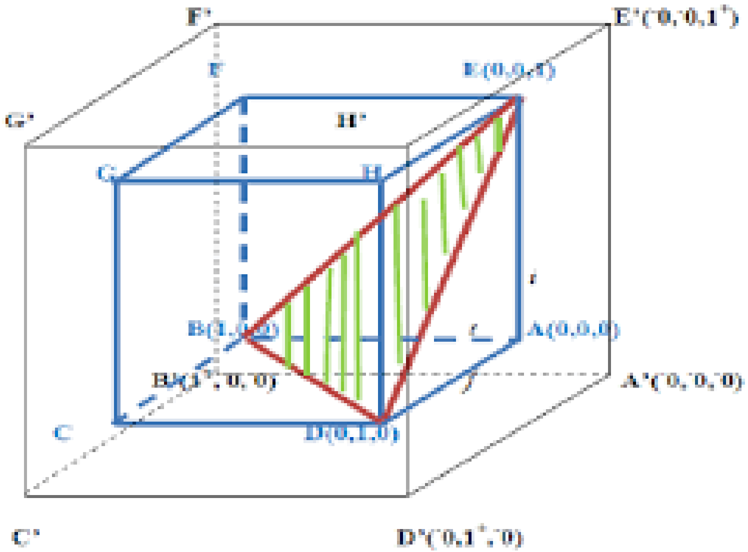
Fuzzy, Intuitionistic and Neutrosophic Set Theories and their Applications in Decision Analysis

Editors

Prof. Dr.Florentin Smarandache
Dr. K. Mohana



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We extend our sincere gratitude to the authors, whose valuable contributions and dedication have enriched this book chapter, "***Fuzzy, Intuitionistic, and Neutrosophic Set Theories and Their Applications in Decision Analysis***". Their insightful research and scholarly efforts have significantly advanced the understanding and application of these mathematical frameworks in decision analysis.

We also express our heartfelt appreciation to the publishers for their unwavering support and commitment to disseminating knowledge. Their efforts in bringing this work to a wider audience have played a crucial role in fostering academic and practical advancements in the field. Furthermore, we acknowledge the pioneering mathematicians whose foundational work in fuzzy, intuitionistic fuzzy, and neutrosophic set theories has laid the groundwork for this research. The contributions of Lotfi Zadeh, Krassimir Atanassov, and many others have provided the theoretical backbone that continues to inspire and drive innovation in decision science.

We would like to express our sincere gratitude to all the Author's for contributing their insightful and thought-provoking chapter, to this collective work. Their expertise and dedication to the subject matter have enriched the overall quality of this volume. We truly appreciate the time, effort, and passion invested in crafting such a meaningful piece, and we are grateful for their collaboration throughout this project.

This book is stronger because of all author's valuable contribution, and we look forward to the continued impact of their work in the field.

With deep gratitude,

Prof. Dr. Florentin Smarandache

&

Dr. K. Mohana

(Editors)

THE PREFACE

This book chapter***, Fuzzy, Intuitionistic, and Neutrosophic Set Theories and Their Applications in Decision Analysis***, explores the development and importance of these mathematical approaches in managing uncertainty and imprecision in decision-making.

Fuzzy set theory, pioneered by Lotfi Zadeh in 1965, introduced the concept of partial membership, moving beyond traditional binary logic to offer a more flexible representation of ambiguity in real-world problems. Expanding on this, Krassimir Atanassov introduced intuitionistic fuzzy sets in 1983, incorporating both membership and non-membership degrees for a more detailed depiction of uncertainty. Later, in 1998, Florentin Smarandache introduced neutrosophic set theory, which added the concept of indeterminacy, enabling a three-fold perspective—truth, falsity, and indeterminacy—to better model complex and uncertain scenarios. The book focuses on the practical applications of these theories in decision-making, demonstrating how they improve uncertainty modeling and facilitate more effective decision-making across various fields. By leveraging these frameworks, decision-makers can better navigate complexity, leading to more accurate and dependable outcomes.

Ultimately, this book highlights the progression from fuzzy to intuitionistic fuzzy to neutrosophic set theories as a significant advancement in capturing and analyzing uncertainty within decision-making contexts.

Prof. Dr. Florentin Smarandache

Dr. K. Mohana

(Editors)

CONTENTS

TANGENT AND COTANGENT SIMILARITY MEASURES OF FERMATEAN QUADRI PARTITIONED NEUTROSOPHIC SETS

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Abstract: In this Chapter, a new tangent and cotangent similarity measures between two Fermatean Quadripartitioned Neutrosophic [FQN] sets with truth membership, falsity membership, ignorance and contradiction membership as Neutrosophic component is proposed and its properties are investigated. Also, the weighted similarity measures are also studied with a decision-making problem.

Keywords: FQN set, Tangent similarity measure, cotangent similarity measure.

1. Introduction

Traditionally, the teaching and learning method uses several exercises fixing, sending and evaluating ideas and information about a subject. Learning is that the method of getting relative permanent changes in understanding, attitude, knowledge, information, capability and skill through expertise. A modification are often set or involuntary, to raised or worse learning. The training method is an enclosed cognitive event. To assist this teaching and learning method, it is necessary the utilization of a laptop tool ready to stimulate these changes. Also, it is necessary that it will operate as validation and serving tool to the college students.

The COVID-19 pandemic has caused important disruption with in the domain of education, that is considered as essential determinant for economic progress of any country. Even developed countries are waging a battle against COVID-19 for minimizing the impact on their economy because of prolonged lockdown. Education sector isn't an exception, and method of educational delivery has been grossly affected. There has been unforeseen and impetuous transition from real classroom to on-line and virtual teaching methodology across the world. There's an enormous question on the sustainability of online mode of teaching post-pandemic and its percussions on world education market. Impact of lockdown on the teaching—learning method has been studied in present paper with the objective to assess the quality of online classes and challenges associated with them. The paper proposes about the benefits of social media in virtual education among College Students

In order to deal with uncertainties, the thought of fuzzy sets and fuzzy set operations was introduced by Zadeh [17]. The speculation of fuzzy topological space was studied and developed by C.L. Chang [3]. The paper of Chang sealed the approach for the subsequent growth of the various fuzzy topological ideas. Since then, a lot of attention has been paid to generalize the fundamental ideas of general topology in fuzzy setting and therefore a contemporary theory of fuzzy topology has been developed. Atanassov and plenty of researchers [1] worked on intuitionistic fuzzy sets within the literature. Florentine Smarandache [15] introduced the idea of Neutrosophic set in 1995 that provides the information of neutral thought by introducing the new issue referred to as uncertainty within the set. Thus neutrosophic set was framed and it includes the parts of truth membership function(T), indeterminacy membership function(I), and falsity membership function(F) severally. Neutrosophic sets deals with non-normal interval of]−0 1+[. Pentapartitioned neutrosophic set and its properties were introduced by Rama Malik and Surpati Pramanik [14]. In this case, indeterminacy is divided into three components: contradiction, ignorance, and an unknown membership function. The concept of Fermatean Quadripartitioned Neutrosophic sets s was initiated by M.Ramya[13].

Similarity measure is an important topic in the current fuzzy, Pythagorean , Neutrosophic and different hybrid environments. Recently, the improved correlation coefficients of Pentapartitioned Neutrosophic Pythagorean sets and Quadripartitioned Neutrosophic Pythagorean sets was introduced by R. Radha and A. Stanis Arul Mary. Pranamik and Mondal [5,6] has also proposed weighted similarity measures based on tangent function and cotangent function and its application on medical diagnosis. In this paper, the weighted similarity measures of Tangent and Cotangent functions has been applied to PNP sets in virtual education during Covid Pandemic.

2. Preliminaries

Definition 2.1 [15]

Let X be a universe. A Neutrosophic set A on X can be defined as follows:

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}$$

Where $T_A\ , I_A, F_A: U \rightarrow [0,1]\ and\ 0 \leq T_A(x) + I_A(x) + F_A\,(x) \leq 3$

Definition 2. 2[9]

Let X be a universe. A Fermatean Quadripartitioned neutrosophic [FQN] set A with neutrosophic components for A on X is an object of the form

$$A = \{< x, T_A, C_A, U_A, F_A >: x \in X\}$$

Where $(T_A)^3 + (C_A)^3 + (U_A)^3 + (F_A)^3 \leq 2$

Here, $T_A(x)$ is the truth membership, $C_A(x)$ is contradiction membership, $U_A(x)$ is ignorance membership and $F_A(x)$ is the false membership.

Definition 2.3 [14]

Let P be a non-empty set. A Pentapartitioned neutrosophic set A over P characterizes each element p in P a truth -membership function T_A, a contradiction membership function C_A, an ignorance membership function G_A, unknown membership function U_A and a false membership function F_A, such that for each p in P.

$$T_A + C_A + G_A + U_A + F_A \leq 5$$

Definition 2. 4 [9]

The complement of a FQN set A on R Denoted by A^C or A^* and is defined as

$A^C = \{< x, F_A(x), U_A(x), C_A(x), T_A(x) > : x \in X\}$

Definition 2.5 [9]

Let A = $< x, T_A(x), C_A(x), U_A(x), F_A(x) >$ and

B = $< x, T_B(x), C_B(x), U_B(x), F_B(x) >$ are FQN sets.

Then

A∪ B=<x,$max(T_A(x), T_B(x)), max(C_A(x), C_B(x)), \min(U_A(x), U_B(x)), \min(F_A(x), F_B(x)), >$

A∩ B=$< x min(T_A(x), T_B(x)), min(C_A(x), C_B(x)), max(U_A(x), U_B(x)), max(F_A(x), F_B(x)) >$

Definition 2.6 [9]

A FQN topology on a nonempty set R is a family of a FQN sets in R satisfying the following axioms.

1) 0,1$\in \tau$
2) $R_1 \cap R_2 \in \tau$ for any $R_1, R_2 \in \tau$
3) $\cup R_i \in \tau$ for any $R_i: i \in I \subseteq \tau$

The complement R^* of FQN open set (FQNOS, in short) in FQN topological space [FQNTS] (R,$\boldsymbol{\tau}$), is called a FQN closed set [FQNCS].

3.Tangent and Cotangent Similarity Measures of FQN Sets

Definition 3.1

Let P = $\{(\mathrm{r}, \mathrm{B1}_P(r), \mathrm{B2}_P(r), \mathrm{B3}_P(r), \mathrm{B4}_P(r)): r \in R\}$ and

Q = $\{(\mathrm{r}, \mathrm{B1}_Q(r), \mathrm{B2}_Q(r), \mathrm{B3}_Q(r), \mathrm{B4}_Q(r), : r \in R\}$ be two Fermatean Quadripartitioned Neutrosophic numbers with Neutrosophic components. Now tangent similarity function which measures the similarity between two vectors based only on the direction, ignoring the impact of the distance between them can be presented as follows

$$T_{FQN}(P,Q) = \frac{1}{n}\sum_{i=1}^{n}[1 - \tan(\frac{\pi}{16}[\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| + \left|B2_P^3(r_i) - B2_Q^3(r_i)\right| + \left|B3_P^3(r_i) - B3_Q^3(r_i)\right| + \left|B4_P^3(r_i) - B4_Q^3(r_i)\right|])]$$

Theorem 3. 2

The defined tangent similarity measure $T_{FQN}(P,Q)$ between FQN set P and Q satisfies the following properties

1. $0 \leq T_{FQN}(P,Q) \leq 1$;
2. $T_{FQN}(P,Q) = 1$ iff P = Q;
3. $T_{FQN}(P,Q) = T_{FQN}(Q,P)$;
4. If T is a FQN set in R and $P \subseteq Q \subseteq T$ then $T_{FQN}(P,T) \leq T_{FQN}(P,Q)$ and $T_{FQN}(P,T) \leq T_{FQN}(Q,T)$.

Proof

1) As the truth membership, contradiction membership, ignorance membership and falsity membership function of the FQN sets and the value of the tangent function also is within [0,1].

Hence $0 \leq T_{FQN}(P,Q) \leq 1$.

2) For any two FQN sets P and Q if P = Q, this implies $B1_P(r_i) = B1_Q(r_i)$, $B2_P(r_i) = B2_Q(r_i)$, $B3_P(r_i) = B3_Q(r_i)$,and $B4_P(r_i) = B4_Q(r_i)$.

Hence $\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| = 0, \left|B2_P^3(r_i) - B2_Q^3(r_i)\right| = 0, \left|B3_P^3(r_i) - B3_Q^3(r_i)\right| = 0$ and $\left|B4_P^3(r_i) - B4_Q^3(r_i)\right| = 0$.

Thus $T_{FQN}(P,Q) = 1$.

Conversely, if $T_{FQN}(P,Q) = 1$, then $\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| = 0, \left|B2_P^3(r_i) - B2_Q^3(r_i)\right| = 0$, $\left|B3_P^3(r_i) - B3_Q^3(r_i)\right| = 0$ and $\left|B4_P^3(r_i) - B4_Q^3(r_i)\right| = 0$ since tan(0) = 0. So we can write $B1_P(r_i) = B1_Q(r_i)$, $B2_P(r_i) = B2_Q(r_i)$, $B3_P(r_i) = B3_Q(r_i)$ And $B4_P(r_i) = B4_Q(r_i)$.

Hence P = Q.

3) The Proof is obvious

4) If $P \subseteq Q \subseteq T$ then $B1_P(r_i) \leq B1_Q(r_i) \leq B1_T(r_i)$, $B2_P(r_i) \leq B2_Q(r_i) \leq B2_T(r_i)$, $B3_P(r_i) \leq B3_Q(r_i) \leq B3_T(r_i)$, $B4_P(r_i) \leq B4_Q(r_i) \leq B4_T(r_i)$,

$$\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| \leq |B1_P^3(r_i) - B1_T^3(r_i)|,$$

$$\left|B1_Q^3(r_i) - B1_T^3(r_i)\right| \leq |B1_P^3(r_i) - B1_T^3(r_i)|,$$

$$\left|B2_P^3(r_i) - B2_Q^3(r_i)\right| \leq |B2_P^3(r_i) - B2_T^3(r_i)|,$$

$$\left|B2_Q^3(r_i) - B2_T^3(r_i)\right| \leq |B2_P^3(r_i) - B2_T^3(r_i)|,$$

$$\left|B3_P^3(r_i) - B3_Q^3(r_i)\right| \leq |B3_P^3(r_i) - B3_T^3(r_i)|,$$

$$\left|B3_Q^3(r_i) - B3_T^3(r_i)\right| \leq |B3_P^3(r_i) - B3_T^3(r_i)|,$$

$$\left|B4_P^3(r_i) - B4_Q^3(r_i)\right| \leq |B4_P^3(r_i) - B4_T^3(r_i)|,$$

$$\left|B4_Q^3(r_i) - B4_T^3(r_i)\right| \leq |B4_P^3(r_i) - B4_T^3(r_i)|$$

Thus,

$T_{FQN}(P,T) \leq T_{FQN}(P,Q)$ and $T_{FQN}(P,T) \leq T_{FQN}(Q,T)$

Since tangent function is increasing in the interval $[0, \frac{\pi}{4}]$.

Definition 3.3

Let P = $\{(\mathrm{r}, \mathrm{B1}_P(r), \mathrm{B2}_P(r), \mathrm{B3}_P(r), \mathrm{B4}_P(r)): r \in R\}$ and

Q = $\{(\mathrm{r}, \mathrm{B1}_Q(r), \mathrm{B2}_Q(r), \mathrm{B3}_Q(r), \mathrm{B4}_Q(r))(r): r \in R\}$ be two FQN with Neutrosophic components. Now weighted tangent similarity function which measures the similarity between two vectors based only on the direction, ignoring the impact of the distance between them can be presented as follows

$T_{WFQN}(P,Q) = \sum_{i=1}^{n} w_i[1 - \tan(\frac{\pi}{16}[\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| + \left|B2_P^3(r_i) - B2_Q^3(r_i)\right| + \left|B3_P^3(r_i) - B3_Q^3(r_i)\right| + \left|B4_P^3(r_i) - B4_Q^3(r_i)\right|)]$

Where $w_i \in [0,1], i = 0,1,2 \dots n$ are the weights and $\sum_{i=1}^{n} w_i = 1$. If we take $w_i = \frac{1}{n}, i = 0,1,2 \dots, n$, then $T_{WFQN}(P,Q) = T_{FQN}(P,Q)$.

Theorem 3. 4

The defined weighted tangent similarity measure $T_{WFQN}(P,Q)$ between FQN set P and Q satisfies the following properties

1) $0 \leq T_{WFQN}(P,Q) \leq 1$;
2) $T_{WFQN}(P,Q) = 1$ iff P = Q;
3) $T_{WFQN}(P,Q) = T_{WFQN}(Q,P)$;
4) If T is a FQN set in R and $P \subseteq Q \subseteq T$ then

   $T_{WFQN}(P,T) \leq T_{WFQN}(P,Q)$ and $T_{WFQN}(P,T) \leq T_{WFQN}(Q,T)$.

Proof

1) As the truth membership, contradiction membership, ignorance membership, falsity membership and the unknown membership function of the FQN sets and the value of the tangent function also is within [0,1] and where $w_i \in [0,1], i = 0,1,2 \dots n$ are the weights and

$\sum_{i=1}^{n} w_i = 1$.

Hence $0 \leq T_{WFQN}(P,Q) \leq 1$.

$\mathrm{B3}_P(r_i) = \mathrm{B3}_Q(r_i)$ and $\mathrm{B4}_P(r_i) = \mathrm{B4}_Q(r_i)$.

Hence $\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| = 0, \left|B2_P^3(r_i) - B2_Q^3(r_i)\right| = 0, \left|B3_P^3(r_i) - B3_Q^3(r_i)\right| = 0$ and

$\left|B4_P^3(r_i) - B4_Q^3(r_i)\right| = 0$.

Thus $T_{WFQN}(P,Q) = 1$.

Conversely, if $T_{WFQN}(P,Q) = 1$, then $\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| = 0, \left|B2_P^3(r_i) - B2_Q^3(r_i)\right| = 0$, $\left|B3_P^3(r_i) - B3_Q^3(r_i)\right| = 0$ *and* $\left|B4_P^3(r_i) - B4_Q^3(r_i)\right| = 0$ since tan(0) = 0. So we can write $\mathrm{B1}_P(r_i) = \mathrm{B1}_Q(r_i)$, $\mathrm{B2}_P(r_i) = \mathrm{B2}_Q(r_i)$, $\mathrm{B3}_P(r_i) = \mathrm{B3}_Q(r_i)$ and $\mathrm{B4}_P(r_i) = \mathrm{B4}_Q(r_i)$.

Hence P = Q.

3) The Proof is obvious

4) If $P \subseteq Q \subseteq T$ then $\mathrm{B1}_P(r_i) \leq \mathrm{B1}_Q(r_i) \leq \mathrm{B1}_T(r_i)$, $\mathrm{B2}_P(r_i) \leq \mathrm{B2}_Q(r_i) \leq \mathrm{B2}_T(r_i)$,

$\mathrm{B3}_P(r_i) \leq \mathrm{B3}_Q(r_i) \leq \mathrm{B3}_T(r_i)$ and $\mathrm{B4}_P(r_i) \leq \mathrm{B4}_Q(r_i) \leq \mathrm{B4}_T(r_i)$ and $\sum_{i=1}^{n} w_i = 1$.

$$\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| \leq \left|B1_P^3(r_i) - B1_T^3(r_i)\right|,$$

$$\left|B1_Q^3(r_i) - B1_T^3(r_i)\right| \leq \left|B1_P^3(r_i) - B1_T^3(r_i)\right|,$$

$$\left|B2_P^3(r_i) - B2_Q^3(r_i)\right| \leq \left|B2_P^3(r_i) - B2_T^3(r_i)\right|,$$

$$\left|B2_Q^3(r_i) - B2_T^3(r_i)\right| \leq \left|B2_P^3(r_i) - B2_T^3(r_i)\right|,$$

$$\left|B3_P^3(r_i) - B3_Q^3(r_i)\right| \leq \left|B3_P^3(r_i) - B3_T^3(r_i)\right|,$$

$$\left|B3_Q^3(r_i) - B3_T^3(r_i)\right| \leq \left|B3_P^3(r_i) - B3_T^3(r_i)\right|,$$

$$\left|B4_P^3(r_i) - B4_Q^3(r_i)\right| \leq \left|B4_P^3(r_i) - B4_T^3(r_i)\right|,$$

$$\left|B4_Q^3(r_i) - B4_T^3(r_i)\right| \leq \left|B4_P^3(r_i) - B4_T^3(r_i)\right|$$

Thus,

$T_{WFQN}(P,T) \leq T_{WFQN}(P,Q)$ and $T_{WFQN}(P,T) \leq T_{WFQN}(Q,T)$

Since tangent function is increasing in the interval $[0, \frac{\pi}{4}]$.

Definition 3.5

Assume that P = {(r, $\mathrm{B1}_P(r), \mathrm{B2}_P(r), \mathrm{B3}_P(r), \mathrm{B4}_P(r)): r \in R\}$ and

Q = {(r, $B1_Q(r), B2_Q(r), B3_Q(r), B4_Q(r))(r){:}\, r \in R\}$ are two FQN set with dependent Neutrosophic components. A cotangent similarity measure between two FQN sets P and Q is proposed as follows

$COT_{FQN}(P,Q) = \frac{1}{n}\sum_{i=1}^{n} [\cot(\frac{\pi}{16}[4 + |B1_P^3(r_i) - B1_Q^3(r_i)| + |B2_P^3(r_i) - B2_Q^3(r_i)| + |B3_P^3(r_i) - B3_Q^3(r_i)| + |B4_P^3(r_i) - B4_Q^3(r_i)|])]$

Theorem 3. 6

The cotangent similarity measure $COT_{FQN}(P,Q)$ between FQN set P and Q also satisfies the following properties

1) $0 \le COT_{FQN}(P,Q) \le 1$;
2) $COT_{FQN}(P,Q) = 1$ iff P = Q;
3) $COT_{FQN}(P,Q) = COT_{FQN}(Q,P)$;
4) If T is a PNP set in R and P $\subseteq Q \subseteq T$ then

$COT_{FQN}(P,T) \le COT_{FQN}(P,Q)$ and $COT_{FQN}(P,T) \le COT_{FQN}(Q,T)$.

Definition 3. 7

Assume that P = {(r, $B1_P(r), B2_P(r), B3_P(r), B4_P(r)){:}\, r \in R\}$ and

Q = {(r, $B1_Q(r), B2_Q(r), B3_Q(r), B4_Q(r)){:}\, r \in R\}$ are two FQN numbers as Neutrosophic components. A weighted cotangent similarity measure between two FQN sets P and Q is proposed as follows

$COT_{WFQN}(P,Q) = \sum_{i=1}^{n} w_i[\cot(\frac{\pi}{16}[4 + |B1_P^3(r_i) - B1_Q^3(r_i)| + |B2_P^3(r_i) - B2_Q^3(r_i)| + |B3_P^3(r_i) - B3_Q^3(r_i)| + |B4_P^3(r_i) - B4_Q^3(r_i)|])]$

Where $w_i \in [0,1], i = 0,1,2 \dots n$ are the weights and $\sum_{i=1}^{n} w_i = 1$. If we take $w_i = \frac{1}{n}, i = 0,1,2 \dots, n$, then $COT_{WFQN}(P,Q) = COT_{FQN}(P,Q)$.

Theorem 3. 8

The weighted cotangent similarity measure $COT_{PNP}(P,Q)$ between PNP set P and Q also satisfies the following properties

1) $0 \le COT_{WFQN}(P,Q) \le 1$;
2) $COT_{WFQN}(P,Q) = 1$ iff P = Q;
3) $COT_{WFQN}(P,Q) = COT_{WFQN}(Q,P)$;
4) If T is a PNP set in R and P $\subseteq Q \subseteq T$ then

$COT_{WFQN}(P,T) \le COT_{WFQN}(P,Q)$ and $COT_{WFQN}(P,T) \le COT_{WFQN}(Q,T)$.

Proof

1) As the truth membership, contradiction membership, ignorance membership and falsity membership function of the FQN sets and the value of the tangent function also is within [0,1] and $\sum_{i=1}^{n} w_i = 1$.

Hence 0≤ $COT_{WFQN}(P,Q) \leq 1$.

2) For any two FQN sets P and Q if P = Q, this implies $\mathrm{B1}_P(r_i) = \mathrm{B1}_Q(r_i)$, $\mathrm{B2}_P(r_i) = \mathrm{B2}_Q(r_i)$, $\mathrm{B3}_P(r_i) = \mathrm{B3}_Q(r_i)$, $\mathrm{B4}_P(r_i) = \mathrm{B4}_Q(r_i)$.

Hence $\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| = 0, \left|B2_P^3(r_i) - B2_Q^3(r_i)\right| = 0, \left|B3_P^3(r_i) - B3_Q^3(r_i)\right| = 0,$

$\left|B4_P^3(r_i) - B4_Q^3(r_i)\right| = 0$.

Thus $COT_{WFQN}(P,Q)$ = 1.

Conversely, if $T_{WPNP}(P,Q) = 1$, then $\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| = 0, \left|B2_P^3(r_i) - B2_Q^3(r_i)\right| = 0,$ $\left|B3_P^3(r_i) - B3_Q^3(r_i)\right| = 0$ And $\left|B4_P^3(r_i) - B4_Q^3(r_i)\right| = 0$ since tan(0) = 0. So we can write $\mathrm{B1}_P(r_i) = \mathrm{B1}_Q(r_i)$, $\mathrm{B2}_P(r_i) = \mathrm{B2}_Q(r_i)$, $\mathrm{B3}_P(r_i) = \mathrm{B3}_Q(r_i)$ *and* $\mathrm{B4}_P(r_i) = \mathrm{B4}_Q(r_i)$.

Hence P = Q.

3) The Proof is obvious

4) If P ⊆ Q ⊆ T then $\mathrm{B1}_P(r_i) \leq \mathrm{B1}_Q(r_i) \leq \mathrm{B1}_T(r_i)$, $\mathrm{B2}_P(r_i) \leq \mathrm{B2}_Q(r_i) \leq \mathrm{B2}_T(r_i)$,

$\mathrm{B3}_P(r_i) \geq \mathrm{B3}_Q(r_i) \geq \mathrm{B3}_T(r_i)$, $\mathrm{B4}_P(r_i) \geq \mathrm{B4}_Q(r_i) \geq \mathrm{B4}_T(r_i)$ and $\sum_{i=1}^{n} w_i = 1$.

$$\left|B1_P^3(r_i) - B1_Q^3(r_i)\right| \leq |B1_P^3(r_i) - B1_T^3(r_i)|,$$

$$\left|B1_Q^3(r_i) - B1_T^3(r_i)\right| \leq |B1_P^3(r_i) - B1_T^3(r_i)|,$$

$$\left|B2_P^3(r_i) - B2_Q^3(r_i)\right| \leq |B2_P^3(r_i) - B2_T^3(r_i)|,$$

$$\left|B2_Q^3(r_i) - B2_T^3(r_i)\right| \leq |B2_P^3(r_i) - B2_T^3(r_i)|,$$

$$\left|B3_P^3(r_i) - B3_Q^3(r_i)\right| \leq |B3_P^3(r_i) - B3_T^3(r_i)|,$$

$$\left|B3_Q^3(r_i) - B3_T^3(r_i)\right| \leq |B3_P^3(r_i) - B3_T^3(r_i)|,$$

$$\left|B4_P^3(r_i) - B4_Q^3(r_i)\right| \leq |B4_P^3(r_i) - B4_T^3(r_i)|,$$

$$\left|B4_Q^3(r_i) - B4_T^3(r_i)\right| \leq |B4_P^3(r_i) - B4_T^3(r_i)|$$

The cotangent function is decreasing function within the interval [0, $\frac{\pi}{4}$].

Hence $\sum_{i=1}^{n} w_i = 1$.

Hence, we can write

$COT_{WFQN}(P,T) \leq COT_{WFQN}(P,Q)$ and $COT_{WFQN}(P,T) \leq COT_{WFQN}(Q,T)$

4. Decision Making Based on Tangent and Cotangent Similarity Measures

Let $A_1, A_2, \dots A_m$ be a discrete set of candidates, $C_1, C_2, .. C_n$ be the set of criteria for each candidate and $D_1, D_2, \dots., D_k$ are the alternatives of each candidate. The decision -maker provides the ranking of alternatives with respect to each candidate. The ranking presents the performance of candidates $A_i(i = 1,2, \dots m)$ against the criteria $C_j(j = 1,2 \dots, n)$.The values associated with the alternatives for MADM problem can be presented in the following decision matrix(see Table 1 and Table 2). The relation between candidates and attributes are given in Tab 1. The relation between attributes and alternatives are given in the Tab 2.

Table 1 : The relation between candidates and attributes

R_1	C_1	C_2	$\cdots$	C_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{13}	$\cdots$	a_{2n}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Table 2: The relation between attributes and alternatives

R_2	D_1	D_2	$\cdots$	D_k
C_1	c_{11}	c_{12}	$\cdots$	c_{1k}
C_2	c_{21}	c_{22}	$\cdots$	c_{2k}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
C_n	c_{n1}	c_{n2}	$\cdots$	c_{nk}

Here a_{ij} and c_{ij} are all FQN numbers.

The steps corresponding to FQN number based on tangent and cotangent functions are presented following steps.

Step 1: Determination of the relation between candidates and attributes

The relation between candidate $A_i(i = 1,2, \dots m)$ and the attribute $C_j(j = 1,2 \dots n)$ is presented in Table 3.

Table 3 : The relation between candidates and attributes in terms of FQN sets

R_1	C_1	C_2	$\cdots$	C_n
A_1	$(b1_{11}, b2_{11}, b3_{11}, b4_{11})$	$(b1_{12}, b2_{12}, b3_{12}, b4_{12})$	$\cdots$	$(b1_{1n}, b2_{1n}, b3_{1n}, b4_{1n})$
A_2	$(b1_{21}, b2_{21}, b3_{21}, b4_{21})$	$(b1_{22}, b2_{22}, b3_{22}, b4_{22})$	$\cdots$	$(b1_{2n}, b2_{2n}, b3_{2n}, b4_{2n})$

…	…	…	…	…
A_m	$(b1_{m1}b2_{m1}, b3_{m1}, b4_{m1})$	$(b1_{m2},b2_{m2}, b3_{m2}, b4_{m2})$	…	$(b1_{mn},b2_{mn}, b3_{mn}, b4_{mn})$

Table 4 : The relation between attributes and alternatives in terms of FQN sets

R_2	D_1	D_2	…	D_k
C_1	$(c1_{11},cb2_{11}, c3_{11}, c4_{11})$	$(c1_{12},c2_{12}, c3_{12}, c4_{12})$	…	$(c1_{1k},c2_{1k}, c3_{1k}, c4_{1k})$
C_2	$(b1_{21},b2_{21}, b3_{21}, b4_{21})$	$(c1_{22},c2_{22}, c3_{22}, c4_{22})$	…	$(c1_{2k},c2_{2k}, c3_{2k}, c4_{2k})$
…	…	…	…	…
C_n	$(c1_{n1},c2_{n1}, c3_{n1}, c4_{n1})$	$(c1_{n2},c2_{n2}, c3_{n2}, c4_{n2})$	…	$(c1_{nk},c2_{nk}, c3_{nk}, c4_{nk})$

Step 3: Determination of the relation between attributes and alternatives

Determine the similarity measure between the Tab 3 and Tab 4 using $T_{FQN}(P,Q)$ $T_{WFQN}(P,Q)$, $COT_{FQN}(P,Q)$ and $COT_{WFQN}(P,Q)$.

Step 4: Ranking the alternatives

Ranking the alternatives is prepared based on the descending order of the similarity measures.

Highest value reflects the best alternative.

Step 5: End

5. Application

Higher education institutions have faced various challenges in adapting online education to control the pandemic spread of COVID. The present work aims to apply similaty measures between social media and its benefits of students. Let D = {R1, R2, R3} be a set of college student respondents, E = {YouTube, Facebook, WhatsApp, Blog} be social medias and H = {Communication Tool, Online Learning, connecting with experts, Global exposure} be its benefits. The solution strategy is to determine the student regarding the relation between student respondents and its benefits in virtual education (see Tab 5) and the relation between social media and its benefits in Table 6 . Further we have calculated Tangent and Cotangent similarity measures can be calculated in Table 7 and 8. Also the weighted similarity measures of the tangent and cotangent functions of PNP sets be calculated in Table 9 and 10. w = (0.3,.4,.3)

Table 5 : (P1) The relation between respondents and benefits in Virtual Education

P1	*Online Learning*	*Communication Tool*	*Connecting with Experts*
R1	*(0.7,0.2,0.4,0.3)*	*(0.1,0.2,0.4,0.7)*	*(0.4,0.6,0.3,0.6)*
R2	*(0.3,0.5,0.4,0.6)*	*(0.6,0.5,0.7,0.4)*	*(0.6,0.7,0.3,0.4)*
R3	*(0.1,0.4,0.3,0.5)*	*(0.6,0.6,0.3,0.4)*	*(0.6,0.1,0.9,0.4)*

Table 6: (P2) The relation between social media and its benefits

P2	WhatsApp	YouTube	Facebook
Online Learning	(0.4,0.2,0.3,0.1)	(0.1, 0.2, 0.3, 0.5)	(0.2,0.2,0.3,0.4)
Communication Tool	(0.7,0.2,0.3,0.3)	(0.5,0.2,0.3,0.5)	(0.7,0.2,0.3,0.2)
Connecting with Experts	(0.1,0.2,0.3,0.7)	(0.8,0.2.0.3,0.2)	(0.6,0.2,0.3,0.4)

Table 7: The Tangent Similarity Measure between P1 and P2

Tangent Similarity Measure	WhatsApp	YouTube	Facebook
R1	**0.8467**	0.8430	0.8148
R2	0.8583	**0.8792**	0.8603
R3	0.8595	**0.8791**	0.8504

Table 8: The Weighted Tangent Similarity Measure between P1 and P2

Weighted Tangent Similarity Measure	WhatsApp	YouTube	Facebook
R1	**0.8484**	0.8409	0.8219
R2	0.8547	**0.8647**	0.8497
R3	0.8635	**0.8961**	0.8565

Table 9: The Cotangent Similarity Measure between P1 and P2

Cotangent Similarity Measure	WhatsApp	YouTube	Facebook
R1	**0.8959**	0.8927	0.8504
R2	0.8583	**0.8567**	0.8195
R3	0.8244	**0.8496**	0.8092

Table 10: The Weighted Cotangent Similarity Measure between P1 and P2

Weighted Cotangent Similarity Measure	WhatsApp	YouTube	Facebook
R1	***0.8977***	*0.889*	*0.8587*
R2	*0.8547*	***0.8761***	*0.8185*
R3	*0.8308*	***0.8545***	*0.8207*

The highest similarity measures reflect the benefits of social media among College Students. Therefore, Student R2 and R3 gains knowledge more from YouTube and R1 from WhatsApp.

6. Conclusion

In this paper, we have proposed tangent and cotangent similarity measures for Fermatean Quadripartitioned Neutrosophic set with Neutrosophic components and proved some of its basic properties. Furthermore, we have also investigated about the weighted similarity measures in Decision

Making and illustrated with an example. In future, we can study about the improved similarity measure for the above set and can be used in Medical Diagnosis, Data mining. Clustering Analysis etc.

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NEUTROSOPHIC LINEAR REGRESSION-TYPE ESTIMATOR FOR ESTIMATING THE POPULATION MEAN

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Abstract

In traditional statistics all data are determined and it is used to estimate the mean of the population when auxiliary information is available. Those estimators often are biased. The main aim is to find the best estimator for the unknown value of the population mean with minimum variance/mean square error (MSE). The neutrosophic statistics, generalization of classical statistics deal with indistinct, indeterminate, uncertain information. The neutrosophic observation is of the form $Z_N = Z_L + Z_U I_N$ where $I_N \in [I_L, I_U]$, $Z_N \in [Z_L, Z_U]$. In this paper neutrosophic linear regression type estimator and modified neutrosophic linear regression type estimators for estimation of population mean of the study variable using the known parameters of the auxiliary variable have been proposed. The variance/mean squared error of the proposed estimators is derived up to first order of approximation. The efficiency of the proposed neutrosophic linear regression-type estimators is evaluated using natural population and also by using simulation study. A comparison is also carried out to illustrate the usefulness of proposed neutrosophic linear regression-type estimators over the classical estimator.

Keywords: Parameters, Auxiliary variable, Regression type, Neutrosophic statistics

1. Introduction

Consider a finite population $\mathrm{U} = \{\mathrm{U}_1, \mathrm{U}_2, \dots, \mathrm{U}_\mathrm{N}\}$ of N distinct and identifiable units. Let Yis a study variable with value Y_i measured on U_i, $\mathrm{i} = 1,2,3, \dots, \mathrm{N}$ giving a vector$\mathrm{Y} = \{\mathrm{Y}_1, \mathrm{Y}_2, \dots, \mathrm{Y}_\mathrm{N}\}$. In general the population mean $\overline{\mathrm{Y}} = \frac{1}{\mathrm{N}} \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{Y}_\mathrm{i}$ and the population variance $\mathrm{S}_\mathrm{y}^2 = \frac{1}{(\mathrm{N}-1)} \sum_{\mathrm{i}=1}^{\mathrm{N}} (\mathrm{Y}_\mathrm{i} - \overline{\mathrm{Y}})^2$ are unknown. Now, in this article the problem is to estimate the population mean $\overline{\mathrm{Y}}$ on the basis of a random sample of sizen, selected from the population U with some desirable properties like:

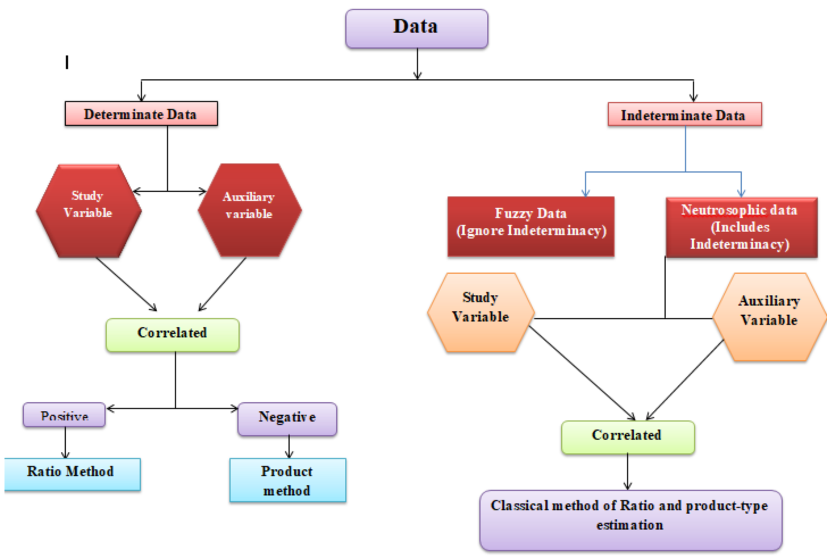
- Unbiasedness / Minimum Bias
- Minimum Variance / Mean squared error(MSE)

It is often the case that an auxiliary variable X closely related to the study variable Y is available. Then, one can improve the performance of the estimator of the study variable by using the known values of the population parameters of the auxiliary variable like Population mean $\overline{X}$, Coefficient of variation C_X, Skewness $\beta_{1(x)}$, Kurtosis $\beta_{2(x)}$, etc. For further discussion about ratio estimators and its modification for estimating population mean one may refer to Al-Omari et al. (2009), Cingi and Kadilar (2009), Cochran (1940, 1977), Kadilar and Cingi (2004,2005,2006a,2006b), Murthy(1967), Sen (1993), Singh and Chaudhary (1986), Singh (2003), Singh and Tailor (2003), Singh et al. (2004), Sisodia and Dwivedi (1981), Sukhatme (1970), Upadhyaya and Singh (1999b) and Yan and Tian (2010).

While the classical method of statistics deals with determinate inference method or randomness so that when the data is indeterminate or ambiguous or vague the classical method of estimation would not give the required result. In that situation neutrosophic statistics gives us promising results. The neutrosophic statistics is an extension of classical statistics (Indeterminacy is zero). The main difference between classical and neutrosophic statistics, is the total of number of sample size would not be an exact number in neutrosophic. In other words, neutrosophic statistics is a set analysis. The probability distribution of neutrosophic data shall be presented in three curves and they are:

- Probability of event that occur
- Probability of event that do not occur
- Intermediate chance of the event occur or not

For further discussion about neutrosophic statistics one may refer to Smarandache(1998, 2014,2015), Alblowi et.al (2014), Smarandache and Pramanik(2016), Alhabib(2018), Aslam(2018), Smarandache et.al (2019), Olgun and Hatip(2020) and Tahir et.al(2021). The following flow chart explains the way of using proposed methods under neutrosophic statistics



2. Notation and Terminology

Consider a neutrosophic random sample of size $n_N \in [n_L, n_U]$, which is drawn from a finite population of N units $(T_1, T_2, \ldots, T_N)$. Let $y_N(i)$ is the i^{th} sample observation of our neutrosophic data, which is of the form $y_N(i) \in [y_L, y_U]$ and similarly for auxiliary variable we have $x_N(i) \in [x_L, x_U]$ which is correlated to our study variable. Let $\bar{y}_N(i) \in [\bar{y}_L, \bar{y}_U]$ is mean of neutrosophic variable of interest, and $\bar{x}_N(i) \in [\bar{x}_L, \bar{x}_U]$is mean of auxiliary neutrosophic variable. In addition,$\mathrm{S}^2_{\mathrm{yN}} \in [\mathrm{S}^2_{\mathrm{yNL}}, \mathrm{S}^2_{\mathrm{yNU}}]$ and $\mathrm{S}^2_{\mathrm{xN}} \in [\mathrm{S}^2_{\mathrm{xNL}}, \mathrm{S}^2_{\mathrm{xNU}}]$ are the variance of the neutrosophic set of data. $\mathrm{C}_{\mathrm{yN}} \in [\mathrm{C}_{\mathrm{yNL}}, \mathrm{C}_{\mathrm{yNU}}]$ and $\mathrm{C}_{\mathrm{xN}} \in [\mathrm{C}_{\mathrm{xNL}}, \mathrm{C}_{\mathrm{xNU}}]$ are neutrosophic coefficients of variation for Y_N and X_N , respectively. ρ_N is the neutrosophic correlation between Y_N and X_N (neutrosophic variables). In addition, $\beta_{2(\mathrm{xN})} \in [\beta_{2(\mathrm{xNL})}, \beta_{2(\mathrm{xNU})}]$ is the neutrosophic coefficient of kurtosis for auxiliary variable X_N. $\bar{e}_{yN}(i) \in [\bar{e}_{yL}, \bar{e}_{yU}]$and $\bar{e}_{xN}(i) \in [\bar{e}_{xL}, \bar{e}_{xU}]$ are the neutrosophic mean errors.. Similarly, $\mathrm{MSE}(.) \in [\mathrm{MSE_L}, \mathrm{MSE_U}]$ belong to the MSE of neutrosophic sets were also computed for the analysis:

These terms can be computed by the following notations to be used in this article are defined below:

- N − Population Size
- n − Sample Size
- $\mathrm{f} = \mathrm{n}/\mathrm{N}$, - Sampling Fraction
- $\delta = \frac{1-\mathrm{f}}{\mathrm{n}}$, - Finite Population Correction
- $\mathrm{X_N}$,Y_N-Population Totals
- $\bar{\mathrm{X}}_N$, $\bar{\mathrm{Y}}_N$ - Population Means
- x_N, y_N - Sample Totals
- $\bar{\mathrm{x}}_N$, $\bar{\mathrm{y}}_N$Sample Means
- $\mathrm{S_{XN}}$, $\mathrm{S_{yN}}$ − Population Standard Deviations
- $\mathrm{S_{xyN}} = \mathrm{E}(X_N - \bar{\mathrm{X}}_N)(\mathrm{Y}_N - \bar{\mathrm{Y}}_N)$ − Population Covariance between $\mathrm{X_N}$ and Y_N
- $\mathrm{C_{XN}} = \frac{\mathrm{S_{xN}}}{\overline{\mathrm{XN}}}$ &$\mathrm{C_{YN}} = \frac{\mathrm{S_{yN}}}{\overline{\mathrm{YN}}}$ − Co-efficient of Variations
- $\rho_\mathrm{N} = \frac{\mathrm{S_{xyN}}}{\mathrm{S_{xN}}\mathrm{S_{yN}}}$ − Co-efficient of Correlation between $\mathrm{X_N}$ and Y_N
- β_N -Population regression coefficient of Y_N on $\mathrm{X_N}$
- $\mu_{\mathrm{rsN}} = \frac{1}{\mathrm{N}}\sum_{\mathrm{i}=1}^{\mathrm{N}}(\mathrm{Y_{iN}} - \bar{\mathrm{Y}}_\mathrm{N})^\mathrm{r}\,(\mathrm{X_{iN}} - \bar{\mathrm{X}}_\mathrm{N})^\mathrm{s}$
- $\beta_{1(\mathrm{xN})} = \frac{\mu^2_{03\mathrm{N}}}{\mu^3_{02\mathrm{N}}}$, Skewness of the Auxiliary Variable
- $\beta_{2(\mathrm{xN})} = \frac{\mu_{04\mathrm{N}}}{\mu^2_{02\mathrm{N}}}$, Kurtosis of the Auxiliary Variable
- $\beta_{2(\mathrm{yN})} = \frac{\mu_{40\mathrm{N}}}{\mu^2_{20\mathrm{N}}}$, Kurtosis of the Study Variable
- $\mathrm{M_{dN}}$ −Median of the Auxiliary Variable

- $\mathrm{Q}_{1\mathrm{N}}$ −First (lower) Quartile of the Auxiliary Variable
- $\mathrm{Q}_{3\mathrm{N}}$ −Third (upper) Quartile of the Auxiliary Variable
- $\mathrm{Q}_{\mathrm{rN}} = \mathrm{Q}_3 - \mathrm{Q}_1$, Inter-Quartile Range of the Auxiliary Variable
- $\mathrm{Q}_{\mathrm{dN}} = \frac{\mathrm{Q}_3 - \mathrm{Q}_1}{2}$, Semi-Quartile Range of the Auxiliary Variable
- $\mathrm{Q}_{\mathrm{aN}} = \frac{\mathrm{Q}_3 + \mathrm{Q}_1}{2}$, Semi-Quartile Average of the Auxiliary Variable
- $\mathrm{D}_{\mathrm{mN}} -$ m^{th}Decile of the Auxiliary Variable
- $\mathrm{MSE}(.)$ − Mean Squared Error (MSE) of the Estimator
- $\mathrm{V}(.)$ − Variancer of the Estimator
- $\bar{\mathrm{y}}_N$ - Neutrosophic Simple Random Sampling Without Replacement (SRSWOR) Sample mean
- $\widehat{\bar{\mathrm{Y}}}_{\mathrm{RN}}$ − Neutrosophic Ratio Type Estimator of $\bar{Y}$
- $\widehat{\bar{\mathrm{Y}}}_{\mathrm{LRN}}$- Neutrosophic Linear Regression Type Estimator of $\bar{Y}$
- $\widehat{\bar{\mathrm{Y}}}_{\mathrm{MLRN}_{\mathrm{j}}}$ − j^{th} Proposed Modified Neutrosophic Linear regression Type Estimator of $\bar{Y}$

When there is no auxiliary information available, the simplest estimator of population mean is the sample mean obtained by using simple random sampling without replacement. In case of simple random sampling without replacement (SRSWOR), the sample mean $\bar{\mathrm{y}}_N$ is used to estimate population mean $\bar{Y}$ which is an unbiased estimator and its variance is given below:

$$\mathrm{V}(\bar{\mathrm{y}}_N) = \delta \mathrm{S}_{\mathrm{yN}}^2 \tag{1}$$

In the presence of an auxiliary variable X and is positively correlated with the study variable Y, Tahir et.al (2021) has introduced the classical neutrosophic ratio estimator for estimating the population mean of the study variable as given below:

$$\widehat{\bar{\mathrm{Y}}}_{\mathrm{RN}} = \frac{\bar{\mathrm{y}}_N}{\bar{\mathrm{x}}_N} \bar{\mathrm{X}}_N \tag{2}$$

The mean squared error of $\widehat{\bar{\mathrm{Y}}}_{\mathrm{RN}}$ to the first order of approximation is given below:

$$\mathrm{MSE}\left(\widehat{\bar{\mathrm{Y}}}_{\mathrm{RN}}\right) = \delta \bar{\mathrm{Y}}_{\mathrm{N}}^2 \left(\mathrm{C}_{\mathrm{yN}}^2 + \mathrm{C}_{\mathrm{xN}}^2 - 2\rho_{\mathrm{N}} \mathrm{C}_{\mathrm{xN}} \mathrm{C}_{\mathrm{yN}}\right) \tag{3}$$

Motivated by Tahir et.al (2021), neutrosophic linear regression type estimator for estimating population mean has been proposed. Further improvements shall be made on the neutrosophic linear regression type estimator by introducing a large number of modified neutrosophic linear regression type estimators with known neutrosophic Co-efficient of Variation, Kurtosis, Skewness and Population Correlation Coefficient, First quartile, Third quartile etc of auxiliary variable.

3. Proposed Neutrosophic Linear Regression Type Estimator for Estimating Population Mean

The proposed Neutrosophic linear regression type estimator and its variance are given as

$$\hat{\hat{\mathrm{Y}}}_{\mathrm{LRN}} = \bar{y}_N + \beta_N(\bar{X}_N - \bar{x}_N) \tag{4}$$

$$V\left(\hat{\hat{\mathrm{Y}}}_{\mathrm{LRN}}\right) = \delta S_{yN}^2(1-\rho_N^2) \tag{5}$$

Motivated by Kadilar and Cingi (2004), modified neutrosophic linear regression type estimator has been proposed for estimating population mean by replacing SRSWOR sample mean $\bar{y}_N$ by regression estimator $\hat{\hat{\mathrm{Y}}}_{\mathrm{LRN}}$ in $\hat{\hat{\mathrm{Y}}}_{\mathrm{RN}}$ as given below:

$$\hat{\hat{\mathrm{Y}}}_{\mathrm{MLRN}_{\mathrm{KC}}} = [\bar{y}_N + \beta_N(\bar{X}_N - \bar{x}_N)]\left[\frac{\bar{X}_N}{\bar{x}_N}\right] \tag{6}$$

The mean squared error of the proposed estimators $\hat{\hat{\mathrm{Y}}}_{\mathrm{MLRN}_{\mathrm{KC}}}$ have been derived and given below:

$$\mathrm{MSE}\left(\hat{\hat{\mathrm{Y}}}_{\mathrm{MLRN}_{\mathrm{KC}}}\right) = \delta\left(R_N^2 S_{xN}^2 + S_{yN}^2(1-\rho_N^2)\right) \tag{7}$$

A class of modified neutrosophic linear regression type estimator using the known parameters of the auxiliary variable for estimating the neutrosophic population mean of the study variable $Y_N \in [Y_L, Y_U]$ have been suggested. The proposed modified neutrosophic linear regression type estimator $\hat{\hat{\mathrm{Y}}}_{\mathrm{MLRN}_j}$, $j = 1, \ldots, 36$ for estimating the neutrosophic population mean $\bar{Y}_N$ is given below:

$$\hat{\hat{\mathrm{Y}}}_{\mathrm{MLRN}_j} = [\bar{y}_N + \beta_N(\bar{X}_N - \bar{x}_N)]\left[\frac{\bar{X}_N+\omega_j}{\bar{x}_N+\omega_j}\right] \tag{8}$$

The mean squared error of the proposed estimators $\hat{\hat{\mathrm{Y}}}_{\mathrm{MLRN}_j}$, $i = 1,2,\ldots,36$ have been derived and are given below:

$$\mathrm{MSE}\left(\hat{\hat{\mathrm{Y}}}_{\mathrm{MLRN}_j}\right) = \delta\left(R_{N_j}^2 S_{xN}^2 + S_{yN}^2(1-\rho_N^2)\right) \tag{9}$$

where $R_{N_j} = \frac{\bar{Y}}{\bar{X}+\omega_j}$, $\omega_1 = C_{xN}$, $\omega_2 = \beta_{2(xN)}$, $\omega_3 = \frac{C_{xN}}{\beta_{2(xN)}}$, $\omega_4 = \frac{\beta_{2(xN)}}{C_{xN}}$, $\omega_5 = \rho_N$, $\omega_6 = \frac{\rho_N}{C_{xN}}$

$\omega_7 = \frac{C_{xN}}{\rho_N}$, $\omega_8 = \frac{\rho_N}{\beta_{2(xN)}}$, $\omega_9 = \frac{\beta_{2(xN)}}{\rho_N}$, $\omega_{10} = \beta_{1(xN)}$, $\omega_{11} = \frac{\beta_{2(xN)}}{\beta_{1(xN)}}$, $\omega_{12} = M_{dN}$, $\omega_{13} = \frac{M_{dN}}{C_{xN}}$,

$\omega_{14} = \frac{M_{dN}}{\beta_{2(xN)}}$, $\omega_{15} = \frac{M_{dN}}{\beta_{1(xN)}}$, $\omega_{16} = \frac{M_{dN}}{\rho_N}$, $\omega_{17} = Q_{1N}$, $\omega_{18} = Q_{3N}$, $\omega_{19} = Q_{rN}$, $\omega_{20} = Q_{dN}$,

$\omega_{21} = Q_{aN}$, $\omega_{22} = \frac{Q_{1N}}{C_{xN}}$, $\omega_{23} = \frac{Q_{3N}}{C_{xN}}$, $\omega_{24} = \frac{Q_{rN}}{C_{xN}}$, $\omega_{25} = \frac{Q_{dN}}{C_{xN}}$, $\omega_{26} = \frac{Q_{aN}}{C_{xN}}$, $\omega_{27} = D_{1N}$,

$\omega_{28} = D_{2N}$, $\omega_{29} = D_{3N}$, $\omega_{30} = D_{4N}$, $\omega_{31} = D_{5N}$, $\omega_{32} = D_{6N}$, $\omega_{33} = D_{7N}$, $\omega_{34} = D_{8N}$,

$\omega_{35} = D_{9N}$, $\omega_{36} = D_{10N}$

Remark 3.1: When the study variable $\mathbf{Y_N} \in [\mathbf{Y_L}, \mathbf{Y_U}]$ and auxiliary variable $\mathbf{X_N} \in [\mathbf{X_L}, \mathbf{X_U}]$ are negatively correlated and the population parameters of the auxiliary variable are known, the following modified neutrosophic regression type variance estimator can be proposed:

$$\widehat{\widehat{\mathrm{Y}}}_{\mathrm{MLPRN_j}} = [\bar{\mathrm{y}}_N + \beta_{\mathrm{N}}(\bar{\mathrm{X}}_N - \bar{\mathrm{x}}_N)]\left[\frac{\bar{\mathrm{x}}_N+\omega_{\mathrm{j}}}{\bar{\mathrm{X}}_N+\omega_{\mathrm{j}}}\right]$$

4. Efficiency of the Proposed Estimators

Comparing (1) and (8) we have derived (see Appendix A) the condition for which the proposed modified neutrosophic regression type estimators $\widehat{\widehat{\mathbf{Y}}}_{\mathbf{MLRN_j}}$,is more efficient than the neutrosophic SRSWOR sample mean $\bar{\boldsymbol{y}}_{\boldsymbol{N}}$ and it is given below:

$$\mathbf{MSE}\left(\widehat{\widehat{\mathbf{Y}}}_{\mathbf{MLRN_j}}\right) \leq \mathbf{V}(\bar{\mathbf{y}}_N) \text{ if } \mathbf{R_{N_j}} \leq \boldsymbol{\rho}_{\mathbf{N}}\frac{\mathbf{S_{yN}}}{\mathbf{S_{xN}}}; \mathbf{j} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{36} \qquad (10)$$

Comparing (2) and (8) we have derived (see Appendix B) the conditions for which the proposed modified neutrosophic regression type estimators $\widehat{\widehat{\mathbf{Y}}}_{\mathbf{MLRN_j}}$ is more efficient than the neutrosophic ratio type estimator $\widehat{\widehat{\mathbf{Y}}}_{\mathbf{RN}}$ and it is given below:

$$\mathbf{MSE}\left(\widehat{\widehat{\mathbf{Y}}}_{\mathbf{MLRN_j}}\right) \leq \mathbf{MSE}\left(\widehat{\widehat{\mathbf{Y}}}_{\mathbf{RN}}\right) \text{ if } \bar{\mathbf{Y}}_N\left(\frac{\mathbf{C_{xN}} - \boldsymbol{\rho}_{\mathbf{N}}\mathbf{C_{yN}}}{\mathbf{S_{xN}}}\right) \leq \mathbf{R_{N_j}} \leq \bar{\mathbf{Y}}_N\left(\frac{\boldsymbol{\rho}_{\mathbf{N}}\mathbf{C_{yN}} - \mathbf{C_{xN}}}{\mathbf{S_{xN}}}\right) \text{ or}$$

$$\bar{\mathbf{Y}}_N\left(\frac{\boldsymbol{\rho}_{\mathbf{N}}\mathbf{C_{yN}} - \mathbf{C_{xN}}}{\mathbf{S_{xN}}}\right) \leq \mathbf{R_{N_j}} \leq \bar{\mathbf{Y}}_N\left(\frac{\mathbf{C_{xN}} - \boldsymbol{\rho}_{\mathbf{N}}\mathbf{C_{yN}}}{\mathbf{S_{xN}}}\right) \qquad (11)$$

5. Numerical Study

The performance of the proposed Neutrosophic Linear Regression type estimator for mean are compared with that of Neutrosophic SRSWOR sample mean, Neutrosophic ratio type mean estimator and Modified Neutrosophic Linear Regression type estimator using a natural population. For numerical study, we have considered the daily stock prices of Samsung Electronics Co., Ltd. from 1st September 2020 to 30th September 2021 as the neutrosophic variable. We are estimating this low price and high price interval (Y_L, Y_U) within which the price of the stock lies using daily opening price and closing price as an neutrosophic auxiliary variable (X_L, X_U).

