$\mu_{\dot{A}}(t,x) = \mu_A(x, p_1(t), \ldots, p_m(t)),$$

where each parameter $p_i(t)$ is a time-dependent perturbation defined by:

$$p_i(t) = p_i + k_i f_i(t), \quad i = 1, \ldots, m,$$

with k_i as a constant and $f_i(t)$ as a perturbation function for the parameter p_i.

The Non-Stationary Fuzzy Set $\dot{A}$ is formally expressed as:

$$\dot{A} = \int_{t \in T} \int_{x \in X} \mu_{\dot{A}}(t,x)/x/t.$$

Theorem 3.142. *A Non-Stationary Fuzzy Set $\dot{A}$ can be transformed into a standard Fuzzy Set A by fixing the time parameter t to a constant value t_0.*

Proof. Let $\dot{A}$ be a Non-Stationary Fuzzy Set characterized by the membership function:

$$\mu_{\dot{A}} : T \times X \to [0,1],$$

where $\mu_{\dot{A}}(t,x) = \mu_A(x, p_1(t), \ldots, p_m(t))$, and each parameter $p_i(t)$ is time-dependent:

$$p_i(t) = p_i + k_i f_i(t), \quad i = 1, \ldots, m.$$

To transform $\dot{A}$ into a standard Fuzzy Set A, we fix the time parameter t to a constant t_0. Substituting t_0 into the time-dependent parameters, we obtain:

$$p_i(t_0) = p_i + k_i f_i(t_0), \quad i = 1, \ldots, m.$$

The membership function for the fixed time t_0 becomes:

$$\mu_{\dot{A}}(t_0, x) = \mu_A(x, p_1(t_0), \ldots, p_m(t_0)).$$

Define a new membership function for A as:

$$\mu'_A(x) = \mu_{\dot{A}}(t_0, x).$$

This membership function $\mu'_A(x)$ is independent of t and satisfies the conditions of a standard fuzzy set:

$$\mu'_A(x) \in [0,1], \quad \forall x \in X.$$

Therefore, A is a standard Fuzzy Set, and we can express it as:

$$A = \int_{x \in X} \mu'_A(x)/x.$$

This shows that by fixing the time parameter t, the Non-Stationary Fuzzy Set $\dot{A}$ reduces to a standard Fuzzy Set A. □

Definition 3.143 (Non-Stationary Neutrosophic Set)**.** Let X be a given set, and let T be a set of time points t_i (potentially infinite). A *Non-Stationary Neutrosophic Set* $\dot{A}$ on X is characterized by three non-stationary membership functions:

$$T_{\dot{A}} : T \times X \to [0,1], \quad I_{\dot{A}} : T \times X \to [0,1], \quad F_{\dot{A}} : T \times X \to [0,1],$$

where for each $(t,x) \in T \times X$, $T_{\dot{A}}(t,x)$, $I_{\dot{A}}(t,x)$, and $F_{\dot{A}}(t,x)$ represent the time-specific degrees of truth, indeterminacy, and falsity, respectively.

The membership functions are defined as:

$$T_{\dot{A}}(t,x) = T_A(x, p_1(t), \ldots, p_m(t)), \quad I_{\dot{A}}(t,x) = I_A(x, q_1(t), \ldots, q_n(t)), \quad F_{\dot{A}}(t,x) = F_A(x, r_1(t), \ldots, r_k(t)),$$

where $T_A(x)$, $I_A(x)$, and $F_A(x)$ are the underlying membership functions of a standard neutrosophic set, and each parameter $p_i(t)$, $q_j(t)$, and $r_k(t)$ is a time-dependent perturbation given by:

$$p_i(t) = p_i + k_i f_i(t), \quad q_j(t) = q_j + l_j g_j(t), \quad r_k(t) = r_k + m_k h_k(t),$$

with $f_i(t)$, $g_j(t)$, and $h_k(t)$ as perturbation functions, and k_i, l_j, and m_k as constants.

The membership values satisfy the following condition for all $(t,x) \in T \times X$:

$$0 \leq T_{\dot{A}}(t,x) + I_{\dot{A}}(t,x) + F_{\dot{A}}(t,x) \leq 3.$$

The Non-Stationary Neutrosophic Set $\dot{A}$ can be formally expressed as:

$$\dot{A} = \int_{t \in T} \int_{x \in X} \big(T_{\dot{A}}(t,x), I_{\dot{A}}(t,x), F_{\dot{A}}(t,x)\big)/x/t.$$

Theorem 3.144. *A Non-Stationary Neutrosophic Set generalizes the concept of a Non-Stationary Fuzzy Set.*

Proof. Let $\dot{A}$ be a Non-Stationary Neutrosophic Set characterized by the membership functions:

$$T_{\dot{A}} : T \times X \to [0,1], \quad I_{\dot{A}} : T \times X \to [0,1], \quad F_{\dot{A}} : T \times X \to [0,1],$$

where $T_{\dot{A}}(t,x), I_{\dot{A}}(t,x), F_{\dot{A}}(t,x)$ represent the time-specific degrees of truth, indeterminacy, and falsity, respectively.

If we set $I_{\dot{A}}(t,x) = 0$ and $F_{\dot{A}}(t,x) = 1 - T_{\dot{A}}(t,x)$, then the Non-Stationary Neutrosophic Set reduces to a Non-Stationary Fuzzy Set $\dot{A}_{\text{fuzzy}}$, characterized by:

$$\mu_{\dot{A}}(t,x) = T_{\dot{A}}(t,x),$$

which corresponds to the time-specific membership function of a Non-Stationary Fuzzy Set. Thus, the Non-Stationary Fuzzy Set is a special case of the Non-Stationary Neutrosophic Set, proving the generalization. □

Theorem 3.145. *A Non-Stationary Neutrosophic Set generalizes the concept of a Neutrosophic Set.*

Proof. Let $\dot{A}$ be a Non-Stationary Neutrosophic Set defined by:

$$T_{\dot{A}} : T \times X \to [0,1], \quad I_{\dot{A}} : T \times X \to [0,1], \quad F_{\dot{A}} : T \times X \to [0,1].$$

If we consider the case where the time set T consists of a single time point, $T = \{t_0\}$, the membership functions reduce to:

$$T_{\dot{A}}(t_0,x) = T_A(x), \quad I_{\dot{A}}(t_0,x) = I_A(x), \quad F_{\dot{A}}(t_0,x) = F_A(x),$$

which are the membership functions of a standard Neutrosophic Set A on X. Therefore, the Neutrosophic Set is a special case of the Non-Stationary Neutrosophic Set, proving the generalization. □

3.2.20 Cosine Neutrosophic Set

The Cosine Neutrosophic Set, an extension of the Cosine Fuzzy Set, is defined here. It is anticipated that further studies will explore its mathematical structure and potential applications in greater depth.

Definition 3.146 (Cosine Fuzzy Set)**.** [330] A *Cosine Fuzzy Set* A is defined on a closed interval $x = [a, b] \subseteq \mathbb{R}$, where each element $x \in x$ can be scaled to a corresponding value $z \in z = [-\pi, \pi]$ using the following transformations:

1. *Center*:
$$c = \frac{a+b}{2},$$
2. *Scale Factor*:
$$s = \frac{b-a}{2},$$
3. *Transformation*:
$$z = \frac{x-c}{s}\pi, \quad z \in [-\pi, \pi].$$

The membership function of the cosine fuzzy set A is defined as:

$$\mu_A(z) = \left(\frac{1+\cos(z)}{2}\right)^{2b}, \quad z \in [-\pi, \pi], \quad b \geq 0.$$

This membership function maps the scaled variable z to a fuzzy membership degree in $[0, 1]$ using a generalized cosine function.

Definition 3.147 (Cosine Neutrosophic Set)**.** A *Cosine Neutrosophic Set* A on a closed interval $x = [a, b] \subseteq \mathbb{R}$ is characterized by three membership functions, $T_A(z)$, $I_A(z)$, and $F_A(z)$, which represent the truth, indeterminacy, and falsity degrees, respectively. These membership functions are scaled and defined based on the generalized cosine fuzzy set as follows:

1. *Transformation*:
$$z = \frac{x-c}{s}\pi, \quad z \in [-\pi, \pi],$$
where $c = \frac{a+b}{2}$ is the center and $s = \frac{b-a}{2}$ is the scale factor.
2. *Membership Functions*:
$$T_A(z) = \left(\frac{1+\cos(z)}{2}\right)^{2b_T}, \quad z \in [-\pi, \pi], \quad b_T \geq 0,$$
$$I_A(z) = \left(\frac{1+\cos(z)}{2}\right)^{2b_I}, \quad z \in [-\pi, \pi], \quad b_I \geq 0,$$
$$F_A(z) = \left(\frac{1+\cos(z)}{2}\right)^{2b_F}, \quad z \in [-\pi, \pi], \quad b_F \geq 0.$$
3. *Condition*: The membership values satisfy:
$$0 \leq T_A(z) + I_A(z) + F_A(z) \leq 3, \quad \forall z \in [-\pi, \pi].$$

This definition extends the cosine fuzzy set to incorporate neutrosophic components, allowing the representation of truth, indeterminacy, and falsity in a single framework.

Theorem 3.148. *The Cosine Neutrosophic Set generalizes both the Cosine Fuzzy Set and the Neutrosophic Set.*

Proof. Let A be a Cosine Neutrosophic Set defined on the interval $x = [a, b] \subseteq \mathbb{R}$. The membership functions $T_A(z)$, $I_A(z)$, and $F_A(z)$ are given as:

$$T_A(z) = \left(\frac{1 + \cos(z)}{2}\right)^{2b_T}, \quad I_A(z) = \left(\frac{1 + \cos(z)}{2}\right)^{2b_I}, \quad F_A(z) = \left(\frac{1 + \cos(z)}{2}\right)^{2b_F}, \quad z \in [-\pi, \pi].$$

To recover a Cosine Fuzzy Set from A, we set:

$$b_T = b, \quad b_I = 0, \quad b_F = 0.$$

This results in:

$$T_A(z) = \left(\frac{1 + \cos(z)}{2}\right)^{2b}, \quad I_A(z) = 0, \quad F_A(z) = 0.$$

Here, $T_A(z)$ reduces to the membership function of a Cosine Fuzzy Set, and the indeterminacy and falsity components vanish. Thus, the Cosine Neutrosophic Set A generalizes the Cosine Fuzzy Set.

And let A be a Cosine Neutrosophic Set. The membership functions $T_A(z)$, $I_A(z)$, and $F_A(z)$ are defined as functions of z. To recover a standard Neutrosophic Set, we fix $z = 0$ (or equivalently $x = c$, the center of the interval):

$$z = \frac{x - c}{s}\pi = 0.$$

At $z = 0$, the cosine function evaluates to:

$$\cos(z) = \cos(0) = 1.$$

Substituting this into the membership functions:

$$T_A(0) = \left(\frac{1 + \cos(0)}{2}\right)^{2b_T} = \left(\frac{2}{2}\right)^{2b_T} = 1,$$

$$I_A(0) = \left(\frac{1 + \cos(0)}{2}\right)^{2b_I} = \left(\frac{2}{2}\right)^{2b_I} = 1,$$

$$F_A(0) = \left(\frac{1 + \cos(0)}{2}\right)^{2b_F} = \left(\frac{2}{2}\right)^{2b_F} = 1.$$

By allowing b_T, b_I, b_F to vary, we can reconstruct the membership functions $T(x)$, $I(x)$, and $F(x)$ of any Neutrosophic Set. Therefore, the Cosine Neutrosophic Set generalizes the Neutrosophic Set. □

3.3 Hyper Concepts and Superhyper concepts

In this subsection, we examine several Hyper concepts and Superhyper concepts. Note that in some fields, multiple definitions exist, and the meanings of "hyper" or "super" may vary.

3.3.1 HyperNeutrosophic Set

The HyperNeutrosophic Set is an extended set concept based on the Hyperfuzzy Set. The definitions and theorems are provided below.

Definition 3.149 (HyperNeutrosophic Set)**.** Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0,1]^3)$ is called a *HyperNeutrosophic Set* over X, where $\tilde{P}([0,1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ represents a set of neutrosophic membership degrees, each consisting of truth (T), indeterminacy (I), and falsity (F) components, satisfying:

$$0 \leq T + I + F \leq 3.$$

Theorem 3.150. *A HyperNeutrosophic Set generalizes a Neutrosophic Set.*

Proof. To show that a Neutrosophic Set is a special case of a HyperNeutrosophic Set, we construct a specific HyperNeutrosophic Set HNS' such that:

$$\tilde{\mu}'(x) = \{\mu(x)\}.$$

Here, $\tilde{\mu}'(x)$ is a singleton subset of $[0,1]^3$, containing exactly the membership triplet $\mu(x)$ from the Neutrosophic Set. For every $x \in X$, the singleton $\tilde{\mu}'(x) \subseteq [0,1]^3$ satisfies the conditions for a HyperNeutrosophic Set. Consequently, HNS' is a valid HyperNeutrosophic Set.

Thus, any Neutrosophic Set NS can be represented as a special case of a HyperNeutrosophic Set where the membership subset is always a singleton. Therefore, the HyperNeutrosophic Set generalizes the Neutrosophic Set. □

Theorem 3.151. *Every HyperNeutrosophic Set can be transformed into an equivalent HyperFuzzy Set.*

Proof. Given a HyperNeutrosophic Set HNS with mapping $\tilde{\mu} : X \to \tilde{P}([0,1]^3)$, we construct a HyperFuzzy Set HFS over X by defining:

$$\tilde{\mu}'(x) = \left\{ \frac{T + (1 - F)}{2} \mid (T, I, F) \in \tilde{\mu}(x) \right\}.$$

This mapping converts each neutrosophic membership triplet into a single membership degree in $[0,1]$ by averaging the truth and complement of the falsity degrees.

Since $T, I, F \in [0,1]$, and $0 \leq T + I + F \leq 3$, it follows that $\tilde{\mu}'(x) \subseteq [0,1]$.

Thus, HFS is a HyperFuzzy Set corresponding to HNS. □

Definition 3.152 (HyperPlithogenic Set)**.** Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set* HPS over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : \left(\bigcup_{i=1}^n Pv_i\right) \times \left(\bigcup_{i=1}^n Pv_i\right) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

Theorem 3.153. *A HyperPlithogenic Set generalizes a Plithogenic Set.*

Proof. To show that PS is a special case of HPS, consider the following construction:

$$n = 1, \quad \{v_i\}_{i=1}^n = \{v\}, \quad \{Pv_i\}_{i=1}^n = \{Pv\}.$$

Define $\tilde{pdf}_1$ for HPS such that:

$$\tilde{pdf}_1(x, a) = \{pdf(x, a)\}, \quad \text{for all } x \in P, \ a \in Pv.$$

Here, $\tilde{pdf}_1(x, a)$ is a singleton subset of $[0, 1]^s$, containing exactly the membership degree $pdf(x, a)$ from the Plithogenic Set.

Since pCF in HPS is identical to that in PS, PS can be viewed as a HyperPlithogenic Set where:

$$\tilde{pdf}_i(x, a) \text{ are singletons for all } x \in P, \ a \in Pv.$$

Thus, the HyperPlithogenic Set HPS generalizes the Plithogenic Set PS. □

Theorem 3.154. *Every HyperPlithogenic Set with $s = 1$ and $t = 1$ can be transformed into an equivalent HyperFuzzy Set.*

Proof. Given a HyperPlithogenic Set $HPS = (P, v, Pv, \tilde{pdf}, pCF)$ with $s = 1$, the Hyper Degree of Appurtenance Function $\tilde{pdf} : P \times Pv \to \tilde{P}([0, 1])$ assigns to each (x, a) a subset of $[0, 1]$.

We construct a HyperFuzzy Set HFS over P by defining the mapping $\tilde{\mu} : P \to \tilde{P}([0, 1])$ as:

$$\tilde{\mu}(x) = \bigcup_{a \in Pv} \tilde{pdf}(x, a).$$

Since $\tilde{pdf}(x, a) \subseteq [0, 1]$ for each $a \in Pv$, it follows that $\tilde{\mu}(x) \subseteq [0, 1]$. Thus, HFS is a HyperFuzzy Set.

This mapping effectively aggregates all the hyper degrees of appurtenance of x across all attribute values into a single hyperfuzzy membership set. The DCF pCF is not needed in the HyperFuzzy Set and is thus omitted.

Therefore, the HyperPlithogenic Set reduces to a HyperFuzzy Set when $s = 1$ and $t = 1$. □

Theorem 3.155. *Every HyperPlithogenic Set with $s = 3$ and $t = 1$ can be transformed into an equivalent HyperNeutrosophic Set.*

Proof. Given a HyperPlithogenic Set $HPS = (P, v, Pv, \tilde{pdf}, pCF)$ with $s = 3$, the Hyper Degree of Appurtenance Function $\tilde{pdf} : P \times Pv \to \tilde{P}([0, 1]^3)$ assigns to each (x, a) a subset of $[0, 1]^3$.

We construct a HyperNeutrosophic Set HNS over P by defining the mapping $\tilde{\mu} : P \to \tilde{P}([0, 1]^3)$ as:

$$\tilde{\mu}(x) = \bigcup_{a \in Pv} \tilde{pdf}(x, a).$$

Since $\tilde{pdf}(x, a) \subseteq [0, 1]^3$ for each $a \in Pv$, it follows that $\tilde{\mu}(x) \subseteq [0, 1]^3$. Each element in $\tilde{\mu}(x)$ is a triplet (T, I, F) satisfying $0 \leq T + I + F \leq 3$.

This mapping consolidates all hyper degrees of appurtenance of x across all attribute values into a single hyperneutrosophic membership set.

Therefore, the HyperPlithogenic Set reduces to a HyperNeutrosophic Set when $s = 3$ and $t = 1$. □

3.3.2 HyperVague Offset/Overset/Underset

We proceed to further define the concepts of the HyperVague Set, as well as Vague Offset, Vague Overset, and Vague Underset. Additionally, we will consider and define HyperVague Offset, HyperVague Overset, and HyperVague Underset. Some of these definitions are provided below.

Definition 3.156 (HyperVague Set)**.** Let U be a universe of discourse, and let $\tilde{P}([0,1]^2)$ denote the power set of the unit square $[0,1]^2$, excluding the empty set. A *HyperVague Set* $\tilde{A}$ in U is defined as a mapping:

$$\tilde{A} : U \to \tilde{P}([0,1]^2),$$

such that for each $u \in U$, $\tilde{A}(u) \subseteq [0,1]^2$ and each pair $(t, f) \in \tilde{A}(u)$ satisfies:

$$0 \leq t + f \leq 1.$$

Theorem 3.157. *A HyperVague Set is a generalization of a HyperFuzzy Set.*

Proof. For a given set X, a HyperFuzzy Set $\tilde{\mu} : X \to \tilde{P}([0,1])$ is a mapping where each element $x \in X$ is associated with a non-empty subset $\tilde{\mu}(x) \subseteq [0,1]$. Here, each subset represents degrees of membership within the interval $[0,1]$.

For a universe U, a HyperVague Set $\tilde{A} : U \to \tilde{P}([0,1]^2)$ maps each element $u \in U$ to a non-empty subset $\tilde{A}(u) \subseteq [0,1]^2$. Each pair $(t, f) \in \tilde{A}(u)$, representing degrees of membership t and non-membership f, must satisfy $0 \leq t + f \leq 1$.

Since each pair (t, f) in a HyperVague Set provides both a membership degree t and a non-membership degree f, the HyperVague Set extends the structure of the HyperFuzzy Set by accommodating additional uncertainty and dual-valued degrees.

To clarify this relationship:

- If we restrict the HyperVague Set $\tilde{A}$ to cases where each (t, f) satisfies $f = 0$, then $\tilde{A}$ effectively behaves as a HyperFuzzy Set, as it only represents membership degrees within the range $[0,1]$.
- Conversely, every HyperFuzzy Set can be viewed as a special case of a HyperVague Set by interpreting the membership values $\tilde{\mu}(x) \subseteq [0,1]$ as pairs $(t, f) = (t, 0)$ within the HyperVague framework.

Thus, the HyperVague Set encompasses the structure of the HyperFuzzy Set as a specific case, proving that it is a generalization of the HyperFuzzy Set. □

Theorem 3.158. *A HyperVague Set is a generalization of a Vague Set.*

Proof. If we restrict a HyperVague Set $\tilde{A}$ such that for each $u \in U$, $\tilde{A}(u)$ contains only a single pair $(t_A(u), f_A(u))$ that satisfies $t_A(u) + f_A(u) \leq 1$, then $\tilde{A}$ corresponds exactly to a Vague Set. Here, the truth-membership and false-membership degrees are fixed as in the Vague Set structure, with each element having one specific pair of truth and false values.

Conversely, the HyperVague Set allows multiple pairs (t, f) for each element u, where each pair meets the constraint $0 \leq t + f \leq 1$. This flexibility extends the Vague Set framework by permitting a range of truth and falsity degrees, effectively broadening the scope of possible membership representations and accommodating greater uncertainty.

Thus, the Vague Set is a specific instance of a HyperVague Set where each element u in the universe U has a unique pair (t, f). The HyperVague Set, by allowing multiple pairs per element, generalizes the Vague Set concept. □

Theorem 3.159. *Every HyperPlithogenic Set with $s = 2$ and $t = 1$ can be transformed into an equivalent HyperVague Set.*

Proof. The proof can be established in a similar manner as in the cases of Hyperfuzzy Sets and Hyperneutrosophic Sets. □

Definition 3.160 (Vague Overset)**.** A *Vague Overset A* in *U* is characterized by functions:

$$t_A : U \to [0, \Omega], \quad f_A : U \to [0, \Omega],$$

where $\Omega > 1$ and for each $u \in U$, the pair $(t_A(u), f_A(u))$ satisfies:

$$0 \leq t_A(u) + f_A(u) \leq \Omega,$$

with $t_A(u)$ or $f_A(u)$ potentially exceeding 1.

Theorem 3.161. *Every Vague OverSet can be transformed into an equivalent Fuzzy OverSet.*

Proof. The proof can be conducted using the same method as in the case of the Neutrosophic OverSet. □

Definition 3.162 (Vague Underset)**.** A *Vague Underset A* in *U* is characterized by functions:

$$t_A : U \to [\Psi, 1], \quad f_A : U \to [\Psi, 1],$$

where $\Psi < 0$ and for each $u \in U$, the pair $(t_A(u), f_A(u))$ satisfies:

$$\Psi \leq t_A(u) + f_A(u) \leq 1,$$

allowing $t_A(u)$ or $f_A(u)$ to be less than 0.

Theorem 3.163. *Every Vague UnderSet can be transformed into an equivalent Fuzzy UnderSet.*

Proof. The proof can be conducted using the same method as in the case of the Neutrosophic UnderSet. □

Definition 3.164 (Vague Offset)**.** A *Vague Offset A* in *U* is characterized by functions:

$$t_A : U \to [\Psi, \Omega], \quad f_A : U \to [\Psi, \Omega],$$

where $\Omega > 1$, $\Psi < 0$, and for each $u \in U$, the pair $(t_A(u), f_A(u))$ satisfies:

$$\Psi \leq t_A(u) + f_A(u) \leq \Omega,$$

permitting $t_A(u)$ and $f_A(u)$ to exceed 1 or be less than 0.

Theorem 3.165. *Every Vague OffSet can be transformed into an equivalent Fuzzy OffSet.*

Proof. The proof can be conducted using the same method as in the case of the Neutrosophic OffSet. □

Theorem 3.166. *A Vague Offset can generalize both Vague Overset and Vague Underset.*

Proof. The statement follows directly from the definitions. □

Definition 3.167 (HyperVague Overset)**.** A *HyperVague Overset* $\tilde{A}$ in *U* is a mapping:

$$\tilde{A} : U \to \tilde{P}([0, \Omega]^2),$$

where $\Omega > 1$, and for each $u \in U$, $\tilde{A}(u) \subseteq [0, \Omega]^2$ with each pair $(t, f) \in \tilde{A}(u)$ satisfying:

$$0 \leq t + f \leq \Omega,$$

and t or f may exceed 1.

Theorem 3.168. *A HyperVague OverSet is a generalization of a Vague OverSet.*

Proof. This is evident. □

Theorem 3.169. *Every HyperVague OverSet can be transformed into an equivalent HyperFuzzy OverSet.*

Proof. The proof can be conducted using the same method as in the case of the Neutrosophic OverSet. □

Definition 3.170 (HyperVague Underset)**.** A *HyperVague Underset* $\tilde{A}$ in U is a mapping:

$$\tilde{A} : U \to \tilde{P}([\Psi, 1]^2),$$

where $\Psi < 0$, and for each $u \in U$, $\tilde{A}(u) \subseteq [\Psi, 1]^2$ with each pair $(t, f) \in \tilde{A}(u)$ satisfying:

$$\Psi \leq t + f \leq 1,$$

allowing t or f to be less than 0.

Theorem 3.171. *A HyperVague UnderSet is a generalization of a Vague UnderSet.*

Proof. This is evident. □

Theorem 3.172. *Every HyperVague UnderSet can be transformed into an equivalent HyperFuzzy UnderSet.*

Proof. The proof can be conducted using the same method as in the case of the Neutrosophic UnderSet. □

Definition 3.173 (HyperVague Offset)**.** A *HyperVague Offset* $\tilde{A}$ in U is a mapping:

$$\tilde{A} : U \to \tilde{P}([\Psi, \Omega]^2),$$

where $\Omega > 1$, $\Psi < 0$, and for each $u \in U$, $\tilde{A}(u) \subseteq [\Psi, \Omega]^2$ with each pair $(t, f) \in \tilde{A}(u)$ satisfying:

$$\Psi \leq t + f \leq \Omega,$$

permitting t and f to exceed 1 or be less than 0.

Theorem 3.174. *A HyperVague OffSet is a generalization of a Vague OffSet.*

Proof. This is evident. □

Theorem 3.175. *Every HyperVague OffSet can be transformed into an equivalent HyperFuzzy OffSet.*

Proof. The proof can be conducted using the same method as in the case of the Neutrosophic OffSet. □

Theorem 3.176. *A HyperVague Offset can generalize both HyperVague Overset and HyperVague Underset.*

Proof. The statement follows directly from the definitions. □

3.3.3 IndetermSuperHyperSoft Set

The definition of the IndetermSuperHyperSoft Set is provided below.

Definition 3.177 (IndetermSuperHyperSoft Set)**.** Let U be a universe of discourse, H a non-empty subset of U, and $\mathcal{P}(H)$ the power set of H. Let $a_1, a_2, \ldots, a_n$ be n distinct attributes, with corresponding attribute value sets $A_1, A_2, \ldots, A_n$, where $A_i \cap A_j = \emptyset$ for all $i \neq j$.

For each attribute a_i, let $\mathcal{P}(A_i)$ denote the power set of A_i. Then, the pair

$$(F, \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n)),$$

where $F : \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n) \to \mathcal{P}(H)$, is called an *IndetermSuperHyperSoft Set* over U if at least one of the following conditions holds:

1. At least one of the sets $A_1, A_2, \ldots, A_n$ has some indeterminacy.
2. At least one of the sets H or $\mathcal{P}(H)$ has some indeterminacy.
3. The function F has some indeterminacy; that is, there exists at least one mapping $F(S_1, S_2, \ldots, S_n) = M$, where some of $S_1, S_2, \ldots, S_n$ or M are indeterminate (non-unique, unclear, incomplete, or unknown).

An IndetermSuperHyperSoft Set extends both the IndetermHyperSoft Set and the SuperHyperSoft Set by allowing indeterminacy in the attribute sets, the universe, or the mapping function, while also utilizing the power sets of the attribute value sets.

Theorem 3.178. *Every IndetermHyperSoft Set and every SuperHyperSoft Set can be considered as a special case of an IndetermSuperHyperSoft Set. Therefore, the IndetermSuperHyperSoft Set generalizes both the IndetermHyperSoft Set and the SuperHyperSoft Set.*

Proof. An IndetermHyperSoft Set is defined as follows:

Let U be a universe of discourse, $H \subseteq U$, and $\mathcal{P}(H)$ the power set of H. Let $a_1, a_2, \ldots, a_n$ be attributes with attribute value sets $A_1, A_2, \ldots, A_n$, where $A_i \cap A_j = \emptyset$ for $i \neq j$. The function $F : A_1 \times A_2 \times \cdots \times A_n \to \mathcal{P}(H)$ defines the IndetermHyperSoft Set $(F, A_1 \times A_2 \times \cdots \times A_n)$, with indeterminacy in A_i, H, or F.

To represent this as an IndetermSuperHyperSoft Set, we proceed as follows:

For each attribute a_i, consider the power set $\mathcal{P}(A_i)$. Since $A_i \subseteq \mathcal{P}(A_i)$, we can identify each element $e_i \in A_i$ with the singleton set $\{e_i\} \in \mathcal{P}(A_i)$. Define a new function $F' : \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n) \to \mathcal{P}(H)$ such that for all $(S_1, S_2, \ldots, S_n) \in \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$:

$$F'(S_1, S_2, \ldots, S_n) = F(e_1, e_2, \ldots, e_n),$$

where $e_i \in S_i$ and $S_i = \{e_i\}$. The indeterminacy in A_i, H, or F remains the same in this representation.

Thus, the IndetermHyperSoft Set is a special case of an IndetermSuperHyperSoft Set where the inputs to F' are singleton subsets of A_i.

A SuperHyperSoft Set is defined as:

Let U be a universe of discourse, and $a_1, a_2, \ldots, a_n$ be attributes with attribute value sets $A_1, A_2, \ldots, A_n$, where $A_i \cap A_j = \emptyset$ for $i \neq j$. The function $F : \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n) \to \mathcal{P}(U)$ defines the SuperHyperSoft Set $(F, \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n))$, typically without indeterminacy.

Since the SuperHyperSoft Set allows for all possible subsets of A_i as inputs, and maps them to subsets of U, we can consider $H = U$ and $\mathcal{P}(H) = \mathcal{P}(U)$. If there is no indeterminacy in A_i, H, or F, we can define the IndetermSuperHyperSoft Set with no indeterminacy, satisfying the conditions of an IndetermSuperHyperSoft Set trivially. Thus, a SuperHyperSoft Set is an IndetermSuperHyperSoft Set where no indeterminacy is present.

Since both the IndetermHyperSoft Set and the SuperHyperSoft Set can be viewed as special cases of the IndetermSuperHyperSoft Set (with singleton attribute subsets or without indeterminacy, respectively), it follows that the IndetermSuperHyperSoft Set generalizes both concepts. □

3.3.4 HyperRough Set and HyperRough Graph

We now define the HyperRough Set by combining the concepts of the HyperConcepts and the Rough Set.

Definition 3.179 (HyperRough Set)**.** Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with domains $J_1, J_2, \ldots, J_n$. Define $J = J_1 \times J_2 \times J_n$. Let $R \subseteq X \times X$ be an equivalence relation on X.

A *HyperRough Set* over X is a pair (F, J), where F is a mapping:

$$F : J \to \mathcal{P}(X),$$

such that for each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$, $F(a)$ is associated with a rough set $(\underline{F(a)}, \overline{F(a)})$ defined by:

$$\underline{F(a)} = \{x \in X \mid R(x) \subseteq F(a)\},$$
$$\overline{F(a)} = \{x \in X \mid R(x) \cap F(a) \neq \emptyset\}.$$

Theorem 3.180. *Every Rough Set is a special case of a HyperRough Set when the number of attributes* $n = 1$.

Proof. Let $U \subseteq X$ be a target set, and let R be an equivalence relation on X. The Rough Set approximates U by $(\underline{U}, \overline{U})$, where:

$$\underline{U} = \{x \in X \mid R(x) \subseteq U\},$$
$$\overline{U} = \{x \in X \mid R(x) \cap U \neq \emptyset\}.$$

Define a single attribute T_1 with domain $J_1 = \{a\}$. Then $J = J_1 = \{a\}$, and define the mapping $F : J \to \mathcal{P}(X)$ by $F(a) = U$.

In this case, the HyperRough Set (F, J) with $n = 1$ reduces to the Rough Set approximation of U, since:

$$\underline{F(a)} = \{x \in X \mid R(x) \subseteq F(a)\} = \underline{U},$$
$$\overline{F(a)} = \{x \in X \mid R(x) \cap F(a) \neq \emptyset\} = \overline{U}.$$

Therefore, the Rough Set is a special case of the HyperRough Set when $n = 1$, which proves that the HyperRough Set generalizes the Rough Set. □

Definition 3.181 (HyperRough Graph)**.** Let $G = (V, E)$ be a graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. Let $T_1, T_2, \ldots, T_n$ be n attributes with domains $J_1, J_2, \ldots, J_n$, and define $J = J_1 \times J_2 \times \cdots \times J_n$.

A *HyperRough Graph* is a pair (F, J), where $F : J \to \mathcal{P}(V)$, such that for each attribute combination $a \in J$, the subgraph induced by $F(a)$ is associated with rough approximations:

$$\underline{F(a)} = \{v \in V \mid R(v) \subseteq F(a)\},$$
$$\overline{F(a)} = \{v \in V \mid R(v) \cap F(a) \neq \emptyset\},$$

and the edge sets are approximated as:

$$\underline{E(a)} = \{e = (u, v) \in E \mid u, v \in \underline{F(a)}\},$$
$$\overline{E(a)} = \{e = (u, v) \in E \mid u, v \in \overline{F(a)}\}.$$

Theorem 3.182. *Every Rough Graph is a special case of a HyperRough Graph when the number of attributes* $n = 1$.

Proof. Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let R be an equivalence relation on V. The Rough Graph approximates V and E by their lower and upper approximations.

Define a single attribute T_1 with domain $J_1 = \{a\}$. Then $J = J_1 = \{a\}$, and define $F : J \to \mathcal{P}(V)$ by $F(a) = V$.

In this case, the HyperRough Graph (F, J) with $n = 1$ reduces to the Rough Graph, since:

$$\begin{aligned}
\underline{F(a)} &= \{v \in V \mid R(v) \subseteq F(a)\} = \{v \in V \mid R(v) \subseteq V\} = V, \\
\overline{F(a)} &= \{v \in V \mid R(v) \cap F(a) \neq \emptyset\} = V, \\
\underline{E(a)} &= \{e = (u, v) \in E \mid u, v \in \underline{F(a)}\} = E, \\
\overline{E(a)} &= \{e = (u, v) \in E \mid u, v \in \overline{F(a)}\} = E.
\end{aligned}$$

Thus, the Rough Graph is a special case of the HyperRough Graph when $n = 1$, which shows that the HyperRough Graph generalizes the Rough Graph. □

Alongside the HyperRough Set, the SuperHyperRough Set is also considered. The definitions are provided below. It is hoped that further exploration of their mathematical structures and the validity of these definitions will be advanced in future studies. This is defined based on the SuperHyperSoft Set.

Definition 3.183 (SuperHyperRough Set)**.** Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote the power set of J_i. Define the set of all possible attribute value combinations as the Cartesian product of the power sets:

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Let $R \subseteq X \times X$ be an equivalence relation on X.

A *SuperHyperRough Set* over X is a pair (F, J), where F is a mapping:

$$F : J \to \mathcal{P}(X),$$

such that for each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$, where $A_i \subseteq J_i$, $F(A)$ is associated with a rough set $(\underline{F(A)}, \overline{F(A)})$ defined by:

$$\begin{aligned}
\underline{F(A)} &= \{x \in X \mid R(x) \subseteq F(A)\}, \\
\overline{F(A)} &= \{x \in X \mid R(x) \cap F(A) \neq \emptyset\}.
\end{aligned}$$

Theorem 3.184. *Every HyperRough Set is a special case of a SuperHyperRough Set when the attribute value subsets are singleton sets.*

Proof. In a HyperRough Set, the set of attribute value combinations J is defined as $J = J_1 \times J_2 \times \cdots \times J_n$, where each J_i is the domain of attribute T_i.

Consider the SuperHyperRough Set where we restrict each attribute value subset $A_i \subseteq J_i$ to be a singleton set $\{a_i\}$. Then, the Cartesian product of the power sets simplifies to:

$$J = \{\{a_1\}\} \times \{\{a_2\}\} \times \cdots \times \{\{a_n\}\} \cong J_1 \times J_2 \times \cdots \times J_n.$$

Under this restriction, the mapping $F : J \to \mathcal{P}(X)$ in the SuperHyperRough Set operates on individual attribute values, and the definitions of $\underline{F(A)}$ and $\overline{F(A)}$ reduce to those in the HyperRough Set.

Therefore, the HyperRough Set is a special case of the SuperHyperRough Set when attribute value subsets are singleton sets. □

For reference, the relationships between the SuperhyperSoft set and the Superhyperrough set are illustrated in Figure 9.

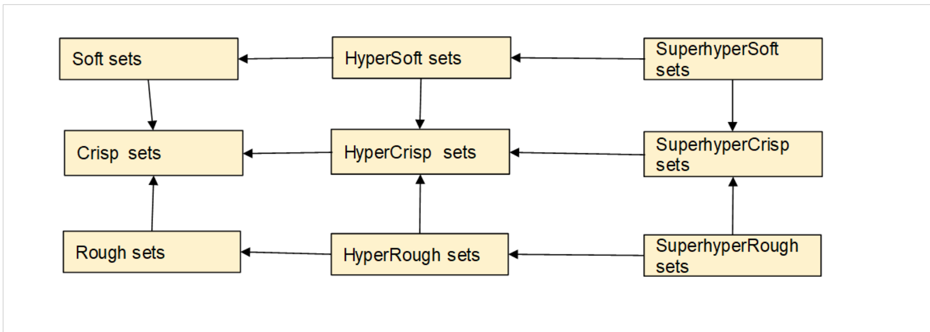


Figure 9: Some the SuperhyperSoft sets and the Superhyperrough sets Hierarchy. The set class at the origin of an arrow contains the set class at the destination of the arrow.

3.3.5 Neutrosophic Over Hypergraph, Off Hypergraph, and Under Hypergraph

The definitions of Neutrosophic Over Hypergraph, Off Hypergraph, and Under Hypergraph are provided below. These are extended graph concepts that generalize the Neutrosophic Over Graph using Hypergraph structures.

Definition 3.185. A *Single-Valued Neutrosophic Over Hypergraph* $H = (V, E)$ is a hypergraph where:

1. Each vertex $v \in V$ is associated with three membership degrees:
$$T(v) \in [0, \Omega], \quad I(v) \in [0, \Omega], \quad F(v) \in [0, \Omega], \quad \text{with } \Omega > 1.$$
2. Each hyperedge $e \in E$ is associated with three membership degrees:
$$T(e) \in [0, \Omega], \quad I(e) \in [0, \Omega], \quad F(e) \in [0, \Omega], \quad \text{with } \Omega > 1.$$
3. For all $v \in V$, the sum satisfies:
$$T(v) + I(v) + F(v) \leq 3\Omega.$$

Definition 3.186. A *Single-Valued Neutrosophic Under Hypergraph* $H = (V, E)$ is a hypergraph where:

1. Each vertex $v \in V$ is associated with three membership degrees:
$$T(v) \in [\Psi, 1], \quad I(v) \in [\Psi, 1], \quad F(v) \in [\Psi, 1], \quad \text{with } \Psi < 0.$$
2. Each hyperedge $e \in E$ is associated with three membership degrees:
$$T(e) \in [\Psi, 1], \quad I(e) \in [\Psi, 1], \quad F(e) \in [\Psi, 1], \quad \text{with } \Psi < 0.$$
3. For all $v \in V$, the sum satisfies:
$$T(v) + I(v) + F(v) \leq 3.$$

Definition 3.187. A *Single-Valued Neutrosophic Off Hypergraph* $H = (V, E)$ is a hypergraph where:

1. Each vertex $v \in V$ is associated with three membership degrees:
$$T(v) \in [\Psi, \Omega], \quad I(v) \in [\Psi, \Omega], \quad F(v) \in [\Psi, \Omega] \text{with } \Omega > 1 and \Psi < 0.$$
2. Each hyperedge $e \in E$ is associated with three membership degrees:
$$T(e) \in [\Psi, \Omega], \quad I(e) \in [\Psi, \Omega], \quad F(e) \in [\Psi, \Omega], \quad \text{with } \Omega > 1 and \Psi < 0.$$
3. For all $v \in V$, the sum satisfies:
$$T(v) + I(v) + F(v) \leq 3\Omega.$$

Theorem 3.188. *The Single-Valued Neutrosophic Over Hypergraph, Single-Valued Neutrosophic Under Hypergraph, and Single-Valued Neutrosophic Off Hypergraph generalize the concept of a Hypergraph.*

Proof. This is self-evident. □

Theorem 3.189. *Every Single-Valued Neutrosophic Over Graph is a special case of a Single-Valued Neutrosophic Over Hypergraph when all hyperedges are of size two.*

Proof. Consider a Single-Valued Neutrosophic Over Hypergraph $H = (V, E)$ where each hyperedge $e \in E$ is constrained to size two, i.e., $|e| = 2$. In this configuration, H reduces to a standard graph $G = (V, E')$ where E' consists of edges connecting pairs of vertices.

In this reduced form, each vertex $v \in V$ and edge $e \in E$ has associated neutrosophic membership degrees $T(v)$, $I(v)$, $F(v)$ for vertices and $T(e)$, $I(e)$, $F(e)$ for edges, which align with the definition of a Single-Valued Neutrosophic Over Graph. Additionally, the condition $\Omega > 1$ allows membership degrees to exceed the typical bound of 1, thus satisfying the overlimit property.

Thus, when hyperedges are limited to size two, the Single-Valued Neutrosophic Over Hypergraph becomes equivalent to a Single-Valued Neutrosophic Over Graph, showing that the former is a generalization of the latter. □

Theorem 3.190. *Every Single-Valued Neutrosophic Under Graph is a special case of a Single-Valued Neutrosophic Under Hypergraph when all hyperedges are of size two.*

Proof. Similarly, consider a Single-Valued Neutrosophic Under Hypergraph $H = (V, E)$ where each hyperedge $e \in E$ is restricted to size two, i.e., $|e| = 2$. This reduction results in a standard graph $G = (V, E')$ with edges connecting pairs of vertices.

Each vertex $v \in V$ and edge $e \in E$ is assigned membership degrees $T(v)$, $I(v)$, $F(v)$ for vertices and $T(e)$, $I(e)$, $F(e)$ for edges, consistent with those in a Single-Valued Neutrosophic Under Graph. Furthermore, the condition $\Psi < 0$ permits membership values to fall below zero, satisfying the underlimit property.

Consequently, when hyperedges are of size two, the Single-Valued Neutrosophic Under Hypergraph reduces to a Single-Valued Neutrosophic Under Graph, proving that the hypergraph form generalizes the graph form. □

Theorem 3.191. *Every Single-Valued Neutrosophic Off Graph is a special case of a Single-Valued Neutrosophic Off Hypergraph when all hyperedges are of size two.*

Proof. Consider a Single-Valued Neutrosophic Off Hypergraph $H = (V, E)$ where each hyperedge $e \in E$ has size two, i.e., $|e| = 2$. Under this constraint, H reduces to a standard graph $G = (V, E')$ with edges connecting pairs of vertices.

In this scenario, each vertex $v \in V$ and edge $e \in E$ has associated neutrosophic membership degrees $T(v)$, $I(v)$, $F(v)$ for vertices and $T(e)$, $I(e)$, $F(e)$ for edges, consistent with the Single-Valued Neutrosophic Off Graph definition. The ranges $[\Psi, \Omega]$ for T, I, and F, where $\Omega > 1$ and $\Psi < 0$, allow for values exceeding 1 or below 0, satisfying the offset condition.

Therefore, by limiting hyperedges to size two, the Single-Valued Neutrosophic Off Hypergraph simplifies to a Single-Valued Neutrosophic Off Graph, showing that the hypergraph structure generalizes the graph structure. □

Theorem 3.192. *A Single-Valued Neutrosophic Off Hypergraph generalizes both Single-Valued Neutrosophic Over Hypergraph and Single-Valued Neutrosophic Under Hypergraph.*

Proof. Let $H = (V, E)$ be a Single-Valued Neutrosophic Off Hypergraph where:

$$T(v), I(v), F(v) \in [\Psi, \Omega] \quad \text{with } \Psi < 0,\ \Omega > 1,$$

and the sum satisfies:

$$T(v) + I(v) + F(v) \leq 3\Omega \quad \forall v \in V.$$

If $\Psi = 0$, the ranges of $T(v), I(v), F(v)$ reduce to $[0, \Omega]$. Under this restriction, the definitions of vertex and edge membership degrees in the Single-Valued Neutrosophic Off Hypergraph coincide with those in the Single-Valued Neutrosophic Over Hypergraph:

$$T(v), I(v), F(v) \in [0, \Omega], \quad T(v) + I(v) + F(v) \leq 3\Omega.$$

Thus, a Single-Valued Neutrosophic Off Hypergraph becomes a Single-Valued Neutrosophic Over Hypergraph when $\Psi = 0$.

If $\Omega = 1$, the ranges of $T(v), I(v), F(v)$ reduce to $[\Psi, 1]$. Under this restriction, the definitions of vertex and edge membership degrees in the Single-Valued Neutrosophic Off Hypergraph coincide with those in the Single-Valued Neutrosophic Under Hypergraph:

$$T(v), I(v), F(v) \in [\Psi, 1], \quad T(v) + I(v) + F(v) \leq 3.$$

Thus, a Single-Valued Neutrosophic Off Hypergraph becomes a Single-Valued Neutrosophic Under Hypergraph when $\Omega = 1$.

Since the Single-Valued Neutrosophic Off Hypergraph allows for arbitrary $\Psi < 0$ and $\Omega > 1$, it encompasses both the Single-Valued Neutrosophic Over Hypergraph ($\Psi = 0$) and the Single-Valued Neutrosophic Under Hypergraph ($\Omega = 1$) as special cases. Therefore, the Single-Valued Neutrosophic Off Hypergraph generalizes both structures. □

3.3.6 Superhypercrisp Set and SuperHyperFuzzy Set

The definitions of a HyperCrisp Set, SuperHyperCrisp Set, and SuperHyperFuzzy Set are provided below. The concept of the SuperHyperFuzzy Set serves as a generalization of the HyperFuzzy Set.

Definition 3.193 (HyperCrisp Set)**.** Let X be a non-empty set. A *HyperCrisp Set* $\tilde{C}$ over X is defined as a mapping:

$$\tilde{C} : X \to \tilde{\mathcal{P}}(\{0, 1\}),$$

where $\tilde{\mathcal{P}}(\{0, 1\})$ denotes the family of all non-empty subsets of $\{0, 1\}$. For each element $x \in X$, $\tilde{C}(x) \subseteq \{0, 1\}$, meaning that each element of X is assigned a subset of $\{0, 1\}$ as its membership values. This allows each element x to either fully belong to the set (membership degree 1) or not belong at all (membership degree 0).

Theorem 3.194. *A HyperCrisp Set generalizes the concept of a Crisp Set.*

Proof. A Crisp Set $A \subseteq X$ is defined by a characteristic function $\chi_A : X \to \{0, 1\}$, where:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

In the case of a HyperCrisp Set $\tilde{C} : X \to \tilde{\mathcal{P}}(\{0, 1\})$, each $x \in X$ is assigned a subset of $\{0, 1\}$. If we restrict $\tilde{C}(x)$ to be either $\{1\}$ or $\{0\}$ for all $x \in X$, then $\tilde{C}$ reduces to the characteristic function of a Crisp Set. Specifically:

- If $\tilde{C}(x) = \{1\}$, this corresponds to $\chi_A(x) = 1$, meaning $x \in A$.
- If $\tilde{C}(x) = \{0\}$, this corresponds to $\chi_A(x) = 0$, meaning $x \notin A$.

Thus, the Crisp Set is a special case of the HyperCrisp Set where the assigned subsets of $\{0, 1\}$ are restricted to singletons. Therefore, the HyperCrisp Set generalizes the concept of a Crisp Set. □

Theorem 3.195. *A HyperFuzzy Set is a generalization of a HyperCrisp Set.*

Proof. Let X be a non-empty set, and let $\tilde{C} : X \to \tilde{\mathcal{P}}(\{0, 1\})$ be a HyperCrisp Set over X, where each element $x \in X$ is assigned a subset $\tilde{C}(x) \subseteq \{0, 1\}$ as its membership values. Specifically, $\tilde{C}(x) = \{0\}$ if $x \notin \tilde{C}$ and $\tilde{C}(x) = \{1\}$ if $x \in \tilde{C}$.

Now, consider a HyperFuzzy Set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1])$, where each element $x \in X$ is assigned a non-empty subset $\tilde{\mu}(x) \subseteq [0, 1]$, allowing partial membership values.

To show that a HyperCrisp Set is a special case of a HyperFuzzy Set, we define $\tilde{\mu}(x) = \tilde{C}(x)$, where $\tilde{C}(x) \subseteq \{0, 1\} \subseteq [0, 1]$. Here:

- If $\tilde{C}(x) = \{1\}$, then $\tilde{\mu}(x) = \{1\}$, indicating full membership in the HyperFuzzy Set.
- If $\tilde{C}(x) = \{0\}$, then $\tilde{\mu}(x) = \{0\}$, indicating non-membership.

Thus, every HyperCrisp Set can be represented as a HyperFuzzy Set where membership values are restricted to the discrete set $\{0, 1\}$, proving that the HyperFuzzy Set generalizes the HyperCrisp Set. □

Definition 3.196 (SuperHyperCrisp Set)**.** Let X be a non-empty set. A *SuperHyperCrisp Set* $\tilde{C}$ over X is a mapping:

$$\tilde{C} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}(\{0, 1\}),$$

where $\tilde{\mathcal{P}}(X)$ denotes the family of all non-empty subsets of X and $\tilde{\mathcal{P}}(\{0, 1\})$ denotes the family of all non-empty subsets of $\{0, 1\}$. For each subset $A \in \tilde{\mathcal{P}}(X)$, $\tilde{C}(A) \subseteq \{0, 1\}$, representing the membership degree of the entire subset A. This setup allows each subset of X to either fully belong to the SuperHyperCrisp Set or not.

Theorem 3.197. *A HyperCrisp Set is a generalization of a Crisp Set, and a SuperHyperCrisp Set is a further generalization of a HyperCrisp Set.*

Proof. We establish the relationship between Crisp Sets, HyperCrisp Sets, and SuperHyperCrisp Sets as follows.

1. *HyperCrisp Set as a Generalization of Crisp Set:*

A Crisp Set $C \subseteq X$ is defined by a characteristic function:

$$\chi_C : X \to \{0, 1\},$$

where $\chi_C(x) = 1$ if $x \in C$ and $\chi_C(x) = 0$ if $x \notin C$. This function assigns a unique value, either 0 or 1, to each element $x \in X$, indicating its non-membership or membership in C, respectively.

A HyperCrisp Set $\tilde{C}$ over X generalizes this concept by allowing each element $x \in X$ to be mapped to a subset of $\{0, 1\}$. Formally:

$$\tilde{C} : X \to \tilde{\mathcal{P}}(\{0, 1\}),$$

where $\tilde{\mathcal{P}}(\{0, 1\})$ represents the set of all non-empty subsets of $\{0, 1\}$. For each $x \in X$, $\tilde{C}(x) \subseteq \{0, 1\}$ allows for three possible values for membership:

- $\tilde{C}(x) = \{1\}$: x fully belongs to $\tilde{C}$.
- $\tilde{C}(x) = \{0\}$: x does not belong to $\tilde{C}$.
- $\tilde{C}(x) = \{0, 1\}$: membership is indeterminate.

Thus, a Crisp Set is a special case of a HyperCrisp Set where each element is mapped to a single value, either 0 or 1, without the possibility of indeterminacy.

2. *SuperHyperCrisp Set as a Generalization of HyperCrisp Set:*

A SuperHyperCrisp Set extends the concept of HyperCrisp Sets to operate on the power set of X, assigning a subset of $\{0, 1\}$ as the membership for each subset of X. Formally, a SuperHyperCrisp Set $\tilde{S}$ over X is defined as:

$$\tilde{S} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}(\{0, 1\}),$$

where $\tilde{\mathcal{P}}(X)$ denotes the family of all non-empty subsets of X. For each subset $A \in \tilde{\mathcal{P}}(X)$, $\tilde{S}(A) \subseteq \{0, 1\}$, allowing each subset to fully belong, not belong, or have indeterminate membership in $\tilde{S}$.

A HyperCrisp Set is then a special case of a SuperHyperCrisp Set where the mapping $\tilde{S}$ is restricted to singleton subsets of X. In this way, the SuperHyperCrisp Set provides a more flexible structure that includes both the individual membership information of elements and the collective membership of subsets.

In conclusion, the SuperHyperCrisp Set generalizes the HyperCrisp Set, which in turn generalizes the Crisp Set, thus completing the proof. □

Theorem 3.198. *The SuperHyperRough Set generalizes the SuperHyperCrisp Set.*

Proof. Let X be a finite non-empty universe. A SuperHyperCrisp Set $\tilde{C}$ over X is defined as a mapping:

$$\tilde{C} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}(\{0, 1\})$$

.

A SuperHyperRough Set (F, J) is defined over X using the attribute domain $J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n)$ and a mapping:

$$F : J \to \mathcal{P}(X),$$

such that for each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$, $F(A)$ is associated with a rough set $(\underline{F(A)}, \overline{F(A)})$, where:

$$\underline{F(A)} = \{x \in X \mid R(x) \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid R(x) \cap F(A) \neq \emptyset\}.$$

For any $A \in J$, if $R(x) = \{x\}$ for all $x \in X$, the lower and upper approximations satisfy:

$$\underline{F(A)} = F(A), \quad \overline{F(A)} = F(A).$$

In this case, the SuperHyperRough Set reduces to a SuperHyperCrisp Set where each subset A maps to a crisp membership degree (0 or 1).

The SuperHyperRough Set allows for $R(x)$ to represent equivalence classes, enabling uncertainty in set membership. This is not possible in a SuperHyperCrisp Set, which only maps subsets to binary membership values.

A SuperHyperCrisp Set can be viewed as a special case of a SuperHyperRough Set where all attribute domains J_i are binary ($\{0, 1\}$) and the equivalence relation R is trivial (identity mapping).

Thus, the SuperHyperRough Set framework extends the SuperHyperCrisp Set by introducing equivalence relations and rough approximations, allowing for the modeling of uncertainty and granularity in set membership. □

Corollary 3.199. *The HyperRough Set generalizes the HyperCrisp Set.*

Proof. The proof can be established in a manner similar to that of the SuperHyperRough Set, but with a simplified approach. □

Theorem 3.200. *The SuperHyperSoft Set generalizes the SuperHyperCrisp Set.*

Proof. Let U be a universe of discourse. A *SuperHyperCrisp Set* $\tilde{C}$ over U is a mapping:

$$\tilde{C} : \tilde{\mathcal{P}}(U) \to \tilde{\mathcal{P}}(\{0, 1\}),$$

where $\tilde{\mathcal{P}}(U)$ represents the family of all non-empty subsets of U and $\tilde{\mathcal{P}}(\{0, 1\})$ denotes the family of all non-empty subsets of $\{0, 1\}$. In this setup, each subset $A \in \tilde{\mathcal{P}}(U)$ is mapped to $\{0\}$, $\{1\}$, or $\{0, 1\}$, ensuring that each subset either fully belongs to or does not belong to $\tilde{C}$.