The population parameters of the above population are given below:

X_L – Opening Price	X_U – Closing Price		
Y_L – Low Price	Y_U – High Price		
N=267	n=120	$\bar{Y}_L$= 751.70	$\bar{Y}_U$ = 765.15
$\bar{X}_L$= 758.09	$\bar{X}_U$= 758.01	S_{yL}= 91.7629	S_{yU} = 93.7754

S_{xL}= 92.8418	S_{xU}= 92.1824	$\beta_{1(xL)}$ = 0.7859	$\beta_{1(xU)}$ =0.7934
$\beta_{1(yL)}$ = 0.8362	$\beta_{1(yU)}$ = 0.6983	$\beta_{2(xL)}$ = 2.4808	$\beta_{2(xU)}$ = 2.5018
$\beta_{2(yL)}$= 2.4736	$\beta_{2(yU)}$ = 2.5296	C_{xL}= 0.1225	C_{xU}= 0.1216
λ_{22L} = 2.4708	λ_{22U} = 2.5008	Md_L = 797	Md_U =797
Q_{1L}= 723	Q_{1U}= 725	Q_{3L}= 823.5	Q_{3U}=821
D_{1L}=598.6	D_{1U}= 596.2	D_{2L}= 649.8	D_{2U}= 658.2
D_{3L}= 738.8	D_{3U}=739	D_{4L}= 773.4	D_{4U}=773
D_{5L}= 797	D_{5U}= 797	D_{6L}=810	D_{6U}= 809
D_{7L}= 818.2	D_{7U}= 819	D_{8L}= 828	D_{8U}=826
D_{9L}=840	D_{9U}= 840	D_{10L}= 903	D_{10U}= 910

The variance of Neutrosophic SRSWOR sample mean, the MSE of the Neutrosophic Ratio type estimator and the variance of Neutrosophic Linear Regression estimator for the natural population is given below:

Table 1: Variance of Neutrosophic SRSWOR sample mean, MSE of the Neutrosophic Ratio Type estimator and Variance of Neutrosophic Linear Regression Estimator

Estimators	**MSE / Variance**
Neutrosophic SRSWOR sample mean, $\bar{y}_N$	[38.6331,40.3462]
Neutrosophic Ratio Type Estimator, $\bar{Y}_{RN}$	[0.1185, 0.1966]
Neutrosophic Linear Regression Estimator, $\bar{Y}_{LRN}$	[0.1177, 0.1954]

The MSE of the proposed Neutrosophic Modified Linear Regression Type Ratio (NMLRR) estimator are given below:

Table 2: MSE of the proposed Neutrosophic Modified Linear Regression Type Ratio estimator

Proposed Neutrosophic Estimators	**MSE values**
$\bar{Y}_{MLRN_{KC}}$	[38.9981, 39.9191]
$\bar{Y}_{MLRN_1}$	[38.9855, 39.9063]
$\bar{Y}_{MLRN_2}$	[38.7448, 39.6581]
$\bar{Y}_{MLRN_3}$	[38.9930, 39.9139]
$\bar{Y}_{MLRN_4}$	[37.0010, 37.8473]

$\bar{Y}_{MLRN_5}$	[38.8958, 39.8147]
$\bar{Y}_{MLRN_6}$	[38.1752, 39.0729]
$\bar{Y}_{MLRN_7}$	[38.9855, 39.9063]
$\bar{Y}_{MLRN_8}$	[38.9568, 39.8773]
$\bar{Y}_{MLRN_9}$	[38.7445, 39.6575]
$\bar{Y}_{MLRN_{10}}$	[38.9176, 39.8360]
$\bar{Y}_{MLRN_{11}}$	[38.6763, 39.5906]
$\bar{Y}_{MLRN_{12}}$	[9.3561, 9.6330]
$\bar{Y}_{MLRN_{13}}$	[0.5393, 0.6203]
$\bar{Y}_{MLRN_{14}}$	[19.2965, 19.8871]
$\bar{Y}_{MLRN_{15}}$	[7.2306, 7.5409]
$\bar{Y}_{MLRN_{16}}$	[9.3419, 9.609]
$\bar{Y}_{MLRN_{17}}$	[10.3025, 10.5718]
$\bar{Y}_{MLRN_{18}}$	[9.0490, 9.3482]
$\bar{Y}_{MLRN_{19}}$	[30.4283, 31.4898]
$\bar{Y}_{MLRN_{20}}$	[34.3142, 35.3284]
$\bar{Y}_{MLRN_{21}}$	[9.6449, 9.9313]
$\bar{Y}_{MLRN_{22}}$	[0.6196, 0.6989]
$\bar{Y}_{MLRN_{23}}$	[0.5151, 0.598]
$\bar{Y}_{MLRN_{24}}$	[9.0840, 9.7251]
$\bar{Y}_{MLRN_{25}}$	[16.4874, 17.3706]
$\bar{Y}_{MLRN_{26}}$	[0.5628, 0.6443]
$\bar{Y}_{MLRN_{27}}$	[12.2562, 12.6399]
$\bar{Y}_{MLRN_{28}}$	[11.3893, 11.5739]
$\bar{Y}_{MLRN_{29}}$	[10.0886, 10.3786]
$\bar{Y}_{MLRN_{30}}$	[9.6430, 9.9313]
$\bar{Y}_{MLRN_{31}}$	[9.3561, 9.6330]
$\bar{Y}_{MLRN_{32}}$	[9.2035, 9.4889]
$\bar{Y}_{MLRN_{33}}$	[9.1092, 9.3715]

$\bar{Y}_{MLRN_{34}}$	[8.9984, 9.2905]
$\bar{Y}_{MLRN_{35}}$	[8.8655, 9.1318]
$\bar{Y}_{MLRN_{36}}$	[8.2144, 8.3974]

From the last column of Table 1 and 2, we can see that the variance of the Neutrosophic Linear Regression Estimator is minimum when compared with the variance of SRSWOR sample mean, MSE of Neutrosophic ratio type estimators and proposed Neutrosophic modified linear regression type ratio estimators. Hence, the Neutrosophic Linear Regression Estimator is performed better than other estimators for this available data.

6. Simulation Study

To evaluate more about the efficiency of the proposed neutrosophic estimators, we have undertaken a simulation study as given below:

For simulating 1000 normal random variates from a Bi-variate normal distribution we took $X_N \sim NN$ ([171.2,180.4], [$(5.8)^2$, $(6.7)^2$]) and $Y_N \sim NN$ ([76.0, 84.9], [$(12.9)^2$, $(17.2)^2$]). The correlation coefficient is fixed at value [0.992, 0.996]. Simple random sampling without replacement has been considered for sample size, n=100.

Variance of Neutrosophic SRSWOR sample mean and the MSE of the Neutrosophic Ratio type estimator and the variance of Neutrosophic Linear Regression estimator for simulated data are given below:

Table 4: Variance of Neutrosophic SRSWOR sample mean, MSE of the Neutrosophic Ratio Type estimator and Variance of Neutrosophic Linear Regression Estimator

Estimators	**MSE / Variance**
Neutrosophic SRSWOR sample mean, $\bar{y}_N$	[1.4962, 2.6599]
Neutrosophic Ratio Type Estimator, $\bar{Y}_{RN}$	[0.9671, 1.7835]
Neutrosophic Linear Regression Estimator, $\bar{Y}_{LRN}$	[0.0238, 0.0212]

Table 5 shows the MSE of the proposed Neutrosophic Modified Linear Regression Type Ratio (NMLRR) estimators.

Table 5: MSE of the proposed neutrosophic modified linear regression type ratio estimators for ρ=[0.992,0.996]

Proposed Neutrosophic Estimators	**MSE values**
$\bar{Y}_{MLRN_{KC}}$	[0.0825, 0.1094]
$\bar{Y}_{MLRN_1}$	[0.0825, 0.1093]
$\bar{Y}_{MLRN_2}$	[0.0803, 0.1064]
$\bar{Y}_{MLRN_3}$	[0.0825, 0.1094]
$\bar{Y}_{MLRN_4}$	[0.0474, 0.0623]
$\bar{Y}_{MLRN_5}$	[0.0818, 0.1084]
$\bar{Y}_{MLRN_6}$	[0.0665, 0.0879]
$\bar{Y}_{MLRN_7}$	[0.0825, 0.1093]
$\bar{Y}_{MLRN_8}$	[0.0823, 0.1091]
$\bar{Y}_{MLRN_9}$	[0.0803, 0.1064]
$\bar{Y}_{MLRN_{10}}$	[0.0825, 0.1094]
$\bar{Y}_{MLRN_{11}}$	[0.0319, 0.0390]
$\bar{Y}_{MLRN_{12}}$	[0.0385, 0.0433]
$\bar{Y}_{MLRN_{13}}$	[0.0239, 0.0213]
$\bar{Y}_{MLRN_{14}}$	[0.0585, 0.0716]
$\bar{Y}_{MLRN_{15}}$	[0.0239, 0.0213]
$\bar{Y}_{MLRN_{16}}$	[0.0384, 0.0432]
$\bar{Y}_{MLRN_{17}}$	[0.0388, 0.0438]
$\bar{Y}_{MLRN_{18}}$	[0.0382, 0.0428]
$\bar{Y}_{MLRN_{19}}$	[0.0776, 0.1014]
$\bar{Y}_{MLRN_{20}}$	[0.0799, 0.1053]
$\bar{Y}_{MLRN_{21}}$	[0.0385, 0.0433]
$\bar{Y}_{MLRN_{22}}$	[0.0239, 0.0214]
$\bar{Y}_{MLRN_{23}}$	[0.0239, 0.0213]
$\bar{Y}_{MLRN_{24}}$	[0.0347, 0.0378]
$\bar{Y}_{MLRN_{25}}$	[0.0451, 0.0534]
$\bar{Y}_{MLRN_{26}}$	[0.0239, 0.0213]

$\bar{Y}_{MLRN\,27}$	[0.0391, 0.0443]
$\bar{Y}_{MLRN\,28}$	[0.0389, 0.0439]
$\bar{Y}_{MLRN\,29}$	[0.0388, 0.0437]
$\bar{Y}_{MLRN\,30}$	[0.0387, 0.0435]
$\bar{Y}_{MLRN\,31}$	[0.0385, 0.0433]
$\bar{Y}_{MLRN\,32}$	[0.0384, 0.0431]
$\bar{Y}_{MLRN\,33}$	[0.0383, 0.0429]
$\bar{Y}_{MLRN\,34}$	[0.0381, 0.0426]
$\bar{Y}_{MLRN\,35}$	[0.0379, 0.0423]
$\bar{Y}_{MLRN\,36}$	[0.0369, 0.0409]

From the above table, the variance of Neutrosophic Linear Regression Estimator is minimum when comparing with the variance of SRSWOR sample mean, MSE of Neutrosophic ratio type estimators and proposed Neutrosophic modified linear regression type ratio estimators. So we conclude that the performance efficiency of the Neutrosophic Linear Regression Estimator is better than the other estimators.

7. Conclusion:

In this paper a neutrosophic linear regression type estimator and modified neutrosophic linear regression type estimators for estimating population mean using the known parameters of the auxiliary variable has been proposed. The mean squared error of the proposed estimators is derived. The performances of the proposed estimators with that of the neutrosophic SRSWOR sample mean and neutrosophic ratio type estimator for simulated data and natural population have been assessed. It is observed from the numerical comparison that the variance of the proposed neutrosophic linear regression type estimator and MSE of proposed modified neutrosophic linear regression type estimator is less than the variance of the neutrosophic SRSWOR sample mean and MSE of neutrosophic ratio type estimator. Hence, we strongly recommend that the proposed neutrosophic linear regression type estimator and modified neutrosophic linear regression type estimators for the use of practical applications for estimation of population mean of neutrosophic statistics.

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A STUDY ON APPLICATION OF NEUTROSOPHIC SOFT MATRICES TO DECISION MAKING

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Abstract:

Decision-making processes often involve selecting the most suitable candidate from a pool of alternatives, which can be challenging due to uncertainties and subjective criteria. Soft set theory has proven effective in handling such complexities. In this paper we introduced neutrosophic soft matrices as a comprehensive frame work for managing uncertain and imprecise data in decision making. Specifically, we proposed the integration of score matrices within the neutrosophic soft matrix framework to facilitate the selection of the best candidates. Score matrices provide a structured approach for evaluating candidates based on multiple criteria, accommodating vague and conflicting information. By incorporating score matrices into neutrosophic soft matrices, we enhance the decision-making process, enabling a more comprehensive assessment of candidate suitability. Through case studies and examples, we demonstrate the practical application of this approach in selecting the best candidate across various domains. Our research contributes to the advancement of decision-making methodologies, offering a robust tool for navigating uncertainty and ambiguity in candidate selection processes.

Keywords: Fuzzy soft set, Neutrosophic soft set, Neutrosophic soft matrix, Decision- Making and Score matrix.

1. Introduction

A multitude of academics utilize various approaches to address uncertainties and challenges in a variety of fields, such as engineering, business administration, environmental sciences, and medical sciences. Often, traditional mathematical tools are insufficient to adequately address these issues. To overcome these obstacles, researchers hunt for other approaches such fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, and so forth.

The journey began in 1965 when Lotfi A. Zadeh [16] proposed the groundbreaking fuzzy set theory to tackle uncertain problems. Subsequently, in 1975, Yang [15] introduced the interval-valued fuzzy set (IVFS), offering a broader scope than conventional fuzzy sets. In 1982, Pawlak introduced the rough set theory [12], providing another valuable tool for handling uncertainty. Building upon thesefoundations, Atanassov [1] coined the intuitionistic fuzzy set theory in 1983, followed by Florentin Smarandache's [14] proposal of the neutrosophic fuzzy set in 1995. Molodtsov [11] created the soft set theory in 1999 as a result, and it is a crucial mathematical tool for handling decision-making issues in ambiguous situations. The idea of Molodtsov [11] was expanded upon and many foundations of soft sets were defined by Maji et al. [6] in 2001. Intuitionistic fuzzy soft sets were introduced later in 2004 by Maji et al. [7]. Fuzzy soft matrices were first proposed in 2010 by Cagman [3].

Neutrosophic soft matrices find applications in various real-world decision-making domains including, but not limited to engineering, finance, healthcare, and environmental management. They enable decision-makers to make informed decisions in complex and uncertain environments. Researchers have developed algorithms and computational techniques for solving decision-making problems using neutrosophic soft matrices. These algorithms assist in analyzing and processing the information contained in the matrices to derive optimal or satisfactory solutions.

The main objective of this article is to utilize Neutrosophic soft matrices in decision-making, supported by a scoring system, to identify top candidates with varied skill sets suitable for specific positions within the company.

2. Preliminaries

Definition 2.1:[11]

Let U be an initial universe set and E be a set of parameters. Let P(U) denote the power set of U. Let A ⊆ E. A pair (F_A, E) is called a soft set over U, where F_A is a mapping given by F_A: E⟶ P(U) Such that F_A(e) = φ if e∉A. Here F_A is called approximate function of the soft set (F_A, E). The set F_A(e) is called e- approximate value set which consist of related objects of the parameter e ∈ E. In other words, a soft set over U is a parameterized family of subsets of the universe U.

Definition 2.2:[8]

Let U be an initial Universe set and E be the set of parameters. Let **A** ⊆E . A pair (F, A) is called fuzzy soft set over U where F is a mapping given by F: A⟶I^U, where I^U denotes the collection of all fuzzy subsets of U.

Definition 2.3:

Let (F_A,E) be a fuzzy soft set over U. Then a subset of U×E is uniquely defined by R_A={(u,e): e ∈ A, u ∈F_A (e)} which is called relation form of (F_A,E). The characteristic function of R_A is written by μ_{RA} : U×E⟶ [0,1],whereμ_{RA}(u,e) ∈ [0,1] is the membership value of u ∈ U for each e ∈ U.

If [μ_{ij}] = μ_{RA} (u_i ,e_j), we can define a matrix

$$[\mu_{ij}]_{m\times n} = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{m1} & \mu_{m2} & \cdots & \mu_{mn} \end{bmatrix}$$

which is called an m × n soft matrix of the soft set (F_A,E) over U. Therefore, we can say that a fuzzy soft set (F_A ,E) is uniquely characterized by the matrix$[\mu_{ij}]_{m\times n}$ and both concepts are interchangeable.

Definition 2.4: [13]
Let U be the Universe of discourse. The neutrosophic set A on the Universe of discourse U is defined as A={<$\boldsymbol{T_A(X), I_A(X), F_A(X)}$>: x∈U}, where the characteristic functions T,I,F:U→[0,1] and 0≤ T+I+F ≤3 ; T, I, F are neutrosophic components which defines the degree of membership, the degree of indeterminacy and the degree of non membership respectively.

Definition 2.5:[4]

Suppose K be a universe with an element in K denoted by f and D be a set of attributes. A neutrosophic set N over K is characterized by a truthiness$\boldsymbol{T_A}$, indeterminacy $\boldsymbol{I_A}$, and a falsity value $\boldsymbol{F_A}$ where $\boldsymbol{T_A}$,$\boldsymbol{I_A}$ and $\boldsymbol{F_A}$are real standard subsets of [0,1]. And $\boldsymbol{f_N}: \boldsymbol{D} \to \boldsymbol{N(K)}$.

A = {(e, {< f, (T_A(f), $I_A(f), F_A(f))$} >): $f \in U, e \in$D, $T_A(f), I_A(f), F_A(f) \in$ [0,1]}

There is no restriction on the sum of $T_A(f), I_A(f), F_A(f)$, $0 \le T_A(f) + I_A(f) + F_A(f) \le 3^+$

Definition 2.6:[2]

Suppose K = {$\dot{\boldsymbol{f}}_{\boldsymbol{1}}$,$\dot{\boldsymbol{f}}_{\boldsymbol{2}}$,$\dot{\boldsymbol{f}}_{\boldsymbol{3}}$ …..} be the univers and D = {$\boldsymbol{e_1}$,$\boldsymbol{e_2}$,$\boldsymbol{e_3}$ ……} be a set of attributes and A ⊆D. A set (F ,A) be an NSS over K. Then the subset of K×D is defined as $\boldsymbol{R_A}$ = {($\dot{\boldsymbol{f}}, \boldsymbol{e}$); e∈ A, $\dot{\boldsymbol{f}} \in \boldsymbol{F_A}$(e)} which is the relation form of ($\boldsymbol{F_A}, \boldsymbol{D}$). The truithiness, indeterminacy and falsity value are :

$$T_{R_A} : \mathrm{K} \times \mathrm{D} \to [0,1], \qquad I_{R_A} : \mathrm{K} \times \mathrm{D} \to [0,1], \qquad F_{R_A} : \mathrm{K} \times \mathrm{D} \to [0,1],$$

$T_{RA}(f\ , e) \in [0,1]$, $I_{RA}(f, e) \in [0,1]$,$F_{RA}(f, e) \in [0,1]$ are the truithiness , indeterminacy and falsity of f∈ K for each e ∈ D

If $[(T_{ij}, I_{ij}, F_{ij})] = [T_{ij}(\dot{f}_i, e_j), I_{ij}(\dot{f}_i, e_j), F_{ij}(\dot{f}_i, e_j)]$, then

$$[(T_{ij}, I_{ij}, F_{ij})]_{m\times n} = \begin{bmatrix} (T_{11}, I_{11}, F_{11}) & (T_{12}, I_{12}, F_{12}) & \cdots & (T_{1n}, I_{1n}, F_{1n}) \\ (T_{21}, I_{21}, F_{21}) & (T_{22}, I_{22}, F_{22}) & \cdots & (T_{2n}, I_{2n}, F_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (T_{m1}, I_{m1}, F_{m1}) & (T_{m2}, I_{m2}, F_{m2}) & \cdots & (T_{mn}, I_{mn}, F_{mn}) \end{bmatrix}$$

That's known as an m × n neutrosophic soft matrix over K.

Definition 2.7: [5]

Suppose A = [(T_{ij}, I_{ij}, F_{ij})] ∈ $NSM_{\mathrm{m\times n}}$ .Then the complement of A is denoted byA° and is defined as A°= [(F_{ij},1−I_{ij}, T_{ij})] for all i and j.

Definition 2.8: [10]

If A = [(T_{ij}^{A},I_{ij}^{A}, F_{ij}^{A})] ∈NS$M_{m\times n}$, B = [(T_{ij}^{B},I_{ij}^{B}, F_{ij}^{B})] ∈NS$M_{m\times n}$then C=[(T_{ij}^{C},I_{ij}^{C}, F_{ij}^{C})] ∈NS$M_{m\times n}$ .Then the addition of A and B as

A+B = C = (max (T_{ij}^{A},T_{ij}^{B}), $\frac{I_{ij}^{A}+I_{ij}^{B}}{2}$, min ($F_{ij}^{A}$,$F_{ij}^{B}$)) ∀i and j.

Definition 2.9: [8]

If A = [T_{ij}^{A},I_{ij}^{A}, F_{ij}^{A}] ∈NS$M_{m\times n}$, B = [(T_{ij}^{B},I_{ij}^{B}, F_{ij}^{B})] ∈NS$M_{m\times n}$then C = [(T_{ij}^{C},I_{ij}^{C}, F_{ij}^{C})] ∈NS$M_{m\times n}$ .Then the subtraction of A and B as

A−B = C = ($T_{ij}^{A} - T_{ij}^{B}, I_{ij}^{A} - I_{ij}^{B}, F_{ij}^{A} - F_{ij}^{B}$)

Definition 2.10: [9]

Let A = [T_{ij}^{A},I_{ij}^{A}, F_{ij}^{A}] and$B$ = [(T_{ij}^{B},I_{ij}^{B}, F_{ij}^{B})] be two neutrosophic soft matrices. Then the max-min produuct of the two neutrosophic soft matrices A and B is denoted as A*B is defined as,

A*B = [max min ((T_{ij}^{A},T_{ij}^{B}), min max (I_{ij}^{A}, I_{ij}^{B}), min max (F_{ij}^{A},F_{ij}^{B})] ∀i and j.

3.Neutrosophic Soft Matrix Theory in Decision Making

Definition 3.1:

Suppose A = [(T_{ij}^{A}, I_{ij}^{A}, F_{ij}^{A})] ∈ NS$M_{m\times n}$. Then A is called the value of NSM denoted by V(A) and is defined by V(A) = [($T_{ij}^{A} + I_{ij}^{A} - F_{ij}^{A}$)] for all i and j respectively, where i=1,2,3,…m and j=1,2,3....n.

Definition 3.2:

If A = [(T_{ij}^{A}, I_{ij}^{A}, F_{ij}^{A})] ∈ NS$M_{m\times n}$, B = [(T_{ij}^{B}, I_{ij}^{B}, F_{ij}^{B})] ∈ NS$M_{m\times n}$. Then the score matrix of A and B is denoted by $S_{(A,B)}$ and is defined as $S_{(A,B)} = V(A) -$ V(B).

Definition 3.3:

If A = [(T_{ij}^{A}, I_{ij}^{A}, F_{ij}^{A})] ∈ NS$M_{m\times n}$, B = [(T_{ij}^{B},I_{ij}^{B}, F_{ij}^{B})] ∈ NS$M_{m\times n}$. Then, their corresponding value matrix be V(A),V(B) and their score matrix be $S_{(A,B)}$ .Then the total score for each u_i in U as

$$S_i = \sum_{j=1}^{n} ((V(A) - V(B))$$

Methodology

Take a set U of applicants that are going through interviews to be hired by a corporation for managerial roles. Let E be a collection of criteria related to the managerial experience of candidates. Two Neutrosophic Soft Sets (NSS) are established: (F_A, E) over U, which represents candidate selection by field expert X, and (G_B, E) over U, which represents candidate selection by field expert Y. F is a mapping defined by F: A→I^U, and I^U signifies the collection of all fuzzy subsets comprising U. The corresponding NSS (F_A, E) and (G_B, E) are represented by matrices A and B. The maximum membership for the judges' candidate selection process is determined by evaluating A+B, which is computed by taking the complements (F_A, E)° and (G_B, E)° and calculating their corresponding matrices A° and B°. Additionally, we also compute A° + B°. For each candidate in U, we compute V(A+B), V(A°+B°), $S_{((A+B),(A°+B°))}$, and the sum of the scores, S_i. In order to determine that candidate C_i has been chosen by the judges, we finally find S_k = max(S_i). Re-evaluating the parameters is the process that is repeated if S_k has more than one value.

4. Algorithm

Step 1 : Provide the Neutrosophic soft sets (F_A, E) and (G_B, E), then derive the corresponding Neutrosophic soft matrices A and B for (F_A, E) and (G_B, E) respectively.

Step 2 : Formulate the complement sets of the Neutrosophic soft sets (F_A, E) and (G_B, E) as (F_A, E)° and (G_B, E)° respectively, then derive the corresponding Neutrosophic soft matrices A° and B° for (F_A, E)° and (G_B, E)° accordingly.

Step 3 : Determine (A+B), (A°+B °), V(A+B), V(A°+B °) and $S_{((A+B),(A°+B°))}$.

Step 4 : Work out the total score S_i for each u_i in U.

Step 5 : Determine S_K= max(S_i).

In conclusion, candidate C_i is deemed suitable for the position.

Step 6 : If multiple instances yield the maximum value S_K, continue by reassessing the parameters and iterating the process.

5. Technology in a Decision-Making Problem

In the context provided, let (F_A, E) and (G_B, E) denote two neutrosophic soft sets representing the selection of four candidates from the universal set U = {C_1, C_2, C_3, C_4} by the experts X and Y, respectively.

Here, E = $\{e_1, e_2, e_3\}$ represents the set of parameters, symbolising different types of qualities such as stand for confident, presence of mind and willingness to take risk.

$(F_A, E) = \{F_A(e_1), F_A(e_2), F_A(e_3)\}$

$F_A(e_1) = \{(C_1,0.6,0.2,0.1),(C_2,0.4,0.3,0.1),(C_3,0.5,0.2,0.2),(C_4,0.3,0.5,0.2)\}$

$F_A(e_2)= \{(C_1,0.4,0.5,0.1),(C_2,0.7,0.2,0.1),(C_3,0.2,0.4,0.3),(C_4,0.2,0.5,0.3)\}$

$F_A(e_3)= \{(C_1,0.7,0.1,0.2),(C_2,0.5,0.3,0.2),(C_3,0.6,0.1,0.2),(C_4,0.3,0.5,0.2)\}$

$(G_B, E) = \{G_B(e_1), G_B(e_2), G_B(e_3)\}$

$G_B(e_1) = \{(C_1,0.1,0.5,0.2),(C_2,0.4,0.3,0.2),(C_3,0.6,0.3,0.1),(C_4,0.2,0.6,0.1)\}$

$G_B(e_2) = \{(C_1,0.6,0.2,0.1),(C_2,0.7,0.2,0.1),(C_3,0.5,0.3,0.2),(C_4,0.4,0.3,0.2)\}$

$G_B(e_3)= \{(C_1,0.7,0.2,0.1),(C_2,0.2,0.5,0.3),(C_3,0.4,0.2,0.4),(C_4,0.6,0.2,0.2)\}$

The following neutrosophic fuzzy soft matrices represent these two neutrosophic fuzzy soft sets, respectively.

$$A = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.6,0.2,0.1) & (0.4,0.5,0.1) & (0.7,0.1,0.2) \\ (0.4,0.3,0.1) & (0.7,0.2,0.1) & (0.5,0.3,0.2) \\ (0.5,0.2,0.2) & (0.2,0.4,0.3) & (0.6,0.1,0.2) \\ (0.3,0.5,0.2,) & (0.2,0.5,0.3) & (0.3,0.5,0.2) \end{bmatrix} \end{array}$$

$$B = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.1,0.5,0.2) & (0.6,0.2,0.1) & (0.7,0.2,0.1) \\ (0.4,0.3,0.2) & (0.7,0.2,0.1) & (0.2,0.5,0.3) \\ (0.6,0.3,0.1) & (0.5,0.3,0.2) & (0.4,0.2,0.4) \\ (0.2,0.6,0.1) & (0.4,0.3,0.2) & (0.6,0.2,0.2) \end{bmatrix} \end{array}$$

Then the neutrosophic fuzzy soft complement matrices are,

$$A^{\circ} = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.1,0.8,0.6) & (0.1,0.5,0.4) & (0.2,0.9,0.7) \\ (0.1,0.7,0.4) & (0.1,0.8,0.7) & (0.2,0.7,0.5) \\ (0.2,0.8,0.5) & (0.3,0.6,0.2) & (0.2,0.9,0.6) \\ (0.2,0.5,0.3) & (0.3,0.5,0.2) & (0.2,0.5,0.3) \end{bmatrix} \end{array}$$

$$B^{\circ} = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.2,0.5,0.1) & (0.1,0.8,0.6) & (0.1,0.8,0.7) \\ (0.2,0.7,0.4) & (0.1,0.8,0.7) & (0.3,0.5,0.2) \\ (0.1,0.7,0.6) & (0.2,0.7,0.5) & (0.3,0.8,0.4) \\ (0.1,0.4,0.2) & (0.2,0.7,0.4) & (0.2,0.8,0.6) \end{bmatrix} \end{array}$$

Following that, we compute the addition matrices.

$$(A+B)=\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array}\begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.6,0.35,0.1) & (0.6,0.35,0.1) & (0.7,0.15,0.1) \\ (0.4,0.3,0.1) & (0.7,0.2,0.1) & (0.5,0.4,0.2) \\ (0.6,0.25,0.1) & (0.5,0.35,0.2) & (0.6,0.15,0.2) \\ (0.3,0.55,0.1) & (0.4,0.4,0.2) & (0.6,0.35,0.2) \end{bmatrix} \end{array}$$

$$(A^{\circ}+B^{\circ})=\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array}\begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} (0.2,0.65,0.1) & (0.1,0.65,0.4) & (0.2,0.85,0.7) \\ (0.2,0.7,0.4) & (0.1,0.8,0.7) & (0.3,0.6,0.2) \\ (0.2,0.75,0.5) & (0.3,0.65,0.2) & (0.3,0.85,0.4) \\ (0.2,0.65,0.1) & (0.1,0.65,0.4) & (0.2,0.85,0.7) \end{bmatrix} \end{array}$$

$$V(A+B)=\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array}\begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 0.85 & 0.85 & 0.75 \\ 0.6 & 0.8 & 0.7 \\ 0.75 & 0.65 & 0.55 \\ 0.75 & 0.6 & 0.75 \end{bmatrix} \end{array}$$

$$V(A^{\circ}+B^{\circ})=\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array}\begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 0.75 & 0.35 & 0.35 \\ 0.5 & 0.2 & 0.7 \\ 0.45 & 0.75 & 0.75 \\ 0.6 & 0.7 & 0.55 \end{bmatrix} \end{array}$$

Compute the score matrix and total score for the selection

$$S_{((A+B),(A^{\circ}+B^{\circ}))}=\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array}\begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.1 & 0.6 & 0.0 \\ 0.3 & -0.1 & -0.2 \\ 0.15 & -0.1 & 0.2 \end{bmatrix} \end{array}$$

The aggregate score for the top-candidate : $\begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \end{array}\begin{bmatrix} 1.0 \\ 0.7 \\ 0.0 \\ 0.25 \end{bmatrix}$

Upon observing that the second candidate possesses the highest value, we deduce that, based on the opinions of both experts, candidate c_2 is chosen for the position.

Product of Neutrosophic Soft Matrices

Here we have defined the Neutrosophic soft matrices with suitable examples and the matrices is compared with other existing matrices like Neutrosophic soft matrices, Neutrosophic complement matrices, Neutrosophic Square matrices and somebinary operators and so on. The product and complement of two Neutrosophic Soft Matrices were examined and total score matrices are derived.

Case Study:

Suppose the test results of four candidates C ={c1,c2,c3 and c4} as the universal set where c1,c2,c3and c4 represents Swathi, Sowndarya, Saranya, Aruna with systems.

Q = {q1, q2, q3, q4, q5} represents the set of parameters, symbolizing different types of qualities such as standing for confidence, presence of mind, willingness to take risks, communication, Innovative ideas, and experience leadership respectively. Let the possible qualities relating to the above posting P ={P1,P2, P3}be Manager, HR, TL.

Suppose that NSS (F, Q) over C, where F is mapping F: Q→ gives a collection of an approximate description of candidate selection in the company.

Algorithm

Step1:Input Neutrosophic soft set (F,Q) and (G,P) and obtain Neutrosophics of matrices A and B.

Step2: Write the Neutrosophic soft complement set(,) and obtain Neutrosophics complement matrix B^c.

Step3: Compute candidate posting quality A∗B.

Step4: Compute the candidate's quality non-posting matrix $A*B^C$.

Step 5: Compute V, W.

Step6: Compute the score matrix$_{(*,*)}$.

Step7:Identify the maximum score for candidate c_i and conclude that candidate c_i and conclude that candidate C_i has the posting P_i.

(F, Q) ={F(q1) ={(c1,0.2,0.8,0.2),(c2,0.6,0.1,0.3), (c3,0.0,0.6,0.4), (c4,0.3,0.4,0.5)}

{F(q2) ={(c1,0.7,0.1,0.2), (c2,0.1,0.8,0.1), (c3,0.6,0.1,0.4), (c4,0.5,0.2,0.4)}

{F(q3) ={(c1,0.6,0.1,0.3), (c2,0.4,0.4,0.8), (c3,0.8,0.1,0.2), (c4,0.5,0.4,0.1)}

{F(q4) ={(c1,0.1,0.6,0.3), (c2,0.1,0.8,0.2), (c3,0.6,0.5,0.1), (c4,0.3,0.4,0.4)}

{F(q_5) ={(c1,0.1,0.6,0.2), (c2,0.1,0.7,0.2), (c3,0.5,0.3,0.3), (c4,0.7,0.2,0.1)}

The Neutrosophic soft set is represented by the following Neutrosophic soft matrix to describe the candidate's qualities relationship.

$$
\begin{array}{cc}
 & \begin{array}{ccccc} q_1 & q_2 & q_3 & q_4 & q_5 \end{array} \\
A= & \begin{bmatrix}
(0.2,0.8,0.2) & (0.7,0.1,0.2) & (0.6,0.1,0.3) & (0.1,0.6,0.3) & (0.1,0.6,0.2) \\
(0.6,0.1,0.3) & (0.1,0.8,0.1) & (0.4,0.4,0.5) & (0.1,0.8,0.2) & (0.1,0.7,0.2) \\
(0.0,0.6,0.4) & (0.6,0.1,0.4) & (0.8,0.1,0.2) & (0.6,0.5,0.1) & (0.5,0.3,0.3) \\
(0.3,0.4,0.5) & (0.5,0.2,0.4) & (0.5,0.4,0.1) & (0.3,0.4,0.4) & (0.7,0.2,0.1)
\end{bmatrix}
\end{array}
$$

Again, let the set Q = {q1, q2, q3, q4, q5} be a universal set where q1, q2, q3, q4 and q5 represent the qualities confidence, presence of mind, willingness to take risks, communication, Innovative ideas and experience leadership respectively.

Let the possible qualities relating to the above posting P = {P1, P2, P3} be Manager, HR,TL.

Suppose that Neutrosophic of the set (G,P) over Q, where G is mapping gives collection of an approximate description of the selection process of the three postings and their qualities.

(G,P)={G(P1)={(q1,0.3,0.4,0.3),(q2,0.7,0.2,0.4),(q3,0.7,0.2,0.3),
(q4,0.3,0.4,0.4),(q5,0.2,0.7,0.3)}

{G(P2)={(q1,0.6,0.2,0.2),(q2,0.2,0.6,0.3),(q3,0.5,0.4,0.3),
(q4,0.7,0.2,0.1),(q5,0.1,0.8,0.2)}

{G(P3)={(q1,0.6,0.2,0.3),(q2,0.3,0.5,0.4),(q3,0.1,0.8,0.1),
(q4,0.4,0.5,0.3),(q5,0.7,0.4,0.2)}

The Neutrosophic soft set can be represented by the following Neutrosophic matrix.

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
B= \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix}
(0.3,0.4,0.3) & (0.6,0.2,0.2) & (0.6,0.2,0.3) \\
(0.7,0.2,0.4) & (0.2,0.6,0.3) & (0.3,0.5,0.4) \\
(0.7,0.2,0.3) & (0.5,0.4,0.3) & (0.1,0.8,0.1) \\
(0.3,0.4,0.4) & (0.7,0.2,0.1) & (0.4,0.5,0.3) \\
(0.2,0.7,0.3) & (0.1,0.8,0.2) & (0.7,0.4,0.2)
\end{bmatrix}
\end{array}
$$

Neutrosophic of complement matrix

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
B^{\circ}= \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix}
(0.3,0.4,0.3) & (0.2,0.2,0.6) & (0.3,0.2,0.6) \\
(0.4,0.2,0.7) & (0.3,0.6,0.2) & (0.4,0.5,0.3) \\
(0.3,0.2,0.7) & (0.3,0.4,0.5) & (0.1,0.8,0.1) \\
(0.4,0.4,0.3) & (0.1,0.2,0.7) & (0.3,0.5,0.4) \\
(0.3,0.7,0.2) & (0.2,0.8,0.1) & (0.2,0.4,0.7)
\end{bmatrix}
\end{array}
$$

Max-min compositions of two Neutrosophics of matrices will produces the following results. Let us suppose $A*B = \left[{}_{i}\right]_{\times}$ where,

C11=[max(0.2,0.7,0.6,0.1,0.1),min(0.8,0.2,0.2,0.6,0.7),min(0.3,0.4,0.3,0.4,0.3)
C1=(0.7,0.2,0.3)

C12=[max(0.2,0.2,0.5,0.1,0.1),min(0.8,0.6,0.4,0.6,0.8),min(0.3,0.4,0.3,0.3,0.2)]

C12=(0.5,0.4,0.2)

C13=[max(0.2,0.3,0.1,0.1,0.1),min(0.8,0.5,0.8,0.6,0.6),min(0.3,0.4,0.3,0.3,0.2)]

C13=(0.3,0.5,0.2)

C21=[max(0.3,0.1,0.4,0.1,0.1),min(0.4,0.8,0.4,0.8,0.7),min(0.3,0.4,0.3,0.4,0.3)]

C21=(0.4,0.4,0.3)

C22=[max(0.6,0.1,0.4,0.1,0.1),min(0.2,0.8,0.4,0.8,0.8),min(0.3,0.3,0.5,0.2,0.2)]

C22=(0.6,0.2,0.2)

C23=[max(0.6,0.1,0.1,0.1,0.1),min(0.2,0.8,0.8,0.8,0.7),min(0.3,0.4,0.5,0.3,0.2)]

C23=(0.6,0.2,0.2)

C31=[max(0.0,0.6,0.7,0.3,0.2),min(0.6,0.2,0.2,0.5,0.7),min(0.4,0.4,0.3,0.4,0.3)]

C31=(0.7,0.2,0.3)

C32=[max(0.0,0.2,0.5,0.6,0.1),min(0.6,0.6,0.4,0.5,0.3),min(0.4,0.4,0.3,0.1,0.3)]

C32=(0.6,0.3,0.1)

C33=[max(0.0,0.3,0.1,0.4,0.5),min(0.6,0.5,0.8,0.5,0.4),min(0.4,0.4,0.2,0.3,0.3)]

C33=(0.5,0.4,0.2)

C41=[max(0.3,0.5,0.5,0.3,0.2),min(0.4,0.2,0.4,0.4,0.7),min(0.5,0.4,0.3,0.4,0.3)]

C41=(0.5,0.2,0.3)

C42=[max(0.3,0.2,0.5,0.3,0.7),min(0.4,0.6,0.4,0.4,0.8),min(0.5,0.4,0.3,0.4,0.2)]

C42=(0.5,0.4,0.2)

C43=[max(0.3,0.3,0.1,0.3,0.7),min(0.4,0.5,0.8,0.5,0.4),min(0.5,0.4,0.1,0.4,0.2)]

C43=(0.7,0.4,0.1)

Then $A*B=\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} (0.4,0.2,0.2) & (0.3,0.4,0.2) & (0.1,0.5,0.3) \\ (0.3,0.4,0.3) & (0.3,0.2,0.2) & (0.3,0.2,0.3) \\ (0.4,0.2,0.3) & (0.3,0.4,0.3) & (0.4,0.4,0.2) \\ (0.4,0.2,0.2) & (0.3,0.4,0.1) & (0.4,0.4,0.1) \end{bmatrix} \end{array}$

Hence

$$W = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0.4 & 0.5 & 0.6 \\ 0.3 & 0.3 & 0.2 \\ 0.3 & 0.4 & 0.6 \\ 0.4 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

and finally, it is observed that,

$$\text{V-W} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.0 \\ 0.2 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.1 \\ 0.0 & 0.1 & 0.3 \end{bmatrix} \end{array}$$

6. Conclusion

The literature on Neutrosophic soft matrices, first introduced by Florentine Smarandache in 1995 and expanded by Molodtsov in 1999, reveals multiple methods for decision-making selection processes. This article compares various Neutrosophic soft matrices operations, including addition, subtraction, product, union, and intersection, with other existing matrices. It highlights the potential of these matrices in addressing uncertainties through new operations and demonstrates their application in real-life decision-making problems. The study underscores the significance of Neutrosophic soft matrices in tackling uncertainties and facilitating decision-making, presenting a novel solution procedure with practical utility. It contributes to advancing decision-making techniques in fields requiring uncertainty management, suggesting broader applications and future exploration.

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ON CHARACTERISTICS OF ϑ −TRANSLATION AND ϑ −MULTIPLICATION IN DOUBT FUZZY Z– ALGEBRA

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Abstract:

In this paper, we define the new notation of Algebraic structures of ϑ −T and $\vartheta - M$ in Doubt Fuzzy Z −Sub Algebra of Z −Aalgebras. And also defined the $\vartheta - T$ and $\vartheta - M$ in Doubt Fuzzy Z −Ideal of Z −Algebra and discussed some of their properties in detail by Z −Aalgebras.

Keywords:

Fuzzy Set(FS), Fuzzy Subset(FSb), Doubt Fuzzy Z–Ideal (DFZI), Doubt Fuzzy Z −Sub Algebra(DFZSA), Doubt Fuzzy $\vartheta -$ Translation $(DF\,\vartheta - T)$, Doubt Fuzzy $\vartheta -$ Multiplication $(DF\,\vartheta - M)$.

1. Introduction

In 1965, Zadeh L A [14], initiated by the concept of fuzzy sets. Several researchers explored on the generalization of the notion of fuzzy subset. The study of fuzzy subsets and its applications to various mathematical contents has given rise to what is now commonly called fuzzy mathematics. Iseki K and Tanaka S [3], introduced the concept of an introduction to the theory of BCK-algebras in 1978. In1980, Iseki K [4], first introduced the notation on BCI-algebras. Kyoungja Lee, Young Bae Jun and Myung ImDoh [5], introduced the concept of fuzzy translations and fuzzy multiplication of BCK/BCI-algebras in 2009. Abu Ayub Ansari and Chandramouleeswaran M [1], introduced the concept of fuzzy translation of fuzzy $\beta -$ ideals of β −algebras in 2014. In 2014, Priya and Ramachandran T [11], introduced the new notation of fuzzy translation and multiplication on PS-algebras. Prasanna A, Premkumar M and Ismail Mohideen S [6] & [7], introduced the concept of fuzzy translation and multiplication on B-algebras in 2018 and also derived from Fuzzy Translation and Fuzzy Multiplication in BG – Algebras in 2019. In 2021, Premkumar [8] derived the new notation of Algebraic Properties on Fuzzy Translation and Multiplication in BP– Algebras. Premkumar [9] & [10], introduced the new concept of Algebraic Properties on ω − Fuzzy Translation and Multiplication in $BH-$ Algebras in 2020 and also derived from the concept of

Characteristics of $\kappa - Q -$ Fuzzy Translation and Fuzzy Multiplication in T-Ideals in T-Algebra in 2022. Sowmiya[12] & [13] initiated by the concept on Fuzzy Z-ideals in Z-algebras and also Fuzzy Algebraic Structure in Z-Algebras in 2019.

We define the new notation of Algebraic structures of DF$\vartheta -$T and DF$\vartheta -$M in $DFZSA$ of $Z -$Aalgebras. And also defined the DF$\vartheta -$T and DF$\vartheta -$Tin $DFZI$ of $Z -$Algebra and discussed some of their properties

2. Preliminaries:

Definition 2.1:

A Z-algebra $(\tilde{\omega},*,0)$ be a Z-algebra. A FS A in $\tilde{\omega}$ with a membership function $\check{\mathfrak{z}}_A$ is said to be a FZSA of a Z-algebra $\tilde{\omega}$ if , for all $ŕ, š$ in $\tilde{\omega}$ the following condition is satisfied

$$\check{\mathfrak{z}}(ŕ * š) \geq \{\check{\mathfrak{z}}(ŕ) \wedge \check{\mathfrak{z}}(š)\}$$

Definition 2.2:

A Z-algebra $(\tilde{\omega},*,0)$ be a Z-algebra. A FS V in $\tilde{\omega}$ with a membership function $\check{\mathfrak{z}}_A$ is said to be a FZI of a Z-algebra $\tilde{\omega}$ if , for all $ŕ, š$ in $\tilde{\omega}$ the following condition is satisfied

(i) $\check{\mathfrak{z}}(0) \geq \check{\mathfrak{z}}(ŕ)$

(ii) $\check{\mathfrak{z}}(ŕ) \geq \{\check{\mathfrak{z}}(ŕ * š) \wedge \check{\mathfrak{z}}(š)\}$

3. Algebraic Structures of $\vartheta -$ Translation and $\vartheta -$ Multiplication in Doubt Fuzzy Z-Subalgebra

Let, $\tilde{\omega}$ be a Z-algebra. For any DFS $\check{\mathfrak{z}}$ of $\tilde{\omega}$, we define T=1-$sup\{\check{\mathfrak{z}}(ŕ)/ŕ \in \tilde{\omega}\}$, unless otherwise we specified.

Definition: 3.1

Let, $\check{\mathfrak{z}}$ be a DFSb of $\tilde{\omega}$ and $\vartheta \in [0, T]$. A mapping $\check{\mathfrak{z}}_{\vartheta}^{T}: \tilde{\omega} \to [0,1]$ is said to be a DF $\vartheta -$ T of $\check{\mathfrak{z}}$ if it satisfies $\check{\mathfrak{z}}_{\vartheta}^{T} = \check{\mathfrak{z}}(ŕ) + \vartheta, \forall ŕ \in \tilde{\omega}$.

Definition: 3.2

Let, $\check{\mathfrak{z}}$ be a DFSb of $\tilde{\omega}$ and $\vartheta \in [0,1]$. A mapping $\check{\mathfrak{z}}_{\vartheta}^{M}: \tilde{\omega} \to [0,1]$ is said to be a DF $\vartheta -$ M of $\check{\mathfrak{z}}$ if it satisfies $\check{\mathfrak{z}}_{\vartheta}^{M} = \vartheta\, \check{\mathfrak{z}}(ŕ), \forall ŕ \in \tilde{\omega}$.

Example: 3.2.1

Let $\tilde{\omega}$ ={0,1,2,3} be the set with the following table.

*	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then $(\tilde{\omega}, *, 0)$ is a Z – algebra.

Define DFS $\check{ʒ}$ is of $\tilde{\omega}$ by $\check{ʒ}(ŕ) = \begin{Bmatrix} 0.4 & if\ ŕ \neq 1 \\ 0.3 & if\ ŕ = 1 \end{Bmatrix}$

Thus $\check{ʒ}$ is a $DFZSA$ of X.

Hence $T = 1 - sup\{\check{ʒ}(ŕ)/ŕ \in \tilde{\omega}\}$ = 1-0.4 = 0.6,

Choose $\vartheta = 0.2 \in [0,1]$ and $\vartheta = 0.3 \in [0,1]$.

Then the mapping $\check{ʒ}_{0.2^T} : \tilde{\omega} \to [0,1]$ is defined by

$$\check{ʒ}_{0.2^T} = \begin{cases} 0.2 + 0.4 = 0.6 & if\ ŕ \neq 1 \\ 0.2 + 0.3 = 0.5 & if\ ŕ = 1 \end{cases}$$

Which satisfies $\check{ʒ}_{0.2^T}(ŕ) = \check{ʒ}(ŕ) + 0.2,\ \forall\, ŕ \in \tilde{\omega}$, is a DF 0.2-T.

The mapping $\gamma_{0.3^M} : \tilde{\omega} \to [0,1]$ is defined by

$$\check{ʒ}_{0.3^M} = \begin{cases} 0.3 * 0.4 = 0.12 & if\ ŕ \neq 1 \\ 0.3 * 0.3 = 0.09 & if\ ŕ = 1 \end{cases}$$

Which satisfies $\check{ʒ}_{0.3^M}(ŕ) = \check{ʒ}(ŕ)(0.3), \forall\, ŕ \in \tilde{\omega}$, is aDF0.3-M.

Theorem: 3.3

If $\check{ʒ}$ of $\tilde{\omega}$ is a $DFZSA$ and $\vartheta \in [0,1]$, then the $DF\ \vartheta - T$. ${\check{ʒ}_{\vartheta}}^T(ŕ)$of $\check{ʒ}$ is also a $DFZSA$ of $\tilde{\omega}$.

Proof:

Let, $ŕ, š \in \tilde{\omega}$ and $\vartheta \epsilon [0, T]$

Then, $\check{ʒ}(ŕ * š) \leq \check{ʒ}(ŕ) \vee \check{ʒ}(š)$

Now,

$$\begin{aligned} {\check{ʒ}_{\vartheta}}^T(ŕ * š) &= \check{ʒ}(ŕ * š) + \vartheta \\ &\leq [\check{ʒ}(ŕ) \vee \check{ʒ}(š)] + \vartheta \\ &= [(\check{ʒ}(ŕ) + \vartheta) \vee (\check{ʒ}(š) + \vartheta)] \\ &= \left[{\check{ʒ}_{\vartheta}}^T(ŕ) \vee {\check{ʒ}_{\vartheta}}^T(š)\right]. \forall\, ŕ, š \in \tilde{\omega}. \end{aligned}$$

Theorem: 3.4

Let, $\check{ʒ}$ be a DFSb of $\tilde{\omega}$ such that the $DF\ \vartheta - T\ {\check{ʒ}_{\vartheta}}^T(ŕ)$ of $\check{ʒ}$ is a $DFZSA$ of $\tilde{\omega}$, for some $\vartheta \epsilon [0, T]$, then $\check{ʒ}$ is a $DFZSA$ of $\tilde{\omega}$.

Proof:

Assume that ${\check{ʒ}_{\vartheta}}^T(ŕ)$is a $DFZSA$ of $\tilde{\omega}$ for some $\vartheta \epsilon [0, T]$

Let $ŕ, š \in \tilde{\omega}$, we have

$$\begin{aligned} \check{ʒ}(ŕ * š) + \vartheta &= {\check{ʒ}_{\vartheta}}^T(ŕ * š) \\ &\leq \left[{\check{ʒ}_{\vartheta}}^T(ŕ) \vee {\check{ʒ}_{\vartheta}}^T(š)\right] \\ &= [(\check{ʒ}(ŕ) + \vartheta) \vee (\check{ʒ}(š) + \vartheta)] \\ &= [\check{ʒ}(ŕ) \vee \check{ʒ}(š)] + \vartheta \end{aligned}$$

$$\Longrightarrow \check{ʒ}(ŕ * š) \geq [\check{ʒ}(ŕ) \vee \check{ʒ}(š)], \forall ŕ, š \in \tilde{\omega}$$

Hence, $\check{ʒ}$ is $DFZSA$ of $\tilde{\omega}$.

Theorem: 3.5

For any $DFZSA$ $\breve{\mathfrak{z}}$ of $\tilde{\omega}$ and $\vartheta\epsilon[0,1]$, if the DF $\vartheta - M$ $\breve{\mathfrak{z}}_{\vartheta}{}^{M}(ŕ)$ of $\breve{\mathfrak{z}}$ is a $DFZSA$ of $\tilde{\omega}$.

Proof:

Let $ŕ, š \in \tilde{\omega}$ and $\vartheta\epsilon[0, T]$

Then $\breve{\mathfrak{z}}(ŕ * š) \geq \breve{\mathfrak{z}}(ŕ) \wedge \breve{\mathfrak{z}}(š)$

Now,

$$\begin{aligned}\breve{\mathfrak{z}}_{\vartheta}{}^{M}(ŕ * š) &= \vartheta\breve{\mathfrak{z}}(ŕ * š)\\ &\leq \vartheta\,[\breve{\mathfrak{z}}(ŕ) \vee \breve{\mathfrak{z}}(š)]\\ &\leq \left[\vartheta\,\breve{\mathfrak{z}}(ŕ) \vee \vartheta\,\breve{\mathfrak{z}}(š)\right]\\ &= \left[\breve{\mathfrak{z}}_{\vartheta}{}^{M}(ŕ) \vee \breve{\mathfrak{z}}_{\vartheta}{}^{M}(š)\right]\\ \Longrightarrow \breve{\mathfrak{z}}_{\vartheta}{}^{M}(ŕ * š) &\leq \left[\breve{\mathfrak{z}}_{\vartheta}{}^{M}(ŕ) \vee \breve{\mathfrak{z}}_{\vartheta}{}^{M}(š)\right]\end{aligned}$$

Therefore, $\breve{\mathfrak{z}}_{\vartheta}{}^{M}$ is a $DFZSA$ of $\tilde{\omega}$.

Theorem: 3.6

For any DFSB, $\breve{\mathfrak{z}}$ of $\tilde{\omega}$ and $\vartheta\epsilon[0,1]$, if the DF $\vartheta - M$ $\breve{\mathfrak{z}}_{\vartheta}{}^{M}(ŕ)$ of $\breve{\mathfrak{z}}$ is a $DFZSA$ of $\tilde{\omega}$, then so in $\breve{\mathfrak{z}}$.

Proof:

Assume that, $\breve{\mathfrak{z}}_{\vartheta}{}^{M}(ŕ)$ of $\breve{\mathfrak{z}}$ is a $DFZSA$ of $\tilde{\omega}$ for some $\vartheta\epsilon[0, T]$

Let $ŕ, š \in \tilde{\omega}$, we have

$$\begin{aligned}\vartheta\breve{\mathfrak{z}}(ŕ * š) &= \breve{\mathfrak{z}}_{\vartheta}{}^{M}(ŕ * š)\\ &\leq \left[\breve{\mathfrak{z}}_{\vartheta}{}^{M}(ŕ) \vee \breve{\mathfrak{z}}_{\vartheta}{}^{M}(š)\right]\\ &= \left[\vartheta\,\breve{\mathfrak{z}}(ŕ) \vee \vartheta\,\breve{\mathfrak{z}}(š)\right]\\ &= \vartheta\,\,[\breve{\mathfrak{z}}(ŕ) \vee \breve{\mathfrak{z}}(š)]\\ \Longrightarrow \breve{\mathfrak{z}}(ŕ * š) &\leq \vartheta\,\,[\breve{\mathfrak{z}}(ŕ) \vee \breve{\mathfrak{z}}(š)]\end{aligned}$$

Hence, $\breve{\mathfrak{z}}$ is a $DFZSA$ of $\tilde{\omega}$.

4. Algebraic Structures of $\boldsymbol{\vartheta}$ − Translation and $\boldsymbol{\vartheta}$ − Multiplication in Doubt Fuzzy Z-Ideal of Z-Algebra

Theorem: 4.1

If the DF $\vartheta - T$ $\breve{\mathfrak{z}}_{\vartheta}{}^{T}(ŕ)$ of $\breve{\mathfrak{z}}$ is a $DFZI$, then it satisfies the condition $\breve{\mathfrak{z}}_{\vartheta}{}^{T}(š * (ŕ * š)) \leq \breve{\mathfrak{z}}_{\vartheta}{}^{T}(ŕ)$.

Proof:

$$\begin{aligned}\breve{\mathfrak{z}}_{\vartheta}{}^{T}\big(š * (ŕ * š)\big) &= \breve{\mathfrak{z}}\big(š * (ŕ * š)\big) + \vartheta\\ &\leq \left\{\breve{\mathfrak{z}}(0 * \big(š * (ŕ * š)\big) + \vartheta \,\vee\, \breve{\mathfrak{z}}(0) + \vartheta\right\}\\ &\leq \left\{\breve{\mathfrak{z}}(0 * \big(š * (š * ŕ)\big) + \vartheta \,\vee\, \breve{\mathfrak{z}}(0) + \vartheta\right\}\\ &= \left\{\breve{\mathfrak{z}}(0 * \big((š * š) * ŕ\big) + \vartheta \,\vee\, \breve{\mathfrak{z}}(0) + \vartheta\right\}\\ &= \left\{\breve{\mathfrak{z}}(0 * (š * ŕ) + \vartheta \,\vee \breve{\mathfrak{z}}(0) + \vartheta\right\}\\ &= \left\{\breve{\mathfrak{z}}((š * ŕ) * 0 + \vartheta \vee \breve{\mathfrak{z}}(0) + \vartheta\right\}\\ &\leq \left\{\breve{\mathfrak{z}}((š * ŕ) * 0 + \vartheta \vee \breve{\mathfrak{z}}(ŕ) + \vartheta\right\}\end{aligned}$$

$$\leq \left\{ ǯ_{\vartheta}{}^{T}(0) \ \vee \ ǯ_{\vartheta}{}^{T}(ŕ) \right\}$$
$$= ǯ_{\vartheta}{}^{T}(ŕ).$$
$$\Rightarrow ǯ_{\vartheta}{}^{T}(š * (ŕ * š)) \leq ǯ_{\vartheta}{}^{T}(ŕ) \ \ \forall\, ŕ, š \in \tilde{ω}$$

Theorem: 4.2

If, $ǯ$ is a $DFZI$ of $\tilde{ω}$, then the $DF\ \vartheta - T\ ǯ_{\vartheta}{}^{T}(ŕ)$ of $ǯ$ is a $DFZI$ of $\tilde{ω}$, for all $\vartheta\epsilon[0,T]$.

Proof:

Let, $ǯ$ be a $DFZI$ of $\tilde{ω}$ and let $\vartheta\epsilon[0,T]$

Then,

$$\text{(i)}\, ǯ_{\vartheta}{}^{T}(0) = ǯ(0) + \vartheta$$
$$\leq ǯ(ŕ) + \vartheta$$
$$= ǯ_{\vartheta}{}^{T}(ŕ)$$
$$\text{(ii)} \ \ ǯ_{\vartheta}{}^{T}(ŕ) = ǯ(ŕ) + \vartheta$$
$$\leq \{ǯ(ŕ * š) \ \vee \ ǯ(š)\} + \vartheta$$
$$= \{(ǯ(ŕ * š) + \vartheta) \vee (ǯ(š) + \vartheta)\}$$
$$= \left\{ǯ_{\vartheta}{}^{T}(ŕ * š) \vee ǯ_{\vartheta}{}^{T}(š)\right\}$$
$$\Rightarrow ǯ_{\vartheta}{}^{T}(ŕ) \leq \left\{ǯ_{\vartheta}{}^{T}(ŕ * š) \vee ǯ_{\vartheta}{}^{T}(š)\right\}$$

Hence $ǯ_{\vartheta}{}^{T}(ŕ)$ of $ǯ$ is a $DFZI$ of $\tilde{ω}$, $\forall\vartheta\epsilon[0,T]$

Theorem: 4.3

Let, $ǯ$ is a $DFSb$ of $\tilde{ω}$ such that the $F\ \vartheta - T\ ǯ_{\vartheta}{}^{T}(ŕ)$ of $ǯ$ is a $DFZI$ of $\tilde{ω}$, for some $\vartheta\epsilon[0,T]$, then $ǯ$ is a $DFZI$ of $\tilde{ω}$.

Proof:

Assume that, $ǯ_{\vartheta}{}^{T}$ is a $DFZI$ of $\tilde{ω}$ for some $\vartheta\epsilon[0,T]$.

Let $ŕ, š \in \tilde{ω}$

Then,

$$\text{(i)} \qquad ǯ(0) + \vartheta = ǯ_{\vartheta}{}^{T}(0)$$
$$\leq ǯ_{\vartheta}{}^{T}(ŕ)$$
$$= ǯ(ŕ) + \vartheta$$
$$\text{And so} \qquad \Rightarrow ǯ(0) \leq ǯ(ŕ)$$
$$\text{(ii)} \qquad ǯ(ŕ) + \vartheta = ǯ_{\vartheta}{}^{T}(ŕ)$$
$$\leq \left\{ǯ_{\vartheta}{}^{T}(ŕ * š) \vee ǯ_{\vartheta}{}^{T}(š)\right\}$$
$$= \{(ǯ(ŕ * š) + \vartheta) \vee (ǯ(š) + \vartheta)\}$$
$$= \{ǯ(ŕ * š) \ \vee \ ǯ(š)\} + \vartheta$$
$$\text{and so } ǯ(ŕ) \leq \{(ŕ * š) \ \vee \ ǯ(š)\}$$

Hence, $ǯ$ is a $DFZI$ of $\tilde{ω}$.

Theorem: 4.4

Let, $\vartheta \epsilon [0, T]$ and let $\breve{ʒ}$ be a $DFZI$ of $\tilde{\tilde{\omega}}$. If $\tilde{\tilde{\omega}}$ is a Z-algebra, then the fuzzy $DF\ \vartheta - T\ \breve{ʒ}_{\vartheta}{}^{T}$ of $\breve{ʒ}$ is a $DFZSA$ of $\tilde{\tilde{\omega}}$.

Proof:

Let, ŕ, š ∈ $\tilde{\tilde{\omega}}$.

Now, we have,
$$\begin{aligned}\breve{ʒ}_{\vartheta}{}^{T}(ŕ * š) &= \breve{ʒ}(ŕ * š) + \vartheta\\ &\le \{\breve{ʒ}((ŕ * š) * š) \vee \breve{ʒ}(š)\} + \vartheta\\ &= \{\breve{ʒ}(š * (ŕ * š)) \vee \breve{ʒ}(š)\} + \vartheta \text{ by Theorem 3.7}\\ &\ge \{\breve{ʒ}(0) \vee \breve{ʒ}(š)\} + \vartheta\\ &\le \{\breve{ʒ}(ŕ) \vee \breve{ʒ}(š)\} + \vartheta\\ &\le \{(\breve{ʒ}(ŕ) + \vartheta) \vee (\breve{ʒ}(š) + \vartheta)\}\\ &= \{\breve{ʒ}_{\vartheta}{}^{T}(ŕ) \vee \breve{ʒ}_{\vartheta}{}^{T}(š)\}\end{aligned}$$

Hence $\breve{ʒ}_{\vartheta}{}^{T}$ is a $DFZSA$ of $\tilde{\tilde{\omega}}$.