Now consider a *SuperHyperSoft Set* $(F, \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n))$, where:

$$F : \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n) \to \mathcal{P}(U).$$

Here, F assigns each n-tuple $(S_1, S_2, \ldots, S_n)$ of attribute value subsets to a subset of U.

To show that the SuperHyperSoft Set generalizes the SuperHyperCrisp Set, consider a specific case of the SuperHyperSoft Set where the mappings F and $\mathcal{P}(A_i)$ are restricted as follows:

- For each i, $\mathcal{P}(A_i) = \{\emptyset, \{a_i\}\}$, where a_i represents a single attribute value.
- The mapping F is defined such that $F(S_1, S_2, \ldots, S_n) \in \{\emptyset, U\}$, corresponding to whether the tuple $(S_1, S_2, \ldots, S_n)$ belongs to the SuperHyperSoft Set.

Under these restrictions, the SuperHyperSoft Set reduces to a structure equivalent to a SuperHyperCrisp Set. Specifically, each subset $A \in \tilde{\mathcal{P}}(U)$ is mapped to $\{0\}$, $\{1\}$, or $\{0, 1\}$, as in the SuperHyperCrisp Set definition.

Therefore, the SuperHyperSoft Set generalizes the SuperHyperCrisp Set by allowing more flexible mappings F and attribute power sets $\mathcal{P}(A_i)$, which enable a broader range of input-output relationships. □

Corollary 3.201. *The HyperSoft Set generalizes the HyperCrisp Set.*

Proof. The proof can be established in a manner similar to that of the SuperHyperSoft Set, but with a simplified approach. □

Theorem 3.202. *A SuperHyperCrisp Set $\tilde{C} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}(\{0, 1\})$ inherently includes structures isomorphic to levels of the n-th PowerSet $\mathcal{P}_n^*(X)$, for certain subsets of X and mappings $\tilde{C}$.*

Proof. We consider each cases.

1. **Base Case:** $n = 0$**:**
   - In the n-th PowerSet definition, $\mathcal{P}_0^*(X) := X$.
   - For the SuperHyperCrisp Set, the domain $\tilde{\mathcal{P}}(X)$ includes all non-empty subsets of X, which naturally contains X itself as an element.
   - Thus, $\mathcal{P}_0^*(X) \subseteq \tilde{\mathcal{P}}(X)$.
2. **First-Level PowerSet:** $n = 1$**:**
   - The first-level PowerSet is $\mathcal{P}_1^*(X) = \mathcal{P}(X)$, the power set of X.
   - The domain of the SuperHyperCrisp Set, $\tilde{\mathcal{P}}(X)$, is defined as the set of all non-empty subsets of X, which aligns with $\mathcal{P}(X) \setminus \{\emptyset\}$.
   - The mapping $\tilde{C}$ assigns membership values $\tilde{C}(A) \subseteq \{0, 1\}$ to each $A \in \mathcal{P}(X) \setminus \{\emptyset\}$.
   - Hence, $\mathcal{P}_1^*(X) \setminus \{\emptyset\} \subseteq \tilde{\mathcal{P}}(X)$.
3. **Recursive Case:** $n \geq 2$**:**

- By definition, the n-th PowerSet $\mathcal{P}_n^*(X)$ is recursively defined as $\mathcal{P}_n^*(X) = \mathcal{P}(\mathcal{P}_{n-1}^*(X))$.
- For $n = 2$, $\mathcal{P}_2^*(X) = \mathcal{P}(\mathcal{P}(X))$, representing the set of all subsets of the power set of X. Similarly, higher levels involve recursive applications of $\mathcal{P}$.
- The domain $\tilde{\mathcal{P}}(X)$ inherently supports recursive structures, as $\tilde{\mathcal{P}}(X)$ is defined over subsets of subsets, aligning with $\mathcal{P}(\mathcal{P}(X))$ and higher-order sets.
- The mapping $\tilde{C}$ extends to these subsets by defining membership degrees $\tilde{C}(A) \subseteq \{0, 1\}$, preserving the recursive nature.

4. **Isomorphic Structures:**
   - The recursive structure of $\tilde{\mathcal{P}}(X)$ ensures that for any n, the elements of $\mathcal{P}_n^*(X)$ can be mapped to elements in $\tilde{\mathcal{P}}(X)$, maintaining set-theoretic relationships and hierarchical nesting.

5. **Exclusion of Empty Sets:**
   - The n-th PowerSet excludes $\emptyset$ at every level. This is consistent with $\tilde{\mathcal{P}}(X)$, where only non-empty subsets are considered.

The SuperHyperCrisp Set $\tilde{C}$ inherently possesses structural properties consistent with the n-th PowerSet $\mathcal{P}_n^*(X)$ due to its recursive definition over subsets of X and the alignment of its domain $\tilde{\mathcal{P}}(X)$ with $\mathcal{P}_n^*(X)$. The mappings $\tilde{C}(A)$ ensure that membership relationships are maintained, further establishing the correspondence. □

As discussed above, if we explicitly define the n-SuperHyperCrisp Set, it would be as follows.

Definition 3.203 (n-SuperHyperCrisp Set)**.** Let X be a non-empty set, and $n \geq 0$ be an integer. An *n-SuperHyperCrisp Set* $\tilde{C}_n$ over X is a mapping:

$$\tilde{C}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}(\{0, 1\}),$$

where $\tilde{\mathcal{P}}_n^*(X)$ denotes the family of all non-empty elements of the n-th PowerSet $\mathcal{P}_n^*(X)$, and $\tilde{\mathcal{P}}(\{0, 1\})$ denotes the family of all non-empty subsets of $\{0, 1\}$.

For each element $A \in \tilde{\mathcal{P}}_n^*(X)$, $\tilde{C}_n(A) \subseteq \{0, 1\}$, representing the membership degree(s) of A.

Theorem 3.204. *An n-SuperHyperCrisp Set $\tilde{C}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}(\{0, 1\})$ aligns with the n-th PowerSet $\mathcal{P}_n^*(X)$ and generalizes the SuperHyperCrisp Set to include higher-order structures.*

Proof. We will demonstrate that the domain of $\tilde{C}_n$ is the set of all non-empty elements of $\mathcal{P}_n^*(X)$, which inherently includes the structures of the n-th PowerSet.

1. **Base Case:** $n = 0$**.**
   - $\mathcal{P}_0^*(X) = X$.
   - The mapping $\tilde{C}_0 : X \to \tilde{\mathcal{P}}(\{0, 1\})$.
   - This reduces to the definition of a *HyperCrisp Set*, where each element $x \in X$ is mapped to $\tilde{C}_0(x) \subseteq \{0, 1\}$.

2. **First-Level PowerSet:** $n = 1$**.**
   - $\mathcal{P}_1^*(X) = \mathcal{P}(X)$.
   - The mapping $\tilde{C}_1 : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}(\{0, 1\})$, where $\tilde{\mathcal{P}}(X) = \mathcal{P}(X) \setminus \{\emptyset\}$.
   - This corresponds to the *SuperHyperCrisp Set* as previously defined.

3. **Recursive Case:** $n \geq 2$**.**
   - $\mathcal{P}_n^*(X) = \mathcal{P}(\mathcal{P}_{n-1}^*(X))$.

- The mapping $\tilde{C}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}(\{0, 1\})$.
- Each non-empty element $A \in \mathcal{P}_n^*(X)$ is assigned a subset $\tilde{C}_n(A) \subseteq \{0, 1\}$, capturing higher-order relationships.
- This extends the SuperHyperCrisp Set to higher-order structures corresponding to the n-th PowerSet.

Thus, the n-SuperHyperCrisp Set aligns with the n-th PowerSet $\mathcal{P}_n^*(X)$ for all $n \geq 0$, generalizing the SuperHyperCrisp Set to include hierarchical and recursive structures. □

Definition 3.205 (SuperHyperFuzzy Set)**.** Let X be a non-empty set. A mapping $\tilde{\mu} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0, 1])$ is called a *SuperHyperFuzzy Set* over X, where $\tilde{\mathcal{P}}(X)$ denotes the family of all non-empty subsets of X, and $\tilde{\mathcal{P}}([0, 1])$ denotes the family of all non-empty subsets of the interval $[0, 1]$.

In this structure:

- Each element $A \in \tilde{\mathcal{P}}(X)$ is a non-empty subset of X.
- The mapping $\tilde{\mu}$ assigns to each $A \in \tilde{\mathcal{P}}(X)$ a non-empty subset $\tilde{\mu}(A) \subseteq [0, 1]$, representing the degrees of membership associated with the subset A.

Example 3.206. A *SuperHyperFuzzy Set* $\tilde{B}$ over X is defined by a mapping $\tilde{\mu}_{\tilde{B}} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0, 1])$, where $\tilde{\mathcal{P}}(X)$ denotes the family of all non-empty subsets of X.

The non-empty subsets of X are:

$$\tilde{\mathcal{P}}(X) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\}.$$

We define $\tilde{\mu}_{\tilde{B}}$ as:

$$\begin{aligned}
\tilde{\mu}_{\tilde{B}}(\{x_1\}) &= \{0.1, 0.2, 0.3\},\\
\tilde{\mu}_{\tilde{B}}(\{x_2\}) &= \{0.4, 0.5, 0.6\},\\
\tilde{\mu}_{\tilde{B}}(\{x_3\}) &= \{0.6, 0.7, 0.8\},\\
\tilde{\mu}_{\tilde{B}}(\{x_1, x_2\}) &= \{0.3, 0.5\},\\
\tilde{\mu}_{\tilde{B}}(\{x_1, x_3\}) &= \{0.4, 0.6\},\\
\tilde{\mu}_{\tilde{B}}(\{x_2, x_3\}) &= \{0.5, 0.7\},\\
\tilde{\mu}_{\tilde{B}}(\{x_1, x_2, x_3\}) &= \{0.6, 0.8\}.
\end{aligned}$$

This means:

- Each non-empty subset of X is assigned a set of membership degrees in $\tilde{B}$.
- For singleton subsets, the membership degrees correspond to those in the hyperfuzzy set $\tilde{A}$.
- For subsets with more elements, the membership degrees represent the collective membership of the group.

Theorem 3.207. *A SuperHyperFuzzy Set is a generalization of a SuperHyperCrisp Set.*

Proof. Let X be a non-empty set, and let $\tilde{C} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}(\{0, 1\})$ be a SuperHyperCrisp Set over X, where each subset $A \in \tilde{\mathcal{P}}(X)$ is assigned a subset $\tilde{C}(A) \subseteq \{0, 1\}$ as its membership degrees. Specifically, $\tilde{C}(A) = \{0\}$ if $A \notin \tilde{C}$ and $\tilde{C}(A) = \{1\}$ if $A \in \tilde{C}$.

Now, consider a SuperHyperFuzzy Set $\tilde{\mu} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0, 1])$, where each subset $A \in \tilde{\mathcal{P}}(X)$ is assigned a non-empty subset $\tilde{\mu}(A) \subseteq [0, 1]$, allowing partial membership values.

To show that a SuperHyperCrisp Set is a special case of a SuperHyperFuzzy Set, we define $\tilde{\mu}(A) = \tilde{C}(A)$, where $\tilde{C}(A) \subseteq \{0, 1\} \subseteq [0, 1]$. Here:

- If $\tilde{C}(A) = \{1\}$, then $\tilde{\mu}(A) = \{1\}$, indicating full membership in the SuperHyperFuzzy Set.
- If $\tilde{C}(A) = \{0\}$, then $\tilde{\mu}(A) = \{0\}$, indicating non-membership.

Thus, every SuperHyperCrisp Set can be represented as a SuperHyperFuzzy Set where membership values are restricted to the discrete set $\{0, 1\}$, proving that the SuperHyperFuzzy Set generalizes the SuperHyperCrisp Set. □

Theorem 3.208. *The SuperHyperFuzzy Set generalizes both the HyperFuzzy Set and the traditional Fuzzy Set. Specifically:*

1. *Every Fuzzy Set can be represented as a SuperHyperFuzzy Set.*
2. *Every HyperFuzzy Set can be represented as a SuperHyperFuzzy Set.*

Proof. A Fuzzy Set is a mapping $\mu : X \to [0, 1]$ that assigns to each element $x \in X$ a membership degree $\mu(x) \in [0, 1]$.

To represent a Fuzzy Set as a SuperHyperFuzzy Set $\tilde{\mu} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0, 1])$, we define:

$$\tilde{\mu}(A) = \{\mu(a) \mid a \in A\}$$

for all $A \in \tilde{\mathcal{P}}(X)$.

This means:

- For singleton subsets $\{x\}$, we have $\tilde{\mu}(\{x\}) = \{\mu(x)\}$.
- For larger subsets $A \subseteq X$, $\tilde{\mu}(A)$ collects the membership degrees of all elements in A.

Thus, the Fuzzy Set is embedded into the SuperHyperFuzzy Set by considering the membership degrees over subsets of X.

A HyperFuzzy Set is a mapping $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0, 1])$ that assigns to each element $x \in X$ a non-empty subset $\tilde{\mu}(x) \subseteq [0, 1]$.

We construct a SuperHyperFuzzy Set $\tilde{\mu}' : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0, 1])$ by defining:

$$\tilde{\mu}'(A) = \bigcup_{a \in A} \tilde{\mu}(a)$$

for all $A \in \tilde{\mathcal{P}}(X)$.

This implies:

- For singleton subsets $\{x\}$, $\tilde{\mu}'(\{x\}) = \tilde{\mu}(x)$.
- For subsets $A \subseteq X$, $\tilde{\mu}'(A)$ is the union of the membership degree sets of all elements in A.

Therefore, the HyperFuzzy Set is a special case of the SuperHyperFuzzy Set when restricted to singleton subsets of X.

Since both Fuzzy Sets and HyperFuzzy Sets can be represented within the framework of SuperHyperFuzzy Sets, it follows that the SuperHyperFuzzy Set generalizes these concepts. □

As discussed above, if we explicitly define the n-SuperHyperFuzzy Set, it would be as follows.

Definition 3.209 (n-SuperHyperFuzzy Set)**.** Let X be a non-empty set, and $n \geq 0$ be an integer. An *n-SuperHyperFuzzy Set* is a mapping:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}([0, 1]),$$

where:

- $\tilde{\mathcal{P}}_n^*(X)$ denotes the family of all non-empty elements of the n-th PowerSet $\mathcal{P}_n^*(X)$, defined recursively as:

$$\mathcal{P}_0^*(X) = X, \quad \mathcal{P}_1^*(X) = \mathcal{P}(X), \quad \mathcal{P}_n^*(X) = \mathcal{P}(\mathcal{P}_{n-1}^*(X)), \text{ for } n \geq 2,$$

  with $\tilde{\mathcal{P}}_n^*(X) = \mathcal{P}_n^*(X) \setminus \{\emptyset\}$.

- $\tilde{\mathcal{P}}([0, 1])$ denotes the family of all non-empty subsets of the interval $[0, 1]$.

Structure:

1. Each element $A \in \tilde{\mathcal{P}}_n^*(X)$ is a non-empty subset within the n-th PowerSet hierarchy of X.
2. The mapping $\tilde{\mu}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n^*(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0, 1]$, representing the degrees of membership associated with the subset A.

Properties:

- If $n = 0$, $\tilde{\mathcal{P}}_0^*(X) = X$, and the structure reduces to a standard fuzzy set:

$$\tilde{\mu}_0 : X \to [0, 1].$$

- For $n = 1$, $\tilde{\mathcal{P}}_1^*(X) = \tilde{\mathcal{P}}(X)$, and the structure represents a SuperHyperFuzzy Set:

$$\tilde{\mu}_1 : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0, 1]).$$

- For $n \geq 2$, the structure recursively extends to higher-order fuzzy relationships:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}([0, 1]).$$

The n-SuperHyperFuzzy Set generalizes the concept of fuzzy sets to hierarchical and recursive levels of membership, allowing for higher-order relationships and fuzzy degrees associated with subsets of subsets, and so on, up to the n-th PowerSet hierarchy of X.

Definition 3.210 (SuperHyperVague Set)**.** Let X be a non-empty set. A mapping $\tilde{A} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0, 1]^2)$ is called a *SuperHyperVague Set* over X, where:

- $\tilde{\mathcal{P}}(X)$ denotes the family of all non-empty subsets of X.
- $\tilde{\mathcal{P}}([0, 1]^2)$ denotes the family of all non-empty subsets of the unit square $[0, 1]^2$.

For each $A \in \tilde{\mathcal{P}}(X)$, $\tilde{A}(A) \subseteq [0, 1]^2$, and for each pair $(t, f) \in \tilde{A}(A)$, it satisfies:

$$0 \leq t + f \leq 1,$$

where t represents the degree of truth membership and f represents the degree of falsity membership for the subset A.

Definition 3.211 (SuperHyperNeutrosophic Set)**.** Let X be a non-empty set. A mapping $\tilde{A} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0, 1]^3)$ is called a *SuperHyperNeutrosophic Set* over X, where:

- $\tilde{\mathcal{P}}(X)$ denotes the family of all non-empty subsets of X.
- $\tilde{\mathcal{P}}([0,1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0,1]^3$.

For each $A \in \tilde{\mathcal{P}}(X)$, $\tilde{A}(A) \subseteq [0,1]^3$, and for each triplet $(T, I, F) \in \tilde{A}(A)$, it satisfies:

$$0 \le T + I + F \le 3,$$

where T represents the degree of truth membership, I represents the degree of indeterminacy, and F represents the degree of falsity membership for the subset A.

Definition 3.212 (SuperHyperPlithogenic Set)**.** Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes, each with a set of possible values P_{v_i}. A *SuperHyperPlithogenic Set SHPS* over X is defined as:

$$SHPS = (P, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF),$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, P_{v_i} is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in P_{v_i}$ a non-empty subset of $[0,1]^s$.
- $pCF : \left(\bigcup_{i=1}^n P_{v_i}\right) \times \left(\bigcup_{i=1}^n P_{v_i}\right) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.
- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

Theorem 3.213. *Every SuperHyperVague Set can be transformed into a HyperVague Set and a SuperHyper-Fuzzy Set.*

Proof. Let $\tilde{A} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1]^2)$ be a SuperHyperVague Set over X.

Consider the restriction of $\tilde{A}$ to singleton subsets of X:

$$\tilde{A}|_X : X \to \tilde{\mathcal{P}}([0,1]^2), \quad \tilde{A}|_X(x) = \tilde{A}(\{x\}).$$

This mapping assigns to each element $x \in X$ a non-empty subset $\tilde{A}|_X(x) \subseteq [0,1]^2$, satisfying $0 \le t + f \le 1$ for each $(t, f) \in \tilde{A}|_X(x)$. Therefore, $\tilde{A}|_X$ is a HyperVague Set over X.

Define a mapping $\tilde{\mu} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1])$ by:

$$\tilde{\mu}(A) = \{t \mid (t, f) \in \tilde{A}(A),\ f \in [0,1]\}.$$

For each $A \in \tilde{\mathcal{P}}(X)$, $\tilde{\mu}(A) \subseteq [0,1]$ represents the degrees of truth membership extracted from $\tilde{A}(A)$. Thus, $\tilde{\mu}$ is a SuperHyperFuzzy Set over X. □

Theorem 3.214. *Every SuperHyperNeutrosophic Set can be transformed into a SuperHyperVague Set and a HyperNeutrosophic Set.*

Proof. Let $\tilde{A} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1]^3)$ be a SuperHyperNeutrosophic Set over X.

Define a mapping $\tilde{B} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1]^2)$ by projecting the neutrosophic membership degrees onto the truth and falsity components:

$$\tilde{B}(A) = \{(T, F) \mid (T, I, F) \in \tilde{A}(A),\ I \in [0,1]\}.$$

For each $A \in \tilde{\mathcal{P}}(X)$, $\tilde{B}(A) \subseteq [0,1]^2$, and $0 \le T + F \le 2$ (since $0 \le T + I + F \le 3$).

Adjust the normalization by defining:

$$\tilde{B}'(A) = \left\{ \left(\frac{T}{T+F}, \frac{F}{T+F} \right) \middle| (T, F) \in \tilde{B}(A),\ T + F > 0 \right\}.$$

Now, for each $(t, f) \in \tilde{B}'(A)$, $0 \le t + f = 1$.

Thus, $\tilde{B}'$ is a SuperHyperVague Set over X.

Consider the restriction of $\tilde{A}$ to singleton subsets of X:

$$\tilde{A}|_X : X \to \tilde{\mathcal{P}}([0,1]^3), \quad \tilde{A}|_X(x) = \tilde{A}(\{x\}).$$

This mapping assigns to each $x \in X$ a non-empty subset $\tilde{A}|_X(x) \subseteq [0,1]^3$, satisfying $0 \le T + I + F \le 3$.

Therefore, $\tilde{A}|_X$ is a HyperNeutrosophic Set over X. □

Theorem 3.215. *Every SuperHyperPlithogenic Set can be transformed into a HyperPlithogenic Set and the following SuperHyper Sets:*

- *SuperHyperFuzzy Set with* $s = 1$ *and* $t = 1$.
- *SuperHyperVague Set with* $s = 2$ *and* $t = 1$.
- *SuperHyperNeutrosophic Set with* $s = 3$ *and* $t = 1$.

Proof. Let $SHPS = (P, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$ be a SuperHyperPlithogenic Set over X, with s and t representing the dimensions of the membership degrees and contradiction degrees, respectively.

Transformation to HyperPlithogenic Set:

Consider the restriction of $SHPS$ to singleton subsets of P:

$$HPS = (P, V, \{P_{v_i}\}_{i=1}^n, \{pdf_i\}_{i=1}^n, pCF),$$

where $pdf_i : P \times P_{v_i} \to [0,1]^s$ is defined by:

$$pdf_i(x, a_i) = \tilde{pdf}_i(\{x\}, a_i).$$

Since $\tilde{pdf}_i$ assigns a subset of $[0,1]^s$ to each element, we can select representative values or aggregate them to obtain pdf_i.

Therefore, HPS is a HyperPlithogenic Set over X.

Transformation to SuperHyperFuzzy Set ($s = 1, t = 1$*)*:

If $s = 1$ and $t = 1$, the membership degrees are single-dimensional, and the contradiction degrees are also single-dimensional.

Define $\tilde{\mu} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1])$ by:

$$\tilde{\mu}(A) = \bigcup_{x \in A} \bigcup_{i=1}^{n} \bigcup_{a_i \in P_{v_i}} \tilde{pdf}_i(x, a_i).$$

This mapping assigns to each subset $A \subseteq X$ a non-empty subset of $[0,1]$, forming a SuperHyperFuzzy Set.

Transformation to SuperHyperVague Set ($s = 2, t = 1$*)*:

If $s = 2$, the membership degrees are two-dimensional, corresponding to the truth and falsity components.

Define $\tilde{A} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1]^2)$ by:

$$\tilde{A}(A) = \bigcup_{x \in A} \bigcup_{i=1}^{n} \bigcup_{a_i \in P_{v_i}} \tilde{pdf}_i(x, a_i),$$

where $\tilde{pdf}_i(x, a_i) \subseteq [0,1]^2$.

This forms a SuperHyperVague Set over X.

Transformation to SuperHyperNeutrosophic Set ($s = 3, t = 1$):

If $s = 3$, the membership degrees have three components, representing truth, indeterminacy, and falsity.

Define $\tilde{A} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1]^3)$ similarly:

$$\tilde{A}(A) = \bigcup_{x \in A} \bigcup_{i=1}^{n} \bigcup_{a_i \in P_{v_i}} \tilde{pdf}_i(x, a_i),$$

with $\tilde{pdf}_i(x, a_i) \subseteq [0,1]^3$.

This mapping constitutes a SuperHyperNeutrosophic Set over X. □

As discussed above, if we explicitly define the n-SuperHyperVague Set n-SuperHyperNeutrosophic Set, and SuperHyperPlithogenic set, it would be as follows.

Definition 3.216 (n-SuperHyperVague Set)**.** Let X be a non-empty set, and $n \geq 0$ be an integer. An *n-SuperHyperVague Set* is a mapping:

$$\tilde{V}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}([0,1]^2),$$

where:

- $\tilde{\mathcal{P}}_n^*(X)$ denotes the family of all non-empty elements of the n-th PowerSet $\mathcal{P}_n^*(X)$, defined recursively as:

  $$\mathcal{P}_0^*(X) = X, \quad \mathcal{P}_1^*(X) = \mathcal{P}(X), \quad \mathcal{P}_n^*(X) = \mathcal{P}(\mathcal{P}_{n-1}^*(X)), \text{ for } n \geq 2,$$

  with $\tilde{\mathcal{P}}_n^*(X) = \mathcal{P}_n^*(X) \setminus \{\emptyset\}$.
- $\tilde{\mathcal{P}}([0,1]^2)$ denotes the family of all non-empty subsets of the unit square $[0,1]^2$.

Structure:

1. Each element $A \in \tilde{\mathcal{P}}_n^*(X)$ is a non-empty subset within the n-th PowerSet hierarchy of X.
2. The mapping $\tilde{V}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n^*(X)$ a non-empty subset $\tilde{V}_n(A) \subseteq [0,1]^2$, representing pairs (t, f), where:

   $$0 \leq t + f \leq 1,$$

   t represents the degree of truth membership, and f represents the degree of falsity membership for the subset A.

Properties:

- If $n = 0$, $\tilde{\mathcal{P}}_0^*(X) = X$, and the structure reduces to a standard vague set:

$$\tilde{V}_0 : X \to [0,1]^2.$$

- For $n = 1$, $\tilde{\mathcal{P}}_1^*(X) = \tilde{\mathcal{P}}(X)$, and the structure represents a SuperHyperVague Set:

$$\tilde{V}_1 : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1]^2).$$

- For $n \geq 2$, the structure recursively extends to higher-order vague relationships:

$$\tilde{V}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}([0,1]^2).$$

The n-SuperHyperVague Set generalizes the concept of vague sets to hierarchical and recursive levels, allowing for higher-order relationships and vague degrees associated with subsets of subsets, and so on, up to the n-th PowerSet hierarchy of X. The use of pairs (t, f) ensures the balance between truth and falsity membership for each subset at every level.

Definition 3.217 (n-SuperHyperNeutrosophic Set)**.** Let X be a non-empty set, and $n \geq 0$ be an integer. An *n-SuperHyperNeutrosophic Set* is a mapping:

$$\tilde{N}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}([0,1]^3),$$

where:

- $\tilde{\mathcal{P}}_n^*(X)$ denotes the family of all non-empty elements of the n-th PowerSet $\mathcal{P}_n^*(X)$, defined recursively as:

$$\mathcal{P}_0^*(X) = X, \quad \mathcal{P}_1^*(X) = \mathcal{P}(X), \quad \mathcal{P}_n^*(X) = \mathcal{P}\left(\mathcal{P}_{n-1}^*(X)\right), \quad \text{for } n \geq 2,$$

  with $\tilde{\mathcal{P}}_n^*(X) = \mathcal{P}_n^*(X) \setminus \{\emptyset\}$.

- $\tilde{\mathcal{P}}([0,1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0,1]^3$.

Structure:

1. Each element $A \in \tilde{\mathcal{P}}_n^*(X)$ is a non-empty subset within the n-th PowerSet hierarchy of X.
2. The mapping $\tilde{N}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n^*(X)$ a non-empty subset $\tilde{N}_n(A) \subseteq [0,1]^3$, representing triplets (T, I, F), where T, I, and F satisfy:

$$T, I, F \in [0,1], \quad 0 \leq T + I + F \leq 3.$$

   T represents the degree of truth membership, I represents the degree of indeterminacy, and F represents the degree of falsity membership for the subset A.

Properties:

- If $n = 0$, $\tilde{\mathcal{P}}_0^*(X) = X$, and the structure reduces to a standard neutrosophic set:

$$\tilde{N}_0 : X \to [0,1]^3.$$

- For $n = 1$, $\tilde{\mathcal{P}}_1^*(X) = \tilde{\mathcal{P}}(X)$, and the structure represents a SuperHyperNeutrosophic Set:

$$\tilde{N}_1 : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1]^3).$$

- For $n \geq 2$, the structure recursively extends to higher-order neutrosophic relationships:

$$\tilde{N}_n : \tilde{\mathcal{P}}_n^*(X) \to \tilde{\mathcal{P}}([0,1]^3).$$

The n-SuperHyperNeutrosophic Set generalizes the concept of neutrosophic sets to hierarchical and recursive levels, allowing for higher-order relationships and degrees associated with subsets of subsets, and so on, up to the n-th PowerSet hierarchy of X.

Definition 3.218 (n-SuperHyperPlithogenic Set)**.** Let X be a non-empty set, and $n \geq 0$ be an integer. Let $V = \{v_1, v_2, \ldots, v_m\}$ be a set of attributes, each with a set of possible values P_{v_i}.

An *n-SuperHyperPlithogenic Set* $SHPS_n$ over X is defined as:

$$SHPS_n = (P, V, \{P_{v_i}\}_{i=1}^m, \{\tilde{pdf}_{i,n}\}_{i=1}^m, pCF_n),$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, P_{v_i} is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_{i,n} : \tilde{\mathcal{P}}_n^*(P) \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *n-th order Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $A \in \tilde{\mathcal{P}}_n^*(P)$ and attribute value $a_i \in P_{v_i}$ a non-empty subset of $[0,1]^s$.
- $pCF_n : \left(\bigcup_{i=1}^m P_{v_i}\right) \times \left(\bigcup_{i=1}^m P_{v_i}\right) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.
- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

The n-SuperHyperPlithogenic Set generalizes plithogenic sets to higher-order structures, incorporating multiple attributes and their possible values, along with higher-order degrees of membership and contradiction over the n-th PowerSet hierarchy of X.

Theorem 3.219. *An n-SuperHyperPlithogenic Set generalizes the n-SuperHyperFuzzy Set, n-SuperHyperVague Set, and n-SuperHyperNeutrosophic Set.*

Proof. Let $SHPS_n = (P, V, \{P_{v_i}\}_{i=1}^m, \{\tilde{pdf}_{i,n}\}_{i=1}^m, pCF_n)$ be an n-SuperHyperPlithogenic Set over X, with s and t representing the dimensions of the membership degrees and contradiction degrees, respectively.

We will show that:

1. When $s = 1$ and $t = 1$, $SHPS_n$ reduces to an n-SuperHyperFuzzy Set.
2. When $s = 2$ and $t = 1$, $SHPS_n$ reduces to an n-SuperHyperVague Set.
3. When $s = 3$ and $t = 1$, $SHPS_n$ reduces to an n-SuperHyperNeutrosophic Set.

Case 1: $s = 1$, $t = 1$ (Reduction to n-SuperHyperFuzzy Set) In this case, the membership degrees are single-dimensional, and the contradiction degrees are also single-dimensional.

Define a mapping $\tilde{\mu}_n : \tilde{\mathcal{P}}_n^*(P) \to \tilde{\mathcal{P}}([0,1])$ by:

$$\tilde{\mu}_n(A) = \bigcup_{i=1}^m \bigcup_{a_i \in P_{v_i}} \tilde{pdf}_{i,n}(A, a_i),$$

for each $A \in \tilde{\mathcal{P}}_n^*(P)$.

Since $\tilde{pdf}_{i,n}(A, a_i) \subseteq [0,1]$, their union is a subset of $[0,1]$.

Thus, $\tilde{\mu}_n$ is an n-SuperHyperFuzzy Set over P.

Case 2: $s = 2$**,** $t = 1$ **(Reduction to** n**-SuperHyperVague Set)** Here, the membership degrees are two-dimensional, corresponding to the truth and falsity components.

Define a mapping $\tilde{V}_n : \tilde{\mathcal{P}}_n^*(P) \to \tilde{\mathcal{P}}([0,1]^2)$ by:

$$\tilde{V}_n(A) = \bigcup_{i=1}^{m} \bigcup_{a_i \in P_{v_i}} \tilde{pdf}_{i,n}(A, a_i),$$

for each $A \in \tilde{\mathcal{P}}_n^*(P)$.

Since $\tilde{pdf}_{i,n}(A, a_i) \subseteq [0,1]^2$, $\tilde{V}_n$ is an n-SuperHyperVague Set over P.

Case 3: $s = 3, t = 1$ **(Reduction to** n**-SuperHyperNeutrosophic Set)** In this case, the membership degrees are three-dimensional, representing truth, indeterminacy, and falsity.

Define a mapping $\tilde{N}_n : \tilde{\mathcal{P}}_n^*(P) \to \tilde{\mathcal{P}}([0,1]^3)$ by:

$$\tilde{N}_n(A) = \bigcup_{i=1}^{m} \bigcup_{a_i \in P_{v_i}} \tilde{pdf}_{i,n}(A, a_i),$$

for each $A \in \tilde{\mathcal{P}}_n^*(P)$.

Since $\tilde{pdf}_{i,n}(A, a_i) \subseteq [0,1]^3$, $\tilde{N}_n$ is an n-SuperHyperNeutrosophic Set over P.

In each case, by choosing appropriate dimensions s and t for the membership and contradiction degrees, the n-SuperHyperPlithogenic Set $SHPS_n$ reduces to the corresponding n-SuperHyperFuzzy, n-SuperHyperVague, or n-SuperHyperNeutrosophic Set.

Therefore, the n-SuperHyperPlithogenic Set generalizes these sets. □

For reference, the relationships between the SuperhyperUncertain set are illustrated in Figure 10.

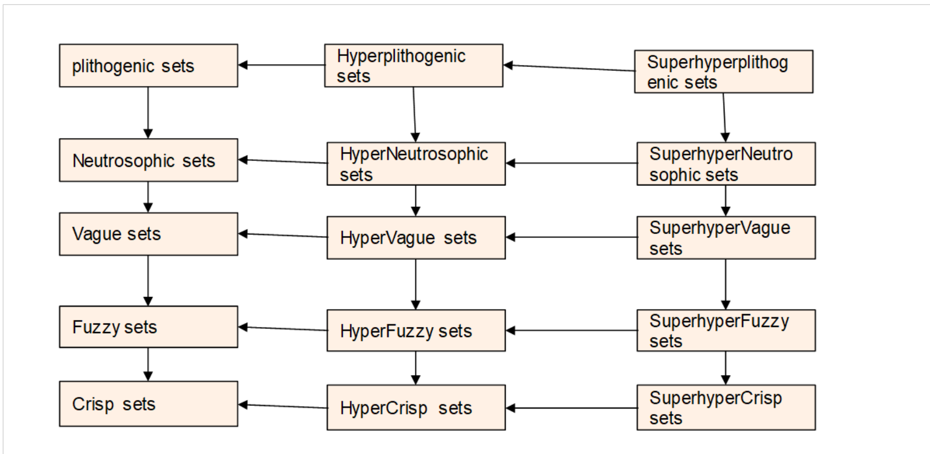


Figure 10: Some the SuperhyperUncertain sets Hierarchy. The set class at the origin of an arrow contains the set class at the destination of the arrow.

3.3.7 Hypersoft offgraph/overgraph/undergraph

The definitions of Neutrosophic Hypersoft offgraph, overgraph, and undergraph are provided below. These extend the concepts of Soft offgraph, overgraph, and undergraph. In the future, we aim to examine the mathematical validity of these definitions and explore whether more practical and applicable formulations can be devised.

Definition 3.220. Let $G = (V, E)$ be a simple graph where V is the set of vertices and E is the set of edges. Let $\{T_i\}_{i=1}^n$ be n distinct attributes, each with corresponding domains J_i (non-empty finite sets), and let $J = J_1 \times J_2 \times \cdots \times J_n$.

Define a mapping $F : J \to \mathcal{P}(V)$ such that for each attribute combination $a = (a_1, a_2, \ldots, a_n) \in J$, $F(a) \subseteq V$.

For each vertex $v \in F(a)$ and each attribute combination $a \in J$, assign neutrosophic membership degrees:

$$\begin{aligned} T(v, a) &\in [\Psi, \Omega], \quad \Psi < 0, \quad \Omega > 1, \\ I(v, a) &\in [\Psi, \Omega], \quad \Psi < 0, \quad \Omega > 1, \\ F(v, a) &\in [\Psi, \Omega], \quad \Psi < 0, \quad \Omega > 1, \\ T(v, a) + I(v, a) + F(v, a) &\le 3\Omega. \end{aligned}$$

Similarly, for each edge $e = (u, v) \in E$ with $u, v \in F(a)$, assign neutrosophic membership degrees $T(e, a)$, $I(e, a)$, $F(e, a)$ satisfying:

$$\begin{aligned} T(e, a) &\in [\Psi, \Omega], \\ I(e, a) &\in [\Psi, \Omega], \\ F(e, a) &\in [\Psi, \Omega], \\ T(e, a) + I(e, a) + F(e, a) &\le 3\Omega. \end{aligned}$$

The structure $G_{\text{HSO}} = (G, F, T, I, F)$ is called a *Neutrosophic Hypersoft OffGraph*.

Definition 3.221. Using the same setup as above, but with the following membership degree conditions for vertices and edges:

$$\begin{aligned} T(v, a), T(e, a) &\in [0, \Omega], \quad \Omega > 1, \\ I(v, a), I(e, a) &\in [0, \Omega], \quad \Omega > 1, \\ F(v, a), F(e, a) &\in [0, \Omega], \quad \Omega > 1, \\ T(v, a) + I(v, a) + F(v, a) &\le 3\Omega. \end{aligned}$$

The structure $G_{\text{HSOver}} = (G, F, T, I, F)$ is called a *Neutrosophic Hypersoft OverGraph*.

Definition 3.222. Again, using the same setup, but with the following membership degree conditions:

$$\begin{aligned} T(v, a), T(e, a) &\in [\Psi, 1], \quad \Psi < 0, \\ I(v, a), I(e, a) &\in [\Psi, 1], \quad \Psi < 0, \\ F(v, a), F(e, a) &\in [\Psi, 1], \quad \Psi < 0, \\ T(v, a) + I(v, a) + F(v, a) &\le 3. \end{aligned}$$

The structure $G_{\text{HSUnder}} = (G, F, T, I, F)$ is called a *Neutrosophic Hypersoft UnderGraph*.

Theorem 3.223. *Every Neutrosophic Hypersoft OffGraph can be transformed into a Neutrosophic Soft OffGraph.*

Proof. Let $G_{\text{HSO}} = (G, F, T, I, F)$ be a Neutrosophic Hypersoft OffGraph, where:

- $G = (V, E)$ is the classical graph with vertices V and edges E,

- $F : J \to \mathcal{P}(V)$ maps attribute combinations J to subsets of V,
- $T(v, a), I(v, a), F(v, a)$ represent neutrosophic membership degrees for vertices in context $a \in J$,
- $T(e, a), I(e, a), F(e, a)$ represent neutrosophic membership degrees for edges in context $a \in J$.

To transform G_{HSO} into a Neutrosophic Soft OffGraph $G_{\text{NSO}} = (G^*, J', K', A)$, follow these steps:

1. Define $A = J$, the set of attribute combinations.

2. For each $a \in A$:
$$J'(a) = F(a), \quad K'(a) = \{e \in E \mid \text{endpoints of } e \in F(a)\}.$$

3. For each vertex $v \in V$, aggregate membership degrees over contexts $a \in A$ where $v \in F(a)$:
$$T(v) = \sup_{a \in A_v} T(v, a), \quad I(v) = \sup_{a \in A_v} I(v, a), \quad F(v) = \inf_{a \in A_v} F(v, a),$$
where $A_v = \{a \in A \mid v \in F(a)\}$.

4. Similarly, for each edge $e \in E$, aggregate membership degrees over contexts $a \in A$ where $e \in K'(a)$:
$$T(e) = \sup_{a \in A_e} T(e, a), \quad I(e) = \sup_{a \in A_e} I(e, a), \quad F(e) = \inf_{a \in A_e} F(e, a),$$
where $A_e = \{a \in A \mid \text{endpoints of } e \in F(a)\}$.

By definition of the supremum and infimum operations, the aggregated degrees satisfy:
$$T(v) + I(v) + F(v) \leq 3\Omega, \quad T(e) + I(e) + F(e) \leq 3\Omega,$$
and $T(v), I(v), F(v), T(e), I(e), F(e) \in [\Psi, \Omega]$. Hence, G_{NSO} is a Neutrosophic Soft OffGraph. □

Theorem 3.224. *Every Neutrosophic Hypersoft OverGraph can be transformed into a Neutrosophic Soft OverGraph.*

Proof. Let $G_{\text{HSOver}} = (G, F, T, I, F)$ be a Neutrosophic Hypersoft OverGraph. Transform G_{HSOver} into $G_{\text{NSOver}} = (G^*, J', K', A)$ as follows:

1. Define $A = J$.

2. For each $a \in A$:
$$J'(a) = F(a), \quad K'(a) = \{e \in E \mid \text{endpoints of } e \in F(a)\}.$$

3. Aggregate membership degrees for vertices and edges as in the proof for OffGraphs.

Since the aggregated degrees satisfy:
$$T(v) + I(v) + F(v) \leq 3\Omega, \quad T(e) + I(e) + F(e) \leq 3\Omega,$$
with $T(v), T(e) \in [0, \Omega]$, $I(v), F(v), I(e), F(e) \in [0, 1]$, G_{NSOver} is a Neutrosophic Soft OverGraph. □

Theorem 3.225. *Every Neutrosophic Hypersoft UnderGraph can be transformed into a Neutrosophic Soft UnderGraph.*

Proof. Let $G_{\text{HSUnder}} = (G, F, T, I, F)$ be a Neutrosophic Hypersoft UnderGraph. Transform G_{HSUnder} into $G_{\text{NSUnder}} = (G^*, J', K', A)$ as follows:

1. Define $A = J$.

2. For each $a \in A$:
$$J'(a) = F(a), \quad K'(a) = \{e \in E \mid \text{endpoints of } e \in F(a)\}.$$

3. Aggregate membership degrees for vertices and edges using:

$$T(v) = \sup_{a \in A_v} T(v, a), \quad I(v) = \sup_{a \in A_v} I(v, a), \quad F(v) = \inf_{a \in A_v} F(v, a).$$

Since:
$$T(v) + I(v) + F(v) \leq 3, \quad T(v), I(v), F(v) \in [\Psi, 1],$$

G_{NSUnder} satisfies the conditions for a Neutrosophic Soft UnderGraph. □

Theorem 3.226. *Every Neutrosophic Hypersoft OffGraph, OverGraph, and UnderGraph is a generalization of the Hypersoft Graph.*

Proof. Let $G = (V, E)$ be a simple connected graph, where V is the set of vertices and E is the set of edges. Define the Hypersoft Graph $G_{HS} = (F, J)$ with a mapping $F : J \to \mathcal{P}(V)$ such that $F(a) \subseteq V$ induces a connected subgraph of G for each $a \in J$.

The Neutrosophic Hypersoft OffGraph $G_{\text{HSO}} = (G, F, T, I, F)$ extends the concept of the Hypersoft Graph by assigning neutrosophic membership degrees T, I, F to vertices and edges. Specifically:

$$T(v, a), I(v, a), F(v, a) \in [\Psi, \Omega], \quad \Psi < 0, \quad \Omega > 1,$$
$$T(v, a) + I(v, a) + F(v, a) \leq 3\Omega,$$

where $T(v, a), I(v, a), F(v, a)$ provide a richer representation of uncertainty, indeterminacy, and falsity compared to the binary inclusion in $F(a) \subseteq V$ of the Hypersoft Graph.

For any G_{HSO}, if $\Psi = 0$ and $\Omega = 1$, the neutrosophic degrees $T(v, a), I(v, a), F(v, a)$ reduce to binary values, recovering the structure of a Hypersoft Graph. Thus, G_{HSO} generalizes G_{HS}.

The Neutrosophic Hypersoft OverGraph $G_{\text{HSOver}} = (G, F, T, I, F)$ further extends the Hypersoft Graph by allowing $T(v, a), I(v, a), F(v, a)$ to take values in $[0, \Omega]$ with $\Omega > 1$. This allows degrees to exceed the classical upper bound of 1:

$$T(v, a), I(v, a), F(v, a) \in [0, \Omega], \quad T(v, a) + I(v, a) + F(v, a) \leq 3\Omega.$$

If $\Omega = 1$, the neutrosophic membership degrees revert to the standard interval $[0, 1]$, recovering the Hypersoft Graph structure. Therefore, G_{HSOver} is also a generalization of G_{HS}.

The Neutrosophic Hypersoft UnderGraph $G_{\text{HSUnder}} = (G, F, T, I, F)$ introduces the possibility of negative membership degrees ($\Psi < 0$) for T, I, F:

$$T(v, a), I(v, a), F(v, a) \in [\Psi, 1], \quad \Psi < 0,$$
$$T(v, a) + I(v, a) + F(v, a) \leq 3.$$

For $\Psi = 0$, the degrees revert to the standard range $[0, 1]$, matching the Hypersoft Graph structure. Hence, G_{HSUnder} is a generalization of G_{HS}.

All three structures—Neutrosophic Hypersoft OffGraph, OverGraph, and UnderGraph—extend the Hypersoft Graph by incorporating neutrosophic degrees T, I, F over extended ranges $[\Psi, \Omega]$ with additional flexibility in representing uncertainty, indeterminacy, and falsity. By appropriate selection of Ψ and Ω, each structure reduces to the Hypersoft Graph, confirming the generalization. □

3.3.8 Hyperbinary fuzzy set

The definition of a Hyperbinary fuzzy set is provided below. While the concept of Hyperbinary differs somewhat from other hyperconcepts [170,283,485], this definition aims to clarify its unique structure. For reference, we also define the concepts of Hyperbinary fuzzy and Hyperbinary neutrosophic sets using this foundational idea.

Definition 3.227. Let X be a non-empty set. A *hyperbinary set* H on X is a multiset characterized by a multiplicity function $\mu_H : X \to \{0,1,2\}$, where for each element $x \in X$, $\mu_H(x)$ denotes the number of times x appears in H, with $\mu_H(x) \leq 2$.

Formally, the hyperbinary set H is represented as:

$$H = \{(x, \mu_H(x)) \mid x \in X,\ \mu_H(x) \in \{0,1,2\}\}.$$

Theorem 3.228. *Every binary set is a special case of a hyperbinary set.*

Proof. A binary set B on X is defined by a membership function $\mu_B : X \to \{0,1\}$, where $\mu_B(x) = 1$ if x is a member of B, and $\mu_B(x) = 0$ otherwise.

Since the set $\{0,1\}$ is a subset of $\{0,1,2\}$, we can consider the multiplicity function of the binary set B as a special case of the hyperbinary set's multiplicity function μ_H, where $\mu_H(x) = \mu_B(x)$ for all $x \in X$.

Therefore, every binary set B is a hyperbinary set H with multiplicities restricted to $\{0,1\}$. □

Definition 3.229. Let X be a non-empty set. A *hyperbinary fuzzy set* H on X is defined by a membership function $\mu_H : X \to [0,2]$, where for each element $x \in X$, $\mu_H(x)$ represents the degree of membership of x in H, and $\mu_H(x) \in [0,2]$.

The hyperbinary fuzzy set H is formally expressed as:

$$H = \{(x, \mu_H(x)) \mid x \in X,\ \mu_H(x) \in [0,2]\}.$$

Theorem 3.230. *Every hyperbinary fuzzy set generalizes both the binary fuzzy set and the hyperbinary set.*

Proof. 1. *Binary Fuzzy Set as a Special Case:*

When $\mu_H(x) \in [0,1]$ for all $x \in X$, the hyperbinary fuzzy set H reduces to a binary fuzzy set B with membership function $\mu_B(x) = \mu_H(x)$.

2. *Hyperbinary Set as a Special Case:*

When $\mu_H(x) \in \{0,1,2\}$ for all $x \in X$, the hyperbinary fuzzy set H becomes a hyperbinary set M with multiplicity function $\mu_M(x) = \mu_H(x)$.

Therefore, the hyperbinary fuzzy set encompasses both binary fuzzy sets and hyperbinary sets as special cases. □

Definition 3.231 (Hyperbinary Neutrosophic Set (HBNS))**.** Let (X,Y) be a non-empty fixed space, where X and Y are distinct sets. A *Hyperbinary Neutrosophic Set* (HBNS) H in (X,Y) is represented as an ordered pair:

$$H = \left((H_{11}, H_{12}, H_{13}), (H_{21}, H_{22}, H_{23})\right),$$

where each $H_{ij} \subseteq X$ or Y is defined by a multiplicity function $\mu_{H_{ij}} : H_{ij} \to \{0,1,2\}$, which assigns each element a multiplicity of 0, 1, or 2. This multiplicity function allows each element to appear up to twice in each subset H_{ij}.

Thus, the hyperbinary neutrosophic set H can be formally written as:

$$H = \left\{\left(x, \mu_{H_{11}}(x), \mu_{H_{12}}(x), \mu_{H_{13}}(x)\right), \left(y, \mu_{H_{21}}(y), \mu_{H_{22}}(y), \mu_{H_{23}}(y)\right) \mid x \in X,\ y \in Y,\ \mu_{H_{ij}}(x), \mu_{H_{ij}}(y) \in \{0,1,2\}\right\}.$$

Each subset H_{ij} represents values associated with the degrees of truth, indeterminacy, and falsity, respectively, within the context of sets X and Y.

Theorem 3.232. *Any Hyperbinary Neutrosophic Set (HBNS) can be transformed into a Binary Neutrosophic Set (BNCS) by restricting the multiplicity function $\mu_{H_{ij}}$ to $\{0,1\}$ for each subset H_{ij}.*

Proof. Let $H = ((H_{11}, H_{12}, H_{13}), (H_{21}, H_{22}, H_{23}))$ be a Hyperbinary Neutrosophic Set with a multiplicity function $\mu_{H_{ij}} : H_{ij} \to \{0, 1, 2\}$.

By restricting $\mu_{H_{ij}}$ to take values in $\{0, 1\}$ only, we effectively limit each element to appear at most once in each subset H_{ij}. Therefore, each H_{ij} becomes a subset in which elements can either be present or absent, without multiplicity higher than 1.

This transformation reduces H to a structure that corresponds to the definition of a Binary Neutrosophic Set $C = ((C_{11}, C_{12}, C_{13}), (C_{21}, C_{22}, C_{23}))$, where each $C_{ij} \subseteq X$ or Y contains elements with no multiplicity (i.e., $\mu_{C_{ij}} : C_{ij} \to \{0, 1\}$).

Hence, the HBNS H reduces to a BNCS C under this constraint, completing the proof. □

Theorem 3.233. *A Hyperbinary Neutrosophic Set (HBNS) generalizes a Hyperbinary Fuzzy Set (HBFS).*

Proof. To prove that a Hyperbinary Neutrosophic Set (HBNS) generalizes a Hyperbinary Fuzzy Set (HBFS), consider the following.

For any HBNS H, let the indeterminacy and falsity membership degrees satisfy:

$$I_H(x) = 0, \quad F_H(x) = 0 \quad \text{for all } x \in X.$$

In this case, the HBNS reduces to:

$$H = \{(x, T_H(x), 0, 0) \mid x \in X,\ T_H(x) \in [0, 2]\}.$$

This can be equivalently written as:

$$F = \{(x, T_H(x)) \mid x \in X,\ T_H(x) \in [0, 2]\},$$

which is exactly the definition of an HBFS.

An HBNS introduces two additional membership degrees, $I_H(x)$ (indeterminacy) and $F_H(x)$ (falsity), which are not present in an HBFS. These components provide more detailed information about the membership state of each element x in H, allowing for richer representations. When $I_H(x) \neq 0$ or $F_H(x) \neq 0$, the HBNS cannot be reduced to an HBFS.

Therefore, the HBNS generalizes the HBFS by encompassing it as a special case where indeterminacy and falsity components are zero, while also extending it to include these additional components. □

Theorem 3.234. *A Hyperbinary Neutrosophic Set (HBNS) can be transformed into a Neutrosophic Set (NS) by constraining its multiplicity function.*

Proof. To transform an Hyperbinary Neutrosophic Set (HBNS) H into an Neutrosophic Set (NS) N, we normalize the membership degrees of H as follows:

$$T_N(x) = \frac{T_H(x)}{2}, \quad I_N(x) = \frac{I_H(x)}{2}, \quad F_N(x) = \frac{F_H(x)}{2}, \quad \forall x \in X.$$

Since $T_H(x), I_H(x), F_H(x) \in [0, 2]$, dividing by 2 ensures that:

$$T_N(x), I_N(x), F_N(x) \in [0, 1].$$

For any $x \in X$, the sum of the normalized values satisfies:

$$T_N(x) + I_N(x) + F_N(x) = \frac{T_H(x)}{2} + \frac{I_H(x)}{2} + \frac{F_H(x)}{2} = \frac{T_H(x) + I_H(x) + F_H(x)}{2}.$$

Since $T_H(x) + I_H(x) + F_H(x) \leq 6$ in an HBNS, it follows that:

$$T_N(x) + I_N(x) + F_N(x) \leq 3.$$

This satisfies the constraints of a Neutrosophic Set.

The mapping:

$$(T_H(x), I_H(x), F_H(x)) \mapsto (T_N(x), I_N(x), F_N(x))$$

is well-defined and preserves the structure of the set. Each element in H is uniquely transformed into an element in N.

Hence, the Hyperbinary Neutrosophic Set H can be transformed into a Neutrosophic Set N by the normalization procedure. This transformation is reversible if the scaling factor 2 is known. □

3.3.9 Ranked Hypersoft Set

The definition of the Ranked Hypersoft Set, which extends the concept of the Ranked Soft Set, is provided below. It is hoped that further research will continue to explore the mathematical structure and applications of these concepts.

Definition 3.235 (Ranked Hypersoft Set)**.** A *Ranked Hypersoft Set* (H, J, E) over a universe X is defined by three components: a finite set of attributes $E = \{T_1, T_2, \ldots, T_n\}$, disjoint sets $J_1, J_2, \ldots, J_n$ associated with each attribute, and a mapping H that assigns to each tuple of attribute values a ranked partition over X.

Formally, let $J = J_1 \times J_2 \times \cdots \times J_n$ be the set of all possible combinations of attribute values. Then (H, J, E) is defined as:

$$H : J \to \mathcal{R}(X),$$

where each $j = (j_1, j_2, \ldots, j_n) \in J$ is associated with a ranked partition $H(j) = (V_0, V_1, \ldots, V_k)$ of X according to the parameters in E.