Theorem: 4.5

If the $DF\ \vartheta - T\ \breve{ʒ}_{\vartheta}{}^{T}$ of $\breve{ʒ}$ is a $DFZSA$ of $\tilde{\tilde{\omega}}$, $\vartheta \epsilon [0, T]$, then $\breve{ʒ}$ is a $DFZSA$ of $\tilde{\tilde{\omega}}$.

Proof:

Let us assume that, $\breve{ʒ}_{\vartheta}{}^{T}$ of $\breve{ʒ}$ is a $DFZI$ of $\tilde{\tilde{\omega}}$.

Then

$$\begin{aligned}\breve{ʒ}(ŕ * š) + \vartheta &= \breve{ʒ}_{\vartheta}{}^{T}(ŕ * š)\\ &\le \{\breve{ʒ}_{\vartheta}{}^{T}((ŕ * š) * š) \vee \breve{ʒ}_{\vartheta}{}^{T}(š)\}\\ &= \{\breve{ʒ}_{\vartheta}{}^{T}(š * (ŕ * š)) \vee \breve{ʒ}_{\vartheta}{}^{T}(š)\}\\ &\le \{\breve{ʒ}_{\vartheta}{}^{T}(0) \vee \breve{ʒ}_{\vartheta}{}^{T}(š)\}\\ &\le \{\breve{ʒ}_{\vartheta}{}^{T}(ŕ) \vee \breve{ʒ}_{\vartheta}{}^{T}(š)\}\\ &= \{(\breve{ʒ}(ŕ) + \vartheta) \vee (\breve{ʒ}(š) + \vartheta)\}\\ &= \{\breve{ʒ}(ŕ) \vee \breve{ʒ}(š)\} + \vartheta\\ &\Rightarrow \breve{ʒ}(ŕ * š) \le \{\breve{ʒ}(ŕ) \vee \breve{ʒ}(š)\}\end{aligned}$$

Hence $\breve{ʒ}$ is a $DFZSA$ of $\tilde{\tilde{\omega}}$.

Theorem: 4.6

Let, $\breve{ʒ}$ is a $DFSb$ of $\tilde{\tilde{\omega}}$ such that the $DF\ \vartheta - M\ \breve{ʒ}_{\vartheta}{}^{M}(ŕ)$ of $\breve{ʒ}$ is a $DFZI$ of $\tilde{\tilde{\omega}}$, for some $\vartheta \epsilon (0,1]$, then $\breve{ʒ}$ is a $DFZI$ of $\tilde{\tilde{\omega}}$.

Proof:

Assume that, $\breve{ʒ}_{\vartheta}{}^{M}$ is a $DFZI$ of $\tilde{\tilde{\omega}}$ for some $\vartheta \epsilon [0, T]$.

Let ŕ, š ∈ $\tilde{\tilde{\omega}}$

(i) $$\begin{aligned}\vartheta\breve{ʒ}(ŕ) &= \breve{ʒ}_{\vartheta}{}^{M}(0)\\ &\le \breve{ʒ}_{\vartheta}{}^{M}(ŕ)\\ &= \vartheta\breve{ʒ}(ŕ)\end{aligned}$$

And so $\Rightarrow \breve{ʒ}(0) \le \breve{ʒ}(ŕ)$

(ii) $\vartheta\breve{ʒ}(ŕ) = \breve{ʒ}_{\vartheta}{}^{M}(ŕ)$

$$\leq \left\{\check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r} * \check{s}) \vee \check{\mathfrak{z}}_{\vartheta}{}^{M}(\check{s})\right\}$$
$$= \left\{\left(\vartheta\check{\mathfrak{z}}(\acute{r} * \check{s})\right) \vee (\vartheta\check{\mathfrak{z}}(\check{s}))\right\}$$
$$= \vartheta\{\check{\mathfrak{z}}(\acute{r} * \check{s}) \vee \check{\mathfrak{z}}(\check{s})\}$$
and so $\check{\mathfrak{z}}(\acute{r}) \leq \{(\acute{r} * \check{s}) \vee \check{\mathfrak{z}}(\check{s})\}$

Hence, $\check{\mathfrak{z}}$ is a $DFZI$ of $\tilde{\ddot{\omega}}$.

Theorem: 4.7

If, $\check{\mathfrak{z}}$ is a $DFZI$ of $\tilde{\ddot{\omega}}$, then the $DF\ \vartheta - M\ \check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r})$ of $\check{\mathfrak{z}}$ is a $DFZI$ of $\tilde{\ddot{\omega}}$, for all $\vartheta\epsilon(0,1]$.

Proof:

Let, $\check{\mathfrak{z}}$ be a $DFZI$ of $\tilde{\ddot{\omega}}$ and let $\vartheta\epsilon(0,1]$

Then

(i) $$\check{\mathfrak{z}}_{\vartheta}{}^{M}(0) = \vartheta\check{\mathfrak{z}}(\acute{r})$$
$$\leq \vartheta\check{\mathfrak{z}}(\acute{r})$$
$$= \check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r})$$
$$\Rightarrow \check{\mathfrak{z}}_{\vartheta}{}^{M}(0) \leq \check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r})$$

(ii) $$\check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r}) = \vartheta\check{\mathfrak{z}}(\acute{r})$$
$$\leq \vartheta\{\check{\mathfrak{z}}(\acute{r} * \check{s}) \vee \check{\mathfrak{z}}(\check{s})\}$$
$$= \vartheta\{\check{\mathfrak{z}}(\acute{r} * \check{s}) \vee \check{\mathfrak{z}}(\check{s})\}$$
$$= \left\{\left(\vartheta\check{\mathfrak{z}}(\acute{r} * \check{s})\right) \vee (\vartheta\check{\mathfrak{z}}(\check{s}))\right\}$$
$$\leq \left\{\check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r} * \check{s}) \vee \check{\mathfrak{z}}_{\vartheta}{}^{M}(\check{s})\right\}$$
$$\Rightarrow \check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r}) \leq \left\{\check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r} * \check{s}) \vee \check{\mathfrak{z}}_{\vartheta}{}^{M}(\check{s})\right\}$$

Hence, $\check{\mathfrak{z}}_{\vartheta}{}^{M}$ of $\check{\mathfrak{z}}$ is a $DFZI$ of $\tilde{\ddot{\omega}}$, $\forall \acute{r}, \check{s} \in (0,1]$.

Theorem: 4.8

Let, $\vartheta\epsilon(0,1]$ and let, $\check{\mathfrak{z}}$ be a $DFZI$ of a Z-algebra $\tilde{\ddot{\omega}}$. Then the $DF\ \vartheta - M\ \check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r})$ of $\check{\mathfrak{z}}$ is a $DFZSA$ of $\tilde{\ddot{\omega}}$.

Proof:

Let, $\acute{r}, \check{s} \in \tilde{\ddot{\omega}}$.

Now, we have

$$\check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r} * \check{s}) = \vartheta\check{\mathfrak{z}}(\acute{r} * \check{s})$$
$$\leq \vartheta\{\check{\mathfrak{z}}((\acute{r} * \check{s}) * \check{s}) \vee \check{\mathfrak{z}}(\check{s})\}$$
$$= \{\vartheta\check{\mathfrak{z}}(\check{s} * (\acute{r} * \check{s})) \vee \vartheta\check{\mathfrak{z}}(\check{s})\}$$
$$= \vartheta\{\check{\mathfrak{z}}(0) \vee \check{\mathfrak{z}}(\check{s})\}$$
$$\leq \vartheta\{\check{\mathfrak{z}}(\acute{r}) \vee \check{\mathfrak{z}}(\check{s})\}$$
$$\leq \left\{\left(\vartheta\check{\mathfrak{z}}(\acute{r})\right) \vee \left(\vartheta\check{\mathfrak{z}}(\check{s})\right)\right\}$$
$$= \left\{\check{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r}) \vee \check{\mathfrak{z}}_{\vartheta}{}^{M}(\check{s})\right\}$$

Hence $\check{\mathfrak{z}}_{\vartheta}{}^{M}$ is a $DFZSA$ of $\tilde{\ddot{\omega}}$, $\forall \acute{r}, \check{s} \in (0,1]$.

Theorem: 4.9

If the $DF\ \vartheta - T\ \breve{\mathfrak{z}}_{\vartheta}{}^{M}$ of $\breve{\mathfrak{z}}$ is a $DFZSA$ of $\tilde{\tilde{\omega}}$, $\vartheta\epsilon(0,1]$, then $\breve{\mathfrak{z}}$ is a $DFZSA$ of $\tilde{\tilde{\omega}}$.

Proof:

Let us assume that, $\breve{\mathfrak{z}}_{\vartheta}{}^{M}$ of $\breve{\mathfrak{z}}$ is a $DFZI$ of $\tilde{\tilde{\omega}}$.

Then

$$\vartheta\breve{\mathfrak{z}}(\acute{r} * \check{s}) = \breve{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r} * \check{s})$$
$$\leq \left\{\breve{\mathfrak{z}}_{\vartheta}{}^{M}\big((\acute{r} * \check{s}) * \check{s}\big) \vee \breve{\mathfrak{z}}_{\vartheta}{}^{M}(\check{s})\right\}$$
$$= \left\{\breve{\mathfrak{z}}_{\vartheta}{}^{M}\big(\check{s} * (\acute{r} * \check{s})\big) \vee \breve{\mathfrak{z}}_{\vartheta}{}^{M}(\check{s})\right\}$$
$$= \left\{\breve{\mathfrak{z}}_{\vartheta}{}^{M}(0) \vee \breve{\mathfrak{z}}_{\vartheta}{}^{M}(\check{s})\right\}$$
$$\leq \left\{\breve{\mathfrak{z}}_{\vartheta}{}^{M}(\acute{r}) \vee \breve{\mathfrak{z}}_{\vartheta}{}^{M}(\check{s})\right\}$$
$$= \left\{\left(\vartheta\breve{\mathfrak{z}}(\acute{r})\right) \vee \left(\vartheta\breve{\mathfrak{z}}(\check{s})\right)\right\}$$
$$\Rightarrow \breve{\mathfrak{z}}(\acute{r} * \check{s}) \leq \{\breve{\mathfrak{z}}(\acute{r}) \vee \breve{\mathfrak{z}}(\check{s})\}$$

Hence $\breve{\mathfrak{z}}$ is a $DFZSA$ of $\tilde{\tilde{\omega}}$.

Theorem: 4.10

Intersection and union of any two $\vartheta - \mathrm{T}$ of a $DFZI$ of $\breve{\mathfrak{z}}$ of $\tilde{\tilde{\omega}}$ is also a $DFZI$ of $\tilde{\tilde{\omega}}$.

Proof:

Let $\breve{\mathfrak{z}}_{\vartheta}{}^{T}$ and $\breve{\mathfrak{z}}_{\delta}{}^{T}$ be two $DF\ \vartheta - \mathrm{T}$ of a $DFZI$ of $\breve{\mathfrak{z}}$ of $\tilde{\tilde{\omega}}$, where $\vartheta, \delta \in [0,1]$.

Assume that $\vartheta \leq \delta$.

Then by theorem 3.14, $\breve{\mathfrak{z}}_{\vartheta}{}^{T}$ and $\breve{\mathfrak{z}}_{\delta}{}^{T}$ are $DFZIs$ of $\tilde{\tilde{\omega}}$.

Now, $(\breve{\mathfrak{z}}_{\vartheta}{}^{T} \cap \breve{\mathfrak{z}}_{\delta}{}^{T})(\acute{r}) = \left\{\breve{\mathfrak{z}}_{\vartheta}{}^{T}(\acute{r}) \vee \breve{\mathfrak{z}}_{\delta}{}^{T}(\acute{r})\right\}$

$$= \{(\breve{\mathfrak{z}}(\acute{r}) + \vartheta) \vee (\breve{\mathfrak{z}}(\acute{r}) + \delta)\}$$
$$= \breve{\mathfrak{z}}(\acute{r}) + \vartheta$$
$$= \breve{\mathfrak{z}}_{\vartheta}{}^{T}(\acute{r})$$

And $(\breve{\mathfrak{z}}_{\vartheta}{}^{T} \cup \breve{\mathfrak{z}}_{\delta}{}^{T})(\acute{r}) = \left\{\breve{\mathfrak{z}}_{\vartheta}{}^{T}(\acute{r}) \wedge \breve{\mathfrak{z}}_{\delta}{}^{T}(\acute{r})\right\}$

$$= \{(\breve{\mathfrak{z}}(\acute{r}) + \vartheta) \wedge (\breve{\mathfrak{z}}(\acute{r}) + \delta)\}$$
$$= \breve{\mathfrak{z}}(\acute{r}) + \delta$$
$$= \breve{\mathfrak{z}}_{\delta}{}^{T}(\acute{r})$$

Hence $\breve{\mathfrak{z}}_{\vartheta}{}^{T} \cap \breve{\mathfrak{z}}_{\delta}{}^{T}$ and $\breve{\mathfrak{z}}_{\vartheta}{}^{T} \cup \breve{\mathfrak{z}}_{\delta}{}^{T}$ are $DFZIs$ of $\tilde{\tilde{\omega}}$.

5. Conclusion

In this paper we have discussed $\vartheta - T$ and $\vartheta - M$ on Z-Algebras through DFZSA and discussed with some other properties. And also derived from the $\vartheta - T$ and $\vartheta - M$ on DFZI of FZA.

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PENTA PARTITIONED NEUTROSOPHIC SOFT TOPOLOGICAL SPACE

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Abstract :

In this study, we introduce and explore the concept of a Penta-Partitioned Neutrosophic Soft Topological Space (PPNSTS), an innovative framework combining the principles of neutrosophic logic, soft set theory, and topology. The PPNSTS is designed to model uncertainty, indeterminacy, and vagueness in complex systems with enhanced granularity through five distinct partitions of the neutrosophic domain.

The construction of PPNSTS integrates truth-membership, indeterminacy-membership, and falsity-membership functions, partitioned into five subsets, allowing for a multi-layered approach to handling imprecise and ambiguous data. Key properties such as open sets, closed sets, neighbourhood systems, bases, and subspaces are defined and analyzed within this novel framework. Additionally, the study investigates the interrelationships between penta-partitioned neutrosophic soft topological spaces and existing soft and neutrosophic topological spaces.

Applications of PPNSTS in decision-making, artificial intelligence, and data classification are presented to highlight its utility in real-world problem-solving scenarios where conventional methods fall short. This work extends the theoretical foundation of soft topology and neutrosophic systems, providing a robust mathematical tool for researchers and practitioners dealing with uncertainty in multi-dimensional environments.

Keywords*: Soft set, Penta partitioned Neutrosophic set, Penta partitioned neutrosophic topological space.*

1.Introduction

The concept of fuzzy sets was introduced by Zadeh [25] in 1965 to handle data uncertainty and imprecision in mathematical models. This groundbreaking idea paved the way for various generalized set theories aimed at addressing increasingly complex problems involving vagueness and ambiguity. Among these advancements, the Neutrosophic Set, proposed by F. Smarandache[20], has emerged as a versatile mathematical framework for managing imprecise, indeterminate, and inconsistent data. Unlike fuzzy sets, neutrosophic sets offer a distinct feature where the indeterminacy membership function operates independently of the truth and falsity membership functions. This flexibility allows neutrosophic theory to be effectively applied in solving real-world problems characterized by significant uncertainty and contradictions.

Building upon the neutrosophic framework, researchers have further explored its practical applications in decision-making, classification, and artificial intelligence. Neutrosophic sets have been successfully integrated into a wide array of mathematical and computational theories, proving their robustness in diverse domains.

Similarly, the concept of soft sets, first introduced by Molodtsov [6], serves as another powerful tool for addressing uncertainty. A soft set is defined as a parameterized family of subsets of a universal set, where each parameter represents a set of approximate elements. Soft set theory has garnered significant attention over the years, with its fundamentals and extensions being extensively studied by researchers. It has shown tremendous potential in applications requiring flexible and generalized solutions for uncertain data. In 2020, Rama Malik and Surapati Pramanik [14] introduced the concept of Pentapartitioned Neutrosophic set and its properties. The researchers [15],[16] introduced the new concept Quadri partitioned Neutrosophic soft set and its topological space.

In our previous work, the researchers [17] introduced the concept of the Penta Partitioned Neutrosophic Soft Set (PPNSS), an extension that combines the strengths of neutrosophic and soft set theories while partitioning the neutrosophic domain into five distinct subsets. This structure provides a more granular approach to managing uncertainty and indeterminacy. We established foundational properties of PPNSS, demonstrating its efficacy in theoretical and application-oriented scenarios.

Building on this foundation, we now extend our research by incorporating PPNSS into the realm of topological spaces. This extension, termed the Penta Partitioned Neutrosophic Soft Topological Space (PPNSTS), introduces a novel framework for applying the principles of topology to neutrosophic soft sets. By defining open and closed sets, neighborhoods, and other fundamental topological concepts within the PPNSTS framework, we aim to explore its potential in providing structured solutions to problems in mathematics and applied sciences. This work not only enriches the theory of soft topology but also contributes to the broader understanding of neutrosophic systems as a versatile tool for handling uncertainty.

2. Preliminaries

Definition: 2. 1[14]

Let P be a non-empty set. A Pentapartitioned neutrosophic set A over P characterizes each element p in P a truth -membership function T_A , a contradiction membership function C_A, an ignorance membership function G_A, unknown membership function U_A and a false membership function F_A , such that for each p in P $T_A + C_A + G_A + U_A + F_A \leq 5$

Definition:2. 2[15]

Let X be the initial universe set and E be a set of parameters. Consider a non-empty set A and A ⊆ E. Let P(X) denote the set of all quadri partitioned neutrosophic sets of X. The collection (F, A) is termed to be the quadri partitioned neutrosophic soft set (QNSS) over X, where F is a mapping given by F : A⟶ P(X). Where $A = \{< x, T_A(x), C_A(x), U_A(x), F_A(x) >: x \in X\}$ Where $T_A , F_A, C_A , U_A : X \longrightarrow [0,1]$ *and*

$0 \leq T_A(x) + C_A(x) + U_A(x) + F_A(x) \leq 4$ Here $T_A(x)$ is the truth membership, $C_A(x)$ is contradiction membership, $U_A(x)$ is ignorance membership and $F_A(x)$ is the false membership.

Definition:2. 3[17]

Let X be the initial universe set and E be set of parameters. Consider a non-empty set A on E, Let P(X) denote the set of all penta partitioned neutrosophic sets of U. The collection (F, A) is termed to be penta partitioned neutrosophic soft set over U, where F is a mapping given by F: A → P(X). Where $A = \{< x, T_A(x), C_A(x), G_A(x), U_A(x), F_A(x) >: x \in X\}$ Where $T_A\ , F_A, C_A\ , U_A\ : X\ \longrightarrow [0,1]\ and$

$0 \leq T_A(x) + C_A(x) + U_A(x) + F_A\ (x) \leq 4$ Here $T_A(x)$ is the truth membership, $C_A(x)$ is contradiction membership, $G_A(x)$ is ignorance membership $U_A(x)$ is unknown membership and $F_A(x)$ is the false membership.

3. Penta Partitioned Neutrosophic Soft Topological Space [PPNSTS]

Definition : 3. 1

Let (K, M) be Penta Partitioned Neutrosophic set on (X, R) and τ be a collection of Penta partitioned neutrosophic soft subsets of (K, M). Then (K, M) is called Penta Partitioned Neutrosophic Soft Topology if the following conditions are satisfied.

i) $\phi_M, X_M \in \tau$

ii) The union of the elements of any sub collection of τ is in τ

iii) The intersection of the elements of any finite sub collection τ is in τ

The triplet (X, τ, M) is called an Penta Partitioned Neutrosophic Soft Topological Space over X.

Note :3. 2

1. Every member of τ is called a Penta Partitioned Neutrosophic Soft open set in X.
2. The set $A_M$is called a Penta Partitioned Neutrosophic Soft closed set in X if $A_M \in \tau^c$, where $\tau^c = \{{A_M}^c : A_M \in \tau\}$.

Example : 3. 3

Let X = $\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$, M = $\{m_1, m_2\}$ and Let A_M, B_M, C_M, D_M be Penta Partitioned Neutrosophic Soft where

A(m_1) = $\{< \varsigma_1, .5, .6, .1, .7, .2 > < \varsigma_2, .7, .5, .2, .4, .1 > < \varsigma_3, .6, .5, .3, .4, .3 > < \varsigma_4, .3, .2, .4, .6, .1 >\}$

A(m_2) = $\{< \varsigma_1, .8, .7, .4, .6, .3 > < \varsigma_2, .2, .3, .5, .6, .7 > < \varsigma_3, .9, .8, .6, .7, .1 > < \varsigma_4, .7, .5, .7, .4, .3 >\}$

B(m_1) = $\{< \varsigma_1, .2, .3, .5, .5, .1 > < \varsigma_2, .6, .4, .6, .3, .2 > < \varsigma_3, .2, .3, .7, .6, .5 > < \varsigma_4, .1, .8, .8, .9, .5 >\}$

B(m_2) = $\{< \varsigma_1, .5, .8, .9, .2, .4 > < \varsigma_2, .5, .8, .8, .7, .9 > < \varsigma_3, .4, .5, .7, .8, .6 > < \varsigma_4, .5, .6, .7, .8, .9 >\}$

C(m_1) = $\{< \varsigma_1, .1, .2, .6, .3, .4 > < \varsigma_2, .7, .8, .5, .9, .1 > < \varsigma_3, .3, .2, .4, .4, .5 > < \varsigma_4, .9, .3, .3, .4, .5 >\}$

C(m_2) = $\{< \varsigma_1, .3, .8, .2, .7, .6 > < \varsigma_2, .7, .6, .1, .5, .4 > < \varsigma_3, .2, .3, .2, .4, .5 > < \varsigma_4, .7, .2, .3, .8, .5 >\}$

D(m_1) = $\{< \varsigma_1, .9, .2, .4, .7, .3 > < \varsigma_2, .2, .4, .5, .6, .8 > < \varsigma_3, .1, .3, .6, .5, .7 > < \varsigma_4, .3, .6, .7, .9, .1 >\}$

D(m_2) = $\{< \varsigma_1, .8, .6, .5, .4, .2 > < \varsigma_2, .9, .7, .4, .5, .3 > < \varsigma_3, .2, .3, .2, .4, .5 > < \varsigma_4, .7, .8, .1, .9, .2 >\}$

$\tau = \{A_M, B_M, C_M, D_M, \phi_M, X_M\}$ is an PPNST on X.

Preposition : 3. 4

Let (X, τ_1, M) and (X, τ_2, M) be two PPNSTS on X. Then $\tau_1 \cap \tau_2$ is an Penta Partitioned Neutrosophic Soft topology on X. where $\tau_1 \cap \tau_2 = \{A_M : A_M \in \tau_1$ and $A_M \in \tau_2\}$

Proof :

Obviously $\phi_M, X_M \in \tau$.

Let $A_M, B_M \in \tau_1 \cap \tau_2$

Then $A_M, B_M \in \tau_1$ and $A_M, B_M \in \tau_2$

We know that τ_1 and τ_2 are two PPNSTS on X.

Then $A_M \cap B_M \in \tau_1$ and $A_M \cap B_M \in \tau_2$

Hence $A_M \cap B_M \in \tau_1 \cap \tau_2$.

Let τ_1 and τ_2 are two PPNSTS on X.

Denote $\tau_1 \vee \tau_2 = \{A_M \sqcup B_M : A_M \in \tau_1$ and $A_M \in \tau_2\}$

$\tau_1 \wedge \tau_2 = \{A_M \sqcap B_M : A_M \in \tau_1$ and $A_M \in \tau_2\}$.

Example : 3. 5

Let A_M *and* B_M be two PPNSTS on X.

Define $\tau_1 = \{\phi_M,\ X_M,\ A_M\}$

$\tau_2 = \{\phi_M,\ X_M,\ B_M\}$

Then $\tau_1 \cap \tau_2 = \{\phi_M,\ X_M\}$ is a PPNSTS on X.

But $\tau_1 \cup \tau_2 = \{\phi_M, A_M, B_M, X_M\}$, $\tau_1 \vee \tau_2 = \{\phi_M, A_M, B_M, X_M, A_M \sqcup B_M\}$ *and*

$\tau_1 \wedge \tau_2 = \{\phi_M, A_M, B_M, X_M, A_M \sqcap B_M\}$ are not PPNSTS on X.

Theorem : 3. 6

Let (X, τ, M) be a PPNSTS on X and let m ∈ M,

$\{\tau(m) = \{A(m): A_M \in \tau\}$ is an PPNSTS on X.

Proof:

Let m ∈ M.

i) $\phi_M, X_M \in \tau$ 0_N^c = $\phi(m)$ and 1_N^c = X(m)

we have $0_N^c, 1_N^c \in \tau(m)$

ii) Let V, W ∈ $\tau(m)$. Then there exist $A_M, B_M \in \tau$ such that V =A(m) and W =G(m)

By τ is an PPNSTS on X, $A_M \sqcap B_M \in \tau$

Take $C_M = A_M \sqcap B_M$

Then $C_M \in \tau$

Note that V∩ W = $A_M \cap B_M = C_M$ and $\{\tau(m) = \{A(m): A_M \in \tau\}$

Then V∩ W = $\tau(m)$

Definition : 3. 7

Let (X, τ, M) be a PPNSTS on X and let $\mathfrak{B} \subseteq \tau, \mathfrak{B}$ is a basis on τ if for each $A_M \in \tau$, there exist $\mathfrak{B}' \subseteq \mathfrak{B}$ such that $A_M \sqcup \mathfrak{B}'$

Example : 3. 8

Let (X, τ, M) be a PPNSTS on X as in Example:3.3

Then $\mathfrak{B} = \{A_M, B_M, C_M, \phi_M, X_M\}$ is a basis for τ.

Theorem : 3. 9

Let $\mathfrak{B}$ be a basis for PPNSTS on τ. Define $\mathfrak{B}_m = \{A(m): A_M \in \mathfrak{B}\}$and $\tau(m) = \{A(m): A_M \in \tau\}$ for and m ∈ M. Then $\mathfrak{B}_m$ is a basis PPNST $\tau(m)$.

Proof :

Let m ∈ M. For any V= $\tau(m)$, V= B(m) , for $B_M \in \tau$.

Now $\mathfrak{B}$ is a basis for τ.

Then there exists $\mathfrak{B}' \subseteq \mathfrak{B}$ such that $B_M = \sqcup \mathfrak{B}'$ where $\mathfrak{B}_m' = \{A(m): A_M \in \mathfrak{B}''\} \subseteq \mathfrak{B}_m$.

Thus $\mathfrak{B}_m$is a basis for PPNST $\tau(m)$.

4. PROPERTIES OF PPNSTS

Definition : 4. 1

Let (X, τ, M) be a PPNSTS on X and let A_M belongs to PPNSS on X_M. Then the interior of A_M is denoted as PPNSint (A_M). It is defined by PPNSint (A_M) = $\sqcup$ $\{B_M \in \tau: B_M \subseteq A_m\}$

Definition : 4. 2

Let (X, τ, M) be a PPNSTS on X and let A_M belongs to PPNSS on X_M. Then the closure of A_M is denoted as PPNScl (A_M). It is defined by PPNScl (A_M) = $\sqcap$ $\{B_M \in \tau^c : A_M \subseteq B_m\}$

Theorem : 4. 3

Let (X, τ, M) be a PPNSTS over X. Then the following properties are hold.

i) ϕ_M and X_M are PPNS closed sets over X

ii) The intersection of any number of PPNS closed set is a PPNS closed set over X.

iii) The union of any two PPNS closed set is an PPNS closed set over X.

Proof:

It is obviously true.

Theorem : 4. 4

Let (X, τ, M) be a be a PPNSTS over X.and Let $A_M \in$ Penta Partitioned Neutrosophic Soft topological space .Then the following properties hold.

(i) PPNSInt $(A_M) \subseteq A_M$

(ii) $A_M \subseteq B_M$ implies PPNSInt $(A_M) \subseteq$ PPNSInt (B_M) .

(iii) PPNSInt $(A_M) \in \tau$.

(iv) A_M is a PPNS open set implies PPNSInt $(A_M) = A_M$.

(v) Pinsent (PPNSInt (A_M)) = PPNSInt (A_M)

(vi) PPNSInt $(\phi_M) = \phi_M$, QNSInt $(X_M) = X_M$.

Proof:

(i) and (ii) are obviously true.

(iii) Obviously $\sqcup$ $\{B_M \in \tau: B_M \subseteq A_m\} \in \tau$

Note that $\sqcup$ $\{B_M \in \tau: B_M \subseteq A_m\}$ = PPNSInt (A_M)

$\therefore$ PPNSInt $(A_M) \in \tau$

(iv) Necessity: Let A_M be a PPNS open set. ie., $A_M \in \tau$. By (i) and (ii) PPNSInt $(A_M) \subseteq A_m$.

Since $A_M \in \tau$ and $A_M \subseteq A_m$

Then $A_M \subseteq \sqcup \{B_M \in \tau: B_M \subseteq A_m\}$= PPNSInt (A_M)

$A_M \subseteq$ PPNSInt (A_M)

Thus PPNSInt = A_m.

Sufficiency: Let PPNSInt(A_m) = A_m

By (iii) PPNSInt$(A_m) \in \tau$, ie., A_m is a PPNS open set.

(v) To prove PPNSInt (PPNSInt (A_m)) = PPNSInt (A_m)

By (iii) PPNSInt $(A_m) \in \tau$.

By (iv) PPNSInt (QNSInt (A_m)) = PPNSInt (A_m).

(vi) We know that ϕ_M and X_M are in τ

By (iv) PPNSInt $(\phi_M) = \phi_M$, PPNSInt $(X_M) = X_M$.

Theorem : 4. 5

Let (X, τ, M) be a be a PPNSTS over X and Let A_M is in the PPNSTS .Then the following properties hold.

(i) $A_M \subseteq$ PPNSCl(A_M)

(ii) $A_M \subseteq B_M$ implies PPNSCl $(A_M) \subseteq$ PPNSCl (B_M) .

(iii) PPNSCl $(A_M)^c \in \tau$.

(iv) A_M is a PPNS closed set implies PPNSCl $(A_M) = A_M$.

(v) PPNSCl (PPNSCl (A_M)) = PPNSCl (A_M)

(vi) PPNSCl $(\phi_M) = \phi_M$, PPNSCl $(X_M) = X_M$.

Proof:

(i) and (ii) are obviously true.

(iii) By theorem, PPNSInt $({A_M}^c) \in \tau$

Therefore [PPNSCl $(A_M)]^c = (\sqcap \{B_M \in \tau^c: B_M \subseteq A_m\})^c$

$= \sqcup \{B_M \in \tau : B_M \subseteq {A_m}^c\}$ = PPNSInt $({A_M}^c)$

∴ [PPNSCl $(A_M)]^c \in \tau$

(iv) Necessity:

By theorem, $A_M \subseteq$ PPNSCl(A_M)

Let A_M be a PPNS closed set. ie., $A_M \in \tau^c$.

Since $A_M \in \tau \; and \; A_M \subseteq A_m$

[PPNSCl $(A_M)] = \sqcap \{B_M \in \tau^c : A_M \subseteq B_m\} \subseteq \{B_M \in \tau^c : A_M \subseteq A_m\}$

PPNSCl $(A_M) \subseteq A_m$.

Thus A_m =PPNSCl(A_m)

Sufficiency: This is obviously true by (iii)

(v) and (vi) can be proved by (iii) and (iv) .

Theorem :4. 6

Let (X, τ, M) be a be a PPNSTS over X and Let A_M, B_M are in PPNSTS X_M .Then the following properties hold.

(i) PPNSInt $(A_M) \sqcap$ PPNSInt (B_M) = PPNSInt $(A_M \sqcap B_M)$

(ii) PPNSInt $(A_M) \sqcup$ PPNSInt $(B_M) \subseteq$ PPNSInt $(A_M \sqcup B_M)$

(iii) PPNSCl $(A_M) \sqcup$ PPNSCl $(B_M) \subseteq$ PPNSCl $(A_M \sqcup B_M)$

(iv) PPNSCl $(A_M \sqcup B_M) \subseteq$ PPNSCl $(A_M) \sqcap$ PPNSCl (B_M)

(v) (PPNSInt $(F_E))^c$ = PPNSCl $({F_E}^c)$

(vi) (PPNSCl $(F_E))^c$ = PPNSInt $({F_E}^c)$

Proof

(i) Since $A_M \sqcap B_M \subseteq A_m$ for any m in M

By theorem, PPNSInt $(A_M \sqcap B_M) \subseteq$ PPNSInt (A_M)

Similarly, PPNSInt $(A_M \sqcap B_M) \subseteq$ PPNSInt (B_M)

PPNSInt $(A_M \sqcap B_M) \subseteq$ PPNSInt $(A_M) \sqcap$ PPNSInt(B_M)

By theorem, PPNSInt $(A_M) \subseteq A_M$ and PPNSInt $(B_M) \subseteq B_M$

Thus PPNSInt $(A_M \sqcap B_M) \subseteq A_M \sqcap B_M$

Therefore, PPNSInt $(A_M) \sqcap$ PPNSInt (B_M) = PPNSInt $(A_M \sqcap B_M)$

Similarly we can prove (ii),(iii) and (iv).

v) (PPNSInt $(F_E))^c = (\sqcup \{B_M \in \tau : B_M \subseteq A_m\})^c$

$= \sqcap \{B_M \in \tau^c: {A_M}^c \subseteq B_m\}$

$= \text{PPNSCl}\,({A_M}^c)$

Similarly we can prove (vi)

Example : 4. 7

Let X = { ς_1, ς_2}, M = {m_1, m_2} and Let A_M, B_M, C_M, D_M be PPNS where

A(m_1) = $\{< \varsigma_1, .5, .6, .1, .7, .2 > < \varsigma_2, .7, .5, .6, .4, .1 >\}$

A(m_2) = $\{< \varsigma_1, .8, .7, .5, .6, .3 > < \varsigma_2, .2, .3, .4, .6, .7 >\}$

B(m_1) = $\{< \varsigma_1, .2, .3, .4, .5, .1 > < \varsigma_2, .6, .4, .5, .3, .2 >\}$

B(m_2) = $\{< \varsigma_1, .5, .8, .3, .2, .4 > < \varsigma_2, .5, .8, .2, .7, .9 >\}$

C(m_1) = $\{< \varsigma_1, .1, .2, .5, .3, .4 > < \varsigma_2, .7, .8, .6, .9, .1 >\}$

C(m_2) = $\{< \varsigma_1, .3, .8, .4, .7, .6 > < \varsigma_2, .7, .6, .3, .5, .4 >\}$

$\tau = \{A_M, B_M, C_M, D_M, \phi_M, X_M\}$ is an PPNST on X.

i) PPNSInt (B_M) = ϕ_M= QNSInt (C_M)

Then $B_M \sqcup C_M = A_M$

PPNSInt (B_M) ⊔ PPNSInt (C_M) =$\phi_M \sqcup \phi_M$=ϕ_M

And QNSInt ($B_M \sqcup C_M$) = PPNSInt (A_M)= A_M

PPNSInt (B_M) ⊔ PPNSInt (C_M) $\neq$ PPNSInt ($B_M \sqcup C_M$)

ii) PPNSCl $(B_M)^c$ = (PPNSCl $(B_M))^c$ = ${\phi_M}^c = X_M$

Similarly PPNSCl $(C_M)^c$= X_M

PPNSCl $(B_M)^c$ ⊓ PPNSCl $(C_M)^c$= $X_M \sqcap X_M = X_M$

Similarly PPNSCl (${B_M}^c \sqcap {C_M}^c$) = PPNSCl $(B_M \sqcap C_M)^c$

= PPNSInt $(B_M \sqcup C_M)^c$

= ${A_M}^c$

QNSCl (${B_M}^c \sqcap {C_M}^c$) ≠ PPNSCl $(B_M)^c$ ⊓ (PPNSCl $(B_M)]^c$

5. CONCLUSION

In this study, we have successfully extended the concept of Penta Partitioned Neutrosophic Soft Set (PPNSS) into the framework of Penta Partitioned Neutrosophic Soft Topological Space (PPNSTS). By integrating the principles of topology with the advanced structure of neutrosophic soft sets, we have established a robust mathematical model capable of addressing complex problems involving uncertainty, indeterminacy, and inconsistency.

The properties and foundational elements of PPNSTS, such as open and closed sets, neighbourhood systems, bases, and subspaces, have been rigorously defined and analyzed. These theoretical constructs not only enhance the flexibility and applicability of neutrosophic soft sets but also provide a deeper understanding of their behaviour in a topological context.

The PPNSTS framework opens new avenues for research and practical applications, particularly in fields like decision-making, data classification, and artificial intelligence, where uncertainty is a predominant challenge. By offering a systematic and granular approach to uncertainty modeling, this work lays a solid foundation for future studies in soft topology and neutrosophic systems.

In conclusion, the introduction of PPNSTS marks a significant advancement in the intersection of neutrosophic theory and topology, providing a versatile tool for both theoretical exploration and real-world problem-solving. Further research could focus on extending this framework to dynamic systems, hybrid models, and interdisciplinary applications.

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PERFORMANCE ANALYSIS OF IMAGE FILTERING TECHNIQUES FOR MEDICAL IMAGE DENOISING USING NEUTROSOPHIC DOMAIN

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Abstract:
Image processing is the process of enhancing images that are digital or extracting information from them. The proposed approach initially denoises the image in the normal domain using all three filter types. The image is then turned into a Neutrosophic Set, which is classed as True (T), Indeterminacy (I), or False (F). The entropy of the Neutrosophic Set is used to assess the image's level of indeterminacy. PSNR, RMSE, and MSE are used to assess performance on datasets from the brain, eye, and lung. The results show that Neutrosophic filters outperform standard approaches, notably in dealing with mixed noise and maintaining image quality. Neutrosophic sets improve denoising by resolving uncertainty in noisy data, making them suitable for medical imaging applications. These methods demonstrate the efficiency of Neutrosophic filters in improving image clarity and diagnostic accuracy.

***Keywords:** Image Processing, Denoising, Filtration, Neutrosophic Sets, Medical Imaging.*

1. Introduction

Image processing is a significant technique in many areas, particularly medical imaging, where image quality has a substantial impact on diagnosis and treatment plans[1]. Medical imaging of the brain, eye, and lungs frequently contains noise added during acquisition, which can make precise interpretation difficult. Noise, particularly Gaussian noise, is one of the most common forms seen in medical images. To solve this issue, image denoising techniques are used to improve image quality and detail. Filters such as Average, Median, and Wiener have been frequently utilized for noise reduction, with each providing distinct benefits in terms of performance and computational efficiency[2]. The filter performance is evaluated in two domains: the original domain and the Neutrosophic Set (NS) domain, which has a more precise representation of the image's truth, false, and indeterminacy components[3]. The analysis evaluates the performance of three image denoising filters Average, Median, and Wiener, on medical image datasets (brain, eye, and lung) with Gaussian noise. It analyzes the performance of these filters in two domains: the ordinary domain and the Neutrosophic Set (NS) domain (where image data is represented with added uncertainty and indeterminacy). The objective is to determine the optimum image processing approach for improving medical images by assessing domain combinations using metrics such as Peak Signal-to-Noise Ratio (PSNR), Root Mean Square Error (RMSE), and Mean Square Error (MSE). These measures aid in determining the most effective method for reducing noise and improving image quality in diagnostic applications.

2. Preprocessing

Suitable preprocessing plays an essential role in image processing, especially for medical imaging, in which accurate diagnosis requires clear, high-quality images [4]. Initial processing prepares images to provide filtration by classifying them, allowing them to be used in subsequent analyses. To maintain consistency across datasets, each image in the brain, lung, and eye datasets was pre-processed, which included resizing, normalization, and controlled noise addition [5].

2.1.NOISE REDUCTION FILTERS

2.1.1. GAUSSIAN NOISE

Gaussian noise, which is created by random variations in the signal, is represented by adding random numbers to an image [6]. This noise follows a normal Probability Density Function (pdf). It might also be known as the Gaussian distribution.

3. NEUTROSOPHIC SET

A neutrosophic set is a fuzzy set extension that uses truth, indeterminacy, and falsehood as its three membership functions to describe uncertainty [7]. It is used to simulate situations involving inadequate, inconsistent, or uncertain data. The neutrosophic set for an element x in a universe U is defined by three values:

- True (T(x)) is the degree of truth, indicating how much x belongs to the set.
- Falsity (F(x)) is the degree to which x does not belong in the set.
- Indeterminacy (I(x)) refers to the uncertainty or undecidability around x's membership.

These values vary from 0 to 1, and the total of truth, indeterminacy, and falsity does not always equal 1[8]. The neutrosophic set is especially helpful in image processing and decision-making situations where there is ambiguity or uncertainty, like in medical image analysis.

A. Convert the image into a Neutrosophic

To convert an image into neutrosophic, each of the pixels is assigned three membership values: truth (T(p,q)), indeterminacy (I(p,q)), and falsity (F(p,q). The truth is computed using the local mean intensity of the pixel's neighborhood, the indeterminacy is calculated using the difference between the pixel intensity and the local mean, and the falsity is the inverse of the truth. The values constitute a 3D matrix with dimensions ($m \times n \times 3$). Each pixel has a triplet reflecting its truth, indeterminacy, and falsity. These are defined as follows:

$$\mathrm{T}\,(p,q) = \frac{\bar{L}(p,q) - \bar{L}_{min}}{\bar{L}_{max} - \bar{L}_{min}} \quad (1)$$

$$\bar{L}_{max} = \max \bar{L}(p,q) \quad (2)$$

$$\bar{L}_{min} = \min \bar{L}(p,q) \quad (3)$$

$$\bar{L}(p,q) = \frac{1}{v\,X\,v} \sum_{r=p-\frac{v}{2}}^{p+\frac{v}{2}} \sum_{s=q-\frac{v}{2}}^{q+\frac{v}{2}} L(r,s) \quad (4)$$

$$\mathrm{M}\,(p,q) = \frac{\epsilon(p,q) - \epsilon_{min}}{\epsilon_{max} - \epsilon_{min}} \quad (5)$$

$$\epsilon(p,q) = \mathrm{abs}\,(\mathrm{L}\,(p,q) - \mathrm{L}(p,q)) \quad (6)$$

$$\epsilon_{max} = \max \epsilon(p,q) \quad (7)$$

$$\epsilon_{min} = \min \epsilon(p,q) \quad (8)$$

$$\mathrm{F}(p,q) = 1 - \mathrm{T}(p,q) \quad (9)$$

where $\bar{L}(p,q)$ is the average intensity of the input data in a window that occupies the position. $\bar{L}_{min}$ and $\bar{L}_{max}$ are the minimum and maximum average intensity in the region. U(p, q) function calculates the complement of the normalized intensity. It is commonly employed in applications where the opposite scale is required.

4. FILTERING TECHNIQUES

4.1. MEDIAN FILTER

A Median Filter represents a non-linear filtration process that reduces noise, particularly random noise as salt and pepper, preserving its edges. This operates by substituting the value of each pixel with the median of each from its neighbours across a specified frame. Using this technique effectively removes both minor and large amounts of noise without causing image edges to blur [9]. Yet, it might not be best suited for captures containing individual components because it may cause some loss of minor elements. Regardless, it is straightforward to build and is frequently employed during preprocessing to improve the quality of images while preserving critical components [10].

4.2. AVERAGE FILTER

In image processing, an average filter, sometimes referred to as a mean filter, is a kind of linear filter that smoothes and reduces noise. It lowers sharpness and becomes flatter regional variations [11]. The algorithm replaces each pixel with the average of pixels in a square window surrounding it. The kernel, which usually occupies an odd dimension (e.g., 3x3 or 5x5), moves over the image and computes the average of the pixels that are adjacent to each place. This procedure reduces random noise, such as salt-and-pepper noise, by blurring the image and smoothing sharp transitions. While successful at reducing noise, the average filter can create blurring, resulting in a loss of clarity and sharpness in the image, particularly around the edges. It is computationally simple and frequently used, however, it may not maintain crucial features like edges or more complicated filters.

4.3.WIENER FILTER

By comparing a signal's noise level to an approximation of the perfect noiseless signal, the Wiener filter a ims to reduce noise in the signal [12]. The Wiener filter reduces noise by determining the target and predicted procedure and lowering the difference between them [13]. The Wiener filter filters out noise that might degrade signal quality. This filter removes additive noise while also inverting blurring [14].

5. THE INDICATORS OF EVALUATION

The PSNR, MSE, and RMSE are used in this work to assess the filter's efficacy.

5.1. PEAK SIGNAL TO NOISE RATIO (PSNR)

The PSNR is a parameter used to evaluate the image quality of regenerated or denoised representations [15]. This contrasts with the strongest achievable signals with noise created through the processing. A higher essential PSNR value signifies higher image quality. The following is an expression for the PSNR:

$$\mathrm{PSNR} = 10.log_{10}\left(\frac{MAX^2}{MSE}\right) \qquad (10)$$

That is, MAX represents the image's maximum potential pixel value.

The Mean Squared Error (MSE) between original and processed images [16]. A higher PSNR value denotes superior image quality.

5.2. MEAN SQUARE ERROR (MSE)

The MSE is a statistic that determines the average squared difference between the original and processed images. It quantifies the difference (or inaccuracy) between the two photographs [17]. A lesser mean square error suggests that the result of processing is more similar to the actual[18].

$$\text{MSE} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n} \qquad (11)$$

5.3. THE ROOT MEAN SQUARE ERROR (RMSE)

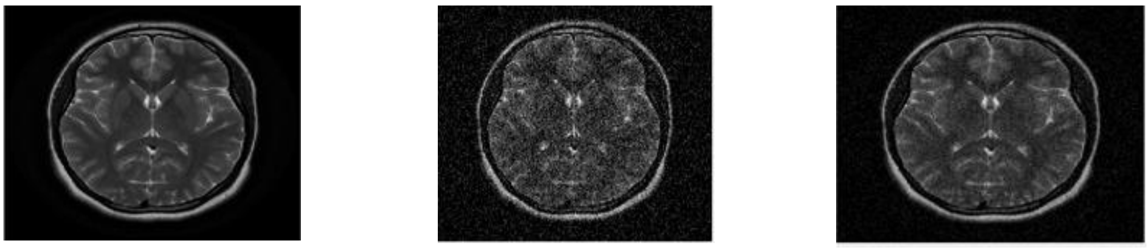
The RMSE is the square root of the MSE, which provides an absolute measure of error. It determines the value of the error to an identical measure with the image's pixels[19]. A lower RMSE indicates that the original and denoised photos are more similar. The formula for RMSE is:

$$\text{RMSE} = \sqrt{\frac{\sum(y_i - \hat{y}_i)^2}{N - P}} \qquad (12)$$

Using the Root Mean Square Error (RMSE), we compare the observed value y_i to the expected value $\hat{y}_i$ [20]. Because the root mean square error is always positive, a lower RMSE number suggests that the model fits the data more well.

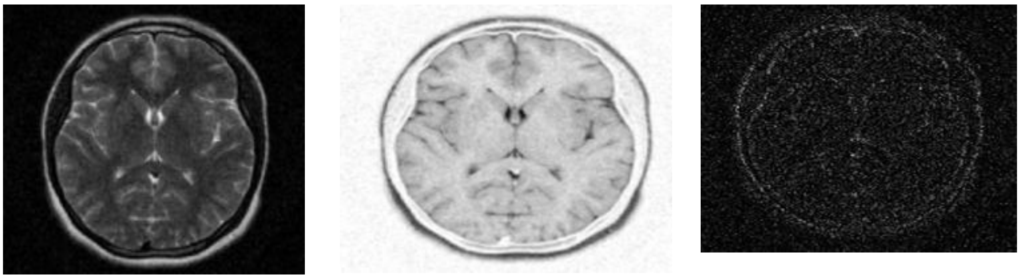
6. EXPERIMENTAL RESULTS AND COMPARATIVE ANALYSIS

The image undergoes denoising in both the normal domain and the neutrosophic domain through the use of the Median Filter as well as average and wiener filters. After the medical image is tainted by the addition of Gaussian noise, it is subjected to denoising processes. The tasks were performed using MATLAB. Once the image preprocessing is complete, it is altered into the Neutrosophic domain, as shown in the results. Figure 1 displays the brain image after applying the median filter during preprocessing. Figure 2 shows the brain image in the Neutrosophic Set (NS) domain, also with the median filter applied. Figure 3 illustrates the pre-processed lung image after the median filter has been used, highlighting its noise reduction effectiveness. Following the median filter application, Figure 4 presents the lung image transformed into the Neutrosophic Set (NS) domain, emphasizing the components of truth, falsehood, and indeterminacy. Figure 5 exhibits the pre-processed eye image containing Gaussian noise, with the median filter applied. Figure 6 shows the eye image in the NS domain after median filtering. Figures 7 and 8 feature a brain image processed with the Average filter and its representation in the NS domain. Figures 9 and 10 present an eye image processed with the Average filter alongside its NS domain representation. Figures 11 and 12 showcase a lung image filtered using the Average filter, including its NS domain representation. Figures 13 and 14 display a brain image processed by the Wiener filter, along with its NS domain representation. Figures 15 and 16 depict a lung image processed with the Wiener filter and its corresponding NS domain representation. Finally, Figures 17 and 18 illustrate an eye image processed through the Wiener filter, along with its NS domain representation are shown.



(a) Original Image (b) Gaussian Noise Image (c) Median Filtered Image

Fig 1 demonstrates a pre-processed brain image using the median filter.

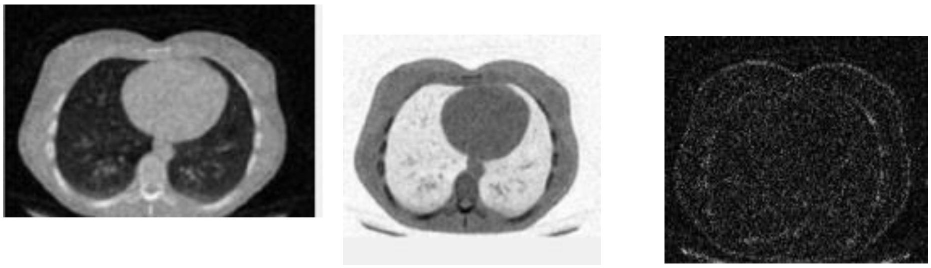


(a) True (b) False (c) Indeterminacy

Fig 2 demonstrates the brain image in the NS Domain with the median filter

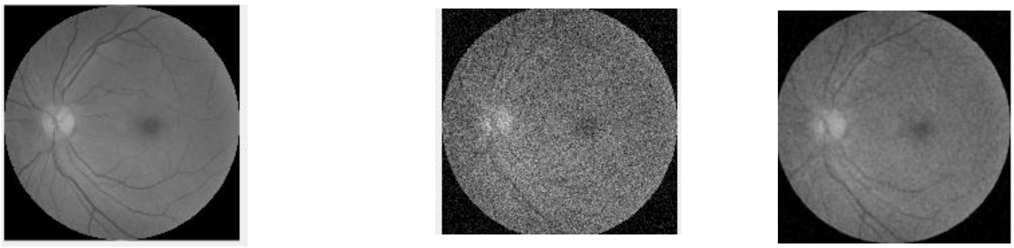
(a) Original Image (b) Gaussian Noise Image (c) Median Filtered Image

Fig 3 demonstrates a pre-processed lung image using the median filter



(a) True (b) False (c) Indeterminacy

Fig 4 demonstrates the lung image in the NS Domain with the median filter



(a)Original Image (b) Gaussian Noise Image (c) Median Filtered Image

Fig 5 demonstrates a pre-processing eye image using the median filter

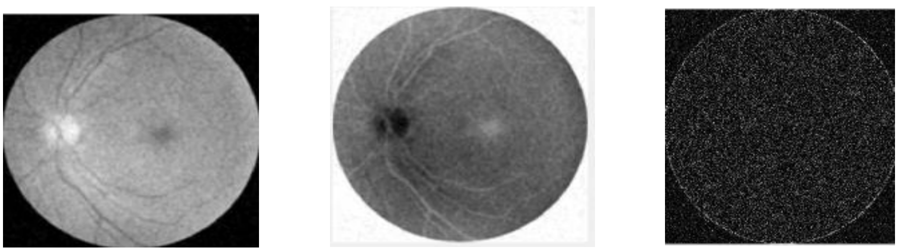
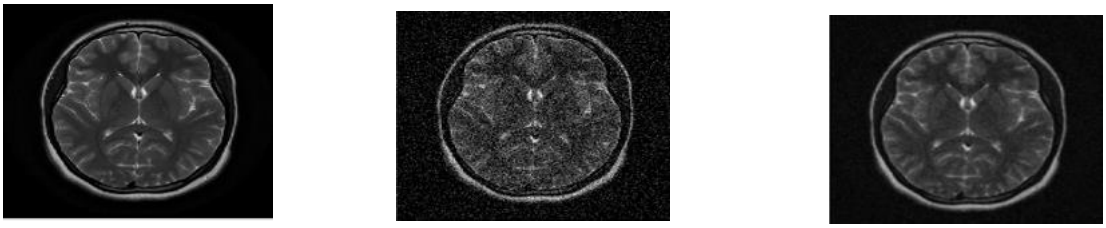
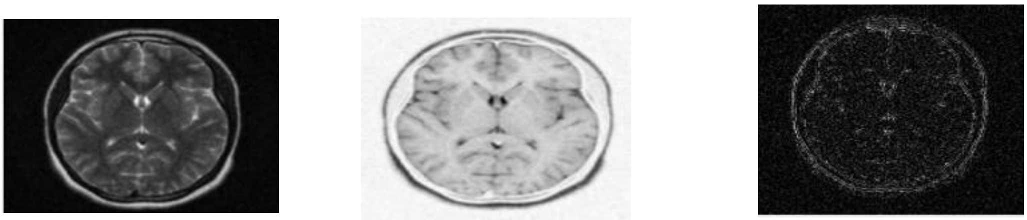


Fig 6 demonstrates the eye image in the NS Domain with the median filter



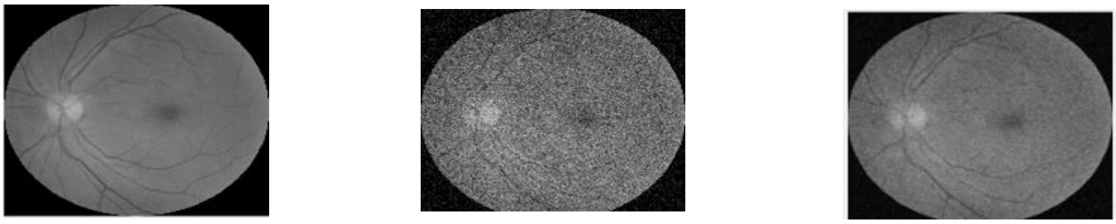
(a)Original Image (b) Gaussian Noise Image (c) Average Filtered Image

Fig 7 demonstrates a pre-processing brain image using the average filter



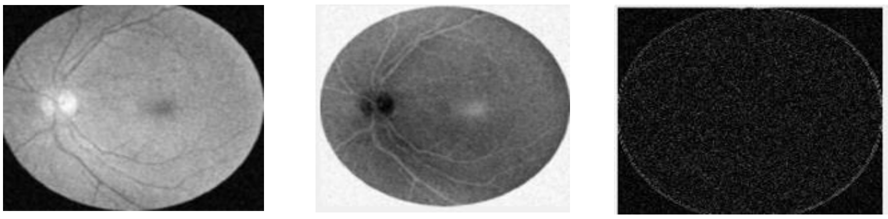
(a) True (b) False (c) Indeterminacy

Fig 8 demonstrates the brain image in the NS Domain with the average filter



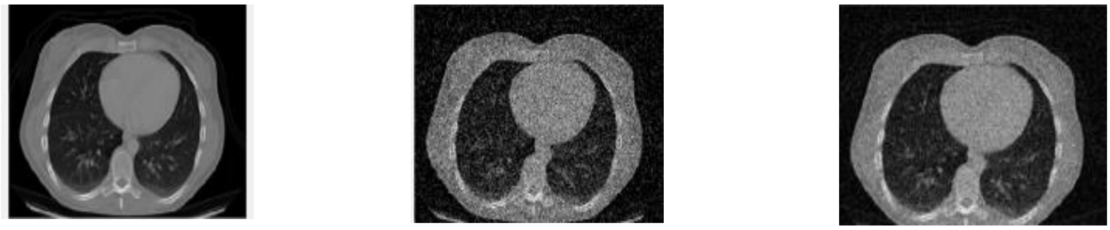
(a) Original Image (b) Gaussian Noisy Image (c) Average Filtered Image

Fig 9 demonstrates pre-processing eye image using the average filter



(a) True (b) False (c) Indeterminacy

Fig 10 demonstrates the eye image in the NS Domain with the average filter



(a) Original Image (b) Gaussian Noise Image (c) Average Filtered Image

(a) True (b) False (c) Indeterminacy

Fig 11 demonstrates pre-processing lung image using the average filter

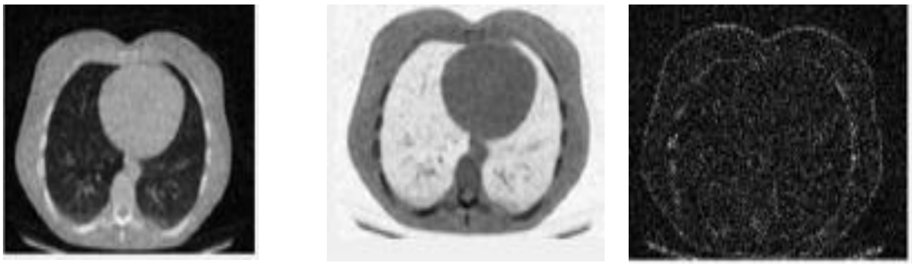
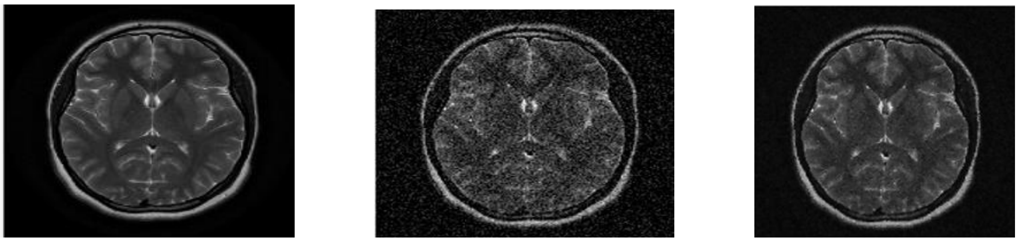
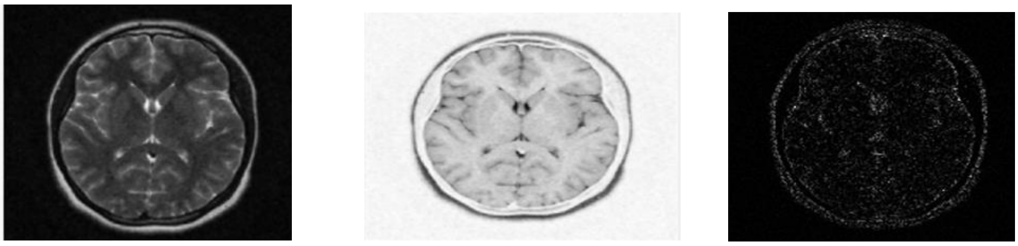


Fig 12 demonstrates the lung image in the NS Domain with the wiener filter



(a) Original Image (b) Gaussian Noise Image (c) Wiener Filtered Image

Fig 13 demonstrates pre-processing brain image using the wiener filter



(a) True (b) False (c) Indeterminacy

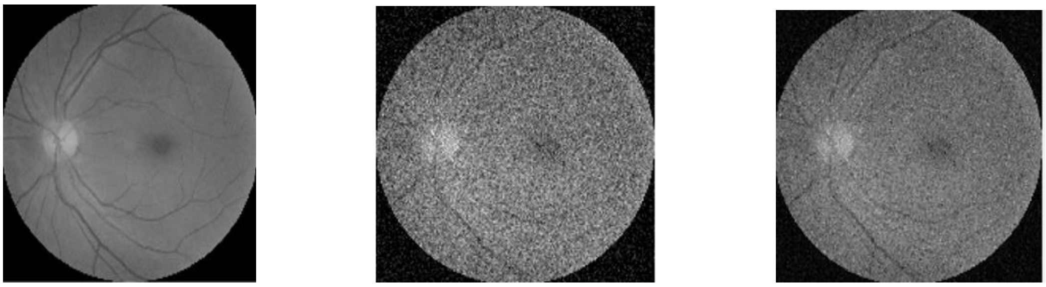
Fig 14 demonstrates the brain image in the NS Domain with the wiener filter

(a) Original Image (b) Gaussian Noise Image (c) Wiener Filtered Image

Fig 15 demonstrates pre-processing lung image using the wiener filter

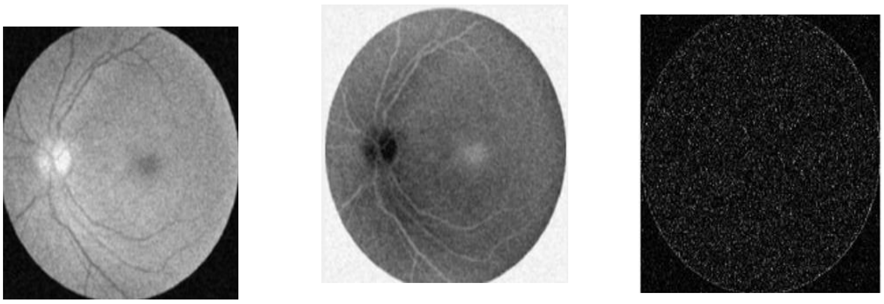
(a) True (b) False (c) Indeterminacy

Fig 16 demonstrates the lung image in the NS Domain with the wiener filter



(a) Original Image (b) Gaussian Noisy Image (c) Wiener Filtered Image

Fig 17 demonstrates pre-processing lung image using the wiener filter



(a) True (b) False (c) Indeterminacy

Fig 18 demonstrates the eye image in the NS Domain with the wiener filter

We analyzed each filter's efficiency across several window sizes (ranging from 2x2 to 6x6) for each dataset, and the PSNR values are provided in Table 1. The noise utilized in the evaluation was Gaussian. Table 2 provides the RMSE values, whereas Table 3 shows the MSE values, which provide further information about the performance of each filtering strategy in terms of error metrics. The table displays the results acquired using several filters in two distinct domains, demonstrating each filter's performance across a range of window sizes and domains.

Filters	Window size	Normal			Neutrosophic		
		Brain	Eye	Lung	Brain	Eye	Lung
Median	2x2	21.5601	21.0702	21.0442	25.9562	25.2223	24.9792
	3x3	20.9360	20.4766	20.6291	28.7297	27.9516	27.7868
	4x4	20.7023	20.2731	20.1861	29.8540	29.3192	27.7187

	5x5	20.6711	20.2541	20.2492	31.2066	31.1950	28.9539
	6x6	20.4947	20.1440	19.9279	30.5022	31.0372	27.7699
Average	2x2	20.1099	20.2577	20.0014	25.9043	25.6052	25.2782
	3x3	20.7619	20.9141	20.8008	28.1631	28.1660	27.8916
	4x4	20.8231	21.0671	20.6351	28.2332	28.9456	27.1436
	5x5	20.9143	21.1547	20.7396	28.6503	29.5537	27.3213
	6x6	20.8168	21.1423	20.4832	27.9753	29.4241	26.2270
Wiener	2x2	24.4927	24.0731	24.0409	24.5824	24.2481	24.3484
	3x3	22.8537	22.3531	22.3715	26.4539	26.7064	26.6232
	4x4	22.1752	21.6951	21.6807	27.1806	28.1550	27.4650
	5x5	21.8629	21.3299	21.3640	27.6451	29.1309	27.8725
	6x6	21.6434	21.0645	21.1542	27.7915	29.7406	27.8200

Table 1: PSNR values of brain, eye, and lung datasets

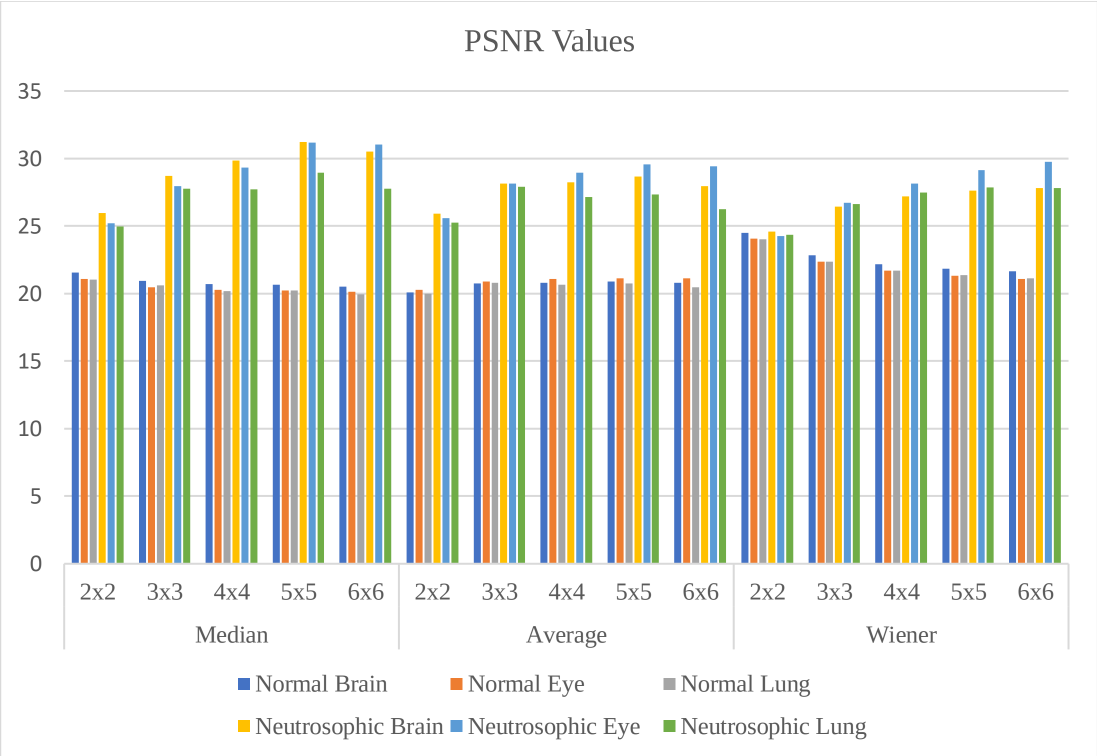


Fig. 19. PSNR values are graphically shown.

Filters	Window size	Normal			Neutrosophic		
		Brain	Eye	Lung	Brain	Eye	Lung
Median	2x2	4.6432	4.5902	4.5873	2.8447	3.9773	1.3257
	3x3	4.5756	4.5251	4.5419	3.3338	1.2085	2.4039
	4x4	4.5499	4.5025	4.4928	3.2004	3.7212	1.4858
	5x5	4.5465	4.5004	4.4999	3.0178	3.0272	3.9579
	6x6	4.5271	4.4882	4.4640	3.6107	3.1561	1.4242
	2x2	3.0987	4.4970	3.0999	2.9217	1.3745	1.8876

Average	3x3	3.9216	6.0900	4.0911	2.9629	3.9595	2.2791
	4x4	4.4369	4.0884	5.0929	3.8828	3.1045	4.2036
	5x5	5.3965	3.0875	2.0918	3.4194	2.4889	2.9766
	6x6	4.0910	2.0876	4.9458	2.1806	1.6165	1.4505
Wiener	2x2	4.9490	5.9064	3.9031	1.0458	1.6362	1.4568
	3x3	4.7805	3.7279	2.7298	1.1294	2.7819	2.8953
	4x4	4.7091	4.6577	6.6562	1.1560	3.9721	2.7966
	5x5	4.6757	4.6184	5.6221	2.5750	3.9123	2.3318
	6x6	4.6522	4.5896	2.5993	2.3982	2.3082	1.3643

Table 2: RMSE values of brain, eye, and lung datasets

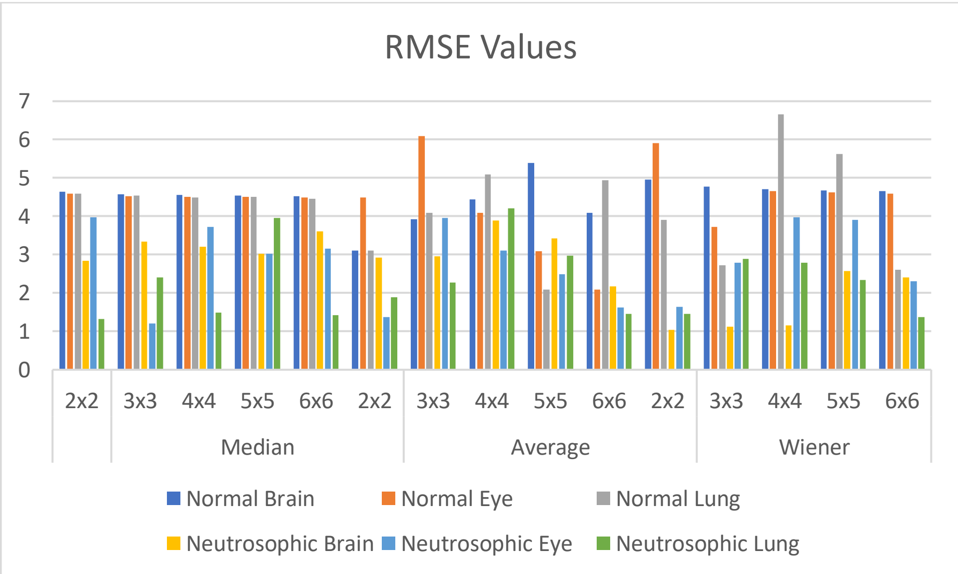


Fig. 20. RMSE values are graphically shown.

Filters	Window size	Normal			Neutrosophic		
		Brain	Eye	Lung	Brain	Eye	Lung
Median	2x2	4.5401	5.0823	5.1127	1.6498	1.9536	2.0522
	3x3	5.2415	5.8266	5.6255	4.1193	1.2132	1.0824
	4x4	5.5314	6.1061	6.2296	4.2480	4.0605	1.0995
	5x5	5.5714	6.1329	6.1397	4.2508	4.3825	2.7335
	6x6	5.8023	6.2904	6.6113	3.9234	5.2101	1.0866
Average	2x2	0.0097	4.3094	2.3099	1.6697	2.7887	1.9286
	3x3	3.3483	3.0081	4.3083	2.2587	3.1923	1.0566
	4x4	4.3082	2.0078	2.2186	2.6697	1.8925	1.2552
	5x5	3.0081	2.2376	3.6584	2.7258	1.0617	1.2048
	6x6	4.0082	7.0076	2.9889	2.0364	4.2443	1.5501
Wiener	2x2	2.3110	5.5453	5.5643	1.2637	4.4449	2.3891
	3x3	3.3709	3.7823	3.7664	1.4712	1.3881	1.4149
	4x4	3.9399	4.4011	4.4157	1.2445	3.4429	1.1656

	5x5	4.2343	4.7871	4.7498	1.1183	3.4303	1.0612
	6x6	4.4538	5.0889	4.9848	1.0812	4.0271	1.0741

Table 3: MSE values of brain, eye, and lung datasets

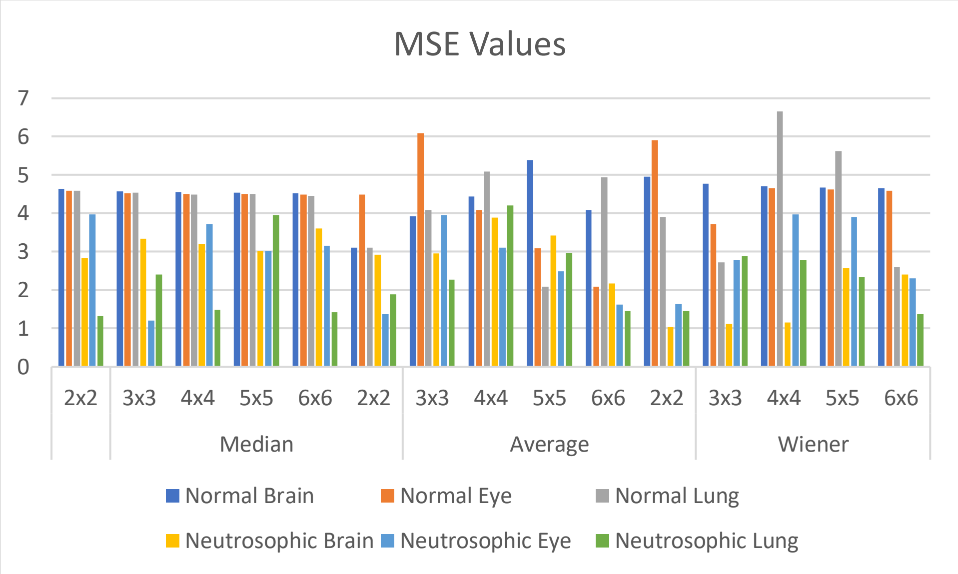


Fig 21 MSE values are graphically shown.

7. CONCLUSION

The image denoising method utilizes the Median filter, Average filter, and Wiener filter to handle Gaussian noise in three datasets: brain, eye, and lung. The filters are used in both the Neutrosophic and Normal domains. However, the results show that denoising is more effective in the Neutrosophic domain, resulting in better noise reduction. The use of Neutrosophic sets improved the results by addressing the uncertainty and indeterminacy inherent in noisy data, making them especially useful for medical imaging with mixed noise types. The addition of Neutrosophic sets improved performance even more by addressing uncertainty and indeterminacy in noisy data, making them especially useful for medical imaging with complex or mixed noise. Neutrosophic filtering not only reduced noise more effectively, but it also preserved small features, resulting in enhanced image quality. Thus, Neutrosophic-based filters are the most effective way to improve medical image, improving the capacity for diagnosis by providing clearer and better-quality images.

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FERMATEAN QUADRIPARTITIONED NEUTROSOPHIC LIE ALGEBRA

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Abstract: A Fermatean Quadripartitioned Neutrosophic Set is a powerful general formal framework that generalizes the concept of Fermatean Neutrosophic sets and Quadripartitioned Neutrosophic Sets. In this paper, we apply the notion of Fermatean Quadripartitioned Neutrosophic (FQN) sets to Lie algebras. We develop the concepts of FQN Lie subalgebras and FQN Lie ideals. We describe some interesting results of FQN Lie ideals.

Keywords: FQN Lie ideal, FQN sets, FQN Lie subalgebra

1. Introduction

The concept of Lie groups was first introduced by Sophus Lie in nineteenth century through his studies in geometry and integration methods for differential equations. Lie algebras were also discovered by him when he attempted to classify certain smooth subgroups of a general linear group. The importance of Lie algebras in mathematics and physics has become increasingly evident in recent years. In mathematics, Lie theory remains a robust tool for studying differential equations, special functions and perturbation theory. It's noted that Lie theory has applications not only in mathematics and physics but also in diverse fields like continuum mechanics, cosmology and life sciences. Lie algebra has been utilized by electrical engineers, mainly within the mobile robot control [6]. Lie algebra has also been accustomed solve the problems of computer vision. Fuzzy structures are related to theoretical soft computing, especially Lie algebras and their different classifications, have numerous applications to the spectroscopy of molecules, atoms and nuclei. One amongst the key concepts within the applying of Lie algebraic method in physics is that of spectrum generating algebras and their associated dynamic symmetries. The most important advancements within the fascinating world of fuzzy sets started with the work of renowned scientist Zadeh [19] with new directions and ideas. Smarandache and Wang et al. [7] defined SVN sets as a generalization of fuzzy sets and intuitionistic fuzzy sets [4]. Algebraic structures have a major place with vast applications in various disciplines. Neutrosophic set has been applied to algebraic structures. Fuzzification of Lie algebras has been discussed in [1–3]. The idea of single valued neutrosophic Lie algebra was investigated by Muhammad Akram, Hina Gulzar and Kar Ping Shum[8]. Pentapartitioned neutrosophic set and its properties were introduced by Rama Malik and Surpati Pramanik [17]. During this case, indeterminacy is split into three components: contradiction, ignorance, and an unknown membership function. We have now extended our research during this Pentapartitioned neutrosophic set as a space. Also we introduced the concept of Fermatean Quadripartitioned Neutrosophic set and establish variety of its properties in our previous work. During this paper, we apply the notion of Fermatean Quadripartitioned Neutrosophic (FQN) sets to Lie algebras.

In this paper, We develop the concepts of FQN Lie subalgebras and investigated some of its properties. Furthermore, we have also studied the concept of FQN Lie ideals. We describe some interesting results of FQN Lie ideals.

2. Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper. A Lie algebra[1] is a vector space L over a field F (equal to R or C) on which $L \times L \rightarrow L$ denoted by $(x, y) \rightarrow [x, y]$ is defined satisfying the following axioms:

(L1) $[x, y]$ is bilinear,

(L2) $[x, x] = 0$ for all $x \in L$,

(L3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in L$ (Jacobi identity).

Throughout this paper, L is a Lie algebra and F is a field. We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x, y], z] = [x, [y, z]]$. But it is anti-commutative, i.e., $[x, y] = -[y, x]$. A subspace H of L closed under $[\cdot, \cdot]$ will be called a Lie subalgebra.

A fuzzy set $\mu : L \rightarrow [0, 1]$ is called a fuzzy Lie ideal [1] of L if

(a) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,

(b) $\mu(\alpha x) \geq \mu(x)$,

(c) $\mu([x, y]) \geq \mu(x)$

hold for all $x, y \in L$ and $\alpha \in F$.

Definition: 2. 1[10]

Let R be a space of points(objects). A Fermatean Quadripartitioned Neutrosophic (FQN) set on a non-empty R is characterized by truth membership function $A1: R \rightarrow [0, 1]$, contradiction membership function $A2: R \rightarrow [0, 1]$, ignorance membership function $A3: R \rightarrow [0, 1]$ and false membership function $A4: R \rightarrow [0, 1]$.

Thus, R = {<r, A1(r), A2(r), A3(r),A4 (r)>} satisfies with the following conditions $(A1)^3+(A2)^3+(A3)^3+(A4)^3 \leq 2$.

Definition: 2.2 [8]

An SVN set N = (TN, IN, FN) on Lie algebra L is called an SVN Lie subalgebra

if the following conditions are satisfied*:*

(1) $TN(x + y) \geq \min(TN(x), TN(y))$, $IN(x + y) \geq \min(IN(x), IN(y))$ and

$FN(x + y) \leq \max(FN(x), FN(y))$,

(2) $TN(\alpha x) \geq TN(x)$, $IN(\alpha x) \geq IN(x)$ and $FN(\alpha x) \leq FN(x)$,

(3) $TN([x, y]) \geq \min\{TN(x), TN(y)\}$, $IN([x, y]) \geq \min\{IN(x), IN(y)\}$ and $FN([x, y]) \leq \max\{FN(x), FN(y)\}$

for all $x, y \in L$ and $\alpha \in F$.

Definition: 2.3 [8]

A SVN set N = (TN, IN, FN) on L *is* called an SVN Lie ideal if it satisfies

the conditions (1), (2) and the following additional condition:

Single-valued Neutrosophic Lie algebras

(1) $TN([x, y]) \geq TN(x)$, $IN([x, y]) \geq IN(x)$ and $FN([x, y]) \leq FN(x)$

for all $x, y \in L$.

From (2) it follows that:

(2) $TN(0) \geq TN(x)$, $IN(0) \geq IN(x)$, $FN(0) \leq FN(x)$,

(3) $TN(-x) \geq TN(x)$, $IN(-x) \geq IN(x)$, $FN(-x) \leq FN(x)$.

3. Fermatean Quadripartitioned Neutrosophic Lie subalgebra

We define here Fermatean Quadripartitioned Neutrosophic (FQN) Lie subalgebras and Fermatean Quadripartitioned Neutrosophic Lie ideal.

Definition : 3.1

A FQN set $R = (A1_R, A2_R, A3_R, A4_R)$ is called an FQN Lie subalgebra $\mathcal{L}$ if the following conditions are satisfied:

1) $A1_R(a + b) \geq \min(A1_R(a), A1_R(b))$, $A2_R(a + b) \geq \min(A2_R(a), A2_R(b))$.

   $A3_R(a + b) \leq \max(A3_R(a), A3_R(b))$, $A4_R(a + b) \leq \max(A4_R(a), A4_R(b))$.
2) $A1_R(\beta a) \geq A1_R(a)$, $A2_R(\beta a) \geq A2_R(a)$, $A3_R(\beta a) \leq A3_R(a)$, $A4_R(\beta a) \leq A4_R(a)$ and
3) $A1_R([a, b]) \geq \min(A1_R(a), A1_R(b))$, $A2_R([a, b]) \geq \min(A2_R(a), A2_R(b))$,

$A3_R([a, b]) \leq \max(A3_R(a), A3_R(b))$, $A4_R([a, b]) \leq \max(A4_R(a), A4_R(b))$.

For all a, b $\in \mathcal{L}$ and $\beta \in \mathcal{F}$.

Definition: 3.2

A FQN set R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) on $\mathcal{L}$ is called an FQN Lie ideal if it satisfies the following conditions (1) and (2) and the following additional conditions:

1) $A1_R([a, b]) \geq A1_R(a)$, $A2_R([a, b]) \geq A2_R(a)$, $A3_R([a, b]) \leq A3_R(a)$ and
   a. $A4_R([a, b]) \leq A4_R(a)$

   From (2), it follows that:
2) $A1_R(0) \geq A1_R(a)$, $A2_R(0) \geq A2_R(a)$, $A3_R(0) \leq A3_R(a)$ and $A4_R(0) \leq A4_R(a)$
3) $A1_R(-a) \geq A1_R(a)$, $A2_R(-a) \geq A2_R(a)$, $A3_R(-a) \leq A3_R(a)$ and $A4_R(-a) \leq A4_R(a)$.

Proposition: 3.3

Every FQN Lie ideal is an FQN Lie subalgebra.

We note here that the converse of the above proposition does not hold in general as it can be seen in the following example.

Example: 3.4

Consider $\mathcal{F} = \mathbb{R}$. Let $\mathcal{L} = \mathfrak{R}^3$ = {(a, b, c): a, b, c $\in \mathbb{R}$ } be the set of all three-dimensional real vectors which forms a FQN Lie algebra and define

$\mathfrak{R}^3 \times \mathfrak{R}^3 \to \mathfrak{R}^3$

$[a, b] \to a \times b$,

Where x is the usual cross product. We define an FQN set R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) : $\mathfrak{R}^3 \to [0,1] \times [0,1] \times [0,1] \times [0,1]$ by

$$A1_R(a, b, c) = \begin{cases} 1, & if\ a = b = c = 0, \\ 0.3, & if\ a \neq 0, b = c = 0, \\ 0, & otherwise \end{cases}$$

$$A2_R(a, b, c) = \begin{cases} 1, & if\ a = b = c = 0, \\ 0.2, & if\ a \neq 0, b = c = 0, \\ 0, & otherwise \end{cases}$$

$$A3_R(a, b, c) = \begin{cases} 0, & if\ a = b = c = 0, \\ 0.5, & if\ a \neq 0, b = c = 0, \\ 1, & otherwise \end{cases}$$

$$A4_R(a, b, c) = \begin{cases} 0, & if\ a = b = c = 0, \\ 0.3, & if\ a \neq 0, b = c = 0 \\ 1, & otherwise \end{cases}$$

Then R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) is an FQN Lie subalgebra of $\mathcal{L}$ but R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) is not an FQN Lie ideal of $\mathcal{L}$ since

$A1_R$ ([1,0,0) (1,1,1)]) = $A1_R$ (0, -1, 1) = 0,

$A2_R$ ([1,0,0) (1,1,1)]) = $A2_R$ (0, -1, 1) = 0,

$A3_R$ ([1,0,0) (1,1,1)]) = $A3_R$ (0, -1, 1) = 1,

$A4_R$ ([1,0,0) (1,1,1)]) = $A4_R$ (0, -1, 1) = 1

$A1_R$ (1,0,0) = 0.2, $A2_R$ (1,0,0) = 0.3, $A3_R$ (1,0,0) = 0.5, $A4_R$ (1,0,0) = 0.3.

That is,

$A1_R$ ([1,0,0) (1,1,1)]) $\ngeq$ $A1_R$ (1,0,0),

$A2_R$ ([1,0,0) (1,1,1)]) $\ngeq$ $A2_R$ (1,0,0),

$A3_R$ ([1,0,0) (1,1,1)]) $\nleq$ $A3_R$ (1, 0,0),

$A4_R$ ([1,0,0) (1,1,1)]) $\nleq$ $A4_R$ (1,0,0)

Proposition: 3.5

If R is an FQN Lie ideal of $\mathcal{L}$, then

1) $A1_R$ (0) $\geq$ $A1_R$ (a), $A2_R$ (0) $\geq$ $A2_R$ (a), $A3_R$ (0) $\leq$ $A3_R$ (a), $A4_R$ (0) $\leq$ $A4_R$ (a).
2) $A1_R$ ([a, b]) $\geq$ max {$A1_R$ (a), $A1_R$ (b)},
3) $A2_R$ ([a, b]) $\geq$ max {$A2_R$ (a), $A2_R$ (b)},
4) $A3_R$ ([a, b]) $\leq$ min {$A3_R$ (a), $A3_R$ (b)},
5) $A4_R$ ([a, b]) $\leq$ min {$A4_R$ (a), $A4_R$ (b)},
6) $A1_R$ ([a, b]) = $A1_R$ (- [b, a]) = $A1_R$ ([b, a]),
7) $A2_R$ ([a, b]) = $A2_R$ (- [b, a]) = $A2_R$ ([b, a]),
8) $A3_R$ ([a, b]) = $A3_R$ (- [b, a]) = $A3_R$ ([b, a]),
9) $A4_R$ ([a, b]) = $A4_R$ (- [b, a]) = $A4_R$ ([b, a])

For all a, b $\in \mathcal{L}$.

Proof:

The proof follows from Definition 3.2.

Proposition: 3.6

If {R_i : $i \in J$} is a family of FQN Lie algebra of $\mathcal{L}$, then $\bigcap R_i$ = ($\wedge A1_{Ri} \wedge A2_{Ri} \vee A3_{Ri} \vee A4_{Ri}$) is an FQN Lie ideal of $\mathcal{L}$ where,

$\wedge$ A1 $_{Ri}$ (a) = inf {$\wedge$ A1 $_{Ri}$ (a) : $i \in J,\ a \in \mathcal{L}$ },

$\wedge$ A2 $_{Ri}$ (a) = inf {$\wedge$ A2 $_{Ri}$ (a) : $i \in J,\ a \in \mathcal{L}$ },

$\vee$A3 $_{Ri}$ (a) = sup {$\vee$ A3 $_{Ri}$ (a) : $i \in J,\ a \in \mathcal{L}$ },

$\vee$A4 $_{Ri}$ (a) = sup {$\vee$ A4 $_{Ri}$ (a) : $i \in J,\ a \in \mathcal{L}$ }

Proof:

The proof follows from definition 3.2

Definition: 3.7

Let R = (A1 $_R$, A2 $_R$, A3 $_R$, A4 $_R$) be an FQN Lie subalgebra of $\mathcal{L}$ and let $(\alpha, \beta, \gamma, \delta)$ [0,1] X [0,1] X [0,1] X [0,1] with $\alpha + \beta + \gamma + \delta \leq 2$. Then level subset of R is defined as

$R^{(\alpha,\beta,\gamma,\delta,\vartheta)}$ = { $a \in \mathcal{L}$: A1(a) ≥ α ,A2(a) ≥ β, A3(a) ≤ γ, A 4(a) ≤ δ

are called $(\alpha, \beta, \gamma, \delta)$ level subsets of FQN set R. The set of all $(\alpha, \beta, \gamma, \delta)$ ∈ Im(A1$_R$) X Im(A2$_R$) X Im(A3$_R$) X Im(A4$_R$) such that $\alpha + \beta + \gamma + \delta \leq 2$ is known as image of R = (A1 $_R$, A2 $_R$, A3 $_R$, A4 $_R$) .

Note :

$R^{(\alpha,\beta,\gamma,\delta)}$ = { $a \in \mathcal{L}$: A1(a) ≥ α ,A2(a) ≥ β, A3(a) ≤ γ, A 4(a) ≤ δ },

$R^{(\alpha,\beta,\gamma,\delta)}$ ={ $a \in \mathcal{L}$: A1(a) ≥ α} ∩ { $a \in \mathcal{L}$: $A2$(a) ≥ β} ∩ { $a\in\mathcal{L}$: A3(a) ≤ γ} ∩ { $a \in \mathcal{L}$: A 4(a) ≤ δ} },

$R^{(\alpha,\beta,\gamma,\delta)}$ = U(A1(a),α) ∩ U$'$(A2(a), β) ∩ L(A3(a),γ) ∩ L'(A 4(a), δ).

Theorem: 3.8

An FQN set R = (A1 $_R$, A2 $_R$, A3 $_R$, A4 $_R$) of $\mathcal{L}$ is an FQN lie ideal of $\mathcal{L}$ iff $R^{(\alpha,\beta,\gamma,\delta)}$ is a FQN Lie ideal of $\mathcal{L}$ for every $(\alpha, \beta, \gamma, \delta)$ [0,1] X [0,1] X [0,1] X [0,1] with $\alpha + \beta + \gamma + \delta \leq 2$.

Proposition: 3.9

Let R = (A1 $_R$, A2 $_R$, A3 $_R$, A4 $_R$) be an FQN Lie ideal of $\mathcal{L}$ and (r_1, s_1, t_1, u_1), (r_2, s_2, t_2, u_2) ∈ Im(A1$_R$) X Im(A2$_R$) X Im(A3$_R$) X Im(A4$_R$) with $r_i + s_i + t_i + u_i \leq 2$ for i = 1,2. Then $\mathcal{L}_R^{(r1,s1,t1,u1)} = \mathcal{L}_R^{(r2,s2,t2,u2)}$ if and only if (r_1, s_1, t_1, u_1) = (r_2, s_2, t_2, u_2).

Theorem: 3.10

Let $K_0 \subset K_1 \subset K_2 \subset K_3 \ldots\ldots\ldots \subset K_n$ = L be a chain of FQN Lie ideals of a FQN Lie algebra $\mathcal{L}$. Then there exists an FQN ideal A1$_R$ of $\mathcal{L}$ for which level subsets U(A1(a),α), U$'$(A2(a), β), L(A3(a),γ) and L'(A 4(a), δ) coincide with this chain.

Proof

Let $\{ r_k : k = 0,1,2\ldots,n\}$, $\{s_k : k = 0,1,\ldots.n\}$ and $\{t_k: k = 0,1,2\ldots n\}$, $\{u_k: k = 0,1,2\ldots n\}$, $\{v_k: k = 0,1,2\ldots n\}$be finite decreasing and increasing sequences in [0,1]. Let R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$,$A5_R$) be a FQN set in $\mathcal{L}$ defined by $A1_R(K_0) = r_0$, $A2_R(K_0) = s_0$, $A3_R(K_0) = t_0$, $A4_R(K_0) = u_0$,

$A5_R(K_0) = v_0$, $A1_R(K_l:K_{l-1}) = r_l$, $A2_R(K_l\backslash K_{l-1}) = s_l$, $A3_R(K_l:K_{l-1}) = t_l$, $A4_R(K_l\backslash K_{l-1}) = u_l\ K_l$

$A1_R\ (a + b) \geq r_k = \min\{A1_R\ (a), A1_R\ (b)\}$,

$A2_R\ (a + b) \geq s_k = \min\{A2_R\ (a), A2_R\ (b)\}$,

$A3_R\ (a + b) \leq t_k = \max\{A3_R\ (a), A3_R\ (b)\}$,

$A4_R\ (a + b) \leq u_k = \max\{A4_R\ (a), A4_R\ (b)\}$.

$A1_R\ (\alpha a) \geq r_k = A1_R\ (a)$, $A2_R\ (\alpha a) \geq s_k = A2_R\ (a)$, $A3_R\ (\alpha a) \leq t_k = A3_R\ (a)$,

$A4_R\ (\alpha a) \leq u_k = A4_R\ (a)$.

$A1_R\ ([a, b]) \geq r_k = A1_R\ (a)$, $A2_R\ ([a, b]) \geq s_k = A2_R\ (a)$, $A3_R\ ([a, b]) \leq t_k = A3_R\ (a)$,

$A4_R\ ([a, b]) \leq u_k = A4_R\ (a)$.

For i > j, if $a \in K_i \setminus K_{i-1}$ and $b \in K_j \setminus K_{j-1}$, then $A1_R\ (a) = r_i = A1_R\ (b)$, $A2_R\ (a) = s_i = A2_R\ (b)$,

$A3_R\ (a) = t_j = A3_R\ (b)$, $A4_R\ (a) = u_j = A4_R\ (b)$ and a +b, αa, $[a, b] \in K_I$. Thus

$A1_R\ (a + b) \geq r_i = \min\{A1_R\ (a), A1_R\ (b)\}$,

$A2_R\ (a + b) \geq s\ i = \min\{A2_R\ (a), A2_R\ (b)\}$,

$A3_R\ (a + b) \leq t_j = \max\{A3_R\ (a), A3_R\ (b)\}$,

$A4_R\ (a + b) \leq u_j = \max\{A4_R\ (a), A4_R\ (b)\}$

$A1_R\ (\alpha a) \geq r_i = A1_R\ (a)$, $A2_R\ (\alpha a) \geq s\ i = A2_R\ (a)$, $A3_R\ (\alpha a) \leq t\ j = A3_R\ (a)$,

$A4_R\ (\alpha a) \leq u_j = A4_R\ (a)$.

$A1_R\ ([a, b]) \geq r_i = A1_R\ (a)$, $A2_R\ ([a, b]) \geq s_i = A2_R\ (a)$, $A3_R\ ([a, b]) \leq t_j = A3_R\ (a)$,

$A4_R\ ([a, b]) \leq u_j = A4_R\ (a)$.

Thus, we conclude that R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) is an FQN Lie ideal of a FQN Lie algebra $\mathcal{L}$ and all its non-empty level subsets are FQN Lie ideals.

Since Im ($A1_R$) = $\{r_0, r_1, r_2\ldots.., r_n\}$, Im ($A2_R$) = $\{s_0, s_1, s_2\ldots.., s_n\}$,

Im ($A3_R$) = $\{t_0, t_1, t_2\ldots.., t_n\}$,

Im ($A4_R$) = $\{u_0, u_1, u_2\ldots.., u_n\}$ level subsets of R forms chains:

$U(A1_R, r_0) \subset U(A1_R, r_1) \subset \ldots.. \subset U(A1_R, r_n) = L$,

$U'(A2_R, s_0) \subset U'(A2_R, s_1) \subset \ldots.. \subset U'(A2_R, s_n) = L$,

$L(A3_R, t_0) \subset L(A3_R, t_1) \subset \ldots.. \subset L(A3_R, t_n) = L$,

$L'(A4_R, u_0) \subset L'(A4_R, u_1) \subset \ldots.. \subset L'(A4_R, u_n) = LL$,

Respectively. Indeed

$U(A1_R, r_0) = \{ a \in \mathcal{L} : A1_R\ (a) \geq r_0\} = K_0$,

$U'(A2_R, s_0) = \{ a \in \mathcal{L} : A2_R(a) \geq s_0 \} = K_0$,

$L(A3_R, t_0) = \{ a \in \mathcal{L} : A3_R(a) \leq t_0 \} = K_0$,

$L'(A4_R, u_0) = \{ a \in \mathcal{L} : A4_R(a) \leq u_0 \} = K_0$..

We prove that

$U(A1_R, r_l) = U'(A2_R, s_l) = L(A3_R, t_l) = L'(A4_R, u_l) = K_l$ for $0 \leq l \leq n$.

Clearly, $K_l \subseteq U(A1_R, r_l)$, $K_l \subseteq U'(A2_R, s_l)$, $K_l \subseteq L(A3_R, t_l)$, $K_l \subseteq L'(A4_R, u_l)$.

If $a \in U(A1_R, r_l)$, then $A1_R(a) \geq r_l$ and for $a \notin K_j$, for $j > l$. Hence $A1_R(a) \in \{r_0, r_1, r_2.....,r_l\}$,

Which implies $a \in K_j$ for some $j \leq l$. Since $K_j \subset K_l$, it follows that $a \in K_l$. Consequently,

$U(A1_R, r_l) = K_l$ for some $0 < l \leq n$.

If $a \in U'(A2_R, s_l)$, then $A2_R(a) \geq s_l$ and for $a \notin K_j$, for $j > l$. Hence $A2_R(a) \in \{s_0, s_1, s_2.....,s_l\}$,

Which implies $a \in K_j$ for some $j \leq l$. Since $K_j \subset K_l$, it follows that $a \in K_l$. Consequently,

$U'(A2_R, s_l) = K_l$ for some $0 < l \leq n$.

If $a \in L(A3_R, t_l)$, then $A3_R(a) \leq t_l$ and for $a \notin K_m$, for $m > l$. Hence $A3_R(a) \in \{t_0, t_1, t_2.....,t_l\}$,

Which implies $a \in K_m$ for some $m \leq l$. Since $K_m \subset K_l$, it follows that $a \in K_l$. Consequently,

$L(A3_R, t_l) = K_l$ for some $0 < l \leq n$.

If $a \in L'(A4_R, u_l)$, then $A4_R(a) \leq u_l$ and for $a \notin K_m$, for $m>l$. Hence $A4_R(a) \in \{u_0, u_1, u_2.....,u_l\}$,

Which implies $a \in K_m$ for some $m \leq l$. Since $K_m \subset K_l$, it follows that $a \in K_l$.

Consequently, $L'(A4_R, u_l) = K_l$ for some $0 < l \leq n$.

This completes the proof.

Theorem: 3.11

If $R = (A1_R, A2_R, A3_R, A4_R)$ is an FQN Lie ideal of a FQN Lie algebra $\mathcal{L}$, then

$A1_R(a) = \sup \{r \in [0,1] \setminus a \in U(A1_R, r)\}$,

$A2_R(a) = \sup \{s \in [0,1] \setminus a \in U'(A2_R, s)\}$,

$A3_R(a) = \inf \{t \in [0,1] \setminus a \in L(A3_R, t)\}$,

$A4_R(a) = \inf \{u \in [0,1] \setminus a \in L'(A4_R, u)\}$.

For every $a \in \mathcal{L}$.

Proof

The proof follows from definition 3.2.

Definition:3.12

Let f be a map from a set $\mathcal{L}_1$ to a set $\mathcal{L}_2$. If $R1 = (A1_{R1}, A2_{R1}, A3_{R1}, A4_{R1})$ and

R2 = ($A1_{R2}$, $A2_{R2}$, $A3_{R2}$, $A4_{R2}$) are FQN sets in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, then the preimage of R2 under f, denoted by f^{-1}(R2), is a FQN set defined by

f^{-1}(R2) = (f^{-1}($A1_{R2}$), f^{-1}($A2_{R2}$), f^{-1}($A3_{R2}$), f^{-1}($A4_{R2}$)).

Theorem: 3.13

Let f : $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ be an onto homomorphisms of Lie algebras. If R2 = ($A1_{R2}$, $A2_{R2}$, $A3_{R2}$, $A4_{R2}$) is a FQN Lie ideal of $\mathcal{L}_2$, then the preimage

f^{-1}(R2) = (f^{-1}($A1_{R2}$), f^{-1}($A2_{R2}$), f^{-1}($A3_{R2}$), f^{-1}($A4_{R2}$)) under f is a FQN Lie ideal of $\mathcal{L}_1$.

Proof

The proof follows from definition 3.2 and 3.12

Theorem: 3.14

Let f : $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ be an epimorphisms of FQN Lie algebras. If R1 = ($A1_{R1}$, $A2_{R1}$, $A3_{R1}$, $A4_{R1}$) is a FQN Lie ideal of $\mathcal{L}_2$, then the preimage f^{-1}((R1)^C) = (f^{-1}(R1))^C

Proof

The proof follows from definition 3.2 and 3.12.

Theorem: 3.15

Let f : $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ be an epimorphisms of FQN Lie algebras. If R1 = ($A1_{R1}$, $A2_{R1}$, $A3_{R1}$, $A4_{R1}$, $A5_{R1}$) is a FQN Lie ideal of $\mathcal{L}_2$ and R2 = ($A1_{R2}$, $A2_{R2}$, $A3_{R2}$, $A4_{R2}$, $A5_{R2}$) is the preimage of R1 = ($A1_{R1}$, $A2_{R1}$, $A3_{R1}$, $A4_{R1}$, $A5_{R1}$) under f. Then R2 is a FQN Lie ideal of $\mathcal{L}_1$.

Proof

The proof follows from definition 3.2 and 3.12.

Definition: 3.16

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two FQN Lie algebras and f be a mapping of $\mathcal{L}_1$ into $\mathcal{L}_2$.

If R1 = ($A1_{R1}$, $A2_{R1}$, $A3_{R1}$, $A4_{R1}$) is a FQN set of $\mathcal{L}_1$, then the image of R1 under f is the FQN set in $\mathcal{L}_2$ defined by

$$f(A1_{R1})(b) = \begin{cases} sup_{a\in f^{-1}(b)}\, A1_{R1}(a), & if\ f^{-1}(b) \neq 0, \\ 0, & otherwise \end{cases}$$

$$f(A2_{R1})(b) = \begin{cases} sup_{a\in f^{-1}(b)}\, A2_{R1}(a), & if\ f^{-1}(b) \neq 0, \\ 0, & otherwise \end{cases}$$

$$f(A3_{R1})(b) = \begin{cases} inf_{a\in f^{-1}(b)}\, A3_{R1}(a), & if\ f^{-1}(b) \neq 0, \\ 1, & otherwise \end{cases}$$

f(A4 $_{R1}$)(b) = $\begin{cases} inf_{a\in f^{-1}(b)} A4_{R1}(a), & if\ f^{-1}(b) \neq 0 \\ 1, & otherwise \end{cases}$

for each b ∈ $\mathcal{L}_2$

Theorem: 3.17

Let f : $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ be an epimorphisms of FQN Lie algebras. If R1 = (A1 $_{R1}$, A2 $_{R1}$, A3 $_{R1}$, A4 $_{R1}$) is a FQN Lie ideal of $\mathcal{L}_1$, then f(R1) is a FQN Lie ideal of $\mathcal{L}_2$.

Proof

The proof follows from definition 3.2 and 3.16.

Definition: 3.18

Let f : $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ be an homomorphisms of FQN Lie algebras, For any FQN set, If R = (A1 $_R$, A2 $_R$, A3 $_R$, A4 $_R$) is a FQN Lie ideal of $\mathcal{L}_2$, we define a PN set $\boldsymbol{R^f} = (\boldsymbol{A1^f_R}, \boldsymbol{A2^f_R}, \boldsymbol{A3^f_R}, \boldsymbol{A4^f_R})$ in $\mathcal{L}_1$ by

$\mathbf{A1^f_R}$ (a) = A1 $_R$ (f(a)), $\mathbf{A2^f_R}$ (a) = A2 $_R$ (f(a)), $\mathbf{A3^f_R}$ (a) = A3 $_R$ (f(a)), $\mathbf{A4^f_R}$ (a) = A4 $_R$ (f(a)), for all a ∈ $\mathcal{L}_1$ **.**

Lemma: 3.19

Let f : $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ be an homomorphisms of FQN Lie algebras, If R = (A1 $_R$, A2 $_R$, A3 $_R$, A4 $_R$) is a FQN Lie ideal of $\mathcal{L}_2$, then $\boldsymbol{R^f} = (\boldsymbol{A1^f_R}, \boldsymbol{A2^f_R}, \boldsymbol{A3^f_R}, \boldsymbol{A4^f_R})$ is a FQN Lie ideal in $\mathcal{L}_1$.

Proof

Let a, b ∈ $\mathcal{L}_1$ **and β** ∈ $\mathcal{F}$. Then

$\mathbf{A1^f_R}$ (a + b) = A1 $_R$ (f(a + b)) = A1 $_R$ (f(a) + f(b)) ≥ min{ A1 $_R$ (f(a)), A1 $_R$ (f(b))} = min{$\mathbf{A1^f_R}$(a), $\mathbf{A1^f_R}$ (b)},

$\mathbf{A2^f_R}$ (a + b) = A2 $_R$ (f(a + b)) = A2 $_R$ (f(a) + f(b)) ≥ min{ A2 $_R$ (f(a)), A2 $_R$ (f(b))} = min{$\boldsymbol{A2^f_R}$ (a), $\boldsymbol{A2^f_R}$ (b)},

$\mathbf{A3^f_R}$ (a + b) = A3 $_R$ (f(a + b)) = A3 $_R$ (f(a) + f(b)) ≤ min{ A3 $_R$ (f(a)), A3 $_R$ (f(b))} = min{$\mathbf{A3^f_R}$(a), $\mathbf{A3^f_R}$ (b)},

$\mathbf{A4^f_R}$ (a + b) = A4 $_R$ (f(a + b)) = A4 $_R$ (f(a) + f(b)) ≤ min{ A4 $_R$ (f(a)), A4 $_R$ (f(b))} = min{$\mathbf{A4^f_R}$(a), $\mathbf{A4^f_R}$ (b)}.

$\mathbf{A1^f_R}$ (**β** a) = A1 $_R$ (f(**β** a)) = A1 $_R$ (**β** f(a)) ≥ A1 $_R$ (f(a)) = $\mathbf{A1^f_R}$(a),

$\mathbf{A2^f_R}$ (**β** a) = A2 $_R$ (f(**β** a)) = A2 $_R$ (**β** f(a)) ≥ A2 $_R$ (f(a)) = $\mathbf{A2^f_R}$(a),

$\mathbf{A3^f_R}$ (**β** a) = A3 $_R$ (f(**β** a)) = A3 $_R$ (**β** f(a)) ≤ A3 $_R$ (f(a)) = $\mathbf{A3^f_R}$(a),

$\mathbf{A4^f_R}$ (**β** a) = A4 $_R$ (f(**β** a)) = A4 $_R$ (**β** f(a)) ≤ A4 $_R$ (f(a)) = $\mathbf{A4^f_R}$(a).

Similarly,

$\mathbf{A1^f_R}$ ([a, b]) = A1 $_R$ (f[a, b]) = A1 $_R$ ([f(a), f(b]) ≥ A1 $_R$ (f(a)) = $\mathbf{A1^f_R}$(a),

$\mathbf{A2^f_R}$ ([a, b]) = A2 $_R$ (f([a,b]) = A2 $_R$ ([f(a), f(b)]) ≥ A2 $_R$ (f(a)) = $\mathbf{A2^f_R}$(a),

$\mathbf{A3^f_R}$ ([a, b]) = A3 $_R$ (f([a, b]) = A3 $_R$ ([f(a), f(b)]) ≤ A3 $_R$ (f(a)) = $\mathbf{A3^f_R}$(a),

$\mathbf{A4_R^f}$ ([a, b]) = A4 R (f([a, b]) = A4 R ([f(a), f(b)]) ≤ A4 R (f(a)) = $\mathbf{A4_R^f}$(a).

This proves that $\boldsymbol{R^f} = (\boldsymbol{A1_R^f}, \boldsymbol{A2_R^f}, \boldsymbol{A3_R^f}, \boldsymbol{A4_R^f})$ is a FQN Lie ideal in $\mathcal{L}_1$.

We now characterize the FQN Lie ideals of Lie algebras.

Theorem: 3.20

Let f : $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ be an epimorphisms of FQN Lie algebras. Then $\boldsymbol{R^f} = (\boldsymbol{A1_R^f}, \boldsymbol{A2_R^f}, \boldsymbol{A3_R^f}, \boldsymbol{A4_R^f})$ is a FQN Lie ideal in $\mathcal{L}_1$ iff R = (A1 R, A2 R, A3 R , A4 R) is a FQN Lie ideal of $\mathcal{L}_2$.

Definition: 3.21

Let R = (A1 R, A2 R, A3 R , A4 R) be a FQN Lie ideal in $\mathcal{L}$. Define a inductively a sequences of FQN Lie ideals in $\mathcal{L}$ by $\mathbf{R^0}$ = R, $\mathbf{R^1}$ = [$\mathbf{R^0}, \mathbf{R^0}$], $\mathbf{R^2}$ = [$\mathbf{R^1}, \mathbf{R^1}$],….. $\boldsymbol{R^n}$ = [$\boldsymbol{R^{n-1}}, \mathbf{R^{n-1}}$].

$\boldsymbol{R^n}$ is called the n th derived FQN Lie ideal of $\mathcal{L}$. A series $\boldsymbol{R^0} \supseteq \mathbf{R^1} \supseteq \mathbf{R^2} \supseteq \ldots\ldots \supseteq \boldsymbol{R^n} \supseteq \cdots$is called derived series of a FQN Lie ideal R in $\mathcal{L}$.

Definition: 3.22

A FQN Lie ideal R in is called a solvable FQN Lie ideal, if there exists a positive integer n such that

$\boldsymbol{R^0} \supseteq \mathbf{R^1} \supseteq \mathbf{R^2} \supseteq \ldots\ldots \supseteq \boldsymbol{R^n}$ = (0,0,0).

Theorem: 3.23

Homomorphic images of solvable FQN Lie ideals are solvable FQN Lie ideals.

Proof

Let f : $\mathcal{L}_1 \rightarrow \mathcal{L}_2$ be homomorphisms of FQN Lie algebras. Suppose that R = (A1 R, A2 R, A3 R , A4 R) is a FQN Lie ideal of $\mathcal{L}_1$. We prove by induction on n that f($\boldsymbol{R^n}$) $\supseteq [\boldsymbol{f(R)}]^{\boldsymbol{n}}$, where n is any positive integer. First we claim that f([R, A]) ⊇ [f(R), f(R)]. Let y ∈ $\mathcal{L}_2$. Then

f(<<A1 R , A1 R >>)(y) = sup { <<A1 R, A1 R >>(y)\f(x) = y}

= sup{sup{min(A1 R(a), A1 R(b))\a, b ∈ $\mathcal{L}_1$,[a, b] = x, f(x) = y}}

= sup{min(A1 R(a), A1 R(b))\a, b ∈ $\mathcal{L}_1$,,[a, b] = x, f(x) = y}}

= sup{min(A1 R(a), A1 R(b))\a, b ∈ $\mathcal{L}_1$,,[f(a), f(b)] = x}

= sup{min(A1 R(a), A1 R(b))\a, b ∈ $\mathcal{L}_1$,f(a) = u, f(b) = v,[u, v] = y}}

≥sup{min(su$\boldsymbol{p}_{\boldsymbol{a}\in\boldsymbol{f}^{-1}(\boldsymbol{u})}$ A1 R(a), min(su$\boldsymbol{p}_{\boldsymbol{b}\in\boldsymbol{f}^{-1}(\boldsymbol{v})}$ A1 R(b)\[u, v] = y}

= sup{min{f(A1 R)(u),f(A1 R)(v))\[u, v] = y}

= << f(A1 R), f(A1 R)>>(y),

f(<<A2 $_R$, A2 $_R$ >>)(y) = sup { <<A2 $_R$, A2 $_R$ >>(y)\f(x) = y}

= sup{sup{min(A2 $_R$(a), A2 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,[a, b] = x, f(x) = y}}

= sup{min(A2 $_R$(a), A2 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,,[a, b] = x, f(x) = y}}

= sup{min(A2 $_R$(a), A2 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,,[f(a), f(b)] = x}

= sup{min(A2 $_R$(a), A2 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,,f(a) = u, f(b) = v,[u, v] = y}}

≥sup{min(su$\boldsymbol{p_{a\in f^{-1}(u)}}$ A2 $_R$(a), min(su$\boldsymbol{p_{b\in f^{-1}(v)}}$ A2 $_R$(b)\[u, v] = y}

= sup{min{f(A2 $_R$)(u),f(A2 $_R$)(v))\[u, v] = y}

= << f(A2 $_R$), f(A2 $_R$)>>(y),

f(<<A3 $_R$, A3 $_R$ >>)(y) = inf { <<A3 $_R$, A3 $_R$ >>(y)\f(x) = y}

= inf{inf{max(A3 $_R$(a), A3 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,,[a, b] = x, f(x) = y}}

= inf{max(A3 $_R$(a), A3 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,[a, b] = x, f(x) = y}}

= inf {max(A3 $_R$(a), A3 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,[f(a), f(b)] = x}

= inf{max(A3 $_R$(a), A3 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,f(a) = u, f(b) = v,[u, v] = y}}

≤inf{max(in$\mathbf{f_{a\in f^{-1}(u)}}$ A3 $_R$(a), min(in$\mathbf{f_{b\in f^{-1}(v)}}$ A3 $_R$(b)\[u, v] = y}

= inf{max{f(A3 $_R$)(u),f(A3 $_R$)(v))\[u, v] = y}

= << f(A3 $_R$), f(A3 $_R$)>>(y),

f(<<A4 $_R$, A4 $_R$ >>)(y) = inf { <<A4 $_R$, A4 $_R$ >>(y)\f(x) = y}

= inf{inf{max(A4$_R$(a), A4 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,[a, b] = x, f(x) = y}}

= inf{max(A4 $_R$(a), A4 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,[a, b] = x, f(x) = y}}

= inf {max(A4 $_R$(a), A4 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,[f(a), f(b)] = x}

= inf{max(A4 $_R$(a), A4 $_R$(b))\a, b ∈ $\boldsymbol{\mathcal{L}_1}$,f(a) = u, f(b) = v,[u, v] = y}}

≤inf{max(in$\mathbf{f_{a\in f^{-1}(u)}}$ A4 $_R$(a), min(in$\mathbf{f_{b\in f^{-1}(v)}}$ A4 $_R$(b)\[u, v] = y}

= inf{max{f(A4 $_R$)(u),f(A4 $_R$)(v))\[u, v] = y}

= << f(A4 $_R$), f(A4 $_R$)>>(y),

Thus f([R, R]) ⊇ f(<<A,A>>) ⊇ <<f(R), f(R)>> = [f(R), f(R)].

Now for n > 1, we get f($\boldsymbol{R^n}$) = f([$\boldsymbol{R^{n-1}}$, $\boldsymbol{R^{n-1}}$]) ⊇ [f($\boldsymbol{R^{n-1}}$), f($\boldsymbol{R^{n-1}}$)].

This completes the proof.

Definition: 3.24

Let R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) be a FQN Lie ideal in $\mathcal{L}$. We define a inductively a sequences of FQN Lie ideals in $\mathcal{L}$ by $\mathbf{R_0}$= R, $\mathbf{R_1}$ = [R, $\mathbf{R_0}$], $\boldsymbol{R_2}$= [$\boldsymbol{R}, \boldsymbol{R_1}$] .... $\mathbf{R_n}$ = [$\mathbf{R}, \mathbf{R_{n-1}}$]. A series

$\boldsymbol{R_0} \supseteq \boldsymbol{R_1} \supseteq \boldsymbol{R_2} \supseteq \ldots . \supseteq \boldsymbol{R_n} \supseteq \cdots$is called descending central series of a FQN Lie ideal R in $\mathcal{L}$.

Definition: 3.25

An FQN Lie ideal R is called a nilpotent FQN Lie ideal in $\mathcal{L}$, if there exists a positive integer n such that $\boldsymbol{R_0} \supseteq \boldsymbol{R_1} \supseteq \boldsymbol{R_2} \supseteq \ldots . \supseteq \boldsymbol{R_n}$= (0,0,0).

Theorem: 3.26

Homomorphic image of a nilpotent FQN Lie ideal is a nilpotent FQN Lie ideal.

Proof: It is obvious

Theorem: 3.27

Let K be a FQN Lie ideal of a FQN Lie algebra $\mathcal{L}$. If R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) is a FQN Lie ideal of $\mathcal{L}$, then the FQN set *R = (*$A1_R$, *$A2_R$, *$A3_R$, *$A4_R$) of $\mathcal{L}$ /K defined by

*$A1_R$(a + K) = $\mathbf{sup_{x\in K}\, A1_R(a + x)}$,

*$A2_R$(a + K) = $\mathbf{sup_{x\in K}\, A2_R(a + x)}$,

*$A3_R$(a + K) = $\boldsymbol{inf_{x\in K}\, A3_R}(\mathbf{a + x})$,

*$A4_R$(a + K) = $\boldsymbol{inf_{x\in K}\, A4_R}(\mathbf{a + x})$

is a FQN Lie ideal of the quotient FQN Lie algebra $\mathcal{L}$ /K of $\mathcal{L}$ with respect to K.

Proof

Clearly, *R is defined. Let x + K, y + K ∈ $\mathcal{L}$/ K. Then

*$A1_R$((x + K) + (y + K)) = *$A1_R$((x + y) + K)

= $\mathbf{sup_{z\in K}\, A1_R\big((x + y) + z\big)}$,

= $\mathbf{sup_{z=s+t\in K}\, A1_R\big((x + y) + (s + t)\big)}$,

$\geq \mathbf{sup_{s,t\in K}\, min\{A1_R(x + s), A1_R(y + t)\}}$,

= min { $\mathbf{sup_{s\in K}\ A1_R(x + s)}$,$\mathbf{sup_{t\in K}\ A1_R(y + t)}$},

= min{$*\,\mathbf{A1_R}(\boldsymbol{x + s}), *\,\mathbf{A1_R}(\boldsymbol{y + t})$},

*$A1_R$($\boldsymbol{\beta}$(x + K) = *$A1_R$($\boldsymbol{\beta}$x + K) = $\mathbf{sup_{z\in K}\, A1_R(\beta x + z) \geq sup_{z\in K}\, A1_R(x + z)}$=*$A1_R$(x + K).

*$A1_R$([x + K, *$A1_R$(a + K) = $\mathbf{sup_{x\in K}}$ $A1_R$(a + x),y + K]) = *$A1_R$([x , y] + K) = $\mathbf{sup_{z\in K}\, A1_R([x, y] + z) \geq}$ $\mathbf{sup_{z\in K}\, A1_R([x, y] + z)}$=*$A1_R$(x + K).

Thus *$A1_R$ is a FQN Lie ideal of $\mathcal{L}$/ K. In a similar way, we can verify that *$A2_R$, *$A3_R$, *$A4_R$ FQN Lie ideals of $\mathcal{L}$ /K. Hence *R = (*$A1_R$, *$A2_R$, *$A3_R$, *$A4_R$) is a FQN Lie ideal of

$\mathcal{L}$ /K

Theorem: 3.28

Let K be a FQN Lie ideal of a FQN Lie algebra $\mathcal{L}$. Then there is a one-to=one correspondence between the set of FQN Lie ideals R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) of $\mathcal{L}$ such that R(0) = A(s) for all s ∈ K and the set of all FQN Lie ideals *R = ($*A1_R$, $*A2_R$, $*A3_R$, $*A4_R$) of $\mathcal{L}$ /K.

Proof

Let R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) be FQN Lie ideal of $\mathcal{L}$. Using Theorem 3.27, we prove that

$*A1_R$, $*A2_R$, $*A3_R$, $*A4_R$, $*A5_R$ defined by

$*A1_R(a + K) = \mathbf{sup}_{\mathbf{x \in K}} A1_R(a + x)$,

$*A2_R(a + K) = \mathbf{sup}_{\mathbf{x \in K}} A2_R(a + x)$,

$*A3_R(a + K) = \mathbf{inf}_{\mathbf{x \in K}} A3_R(a + x)$,

$*A4_R(a + K) = \boldsymbol{inf}_{\mathbf{x \in K}} A4_R(a + x)$

are FQN Lie ideals of $\mathcal{L}$ /K. Since $A1_R(0) = A1_R(s)$, $A2_R(0) = A2_R(s)$, $A3_R(0) = A3_R(s)$, $A4_R(0) = A4_R(s)$ for all s ∈ K,

$A1_R(a + s) \geq \min(A1_R(a), A1_R(s)) = A1_R(a)$,

$A2_R(a + s) \geq \min(A2_R(a), A2_R(s)) = A2_R(a)$,

$A3_R(a + s) \leq \max(A3_R(a), A3_R(s)) = A3_R(a)$,

$A4_R(a + s) \leq \max(A4_R(a), A4_R(s)) = A4_R(a)$.

Again,

$A1_R(a) = A1_R(a + s - s) \geq \min(A1_R(a + s), A1_R(s)) = A1_R(a + s)$,

$A2_R(a) = A2_R(a + s - s) \geq \min(A2_R(a + s), A2_R(s)) = A2_R(a + s)$,

$A3_R(a) = A3_R(a + s - s) \leq \max(A3_R(a + s), A3_R(s)) = A3_R(a + s)$,

$A4_R(a) = A4_R(a + s - s) \leq \max(A4_R(a + s), A4_R(s)) = A4_R(a + s)$.