In this structure:

1. $V_0, V_1, \ldots, V_k \subseteq X$ are disjoint subsets of X, with V_0 representing elements that do not satisfy the attribute combination j, and $V_1, \ldots, V_k$ representing increasingly higher levels of satisfaction or relevance.
2. The union of all subsets covers X: $V_0 \cup V_1 \cup \cdots \cup V_k = X$.

The Ranked Hypersoft Set thus captures the hierarchical levels of satisfaction or relevance across multiple attributes, allowing for a structured, multi-parameter assessment of the elements in X.

Theorem 3.236. *A Ranked Hypersoft Set generalizes both Ranked Soft Sets and Hypersoft Sets.*

Proof. To show that the Ranked Hypersoft Set generalizes Ranked Soft Sets, consider the case where $n = 1$, so that there is only one attribute in E. In this case, $J = J_1$ corresponds to the set of possible values for this attribute, and the mapping $H : J \to \mathcal{R}(X)$ associates each attribute value with a ranked partition of X. This is consistent with the definition of a Ranked Soft Set, where each parameter (single attribute) maps to a ranked partition over X.

To show that the Ranked Hypersoft Set generalizes Hypersoft Sets, observe that if the ranked partition $(V_0, V_1, \ldots, V_k)$ is restricted to $k = 1$, meaning no ranking is applied, the ranked partition reduces to a simple partition of X as in the Hypersoft Set model. Thus, by allowing ranked partitions, the Ranked Hypersoft Set extends the Hypersoft Set concept to incorporate ranked levels of satisfaction for each attribute combination.

Therefore, the Ranked Hypersoft Set encompasses both the ranked hierarchy of satisfaction in Ranked Soft Sets and the multi-attribute combinations in Hypersoft Sets, achieving a generalization of both concepts. □

3.3.10 TreeHyperSoft Set

We define the TreeHyperSoft Set as follows.

Definition 3.237. Let U be a non-empty universe of discourse. Let $\text{Tree}(A)$ be a hierarchical attribute tree, where each node represents an attribute a_i or an attribute value. Each attribute a_i at any level in the tree has an associated set of attribute values J_i, and may have sub-attributes forming the tree structure.

We define J as the set of all possible combinations of attribute values along the paths from the root to the leaves in the tree, representing the different combinations of attribute values across various levels.

A *TreeHyperSoft Set* is a pair (F, J), where F is a mapping defined by:

$$F : J \to \mathcal{P}(U),$$

where $\mathcal{P}(U)$ denotes the power set of U, and J is the set of all possible attribute value combinations along the paths in $\text{Tree}(A)$.

For each combination $x \in J$, $F(x) \subseteq U$ represents the subset of U corresponding to that particular combination of attribute values.

Theorem 3.238. *The TreeHyperSoft Set generalizes both the Hypersoft Set and the TreeSoft Set. Specifically:*

1. *When the attribute tree $\text{Tree}(A)$ has only one level (i.e., no sub-attributes), the TreeHyperSoft Set reduces to the Hypersoft Set.*
2. *When each attribute in $\text{Tree}(A)$ has only one attribute value (i.e., attributes are not associated with multiple values), the TreeHyperSoft Set reduces to the TreeSoft Set.*

Proof. We prove each part separately.

Part 1: Reduction to Hypersoft Set

Assume that the attribute tree $\text{Tree}(A)$ has only one level. That is, $\text{Tree}(A) = \{a_1, a_2, \ldots, a_n\}$, where each attribute a_i has associated attribute values J_i.

In this case, the set J of all possible combinations of attribute values is simply the Cartesian product:

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Therefore, the mapping F becomes:

$$F : J_1 \times J_2 \times \cdots \times J_n \to \mathcal{P}(U),$$

which is precisely the definition of a *Hypersoft Set* as defined in [861].

Thus, when the attribute tree has only one level, the TreeHyperSoft Set reduces to the Hypersoft Set.

Part 2: Reduction to TreeSoft Set

Assume that each attribute in $\text{Tree}(A)$ has only one attribute value, i.e., for all attributes a_i, $J_i = \{v_i\}$, where v_i is a single value.

In this case, the set J consists of all possible combinations of these single attribute values along the tree structure.

Since each attribute has only one value, the combinations are determined solely by the structure of the tree.

The mapping F becomes:

$$F : P(\text{Tree}(A)) \to \mathcal{P}(U),$$

where $P(\text{Tree}(A))$ is the power set of the nodes in the tree (excluding attribute values since they are singular and fixed).

This matches the definition of a *TreeSoft Set* as defined in [880], where the mapping is from subsets of the tree's nodes to subsets of U.

Therefore, when each attribute has only one attribute value, the TreeHyperSoft Set reduces to the TreeSoft Set. □

3.3.11 Hyperweighted Graphs and Superhyperweighted Graphs

We define a Hyperweighted Graph as follows.To define a *Hyperweighted Graph*, we first introduce the concept of a *Hyperweighted Set*.A *Hyperweighted Set* is a set S where each element $s \in S$ is associated with a set of weights, rather than a single weight. Formally, we define:

Definition 3.239. Let S be a non-empty set. A *Hyperweighted Set* is a pair (S, W), where $W : S \to \mathcal{P}(\mathbb{R})$ is a function that assigns to each element $s \in S$ a non-empty set of real-valued weights $W(s) \subseteq \mathbb{R}$.

In this context, each element $s \in S$ carries multiple weights, capturing various attributes or measurements associated with that element.

Definition 3.240. A *Hyperweighted Graph* $G = (V, E, W)$ consists of:

- A set of *vertices* V.
- A set of *edges* $E \subseteq V \times V$.
- A *hyperweight function* $W : E \to \mathcal{P}(\mathbb{R})$, which assigns to each edge $e \in E$ a non-empty set of real-valued weights $W(e) \subseteq \mathbb{R}$.

Example 3.241. Consider a hyperweighted graph $G = (V, E, W)$ defined as follows:

- *Vertices*:
$$V = \{v_1, v_2, v_3\}.$$
- *Edges*:
$$E = \{e_1, e_2\},$$
where:
$$e_1 = (v_1, v_2),$$
$$e_2 = (v_2, v_3).$$
- *Hyperweight Function W*:
$$W(e_1) = \{5, 7\},$$
$$W(e_2) = \{3, 4, 6\}.$$

In this example:

- Edge e_1 connects vertices v_1 and v_2 and has multiple weights $W(e_1) = \{5, 7\}$.
- Edge e_2 connects vertices v_2 and v_3 and has multiple weights $W(e_2) = \{3, 4, 6\}$.

These multiple weights could represent different metrics such as:

- For e_1:
  - 5: Distance in kilometers.
  - 7: Time in minutes.
- For e_2:
  - 3: Cost in dollars.
  - 4: Risk level on a scale from 1 to 5.

– 6: Traffic congestion level.

Theorem 3.242. *A Weighted Graph is a special case of a Hyperweighted Graph where each edge has exactly one weight.*

Proof. Let $G' = (V', E', w')$ be a standard Weighted Graph, where:

- V' is the set of vertices.
- $E' \subseteq V' \times V'$ is the set of edges.
- $w' : E' \to \mathbb{R}$ is the weight function assigning a single real-valued weight to each edge.

We construct a Hyperweighted Graph $G = (V, E, W)$ as follows:

- $V = V'$.
- $E = E'$.
- $W : E \to \mathcal{P}(\mathbb{R})$ is defined by $W(e) = \{w'(e)\}$ for all $e \in E$.

In this construction, each edge $e \in E$ is associated with a singleton set $W(e) = \{w'(e)\}$, containing exactly one weight.

Therefore, the standard Weighted Graph G' can be viewed as a Hyperweighted Graph G where the hyperweight function W assigns a singleton set of weights to each edge. This shows that the Hyperweighted Graph is a generalization of the Weighted Graph, accommodating multiple weights per edge. □

Definition 3.243 (Superhyperweighted Graph)**.** A *Superhyperweighted Graph* $G = (V, E, \mathcal{W})$ consists of:

- A set of *vertices* V.
- A set of *edges* $E \subseteq V \times V$.
- A *superhyperweight function* $\mathcal{W} : E \to \mathcal{P}(\mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}) \setminus \{\emptyset\}$, which assigns to each edge $e \in E$ a non-empty set of hyperweights, where each hyperweight is a non-empty set of real numbers.

Theorem 3.244. *Every Hyperweighted Graph is a special case of a Superhyperweighted Graph. Specifically, if all superhyperweights in a Superhyperweighted Graph are singleton sets containing singleton hyperweights, then it reduces to a Hyperweighted Graph.*

Proof. Let $G_H = (V, E, W)$ be a Hyperweighted Graph, where $W : E \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$.

We can construct a Superhyperweighted Graph $G_S = (V, E, \mathcal{W})$ corresponding to G_H by defining:

$$\mathcal{W}(e) = \{W(e)\}, \quad \forall e \in E.$$

Here, each superhyperweight $\mathcal{W}(e)$ is a singleton set containing the hyperweight $W(e)$, which is itself a non-empty set of real numbers.

Conversely, if in a Superhyperweighted Graph $G_S = (V, E, \mathcal{W})$, all superhyperweights $\mathcal{W}(e)$ are singleton sets containing singleton hyperweights, that is:

$$\mathcal{W}(e) = \{W(e)\}, \quad \text{with } W(e) \in \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\},$$

then we can define a hyperweight function:

$$W : E \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}, \quad W(e) = \text{the unique element of } \mathcal{W}(e),$$

thereby obtaining a Hyperweighted Graph $G_H = (V, E, W)$.

Therefore, Superhyperweighted Graphs generalize Hyperweighted Graphs by allowing multiple hyperweights (which are sets of weights) for each edge, whereas Hyperweighted Graphs are the special case where superhyperweights are singleton sets containing a single hyperweight. □

Theorem 3.245. *The set defined by the superhyperweight function in a Superhyperweighted Graph generalizes the SuperHyperCrisp Set.*

Proof. To prove or disprove the generalization, we analyze the definitions of the SuperHyperCrisp Set and the superhyperweight function in a Superhyperweighted Graph.

A SuperHyperCrisp Set $\tilde{C}$ over a universe X is defined as:

$$\tilde{C} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}(\{0, 1\}).$$

In a Superhyperweighted Graph $G = (V, E, \mathcal{W})$, the superhyperweight function $\mathcal{W}$ assigns to each edge $e \in E$ a non-empty subset of $\mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$. Formally:

$$\mathcal{W} : E \to \mathcal{P}\left(\mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}\right) \setminus \{\emptyset\}.$$

This mapping associates higher-order hierarchical weights (sets of subsets of real numbers) with edges.

The SuperHyperCrisp Set maps subsets of X to subsets of $\{0, 1\}$, which is equivalent to assigning binary membership values. The superhyperweight function $\mathcal{W}$ maps edges to higher-order structures, specifically sets of subsets of real numbers.

If we restrict $\mathcal{W}$ such that all weights are binary (i.e., each weight is a subset of $\{0, 1\}$), then $\mathcal{W}$ behaves like a SuperHyperCrisp Set, mapping elements to subsets of binary values. Conversely, the SuperHyperCrisp Set does not have the capacity to represent hierarchical weights involving subsets of real numbers as defined in $\mathcal{W}$. Thus, $\mathcal{W}$ is strictly more general.

The superhyperweight function in a Superhyperweighted Graph generalizes the SuperHyperCrisp Set by allowing the representation of hierarchical structures involving subsets of real numbers, which extends beyond binary membership. □

Corollary 3.246. *The set defined by the hyperweight function in a hyperweighted Graph generalizes the HyperCrisp Set.*

Proof. The proof can be established in a manner similar to that of the SuperHyperWeighted Graph, but with a simplified approach. □

3.3.12 Hyperlabeling graph and superhyperlabeling graph

A labeling graph is a graph where each vertex and edge is assigned a label from defined sets, often following specific rules [389, 390]. Several related concepts include Graceful Labeling [664], Edge-Graceful Labeling [250, 757], Harmonious Labeling [331, 603], and Lucky Labeling [235, 661]. The following describes an extended graph concept based on the labeling graph.

Definition 3.247 (Hyperlabeling Graph)**.** Let $G = (V, E)$ be a classical graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *hyperlabeling graph*, denoted as $G_H = (V, \sigma, \mu)$, consists of the following components:

1. *Vertex Hyperlabeling:*
$$\sigma : V \to \mathcal{P}(L_V) \setminus \{\emptyset\},$$
where L_V is the set of possible labels for vertices, and $\mathcal{P}(L_V) \setminus \{\emptyset\}$ denotes the set of all non-empty subsets of L_V. The function σ assigns a non-empty set of labels $\sigma(v) \subseteq L_V$ to each vertex $v \in V$.

2. *Edge Hyperlabeling:*
$$\mu : E \to \mathcal{P}(L_E) \setminus \{\emptyset\},$$
where L_E is the set of possible labels for edges, and $\mathcal{P}(L_E) \setminus \{\emptyset\}$ denotes the set of all non-empty subsets of L_E. The function μ assigns a non-empty set of labels $\mu(e) \subseteq L_E$ to each edge $e \in E$.

3. *Labeling Rules:* The hyperlabeling functions σ and μ may satisfy specific rules or constraints depending on the context of the hyperlabeling graph. These rules can include, but are not limited to:
   - *Consistency Rules:* Relationships between vertex hyperlabels and edge hyperlabels. For example, for an edge $e = (u, v)$, the edge hyperlabel $\mu(e)$ may depend on the vertex hyperlabels $\sigma(u)$ and $\sigma(v)$.
   - *Constraints on Hyperlabels:* Conditions on the hyperlabels, such as cardinality constraints, intersection properties, or other logical conditions relevant to the application.

Theorem 3.248. *Every labeling graph is a special case of a hyperlabeling graph. Specifically, if all hyperlabels in a hyperlabeling graph are singleton sets, then it reduces to a standard labeling graph.*

Proof. Let $G_L = (V, \sigma', \mu')$ be a labeling graph, where:

- $\sigma' : V \to L_V$ is the vertex labeling function.
- $\mu' : E \to L_E$ is the edge labeling function.

We can construct a hyperlabeling graph $G_H = (V, \sigma, \mu)$ corresponding to G_L by defining:

$$\sigma(v) = \{\sigma'(v)\}, \quad \forall v \in V,$$
$$\mu(e) = \{\mu'(e)\}, \quad \forall e \in E.$$

Here, each hyperlabel is a singleton set containing the label from the original labeling graph. Since all hyperlabels are singleton sets, G_H effectively replicates the labeling scheme of G_L.

Conversely, if in a hyperlabeling graph $G_H = (V, \sigma, \mu)$, all hyperlabels $\sigma(v)$ and $\mu(e)$ are singleton sets, then we can define functions:

$$\sigma' : V \to L_V, \quad \sigma'(v) = \text{the unique element of } \sigma(v),$$
$$\mu' : E \to L_E, \quad \mu'(e) = \text{the unique element of } \mu(e),$$

thereby obtaining a labeling graph $G_L = (V, \sigma', \mu')$.

Therefore, hyperlabeling graphs generalize labeling graphs by allowing multiple labels (hyperlabels) for vertices and edges, and labeling graphs are the special case where hyperlabels are singleton sets. □

Example 3.249. Consider a graph $G = (V, E)$ where:

- $V = \{v_1, v_2, v_3\}$.
- $E = \{e_1, e_2\}$, with $e_1 = (v_1, v_2)$ and $e_2 = (v_2, v_3)$.

Let the sets of possible labels be:

- $L_V = \{\text{red}, \text{blue}, \text{green}\}$.
- $L_E = \{\text{solid}, \text{dashed}\}$.

Define the vertex hyperlabeling function σ and edge hyperlabeling function μ as:

$$\sigma(v_1) = \{\text{red}, \text{blue}\},$$
$$\sigma(v_2) = \{\text{blue}\},$$
$$\sigma(v_3) = \{\text{green}, \text{blue}\},$$
$$\mu(e_1) = \{\text{solid}, \text{dashed}\},$$
$$\mu(e_2) = \{\text{dashed}\}.$$

This hyperlabeling graph assigns multiple labels to vertices and edges, offering more flexibility than a standard labeling graph.

We have defined the Superhyperlabeling Graph as a generalization of the hyperlabeling graph. In a hyperlabeling graph, each vertex and edge is assigned a non-empty set of labels (hyperlabels).

Definition 3.250 (Superhyperlabeling Graph)**.** Let $G = (V, E)$ be a classical graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Superhyperlabeling Graph*, denoted as $G_S = (V, \Sigma, M)$, consists of the following components:

1. *Vertex Superhyperlabeling:*
$$\Sigma : V \to \mathcal{P}\left(\mathcal{P}(L_V) \setminus \{\emptyset\}\right) \setminus \{\emptyset\},$$
where L_V is the set of possible labels for vertices, and $\mathcal{P}\left(\mathcal{P}(L_V) \setminus \{\emptyset\}\right) \setminus \{\emptyset\}$ denotes the set of all non-empty subsets of the power set of L_V excluding the empty set. The function Σ assigns to each vertex $v \in V$ a non-empty set of hyperlabels, where each hyperlabel is a non-empty subset of L_V.

2. *Edge Superhyperlabeling:*
$$M : E \to \mathcal{P}\left(\mathcal{P}(L_E) \setminus \{\emptyset\}\right) \setminus \{\emptyset\},$$
where L_E is the set of possible labels for edges, and $\mathcal{P}\left(\mathcal{P}(L_E) \setminus \{\emptyset\}\right) \setminus \{\emptyset\}$ denotes the set of all non-empty subsets of the power set of L_E excluding the empty set. The function M assigns to each edge $e \in E$ a non-empty set of hyperlabels.

3. *Labeling Rules:* The superhyperlabeling functions Σ and M may satisfy specific rules or constraints depending on the context of the Superhyperlabeling Graph. These rules can include, but are not limited to:
   - *Consistency Rules:* Relationships between vertex superhyperlabels and edge superhyperlabels. For example, for an edge $e = (u, v)$, the edge superhyperlabel $M(e)$ may depend on the vertex superhyperlabels $\Sigma(u)$ and $\Sigma(v)$.
   - *Constraints on Superhyperlabels:* Conditions on the superhyperlabels, such as cardinality constraints, intersection properties, or other logical conditions relevant to the application.

Theorem 3.251. *Every hyperlabeling graph is a special case of a Superhyperlabeling Graph. Specifically, if all superhyperlabels in a Superhyperlabeling Graph are singleton sets containing singleton hyperlabels, then it reduces to a hyperlabeling graph.*

Proof. Let $G_H = (V, \sigma, \mu)$ be a hyperlabeling graph, where:

- $\sigma : V \to \mathcal{P}(L_V) \setminus \{\emptyset\}$ is the vertex hyperlabeling function.
- $\mu : E \to \mathcal{P}(L_E) \setminus \{\emptyset\}$ is the edge hyperlabeling function.

We can construct a Superhyperlabeling Graph $G_S = (V, \Sigma, M)$ corresponding to G_H by defining:

$$\Sigma(v) = \{\sigma(v)\}, \quad \forall v \in V,$$
$$M(e) = \{\mu(e)\}, \quad \forall e \in E.$$

Here, each superhyperlabel $\Sigma(v)$ is a singleton set containing the hyperlabel $\sigma(v)$, which is itself a non-empty subset of L_V. Similarly for $M(e)$ and $\mu(e)$.

Conversely, if in a Superhyperlabeling Graph $G_S = (V, \Sigma, M)$, all superhyperlabels $\Sigma(v)$ and $M(e)$ are singleton sets containing singleton hyperlabels, that is:

$$\Sigma(v) = \{\sigma(v)\}, \quad \text{with } \sigma(v) \in \mathcal{P}(L_V) \setminus \{\emptyset\},$$
$$M(e) = \{\mu(e)\}, \quad \text{with } \mu(e) \in \mathcal{P}(L_E) \setminus \{\emptyset\},$$

then we can define hyperlabeling functions:

$$\sigma : V \to \mathcal{P}(L_V) \setminus \{\emptyset\}, \quad \sigma(v) = \text{the unique element of } \Sigma(v),$$
$$\mu : E \to \mathcal{P}(L_E) \setminus \{\emptyset\}, \quad \mu(e) = \text{the unique element of } M(e),$$

thereby obtaining a hyperlabeling graph $G_H = (V, \sigma, \mu)$.

Therefore, Superhyperlabeling Graphs generalize hyperlabeling graphs by allowing multiple hyperlabels (which are sets of labels) for vertices and edges, whereas hyperlabeling graphs are the special case where superhyperlabels are singleton sets containing a single hyperlabel. □

Definition 3.252 (Isomorphic)**.** Two mathematical structures A and B are said to be *isomorphic*, denoted as $A \cong B$, if there exists a bijective mapping $f : A \to B$ such that:

1. f is a one-to-one correspondence between the elements of A and B.
2. f preserves the structure of A in B, meaning that for all elements x, y in A, the relationships or operations between x and y in A are reflected in the corresponding elements $f(x)$ and $f(y)$ in B.

If such a mapping f exists, the structures A and B are considered structurally identical.

Theorem 3.253. *The structures of Vertex Superhyperlabeling and Edge Superhyperlabeling in a Superhyperlabeling Graph $G_S = (V, \Sigma, M)$ are isomorphic to the structure of a SuperHyperCrisp Set.*

Proof. To establish the isomorphism, we compare the formal definitions of Vertex Superhyperlabeling, Edge Superhyperlabeling, and SuperHyperCrisp Set.

A SuperHyperCrisp Set $\tilde{C}$ over a universe X is defined as:

$$\tilde{C} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}(\{0, 1\}).$$

The Vertex Superhyperlabeling function Σ in a Superhyperlabeling Graph is defined as:

$$\Sigma : V \to \mathcal{P}\left(\mathcal{P}(L_V) \setminus \{\emptyset\}\right) \setminus \{\emptyset\}.$$

Here:

- V is the set of vertices.
- L_V is the set of possible labels for vertices.
- $\mathcal{P}\left(\mathcal{P}(L_V) \setminus \{\emptyset\}\right) \setminus \{\emptyset\}$ is the family of all non-empty subsets of the power set of L_V, excluding the empty set.

The function Σ assigns each vertex $v \in V$ a non-empty subset of the power set of L_V, enabling hierarchical and recursive labeling structures.

Similarly, the Edge Superhyperlabeling function M is defined as:

$$M : E \to \mathcal{P}\left(\mathcal{P}(L_E) \setminus \{\emptyset\}\right) \setminus \{\emptyset\},$$

where L_E is the set of possible labels for edges. The function M assigns to each edge $e \in E$ a non-empty subset of the power set of L_E.

The Vertex Superhyperlabeling function Σ maps vertices $v \in V$ to non-empty subsets of $\mathcal{P}(L_V)$, analogous to how a SuperHyperCrisp Set maps subsets of X to binary memberships $\{0, 1\}$. Similarly, the Edge Superhyperlabeling function M maps edges $e \in E$ to hierarchical structures of labels, akin to the mappings in a SuperHyperCrisp Set. Both Σ and M satisfy the structural requirements of a SuperHyperCrisp Set by representing elements (vertices or edges) using higher-order mappings, where the output values (hyperlabels) generalize binary membership to hierarchical subsets.

The mapping structures in Vertex Superhyperlabeling Σ and Edge Superhyperlabeling M mirror the structure of a SuperHyperCrisp Set, with the added generalization of handling higher-order hierarchical relationships. Hence, Vertex Superhyperlabeling and Edge Superhyperlabeling are isomorphic to the structure of a SuperHyperCrisp Set. □

Furthermore, since labeling is closely related to coloring in graph theory, concepts such as hypercoloring and superhypercoloring can also be defined, similar to hyperlabeling and superhyperlabeling.

Question 3.254. What is the relationship between hyperlabeling and multilabeling, and between hypercoloring and multi-coloring [119, 131, 623, 690, 702]?

4 Discussion: Procedure of Graphization, Hyperization, and Uncertainization

In this section, we present an example of the Procedure of Graphization, Hyperization, and Uncertainization to facilitate future research efforts. It should be noted that this is merely an illustrative example. The meanings of "hyper" and "superhyper" can vary significantly across different mathematical fields, and the steps provided here represent only one approach, without verified accuracy. Other perspectives beyond these procedures undoubtedly exist, and thus, this example is intended solely as a reference.

Note 1 (Example Procedure of Graphization). *Transforming set concepts into graph concepts involves mapping the properties and elements of a set-based structure into the components and attributes of a graph. This allows the rich theoretical framework of sets to be utilized within graph theory, enabling new insights and applications.*

Below is a step-by-step procedure to transform a set concept into a graph concept:

1. Identify the Set Concept and Its Elements

- *Definition: Start by clearly defining the set concept, including its elements, operations, and properties.*
- *Elements: Determine what the elements of the set represent (e.g., membership degrees in fuzzy sets).*

2. Define Corresponding Graph Components

- *Vertices and Edges: Decide how the elements of the set will correspond to the vertices and edges of the graph.*
- *Attributes: Determine how the properties or operations in the set concept translate to graph attributes such as weights, labels, or edge relations.*

3. Establish Mapping Between Set and Graph

- *Vertices Mapping: Map the elements of the set to the vertices of the graph.*
- *Edges Mapping: Define edges based on relationships or interactions between set elements.*
- *Weights and Labels: Assign weights or labels to vertices or edges to represent set properties (e.g., membership degrees).*

4. Define Graph Operations and Properties

- *Adjacency and Connectivity: Establish how adjacency in the graph reflects relationships in the set.*
- *Graph Properties: Extend set operations (e.g., union, intersection) to graph operations (e.g., edge union, vertex intersection).*

5. Ensure Consistency and Validity

- *Consistency: Verify that the graph structure accurately reflects the original set concept.*
- *Validity: Ensure that all graph definitions comply with graph theory standards.*

Example 4.1 (Fuzzy Set to Fuzzy Graph)**.** *Fuzzy Set Definition*: A fuzzy set $\tilde{A}$ in a universe U is defined by a membership function $\mu_{\tilde{A}} : U \to [0, 1]$.

Fuzzy Graph Definition: A fuzzy graph $G = (V, \tilde{E})$ consists of a set of vertices V and a fuzzy edge set $\tilde{E}$, where the edge membership function $\mu_{\tilde{E}} : V \times V \to [0, 1]$ assigns a degree of adjacency between vertex pairs.

Transformation Procedure:

- *Vertices*: Let the elements of the universe U correspond to the vertices V.
- *Edges*: Define edges between vertices based on a relation that reflects the fuzzy set's context.
- *Edge Weights*: Assign edge weights using a function derived from the fuzzy set's membership function or other relevant criteria.

Example 4.2 (Neutrosophic Set to Neutrosophic Graph)**.** *Neutrosophic Set Definition*: A neutrosophic set A in a universe U is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A, and a falsity-membership function F_A, mapping U to the interval $[0, 1]$.

Neutrosophic Graph Definition: A neutrosophic graph $G = (V, E, T, I, F)$ includes functions assigning degrees of truth, indeterminacy, and falsity to vertices and/or edges.

Transformation Procedure:

- *Vertices*: Map elements of U to vertices V.
- *Edges*: Define edges based on relationships in the neutrosophic context.
- *Vertex and Edge Attributes*: Assign T, I, and F values to vertices and edges based on the neutrosophic set's membership functions.

Note 2 (Example Procedure of Hyperization and Superhyperization)**.** *Let us consider a mathematical concept characterized by a set S and a function f that assigns attributes, labels, weights, or other properties to elements of S. The superhyperextension of this concept involves the following steps:*

1. *Identify the Original Structure:*

   Begin with a mathematical structure defined by a set S and a mapping:

$$f : S \to T,$$

   where T is a set representing attributes, labels, weights, etc.

2. *First-Level Hyperextension:*
   Extend the mapping f to a function $f' : S \to \mathcal{P}(T) \setminus \{\emptyset\}$, where $\mathcal{P}(T)$ denotes the power set of T:
$$f' : S \to \mathcal{P}(T) \setminus \{\emptyset\}, \quad \text{such that} \quad f'(s) = \{t_1, t_2, \dots\} \subseteq T.$$
   This creates a hyperstructure, where each element $s \in S$ is associated with a non-empty set of attributes from T.

3. *Second-Level Superhyperextension:*
   Further extend the mapping to a function $f'' : S \to \mathcal{P}(\mathcal{P}(T) \setminus \{\emptyset\}) \setminus \{\emptyset\}$:
$$f'' : S \to \mathcal{P}\left(\mathcal{P}(T) \setminus \{\emptyset\}\right) \setminus \{\emptyset\}, \quad \text{such that} \quad f''(s) = \left\{\{t_{11}, t_{12}, \dots\}, \{t_{21}, t_{22}, \dots\}, \dots\right\}.$$
   This results in a superhyperstructure, where each element $s \in S$ is associated with a non-empty set of hyperattributes (sets of attributes).

4. *Define Extension Rules:*
   Specify any necessary rules or constraints for the superhyperstructure to maintain consistency and meaningful interpretation. These may include:
   - *Hierarchical Relationships: Define how the multiple levels of attributes relate to each other.*
   - *Consistency Conditions: Ensure that the superhyperstructure reduces to the hyperstructure or original structure under certain conditions.*
   - *Operations and Interactions: Extend operations, relations, or functions defined on S and T to accommodate the superhyperstructure.*

5. *Generalize Existing Theorems and Properties:*
   Examine how existing theorems, properties, or algorithms applicable to the original structure can be generalized or adapted to the superhyperstructure.

Note 3 (Example Procedure of Uncertainization)**.** *The following provides a concise procedure for extending a mathematical concept to its fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic forms. Note that fuzzy is a generalization of classical concepts, intuitionistic fuzzy is a further generalization of fuzzy, neutrosophic generalizes intuitionistic fuzzy, and plithogenic further generalizes neutrosophic.*

1. Identify Core Concept *Define the original concept (e.g., set, function, graph) along with its properties and foundational structure.*

2. Analyze Elements for Uncertainty *Determine which elements in the original concept are suitable for incorporating degrees of uncertainty, such as membership, indeterminacy, or multi-valued attributes.*

3. Fuzzy Extension *Introduce partial membership by defining a membership function*
$$\mu_A : U \to [0, 1],$$
where U is the universe of discourse. Adjust the definitions and operations to reflect partial membership values, thereby converting the original concept into its fuzzy counterpart.

4. Intuitionistic Fuzzy Extension *Define both membership and non-membership degrees with functions*
$$\begin{cases} \mu_A : U \to [0, 1], \\ \nu_A : U \to [0, 1], \end{cases}$$
where, for each $x \in U$,
$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$
The hesitation degree is given by
$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$
In this way, the concept is extended to an intuitionistic fuzzy form by incorporating both membership and non-membership degrees.

5. Neutrosophic Extension *Generalize further by allowing the degrees of truth, indeterminacy, and falsity to be independent, potentially ranging beyond the interval* $[0, 1]$. *This step transforms the concept into a neutrosophic form.*

6. Plithogenic Extension *Extend the concept to a plithogenic form by incorporating attributes with possibly contradictory values, enabling multi-valued characteristics and capturing complex, real-world phenomena.*

7. Validation *Verify that each extension aligns with the original concept's theoretical framework, ensuring internal consistency and applicability to the uncertainty domains introduced.*

5 Future Tasks of this research

This section outlines the future prospects of this research.

5.1 Extensions Using Related Set and Graph Concepts

This paper introduces several set concepts, and We foresee further advancements by extending these concepts using related set and graph theories. Examples include Superhypersoft Sets [350,666,878], Hesitant Fuzzy Sets [942,943], Spherical Fuzzy Sets [94,600,601], Neutrosophic SuperHypersoft Sets [878,885,899], IndetermSoft Sets [382, 772, 880, 882], IndetermHyperSoft Sets [880, 882], TreeSoft Sets [13, 77, 270, 271, 382, 407, 520, 687, 719, 727, 824, 880, 882, 887], soft rough sets [959], Meta Set [553, 554, 904–910], Intersectional Soft Sets [781, 901], and Plithogenic Sets [864, 895].

We hope that these extensions will enrich the theoretical framework established in this paper.[1]

Additionally, We are interested in exploring new types of sets. An example is provided below.

5.1.1 Multihypersoft Graphs

Future research may investigate the definition and graph representation of Multihypersoft Graphs and Multi-superhypersoft Graphs, as extensions of Multisoft Graphs. Although still in the conceptual phase, the initial definition of Multihypersoft Graphs is provided below.

Definition 5.1. Let $G^* = (V, E)$ be a simple graph where V is the set of vertices and E is the set of edges. Let $T_1, T_2, \ldots, T_n$ be n distinct attributes, each with corresponding attribute values E_i, and define $E = E_1 \times E_2 \times \cdots \times E_n$. Consider a subset $A \subseteq \mathcal{P}(E)$, where $\mathcal{P}(E)$ is the power set of E.

A *MultiHypersoft Graph* $G = (G^*, F, K, A)$ over the graph $G^* = (V, E)$ consists of the following components:

1. $G^* = (V, E)$ is the underlying simple graph.
2. A is a set of attribute combinations where $A \subseteq \mathcal{P}(E)$.
3. $F : A \to \mathcal{P}(V)$ is a mapping from attribute combinations in A to subsets of vertices in V.
4. $K : A \to \mathcal{P}(E)$ is a mapping from attribute combinations in A to subsets of edges in E.

For each $a \in A$, the pair $(F(a), K(a))$ forms a subgraph of G^*, denoted by $H(a)$. The MultiHypersoft Graph G can thus be represented as:

$$G = (F, K, A) = \{H(a) \mid a \in A\}.$$

[1] It is noteworthy that the Supersoft set concept generalizes the soft set, and the Superhypersoft set extends the Supersoft set, much like the relationship between supergraphs and superhypergraphs in graph theory [867, 868]. Additionally, we aim to define the IndetermSuperSoft set as well.Future work will explore the mathematical properties and practical applications of these advanced set structures.

Theorem 5.2. *Let $G = (G^*, F, K, A)$ be a MultiHypersoft Graph. If the attribute set A is restricted to a single attribute space, where each attribute has a unique value, then G can be transformed into a Hypersoft Graph.*

Proof. A Hypersoft Graph is defined based on a specific single attribute or attribute space, without considering attribute multiplicity. In a Hypersoft Graph, each vertex and edge is uniquely associated with an attribute, focusing on the attribute itself rather than combinations of attributes.

If A is restricted to a single attribute space with unique values for each attribute (i.e., each element a in A has a unique value for each attribute), the structure of a MultiHypersoft Graph aligns with that of a Hypersoft Graph. This is because the set of attributes A becomes a unique set over the attribute space, making each $F(a)$ and $K(a)$ correspond to a unique combination, thereby satisfying the definition of a Hypersoft Graph.

Thus, under the specified conditions, any MultiHypersoft Graph G can be redefined as a Hypersoft Graph. □

Theorem 5.3. *A MultiHypersoft Graph generalizes a Multisoft Graph.*

Proof. By definition, a MultiHypersoft Graph allows attributes $T_1, T_2, \ldots, T_n$ to be associated with corresponding value domains $E_1, E_2, \ldots, E_n$, and the set $E = E_1 \times E_2 \times \cdots \times E_n$ defines the space of all possible combinations of these attributes. The attribute combinations $A \subseteq \mathcal{P}(E)$ in the MultiHypersoft Graph are thus more general than the single-attribute mappings in a Multisoft Graph.

To demonstrate that a Multisoft Graph is a special case of a MultiHypersoft Graph, consider the following restriction: 1. Let $n = 1$, so there is only a single attribute T with a corresponding value domain $E_1 = E$. 2. Let $A \subseteq \mathcal{P}(E)$, where each element of A corresponds to a single attribute value a in the domain E_1.

Under this restriction:

$$F : A \to \mathcal{P}(V), \quad K : A \to \mathcal{P}(E),$$

and the pair $(F(a), K(a))$ for each $a \in A$ forms a subgraph of G^*, identical to the structure of a Multisoft Graph.

Conversely, a MultiHypersoft Graph allows attributes $T_1, T_2, \ldots, T_n$ with multiple distinct value domains, creating a Cartesian product space $E = E_1 \times E_2 \times \cdots \times E_n$. The attribute combinations in $A \subseteq \mathcal{P}(E)$ represent a more complex and higher-dimensional attribute space, which cannot always be reduced to the single-attribute structure of a Multisoft Graph.

The MultiHypersoft Graph generalizes the Multisoft Graph by introducing a multi-attribute structure where attributes can take values from multiple domains, forming complex combinations. A Multisoft Graph is thus a special case of a MultiHypersoft Graph under the restriction to a single attribute space. □

5.1.2 Neutrosophic Axial Graph and Partner Multineutrosophic Graph

The definitions of a Neutrosophic Axial Graph and a Partner Multineutrosophic Graph are provided below, though they remain at the conceptual stage. Further exploration and analysis of these definitions are planned for the future.

Definition 5.4 (Neutrosophic Axial Graph)**.** Let X be a universe, and let $\mathfrak{N}\mathfrak{A}A$ denote a Neutrosophic Axial Set as previously defined.

A *Neutrosophic Axial Graph* is a graph $G = (V, E)$ where:

1. *Vertex Association*: Each vertex $v \in V$ is associated with a Neutrosophic Axial Set $\mathfrak{N}\mathfrak{A}A_v$, defined as:

$$\mathfrak{N}\mathfrak{A}A_v = \{\langle A_v, A_{v1}, A_{v2}\rangle \mid A_v \cap A_{vi} = \emptyset,\ i = 1, 2\},$$

where $A_v, A_{v1}, A_{v2} \subseteq X$.

2. *Edge Association*: Each edge $e = (u, v) \in E$ is associated with a Neutrosophic Axial Set $\mathfrak{NA}A_e$, defined as:

$$\mathfrak{NA}A_e = \{\langle A_e, A_{e1}, A_{e2}\rangle \mid A_e \cap A_{ei} = \emptyset,\ i = 1, 2\},$$

where $A_e, A_{e1}, A_{e2} \subseteq X$.

3. *Consistency Conditions*: The Neutrosophic Axial Sets of the vertices and edges satisfy specified consistency conditions. For example:

   - *Adjacency Condition*: An edge $e = (u, v)$ exists between vertices u and v if and only if there is a relation between their associated Neutrosophic Axial Sets, such as:

   $$A_e = A_u \cup A_v, \quad A_{e1} = A_{u1} \cup A_{v1}, \quad A_{e2} = A_{u2} \cup A_{v2}.$$

   - *Intersection Condition*: For each edge $e = (u, v)$, the associated sets satisfy:

   $$A_u \cap A_v = \emptyset, \quad A_{u1} \cap A_{v1} = \emptyset, \quad A_{u2} \cap A_{v2} = \emptyset.$$

Definition 5.5 (Partner Multineutrosophic Graph)**.** Let $G = (V, E)$ be a graph, where:

- V is a non-empty set of vertices.
- $E \subseteq V \times V$ is the set of edges.

Let $n \in \mathbb{N}$ be a positive integer, and $r, s, t \in \mathbb{N}$ such that $r + s + t = n$.

A *Partner Multineutrosophic Graph* $G_p = (V, E, f_V, f_E)$ is defined as follows:

1. *Vertex Association*:

   - Each vertex $v \in V$ is associated with a Multineutrosophic Set:

   $$M_n(v) = \langle T_1(v), \dots, T_r(v);\ I_1(v), \dots, I_s(v);\ F_1(v), \dots, F_t(v)\rangle,$$

   where:

     - $T_i(v) \in [0, 1]$ for $i = 1, \dots, r$ are the truth-membership degrees of v.
     - $I_j(v) \in [0, 1]$ for $j = 1, \dots, s$ are the indeterminacy-membership degrees of v.
     - $F_k(v) \in [0, 1]$ for $k = 1, \dots, t$ are the falsity-membership degrees of v.

   - The *partner set* of vertex v is defined as:

   $$M_n^p(v) = \left\langle v, f_{M_n}(v)\right\rangle,$$

   where $f_{M_n} : V \to [0, 1]$ is given by:

   $$f_{M_n}(v) = \frac{1}{n}\left(\sum_{i=1}^{r} T_i(v) + \sum_{j=1}^{s} I_j(v) + \sum_{k=1}^{t} F_k(v)\right).$$

2. *Edge Association*:

   - Each edge $e \in E$ is associated with a Multineutrosophic Set:

   $$M_n(e) = \langle T_1(e), \dots, T_r(e);\ I_1(e), \dots, I_s(e);\ F_1(e), \dots, F_t(e)\rangle,$$

   where:

     - $T_i(e) \in [0, 1]$ for $i = 1, \dots, r$ are the truth-membership degrees of e.
     - $I_j(e) \in [0, 1]$ for $j = 1, \dots, s$ are the indeterminacy-membership degrees of e.
     - $F_k(e) \in [0, 1]$ for $k = 1, \dots, t$ are the falsity-membership degrees of e.

- The *partner set* of edge e is defined as:

$$M_n^p(e) = \langle e, f_{M_n}(e) \rangle,$$

where $f_{M_n} : E \to [0,1]$ is given by:

$$f_{M_n}(e) = \frac{1}{n}\left(\sum_{i=1}^{r} T_i(e) + \sum_{j=1}^{s} I_j(e) + \sum_{k=1}^{t} F_k(e)\right).$$

3. *Membership Functions*:

- The function $f_V : V \to [0,1]$ assigns to each vertex $v \in V$ the aggregated membership degree $f_{M_n}(v)$.
- The function $f_E : E \to [0,1]$ assigns to each edge $e \in E$ the aggregated membership degree $f_{M_n}(e)$.

Therefore, the Partner Multineutrosophic Graph G_p incorporates multineutrosophic information by associating each vertex and edge with a partner set, resulting in a weighted graph where weights are derived from the aggregated neutrosophic membership degrees.

Question 5.6. Can the Partner Multineutrosophic Graph or the Neutrosophic Axial Graph generalize any existing graph concepts? What potential applications could be considered for these graphs?

5.2 Derived form or variant of the Nonstandard Real Set

I look forward to future research on derived concepts of the Nonstandard Real Set. This paper provides one such example.

5.2.1 Nonstandard Rational Set

In this paper, we primarily focused on real numbers; however, numerous other types of numbers are well-known, including complex numbers, rational numbers, integers, and natural numbers. I personally hope that future research will explore derived concepts of the Nonstandard Real Set using these various number sets. Although some are still in the conceptual stage, several definitions are provided below.

Definition 5.7. (cf. [310, 395, 604, 779, 784]) The following are the mathematical definitions of the complex numbers, rational numbers, integers, and natural numbers:

1. *Complex Numbers* ($\mathbb{C}$): The set of complex numbers $\mathbb{C}$ is defined as

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R},\ i = \sqrt{-1}\},$$

where a and b are real numbers, and i is the imaginary unit satisfying $i^2 = -1$.

2. *Rational Numbers* ($\mathbb{Q}$): The set of rational numbers $\mathbb{Q}$ is defined as

$$\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z},\ q \neq 0\right\},$$

where p and q are integers, and q is nonzero. Rational numbers are precisely those numbers that can be expressed as a fraction of two integers.

3. *Integers* ($\mathbb{Z}$): The set of integers $\mathbb{Z}$ is defined as

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

This includes all whole numbers, both positive and negative, as well as zero.

4. *Natural Numbers* ($\mathbb{N}$): The set of natural numbers $\mathbb{N}$ is typically defined as

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\},$$

representing the set of all positive integers. In some contexts, $\mathbb{N}$ may also include 0, i.e., $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Definition 5.8. The *Nonstandard Rational Set*, denoted as $\mathbb{Q}^*$, is an extension of the standard set of rational numbers $\mathbb{Q}$ within the framework of nonstandard analysis. This set includes infinitesimal and infinitely large rational elements that do not exist within the standard rational numbers. We define it as follows:

$$\mathbb{Q}^* = \mathbb{Q} \cup \mathbb{Q}^{+*} \cup \mathbb{Q}^{-*},$$

where:

- $\mathbb{Q}$ represents the set of standard rational numbers.
- $\mathbb{Q}^{+*}$ denotes the set of positive infinitesimal rationals. An element $\epsilon \in \mathbb{Q}^{+*}$ is such that $0 < \epsilon < \frac{1}{n}$ for any positive integer n.
- $\mathbb{Q}^{-*}$ denotes the set of negative infinitesimal rationals, where $-\epsilon$ (for $\epsilon \in \mathbb{Q}^{+*}$) represents values closer to zero from the negative side.

Additionally, $\mathbb{Q}^*$ encompasses rational numbers of infinitely large magnitude, which we denote by $\infty^*_{\mathbb{Q}}$. These elements are greater in absolute value than any standard rational number and are used to model quantities that go beyond the standard finite framework.

Within $\mathbb{Q}^*$, we observe that the set includes both infinitesimal and infinite quantities, allowing for more detailed distinctions among rational numbers. This extended set maintains the basic arithmetic operations (addition, subtraction, multiplication, and division, where applicable) of $\mathbb{Q}$ and adheres to the ordering relations in a manner that extends the standard rational number properties. For example:

$$0 < \epsilon < \frac{1}{n} \quad \text{for any } \epsilon \in \mathbb{Q}^{+*} \text{ and all } n \in \mathbb{N},$$

$$q < \infty^*_{\mathbb{Q}} \quad \text{for any } q \in \mathbb{Q}.$$

Theorem 5.9. *The Nonstandard Rational Set $\mathbb{Q}^*$ is a subset of the Nonstandard Real Set $\mathbb{R}^*$.*

Proof. To show that $\mathbb{Q}^* \subset \mathbb{R}^*$, we need to demonstrate that every element of $\mathbb{Q}^*$ is also an element of $\mathbb{R}^*$. By definition, we have:

$$\mathbb{Q}^* = \mathbb{Q} \cup \mathbb{Q}^{+*} \cup \mathbb{Q}^{-*},$$

where:

- $\mathbb{Q} \subset \mathbb{R}$, as every standard rational number is also a standard real number.
- $\mathbb{Q}^{+*}$ consists of positive infinitesimal rational values, which are real numbers closer to zero than any positive standard real number, hence $\mathbb{Q}^{+*} \subset \mathbb{R}^{+*}$.
- $\mathbb{Q}^{-*}$ consists of negative infinitesimal rational values, which are real numbers closer to zero from the negative side than any negative standard real number, hence $\mathbb{Q}^{-*} \subset \mathbb{R}^{-*}$.

Thus, $\mathbb{Q} \subset \mathbb{R}$, $\mathbb{Q}^{+*} \subset \mathbb{R}^{+*}$, and $\mathbb{Q}^{-*} \subset \mathbb{R}^{-*}$. Therefore, every component of $\mathbb{Q}^*$ is contained within $\mathbb{R}^*$, implying that $\mathbb{Q}^* \subset \mathbb{R}^*$ as required.

Additionally, any infinitely large rational number in $\mathbb{Q}^*$, denoted $\infty^*_{\mathbb{Q}}$, is also a real number that exceeds the magnitude of any standard real number, so $\infty^*_{\mathbb{Q}} \subset \infty^*_{\mathbb{R}}$.

Hence, we conclude that $\mathbb{Q}^* \subset \mathbb{R}^*$. □

Definition 5.10. The *Nonstandard Complex Set*, denoted as $\mathbb{C}^*$, is an extension of the standard set of complex numbers $\mathbb{C}$ within the nonstandard analysis framework. This set includes infinitesimal and infinitely large complex elements, which are beyond the scope of the standard complex numbers. We define it as:

$$\mathbb{C}^* = \mathbb{C} \cup \mathbb{C}^{+*} \cup \mathbb{C}^{-*},$$

where:

- $\mathbb{C}$ represents the standard complex numbers.
- $\mathbb{C}^{+*}$ denotes the set of positive infinitesimal complex values. For an element $\epsilon \in \mathbb{C}^{+*}$, both the real and imaginary components satisfy $0 < |\epsilon| < \frac{1}{n}$ for any positive integer n.
- $\mathbb{C}^{-*}$ denotes the set of negative infinitesimal complex values, where $-\epsilon$ for $\epsilon \in \mathbb{C}^{+*}$ represents values infinitesimally close to zero in both the real and imaginary parts.

The nonstandard complex set $\mathbb{C}^*$ also encompasses infinitely large complex numbers with magnitudes exceeding any standard complex number's modulus.

Theorem 5.11. *The Nonstandard Real Set $\mathbb{R}^*$ is a subset of the Nonstandard Complex Set $\mathbb{C}^*$.*

Proof. To show that $\mathbb{R}^* \subset \mathbb{C}^*$, we need to demonstrate that every element of $\mathbb{R}^*$ is also an element of $\mathbb{C}^*$. By definition, we have:

$$\mathbb{C}^* = \mathbb{C} \cup \mathbb{C}^{+*} \cup \mathbb{C}^{-*},$$

where:

- $\mathbb{C}$ represents the set of standard complex numbers, which includes all real numbers as complex numbers with an imaginary part of zero. Thus, $\mathbb{R} \subset \mathbb{C}$.
- $\mathbb{C}^{+*}$ represents the set of positive infinitesimal complex values, and $\mathbb{R}^{+*} \subset \mathbb{C}^{+*}$ because any positive infinitesimal real number can be viewed as a complex number with zero imaginary part.
- $\mathbb{C}^{-*}$ represents the set of negative infinitesimal complex values, and $\mathbb{R}^{-*} \subset \mathbb{C}^{-*}$ since any negative infinitesimal real number can similarly be seen as a complex number with zero imaginary part.

Furthermore, $\mathbb{R}^*$ also includes infinitely large real values, all of which are complex numbers within $\mathbb{C}^*$, with zero imaginary part and modulus greater than any finite complex number.

Thus, all components of $\mathbb{R}^*$ (standard reals, positive and negative infinitesimals, and infinitely large reals) are included within $\mathbb{C}^*$. Therefore, we conclude that $\mathbb{R}^* \subset \mathbb{C}^*$. □

Definition 5.12. The *Nonstandard Integer Set*, denoted as $\mathbb{Z}^*$, is an extension of the set of standard integers $\mathbb{Z}$, incorporating both infinitesimal and infinitely large integer-like values. It is defined as follows:

$$\mathbb{Z}^* = \mathbb{Z} \cup \mathbb{Z}^{+*} \cup \mathbb{Z}^{-*},$$

where:

- $\mathbb{Z}$ denotes the set of standard integers.
- $\mathbb{Z}^{+*}$ includes positive infinitesimal values that behave like integers but are closer to zero than any positive integer $\frac{1}{n}$ for all $n \in \mathbb{N}$.

- $\mathbb{Z}^{-*}$ includes negative infinitesimal values behaving like integers, satisfying $-\frac{1}{n} < \epsilon < 0$ for all $n \in \mathbb{N}$.

This set also includes infinitely large values that are integer-like but exceed the magnitude of any standard integer.

Theorem 5.13. *The Nonstandard Integer Set $\mathbb{Z}^*$ is a subset of the Nonstandard Rational Set $\mathbb{Q}^*$.*

Proof. To show that $\mathbb{Z}^* \subset \mathbb{Q}^*$, we need to demonstrate that every element of $\mathbb{Z}^*$ is also an element of $\mathbb{Q}^*$. By definition, we have:

$$\mathbb{Q}^* = \mathbb{Q} \cup \mathbb{Q}^{+*} \cup \mathbb{Q}^{-*},$$

where:

- $\mathbb{Q}$ represents the set of standard rational numbers, which includes all integers since $\mathbb{Z} \subset \mathbb{Q}$.
- $\mathbb{Q}^{+*}$ and $\mathbb{Q}^{-*}$ represent the sets of positive and negative infinitesimal rational values, respectively. Since any infinitesimal integer-like value (positive or negative) behaves as an infinitesimal rational number, we have $\mathbb{Z}^{+*} \subset \mathbb{Q}^{+*}$ and $\mathbb{Z}^{-*} \subset \mathbb{Q}^{-*}$.
- $\mathbb{Z}^*$ also includes infinitely large integer-like values, all of which can be expressed as rational numbers by setting the denominator to 1, and therefore they belong to $\mathbb{Q}^*$.

Thus, each part of $\mathbb{Z}^*$ (standard integers, infinitesimal integers, and infinitely large integers) is included within the corresponding components of $\mathbb{Q}^*$. Therefore, we conclude that $\mathbb{Z}^* \subset \mathbb{Q}^*$. □

Definition 5.14. The *Nonstandard Natural Set*, denoted as $\mathbb{N}^*$, extends the standard set of natural numbers $\mathbb{N}$ by incorporating infinitesimal and infinitely large values. It is defined as:

$$\mathbb{N}^* = \mathbb{N} \cup \mathbb{N}^{+*} \cup \mathbb{N}^{-*},$$

where:

- $\mathbb{N}$ represents the set of standard natural numbers.
- $\mathbb{N}^{+*}$ includes positive infinitesimal values, defined such that $0 < \epsilon < \frac{1}{n}$ for any positive integer n.
- $\mathbb{N}^{-*}$ typically does not contain elements as negative values are not included in the traditional natural number set.

Additionally, $\mathbb{N}^*$ includes infinitely large natural numbers that are greater than any standard natural number, used to model nonstandard finite yet unbounded values within $\mathbb{N}^*$.

Theorem 5.15. *The Nonstandard Natural Set $\mathbb{N}^*$ is a subset of the Nonstandard Integer Set $\mathbb{Z}^*$.*

Proof. To prove $\mathbb{N}^* \subset \mathbb{Z}^*$, we need to verify that each element in $\mathbb{N}^*$ is also an element of $\mathbb{Z}^*$. By definition:

$$\mathbb{N}^* = \mathbb{N} \cup \mathbb{N}^{+*} \cup \mathbb{N}^{-*},$$

where:

- $\mathbb{N} \subset \mathbb{Z}$, meaning all standard natural numbers are also standard integers.
- $\mathbb{N}^{+*} \subset \mathbb{Z}^{+*}$, as any positive infinitesimal natural number is also a positive infinitesimal integer.

- $\mathbb{N}^{-*}$ is typically empty because natural numbers do not include negative values. Thus, there is no conflict with the structure of $\mathbb{Z}^*$, which can contain negative infinitesimals within $\mathbb{Z}^{-*}$.

Additionally, any infinitely large natural number in $\mathbb{N}^*$ is also an infinitely large integer in $\mathbb{Z}^*$ since it is integer-like and unbounded. Therefore, all components of $\mathbb{N}^*$ are contained within the corresponding components of $\mathbb{Z}^*$.

Consequently, we conclude that $\mathbb{N}^* \subset \mathbb{Z}^*$. □

From the above discussion, we establish the following results. Moving forward, we aim to delve deeper into their mathematical structures and anticipate further research on the integration of these sets with concepts such as fuzzy sets and neutrosophic sets.

Theorem 5.16. *In the framework of nonstandard analysis, the Nonstandard Natural Set $\mathbb{N}^*$, the Nonstandard Integer Set $\mathbb{Z}^*$, the Nonstandard Rational Set $\mathbb{Q}^*$, and the Nonstandard Real Set $\mathbb{R}^*$ are all subsets of the Nonstandard Complex Set $\mathbb{C}^*$. Specifically, we have:*

$$\mathbb{N}^* \subset \mathbb{Z}^* \subset \mathbb{Q}^* \subset \mathbb{R}^* \subset \mathbb{C}^*.$$

For reference, the relationships between Nonstandard sets and Standard sets are illustrated in Figure 11.