Thus R(a + s) = R(a) for all s ∈ K. Hence the correspondence R → *R is one- to -one. Let *R be a FQN Lie ideal of $\mathcal{L}$ / K and define a FQN set R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) in $\mathcal{L}$ by

$A1_R(a) = * A1_R(a + K)$, $A2_R(a) = * A2_R(a + K)$, $A3_R(a) = * A3_R(a + K)$, $A4_R(a) = * A4_R(a + K)$

For a, b ∈ $\mathcal{L}$, we have

$A1_R(a + b) = *A1_R((a + b) + K) = * A1_R((a + K) + (b + K))$

$\geq \min\{*A1_R(a + K), *A1_R(b + K)\}$,

$= \min \{ A1_R(a), A1_R(b)\}$,

$A1_R(\boldsymbol{\beta} a) = * A1_R(\boldsymbol{\beta} a + K) \geq * A1_R(a + K) = A1_R(a)$,

$A1_R([a, b]) = * A1_R([a, b] + K) = * A1_R([a + K, b + K])$

$\geq * A1_R(a + K) = A1_R(a)$.

Thus $A1_R$ is a FQN lie ideal of $\mathcal{L}$. In a similar way, we can verify that $A2_R$, $A3_R$ and $A4_R$ are FQN Lie ideals of $\mathcal{L}$. Hence R = ($A1_R$, $A2_R$, $A3_R$, $A4_R$) is a FQN Lie ideal of $\mathcal{L}$.

Note that $A1_R(a) = {}^* A1_R(a+K)$, $A2_R(a) = {}^* A2_R(a+K)$, $A3_R(a) = {}^* A3_R(a+K)$, $A4_R(a) = {}^* A4_R(a+K)$. For $a \in \mathbf{K}$, which shows that R(a) = R(0) for all $a \in \mathbf{K}$.This completes the proof.

4. Conclusion

In this article, we have discussed above FQN Lie subalgebra and FQN Lie ideals of a FQN Lie Algebra. We have also investigated some of its properties of Fermatean Quadripartitioned Neutrosophic Lie ideals. In future, we are Planned to study on Lie rings. We may also develop for heptapartitioned neutrosophic sets and other hybrid sets.

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QUADRIPARTITIONED NEUTROSOPHIC PRODUCT SPACE

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Abstract:

In this paper we introduce the concept of quadripartitioned neutrosophic product spaces and investigate some of their properties.

Keywords : Fuzzy Neutrosophic set, Fuzzy Neutrosophic topological space and Fuzzy Neutrosophic product space.

1 Introduction

The concept of neutrosophic set was introduced by Smarnandache [28, 29]. The traditional neutrosophic sets is characterized by the truth value, indeterminate value and false value. Neutrosophic set is a mathematically tool for handling problems involving imprecise, indeterminacy inconsistent data and inconsistent information which exits in belief system. The concept of neutrosophic set which overcomes the inherent difficulties that existed in fuzzy sets and intuitionistic fuzzy sets. Following this, the neutrosophic sets are explored to differ- ent heights in all fields of science and engineering. A.A.Salama [9] - [26]applied neutrosphic set in various prospects. Many researchers [3, 4, 5, 6, 7, 8, 30] applied the concept of fuzzy sets and intuitionistic fuzzy sets to topologies. In this paper we initiate the concept of fuzzy neutrosophic product and some of its properties are discussed.

2 Preliminary Notes

Definition 2.1. [1] A Fuzzy Neutrosophic set A on the universe of discourse X is defined as $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$, $x\in X$ where $T, I, F : X \longrightarrow [0, 1]$ and $0\leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.2. [1] Let X be a non empty set, and $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$, $B = \langle x, T_B(x), I_B(x), F_B(x)\rangle$ are fuzzy neutrosophic sets. Then A is a subset of B if $\forall\ x \in X$, $T_A(x) \leq T_B(x), I_A(x) \leq I_B(x)), F_A(x) \geq F_B(x))$.

Definition 2.3. [1] Let X be a non-empty set, and $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$,

$B = \langle x, T_B(x), I_B(x), F_B(x)\rangle$ are fuzzy neutrosophic sets. Then

$A \cup B = \langle x, \max(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \min(F_A(x), F_B(x))\rangle$

$A \cap B = \langle x, \min(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \max(F_A(x), F_B(x))\rangle$

Definition 2.4. [1] A Fuzzy neutrosophic set A over the non-empty set X is said to be empty fuzzy neutrosophic set if $T_A(x) = 0$, $I_A(x) = 0$, $F_A(x) = 1, \forall x \in X$. *It is denoted by* 0_N.

A Fuzzy neutrosophic set A over the non-empty set X is said to be universe fuzzy neutrosophic set if $T_A(x) = 1$, $I_A(x) = 1$, $F_A(x) = 0, \forall x \in X$. *It is de- noted by* 1_N.

Definition 2.5. [1] The complement of Fuzzy neutrosophic set A denoted by

A^c and is defined as

$A^c(x) = \langle x, T_{A^c}(x) = F_A(x), I_{A^c}(x) = 1 - I_A(x), F_{A^c}(x) = T_A(x)\rangle$

Definition 2.6. [2] Let X and Y be a non-empty sets and let f be a map- ping from a set X to a set Y. Let $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle / x \in X\}$, $B = \{\langle y, T_B(y), I_B(y), F_B(y)\rangle / y \in Y\}$ be fuzzy neutrosophic set in X and Y respectively,

(a) then the preimage of B under f denoted by $f^{-1}(B)$ is the fuzzy neutrosophic set in X defined by

$f^{-1} = \{< x, f^{-1}(T_B)(x), f^{-1}(I_B)(x), f^{-1}(F_B)(x) > / x \in X\}$ where

$f^{-1}(T_B)(x) = T_B(f(x))$, $f^{-1}(I_B)(x) = I_B(f(x))$ and $f^{-1}(F_B)(x) = F_B(f(x))$ for all $x \in X$.

(b) the image of A under f, denoted by f(A) is the fuzzy neutrosophic set in Y defined by $f(A) = (f(T_A, f(I_A, f(F_A))$, where for each $y \in Y$.

$$f(T_A)(y) = \begin{cases} \vee_{x \in f^{-1}(y)} T_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$f(I_A)(y) = \begin{cases} \vee_{x \in f^{-1}(y)} I_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$f(F_A)(y) = \begin{cases} \vee_{x \in f^{-1}(y)} F_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2.7. [2] Let A, $A_i (i \in I)$be fuzzy neutrosophic sets in X let B, $B_j(j \in J)$be fuzzy neutrosophic sets in Y and let $f : X \to Y$ a mapping. Then

1. $A_1 \subset A_2$ implies $f(A_1) \subset f(A_2)$.
2. $B_1 \subset B_2$ implies $f^{-1}(B_1) \subset f^{-1}(B_2)$.
3. $A \subset f^{-1}(f(A))$. If f is injective, then $A = f^{-1}(f(A))$.

4. $f(f^{-1}(B)) \subset B$. If f is surjective, then $f(f^{-1}(B))=B$.
5. $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$.
6. $f^{-1}(\cap B_j) = \cap f^{-1}(B_j)$.
7. $f(\cup A_i) = \cup f(A_i)$.
8. $f(\cap A_i) \subset \cap f(A_i)$.
9. $f(1_N) = 1_N$, if f is surjective and $f(0_N) = 0_N$.
10. $f^{-1}(1_N) = 1_N$ and $f^{-1}(0_N) = 0_N$.
11. $[f(A)]^c \subset f(A^c)$ if f is surjective.
12. $f^{-1}(B^c) = [f^{-1}(B)]^c$

3 Quadripartitioned Neutrosophic product space

Definition 3.1 . Let p, q, r, s $\in [0,1]$ and $p+q+r+s \leq 4$. of Quadripartitioned Neutrosophic soft point $x_{(p,q,r,s)}$ of X is of quadripartitioned Neutrosophic soft set in X defined by ,

$$X_{(p,q,r,s)}(y) = \begin{cases} (p,q,r,s) \text{ if } x = y, \text{ for each } y \in X. \\ \quad (0,0,1,1) \quad \text{if } y \neq x \end{cases}$$

A quadripartitioned Neutrosophic point $X_{(p,q,r,s)}$ is said to belong to an quadripartitioned Neutrosophic soft set **A** = $\langle T_A, C_A, U_A, F_A \rangle$ in X denoted by $X_{(p,q,r,s)} \in A$ if

P $\leq T_A(x)$, $q \leq C_A(x)$, $r \leq U_A(x)$ and $s \leq F_A(x)$. We denote the set of all quadripartitioned Neutrosophic points in X as FNP(X).

Theorem 3.2. Let A = $\langle T_A, C_A, U_A, F_A \rangle$ and B = $\langle T_A, C_A, U_A, F_A \rangle$ be quadripartitioned Neutrosophic set in X, then $A \subset B$ if and only for each $x_{(p,q,r,s)} \in A$ implies $x_{(p,q,r,s)} \in B$.

Proof: Let $A \subseteq B$ and $x_{(p,q,r,s)} \in A$, then $P \leq T_A(x) \leq T_B(x)$, $q \leq C_A(x) \leq C_B(x)$, $r \geq U_A(x) \geq U_B(x)$ and $s \geq F_A(x) \geq F_B(x)$. thus $x_{(p,q,r,s)} \in B$.

Conversely, take and $x \in X$. let $p = T_A(x)$, $q = C_A(x)$, $r = U_A(x)$ and $s = F_A(x)$.then $x_{(p,q,r,s)}$ is a quadripartitioned Neutrosophic point in X and $x_{(p,q,r,s)} \in A$. by the hypothesis, $x_{(p,q,r,s)} \in B$. thus $T_A = p \leq T_B(x)$, $C_A = C_B(x)$, , $U_A = \geq U_B(x)$ and $F_A \geq F_B(x)$. Hence $A \subseteq B$.

Theorem 3.3. let A = $\langle T_A, C_A, U_A, F_A \rangle$ be a quadripartitioned Neutrosophic set in X. then $A = \cup \{ x_{(p,q,r,s)}: x_{(p,q,r,s)} \in A\}$.

Definition 3.4. let X be a set and let p,q,r,s ∈ [0,1] with $0 \leq p+q+r+s \leq 4$. Then the quadripartitioned Neutrosophic set $C_{(p,q,r,s)} \in X$ is defined by for each $x \in x$, $C_{(p,q,r,s)}(x) = (p,q,r,s)$ ie., $T_{C_{(p,q,r,s)}(x)} = p, C_{C_{(p,q,r,s)}(x)} = q, U_{C_{(p,q,r,s)}(x)} = r$ and $F_{C_{(p,q,r,s)}(x)} = s$.

Definition 3.5.let X be a non-empty set and let $\mathscr{g} \subset FNS(X)$.then $\mathscr{g}$ is called a quadripartitioned Neutrosophic topology (FNT) on X in the sense of Lowen[69], if it satisfies the following axioms:

i. For each $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma \leq 3, C_{(\alpha,\beta,\gamma)} \in \mathscr{g}$
ii. For any $A_1, A_2 \in \mathscr{g}$, $A_1 \cap A_2 \in \mathscr{g}$
iii. For any $\{A_k\}_{k \in K} \subset \mathscr{g}$, $\bigcup_{k\in K} \mathscr{g}$

Definition 3.6. let A be a quadripartitioned Neutrosophic set in a quadripartitioned Neutrosophic topology space (X, $\mathscr{g}$) ,then the induced quadripartitioned Neutrosophic topology (IFNT in short) on A is the family of quadripartitioned Neutrosophic set in A which are the intersection with A quadripartitioned Neutrosophic open sets in X. the IFNT is denoted by $\mathscr{g}_A$ and the pair (A, $\mathscr{g}_A$) is called a quadripartitioned Neutrosophic subspace of (X, $\mathscr{g}$).

Definition 3.7.Let (X, $\mathscr{g}$) and (Y, $\mathfrak{A}$) be two quadripartitioned Neutrosophic topological spaces. A mapping f: (X, $\mathscr{g}$) → (Y, $\mathscr{g}$) is said to be quadripartitioned Neutrosophic continuous if the preimage of each quadripartitioned Neutrosophic set in $\mathfrak{A}$ is a quadripartitioned Neutrosophic set in $\mathscr{g}$, and f is said to be quadripartitioned Neutrosophic open if the image of each quadripartitioned Neutrosophic set in $\mathscr{g}$ is a quadripartitioned Neutrosophic set in $\mathfrak{A}$.

Definition 3.8. Let (A, $\mathscr{g}_A$) and $(B, \mathfrak{A}_A)$ be a quadripartitioned Neutrosophic subspace of quadripartitioned Neutrosophic topological spaces (X, $\mathscr{g}$) and (Y, $\mathfrak{A}$)respectively and let f : (X, $\mathscr{g}$) → (Y, $\mathscr{g}$) be a mapping. Then f is a mapping of (A, $\mathscr{g}_A$) into $(B, \mathfrak{A}_B)$ if f(A)⊂ B.
Furthermore f is said to be relatively quadripartitioned Neutrosophic continuous if for each quadripartitioned Neutrosophic set V_B in $\mathfrak{A}_B$, the intersection $f^{-1}(V_B) \cap A$ is a quadripartitioned Neutrosophic set in $\mathscr{g}_A$ and f is said to be relatively quadripartitioned Neutrosophic open if for each quadripartitioned Neutrosophic set U_A in $\mathscr{g}_A$,the image $f(U_A)$ is the quadripartitioned Neutrosophic set in $\mathfrak{A}_B$.

Proposition 3.9. Let (A, $\mathscr{g}_A$) and $(B, \mathfrak{A}_A)$ be a quadripartitioned Neutrosophic subspace of quadripartitioned Neutrosophic topological spaces (X, $\mathscr{g}$) and (Y, $\mathfrak{A}$)respectively and let f be a quadripartitioned Neutrosophic continuous mapping of $(X, \mathcal{g})$ into $(Y, \mathfrak{A})$ such that $f(A) \subset B$ then f is relatively quadripartitioned Neutrosophic continuous mapping of $(A, \mathcal{g})$ into $(B, \mathfrak{A})$.

Proof: let V_B be a quadripartitioned Neutrosophic set in $\mathfrak{A}_B$.then there exist $V \in \mathfrak{A}$ such that $V_B = V \cap B$. since f is quadripartitioned Neutrosophic continuous it follows that $f^{-1}(V)$ is a quadripartitioned Neutrosophic set in $\mathcal{g}$. Hence $f^{-1}(V_B) \cap A = f^{-1}(V_B) \cap f^{-1}(B) \cap A = f^{-1}(V) \cap A$ is a quadripartitioned Neutrosophic set in $\mathcal{g}_A$. Hence the proof.

Definition 3.10. A bijective mapping f of a quadripartitioned Neutrosophic topological spaces $(X, \mathcal{g})$ into a quadripartitioned Neutrosophic topological spaces $(Y, \mathfrak{A})$ is a quadripartitioned Neutrosophic homeomorphism iff it is quadripartitioned Neutrosophic continuous and quadripartitioned Neutrosophic open. A bijective mapping f of a quadripartitioned Neutrosophic subspace $(A, \mathcal{g}_A)$ of $(X, \mathcal{g})$into a quadripartitioned Neutrosophic spaces $(B, \mathfrak{A}_B)$ of $(Y, \mathfrak{A})$ is relative quadripartitioned Neutrosophic homeomorphism iff f[A]=B and f is relatively quadripartitioned Neutrosophic continuous and relatively quadripartitioned Neutrosophic open.

Proposition 3.11. Let f be a quadripartitioned Neutrosophic continuous (resp. quadripartitioned Neutrosophic open) mapping of a quadripartitioned Neutrosophic space $(X, \mathcal{g})$ into a quadripartitioned Neutrosophic space $(Y, \mathfrak{A})$ and g a quadripartitioned Neutrosophic continuous (resp. quadripartitioned Neutrosophic open) mapping of $(Y, \mathfrak{A})$ into a quadripartitioned Neutrosophic topological spaces $(Z, \mathcal{W})$. Then the composition $g \circ f$ is a quadripartitioned Neutrosophic continuous (resp. quadripartitioned Neutrosophic open) mapping of $(X, \mathcal{g})$ into $(Z, \mathcal{W})$.

Proof: consider a quadripartitioned Neutrosophic set W in $\mathcal{W}$, then $g^{-1}(W)$ is quadripartitioned Neutrosophic open in $\mathfrak{A}$ (since g is quadripartitioned Neutrosophic continuous). Let $g^{-1}(W)$ be quadripartitioned Neutrosophic open in $\mathfrak{A}$, then $(f^{-1}\ g^{-1}(W)) = (g \circ f)^{-1})(W)$ is quadripartitioned Neutrosophic open in $\mathcal{g}$(since f is quadripartitioned neutrosophic continuous). Hence $g \circ f$ is quadripartitioned neutrosophic continuous mapping of $(X, \mathcal{g})$ into $(Z, \mathcal{W})$. Similarly we can prove for quadripartitioned neutrosophic open mapping.

Proposition 3.12. Let $(A, \mathcal{g}_A)$ and $(B, \mathfrak{A}_B)$ and $(C, \mathcal{W}_C)$ be a quadripartitioned Neutrosophic subspace of quadripartitioned Neutrosophic topological spaces $(X, \mathcal{g})$ and $(Y, \mathfrak{A})$and $(Z, \mathcal{W})$respectively. Let f be a relatively quadripartioned neutrosophic continuous (resp. relatively quadripartioned neutrosophic open) mapping of $(A, \mathcal{g}_A)$ into $(B, \mathfrak{A}_B)$ and g a relatively quadripartioned neutrosophic continuous (resp. relatively quadripartioned neutrosophic open) mapping of $(B, \mathfrak{A}_B)$ into $(C, \mathcal{W}_C)$. Then the composition $g \circ f$ is relatively quadripartitioned neutrosophic continuous (resp. relatively quadripartitioned neutrosophic open) mapping of $(A, \mathcal{g}_C)$ into $(C, \mathcal{W}_C)$.

Proof: Let $W_C \in \mathcal{W}_C$. Since g is relatively quadripartitioned neutrosophic continuous, $g^{-1}(W_C) \cap B \in \mathfrak{U}_B$. Since f is relatively quadripartitioned nuetrosophic continuous f^{-1} $[g^{-1}(W_C) \cap B] \cap A \in \mathcal{g}_A$. Now f^{-1} $[g^{-1}(W_C) \cap B] \cap A = f^{-1}$ $[g^{-1}(W_C) \cap f^{-1}(B) \cap A = (g \circ f)^{-1}(W_C) \cap f^{-1}(B) \cap A = (g \circ f)^{-1}(W_C) \cap A$ (since f (A) ⊂ B).Thus $(g \circ f)^{-1}(W_C) \cap A \in \mathcal{g}_A$. Hence g ∘ f is relatively quadripartitioned nuetrosophic continuous.

Let $U_A \in \mathcal{g}_A$. Since f is relatively quadripartitioned neutrosophic open, $f(U_A) \in \mathfrak{U}_B$. Since g is relatively quadripartitioned neutrosophic open $g(f(U_A)) = (g \circ f)(U_A)$. Thus $(g \circ f)(U_A) \in \mathcal{W}_C$. Hence g ∘ f is relatively quadripartitioned neutrosophic open.

Definition 3.13. Let $\mathcal{g}$ be a quadripartitioned neutrosophic topology on X. A subfamily $\mathfrak{B}$ of $\mathcal{g}$ is a base for $\mathcal{g}$ iff each member of $\mathcal{g}$ can be expressed as the union of members of $\mathfrak{B}$.

Definition 3.14. Let $\mathcal{g}$ be a quadripartitioned neutrosophic topology on X and $\mathcal{g}_A$ the induced quadripartitioned neutrosophic topology on a quadripartitioned neutrosophic subset of A of X. A subfamily $\mathfrak{B}_A$ of $\mathcal{g}_A$ is a base for $\mathcal{g}_A$ iff each member of $\mathcal{g}_A$ can be expressed as the union of members of $\mathfrak{B}$.

If $\mathfrak{B}$ is a base for a quadripartitioned neutrosophic topology $\mathcal{g}$ on a set X, then $\mathfrak{B}_A = \{U \cap A : U \in \mathcal{g}\}$ is a base for the induced quadripartitioned nuetrosophic topology $\mathcal{g}_A$ on the quadripartitioned neutrosophic subset A.

Proposition 3.15. Let f be a mapping from a quadripartitioned neutrosophic topological space (X, $\mathcal{g}$) to a quadripariititioned neutrosophic topological space (Y,$\mathfrak{U}$). Let $\mathfrak{B}$ be a base for $\mathfrak{U}$. Then f is a quadripariititioned neutrosophic continuous iff for each B ∈ $\mathfrak{B}$ the inverse image $f^{-1}(B)$ is in $\mathcal{g}$.

Proof: The only if part is obvious. Suppose the given condition is satisfied. Let V ∈ $\mathfrak{U}$, then there exist $V_{i\in I}$ ∈ $\mathfrak{B}$ such that $V = \cup_{i\in I} V_i$ and $f^{-1}(V_i) \in \mathcal{g}$, i∈ I. Hence $f^{-1}(V) = f^{-1}(\cap V_i) = \cap f^{-1}(V_i) \in$. So f is quadripartitioned neutrosophic continuous.

Proposition 3.16. Let $(A, \mathcal{g}_A), (B, \mathfrak{U}_B)$, be quadripartitioned neutrosophic subspaces of quadripartitioned neutrosophic topologies (X, $\mathcal{g}$) and (Y, $\mathfrak{U}$) respectively. Let $\mathfrak{B}$ be a base for $\mathfrak{U}_B$. Then a mapping f of (A, $\mathcal{g}_A$)into$(B, \mathfrak{U}_B)$ is relatively continuous iff for each B in $\mathfrak{B}$ the intersection $f^{-1}[B] \cap A$ is in $\mathcal{g}_A$.

Proof: Straightforward.

Definition 3.17. Given two quadripartitioned neutrosophic topologies $\mathcal{g}_1$, $\mathcal{g}_2$ on the same set X, then $\mathcal{g}_1$is said to be finer than $\mathcal{g}_2$(or $\mathcal{g}_2$ is coarser than $\mathcal{g}_1$) if the identity mapping of (X, $\mathcal{g}_1$) into (X, $\mathcal{g}_2$) is quadripartitioned neutrosophic continuous , ie., (X, $\mathcal{g}_2$) ⊂ (X, $\mathcal{g}_1$).

Definition 3.18. Let f be a mapping of a set X into a set Y, and let $\mathfrak{U}$ be a quadripartitioned neutrosophic topology on Y. Then the family $\mathcal{g}_{f^{-1}} = \{f^{-1}(U) \in FNS(X); U \in \mathfrak{U}\}$ is called the inverse image of $\mathfrak{U}$ under

f. $\mathcal{g}_{f^{-1}}$ is the coarser quadripartitioned neutrosophic topology on X for which $f : (X, \mathcal{g}_{f^{-1}}) \to (Y, \mathfrak{A})$ is quadripartioned neutrosophic continuous.

Definition 3.19. Let f be a mapping of a set X into a set Y, and let $\mathcal{g}$ be a quadripartitioned neutrosophic topology on X. Then the family $\mathfrak{A}_f = \{U \in FNS(Y); f^{-1}(U) \in \mathcal{g}\}$ is called the image of $\mathcal{g}$ under f. $\mathfrak{A}_f$ is the finest quadripartitioned neutrosophic topology on Y for which $f : (X, \mathcal{g}) \to (Y, \mathfrak{A}_f)$ is quadripartioned neutrosophic continuous.

Definition 3.20. Given a family $\{(X_\lambda, \mathcal{g}_\lambda)\}_{\lambda\in\Lambda}$ of quadripartitioned neutrosophic topologies and let $X = \prod_{\lambda\in\Lambda} X_\lambda$, let (X, $\mathcal{g}$) a quadripartitioned neutrosophic topological space and let $\mathcal{g}$ the coarsest quadripartitioned neutrosophic topology on X for which $\mathcal{p}_\lambda : (X, \mathcal{g}) \to (X_\lambda, \mathcal{g}_\lambda)$ is quadripartitioned neutrosophic continuous for each $\lambda \in \Lambda$, where $\mathcal{p}_\lambda$ is the usual projection. Then $\mathcal{g}$ is called the quadripartitioned neutrosophic product topology on X and denoted by $\prod_{\lambda\in\Lambda} X_\lambda$ and (X, $\mathcal{g}$) a quadripartitioned neutrosophic product space.

From the definition 6.2.13 and 6.2.20 we have the following proposition.

Definition 3.21. Let $\{(X_\lambda, \mathcal{g}_\lambda)\}_{\lambda\in\Lambda}$ be a family of quadripartitioned neutrosophic topological spaces and (X, $\mathcal{g}$) the quadripartitioned neutrosophic product space. Then $\mathcal{g}$ has a base the set of finite intersections of quadripartitioned neutrosophic sets in X of the form $\mathcal{p}_\lambda^{-1}[U_\lambda]$ where $U_\lambda \in \mathcal{g}_\lambda$ for each $\lambda \in \Lambda$.

Definition 3.22. Let $\{X_i\}$, i=1,2,…..n be a finite family of sets and for each i=1,2,….n, let $A_i$be a quadripartitioned neutrosophic set in X_i.We define the product $A = \prod_{i=1}^{n} A_i$ of the family A_i, i=1,2,…n, as the quadripartitioned neutrosophic set in $X = \prod_{i=1}^{n} X_i$ that has membership function, indeterministic function and non-membership function given by for each $(x_1, x_2, \dots \dots x_n) \in X$

$$T_A(x_1, x_2, \dots \dots x_n) = T_{A1}(x_1) \wedge T_{A2}(x_2) \wedge, \dots \dots . \wedge T_{An}(x_n),$$

$$U_A(x_1, x_2, \dots \dots x_n) = U_{A1}(x_1) \wedge U_{A2}(x_2) \wedge, \dots \dots . \wedge U_{An}(x_n),$$

$$C_A(x_1, x_2, \dots \dots x_n) = C_{A1}(x_1) \vee C_{A2}(x_2) \vee, \dots \dots . \vee C_{An}(x_n) \text{ and}$$

$$F_A(x_1, x_2, \dots \dots x_n) = F_{A1}(x_1) \vee F_{A2}(x_2) \vee, \dots \dots . \vee F_{An}(x_n).$$

Remark 3.23. From the definition 6.2.20 and proposition 6.2.21 that if X_i, has quadripartitioned neutrosophic topology $\mathcal{g}_i$, i=1,2,….n, then the quadripartitioned neutrosophic product topology $\mathcal{g}$ on X has the det of quadripartitioned neutrosophic product spaces of the form $\prod_{i=1}^{n} U_i$ where $U_i \in \mathcal{g}_i$ for each i=1,2,…n.

Proposition 3.24. Let $\{X_i\}, i = 1,2,....n$, be a finite family of sets and let A=$\prod_{i=1}^{n} A_i$the quadripartitioned neutrosophic product space in X = $\prod_{i=1}^{n} X_i$, where $A_i \in$ FNS(X_i) for each i= 1,2,….n. then $\wp_i(A) \subset A_i$ for each i=1,2,…n.

Proof : Let $x_i \in X_i$. Then $T_{\wp_{i(A)}}(x_i) = \wp_i(T_A)(x_i) = \vee$ $T_A(z_1, z_2, \ldots\ldots z_n)$ = $\vee$ $[T_{A1}(z_1) \wedge T_{A2}(z_2) \wedge, \ldots\ldots. \wedge T_{An}(z_n)]$

= $\wedge^{z_1,z_2,\ldots\ldots z_n \wp_i{}^{-1}(x_i)} \{\vee_{z_1 \in x_1} T_{A1}(z_1), \ldots\ldots. \vee_{z_n \in x_n} T_{An}(z_n)\} \leq T_{Ai}(x_i)$. similarly we can prove $C_{\wp i}(x_i) \leq C_{Ai}(x_i), U_{\wp i}(x_i) \geq U_{Ai}(x_i)$ and $F_{\wp i}(x_i) \geq F_{Ai}(x_i)$.Hence $\wp_i(A) \subset A_i$ for each i=1,2,….n.

Proposition 3.25. let $\{X_i, \mathcal{g}_i\}, i = 1,2,....n$ be a finite family of quadripartitioned neutrosophic topological spaces, let (X, $\mathcal{g}$) the quadripartitioned neutrosophic product space and let A= $\prod_{i=1}^{n} A_i$ where A_i a quadripartitioned neutrosophic set in X_i for each i = 1,2,……n. then the induced quadripartitioned neutrosophic topology $\mathcal{g}_A$ on A has a base the set of quadripartitioned neutrosophic spaces of the form $\prod_{i=1}^{n} \acute{U}_i$ where $\acute{U}_i \in (\mathcal{g}_i)_{A_i}$, i= 1,2,…..n.

Proof : by the above remark 6.2.23, $\mathcal{g}$ has a base $\mathfrak{B}$ =$\{\prod_{i=1}^{n} U_i : U_i \in \mathcal{g}_i$, i=1,2,….n}. A base for $\mathcal{g}_A$ is therefore by $\mathfrak{B}_A$ = $\{(\prod_{i=1}^{n} U_i) \cap A : U_i \in \mathcal{g}_i$, i=1,2,…n. But $(\prod_{i=1}^{n} U_i) \cap A = (\prod_{i=1}^{n} U_i \cap A_i)$ and $U_i \cap A_i \in (\mathcal{g}_i)_{A_i}$ for i = 1,2,…..n. Let $\acute{U}_i = U_i \cap A_i$ for each i= 1,2,…n.

Then $\mathfrak{B}_A = \{\prod_{i=1}^{n} \acute{U}_i : \acute{U}_i \in (\mathcal{g}_i)_{A_i}$, i= 1,2,…..n} and we denote the quadripartitioned neutrosophic subspace (A, $\mathcal{g}_A$) by $\prod_{i=1}^{n} A_i \,.\, (\mathcal{g}_i)_{A_i}$.

Proposition 3.26. Let $\{(X_\lambda, \mathcal{g}_\lambda)\}_{\lambda \in \Lambda}$ be a family of quadripartitioned neutrosophic topological spaces, let (X, $\mathcal{g}$) the quadripartitioned neutrosophic product space, (Y,$\mathfrak{A}$) an quadripartitioned neutrosophic topological space and let f : (Y,$\mathfrak{A}$)→ (X,$\mathcal{g}$) . Then f is quadripartitioned neutrosophic continuous iff $\wp_\lambda \circ$ f : (Y,$\mathfrak{A}$)→ $(X_\lambda, \mathcal{g}_\lambda)$ is quadripartitioned neutrosophic continuous for each $\lambda \in \Lambda$.

Proof: suppose f : (Y,$\mathfrak{A}$)→ (X,$\mathcal{g}$) is quadripartitioned neutrosophic continuous. For each $\lambda \in \Lambda$., let $U_\lambda \in \mathcal{g}_\lambda$. But $f^{-1}(\wp_\lambda{}^{-1}(U_\lambda)) = (\wp_\lambda \circ f)^{-1}(U_\lambda)$. Thus $(\wp_\lambda \circ f)^{-1}(U_\lambda) \in \mathfrak{A}$. Hence $\wp_\lambda \circ$ f : (Y,$\mathfrak{A}$)→ $(X_\lambda, \mathcal{g}_\lambda)$ is quadripartitioned neutrosophic continuous.

Conversely, let the necessary condition hold and let U ∈ $\mathcal{g}$. By proposition 6.2.21, there exist a finite subset $\acute{\Lambda}$ of Λ such that U = $(\cap_{\lambda \in \acute{\Lambda}}(\wp_\lambda{}^{-1}(U_\lambda))$=$f^{-1}$(U). so $f^{-1}(U) \in \mathfrak{A}$. Hence f is quadripartitioned neutrosophic continuous.

Corollary 3.27. Let $\{(X_\lambda, \mathcal{g}_\lambda)\}_{\lambda \in \Lambda}, \{(Y_\lambda, \mathfrak{A}_\lambda)\}_{\lambda \in \Lambda}$ be two families of quadripartitioned neutrosophic topological spaces and let (X, $\mathcal{g}$) and (Y,$\mathfrak{A}$) the respectively quadripartitioned neutrosophic product spaces, where X = $\prod_{\lambda \in \Lambda} X_\lambda$ and Y =$\prod_{\lambda \in \Lambda} Y_\lambda$. For each $\lambda \in \Lambda$, let f_λ be mapping of $(X_\lambda, \mathcal{g}_\lambda)$ into $(Y_\lambda, \mathfrak{A}_\lambda)$.

Then the product mapping $f = \prod_{\lambda\in\Lambda} f_\lambda : (X, \mathscr{g}) \to (Y, \mathfrak{A})$ is quadripartitioned neutrosophic continuous iff f_λ is quadripartitioned neutrosophic continuous for each $\lambda \in \Lambda$, where $f(x) = (f_\lambda(\mathscr{p}_\lambda(x))$ for each $x \in \prod_{\lambda\in\Lambda} X_\lambda$.

Proof: The proof is obvious from the above Proposition.

Proposition 3.28. Let $(X_i, \mathscr{g}_i)_i$, i=1,2,….n be a finite family of quadripartitioned neutrosophic topological spaces and $(X, \mathscr{g})$ the quadripartitioned neutrosophic product spaces. For each i= 1,2,…..n, let A_i be a quadripartitioned neutrosophic set in X_i and let $A = \prod_{i=1}^{n} A_i$ a quadripartitioned neutrosophic set in X. Let $(Y, \mathfrak{A})$ be a quadripartitioned neutrosophic topological spaces and let B a quadripartitioned neutrosophic set in Y, and $f : (B, \mathfrak{A}_B) \to (A, \mathscr{g}_A)$ is respectively quadripartitioned neutrosophic continuous iff $\mathscr{p}_\lambda \circ f : (B, \mathfrak{A}_B) \to (A_i, (\mathscr{g}_i)_{A_i})$ is relatively quadripartitioned neutrosophic continuous for each i=1,2,….n.

Proof: Suppose $f : (B, \mathfrak{A}_B) \to (A, \mathscr{g}_A)$ is relatively quadripartitioned neutrosophic continuous. $\mathscr{p}: (X, \mathscr{g}) \to (X_i, \mathscr{g}_i)$ is quadripartitioned neutrosophic continuous for each i= 1,2,…n and by Proposition 6.2.24 $\mathscr{p}(A) \subset A_i$ for each i=1,2,…n. Then by proposition 6.2.9 $\mathscr{p}_\lambda : (A, \mathscr{g}_A)\,) \to (A_i, (\mathscr{g}_i)_{A_i})$ is relatively quadripartitioned neutrosophic continuous for each i=1,2,….n. Hence $\mathscr{p}_i \circ f : (B, \mathfrak{A}_B) \to (A_i, (\mathscr{g}_i)_{A_i})$ is relatively quadripartitioned neutrosophic continuous for each i=1,2,….n. conversely, the necessary condition holds. Let $\acute{U} = \acute{U}_1 \times \ldots\ldots \times \acute{U}_n$ where $\acute{U}_\imath \in ((\mathscr{g}_i)_{A_i})$, i=1,2,…..n. By the Proposition the 6.2.25 set of $\acute{U}$ forms a base for $\mathscr{g}_A$ and $f^{-1}(\acute{U}) \cap B = f^{-1}[\,\mathscr{p}_i{}^{-1}(\acute{U}_1) \cap \ldots\ldots\ldots \cap \mathscr{p}_n{}^{-1}(\acute{U}_n)] \cap B = \cap_{i=1}^{n}((\mathscr{p} \circ f)^{-1}[\acute{U}_\imath] \cap B)$. Since $\mathscr{p}_i \circ f : (B, \mathfrak{A}_B) \to (A_i, (\mathscr{g}_i)_{A_i})$ is relatively quadripartitioned neutrosophic continuous for each i=1,2,….n, $f^{-1}(\acute{U}) \cap B \in \mathfrak{A}_B$. Hence by Proposition 6.2.16 $f : (B, \mathfrak{A}_B) \to (A, \mathscr{g}_A)$ is relatively quadripartitioned neutrosophic continuous.

Corollary 3.29. Let $\{\{X_i, \mathscr{g}_i\}, \{\{Y_i, \mathfrak{A}_i\}$, $i = 1,2,\ldots n$ be two finite families of quadripartitioned neutrosophic topological spaces and $(X, \mathscr{g})$ and $(Y, \mathfrak{A})$ the respective quadripartitioned neutrosophic product spaces. For each i=1,2,….n let A_i be a quadripartitioned neutrosophic set in X_i, B_i a quadriparititioned neutrosophic set in Y_i and $f_i : (A_i, (\mathscr{g}_i)_{A_i}) \to (B_i, (\mathfrak{A}_i)_{B_i})$. Let $A = \prod_{i=1}^{n} A_i$, $B = \prod_{i=1}^{n} B_i$ be the quadripartitioned neutrosophic product space in X,Y respectively. Then the product mapping $f = \prod_{i=1}^{n} f_i : (A_i, (\mathscr{g}_i)_{A_i}) \to (B_i, (\mathfrak{A}_i)_{B_i})$ is relatively quadripartitioned neutrosophic continuous if f_i is relatively quadripartitioned neutrosophic continuous for each i=1,2,…n.

Proposition 3.30. Let $\{(X_i, \mathscr{g}_i), \{(Y_i, \mathfrak{A}_i)\}$, $i = 1,2,\ldots n$ be two finite families of quadripartitioned neutrosophic topological spaces and $(X, \mathscr{g})$ and $(Y, \mathfrak{A})$ the respective quadripartitioned neutrosophic product spaces. For each i=1,2,….n, let $f_i : (X_i, \mathscr{g}_i) \to (Y_i, \mathfrak{A}_i)$.Then the product mapping $f = \prod_{i=1}^{n} f_i : (X, \mathscr{g}) \to (Y, \mathfrak{A})$ is quadripartitioned neutrosophic open if f_i is quadripartitioned neutrosophic for each i=1,2,….n.

Proof : Let U be open in $\mathscr{g}$.Let $\mathfrak{B} = \{\prod_{i=1}^{n} U_i$ a quadripartitioned neutrosophic set in X : $U_i \in \mathscr{g}_i$ for each i=1,2,….n}. Since $\mathfrak{B}$ is a base for $\mathscr{g}$, there is a $\acute{B} \subset \mathfrak{B}$ such that U = $\cup \acute{B}$. Since each member of $\acute{B}$ is the form $\prod_{i=1}^{n} U_i$, we can consider $\acute{B} = \{\prod_{i=1}^{n} U_{i\lambda}\}_{\lambda\in\Lambda}$. Then U= $\cup_{\lambda\in\Lambda}\prod_{i=1}^{n} U_{i\lambda}$.Let $y \in Y$ such that $f^{-1}(y) \neq \emptyset$. Then $T_{f(U)}(y) = f(T_U)(y) = \vee_{z\in f^{-1}(y)} T_U(z)$

$= \vee_{z\in f^{-1}(y)} T_{\cup_{\lambda\in\Lambda}\prod_{i=1}^{n} U_{i\lambda}}(z) = \vee_{z\in f^{-1}(y)} \vee_{\lambda\in\Lambda} T_{\prod_{i=1}^{n} U_{i\lambda}}(z)$

$=\vee_{\lambda\in\Lambda} \vee_{z_1\in {f^{-1}}_1(y_1)} \ldots\ldots \vee_{z_n\in {f^{-1}}_n(y_n)} [T_{U_{1\lambda}}(z_1) \wedge \ldots\ldots \wedge T_{U_{n\lambda}}(z_n)]$

$=\vee_{\lambda\in\Lambda}[\vee_{z_1\in {f^{-1}}_1(y_1)} T_{U_{1\lambda}}(z_1) \wedge \ldots\ldots \wedge \vee_{z_n\in {f^{-1}}_n(y_n)} T_{U_{n\lambda}}(z_n)]$

$=\vee_{\lambda\in\Lambda}[T_{f_{1(U_{1,\lambda})}}(y_1) \wedge \ldots\ldots \wedge T_{f_{n(U_{n,\lambda})}}(y_n)] = \vee_{\lambda\in\Lambda} T_{\prod_{i=1}^{n} U_{i\lambda}}(y) = T_{\cup_{\lambda\in\Lambda}\prod_{i=1}^{n} f_i(U_{i\lambda})}(y)$

$F_{f(U)}(y) = f(T_U)(y) = \wedge_{z\in f^{-1}(y)} F_U(z) = \wedge_{z\in f^{-1}(y)} F_{\cup_{\lambda\in\Lambda}\prod_{i=1}^{n} U_{i\lambda}}(z) = \wedge_{z\in f^{-1}(y)} \wedge_{\lambda\in\Lambda} F_{\prod_{i=1}^{n} U_{i\lambda}}(z)$

$= \wedge_{\lambda\in\Lambda} \wedge_{z_1\in {f^{-1}}_1(y_1)} \ldots\ldots \wedge_{z_n\in {f^{-1}}_n(y_n)} [F_{U_{1\lambda}}(z_1) \vee \ldots\ldots \vee F_{U_{n\lambda}}(z_n)]$

$=\wedge_{\lambda\in\Lambda} [\wedge_{z_1\in {f^{-1}}_1(y_1)} F_{U_{1\lambda}}(z_1) \vee \ldots\ldots \vee \wedge_{z_n\in {f^{-1}}_n(y_n)} F_{U_{n\lambda}}(z_n)]$

$=\wedge_{\lambda\in\Lambda} [F_{f_{1(U_{1,\lambda})}}(y_1) \vee \ldots\ldots \vee F_{f_{n(U_{n,\lambda})}}(y_n)] = \wedge_{\lambda\in\Lambda} F_{\prod_{i=1}^{n} U_{i\lambda}}(y) = F_{\cup_{\lambda\in\Lambda}\prod_{i=1}^{n} f_i(U_{i\lambda})}(y).$

Thus f(U) = $\cup_{\lambda\in\Lambda}\prod_{i=1}^{n} f_i(U_{i\lambda})$. Since f_i is quadripartitioned neutrosophic open for each i=1,2,….n, $f_i(U_{i\lambda})$ is quadripartitioned neutrosophic open X_i for each i=1,2,….n. Then $\prod_{i=1}^{n} f_i(U_{i\lambda})$ is quadripartitioned neutrosophic open in Y. So $\cup_{\lambda\in\Lambda}\prod_{i=1}^{n} f_i(U_{i\lambda})$ is a quadripartitioned neutrosophic open in Y. Hence f is quadripartitioned neutrosophic open.

Proposition 3.31. Let $\{(X_i,\mathscr{g}_i),\{(Y_i,\mathfrak{A}_i)\}$, $i = 1,2,\ldots n$ be two finite families of quadripartitioned neutrosophic topological spaces and (X, $\mathscr{g}$) and (Y,$\mathfrak{A}$) the respective quadripartitioned neutrosophic product spaces. For each i=1,2,….n, let A_i a quadripartitioned neutrosophic set in X_i, B_i a quadripartitioned neutrosophic set in Y_i and let A = $\prod_{i=1}^{n} A_i$, B =$\prod_{i=1}^{n} B_i$ be the quadripartitioned neutrosophic product spaces in X,Y respectively. If f_i: $A_i \to B_i$ is relatively quadripartitioned neutrosophic open for each i=1,2,…n, then the product mapping f = $\prod_{i=1}^{n} f_i$: : $(A,\mathscr{g}_A)$) $\to$ $(B,\mathfrak{A}_B)$ is relatively quadripartitioned neutrosophic open.

Proof : Let $\mathfrak{B} = \{\prod_{i=1}^{n} U_i$ a quadripartitioned neutrosophic set in A : $U_i \in (\mathscr{g}_i)_{A_i}$ for each i=1,2,….n}. Then by proposition 6.2.30 , $\mathfrak{B}$ is a base for $\mathscr{g}_A$. Let U$\in$ $\mathscr{g}_A$. Then there is $\acute{\mathfrak{B}} \subset \mathfrak{B}$ such that $\cup\acute{\mathfrak{B}} = U$. We can consider $\acute{\mathfrak{B}}$ as $\{\prod_{i=1}^{n} U_{i\lambda}\}_{\lambda\in\Lambda}$. Then U=$\cup_{\lambda\in\Lambda}\prod_{i=1}^{n} U_{i\lambda}$. As in the above proposition 6.2.30 we get f(U) = $\cup_{\lambda\in\Lambda}\prod_{i=1}^{n} f_i\ (U_{i\lambda})$. Since f_i is relatively quadripartitioned neutrosophic open for each i=1,2,….n, f(U)$\in \mathfrak{A}_B$. Hence f is relatively quadripartitioned neutrosophic open.

Lemma 3.32. Let $(X_1,\mathscr{g}_1),(Y_2,\mathfrak{A}_2)$ be a quadripartitioned neutrosophic topological spaces. Then the constant mapping f : $(X_2,\mathscr{g}_2) \longrightarrow (X_1,\mathscr{g}_1)$ given by $f(x_2) = x_0 \in X_1$for each $x_2 \in X_2$, is quadripartitioned neutrosophic continuous.

Proof: Let $U \in \mathcal{g}_1$ and let $x_2 \in X_2$. Then $T_{f^{-1}(U)}(x_2) = f^{-1}(T_U)(x_2) = T_U f(x_2) = T_U(x_0)$. Similarly we have $C_{f^{-1}(U)}(x_2) = f^{-1}(C_U)(x_2) = C_U f(x_2) = C_U(x_0), U_{f^{-1}(U)}(x_2) = U_U(x_0)$ and $F_{f^{-1}(U)}(x_2) = F_U(x_0)$. Let $T_U(x_0) = \alpha, C_U(x_0) = \beta, U_U(x_0) = \gamma$ and $F_U(x_0) = \delta$. Consider $S_{\alpha,\beta,\gamma,\delta}$. Since $U \in FNS(X_1), \alpha + \beta + \gamma + \delta \leq 4$. Then $S_{(\alpha,\beta,\gamma,\delta)}$ is quadripartitioned neutrosophic open in X_2. Thus $T_{f^{-1}(U)}(x_2) = \alpha = T_{S_{(\alpha,\beta,\gamma,\delta)}}(x_2), C_{f^{-1}(U)}(x_2) = \beta = C_{S_{(\alpha,\beta,\gamma,\delta)}}(x_2)$, $U_{f^{-1}(U)}(x_2) = \gamma = U_{S_{(\alpha,\beta,\gamma,\delta)}}(x_2)$ and $F_{f^{-1}(U)}(x_2) = \delta = F_{S_{(\alpha,\beta,\gamma,\delta)}}(x_2)$ implies $f^{-1}(U) = S_{(\alpha,\beta,\gamma,\delta)}$. So $f^{-1}(U)$ is quadripartitioned neutrosophic open in X_2. Hence f is quadripartitioned neutrosophic continuous.

Proposition 3.33. Let $(X_1, \mathcal{g}_1), (X_2, \mathcal{g}_2)$ be a quadripartitioned neutrosophic topological spaces and let $(X, \mathcal{g})$ the quadripartitioned neutrosophic product space. Then for each $x_1 \in X_1$ the mapping $i : (X_2, \mathcal{g}_2) \longrightarrow (X, \mathcal{g})$ defined by $i(x_2) = (x_1, x_2)$ for each $x_2 \in X_2$ is quadripartitioned neutrosophic continuous.

Proof: By Lemma 6.2.32 the constant mapping $i_1 : (X_2, \mathcal{g}_2) \longrightarrow (X_1, \mathcal{g})$ given by $i(x_2) = x_1$ for each $x_2 \in X_2$ is quadripartiioned neutrosophic continuous. Then identity mapping $i_2 : (X_2, \mathcal{g}_2) \longrightarrow (X_1, \mathcal{g}_2)$ is quadriopartitioned neutrosophic continuous. Hence by proposition 6.2.26 i is quadripartitioned neutrosophic continuous.

Proposition 3.34. Let $(X_1, \mathcal{g}_1), (X_2, \mathcal{g}_2)$ be a quadripartitioned neutrosophic topological spaces and let $(X, \mathcal{g})$ the quadripartitioned neutrosophic product space. Let A_1, A_2 be quadripartitioned neutrosophic sets in X_1, X_2 respectively and let A the quadripartitioned neutrosophic product space in X. Let $a_1 \in X_1$ such that $T_{A_1}(a_1) \geq T_{A_2}(a_2), C_{A_1}(a_1) \geq C_{A_2}(x_2), U_{A_1}(a_1) \leq U_{A_2}(x_2)$ and $F_{A_1}(a_1) \leq F_{A_2}(x_2)$ for each $x_2 \in X_2$. Then the mapping $i : (A_2, (\mathcal{g}_2)_{A_2}) \longrightarrow (A_1, \mathcal{g}_A)$ given by $i(x_2) = (a_1, x_2)$ for each $x_2 \in X_2$ is relatively quadripartitioned neutrosophic continuous.

Proof : Let $(x_1, x_2) \in X$. Then

$$T_{i(A_2)}(x_1, x_2) = \begin{cases} \vee_{x_2' \in i^{-1}(x_1,x_2)} T_{A_2}(x_2') & \text{if } i^{-1}(x_1, x_2) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} = \begin{cases} T_{A_2}(x_2) & \text{if } x_1 = a_1 \\ 0 & \text{otherwise} \end{cases}$$

$$C_{i(A_2)}(x_1, x_2) = \begin{cases} \vee_{x_2' \in i^{-1}(x_1,x_2)} C_{A_2}(x_2') & \text{if } i^{-1}(x_1, x_2) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} = \begin{cases} C_{A_2}(x_2) & \text{if } x_1 = a_1 \\ 0 & \text{otherwise} \end{cases}$$

$$U_{i(A_2)}(x_1, x_2) = \begin{cases} \wedge_{x_2' \in i^{-1}(x_1,x_2)} U_{A_2}(x_2') & \text{if } i^{-1}(x_1, x_2) \neq \emptyset \\ 1 & \text{otherwise} \end{cases} = \begin{cases} U_{A_2}(x_2) & \text{if } x_1 = a_1 \\ 1 & \text{otherwise} \end{cases}$$

$$F_{i(A_2)}(x_1, x_2) = \begin{cases} \wedge_{x_2' \in i^{-1}(x_1,x_2)} F_{A_2}(x_2') & \text{if } i^{-1}(x_1, x_2) \neq \emptyset \\ 1 & \text{otherwise} \end{cases} = \begin{cases} F_{A_2}(x_2) & \text{if } x_1 = a_1 \\ 1 & \text{otherwise} \end{cases}$$ and $T_A(x_1, x_2) = T_A(x_1) \wedge T_A(x_2), C_A(x_1, x_2) = C_A(x_1) \wedge C_A(x_2), U_A(x_1, x_2) = U_A(x_1) \vee U_A(x_2)$ and $F_A(x_1, x_2) = F_A(x_1) \vee F_A(x_2)$. By the assumption, $T_A(x_1, x_2) \geq T_A(x_2), C_A(x_1, x_2) \geq C_A(x_2), U_A(x_1, x_2) \leq$

$U_A(x_2)$ and $F_A(x_1, x_2) \leq F_A(x_2)$. Hence i(A) ⊂ A. The proof of relative continuity of i is similar to the proof of quadripartitioned neutrosophic continuity of i in proposition 6.2.33.

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ON ALGEBRAIC PROPERTIES OF BIPOLAR $\kappa - Q$ –FUZZY ORDER SUBGROUPS AND $\kappa - Q$ – Fuzzy Order NORMAL SUBGROUPS

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Abstract:

In this paper we introduce the Bipolar $\kappa - Q -$ fuzzy subsets and show that Bipolar $\kappa - Q -$ fuzzy Subgroups and Normal Subgroups. Furthermore, over we initiate the Bipolar $\kappa - Q -$ Fuzzy order subgroups and Normal subgroups of the various group theoretical proofs.

Key Words:

Fuzzy Set (FS), fuzzy subset (FSb) , $\kappa - Q$ –fuzzy subset $(\kappa - Q - FSb)$, fuzzy orders (FO), fuzzy group (FG),fuzzy subgroup(FSG), $\kappa - Q$ –fuzzy orders $(\kappa - Q - FO)$, $\kappa - Q$ –fuzzy group$(\kappa - Q - FG)$, $\kappa - Q$ –fuzzy subgroup$(\kappa - Q - FSG)$, κ $-Q$ –fuzzy normal subgroup$(\kappa - Q - FNSG)$ *and* $\kappa - Q$ –Cyclic group.

1. Introduction

Abou-Zaid S[1] initiated by On generalized characteristic fuzzy subgroups of a finite group in 1991. In 1988, Some properties of fuzzy groups developed by Akgul M[2]. Asaad M[3] introduced by the concepts of Groups and fuzzy subgroups 1991. Atanassov K T[4], described the notation Intuitionistic fuzzy sets 1986. In 1981, described the notation of fuzzy groups and level subgroups in Das P S[5]. Lee K J[5 & 6] introduced by the new notation of Bipolar valued fuzzy sub algebras and bipolar fuzzy ideals of BCK/BCI-algebras in 2009 and also introduced the concept of Bipolar valued fuzzy sets and their operations. Rosenfeld A[14] described the new concept of Fuzzy groups in 1971. In 1982, Fuzzy invariant subgroups and fuzzy ideals developed by Liu W J[8]. Jae-Gyeom Kim, Fuzzy orders Relative to fuzzy subgroups 1994. In 1984 initated by the concept of Fuzzy Normal subgroups and fuzzy cosets by Mukherjee N P[10]. Prasanna[11] et.al.. introduced by the concept of $\kappa - Q -$Fuzzy Orders Relative to $\kappa - Q -$Fuzzy Subgroups and Cyclic group on various fundamental aspects in 2020 and also Fundamental Algebraic Properties on $\kappa - Q -$ Anti Fuzzy Normed Prime Ideal and $\kappa - Q -$ Anti Fuzzy Normed Maximal Ideal in 2021[12]. Solairaju A and Nagarajan R[13] developed by the new structure and construction of Q- fuzzy groups in 2009. Zadeh L A[15] introduced the concept of fuzzy sets 1965. In 2004, Zhang W R[16] developed the new notation of Bipolar logic and bipolar fuzzy logic. Zimmermann H J[17] described by the notation of fuzzy set theory and its applications in 1985.

In this paper we introduce the Bipolar $\kappa - Q -$ fuzzy subsets and show that Bipolar $\kappa - Q -$ fuzzy Subgroups and Normal Subgroups. Furthermore, over we initiate the Bipolar $\kappa - Q -$ Fuzzy order subgroups and Normal subgroups of the various group theoretical proofs.

2. Preliminaries

Definition: 2.1[15]

If X may be a nonempty set, then a function $\mu: X \to [0,1]$ may be a FSb of X.

Definition: 2.2[14]

Let G be a group. A FSb μ of G may be a FSG of G if

(i) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$

(ii) $\mu(x^{-1}) \geq \mu(x)$, for all $x, y \in G$.

(iii) From this definition, we obviously have $\mu(x^{-1}) = \mu(x)$, for all $x, y \in G$.

Definition: 2.3[14]

Let G be a group. A FSG μ of G is normal in G if $\mu(xy) = (yx)$ for all $x, y \in G$.

Definition: 2.4[11]

Let G and Q be any two nonempty sets and $\kappa \in [0,1]$ and μ be a $\bar{Q} - FSb$ of a set G. The FS μ^{κ} of G is called the $\kappa - Q - FSb$ of G is defined by

$\mu^{\kappa}(x, q) = (\mu(x, q), \kappa), \forall\, x \in G \text{ and } q \in Q.$

3. On Algebraic Aspects of Bipolar $\kappa - Q -$ fuzzy Order subgroups and Normal Subgroups

Definition: 3.1

Let G and Q be any two nonempty sets and $\kappa \in [0,1]$ and A^k be a $Q - FSb$ of a set G. The fuzzy set A^κ of G is called the Bipolar $\kappa - Q - FSb$ of G is defined by

(i) $\gamma_{A^{\kappa -}}(\theta, q) \geq min\{\gamma_{A^-}((\theta, q), \kappa)\}$
(ii) $\gamma_{A^{\kappa +}}(\theta, q) \leq max\{\gamma_{A^+}((\theta, q), \kappa)\}$, $\forall\, \theta \in G$ *and* $q \in Q$.

Definition: 3.2

Let A^k be a Bipolar $\kappa - Q$-fuzzy subgroup of a group G. For a given $\theta \in G$ and $q \in Q$, the positive integer n such that the following conditions are

(i) $\gamma_{A^{\kappa -}}(\theta^n, q) \geq min\{\gamma_{A^-}((e, q), \kappa)\}$
(ii) $\gamma_{A^{\kappa +}}(\theta^n, q) \leq max\{\gamma_{A^+}((e, q), \kappa)\}$, $\forall\, \theta \in G$ *and* $q \in Q$.

Definition: 3.3

The above definition of the two conditions is Bipolar $\kappa - Q$-fuzzy order of θ with reference to $A^k$is $FO_{A^{\kappa -}}(\theta, q)$ and $FO_{A^{\kappa +}}(\theta, q)$. If no such n exists θ is infinite Bipolar $\kappa - Q$-fuzzy order reference to A^k.
$\Rightarrow$ $O(\theta, q)$ and $O(\varphi, q)$ doesn't imply that of $FO_{A^{\kappa -}}(\theta, q), FO_{A^{\kappa -}}(\varphi, q)$ and $FO_{A^{\kappa +}}(\theta, q)$, $FO_{A^{\kappa +}}(\varphi, q)$.

Example: 3.3.1

Let $G = \{a, b / a^2 = b^2 = e\}$ be the 4 group.
Define Bipolar $\kappa - Q$- fuzzy subgroup A^k of G by $\gamma_{A^{\kappa -}}(e, q) = \gamma_{A^-}((ab, q), \kappa)$ *and equal to* $\gamma_{A^{\kappa +}}(e, q) = \gamma_{A^+}((ab, q), \kappa) = t_o$ and $\gamma_{A^{\kappa -}}(a, q) = \gamma_{A^-}((a, q), \kappa)$ *and equal to* $\gamma_{A^{\kappa +}}(a, q) = \gamma_{A^+}((a, q), \kappa) = t_1$, Where $t_o > t_1$ and $q \in Q$.

Clearly $O(a, q) = O(ab, q) = 2$ but $FO_{A^{\kappa -}}(a, q) = FO_{A^{\kappa +}}(a, q) = 2$ and $FO_{A^{\kappa -}}(ab, q) = FO_{A^{\kappa +}}(ab, q) = 1$.

Theorem: 3.4

Let A^k be a Bipolar $\kappa - Q$-fuzzy subgroup G. For $\theta \in G$ and $q \in Q$, if
(i) $\gamma_{A^{\kappa -}}(\theta^m, q) \geq min\{\gamma_{A^-}((e, q), \kappa)\}$
(ii) $\gamma_{A^{\kappa +}}(\theta^m, q) \leq max\{\gamma_{A^+}((e, q), \kappa)\}$, for a few integer m then $FO_{A^{\kappa -}}(\theta, q)$ *and* $FO_{A^{\kappa +}}(\theta, q)$ divides m.

Proof:

Let $FO_{A^{\kappa -}}(\theta, q) = FO_{A^{\kappa +}}(\theta, q)$.
If there exist integers s *and* t such that $m = ns + t$, where $0 \leq t < n$ and $q \in Q$.
Then,

(i)
$$\begin{aligned} \gamma_{A^{\kappa -}}(\theta^n, q) &= \gamma_{A^{\kappa -}}(\theta^{m-ns}, q) \\ &= \gamma_{A^{\kappa -}}(\theta^m (\theta^n)^{-s}, q) \\ &\geq min\{\, \gamma_{A^-}((\theta^m, q), \kappa), \gamma_{A^-}(((\theta^n)^{-s}, q), \kappa)\, \} \end{aligned}$$

$$\geq min\{\gamma_{A^-}((e,q),\kappa),\gamma_{A^-}((e,q),\kappa)\}$$
$$\gamma_{A^{\kappa-}}(\theta^n,q)=min\{\gamma_{A^-}((e,q),\kappa)\}$$

(ii) $\gamma_{A^{\kappa+}}(\theta^n,q)=\ \gamma_{A^{\kappa+}}(\theta^{m-ns},q)$

$$=\ \gamma_{A^{\kappa+}}(\theta^m(\theta^n)^{-s},q)$$
$$\leq max\{\ \gamma_{A^+}((\theta^m,q),\kappa),\gamma_{A^+}(((\theta^n)^{-s},q),\kappa)\ \}$$
$$\leq max\{\gamma_{A^+}((e,q),\kappa),\gamma_{A^+}((e,q),\kappa)\}$$
$$\gamma_{A^{\kappa+}}(\theta^n,q)=max\{\gamma_{A^+}((e,q),\kappa)\}$$

Hence $t=0$ by the choice of n.

If $O(\theta,q)$is finite then $FO_{A^{\kappa-}}(\theta,q)\ and\ FO_{A^{\kappa+}}(\theta,q)$ is clearly finite from for all Bipolar $\kappa-Q$-fuzzy subset A^k of G.

If $O(\theta,q)$ is infinite then for every positive integer n there exists a Bipolar $\kappa-Q$-fuzzy subgroup γ_{A^κ} of G such that $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta,q)=FO_{\gamma_{A_n}{}^{\kappa+}}(\theta,q)=n$.

Theorem: 3.5

Let γ_{A^κ} be a Bipolar $\kappa-Q$-fuzzy subgroup of a group G and let $\theta\ and\ \varphi$ be two elements of G such that $\left[FO_{\gamma_{A_n}{}^{\kappa-}}(\theta,q),FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi,q)\right]=1=\left[FO_{\gamma_{A_n}{}^{\kappa+}}(\theta,q),FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi,q)\right]$ and $\theta\varphi=\varphi\theta$. If $\gamma_{A^{\kappa-}}(\theta\varphi,q)=\gamma_{A^-}((e,q),\kappa)$ and $\gamma_{A^{\kappa+}}(\theta\varphi,q)=\gamma_{A^+}((e,q),\kappa)$ then $\gamma_{A^{\kappa-}}(\theta,q)=\gamma_{A^-}((e,q),\kappa)=\gamma_{A^{\kappa-}}(\varphi,q)$ and $\gamma_{A^{\kappa+}}(\theta,q)=\gamma_{A^+}((e,q),\kappa)=\gamma_{A^{\kappa+}}(\varphi,q)$.

Proof:

Let $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta,q)=FO_{\gamma_{A_n}{}^{\kappa+}}(\theta,q)=n$ and $FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi,q)=FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi,q)=m$.

Then

(i) $min\{\gamma_{A^-}((e,q),\kappa)\}=\{\gamma_{A^{\kappa-}}(\theta\varphi,q)\}$

$$\geq\{\gamma_{A^{\kappa-}}((\theta\varphi)^m,q)\}$$
$$\geq\{\gamma_{A^{\kappa-}}(\theta^m\varphi^m,q)\}.$$

(ii) $max\{\gamma_{A^+}((e,q),\kappa)\}=\{\gamma_{A^{\kappa+}}(\theta\varphi,q)\}$

$$\leq\{\gamma_{A^{\kappa+}}(\theta\varphi)^m,q\}$$
$$=\{\gamma_{A^{\kappa+}}(\theta^m\varphi^m,q)\}.$$

$\Rightarrow\gamma_{A^{\kappa-}}(\theta^m,q)=\gamma_{A^{\kappa-}}(\varphi^m,q)=min\{\gamma_{A^-}((e,q),\kappa)\}$ and

$\gamma_{A^{\kappa+}}(\theta^m,q)=\gamma_{A^{\kappa+}}(\varphi^m,q)=max\{\gamma_{A^+}((e,q),\kappa)\}$

Therefore, n/m

But $(n,m)=1$. If $n=1.,ie\ \gamma_{A^{\kappa-}}(\theta,q)=min\{\gamma_{A^-}((e,q),\kappa)\}=\gamma_{A^{\kappa-}}(\varphi,q)$ and

$\gamma_{A^{\kappa+}}(\theta,q)=max\{\gamma_{A^+}((e,q),\kappa)\}=\gamma_{A^{\kappa+}}(\varphi,q)$.

Hence,

$\Rightarrow \gamma_{A^{\kappa}}$ is normal the belief $\theta\varphi = \varphi\theta$ may not be omitted.

Theorem: 3.6

Let $\gamma_{A^{\kappa}}$ be a Bipolar $\kappa - Q$-fuzzy subgroup of a group G. Let $FO_{\gamma_{A_n}\kappa^-}(\theta, q) = n - FO_{\gamma_{A_n}\kappa^+}(\theta, q)$, where $\theta \in G$ and $q \in Q$. If m may be an integer with $d = (m, n)$ then $FO\gamma_{A_n}{}^{\kappa^-}(\theta^m, q) = \frac{n}{d} = FO\gamma_{A_n}{}^{\kappa^+}(\theta^m, q)$.

Proof:

Let $FO\gamma_{A_n}{}^{\kappa^-}(\theta^m, q) = FO\gamma_{A_n}{}^{\kappa^+}(\theta^m, q) = t$

First we have

(i) $\gamma_{A^{\kappa^-}}\left((\theta^m, q)^{\frac{n}{d}}\right) = \gamma_{A^{\kappa^-}}(\theta^{np}, q)$

$\geq min\{\gamma_{A^{\kappa^-}}(\theta^n, q)\}$

$\geq min\{\gamma_{A^-}((e, q), \kappa)\}$, $\forall\, p$ is integer and $q \in Q$.

(ii) $\gamma_{A^{\kappa^+}}\left((\theta^m, q)^{\frac{n}{d}}\right) = \gamma_{A^{\kappa^+}}(\theta^{np}, q)$

$\leq max\{\gamma_{A^{\kappa^+}}(\theta^n, q)\}$

$\leq max\{\gamma_{A^+}((e, q), \kappa)\}$, $\forall\, p$ is integer and $q \in Q$

Thus n/d becomes $d = (m, n)$ there exist integer i and j such that $n_i + m_j = d$.

Now

(i) $\gamma_{A^{\kappa^-}}(\theta^{td}, q) = \gamma_{A^{\kappa^-}}\left(\theta^{t(n_i+m_j)}, q\right)$

$\geq min\{\gamma_{A^{\kappa^-}}((\theta^n)^{t_i}(\theta^m)^{t_j}, q)\}$

$\geq min\{min\{\gamma_{A^{\kappa^-}}((\theta^n)^{t_i}, q), \gamma_{A^{\kappa^-}}((\theta^m)^{t_j}, q)\}\}$

$\geq min\{min\{\gamma_{A^{\kappa^-}}((\theta^n)^{t}, q), \gamma_{A^{\kappa^-}}((\theta^m)^{t}, q)\}\}$

$= min\left\{min\{\gamma_{A^-}((e, q), \kappa), \gamma_{A^-}((e, q), \kappa)\}\right\}$

$= min\{\gamma_{A^-}((e, q), \kappa)\}$.

(ii) $\gamma_{A^{\kappa^+}}(\theta^{td}, q) = \gamma_{A^{\kappa^+}}\left(\theta^{t(n_i+m_j)}, q\right)$

$\leq max\{\gamma_{A^{\kappa^+}}((\theta^n)^{t_i}(\theta^m)^{t_j}, q)\}$

$\leq max\{max\{\gamma_{A^{\kappa^+}}((\theta^n)^{t_i}, q), \gamma_{A^{\kappa^+}}((\theta^m)^{t_j}, q)\}\}$

$\leq max\{max\{\gamma_{A^{\kappa^+}}((\theta^n)^{t}, q), \gamma_{A^{\kappa^+}}((\theta^m)^{t}, q)\}\}$

$= max\left\{max\{\gamma_{A^+}((e, q), \kappa), \gamma_{A^+}((e, q), \kappa)\}\right\}$

$= max\{\gamma_{A^+}((e, q), \kappa)\}$.

$\Rightarrow n/td.$, $n/d/t$. Consequently $t = nd$.

Theorem: 3.7

Let $\gamma_{A^{\kappa}}$ be a Bipolar $\kappa - Q$-fuzzy subgroup of a group G. Let $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta, q) = n = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta, q)$, where $\theta \in G$ and $q \in Q$. If m may be an integer with $(n, m) = 1$ then $\gamma_{A^{\kappa-}}(\theta^m, q) = \gamma_{A^{\kappa-}}(\theta, q)$ and $\gamma_{A^{\kappa+}}(\theta^m, q) = \gamma_{A^{\kappa+}}(\theta, q)$.

Proof:

Let $(n, m) = 1$ there exist s and t such that $ns + mt = 1, \forall\, q \in Q$.

(i) $\gamma_{A^{\kappa-}}(\theta, q) = \gamma_{A^{\kappa-}}(\theta^{ns+mt}, q)$

$$\geq min\{\gamma_{A^{\kappa-}}\big((\theta^s, q), (\theta^{mt}, q)\big)\}$$

$$\geq min\{\gamma_{A^{\kappa-}}\big(((\theta^n)^s, q), ((\theta^m)^t, q)\big)\}$$

$$\geq min\left\{min\{\gamma_{A^-}\big((e, q), \kappa\big), \gamma_{A^{\kappa-}}\ (\theta^m, q)\}\right\}$$

$\Rightarrow \gamma_{A^{\kappa-}}(\theta, q) = min\{\gamma_{A^{\kappa-}}\ (\theta^m, q)\} = \gamma_{A^{\kappa-}}\ (\theta^m, q)$.

(ii) $\gamma_{A^{\kappa+}}(\theta, q) = \gamma_{A^{\kappa+}}(\theta^{ns+mt}, q)$

$$\leq max\{\gamma_{A^{\kappa+}}\big((\theta^s, q), (\theta^{mt}, q)\big)\}$$

$$\leq max\{\gamma_{A^{\kappa+}}\big(((\theta^n)^s, q), ((\theta^m)^t, q)\big)\}$$

$$\leq max\left\{max\{\gamma_{A^+}\big((e, q), \kappa\big), \gamma_{A^{\kappa+}}\ (\theta^m, q)\}\right\}$$

$\Rightarrow \gamma_{A^{\kappa+}}(\theta, q) = max\{\gamma_{A^{\kappa+}}\ (\theta^m, q)\} = \gamma_{A^{\kappa+}}\ (\theta^m, q)$.

Theorem: 3.7

Let $\gamma_{A^{\kappa}}$ be a Bipolar $\kappa - Q$-fuzzy subgroup of a group G. Let $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta, q) = n \text{ and } FO_{\gamma_{A_n}{}^{\kappa+}}(\theta, q) = n$, where $\theta \in G$ and $q \in Q$, if $i = j(modn)$, where $i, j \in Z$ then $FO_{\gamma_{A_n}{}^{\kappa-}}\big(\theta^i, q\big) = FO_{\gamma_{A_n}{}^{\kappa-}}\big(\theta^j, q\big)$ and $FO_{\gamma_{A_n}{}^{\kappa+}}\big(\theta^i, q\big) = FO_{\gamma_{A_n}{}^{\kappa+}}\big(\theta^j, q\big)$.

Proof:

Let $FO_{\gamma_{A_n}{}^{\kappa-}}\big(\theta^i, q\big) = t = FO_{\gamma_{A_n}{}^{\kappa+}}\big(\theta^i, q\big)$ and $FO_{\gamma_{A_n}{}^{\kappa-}}\big(\theta^j, q\big) = t = FO_{\gamma_{A_n}{}^{\kappa+}}\big(\theta^j, q\big) = s$

by the assumption $i = j + nk$ for all integer k and $q \in Q$

(i) $\gamma_{A^{\kappa-}}\big((\theta^i)^s, q\big) = \gamma_{A^{\kappa-}}\big((\theta^{(j+nk)})^s, q\big)$

$$= min\{\gamma_{A^{\kappa-}}\big((\theta^j)^s (\theta^n)^{ks}, q\big)\}$$

$$\geq min\left\{min\{\gamma_{A^{\kappa-}}\big((\theta^j)^s, q\big), \gamma_{A^{\kappa-}}((\theta^n)^{ks}, q)\}\right\}$$

$$\geq min\left\{min\{\gamma_{A^-}\big((e, q), \kappa\big), \gamma_{A^{\kappa-}}((\theta^n)^s, q)\}\right\}$$

$$\geq min\left\{min\{\gamma_{A^-}\big((e, q), \kappa\big), \gamma_{A^-}\big((e, q), \kappa\big)\}\right\}$$

$$\geq min\{\gamma_{A^-}((e,q),\kappa)\}.$$

$$\Rightarrow \gamma_{A^{\kappa-}}\left(\left(\theta^i\right)^s,q\right) \geq min\{\gamma_{A^-}((e,q),\kappa)\}$$

(ii) $\gamma_{A^{\kappa+}}\left(\left(\theta^i\right)^s,q\right) = \gamma_{A^{\kappa+}}\left(\left(\theta^{(j+nk)}\right)^s,q\right)$

$$= max\{\gamma_{A^{\kappa+}}\left(\left(\theta^j\right)^s(\theta^n)^{ks},q\right)\}$$

$$\leq max\left\{max\{\gamma_{A^{\kappa+}}\left(\left(\theta^j\right)^s,q\right),\gamma_{A^{\kappa+}}((\theta^n)^{ks},q)\}\right\}$$

$$\leq max\left\{max\{\gamma_{A^+}((e,q),\kappa),\gamma_{A^{\kappa+}}((\theta^n)^s,q)\}\right\}$$

$$\leq max\left\{max\{\gamma_{A^+}((e,q),\kappa),\gamma_{A^+}((e,q),\kappa)\}\right\}$$

$$\leq max\{\gamma_{A^+}((e,q),\kappa)\}.$$

$$\Rightarrow \gamma_{A^{\kappa+}}\left(\left(\theta^i\right)^s,q\right) \leq max\{\gamma_{A^+}((e,q),\kappa)\}$$

And also t/s, similarly s/t.

Thus we have $t = s$.

Theorem: 3.8

Let γ_{A^κ} be a Bipolar $\kappa - Q$-fuzzy subgroup of a group G, and let θ and φ be element of G and $q \in Q$ there exists $\theta\varphi = \varphi\theta$ and $\left[FO_{\gamma_{A_n}{}^{\kappa-}}(\theta,q),FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi,q)\right] = 1$ and $\left[FO_{\gamma_{A_n}{}^{\kappa+}}(\theta,q),FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi,q)\right] = 1$. Then $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta\varphi,q) = FO_{\gamma_{A_n}{}^{\kappa-}}(\theta,q) \times FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi,q)$ and $FO_{\gamma_{A_n}{}^{\kappa+}}(\theta\varphi,q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta,q) \times FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi,q)$.

Proof:

Let $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta\varphi,q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta\varphi,q) = n$, $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta,q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta,q) = s$ and

$$FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi,q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi,q) = t$$

Then,

(i) $\gamma_{A^{\kappa-}}((\theta\varphi)^{st},q) = \gamma_{A^{\kappa-}}(\theta^{st}\varphi^{st},q)$

$$\geq min\{\gamma_{A^{\kappa-}}((\theta^s)^t(\varphi^s)^t,q)\}$$

$$\geq min\{min\{\gamma_{A^{\kappa-}}((\theta^s)^t,q),\gamma_{A^{\kappa-}}((\varphi^s)^t,q)\}\}$$

$$\geq min\{min\{\gamma_{A^{\kappa-}}(\theta^s,q),\gamma_{A^{\kappa-}}(\varphi^s,q)\}\}$$

$$\geq min\left\{min\{\gamma_{A^-}((e,q),\kappa),\gamma_{A^-}((e,q),\kappa)\}\right\}$$

$$\geq min\{\gamma_{A^-}((e,q),\kappa)\}$$

Thus n/st,

$$\Rightarrow \gamma_{A^{\kappa-}}((\theta\varphi)^n,q) = \gamma_{A^{\kappa-}}(\theta^n\varphi^n,q)$$

$$= min\{\gamma_{A^-}((e,q),\kappa)\}.$$

(ii) $\gamma_{A^{\kappa+}}((\theta\varphi)^{st}, q) = \gamma_{A^{\kappa+}}(\theta^{st}\varphi^{st}, q)$

$$\leq max\{\gamma_{A^{\kappa-}}((\theta^s)^t(\varphi^s)^t, q)\}$$

$$\leq max\{max\{\gamma_{A^{\kappa+}}((\theta^s)^t, q), \gamma_{A^{\kappa+}}((\varphi^s)^t, q)\}\}$$

$$\leq max\{max\{\gamma_{A^{\kappa+}}(\theta^s, q), \gamma_{A^{\kappa+}}(\varphi^s, q)\}\}$$

$$\leq max\left\{max\{\gamma_{A^+}\big((e,q),\kappa\big), \gamma_{A^+}\big((e,q),\kappa\big)\}\right\}$$

$$\leq max\{\gamma_{A^+}\big((e,q),\kappa\big)\}$$

Besides $\left[FO_{\gamma_{A_n}{}^{\kappa-}}(\theta^n, q) = FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi^n, q)\right] = 1$ and $\left[FO_{\gamma_{A_n}{}^{\kappa+}}(\theta^n, q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi^n, q)\right] = 1$.

Therefore

$\gamma_{A^{\kappa-}}(\theta^n, q) = \gamma_{A^{\kappa-}}(\varphi^n, q) = min\{\gamma_{A^-}\big((e,q),\kappa\big)\}$ and$\gamma_{A^{\kappa+}}(\theta^n, q) = \gamma_{A^{\kappa+}}(\varphi^n, q) = min\{\gamma_{A^+}\big((e,q),\kappa\big)\}$

then both $s\ and\ t$ divide on n.

Therefore st/n because $(s,t) = 1$.

$$\Rightarrow n = st.$$

Corollary: 3.8.1

Let γ_{A^κ} be a Bipolar $\kappa - Q$-fuzzy subgroup of a group G, and let θ and φ be element of G and $q \in Q$ there exists $O(\theta\varphi) = O(\varphi\theta)$=1. Then $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta\varphi, q) = FO_{\gamma_{A_n}{}^{\kappa-}}(\theta, q) \times FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi, q)$ and $FO_{\gamma_{A_n}{}^{\kappa+}}(\theta\varphi, q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta, q) \times FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi, q)$. Since γ_{A^κ} is normal subgroup the assumption $\theta\varphi = \varphi\theta$ may not be omitted.

Result: 3.8.2

Let γ_{A^κ} be a Bipolar $\kappa - Q$-normal fuzzy subgroup of a symmetric group S_4.

$$\gamma_{A^{\kappa-}}(\theta, q) = \gamma_{A^{\kappa+}}(\theta, q) = \begin{cases} t_o, & if\ \theta = e \\ t_1, if\ otherwise \end{cases}$$

Where $t_o > t_1$ and $q \in Q$.

Now , let $\theta = (1{,}2)$ and $\varphi = (1{,}2{,}3)$

Then $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta, q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta, q) = 2$, $FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi, q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi, q) = 3$, $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta\varphi, q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta\varphi, q) = 4$ and $\theta\varphi \neq \varphi\theta$.

Theorem: 3.9

Let γ_{A^κ} be a Bipolar $\kappa - Q$-fuzzy subgroup of a group G. For $Z \in G$ and $q \in Q$. If $FO_{\gamma_{A_n}{}^{\kappa-}}(Z, q) = nm$ with $(n, m) = 1$ then there exists θ and φ in $G\ and\ q \in Q$ such that $Z = \theta\varphi = \varphi\theta, FO_{\gamma_{A_n}{}^{\kappa-}}(\theta, q) = m = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta, q)$ and $FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi, q) = n = FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi, q)$. Furthermore, explain for Z is exclusive with in the Bipolar $\kappa - Q$-fuzzy grades. If $(\theta,\varphi),(\theta_1,\varphi_1)$ and $q \in Q$ are such pair then $\gamma_{A^{\kappa-}}(\theta,q) = \gamma_{A^{\kappa+}}(\theta,q) = \gamma_{A^{\kappa-}}(\theta_1,q) = \gamma_{A^{\kappa+}}(\theta_1,q)$ and $\gamma_{A^{\kappa-}}(\varphi,q) = \gamma_{A^{\kappa+}}(\varphi,q) = \gamma_{A^{\kappa-}}(\varphi_1,q) = \gamma_{A^{\kappa+}}(\varphi_1,q)$.

Proof:

Let $(n,m) = 1$ there exists integer $s\ and\ t$ such that $ms + nt = 1\ and\ q \in Q$.

Here $(m,t) = (n,s) = 1$.

Let $\theta = Z^{nt}$, $\varphi = Z^{ms}$ and $q \in Q$.

Then $Z = \theta\varphi = \varphi\theta$ by theorem,

$$FO_{\gamma_{A_n}{}^{\kappa-}}(\theta,q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta,q) = FO_{\gamma_{A_n}{}^{\kappa-}}(Z^{nt},q) = FO_{\gamma_{A_n}{}^{\kappa+}}(Z^{nt},q) = m$$

and $FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi,q) = n = FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi,q) = FO_{\gamma_{A_n}{}^{\kappa-}}(Z^{ms},q) = FO_{\gamma_{A_n}{}^{\kappa+}}(Z^{ms},q)$

Let $(\theta,\varphi),(\theta_1,\varphi_1)$ and $q \in Q$ be pair satisfied.

$$\Rightarrow FO_{\gamma_{A_n}{}^{\kappa-}}(\theta,q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta,q) = FO_{\gamma_{A_n}{}^{\kappa-}}(\theta_1,q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\theta_1,q) = m$$

and $FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi,q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi,q) = FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi_1,q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi_1,q) = n$

We obtain

(i)
$$\begin{aligned}
\gamma_{A^{\kappa-}}(\theta,q) &= min\{\gamma_{A^{\kappa-}}(\theta^{1-ms},q)\} \\
&= min\{\gamma_{A^{\kappa-}}(\theta^{nt},q)\} \\
&= min\{\gamma_{A^{\kappa-}}(\theta^{nt}\varphi^{nt},q)\} \\
&= min\{\gamma_{A^{\kappa-}}(\theta\varphi)^{nt},q\} \\
&\geq min\{min\{\gamma_{A^{\kappa-}}(\theta^{nt},q),\gamma_{A^{\kappa-}}(\varphi^{nt},q)\}\} \\
&= min\{\gamma_{A^{\kappa-}}({\theta_1}^{nt},q)\} \\
&= min\{\gamma_{A^{\kappa-}}({\theta_1}^{1-ms},q)\}
\end{aligned}$$

$$\Rightarrow \gamma_{A^{\kappa-}}(\theta,q) \geq min\{\gamma_{A^{\kappa-}}(\theta_1,q)\}.$$

(ii)
$$\begin{aligned}
\gamma_{A^{\kappa+}}(\theta,q) &= max\{\gamma_{A^{\kappa+}}(\theta^{1-ms},q)\} \\
&= max\{\gamma_{A^{\kappa+}}(\theta^{nt},q)\} \\
&= max\{\gamma_{A^{\kappa+}}(\theta^{nt}\varphi^{nt},q)\} \\
&= max\{\gamma_{A^{\kappa+}}(\theta\varphi)^{nt},q\} \\
&\leq max\{max\{\gamma_{A^{\kappa+}}(\theta^{nt},q),\gamma_{A^{\kappa+}}(\varphi^{nt},q)\}\} \\
&= max\{\gamma_{A^{\kappa+}}({\theta_1}^{nt},q)\} \\
&= max\{\gamma_{A^{\kappa+}}({\theta_1}^{1-ms},q)\}
\end{aligned}$$

$$\Rightarrow \gamma_{A^{\kappa+}}(\theta,q) \leq max\{\gamma_{A^{\kappa+}}(\theta_1,q)\}.$$

(iii)
$$\begin{aligned}
\gamma_{A^{\kappa-}}(\varphi,q) &= min\{\gamma_{A^{\kappa-}}(\varphi^{1-ms},q)\} \\
&= min\{\gamma_{A^{\kappa-}}(\varphi^{nt},q)\} \\
&= min\{\gamma_{A^{\kappa-}}(\varphi^{nt}\theta^{nt},q)\}
\end{aligned}$$

$$= min\{\gamma_{A^{\kappa-}}(\varphi\theta)^{nt}, q\}$$

$$\geq min\{min\{\gamma_{A^{\kappa-}}(\varphi^{nt}, q), \gamma_{A^{\kappa-}}(\theta^{nt}, q)\}\}$$

$$= min\{\gamma_{A^{\kappa-}}({\varphi_1}^{nt}, q)\}$$

$$= min\{\gamma_{A^{\kappa-}}({\varphi_1}^{1-ms}, q)\}$$

$$\Rightarrow \gamma_{A^{\kappa-}}(\varphi, q) \geq min\{\gamma_{A^{\kappa-}}(\varphi_1, q)\}.$$

(iv) $\gamma_{A^{\kappa-}}(\varphi, q) = max\{\gamma_{A^{\kappa-}}(\varphi^{1-ms}, q)\}$

$$= max\{\gamma_{A^{\kappa-}}(\varphi^{nt}, q)\}$$

$$= max\{\gamma_{A^{\kappa-}}(\varphi^{nt}\theta^{nt}, q)\}$$

$$= max\{\gamma_{A^{\kappa-}}(\varphi\theta)^{nt}, q\}$$

$$\leq max\{max\{\gamma_{A^{\kappa-}}(\varphi^{nt}, q), \gamma_{A^{\kappa-}}(\theta^{nt}, q)\}\}$$

$$= max\{\gamma_{A^{\kappa-}}({\varphi_1}^{nt}, q)\}$$

$$= max\{\gamma_{A^{\kappa-}}({\varphi_1}^{1-ms}, q)\}$$

$$\Rightarrow \gamma_{A^{\kappa-}}(\varphi, q) \leq max\{\gamma_{A^{\kappa-}}(\varphi_1, q)\}.$$

Theorem: 3.10

Let γ_{A^κ} be a Bipolar $\kappa - Q$-fuzzy normal subgroup of a group G. Then $FO_{\gamma_{A_n}{}^{\kappa-}}(\theta, q) = FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi^{-1}\theta\varphi, q)$ and $FO_{\gamma_{A_n}{}^{\kappa+}}(\theta, q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi^{-1}\theta\varphi, q), \forall\, \theta, \varphi \in G \ and\ q \in Q$.

Proof:

Let $\theta, \varphi \in G \ and\ q \in Q$.

We have

(i) $\gamma_{A^{\kappa-}}(\theta^n, q) = min\{\gamma_{A^{\kappa-}}(\varphi^{-1}\theta\varphi, q)\}$

$$\geq min\{\gamma_{A^{\kappa-}}((\varphi^{-1}\theta\varphi)^n, q)\}$$

$$\Rightarrow \gamma_{A^{\kappa-}}(\theta^n, q) \geq min\{\gamma_{A^{\kappa-}}((\varphi^{-1}\theta\varphi)^n, q)\}$$

(ii) $\gamma_{A^{\kappa+}}(\theta^n, q) = max\{\gamma_{A^{\kappa-}}(\varphi^{-1}\theta\varphi, q)\}$

$$\leq max\{\gamma_{A^{\kappa-}}((\varphi^{-1}\theta\varphi)^n, q)\}$$

$$\Rightarrow \gamma_{A^{\kappa+}}(\theta^n, q) \leq max\{\gamma_{A^{\kappa-}}((\varphi^{-1}\theta\varphi)^n, q)\}, \forall\, n \in Z \ and\ q \in Q$$

$\Rightarrow FO_{\gamma_{A_n}{}^{\kappa-}}(\theta, q) = FO_{\gamma_{A_n}{}^{\kappa-}}(\varphi^{-1}\theta\varphi, q)$ and $FO_{\gamma_{A_n}{}^{\kappa+}}(\theta, q) = FO_{\gamma_{A_n}{}^{\kappa+}}(\varphi^{-1}\theta\varphi, q)$

$\Rightarrow \gamma_{A^\kappa}$ is not normal in G.

Example: 3.9.1

Let $\omega_3 = \{a, b / a^3 = b^3 = e, ba = a^2 b\}$ be the group with t elements.

Now define $\kappa - Q$-fuzzy subgroup of a group γ_{A^κ} of ω_3

$$\gamma_{A^{\kappa-}}(\theta, q) = \gamma_{A^{\kappa+}}(\theta, q) = \begin{cases} t_0, if\ \theta \in \langle b \rangle \\ t_1, otherwise \end{cases}$$

Where $t_0 > t_1$ and $q \in Q$.

Then $a^{-1}ba \in \langle b \rangle$ and so $FO_{\gamma_{A_n}}{}^{\kappa-}(b,q) = 1 = FO_{\gamma_{A_n}}{}^{\kappa+}(b,q) \neq FO_{\gamma_{A_n}}{}^{\kappa-}(a^{-1}ba,q) = FO_{\gamma_{A_n}}{}^{\kappa+}(a^{-1}ba,q)$.

4. Conclusion

In present work, Bipolar $\kappa - Q -$ fuzzy subsets and show that Bipolar $\kappa - Q -$ fuzzy Subgroups and Normal Subgroups. Moreover we define the properties of Bipolar $\kappa - Q -$ Fuzzy order subgroups and Normal subgroups has been innovated and that we have established several fundamental characteristics of this notion. For fresh findings in upcoming research, this notion can be further generalised to intuitionistic fuzzy sets, interval valued fuzzy sets, and Doubt bipolar fuzzy sets.

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APPLCATION OF NEUTROSOPHIC SOFT RELATION IN THE ANALYSIS OF SUPPORTING FACTORS FOR MARRIED STUDENTS HAVING CHILDREN

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Abstract

The aim of this paper is to explore neutrosophic soft sets (NSS) in decision making. Neutrosophic soft set is a combination of neutrosophic set and soft set. We use neutrosophic soft sets on three components (t, f, i) = (truth, falsehood, indeterminacy) to deal with exact state of data in several directions. A decision making method is developed based on neutrosophic soft relations(NSR) related to the problems encountered in day-to-day life by married girl students having children is the prime focus of the work. The scrutinized women are pursing higher education in the colleges of Coimbatore city, Tamil Nadu, India. As an application we provide an algorithm for the decision-making problem under neutrosophic soft relation environment by using comparison matrix for the data collected from the sample.

Keywords: Soft sets, Neutrosophic sets, Neutrosophic soft sets, neutrosophic soft relation, algorithm, decision making.

1. Introduction

The fuzzy set theory was introduced by L. A. Zadeh [17] in 1965. Many researchers have extended the concept of fuzzy set in multi directions. The traditional fuzzy set is characterized by the membership value or the grade of membership value. In some real life problems in expert system, belief system, information fusion and so on, we must consider the truth-membership as well as the falsity-membership for proper description of an object in uncertain, ambiguous environment. Intuitionistic fuzzy set is introduced by Atanassov [1] is appropriate for such

a situation. But it does not handle the indeterminate and inconsistent information which exists in belief system. F.Smarandache [15] proposed the concept of neutrosophic set which is a mathematical tool for handling problems

involving imprecise, indeterminacy and inconsistent data. Neutrosophic set is the generalization of many theory such as; fuzzy set, intuitionistic fuzzy set.

In 1999, Molodtsov[11] introduced the theory of soft set which is free from the parameterization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory for dealing with uncertainty. Using the concept of soft set theory Maji in 2013 introduced neutrosophic soft set [10]. Soft set theory have been expanded by embedding the ideas of fuzzy sets (e.g. [5,4]), intuitionistic fuzzy sets (e.g. [3,7,8]), neutrosophic sets (e.g. [9]). Also, many authors studied on relations in soft set [2,13], in fuzzy soft set[16] and in intuitionistic fuzzy soft set[6,12]. Presently, work on this NSS theory is progressing rapidly in different branches of Mathematics.

This paper proposes a model for analyzing the supporting factors for student mothers who are facing difficulties in everyday life based on neutrosophic soft relations. An illustrative example demonstrates the application of proposed decision making method in a real life problem. In section 2, we provide the basic definitions of neutrosophic set theory soft set theory and neutrosophic soft set theory that are useful for subsequent discussions. In section 3, a decision making method on neutrosophic soft sets is established with an example in a real life problem. In section 4, the conclusion for the proposed model is given.

2. Preliminaries

Definition 2.1 [15] A neutrosophic set A on the universe of discourse X is defined as $A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in X\}$, where $T, I, F : X \to]^-0, 1^+[$ and $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of [0, 1].

Definition 2.2 [11] Let U be an initial universe set and E be a set of parameters. Let P(U) denotes the power set of U. Consider a nonempty set A, $A \subset E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.3 [10] Let U be an initial universe set and E be a set of parameters. Consider $A \subset E$. Let P(U) denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the soft neutrosophic set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.4 [**10**] Let N_1 and N_2 be two neutrosophic soft sets over neutrosophic soft universes (U,A) and (U,B), respectively.

(1) N_1 is said to be neutrosophic soft subset of N_2 if $A \subseteq B$ and

$T_{f_{N_1(x)}}(u) \leq T_{f_{N_2(x)}}(u)$, $I_{f_{N_1(x)}}(u) \leq I_{f_{N_2(x)}}(u)$, $F_{f_{N_1(x)}}(u) \geq F_{f_{N_2(x)}}(u)$, $\forall x \in A$, $u \in U$.

(2) N_1 and N_2 are said to be equal if N_1 neutrosophic soft subset of N_2 and N_2 neutrosophic soft subset of N_2.

Definition 2.5 [**10**] Let N_1 and N_2 be two neutrosophic soft sets over soft universes (U,A) and (U,B), respectively,

(1) The complement of a neutrosophic soft set N_1 denoted by N_1^C and is defined by a set valued function f_N^C representing a mapping $f_{N_1}^c : E \rightarrow N(U)$

$f_{N_1}^c = \{(x, < F_{f_{N_1(x)}}(u), I_{f_{N_1(x)}}(u), T_{f_{N_1(x)}}(u) >) : x \in E, u \in U\}$.

(2) Then the union of N_1 and N_2 is denoted by $N_1 \cup N_2$ and is defined by $N_3(C = A \cup B)$, where the truth-membership, indeterminacy-membership and falsity membership of N_3 are as follows: $\forall u \in U$,

$$T_{f_{N_3(x)}}(u) = \begin{cases} T_{f_{N_1(x)}}(u), & if x \in A - B \\ T_{f_{N_2(x)}}(u), & if x \in B - A \\ \max\left\{T_{f_{N_1(x)}}(u), T_{f_{N_1(x)}}(u)\right\}, & if x \in A \cap B \end{cases}$$

$$I_{f_{N_3(x)}}(u) = \begin{cases} I_{f_{N_1(x)}}(u), & if x \in A - B \\ I_{f_{N_2(x)}}(u), & if x \in B - A \\ \dfrac{I_{f_{N_1(x)}}(u) + I_{f_{N_2(x)}}(u)}{2} & if x \in A \cap B \end{cases}$$

$$F_{f_{N_3(x)}}(u) = \begin{cases} F_{f_{N_1(x)}}(u), & if x \in A - B \\ F_{f_{N_2(x)}}(u), & if x \in B - A \\ \min\left\{F_{f_{N_1(x)}}(u), F_{f_{N_1(x)}}(u)\right\}, & if x \in A \cap B \end{cases}$$

(3) Then the intersection of N_1 and N_2 is denoted by $N_1 \cap N_2$ and is defined by $N_3(C = A \cap B)$, where the truth-membership, indeterminacy-membership and falsity-membership of N_3 are as follows: $\forall u \in U$,

$T_{f_{N_3(x)}}(u) = min\{ T_{f_{N_1(x)}}(u) , T_{f_{N_2(x)}}(u)\}$, $I_{f_{N_3(x)}}(u) = \dfrac{(I_{f_{N_1(x)}}(u) + I_{f_{N_2(x)}}(u))}{2}$

and $F_{f_{N_3(x)}}(u) = max\{ F_{f_{N_1(x)}}(u) , F_{f_{N_2(x)}}(u) \}$, $\forall x \in C$

Definition 2.6 [10] Let N_1 and N_2 be two neutrosophic soft sets over neutrosophic soft universes (U,A) and (U,B), respectively. Then the cartesian product of N_1 and N_2 is denoted by $N_1 \times N_2 = N_3$ is defined by

$N_3 = \{((x, y), f_{N_3}(x, y)) : (x, y) \in A \times B\}$

where the truth-membership, indeterminacy-membership and falsity-membership of N_3 are as follows:

$\forall u \in U;\ \forall (x, y) \in A \times B,$

$T_{f_{N_3\,(x,y)}}(u) = min\{T_{f_{N_1\,(x)}}(u)\ ,\ T_{f_{N_2\,(y)}}(u)\},$

$I_{f_{N_3\,(x,y)}}(u) = \frac{(I_{f_{N_1\,(x)}}(u), I_{f_{N_2\,(y)}}(u))}{2}$ and

$F_{f_{N_3\,(x,y)}}(u) = \max\{\ F_{f_{N_1\,(x)}}(u)\ ,\ F_{f_{N_2\,(y)}}(u)\ \}.$

3. An Application of Neutrosophic Soft Relation in Decision Making Method:

Under this section, we construct a decision making by using neutrosophic soft method on relations to analyze which factor supports the married girl students having children.

Now, we present an algorithm to form the decision making based on neutrosophic soft relation. The algorithm is as follows:

Algorithm:

1) Enter the Neutrosophic Soft N_1 and N_2.
2) Determine the matrix for the Neutrosophic Soft (relation table I) which corresponds to the Cartesian product of the neutrosophic soft sets N_1 and N_2.
3) Calculate the comparison table using the formula, T+I-F
4) Choose the greatest numerical value from the comparison table for each row.
5) Compute the score table in the form,

	(x_1, y_1)	...	...	(x_n, y_n)
Objects	h_i	...	...	...
Greatest Numerical Value	...	...	...	...

Where x_n denoted the parameter of N_1 and y_n denotes the parameter of N_2.

6) Take the sum of these numerical value from the score table and compute the score of each grade.
7) Find m, for which $s_m = max s_j$. Then s_m is the greatest score, if m has more than one values, you can choose any one value s_j.

Now, we use this algorithm to find the best supporting factor in decision making.

Example 3.1: Let U = {child care, home management, college work support, Moral support and financial support} denoted by $\{u_1, u_2, u_3, u_4, u_5\}$ the set of 5 supporting factors. Now, we consider two neutrosophic soft sets N_1 and N_2 over U. Suppose N_1 describes "Student mothers having children of age below 5" and N_2 describes "Student mothers having children of age above 5".

Let $E_1 = \{x_1, x_2, x_3, x_4\}$ ={Changes in family environment, Changes in economic condition, Burdened with family responsibility, Changes in academic performance} and $E_2 = \{y_1, y_2, y_3, y_4\}$ = **{**lack of concentration in studies, lack in conveyance and time, changes on routine study hours, changes in health condition} respectively. Then we find a supporting factor based on the sets of mother's parameters with the help of neutrosophic soft relation decision making method.

1) We enter the Neutrosophic Soft N_1 and N_2 as,

$$N_1 = \begin{Bmatrix} x_1, \frac{u_1}{(0.9,0.8,0.5)}, \frac{u_2}{(0.2,0.7,0.3)}, \frac{u_3}{(0.4,0.6,0.8)}, \frac{u_4}{(0.1,0.8,0.5)}, \frac{u_5}{(0.3,0.8,0.4)} \\ x_2, \frac{u_1}{(0.2,0.9,0.1)}, \frac{u_2}{(0.5,0.7,0.8)}, \frac{u_3}{(0.6,0.5,0.3)}, \frac{u_4}{(0.2,0.8,0.9)}, \frac{u_5}{(0.1,0.6,0.7)} \\ x_3, \frac{u_1}{(0.7,0.5,0.4)}, \frac{u_2}{(0.4,0.5,0.5)}, \frac{u_3}{(0.2,0.4,0.7)}, \frac{u_4}{(0.8,0.9,0.1)}, \frac{u_5}{(0.1,0.9,0.7)} \\ x_4, \frac{u_1}{(0.2,0.3,0.3)}, \frac{u_2}{(0.3,0.9,0.6)}, \frac{u_3}{(0.4,0.4,0.8)}, \frac{u_4}{(0.8,0.8,0.1)}, \frac{u_5}{(0.4,0.6,0.5)} \end{Bmatrix}$$

and

$$N_2 = \begin{Bmatrix} y_1, \frac{u_1}{(0.3,0.7,0.5)}, \frac{u_2}{(0.2,0.8,0.7)}, \frac{u_3}{(0.1,0.9,0.8)}, \frac{u_4}{(0.5,0.8,0.9)}, \frac{u_5}{(0.3,0.8,0.1)} \\ y_2, \frac{u_1}{(0.4,0.5,0.2)}, \frac{u_2}{(0.3,0.1,0.9)}, \frac{u_3}{(0.2,0.6,0.4)}, \frac{u_4}{(0.5,0.8,0.1)}, \frac{u_5}{(0.9,0.4,0.9)} \\ y_3, \frac{u_1}{(0.7,0.5,0.4)}, \frac{u_2}{(0.2,0.5,0.7)}, \frac{u_3}{(0.8,0.2,0.5)}, \frac{u_4}{(0.9,0.2,0.4)}, \frac{u_5}{(0.7,0.9,0.2)} \\ y_4, \frac{u_1}{(0.2,0.4,0.1)}, \frac{u_2}{(0.4,0.5,0.9)}, \frac{u_3}{(0.3,0.1,0.7)}, \frac{u_4}{(0.6,0.9,0.8)}, \frac{u_5}{(0.2,0.2,0.8)} \end{Bmatrix}$$

2) In table I, we determine the matrix R for the Neutrosophic Soft (relation table I) which corresponds to the Cartesian product of the neutrosophic soft sets N_1 and N_2 respectively.

R	u_1	u_2	u_3	u_4	u_5
(x_1, y_1)	(0.3,0.75,0.5)	(0.2,0.75,0.7)	(0.1,0.75,0.8)	(0.1,0.8,0.9)	(0.3,0.8,0.4)
(x_1, y_2)	(0.4,0.65,0.5)	(0.2,0.4,0.9)	(0.2,0.6,0.8)	(0.1,0.8,0.5)	(0.3,0.6,0.9)
(x_1, y_3)	(0.7,0.65,0.5)	(0.2,0.6,0.7)	(0.4,0.4,0.8)	(0.1,0.5,0.5)	(0.3,0.85,0.4)
(x_1, y_4)	(0.2,0.6,0.5)	(0.2,0.65,0.9)	(0.3,0.35,0.8)	(0.1,0.85,0.8)	(0.2,0.5,0.8)
(x_2, y_1)	(0.2,0.8,0.5)	(0.2,0.75,0.8)	(0.1,0.7,0.8)	(0.2,0.8,0.9)	(0.1,0.9,0.9)
(x_2, y_2)	(0.2,0.7,0.2)	(0.3,0.4,0.9)	(0.2,0.55,0.4)	(0.2,0.8,0.9)	(0.1,0.5,0.9)
(x_2, y_3)	(0.2,0.7,0.4)	(0.2,0.6,0.8)	(0.6,0.35,0.5)	(0.2,0.5,0.9)	(0.1,0.75,0.7)
(x_2, y_4)	(0.2,0.65,0.1)	(0.4,0.65,0.9)	(0.3,0.3,0.7)	(0.2,0.85,0.9)	(0.1,0.4,0.8)
(x_3, y_1)	(0.3,0.6,0.5)	(0.2,0.65,0.7)	(0.1,0.65,0.8)	(0.5,0.85,0.9)	(0.1,0.85,0.7)
(x_3, y_2)	(0.4,0.5,0.4)	(0.3,0.3,0.9)	(0.2,0.5,0.7)	(0.5,0.85,0.1)	(0.1,0.65,0.9)

(x_3, y_3)	(0.7,0.5,0.4)	(0.2,0.5,0.7)	(0.2,0.3,0.7)	(0.8,0.55,0.4)	(0.1,0.9,0.7)
(x_3, y_4)	(0.2,0.45,0.4)	(0.4,0.55,0.9)	(0.2,0.25,0.7)	(0.6,0.9,0.8)	(0.1,0.55,0.8)
(x_4, y_1)	(0.2,0.5,0.5)	(0.2,0.85,0.7)	(0.1,0.65,0.8)	(0.5,0.8,0.9)	(0.3,0.7,0.5)
(x_4, y_2)	(0.2,0.4,0.3)	(0.3,0.5,0.9)	(0.2,0.5,0.8)	(0.5,0.8,0.1)	(0.4,0.5,0.9)
(x_4, y_3)	(0.2,0.4,0.4)	(0.2,0.7,0.7)	(0.4,0.3,0.8)	(0.8,0.5,0.4)	(0.4,0.75,0.5)
(x_4, y_4)	(0.2,0.35,0.3)	(0.2,0.75,0.9)	(0.3,0.25,0.8)	(0.6,0.85,0.8)	(0.2,0.4,0.8)

Table I : Neutrosophic Soft Matrix R (Relation Table)

3) With the help of Table I, we calculate the comparison table II as follows;

R	u_1	u_2	u_3	u_4	u_5
(x_1, y_1)	0.55	0.25	0.05	0	0.7
(x_1, y_2)	0.55	-0.3	0	0.4	0
(x_1, y_3)	0.85	0.1	0	0.1	0.75
(x_1, y_4)	0.3	-0.05	-0.15	0.15	-0.1
(x_2, y_1)	0.5	0.15	0	0.1	0.1
(x_2, y_2)	0.7	-0.2	0.35	0.1	-0.3
(x_2, y_3)	0.5	0	0.45	-0.2	0.15
(x_2, y_4)	0.75	0.15	-0.1	0.15	-0.3
(x_3, y_1)	0.4	0.15	-0.05	0.45	0.25
(x_3, y_2)	0.5	-0.3	0	1.25	-0.15
(x_3, y_3)	0.8	0	-0.2	0.95	0.3
(x_3, y_4)	0.25	0.05	-0.25	0.7	-0.15
(x_4, y_1)	0.2	0.35	-0.05	0.4	0.5
(x_4, y_2)	0.3	-0.1	0	1.2	0
(x_4, y_3)	0.2	0.2	-0.1	0.9	0.65
(x_4, y_4)	0.25	0.05	-0.25	0.65	-0.2

Table II: Comparison Matrix Table.

4) In Table 3, we choose the greatest numerical value from the comparison table II for each row.

R	u_1	u_2	u_3	u_4	u_5
(x_1, y_1)	0.55	0.25	0.05	0	**0.7**
(x_1, y_2)	**0.55**	-0.3	0	0.4	0
(x_1, y_3)	**0.85**	0.1	0	0.1	0.75

(x_1, y_4)	**0.3**	-0.05	-0.15	0.15	-0.1
(x_2, y_1)	**0.5**	0.15	0	0.1	0.1
(x_2, y_2)	**0.7**	-0.2	0.35	0.1	-0.3
(x_2, y_3)	**0.5**	0	0.45	-0.2	0.15
(x_2, y_4)	**0.75**	0.15	-0.1	0.15	-0.3
(x_3, y_1)	0.4	0.15	-0.05	**0.45**	0.25
(x_3, y_2)	0.5	-0.3	0	**1.25**	-0.15
(x_3, y_3)	0.8	0	-0.2	**0.95**	0.3
(x_3, y_4)	0.25	0.05	-0.25	**0.7**	-0.15
(x_4, y_1)	0.2	0.35	-0.05	0.4	**0.5**
(x_4, y_2)	0.3	-0.1	0	**1.2**	0
(x_4, y_3)	0.2	0.2	-0.1	**0.9**	0.65
(x_4, y_4)	0.25	0.05	-0.25	**0.65**	-0.2

Table III

5) We compute the score table in the following form;

R	(x_1, y_1)	(x_1, y_2)	(x_1, y_3)	(x_1, y_4)
u_i	u_5	u_1	u_1	u_1
	0.7	0.55	0.85	0.3
	(x_2, y_1)	(x_2, y_2)	(x_2, y_3)	(x_2, y_4)
	u_1	u_1	u_1	u_1
	0.5	0.7	0.5	0.75
	(x_3, y_1)	(x_3, y_2)	(x_3, y_3)	(x_3, y_4)
	u_4	u_4	u_4	u_4
	0.45	1.25	0.95	0.7
	(x_4, y_1)	(x_4, y_2)	(x_4, y_3)	(x_4, y_4)
	u_5	u_4	u_4	u_4
	0.5	1.2	0.9	0.65

Table IV: Score Table

6) Now, we compute the score of each grade by taking the sum of the numerical value from the score table IV.

u_1: 0.55+0.85+0.3+0.5+0.7+0.5+0.75 = 4.15

u_4: 0.45+1.26+0.95+0.7+1.2+0.9+0.65 = 6.1

u_5: 0.7+0.5 = 1.2

The score value calculated is neatly tabulated for easy reference.

R	Score
Moral support (u_1)	4.61
Child care (u_4)	6.1
Financial Support (u_5)	1.2

Clearly the maximum score, s_m= 6.1

Therefore from the obtained maximum value we conclude that Moral Support (u_4) is the most needed factor expected by the student mothers during their period of study.

4. Conclusion:

In this paper we examine the challenges faced by student mothers during their academic period by using neutrosophic soft relation. Finally the support they need from family and society to live a peaceful and healthy life is found. We use this concept in soft sets considering the fact that the parameters (which are words or sentences) are mostly neutrosophic set. The neutrosophic soft relation concept may be applied in operations research, data analytics, medical sciences, etc. Industry may adopt this technique to minimize the cost of investment and maximize the profit.

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ON CARTESIAN PRODUCT OVER INTERVAL VALUED INTUITIONISTIC FUZZY MATRIX SETS USING PYTHON PROGRAM

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Abstract: In this paper, the conceptions of an operations and relations on the Cartesian product over interval valued intuitionistic fuzzy matrices set are introduced and its some properties are explored. We prove some equality based on the operation and the relation over IVIFSs. Finally, we introducing some Cartesian formulas x_4, x_5 in Cartesian product of interval valued intuitionistic fuzzy matrix sets.

Keywords: Fuzzy sets, intuitionistic fuzzy sets (IFS), Cartesian product over intuitionistic fuzzy sets, operation, geometric interpretation and interval valued intuitionistic fuzzy set (IVIFS).

1. Introduction

A lot of operations are introduced and proved over the intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets. In 1994, Atanassov Krassimir T [5] has proposed new operations defined over the intuitionistic fuzzy sets. In 2011, Wei, Cui-Ping, Pei Wang, and Yu-Zhong Zhang [24] have been proposed “Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications.” and In 2007, Xu, Zeshui [25] have been proposed “Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making.” In 2006, Riecan, Beloslav and Atanassov Krassimir T [5] have proposed n-extraction operation over intuitionistic fuzzy sets. In 2012, Parvathi, Rangaswamy, Beloslav Riecan, and Krassimir T. Atanassov [16] Have been proposed “Properties of some operations defined over intuitionistic fuzzy sets. In 2019, S. Senthilkumar, Eswari Prem, and C. Ragavan [18] have proposed Cartesian products over a contrary intuitionistic fuzzy α-translation of H-ideals in division BG-algebras. In 2008, Liu Q, Ma C, and Zhou X. [14] have proposed on the properties of some intuitionistic fuzzy set operators. In 2010, Riecan, Beloslav and Atanassov Krassimir T [2, 3] have proposed operation division by n over intuitionistic fuzzy sets. In 2012, Rangaswamy Parvathi, Beloslav Riecan and Atanassov Krassimir T [4] have proposed properties of some operations defined over intuitionistic fuzzy sets. In 1989, the notion of interval valued intuitionistic fuzzy sets which are a generalization of both intuitionistic fuzzy sets and interval valued fuzzy sets and interval valued fuzzy sets were proposed by Atanassov Krassimir T and Gargov G [4]. After the introduction of interval valued intuitionistic fuzzy sets, many researchers have shown intervals in the interval valued intuitionistic fuzzy set theory and applied it to the various fields. In 1994, Operators over an interval valued intuitionistic fuzzy sets was proposed by Atanassov Krassimir T

[5]. In 2007, methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making was proposed by Xu, Zeshui [25, 26]. In 2007, some geometric aggression operators based on interval valued intuitionistic fuzzy sets and their application to group decision making were proposed by Wei, Gui Wu, and Xiao-Rong Wang [21]. In 2012, some results on generalized interval- valued intuitionistic fuzzy sets were proposed by Bhowmik, Monoranjan, and Madhumangal pal [6]. In 2013, interval valued intuitionistic hesitant fuzzy aggregation operators and their application in group decision making were proposed by Zhiming Zhang [32]. In 2014, the new operations over an interval valued intuitionistic hesitant fuzzy sets were proposed by Broumi, Said, and Florentine Smarandache [8]. In this paper the formula for Cartesian product over interval- valued intuitionistic fuzzy matrix sets are investigated. Finally, the Cartesian product of two interval- valued intuitionistic fuzzy matrix sets in was derived.

2. Preliminaries

Definition 2.1: Fuzzy Sets: A fuzzy set is any set that allows its members to have different degree of membership function, having interval [0, 1]

Definition 2.2: Fuzzy Matrix set: Fuzzy matrices play a vital role in scientific development. A Fuzzy matrix may be matrix that has its parts from [0, 1]. Consider a matrix $A = [a_{ij}]_{3\times 3}$ where $a_{ij} \in [0,1], 1 \le j \le n.$ Then A is a Fuzzy Matrix [FM].

Definition 2.3: Fuzzy Rectangular Matrix: Let $A = [a_{ij}]_{m\times n}\,(m \ne n)$ where $a_{ij} \in [0,1], 1 \le i \le n, 1 \le j \le m.$ then A is a Fuzzy Rectangular Matrix.

Definition 2.4: fuzzy square matrix: Let $A = \left[a_{11}a_{12} \cdots a_{1j} \cdots a_{1n}\ a_{21}a_{22} \cdots a_{2j} \cdots a_{2n} \vdots \vdots \vdots \vdots \vdots \vdots\ a_{i1}a_{i2} \cdots a_{ij} \cdots a_{in} \vdots \vdots \vdots \vdots \vdots \vdots\ a_{n1}a_{n2} \cdots a_{nj} \cdots a_{nn}\right]$ .where, $a_{ij} \in [0,1], 1 \le i,j \le n$. Then A is a fuzzy square matrix.

Definition 2.5: fuzzy row matrix: Let A= $[a_1, a_2, a_3, \cdots, a_n]$ where $a_{ij} \in [0,1], j = 1,2,\dots,n$.Then A is called $1 \times n$ a fuzzy row matrix or row vector.

Definition 2.6: fuzzy column matrix: Let $A = [b_1\ b_2\ \ \vdots\ \ b_m]$ where $a_i[0,1], i = 1,2,\dots,n$.Then A is called $m \times 1$ a fuzzy column matrix.

Definition 2.7: fuzzy diagonal matrix: A Fuzzy square matrix A= $[a_{ij}]_{m\times n}$ is said to fuzzy diagonal matrix. If $a_{ij} = 0$ when$i \neq j, a_{ij} \in [0,1], 1 \leq I$.

Definition 2.8: fuzzy relation: A fuzzy relation is the Cartesian product of mathematical fuzzy sets. Two fuzzy sets are taken as input; the fuzzy relation is then equal to the cross product of the sets which is created by vector multiplication.

Definition 2.9: Cartesian product: Consider two sets A and B. The set of all ordered pairs $\{a, b\}$ where $a \in A$ & $b \in B$ is called Cartesian product. It is denoted by, $A \times B = \{(a, b): (a \in A \text{ } and \text{ } b \in B)\}$.

Definition 2.10: Membership function: The membership function of a fuzzy set A is denoted by μ_A, $\mu_A: E \to [0,1]$.The most commonly used range of value of membership function is the unit interval [a, b].

Definition 2.11: Degree of membership: Membership function for an intuitionistic fuzzy set A on the universe of discourse is defined as$\mu_A: X \to [0,1]$. where each element X is mapped to a value between 0 and 1. The value $\mu_A(x), x \in X$ is called Membership value or degree of membership function. The most commonly used range of value of membership functions is the interval [a, b].

Definition 2.12: Degree of Non- Membership function: non-membership function for an intuitionistic fuzzy set Aon the universe of discourse is defined as $\vartheta_A : X \to [0,1]$ Where each element X is mapped to a value between 0 and 1. The value $\mu_A(x), x \in X$ is called non-Membership value or degree of non-Membership function.

Definition 2.13: intuitionistic fuzzy set: An Intuitionistic Fuzzy Set (IFs) A in E is stated as an particular of the following form $A = \{< X, \mu_A(x), \vartheta_A(x) > | x \in E\}$ Here the functions: $\mu_A: E \to [0,1]$ and $\vartheta_A: E \to [0,1]$.

Definition 2.14: intuitionistic fuzzy matrix: An intuitionistic fuzzy matrix is a pair of fuzzy matrices, namely, a membership and non-membership function which represent positive and negative aspects. The concept of intuitionistic fuzzy matrices was introduced by pal et al.

Definition 2.15: interval valued intuitionistic fuzzy set: An Interval valued intuitionistic fuzzy set A in the finite universe X is defined as A= $\{[\, x, \mu_A(x), \gamma_A(x)] | x \epsilon X\}$. The intervals $\mu_A(x)$ and$\gamma_A(x)$ denote the degree of membership function and the degree of non-membership of the element x in the set A.For every $x \epsilon$ X,

$\mu_A(x)$ and $\gamma_A(x)$ are closed intervals and their left and right end points are denoted by$\mu_A^L(x)$, $\mu_A^R(x), \mu_A^L(x)$and$\gamma_A^R(x)$. Let as denote A = {[x,($\mu_A^L(x)$, $\mu_A^R(x)$,($\gamma_A^L(x), \gamma_A^R(x)$))]|$x \epsilon X$} where $0 \leq \mu_A^R(x) + \gamma_A^R(x) \leq 1, \mu_A^L(x) \geq 0, \gamma_A^L(x) \geq 0$. Especially if $\mu_A(x) = \mu_A^L(x) = \mu_A^R(x)$ and $\gamma_A(x) = \gamma_A^L(x) = \gamma_A^R(x)$ then the given interval valued intuitionistic fuzzy set A is reduced to an ordinary intuitionistic fuzzy set.

Definition 2.16: operations on intuitionistic fuzzy sets: Let A and B be two intuitionistic fuzzy sets on the universe X. Where, A ={[$x, \mu_A(x), \gamma_A(x)$]|$x \epsilon X$} and B= {[$x, \mu_B(x), \gamma_B(x)$]|$x \epsilon X$}.

Definition 2.17: The five Cartesian products of two IFSs A and B are defined as follows:

Let A and B are two intuitionistic fuzzy sets of the universes A_E and B_F, then the Cartesian product of two IFSs is defined by the

The Cartesian product "X_4" is defined by

$$(A \times_4 B) = \left\{(x,y), \begin{pmatrix} [\max(\alpha, min(inf\mu_A(x), inf\mu_B(x)), \max(\alpha, \min(sup\mu_A(x), sup\mu_B(x))] \\ [min(\beta, max(inf\gamma_A(x), inf\gamma_B(x)) \ \ , min(\beta, max(sup\gamma_A(x), sup\gamma_B(x))] \end{pmatrix}; x \in A \ and \ y \in B\right\}$$

The Cartesian product "X_5" is defined by

$$(A \times_5 B) = \left\{(x,y), \begin{pmatrix} [\max(\alpha, max(inf\mu_A(x), inf\mu_B(x)), \max(\alpha, \max(sup\mu_A(x), sup\mu_B(x))] \\ [min(\beta, min(inf\gamma_A(x), inf\gamma_B(x)) \ \ , min(\beta, min(sup\gamma_A(x), sup\gamma_B(x))] \end{pmatrix}; x \in A \ and \ y \in B\right\}$$

Theorem 2.1: If A_E and B_F are two intervals valued intuitionistic fuzzy matrix set, then $A_E X_4 B_F$ is also an interval valued intuitionistic fuzzy matrix set.