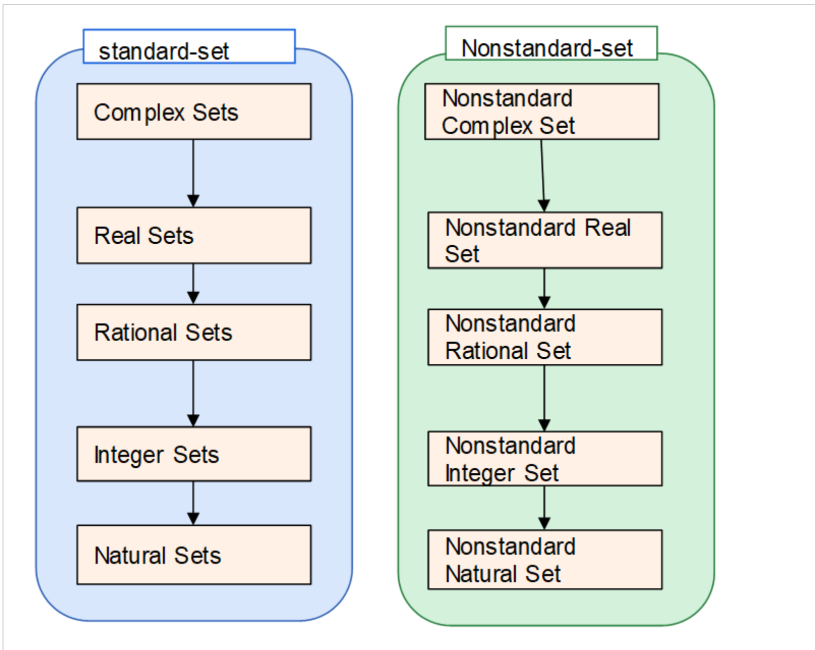


Figure 11: Some Standard sets and Nonstandard sets Hierarchy. The set class at the origin of an arrow contains the set class at the destination of the arrow.

Building on the concepts discussed above, there is significant potential to extend nonstandard sets by incorporating structures such as Fuzzy Numbers [207, 213, 288, 387, 397, 608, 747, 1038], Neutrosophic Numbers [206, 259, 438, 625, 671, 920], ZE-numbers [301, 474, 937] [2], and Z-numbers [88, 257, 750, 936, 972, 997]. While further validation is needed, I hypothesize that these extensions may align with or resemble the concept of a bipolar (infinite) set. To illustrate this, I present an example of a Nonstandard Z-number Set below.

I am hopeful that these concepts, along with the development of set-based frameworks and their graph representations, will continue to be explored and receive rigorous mathematical analysis in future studies.

[2] As an extension of Z-numbers, ZE-numbers could potentially be expanded through graph representations or applied within existing set theories [301, 474, 937].

Definition 5.17. [1020] A *Z-number* is an ordered pair of fuzzy numbers, represented as $Z = (A, R)$. This unique construct allows for the representation of uncertain information by capturing both a value's restriction and its reliability.

- A: The first component, A, is a fuzzy number representing a restriction on the possible values of a real-valued uncertain variable X. This component reflects the range of possible values that X may assume under the conditions specified.
- R: The second component, R, is a fuzzy number that measures the reliability, confidence, or sureness of the restriction represented by A. Unlike probability, which measures the likelihood of a specific event, R measures the certainty or confidence level associated with the values represented by A.

In other words, a Z-number (A, R) describes a scenario where A provides a fuzzy restriction on the values that X might take, while R represents the confidence in that restriction. For example, in risk analysis, a Z-number $Z = (\text{``very low''}, \text{``very likely''})$ might represent that the severity of a risk is restricted to a "very low" value with a confidence level of "very likely."

Definition 5.18. A *Nonstandard Z-number Set*, denoted as Z^*, is an extension of the standard Z-number set framework within nonstandard analysis. This set includes not only standard Z-numbers but also their infinitesimal and infinitely large extensions. Formally, we define the Nonstandard Z-number Set Z^* as follows:

$$Z^* = Z \cup Z^{+*} \cup Z^{-*} \cup \infty_Z^*,$$

where:

- Z represents the set of standard Z-numbers, each of which is an ordered pair of fuzzy numbers, $Z = (A, R)$.
- Z^{+*} denotes the set of positive infinitesimal Z-numbers, where each Z-number $(A^*, R^*) \in Z^{+*}$ has components A^* and R^* that are positive infinitesimal fuzzy numbers. Here, A^* represents an infinitesimal restriction on the values of an uncertain variable X, while R^* provides an infinitesimal degree of reliability associated with this restriction.
- Z^{-*} represents the set of negative infinitesimal Z-numbers, where each Z-number $(A^*, R^*) \in Z^{-*}$ has components A^* and R^* as negative infinitesimal fuzzy numbers, respectively representing an infinitesimal restriction and an infinitesimal level of confidence.
- ∞_Z^* denotes the set of infinitely large Z-numbers. Here, each Z-number $(A^*, R^*) \in \infty_Z^*$ includes components A^* and R^* that are infinitely large fuzzy numbers, modeling situations where both the restriction A and the reliability R are of an unbounded nature.

Thus, the Nonstandard Z-number Set Z^* allows for a comprehensive modeling of uncertain information by extending the range of Z-numbers to include infinitesimal and infinitely large restrictions and reliabilities, broadening the traditional framework.

Each element of Z^* can be expressed as an ordered pair (A^*, R^*), where:

- A^* is a fuzzy number that may represent standard, infinitesimal, or infinitely large restrictions on the possible values of an uncertain variable X.
- R^* is a fuzzy number representing the reliability, which can also be standard, infinitesimal, or infinitely large, associated with the values described by A^*.

Question 5.19. As a proposed framework for new set concepts in the context of Nonstandard Sets, the following structure is suggested. The question remains: are these concepts mathematically valid and applicable across various types of sets?

The *Nonstandard Set Construction Framework* aims to establish a rigorous structure for defining and organizing nonstandard extensions of traditional sets. This framework includes the following key components:

1. *Standard Set*: A set S (e.g., the set of real numbers $\mathbb{R}$, integers $\mathbb{Z}$, or rational numbers $\mathbb{Q}$) that serves as the classical foundation from which nonstandard extensions are derived.

2. *Infinitesimal Extensions*:

   - The set of positive infinitesimal elements S^{+*} associated with S, consisting of elements $\epsilon \in S^{+*}$ where, for any positive integer n, $0 < \epsilon < \frac{1}{n}$.
   - Similarly, the set of negative infinitesimal elements S^{-*} includes elements $\delta \in S^{-*}$ such that $-\frac{1}{n} < \delta < 0$.

3. *Infinite Extensions*:

   - The set of infinitely large elements ∞_S^* associated with S, defined as elements greater in magnitude than any standard element in S. Formally, for any $x \in S$, there exists $y \in \infty_S^*$ such that $|y| > |x|$.

4. *Nonstandard Set Definition*:

   - The *Nonstandard Set* S^* associated with a standard set S is defined as:

   $$S^* = S \cup S^{+*} \cup S^{-*} \cup \infty_S^*,$$

   where:
     - S represents the standard elements in the original set S,
     - S^{+*} and S^{-*} extend S to include infinitesimal values, and
     - ∞_S^* includes infinitely large elements in the extended set.

5. *Ordering and Arithmetic Consistency*:

   - The framework preserves the original set's ordering and arithmetic operations (addition, subtraction, multiplication, and division, as applicable), extending these operations to handle infinitesimal and infinitely large values within S^*.

Question 5.20. Is it possible to define a nonstandard set using algebraic numbers(cf. [287,588,922]), imaginary numbers [450, 699], Eisenstein integers, or Gaussian integers(cf. [428])? Additionally, what would be the mathematical structure and relationships of such a set with other sets? If such a definition is feasible, is it also possible to represent it graphically?

5.2.2 Augmented Nonstandard Real Set

As a derived concept of infinitesimals, hyper-infinitesimals are well known, and definitions of the Nonstandard Real Set based on these are presented here [125, 336]. Since there are also multiple approaches to defining derivatives of infinite values, I note that future research will likely involve active exploration of similar definitions, mathematical structures, and applications based on these concepts.

Definition 5.21. (cf. [243, 929]) The set of hyperreal numbers, denoted ${}^*\mathbb{R}$, is an extension of the real number set $\mathbb{R}$. This set ${}^*\mathbb{R}$ includes not only real numbers but also infinitesimal and infinite numbers, which are defined using equivalence classes of sequences of real numbers. For example, the sequences $\{1/n\}$ and $\{1/n^2\}$ both converge to zero in $\mathbb{R}$ but have different convergence rates. In ${}^*\mathbb{R}$, these sequences belong to different equivalence classes, capturing the rate of convergence.

Additionally, the hyperreal numbers allow for the existence of multiple distinct infinitesimal numbers (numbers that are smaller in absolute value than any positive real number but greater than zero), unlike $\mathbb{R}$, which has only a single infinitesimal element, zero.

Definition 5.22. A *hyper-infinitesimal* is a type of infinitesimal element within the hyperreal number system ${}^*\mathbb{R}$, which is defined to represent values closer to zero than any standard real positive number, but with properties that extend beyond the bounds of conventional infinitesimal behavior in nonstandard analysis. Formally, let $x \in^* \mathbb{R}$ denote a hyperreal number. Then x is termed a hyper-infinitesimal if for every real number $r \in \mathbb{R}^+$ (where $\mathbb{R}^+$ denotes the positive real numbers), we have

$$0 < |x| < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

To define hyper-infinitesimals with even more nuanced properties, we introduce classes of hyper-infinitesimals that capture distinct orders of magnitude, each of which is smaller than any preceding infinitesimal class in the nonstandard framework. Specifically:

1. First-order Hyper-Infinitesimals: Defined as above, these are the smallest quantities in the hyperreal system that remain greater than zero but smaller than any real number.

2. Higher-order Hyper-Infinitesimals: For an n-th order hyper-infinitesimal x, it holds that

$$|x| < \frac{1}{k^n} \quad \text{for any } k \in \mathbb{N}.$$

These elements represent hyper-infinitesimals that decrease in magnitude with each increase in order, achieving values infinitesimally close to zero at a rate faster than that of the previous order.

Definition 5.23. The Augmented Nonstandard Real Set $\mathbb{R}^{*+}$ incorporates hyper-infinitesimals and hyper-infinities, designed to represent values even smaller than typical infinitesimals or larger than typical infinite values in the nonstandard real framework. We define it as:

$$\mathbb{R}^{*+} = \mathbb{R} \cup \mathbb{R}^{+*+} \cup \mathbb{R}^{-*+},$$

where:

- $\mathbb{R}^{+*+}$ represents hyper-infinitesimal values closer to zero than any value in $\mathbb{R}^{+*}$,
- $\mathbb{R}^{-*+}$ represents hyper-infinitesimal values closer to zero from the negative side, and
- Infinite values in $\mathbb{R}^{*+}$ can extend to hyper-infinite magnitudes, denoted ∞^{+}.

Question 5.24. The concept of a super-infinitesimal [128,913] is known as a related extension of infinitesimals. Is it possible to define an expanded framework or a set that combines the concepts of super-infinitesimals and hyper-infinitesimals?

Question 5.25. Is it possible to define an Augmented Nonstandard Set for hypernatural numbers [984] and hypercomplex numbers [444,445] in a similar manner? Additionally, what are the mathematical characteristics of such sets?

Question 5.26. Is it possible to define sets using dual numbers [332] or surreal numbers [74, 430, 584, 790]? Additionally, what are the mathematical characteristics of such sets?

5.3 Filters and matroids on an Offset, Overset, and Underset

In this paper, we introduce the concepts of neutrosophic Offset, Overset, and Underset sets, and we anticipate that future studies will develop filters and matroids based on these sets. This will enable a more comprehensive exploration of their relationships and underlying mathematical structures. These concepts are of considerable mathematical interest, and we hope they will reveal distinctive characteristics within the framework of neutrosophic Offset, Overset, and Underset sets.

In set theory, a filter is a collection of sets closed under intersection that includes all supersets of its elements [87,153,362,381]. A maximal filter, known as an ultrafilter, is a fundamental concept with applications across various mathematical disciplines [153,223,276,381,420,489,815,900] and even in social choice theory [672]. Related concepts include weak filter [592–594], ideal [155, 345, 357], bramble [146, 147, 196, 442, 540], and quasi filter [192,367].

A matroid is a combinatorial structure that generalizes the concept of linear independence from vector spaces to broader sets [709, 982]. Due to their robust mathematical properties, matroids play a fundamental role in discrete optimization and find applications in fields such as graph theory and computer science [357,491,643, 709, 768, 946, 983]. Related concepts include antimatroids [58, 218, 360, 456, 643], fuzzy matroids [39, 95, 417–419, 447, 617, 704, 821], Neutrosophic matroids [361, 369], greedoids [143, 590, 591], regular matroids

[130, 162, 832], ultra matroid [343], linear matroid [209, 630], hypermatroids [482], and quasi matroids [321, 344, 571, 572].

Research has also focused on Tangles [281, 282, 381, 776], which are closely related to Ultrafilters. Studies on these concepts often involve Tree Sets [278, 421] and Abstract Separation Systems [54, 277, 280, 308], representing a high level of generalization. There is an anticipation of further exploration into extending these structures using fuzzy sets, Neutrosophic Sets, Plithogenic Sets, and hyper or superhyper concepts.

We also hope to see advancements in extending these ideas to hypersets and superhypersets. For instance, the definitions of a hyperfilter and a superhyperfilter are presented below. These definitions build upon the concepts of hypersets and superhypersets. Moving forward, we aim to investigate their potential applications and underlying mathematical structures. Note that it is inferred that the superhyperset and superhyperultrafilter are not entirely new concepts but can be generalized from existing ones.

Definition 5.27 (Filter)**.** Let X be a universal set. A *filter* on a set X is a non-empty family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ such that:

1. $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
3. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

Definition 5.28 (Ultrafilter)**.** (cf. [420]) Let X be a universal set. An *ultrafilter* $\mathcal{U}$ on X is a filter $\mathcal{F}$ that satisfies:

1. For every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Definition 5.29 (Hyperfilter)**.** Let U be a universal set, and $\mathcal{P}(U)$ its power set. A subset $HF \subseteq \mathcal{P}(U)$ is called a *hyperfilter* if:

1. $HF \neq \emptyset$,
2. For every $X, Y \in HF$, there exists $Z \in HF$ such that $Z \subseteq X \cap Y$,
3. For every $X \in HF$ and $Y \in \mathcal{P}(U)$, if $X \subseteq Y$, then $Y \in HF$.

Definition 5.30 (Hyperultrafilter)**.** Let U be a universal set, and $HF \subseteq \mathcal{P}(U)$ a hyperfilter. A subset $HUF \subseteq \mathcal{P}(U)$ is called a *hyperultrafilter* if:

1. HUF is a hyperfilter,
2. For every $X \in \mathcal{P}(U)$, either $X \in HUF$ or $U \setminus X \in HUF$.

Definition 5.31 (Superhyperultrafilter)**.** Let U be a universal set, and $SHF \subseteq \mathcal{P}(\mathcal{P}(U))$ a superhyperfilter. A subset $SHUF \subseteq \mathcal{P}(\mathcal{P}(U))$ is called a *superhyperultrafilter* if:

1. $SHUF$ is a superhyperfilter,
2. For every $X \in \mathcal{P}(\mathcal{P}(U))$, either $X \in SHUF$ or $\mathcal{P}(U) \setminus X \in SHUF$.

Theorem 5.32. *A hyperultrafilter $\mathcal{H}$ on a hyperset $H \subseteq \mathcal{P}(X)$ can be transformed into an ultrafilter $\mathcal{U}$ on X.*

Proof. Let $\mathcal{H}$ be a hyperultrafilter on $H \subseteq \mathcal{P}(X)$. Define $\mathcal{U} = \{A \subseteq X \mid \{A\} \in \mathcal{H}\}$.

1. **Non-emptiness:** Since $H \in \mathcal{H}$, there exists $A \subseteq X$ such that $\{A\} \in \mathcal{H}$, so $A \in \mathcal{U}$.
2. **Closure under intersection:** If $A, B \in \mathcal{U}$, then $\{A\}, \{B\} \in \mathcal{H}$. Thus, $\{A \cap B\} \in \mathcal{H}$, and $A \cap B \in \mathcal{U}$.

3. **Maximality:** For every $A \subseteq X$, either $\{A\} \in \mathcal{H}$ or $\{X \setminus A\} \in \mathcal{H}$, so $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Thus, $\mathcal{U}$ is an ultrafilter on X. □

Theorem 5.33. *A superhyperultrafilter $\mathcal{S}$ on a superhyperset $SH \subseteq \mathcal{P}(\mathcal{P}(X))$ can be transformed into a hyperultrafilter $\mathcal{H}$ on $H \subseteq \mathcal{P}(X)$.*

Proof. Let $\mathcal{S}$ be a superhyperultrafilter. Define $\mathcal{H} = \{A \subseteq H \mid \{A\} \in \mathcal{S}\}$, where $H = \bigcup SH$. The same reasoning as above applies to verify the properties of $\mathcal{H}$, making it a hyperultrafilter. □

5.4 Soft intersection graph and Plithogenic Intersection Over Graph

An intersection graph represents relationships between sets where each vertex corresponds to a set, and edges exist between vertices if the sets intersect. In graph theory, the concept of an Intersection Graph is well-established [135, 144, 373, 403, 651, 653, 698, 767, 831, 847], with examples such as Interval Graphs [422, 455, 480], Unit Disk Graphs [51, 205, 220, 379, 556, 596, 607], Circular-arc Graphs [185, 359], String Graphs [156, 573, 595, 782], Unit Ball Graphs [490], and Permutation Graphs [364, 1007]. By applying Soft Sets, we aim to extend this concept to define Soft Intersection Graphs and Hypersoft Intersection Graphs, providing an overview of these extensions. Although still in the conceptual phase, definitions for these and related concepts are presented below. Readers seeking more details on Intersection Graphs may find relevant surveys or introductory notes helpful [373, 713, 714].

Definition 5.34 (Intersection Graph)**.** [403] Let $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ be a family of sets. The *intersection graph $G = (V, E)$* of $\mathcal{F}$ is defined as follows:

- The vertex set V of G corresponds to the sets in $\mathcal{F}$; that is, each vertex $v_i \in V$ represents a set $S_i \in \mathcal{F}$.
- There is an edge $(v_i, v_j) \in E$ between two vertices v_i and v_j if and only if the corresponding sets S_i and S_j have a non-empty intersection, i.e., $S_i \cap S_j \neq \emptyset$.

Formally, we define:

$$V = \{v_i \mid S_i \in \mathcal{F}\}$$

and

$$E = \{(v_i, v_j) \mid S_i, S_j \in \mathcal{F}, i \neq j, \text{ and } S_i \cap S_j \neq \emptyset\}.$$

Thus, the intersection graph G represents the pairwise intersections within the family of sets $\mathcal{F}$.

Example 5.35. Let $\mathcal{F} = \{S_1, S_2, S_3, S_4\}$ be a family of sets, where:

$$S_1 = \{a, b, c\}, \quad S_2 = \{b, d\}, \quad S_3 = \{c, d, e\}, \quad S_4 = \{f\}.$$

The *intersection graph $G = (V, E)$* is constructed as follows:

- The vertex set is $V = \{v_1, v_2, v_3, v_4\}$, where v_i corresponds to S_i.
- An edge exists between v_i and v_j if and only if $S_i \cap S_j \neq \emptyset$.

From the intersections:

$$S_1 \cap S_2 = \{b\}, \quad S_1 \cap S_3 = \{c\}, \quad S_2 \cap S_3 = \{d\},$$

and $S_1 \cap S_4 = S_2 \cap S_4 = S_3 \cap S_4 = \emptyset$, the edge set is:

$$E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3)\}.$$

Thus, the intersection graph is:

$$V = \{v_1, v_2, v_3, v_4\}, \quad E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3)\}.$$

Definition 5.36 (Soft Intersection Graph)**.** Let U be a universal set, E be a set of parameters, and let $F : E \to \mathcal{P}(U)$ be a soft set over U, where $\mathcal{P}(U)$ denotes the power set of U.

The *Soft Intersection Graph* $G = (V, E')$ is defined as follows:

- The vertex set V is the set of parameters E.
- There is an edge between two vertices $e_i, e_j \in V$ if and only if the corresponding soft set values intersect, that is:
$$(e_i, e_j) \in E' \quad \text{if and only if} \quad F(e_i) \cap F(e_j) \neq \emptyset.$$

Theorem 5.37. *Every Soft Graph can be represented as a Soft Intersection Graph.*

Proof. Let $G = (G^*, F, K, A)$ be a Soft Graph, where:

- $G^* = (V, E)$ is a simple graph.
- A is a non-empty set of parameters.
- (F, A) is a soft set over V, with $F : A \to \mathcal{P}(V)$.
- (K, A) is a soft set over E, with $K : A \to \mathcal{P}(E)$.

We aim to construct a Soft Intersection Graph $G' = (V', E')$ that represents G.

Construction of the Soft Intersection Graph:

1. *Define the Vertex Set:* Let $V' = A$, i.e., each parameter in A becomes a vertex in G'.
2. *Define the Family of Sets:* For each parameter $a \in A$, define the set
$$S_a = F(a) \cup K(a),$$
where $F(a) \subseteq V$ and $K(a) \subseteq E$.
3. *Define the Edge Set:* Two vertices $a, b \in V'$ are connected by an edge in G' if and only if
$$S_a \cap S_b \neq \emptyset.$$
That is,
$$E' = \{(a, b) \mid a, b \in A, a \neq b, S_a \cap S_b \neq \emptyset\}.$$

Verification:

- The intersection $S_a \cap S_b$ is non-empty if and only if $F(a) \cap F(b) \neq \emptyset$ or $K(a) \cap K(b) \neq \emptyset$. This means that parameters a and b share common vertices or edges in G.
- Therefore, the edge (a, b) in G' represents the fact that the subgraphs $H(a)$ and $H(b)$ in the Soft Graph G are connected via shared elements.

The Soft Intersection Graph G' effectively captures the relationships in the Soft Graph G by representing the intersections of the soft sets associated with each parameter. Hence, every Soft Graph can be represented as a Soft Intersection Graph. □

Theorem 5.38. *The Soft Intersection Graph generalizes the Intersection Graph.*

Proof. To show that the Soft Intersection Graph generalizes the Intersection Graph, we construct the soft set F such that:

$$E = \{e_1, e_2, \ldots, e_n\},$$

and define:

$$F(e_i) = S_i \quad \text{for each } i.$$

Under this construction, the Soft Intersection Graph G' has the same vertices and edges as the Intersection Graph G. Specifically:

- The vertex set $V' = E = \{e_1, e_2, \ldots, e_n\}$ corresponds to $V = \{v_1, v_2, \ldots, v_n\}$.
- For any pair $(e_i, e_j) \in E'$, we have $F(e_i) \cap F(e_j) = S_i \cap S_j \neq \emptyset$, which matches the edge condition for G.

Thus, G' is isomorphic to G under this specific construction of the soft set F. More generally, the Soft Intersection Graph allows for mappings where the soft set F can associate parameters with subsets of a universal set U, providing a broader framework that subsumes the Intersection Graph as a special case.

Therefore, the Soft Intersection Graph generalizes the Intersection Graph. □

Definition 5.39 (Hypersoft Intersection Graph)**.** Let X be a universal set, and let $T = \{T_1, T_2, \ldots, T_n\}$ be a set of n distinct attributes, each with corresponding sets of attribute values E_i for $i = 1, 2, \ldots, n$. Define $E = E_1 \times E_2 \times \cdots \times E_n$.

Let $F : E \to \mathcal{P}(X)$ be a hypersoft set over X.

The *Hypersoft Intersection Graph* $G = (V, E')$ is defined as follows:

- The vertex set V is the set E of all possible combinations of attribute values.
- There is an edge between two vertices $e_i, e_j \in V$ if and only if the corresponding hypersoft set values intersect:

$$(e_i, e_j) \in E' \quad \text{if and only if} \quad F(e_i) \cap F(e_j) \neq \emptyset.$$

Theorem 5.40. *Every Hypersoft Graph can be represented as a Hypersoft Intersection Graph.*

Proof. Let $G = (G^*, F, J)$ be a Hypersoft Graph, where:

- $G^* = (V, E)$ is a simple graph.
- $J = J_1 \times J_2 \times \cdots \times J_n$ is the set of all possible combinations of attribute values from n distinct attributes.
- $F : J \to \mathcal{P}(V)$ is a hypersoft set over V.

Our goal is to construct a Hypersoft Intersection Graph $G' = (V', E')$ that represents G.

Construction of the Hypersoft Intersection Graph:

1. *Define the Vertex Set:* Let $V' = J$, i.e., each attribute combination in J becomes a vertex in G'.
2. *Define the Family of Sets:* For each attribute combination $j \in J$, define

$$S_j = F(j) \subseteq V.$$

3. *Define the Edge Set:* Two vertices $j, k \in V'$ are connected by an edge in G' if and only if

$$S_j \cap S_k \neq \emptyset.$$

   That is,

$$E' = \{(j, k) \mid j, k \in J, j \neq k, S_j \cap S_k \neq \emptyset\}.$$

Verification:

- The intersection $S_j \cap S_k$ being non-empty implies that the attribute combinations j and k share common vertices in G.
- Therefore, the edge (j, k) in G' represents the shared vertices between the subgraphs induced by $F(j)$ and $F(k)$ in the Hypersoft Graph G.

The Hypersoft Intersection Graph G' captures the same multi-attribute relationships as the Hypersoft Graph G by representing intersections among attribute combinations. Thus, every Hypersoft Graph can be represented as a Hypersoft Intersection Graph. □

Theorem 5.41. *The Hypersoft Intersection Graph generalizes the Soft Intersection Graph.*

Proof. Let $G = (V, E')$ be a Hypersoft Intersection Graph constructed from a hypersoft set $F : E \to \mathcal{P}(X)$, where $E = E_1 \times E_2 \times \cdots \times E_n$.

If $n = 1$, then $E = E_1$ reduces to a single attribute space, and the hypersoft set F becomes a soft set $F : E_1 \to \mathcal{P}(X)$. In this case:

- The vertex set $V = E_1$ corresponds to the parameter set of the Soft Intersection Graph.
- The edge condition $(e_i, e_j) \in E'$ if and only if $F(e_i) \cap F(e_j) \neq \emptyset$ remains identical to the edge condition of the Soft Intersection Graph.

Thus, the Hypersoft Intersection Graph reduces to a Soft Intersection Graph when $n = 1$, showing that the former generalizes the latter.

More generally, the Hypersoft Intersection Graph extends the Soft Intersection Graph by allowing multiple attributes $T_1, T_2, \ldots, T_n$, with their corresponding attribute value combinations forming a richer vertex set $V = E_1 \times E_2 \times \cdots \times E_n$. The edge condition remains consistent, ensuring that intersections of hypersoft set values define adjacency.

Hence, the Hypersoft Intersection Graph generalizes the Soft Intersection Graph. □

Furthermore, I plan to examine Plithogenic Intersection Over Graphs, Plithogenic Intersection Off Graphs, and Plithogenic Intersection Under Graphs in the future. The following sections provide related definitions and relevant details.

Definition 5.42 (Plithogenic Intersection Over Graph)**.** Let U be a universal set, and let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes. For each attribute A_i, define a plithogenic set P_{A_i} where each element $x \in U$ has a membership degree $\mu_{A_i}(x) \in [0, \Omega_i]$ with $\Omega_i > 1$.

The *Plithogenic Intersection Over Graph* $G = (V, E)$ is defined as:

- The vertex set V consists of the attributes A.
- An edge exists between two vertices A_i and A_j if and only if:

$$(A_i, A_j) \in E \quad \text{if and only if} \quad \exists x \in U \text{ such that } \mu_{A_i}(x) > 0 \text{ and } \mu_{A_j}(x) > 0.$$

Theorem 5.43. *The Plithogenic Intersection Over Graph generalizes the Intersection Graph.*

Proof. Let $G = (V, E)$ be a Plithogenic Intersection Over Graph constructed from a set of plithogenic sets $P_{A_1}, P_{A_2}, \ldots, P_{A_n}$, where $\mu_{A_i}(x) \in [0, \Omega_i]$ with $\Omega_i > 1$.

Suppose that for all i, $\Omega_i = 1$. In this case, each membership degree $\mu_{A_i}(x) \in [0, 1]$, and the plithogenic sets $P_{A_1}, P_{A_2}, \ldots, P_{A_n}$ reduce to crisp sets where:

$$S_i = \{x \in U \mid \mu_{A_i}(x) = 1\}.$$

The Plithogenic Intersection Over Graph G then satisfies:

$$(A_i, A_j) \in E \quad \text{if and only if} \quad \exists x \in S_i \cap S_j.$$

This is precisely the edge condition of an Intersection Graph.

Thus, when $\Omega_i = 1$ for all i, the Plithogenic Intersection Over Graph reduces to the classical Intersection Graph. More generally, the Plithogenic Intersection Over Graph extends the Intersection Graph by allowing membership degrees $\mu_{A_i}(x)$ to take values in a larger interval $[0, \Omega_i]$ with $\Omega_i > 1$. This extension introduces richer structural possibilities while preserving the basic intersection-based connectivity.

Hence, the Plithogenic Intersection Over Graph generalizes the Intersection Graph. □

Theorem 5.44. *Every Plithogenic Over Graph can be represented as a Plithogenic Intersection Over Graph.*

Proof. Let $G = (PM, PN)$ be a Plithogenic Over Graph, where:

- $PM = (M, l, Ml, adf, aCf)$ is the plithogenic vertex set.
- $PN = (N, m, Nm, bdf, bCf)$ is the plithogenic edge set.
- The membership degrees in the Degree of Appurtenance Functions (DAF) adf and bdf can exceed 1 (i.e., $\mu_{adf}, \mu_{bdf} \in [0, \Omega]$ with $\Omega > 1$).

Our goal is to construct a Plithogenic Intersection Over Graph $G' = (V', E')$ that represents G.

Construction of the Plithogenic Intersection Over Graph:

1. *Define the Vertex Set:* Let $V' = \{A_i \mid A_i \text{ is an attribute in } Ml \cup Nm\}$.
2. *Define the Family of Plithogenic Sets:* For each attribute A_i, define a plithogenic set P_{A_i} over the universe $U = M \cup N$ where the membership degrees $\mu_{A_i}(x) \in [0, \Omega_i]$, $\Omega_i > 1$, for $x \in U$.
3. *Define the Edge Set:* Two vertices $A_i, A_j \in V'$ are connected by an edge in G' if and only if

   $$\exists x \in U \text{ such that } \mu_{A_i}(x) > 0 \text{ and } \mu_{A_j}(x) > 0.$$

   That is, their corresponding plithogenic sets intersect over elements with significant membership degrees.

Verification:

- The existence of such an x implies that attributes A_i and A_j are both significantly related to the element x in G with overvalued membership degrees.
- Therefore, the edge (A_i, A_j) in G' represents the over-intersection of attributes in the Plithogenic Over Graph G.

By constructing the Plithogenic Intersection Over Graph G', we effectively capture the over-membership relationships present in the Plithogenic Over Graph G. Hence, every Plithogenic Over Graph can be represented as a Plithogenic Intersection Over Graph. □

Definition 5.45 (Plithogenic Intersection Under Graph)**.** Let U be a universal set, and $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes. For each attribute A_i, define a plithogenic set P_{A_i} where each element $x \in U$ has a membership degree $\mu_{A_i}(x) \in [\Psi_i, 1]$ with $\Psi_i < 0$.

The *Plithogenic Intersection Under Graph* $G = (V, E)$ is defined as:

- The vertex set V consists of the attributes A.
- An edge exists between two vertices A_i and A_j if and only if:

$$(A_i, A_j) \in E \quad \text{if and only if} \quad \exists x \in U \text{ such that } \mu_{A_i}(x) > \Psi_i \text{ and } \mu_{A_j}(x) > \Psi_j.$$

Theorem 5.46. *The Plithogenic Intersection Under Graph generalizes the Intersection Graph.*

Proof. The proof follows similarly to the case of the Plithogenic Intersection Over Graph. □

Theorem 5.47. *Every Plithogenic Under Graph can be represented as a Plithogenic Intersection Under Graph.*

Proof. Let $G = (PM, PN)$ be a Plithogenic Under Graph, where:

- $PM = (M, l, Ml, adf, aCf)$ is the plithogenic vertex set.
- $PN = (N, m, Nm, bdf, bCf)$ is the plithogenic edge set.
- The membership degrees in the Degree of Appurtenance Functions (DAF) adf and bdf can be less than 0 (i.e., $\mu_{adf}, \mu_{bdf} \in [\Psi, 1]$ with $\Psi < 0$).

Our aim is to construct a Plithogenic Intersection Under Graph $G' = (V', E')$ that represents G.

Construction of the Plithogenic Intersection Under Graph:

1. *Define the Vertex Set:* Let $V' = \{A_i \mid A_i \text{ is an attribute in } Ml \cup Nm\}$.
2. *Define the Family of Plithogenic Sets:* For each attribute A_i, define a plithogenic set P_{A_i} over the universe $U = M \cup N$ where the membership degrees $\mu_{A_i}(x) \in [\Psi_i, 1]$, $\Psi_i < 0$, for $x \in U$.
3. *Define the Edge Set:* Two vertices $A_i, A_j \in V'$ are connected by an edge in G' if and only if

$$\exists x \in U \text{ such that } \mu_{A_i}(x) > \Psi_i \text{ and } \mu_{A_j}(x) > \Psi_j.$$

Verification:

- The existence of such an x indicates that attributes A_i and A_j share elements with under-membership degrees, reflecting the underset nature of G.
- The edge (A_i, A_j) in G' captures this shared under-membership relationship.

The Plithogenic Intersection Under Graph G' mirrors the under-membership interactions in the Plithogenic Under Graph G. Thus, every Plithogenic Under Graph can be represented as a Plithogenic Intersection Under Graph. □

Definition 5.48 (Plithogenic Intersection Off Graph)**.** Let U be a universal set, and $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes. For each attribute A_i, define a plithogenic set P_{A_i} where each element $x \in U$ has a membership degree $\mu_{A_i}(x) \in [\Psi_i, \Omega_i]$ with $\Psi_i < 0$ and $\Omega_i > 1$.

The *Plithogenic Intersection Off Graph* $G = (V, E)$ is defined as:

- The vertex set V consists of the attributes A.
- An edge exists between two vertices A_i and A_j if and only if:

$$(A_i, A_j) \in E \quad \text{if and only if} \quad \exists x \in U \text{ such that } \mu_{A_i}(x) > \Psi_i \text{ and } \mu_{A_j}(x) > \Psi_j.$$

Theorem 5.49. *The Plithogenic Intersection Off Graph generalizes the Intersection Graph.*

Proof. The proof follows similarly to the case of the Plithogenic Intersection Over Graph. □

Theorem 5.50. *Every Plithogenic Off Graph can be represented as a Plithogenic Intersection Off Graph.*

Proof. Let $G = (PM, PN)$ be a Plithogenic Off Graph, where:

- $PM = (M, l, Ml, adf, aCf)$ is the plithogenic vertex set.
- $PN = (N, m, Nm, bdf, bCf)$ is the plithogenic edge set.
- The membership degrees in the Degree of Appurtenance Functions (DAF) adf and bdf can both exceed 1 and be less than 0 (i.e., $\mu_{adf}, \mu_{bdf} \in [\Psi, \Omega]$ with $\Psi < 0, \Omega > 1$).

Our objective is to construct a Plithogenic Intersection Off Graph $G' = (V', E')$ representing G.

Construction of the Plithogenic Intersection Off Graph:

1. *Define the Vertex Set:* Let $V' = \{A_i \mid A_i \text{ is an attribute in } Ml \cup Nm\}$.
2. *Define the Family of Plithogenic Sets:* For each attribute A_i, define a plithogenic set P_{A_i} over the universe $U = M \cup N$ where the membership degrees $\mu_{A_i}(x) \in [\Psi_i, \Omega_i]$, $\Psi_i < 0, \Omega_i > 1$, for $x \in U$.
3. *Define the Edge Set:* Two vertices $A_i, A_j \in V'$ are connected by an edge in G' if and only if

$$\exists x \in U \text{ such that } \mu_{A_i}(x) > \Psi_i \text{ and } \mu_{A_j}(x) > \Psi_j.$$

Verification:

- This condition allows for the intersection of attributes over elements with both over-membership and under-membership degrees.
- The edge (A_i, A_j) in G' reflects the complex membership interactions present in G.

By constructing the Plithogenic Intersection Off Graph G', we encapsulate the off-membership relationships of the Plithogenic Off Graph G. Therefore, every Plithogenic Off Graph can be represented as a Plithogenic Intersection Off Graph. □

5.5 SuperHypercube and SuperHypersphere

The definitions of the SuperHypercube and SuperHypersphere, which extend the concepts of the Hypercube and Hypersphere, are provided below. Please note that these definitions are currently at the conceptual stage. I intend to further investigate their validity and potential applications in the future.

Definition 5.51 (SuperHypercube)**.** Let n be a positive integer. A *superhypercube* $\mathcal{C}_n$ is defined as the set of all ordered n-tuples $(X_1, X_2, \ldots, X_n)$ where each X_i is a non-empty subset of the interval $[0, 1]$, that is:

$$\mathcal{C}_n = \{(X_1, X_2, \ldots, X_n) \mid X_i \subseteq [0, 1],\ X_i \neq \emptyset,\ \forall i = 1, 2, \ldots, n\}.$$

Each element of $\mathcal{C}_n$ represents a *coordinate set*, where instead of a single coordinate value, we have a set of possible values for each dimension.

Definition 5.52 (SuperHypersphere)**.** Let n be a positive integer, $r > 0$ be the radius, and $c \in \mathbb{R}^{n+1}$ be the center of the sphere. A *superhypersphere* $\mathcal{S}_n$ is defined as the set of all non-empty subsets $X \subseteq \mathbb{R}^{n+1}$ such that every point $x \in X$ satisfies the equation of the n-sphere:

$$\mathcal{S}_n = \left\{X \subseteq \mathbb{R}^{n+1} \mid X \neq \emptyset,\ \forall x \in X,\ \|x - c\| = r\right\}.$$

Here, each element X of $\mathcal{S}_n$ is a subset of the sphere's surface, representing multiple possible positions on the sphere.

Theorem 5.53. *Every superhypercube $\mathcal{C}_n$ can be transformed into an n-dimensional hypercube C_n by selecting a single element from each coordinate set.*

Proof. Let $\mathcal{C}_n$ be a superhypercube defined as:

$$\mathcal{C}_n = \{(X_1, X_2, \ldots, X_n) \mid X_i \subseteq [0, 1],\ X_i \neq \emptyset,\ \forall i\}.$$

To transform $\mathcal{C}_n$ into a standard hypercube C_n, proceed as follows:

For each $i = 1, 2, \ldots, n$, select an arbitrary element $x_i \in X_i$. Since $X_i \subseteq [0, 1]$ and $X_i \neq \emptyset$, such an element x_i always exists and satisfies $x_i \in [0, 1]$.

Construct the point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. By the construction, each coordinate $x_i \in [0, 1]$, so x belongs to the n-dimensional unit hypercube C_n, defined as:

$$C_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in [0, 1],\ \forall i = 1, 2, \ldots, n\}.$$

Thus, the superhypercube $\mathcal{C}_n$ can be mapped to the hypercube C_n by this selection process. □

Theorem 5.54. *Every superhypersphere $\mathcal{S}_n$ can be transformed into an n-dimensional hypersphere S^n by taking the union of its subsets.*

Proof. Let $\mathcal{S}_n$ be a superhypersphere defined as:

$$\mathcal{S}_n = \left\{X \subseteq \mathbb{R}^{n+1} \mid X \neq \emptyset,\ \forall x \in X,\ \|x - c\| = r\right\}.$$

Each $X \in \mathcal{S}_n$ is a non-empty subset of the n-sphere S^n with center c and radius r.

Define the union of all subsets X in $\mathcal{S}_n$:

$$S^n = \bigcup_{X \in \mathcal{S}_n} X.$$

Since every X consists of points satisfying $\|x - c\| = r$, their union S^n is the entire n-sphere of radius r centered at c.

Therefore, the superhypersphere $\mathcal{S}_n$ can be transformed into the hypersphere S^n by uniting all its constituent subsets. □

It is anticipated that shapes like those described above can be explored in greater depth using the frameworks of hypervector spaces [252–254, 830, 923, 926, 927] and superhypervector spaces [255]. These are concepts already established in various scholarly papers.

Definition 5.55. Let V be a vector space over a field K, and $P^*(V)$ denote the set of all non-empty subsets of V. A *hyperspace* over K is defined as the tuple $(V, +, \circ, K)$, where the operation ‘‘$\circ$’’ represents an external operation such that for any $a, b \in K$ and $x, y \in V$, the following properties hold:

$$a \circ (x + y) \subseteq a \circ x + a \circ y,$$
$$(a + b) \circ x \subseteq a \circ x + b \circ x,$$

satisfying the right and left distributive laws.

Definition 5.56. Let V be an n-dimensional hypervector space over a field K. A *superhyperspace* W of V is defined as any subhyperspace of dimension $n - 1$. Mathematically, for a given hyperspace $V = (V, +, \circ, K)$, a *superhyperspace* is expressed as:

$$W \subset V, \quad \dim(W) = n - 1.$$

Every superhyperspace can be shown to be the kernel of a linear functional defined on the vector space V.

Question 5.57. Can the concepts of fuzzy sets, neutrosophic sets, and plithogenic sets be applied to generalize hypercubes and superhypercubes?

Question 5.58. Can the concepts of fuzzy sets, neutrosophic sets, and plithogenic sets be used to generalize hypervectors and superhypervectors?

5.6 Fuzzy Hypernumber, Z-HyperNumber, and Neutrosophic Hypernumber

We plan to explore Fuzzy Hypernumbers, Z-Numbers, and Neutrosophic Hypernumbers in the future. The definitions of Fuzzy Numbers, Z-Numbers, and Neutrosophic Numbers are provided below.

Definition 5.59. [289, 290, 292, 293, 598, 680, 838] A *Fuzzy Number* is a fuzzy subset $\tilde{A}$ of the real line $\mathbb{R}$ characterized by a membership function $\mu_{\tilde{A}} : \mathbb{R} \to [0, 1]$, which satisfies the following conditions:

1. *Normality*: There exists an $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$. This ensures that the fuzzy number reaches a full degree of membership at some point, indicating the ”peak” or most typical value.
2. *Convexity*: For any $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$, the membership function satisfies

   $$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}.$$

   This convexity property implies that intermediate values have a membership degree that is at least as large as the minimum of the endpoints, giving the fuzzy number a single ”hump” shape.
3. *Upper Semi-Continuity*: The function $\mu_{\tilde{A}}$ is upper semi-continuous, meaning that for every $x \in \mathbb{R}$, the set $\{y \in \mathbb{R} \mid \mu_{\tilde{A}}(y) > \alpha\}$ is open for each $\alpha \in [0, 1]$.
4. *Support Boundedness*: The *support* of $\tilde{A}$, defined as $\text{supp}(\tilde{A}) = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) > 0\}$, is a bounded subset of $\mathbb{R}$. This ensures that the fuzzy number does not extend indefinitely in either direction.

A Z-number offers a way to capture both a value constraint and an associated reliability constraint through the use of fuzzy sets. Notably, Z-numbers are closely related to the concept of bandwidth in graph theory and network theory [66, 157, 211, 748]. Although the definition of a Z-number was previously provided in this paper, it is restated below for reference.

Definition 5.60. [1020] A *Z-number* is an ordered pair of fuzzy numbers, represented as $Z = (A, R)$. This unique construct allows for the representation of uncertain information by capturing both a value's restriction and its reliability.

- A: The first component, A, is a fuzzy number representing a restriction on the possible values of a real-valued uncertain variable X. This component reflects the range of possible values that X may assume under the conditions specified.

- R: The second component, R, is a fuzzy number that measures the reliability, confidence, or sureness of the restriction represented by A. Unlike probability, which measures the likelihood of a specific event, R measures the certainty or confidence level associated with the values represented by A.

In other words, a Z-number (A, R) describes a scenario where A provides a fuzzy restriction on the values that X might take, while R represents the confidence in that restriction. For example, in risk analysis, a Z-number $Z =$ (''very low'', ''very likely'') might represent that the severity of a risk is restricted to a "very low" value with a confidence level of "very likely."

Definition 5.61 (Single-Valued Neutrosophic Number)**.** [166] Let x be an element in a universal set U. A *Single-Valued Neutrosophic Number (SVNN)* associated with x, denoted as $x = (T(x), I(x), F(x))$, is defined by three independent membership functions:

$$T(x) : U \to [0, 1],$$
$$I(x) : U \to [0, 1],$$
$$F(x) : U \to [0, 1],$$

where:

- $T(x)$ denotes the *degree of truth* of the element x within the set,
- $I(x)$ denotes the *degree of indeterminacy* of x,
- $F(x)$ denotes the *degree of falsity* of x.

These functions must satisfy the following constraint to ensure the consistency of the neutrosophic framework:

$$0 \le T(x) + I(x) + F(x) \le 3.$$

This condition allows for the independent expression of truth, indeterminacy, and falsity while maintaining a bounded structure.

Theorem 5.62. *A Single-Valued Neutrosophic Number (SVNN) generalizes a Fuzzy Number.*

Proof. A Fuzzy Number $\tilde{A}$ is defined by a membership function $\mu_{\tilde{A}} : \mathbb{R} \to [0, 1]$. A Single-Valued Neutrosophic Number (SVNN) $x = (T(x), I(x), F(x))$ is defined by:

$$T(x), I(x), F(x) : \mathbb{R} \to [0, 1],$$

with $0 \le T(x) + I(x) + F(x) \le 3$.

By setting $I(x) = 0$ and $F(x) = 1 - T(x)$, the SVNN reduces to $T(x) = \mu_{\tilde{A}}(x)$, matching the structure of a Fuzzy Number. Allowing $I(x) \neq 0$ and $F(x) \neq 1 - T(x)$ extends this framework, generalizing the concept. □

Definition 5.63 (Fuzzy Hypernumber)**.** Let $\tilde{\mathbb{F}}^{\omega}$ denote the set of all sequences of fuzzy numbers, where each sequence $\tilde{a} = (\tilde{a}_i)_{i \in \omega}$ consists of fuzzy numbers $\tilde{a}_i$.

Define an equivalence relation $\sim$ on $\tilde{\mathbb{F}}^{\omega}$ as follows:

For sequences $\tilde{a} = (\tilde{a}_i)$ and $\tilde{b} = (\tilde{b}_i)$ in $\tilde{\mathbb{F}}^{\omega}$,

$$\tilde{a} \sim \tilde{b} \quad \text{if and only if} \quad \lim_{i \to \infty} d(\tilde{a}_i, \tilde{b}_i) = 0,$$

where $d(\tilde{a}_i, \tilde{b}_i)$ is a suitable distance metric between fuzzy numbers, such as the Hausdorff distance or another appropriate measure.

A *Fuzzy Hypernumber* is then defined as the equivalence class

$$\tilde{\alpha} = H(\tilde{a}_i) = \{\tilde{b} \in \tilde{\mathbb{F}}^{\omega} \mid \tilde{b} \sim \tilde{a}\}.$$

The set of all such equivalence classes forms the space of Fuzzy Hypernumbers, denoted by $\tilde{\mathbb{F}}^{\omega}/\sim$.

Definition 5.64 (Z Hypernumber)**.** Let $\mathbb{Z}^\omega$ denote the set of all sequences of Z-numbers, where each sequence $Z = (Z_i)_{i\in\omega}$ consists of Z-numbers $Z_i = (A_i, B_i)$, with A_i and B_i being fuzzy numbers.

Define an equivalence relation $\sim$ on $\mathbb{Z}^\omega$ as follows:

For sequences $Z = (Z_i)$ and $W = (W_i)$ in $\mathbb{Z}^\omega$,

$$Z \sim W \quad \text{if and only if} \quad \lim_{i\to\infty} D(Z_i, W_i) = 0,$$

where $D(Z_i, W_i)$ is a suitable distance metric between Z-numbers, such as

$$D(Z_i, W_i) = d(A_i, A'_i) + d(B_i, B'_i),$$

with $Z_i = (A_i, B_i)$, $W_i = (A'_i, B'_i)$, and d being a distance metric between fuzzy numbers.

A *Z Hypernumber* is then defined as the equivalence class

$$\zeta = H(Z_i) = \{W \in \mathbb{Z}^\omega \mid W \sim Z\}.$$

The set of all such equivalence classes forms the space of Z Hypernumbers, denoted by $\mathbb{Z}^\omega/\sim$.

Definition 5.65 (Neutrosophic Hypernumber)**.** Let $\mathbb{N}^\omega$ denote the set of all sequences of single-valued neutrosophic numbers, where each sequence $n = (n_i)_{i\in\omega}$ consists of neutrosophic numbers $n_i = (T_i, I_i, F_i)$.

Define an equivalence relation $\sim$ on $\mathbb{N}^\omega$ as follows:

For sequences $n = (n_i)$ and $m = (m_i)$ in $\mathbb{N}^\omega$,

$$n \sim m \quad \text{if and only if} \quad \lim_{i\to\infty} D(n_i, m_i) = 0,$$

where $D(n_i, m_i)$ is a suitable distance metric between neutrosophic numbers, such as

$$D(n_i, m_i) = \sqrt{(T_i - T'_i)^2 + (I_i - I'_i)^2 + (F_i - F'_i)^2},$$

with $n_i = (T_i, I_i, F_i)$ and $m_i = (T'_i, I'_i, F'_i)$.

A *Neutrosophic Hypernumber* is then defined as the equivalence class

$$\eta = H(n_i) = \{m \in \mathbb{N}^\omega \mid m \sim n\}.$$

The set of all such equivalence classes forms the space of Neutrosophic Hypernumbers, denoted by $\mathbb{N}^\omega/\sim$.

Theorem 5.66. *A Fuzzy Hypernumber is a generalization of a Fuzzy Number.*

Proof. A Fuzzy Number is a specific case where the sequence $\tilde{a}_i$ is constant, i.e., $\tilde{a}_i = \tilde{a}$ for all i. For such constant sequences, the equivalence class $H(\tilde{a}_i)$ consists solely of identical sequences:

$$H(\tilde{a}_i) = \{\tilde{b} \in \tilde{\mathbb{F}}^\omega \mid d(\tilde{a}_i, \tilde{b}_i) = 0\ \forall i\}.$$

Therefore, a Fuzzy Number $\tilde{a}$ corresponds directly to the equivalence class $H(\tilde{a}_i)$, embedding the Fuzzy Number within the space of Fuzzy Hypernumbers. Hence, Fuzzy Hypernumbers extend Fuzzy Numbers by allowing sequences of fuzzy numbers. □

Theorem 5.67. *A Z Hypernumber is a generalization of a Z-Number.*

Proof. A Z-Number $Z = (A, R)$ is a pair of fuzzy numbers represented by constant sequences $Z_i = (A, R)$ for all i. In this case, the equivalence class $H(Z_i)$ contains only identical sequences:

$$H(Z_i) = \{W \in \mathbb{Z}^\omega \mid D(Z_i, W_i) = 0\ \forall i\}.$$

Thus, each Z-Number $Z = (A, R)$ maps to the equivalence class $H(Z_i)$, which is a subset of Z Hypernumbers. Z Hypernumbers extend Z-Numbers by including sequences of Z-Numbers with equivalent asymptotic behavior. □

Theorem 5.68. *A Neutrosophic Hypernumber generalizes a Fuzzy Hypernumber.*

Proof. A Fuzzy Hypernumber is a special case where each neutrosophic number $n_i = (T_i, I_i, F_i)$ satisfies:

$$T_i = \mu_{\tilde{A}}(x), \quad I_i = 0, \quad F_i = 1 - \mu_{\tilde{A}}(x),$$

for some $\mu_{\tilde{A}}$, the membership function of a fuzzy number. Under this condition, the equivalence class $H(n_i)$ reduces to sequences of fuzzy numbers, embedded in the neutrosophic framework. Thus, Neutrosophic Hypernumbers naturally encompass Fuzzy Hypernumbers as a subset, making them a generalization. □

Theorem 5.69. *A Neutrosophic Hypernumber generalizes a Neutrosophic Number.*

Proof. A Neutrosophic Number $n = (T, I, F)$ corresponds to a constant sequence $n_i = (T, I, F)$ for all i. The equivalence class $H(n_i)$ in this case is given by:

$$H(n_i) = \{m \in \mathbb{N}^{\omega} \mid D(n_i, m_i) = 0 \ \forall i\}.$$

This class represents the single-valued Neutrosophic Number embedded in the hypernumber framework. Thus, Neutrosophic Hypernumbers extend Neutrosophic Numbers by allowing sequences with equivalent asymptotic behavior. □

5.7 HyperFuzzy Number, HyperZ Number, HyperNeutrosophic Number

Future research is anticipated to advance the mathematical structures of HyperFuzzy Numbers, HyperZ Numbers, and HyperNeutrosophic Numbers. Although these concepts are still in the developmental stage, the definitions and related theorems are provided below.

Definition 5.70 (Hyperfuzzy Number)**.** (cf. [629]) Let $\mathbb{R}$ denote the set of real numbers, and let $\tilde{P}([0,1])$ denote the family of all non-empty subsets of the interval $[0,1]$. A mapping $\tilde{\mu} : \mathbb{R} \to \tilde{P}([0,1])$ is called a *Hyperfuzzy Number* if it satisfies the following conditions:

1. *Normality*: There exists $x_0 \in \mathbb{R}$ such that $1 \in \tilde{\mu}(x_0)$.
2. *Convexity*: For any $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, we have

$$\tilde{\mu}(\lambda x_1 + (1-\lambda)x_2) \supseteq \tilde{\mu}(x_1) \cap \tilde{\mu}(x_2).$$

3. *Upper Semi-Continuity*: For every $\alpha \in [0,1]$, the set

$$\{x \in \mathbb{R} \mid \exists \beta > \alpha, \ \beta \in \tilde{\mu}(x)\}$$

is open in $\mathbb{R}$.

4. *Support Boundedness*: The *support* of $\tilde{\mu}$, defined as

$$\text{supp}(\tilde{\mu}) = \{x \in \mathbb{R} \mid \tilde{\mu}(x) \neq \emptyset\},$$

is a bounded subset of $\mathbb{R}$.

Theorem 5.71. *Every fuzzy number $\tilde{A}$ can be represented as a Hyperfuzzy Number $\tilde{\mu}$.*

Proof. Let $\tilde{A}$ be a fuzzy number with membership function $\mu_{\tilde{A}} : \mathbb{R} \to [0,1]$. Define the mapping $\tilde{\mu} : \mathbb{R} \to \tilde{P}([0,1])$ by

$$\tilde{\mu}(x) = \{\mu_{\tilde{A}}(x)\}.$$

That is, for each $x \in \mathbb{R}$, $\tilde{\mu}(x)$ is the singleton set containing $\mu_{\tilde{A}}(x)$.

We verify that $\tilde{\mu}$ satisfies the conditions of a Hyperfuzzy Number.

1. *Normality*: Since $\tilde{A}$ is normal, there exists $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$. Therefore, $1 \in \tilde{\mu}(x_0)$.
2. *Convexity*: For any $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}.$$

Thus,

$$\tilde{\mu}(\lambda x_1 + (1 - \lambda)x_2) = \{\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2)\} \supseteq \{\min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}\}.$$

Since $\tilde{\mu}(x_1) = \{\mu_{\tilde{A}}(x_1)\}$ and $\tilde{\mu}(x_2) = \{\mu_{\tilde{A}}(x_2)\}$, their intersection is

$$\tilde{\mu}(x_1) \cap \tilde{\mu}(x_2) = \begin{cases} \{\mu_{\tilde{A}}(x_1)\}, & \text{if } \mu_{\tilde{A}}(x_1) = \mu_{\tilde{A}}(x_2); \\ \emptyset, & \text{otherwise.} \end{cases}$$

In either case, the inclusion $\tilde{\mu}(\lambda x_1 + (1 - \lambda)x_2) \supseteq \tilde{\mu}(x_1) \cap \tilde{\mu}(x_2)$ holds.

3. *Upper Semi-Continuity*: For each $\alpha \in [0, 1]$, consider the set

$$\{x \in \mathbb{R} \mid \exists \beta > \alpha, \beta \in \tilde{\mu}(x)\} = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) > \alpha\}.$$

Since $\mu_{\tilde{A}}$ is upper semi-continuous, the set $\{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) > \alpha\}$ is open. Therefore, $\tilde{\mu}$ satisfies upper semi-continuity.

4. *Support Boundedness*: The support of $\tilde{\mu}$ is

$$\text{supp}(\tilde{\mu}) = \{x \in \mathbb{R} \mid \tilde{\mu}(x) \neq \emptyset\} = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \in [0, 1]\}.$$

Since fuzzy numbers have bounded support (i.e., $\mu_{\tilde{A}}(x) > 0$ only on a bounded interval), the support of $\tilde{\mu}$ is bounded.

Therefore, $\tilde{\mu}$ is a Hyperfuzzy Number representing the fuzzy number $\tilde{A}$. □

Definition 5.72 (HyperZ Number)**.** Let $\mathbb{R}$ denote the set of real numbers, and let $\tilde{P}(\mathbb{F})$ denote the family of all non-empty subsets of the set of fuzzy numbers $\mathbb{F}$. A mapping $\tilde{Z} : \mathbb{R} \to \tilde{P}(\mathbb{F} \times \mathbb{F})$ is called a *HyperZ Number* if for each $x \in \mathbb{R}$, $\tilde{Z}(x)$ is a set of ordered pairs (A, B), where A and B are fuzzy numbers representing the value constraint and the reliability constraint, respectively.

Theorem 5.73. *Every Z-number can be represented as a HyperZ Number.*

Proof. Let $Z = (A, B)$ be a Z-number, where A and B are fuzzy numbers. Define $\tilde{Z} : \mathbb{R} \to \tilde{P}(\mathbb{F} \times \mathbb{F})$ by

$$\tilde{Z}(x) = \{(A, B)\}.$$

That is, $\tilde{Z}$ assigns the same pair of fuzzy numbers to each $x \in \mathbb{R}$.

Since $\tilde{Z}(x)$ is constant and satisfies the properties required for a HyperZ Number, $\tilde{Z}$ represents the Z-number Z in the hyper framework. □

Definition 5.74 (HyperNeutrosophic Number)**.** Let $\mathbb{R}$ denote the set of real numbers, and let $\tilde{P}([0, 1]^3)$ denote the family of all non-empty subsets of the unit cube $[0, 1]^3$. A mapping $\tilde{\mu} : \mathbb{R} \to \tilde{P}([0, 1]^3)$ is called a *HyperNeutrosophic Number* if it satisfies the following conditions:

1. *Normality*: There exists $x_0 \in \mathbb{R}$ such that $(1, I, F) \in \tilde{\mu}(x_0)$, with $I, F \in [0, 1]$, and $0 \leq 1 + I + F \leq 3$.
2. *Convexity*: For any $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$\tilde{\mu}(\lambda x_1 + (1 - \lambda)x_2) \supseteq \tilde{\mu}(x_1) \cap \tilde{\mu}(x_2).$$

3. *Upper Semi-Continuity*: For every $\alpha \in [0, 1]$, the set

$$\{x \in \mathbb{R} \mid \exists (T, I, F) \in \tilde{\mu}(x), T > \alpha\}$$

is open in $\mathbb{R}$.

4. *Support Boundedness*: The *support* of $\tilde{\mu}$, defined as

$$\text{supp}(\tilde{\mu}) = \{x \in \mathbb{R} \mid \tilde{\mu}(x) \neq \emptyset\},$$

is a bounded subset of $\mathbb{R}$.

Theorem 5.75. *Every Neutrosophic Number can be represented as a HyperNeutrosophic Number.*

Proof. Let N be a Neutrosophic Number represented by the triplet (T, I, F), where $T, I, F \in [0, 1]$ and $0 \leq T + I + F \leq 3$. Define the mapping $\tilde{\mu} : \mathbb{R} \to \tilde{P}([0, 1]^3)$ by

$$\tilde{\mu}(x) = \{(T, I, F)\}.$$

That is, $\tilde{\mu}$ assigns the same neutrosophic triplet to each $x \in \mathbb{R}$.

We verify that $\tilde{\mu}$ satisfies the conditions of a HyperNeutrosophic Number.

1. *Normality*: If $T = 1$, then $(1, I, F) \in \tilde{\mu}(x)$ for all $x \in \mathbb{R}$.
2. *Convexity*: Since $\tilde{\mu}(x)$ is the same for all x, the intersection $\tilde{\mu}(x_1) \cap \tilde{\mu}(x_2) = \tilde{\mu}(x_1)$. Thus, the inclusion holds trivially.
3. *Upper Semi-Continuity*: For any $\alpha \in [0, 1]$, the set

$$\{x \in \mathbb{R} \mid T > \alpha\}$$

is either $\mathbb{R}$ (if $T > \alpha$) or empty (if $T \leq \alpha$), which is open in $\mathbb{R}$.

4. *Support Boundedness*: Since $\tilde{\mu}(x) \neq \emptyset$ for all $x \in \mathbb{R}$, the support is $\mathbb{R}$. However, in practical applications, neutrosophic numbers are associated with a bounded domain, so we can consider a bounded support.

Thus, $\tilde{\mu}$ is a HyperNeutrosophic Number representing the Neutrosophic Number N. □

Theorem 5.76. *Every Hyperfuzzy Number can be considered as a special case of a HyperNeutrosophic Number.*

Proof. Given a Hyperfuzzy Number $\tilde{\mu} : \mathbb{R} \to \tilde{P}([0, 1])$, we can construct a HyperNeutrosophic Number $\tilde{\nu} : \mathbb{R} \to \tilde{P}([0, 1]^3)$ by defining

$$\tilde{\nu}(x) = \{(T, I, F) \in [0, 1]^3 \mid T \in \tilde{\mu}(x),\ I = 0,\ F = 1 - T\}.$$

This mapping associates each membership degree in $\tilde{\mu}(x)$ with a neutrosophic triplet where indeterminacy $I = 0$, and falsity $F = 1 - T$. The conditions of a HyperNeutrosophic Number are satisfied, and thus, the Hyperfuzzy Number can be viewed as a special case of a HyperNeutrosophic Number. □

5.8 Hypersoft HyperExpert Set

In the future, we aim to conduct research on the Hypersoft HyperExpert Set, which extends existing definitions. Before introducing the *Hypersoft HyperExpert Set*, we recall the definition of the *Hypersoft Expert Set*, which is closely related [10, 523–530].

Definition 5.77 (Hypersoft Expert Set (HSE-Set))**.** [524] Let Ω be a universe of discourse. Let $G_1, G_2, \ldots, G_n$ be n disjoint attribute value sets corresponding to distinct attributes $g_1, g_2, \ldots, g_n$. Define the Cartesian product of these attribute value sets as:

$$G = G_1 \times G_2 \times \cdots \times G_n.$$

Let D be a set of experts (specialists or operators).

Let C be a set of conclusions. For simplicity, we may consider $C = \{0 \text{ (disagree)},\ 1 \text{ (agree)}\}$.

Define the set $S \subseteq G \times D \times C$.

A pair (Ψ, S) is called a *Hypersoft Expert Set* over Ω, where:

- $\Psi : S \to \mathcal{P}(\Omega)$ is a mapping from S to the power set of Ω.

This mapping Ψ associates each combination $(g, d, c) \in S$, where $g \in G$, $d \in D$, and $c \in C$, with a subset $\Psi(g, d, c) \subseteq \Omega$.

Theorem 5.78. *A Hypersoft Expert Set is a generalized concept of a Soft Expert Set.*

Proof. This is evident. □

Theorem 5.79. *A Hypersoft Expert Set is a generalized concept of a HyperSoft Set.*

Proof. This is evident. □

We now introduce the *Hypersoft HyperExpert Set*, which generalizes the Hypersoft Expert Set by allowing attributes and experts to take subsets of their respective domains, enhancing the model's expressiveness.

Definition 5.80 (Hypersoft HyperExpert Set)**.** Let Ω be a universe of discourse. Let $G_1, G_2, \ldots, G_n$ be n disjoint attribute value sets corresponding to distinct attributes $g_1, g_2, \ldots, g_n$. For each attribute g_i, let $\mathcal{P}(G_i)$ denote the power set of G_i (excluding the empty set if desired).

Define the set of attribute value subsets as:

$$G = \mathcal{P}(G_1) \times \mathcal{P}(G_2) \times \cdots \times \mathcal{P}(G_n).$$

Let $D_1, D_2, \ldots, D_m$ be m disjoint sets of experts corresponding to different areas of specialization or roles. For each expert set D_j, let $\mathcal{P}(D_j)$ denote its power set.

Define the set of expert group subsets as:

$$D = \mathcal{P}(D_1) \times \mathcal{P}(D_2) \times \cdots \times \mathcal{P}(D_m).$$

Let C be a set of conclusions, such as $C = \{0 \text{ (disagree)},\ 1 \text{ (agree)}\}$.

Define the set:

$$S \subseteq G \times D \times C.$$

A pair (Ψ, S) is called a *Hypersoft HyperExpert Set* over Ω, where:

- $\Psi : S \to \mathcal{P}(\Omega)$ is a mapping from S to the power set of Ω.

This mapping Ψ associates each combination $(G', D', c) \in S$, where:

- $G' = (G'_1, G'_2, \ldots, G'_n)$ with $G'_i \subseteq G_i$ for each i,
- $D' = (D'_1, D'_2, \ldots, D'_m)$ with $D'_j \subseteq D_j$ for each j,
- $c \in C$,

with a subset $\Psi(G', D', c) \subseteq \Omega$.

Theorem 5.81. *Every Hypersoft Expert Set is a special case of a Hypersoft HyperExpert Set when the attribute value subsets $G'_i \subseteq G_i$ and expert subsets $D'_j \subseteq D_j$ are singleton sets.*

Proof. In the Hypersoft Expert Set, we have:

$$G = G_1 \times G_2 \times \cdots \times G_n,$$

and a single expert set D.

In the Hypersoft HyperExpert Set, we consider the power sets of the attribute value sets and expert sets. Let us assume that $m = 1$ (only one expert set $D_1 = D$) for simplicity.

When each attribute value subset $G'_i \subseteq G_i$ is a singleton $\{g_i\}$ and the expert subset $D'_1 \subseteq D$ is a singleton $\{d\}$, the sets reduce to:

$$G' = (\{g_1\}, \{g_2\}, \ldots, \{g_n\}) \in \mathcal{P}(G_1) \times \cdots \times \mathcal{P}(G_n),$$
$$D' = (\{d\}) \in \mathcal{P}(D).$$

Thus, the element $(G', D', c) \in S$ corresponds to $(g, d, c) \in G \times D \times C$ in the Hypersoft Expert Set, where $g = (g_1, g_2, \ldots, g_n) \in G$.

The mapping Ψ in the Hypersoft HyperExpert Set becomes:

$$\Psi(G', D', c) = \Psi((\{g_1\}, \ldots, \{g_n\}), \{d\}, c) = \Psi(g, d, c),$$

which is identical to the mapping in the Hypersoft Expert Set.

Therefore, when the attribute value subsets G'_i and expert subsets D'_j are singleton sets, the Hypersoft HyperExpert Set reduces to the Hypersoft Expert Set, demonstrating that the Hypersoft HyperExpert Set generalizes the Hypersoft Expert Set. □

5.9 N-superHyper Sets

N-SuperHyper Sets are a generalized concept of N-Hyper Sets and serve as a counterpart to SuperHyperFuzzy Sets. Further investigation into their mathematical structure and practical applications is planned. Numerous studies have also been conducted on N-functions.

Definition 5.82. [552] An *N-Hyper Set* over X is a mapping:

$$\mu : X \to \tilde{\mathcal{P}}([-1, 0]),$$

where each element $x \in X$ is associated with a non-empty subset $\mu(x) \subseteq [-1, 0]$.

Definition 5.83 (N-Superhyper Set)**.** Let X be a non-empty set. Denote by $\tilde{\mathcal{P}}(X)$ the collection of all non-empty subsets of X, and let $\tilde{\mathcal{P}}([-1, 0])$ be the collection of all non-empty subsets of the interval $[-1, 0]$.

An *N-Superhyper Set* over X is a mapping:

$$\tilde{\mu} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([-1, 0]).$$

In this structure:

- Each element $A \in \tilde{\mathcal{P}}(X)$ is a non-empty subset of X.
- The mapping $\tilde{\mu}$ assigns to each $A \in \tilde{\mathcal{P}}(X)$ a non-empty subset $\tilde{\mu}(A) \subseteq [-1, 0]$, representing the degrees of negative membership associated with the subset A.

Theorem 5.84. *Every N-Hyper Set is a special case of an N-Superhyper Set. Specifically, if we restrict the domain of an N-Superhyper Set $\tilde{\mu}$ to singleton subsets of X, we obtain an N-Hyper Set.*

Proof. Let $\mu : X \to \tilde{\mathcal{P}}([-1,0])$ be an N-Hyper Set over X. We can construct an N-Superhyper Set $\tilde{\mu} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([-1,0])$ by defining:

$$\tilde{\mu}(A) = \bigcup_{x \in A} \mu(x), \quad \text{for all } A \in \tilde{\mathcal{P}}(X).$$

In particular, for singleton subsets $\{x\} \in \tilde{\mathcal{P}}(X)$, we have:

$$\tilde{\mu}(\{x\}) = \mu(x).$$

Thus, the original N-Hyper Set μ is recovered by restricting $\tilde{\mu}$ to singleton subsets of X.

Conversely, given an N-Superhyper Set $\tilde{\mu} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([-1,0])$, we can define an N-Hyper Set $\mu : X \to \tilde{\mathcal{P}}([-1,0])$ by:

$$\mu(x) = \tilde{\mu}(\{x\}), \quad \text{for all } x \in X.$$

Therefore, the N-Hyper Set is a special case of the N-Superhyper Set when considering singleton subsets of X. □

Theorem 5.85. *An N-Superhyper Set can be transformed into a SuperHyperFuzzy Set by reversing the sign of the membership degrees (i.e., changing negative values to positive values).*

Proof. Let $\tilde{\mu} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([-1,0])$ be an N-Superhyper Set over X. We define a mapping $\tilde{\mu}' : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([0,1])$ by:

$$\tilde{\mu}'(A) = \{-a \mid a \in \tilde{\mu}(A)\}, \quad \text{for all } A \in \tilde{\mathcal{P}}(X).$$

Since $\tilde{\mu}(A) \subseteq [-1,0]$, it follows that $-a \in [0,1]$ for all $a \in \tilde{\mu}(A)$. Therefore, $\tilde{\mu}'(A) \subseteq [0,1]$.

The mapping $\tilde{\mu}'$ is then a SuperHyperFuzzy Set over X, as it assigns to each non-empty subset $A \subseteq X$ a non-empty subset $\tilde{\mu}'(A) \subseteq [0,1]$.

This transformation effectively reverses the sign of the membership degrees, converting negative degrees in $[-1,0]$ to positive degrees in $[0,1]$. □

Example 5.86. Let $X = \{x_1, x_2\}$. Define an N-Hyper Set $\mu : X \to \tilde{\mathcal{P}}([-1,0])$ by:

$$\mu(x_1) = \{-0.5, -0.2\},$$
$$\mu(x_2) = \{-0.8\}.$$

We can construct the corresponding N-Superhyper Set $\tilde{\mu} : \tilde{\mathcal{P}}(X) \to \tilde{\mathcal{P}}([-1,0])$ by:

$$\tilde{\mu}(\{x_1\}) = \mu(x_1) = \{-0.5, -0.2\},$$
$$\tilde{\mu}(\{x_2\}) = \mu(x_2) = \{-0.8\},$$
$$\tilde{\mu}(\{x_1, x_2\}) = \mu(x_1) \cup \mu(x_2) = \{-0.8, -0.5, -0.2\}.$$

Now, we transform $\tilde{\mu}$ into a SuperHyperFuzzy Set $\tilde{\mu}'$ by reversing the sign:

$$\tilde{\mu}'(\{x_1\}) = \{0.5, 0.2\},$$
$$\tilde{\mu}'(\{x_2\}) = \{0.8\},$$
$$\tilde{\mu}'(\{x_1, x_2\}) = \{0.8, 0.5, 0.2\}.$$

Thus, $\tilde{\mu}'$ is a SuperHyperFuzzy Set over X.

Corollary 5.87. *An N-hyper Set can be transformed into a HyperFuzzy Set by reversing the sign of the membership degrees (i.e., changing negative values to positive values).*

Proof. This is evident. □

Furthermore, as frequently discussed in this paper, we also intend to examine the following using the concept of Overset.

Definition 5.88 (N-Hyper Overset)**.** Let X be a universe of discourse. An *N-Hyper Overset* A_{over} in X is defined as:

$$A_{\text{over}} = \{(x, \mu(x)) \mid x \in X,\ \mu(x) \subseteq [-1, \Omega]\},$$

where $\Omega > 1$ is the *Overlimit*.

Theorem 5.89. *An N-Hyper Overset becomes an N-Hyper Set when the Overlimit* Ω *is replaced by 0.*

Proof. Let A_{over} be an N-Hyper Overset defined over X with the membership function:

$$\mu : X \to \tilde{\mathcal{P}}([-1, \Omega]),$$

where $\Omega > 1$ and for all $x \in X$, $\mu(x) \subseteq [-1, \Omega]$.

If we replace Ω with 0, the membership function becomes:

$$\mu : X \to \tilde{\mathcal{P}}([-1, 0]),$$

which restricts each subset $\mu(x)$ to $[-1, 0]$. This coincides with the definition of an N-Hyper Set, as defined by:

$$\mu : X \to \tilde{\mathcal{P}}([-1, 0]).$$

Thus, when $\Omega = 0$, the structure of A_{over} reduces to that of an N-Hyper Set, as required. □

5.10 HyperNested Set

A nested set is a collection of subsets organized hierarchically, where each subset is either contained within or disjoint from another. Hypernested sets and superhypernested sets are proposed as defined above. It is anticipated that the validity and applications of these mathematical definitions will be further explored in the future.

Definition 5.90 (Nested Set)**.** Let B be a non-empty set, and let C be a collection of subsets of B. The collection C is called a *nested set collection* if it satisfies the following properties:

1. $B \in C$ (the whole set B is included in the collection).
2. For all $H, K \in C$, if $H \cap K \neq \emptyset$, then either $H \subseteq K$ or $K \subseteq H$.

Definition 5.91 (Hypernested Set)**.** Let B be a non-empty set, and let C be a collection of subsets of B. The collection C is called a *Hypernested Set Collection* if it satisfies the following properties:

1. $B \in C$ (the whole set B is included in the collection).
2. For every subset $H \in C$, there exists a collection $\mathcal{D}_H \subseteq \mathcal{P}(H)$ (the power set of H), such that:
   (a) $H \in \mathcal{D}_H$, and
   (b) For all $K_1, K_2 \in \mathcal{D}_H$, if $K_1 \cap K_2 \neq \emptyset$, then either $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$.

The collection C is called a *Hypernested Set* if, for every $H \in C$, $\mathcal{D}_H$ satisfies the above conditions.

Definition 5.92 (Superhypernested Set)**.** Let B be a non-empty set, and let C be a collection of subsets of B. The collection C is called a *Superhypernested Set* if it satisfies the following properties:

1. $B \in C$ (the entire set B is included in the collection).

2. For every subset $H \in C$, there exists a family of collections $\mathcal{D}_H = \{\mathcal{D}_H^{(1)}, \mathcal{D}_H^{(2)}, \ldots, \mathcal{D}_H^{(m)}\}$, where each $\mathcal{D}_H^{(k)}$ is a collection of subsets of H, and:
   (a) $H \in \mathcal{D}_H^{(k)}$ for all $k \in \{1, 2, \ldots, m\}$,
   (b) For every $\mathcal{D}_H^{(k)}$, if $K_1, K_2 \in \mathcal{D}_H^{(k)}$ and $K_1 \cap K_2 \neq \emptyset$, then either $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$,
   (c) The families $\mathcal{D}_H^{(k)}$ are hierarchically nested within each other, such that for any $k_1, k_2 \in \{1, 2, \ldots, m\}$ with $k_1 < k_2$, we have:
$$\bigcup_{K \in \mathcal{D}_H^{(k_2)}} K \subseteq \bigcup_{K \in \mathcal{D}_H^{(k_1)}} K.$$

The collection C is called a *Superhypernested Set Collection* if, for every $H \in C$, the family $\mathcal{D}_H$ satisfies the above properties.

Theorem 5.93. *A nested set is a special case of a hypernested set.*

Proof. To show that a nested set is a special case of a hypernested set, consider the following:

- In a nested set, for every $H \in C$, there is no additional family $\mathcal{D}_H$. Instead, each subset H directly satisfies the nested condition $H \subseteq K$ or $K \subseteq H$ for all $K \in C$.
- Setting $\mathcal{D}_H = \{\{H\}\}$ for each $H \in C$, we observe that the hypernested condition reduces to the nested condition.

Thus, a nested set is a special case of a hypernested set where $\mathcal{D}_H = \{\{H\}\}$ for all $H \in C$. □

Theorem 5.94. *A hypernested set is a special case of a superhypernested set.*

Proof. To show that a hypernested set is a special case of a superhypernested set:

- In a hypernested set, the hierarchy is trivial; each family $\mathcal{D}_H$ consists of a single level $\mathcal{D}_H^{(1)}$, and there are no higher levels $\mathcal{D}_H^{(k)}$ for $k > 1$.
- Setting $\mathcal{D}_H^{(k)} = \emptyset$ for all $k > 1$, the hierarchical condition of a superhypernested set is satisfied trivially.

Thus, a hypernested set is a special case of a superhypernested set where the hierarchy is restricted to a single level. □

Corollary 5.95. *A nested set is a special case of a superhypernested set.*

Proof. By combining the results of the above two theorems, a nested set can be seen as a hypernested set with no additional nested families $\mathcal{D}_H$, and a hypernested set is a superhypernested set with a trivial hierarchy. Therefore, a nested set is a special case of a superhypernested set. □

5.11 ProperSubset-Valued Neutrosophic Set

The ProperSubset-Valued Neutrosophic Set is defined below by applying the concept of Proper Subset to the Subset-Valued Neutrosophic Set. It is hoped that further exploration of its mathematical structure and potential applications will be pursued as needed.

Definition 5.96 (Proper Subset)**.** Let A and B be two sets. The set A is called a *proper subset* of the set B, denoted by $A \subset B$, if and only if:

$$A \subseteq B \quad \text{and} \quad A \neq B.$$

Definition 5.97 ((Revisit) Subset-Valued Neutrosophic Set)**.** [888] Let X be a given set. A *Subset-Valued Neutrosophic Set (SVNS)* A on X is defined by three membership functions:

$$T_A : X \to \mathcal{P}([0,1]), \quad I_A : X \to \mathcal{P}([0,1]), \quad F_A : X \to \mathcal{P}([0,1]),$$

where $\mathcal{P}([0,1])$ denotes the power set of the interval $[0,1]$. For each $x \in X$, $T_A(x), I_A(x), F_A(x)$ are subsets of $[0,1]$ representing the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq \inf(T_A(x)) + \inf(I_A(x)) + \inf(F_A(x)) \leq \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \leq 3,$$

where inf and sup represent the infimum and supremum, respectively. The values $T_A(x), I_A(x), F_A(x)$ provide a subset-based representation of uncertainty.

Definition 5.98 (ProperSubset-Valued Neutrosophic Set)**.** Let X be a given set. A *ProperSubset-Valued Neutrosophic Set (PSVNS)* A on X is defined by three membership functions:

$$T_A : X \to \mathcal{P}([0,1]), \quad I_A : X \to \mathcal{P}([0,1]), \quad F_A : X \to \mathcal{P}([0,1]),$$

where $\mathcal{P}([0,1])$ denotes the power set of the interval $[0,1]$. For each $x \in X$, $T_A(x), I_A(x), F_A(x)$ are *proper subsets* of $[0,1]$, satisfying:

$$T_A(x) \subset [0,1], \quad I_A(x) \subset [0,1], \quad F_A(x) \subset [0,1],$$

and

$$T_A(x) \neq [0,1], \quad I_A(x) \neq [0,1], \quad F_A(x) \neq [0,1].$$

Additionally, the degrees of truth, indeterminacy, and falsity satisfy the condition:

$$0 \leq \inf(T_A(x)) + \inf(I_A(x)) + \inf(F_A(x)) \leq \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \leq 3,$$

where inf and sup represent the infimum and supremum of the respective subsets.

This definition ensures that the membership degrees of truth, indeterminacy, and falsity are represented as proper subsets of $[0,1]$, capturing uncertainty in a more granular form.

Theorem 5.99. *A ProperSubset-Valued Neutrosophic Set (PSVNS) is a subset of the Subset-Valued Neutrosophic Set (SVNS).*

Proof. Let X be a given set. A Subset-Valued Neutrosophic Set A is defined by three membership functions.

A ProperSubset-Valued Neutrosophic Set B is defined similarly but with an additional restriction: the subsets $T_B(x), I_B(x), F_B(x)$ must be proper subsets of $[0,1]$. That is:

$$T_B(x) \subset [0,1], \quad T_B(x) \neq [0,1],$$

$$I_B(x) \subset [0,1], \quad I_B(x) \neq [0,1],$$

$$F_B(x) \subset [0,1], \quad F_B(x) \neq [0,1].$$

Clearly, every proper subset of $[0,1]$ is also a subset of $[0,1]$. Therefore, for all $x \in X$:

$$T_B(x) \subseteq T_A(x), \quad I_B(x) \subseteq I_A(x), \quad F_B(x) \subseteq F_A(x).$$

Additionally, the membership functions of a ProperSubset-Valued Neutrosophic Set satisfy the same infimum and supremum condition:

$$0 \leq \inf(T_B(x)) + \inf(I_B(x)) + \inf(F_B(x)) \leq \sup(T_B(x)) + \sup(I_B(x)) + \sup(F_B(x)) \leq 3.$$

Thus, B, as a ProperSubset-Valued Neutrosophic Set, fulfills all the conditions of a Subset-Valued Neutrosophic Set, with the added constraint that the subsets $T_B(x), I_B(x), F_B(x)$ are proper. Therefore, every ProperSubset-Valued Neutrosophic Set is also a Subset-Valued Neutrosophic Set.

Hence, the set of all ProperSubset-Valued Neutrosophic Sets is a subset of the set of all Subset-Valued Neutrosophic Sets, completing the proof. □

Additionally, therelated concept of a ProperSubset-Valued Fuzzy Set can also be defined as follows.

Definition 5.100 ((Revisit) Subset-Valued Fuzzy Set)**.** Let X be a given set. A *Subset-Valued Fuzzy Set (SVFS) A* on X is defined by a membership function:

$$\mu_A : X \to \mathcal{P}([0, 1]),$$

where $\mathcal{P}([0, 1])$ denotes the power set of the interval $[0, 1]$. For each $x \in X$, $\mu_A(x)$ is a subset of $[0, 1]$ that represents the degrees of membership of x to the set A.

The degrees of membership satisfy the following condition for all $x \in X$:

$$0 \leq \inf(\mu_A(x)) \leq \sup(\mu_A(x)) \leq 1,$$

where inf and sup denote the infimum and supremum of the subset $\mu_A(x)$, respectively.

This definition provides a flexible framework for modeling uncertainty, where an element $x \in X$ can have multiple degrees of membership represented by a subset of $[0, 1]$.

Definition 5.101 (ProperSubset-Valued Fuzzy Set)**.** Let X be a given set. A *ProperSubset-Valued Fuzzy Set (PSVFS) A* on X is defined by a membership function:

$$\mu_A : X \to \mathcal{P}([0, 1]),$$

where $\mathcal{P}([0, 1])$ denotes the power set of the interval $[0, 1]$. For each $x \in X$, $\mu_A(x)$ is a *proper subset* of $[0, 1]$, satisfying:

$$\mu_A(x) \subset [0, 1], \quad \mu_A(x) \neq [0, 1].$$

The degrees of membership satisfy the following condition for all $x \in X$:

$$0 \leq \inf(\mu_A(x)) \leq \sup(\mu_A(x)) \leq 1,$$

where inf and sup denote the infimum and supremum of the proper subset $\mu_A(x)$, respectively.

This definition ensures that the membership of each element $x \in X$ is represented by a proper subset of $[0, 1]$, allowing for a more granular representation of uncertainty while excluding the possibility of full coverage of the interval $[0, 1]$ as a membership degree.

Corollary 5.102. *A ProperSubset-Valued Fuzzy Set is a subset of the Subset-Valued Fuzzy Set.*

Proof. This is evident. □

Corollary 5.103. *A ProperSubset-Valued Fuzzy Set can be generalized using the concept of a ProperSubset-Valued Neutrosophic Set.*

Proof. This is evident. □

Additionally, other derived definitions, such as the Time-Dependent Subset-Valued Neutrosophic Set, where the membership functions depend on time, and the Weighted Subset-Valued Neutrosophic Set, which incorporates weights, are defined below. It is hoped that research on these types of sets will progress as needed in the future.

Definition 5.104 (Time-Dependent Subset-Valued Neutrosophic Set)**.** Let X be a given set and $\mathbb{T}$ a set representing time (e.g., $\mathbb{R}^+$ for non-negative real numbers). A *Time-Dependent Subset-Valued Neutrosophic Set (TD-SVNS) A* on X is defined by three time-dependent membership functions:

$$T_A : X \times \mathbb{T} \to \mathcal{P}([0,1]), \quad I_A : X \times \mathbb{T} \to \mathcal{P}([0,1]), \quad F_A : X \times \mathbb{T} \to \mathcal{P}([0,1]),$$

where $\mathcal{P}([0,1])$ denotes the power set of the interval $[0,1]$. For each $x \in X$ and $t \in \mathbb{T}$, $T_A(x,t), I_A(x,t), F_A(x,t)$ are subsets of $[0,1]$ representing the time-dependent degrees of truth, indeterminacy, and falsity, respectively.

These membership values satisfy the following condition for all $x \in X$ and $t \in \mathbb{T}$:

$$0 \le \inf(T_A(x,t)) + \inf(I_A(x,t)) + \inf(F_A(x,t)) \le \sup(T_A(x,t)) + \sup(I_A(x,t)) + \sup(F_A(x,t)) \le 3,$$

where inf and sup represent the infimum and supremum of the subsets $T_A(x,t), I_A(x,t), F_A(x,t)$.

This definition allows the representation of uncertainty and variation of the membership degrees over time.

Definition 5.105 (Weighted Subset-Valued Neutrosophic Set)**.** Let X be a given set. A *Weighted Subset-Valued Neutrosophic Set (W-SVNS) A* on X is defined by three membership functions that assign both a subset of $[0,1]$ and a weight to each element:

$$T_A : X \to \mathcal{P}([0,1]) \times [0, w_T], \quad I_A : X \to \mathcal{P}([0,1]) \times [0, w_I], \quad F_A : X \to \mathcal{P}([0,1]) \times [0, w_F],$$

where $\mathcal{P}([0,1])$ denotes the power set of the interval $[0,1]$, and $w_T, w_I, w_F > 0$ are the maximum weights for truth, indeterminacy, and falsity, respectively.

For each $x \in X$, the membership values $(T_A(x), w_T(x))$, $(I_A(x), w_I(x))$, and $(F_A(x), w_F(x))$ satisfy:

$$T_A(x) \subseteq [0,1], \quad I_A(x) \subseteq [0,1], \quad F_A(x) \subseteq [0,1],$$

and

$$0 \le \inf(T_A(x)) + \inf(I_A(x)) + \inf(F_A(x)) \le \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \le 3.$$

The weights $w_T(x), w_I(x), w_F(x)$ provide a measure of the relative importance of the membership degrees for each element $x \in X$.

Theorem 5.106. *The Time-Dependent Subset-Valued Neutrosophic Set (TDSVNS) and Weighted Subset-Valued Neutrosophic Set (WSVNS) generalize the Subset-Valued Neutrosophic Set (SVNS).*

Proof. To prove that TDSVNS and WSVNS generalize SVNS, we demonstrate that each is a proper extension of SVNS by introducing additional structural elements (time-dependency and weights) while retaining the fundamental properties of SVNS.

Case 1: Time-Dependent Subset-Valued Neutrosophic Set (TDSVNS) By definition, a TDSVNS A is characterized by time-dependent membership functions:

$$T_A : X \times \mathbb{T} \to \mathcal{P}([0,1]), \quad I_A : X \times \mathbb{T} \to \mathcal{P}([0,1]), \quad F_A : X \times \mathbb{T} \to \mathcal{P}([0,1]),$$

where $\mathbb{T}$ represents the time domain. For a fixed time $t \in \mathbb{T}$, the membership functions reduce to:

$$T_A(\cdot,t),\ I_A(\cdot,t),\ F_A(\cdot,t) : X \to \mathcal{P}([0,1]),$$

which satisfy the SVNS conditions:

$$0 \le \inf(T_A(x,t)) + \inf(I_A(x,t)) + \inf(F_A(x,t)) \le \sup(T_A(x,t)) + \sup(I_A(x,t)) + \sup(F_A(x,t)) \le 3.$$

Thus, for any fixed $t \in \mathbb{T}$, a TDSVNS reduces to an SVNS. Hence, a TDSVNS generalizes the SVNS by incorporating time-dependency.

Case 2: Weighted Subset-Valued Neutrosophic Set (WSVNS) By definition, a WSVNS A is characterized by weighted membership functions:

$$T_A : X \to \mathcal{P}([0,1]) \times \mathbb{R}, \quad I_A : X \to \mathcal{P}([0,1]) \times \mathbb{R}, \quad F_A : X \to \mathcal{P}([0,1]) \times \mathbb{R},$$

where the additional weight component represents the relative importance of the membership degrees. For $x \in X$, let $T_A(x) = (S_T(x), w_T(x))$, where $S_T(x) \subseteq [0,1]$ and $w_T(x) \in \mathbb{R}$. The subset $S_T(x)$ satisfies:

$$0 \leq \inf(S_T(x)) \leq \sup(S_T(x)) \leq 1.$$

Similarly, $S_I(x), S_F(x) \subseteq [0,1]$ satisfy the analogous conditions. When weights are ignored (i.e., $w_T(x) = w_I(x) = w_F(x) = 1$ for all $x \in X$), the WSVNS reduces to an SVNS. Thus, a WSVNS generalizes the SVNS by incorporating weights.

Both TDSVNS and WSVNS retain the structural properties of SVNS and add additional features (time-dependency or weights). Therefore, they are proper generalizations of SVNS. □

At the conceptual stage, the definition of a Probability-Subset-Valued Neutrosophic Set is presented below. Further research into the validity of this definition, its potential applications, and its mathematical structure is anticipated.

Definition 5.107 (Probability-Subset-Valued Neutrosophic Set)**.** Let X be a given set. A *Probability-Subset-Valued Neutrosophic Set (PSVNS) A* on X is defined by three probabilistic subset-valued membership functions:

$$T_A : X \to \mathcal{P}([0,1]) \times \mathcal{D}, \quad I_A : X \to \mathcal{P}([0,1]) \times \mathcal{D}, \quad F_A : X \to \mathcal{P}([0,1]) \times \mathcal{D},$$

where:

- $\mathcal{P}([0,1])$ denotes the power set of the interval $[0,1]$, representing possible degrees of truth, indeterminacy, and falsity.
- $\mathcal{D}$ denotes the set of probability distributions over $\mathcal{P}([0,1])$, representing the likelihood of each subset value within the interval $[0,1]$.

For each $x \in X$, the probabilistic subset membership functions $T_A(x), I_A(x), F_A(x)$ are given as:

$$T_A(x) = (S_T, P_T), \quad I_A(x) = (S_I, P_I), \quad F_A(x) = (S_F, P_F),$$

where:

- $S_T, S_I, S_F \subseteq [0,1]$ are subsets representing the possible degrees of truth, indeterminacy, and falsity.
- P_T, P_I, P_F are probability distributions defined over S_T, S_I, S_F, respectively, such that $P_T : S_T \to [0,1]$, $P_I : S_I \to [0,1]$, $P_F : S_F \to [0,1]$, and satisfy:

$$\int_{S_T} P_T(s)\,ds + \int_{S_I} P_I(s)\,ds + \int_{S_F} P_F(s)\,ds \leq 1.$$

Additionally, the subsets S_T, S_I, S_F satisfy the following condition:

$$0 \leq \inf(S_T) + \inf(S_I) + \inf(S_F) \leq \sup(S_T) + \sup(S_I) + \sup(S_F) \leq 3.$$

This definition incorporates a probability distribution over the subsets of $[0,1]$, providing a nuanced representation of the likelihood for each degree of truth, indeterminacy, and falsity.

5.12 Time-Dependent Neutrosophic Set and Weighted Neutrosophic Set

In relation to the Time-Dependent Subset-Valued Neutrosophic Set and Weighted Subset-Valued Neutrosophic Set, we define the Time-Dependent Neutrosophic Set and Weighted Neutrosophic Set as follows. While similar concepts have been explored in several papers, the definitions presented here aim to provide a mathematical reinterpretation and refinement of these ideas. We hope that further investigations into these concepts will continue in the future.

Definition 5.108 (Time-Dependent Neutrosophic Set)**.** Let X be a universe of discourse and $\mathbb{T}$ represent a time domain. A *Time-Dependent Neutrosophic Set (TDNS) A* on X is defined by three time-dependent membership functions:

$$T_A : X \times \mathbb{T} \to [0,1], \quad I_A : X \times \mathbb{T} \to [0,1], \quad F_A : X \times \mathbb{T} \to [0,1],$$

where $T_A(x,t), I_A(x,t), F_A(x,t)$ represent the truth-membership, indeterminacy-membership, and falsity-membership degrees of $x \in X$ at time $t \in \mathbb{T}$, respectively.

These membership functions satisfy the following condition for all $x \in X$ and $t \in \mathbb{T}$:

$$0 \le T_A(x,t) + I_A(x,t) + F_A(x,t) \le 3.$$

This definition extends the classical Neutrosophic Set by incorporating a temporal component, allowing the membership degrees to vary over time.

Definition 5.109 (Weighted Neutrosophic Set)**.** Let X be a universe of discourse. A *Weighted Neutrosophic Set (WNS) A* on X is defined by three membership functions with associated weights:

$$T_A : X \to [0,1] \times \mathbb{R}, \quad I_A : X \to [0,1] \times \mathbb{R}, \quad F_A : X \to [0,1] \times \mathbb{R},$$

where $T_A(x) = (t_T(x), w_T(x))$, $I_A(x) = (t_I(x), w_I(x))$, $F_A(x) = (t_F(x), w_F(x))$, with:

- $t_T(x), t_I(x), t_F(x) \in [0,1]$ are the truth, indeterminacy, and falsity membership degrees for $x \in X$.
- $w_T(x), w_I(x), w_F(x) \in \mathbb{R}$ are the respective weights representing the relative importance of the membership degrees.

These membership degrees satisfy the condition for all $x \in X$:

$$0 \le t_T(x) + t_I(x) + t_F(x) \le 3.$$

The inclusion of weights allows the Weighted Neutrosophic Set to capture the relative significance of truth, indeterminacy, and falsity membership degrees in the decision-making or evaluation process.

Theorem 5.110. *The Time-Dependent Neutrosophic Set and the Weighted Neutrosophic Set are generalizations of the Neutrosophic Set.*

Proof. Let X be a universe of discourse.

A Neutrosophic Set A on X is defined by the membership functions:

$$T_A, I_A, F_A : X \to [0,1],$$

where $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity for $x \in X$, respectively.

In the Time-Dependent Neutrosophic Set, these membership functions are extended to depend on a temporal parameter $t \in \mathbb{R}$:

$$T_A(x,t) : X \times \mathbb{R} \to [0,1], \quad I_A(x,t) : X \times \mathbb{R} \to [0,1], \quad F_A(x,t) : X \times \mathbb{R} \to [0,1].$$

For any fixed $t \in \mathbb{R}$, $T_A(x,t)$, $I_A(x,t)$, and $F_A(x,t)$ reduce to the membership functions of a standard Neutrosophic Set. Therefore, the Time-Dependent Neutrosophic Set is a generalization of the Neutrosophic Set.

A Weighted Neutrosophic Set extends the Neutrosophic Set by associating weights $w_T, w_I, w_F \in [0,1]$ to the membership functions:

$$T_A(x, w_T) : X \to [0,1], \quad I_A(x, w_I) : X \to [0,1], \quad F_A(x, w_F) : X \to [0,1].$$

For uniform weights $w_T = w_I = w_F = 1$, the Weighted Neutrosophic Set reduces to the standard Neutrosophic Set. Hence, the Weighted Neutrosophic Set generalizes the Neutrosophic Set.

Since both the Time-Dependent Neutrosophic Set and the Weighted Neutrosophic Set include the Neutrosophic Set as a special case, they are generalizations of the Neutrosophic Set. □

5.13 Trice Neutrosophic Set

At the conceptual stage, the definition of the Trice Neutrosophic Set is presented below. It is envisioned as an extension of the Trice Fuzzy Set (cf. [497–503]). The validity and mathematical properties of this definition will be explored as needed in future studies. For details on the operations of the Trice structure, refer to the literature [497–503], which includes some references written in Japanese.

Definition 5.111 (Trice Fuzzy Set)**.** [497] Let X be a non-empty set, and let $T = (L_I, *_1, *_2, *_3)$ be a trice structure, where:

- $L_I = \{(t_1, t_2) \in [0,1] \times [0,1] \mid 0 \leq t_2 \leq t_1 \leq 1\}$ is a set of ordered pairs representing truth degrees,
- $*_1, *_2, *_3$ are semilattice operations defined on L_I, satisfying idempotence, commutativity, and associativity.

A *trice fuzzy set* A over X is defined as a function:

$$\mu_A : X \to L_I,$$

where each element $x \in X$ is mapped to a pair $\mu_A(x) = (t_1(x), t_2(x)) \in L_I$. Here:

- $t_1(x)$ represents the *degree of full truth* of x,
- $t_2(x)$ represents the *degree of partial truth* of x,
- The relationship $0 \leq t_2(x) \leq t_1(x) \leq 1$ must hold for all $x \in X$.

Definition 5.112 (Trice Neutrosophic Set)**.** Let X be a non-empty set, and let $T = (L_I, *_1, *_2, *_3)$ be a trice structure, where:

- $L_I = \{(T, I, F) \in [0,1] \times [0,1] \times [0,1] \mid T + I + F \leq 3\}$,
- $*_1, *_2, *_3$ are semilattice operations on L_I, satisfying idempotence, commutativity, and associativity.

A *Trice Neutrosophic Set* A over X is defined as a function:

$$\mu_A : X \to L_I,$$

where each element $x \in X$ is mapped to an ordered triple $\mu_A(x) = (T(x), I(x), F(x)) \in L_I$. Here:

- $T(x)$ represents the *degree of truth* of x,
- $I(x)$ represents the *degree of indeterminacy* of x,
- $F(x)$ represents the *degree of falsity* of x.

The following conditions must hold for all $x \in X$:

$$0 \le T(x), I(x), F(x) \le 1, \quad T(x) + I(x) + F(x) \le 3.$$

The trice operations $*_1, *_2, *_3$ extend the algebraic structure of neutrosophic sets, enabling enhanced modeling of truth, indeterminacy, and falsity in a trinary framework.

Theorem 5.113. *The Trice Neutrosophic Set generalizes the Trice Fuzzy Set.*

Proof. Let X be a non-empty set, and consider a Trice Fuzzy Set A over X defined as:

$$\mu_A : X \to L_F,$$

where $L_F = \{(t_1, t_2) \in [0,1] \times [0,1] \mid 0 \le t_2 \le t_1 \le 1\}$.

A Trice Neutrosophic Set B over X is defined as:

$$\mu_B : X \to L_N,$$

where $L_N = \{(T, I, F) \in [0,1] \times [0,1] \times [0,1] \mid T + I + F \le 3\}$.

To demonstrate the generalization: Consider any Trice Fuzzy Set A. For each $x \in X$, let $\mu_A(x) = (t_1(x), t_2(x)) \in L_F$. Define a corresponding Trice Neutrosophic Set B by mapping $x \in X$ to:

$$\mu_B(x) = (t_1(x), 0, 1 - t_1(x)) \in L_N.$$

This satisfies the conditions of a Trice Neutrosophic Set:

$$T(x) + I(x) + F(x) = t_1(x) + 0 + (1 - t_1(x)) = 1 \le 3.$$

Since every Trice Fuzzy Set can be embedded within the framework of a Trice Neutrosophic Set by assigning $I(x) = 0$ and $F(x) = 1 - T(x)$, it follows that the Trice Neutrosophic Set generalizes the Trice Fuzzy Set.

Hence, the Trice Neutrosophic Set is a proper generalization of the Trice Fuzzy Set. □

Theorem 5.114. *The Trice Neutrosophic Set generalizes the Neutrosophic Set.*

Proof. Consider any Neutrosophic Set A. For each $x \in X$, let $\mu_A(x) = (T(x), I(x), F(x)) \in L_N$. Define a corresponding Trice Neutrosophic Set B with the same mapping:

$$\mu_B(x) = (T(x), I(x), F(x)) \in L_{TN}.$$

Since $L_N \subseteq L_{TN}$, all elements of L_N are valid elements of L_{TN}. Specifically, for any $(T(x), I(x), F(x)) \in L_N$, the condition $T(x) + I(x) + F(x) \le 1$ automatically satisfies the broader condition $T(x) + I(x) + F(x) \le 3$.

Furthermore, the trice operations $*_1, *_2, *_3$ defined on L_{TN} naturally extend the algebraic structure of L_N, enabling the inclusion of Neutrosophic Sets as a special case where $T(x) + I(x) + F(x) \le 1$.

Thus, every Neutrosophic Set is a specific instance of a Trice Neutrosophic Set, confirming that the latter generalizes the former. □

5.14 Other Graph Class Extension (Revisited)

As revisited in [341], the author is interested in exploring extensions of graph classes and investigating the mathematical structures within specific graph classes. Our aim is to identify suitable graph classes by integrating these various perspectives. Although theoretical generalization is fundamental to mathematics, it does not always lead directly to practical applications. From an applied mathematics perspective, it is equally important to evaluate the practical relevance of such concepts. To support theoretical advancements, experimental approaches and algorithmic explorations are essential. Research in this area often involves expanding or refining graph classes based on the following factors:

- *Classic Graph Properties:* Regular [394, 470, 993], Irregular [393, 715, 834], Complete [37, 135, 496], Perfect [1,144,392], claw-free [216,216,217,322], Tree, Path [134,566,1034], Forest [817], Planar [375], Linear [247, 298], OuterPlanar [538, 789], Median [117, 127, 545], Multigraph [97, 173, 174, 783].
- *Graph Using Operations:* Intersection Graph [373], Product Graph [38,323,555,597,956], Union Graph [85, 934].
- *Subgraph/Hypergraph Properties:* Supergraph [151], Hypergraph [183, 186, 328, 399, 435, 437], Superhypergraph [89, 363, 410, 537, 686, 744, 867, 868, 885, 896], n-Superhypergraph [867, 873], Subgraph, Induced Subgraph [487, 609], Induced Supergraph [373].
- *Graph Directionality:* Undirected, Directed, Mixed [793,794], Bidirected [269,404], Semi-directed [149, 150, 813], Bunch [348, 718, 812].
- *Graph Partition:* Bipartite [96, 299, 314, 483, 614], Tripartite [108, 160, 439], n-partite [53, 468].
- *Uncertain Properties:* Fuzzy, Neutrosophic, Plithogenic, Rough [648, 726, 760, 770, 841], Vague [25, 765, 814], Soft [31, 110, 405, 548, 549, 717, 766],Hypersoft [796, 862, 881, 883, 1008], Weighted [285, 479, 550, 570, 723], Picture Fuzzy [577, 579, 642], Grey [461], Triangular Fuzzy [918], Z-Number [67, 636, 724, 960, 1020], Refined Plithogenic [368], q-Rung Orthopair Fuzzy [739, 991], Quadripartitioned Neutrosophic [519, 752], Pentapartitioned Neutrosophic, Entropy [256], HyperFuzzy [411, 692], Type-2 Fuzzy [190, 193, 245, 568, 585, 655, 712], Hesitant [17, 200, 453, 942, 943, 995], spherical [32], Bipolar [15, 34, 45, 65], Tripolar [762–764] etc.
- *Graph Dimensionality:* 2-dimensional, 3-dimensional [215, 721, 990], 4-dimensional [195, 505, 678], Multidimensional [141, 635, 791] etc.
- *Themes:* Graph Classes Hierarchy [161], Mathematical Structure of Graph Classes [161], Graph Parameters [372, 454, 472, 822], Algorithms [317], Computational Complexity [93, 720], Real-world Applications, Combinatorics [315, 381, 416, 826].

The above is merely one example, and the author believes there are numerous concepts and perspectives that remain unrecognized. It is sincerely hoped that further research in this field will continue to advance.

Funding

This research received no external funding.

Acknowledgments

We humbly extend our heartfelt gratitude to everyone who has provided invaluable support, enabling the successful completion of this paper. We also express our sincere appreciation to all readers who have taken the time to engage with this work. Furthermore, we extend our deepest respect and gratitude to the authors of the references cited in this paper. Thank you for your significant contributions.

Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

References

[1] Perfectly regular and perfectly edge-regular intuitionistic fuzzy graphs. *International Journal of Engineering and Advanced Technology*, 2019.

[2] Shereen .. and Mahmoud M. Ismail. Interval-valued neutrosophic hypersoft sets (ivnhss) for enterprise resource planning selection. *Journal of Neutrosophic and Fuzzy Systems*, 2023.

[3] Scott Aaronson and Alex Arkhipov. The computational complexity of linear optics. In *Proceedings of the forty-third annual ACM symposium on Theory of computing*, pages 333–342, 2011.

[4] Mujahid Abbas, Ghulam Murtaza, and Florentin Smarandache. Basic operations on hypersoft sets and hypersoft point. *Neutrosophic Sets and Systems*, 35:407–421, 2020.

[5] Mujahid Abbas, Ghulam Murtaza, and Florentin Smarandache. *Basic operations on hypersoft sets and hypersoft point*. Infinite Study, 2020.

[6] Mohamed Abdel-Basset, Mohamed El-Hoseny, Abduallah Gamal, and Florentin Smarandache. A novel model for evaluation hospital medical care systems based on plithogenic sets. *Artificial intelligence in medicine*, 100:101710, 2019.

[7] Joshua Ackerman and George V. Cybenko. A survey of neural networks and formal languages. *ArXiv*, abs/2006.01338, 2020.

[8] Isolde Adler, Tomáš Gavenčiak, and Tereza Klimošová. Hypertree-depth and minors in hypergraphs. *Theoretical Computer Science*, 463:84–95, 2012.

[9] Isolde Adler, Georg Gottlob, and Martin Grohe. Hypertree width and related hypergraph invariants. *European Journal of Combinatorics*, 28(8):2167–2181, 2007.

[10] Admin Admin and Abdalla I. Abdalla Musa. Automated credit card risk assessment using fuzzy parameterized neutrosophic hypersoft expert set. *International Journal of Neutrosophic Science*.

[11] Ashok K. Agarwal. Partition theory : Yesterday and today. 2012.

[12] Uzma Ahmad and Muhammad Sabir. Multicriteria decision-making based on the degree and distance-based indices of fuzzy graphs. *Granular Computing*, 8(4):793–807, 2023.

[13] Nouran Ajabnoor. A new approach of tree soft set with mcdm technique for leadership in international climate policy: measuring renewable energy sources and policies. *Neutrosophic Sets and Systems*, 70(1):6, 2024.

[14] M. Akram and K. H. Dar. On n-graphs. *Southeast Asian Bulletin of Mathematics*, 38:35–49, 2014.

[15] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[16] Muhammad Akram. Interval-valued fuzzy line graphs. *Neural Computing and Applications*, 21:145–150, 2012.

[17] Muhammad Akram, Arooj Adeel, and José Carlos R Alcantud. Hesitant fuzzy n-soft sets: A new model with applications in decision-making. *Journal of Intelligent & Fuzzy Systems*, 36(6):6113–6127, 2019.

[18] Muhammad Akram, Ghous Ali, and José Carlos Rodriguez Alcantud. New decision-making hybrid model: intuitionistic fuzzy n-soft rough sets. *Soft Computing*, 23:9853 – 9868, 2019.

[19] Muhammad Akram and Noura Omair Alshehri. Intuitionistic fuzzy cycles and intuitionistic fuzzy trees. *The Scientific World Journal*, 2014(1):305836, 2014.

[20] Muhammad Akram and Noura Omair Alshehri. Tempered interval-valued fuzzy hypergraphs. *University politehnica of bucharest scientific bulletin-series a-applied mathematics and physics*, 77(1):39–48, 2015.

[21] Muhammad Akram and Bijan Davvaz. Strong intuitionistic fuzzy graphs. *Filomat*, 26(1):177–196, 2012.

[22] Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.