Proof: If $A_E =$

$$\begin{pmatrix} [\alpha_{E_{11}}, \beta_{E_{11}}][\gamma_{E_{11}}, \delta_{E_{11}}][\alpha_{E_{12}}, \beta_{E_{12}}][\gamma_{E_{12}}, \delta_{E_{12}}][\alpha_{E_{13}}, \beta_{E_{13}}][\gamma_{E_{13}}, \delta_{E_{13}}][\alpha_{E_{21}}, \beta_{E_{21}}][\gamma_{E_{21}}, \delta_{E_{21}}][\alpha_{E_{22}}, \beta_{E_{22}}] \\ [\gamma_{E_{22}}, \delta_{E_{22}}] \ [\alpha_{E_{23}}, \beta_{E_{23}}][\gamma_{E_{23}}, \delta_{E_{23}}] \ [\alpha_{E_{31}}, \beta_{E_{31}}][\gamma_{E_{31}}, \delta_{E_{31}}] \ [\alpha_{E_{32}}, \beta_{E_{32}}][\gamma_{E_{32}}, \delta_{E_{32}}] \ [\alpha_{E_{33}}, \beta_{E_{33}}][\gamma_{E_{33}}, \delta_{E_{33}}] \end{pmatrix}$$

and

$B_F =$

$$\begin{pmatrix} [\alpha_{F_{11}}, \beta_{F_{11}}][\gamma_{F_{11}}, \delta_{F_{11}}][\alpha_{F_{12}}, \beta_{F_{12}}][\gamma_{F_{12}}, \delta_{F_{12}}][\alpha_{F_{13}}, \beta_{F_{13}}][\gamma_{F_{13}}, \delta_{F_{13}}][\alpha_{F_{21}}, \beta_{F_{21}}][\gamma_{F_{21}}, \delta_{F_{22}}][\alpha_{F_{22}}, \beta_{F_{22}}] \\ [\gamma_{F_{22}}, \delta_{F_{22}}] \ [\alpha_{F_{23}}, \beta_{F_{23}}][\gamma_{F_{23}}, \delta_{F_{23}}] \ [\alpha_{F_{31}}, \beta_{F_{31}}][\gamma_{F_{31}}, \delta_{F_{31}}] \ [\alpha_{F_{32}}, \beta_{F_{32}}][\gamma_{F_{32}}, \delta_{F_{32}}] \ [\alpha_{F_{33}}, \beta_{F_{33}}][\gamma_{F_{33}}, \delta_{F_{33}} \end{pmatrix}$$

are interval valued intuitionistic fuzzy matrix sets. Then

$$A_E X_4 B_F$$

$$= \begin{pmatrix} [\alpha_{E_{11}},\beta_{E_{11}}][\gamma_{E_{11}},\delta_{E_{11}}]\ [\alpha_{E_{12}},\beta_{E_{12}}][\gamma_{E_{12}},\delta_{E_{12}}]\ [\alpha_{E_{13}},\beta_{E_{13}}][\gamma_{E_{13}},\delta_{E_{13}}]\ [\alpha_{E_{21}},\beta_{E_{21}}][\gamma_{E_{21}},\delta_{E_{21}}] \\ [\alpha_{E_{22}},\beta_{E_{22}}] \\ [\gamma_{E_{22}},\delta_{E_{22}}]\ [\alpha_{E_{23}},\beta_{E_{23}}][\gamma_{E_{23}},\delta_{E_{23}}]\ [\alpha_{E_{31}},\beta_{E_{31}}][\gamma_{E_{31}},\delta_{E_{31}}]\ [\alpha_{E_{32}},\beta_{E_{32}}][\gamma_{E_{32}},\delta_{E_{32}}]\ [\alpha_{E_{33}},\beta_{E_{33}}] \\ [\gamma_{E_{33}},\delta_{E_{33}}] \end{pmatrix} X_4$$

$$\begin{pmatrix} [\alpha_{F_{11}},\beta_{F_{11}}][\gamma_{F_{11}},\delta_{F_{11}}][\alpha_{F_{12}},\beta_{F_{12}}][\gamma_{F_{12}},\delta_{F_{12}}][\alpha_{F_{13}},\beta_{F_{13}}][\gamma_{F_{13}},\delta_{F_{13}}][\alpha_{F_{21}},\beta_{F_{21}}][\gamma_{F_{21}},\delta_{F_{22}}][\alpha_{F_{22}},\beta_{F_{22}}] \\ [\gamma_{F_{22}},\delta_{F_{22}}]\ [\alpha_{F_{23}},\beta_{F_{23}}][\gamma_{F_{23}},\delta_{F_{23}}]\ [\alpha_{F_{21}},\beta_{F_{21}}][\gamma_{F_{21}},\delta_{F_{22}}]\ [\alpha_{F_{32}},\beta_{F_{32}}][\gamma_{F_{32}},\delta_{F_{32}}]\ [\alpha_{F_{33}},\beta_{F_{33}}][\gamma_{F_{33}},\delta_{F_{33}}] \end{pmatrix}$$

$A_E \times_4 B_F = (X_{11}\ X_{12}\ X_{13}\ X_{21}\ X_{22}\ X_{23}\ X_{31}\ X_{32}\ X_{33}\ )$ Where,

$X_{11} = ([\alpha_{E_{11}},\beta_{E_{11}}][\gamma_{E_{11}},\delta_{E_{11}}]\ [\alpha_{E_{12}},\beta_{E_{12}}][\gamma_{E_{12}},\delta_{E_{12}}]\ [\alpha_{E_{13}},\beta_{E_{13}}][\gamma_{E_{13}},\delta_{E_{13}}]\ ) \times_4$

$([\alpha_{F_{11}},\beta_{F_{11}}][\gamma_{F_{11}},\delta_{F_{11}}]\ [\alpha_{F_{21}},\beta_{F_{21}}][\gamma_{F_{21}},\delta_{F_{22}}]\ [\alpha_{F_{31}},\beta_{F_{31}}][\gamma_{F_{31}},\delta_{F_{31}}]\ )$

$X_{12} = ([\alpha_{E_{11}},\beta_{E_{11}}][\gamma_{E_{11}},\delta_{E_{11}}]\ [\alpha_{E_{12}},\beta_{E_{12}}][\gamma_{E_{12}},\delta_{E_{12}}]\ [\alpha_{E_{13}},\beta_{E_{13}}][\gamma_{E_{13}},\delta_{E_{13}}]\ )$

$$\times_4 \begin{pmatrix} [\alpha_{F_{12}},\beta_{F_{12}}][\gamma_{F_{12}},\delta_{F_{12}}] \\ [\alpha_{F_{22}},\beta_{F_{22}}][\gamma_{F_{22}},\delta_{F_{22}}]\ [\alpha_{F_{32}},\beta_{F_{32}}][\gamma_{F_{32}},\delta_{F_{32}}] \end{pmatrix}$$

$X_{13} = ([\alpha_{E_{11}},\beta_{E_{11}}][\gamma_{E_{11}},\delta_{E_{11}}]\ [\alpha_{E_{12}},\beta_{E_{12}}][\gamma_{E_{12}},\delta_{E_{12}}]\ [\alpha_{E_{13}},\beta_{E_{13}}][\gamma_{E_{13}},\delta_{E_{13}}]\ ) \times_4$

$([\alpha_{F_{13}},\beta_{F_{13}}][\gamma_{F_{13}},\delta_{F_{13}}]\ [\alpha_{F_{23}},\beta_{F_{23}}][\gamma_{F_{23}},\delta_{F_{23}}]\ [\alpha_{F_{33}},\beta_{F_{33}}][\gamma_{F_{33}},\delta_{F_{33}}]\ )$

$X_{21} = ([\alpha_{E_{21}},\beta_{E_{21}}][\gamma_{E_{21}},\delta_{E_{21}}]\ [\alpha_{E_{22}},\beta_{E_{22}}][\gamma_{E_{22}},\delta_{E_{22}}]\ [\alpha_{E_{23}},\beta_{E_{23}}][\gamma_{E_{23}},\delta_{E_{23}}]\ ) \times_4$

$([\alpha_{F_{11}},\beta_{F_{11}}][\gamma_{F_{11}},\delta_{F_{11}}]\ [\alpha_{F_{21}},\beta_{F_{21}}][\gamma_{F_{21}},\delta_{F_{22}}]\ [\alpha_{F_{31}},\beta_{F_{31}}][\gamma_{F_{31}},\delta_{F_{31}}]\ )$

$X_{22} = ([\alpha_{E_{21}},\beta_{E_{21}}][\gamma_{E_{21}},\delta_{E_{21}}]\ [\alpha_{E_{22}},\beta_{E_{22}}][\gamma_{E_{22}},\delta_{E_{22}}]\ [\alpha_{E_{23}},\beta_{E_{23}}][\gamma_{E_{23}},\delta_{E_{23}}]\ ) \times_4$

$([\alpha_{F_{12}},\beta_{F_{12}}][\gamma_{F_{12}},\delta_{F_{12}}]\ [\alpha_{F_{22}},\beta_{F_{22}}][\gamma_{F_{22}},\delta_{F_{22}}]\ [\alpha_{F_{32}},\beta_{F_{32}}][\gamma_{F_{32}},\delta_{F_{32}}]\ )$

$X_{23} = ([\alpha_{E_{21}},\beta_{E_{21}}][\gamma_{E_{21}},\delta_{E_{21}}]\ [\alpha_{E_{22}},\beta_{E_{22}}][\gamma_{E_{22}},\delta_{E_{22}}]\ [\alpha_{E_{23}},\beta_{E_{23}}][\gamma_{E_{23}},\delta_{E_{23}}]\ ) \times_4$

$([\alpha_{F_{13}},\beta_{F_{13}}][\gamma_{F_{13}},\delta_{F_{13}}]\ [\alpha_{F_{23}},\beta_{F_{23}}][\gamma_{F_{23}},\delta_{F_{23}}]\ [\alpha_{F_{33}},\beta_{F_{33}}][\gamma_{F_{33}},\delta_{F_{33}}]\ )$

$X_{31} = ([\alpha_{E_{31}},\beta_{E_{31}}][\gamma_{E_{31}},\delta_{E_{31}}]\ [\alpha_{E_{32}},\beta_{E_{32}}][\gamma_{E_{32}},\delta_{E_{32}}]\ [\alpha_{E_{33}},\beta_{E_{33}}][\gamma_{E_{33}},\delta_{E_{33}}]\ ) \times_4$

$([\alpha_{F_{11}},\beta_{F_{11}}][\gamma_{F_{11}},\delta_{F_{11}}]\ [\alpha_{F_{21}},\beta_{F_{21}}][\gamma_{F_{21}},\delta_{F_{22}}]\ [\alpha_{F_{31}},\beta_{F_{31}}][\gamma_{F_{31}},\delta_{F_{31}}]\ )$

$X_{32} = ([\alpha_{E_{31}},\beta_{E_{31}}][\gamma_{E_{31}},\delta_{E_{31}}]\ [\alpha_{E_{32}},\beta_{E_{32}}][\gamma_{E_{32}},\delta_{E_{32}}]\ [\alpha_{E_{33}},\beta_{E_{33}}][\gamma_{E_{33}},\delta_{E_{33}}]\ ) \times_4$

$([\alpha_{F_{12}},\beta_{F_{12}}][\gamma_{F_{12}},\delta_{F_{12}}]\ [\alpha_{F_{22}},\beta_{F_{22}}][\gamma_{F_{22}},\delta_{F_{22}}]\ [\alpha_{F_{32}},\beta_{F_{32}}][\gamma_{F_{32}},\delta_{F_{32}}]\ )$

$X_{33} = ([\alpha_{E_{31}},\beta_{E_{31}}][\gamma_{E_{31}},\delta_{E_{31}}]\ [\alpha_{E_{32}},\beta_{E_{32}}][\gamma_{E_{32}},\delta_{E_{32}}]\ [\alpha_{E_{33}},\beta_{E_{33}}][\gamma_{E_{33}},\delta_{E_{33}}]\ ) \times_4$

$([\alpha_{F_{13}},\beta_{F_{13}}][\gamma_{F_{13}},\delta_{F_{13}}]\ [\alpha_{F_{23}},\beta_{F_{23}}][\gamma_{F_{23}},\delta_{F_{23}}]\ [\alpha_{F_{33}},\beta_{F_{33}}][\gamma_{F_{33}},\delta_{F_{33}}]\ )$

$X_{11} = [\alpha_{E_{11}},\beta_{E_{11}}][\gamma_{E_{11}},\delta_{E_{11}}] \times_4 [\alpha_{F_{11}},\beta_{F_{11}}][\gamma_{F_{11}},\delta_{F_{11}}] +$

$[\alpha_{E_{12}},\beta_{E_{12}}][\gamma_{E_{12}},\delta_{E_{12}}] \times_4\ [\alpha_{F_{21}},\beta_{F_{21}}][\gamma_{F_{21}},\delta_{F_{22}}] +$

$[\alpha_{E_{13}},\beta_{E_{13}}][\gamma_{E_{13}},\delta_{E_{13}}] \times_4\ [\alpha_{F_{31}},\beta_{F_{31}}][\gamma_{F_{31}},\delta_{F_{31}}]$

$X_{12} = [\alpha_{E_{11}}, \beta_{E_{11}}][\gamma_{E_{11}}, \delta_{E_{11}}] \times_4 [\alpha_{F_{12}}, \beta_{F_{12}}][\gamma_{F_{12}}, \delta_{F_{12}}] +$
$[\alpha_{E_{12}}, \beta_{E_{12}}][\gamma_{E_{12}}, \delta_{E_{12}}] \times_4 \ [\alpha_{F_{22}}, \beta_{F_{22}}][\gamma_{F_{22}}, \delta_{F_{22}}]+$
$[\alpha_{E_{13}}, \beta_{E_{13}}][\gamma_{E_{13}}, \delta_{E_{13}}] \times_4 \ [\alpha_{F_{32}}, \beta_{F_{32}}][\gamma_{F_{32}}, \delta_{F_{32}}]$

$X_{13} = [\alpha_{E_{11}}, \beta_{E_{11}}][\gamma_{E_{11}}, \delta_{E_{11}}] \times_4 [\alpha_{F_{13}}, \beta_{F_{13}}][\gamma_{F_{13}}, \delta_{F_{13}}] +$
$[\alpha_{E_{12}}, \beta_{E_{12}}][\gamma_{E_{12}}, \delta_{E_{12}}] \times_4 \ [\alpha_{F_{23}}, \beta_{F_{23}}][\gamma_{F_{23}}, \delta_{F_{23}}]+$
$[\alpha_{E_{13}}, \beta_{E_{13}}][\gamma_{E_{13}}, \delta_{E_{13}}] \times_4 \ [\alpha_{F_{33}}, \beta_{F_{33}}][\gamma_{F_{33}}, \delta_{F_{33}}]$

$X_{21} = [\alpha_{E_{21}}, \beta_{E_{21}}][\gamma_{E_{21}}, \delta_{E_{21}}] \times_4 [\alpha_{F_{11}}, \beta_{F_{11}}][\gamma_{F_{11}}, \delta_{F_{11}}] +$
$[\alpha_{E_{22}}, \beta_{E_{22}}][\gamma_{E_{22}}, \delta_{E_{22}}] \times_4 \ [\alpha_{F_{21}}, \beta_{F_{21}}][\gamma_{F_{21}}, \delta_{F_{21}}]+$
$[\alpha_{E_{23}}, \beta_{E_{23}}][\gamma_{E_{23}}, \delta_{E_{23}}] \times_4 \ [\alpha_{F_{31}}, \beta_{F_{31}}][\gamma_{F_{31}}, \delta_{F_{31}}]$

$X_{22} = [\alpha_{E_{21}}, \beta_{E_{21}}][\gamma_{E_{21}}, \delta_{E_{21}}] \times_4 [\alpha_{F_{12}}, \beta_{F_{12}}][\gamma_{F_{12}}, \delta_{F_{12}}] +$
$[\alpha_{E_{22}}, \beta_{E_{22}}][\gamma_{E_{22}}, \delta_{E_{22}}] \times_4 \ [\alpha_{F_{22}}, \beta_{F_{22}}][\gamma_{F_{22}}, \delta_{F_{22}}]+$
$[\alpha_{E_{23}}, \beta_{E_{23}}][\gamma_{E_{23}}, \delta_{E_{23}}] \times_4 \ [\alpha_{F_{32}}, \beta_{F_{32}}][\gamma_{F_{32}}, \delta_{F_{32}}]$

$X_{23} = [\alpha_{E_{21}}, \beta_{E_{21}}][\gamma_{E_{21}}, \delta_{E_{21}}] \times_4 [\alpha_{F_{13}}, \beta_{F_{13}}][\gamma_{F_{13}}, \delta_{F_{13}}] +$
$[\alpha_{E_{22}}, \beta_{E_{22}}][\gamma_{E_{22}}, \delta_{E_{22}}] \times_4 \ [\alpha_{F_{23}}, \beta_{F_{23}}][\gamma_{F_{23}}, \delta_{F_{23}}]+$
$[\alpha_{E_{23}}, \beta_{E_{23}}][\gamma_{E_{23}}, \delta_{E_{23}}] \times_4 \ [\alpha_{F_{33}}, \beta_{F_{33}}][\gamma_{F_{33}}, \delta_{F_{33}}]$

$X_{31} = [\alpha_{E_{31}}, \beta_{E_{31}}][\gamma_{E_{31}}, \delta_{E_{31}}] \times_4 [\alpha_{F_{11}}, \beta_{F_{11}}][\gamma_{F_{11}}, \delta_{F_{11}}] +$
$[\alpha_{E_{32}}, \beta_{E_{32}}][\gamma_{E_{32}}, \delta_{E_{32}}] \times_4 \ [\alpha_{F_{21}}, \beta_{F_{21}}][\gamma_{F_{21}}, \delta_{F_{21}}]+$
$[\alpha_{E_{33}}, \beta_{E_{33}}][\gamma_{E_{33}}, \delta_{E_{33}}] \times_4 \ [\alpha_{F_{31}}, \beta_{F_{31}}][\gamma_{F_{31}}, \delta_{F_{31}}]$

$X_{32} = [\alpha_{E_{31}}, \beta_{E_{31}}][\gamma_{E_{31}}, \delta_{E_{31}}] \times_4 [\alpha_{F_{12}}, \beta_{F_{12}}][\gamma_{F_{12}}, \delta_{F_{12}}] +$
$[\alpha_{E_{32}}, \beta_{E_{32}}][\gamma_{E_{32}}, \delta_{E_{32}}] \times_4 \ [\alpha_{F_{22}}, \beta_{F_{22}}][\gamma_{F_{22}}, \delta_{F_{22}}]+$
$[\alpha_{E_{33}}, \beta_{E_{33}}][\gamma_{E_{33}}, \delta_{E_{33}}] \times_4 \ [\alpha_{F_{32}}, \beta_{F_{32}}][\gamma_{F_{32}}, \delta_{F_{32}}]$

$X_{33} = [\alpha_{E_{31}}, \beta_{E_{31}}][\gamma_{E_{31}}, \delta_{E_{31}}] \times_4 [\alpha_{F_{13}}, \beta_{F_{13}}][\gamma_{F_{13}}, \delta_{F_{13}}] +$
$[\alpha_{E_{32}}, \beta_{E_{32}}][\gamma_{E_{32}}, \delta_{E_{32}}] \times_4 \ [\alpha_{F_{23}}, \beta_{F_{23}}][\gamma_{F_{23}}, \delta_{F_{23}}]+$
$[\alpha_{E_{33}}, \beta_{E_{33}}][\gamma_{E_{33}}, \delta_{E_{33}}] \times_4 \ [\alpha_{F_{33}}, \beta_{F_{33}}][\gamma_{F_{33}}, \delta_{F_{33}}]$

Now, by applying this

$$(A \times_4 B) = \left\{(x,y)\begin{pmatrix}[\max(\alpha, min(inf\mu_A(x), inf\mu_B(x)), \max(\alpha, \min(sup\mu_A(x), sup\mu_B(x))] \ [\min(\beta, \max(inf\gamma_A(x), inf\gamma_B(x)) \ , \\ min(\beta, max(sup\gamma_A(x), sup\gamma_B(x))]\end{pmatrix}; x \in A \ and \ y \in B\right\}$$

$$X_{11} = \left\{[max(\alpha, min(\alpha_{E_{11}}, \alpha_{F_{11}})), max(\alpha, min(\beta_{E_{11}}, \beta_{F_{11}}))][min\left(\beta, max(\gamma_{E_{11}}, \gamma_{F_{11}})\right), \left(\beta, max(\delta_{E_{11}}, \delta_{F_{11}})\right)]\right\}$$
$$+ \left\{\left[\left(\alpha, min(\alpha_{E_{12}}, \alpha_{F_{21}})\right), \left(\alpha, min(\beta_{E_{12}}, \beta_{F_{21}})\right)\right][\left(\beta, max(\gamma_{E_{12}}, \gamma_{F_{21}})\right), \left(\beta, max(\delta_{E_{12}}, \delta_{F_{21}})\right)]\right\} +$$
$$\left\{\left[\left(\alpha, min(\alpha_{E_{13}}, \alpha_{F_{31}})\right), \left(\alpha, min(\beta_{E_{13}}, \beta_{F_{31}})\right)\right][\left(\beta, max(\gamma_{E_{13}}, \gamma_{F_{31}})\right), min(\beta, max(\delta_{E_{13}}, \delta_{F_{31}}))\right\}$$

$$X_{12} = \left\{[max(\alpha, min(\alpha_{E_{11}}, \alpha_{F_{12}})), max(\alpha, min(\beta_{E_{11}}, \beta_{F_{12}}))][min\left(\beta, max(\gamma_{E_{11}}, \gamma_{F_{12}})\right), \left(\beta, max(\delta_{E_{11}}, \delta_{F_{12}})\right)]\right\}$$
$$+ \left\{\left[\left(\alpha, min(\alpha_{E_{12}}, \alpha_{F_{22}})\right), \left(\alpha, min(\beta_{E_{12}}, \beta_{F_{22}})\right)\right][\left(\beta, max(\gamma_{E_{12}}, \gamma_{F_{22}})\right), \left(\beta, max(\delta_{E_{12}}, \delta_{F_{22}})\right)]\right\} +$$
$$\left\{\left[\left(\alpha, min(\alpha_{E_{13}}, \alpha_{F_{32}})\right), \left(\alpha, min(\beta_{E_{13}}, \beta_{F_{32}})\right)\right][\left(\beta, max(\gamma_{E_{13}}, \gamma_{F_{32}})\right), min(\beta, max(\delta_{E_{13}}, \delta_{F_{32}}))\right\}$$

$$X_{13} = \left\{[max(\alpha, min(\alpha_{E_{11}}, \alpha_{F_{13}})), max(\alpha, min(\beta_{E_{11}}, \beta_{F_{13}}))][min\left(\beta, max(\gamma_{E_{11}}, \gamma_{F_{13}})\right), \left(\beta, max(\delta_{E_{11}}, \delta_{F_{13}})\right)]\right\}$$
$$+ \left\{\left[\left(\alpha, min(\alpha_{E_{12}}, \alpha_{F_{23}})\right), \left(\alpha, min(\beta_{E_{12}}, \beta_{F_{23}})\right)\right][\left(\beta, max(\gamma_{E_{12}}, \gamma_{F_{23}})\right), \left(\beta, max(\delta_{E_{12}}, \delta_{F_{23}})\right)]\right\} +$$
$$\left\{\left[\left(\alpha, min(\alpha_{E_{13}}, \alpha_{F_{33}})\right), \left(\alpha, min(\beta_{E_{13}}, \beta_{F_{33}})\right)\right][\left(\beta, max(\gamma_{E_{13}}, \gamma_{F_{33}})\right), min(\beta, max(\delta_{E_{13}}, \delta_{F_{33}}))\right\}$$

$$X_{21} = \left\{[max(\alpha, min(\alpha_{E_{21}}, \alpha_{F_{11}})), max(\alpha, min(\beta_{E_{11}}, \beta_{F_{11}}))][min\left(\beta, max(\gamma_{E_{21}}, \gamma_{F_{11}})\right), \left(\beta, max(\delta_{E_{21}}, \delta_{F_{11}})\right)]\right\}$$
$$+ \left\{\left[\left(\alpha, min(\alpha_{E_{22}}, \alpha_{F_{21}})\right), \left(\alpha, min(\beta_{E_{22}}, \beta_{F_{21}})\right)\right][\left(\beta, max(\gamma_{E_{22}}, \gamma_{F_{21}})\right), \left(\beta, max(\delta_{E_{22}}, \delta_{F_{21}})\right)]\right\} +$$
$$\left\{\left[\left(\alpha, min(\alpha_{E_{23}}, \alpha_{F_{31}})\right), \left(\alpha, min(\beta_{E_{23}}, \beta_{F_{31}})\right)\right][\left(\beta, max(\gamma_{E_{23}}, \gamma_{F_{31}})\right), min(\beta, max(\delta_{E_{23}}, \delta_{F_{31}}))\right\}$$

$$X_{22} = \left\{[max(\alpha, min(\alpha_{E_{21}}, \alpha_{F_{12}})), max(\alpha, min(\beta_{E_{11}}, \beta_{F_{12}}))][min\left(\beta, max(\gamma_{E_{21}}, \gamma_{F_{12}})\right), \left(\beta, max(\delta_{E_{21}}, \delta_{F_{12}})\right)]\right\}$$
$$+ \left\{\left[\left(\alpha, min(\alpha_{E_{22}}, \alpha_{F_{22}})\right), \left(\alpha, min(\beta_{E_{22}}, \beta_{F_{22}})\right)\right][\left(\beta, max(\gamma_{E_{22}}, \gamma_{F_{22}})\right), \left(\beta, max(\delta_{E_{22}}, \delta_{F_{22}})\right)]\right\} +$$
$$\left\{\left[\left(\alpha, min(\alpha_{E_{23}}, \alpha_{F_{32}})\right), \left(\alpha, min(\beta_{E_{23}}, \beta_{F_{32}})\right)\right][\left(\beta, max(\gamma_{E_{23}}, \gamma_{F_{32}})\right), min(\beta, max(\delta_{E_{23}}, \delta_{F_{32}}))\right\}$$

$$X_{23} = \left\{[max(\alpha, min(\alpha_{E_{21}}, \alpha_{F_{13}})), max(\alpha, min(\beta_{E_{11}}, \beta_{F_{13}}))][min\left(\beta, max(\gamma_{E_{21}}, \gamma_{F_{13}})\right), \left(\beta, max(\delta_{E_{21}}, \delta_{F_{13}})\right)]\right\}$$
$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{22}}, \alpha_{F_{23}}\right)\right), \left(\alpha, min\left(\beta_{E_{22}}, \beta_{F_{23}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{22}}, \gamma_{F_{23}}\right)\right), \left(\beta, max\left(\delta_{E_{22}}, \delta_{F_{23}}\right)\right)\right]\right\} +$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{23}}, \alpha_{F_{33}}\right)\right), \left(\alpha, min\left(\beta_{E_{23}}, \beta_{F_{33}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{23}}, \gamma_{F_{33}}\right)\right), min(\beta, max\left(\delta_{E_{23}}, \delta_{F_{33}}\right))\right\}$$

$$X_{31} = \left\{[max(\alpha, min\left(\alpha_{E_{31}}, \alpha_{F_{11}}\right)), max(\alpha, min\left(\beta_{E_{31}}, \beta_{F_{11}}\right))][min\left(\beta, max\left(\gamma_{E_{31}}, \gamma_{F_{11}}\right)\right), \left(\beta, max\left(\delta_{E_{31}}, \delta_{F_{11}}\right)\right)]\right\}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{32}}, \alpha_{F_{21}}\right)\right), \left(\alpha, min\left(\beta_{E_{32}}, \beta_{F_{21}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{32}}, \gamma_{F_{21}}\right)\right), \left(\beta, max\left(\delta_{E_{32}}, \delta_{F_{21}}\right)\right)\right]\right\} +$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{33}}, \alpha_{F_{31}}\right)\right), \left(\alpha, min\left(\beta_{E_{33}}, \beta_{F_{31}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{33}}, \gamma_{F_{31}}\right)\right), min(\beta, max\left(\delta_{E_{33}}, \delta_{F_{31}}\right))\right\}$$

$$X_{32} = \left\{[max(\alpha, min\left(\alpha_{E_{31}}, \alpha_{F_{12}}\right)), max(\alpha, min\left(\beta_{E_{31}}, \beta_{F_{12}}\right))][min\left(\beta, max\left(\gamma_{E_{31}}, \gamma_{F_{12}}\right)\right), \left(\beta, max\left(\delta_{E_{31}}, \delta_{F_{12}}\right)\right)]\right\}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{32}}, \alpha_{F_{22}}\right)\right), \left(\alpha, min\left(\beta_{E_{32}}, \beta_{F_{22}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{32}}, \gamma_{F_{22}}\right)\right), \left(\beta, max\left(\delta_{E_{32}}, \delta_{F_{22}}\right)\right)\right]\right\} +$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{33}}, \alpha_{F_{32}}\right)\right), \left(\alpha, min\left(\beta_{E_{33}}, \beta_{F_{32}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{33}}, \gamma_{F_{32}}\right)\right), min(\beta, max\left(\delta_{E_{33}}, \delta_{F_{32}}\right))\right\}$$

$$X_{33} = \left\{[max(\alpha, min\left(\alpha_{E_{31}}, \alpha_{F_{13}}\right)), max(\alpha, min\left(\beta_{E_{31}}, \beta_{F_{13}}\right))][min\left(\beta, max\left(\gamma_{E_{31}}, \gamma_{F_{13}}\right)\right), \left(\beta, max\left(\delta_{E_{31}}, \delta_{F_{13}}\right)\right)]\right\}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{32}}, \alpha_{F_{23}}\right)\right), \left(\alpha, min\left(\beta_{E_{32}}, \beta_{F_{23}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{32}}, \gamma_{F_{23}}\right)\right), \left(\beta, max\left(\delta_{E_{32}}, \delta_{F_{23}}\right)\right)\right]\right\} +$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{33}}, \alpha_{F_{33}}\right)\right), \left(\alpha, min\left(\beta_{E_{33}}, \beta_{F_{33}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{33}}, \gamma_{F_{33}}\right)\right), min(\beta, max\left(\delta_{E_{33}}, \delta_{F_{33}}\right))\right\}$$

By using A+B= max {A, B}

$$X_{11} = max\begin{pmatrix}\left[\left(\alpha, min\left(\alpha_{E_{11}}, \alpha_{F_{11}}\right)\right), \left(\alpha, min\left(\beta_{E_{11}}, \beta_{F_{11}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{11}}, \gamma_{F_{11}}\right)\right), \left(\beta, max\left(\delta_{E_{11}}, \delta_{F_{11}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{12}}, \alpha_{F_{21}}\right)\right), \left(\alpha, min\left(\beta_{E_{12}}, \beta_{F_{21}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{12}}, \gamma_{F_{21}}\right)\right), \left(\beta, max\left(\delta_{E_{12}}, \delta_{F_{21}}\right)\right)\right]\end{pmatrix}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{13}}, \alpha_{F_{31}}\right)\right), \left(\alpha, min\left(\beta_{E_{13}}, \beta_{F_{31}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{13}}, \gamma_{F_{31}}\right)\right), min(\beta, max\left(\delta_{E_{13}}, \delta_{F_{31}}\right))\right\}$$

$$X_{12} = max\begin{pmatrix}\left[\left(\alpha, min\left(\alpha_{E_{11}}, \alpha_{F_{12}}\right)\right), \left(\alpha, min\left(\beta_{E_{11}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{11}}, \gamma_{F_{12}}\right)\right), \left(\beta, max\left(\delta_{E_{11}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{12}}, \alpha_{F_{22}}\right)\right), \left(\alpha, min\left(\beta_{E_{12}}, \beta_{F_{22}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{12}}, \gamma_{F_{22}}\right)\right), \left(\beta, max\left(\delta_{E_{12}}, \delta_{F_{22}}\right)\right)\right]\end{pmatrix}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{13}}, \alpha_{F_{32}}\right)\right), \left(\alpha, min\left(\beta_{E_{13}}, \beta_{F_{32}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{13}}, \gamma_{F_{32}}\right)\right), min(\beta, max\left(\delta_{E_{13}}, \delta_{F_{32}}\right))\right\}$$

$$X_{13} = max\begin{pmatrix} \left[\left(\alpha, min\left(\alpha_{E_{11}}, \alpha_{F_{13}}\right)\right), \left(\alpha, min\left(\beta_{E_{11}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{11}}, \gamma_{F_{13}}\right)\right), \left(\beta, max\left(\delta_{E_{11}}, \delta_{F_{13}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{12}}, \alpha_{F_{23}}\right)\right), \left(\alpha, min\left(\beta_{E_{12}}, \beta_{F_{23}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{12}}, \gamma_{F_{23}}\right)\right), \left(\beta, max\left(\delta_{E_{12}}, \delta_{F_{23}}\right)\right)\right] \end{pmatrix}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{13}}, \alpha_{F_{33}}\right)\right), \left(\alpha, min\left(\beta_{E_{13}}, \beta_{F_{33}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{13}}, \gamma_{F_{33}}\right)\right), min\left(\beta, max\left(\delta_{E_{13}}, \delta_{F_{33}}\right)\right)\right\}$$

$$X_{21} = max\begin{pmatrix} \left[\left(\alpha, min\left(\alpha_{E_{21}}, \alpha_{F_{11}}\right)\right), \left(\alpha, min\left(\beta_{E_{21}}, \beta_{F_{11}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{21}}, \gamma_{F_{11}}\right)\right), \left(\beta, max\left(\delta_{E_{21}}, \delta_{F_{11}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{22}}, \alpha_{F_{21}}\right)\right), \left(\alpha, min\left(\beta_{E_{22}}, \beta_{F_{21}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{22}}, \gamma_{F_{21}}\right)\right), \left(\beta, max\left(\delta_{E_{22}}, \delta_{F_{21}}\right)\right)\right] \end{pmatrix}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{23}}, \alpha_{F_{31}}\right)\right), \left(\alpha, min\left(\beta_{E_{23}}, \beta_{F_{31}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{23}}, \gamma_{F_{31}}\right)\right), min\left(\beta, max\left(\delta_{E_{23}}, \delta_{F_{31}}\right)\right)\right\}$$

$$X_{22} = max\begin{pmatrix} \left[\left(\alpha, min\left(\alpha_{E_{21}}, \alpha_{F_{12}}\right)\right), \left(\alpha, min\left(\beta_{E_{21}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{21}}, \gamma_{F_{12}}\right)\right), \left(\beta, max\left(\delta_{E_{21}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{22}}, \alpha_{F_{22}}\right)\right), \left(\alpha, min\left(\beta_{E_{22}}, \beta_{F_{22}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{22}}, \gamma_{F_{22}}\right)\right), \left(\beta, max\left(\delta_{E_{22}}, \delta_{F_{22}}\right)\right)\right] \end{pmatrix}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{23}}, \alpha_{F_{32}}\right)\right), \left(\alpha, min\left(\beta_{E_{23}}, \beta_{F_{32}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{23}}, \gamma_{F_{32}}\right)\right), min\left(\beta, max\left(\delta_{E_{23}}, \delta_{F_{32}}\right)\right)\right\}$$

$$X_{23} = max\begin{pmatrix} \left[\left(\alpha, min\left(\alpha_{E_{21}}, \alpha_{F_{13}}\right)\right), \left(\alpha, min\left(\beta_{E_{21}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{21}}, \gamma_{F_{13}}\right)\right), \left(\beta, max\left(\delta_{E_{21}}, \delta_{F_{13}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{22}}, \alpha_{F_{23}}\right)\right), \left(\alpha, min\left(\beta_{E_{22}}, \beta_{F_{23}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{22}}, \gamma_{F_{23}}\right)\right), \left(\beta, max\left(\delta_{E_{22}}, \delta_{F_{23}}\right)\right)\right] \end{pmatrix}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{23}}, \alpha_{F_{33}}\right)\right), \left(\alpha, min\left(\beta_{E_{23}}, \beta_{F_{33}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{23}}, \gamma_{F_{33}}\right)\right), min\left(\beta, max\left(\delta_{E_{23}}, \delta_{F_{33}}\right)\right)\right\}$$

$$X_{31} = max\begin{pmatrix} \left[\left(\alpha, min\left(\alpha_{E_{31}}, \alpha_{F_{11}}\right)\right), \left(\alpha, min\left(\beta_{E_{31}}, \beta_{F_{11}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{31}}, \gamma_{F_{11}}\right)\right), \left(\beta, max\left(\delta_{E_{31}}, \delta_{F_{11}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{32}}, \alpha_{F_{21}}\right)\right), \left(\alpha, min\left(\beta_{E_{32}}, \beta_{F_{21}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{32}}, \gamma_{F_{21}}\right)\right), \left(\beta, max\left(\delta_{E_{32}}, \delta_{F_{21}}\right)\right)\right] \end{pmatrix}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{33}}, \alpha_{F_{31}}\right)\right), \left(\alpha, min\left(\beta_{E_{33}}, \beta_{F_{31}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{33}}, \gamma_{F_{31}}\right)\right), min\left(\beta, max\left(\delta_{E_{33}}, \delta_{F_{31}}\right)\right)\right\}$$

$$X_{32} = max\begin{pmatrix} \left[\left(\alpha, min\left(\alpha_{E_{31}}, \alpha_{F_{12}}\right)\right), \left(\alpha, min\left(\beta_{E_{31}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{31}}, \gamma_{F_{12}}\right)\right), \left(\beta, max\left(\delta_{E_{31}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{32}}, \alpha_{F_{22}}\right)\right), \left(\alpha, min\left(\beta_{E_{32}}, \beta_{F_{22}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{32}}, \gamma_{F_{22}}\right)\right), \left(\beta, max\left(\delta_{E_{32}}, \delta_{F_{22}}\right)\right)\right] \end{pmatrix}$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{33}}, \alpha_{F_{32}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{33}}, \beta_{F_{32}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{33}}, \gamma_{F_{32}}\right)\right)\ , min(\beta, max\left(\delta_{E_{33}}, \delta_{F_{32}}\right))\right\}$$

$$X_{33}$$

$$= max\left(\begin{array}{c}\left[\left(\alpha, min\left(\alpha_{E_{31}}, \alpha_{F_{13}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{31}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{31}}, \gamma_{F_{13}}\right)\right), \left(\beta, max\left(\delta_{E_{31}}, \delta_{F_{13}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{32}}, \alpha_{F_{23}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{32}}, \beta_{F_{23}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{32}}, \gamma_{F_{23}}\right)\right)\ ,\left(\beta, max\left(\delta_{E_{32}}, \delta_{F_{23}}\right)\right)]\end{array}\right)$$

$$+$$

$$\left\{\left[\left(\alpha, min\left(\alpha_{E_{33}}, \alpha_{F_{33}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{33}}, \beta_{F_{33}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{33}}, \gamma_{F_{33}}\right)\right)\ , min(\beta, max\left(\delta_{E_{33}}, \delta_{F_{33}}\right))\right\}$$

By using A+B+C = max$\{max(A, B), C\}$

X_{11} =Max

$$\left(\begin{array}{c} max\left(\begin{array}{c}\left[\left(\alpha, min\left(\alpha_{E_{11}}, \alpha_{F_{11}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{11}}, \beta_{F_{11}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{11}}, \gamma_{F_{11}}\right)\right), \left(\beta, max\left(\delta_{E_{11}}. \delta_{F_{11}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{12}}, \alpha_{F_{21}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{12}}, \beta_{F_{21}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{12}}, \gamma_{F_{21}}\right)\right)\ ,\left(\beta, max\left(\delta_{E_{12}}, \delta_{F_{21}}\right)\right)\right]\end{array}\right), \\ \left[\left(\alpha, min\left(\alpha_{E_{13}}, \alpha_{F_{31}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{13}}, \beta_{F_{31}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{13}}, \gamma_{F_{31}}\right)\right)\ , min(\beta, max\left(\delta_{E_{13}}, \delta_{F_{31}}\right))\end{array}\right)$$

X_{12} =Max

$$\left(\begin{array}{c} max\left(\begin{array}{c}\left[\left(\alpha, min\left(\alpha_{E_{11}}, \alpha_{F_{12}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{11}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{11}}, \gamma_{F_{12}}\right)\right), \left(\beta, max\left(\delta_{E_{11}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{12}}, \alpha_{F_{22}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{12}}, \beta_{F_{22}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{12}}, \gamma_{F_{22}}\right)\right)\ ,\left(\beta, max\left(\delta_{E_{12}}, \delta_{F_{22}}\right)\right)\right]\end{array}\right), \\ \left[\left(\alpha, min\left(\alpha_{E_{13}}, \alpha_{F_{32}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{13}}, \beta_{F_{32}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{13}}, \gamma_{F_{32}}\right)\right)\ , min(\beta, max\left(\delta_{E_{13}}, \delta_{F_{32}}\right))\end{array}\right)$$

X_{13} =Max

$$\left(\begin{array}{c} max\left(\begin{array}{c}\left[\left(\alpha, min\left(\alpha_{E_{11}}, \alpha_{F_{13}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{11}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{11}}, \gamma_{F_{13}}\right)\right), \left(\beta, max\left(\delta_{E_{11}}, \delta_{F_{13}}\right)\right)\right] \\ , \left[\left(\alpha, min\left(\alpha_{E_{12}}, \alpha_{F_{23}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{12}}, \beta_{F_{23}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{12}}, \gamma_{F_{23}}\right)\right)\ ,\left(\beta, max\left(\delta_{E_{12}}, \delta_{F_{23}}\right)\right)\right]\end{array}\right), \\ \left[\left(\alpha, min\left(\alpha_{E_{13}}, \alpha_{F_{33}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{13}}, \beta_{F_{33}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{13}}, \gamma_{F_{33}}\right)\right)\ , min(\beta, max\left(\delta_{E_{13}}, \delta_{F_{33}}\right))\end{array}\right)$$

X_{21} =Max

$$\left(\begin{array}{c} max\left(\begin{array}{c}\left[\left(\alpha, min\left(\alpha_{E_{21}}, \alpha_{F_{11}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{21}}. \beta_{F_{11}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{21}}, \gamma_{F_{11}}\right)\right), \left(\beta, max\left(\delta_{E_{21}}, \delta_{F_{11}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{22}}, \alpha_{F_{21}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{22}}. \beta_{F_{21}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{22}}, \gamma_{F_{21}}\right)\right)\ ,\left(\beta, max\left(\delta_{E_{22}}, \delta_{F_{21}}\right)\right)\right]\end{array}\right), \\ \left[\left(\alpha, min\left(\alpha_{E_{23}}, \alpha_{F_{31}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{23}}. \beta_{F_{31}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{23}}, \gamma_{F_{31}}\right)\right)\ , min(\beta, max\left(\delta_{E_{23}}, \delta_{F_{31}}\right))\end{array}\right)$$

X_{22} =Max

$$\left(\begin{array}{c} max\left(\begin{array}{c}\left[\left(\alpha, min\left(\alpha_{E_{21}}, \alpha_{F_{12}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{21}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{21}}, \gamma_{F_{12}}\right)\right), \left(\beta, max\left(\delta_{E_{21}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{22}}, \alpha_{F_{22}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{22}}, \beta_{F_{22}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{22}}, \gamma_{F_{22}}\right)\right)\ ,\left(\beta, max\left(\delta_{E_{22}}, \delta_{F_{22}}\right)\right)\right]\end{array}\right), \\ \left[\left(\alpha, min\left(\alpha_{E_{23}}, \alpha_{F_{32}}\right)\right)\ ,\left(\alpha, min\left(\beta_{E_{23}}, \beta_{F_{32}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{23}}, \gamma_{F_{32}}\right)\right)\ , min(\beta, max\left(\delta_{E_{23}}, \delta_{F_{32}}\right))\end{array}\right)$$

X_{23} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c}\left[\left(\alpha, min\left(\alpha_{E_{21}}, \alpha_{F_{13}}\right)\right), \left(\alpha, min\left(\beta_{E_{21}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{21}}, \gamma_{F_{13}}\right)\right), \left(\beta, max\left(\delta_{E_{21}}, \delta_{F_{13}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{22}}, \alpha_{F_{23}}\right)\right), \left(\alpha, min\left(\beta_{E_{22}}, \beta_{F_{23}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{22}}, \gamma_{F_{23}}\right)\right), \left(\beta, max\left(\delta_{E_{22}}, \delta_{F_{23}}\right)\right)\right]\end{array}\right), \\ \quad \left[\left(\alpha, min\left(\alpha_{E_{23}}, \alpha_{F_{33}}\right)\right), \left(\alpha, min\left(\beta_{E_{23}}, \beta_{F_{33}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{23}}, \gamma_{F_{33}}\right)\right), min(\beta, max\left(\delta_{E_{23}}, \delta_{F_{33}}\right)) \end{array}\right)$$

X_{31} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c}\left[\left(\alpha, min\left(\alpha_{E_{31}}, \alpha_{F_{11}}\right)\right), \left(\alpha, min\left(\beta_{E_{31}}, \beta_{F_{11}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{31}}, \gamma_{F_{11}}\right)\right), \left(\beta, max\left(\delta_{E_{31}}, \delta_{F_{11}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{32}}, \alpha_{F_{21}}\right)\right), \left(\alpha, min\left(\beta_{E_{32}}, \beta_{F_{21}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{32}}, \gamma_{F_{21}}\right)\right), \left(\beta, max\left(\delta_{E_{32}}, \delta_{F_{21}}\right)\right)\right]\end{array}\right), \\ \quad \left[\left(\alpha, min\left(\alpha_{33}, \alpha_{F_{31}}\right)\right), \left(\alpha, min\left(\beta_{E_{33}}, \beta_{F_{31}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{33}}, \gamma_{F_{31}}\right)\right), min(\beta, max\left(\delta_{E_{33}}, \delta_{F_{31}}\right)) \end{array}\right)$$

X_{32} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c}\left[\left(\alpha, min\left(\alpha_{E_{31}}, \alpha_{F_{12}}\right)\right), \left(\alpha, min\left(\beta_{E_{31}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{31}}, \gamma_{F_{12}}\right)\right), \left(\beta, max\left(\delta_{E_{31}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{32}}, \alpha_{F_{22}}\right)\right), \left(\alpha, min\left(\beta_{E_{32}}, \beta_{F_{22}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{32}}, \gamma_{F_{22}}\right)\right), \left(\beta, max\left(\delta_{E_{32}}, \delta_{F_{22}}\right)\right)\right]\end{array}\right), \\ \quad \left[\left(\alpha, min\left(\alpha_{33}, \alpha_{F_{32}}\right)\right), \left(\alpha, min\left(\beta_{E_{33}}, \beta_{F_{32}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{33}}, \gamma_{F_{32}}\right)\right), min(\beta, max\left(\delta_{E_{33}}, \delta_{F_{32}}\right)) \end{array}\right)$$

X_{33} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c}\left[\left(\alpha, min\left(\alpha_{E_{31}}, \alpha_{F_{13}}\right)\right), \left(\alpha, min\left(\beta_{E_{31}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, max\left(\gamma_{E_{31}}, \gamma_{F_{13}}\right)\right), \left(\beta, max\left(\delta_{E_{31}}, \delta_{F_{13}}\right)\right)\right], \\ \left[\left(\alpha, min\left(\alpha_{E_{32}}, \alpha_{F_{23}}\right)\right), \left(\alpha, min\left(\beta_{E_{32}}, \beta_{F_{23}}\right)\right)\right]\left[\left(\beta, max\left(\gamma_{E_{32}}, \gamma_{F_{23}}\right)\right), \left(\beta, max\left(\delta_{E_{32}}, \delta_{F_{23}}\right)\right)\right]\end{array}\right), \\ \quad \left[\left(\alpha, min\left(\alpha_{33}, \alpha_{F_{33}}\right)\right), \left(\alpha, min\left(\beta_{E_{33}}, \beta_{F_{33}}\right)\right)\right][\left(\beta, max\left(\gamma_{E_{33}}, \gamma_{F_{33}}\right)\right), min(\beta, max\left(\delta_{E_{33}}, \delta_{F_{33}}\right)) \end{array}\right)$$

$A_E \times_4 B_F = (X_{11}\ X_{12}\ X_{13}\ X_{21}\ X_{22}\ X_{23}\ X_{31}\ X_{32}\ X_{33})$. Hence, $A_E \times_4 B_F$ is an interval valued intuitionistic fuzzy matrix set.

Using Python program for $A_E \times_4 B_F$

#intput the values

```
import math
x=float(input("x="))
y=float(input("y="))
a_11=float(input("a_11="))
b_11=float(input("b_11="))
c_11=float(input("c_11="))
d_11=float(input("d_11="))
e_11=float(input("e_11="))
f_11=float(input("f_11="))
g_11=float(input("g_11="))
h_11=float(input("h_11="))
```

```
a_12=float(input("a_12="))
b_12=float(input("b_12="))
c_12=float(input("c_12="))
d_12=float(input("d_12="))
e_12=float(input("e_12="))
f_12=float(input("f_12="))
g_12=float(input("g_12="))
h_12=float(input("h_12="))
a_13=float(input("a_13="))
b_13=float(input("b_13="))
c_13=float(input("c_13="))
d_13=float(input("d_13="))
e_13=float(input("e_13="))
f_13=float(input("f_13="))
g_13=float(input("g_13="))
h_13=float(input("h_13="))
a_21=float(input("a_21="))
b_21=float(input("b_21="))
c_21=float(input("c_21="))
d_21=float(input("d_21="))
e_21=float(input("e_21="))
f_21=float(input("f_21="))
g_21=float(input("g_21="))
h_21=float(input("h_21="))
a_22=float(input("a_22="))
b_22=float(input("b_22="))
c_22=float(input("c_22="))
d_22=float(input("d_22="))
e_22=float(input("e_22="))
f_22=float(input("f_22="))
g_22=float(input("g_22="))
h_22=float(input("h_22="))
a_23=float(input("a_23="))
b_23=float(input("b_23="))
```

```
c_23=float(input("c_23="))
d_23=float(input("d_23="))
e_23=float(input("e_23="))
f_23=float(input("f_23="))
g_23=float(input("g_23="))
h_23=float(input("h_23="))
a_31=float(input("a_31="))
b_31=float(input("b_31="))
c_31=float(input("c_31="))
d_31=float(input("d_31="))
e_31=float(input("e_31="))
f_31=float(input("f_31="))
g_31=float(input("g_31="))
h_31=float(input("h_31="))
a_32=float(input("a_32="))
b_32=float(input("b_32="))
c_32=float(input("c_32="))
d_32=float(input("d_32="))
e_32=float(input("e_32="))
f_32=float(input("f_32="))
g_32=float(input("g_32="))
h_32=float(input("h_32="))
a_33=float(input("a_33="))
b_33=float(input("b_33="))
c_33=float(input("c_33="))
d_33=float(input("d_33="))
e_33=float(input("e_33="))
f_33=float(input("f_33="))
g_33=float(input("g_33="))
h_33=float(input("h_33="))
#creating variables for c_11
a_1=max(x,min(a_11,c_11))
a_2=max(x,min(b_11,d_11))
a_3=min(y,max(e_11,g_11))
```

```
a_4=min(y,max(f_11,h_11))
a_5=max(x,min(a_12,c_21))
a_6=max(x,min(b_12,d_21))
a_7=min(y,max(e_12,g_21))
a_8=min(y,max(f_12,h_21))
a_9=max(x,min(a_13,c_31))
a_10=max(x,min(b_13,d_31))
a_11=min(y,max(e_13,g_31))
a_12=min(y,max(f_13,h_31))
#creating cells
x_11=[max((max(a_1,a_5)),a_9),max(max(a_2,a_6),a_10)]
Y_11=[max((max(a_3,a_7)),a_11),max(max(a_4,a_8),a_12)]
print("C_11=",x_11)
print("D_11=",Y_11)
#creating variables for c_12
b_1=max(x,min(a_11,c_12))
b_2=max(x,min(b_11,d_12))
b_3=min(y,max(e_11,g_12))
b_4=min(y,max(f_11,h_12))
b_5=max(x,min(a_12,c_22))
b_6=max(x,min(b_12,d_22))
b_7=min(y,max(e_12,g_22))
b_8=min(y,max(f_12,h_22))
b_9=max(x,min(a_13,c_32))
b_10=max(x,min(b_13,d_32))
b_11=min(y,max(e_13,g_32))
b_12=min(y,max(f_13,h_32))
#creating cells
x_12=[max((max(b_1,b_5)),b_9),max(max(b_2,b_6),b_10)]
Y_12=[max((max(b_3,b_7)),b_11),max(max(b_4,b_8),b_12)]
print("C_12=",x_12)
print("D_12=",Y_12)
#creating variables for c_13
c_1=max(x,min(a_11,c_13))
```

```
c_2=max(x,min(b_11,d_13))
c_3=min(y,max(e_11,g_13))
c_4=min(y,max(f_11,h_13))
c_5=max(x,min(a_12,c_23))
c_6=max(x,min(b_12,d_23))
c_7=min(y,max(e_12,g_23))
c_8=min(y,max(f_12,h_23))
c_9=max(x,min(a_13,c_33))
c_10=max(x,min(b_13,d_33))
c_11=min(y,max(e_13,g_33))
c_12=min(y,max(f_13,h_33))
#creating cells
x_13=[max((max(c_1,c_5)),c_9),max(max(c_2,c_6),c_10)]
Y_13=[max((max(c_3,c_7)),c_11),max(max(c_4,c_8),c_12)]
print("C_13=",x_13)
print("D_13=",Y_13)
#creating variables for c_21
d_1=max(x,min(a_21,c_11))
d_2=max(x,min(b_21,d_11))
d_3=min(y,max(e_21,g_11))
d_4=min(y,max(f_21,h_11))
d_5=max(x,min(a_22,c_21))
d_6=max(x,min(b_22,d_21))
d_7=min(y,max(e_22,g_21))
d_8=min(y,max(f_22,h_21))
d_9=max(x,min(a_23,c_31))
d_10=max(x,min(b_23,d_31))
d_11=min(y,max(e_23,g_31))
d_12=min(y,max(f_23,h_31))
#creating cells
x_21=[max((max(d_1,d_5)),d_9),max(max(d_2,d_6),d_10)]
Y_21=[max((max(d_3,d_7)),d_11),max(max(d_4,d_8),d_12)]
print("C_21=",x_21)
print("D_21=",Y_21)
```

```
#creating variables for c_22
e_1=max(x,min(a_21,c_12))
e_2=max(x,min(b_21,d_12))
e_3=min(y,max(e_21,g_12))
e_4=min(y,max(f_21,h_12))
e_5=max(x,min(a_22,c_22))
e_6=max(x,min(b_22,d_22))
e_7=min(y,max(e_22,g_22))
e_8=min(y,max(f_22,h_22))
e_9=max(x,min(a_23,c_32))
e_10=max(x,min(b_23,d_32))
e_11=min(y,max(e_23,g_32))
e_12=min(y,max(f_23,h_32))
#creating cells
x_22=[max((max(e_1,e_5)),e_9),max(max(e_2,e_6),e_10)]
Y_22=[max((max(e_3,e_7)),e_11),max(max(e_4,e_8),e_12)]
print("C_22=",x_22)
print("D_22=",Y_22)
#creating variables for c_23
f_1=max(x,min(a_21,c_13))
f_2=max(x,min(b_21,d_13))
f_3=min(y,max(e_21,g_13))
f_4=min(y,max(f_21,h_13))
f_5=max(x,min(a_22,c_23))
f_6=max(x,min(b_22,d_23))
f_7=min(y,max(e_22,g_23))
f_8=min(y,max(f_22,h_23))
f_9=max(x,min(a_23,c_33))
f_10=max(x,min(b_23,d_33))
f_11=min(y,max(e_23,g_33))
f_12=min(y,max(f_23,h_33))
#creating cells
x_23=[max((max(f_1,f_5)),f_9),max(max(f_2,f_6),f_10)]
Y_23=[max((max(f_3,f_7)),f_11),max(max(f_4,f_8),f_12)]
```

```
print("C_23=",x_23)
print("D_23=",Y_23)
#creating variables for c_31
g_1=max(x,min(a_31,c_11))
g_2=max(x,min(b_31,d_11))
g_3=min(y,max(e_31,g_11))
g_4=min(y,max(f_31,h_11))
g_5=max(x,min(a_32,c_21))
g_6=max(x,min(b_32,d_21))
g_7=min(y,max(e_32,g_21))
g_8=min(y,max(f_32,h_21))
g_9=max(x,min(a_33,c_31))
g_10=max(x,min(b_33,d_31))
g_11=min(y,max(e_33,g_31))
g_12=min(y,max(f_33,h_31))
#creating cells
x_31=[max((max(g_1,g_5)),g_9),max(max(g_2,g_6),g_10)]
Y_31=[max((max(g_3,g_7)),g_11),max(max(g_4,g_8),g_12)]
print("C_31=",x_31)
print("D_31=",Y_31)
#creating variables for c_31
h_1=max(x,min(a_31,c_12))
h_2=max(x,min(b_31,d_12))
h_3=min(y,max(e_31,g_12))
h_4=min(y,max(f_31,h_12))
h_5=max(x,min(a_32,c_22))
h_6=max(x,min(b_32,d_22))
h_7=min(y,max(e_32,g_22))
h_8=min(y,max(f_32,h_22))
h_9=max(x,min(a_33,c_32))
h_10=max(x,min(b_33,d_32))
h_11=min(y,max(e_33,g_32))
h_12=min(y,max(f_33,h_32))
#creating cells
```

```
x_32=[max((max(h_1,h_5)),h_9),max(max(h_2,h_6),h_10)]
Y_32=[max((max(h_3,h_7)),h_11),max(max(h_4,h_8),h_12)]
print("C_32=",x_32)
print("D_32=",Y_32)
#creating variables for c_33
i_1=max(x,min(a_31,c_13))
i_2=max(x,min(b_31,d_13))
i_3=min(y,max(e_31,g_13))
i_4=min(y,max(f_31,h_13))
i_5=max(x,min(a_32,c_23))
i_6=max(x,min(b_32,d_23))
i_7=min(y,max(e_32,g_23))
i_8=min(y,max(f_32,h_23))
i_9=max(x,min(a_33,c_33))
i_10=max(x,min(b_33,d_33))
i_11=min(y,max(e_33,g_33))
i_12=min(y,max(f_33,h_33))
#creating cells
x_33=[max((max(i_1,i_5)),i_9),max(max(i_2,i_6),i_10)]
Y_33=[max((max(i_3,i_7)),i_11),max(max(i_4,i_8),i_12)]
print("C_33=",x_33)
print("D_33=",Y_33)
```

Output:

```
x=0.04
y=0.01
a_11=0.05
b_11=0.07
c_11=0.02
d_11=0.06
e_11=0.03
f_11=0.05
g_11=0.03
h_11=0.05
a_12=0.02
```

b_12=0.03

c_12=0.05

d_12=0.07

e_12=0.04

f_12=0.07

g_12=0.02

h_12=0.04

a_13=0.05

b_13=0.06

c_13=0.02

d_13=0.03

e_13=0.01

f_13=0.04

g_13=0.03

h_13=0.06

a_21=0.03

b_21=0.08

c_21=0.04

d_21=0.05

e_21=0.03

f_21=0.04

g_21=0.01

h_21=0.07

a_22=0.04

b_22=0.05

c_22=0.04

d_22=0.08

e_22=0.02

f_22=0.08

g_22=0.02

h_22=0.03

a_23=0.01

b_23=0.03

c_23=0.01

d_23=0.04

e_23=0.04

f_23=0.07

g_23=0.03

h_23=0.08

a_31=0.04

b_31=0.06

c_31=0.02

d_31=0.03

e_31=0.03

f_31=0.09

g_31=0.02

h_31=0.05

a_32=0.01

b_32=0.02

c_32=0.01

d_32=0.03

e_32=0.05

f_32=0.07

g_32=0.05

h_32=0.06

a_33=0.02

b_33=0.03

c_33=0.03

d_33=0.07

e_33=0.08

f_33=0.09

g_33=0.01

h_33=0.05

C_11= [0.04, 0.06]

D_11= [0.01, 0.01]

C_12= [0.04, 0.07]

D_12= [0.01, 0.01]

C_13= [0.04, 0.06]

D_13= [0.01, 0.01]

C_21= [0.04, 0.06]

D_21= [0.01, 0.01]

C_22= [0.04, 0.05]

D_22= [0.01, 0.01]

C_23= [0.04, 0.04]

D_23= [0.01, 0.01]

C_31= [0.04, 0.04]

D_31= [0.01, 0.01]

C_32= [0.04, 0.04]

D_32= [0.01, 0.01]

C_33= [0.04, 0.04]

D_33= [0.01, 0.01]

Theorem 2.2: If A_E and B_F are two intervals valued intuitionistic fuzzy matrix set, then $A_E X_5 B_F$ is also a interval valued intuitionistic fuzzy matrix set.