[23] Muhammad Akram and Wieslaw A. Dudek. Regular bipolar fuzzy graphs. *Neural Computing and Applications*, 21:197–205, 2012.

[24] Muhammad Akram, Feng Feng, Shahzad Sarwar, and Youne Bae Jun. Certain types of vague graphs. *University Politehnica of Bucharest Scientific Bulletin Series A*, 76(1):141–154, 2014.

[25] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[26] Muhammad Akram, MG Karunambigai, K Palanivel, and S Sivasankar. Balanced bipolar fuzzy graphs. *Journal of advanced research in pure mathematics*, 6(4):58–71, 2014.

[27] Muhammad Akram, Sheng-Gang Li, and KP Shum. Antipodal bipolar fuzzy graphs. *Italian Journal of Pure and Applied Mathematics*, 31(56):425–438, 2013.

[28] Muhammad Akram and Anam Luqman. Certain networks models using single-valued neutrosophic directed hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 33(1):575–588, 2017.

[29] Muhammad Akram, Hafsa M Malik, Sundas Shahzadi, and Florentin Smarandache. Neutrosophic soft rough graphs with application. *Axioms*, 7(1):14, 2018.

[30] Muhammad Akram and Saira Nawaz. Operations on soft graphs. *Fuzzy Information and Engineering*, 7:423 – 449, 2015.

[31] Muhammad Akram and Saira Nawaz. Certain types of soft graphs. *UPB Sci. Bull., Ser. A*, 78(4):67–82, 2016.

[32] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[33] Muhammad Akram and Musavarah Sarwar. Novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. *viXra*, 2017.

[34] Muhammad Akram, Musavarah Sarwar, Wieslaw A Dudek, Muhammad Akram, Musavarah Sarwar, and Wieslaw A Dudek. Bipolar neutrosophic graph structures. *Graphs for the Analysis of Bipolar Fuzzy Information*, pages 393–446, 2021.

[35] Muhammad Akram and Gulfam Shahzadi. *Operations on single-valued neutrosophic graphs*. Infinite Study, 2017.

[36] Muhammad Akram and Sundas Shahzadi. Neutrosophic soft graphs with application. *J. Intell. Fuzzy Syst.*, 32:841–858, 2017.

[37] Talal Al-Hawary. Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, 4:26, 2011.

[38] Talal Al-Hawary and Bayan Horani. On product fuzzy graphs. *Annals of fuzzy mathematics and Informatics*, 12(2):279–294, 2016.

[39] Talal Ali Al-Hawary. On fuzzy matroids. *viXra*, 2012.

[40] Talal Ali Al-Hawary. Fuzzy l-closed sets. *Mathematika*, 33:241–246, 2017.

[41] Talal Ali Al-Hawary. Fuzzy m-open sets. *Theory and Applications of Mathematics & Computer Science*, 7:72–77, 2017.

[42] Talal Ali Al-Hawary. Fuzzy w-closed sets. *Cogent Mathematics*, 4, 2017.

[43] Talal Ali Al-Hawary and Ahmed Al-Omari. Quasi b-open sets in bitopological spaces. 2012.

[44] Yousef Al-Qudah and Nasruddin Hassan. Bipolar fuzzy soft expert set and its application in decision making. *Int. J. Appl. Decis. Sci.*, 10:175–191, 2017.

[45] Ashraf Al-Quran, Faisal Al-Sharqi, Kifayat Ullah, Mamika Ujianita Romdhini, Marwa Balti, and Mohammed Alomair. *Bipolar fuzzy hypersoft set and its application in decision making*. Infinite Study, 2023.

[46] Ashraf Al-Quran and Nasruddin Hassan. Neutrosophic vague soft expert set theory. *J. Intell. Fuzzy Syst.*, 30:3691–3702, 2016.

[47] Shuker Mahmood Al-salem. Soft regular generalized b-closed sets in soft topological spaces. *Journal of Linear and Topological Algebra*, 3:196–204, 2014.

[48] Tareq M. Al-shami and Abdelwaheb Mhemdi. Generalized frame for orthopair fuzzy sets: (m,n)-fuzzy sets and their applications to multi-criteria decision-making methods. *Inf.*, 14:56, 2023.

[49] Tuqa AH Al-Tamimi, Luay AA Al-Swidi, Ali HM Al-Obaidi, et al. New concepts in partner multineutrosophic topological space. *International Journal of Neutrosophic Science*, (3):172–72, 2024.

[50] Tuqa AH Al-Tamimi, Luay AA Al-Swidi, Ali HM Al-Obaidi, et al. Partner sets for generalizations of multineutrosophic sets. *International Journal of Neutrosophic Science*, (1):08–8, 2024.

[51] Md Jawaherul Alam, Stephen G Kobourov, Sergey Pupyrev, and Jackson Toeniskoetter. Weak unit disk and interval representation of graphs. In *Graph-Theoretic Concepts in Computer Science: 41st International Workshop, WG 2015, Garching, Germany, June 17-19, 2015, Revised Papers 41*, pages 237–251. Springer, 2016.

[52] Md. Tanvir Alam, Chowdhury Farhan Ahmed, Md. Samiullah, and Carson Kai-Sang Leung. Mining frequent patterns from hypergraph databases. In *Pacific-Asia Conference on Knowledge Discovery and Data Mining*, 2021.

[53] Yousef Alavi, Don R Lick, and Songlin Tian. Randomly complete n-partite graphs. *Mathematica Slovaca*, 39(3):241–250, 1989.

[54] Sandra Albrechtsen. Refining trees of tangles in abstract separation systems i: Inessential parts. 2023.

[55] José Carlos R Alcantud. Ranked hesitant fuzzy sets for multi-criteria multi-agent decisions. *Expert Systems with Applications*, 209:118276, 2022.

[56] José Carlos R Alcantud, Azadeh Zahedi Khameneh, Gustavo Santos-García, and Muhammad Akram. A systematic literature review of soft set theory. *Neural Computing and Applications*, 36(16):8951–8975, 2024.

[57] David Aldous, Alice Contat, Nicolas Curien, and Olivier Hénard. Parking on the infinite binary tree. *Probability Theory and Related Fields*, 187(1-2):481–504, 2023.

[58] E Algaba, Jesús Mario Bilbao, Rene Van den Brink, and Andres Jimenez-Losada. Cooperative games on antimatroids. *Discrete Mathematics*, 282(1-3):1–15, 2004.

[59] M Irfan Ali, Feng Feng, Xiaoyan Liu, Won Keun Min, and Muhammad Shabir. On some new operations in soft set theory. *Computers & Mathematics with Applications*, 57(9):1547–1553, 2009.

[60] Muhammad Irfan Ali. Another view on q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 33:2139 – 2153, 2018.

[61] Mumtaz Ali, Luu Quoc Dat, Le Hoang Son, and Florentin Smarandache. Interval complex neutrosophic set: Formulation and applications in decision-making. *International Journal of Fuzzy Systems*, 20:986 – 999, 2017.

[62] Mumtaz Ali and Florentin Smarandache. Complex neutrosophic set. *Neural Computing and Applications*, 28:1817–1834, 2016.

[63] Mumtaz Ali, Florentin Smarandache, and Mohsin Khan. Study on the development of neutrosophic triplet ring and neutrosophic triplet field. *Mathematics*, 6(4):46, 2018.

[64] Mumtaz Ali, Le Hoang Son, Irfan Deli, and Nguyen Dang Tien. Bipolar neutrosophic soft sets and applications in decision making. *J. Intell. Fuzzy Syst.*, 33:4077–4087, 2017.

[65] Mumtaz Ali, Le Hoang Son, Irfan Deli, and Nguyen Dang Tien. Bipolar neutrosophic soft sets and applications in decision making. *Journal of Intelligent & Fuzzy Systems*, 33(6):4077–4087, 2017.

[66] RA Aliev, OH Huseynov, and RR Aliyev. A sum of a large number of z-numbers. *Procedia computer science*, 120:16–22, 2017.

[67] Rafik Aziz Aliev, Akif Alizadeh, Rashad Rafig Aliyev, and Oleg H Huseynov. *Arithmetic of Z-numbers, the: theory and applications*. World Scientific, 2015.

[68] Moslem Alimohammadlou and Zahra Khoshsepehr. The role of society 5.0 in achieving sustainable development: a spherical fuzzy set approach. *Environmental Science and Pollution Research*, 30:47630–47654, 2023.

[69] Shawkat Alkhazaleh and Emad A. Marei. New soft rough set approximations. *Int. J. Fuzzy Log. Intell. Syst.*, 21:123–134, 2021.

[70] Shawkat Alkhazaleh and Abdul Razak Salleh. Soft expert sets. *Adv. Decis. Sci.*, 2011:757868:1–757868:12, 2011.

[71] Shawkat Alkhazaleh and Abdul Razak Salleh. Fuzzy soft expert set and its application. *Applied Mathematics-a Journal of Chinese Universities Series B*, 5:1349–1368, 2014.

[72] Shawkat Alkhazaleh, Abdul Razak Salleh, Nasruddin Hassan, and Abd Ghafur Ahmad. Multisoft sets. In *Proc. 2nd International Conference on Mathematical Sciences*, pages 910–917, 2010.

[73] Abdulazeez Alkouri and Abdul Razak Salleh. Complex intuitionistic fuzzy sets. 2012.

[74] Norman Larrabee Alling. Conway's field of surreal numbers. *Transactions of the American Mathematical Society*, 287:365–386, 1985.

[75] Noga Alon, Béla Bollobás, and Ingo Wegener. Combinatorics, probability and computing. *Oberwolfach Reports*, 2020.

[76] Uri Alon and Eran Yahav. On the bottleneck of graph neural networks and its practical implications. *arXiv preprint arXiv:2006.05205*, 2020.

[77] Ali Alqazzaz and Karam M Sallam. Evaluation of sustainable waste valorization using treesoft set with neutrosophic sets. *Neutrosophic Sets and Systems*, 65(1):9, 2024.

[78] Marcos Antônio Alves, Petrônio Cândido de Lima e Silva, Carlos Alberto Severiano Junior, Gustavo Linhares Vieira, Frederico Gadelha Guimarães, and Hossein Javedani Sadaei. An extension of nonstationary fuzzy sets to heteroskedastic fuzzy time series. In *ESANN*, 2018.

[79] Reza Ameri, M Amiri-Bideshki, Arsham Borumand Saeid, and Sarka Hoskova-Mayerova. Prime filters of hyperlattices. *Analele ştiinţifice ale Universităţii" Ovidius" Constanţa. Seria Matematică*, 24(2):15–26, 2016.

[80] Reza Ameri, Omid Reza Dehghan, et al. Fuzzy hypervector spaces. *Adv. Fuzzy Syst.*, 2008:295649–1, 2008.

[81] Reza Ameri and OR Dehghan. On dimension of hypervector spaces. *European Journal of Pure and Applied Mathematics*, 1(2):32–50, 2008.

[82] Umair Amin, Aliya Fahmi, Yaqoob Naveed, Aqsa Farid, and Muhammad Arshad Shehzad Hassan. Domination in bipolar fuzzy soft graphs. *J. Intell. Fuzzy Syst.*, 46:6369–6382, 2024.

[83] Omid Amini, Frédéric Mazoit, Nicolas Nisse, and Stéphan Thomassé. Submodular partition functions. *Discrete Mathematics*, 309(20):6000–6008, 2009.

[84] Setyaningsih Ana and P.Elav arasi. Topology. *International Journal of Mathematics Trends and Technology*, 2018.

[85] L Arungalai Anbarasi and VS Selvi. Prime labeling of some union graphs and circulant graphs.

[86] Renzo Angles and Claudio Gutierrez. Survey of graph database models. *ACM Computing Surveys (CSUR)*, 40(1):1–39, 2008.

[87] J Angoa, YF Ortiz-Castillo, and A Tamariz-Mascarúa. Ultrafilters and properties related to compactness. In *Topology Proc*, volume 43, pages 183–200, 2014.

[88] Kushal Anjaria. Knowledge derivation from likert scale using z-numbers. *Inf. Sci.*, 590:234–252, 2022.

[89] Shabana Anwar, Muhammad Azeem, and Muhammad Kamran Jamil. Topological numbers of fuzzy soft graphs and their applications in globalizing the world by mutual trade. *Appl. Soft Comput.*, 159:111642, 2024.

[90] Lakshmanan S Aparna and A Vijayakumar. *Studies on Some Graph Classes*. PhD thesis, Cochin University of Science and Technology, 2008.

[91] Tom M. Apostol. Introduction to analytic number theory. 1976.

[92] Nabeel Ezzulddin Arif et al. Domination (set and number) in neutrosophic soft over graphs. *Wasit Journal for Pure sciences*, 1(3):26–43, 2022.

[93] Sanjeev Arora and Boaz Barak. *Computational complexity: a modern approach*. Cambridge University Press, 2009.

[94] Shahzaib Ashraf, Saleem Abdullah, Tahir Mahmood, Fazal Ghani, and Tariq Mahmood. Spherical fuzzy sets and their applications in multi-attribute decision making problems. *J. Intell. Fuzzy Syst.*, 36:2829–2844, 2019.

[95] Muhammad Asif, Muhammad Akram, and Ghous Ali. Pythagorean fuzzy matroids with application. *Symmetry*, 12:423, 2020.

[96] Armen S Asratian, Tristan MJ Denley, and Roland Häggkvist. *Bipartite graphs and their applications*, volume 131. Cambridge university press, 1998.

[97] Armen S. Asratian and Raffi R. Kamalian. Interval colorings of edges of a multigraph. *ArXiv*, abs/1401.8079, 2014.

[98] Akın Osman Atagün and Hüseyin Kamacı. Strait fuzzy sets, strait fuzzy rough sets and their similarity measures-based decision making systems. *International Journal of Systems Science*, 54(12):2519–2535, 2023.

[99] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[100] Krassimir T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20:87–96, 1986.

[101] Krassimir T. Atanassov. More on intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 33:37–45, 1989.

[102] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[103] Krassimir T. Atanassov. On interval valued intuitionistic fuzzy sets. *Interval-Valued Intuitionistic Fuzzy Sets*, 2019.

[104] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[105] Michael Francis Atiyah and Ian G. MacDonald. Introduction to commutative algebra. 1969.

[106] Muhammad Azeem, Shabana Anwar, Muhammad Kamran Jamil, Muhammad Haris Saeed, and Muhammet Deveci. Topological numbers of fuzzy soft graphs and their application. *Inf. Sci.*, 667:120468, 2024.

[107] Sara Baase and Allen Van Gelder. Computer algorithms: Introduction to design and analysis. 1978.

[108] Rahil Baber, J Robert Johnson, and John Talbot. The minimal density of triangles in tripartite graphs. *LMS Journal of Computation and Mathematics*, 13:388–413, 2010.

[109] Vivek Badhe, RS Thakur, and GS Thakur. Vague set theory for profit pattern and decision making in uncertain data. *International journal of advanced computer science and applications*, 6(6):58–64, 2015.

[110] M Baghernejad and Rajab Ali Borzooei. Results on soft graphs and soft multigraphs with application in controlling urban traffic flows. *Soft Computing*, 27(16):11155–11175, 2023.

[111] M Amin Bahmanian and Mateja Šajna. Hypergraphs: connection and separation. *arXiv preprint arXiv:1504.04274*, 2015.

[112] Mohammad A Bahmanian and Mateja Sajna. Connection and separation in hypergraphs. *Theory and Applications of Graphs*, 2(2):5, 2015.

[113] Wenhui Bai, Juanjuan Ding, and Chao Zhang. Dual hesitant fuzzy graphs with applications to multi-attribute decision making. *International Journal of Cognitive Computing in Engineering*, 1:18–26, 2020.

[114] Raghpat Bains. The shape of space. how to visualise surfaces and three-dimensional manifolds. *Advances in Engineering Software*, 14:237, 1992.

[115] Alexandru T Balaban. Applications of graph theory in chemistry. *Journal of chemical information and computer sciences*, 25(3):334–343, 1985.

[116] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(1):93–107, 1997.

[117] Hans-Jürgen Bandelt and Jean-Pierre Barthélemy. Medians in median graphs. *Discret. Appl. Math.*, 8:131–142, 1984.

[118] Hans-Jurgen Bandelt and Victor Chepoi. Metric graph theory and geometry: a survey. *Contemporary Mathematics*, 453:49–86, 2008.

[119] Amotz Bar-Noy, Magnús M. Halldórsson, Guy Kortsarz, Riyadh Salman, and Hadas Shachnai. Sum multicoloring of graphs. *Journal of Algorithms*, 37(2):422–450, 2000.

[120] Anthony Baraconi. Homological algebra. *Lie Groups, Lie Algebras, and Cohomology. (MN-34), Volume 34*, 2021.

[121] Ildar Z Batyrshin. Fuzzy distribution sets. *Computación y Sistemas*, 26(3):1411–1416, 2022.

[122] Ildar Z Batyrshin, Angelica S Klimova, and Imre J Rudas. Parametric fuzzy distribution sets. In *2023 IEEE 27th International Conference on Intelligent Engineering Systems (INES)*, pages 000205–000208. IEEE, 2023.

[123] Richard E Bellman and Lotfi Asker Zadeh. Decision-making in a fuzzy environment. *Management science*, 17(4):B–141, 1970.

[124] H. Belyadi and Alireza Haghighat. Fuzzy logic. *Machine Learning Guide for Oil and Gas Using Python*, 2018.

[125] Vieri Benci, Mauro Di Nasso, Marco Forti, et al. The eightfold path to nonstandard analysis. *Nonstandard methods and applications in mathematics*, 25:3–44, 2006.

[126] Martin P. Bendsøe and Ole Sigmund. Topology optimization: "theory, methods, and applications". 2011.

[127] Laurine Bénéteau, Jérémie Chalopin, Victor Chepoi, and Yann Vaxès. Medians in median graphs and their cube complexes in linear time. In *International Colloquium on Automata, Languages and Programming*, 2019.

[128] Benjamin Benninghofen and Michael Richter. A general theory of superinfinitesimals. *Fundamenta Mathematicae*, 128(3):199–215, 1987.

[129] Biswajit Bera, Sk Amanathulla, and Sanat Kumar Mahato. A comprehensive study of picture fuzzy planar graphs with real-world applications. *Journal of uncertain Systems*, 16(04):2350009, 2023.

[130] Krist'of B'erczi, Bence Mátravölgyi, and Tamás Schwarcz. Reconfiguration of basis pairs in regular matroids. *Proceedings of the 56th Annual ACM Symposium on Theory of Computing*, 2023.

[131] Claude Berge. Multicoloring of a graph. In *Polyhedral Combinatorics: Dedicated to the memory of DR Fulkerson*, pages 226–234. Springer, 1978.

[132] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[133] Bonnie Berger, Rohit Singht, and Jinbo Xu. Graph algorithms for biological systems analysis. In *Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 142–151, 2008.

[134] Florian Berger. *Path Planning with spatial and temporal Constraints*. PhD thesis, Universitäts-und Landesbibliothek Bonn, 2011.

[135] Isabel Bermejo, I. Garc, and Enrique Reyes. Complete intersection graphs. 2014.

[136] T Bharathi, S Felixia, and S Leo. Intuitionistic felicitous fuzzy graphs.

[137] Anushree Bhattacharya and Madhumangal Pal. A fuzzy graph theory approach to the facility location problem: A case study in the indian banking system. *Mathematics*, 11(13):2992, 2023.

[138] Laxmi N. Bhuyan and Dharma P. Agrawal. Generalized hypercube and hyperbus structures for a computer network. *IEEE Transactions on Computers*, C-33:323–333, 1984.

[139] Chen Bi-hong. The operation of non-standard real number implemented with c++ language. 2000.

[140] Mathew Bibin, John Sunil Jacob, and Garg Harish. Vertex rough graphs. *Complex & Intelligent Systems*, 6(2):347–353, 2020.

[141] Vittorio Bilò, Ioannis Caragiannis, Angelo Fanelli, Michele Flammini, and Gianpiero Monaco. Simple greedy algorithms for fundamental multidimensional graph problems. *Dagstuhl Reports*, 80(125):1–13, 2017.

[142] Anders Björner and Francesco Brenti. Combinatorics of coxeter groups. 2005.

[143] Anders Björner and Günter M Ziegler. Introduction to greedoids. *Matroid applications*, 40:284–357, 1992.

[144] Mindaugas Bloznelis and Tomasz Luczak. Perfect matchings in random intersection graphs. *Acta Mathematica Hungarica*, 138:15–33, 2013.

[145] Lenore Blum, Michael Shub, and Stephen Smale. On a theory of computation and complexity over the real numbers; np-completeness. 1989.

[146] Hans L Bodlaender, Alexander Grigoriev, and Arie MCA Koster. Treewidth lower bounds with brambles. In *Algorithms–ESA 2005: 13th Annual European Symposium, Palma de Mallorca, Spain, October 3-6, 2005. Proceedings 13*, pages 391–402. Springer, 2005.

[147] Hans L Bodlaender, Alexander Grigoriev, and Arie MCA Koster. Treewidth lower bounds with brambles. *Algorithmica*, 51(1):81–98, 2008.

[148] Miklós Bóna. A walk through combinatorics. 2023.

[149] Anthony Bonato, Dejan Delic, and Changping Wang. Universal random semi-directed graphs. *Proceedings of ROGICS'08*, pages 138–148, 2008.

[150] Anthony Bonato, DEJAN DELIĆ, and Changping Wang. The structure and automorphisms of semi-directed graphs. *Journal of Multiple-Valued Logic & Soft Computing*, 27, 2016.

[151] John Adrian Bondy, Uppaluri Siva Ramachandra Murty, et al. *Graph theory with applications*, volume 290. Macmillan London, 1976.

[152] Flavia Bonomo-Braberman, Guillermo Durán, Martín D Safe, and Annegret K Wagler. On some graph classes related to perfect graphs: A survey. *Discrete Applied Mathematics*, 281:42–60, 2020.

[153] David Booth. Ultrafilters on a countable set. *Annals of Mathematical Logic*, 2(1):1–24, 1970.

[154] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[155] Laithun Boro, Madan Mohan Singh, and Jituparna Goswami. Line graph associated to the intersection graph of ideals of rings. *J. Math. Comput. Sci.*, 11(3):2736–2745, 2021.

[156] Prosenjit Bose, Paz Carmi, J Mark Keil, Anil Maheshwari, Saeed Mehrabi, Debajyoti Mondal, and Michiel Smid. Computing maximum independent set on outerstring graphs and their relatives. *Computational Geometry*, 103:101852, 2022.

[157] Julia Böttcher, Klaas P Pruessmann, Anusch Taraz, and Andreas Würfl. Bandwidth, treewidth, separators, expansion, and universality. *Electronic Notes in Discrete Mathematics*, 31:91–96, 2008.

[158] Ferdaous Bouaziz and Naveed Yaqoob. Rough hyperfilters in po-la-semihypergroups. *Discrete Dynamics in Nature and Society*, 2019:1–8, 2019.

[159] John M Boyer and Wendy J Myrvold. Simplified o (n) planarity by edge addition. *Graph Algorithms Appl*, 5:241, 2006.

[160] Peter A Bradshaw. Triangle packing on tripartite graphs is hard. *Rose-Hulman Undergraduate Mathematics Journal*, 20(1):7, 2019.

[161] Andreas Brandstädt, Van Bang Le, and Jeremy P Spinrad. *Graph classes: a survey*. SIAM, 1999.

[162] Nick Brettell, James G. Oxley, Charles Semple, and Geoff Whittle. The excluded minors for 2- and 3-regular matroids. *J. Comb. Theory B*, 163:133–218, 2022.

[163] Alain Bretto. Introduction to hypergraph theory and its use in engineering and image processing. *Advances in Imaging and Electron Physics*, 131:1–64, 2004.

[164] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[165] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.

[166] Said Broumi, Assia Bakali, and Ayoub Bahnasse. Neutrosophic sets: An overview. *Infinite Study*, 2018.

[167] Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache, and Faruk Karaaslan. Interval valued neutrosophic soft graphs. *viXra*, 2018.

[168] Said Broumi and Florentin Smarandache. Several similarity measures of neutrosophic sets. *Infinite Study*, 410(1), 2013.

[169] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[170] Maurizio Brunetti and Alma D'Aniello. On a graph connecting hyperbinary expansions. *arXiv preprint arXiv:1610.00903*, 2016.

[171] A. Buck and James M. Keller. Evaluating path costs in multi-attributed fuzzy weighted graphs. *2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 1–6, 2019.

[172] Dariusz Bugajewski. Fixed-point theorems in hyperconvex spaces revisited. *Mathematical and Computer Modelling*, 32:1457–1461, 2000.

[173] Ryan C. Bunge, M. K. Chwee, Andrew Michael Wokingham Cooper, Saad I. El-Zanati, Katlyn Kennedy, Dan P. Roberts, and C. C. Wilson. The spectrum for a multigraph on 4 vertices and 7 edges. 2018.

[174] Ryan C. Bunge, Joel Jeffries, Julie Kirkpatrick, Dan P. Roberts, and A. L. Sickman. Spectrum for multigraph designs on four vertices and six edges. 2017.

[175] Dietrich Burde. Group theory. *Computers, Rigidity, and Moduli*, 2019.

[176] M Burgin. Hyperfunctionals and generalized distributions. In *Stochastic Processes and Functional Analysis*, pages 109–132. CRC Press, 2004.

[177] M Burgin and A Krinik. Introduction to conditional hyperprobabilities. *Integration*, 2(3):285, 2011.

[178] Mark Burgin. Theory of real hypernumbers properties and axiomatization. *Science Direct Working Paper*, (S1574-0358):04, 2001.

[179] Mark Burgin. Theory of hypernumbers andextrafunctions: Functional spaces and differentiation. *Discrete Dynamics in Nature and Society*, 7(3):201–212, 2002.

[180] Mark Burgin. *Hypernumbers and extrafunctions: extending the classical calculus*. Springer Science & Business Media, 2012.

[181] MS Burgin. Topology in nonlinear extensions of hypernumbers. *Discrete Dynamics in Nature and Society*, 2005(2):145–170, 2005.

[182] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[183] Derun Cai, Moxian Song, Chenxi Sun, Baofeng Zhang, Shenda Hong, and Hongyan Li. Hypergraph structure learning for hypergraph neural networks. In *IJCAI*, pages 1923–1929, 2022.

[184] Derun Cai, Moxian Song, Chenxi Sun, Baofeng Zhang, linda Qiao, and Hongyan Li. Hypergraph structure learning for hypergraph neural networks. In *International Joint Conference on Artificial Intelligence*, 2022.

[185] Tiziana Calamoneri, Saverio Caminiti, Rossella Petreschi, and Stephan Olariu. On the l (h, k)-labeling of co-comparability graphs and circular-arc graphs. *Networks: An International Journal*, 53(1):27–34, 2009.

[186] Florent Capelli, Arnaud Durand, and Stefan Mengel. Hypergraph acyclicity and propositional model counting. In *International Conference on Theory and Applications of Satisfiability Testing*, pages 399–414. Springer, 2014.

[187] Arnaud Carayol and Christof Löding. Mso on the infinite binary tree: Choice and order. In *International Workshop on Computer Science Logic*, pages 161–176. Springer, 2007.

[188] Arnaud Carayol, Christof Löding, Damian Niwinski, and Igor Walukiewicz. Choice functions and well-orderings over the infinite binary tree. *Open Mathematics*, 8(4):662–682, 2010.

[189] Walter Alexandre Carnielli and Marcelo Esteban Coniglio. Paraconsistent set theory by predicating on consistency. *J. Log. Comput.*, 26:97–116, 2016.

[190] Oscar Castillo, Patricia Melin, Janusz Kacprzyk, and Witold Pedrycz. Type-2 fuzzy logic: theory and applications. In *2007 IEEE international conference on granular computing (GRC 2007)*, pages 145–145. IEEE, 2007.

[191] Tiziana Catarci and Laura Tarantino. A hypergraph-based framework for visual interaction with databases. *J. Vis. Lang. Comput.*, 6:135–166, 1995.

[192] Susumu Cato. Quasi-decisiveness, quasi-ultrafilter, and social quasi-orderings. *Social Choice and Welfare*, 41(1):169–202, 2013.

[193] Asli Celikyilmaz and I Burhan Türksen. Spectral learning with type-2 fuzzy numbers for question/answering system. In *IFSA/EUSFLAT Conf.*, pages 1388–1393, 2009.

[194] Nicolo Cesa-Bianchi, Claudio Gentile, Fabio Vitale, Giovanni Zappella, et al. Random spanning trees and the prediction of weighted graphs. *Journal of Machine Learning Research*, 14(1):1251–1284, 2013.

[195] Joe Chaffee and Matt Noble. Dimension 4 and dimension 5 graphs with minimum edge set. *Australas. J Comb.*, 64:327–333, 2016.

[196] Mathieu Chapelle, Frédéric Mazoit, and Ioan Todinca. Constructing brambles. In *International Symposium on Mathematical Foundations of Computer Science*, pages 223–234. Springer, 2009.

[197] Siobhan Chapman. Logic and conversation. 2005.

[198] Ch. A. Charalambides. Enumerative combinatorics. *SIGACT News*, 39:25–27, 2018.

[199] Niladri Chatterjee, Aayush Singha Roy, and Nidhika Yadav. Soft rough set based span for unsupervised keyword extraction. *J. Intell. Fuzzy Syst.*, 42:4379–4386, 2021.

[200] Shshank Chaube, Dheeraj Kumar Joshi, and Chandan Singh Ujarari. Hesitant bifuzzy set (an introduction): A new approach to assess the reliability of the systems. *Mathematics and Computers in Simulation*, 205:98–107, 2023.

[201] P Chellamani, D Ajay, Mohammed M Al-Shamiri, and Rashad Ismail. *Pythagorean Neutrosophic Planar Graphs with an Application in Decision-Making*. Infinite Study, 2023.

[202] P. Chellamani, D. Ajay, Said Broumi, and T. Antony Alphonnse Ligori. An approach to decision-making via picture fuzzy soft graphs. *Granular Computing*, 7:527 – 548, 2021.

[203] P Chellamani, D Ajay, Said Broumi, and T Antony Alphonse Ligori. An approach to decision-making via picture fuzzy soft graphs. *Granular computing*, pages 1–22, 2021.

[204] S Robinson Chellathurai and L Jesmalar. Core in rough graph and weighted rough graph. *International Journal of Contemporary Mathematical Sciences*, 11(6):251–265, 2016.

[205] Jianer Chen, Anxiao Jiang, Iyad A Kanj, Ge Xia, and Fenghui Zhang. Separability and topology control of quasi unit disk graphs. *Wireless Networks*, 17:53–67, 2011.

[206] Jiqian Chen, Jun Ye, and Shigui Du. Scale effect and anisotropy analyzed for neutrosophic numbers of rock joint roughness coefficient based on neutrosophic statistics. *Symmetry*, 9:208, 2017.

[207] Shyi-Ming Chen. Operations on fuzzy numbers with function principal. 1985.

[208] Shyi-Ming Chen. Measures of similarity between vague sets. *Fuzzy sets and Systems*, 74(2):217–223, 1995.

[209] Ho Yee Cheung, Lap Chi Lau, and Kai Man Leung. Algebraic algorithms for linear matroid parity problems. *ACM Transactions on Algorithms (TALG)*, 10(3):1–26, 2014.

[210] Eli Chien, Chao Pan, Jianhao Peng, and Olgica Milenkovic. You are allset: A multiset function framework for hypergraph neural networks. *ArXiv*, abs/2106.13264, 2021.

[211] Phyllis Z Chinn, Jarmila Chvátalová, Alexander K Dewdney, and Norman E Gibbs. The bandwidth problem for graphs and matrices—a survey. *Journal of Graph Theory*, 6(3):223–254, 1982.

[212] Yu-Cheng Chiu and Eric Marberg. Algebraic combinatorics. 2013.

[213] C. C. Chou. The representation of multiplication operation on fuzzy numbers and application to solving fuzzy multiple criteria decision making problems. In *Pacific Rim International Conference on Artificial Intelligence*, 2006.

[214] G. Christoph. Limit theorems of probability theory. 2023.

[215] Marek Chrobak, Michael T Goodrichy, and Roberto Tamassiaz. On the volume and resolution of 3-dimensional convex graph drawing. In *Electronic Proceedings of the 5th MSI-Stony Brook Workshop on Computational Geometry*. Citeseer, 1995.

[216] M. Chudnovsky and Paul D. Seymour. Claw-free graphs. iii. circular interval graphs. *J. Comb. Theory B*, 98:812–834, 2008.

[217] M. Chudnovsky and Paul D. Seymour. Claw-free graphs. v. global structure. *J. Comb. Theory B*, 98:1373–1410, 2008.

[218] Vašek Chvátal. Antimatroids, betweenness, convexity. *Research Trends in Combinatorial Optimization: Bonn 2008*, pages 57–64, 2009.

[219] Yusuf Civan and Demet Taylan. Coloring hypercomplete and hyperpath graphs. *Turkish Journal of Mathematics*, 38(1):1–15, 2014.

[220] Brent N Clark, Charles J Colbourn, and David S Johnson. Unit disk graphs. *Discrete mathematics*, 86(1-3):165–177, 1990.

[221] Jonathan Clive, Max A Woodbury, and Ilene C Siegler. Fuzzy and crisp set-theoretic-based classification of health and disease: A qualitative and quantitative comparison. *Journal of Medical Systems*, 7:317–332, 1983.

[222] Henri Cohen. A course in computational algebraic number theory. In *Graduate texts in mathematics*, 1993.

[223] William Wistar Comfort and Stylianos Negrepontis. *The theory of ultrafilters*, volume 211. Springer Science & Business Media, 2012.

[224] Jonathan Pila Contents. Set theory. *Mathematical Statistics with Applications in R*, 2018.

[225] Thomas H. Cormen, Charles E. Leiserson, and Ronald L. Rivest. Introduction to algorithms. *Journal of the Operational Research Society*, 42:816–817, 1991.

[226] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. Introduction to algorithms, 2nd edition. 2001.

[227] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. Introduction to algorithms, third edition. 2009.

[228] Chris Cornelis, Pieter M. M. De Kesel, and Etienne E. Kerre. Shortest paths in fuzzy weighted graphs. *International Journal of Intelligent Systems*, 19, 2004.

[229] Bruno Courcelle, Joost Engelfriet, and Grzegorz Rozenberg. Context-free handle-rewriting hypergraph grammars. In *Graph-Grammars and Their Application to Computer Science*, 1990.

[230] Bruno Courcelle, Joost Engelfriet, and Grzegorz Rozenberg. Handle-rewriting hypergraph grammars. *J. Comput. Syst. Sci.*, 46:218–270, 1993.

[231] Paolo Crivelli, Timothy Luke Williamson, Gareth Hughes, and Max J. Cresswell. A new introduction to modal logic. 1998.

[232] Bui Cong Cuong. Pythagorean picture fuzzy sets, part 1- basic notions. *Journal of Computer Science and Cybernetics*, 2019.

[233] Bui Cong Cuong. Pythagorean picture fuzzy sets(ppfs), part 2- some main picture logic operators on ppfs and some picture inference processes in ppf systems. *Journal of Computer Science and Cybernetics*, 2022.

[234] Nigel Cutland. *Nonstandard analysis and its applications*, volume 10. Cambridge University Press, 1988.

[235] Sebastian Czerwinski, Jarosaw Grytczuk, and Wiktor Rafa elazny. Lucky labelings of graphs. *Inf. Process. Lett.*, 109:1078–1081, 2009.

[236] Newton C. A. da Costa and Luiz Paulo de Alcantara. On paraconsistent set theories. 1992.

[237] Ajoy Kanti Das and Carlos Granados. Fp-intuitionistic multi fuzzy n-soft set and its induced fp-hesitant n soft set in decision-making. *Decision making: applications in management and engineering*, 5(1):67–89, 2022.

[238] Birojit Das, Baby Bhattacharya, Jayasree Chakraborty, and Binod Chandra Tripathy. Generalized fuzzy closed sets in a fuzzy bitopological space via γ open sets. 2020.

[239] Sujit Das and Samarjit Kar. Intuitionistic multi fuzzy soft set and its application in decision making. In *Pattern Recognition and Machine Intelligence: 5th International Conference, PReMI 2013, Kolkata, India, December 10-14, 2013. Proceedings 5*, pages 587–592. Springer, 2013.

[240] Suman Das, Rakhal Das, and Surapati Pramanik. Single valued pentapartitioned neutrosophic graphs. *Neutrosophic Sets and Systems*, 50(1):225–238, 2022.

[241] Brian A. Davey and Hilary A. Priestley. Introduction to lattices and order. 1990.

[242] Brian A. Davey and Hilary A. Priestley. Introduction to lattices and order: Frontmatter. 2002.

[243] Martin D. Davis. Selected papers of abraham robinson: Nonstandard analysis and philosophy. *Journal of Symbolic Logic*, 47:203–210, 1982.

[244] C Bryan Dawson. *Calculus Set Free: Infinitesimals to the Rescue*. Oxford University Press, 2021.

[245] Arnab Kumar De, Debjani Chakraborty, and Animesh Biswas. Literature review on type-2 fuzzy set theory. *Soft Computing*, 26(18):9049–9068, 2022.

[246] Supriya Kumar De, Ranjit Biswas, and Akhil Ranjan Roy. Some operations on intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 114(3):477–484, 2000.

[247] Philipp de Col, Fabian Klute, and Martin Nöllenburg. Mixed linear layouts: Complexity, heuristics, and experiments. In *International Symposium on Graph Drawing and Network Visualization*, pages 460–467. Springer, 2019.

[248] Somen Debnath. Fuzzy hypersoft sets and its weightage operator for decision making. 2021.

[249] Somen Debnath. Fuzzy hypersoft sets and its weightage operator for decision making. *Journal of fuzzy extension and applications*, 2(2):163–170, 2021.

[250] Mohamed R. Zeen El Deen. Edge even graceful labeling of some graphs. *Journal of the Egyptian Mathematical Society*, 27:1–15, 2019.

[251] Michaël Defferrard, Xavier Bresson, and Pierre Vandergheynst. Convolutional neural networks on graphs with fast localized spectral filtering. *Advances in neural information processing systems*, 29, 2016.

[252] O. R. Dehghan. Study of bipolar fuzzy soft hypervector spaces. 2023.

[253] Omid Reza Dehghan and Mohadeseh Rahbar Nodehi. Some results on soft hypervector spaces. 2020.

[254] Omid Reza Dehghan and Morteza Norouzi. Soft vectors in soft hypervector spaces. *2020 8th Iranian Joint Congress on Fuzzy and intelligent Systems (CFIS)*, pages 217–221, 2020.

[255] OR Dehghan. Linear functionals on hypervector spaces. *Filomat*, 34(9):3031–3043, 2020.

[256] Matthias Dehmer and Abbe Mowshowitz. A history of graph entropy measures. *Information Sciences*, 181(1):57–78, 2011.

[257] Seyyed Jalaladdin Hosseini Dehshiri and Maghsoud Amiri. Evaluation of blockchain implementation solutions in the sustainable supply chain: A novel hybrid decision approach based on z-numbers. *Expert Syst. Appl.*, 235:121123, 2023.

[258] Irfan Deli, Mumtaz Ali, and Florentin Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. *2015 International Conference on Advanced Mechatronic Systems (ICAMechS)*, pages 249–254, 2015.

[259] Irfan Deli and Yusuf Subas. A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. *International Journal of Machine Learning and Cybernetics*, 8:1309 – 1322, 2016.

[260] Irfan Deli, Yusuf Subas, Florentin Smarandache, and Mumtaz Ali. Interval valued bipolar neutrosophic sets and their application in pattern recognition. *viXra*, 2016.

[261] Jayant V. Deshpande. On continuity of a partial order. 1968.

[262] Muhmamet Deveci, Dragan Pamucar, Ilgin Gokasar, Mario Köppen, and Brij Bhooshan Gupta. Personal mobility in metaverse with autonomous vehicles using q-rung orthopair fuzzy sets based opa-rafsi model. *IEEE Transactions on Intelligent Transportation Systems*, 24:15642–15651, 2023.

[263] R. Devi. Minimal domination via neutrosophic over graphs. *1ST INTERNATIONAL CONFERENCE ON MATHEMATICAL TECHNIQUES AND APPLICATIONS: ICMTA2020*, 2020.

[264] R. Devi and G. Muthumari. Dominating energy in neutrosophic over topologized graphs. *2ND INTERNATIONAL CONFERENCE ON MATHEMATICAL TECHNIQUES AND APPLICATIONS: ICMTA2021*, 2022.

[265] R Aruna Devi and K Anitha. Construction of rough graph to handle uncertain pattern from an information system. *arXiv preprint arXiv:2205.10127*, 2022.

[266] R Narmada Devi and R Dhavaseelan. *New Type of Neutrosophic Off Graphs*. Infinite Study.

[267] Keith Devlin. *The joy of sets: fundamentals of contemporary set theory*. Springer Science & Business Media, 1994.

[268] Keith J. Devlin. Fundamentals of contemporary set theory. 1979.

[269] Matt DeVos. Flows on bidirected graphs. *arXiv preprint arXiv:1310.8406*, 2013.

[270] G. Dhanalakshmi, S. Sandhiya, and Florentin Smarandache. Selection of the best process for desalination under a treesoft set environment using the multi-criteria decision-making method. *International Journal of Neutrosophic Science*, 2024.

[271] G Dhanalakshmi, S Sandhiya, Florentin Smarandache, et al. Selection of the best process for desalination under a treesoft set environment using the multi-criteria decision-making method. *International Journal of Neutrosophic Science*, 23(3):140–40, 2024.

[272] Giuseppe Di Maio and Somashekhar Naimpally. Comparison of hypertopologies. 1991.

[273] Scott Dick, Ronald R. Yager, and Omolbanin Yazdanbakhsh. On pythagorean and complex fuzzy set operations. *IEEE Transactions on Fuzzy Systems*, 24:1009–1021, 2016.

[274] Reinhard Diestel. Graduate texts in mathematics: Graph theory.

[275] Reinhard Diestel. Graph theory 3rd ed. *Graduate texts in mathematics*, 173(33):12, 2005.

[276] Reinhard Diestel. Ends and tangles. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 87, pages 223–244. Springer, 2017.

[277] Reinhard Diestel. Abstract separation systems. *Order*, 35:157–170, 2018.

[278] Reinhard Diestel. Tree sets. *Order*, 35(1):171–192, 2018.

[279] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[280] Reinhard Diestel, Joshua Erde, and Daniel Weißauer. Structural submodularity and tangles in abstract separation systems. *J. Comb. Theory A*, 167:155–180, 2018.

[281] Reinhard Diestel and Sang il Oum. Tangle-tree duality: In graphs, matroids and beyond. *Combinatorica*, 39:879 – 910, 2017.

[282] Reinhard Diestel and Sang-il Oum. Tangle-tree duality in abstract separation systems. *Advances in Mathematics*, 377:107470, 2021.

[283] Karl Dilcher and Larry Ericksen. Hyperbinary expansions and stern polynomials. *The Electronic Journal of Combinatorics*, pages P2–24, 2015.

[284] Paul Van Dooren and Luc de Ridder. An adaptive algorithm for numerical integration over the n-cube. *Journal of Computational and Applied Mathematics*, 2:207–217, 1976.

[285] Manfred Droste and Heiko Vogler. Weighted tree automata and weighted logics. *Theoretical Computer Science*, 366(3):228–247, 2006.

[286] Kanghua Du and Yuming Du. Research on performance evaluation of intangible assets operation and management in sports events with double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:2813–2822, 2023.

[287] A. Dubickas. On the distance between two algebraic numbers. *Bulletin of the Malaysian Mathematical Sciences Society*, 43:3049–3064, 2020.

[288] Didier Dubois and Henri Prade. Operations on fuzzy numbers. *International Journal of Systems Science*, 9:613–626, 1978.

[289] Didier Dubois and Henri Prade. Operations on fuzzy numbers. *International Journal of systems science*, 9(6):613–626, 1978.

[290] Didier Dubois and Henri Prade. Ranking fuzzy numbers in the setting of possibility theory. *Information sciences*, 30(3):183–224, 1983.

[291] Didier Dubois and Henri Prade. A review of fuzzy set aggregation connectives. *Information sciences*, 36(1-2):85–121, 1985.

[292] Didier Dubois and Henri Prade. The mean value of a fuzzy number. *Fuzzy sets and systems*, 24(3):279–300, 1987.

[293] Didier Dubois and Henri Prade. Fuzzy numbers: an overview. *Readings in Fuzzy Sets for Intelligent Systems*, pages 112–148, 1993.

[294] Didier Dubois and Henri Prade. Fuzzy sets and systems: theory and applications. In *Mathematics in Science and Engineering*, 2011.

[295] Didier Dubois and Henri Prade. *Fundamentals of fuzzy sets*, volume 7. Springer Science & Business Media, 2012.

[296] Didier Dubois, Henri Prade, and Lotfi A. Zadeh. Fundamentals of fuzzy sets. 2000.

[297] Dwight Duffus and Ivan Rival. A structure theory for ordered sets. *Discrete Mathematics*, 35(1-3):53–118, 1981.

[298] Vida Dujmović and David R Wood. On linear layouts of graphs. *Discrete Mathematics & Theoretical Computer Science*, 6, 2004.

[299] Andrew L Dulmage and Nathan S Mendelsohn. Coverings of bipartite graphs. *Canadian Journal of Mathematics*, 10:517–534, 1958.

[300] Richard Durrett. Probability: Theory and examples. 1993.

[301] Fatih Ecer, Gholamreza Haseli, Raghunathan Krishankumar, and Mostafa Hajiaghaei-Keshteli. Evaluation of sustainable cold chain suppliers using a combined multi-criteria group decision-making framework under fuzzy ze-numbers. *Expert Systems with Applications*, 245:123063, 2024.

[302] Kemal Efe. A variation on the hypercube with lower diameter. *IEEE Trans. Computers*, 40:1312–1316, 1991.

[303] Philip Ehrlich. *Real numbers, generalizations of the reals, and theories of continua*, volume 242. Springer Science & Business Media, 2013.

[304] David Eisenbud. Commutative algebra: with a view toward algebraic geometry. 1995.

[305] PA Ejegwa, SO Akowe, PM Otene, and JM Ikyule. An overview on intuitionistic fuzzy sets. *Int. J. Sci. Technol. Res*, 3(3):142–145, 2014.

[306] Nancy El-Hefenawy, Mohamed A Metwally, Zenat M Ahmed, and Ibrahim M El-Henawy. A review on the applications of neutrosophic sets. *Journal of Computational and Theoretical Nanoscience*, 13(1):936–944, 2016.

[307] S. A. El-Sheikh, S. A. Kandil, and S. H. Shalil. Increasing and decreasing soft rough set approximations. *Int. J. Fuzzy Log. Intell. Syst.*, 23:425–435, 2023.

[308] Christian Elbracht, Jakob Kneip, and Maximilian Teegen. Trees of tangles in infinite separation systems. *Mathematical Proceedings of the Cambridge Philosophical Society*, 173:297 – 327, 2020.

[309] Herbert B. Enderton. A mathematical introduction to logic. 1972.

[310] Herbert B Enderton. *Elements of set theory*. Academic press, 1977.

[311] Joost Engelfriet and Linda Heyker. The string generating power of context-free hypergraph grammars. *J. Comput. Syst. Sci.*, 43:328–360, 1991.

[312] Joost Engelfriet and Linda Heyker. Context-free hypergraph grammars have the same term-generating power as attribute grammars. *Acta Informatica*, 29:161–210, 1992.

[313] Joost Engelfriet and Grzegorz Rozenberg. A comparison of boundary graph grammars and context-free hypergraph grammars. *Inf. Comput.*, 84:163–206, 1990.

[314] Hikoe Enomoto, Tomoki Nakamigawa, and Katsuhiro Ota. On the pagenumber of complete bipartite graphs. *journal of combinatorial theory, Series B*, 71(1):111–120, 1997.

[315] Martin J Erickson. *Introduction to combinatorics*. John Wiley & Sons, 2013.

[316] Raphael Espínola and Mohamed A Khamsi. Introduction to hyperconvex spaces. In *Handbook of metric fixed point theory*, pages 391–435. Springer, 2001.

[317] Shimon Even. *Graph algorithms*. Cambridge University Press, 2011.

[318] Sukhada Fadnavis. On the roots of hypergraph chromatic polynomials. *arXiv preprint arXiv:1509.05950*, 2015.

[319] Wenqi Fan, Yao Ma, Qing Li, Yuan He, Eric Zhao, Jiliang Tang, and Dawei Yin. Graph neural networks for social recommendation. In *The world wide web conference*, pages 417–426, 2019.

[320] Muhammad Farooq, Asghar Khan, and Bijan Davvaz. Characterizations of ordered semihypergroups by the properties of their intersectional-soft generalized bi-hyperideals. *Soft Computing*, 22:3001–3010, 2018.

[321] Oriol Farras and Carles Padró. Extending brickell–davenport theorem to non-perfect secret sharing schemes. *Designs, Codes and Cryptography*, 74:495–510, 2015.

[322] Ralph J. Faudree, Evelyne Flandrin, and Zdenek Ryjavcek. Claw-free graphs - a survey. *Discret. Math.*, 164:87–147, 1997.

[323] Tomás Feder. Product graph representations. *J. Graph Theory*, 16:467–488, 1992.

[324] William Feller. An introduction to probability theory and its applications. 1959.

[325] Feng Feng. Soft rough sets applied to multicriteria group decision making. 2011.

[326] Feng Feng, Xiaoyan Liu, Violeta Leoreanu-Fotea, and Young Bae Jun. Soft sets and soft rough sets. *Information Sciences*, 181(6):1125–1137, 2011.

[327] Yifan Feng, Haoxuan You, Zizhao Zhang, R. Ji, and Yue Gao. Hypergraph neural networks. In *AAAI Conference on Artificial Intelligence*, 2018.

[328] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.

[329] Abdolvahhab Fetanat, Mohsen Tayebi, and Mehran Moteraghi. Technology evaluation for biogas production from animal waste in circular carbon economy: A complex spherical fuzzy set-based decision-making framework. *Bioresource Technology Reports*, 2023.

[330] Juan Carlos Figueroa-García, Roman Neruda, and German Jairo Hernandez-Perez. On cosine fuzzy sets and uncertainty quantification. *Engineering Applications of Artificial Intelligence*, 138:109241, 2024.

[331] Fery Firmansah, Tasari Tasari, and Muhammad Ridlo Yuwono. Odd harmonious labeling of the zinnia flower graphs. *JURNAL ILMIAH SAINS*, 2023.

[332] Ian S. Fischer. Dual-number methods in kinematics, statics and dynamics. 1998.

[333] Philippe Flajolet and Robert Sedgewick. Analytic combinatorics. 2009.

[334] Fedor V Fomin, Mathieu Liedloff, Pedro Montealegre, and Ioan Todinca. Algorithms parameterized by vertex cover and modular width, through potential maximal cliques. *Algorithmica*, 80:1146–1169, 2018.

[335] Fedor V Fomin and Dimitrios M Thilikos. On the monotonicity of games generated by symmetric submodular functions. *Discrete Applied Mathematics*, 131(2):323–335, 2003.

[336] William Freed. Apex infinitesimal calculus, volume i.

[337] Ragnar Freij-Hollanti and Patricija Šapokaitė. Matroidal cycles and hypergraph families. *arXiv preprint arXiv:2410.23932*, 2024.

[338] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[339] Stefan Friedl. Algebraic topology. *Graduate Studies in Mathematics*, 2020.

[340] Takaaki Fujita. General, general weak, anti, balanced, and semi-neutrosophic graph.

[341] Takaaki Fujita. General plithogenic soft rough graphs and some related graph classes.

[342] Takaaki Fujita. Neutrosophic circular-arc graphs and proper circular-arc graphs.

[343] Takaaki Fujita. Relation between ultra matroid and linear decomposition. *Italian Journal of Pure and Applied Mathematics. Accepted*.

[344] Takaaki Fujita. Quasi-matroid on connectivity system. *Preprint*, 2023.

[345] Takaaki Fujita. Revisiting linear width: Rethinking the relationship between single ideal and linear obstacle. *Journal of Advances in Mathematics and Computer Science*, 38(10):167–171, 2023.

[346] Takaaki Fujita. Algorithm and game of ultra matroid and linear tangle on connectivity system. June 2024.

[347] Takaaki Fujita. Bipolar turiyam graph and interval-valued turiyam graph, June 2024.

[348] Takaaki Fujita. Bunch graph in fuzzy, neutrosophic, and plithogenic graphs, June 2024. License: CC BY 4.0.

[349] Takaaki Fujita. Claw-free graph and at-free graph in fuzzy, neutrosophic, and plithogenic graphs, June 2024. License: CC BY 4.0.

[350] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. 2024.

[351] Takaaki Fujita. Contact graphs in fuzzy and neutrosophic graphs. Preprint, May 2024, File available, 2024.

[352] Takaaki Fujita. Extension of bipolar, interval-valued, and complex neutrosophic soft expert graphs to general plithogenic soft rough expert mixed graphs. July 2024.

[353] Takaaki Fujita. General plithogenic soft rough graphs and some related graph classes, June 2024. License: CC BY 4.0.

[354] Takaaki Fujita. General plithogenic soft rough graphs and some related graph classes. 2024.

[355] Takaaki Fujita. Inapproximability of graph width parameters under small set expansion hypothesis. June 2024.

[356] Takaaki Fujita. Interval graphs and proper interval graphs in fuzzy and neutrosophic graphs. Preprint, July 2024, File available, 2024.

[357] Takaaki Fujita. Matroid, ideal, ultrafilter, tangle, and so on: Reconsideration of obstruction to linear decomposition. *International Journal of Mathematics Trends and Technology (IJMTT)*, 70:18–29, 2024.

[358] Takaaki Fujita. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. June 2024.

[359] Takaaki Fujita. Neutrosophic circular-arc graphs and proper circular-arc graphs. 2024.

[360] Takaaki Fujita. Neutrosophic matroid, antimatroid, and greedoid, 2024.

[361] Takaaki Fujita. Neutrosophic matroid, antimatroid, and greedoid, 2024.

[362] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[363] Takaaki Fujita. Obstruction for hypertree width and superhypertree width. *preprint(researchgate)*, 2024.

[364] Takaaki Fujita. Permutation graphs in fuzzy and neutrosophic graphs. *Preprint*, July 2024. File available.

[365] Takaaki Fujita. Plithogenic line graph, star graph, and regular graph. 2024.

[366] Takaaki Fujita. Pythagorean, fermatean, and complex turiyam graphs: Relations with general plithogenic graphs, June 2024. License: CC BY 4.0.

[367] Takaaki Fujita. Quasi-ultrafilter on the connectivity system: Its relationship to branch-decomposition. *International Journal of Mathematics Trends and Technology-IJMTT*, 70, 2024.

[368] Takaaki Fujita. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. *ResearchGate(Preprint)*, 2024.

[369] Takaaki Fujita. Short note of neutrosophic closure matroids. 2024.

[370] Takaaki Fujita. Short note of supertree-width and n-superhypertree-width. *Neutrosophic Sets and Systems*, 77:54–78, 2024.

[371] Takaaki Fujita. Some graph structures in fuzzy, neutrosophic, and plithogenic graphs. 2024.

[372] Takaaki Fujita. Supplemental figure: Comparing graph width parameter (graph parameter hierarchy), August 2024. License: CC BY 4.0.

[373] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *ResearchGate*, July 2024.

[374] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs, 2024.

[375] Takaaki Fujita. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. *ResearchGate*, July 2024.

[376] Takaaki Fujita. Survey of trees, forests, and paths in fuzzy and neutrosophic graphs. July 2024.

[377] Takaaki Fujita. Uncertain automata and uncertain graph grammar, June 2024. License CC BY 4.0.

[378] Takaaki Fujita. Uncertain labeling graphs and uncertain graph classes (with survey for various uncertain sets). July 2024.

[379] Takaaki Fujita. Unit disk graphs in fuzzy and neutrosophic graphs. June 2024. Available under CC BY 4.0 license.

[380] Takaaki Fujita. Vague soft expert graph and complex fuzzy soft expert graph. *ResearchGate*, June 2024. License CC BY 4.0.

[381] Takaaki Fujita. Various properties of various ultrafilters, various graph width parameters, and various connectivity systems. *arXiv preprint arXiv:2408.02299*, 2024.

[382] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[383] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[384] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[385] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[386] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[387] Nagata Furukawa. A parametric total order on fuzzy numbers and a fuzzy shortest route problem. *Optimization*, 30:367–377, 1994.

[388] T. G. Logic in practice. *Nature*, 134:344–344, 1934.

[389] Joseph A Gallian. Graph labeling. *The electronic journal of combinatorics*, pages DS6–Dec, 2012.

[390] Joseph A Gallian. A dynamic survey of graph labeling. *Electronic Journal of combinatorics*, 6(25):4–623, 2022.

[391] Fernando Gama, Joan Bruna, and Alejandro Ribeiro. Stability properties of graph neural networks. *IEEE Transactions on Signal Processing*, 68:5680–5695, 2020.

[392] A. Nagoor Gani and S. Shajitha Begum. Perfect intuitionistic fuzzy graphs. *Bulletin of Pure & Applied Sciences- Mathematics and Statistics*, pages 145–152, 2011.

[393] A Nagoor Gani and SR Latha. On irregular fuzzy graphs. *Applied Mathematical Sciences*, 6(11):517–523, 2012.

[394] A Nagoor Gani and K Radha. On regular fuzzy graphs. 2008.

[395] Jack Ganssle. *Embedded systems dictionary*. CRC press, 2003.

[396] Hongyang Gao and Shuiwang Ji. Graph u-nets. In *international conference on machine learning*, pages 2083–2092. PMLR, 2019.

[397] Shang Gao, Zaiyue Zhang, and Cungen Cao. Multiplication operation on fuzzy numbers. *J. Softw.*, 4:331–338, 2009.

[398] Yue Gao, Yifan Feng, Shuyi Ji, and Rongrong Ji. Hgnn+: General hypergraph neural networks. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 45:3181–3199, 2022.

[399] Yue Gao, Zizhao Zhang, Haojie Lin, Xibin Zhao, Shaoyi Du, and Changqing Zou. Hypergraph learning: Methods and practices. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 44(5):2548–2566, 2020.

[400] Harish Garg and Sukhveer Singh. A novel triangular interval type-2 intuitionistic fuzzy sets and their aggregation operators. *Iranian Journal of Fuzzy Systems*, 15:69–93, 2018.

[401] Jonathan M Garibaldi, Marcin Jaroszewski, and Salang Musikasuwan. New concepts related to non-stationary fuzzy sets. In *2007 IEEE International Fuzzy Systems Conference*, pages 1–6. IEEE, 2007.

[402] Fanica Gavril. Algorithms on circular-arc graphs. *Networks*, 4(4):357–369, 1974.

[403] Fănică Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *Journal of Combinatorial Theory, Series B*, 16(1):47–56, 1974.

[404] Laura Gellert and Raman Sanyal. On degree sequences of undirected, directed, and bidirected graphs. *European Journal of Combinatorics*, 64:113–124, 2017.

[405] Bobin George, Jinta Jose, and Rajesh K Thumbakara. Co-normal products and modular products of soft graphs. *Discrete Mathematics, Algorithms and Applications*, 16(02):2350012, 2024.

[406] Fayed F. M. Ghaleb, Azza Ali Taha, Maryam Hazman, Mahmoud Abd Ellatif, and Mona Abbass. On quasi cycles in hypergraph databases. *IEEE Access*, 8:147560–147568, 2020.

[407] Mona Gharib, Fatima Rajab, and Mona Mohamed. Harnessing tree soft set and soft computing techniques' capabilities in bioinformatics: Analysis, improvements, and applications. *Neutrosophic sets and systems*, 61:579–597, 2023.

[408] Lucantonio Ghionna, Luigi Granata, Gianluigi Greco, and Francesco Scarcello. Hypertree decompositions for query optimization. In *2007 IEEE 23rd International Conference on Data Engineering*, pages 36–45. IEEE, 2006.

[409] M. Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. 2022.

[410] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[411] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[412] Daniela Gifu, Florentin Smarandache, Soft Sets Extensions: Innovating Healthcare Claims Analysis, Preprint, August 2024; DOI: 10.20944/preprints202408.1323.v2, https://www.preprints.org/manuscript/202408.1323/v2

[413] Puspendu Giri, Somnath Paul, and Bijoy Krishna Debnath. A fuzzy graph theory and matrix approach (fuzzy gtma) to select the best renewable energy alternative in india. *Applied Energy*, 358:122582, 2024.

[414] Steven Givant and Hajnal Andréka. Quasi-bijective relation algebras. 2017.

[415] Steven Givant and Hajnal Andréka. Two-quasi-bijective relation algebras. 2017.

[416] Chris Godsil. *Algebraic combinatorics*. Routledge, 2017.

[417] Roy Goetschel and William Voxman. Fuzzy matroids. 1988.

[418] Roy Goetschel and William Voxman. Fuzzy matroid sums and a greedy algorithm. *Fuzzy Sets and Systems*, 37:189–200, 1990.

[419] Roy Goetschel and William Voxman. Spanning properties for fuzzy matroids. *Fuzzy Sets and Systems*, 51:313–321, 1992.

[420] Isaac Goldbring. *Ultrafilters throughout mathematics*, volume 220. American Mathematical Society, 2022.

[421] J Pascal Gollin and Jakob Kneip. Representations of infinite tree sets. *Order*, 38(1):79–96, 2021.

[422] Martin Charles Golumbic. Interval graphs. In *Annals of Discrete Mathematics*, volume 57, pages 171–202. Elsevier, 2004.

[423] Martin Charles Golumbic, Haim Kaplan, and Ron Shamir. On the complexity of dna physical mapping. *Advances in Applied Mathematics*, 15(3):251–261, 1994.

[424] Ana Claudia Golzio. A brief historical survey on hyperstructures in algebra and logic. *South American Journal of Logic*, 4(1):2446–6719, 2018.

[425] S Gomathy, D Nagarajan, S Broumi, and M Lathamaheswari. *Plithogenic sets and their application in decision making*. Infinite Study, 2020.

[426] Ke Gong, Panpan Wang, and Zhi Xiao. Bijective soft set decision system based parameters reduction under fuzzy environments. *Applied Mathematical Modelling*, 37:4474–4485, 2013.

[427] Ke Gong, Zhi Xiao, and Xia Zhang. The bijective soft set with its operations. *Comput. Math. Appl.*, 60:2270–2278, 2010.

[428] Yicheng Gong, Lu Gan, and Hongqing Liu. Multi-channel modulo samplers constructed from gaussian integers. *IEEE Signal Processing Letters*, 28:1828–1832, 2021.

[429] Zengtai Gong and Junhu Wang. Hesitant fuzzy graphs, hesitant fuzzy hypergraphs and fuzzy graph decisions. *Journal of Intelligent & Fuzzy Systems*, 40(1):865–875, 2021.

[430] Harry Gonshor. An introduction to the theory of surreal numbers. 1986.

[431] Igor I Gorban. *Randomness and Hyper-randomness*. Springer, 2018.

[432] Igor I Gorban and Igor I Gorban. Hyper-random events and variables. *The Statistical Stability Phenomenon*, pages 121–142, 2017.

[433] Marco Gori, Gabriele Monfardini, and Franco Scarselli. A new model for learning in graph domains. In *Proceedings. 2005 IEEE international joint conference on neural networks, 2005.*, volume 2, pages 729–734. IEEE, 2005.

[434] Georg Gottlob, Gianluigi Greco, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: Questions and answers. In *Proceedings of the 35th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems*, pages 57–74, 2016.

[435] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.

[436] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: A survey. In *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázne, Czech Republic, August 27–31, 2001 Proceedings 26*, pages 37–57. Springer, 2001.

[437] Georg Gottlob and Reinhard Pichler. Hypergraphs in model checking: Acyclicity and hypertree-width versus clique-width. *SIAM Journal on Computing*, 33(2):351–378, 2004.

[438] Kuppuswami Govindan, Sujatha Ramalingam, Nagarajan Deivanayagampillai, Said Broumi, and Kavikumar Jacob. Markov chain based on neutrosophic numbers in decision making. *Kuwait Journal of Science*, 2021.

[439] Mike J Grannell and Martin Knor. On the number of triangular embeddings of complete graphs and complete tripartite graphs. *Journal of Graph Theory*, 69(4):370–382, 2012.

[440] Salvatore Greco, Benedetto Matarazzo, and Roman Slowinski. Rough sets theory for multicriteria decision analysis. *European journal of operational research*, 129(1):1–47, 2001.

[441] H. Paul Grice. Logic and conversation. *Syntax and Semantics*, 3:41–58, 1975.

[442] Martin Grohe and Dániel Marx. On tree width, bramble size, and expansion. *Journal of Combinatorial Theory, Series B*, 99(1):218–228, 2009.

[443] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.

[444] Ying-Qiu Gu. Clifford algebras, hypercomplex numbers and nonlinear equations in physics. *Geometry, Integrability and Quantization, Papers and Lecture Series*, 25:47–72, 2023.

[445] Yingqiu Gu. Clifford algebra and hypercomplex number as well as their applications in physics. *Journal of Applied Mathematics and Physics*, 10(4):1375–1393, 2022.

[446] LG Gubin, Boris T Polyak, and EV Raik. The method of projections for finding the common point of convex sets. *USSR Computational Mathematics and Mathematical Physics*, 7(6):1–24, 1967.

[447] Fu gui Shi. (l, m)-fuzzy matroids. *Fuzzy Sets Syst.*, 160:2387–2400, 2009.

[448] Rizwana Gul, Muhammad Shabir, Wali Khan Mashwani, and Hayat Ullah. Novel bipolar soft rough-set approximations and their application in solving decision-making problems. *Int. J. Fuzzy Log. Intell. Syst.*, 22:303–324, 2022.

[449] Muhammad Gulistan, Naveed Yaqoob, Zunaira Rashid, Florentin Smarandache, and Hafiz Abdul Wahab. A study on neutrosophic cubic graphs with real life applications in industries. *Symmetry*, 10(6):203, 2018.

[450] Stephen Gull, Anthony Lasenby, and Chris Doran. Imaginary numbers are not real-the geometric algebra of spacetime. *Foundations of Physics*, 23(9):1175–1201, 1993.

[451] Fatma Kutlu Gündogdu and Cengiz Kahraman. A novel vikor method using spherical fuzzy sets and its application to warehouse site selection. *J. Intell. Fuzzy Syst.*, 37:1197–1211, 2019.

[452] Fatma Kutlu Gündogdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *J. Intell. Fuzzy Syst.*, 36:337–352, 2019.

[453] Krishna Kumar Gupta and Sanjay Kumar. Hesitant probabilistic fuzzy set based time series forecasting method. *Granular Computing*, 4:739–758, 2019.

[454] Frank Gurski and Carolin Rehs. Comparing linear width parameters for directed graphs. *Theory of Computing Systems*, 63:1358–1387, 2019.

[455] Marisa Gutierrez and Lía Oubiña. Minimum proper interval graphs. *Discrete mathematics*, 142(1-3):77–85, 1995.

[456] Gregory Gutin and Anders Yeo. Anti-matroids. *Operations research letters*, 30(2):97–99, 2002.

[457] Dae Geun Ha, Tae Wook Ha, Junghyuk Seo, and Myoung-Ho Kim. Index-based searching for isomorphic subgraphs in hypergraph databases. *Journal of KIISE*, 2019.

[458] Mustafa Hasan Hadi and LAA Al-Swidi. The neutrosophic axial set theory. *Neutrosophic Sets and Systems, vol. 51/2022: An International Journal in Information Science and Engineering*, page 295, 2022.

[459] Petr Hájek. Metamathematics of fuzzy logic. In *Trends in Logic*, 1998.

[460] Mohammad Hamidi and Marzieh Rahmati. On binary decision hypertree (hyperdiagram). *AUT Journal of Mathematics and Computing*, 5(2):117–130, 2024.

[461] Mohammad Hamidi, Akbar Rezaei, et al. On grey graphs and their applications in optimization. *Control and Optimization in Applied Mathematics*, 6(2):79–96, 2021.

[462] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.

[463] Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.

[464] Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World*. Infinite Study, 2023.

[465] Will Hamilton, Zhitao Ying, and Jure Leskovec. Inductive representation learning on large graphs. *Advances in neural information processing systems*, 30, 2017.

[466] Asma Hamzeh. Some graph polynomials of the power graph and its supergraphs. *Mathematics Interdisciplinary Research*, 5(1):13–22, 2020.

[467] Asma Hamzeh and Ali Reza Ashrafi. The order supergraph of the power graph of a finite group. *Turkish Journal of Mathematics*, 42(4):1978–1989, 2018.

[468] Heiko Harborth. Parity of numbers of crossings for complete n-partite graphs. *Mathematica Slovaca*, 26(2):77–95, 1976.

[469] Jonathan Harel, Christof Koch, and Pietro Perona. Graph-based visual saliency. *Advances in neural information processing systems*, 19, 2006.

[470] B Mohamed Harif and A Nazeera Begam. On regular spherical fuzzy graphs. *Journal of Hyperstructures*, 12(2):244–256, 2023.

[471] Juris Hartmanis and Richard E Stearns. On the computational complexity of algorithms. *Transactions of the American Mathematical Society*, 117:285–306, 1965.

[472] Daniel J Harvey and David R Wood. Parameters tied to treewidth. *Journal of Graph Theory*, 84(4):364–385, 2017.

[473] Egbert Harzheim. *Ordered sets*, volume 7. Springer Science & Business Media, 2005.

[474] Gholamreza Haseli, Shabnam Rahnamay Bonab, Mostafa Hajiaghaei-Keshteli, Saeid Jafarzadeh Ghoushchi, and Muhammet Deveci. Fuzzy ze-numbers framework in group decision-making using the bcm and cocoso to address sustainable urban transportation. *Information Sciences*, 653:119809, 2024.

[475] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.

[476] Khizar Hayat, Muhammad Irfan Ali, Bing yuan Cao, and Xiaopeng Yang. A new type-2 soft set: Type-2 soft graphs and their applications. *Adv. Fuzzy Syst.*, 2017:6162753:1–6162753:17, 2017.

[477] Pengfei He and Xiaolong Xin. Fuzzy hyperlattices. *Computers & Mathematics with Applications*, 62(12):4682–4690, 2011.

[478] Pengfei He, Xiaolong Xin, and Jianming Zhan. On rough hyperideals in hyperlattices. *Journal of Applied Mathematics*, 2013(1):915217, 2013.

[479] Tong He, Yong Chen, and Kaiquan Shi. Weighted rough graph and its application. In *Sixth International Conference on Intelligent Systems Design and Applications*, volume 1, pages 486–491. IEEE, 2006.

[480] Pinar Heggernes, Daniel Meister, and Yngve Villanger. Induced subgraph isomorphism on interval and proper interval graphs. In *International Symposium on Algorithms and Computation*, pages 399–409. Springer, 2010.

[481] Dorothea Heiss-Czedik. An introduction to genetic algorithms. *Artificial Life*, 3:63–65, 1997.

[482] Thorkell Helgason. Aspects of the theory of hypermatroids. In *Hypergraph Seminar: Ohio State University 1972*, pages 191–213. Springer, 2006.

[483] Pavol Hell and David G Kirkpatrick. Packings by complete bipartite graphs. *SIAM Journal on Algebraic Discrete Methods*, 7(2):199–209, 1986.

[484] R. Hema, R. Sudharani, and M. Kavitha. A novel approach on plithogenic interval valued neutrosophic hypersoft sets and its application in decision making. *Indian Journal Of Science And Technology*, 2023.

[485] Marijn JH Heule, Matti Järvisalo, and Armin Biere. Revisiting hyper binary resolution. In *Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems: 10th International Conference, CPAIOR 2013, Yorktown Heights, NY, USA, May 18-22, 2013. Proceedings 10*, pages 77–93. Springer, 2013.

[486] Nasimeh Heydaribeni, Xinrui Zhan, Ruisi Zhang, Tina Eliassi-Rad, and Farinaz Koushanfar. Hypop: Distributed constrained combinatorial optimization leveraging hypergraph neural networks. *ArXiv*, abs/2311.09375, 2023.

[487] Robert Hickingbotham. Induced subgraphs and path decompositions. *arXiv preprint arXiv:2206.15054*, 2022.

[488] Marco Hien. Abstract algebra. *Mathemagics: A Magical Journey Through Advanced Mathematics*, 2020.

[489] Neil Hindman. Ultrafilters and combinatorial number theory. In *Number Theory Carbondale 1979: Proceedings of the Southern Illinois Number Theory Conference Carbondale, March 30 and 31, 1979*, pages 119–184. Springer, 1979.

[490] Petr Hliněný. Touching graphs of unit balls. In *International Symposium on Graph Drawing*, pages 350–358. Springer, 1997.

[491] Petr Hliněný and Geoff Whittle. Matroid tree-width. *European Journal of Combinatorics*, 27(7):1117–1128, 2006.

[492] Wilfrid Hodges. Basic set theory. *Bulletin of The London Mathematical Society*, 12:158–159, 1980.

[493] Doeko Homan. Very basic set theory. 2023.

[494] Dug Hun Hong and Chang-Hwan Choi. Multicriteria fuzzy decision-making problems based on vague set theory. *Fuzzy sets and systems*, 114(1):103–113, 2000.

[495] Dug Hun Hong and Chul Kim. A note on similarity measures between vague sets and between elements. *Information sciences*, 115(1-4):83–96, 1999.

[496] Snir Hordan, Tal Amir, Steven J Gortler, and Nadav Dym. Complete neural networks for complete euclidean graphs. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 38, pages 12482–12490, 2024.

[497] K Horiuchi. An introduction of trice fuzzy set. In *Proceedings of the World Conference on Soft Computing 2011*, 2011.

[498] KIYOMITSU Horiuchi. Trice and two delegates operation. *Scientiae Mathematicae*, 2(3):373–384, 1999.

[499] Kiyomitsu Horiuchi. An example of trice for fuzzy theory (algebras, languages, algorithms, and computations). *RIMS Kokyuroku (Research Institute for Mathematical Sciences)*, 1769:39–48, 2011.

[500] Kiyomitsu Horiuchi. On triangle constructive trices. *Scientiae Mathematicae Japonicae*, 76(2):209–216, 2013.

[501] Kiyomitsu Horiuchi. Triangle constructive trice 1 (algebra and computer science). *RIMS Kokyuroku (Research Institute for Mathematical Sciences)*, 1873:67–74, 2014.

[502] Kiyomitsu Horiuchi. Triangle constructive trice 2 (logics, algebras and languages in computer science). *RIMS Kokyuroku (Research Institute for Mathematical Sciences)*, 1915:15–21, 2014.

[503] Kiyomitsu Horiuchi, Branimir Šešelja, and Andreja Tepavčević. Trice-valued fuzzy sets: Mathematical model for three-way decisions. *Information Sciences*, 507:574–584, 2020.

[504] Zhe Hou. Automata theory and formal languages. *Texts in Computer Science*, 2021.

[505] Roger F House. A 4-dimensional graph has at least 9 edges. *Discrete Mathematics*, 313(18):1783–1789, 2013.

[506] Karel Hrbacek. Realism, nonstandard set theory, and large cardinals. *Annals of Pure and Applied Logic*, 109(1-2):15–48, 2001.

[507] Karel Hrbacek and Thomas Jech. Introduction to set theory, revised and expanded. 2017.

[508] Krzysztof Hryniewiecki. Basic properties of real numbers. 2004.

[509] Chao Hu, Ruishi Yu, Binqi Zeng, Yu Zhan, Ying Fu, Quan Zhang, Rongkai Liu, and Heyuan Shi. Hyperattack: Multi-gradient-guided white-box adversarial structure attack of hypergraph neural networks. *ArXiv*, abs/2302.12407, 2023.

[510] Weihua Hu, Matthias Fey, Marinka Zitnik, Yuxiao Dong, Hongyu Ren, Bowen Liu, Michele Catasta, and Jure Leskovec. Open graph benchmark: Datasets for machine learning on graphs. *Advances in neural information processing systems*, 33:22118–22133, 2020.

[511] Han-Chen Huang and Xiao-Jun Yang. A comparative investigation of type-2 fuzzy sets, nonstationary fuzzy sets and cloud models. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.*, 24:213–228, 2016.

[512] Han-Chen Huang and Xiaojun Yang. A comparative investigation of type-2 fuzzy sets, nonstationary fuzzy sets and cloud models. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 24(02):213–227, 2016.

[513] Jing Huang and Jie Yang. Unignn: a unified framework for graph and hypergraph neural networks. *ArXiv*, abs/2105.00956, 2021.

[514] Liangsong Huang, Yu Hu, Yuxia Li, PK Kishore Kumar, Dipak Koley, and Arindam Dey. A study of regular and irregular neutrosophic graphs with real life applications. *Mathematics*, 7(6):551, 2019.

[515] R. Jahir Hussain and M. S. Afya Farhana. Fuzzy chromatic number of fuzzy soft cycle and complete fuzzy soft graphs. *AIP Conference Proceedings*, 2023.

[516] S Satham Hussain, N Durga, Muhammad Aslam, G Muhiuddin, and Ganesh Ghorai. New concepts on quadripartitioned neutrosophic competition graph with application. *International Journal of Applied and Computational Mathematics*, 10(2):57, 2024.

[517] S Satham Hussain, N Durga, Rahmonlou Hossein, and Ghorai Ganesh. New concepts on quadripartitioned single-valued neutrosophic graph with real-life application. *International Journal of Fuzzy Systems*, 24(3):1515–1529, 2022.

[518] S Satham Hussain, Hossein Rashmonlou, R Jahir Hussain, Sankar Sahoo, Said Broumi, et al. Quadripartitioned neutrosophic graph structures. *Neutrosophic Sets and Systems*, 51(1):17, 2022.

[519] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[520] Gawaher S Hussein, Khalid A Eldrandaly, AbdelNasser H Zaied, Samar L Elhawy, and Mona Mohamed. Harnessing pliancy tree soft sets in heart diseases for extracting beneficial rules of association rules. *Neutrosophic Sets and Systems*, 67(1):10, 2024.

[521] Hariwan Z. Ibrahim. On some weaker hesitant fuzzy open sets. *Communications Faculty Of Science University of Ankara Series A1Mathematics and Statistics*, 2021.

[522] Muritala A Ibrahim, EO Adeleke, SA Akinleye, et al. On neutrosophic quadruple hypervector spaces. *International Journal of Neutrosophic Science*, 4:20–35, 2020.

[523] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Fuzzy hypersoft expert set with application in decision making for the best selection of product. 2021.

[524] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Hypersoft expert set with application in decision making for recruitment process. 2021.

[525] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Single valued neutrosophic hypersoft expert set with application in decision making. 2021.

[526] Muhammad Ihsan, Muhammad Haris Saeed, Agaeb Mahal Alanzi, and Hamiden Abd El-Wahed Khalifa. An algorithmic multiple attribute decision-making method for heart problem analysis under neutrosophic hypersoft expert set with fuzzy parameterized degree-based setting. *PeerJ Computer Science*, 9, 2023.

[527] Muhammad Ihsan, Muhammad Haris Saeed, Alhanouf Alburaikan, and Hamiden A. Wahed Khalifa. Product evaluation through multi-criteria decision making based on fuzzy parameterized pythagorean fuzzy hypersoft expert set. *AIMS Mathematics*, 2022.

[528] Muhammad Ihsan, Muhammad Haris Saeed, and Atiqe Ur Rahman. An intuitionistic fuzzy hypersoft expert set-based robust decision-support framework for human resource management integrated with modified topsis and correlation coefficient. *Neural Computing and Applications*, pages 1–25, 2023.

[529] Muhammad Ihsan, Muhammad Haris Saeed, Atiqe Ur Rahman, Hüseyin Kamac, Nehad Ali Shah, and Wajaree Weera. An madm-based fuzzy parameterized framework for solar panels evaluation in a fuzzy hypersoft expert set environment. *AIMS Mathematics*, 2022.

[530] Muhammad Ihsan, Muhammad Haris Saeed, Atiqe Ur Rahman, Hamiden A. Wahed Khalifa, and Salwa El-Morsy. An intelligent fuzzy parameterized multi-criteria decision-support system based on intuitionistic fuzzy hypersoft expert set for automobile evaluation. *Advances in Mechanical Engineering*, 14, 2022.

[531] Isao Imai. *Applied hyperfunction theory*, volume 8. Springer Science & Business Media, 2013.

[532] Borislav Iordanov. Hypergraphdb: a generalized graph database. In *Web-Age Information Management: WAIM 2010 International Workshops: IWGD 2010, XMLDM 2010, WCMT 2010, Jiuzhaigou Valley, China, July 15-17, 2010 Revised Selected Papers 11*, pages 25–36. Springer, 2010.

[533] Gurkan Isik. Contextual fuzzy sets. In *5th International Bursa Scientific Researchs Congress*, pages 697–700, Bursa, Turkiye, 2023. Bursa Technical University, Department of Industrial Engineering. ORCID: 0000-0002-5297-3109.

[534] Trevor Jack. On the complexity of properties of partial bijection semigroups. 2021.

[535] Muhammad Naveed Jafar, Muhammad Haris Saeed, Kainat Muniba Khan, Faten S. Alamri, and Hamiden A. Wahed Khalifa. Distance and similarity measures using max-min operators of neutrosophic hypersoft sets with application in site selection for solid waste management systems. *IEEE Access*, PP:1–1, 2022.

[536] S Jagadeesan, KK Myithili, S Thilagavathi, and L Gayathri. A new paradigm on semihypergraph. *Journal of Computational Analysis and Applications (JoCAAA)*, 33(2):514–522, 2024.

[537] Sirus Jahanpanah and Roohallah Daneshpayeh. An outspread on valued logic superhyperalgebras. *Facta Universitatis, Series: Mathematics and Informatics*, pages 427–437, 2024.

[538] Deivanai Jaisankar, Sujatha Ramalingam, Nagarajan Deivanayagampillai, and Tadesse Walelign. Fuzzy outerplanar graphs and its applications. *International Journal of Computational Intelligence Systems*, 17(1):231, 2024.

[539] Chiranjibe Jana, Tapan Senapati, Monoranjan Bhowmik, and Madhumangal Pal. On intuitionistic fuzzy g-subalgebras of g-algebras. *Fuzzy Information and Engineering*, 7(2):195–209, 2015.

[540] Remie Janssen, Mark Jones, Steven Kelk, Georgios Stamoulis, and Taoyang Wu. Treewidth of display graphs: bounds, brambles and applications. *arXiv preprint arXiv:1809.00907*, 2018.

[541] G. J. Jason. The logic of scientific discovery. 1988.

[542] Edwin T. Jaynes, Thomas Boyde Grandy, Ray Smith, Tom Loredo, Myron Tribus, John Skilling, and G Larry Bretthorst. Probability theory: the logic of science. *The Mathematical Intelligencer*, 27:83, 2005.

[543] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[544] Jianwen Jiang, Yuxuan Wei, Yifan Feng, Jingxuan Cao, and Yue Gao. Dynamic hypergraph neural networks. In *International Joint Conference on Artificial Intelligence*, 2019.

[545] Xiaoyi Jiang, Andreas Münger, and Horst Bunke. On median graphs: Properties, algorithms, and applications. *IEEE Trans. Pattern Anal. Mach. Intell.*, 23:1144–1151, 2001.

[546] Chenxia Jin, Jusheng Mi, Fachao Li, and Meishe Liang. A novel probabilistic hesitant fuzzy rough set based multi-criteria decision-making method. *Information Sciences*, 608:489–516, 2022.

[547] Calvin Jongsma. Basic set theory and combinatorics. *Undergraduate Texts in Mathematics*, 2019.

[548] Jinta Jose, Bobin George, and Rajesh K Thumbakara. Soft directed graphs, some of their operations, and properties. *New Mathematics and Natural Computation*, 20(01):129–155, 2024.

[549] Jinta Jose, Bobin George, and Rajesh K Thumbakara. Soft graphs: A comprehensive survey. *New Mathematics and Natural Computation*, pages 1–52, 2024.

[550] Michael Joswig and Georg Loho. Weighted digraphs and tropical cones. *Linear Algebra and its Applications*, 501:304–343, 2016.

[551] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[552] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. N-hyper sets. *Mathematics*, 6(6):87, 2018.

[553] Magdalena Kacprzak and B Starosta. An approach to making decisions with metasets. *Trends in Contemporary Computer Science, Podlasie*, 2014:159–172, 2014.

[554] Magdalena Kacprzak, Bartłomiej Starosta, and Katarzyna Węgrzyn-Wolska. Metasets and opinion mining in new decision support system. In *Artificial Intelligence and Soft Computing: 14th International Conference, ICAISC 2015, Zakopane, Poland, June 14-18, 2015, Proceedings, Part II 14*, pages 625–636. Springer, 2015.

[555] Sai Kiran Kadambari and Sundeep Prabhakar Chepuri. Product graph learning from multi-domain data with sparsity and rank constraints. *IEEE Transactions on Signal Processing*, 69:5665–5680, 2020.

[556] Phisan Kaewprapha, Jing Li, and Nattakan Puttarak. Network localization on unit disk graphs. In *2011 IEEE Global Telecommunications Conference-GLOBECOM 2011*, pages 1–5. IEEE, 2011.

[557] Daniel Kahneman and Amos Tversky. Decision, probability, and utility: Prospect theory: An analysis of decision under risk. 1979.

[558] Cengiz Kahraman. *Fuzzy multi-criteria decision making: theory and applications with recent developments*, volume 16. Springer Science & Business Media, 2008.

[559] Hiroshi Kajino. Molecular hypergraph grammar with its application to molecular optimization. *ArXiv*, abs/1809.02745, 2018.

[560] Shizuo Kamo. Non-standard real number systems with regular gaps. *Tsukuba journal of mathematics*, 5:21–24, 1981.

[561] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[562] Ilanthenral Kandasamy and Florentin Smarandache. Algebraic structure of neutrosophic duplets in neutrosophic rings. 2018.

[563] Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Neutrosophic graphs: a new dimension to graph theory*. Infinite Study, 2015.

[564] Akira Kaneko. *Introduction to the Theory of Hyperfunctions*, volume 3. Springer Science & Business Media, 1989.

[565] Vladimir Kanovei and Michael Reeken. *Nonstandard analysis, axiomatically*. Springer Science & Business Media, 2013.

[566] Haim Kaplan and Ron Shamir. Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques. *SIAM Journal on Computing*, 25(3):540–561, 1996.

[567] Abdullah Kargın, Florentin Smarandache, and Memet Şahin. *New Type Hyper Groups, New Type SuperHyper Groups and Neutro-New Type SuperHyper Groups*. Infinite Study, 2023.

[568] Nilesh N Karnik and Jerry M Mendel. Centroid of a type-2 fuzzy set. *information SCiences*, 132(1-4):195–220, 2001.

[569] M. G. Karunambigai, R. Parvathi, and R. Buvaneswari. Arc in intuitionistic fuzzy graphs. *Notes on Intuitionistic Fuzzy Sets*, 17:37–47, 2011.

[570] Ali Kaveh and Sepehr Beheshti. Weighted triangular and circular graph products for configuration processing. *Periodica Polytechnica Civil Engineering*, 56(1):63–71, 2012.

[571] Yukihito Kawahara. Combinatorics coming from hyperplane arrangements and the orlik-solomon algebra (algebraic combinatorics). *RIMS Kokyuroku (Research Institute for Mathematical Sciences)*, 1394:97–108, 2004.

[572] Yukihito Kawahara. On matroids and orlik-solomon algebras. *Annals of Combinatorics*, 8:63–80, 2004.

[573] J Mark Keil, Joseph SB Mitchell, Dinabandhu Pradhan, and Martin Vatshelle. An algorithm for the maximum weight independent set problem on outerstring graphs. *Computational Geometry*, 60:19–25, 2017.

[574] H Jerome Keisler. *Foundations of infinitesimal calculus*, volume 20. Prindle, Weber & Schmidt Boston, 1976.

[575] Mohamed Khaled. Topics in algebra. *AMS/MAA Textbooks*, 2021.

[576] Mohamed Amine Khamsi, W. A. Kirk, and Carlos Martinez Yañez. Fixed point and selection theorems in hyperconvex spaces. 2000.

[577] Waheed Ahmad Khan, Babir Ali, and Abdelghani Taouti. Bipolar picture fuzzy graphs with application. *Symmetry*, 13:1427, 2021.

[578] Waheed Ahmad Khan, Waqar Arif, Quoc Hung Nguyen, Thanh Trung Le, and Hai Van Pham. Picture fuzzy directed hypergraphs with applications toward decision-making and managing hazardous chemicals. *IEEE Access*, 12:87816–87827, 2024.

[579] Waheed Ahmad Khan, Waqar Arif, Quoc Hung NGUYEN, Thanh Trung Le, and Hai Van Pham. Picture fuzzy directed hypergraphs with applications towards decision-making and managing hazardous chemicals. *IEEE Access*, 2024.

[580] Jasminara Khatun, Sk Amanathulla, and Madhumangal Pal. A comprehensive study on m-polar picture fuzzy graphs and its application. *Journal of Uncertain Systems*, page 2450016, 2024.

[581] J. F. C. Kingman and William Feller. An introduction to probability theory and its applications. *Biometrika*, 130:430–430, 1958.

[582] Thomas N Kipf and Max Welling. Semi-supervised classification with graph convolutional networks. *arXiv preprint arXiv:1609.02907*, 2016.

[583] George J. Klir and Bo Yuan. Fuzzy sets and fuzzy logic - theory and applications. 1995.

[584] Donald Ervin Knuth. Surreal numbers : how two ex-students turned on to pure mathematics and found total happiness : a mathematical novelette. 1974.

[585] Margarita Knyazeva, Stanislav Belyakov, and Janusz Kacprzyk. Topological ordering on interval type-2 fuzzy graph. In *13th International Conference on Theory and Application of Fuzzy Systems and Soft Computing?ICAFS-2018 13*, pages 262–269. Springer, 2019.

[586] Ali NA Koam, Muhammad Akram, and Peide Liu. Decision-making analysis based on fuzzy graph structures. *Mathematical Problems in Engineering*, 2020(1):6846257, 2020.

[587] Alexander W. Kocurek. The logic of hyperlogic. part a: Foundations. *The Review of Symbolic Logic*, 17:244 – 271, 2022.

[588] Wolfram Koepf. Algebraic numbers. *Springer Undergraduate Texts in Mathematics and Technology*, 2021.

[589] Hikosaburo Komatsu. An introduction to the theory of hyperfunctions. In *Hyperfunctions and Pseudo-Differential Equations: Proceedings of a Conference at Katata, 1971*, pages 3–40. Springer, 1973.

[590] Bernhard Korte and László Lovász. Structural properties of greedoids. *Combinatorica*, 3(3):359–374, 1983.

[591] Bernhard Korte, László Lovász, and Rainer Schrader. *Greedoids*, volume 4. Springer Science & Business Media, 2012.

[592] Costas D Koutras, Christos Moyzes, Christos Nomikos, Konstantinos Tsaprounis, and Yorgos Zikos. On weak filters and ultrafilters: Set theory from (and for) knowledge representation. *Logic Journal of the IGPL*, 31(1):68–95, 2023.

[593] Costas D Koutras, Christos Moyzes, Christos Nomikos, and Yorgos Zikos. On the 'in many cases' modality: tableaux, decidability, complexity, variants. In *Artificial Intelligence: Methods and Applications: 8th Hellenic Conference on AI, SETN 2014, Ioannina, Greece, May 15-17, 2014. Proceedings 8*, pages 207–220. Springer, 2014.

[594] Costas D Koutras, Christos Moyzes, and Yorgos Zikos. A modal logic of knowledge, belief and estimation. *Journal of Logic and Computation*, 27(8):2303–2339, 2017.

[595] Jan Kratochvíl. String graphs. ii. recognizing string graphs is np-hard. *Journal of Combinatorial Theory, Series B*, 52(1):67–78, 1991.

[596] Fabian Kuhn, Rogert Wattenhofer, and Aaron Zollinger. Ad-hoc networks beyond unit disk graphs. In *Proceedings of the 2003 joint workshop on Foundations of mobile computing*, pages 69–78, 2003.

[597] N Vinoth Kumar and G Geetha Ramani. Product intuitionistic fuzzy graph. *International Journal of Computer Applications*, 28(1):31–33, 2011.

[598] Rakesh Kumar and Gaurav Dhiman. A comparative study of fuzzy optimization through fuzzy number. *International Journal of Modern Research*, 1(1):1–14, 2021.

[599] S. Udhaya Kumar, H. Hannah Inbarani, and S. Selva Kumar. Bijective soft set based classification of medical data. *2013 International Conference on Pattern Recognition, Informatics and Mobile Engineering*, pages 517–521, 2013.

[600] Fatma Kutlu Gundogdu and Cengiz Kahraman. Extension of waspas with spherical fuzzy sets. *Informatica*, 30(2):269–292, 2019.

[601] Fatma Kutlu Gündoğdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *Journal of intelligent & fuzzy systems*, 36(1):337–352, 2019.

[602] Andrea Lancichinetti, Santo Fortunato, and Filippo Radicchi. Benchmark graphs for testing community detection algorithms. *Physical Review E-Statistical, Nonlinear, and Soft Matter Physics*, 78(4):046110, 2008.

[603] Ahmad Lasim, Ikhsanul Halikin, and Kristiana Wijaya. The harmonious, odd harmonious, and even harmonious labeling. *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, 2022.

[604] Harry Lass. *Elements of pure and applied mathematics*. Courier Corporation, 2009.

[605] Detlef Laugwitz. Definite values of infinite sums: aspects of the foundations of infinitesimal analysis around 1820. *Archive for History of Exact Sciences*, pages 195–245, 1989.

[606] Steven R Lay. *Convex sets and their applications*. Courier Corporation, 2007.

[607] Emmanuelle Lebhar and Zvi Lotker. Unit disk graph and physical interference model: Putting pieces together. In *2009 IEEE International Symposium on Parallel & Distributed Processing*, pages 1–8. IEEE, 2009.

[608] Li-Wei Lee and Shyi-Ming Chen. Fuzzy risk analysis based on fuzzy numbers with different shapes and different deviations. *Expert Syst. Appl.*, 34:2763–2771, 2008.

[609] Benjamin Lévêque, David Y Lin, Frédéric Maffray, and Nicolas Trotignon. Detecting induced subgraphs. *Discrete Applied Mathematics*, 157(17):3540–3551, 2009.

[610] Azriel Levy. *Basic set theory*. Courier Corporation, 2012.

[611] Deng-Feng Li. Multiattribute decision making models and methods using intuitionistic fuzzy sets. *Journal of computer and System Sciences*, 70(1):73–85, 2005.

[612] Deng-Feng Li. Extension of the linmap for multiattribute decision making under atanassov's intuitionistic fuzzy environment. *Fuzzy Optimization and Decision Making*, 7:17–34, 2008.

[613] Hongxing Li and Vincent C Yen. *Fuzzy sets and fuzzy decision-making*. CRC press, 1995.

[614] Shasha Li, Wei Li, and Xueliang Li. The generalized connectivity of complete bipartite graphs. *arXiv preprint arXiv:1012.5710*, 2010.

[615] Xiaonan Li. Three-way fuzzy matroids and granular computing. *International Journal of Approximate Reasoning*, 114:44–50, 2019.

[616] Xiaonan Li, Bingzhen Sun, and Yanhong She. Generalized matroids based on three-way decision models. *International Journal of Approximate Reasoning*, 90:192–207, 2017.

[617] Xiaonan Li and Huangjian Yi. Intuitionistic fuzzy matroids. *J. Intell. Fuzzy Syst.*, 33:3653–3663, 2017.

[618] Meilian Liang, Binmei Liang, Linna Wei, and Xiaodong Xu. Edge rough graph and its application. In *2011 Eighth International Conference on Fuzzy Systems and Knowledge Discovery (FSKD)*, volume 1, pages 335–338. IEEE, 2011.

[619] Thierry Libert. Models for a paraconsistent set theory. *J. Appl. Log.*, 3:15–41, 2005.

[620] Leo Liberti and Carlile Lavor. *Euclidean distance geometry*, volume 3. Springer, 2017.

[621] Robert Lin. Note on fuzzy sets. *Yugoslav Journal of Operations Research*, 24:299–303, 2014.

[622] Tsau Young Lin. Measure theory on granular fuzzy sets. *18th International Conference of the North American Fuzzy Information Processing Society - NAFIPS (Cat. No.99TH8397)*, pages 809–813, 1999.

[623] Wen Lin. Multicoloring and mycielski construction. *Discrete Mathematics*, 308(16):3565–3573, 2008.

[624] Donghai Liu, Xiao hong Chen, and Dan Peng. Some cosine similarity measures and distance measures between q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 34:1572 – 1587, 2019.

[625] Peide Liu and Xi Liu. The neutrosophic number generalized weighted power averaging operator and its application in multiple attribute group decision making. *International Journal of Machine Learning and Cybernetics*, 9:347 – 358, 2015.

[626] Peide Liu and Peng Wang. Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making. *International Journal of Intelligent Systems*, 33, 2018.

[627] Qi Liu, Maximilian Nickel, and Douwe Kiela. Hyperbolic graph neural networks. *Advances in neural information processing systems*, 32, 2019.

[628] Xiaorui Liu, Wei Jin, Yao Ma, Yaxin Li, Hua Liu, Yiqi Wang, Ming Yan, and Jiliang Tang. Elastic graph neural networks. In *International Conference on Machine Learning*, pages 6837–6849. PMLR, 2021.

[629] OV Loginovskiy, OI Dranko, and AV Hollay. Mathematical models for decision-making on strategic management of industrial enterprise in conditions of instability. In *CEUR Workshop Proceedings*, volume 2093, pages 1–12, 2018.

[630] Daniel Lokshtanov, Pranabendu Misra, Fahad Panolan, and Saket Saurabh. Deterministic truncation of linear matroids. *ACM Transactions on Algorithms (TALG)*, 14(2):1–20, 2018.

[631] Juan Lu, Deyu Li, Yanhui Zhai, and Hexiang Bai. Granular structure of type-2 fuzzy rough sets over two universes. *Symmetry*, 9:284, 2017.

[632] Roberto Lucchetti. Hypertopologies and applications. *Recent Developments in Well-Posed Variational Problems*, pages 193–209, 1995.

[633] Wilhelmus AJ Luxemburg. Non-standard analysis. In *Logic, Foundations of Mathematics, and Computability Theory: Part One of the Proceedings of the Fifth International Congress of Logic, Methodology and Philosophy of Science, London, Ontario, Canada-1975*, pages 107–119. Springer, 1968.

[634] M. M., M. Suneetha, Maria Mikhailova, Sripada Nsvsc Ramesh, and Kollati Vijaya Kumar. Leveraging double-valued neutrosophic set for real-time chronic kidney disease detection and classification. *International Journal of Neutrosophic Science*.

[635] Mathieu Mach, Enrico Amico, Raphaël Liégeois, Maria Giulia Preti, Alessandra Griffa, Dimitri Van De Ville, and Mangor Pedersen. Connectome embedding in multidimensional graph spaces. *Network Neuroscience*, pages 1–51, 2024.

[636] Rupkumar Mahapatra, Sovan Samanta, Madhumangal Pal, Tofigh Allahviranloo, and Antonios Kalampakas. A study on linguistic z-graph and its application in social networks. *Mathematics*, 12(18):2898, 2024.

[637] Tahir Mahmood, Kifayat Ullah, Qaisar Khan, and Naeem Jan. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Computing and Applications*, pages 1–13, 2019.

[638] Amir Sabir majeed and Nabeel Ezzulddin Arif. Domination (set and number) in neutrosophic soft over graphs. *Wasit Journal of Pure sciences*, 2022.

[639] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[640] Pinaki Majumdar. Neutrosophic sets and its applications to decision making. In *Computational Intelligence for Big Data Analysis: Frontier Advances and Applications*, pages 97–115. Springer, 2015.

[641] Ebrahim H. Mamdani and S. Assilian. An experiment in linguistic synthesis with a fuzzy logic controller. *Int. J. Hum. Comput. Stud.*, 51:135–147, 1999.

[642] Parimala Mani, Ibtesam Alshammari, and Halimah Alshehri. Shortest path algorithm of a network via spherical fuzzy digraphs. *Journal of Engineering Research*, 9, 2021.

[643] Hua Mao and Sanyang Liu. On antimatroids of infinite character. *Mathematica Pannonica*, 23(2):257–266, 2012.

[644] K. Mariappa and S. Sekar. On regular generalized b-closed set. *International Journal of Mathematical Analysis*, 7:613–624, 2013.

[645] Nivetha Martin, Priya Priya.R, and Florentin Smarandache. Decision making on teachers' adaptation to cybergogy in saturated interval- valued refined neutrosophic overset /underset /offset environment. *International Journal of Neutrosophic Science*, 2020.

[646] Nivetha Martin and Florentin Smarandache. Introduction to combined plithogenic hypersoft sets. *Neutrosophic Sets and Systems*, 35:503–510, 2020.

[647] Christian Massari, Marco Ferrante, Bruno Brunone, and Silvia Meniconi. Is the leak head-discharge relationship in polyethylene pipes a bijective function? *Journal of Hydraulic Research*, 50:409 – 417, 2012.

[648] Bibin Mathew, Sunil Jacob John, and Harish Garg. Vertex rough graphs. *Complex & Intelligent Systems*, 6:347–353, 2020.

[649] Manoj Mathew, Ripon Kumar Chakrabortty, and Michael J. Ryan. A novel approach integrating ahp and topsis under spherical fuzzy sets for advanced manufacturing system selection. *Eng. Appl. Artif. Intell.*, 96:103988, 2020.

[650] David R. Mazur. Combinatorics. *AMS/MAA Textbooks*, 2020.

[651] MLN McAllister. Fuzzy intersection graphs. *Computers & Mathematics with Applications*, 15(10):871–886, 1988.

[652] Colin McDiarmid. Surveys in combinatorics, 1989: On the method of bounded differences. 1989.

[653] Terry A McKee and Fred R McMorris. *Topics in intersection graph theory*. SIAM, 1999.

[654] Toby Meadows. Sets and supersets. *Synthese*, 193:1875–1907, 2016.

[655] Jerry M Mendel and Jerry M Mendel. Type-2 fuzzy sets. *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions, 2nd Edition*, pages 259–306, 2017.

[656] Hamiyet Merkepci and Katy D. Ahmad. On the conditions of imperfect neutrosophic duplets and imperfect neutrosophic triplets. *Galoitica: Journal of Mathematical Structures and Applications*, 2022.

[657] Mesfer Ahmed Mesfer. Blockchain with single-valued neutrosophic hypersoft sets assisted threat detection for secure iot assisted consumer electronics. *International Journal of Neutrosophic Science*.

[658] Ivan Mezzomo, Benjamín Bedregal, Thadeu Milfont, Tiago da Cruz Asmus, and Humberto Bustince. n-dimensional interval uninorms. In *2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 1–6. IEEE, 2019.

[659] Charles F. Miller. Combinatorial group theory. 2002.

[660] Justin J Miller. Graph database applications and concepts with neo4j. In *Proceedings of the southern association for information systems conference, Atlanta, GA, USA*, volume 2324, pages 141–147, 2013.

[661] Mirka Miller, Indra Rajasingh, D. Ahima Emilet, and D. Azubha Jemilet. d-lucky labeling of graphs. *Procedia Computer Science*, 57:766–771, 2015.

[662] Steven J. Miller and Ramin Takloo-Bighash. An invitation to modern number theory. 2020.

[663] Melanie Mitchell. An introduction to genetic algorithms. 1996.

[664] Sarbari Mitra and Soumya Bhoumik. Graceful labeling of triangular extension of complete bipartite graph. *Electron. J. Graph Theory Appl.*, 7:11–30, 2019.

[665] Tzuong-Tsieng Moh. Linear algebra and its applications. In *Series on University Mathematics*, 2020.

[666] Mona Mohamed, Alaa Elmor, Florentin Smarandache, and Ahmed A Metwaly. An efficient super-hypersoft framework for evaluating llms-based secure blockchain platforms. *Neutrosophic Sets and Systems*, 72:1–21, 2024.

[667] SP Mohanty, S Biswal, and G Pradhan. Minimum spanning tree in fuzzy weighted rough graph. *International Journal of Engineering Research and Development*, 1(10):23–28, 2012.

[668] Bojan Mohar. Circular colorings of edge-weighted graphs. *Journal of Graph Theory*, 43(2):107–116, 2003.

[669] Jesper M. Møller. General topology. 2008.

[670] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[671] Kalyan Mondal, Surapati Pramanik, Bibhas Chandra Giri, and Florentin Smarandache. Nn-harmonic mean aggregation operators-based mcgdm strategy in a neutrosophic number environment. *Axioms*, 7:12, 2018.

[672] Bernard Monjardet. On the use of ultrafilters in social choice theory. *Social choice and welfare*, 5:73–78, 1983.

[673] Robert Lee Moore. *Foundations of point set theory*, volume 13. American Mathematical Soc., 1932.

[674] John N Mordeson and Sunil Mathew. *Advanced topics in fuzzy graph theory*, volume 375. Springer, 2019.

[675] John C Morgan. *Point set theory*. CRC Press, 2018.

[676] Christopher Morris, Martin Ritzert, Matthias Fey, William L Hamilton, Jan Eric Lenssen, Gaurav Rattan, and Martin Grohe. Weisfeiler and leman go neural: Higher-order graph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 4602–4609, 2019.

[677] Umberto Mosco. Convergence of convex sets and of solutions of variational inequalities. *Advances in Mathematics*, 3(4):510–585, 1969.

[678] Alexandra Mozgova. Non-singular graph-manifolds of dimension 4. *Algebraic & Geometric Topology*, 5(3):1051–1073, 2005.

[679] Sunil MP and J Suresh Kumar. On intuitionistic hesitancy fuzzy graphs. 2024.

[680] Dilnoz Muhamediyeva and Baxtiyor Tagbayev. Method of converting z-number to classic fuzzy number. *Scientific Collection «InterConf+»*, (21 (109)):348–352, 2022.

[681] P Murugadas and K Lalitha. Implication operator on intuitionistic fuzzy tautological matrix. *Int. Journal of Fuzzy Mathematical Archive*, 5(2):79–87, 2014.

[682] Sagvan Y. Musa and Baravan A. Asaad. Bipolar hypersoft sets. *Mathematics*, 2021.

[683] Sagvan Y. Musa and Baravan A. Asaad. Topological structures via bipolar hypersoft sets. *Journal of Mathematics*, 2022.

[684] C Musès. Hypernumbers-ii. further concepts and computational applications. *Applied Mathematics and Computation*, 4(1):45–66, 1978.

[685] Charles Musès. Applied hypernumbers: computational concepts. *Applied Mathematics and Computation*, 3(3):211–226, 1977.

[686] J Muthuerulappan and S Chelliah. A study of the neutrosophic dominating path-coloring number and multivalued star chromatic number in neutrosophic graphs. *Machine Intelligence Research*, 18(1):585–606, 2024.

[687] M Myvizhi, Ahmed A Metwaly, and Ahmed M Ali. Treesoft approach for refining air pollution analysis: A case study. *Neutrosophic Sets and Systems*, 68(1):17, 2024.

[688] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, and Haitham Rizk Fadlallah. *MADM Strategy Application of Bipolar Single Valued Heptapartitioned Neutrosophic Set*. Infinite Study, 2023.

[689] Somashekhar Naimpally. All hypertopologies are hit-and-miss. *Applied General Topology*, 3(1):45–53, 2002.

[690] Lavanya Narayanan. Channel assignment and graph multicoloring. In *Handbook of wireless networks and mobile computing*, pages 71–94. John Wiley & Sons, 2002.

[691] Hafiza Saba Nawaz, Muhammad Akram, and José Carlos R Alcantud. An algorithm to compute the strength of competing interactions in the bering sea based on pythagorean fuzzy hypergraphs. *Neural Computing and Applications*, 34(2):1099–1121, 2022.

[692] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.

[693] Edward Nelson. Internal set theory: a new approach to nonstandard analysis. *Bulletin of the American Mathematical Society*, 83(6):1165–1198, 1977.

[694] Jaroslav Nešetřil, Patrice Ossona de Mendez, and David R Wood. Characterisations and examples of graph classes with bounded expansion. *European Journal of Combinatorics*, 33(3):350–373, 2012.

[695] Gia Nhu Nguyen, Le Hoang Son, Amira S Ashour, and Nilanjan Dey. A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. *International Journal of Machine Learning and Cybernetics*, 10:1–13, 2019.

[696] Xuan Thao Nguyen, Doan Dong Nguyen, et al. Rough fuzzy relation on two universal sets. *International Journal of Intelligent Systems and Applications*, 6(4):49, 2014.

[697] Maximilian Nickel, Kevin Murphy, Volker Tresp, and Evgeniy Gabrilovich. A review of relational machine learning for knowledge graphs. *Proceedings of the IEEE*, 104(1):11–33, 2015.

[698] S Nikoletseas, Christoforos Raptopoulos, and P Spirakis. Large independent sets in general random intersection graphs. *Theoretical Computer Science*, 406(3):215–224, 2008.

[699] Misha Nikouravan et al. A short history of imaginary numbers: Mathematics. *International Journal of Fundamental Physical Sciences*, 9(1):1–5, 2019.

[700] TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. *Ratio Mathematica*, 47, 2023.

[701] R Nithya and K Anitha. Even vertex zeta-graceful labeling on rough graph. *arXiv preprint arXiv:2208.12047*, 2022.

[702] Christos Nomikos, Aris Pagourtzis, and Stathis Zachos. Routing and path multicoloring. *Information Processing Letters*, 80(5):249–256, 2001.

[703] R Noor, I Irshad, and I Javaid. Soft rough graphs. *arXiv preprint arXiv:1707.05837*, 2017.

[704] Ladislav A. Novak. On goetschel and voxman fuzzy matroids. *Fuzzy Sets Syst.*, 117:407–412, 2001.

[705] L. J. O'Connor. On the distribution of characteristics in bijective mappings. *Journal of Cryptology*, 8:67–86, 1994.

[706] Wendy Olsen and Hisako Nomura. Poverty reduction: fuzzy sets vs. crisp sets compared. *Sociological Theory and Methods*, 24(2):219–246, 2009.

[707] Adriel Ong. Real and complex analysis. 2021.

[708] Izhar Oppenheim. Topological methods in group theory. 2018.

[709] James G Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.

[710] János Pach. Geometric graph theory. *Handbook of Discrete and Computational Geometry, 2nd Ed.*, pages 219–238, 2004.

[711] Alessandro Pagano, Raffaele Giordano, and Ivan Portoghese. A pipe ranking method for water distribution network resilience assessment based on graph-theory metrics aggregated through bayesian belief networks. *Water Resources Management*, 36(13):5091–5106, 2022.

[712] Anita Pal and Saibal Majumder. Searching minimum spanning tree in a type-2 fuzzy graph. *Progress in Nonlinear Dynamics and Chaos*, 5(1):43–58, 2017.

[713] Madhumangal Pal. Intersection graphs: An introduction. *arXiv preprint arXiv:1404.5468*, 2014.

[714] Madhumangal Pal. An introduction to intersection graphs. 2020.

[715] Madhumangal Pal and Hossein Rashmanlou. Irregular interval valued fuzzy graphs. *arXiv preprint arXiv:1407.6190*, 2014.

[716] Madhumangal Pal, Sovan Samanta, and Ganesh Ghorai. *Modern trends in fuzzy graph theory*. Springer, 2020.

[717] K Palani, T Jones, and V Maheswari. Soft graphs of certain graphs. In *Journal of Physics: Conference Series*, volume 1947, page 012045. IOP Publishing, 2021.

[718] Sakshi Dev Pandey, AS Ranadive, and Sovan Samanta. Bipolar-valued hesitant fuzzy graph and its application. *Social Network Analysis and Mining*, 12(1):14, 2022.

[719] Paolo Paolo. A study using treesoft set and neutrosophic sets on possible soil organic transformations in urban agriculture systems. *International Journal of Neutrosophic Science*.

[720] Christos H Papadimitriou. Computational complexity. In *Encyclopedia of computer science*, pages 260–265. 2003.

[721] H Chul Park, S Young Ok, and Hwan-gue Cho. Word extraction in text/graphic mixed image using 3-dimensional graph model. In *ICCPOL*, volume 99, pages 171–176. Citeseer, 1999.

[722] Sehie Park. Fixed point theorems in hyperconvex metric spaces. *Nonlinear Analysis-theory Methods & Applications*, 37:467–472, 1999.

[723] P Parthasarathi and S Shankar. Weighted ternary tree approach for secure group communication among mobile applications. *Wireless Personal Communications*, 117(4):2809–2829, 2021.

[724] Purvag Patel, Shahram Rahimi, and Elham Khorasani. Applied z-numbers. In *2015 Annual Conference of the North American Fuzzy Information Processing Society (NAFIPS) held jointly with 2015 5th World Conference on Soft Computing (WConSC)*, pages 1–6. IEEE, 2015.

[725] T Pathinathan, J Jon Arockiaraj, and J Jesintha Rosline. Hesitancy fuzzy graphs. *Indian Journal of Science and Technology*, 8(35):1–5, 2015.

[726] Vasile Patrascu. Rough sets on four-valued fuzzy approximation space. In *2007 IEEE International Fuzzy Systems Conference*, pages 1–5. IEEE, 2007.

[727] Edwin Collazos Paucar, Jeri G Ramón Ruffner de Vega, Efrén S Michue Salguedo, Agustina C Torres-Rodríguez, and Patricio A Santiago-Saturnino. Analysis using treesoft set of the strategic development plan for extreme poverty municipalities. *Neutrosophic Sets and Systems*, 69(1):3, 2024.

[728] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[729] Zdzislaw Pawlak. Rough set theory and its applications to data analysis. *Cybernetics & Systems*, 29(7):661–688, 1998.

[730] Zdzisław Pawlak. Rough sets and intelligent data analysis. *Information sciences*, 147(1-4):1–12, 2002.

[731] Zdzisław Pawlak. *Rough sets: Theoretical aspects of reasoning about data*, volume 9. Springer Science & Business Media, 2012.

[732] Zdzislaw Pawlak, Jerzy Grzymala-Busse, Roman Slowinski, and Wojciech Ziarko. Rough sets. *Communications of the ACM*, 38(11):88–95, 1995.

[733] Zdzislaw Pawlak, Lech Polkowski, and Andrzej Skowron. Rough set theory. *KI*, 15(3):38–39, 2001.

[734] Zdzisław Pawlak and Andrzej Skowron. Rudiments of rough sets. *Information sciences*, 177(1):3–27, 2007.

[735] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

[736] Nicole Pearcy, Jonathan J Crofts, and Nadia Chuzhanova. Hypergraph models of metabolism. *International Journal of Biological, Veterinary, Agricultural and Food Engineering*, 8(8):752–756, 2014.

[737] Witold Pedrycz. Fuzzy sets engineering. 1995.

[738] Sriram Pemmaraju and Steven Skiena. *Computational discrete mathematics: Combinatorics and graph theory with mathematica®*. Cambridge university press, 2003.

[739] Xindong Peng and Lin Liu. Information measures for q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 34(8):1795–1834, 2019.

[740] Xindong Peng and Lin Liu. Information measures for q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 34:1795 – 1834, 2019.

[741] José M Pérez-Izquierdo. Algebras, hyperalgebras, nonassociative bialgebras and loops. *Advances in Mathematics*, 208(2):834–876, 2007.

[742] Robert Henry Peters. Tautology in evolution and ecology. *The American Naturalist*, 110(971):1–12, 1976.

[743] Tomaz Pisanski and Milan Randic. Bridges between geometry and graph theory. In Catherine A. Gorini, editor, *Geometry at Work*, volume 53 of *MAA Notes*, pages 174–194. Cambridge University Press, 2000.

[744] Stefan Porschen. Cnf-base hypergraphs:(dense) maximal non-diagonality and combinatorial designs. *IAENG International Journal of Computer Science*, 51(3), 2024.

[745] Surapati Pramanik, Partha Pratim Dey, and Bibhas Chandra Giri. Topsis for single valued neutrosophic soft expert set based multi-attribute decision making problems. 2015.

[746] Iu. V. Prokhorov. Probability theory. 1969.

[747] Piotr Prokopowicz. Flexible and simple methods of calculations on fuzzy numbers with the ordered fuzzy numbers model. In *International Conference on Artificial Intelligence and Soft Computing*, 2013.

[748] Andrzej Proskurowski and Jan Arne Telle. From bandwidth to pathwidth k. *_THEORETICAL-E*, page 90, 1996.

[749] Fadya Zahraini Putri, Admi Nazra, and Yanita Yanita. Suatu aplikasi dari penggabungan konsep strait fuzzy set dan strait soft set. *Jurnal Lebesgue: Jurnal Ilmiah Pendidikan Matematika, Matematika dan Statistika*, 5(2):1171–1176, 2024.

[750] Jian qiang Wang, Yong xi Cao, and Hong yu Zhang. Multi-criteria decision-making method based on distance measure and choquet integral for linguistic z-numbers. *Cognitive Computation*, 9:827 – 842, 2017.

[751] Shio Gai Quek, Ganeshsree Selvachandran, D Ajay, P Chellamani, David Taniar, Hamido Fujita, Phet Duong, Le Hoang Son, and Nguyen Long Giang. New concepts of pentapartitioned neutrosophic graphs and applications for determining safest paths and towns in response to covid-19. *Computational and Applied Mathematics*, 41(4):151, 2022.

[752] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.

[753] Hans Rådström. An embedding theorem for spaces of convex sets. *Proceedings of the American mathematical society*, 3(1):165–169, 1952.

[754] Atiqe Ur Rahman, Muhammad Haris Saeed, Ebenezer Bonyah, and Muhammad Arshad. Graphical exploration of generalized picture fuzzy hypersoft information with application in human resource management multiattribute decision-making. *Mathematical Problems in Engineering*, 2022.

[755] Atiqe Ur Rahman, Muhammad Haris Saeed, and Hamiden Abd El-Wahed Khalifa. Multi-attribute decision-making based on aggregations and similarity measures of neutrosophic hypersoft sets with possibility setting. *Journal of Experimental & Theoretical Artificial Intelligence*, 36:161 – 186, 2022.

[756] Atiqe Ur Rahman, Muhammad Haris Saeed, and Florentin Smarandache. Convex and concave hypersoft sets with some properties. 2020.

[757] V. Rajeswari and K. Thiagarajan. Study on binary equivalent decimal edge graceful labeling. *Indian journal of science and technology*, 9, 2016.

[758] Daniel Ramot, Menahem Friedman, Gideon Langholz, Ron Milo, and Abraham Kandel. On complex fuzzy sets. *10th IEEE International Conference on Fuzzy Systems. (Cat. No.01CH37297)*, 3:1160–1163 vol.2, 2001.

[759] Daniel Ramot, Ron Milo, Menahem Friedman, and Abraham Kandel. Complex fuzzy sets. *IEEE Trans. Fuzzy Syst.*, 10:171–186, 2002.

[760] Aruna Devi Ramsankar and Anitha Krishnamoorthy. Exploring metric dimensions for dimensionality reduction and navigation in rough graphs. *Mathematical Modelling of Engineering Problems*, 11(4), 2024.

[761] Pratibha Rani and Arunodaya Raj Mishra. Multi-criteria weighted aggregated sum product assessment framework for fuel technology selection using q-rung orthopair fuzzy sets. *Sustainable Production and Consumption*, 24:90–104, 2020.

[762] M Murali Krishna Rao. Tripolar fuzzy interior ideals and tripolar fuzzy soft interior ideals over semigroups. *Annals of Fuzzy Mathematics and Informatics*, 20(3):243–256, 2020.

[763] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a γ-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[764] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, (422019):731743, 2019.

[765] Hossein Rashmanlou and Rajab Ali Borzooei. Vague graphs with application. *Journal of Intelligent & Fuzzy Systems*, 30(6):3291–3299, 2016.

[766] KP Ratheesh. On soft graphs and chained soft graphs. *International Journal of Fuzzy System Applications (IJFSA)*, 7(2):85–102, 2018.

[767] Sreenanda Raut and Madhumangal Pal. Fuzzy intersection graph: a geometrical approach. *Journal of Ambient Intelligence and Humanized Computing*, pages 1–25, 2022.

[768] András Recski. *Matroid theory and its applications in electric network theory and in statics*, volume 6. Springer Science & Business Media, 2013.

[769] Don Redmond. Number theory. *An Introduction to Probabilistic Number Theory*, 2020.

[770] Noor Rehman, Nasir Shah, Muhammad Irfan Ali, and Choonkil Park. Uncertainty measurement for neighborhood based soft covering rough graphs with applications. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 113:2515–2535, 2019.

[771] Alfréd Rényi. Representations for real numbers and their ergodic properties. *Acta Mathematica Academiae Scientiarum Hungarica*, 8:477–493, 1957.

[772] Akbar Rezae, Karim Ghadimi, and Florentin Smarandache. Associated a nexus with a treesoft sets and vice versa. 2024.

[773] Akbar Rezaei, Tahsin Oner, Tugce Katican, Florentin Smarandache, and N Gandotra. *A short history of fuzzy, intuitionistic fuzzy, neutrosophic and plithogenic sets*. Infinite Study, 2022.

[774] Kaspar Riesen and Horst Bunke. Iam graph database repository for graph based pattern recognition and machine learning. In *Structural, Syntactic, and Statistical Pattern Recognition: Joint IAPR International Workshop, SSPR & SPR 2008, Orlando, USA, December 4-6, 2008. Proceedings*, pages 287–297. Springer, 2008.

[775] David W Roberts. Analysis of forest succession with fuzzy graph theory. *Ecological Modelling*, 45(4):261–274, 1989.

[776] Neil Robertson and Paul D. Seymour. Graph minors. x. obstructions to tree-decomposition. *Journal of Combinatorial Theory, Series B*, 52(2):153–190, 1991.

[777] Abraham Robinson. Non-standard analysis. *Mathematical logic in the 20th century*, pages 385–393, 2003.

[778] Abraham Robinson. *Non-standard analysis*. Princeton University Press, 2016.

[779] Julia Robinson. *The collected works of Julia Robinson*, volume 6. American Mathematical Soc., 1996.

[780] Rosa M Rodríguez, Luis Martínez, Vicenç Torra, ZS Xu, and Francisco Herrera. Hesitant fuzzy sets: state of the art and future directions. *International journal of intelligent systems*, 29(6):495–524, 2014.

[781] Eun Hwan Roh and Young Bae Jun. Positive implicative ideals of bck-algebras based on intersectional soft sets. *Journal of Applied Mathematics*, 2013(1):853907, 2013.

[782] Alexandre Rok and Bartosz Walczak. Outerstring graphs are χ-bounded. *SIAM Journal on Discrete Mathematics*, 33(4):2181–2199, 2019.

[783] Federico Romaniello. On 2-bisections and monochromatic edges in claw-free cubic multigraphs. 2023.

[784] Kenneth H Rosen. *Discrete mathematics and its applications sixth edition*. McGraw-hill, 2007.

[785] Ivo G Rosenberg. An algebraic approach to hyperalgebras. In *Proceedings of 26th IEEE International Symposium on Multiple-Valued Logic (ISMVL'96)*, pages 203–207. IEEE, 1996.

[786] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[787] Jonathan M. Rossiter and Toshiharu Mukai. Learning from uncertain image data using granular fuzzy sets and bio-mimetic applicability functions. In *EUSFLAT Conf.*, 2005.

[788] Carsten Rother, Vladimir Kolmogorov, and Andrew Blake. " grabcut" interactive foreground extraction using iterated graph cuts. *ACM transactions on graphics (TOG)*, 23(3):309–314, 2004.

[789] Bernat Rovira Segu. Outerplanar partial cubes. 2022.

[790] Simon Rubinstein-Salzedo and Ashvin A. Swaminathan. Analysis on surreal numbers. *arXiv: Classical Analysis and ODEs*, 2013.

[791] Michael Rudolf, Hannes Voigt, Christof Bornhövd, and Wolfgang Lehner. Synopsys: Foundations for multidimensional graph analytics. In *Enabling Real-Time Business Intelligence: International Workshops, BIRTE 2013, Riva del Garda, Italy, August 26, 2013, and BIRTE 2014, Hangzhou, China, September 1, 2014, Revised Selected Papers*, pages 159–166. Springer, 2015.

[792] Jan JMM Rutten and Daniele Turi. On the foundations of final semantics: Non-standard sets, metric spaces, partial orders. In *Semantics: Foundations and Applications: REX Workshop Beekbergen, The Netherlands, June 1–4, 1992 Proceedings*, pages 477–530. Springer, 1993.

[793] Kayvan Sadeghi. Stable mixed graphs. 2013.

[794] Kayvan Sadeghi and Steffen Lauritzen. Markov properties for mixed graphs. 2014.

[795] Muhammad Saeed and Muhammad Imran Harl. Fundamentals of picture fuzzy hypersoft set with application. *Neutrosophic Sets and Systems*, 53(1):24, 2023.

[796] Muhammad Saeed, Atiqe Ur Rahman, Muhammad Ahsan, and Florentin Smarandache. An inclusive study on fundamentals of hypersoft set. *Theory and Application of Hypersoft Set*, 1:1–23, 2021.

[797] Muhammad Saeed, Atiqe Ur Rahman, and Muhammad Arshad. A study on some operations and products of neutrosophic hypersoft graphs. *Journal of Applied Mathematics and Computing*, 68(4):2187–2214, 2022.

[798] Muhammad Saeed, Muhammad Khubab Siddique, Muhammad Ahsan, Muhammad Rayees Ahmad, and Atiqe Ur Rahman. A novel approach to the rudiments of hypersoft graphs. *Theory and Application of Hypersoft Set, Pons Publication House, Brussel*, pages 203–214, 2021.

[799] Muhammad Haris Saeed, Atiqe Ur Rahman, and Muhammad Arshad. A novel approach to neutrosophic hypersoft graphs with properties. 2021.

[800] Muhammad Haris Saeed, Atiqe Ur Rahman, and Muhammad Arshad. A novel approach to neutrosophic hypersoft graphs with properties. 2021.

[801] Hasmik Sahakyan. On the set of simple hypergraph degree sequences. *Applied Mathematical Sciences*, 9(5):243–253, 2015.

[802] Hasmik Sahakyan. On nonconvexity of the set of hypergraphic sequences. In *Proc. Computer Science and Information Technologies 2023 Conf*, pages 147–149, 2023.

[803] Hasmik Sahakyan and Levon Aslanyan. Convexity related issues for the set of hypergraphic sequences. *Information Theories and Applications*, 23(1):39–47, 2016.

[804] Mehmet Sahin, Shawkat Alkhazaleh, and Vakkas Ulucay. Neutrosophic soft expert sets. *Applied Mathematics-a Journal of Chinese Universities Series B*, 06:116–127, 2015.

[805] Mehmet Sahin, Vakkas Ulucay, and Merve Meneke. Some new operations of (α,β,γ) interval cut set of interval valued neutrosophic sets. *Journal of Mathematical and Fundamental Sciences*, 2018.

[806] Memet Şahin and Abdullah Kargın. *Neutrosophic triplet metric topology*. Infinite Study, 2019.

[807] Rıdvan Şahin. Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment. *arXiv preprint arXiv:1412.5202*, 2014.

[808] Rıdvan Şahin. An approach to neutrosophic graph theory with applications. *Soft Computing*, 23(2):569–581, 2019.

[809] Kavita Sahu, Fahad Ahmed Al-Zahrani, R. K. Srivastava, and Rajeev Kumar. Hesitant fuzzy sets based symmetrical model of decision-making for estimating the durability of web application. *Symmetry*, 12:1770, 2020.

[810] Tibor Salat. On statistically convergent sequences of real numbers. *Mathematica Slovaca*, 30:139–150, 1980.

[811] Sina Salimian and Seyed Meysam Mousavi. A multi-criteria decision-making model with interval-valued intuitionistic fuzzy sets for evaluating digital technology strategies in covid-19 pandemic under uncertainty. *Arabian Journal for Science and Engineering*, 48:7005 – 7017, 2022.

[812] Sovan Samanta, Vivek Kumar Dubey, and Kousik Das. Coopetition bunch graphs: competition and cooperation on covid19 research. *Information Sciences*, 589:1–33, 2022.

[813] Sovan Samanta, Madhumangal Pal, Rupkumar Mahapatra, Kousik Das, and Robin Singh Bhadoria. A study on semi-directed graphs for social media networks. *Int. J. Comput. Intell. Syst.*, 14:1034–1041, 2021.

[814] Sovan Samanta, Madhumangal Pal, Hossein Rashmanlou, and Rajab Ali Borzooei. Vague graphs and strengths. *Journal of Intelligent & Fuzzy Systems*, 30(6):3675–3680, 2016.

[815] Pierre Samuel. Ultrafilters and compactification of uniform spaces. *Transactions of the American Mathematical Society*, 64(1):100–132, 1948.

[816] Sharad S. Sane. Partition theory of integers. 2013.

[817] S Sangeetha and P Priya. Forests and fuzzy trees fuzzy.

[818] R. Santhi and K. Arun Prakash. On intuitionistic fuzzy semi-generalized closed sets and its applications. 2010.

[819] Gustavo Santos-García and José Carlos R Alcantud. Ranked soft sets. *Expert Systems*, 40(6):e13231, 2023.

[820] Muhammad Saqlain, Naveed Jafar, Sana Moin, Muhammad Farhan Saeed, and Said Broumi. Single and multi-valued neutrosophic hypersoft set and tangent similarity measure of single valued neutrosophic hypersoft sets. *Neutrosophic Sets and Systems*, 32:20, 2020.

[821] Musavarah Sarwar and Muhammad Akram. New applications of m-polar fuzzy matroids. *Symmetry*, 9:319, 2017.

[822] Robert Sasak. Comparing 17 graph parameters. Master's thesis, The University of Bergen, 2010.

[823] S Satham Hussain, Durga Nagarajan, Hossein Rashmanlou, and Farshid Mofidnakhaei. Novel supply chain decision making model under m-polar quadripartitioned neutrosophic environment. *Journal of Applied Mathematics and Computing*, pages 1–26, 2024.

[824] P Sathya, Nivetha Martin, and Florentine Smarandache. Plithogenic forest hypersoft sets in plithogenic contradiction based multi-criteria decision making. *Neutrosophic Sets and Systems*, 73:668–693, 2024.

[825] Bhavanari Satyanarayana, Tumurukota Venkata Pradeep Kumar, and Shaik Mohiddin Shaw. Set theory. *Functional Interpretations*, 2019.

[826] Carla Savage. A survey of combinatorial gray codes. *SIAM review*, 39(4):605–629, 1997.

[827] Franco Scarselli, Marco Gori, Ah Chung Tsoi, Markus Hagenbuchner, and Gabriele Monfardini. The graph neural network model. *IEEE transactions on neural networks*, 20(1):61–80, 2008.

[828] Henrik Schlichtkrull. *Hyperfunctions and harmonic analysis on symmetric spaces*, volume 49. Springer Science & Business Media, 2012.

[829] Bernd SW Schröder. Ordered sets. *Springer*, 29:30, 2003.

[830] Mahlagha Sedghi, Omid Reza Dehghan, and Morteza Norouzi. n-normed hypervector spaces. 2017.

[831] MK Sen, G Chowdhury, and DS Malik. Fuzzy intersection graphs of fuzzy semigroups. *New Mathematics and Natural Computation*, 2(01):1–10, 2006.

[832] Paul D Seymour. Decomposition of regular matroids. *Journal of combinatorial theory, Series B*, 28(3):305–359, 1980.

[833] Muhammad Shabir, Muhammad Irfan Ali, and Tanzeela Shaheen. Another approach to soft rough sets. *Knowl. Based Syst.*, 40:72–80, 2013.

[834] Nasir Shah and Said Broumi. Irregular neutrosophic graphs. *Neutrosophic Sets and Systems*, 13:47–55, 2016.

[835] Nasir Shah and Asim Hussain. Neutrosophic soft graphs. *viXra*, 2016.

[836] Sundas Shahzadi, Musavarah Sarwar, and Muhammad Akram. Decision-making approach with fuzzy type-2 soft graphs. *Journal of Mathematics*, 2020(1):8872446, 2020.

[837] Francina Shalini. Trigonometric similarity measures of pythagorean neutrosophic hypersoft sets. *Neutrosophic Systems with Applications*, 2023.

[838] Tong Shaocheng. Interval number and fuzzy number linear programmings. *Fuzzy sets and systems*, 66(3):301–306, 1994.

[839] Ranu Sharma, Bhagyashri A. Deole, and Smita Verma. Fuzzy open sets in fuzzy tri topological space. 2018.

[840] Alexander Shen and Nikolai K. Vereshchagin. Basic set theory. In *The Student Mathematical Library*, 2002.

[841] Jiao Shi, Xin Zhao, Nan Zhang, Yu Lei, and Lingtong Min. Rough-fuzzy graph learning domain adaptation for fake news detection. *IEEE Transactions on Computational Social Systems*, 2023.

[842] Xiaolong Shi, Saeed Kosari, Hossein Rashmanlou, Said Broumi, and S Satham Hussain. Properties of interval-valued quadripartitioned neutrosophic graphs with real-life application. *Journal of Intelligent & Fuzzy Systems*, 44(5):7683–7697, 2023.

[843] V Shijina, Sunil John, and Anitha Thomas. Multiple sets. *Journal of New Results in Science*, 4(9):18–27, 2015.

[844] V Shijina, Adithya Unni, and Sunil Jacob John. Similarity measure of multiple sets and its application to pattern recognition. *Informatica*, 44(3), 2020.

[845] Sajjan G. Shiva. Introduction to logic design. 2018.

[846] R. K. Shyamasundar. Introduction to algorithms. *Resonance*, 1:14–24, 1996.

[847] Karen B Singer. *Random intersection graphs*. The Johns Hopkins University, 1996.

[848] Prem Kumar Singh. Complex plithogenic set. *International Journal of Neutrosophic Sciences*, 18(1):57–72, 2022.

[849] Sukhveer Singh and Harish Garg. Symmetric triangular interval type-2 intuitionistic fuzzy sets with their applications in multi criteria decision making. *Symmetry*, 10:401, 2018.

[850] P. Sinthamani. Mathematical modeling of optimizing television cable network using fuzzy weighted graph. *Malaya Journal of Matematik*, 2019.

[851] R. Sivasamy and M. Mohammed Jabarulla. Products on interval valued pythagorean fuzzy soft graphs. *Journal of Interdisciplinary Mathematics*, 2024.

[852] Svend-Erik Skaaning. Assessing the robustness of crisp-set and fuzzy-set qca results. *Sociological Methods & Research*, 40(2):391–408, 2011.

[853] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[854] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[855] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[856] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.

[857] Florentin Smarandache. Neutrosophic physics: More problems, more solutions. 2010.

[858] Florentin Smarandache. Neutrosophic set–a generalization of the intuitionistic fuzzy set. *Journal of Defense Resources Management (JoDRM)*, 1(1):107–116, 2010.

[859] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[860] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[861] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic Sets and Systems*, 22:168–170, 2018.

[862] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[863] Florentin Smarandache. Neutrosophic perspectives: Triplets, duplets, multisets, hybrid operators, modal logic, hedge algebras, and applications. *Engineering Educator: Courses*, 2018.

[864] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[865] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[866] Florentin Smarandache. Extended nonstandard neutrosophic logic, set, and probability based on extended nonstandard analysis. *Symmetry*, 11(4):515, 2019.

[867] Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.

[868] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[869] Florentin Smarandache. *NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries (revisited)*, volume 5. Infinite Study, 2021.

[870] Florentin Smarandache. History of superhyperalgebra and neutrosophic superhyperalgebra (revisited again). *Neutrosophic Algebraic Structures and Their Applications*, page 10, 2022.

[871] Florentin Smarandache. Interval-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 117, 2022.

[872] Florentin Smarandache. *Introduction to SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra*. Infinite Study, 2022.

[873] Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today*. Infinite Study, 2022.

[874] Florentin Smarandache. Operators on single-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers*, 9:112, 2022.

[875] Florentin Smarandache. *Practical applications of IndetermSoft Set and IndetermHyperSoft Set and introduction to TreeSoft Set as an extension of the MultiSoft Set*. Infinite Study, 2022.

[876] Florentin Smarandache. *The SuperHyperFunction and the Neutrosophic SuperHyperFunction (revisited again)*, volume 3. Infinite Study, 2022.

[877] Florentin Smarandache. Decision making based on valued fuzzy superhypergraphs. 2023.

[878] Florentin Smarandache. Foundation of the superhypersoft set and the fuzzy extension superhypersoft set: A new vision. *Neutrosophic Systems with Applications*, 11:48–51, 2023.

[879] Florentin Smarandache. *Introduction to the multineutrosophic set*. Infinite Study, 2023.

[880] Florentin Smarandache. New types of soft sets: Hypersoft set, indetermsoft set, indetermhypersoft set, and treesoft set. *International Journal of Neutrosophic Science*, 2023.

[881] Florentin Smarandache. *New types of soft sets "hypersoft set, indetermsoft set, indetermhypersoft set, and treesoft set": an improved version*. Infinite Study, 2023.

[882] Florentin Smarandache. New types of soft sets" hypersoft set, indetermsoft set, indetermhypersoft set, and treesoft set": An improved version. *Neutrosophic Systems with Applications*, 2023.

[883] Florentin Smarandache. New types of topologies and neutrosophic topologies (improved version). *Neutrosophic Sets and Systems*, 57(1):14, 2023.

[884] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[885] Florentin Smarandache. Foundation of revolutionary topologies: An overview, examples, trend analysis, research issues, challenges, and future directions. 2024.

[886] Florentin Smarandache. Foundation of revolutionary topologies: An overview, examples, trend analysis, research issues, challenges, and future directions. 2024.

[887] Florentin Smarandache. Treesoft set vs. hypersoft set and fuzzy-extensions of treesoft sets. *HyperSoft Set Methods in Engineering*, 2024.

[888] Florentin Smarandache. Short introduction to standard and nonstandard neutrosophic set and logic (review paper). *Neutrosophic Sets and Systems*, 77, 2025. Review Paper.

[889] Florentin Smarandache and Mohamed Abdel-Basset. *Optimization Theory Based on Neutrosophic and Plithogenic Sets*. Academic Press, 2020.

[890] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group. *Neural Computing and Applications*, 29(7):595–601, 2018.

[891] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group (revisited). *Neutrosophic sets and Systems*, 26(1):2, 2019.

[892] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[893] Florentin Smarandache and Maissam Jdid. An overview of neutrosophic and plithogenic theories and applications. 2023.

[894] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Applications of bimatrices to some fuzzy and neutrosophic models. 2005.

[895] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

[896] Florentin Smarandache and Nivetha Martin. Plithogenic n-super hypergraph in novel multi-attribute decision making. *International Journal of Neutrosophic Science*, 2020.

[897] Florentin Smarandache, Memet Şahin, Derya Bakbak, Vakkas Uluçay, and Abdullah Kargın. *Neutrosophic SuperHyperAlgebra and New Types of Topologies*. Infinite Study, 2023.

[898] Florentin Smarandache, Memet Şahin, Derya Bakbak, Vakkas Uluçay, and Abdullah Kargın. *Neutrosophic SuperHyperAlgebra and New Types of Topologies*. Infinite Study, 2023.

[899] Florentin Smarandache, A. Saranya, A. Kalavathi, and S. Krishnaprakash. Neutrosophic superhypersoft sets. *Neutrosophic Sets and Systems*, 77:41–53, 2024.

[900] Antonín Sochor and Petr Vopěnka. Ultrafilters of sets. *Commentationes Mathematicae Universitatis Carolinae*, 22(4):689–699, 1981.

[901] Seok Zun Song, Hee Sik Kim, and Young Bae Jun. Ideal theory in semigroups based on intersectional soft sets. *The Scientific World Journal*, 2014(1):136424, 2014.

[902] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[903] Kenneth O. Stanley. A hypercube-based indirect encoding for evolving large-scale neural networks. 2009.

[904] B Starosta. Representing intuitionistic fuzzy sets as metasets. *Developments in fuzzy sets, intuitionistic fuzzy sets, generalized nets and related topics*, 1:185–208, 2010.

[905] Bartłomiej Starosta. Application of meta sets to character recognition. In *Foundations of Intelligent Systems: 18th International Symposium, ISMIS 2009, Prague, Czech Republic, September 14-17, 2009. Proceedings 18*, pages 602–611. Springer, 2009.

[906] Bartlomiej Starosta. Fuzzy sets as metasets. In *Proc. of XI International PhD Workshop (OWD 2009), Conference Archives PTETIS*, volume 26, pages 11–15, 2009.

[907] Bartłomiej Starosta. An approach to cardinality of first order metasets. In *Artificial Intelligence and Soft Computing: 13th International Conference, ICAISC 2014, Zakopane, Poland, June 1-5, 2014, Proceedings, Part II 13*, pages 688–699. Springer, 2014.

[908] Bartłomiej Starosta. Set-theoretic relations for metasets. *Journal of Experimental & Theoretical Artificial Intelligence*, 36(1):71–85, 2024.

[909] Bartłomiej Starosta and Witold Kosiński. Meta sets–another approach to fuzziness. In *Views on Fuzzy Sets and Systems from Different Perspectives: Philosophy and Logic, Criticisms and Applications*, pages 509–532. Springer, 2009.

[910] Bartłomiej Starosta and Witold Kosiński. Metasets, intuitionistic fuzzy sets and uncertainty. In *Artificial Intelligence and Soft Computing: 12th International Conference, ICAISC 2013, Zakopane, Poland, June 9-13, 2013, Proceedings, Part I 12*, pages 388–399. Springer, 2013.

[911] Michael L. Stein. Large sample properties of simulations using latin hypercube sampling. *Technometrics*, 29:143–151, 1987.

[912] Robert S Strichartz. *The way of analysis*. Jones & Bartlett Learning, 2000.

[913] KD Stroyan, B Benninghofen, and MM Richter. Superinfinitesimals in topology and functional analysis. *Proceedings of the London Mathematical Society*, 3(1):153–181, 1989.

[914] Keith D Stroyan. Mathematical background: Foundations of infinitesimal calculus, 1997.

[915] Keith D Stroyan and Wilhelmus Anthonius Josephus Luxemburg. *Introduction to the Theory of Infiniteseimals*. Academic Press, 1977.

[916] A Sudha and P Sundararajan. Robust fuzzy graph. *Ratio Mathematica*, 46, 2023.

[917] S Sudha, Nivetha Martin, and Florentin Smarandache. *Applications of Extended Plithogenic Sets in Plithogenic Sociogram*. Infinite Study, 2023.

[918] N Sujatha. A noval approach on triangular fuzzy unisum labeling in pharmaceutical research. *Journal of Pharmaceutical Negative Results*, pages 2408–2414, 2022.

[919] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.

[920] Aliye Ayca Supciller and Fatih Toprak. Selection of wind turbines with multi-criteria decision making techniques involving neutrosophic numbers: A case from turkey. *Energy*, 207:118237, 2020.

[921] Eulalia Szmidt. *Distances and similarities in intuitionistic fuzzy sets*, volume 307. Springer, 2014.

[922] E. C. T. An introduction to the theory of numbers. *Nature*, 157:390–390, 1946.

[923] Ali Taghavi and Roja Hosseinzadeh. Hahn-banach theorem for functionals on hypervector spaces. 2011.

[924] AL-Hawary Talal and Bayan Hourani. On intuitionistic product fuzzy graphs. *Italian Journal of Pure and Applied Mathematics*, page 113.

[925] M Scafati Tallini. Hypervector spaces. In *Proceeding of the 4th International Congress in Algebraic Hyperstructures and Applications*, pages 167–174, 1991.

[926] Maria Scafati Tallini. Matroidal hypervector spaces. *Journal of Geometry*, 42:132–140, 1991.

[927] Maria Scafati Tallini. Characterization of remarkable hypervector spaces. 2004.

[928] Jian Tang, Xinyang Feng, Bijan Davvaz, and Xiangyun Xie. A further study on ordered regular equivalence relations in ordered semihypergroups. *Open Mathematics*, 16:168 – 184, 2018.

[929] Trevor Tarr. *Leibniz, Calculus, and The Hyperreal Numbers*. PhD thesis, 2024.

[930] Michael B Teitz and Polly Bart. Heuristic methods for estimating the generalized vertex median of a weighted graph. *Operations research*, 16(5):955–961, 1968.

[931] Lev Telyatnikov, Maria Sofia Bucarelli, Guillermo Bernardez, Olga Zaghen, Simone Scardapane, and Pietro Lió. Hypergraph neural networks through the lens of message passing: A common perspective to homophily and architecture design. *ArXiv*, abs/2310.07684, 2023.

[932] Samajh Singh Thakur, Jyoti Pandey Bajpai, and Jyoti Pandey. On intutionistic fuzzy gpr-closed sets. *Fuzzy Information and Engineering*, 4:425–444, 2012.

[933] Samajh Singh Thakur, Anita Singh Banafar, Mahima Thakur, Jyoti Pandey Bajpai, and Archana Kumari Prasad. Operations and similarity measures between (m,n)-fuzzy sets. *Journal of the Indonesian Mathematical Society*, 2024.

[934] T Tharmaraj and PB Sarasija. Square difference labeling of some union graphs. *International Journal of Mathematics Trends and Technology-IJMTT*, 11, 2014.

[935] Dimitrios M Thilikos. Algorithms and obstructions for linear-width and related search parameters. *Discrete Applied Mathematics*, 105(1-3):239–271, 2000.

[936] Ye Tian, Lili Liu, Xiangjun Mi, and Bingyi Kang. Zslf: A new soft likelihood function based on z-numbers and its application in expert decision system. *IEEE Transactions on Fuzzy Systems*, 29:2283–2295, 2021.

[937] Ye Tian, Xiangjun Mi, Yunpeng Ji, and Bingyi Kang. Ze-numbers: a new extended z-numbers and its application on multiple attribute group decision making. *Engineering Applications of Artificial Intelligence*, 101:104225, 2021.

[938] Varun Kumar Tiwari, Prashant Kumar Jain, and Puneet Tandon. A bijective soft set theoretic approach for concept selection in design process. *Journal of Engineering Design*, 28:100 – 117, 2017.

[939] Varun Kumar Tiwari, Prashant Kumar Jain, and Puneet Tandon. An integrated shannon entropy and topsis for product design concept evaluation based on bijective soft set. *Journal of Intelligent Manufacturing*, 30:1645 – 1658, 2017.

[940] Christian Tominski, James Abello, and Heidrun Schumann. Cgv-an interactive graph visualization system. *Computers & Graphics*, 33(6):660–678, 2009.

[941] He Tong, Xue Peijun, and Shi Kaiquan. Application of rough graph in relationship mining. *Journal of Systems Engineering and Electronics*, 19(4):742–747, 2008.

[942] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[943] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[944] Mirko Torrisi, Gianluca Pollastri, and Quan Le. Deep learning methods in protein structure prediction. *Computational and Structural Biotechnology Journal*, 18:1301–1310, 2020.

[945] Nenad Trinajstic. *Chemical graph theory*. CRC press, 2018.

[946] William Thomas Tutte. Matroids and graphs. *Transactions of the American Mathematical Society*, 90(3):527–552, 1959.

[947] Kifayat Ullah, Tahir Mahmood, Zeeshan Ali, and Naeem Jan. On some distance measures of complex pythagorean fuzzy sets and their applications in pattern recognition. *Complex & Intelligent Systems*, 6:15 – 27, 2019.

[948] Vakkas Ulucay. Q-neutrosophic soft graphs in operations management and communication network. *Soft Computing*, 25:8441 – 8459, 2021.

[949] Vakkas Ulucay, Irfan Deli, and Mehmet Sahin. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neural Computing and Applications*, 29:739–748, 2018.

[950] Vakkas Uluçay and Memet Şahin. Intuitionistic fuzzy soft expert graphs with application. *Uncertainty discourse and applications*, 1(1):1–10, 2024.

[951] Shinji Umeyama. An eigendecomposition approach to weighted graph matching problems. *IEEE transactions on pattern analysis and machine intelligence*, 10(5):695–703, 1988.

[952] Adithya Unni and Sunil Jacob John. Entropy of multiple sets. In *AIP Conference Proceedings*, volume 2336. AIP Publishing, 2021.

[953] Robert L Vaught. *Set theory: an introduction*. Springer Science & Business Media, 2001.

[954] AG Rose Venish, L Vidyarani, and M Vigneshwaran. On binary neutrosophic crisp points and binary neutrosophic neighborhoods. *Journal of Neutrosophic and Fuzzy Systems (JNFS) Vol*, 5(01):15–22, 2023.

[955] Nikolai Konstantinovich Vereshchagin and Alexander Shen. *Basic set theory*. Number 17. American Mathematical Soc., 2002.

[956] G Vetrivel, M Mullai, and G Rajchakit. Product operations on pythagorean co-neutrosophic graphs and its application. *Neutrosophic Sets and Systems*, 72:381–407, 2024.

[957] M Vijaya and M Asha Joyce. Operations on interval-valued binary fuzzy graphs. *Journal of Algebraic Statistics*, 13(2):2534–2540, 2022.

[958] M Vijaya and Ms Asha Joyce. Undirected binary fuzzy graphs on composition, tensor and normal products. 2019.

[959] Srinivasan Vijayabalaji. Soft-rough expert set. In *Soft Computing*, pages 238–249. CRC Press, 2024.

[960] P VIJAYALAKSHMI. A novel approach for finding shortest path in z-graphs. 2024.

[961] Dhatchinamoorthy Vinoth and Devarasan Ezhilmaran. An analysis of global and adaptive thresholding for biometric images based on neutrosophic overset and underset approaches. *Symmetry*, 15:1102, 2023.

[962] Miroslav Voechovsky, Jan Masek, and Jan Elias. Distance-based optimal sampling in a hypercube: Analogies to n-body systems. *Adv. Eng. Softw.*, 137, 2019.

[963] Vitaly I. Voloshin. Introduction to graph and hypergraph theory. 2013.

[964] Petr Vopenka and Katerina Trlifajová. Alternative set theory., 2009.

[965] Adam Zsolt Wagner. Constructions in combinatorics via neural networks. *ArXiv*, abs/2104.14516, 2021.

[966] K. S. How Tai Wah. Fault coupling in finite bijective functions. *Software Testing*, 5, 1995.

[967] Chia-Nan Wang, Ngoc-Ai-Thy Nguyen, and Thanh-Tuan Dang. Offshore wind power station (owps) site selection using a two-stage mcdm-based spherical fuzzy set approach. *Scientific Reports*, 12, 2022.

[968] Haibin Wang, Florentin Smarandache, Rajshekhar Sunderraman, and Yan-Qing Zhang. *interval neutrosophic sets and logic: theory and applications in computing: Theory and applications in computing*, volume 5. Infinite Study, 2005.

[969] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[970] Jing Wang, Zichen Liu, Shuai Ma, Nikos Ntarmos, and Peter Triantafillou. Gc: A graph caching system for subgraph/supergraph queries. *Proceedings of the VLDB Endowment*, 11(12):2022–2025, 2018.

[971] Ping Wang, Jie Wang, Guiwu Wei, and Cun Wei. Similarity measures of q-rung orthopair fuzzy sets based on cosine function and their applications. *Mathematics*, 2019.

[972] Tianxing Wang, Huaxiong Li, Xianzhong Zhou, Dun Liu, and Bing Huang. Three-way decision based on third-generation prospect theory with z-numbers. *Inf. Sci.*, 569:13–38, 2021.

[973] Xinyi Wang and Qinghai Wang. Uncertainty measurement of variable precision fuzzy soft rough set model. In *CECNet*, 2022.

[974] Yuxin Wang, Quan Gan, Xipeng Qiu, Xuanjing Huang, and David Paul Wipf. From hypergraph energy functions to hypergraph neural networks. In *International Conference on Machine Learning*, 2023.

[975] Zhao-xia Wang, Quan Wang, Ming Bai, Zeng-qiang Chen, and Zhen Sun. Further exploration on relationship between crisp sets and fuzzy sets. In *2010 The 2nd International Conference on Computer and Automation Engineering (ICCAE)*, volume 2, pages 609–613. IEEE, 2010.

[976] Seth Warner. *Modern algebra*. Courier Corporation, 1990.

[977] Zach Weber. Transfinite numbers in paraconsistent set theory. *The Review of Symbolic Logic*, 3:71 – 92, 2010.

[978] Zach Weber. Transfinite cardinals in paraconsistent set theory. *The Review of Symbolic Logic*, 5:269 – 293, 2012.

[979] Zach Weber. Notes on inconsistent set theory. In *Paraconsistency: Logic and Applications*, 2013.

[980] Lin Wei. An integrated decision-making framework for blended teaching quality evaluation in college english courses based on the double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:3259–3266, 2023.

[981] Tong Wei, Junlin Hou, and Rui Feng. Fuzzy graph neural network for few-shot learning. In *2020 International joint conference on neural networks (IJCNN)*, pages 1–8. IEEE, 2020.

[982] DJA Welsh et al. Matroids: fundamental concepts. *Handbook of combinatorics*, 1:481–526, 1995.

[983] Dominic JA Welsh. *Matroid theory*. Courier Corporation, 2010.

[984] Sylvia Wenmackers. Vieri benci and mauro di nasso. how to measure the infinite: Mathematics with infinite and infinitesimal numbers, 2022.

[985] Douglas Brent West et al. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.

[986] Tomasz Witczak. Neutrosophic borda method. *International Journal of Neutrosophic Science*, 19(1), 2022.

[987] Lingfei Wu, Yu Chen, Kai Shen, Xiaojie Guo, Hanning Gao, Shucheng Li, Jian Pei, Bo Long, et al. Graph neural networks for natural language processing: A survey. *Foundations and Trends® in Machine Learning*, 16(2):119–328, 2023.

[988] Zonghan Wu, Shirui Pan, Fengwen Chen, Guodong Long, Chengqi Zhang, and S Yu Philip. A comprehensive survey on graph neural networks. *IEEE transactions on neural networks and learning systems*, 32(1):4–24, 2020.

[989] Zonghan Wu, Shirui Pan, Guodong Long, Jing Jiang, and Chengqi Zhang. Graph wavenet for deep spatial-temporal graph modeling. *arXiv preprint arXiv:1906.00121*, 2019.

[990] Tong Xia, Junjie Lin, Yong Li, Jie Feng, Pan Hui, Funing Sun, Diansheng Guo, and Depeng Jin. 3dgcn: 3-dimensional dynamic graph convolutional network for citywide crowd flow prediction. *ACM Transactions on Knowledge Discovery from Data (TKDD)*, 15(6):1–21, 2021.

[991] Liming Xiao, Guangquan Huang, Witold Pedrycz, Dragan Pamucar, Luis Martínez, and Genbao Zhang. A q-rung orthopair fuzzy decision-making model with new score function and best-worst method for manufacturer selection. *Information sciences*, 608:153–177, 2022.

[992] Keyulu Xu, Weihua Hu, Jure Leskovec, and Stefanie Jegelka. How powerful are graph neural networks? *arXiv preprint arXiv:1810.00826*, 2018.

[993] Liqiong Xu. On biregular graphs whose energy exceeds the number of vertices. *MATCH Commun. Math. Comput. Chem*, 66:959–970, 2011.

[994] Xiaoyan Xu, Xiaohua Xu, Jin Chen, and Shixun Lin. On forbidden subgraphs of main supergraphs of groups. *Electronic Research Archive*, 32(8), 2024.

[995] Zeshui Xu. *Hesitant fuzzy sets theory*, volume 314. Springer, 2014.

[996] Xingsi Xue, Mahima Poonia, Ghaida Muttashar Abdulsahib, Rakesh Kumar Bajaj, Osamah Ibrahim Khalaf, Himanshu Dhumras, and Varun Shukla. On cohesive fuzzy sets, operations and properties with applications in electromagnetic signals and solar activities. *Symmetry*, 15(3):595, 2023.

[997] Ronald R Yager. On z-valuations using zadeh's z-numbers. *International Journal of Intelligent Systems*, 27(3):259–278, 2012.

[998] Ronald R. Yager. Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 25:1222–1230, 2017.

[999] Xiaofan Yang and Yuanyan Tang. A (4n-9)/3 diagnosis algorithm on n-dimensional cube network. *Inf. Sci.*, 177:1771–1781, 2007.

[1000] Xibei Yang, Dongjun Yu, Jingyu Yang, and Chen Wu. Generalization of soft set theory: from crisp to fuzzy case. In *Fuzzy Information and Engineering: Proceedings of the Second International Conference of Fuzzy Information and Engineering (ICFIE)*, pages 345–354. Springer, 2007.

[1001] Zhilin Yang, William Cohen, and Ruslan Salakhudinov. Revisiting semi-supervised learning with graph embeddings. In *International conference on machine learning*, pages 40–48. PMLR, 2016.

[1002] Yiyu Yao and Jilin Yang. Granular fuzzy sets and three-way approximations of fuzzy sets. *Int. J. Approx. Reason.*, 161:109003, 2023.

[1003] Naveed Yaqoob, Muhammad Gulistan, Jian Tang, and Muhammad Azhar. On generalized fuzzy hyperideals in ordered la-semihypergroups. *Computational and Applied Mathematics*, 38:1–20, 2019.

[1004] Jun Ye. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *J. Intell. Fuzzy Syst.*, 26:2459–2466, 2014.

[1005] Jun Ye and Shigui Du. Some distances, similarity and entropy measures for interval-valued neutrosophic sets and their relationship. *International Journal of Machine Learning and Cybernetics*, 10:347 – 355, 2017.

[1006] Hasan Yeti? and Mehmet Karakose. Modelling type-2 fuzzy systems by optimized nonstationary fuzzy sets with genetic algorithm. *2020 24th International Conference on Information Technology (IT)*, pages 1–4, 2020.

[1007] Eunjeong Yi. The fractional metric dimension of permutation graphs. *Acta Mathematica Sinica, English Series*, 31:367–382, 2015.

[1008] Adem Yolcu and Taha Yasin Ozturk. Fuzzy hypersoft sets and it's application to decision-making. *Theory and application of hypersoft set*, 50, 2021.

[1009] Adem Yolcu, Florentin Smarandache, and Taha Yasin Ozturk. Intuitionistic fuzzy hypersoft sets. 2021.

[1010] Bing Yu, Haoteng Yin, and Zhanxing Zhu. Spatio-temporal graph convolutional networks: A deep learning framework for traffic forecasting. *arXiv preprint arXiv:1709.04875*, 2017.

[1011] Liang-Chih Yu, Jin Wang, K Robert Lai, and Xue-jie Zhang. Predicting valence-arousal ratings of words using a weighted graph method. In *Proceedings of the 53rd Annual Meeting of the Association for Computational Linguistics and the 7th International Joint Conference on Natural Language Processing (Volume 2: Short Papers)*, pages 788–793, 2015.

[1012] Hong yu Zhang, Jian qiang Wang, and Xiao hong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615–627, 2016.

[1013] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[1014] Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.

[1015] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[1016] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[1017] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[1018] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[1019] Lotfi A Zadeh. Fuzzy sets and information granularity. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 433–448. World Scientific, 1996.

[1020] Lotfi A Zadeh. A note on z-numbers. *Information sciences*, 181(14):2923–2932, 2011.

[1021] Shouzhen Zeng, Muhammad Shoaib, Shahbaz Ali, Qamar Abbas, and Muhammad Shahzad Nadeem. Complex vague graphs and their application in decision-making problems. *IEEE access*, 8:174094–174104, 2020.

[1022] Shunqi Zeng, Zhenzhi Lin, Fushuan Wen, and Gerard Ledwich. A new approach for power system black-start decision-making with vague set theory. *International Journal of Electrical Power & Energy Systems*, 34(1):114–120, 2012.

[1023] Jianming Zhan, Muhammad Irfan Ali, and Nayyar Mehmood. On a novel uncertain soft set model: Z-soft fuzzy rough set model and corresponding decision making methods. *Appl. Soft Comput.*, 56:446–457, 2017.

[1024] Jianming Zhan and Kuanyun Zhu. A novel soft rough fuzzy set: Z-soft rough fuzzy ideals of hemirings and corresponding decision making. *Soft Computing*, 21:1923 – 1936, 2016.

[1025] Aiwu Zhang, Jianguo Du, and Hongjun Guan. Interval valued neutrosophic sets and multi-attribute decision-making based on generalized weighted aggregation operator. *J. Intell. Fuzzy Syst.*, 29:2697–2706, 2015.

[1026] Muhan Zhang, Zhicheng Cui, Marion Neumann, and Yixin Chen. An end-to-end deep learning architecture for graph classification. In *Proceedings of the AAAI conference on artificial intelligence*, volume 32, 2018.

[1027] Ping Zhang and Gary Chartrand. Introduction to graph theory. *Tata McGraw-Hill*, 2:2–1, 2006.

[1028] Qian-Sheng Zhang and Sheng-Yi Jiang. A note on information entropy measures for vague sets and its applications. *Information Sciences*, 178(21):4184–4191, 2008.

[1029] Xiaohong Zhang, Qingqing Hu, Florentin Smarandache, and Xiaogang An. On neutrosophic triplet groups: Basic properties, nt-subgroups, and some notes. *Symmetry*, 10(7):289, 2018.

[1030] Xiaohong Zhang, Florentin Smarandache, and Xingliang Liang. Neutrosophic duplet semi-group and cancellable neutrosophic triplet groups. *Symmetry*, 9:275, 2017.

[1031] Hua Zhao, Zeshui Xu, Shousheng Liu, and Zhong Wang. Intuitionistic fuzzy mst clustering algorithms. *Computers & Industrial Engineering*, 62(4):1130–1140, 2012.

[1032] Jiang Zhou and Changjiang Bu. The enumeration of spanning tree of weighted graphs. *Journal of Algebraic Combinatorics*, 54:75–108, 2021.

[1033] Jie Zhou, Ganqu Cui, Shengding Hu, Zhengyan Zhang, Cheng Yang, Zhiyuan Liu, Lifeng Wang, Changcheng Li, and Maosong Sun. Graph neural networks: A review of methods and applications. *AI open*, 1:57–81, 2020.

[1034] Yuhong Zhu, George N Rouskas, and Harry G Perros. A path decomposition approach for computing blocking probabilities in wavelength-routing networks. *IEEE/ACM Transactions on networking*, 8(6):747–762, 2000.

[1035] H-J Zimmermann. Fuzzy set theory and mathematical programming. *Fuzzy sets theory and applications*, pages 99–114, 1986.

[1036] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

[1037] Patrick Zulian, Teseo Schneider, Kai Hormann, and Rolf H. Krause. Parametric finite elements with bijective mappings. *BIT Numerical Mathematics*, 57:1185 – 1203, 2017.

[1038] Nicolás Zumelzu, Benjamín René Callejas Bedregal, Edmundo Mansilla, Humberto Bustince, and R. D'iaz. Admissible orders on fuzzy numbers. *IEEE Transactions on Fuzzy Systems*, 30:4788–4799, 2020.

This book explores the advancement of uncertain combinatorics through innovative methods such as graphization, hyperization, and uncertainization, incorporating concepts from fuzzy, neutrosophic, soft, and rough set theory, among others. Combinatorics and set theory are fundamental mathematical disciplines that focus on counting, arrangement, and the study of collections under specified rules. While combinatorics excels at solving problems involving uncertainty, set theory has expanded to include advanced concepts like fuzzy and neutrosophic sets, which are capable of modeling complex real-world uncertainties by accounting for truth, indeterminacy, and falsehood. These developments intersect with graph theory, leading to novel forms of uncertain sets in "graphized" structures, such as hypergraphs and superhypergraphs. Innovations like Neutrosophic Oversets, Undersets, and Offsets, as well as the Nonstandard Real Set, build upon traditional graph concepts, pushing the boundaries of theoretical and practical advancements. This synthesis of combinatorics, set theory, and graph theory provides a strong foundation for addressing the complexities and uncertainties present in mathematical and real-world systems, paving the way for future research and application.

[Forward by Prof. univ. dr. Florentin Smarandache]

ISBN 978-1-59973-812-3

9 781599 738123 >