Proof: If $A_E =$

$$\begin{pmatrix} [\alpha_{E_{11}},\beta_{E_{11}}][\gamma_{E_{11}},\delta_{E_{11}}][\alpha_{E_{12}},\beta_{E_{12}}][\gamma_{E_{12}},\delta_{E_{12}}][\alpha_{E_{13}},\beta_{E_{13}}][\gamma_{E_{13}},\delta_{E_{13}}][\alpha_{E_{21}},\beta_{E_{21}}][\gamma_{E_{21}},\delta_{E_{21}}][\alpha_{E_{22}},\beta_{E_{22}}] \\ [\gamma_{E_{22}},\delta_{E_{22}}]\ [\alpha_{E_{23}},\beta_{E_{23}}][\gamma_{E_{23}},\delta_{E_{23}}]\ [\alpha_{E_{31}},\beta_{E_{31}}][\gamma_{E_{31}},\delta_{E_{31}}]\ [\alpha_{E_{32}},\beta_{E_{32}}][\gamma_{E_{32}},\delta_{E_{32}}]\ [\alpha_{E_{33}},\beta_{E_{33}}][\gamma_{E_{33}},\delta_{E_{33}}] \end{pmatrix}$$

and

$$B_F =$$

$$\begin{pmatrix} [\alpha_{F_{11}},\beta_{F_{11}}][\gamma_{F_{11}},\delta_{F_{11}}][\alpha_{F_{12}},\beta_{F_{12}}][\gamma_{F_{12}},\delta_{F_{12}}][\alpha_{F_{13}},\beta_{F_{13}}][\gamma_{F_{13}},\delta_{F_{13}}][\alpha_{F_{21}},\beta_{F_{21}}][\gamma_{F_{21}},\delta_{F_{22}}][\alpha_{F_{22}},\beta_{F_{22}}] \\ [\gamma_{F_{22}},\delta_{F_{22}}]\ [\alpha_{F_{23}},\beta_{F_{23}}][\gamma_{F_{23}},\delta_{F_{23}}]\ [\alpha_{F_{31}},\beta_{F_{31}}][\gamma_{F_{31}},\delta_{F_{31}}]\ [\alpha_{F_{32}},\beta_{F_{32}}][\gamma_{F_{32}},\delta_{F_{32}}]\ [\alpha_{F_{33}},\beta_{F_{33}}][\gamma_{F_{33}},\delta_{F_{33}} \end{pmatrix}$$

are interval valued intuitionistic fuzzy matrix sets. Then

$$A_E X_5 B_F = \begin{pmatrix} [\alpha_{E_{11}},\beta_{E_{11}}][\gamma_{E_{11}},\delta_{E_{11}}]\ [\alpha_{E_{12}},\beta_{E_{12}}][\gamma_{E_{12}},\delta_{E_{12}}]\ [\alpha_{E_{13}},\beta_{E_{13}}][\gamma_{E_{13}},\delta_{E_{13}}]\ [\alpha_{E_{21}},\beta_{E_{21}}][\gamma_{E_{21}},\delta_{E_{21}}] \\ [\alpha_{E_{22}},\beta_{E_{22}}][\gamma_{E_{22}},\delta_{E_{22}}]\ [\alpha_{E_{23}},\beta_{E_{23}}][\gamma_{E_{23}},\delta_{E_{23}}]\ [\alpha_{E_{31}},\beta_{E_{31}}][\gamma_{E_{31}},\delta_{E_{31}}]\ [\alpha_{E_{32}},\beta_{E_{32}}][\gamma_{E_{32}},\delta_{E_{32}}]\ [\alpha_{E_{33}},\beta_{E_{33}}][\gamma_{E_{33}},\delta_{E_{33}}] \end{pmatrix} X_5$$

$$\begin{pmatrix} [\alpha_{F_{11}},\beta_{F_{11}}][\gamma_{F_{11}},\delta_{F_{11}}][\alpha_{F_{12}},\beta_{F_{12}}][\gamma_{F_{12}},\delta_{F_{12}}][\alpha_{F_{13}},\beta_{F_{13}}][\gamma_{F_{13}},\delta_{F_{13}}][\alpha_{F_{21}},\beta_{F_{21}}][\gamma_{F_{21}},\delta_{F_{22}}] \\ [\alpha_{F_{22}},\beta_{F_{22}}][\gamma_{F_{22}},\delta_{F_{22}}]\ [\alpha_{F_{23}},\beta_{F_{23}}][\gamma_{F_{23}},\delta_{F_{23}}]\ [\alpha_{F_{21}},\beta_{F_{21}}][\gamma_{F_{21}},\delta_{F_{22}}]\ [\alpha_{F_{32}},\beta_{F_{32}}][\gamma_{F_{32}},\delta_{F_{32}}]\ [\alpha_{F_{33}},\beta_{F_{33}}][\gamma_{F_{33}},\delta_{F_{33}} \end{pmatrix}$$

$A_E \times_5 B_F = (X_{11}\ X_{12}\ X_{13}\ X_{21}\ X_{22}\ X_{23}\ X_{31}\ X_{32}\ X_{33})$ Where,

$X_{11} = ([\alpha_{E_{11}},\beta_{E_{11}}][\gamma_{E_{11}},\delta_{E_{11}}]\ [\alpha_{E_{12}},\beta_{E_{12}}][\gamma_{E_{12}},\delta_{E_{12}}]\ [\alpha_{E_{13}},\beta_{E_{13}}][\gamma_{E_{13}},\delta_{E_{13}}]) \times_5$
$([\alpha_{F_{11}},\beta_{F_{11}}][\gamma_{F_{11}},\delta_{F_{11}}]\ [\alpha_{F_{21}},\beta_{F_{21}}][\gamma_{F_{21}},\delta_{F_{22}}]\ [\alpha_{F_{31}},\beta_{F_{31}}][\gamma_{F_{31}},\delta_{F_{31}}])$

$$X_{12} = ([\alpha_{E_{11}},\beta_{E_{11}}][\gamma_{E_{11}},\delta_{E_{11}}]\ [\alpha_{E_{12}},\beta_{E_{12}}][\gamma_{E_{12}},\delta_{E_{12}}]\ [\alpha_{E_{13}},\beta_{E_{13}}][\gamma_{E_{13}},\delta_{E_{13}}]) \times_5 \begin{pmatrix} [\alpha_{F_{12}},\beta_{F_{12}}][\gamma_{F_{12}},\delta_{F_{12}}][\alpha_{F_{22}},\beta_{F_{22}}][\gamma_{F_{22}},\delta_{F_{22}}] \\ [\alpha_{F_{32}},\beta_{F_{32}}][\gamma_{F_{32}},\delta_{F_{32}}] \end{pmatrix}$$

$$X_{13} = \left(\left[\alpha_{E_{11}}, \beta_{E_{11}}\right]\left[\gamma_{E_{11}}, \delta_{E_{11}}\right] \left[\alpha_{E_{12}}, \beta_{E_{12}}\right]\left[\gamma_{E_{12}}, \delta_{E_{12}}\right] \left[\alpha_{E_{13}}, \beta_{E_{13}}\right]\left[\gamma_{E_{13}}, \delta_{E_{13}}\right]\right) \times_5$$
$$\left(\left[\alpha_{F_{13}}, \beta_{F_{13}}\right]\left[\gamma_{F_{13}}, \delta_{F_{13}}\right] \left[\alpha_{F_{23}}, \beta_{F_{23}}\right]\left[\gamma_{F_{23}}, \delta_{F_{23}}\right] \left[\alpha_{F_{33}}, \beta_{F_{33}}\right]\left[\gamma_{F_{33}}, \delta_{F_{33}}\right]\right)$$

$$X_{21} = \left(\left[\alpha_{E_{21}}, \beta_{E_{21}}\right]\left[\gamma_{E_{21}}, \delta_{E_{21}}\right] \left[\alpha_{E_{22}}, \beta_{E_{22}}\right]\left[\gamma_{E_{22}}, \delta_{E_{22}}\right] \left[\alpha_{E_{23}}, \beta_{E_{23}}\right]\left[\gamma_{E_{23}}, \delta_{E_{23}}\right]\right) \times_5$$
$$\left(\left[\alpha_{F_{11}}, \beta_{F_{11}}\right]\left[\gamma_{F_{11}}, \delta_{F_{11}}\right] \left[\alpha_{F_{21}}, \beta_{F_{21}}\right]\left[\gamma_{F_{21}}, \delta_{F_{22}}\right] \left[\alpha_{F_{31}}, \beta_{F_{31}}\right]\left[\gamma_{F_{31}}, \delta_{F_{31}}\right]\right)$$

$$X_{22} = \left(\left[\alpha_{E_{21}}, \beta_{E_{21}}\right]\left[\gamma_{E_{21}}, \delta_{E_{21}}\right] \left[\alpha_{E_{22}}, \beta_{E_{22}}\right]\left[\gamma_{E_{22}}, \delta_{E_{22}}\right] \left[\alpha_{E_{23}}, \beta_{E_{23}}\right]\left[\gamma_{E_{23}}, \delta_{E_{23}}\right]\right) \times_5$$
$$\left(\left[\alpha_{F_{12}}, \beta_{F_{12}}\right]\left[\gamma_{F_{12}}, \delta_{F_{12}}\right] \left[\alpha_{F_{22}}, \beta_{F_{22}}\right]\left[\gamma_{F_{22}}, \delta_{F_{22}}\right] \left[\alpha_{F_{32}}, \beta_{F_{32}}\right]\left[\gamma_{F_{32}}, \delta_{F_{32}}\right]\right)$$

$$X_{23} = \left(\left[\alpha_{E_{21}}, \beta_{E_{21}}\right]\left[\gamma_{E_{21}}, \delta_{E_{21}}\right] \left[\alpha_{E_{22}}, \beta_{E_{22}}\right]\left[\gamma_{E_{22}}, \delta_{E_{22}}\right] \left[\alpha_{E_{23}}, \beta_{E_{23}}\right]\left[\gamma_{E_{23}}, \delta_{E_{23}}\right]\right) \times_5$$
$$\left(\left[\alpha_{F_{13}}, \beta_{F_{13}}\right]\left[\gamma_{F_{13}}, \delta_{F_{13}}\right] \left[\alpha_{F_{23}}, \beta_{F_{23}}\right]\left[\gamma_{F_{23}}, \delta_{F_{23}}\right] \left[\alpha_{F_{33}}, \beta_{F_{33}}\right]\left[\gamma_{F_{33}}, \delta_{F_{33}}\right]\right)$$

$$X_{31} = \left(\left[\alpha_{E_{31}}, \beta_{E_{31}}\right]\left[\gamma_{E_{31}}, \delta_{E_{31}}\right] \left[\alpha_{E_{32}}, \beta_{E_{32}}\right]\left[\gamma_{E_{32}}, \delta_{E_{32}}\right] \left[\alpha_{E_{33}}, \beta_{E_{33}}\right]\left[\gamma_{E_{33}}, \delta_{E_{33}}\right]\right) \times_5$$
$$\left(\left[\alpha_{F_{11}}, \beta_{F_{11}}\right]\left[\gamma_{F_{11}}, \delta_{F_{11}}\right] \left[\alpha_{F_{21}}, \beta_{F_{21}}\right]\left[\gamma_{F_{21}}, \delta_{F_{22}}\right] \left[\alpha_{F_{31}}, \beta_{F_{31}}\right]\left[\gamma_{F_{31}}, \delta_{F_{31}}\right]\right)$$

$$X_{32} = \left(\left[\alpha_{E_{31}}, \beta_{E_{31}}\right]\left[\gamma_{E_{31}}, \delta_{E_{31}}\right] \left[\alpha_{E_{32}}, \beta_{E_{32}}\right]\left[\gamma_{E_{32}}, \delta_{E_{32}}\right] \left[\alpha_{E_{33}}, \beta_{E_{33}}\right]\left[\gamma_{E_{33}}, \delta_{E_{33}}\right]\right) \times_5$$
$$\left(\left[\alpha_{F_{12}}, \beta_{F_{12}}\right]\left[\gamma_{F_{12}}, \delta_{F_{12}}\right] \left[\alpha_{F_{22}}, \beta_{F_{22}}\right]\left[\gamma_{F_{22}}, \delta_{F_{22}}\right] \left[\alpha_{F_{32}}, \beta_{F_{32}}\right]\left[\gamma_{F_{32}}, \delta_{F_{32}}\right]\right)$$

$$X_{33} = \left(\left[\alpha_{E_{31}}, \beta_{E_{31}}\right]\left[\gamma_{E_{31}}, \delta_{E_{31}}\right] \left[\alpha_{E_{32}}, \beta_{E_{32}}\right]\left[\gamma_{E_{32}}, \delta_{E_{32}}\right] \left[\alpha_{E_{33}}, \beta_{E_{33}}\right]\left[\gamma_{E_{33}}, \delta_{E_{33}}\right]\right) \times_5$$
$$\left(\left[\alpha_{F_{13}}, \beta_{F_{13}}\right]\left[\gamma_{F_{13}}, \delta_{F_{13}}\right] \left[\alpha_{F_{23}}, \beta_{F_{23}}\right]\left[\gamma_{F_{23}}, \delta_{F_{23}}\right] \left[\alpha_{F_{33}}, \beta_{F_{33}}\right]\left[\gamma_{F_{33}}, \delta_{F_{33}}\right]\right)$$

$$X_{11} = \left[\alpha_{E_{11}}, \beta_{E_{11}}\right]\left[\gamma_{E_{11}}, \delta_{E_{11}}\right] \times_5 \left[\alpha_{F_{11}}, \beta_{F_{11}}\right]\left[\gamma_{F_{11}}, \delta_{F_{11}}\right] +$$
$$\left[\alpha_{E_{12}}, \beta_{E_{12}}\right]\left[\gamma_{E_{12}}, \delta_{E_{12}}\right] \times_5 \left[\alpha_{F_{21}}, \beta_{F_{21}}\right]\left[\gamma_{F_{21}}, \delta_{F_{22}}\right] +$$
$$\left[\alpha_{E_{13}}, \beta_{E_{13}}\right]\left[\gamma_{E_{13}}, \delta_{E_{13}}\right] \times_5 \left[\alpha_{F_{31}}, \beta_{F_{31}}\right]\left[\gamma_{F_{31}}, \delta_{F_{31}}\right]$$

$$X_{12} = \left[\alpha_{E_{11}}, \beta_{E_{11}}\right]\left[\gamma_{E_{11}}, \delta_{E_{11}}\right] \times_5 \left[\alpha_{F_{12}}, \beta_{F_{12}}\right]\left[\gamma_{F_{12}}, \delta_{F_{12}}\right] +$$
$$\left[\alpha_{E_{12}}, \beta_{E_{12}}\right]\left[\gamma_{E_{12}}, \delta_{E_{12}}\right] \times_5 \left[\alpha_{F_{22}}, \beta_{F_{22}}\right]\left[\gamma_{F_{22}}, \delta_{F_{22}}\right] +$$
$$\left[\alpha_{E_{13}}, \beta_{E_{13}}\right]\left[\gamma_{E_{13}}, \delta_{E_{13}}\right] \times_5 \left[\alpha_{F_{32}}, \beta_{F_{32}}\right]\left[\gamma_{F_{32}}, \delta_{F_{32}}\right]$$

$$X_{13} = \left[\alpha_{E_{11}}, \beta_{E_{11}}\right]\left[\gamma_{E_{11}}, \delta_{E_{11}}\right] \times_5 \left[\alpha_{F_{13}}, \beta_{F_{13}}\right]\left[\gamma_{F_{13}}, \delta_{F_{13}}\right] +$$
$$\left[\alpha_{E_{12}}, \beta_{E_{12}}\right]\left[\gamma_{E_{12}}, \delta_{E_{12}}\right] \times_5 \left[\alpha_{F_{23}}, \beta_{F_{23}}\right]\left[\gamma_{F_{23}}, \delta_{F_{23}}\right] +$$
$$\left[\alpha_{E_{13}}, \beta_{E_{13}}\right]\left[\gamma_{E_{13}}, \delta_{E_{13}}\right] \times_5 \left[\alpha_{F_{33}}, \beta_{F_{33}}\right]\left[\gamma_{F_{33}}, \delta_{F_{33}}\right]$$

$$X_{21} = \left[\alpha_{E_{21}}, \beta_{E_{21}}\right]\left[\gamma_{E_{21}}, \delta_{E_{21}}\right] \times_5 \left[\alpha_{F_{11}}, \beta_{F_{11}}\right]\left[\gamma_{F_{11}}, \delta_{F_{11}}\right] +$$
$$\left[\alpha_{E_{22}}, \beta_{E_{22}}\right]\left[\gamma_{E_{22}}, \delta_{E_{22}}\right] \times_5 \left[\alpha_{F_{21}}, \beta_{F_{21}}\right]\left[\gamma_{F_{21}}, \delta_{F_{21}}\right] +$$
$$\left[\alpha_{E_{23}}, \beta_{E_{23}}\right]\left[\gamma_{E_{23}}, \delta_{E_{23}}\right] \times_5 \left[\alpha_{F_{31}}, \beta_{F_{31}}\right]\left[\gamma_{F_{31}}, \delta_{F_{31}}\right]$$

$$X_{22} = \left[\alpha_{E_{21}}, \beta_{E_{21}}\right]\left[\gamma_{E_{21}}, \delta_{E_{21}}\right] \times_5 \left[\alpha_{F_{12}}, \beta_{F_{12}}\right]\left[\gamma_{F_{12}}, \delta_{F_{12}}\right] +$$
$$\left[\alpha_{E_{22}}, \beta_{E_{22}}\right]\left[\gamma_{E_{22}}, \delta_{E_{22}}\right] \times_5 \left[\alpha_{F_{22}}, \beta_{F_{22}}\right]\left[\gamma_{F_{22}}, \delta_{F_{22}}\right] +$$
$$\left[\alpha_{E_{23}}, \beta_{E_{23}}\right]\left[\gamma_{E_{23}}, \delta_{E_{23}}\right] \times_5 \left[\alpha_{F_{32}}, \beta_{F_{32}}\right]\left[\gamma_{F_{32}}, \delta_{F_{32}}\right]$$

$X_{23} = [\alpha_{E_{21}}, \beta_{E_{21}}][\gamma_{E_{21}}, \delta_{E_{21}}] \times_5 [\alpha_{F_{13}}, \beta_{F_{13}}][\gamma_{F_{13}}, \delta_{F_{13}}] +$
$[\alpha_{E_{22}}, \beta_{E_{22}}][\gamma_{E_{22}}, \delta_{E_{22}}] \times_5 [\alpha_{F_{23}}, \beta_{F_{23}}][\gamma_{F_{23}}, \delta_{F_{23}}]+$
$[\alpha_{E_{23}}, \beta_{E_{23}}][\gamma_{E_{23}}, \delta_{E_{23}}] \times_5 [\alpha_{F_{33}}, \beta_{F_{33}}][\gamma_{F_{33}}, \delta_{F_{33}}]$

$X_{31} = [\alpha_{E_{31}}, \beta_{E_{31}}][\gamma_{E_{31}}, \delta_{E_{31}}] \times_5 [\alpha_{F_{11}}, \beta_{F_{11}}][\gamma_{F_{11}}, \delta_{F_{11}}] +$
$[\alpha_{E_{32}}, \beta_{E_{32}}][\gamma_{E_{32}}, \delta_{E_{32}}] \times_5 [\alpha_{F_{21}}, \beta_{F_{21}}][\gamma_{F_{21}}, \delta_{F_{21}}]+$
$[\alpha_{E_{33}}, \beta_{E_{33}}][\gamma_{E_{33}}, \delta_{E_{33}}] \times_5 [\alpha_{F_{31}}, \beta_{F_{31}}][\gamma_{F_{31}}, \delta_{F_{31}}]$

$X_{32} = [\alpha_{E_{31}}, \beta_{E_{31}}][\gamma_{E_{31}}, \delta_{E_{31}}] \times_5 [\alpha_{F_{12}}, \beta_{F_{12}}][\gamma_{F_{12}}, \delta_{F_{12}}] +$
$[\alpha_{E_{32}}, \beta_{E_{32}}][\gamma_{E_{32}}, \delta_{E_{32}}] \times_5 [\alpha_{F_{22}}, \beta_{F_{22}}][\gamma_{F_{22}}, \delta_{F_{22}}]+$
$[\alpha_{E_{33}}, \beta_{E_{33}}][\gamma_{E_{33}}, \delta_{E_{33}}] \times_5 [\alpha_{F_{32}}, \beta_{F_{32}}][\gamma_{F_{32}}, \delta_{F_{32}}]$

$X_{33} = [\alpha_{E_{31}}, \beta_{E_{31}}][\gamma_{E_{31}}, \delta_{E_{31}}] \times_5 [\alpha_{F_{13}}, \beta_{F_{13}}][\gamma_{F_{13}}, \delta_{F_{13}}] +$
$[\alpha_{E_{32}}, \beta_{E_{32}}][\gamma_{E_{32}}, \delta_{E_{32}}] \times_5 [\alpha_{F_{23}}, \beta_{F_{23}}][\gamma_{F_{23}}, \delta_{F_{23}}]+$
$[\alpha_{E_{33}}, \beta_{E_{33}}][\gamma_{E_{33}}, \delta_{E_{33}}] \times_5 [\alpha_{F_{33}}, \beta_{F_{33}}][\gamma_{F_{33}}, \delta_{F_{33}}]$

Now, by applying this

$$(A \times_5 B) = \left\{(x,y)\begin{pmatrix}[\max(\alpha, max(inf\,\mu_A(x), inf\,\mu_B(x)), \max(\alpha, \max(sup\mu_A(x), sup\mu_B(x))] \\ [min(\beta, min(inf\gamma_A(x), inf\gamma_B(x)) \ , min(\beta, min(sup\gamma_A(x), sup\gamma_B(x))]\end{pmatrix}; x \in A \; and \; y \in B\right\}$$

$$X_{11} = \begin{Bmatrix}[\max(\alpha, max(\alpha_{E_{11}}, \alpha_{F_{11}})), \max(\alpha, max(\beta_{E_{11}}, \beta_{F_{11}}))][min\left(\beta, min(\gamma_{E_{11}}, \gamma_{F_{11}})\right), \\ \left(\beta, min(\delta_{E_{11}}, \delta_{F_{11}})\right)]\end{Bmatrix} +$$
$$\left\{\left[\left(\alpha, max(\alpha_{E_{12}}, \alpha_{F_{21}})\right), \left(\alpha, max(\beta_{E_{12}}, \beta_{F_{21}})\right)\right][\left(\beta, min(\gamma_{E_{12}}, \gamma_{F_{21}})\right), \left(\beta, min(\delta_{E_{12}}, \delta_{F_{21}})\right)]\right\} +$$
$$\left\{\left[\left(\alpha, max(\alpha_{E_{13}}, \alpha_{F_{31}})\right), \left(\alpha, max(\beta_{E_{13}}, \beta_{F_{31}})\right)\right][\left(\beta, min(\gamma_{E_{13}}, \gamma_{F_{31}})\right), min(\beta, min(\delta_{E_{13}}, \delta_{F_{31}}))\right\}$$

$$X_{12} = \left\{[max(\alpha, max(\alpha_{E_{11}}, \alpha_{F_{12}})), max(\alpha, max(\beta_{E_{11}}, \beta_{F_{12}}))][min\left(\beta, min(\gamma_{E_{11}}, \gamma_{F_{12}})\right), \left(\beta, min(\delta_{E_{11}}, \delta_{F_{12}})\right)]\right\}$$
$$+$$
$$\left\{\left[\left(\alpha, max(\alpha_{E_{12}}, \alpha_{F_{22}})\right), \left(\alpha, max(\beta_{E_{12}}, \beta_{F_{22}})\right)\right][\left(\beta, min(\gamma_{E_{12}}, \gamma_{F_{22}})\right), \left(\beta, min(\delta_{E_{12}}, \delta_{F_{22}})\right)]\right\} +$$
$$\left\{\left[\left(\alpha, max(\alpha_{E_{13}}, \alpha_{F_{32}})\right), \left(\alpha, max(\beta_{E_{13}}, \beta_{F_{32}})\right)\right][\left(\beta, min(\gamma_{E_{13}}, \gamma_{F_{32}})\right), min(\beta, min(\delta_{E_{13}}, \delta_{F_{32}}))\right\}$$

$$X_{13} = \left\{[max(\alpha, max(\alpha_{E_{11}}, \alpha_{F_{13}})), max(\alpha, max(\beta_{E_{11}}, \beta_{F_{13}}))][min\left(\beta, min(\gamma_{E_{11}}, \gamma_{F_{13}})\right), \left(\beta, min(\delta_{E_{11}}, \delta_{F_{13}})\right)]\right\}$$
$$+$$
$$\left\{\left[\left(\alpha, max(\alpha_{E_{12}}, \alpha_{F_{23}})\right), \left(\alpha, max(\beta_{E_{12}}, \beta_{F_{23}})\right)\right][\left(\beta, min(\gamma_{E_{12}}, \gamma_{F_{23}})\right), \left(\beta, min(\delta_{E_{12}}, \delta_{F_{23}})\right)]\right\} +$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{13}}, \alpha_{F_{33}}\right)\right) , \left(\alpha, max\left(\beta_{E_{13}}, \beta_{F_{33}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{13}}, \gamma_{F_{33}}\right)\right) , min(\beta, min\left(\delta_{E_{13}}, \delta_{F_{33}}\right))\right\}$$

$$X_{21}$$

$$= \left\{[max(\alpha, max\left(\alpha_{E_{21}}, \alpha_{F_{11}}\right)), max(\alpha, max\left(\beta_{E_{11}}, \beta_{F_{11}}\right))][min\left(\beta, min\left(\gamma_{E_{21}}, \gamma_{F_{11}}\right)\right), \left(\beta, min\left(\delta_{E_{21}}, \delta_{F_{11}}\right)\right)]\right\}$$

$$+$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{22}}, \alpha_{F_{21}}\right)\right) , \left(\alpha, max\left(\beta_{E_{22}}, \beta_{F_{21}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{22}}, \gamma_{F_{21}}\right)\right) , \left(\beta, min\left(\delta_{E_{22}}, \delta_{F_{21}}\right)\right)]\right\} +$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{23}}, \alpha_{F_{31}}\right)\right) , \left(\alpha, max\left(\beta_{E_{23}}, \beta_{F_{31}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{23}}, \gamma_{F_{31}}\right)\right) , min(\beta, min\left(\delta_{E_{23}}, \delta_{F_{31}}\right))\right\}$$

$$X_{22}$$

$$= \left\{[max(\alpha, max\left(\alpha_{E_{21}}, \alpha_{F_{12}}\right)), max(\alpha, max\left(\beta_{E_{11}}, \beta_{F_{12}}\right))][min\left(\beta, min\left(\gamma_{E_{21}}, \gamma_{F_{12}}\right)\right), \left(\beta, min\left(\delta_{E_{21}}, \delta_{F_{12}}\right)\right)]\right\}$$

$$+$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{22}}, \alpha_{F_{22}}\right)\right) , \left(\alpha, max\left(\beta_{E_{22}}, \beta_{F_{22}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{22}}, \gamma_{F_{22}}\right)\right) , \left(\beta, min\left(\delta_{E_{22}}, \delta_{F_{22}}\right)\right)]\right\} +$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{23}}, \alpha_{F_{32}}\right)\right) , \left(\alpha, max\left(\beta_{E_{23}}, \beta_{F_{32}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{23}}, \gamma_{F_{32}}\right)\right) , min(\beta, min\left(\delta_{E_{23}}, \delta_{F_{32}}\right))\right\}$$

$$X_{23}$$

$$= \left\{[max(\alpha, max\left(\alpha_{E_{21}}, \alpha_{F_{13}}\right)), max(\alpha, max\left(\beta_{E_{11}}, \beta_{F_{13}}\right))][min\left(\beta, min\left(\gamma_{E_{21}}, \gamma_{F_{13}}\right)\right), \left(\beta, min\left(\delta_{E_{21}}, \delta_{F_{13}}\right)\right)]\right\}$$

$$+$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{22}}, \alpha_{F_{23}}\right)\right) , \left(\alpha, max\left(\beta_{E_{22}}, \beta_{F_{23}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{22}}, \gamma_{F_{23}}\right)\right) , \left(\beta, min\left(\delta_{E_{22}}, \delta_{F_{23}}\right)\right)]\right\} +$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{23}}, \alpha_{F_{33}}\right)\right) , \left(\alpha, max\left(\beta_{E_{23}}, \beta_{F_{33}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{23}}, \gamma_{F_{33}}\right)\right) , min(\beta, min\left(\delta_{E_{23}}, \delta_{F_{33}}\right))\right\}$$

$$X_{31}$$

$$= \left\{[max(\alpha, max\left(\alpha_{E_{31}}, \alpha_{F_{11}}\right)), max(\alpha, max\left(\beta_{E_{31}}, \beta_{F_{11}}\right))][min\left(\beta, min\left(\gamma_{E_{31}}, \gamma_{F_{11}}\right)\right), \left(\beta, min\left(\delta_{E_{31}}, \delta_{F_{11}}\right)\right)]\right\}$$

$$+$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{32}}, \alpha_{F_{21}}\right)\right) , \left(\alpha, max\left(\beta_{E_{32}}, \beta_{F_{21}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{32}}, \gamma_{F_{21}}\right)\right) , \left(\beta, min\left(\delta_{E_{32}}, \delta_{F_{21}}\right)\right)]\right\} +$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{33}}, \alpha_{F_{31}}\right)\right) , \left(\alpha, max\left(\beta_{E_{33}}, \beta_{F_{31}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{33}}, \gamma_{F_{31}}\right)\right) , min(\beta, min\left(\delta_{E_{33}}, \delta_{F_{31}}\right))\right\}$$

$$X_{32}$$

$$= \left\{[max(\alpha, max\left(\alpha_{E_{31}}, \alpha_{F_{12}}\right)), max(\alpha, max\left(\beta_{E_{31}}, \beta_{F_{12}}\right))][min\left(\beta, min\left(\gamma_{E_{31}}, \gamma_{F_{12}}\right)\right), \left(\beta, min\left(\delta_{E_{31}}, \delta_{F_{12}}\right)\right)]\right\}$$

$$+$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{32}}, \alpha_{F_{22}}\right)\right) , \left(\alpha, max\left(\beta_{E_{32}}, \beta_{F_{22}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{32}}, \gamma_{F_{22}}\right)\right) , \left(\beta, min\left(\delta_{E_{32}}, \delta_{F_{22}}\right)\right)]\right\} +$$

$$\left\{\left[\left(\alpha, max\left(\alpha_{E_{33}}, \alpha_{F_{32}}\right)\right) , \left(\alpha, max\left(\beta_{E_{33}}, \beta_{F_{32}}\right)\right)\right] [\left(\beta, min\left(\gamma_{E_{33}}, \gamma_{F_{32}}\right)\right) , min(\beta, min\left(\delta_{E_{33}}, \delta_{F_{32}}\right))\right\}$$

X_{33}

$= \left\{[max(\alpha, max(\alpha_{E_{31}}, \alpha_{F_{13}})), max(\alpha, max(\beta_{E_{31}}, \beta_{F_{13}}))][min\left(\beta, min(\gamma_{E_{31}}, \gamma_{F_{13}})\right), \left(\beta, min(\delta_{E_{31}}, \delta_{F_{13}})\right)]\right\}$

$+$

$\left\{\left[\left(\alpha, max(\alpha_{E_{32}}, \alpha_{F_{23}})\right), \left(\alpha, max(\beta_{E_{32}}, \beta_{F_{23}})\right)\right][\left(\beta, min(\gamma_{E_{32}}, \gamma_{F_{23}})\right), \left(\beta, min(\delta_{E_{32}}, \delta_{F_{23}})\right)]\right\} +$

$\left\{\left[\left(\alpha, max(\alpha_{E_{33}}, \alpha_{F_{33}})\right), \left(\alpha, max(\beta_{E_{33}}, \beta_{F_{33}})\right)\right][\left(\beta, min(\gamma_{E_{33}}, \gamma_{F_{33}})\right), min(\beta, min(\delta_{E_{33}}, \delta_{F_{33}}))\right\}$

By using A+B = max {A, B}

X_{11}

$= max\begin{pmatrix}\left[\left(\alpha, max(\alpha_{E_{11}}, \alpha_{F_{11}})\right), \left(\alpha, max(\beta_{E_{11}}, \beta_{F_{11}})\right)\right]\left[min\left(\beta, min(\gamma_{E_{11}}, \gamma_{F_{11}})\right), \left(\beta, min(\delta_{E_{11}}, \delta_{F_{11}})\right)\right] \\ , \left[\left(\alpha, max(\alpha_{E_{12}}, \alpha_{F_{21}})\right), \left(\alpha, max(\beta_{E_{12}}, \beta_{F_{21}})\right)\right][\left(\beta, min(\gamma_{E_{12}}, \gamma_{F_{21}})\right), \left(\beta, min(\delta_{E_{12}}, \delta_{F_{21}})\right)]\end{pmatrix}$

$+$

$\left\{\left[\left(\alpha, max(\alpha_{E_{13}}, \alpha_{F_{31}})\right), \left(\alpha, max(\beta_{E_{13}}, \beta_{F_{31}})\right)\right][\left(\beta, min(\gamma_{E_{13}}, \gamma_{F_{31}})\right), min(\beta, min(\delta_{E_{13}}, \delta_{F_{31}}))\right\}$

X_{12}

$= max\begin{pmatrix}\left[\left(\alpha, max(\alpha_{E_{11}}, \alpha_{F_{12}})\right), \left(\alpha, max(\beta_{E_{11}}, \beta_{F_{12}})\right)\right]\left[min\left(\beta, min(\gamma_{E_{11}}, \gamma_{F_{12}})\right), \left(\beta, min(\delta_{E_{11}}, \delta_{F_{12}})\right)\right], \\ \left[\left(\alpha, max(\alpha_{E_{12}}, \alpha_{F_{22}})\right), \left(\alpha, max(\beta_{E_{12}}, \beta_{F_{22}})\right)\right][\left(\beta, min(\gamma_{E_{12}}, \gamma_{F_{22}})\right), \left(\beta, min(\delta_{E_{12}}, \delta_{F_{22}})\right)]\end{pmatrix}$

$+$

$\left\{\left[\left(\alpha, max(\alpha_{E_{13}}, \alpha_{F_{32}})\right), \left(\alpha, max(\beta_{E_{13}}, \beta_{F_{32}})\right)\right][\left(\beta, min(\gamma_{E_{13}}, \gamma_{F_{32}})\right), min(\beta, min(\delta_{E_{13}}, \delta_{F_{32}}))\right\}$

X_{13}

$= max\begin{pmatrix}\left[\left(\alpha, max(\alpha_{E_{11}}, \alpha_{F_{13}})\right), \left(\alpha, max(\beta_{E_{11}}, \beta_{F_{13}})\right)\right]\left[min\left(\beta, min(\gamma_{E_{11}}, \gamma_{F_{13}})\right), \left(\beta, min(\delta_{E_{11}}, \delta_{F_{13}})\right)\right], \\ \left[\left(\alpha, max(\alpha_{E_{12}}, \alpha_{F_{23}})\right), \left(\alpha, max(\beta_{E_{12}}, \beta_{F_{23}})\right)\right][\left(\beta, min(\gamma_{E_{12}}, \gamma_{F_{23}})\right), \left(\beta, min(\delta_{E_{12}}, \delta_{F_{23}})\right)]\end{pmatrix}$

$+$

$\left\{\left[\left(\alpha, max(\alpha_{E_{13}}, \alpha_{F_{33}})\right), \left(\alpha, max(\beta_{E_{13}}, \beta_{F_{33}})\right)\right][\left(\beta, min(\gamma_{E_{13}}, \gamma_{F_{33}})\right), min(\beta, min(\delta_{E_{13}}, \delta_{F_{33}}))\right\}$

X_{21}

$= max\begin{pmatrix}\left[\left(\alpha, max(\alpha_{E_{21}}, \alpha_{F_{11}})\right), \left(\alpha, max(\beta_{E_{21}}, \beta_{F_{11}})\right)\right]\left[min\left(\beta, min(\gamma_{E_{21}}, \gamma_{F_{11}})\right), \left(\beta, min(\delta_{E_{21}}, \delta_{F_{11}})\right)\right], \\ \left[\left(\alpha, max(\alpha_{E_{22}}, \alpha_{F_{21}})\right), \left(\alpha, max(\beta_{E_{22}}, \beta_{F_{21}})\right)\right][\left(\beta, min(\gamma_{E_{22}}, \gamma_{F_{21}})\right), \left(\beta, min(\delta_{E_{22}}, \delta_{F_{21}})\right)]\end{pmatrix}$

$+$

$\left\{\left[\left(\alpha, max(\alpha_{E_{23}}, \alpha_{F_{31}})\right), \left(\alpha, max(\beta_{E_{23}}, \beta_{F_{31}})\right)\right][\left(\beta, min(\gamma_{E_{23}}, \gamma_{F_{31}})\right), min(\beta, min(\delta_{E_{23}}, \delta_{F_{31}}))\right\}$

$$X_{22} = max\begin{pmatrix} \left[\left(\alpha, max\left(\alpha_{E_{21}}, \alpha_{F_{12}}\right)\right), \left(\alpha, max\left(\beta_{E_{21}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{21}}, \gamma_{F_{12}}\right)\right), \left(\beta, min\left(\delta_{E_{21}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{22}}, \alpha_{F_{22}}\right)\right), \left(\alpha, max\left(\beta_{E_{22}}, \beta_{F_{22}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{22}}, \gamma_{F_{22}}\right)\right), \left(\beta, min\left(\delta_{E_{22}}, \delta_{F_{22}}\right)\right)] \end{pmatrix}$$

$$+ \left\{\left[\left(\alpha, max\left(\alpha_{E_{23}}, \alpha_{F_{32}}\right)\right), \left(\alpha, max\left(\beta_{E_{23}}, \beta_{F_{32}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{23}}, \gamma_{F_{32}}\right)\right), min(\beta, min\left(\delta_{E_{23}}, \delta_{F_{32}}\right))\right\}$$

$$X_{23} = max\begin{pmatrix} \left[\left(\alpha, max\left(\alpha_{E_{21}}, \alpha_{F_{13}}\right)\right), \left(\alpha, max\left(\beta_{E_{21}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{21}}, \gamma_{F_{13}}\right)\right), \left(\beta, min\left(\delta_{E_{21}}, \delta_{F_{13}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{22}}, \alpha_{F_{23}}\right)\right), \left(\alpha, max\left(\beta_{E_{22}}, \beta_{F_{23}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{22}}, \gamma_{F_{23}}\right)\right), \left(\beta, min\left(\delta_{E_{22}}, \delta_{F_{23}}\right)\right)] \end{pmatrix}$$

$$+ \left\{\left[\left(\alpha, max\left(\alpha_{E_{23}}, \alpha_{F_{33}}\right)\right), \left(\alpha, max\left(\beta_{E_{23}}, \beta_{F_{33}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{23}}, \gamma_{F_{33}}\right)\right), min(\beta, min\left(\delta_{E_{23}}, \delta_{F_{33}}\right))\right\}$$

$$X_{31} = max\begin{pmatrix} \left[\left(\alpha, max\left(\alpha_{E_{31}}, \alpha_{F_{11}}\right)\right), \left(\alpha, max\left(\beta_{E_{31}}, \beta_{F_{11}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{31}}, \gamma_{F_{11}}\right)\right), \left(\beta, min\left(\delta_{E_{31}}, \delta_{F_{11}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{32}}, \alpha_{F_{21}}\right)\right), \left(\alpha, max\left(\beta_{E_{32}}, \beta_{F_{21}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{32}}, \gamma_{F_{21}}\right)\right), \left(\beta, min\left(\delta_{E_{32}}, \delta_{F_{21}}\right)\right)] \end{pmatrix}$$

$$+ \left\{\left[\left(\alpha, max\left(\alpha_{E_{33}}, \alpha_{F_{31}}\right)\right), \left(\alpha, max\left(\beta_{E_{33}}, \beta_{F_{31}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{33}}, \gamma_{F_{31}}\right)\right), min(\beta, min\left(\delta_{E_{33}}, \delta_{F_{31}}\right))\right\}$$

$$X_{32} = max\begin{pmatrix} \left[\left(\alpha, max\left(\alpha_{E_{31}}, \alpha_{F_{12}}\right)\right), \left(\alpha, max\left(\beta_{E_{31}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{31}}, \gamma_{F_{12}}\right)\right), \left(\beta, min\left(\delta_{E_{31}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{32}}, \alpha_{F_{22}}\right)\right), \left(\alpha, max\left(\beta_{E_{32}}, \beta_{F_{22}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{32}}, \gamma_{F_{22}}\right)\right), \left(\beta, min\left(\delta_{E_{32}}, \delta_{F_{22}}\right)\right)] \end{pmatrix}$$

$$+ \left\{\left[\left(\alpha, max\left(\alpha_{E_{33}}, \alpha_{F_{32}}\right)\right), \left(\alpha, max\left(\beta_{E_{33}}, \beta_{F_{32}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{33}}, \gamma_{F_{32}}\right)\right), min(\beta, min\left(\delta_{E_{33}}, \delta_{F_{32}}\right))\right\}$$

$$X_{33} = max\begin{pmatrix} \left[\left(\alpha, max\left(\alpha_{E_{31}}, \alpha_{F_{13}}\right)\right), \left(\alpha, max\left(\beta_{E_{31}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{31}}, \gamma_{F_{13}}\right)\right), \left(\beta, min\left(\delta_{E_{31}}, \delta_{F_{13}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{32}}, \alpha_{F_{23}}\right)\right), \left(\alpha, max\left(\beta_{E_{32}}, \beta_{F_{23}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{32}}, \gamma_{F_{23}}\right)\right), \left(\beta, min\left(\delta_{E_{32}}, \delta_{F_{23}}\right)\right)] \end{pmatrix}$$

$$+ \left\{\left[\left(\alpha, max\left(\alpha_{E_{33}}, \alpha_{F_{33}}\right)\right), \left(\alpha, max\left(\beta_{E_{33}}, \beta_{F_{33}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{33}}, \gamma_{F_{33}}\right)\right), min(\beta, min\left(\delta_{E_{33}}, \delta_{F_{33}}\right))\right\}$$

By using A+B+C = max$\{max(A, B), C\}$

X_{11} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c} \left[\left(\alpha, max\left(\alpha_{E_{11}}, \alpha_{F_{11}}\right)\right), \left(\alpha, max\left(\beta_{E_{11}}, \beta_{F_{11}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{11}}, \gamma_{F_{11}}\right)\right), \left(\beta, min\left(\delta_{E_{11}}. \delta_{F_{11}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{12}}, \alpha_{F_{21}}\right)\right), \left(\alpha, max\left(\beta_{E_{12}}, \beta_{F_{21}}\right)\right)\right]\left[\left(\beta, min\left(\gamma_{E_{12}}, \gamma_{F_{21}}\right)\right), \left(\beta, min\left(\delta_{E_{12}}, \delta_{F_{21}}\right)\right)\right] \end{array}\right), \\ \left[\left(\alpha, max\left(\alpha_{E_{13}}, \alpha_{F_{31}}\right)\right), \left(\alpha, max\left(\beta_{E_{13}}, \beta_{F_{31}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{13}}, \gamma_{F_{31}}\right)\right), min(\beta, min\left(\delta_{E_{13}}, \delta_{F_{31}}\right)) \end{array}\right)$$

X_{12} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c} \left[\left(\alpha, max\left(\alpha_{E_{11}}, \alpha_{F_{12}}\right)\right), \left(\alpha, max\left(\beta_{E_{11}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{11}}, \gamma_{F_{12}}\right)\right), \left(\beta, min\left(\delta_{E_{11}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{12}}, \alpha_{F_{22}}\right)\right), \left(\alpha, max\left(\beta_{E_{12}}, \beta_{F_{22}}\right)\right)\right]\left[\left(\beta, min\left(\gamma_{E_{12}}, \gamma_{F_{22}}\right)\right), \left(\beta, min\left(\delta_{E_{12}}, \delta_{F_{22}}\right)\right)\right] \end{array}\right), \\ \left[\left(\alpha, max\left(\alpha_{E_{13}}, \alpha_{F_{32}}\right)\right), \left(\alpha, max\left(\beta_{E_{13}}, \beta_{F_{32}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{13}}, \gamma_{F_{32}}\right)\right), min(\beta, min\left(\delta_{E_{13}}, \delta_{F_{32}}\right)) \end{array}\right)$$

X_{13} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c} \left[\left(\alpha, max\left(\alpha_{E_{11}}, \alpha_{F_{13}}\right)\right), \left(\alpha, max\left(\beta_{E_{11}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{11}}, \gamma_{F_{13}}\right)\right), \left(\beta, min\left(\delta_{E_{11}}, \delta_{F_{13}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{12}}, \alpha_{F_{23}}\right)\right), \left(\alpha, max\left(\beta_{E_{12}}, \beta_{F_{23}}\right)\right)\right]\left[\left(\beta, min\left(\gamma_{E_{12}}, \gamma_{F_{23}}\right)\right), \left(\beta, min\left(\delta_{E_{12}}, \delta_{F_{23}}\right)\right)\right] \end{array}\right), \\ \left[\left(\alpha, max\left(\alpha_{E_{13}}, \alpha_{F_{33}}\right)\right), \left(\alpha, max\left(\beta_{E_{13}}, \beta_{F_{33}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{13}}, \gamma_{F_{33}}\right)\right), min(\beta, min\left(\delta_{E_{13}}, \delta_{F_{33}}\right)) \end{array}\right)$$

X_{21} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c} \left[\left(\alpha, max\left(\alpha_{E_{21}}, \alpha_{F_{11}}\right)\right), \left(\alpha, max\left(\beta_{E_{21}}. \beta_{F_{11}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{21}}, \gamma_{F_{11}}\right)\right), \left(\beta, min\left(\delta_{E_{21}}, \delta_{F_{11}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{22}}, \alpha_{F_{21}}\right)\right), \left(\alpha, max\left(\beta_{E_{22}}. \beta_{F_{21}}\right)\right)\right]\left[\left(\beta, min\left(\gamma_{E_{22}}, \gamma_{F_{21}}\right)\right), \left(\beta, min\left(\delta_{E_{22}}, \delta_{F_{21}}\right)\right)\right] \end{array}\right), \\ \left[\left(\alpha, max\left(\alpha_{E_{23}}, \alpha_{F_{31}}\right)\right), \left(\alpha, max\left(\beta_{E_{23}}. \beta_{F_{31}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{23}}, \gamma_{F_{31}}\right)\right), min(\beta, min\left(\delta_{E_{23}}, \delta_{F_{31}}\right)) \end{array}\right)$$

X_{22} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c} \left[\left(\alpha, max\left(\alpha_{E_{21}}, \alpha_{F_{12}}\right)\right), \left(\alpha, max\left(\beta_{E_{21}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{21}}, \gamma_{F_{12}}\right)\right), \left(\beta, min\left(\delta_{E_{21}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{22}}, \alpha_{F_{22}}\right)\right), \left(\alpha, max\left(\beta_{E_{22}}, \beta_{F_{22}}\right)\right)\right]\left[\left(\beta, min\left(\gamma_{E_{22}}, \gamma_{F_{22}}\right)\right), \left(\beta, min\left(\delta_{E_{22}}, \delta_{F_{22}}\right)\right)\right] \end{array}\right), \\ \left[\left(\alpha, max\left(\alpha_{E_{23}}, \alpha_{F_{32}}\right)\right), \left(\alpha, max\left(\beta_{E_{23}}, \beta_{F_{32}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{23}}, \gamma_{F_{32}}\right)\right), min(\beta, min\left(\delta_{E_{23}}, \delta_{F_{32}}\right)) \end{array}\right)$$

X_{23} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c} \left[\left(\alpha, max\left(\alpha_{E_{21}}, \alpha_{F_{13}}\right)\right), \left(\alpha, max\left(\beta_{E_{21}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{21}}, \gamma_{F_{13}}\right)\right), \left(\beta, min\left(\delta_{E_{21}}, \delta_{F_{13}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{22}}, \alpha_{F_{23}}\right)\right), \left(\alpha, max\left(\beta_{E_{22}}, \beta_{F_{23}}\right)\right)\right]\left[\left(\beta, min\left(\gamma_{E_{22}}, \gamma_{F_{23}}\right)\right), \left(\beta, min\left(\delta_{E_{22}}, \delta_{F_{23}}\right)\right)\right] \end{array}\right), \\ \left[\left(\alpha, max\left(\alpha_{E_{23}}, \alpha_{F_{33}}\right)\right), \left(\alpha, max\left(\beta_{E_{23}}, \beta_{F_{33}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{23}}, \gamma_{F_{33}}\right)\right), min(\beta, min\left(\delta_{E_{23}}, \delta_{F_{33}}\right)) \end{array}\right)$$

X_{31} =Max

$$\left(\begin{array}{l} max\left(\begin{array}{c} \left[\left(\alpha, max\left(\alpha_{E_{31}}, \alpha_{F_{11}}\right)\right), \left(\alpha, max\left(\beta_{E_{31}}, \beta_{F_{11}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{31}}, \gamma_{F_{11}}\right)\right), \left(\beta, min\left(\delta_{E_{31}}, \delta_{F_{11}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{32}}, \alpha_{F_{21}}\right)\right), \left(\alpha, max\left(\beta_{E_{32}}, \beta_{F_{21}}\right)\right)\right]\left[\left(\beta, min\left(\gamma_{E_{32}}, \gamma_{F_{21}}\right)\right), \left(\beta, min\left(\delta_{E_{32}}, \delta_{F_{21}}\right)\right)\right] \end{array}\right), \\ \left[\left(\alpha, max\left(\alpha_{33}, \alpha_{F_{31}}\right)\right), \left(\alpha, max\left(\beta_{E_{33}}, \beta_{F_{31}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{33}}, \gamma_{F_{31}}\right)\right), min(\beta, min\left(\delta_{E_{33}}, \delta_{F_{31}}\right)) \end{array}\right)$$

X_{32} =Max

$$\left(\begin{array}{c} max\left(\begin{array}{c} \left[\left(\alpha, max\left(\alpha_{E_{31}}, \alpha_{F_{12}}\right)\right), \left(\alpha, max\left(\beta_{E_{31}}, \beta_{F_{12}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{31}}, \gamma_{F_{12}}\right)\right), \left(\beta, min\left(\delta_{E_{31}}, \delta_{F_{12}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{32}}, \alpha_{F_{22}}\right)\right), \left(\alpha, max\left(\beta_{E_{32}}, \beta_{F_{22}}\right)\right)\right]\left[\left(\beta, min\left(\gamma_{E_{32}}, \gamma_{F_{22}}\right)\right), \left(\beta, min\left(\delta_{E_{32}}, \delta_{F_{22}}\right)\right)\right] \end{array} \right), \\ \left[\left(\alpha, max\left(\alpha_{33}, \alpha_{F_{32}}\right)\right), \left(\alpha, max\left(\beta_{E_{33}}, \beta_{F_{32}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{33}}, \gamma_{F_{32}}\right)\right), min(\beta, min\left(\delta_{E_{33}}, \delta_{F_{32}}\right)) \end{array} \right)$$

X_{33} =Max

$$\left(\begin{array}{c} max\left(\begin{array}{c} \left[\left(\alpha, max\left(\alpha_{E_{31}}, \alpha_{F_{13}}\right)\right), \left(\alpha, max\left(\beta_{E_{31}}, \beta_{F_{13}}\right)\right)\right]\left[min\left(\beta, min\left(\gamma_{E_{31}}, \gamma_{F_{13}}\right)\right), \left(\beta, min\left(\delta_{E_{31}}, \delta_{F_{13}}\right)\right)\right], \\ \left[\left(\alpha, max\left(\alpha_{E_{32}}, \alpha_{F_{23}}\right)\right), \left(\alpha, max\left(\beta_{E_{32}}, \beta_{F_{23}}\right)\right)\right]\left[\left(\beta, min\left(\gamma_{E_{32}}, \gamma_{F_{23}}\right)\right), \left(\beta, min\left(\delta_{E_{32}}, \delta_{F_{23}}\right)\right)\right] \end{array} \right), \\ \left[\left(\alpha, max\left(\alpha_{33}, \alpha_{F_{33}}\right)\right), \left(\alpha, max\left(\beta_{E_{33}}, \beta_{F_{33}}\right)\right)\right][\left(\beta, min\left(\gamma_{E_{33}}, \gamma_{F_{33}}\right)\right), min(\beta, min\left(\delta_{E_{33}}, \delta_{F_{33}}\right)) \end{array} \right)$$

$A_E \times_5 B_F = (X_{11}\ X_{12}\ X_{13}\ X_{21}\ X_{22}\ X_{23}\ X_{31}\ X_{32}\ X_{33})$. Hence, $A_E \times_5 B_F$ is an interval valued intuitionistic fuzzy matrix set.

Using Python program for $A_E \times_5 B_F$

#intput the values

```
import math
x=float(input("x="))
y=float(input("y="))
a_11=float(input("a_11="))
b_11=float(input("b_11="))
c_11=float(input("c_11="))
d_11=float(input("d_11="))
e_11=float(input("e_11="))
f_11=float(input("f_11="))
g_11=float(input("g_11="))
h_11=float(input("h_11="))
a_12=float(input("a_12="))
b_12=float(input("b_12="))
c_12=float(input("c_12="))
d_12=float(input("d_12="))
e_12=float(input("e_12="))
f_12=float(input("f_12="))
g_12=float(input("g_12="))
h_12=float(input("h_12="))
a_13=float(input("a_13="))
b_13=float(input("b_13="))
```

```
c_13=float(input("c_13="))
d_13=float(input("d_13="))
e_13=float(input("e_13="))
f_13=float(input("f_13="))
g_13=float(input("g_13="))
h_13=float(input("h_13="))
a_21=float(input("a_21="))
b_21=float(input("b_21="))
c_21=float(input("c_21="))
d_21=float(input("d_21="))
e_21=float(input("e_21="))
f_21=float(input("f_21="))
g_21=float(input("g_21="))
h_21=float(input("h_21="))
a_22=float(input("a_22="))
b_22=float(input("b_22="))
c_22=float(input("c_22="))
d_22=float(input("d_22="))
e_22=float(input("e_22="))
f_22=float(input("f_22="))
g_22=float(input("g_22="))
h_22=float(input("h_22="))
a_23=float(input("a_23="))
b_23=float(input("b_23="))
c_23=float(input("c_23="))
d_23=float(input("d_23="))
e_23=float(input("e_23="))
f_23=float(input("f_23="))
g_23=float(input("g_23="))
h_23=float(input("h_23="))
a_31=float(input("a_31="))
b_31=float(input("b_31="))
c_31=float(input("c_31="))
d_31=float(input("d_31="))
```

```
e_31=float(input("e_31="))
f_31=float(input("f_31="))
g_31=float(input("g_31="))
h_31=float(input("h_31="))
a_32=float(input("a_32="))
b_32=float(input("b_32="))
c_32=float(input("c_32="))
d_32=float(input("d_32="))
e_32=float(input("e_32="))
f_32=float(input("f_32="))
g_32=float(input("g_32="))
h_32=float(input("h_32="))
a_33=float(input("a_33="))
b_33=float(input("b_33="))
c_33=float(input("c_33="))
d_33=float(input("d_33="))
e_33=float(input("e_33="))
f_33=float(input("f_33="))
g_33=float(input("g_33="))
h_33=float(input("h_33="))
#creating variables for c_11
a_1=max(x,max(a_11,c_11))
a_2=max(x,max(b_11,d_11))
a_3=min(y,min(e_11,g_11))
a_4=min(y,min(f_11,h_11))
a_5=max(x,max(a_12,c_21))
a_6=max(x,max(b_12,d_21))
a_7=min(y,min(e_12,g_21))
a_8=min(y,min(f_12,h_21))
a_9=max(x,max(a_13,c_31))
a_10=max(x,max(b_13,d_31))
a_11=min(y,min(e_13,g_31))
a_12=min(y,min(f_13,h_31))
#creating cells
```

```
x_11=[max((max(a_1,a_5)),a_9),max(max(a_2,a_6),a_10)]
Y_11=[max((max(a_3,a_7)),a_11),max(max(a_4,a_8),a_12)]
print("C_11=",x_11)
print("D_11=",Y_11)
#creating variables for c_12
b_1=max(x,max(a_11,c_12))
b_2=max(x,max(b_11,d_12))
b_3=min(y,min(e_11,g_12))
b_4=min(y,min(f_11,h_12))
b_5=max(x,max(a_12,c_22))
b_6=max(x,max(b_12,d_22))
b_7=min(y,min(e_12,g_22))
b_8=min(y,min(f_12,h_22))
b_9=max(x,max(a_13,c_32))
b_10=max(x,max(b_13,d_32))
b_11=min(y,min(e_13,g_32))
b_12=min(y,min(f_13,h_32))
#creating cells
x_12=[max((max(b_1,b_5)),b_9),max(max(b_2,b_6),b_10)]
Y_12=[max((max(b_3,b_7)),b_11),max(max(b_4,b_8),b_12)]
print("C_12=",x_12)
print("D_12=",Y_12)
#creating variables for c_13
c_1=max(x,max(a_11,c_13))
c_2=max(x,max(b_11,d_13))
c_3=min(y,min(e_11,g_13))
c_4=min(y,min(f_11,h_13))
c_5=max(x,max(a_12,c_23))
c_6=max(x,max(b_12,d_23))
c_7=min(y,min(e_12,g_23))
c_8=min(y,min(f_12,h_23))
c_9=max(x,max(a_13,c_33))
c_10=max(x,max(b_13,d_33))
c_11=min(y,min(e_13,g_33))
```

```
c_12=min(y,min(f_13,h_33))
#creating cells
x_13=[max((max(c_1,c_5)),c_9),max(max(c_2,c_6),c_10)]
Y_13=[max((max(c_3,c_7)),c_11),max(max(c_4,c_8),c_12)]
print("C_13=",x_13)
print("D_13=",Y_13)
#creating variables for c_21
d_1=max(x,max(a_21,c_11))
d_2=max(x,max(b_21,d_11))
d_3=min(y,min(e_21,g_11))
d_4=min(y,min(f_21,h_11))
d_5=max(x,max(a_22,c_21))
d_6=max(x,max(b_22,d_21))
d_7=min(y,min(e_22,g_21))
d_8=min(y,min(f_22,h_21))
d_9=max(x,max(a_23,c_31))
d_10=max(x,max(b_23,d_31))
d_11=min(y,min(e_23,g_31))
d_12=min(y,min(f_23,h_31))
#creating cells
x_21=[max((max(d_1,d_5)),d_9),max(max(d_2,d_6),d_10)]
Y_21=[max((max(d_3,d_7)),d_11),max(max(d_4,d_8),d_12)]
print("C_21=",x_21)
print("D_21=",Y_21)
#creating variables for c_22
e_1=max(x,max(a_21,c_12))
e_2=max(x,max(b_21,d_12))
e_3=min(y,min(e_21,g_12))
e_4=min(y,min(f_21,h_12))
e_5=max(x,max(a_22,c_22))
e_6=max(x,max(b_22,d_22))
e_7=min(y,min(e_22,g_22))
e_8=min(y,min(f_22,h_22))
e_9=max(x,max(a_23,c_32))
```

```
e_10=max(x,max(b_23,d_32))
e_11=min(y,min(e_23,g_32))
e_12=min(y,min(f_23,h_32))
#creating cells
x_22=[max((max(e_1,e_5)),e_9),max(max(e_2,e_6),e_10)]
Y_22=[max((max(e_3,e_7)),e_11),max(max(e_4,e_8),e_12)]
print("C_22=",x_22)
print("D_22=",Y_22)
#creating variables for c_23
f_1=max(x,max(a_21,c_13))
f_2=max(x,max(b_21,d_13))
f_3=min(y,min(e_21,g_13))
f_4=min(y,min(f_21,h_13))
f_5=max(x,max(a_22,c_23))
f_6=max(x,max(b_22,d_23))
f_7=min(y,min(e_22,g_23))
f_8=min(y,min(f_22,h_23))
f_9=max(x,max(a_23,c_33))
f_10=max(x,max(b_23,d_33))
f_11=min(y,min(e_23,g_33))
f_12=min(y,min(f_23,h_33))
#creating cells
x_23=[max((max(f_1,f_5)),f_9),max(max(f_2,f_6),f_10)]
Y_23=[max((max(f_3,f_7)),f_11),max(max(f_4,f_8),f_12)]
print("C_23=",x_23)
print("D_23=",Y_23)
#creating variables for c_31
g_1=max(x,max(a_31,c_11))
g_2=max(x,max(b_31,d_11))
g_3=min(y,min(e_31,g_11))
g_4=min(y,min(f_31,h_11))
g_5=max(x,max(a_32,c_21))
g_6=max(x,max(b_32,d_21))
g_7=min(y,min(e_32,g_21))
```

```
g_8=min(y,min(f_32,h_21))
g_9=max(x,max(a_33,c_31))
g_10=max(x,max(b_33,d_31))
g_11=min(y,min(e_33,g_31))
g_12=min(y,min(f_33,h_31))
#creating cells
x_31=[max((max(g_1,g_5)),g_9),max(max(g_2,g_6),g_10)]
Y_31=[max((max(g_3,g_7)),g_11),max(max(g_4,g_8),g_12)]
print("C_31=",x_31)
print("D_31=",Y_31)
#creating variables for c_32
h_1=max(x,max(a_31,c_12))
h_2=max(x,max(b_31,d_12))
h_3=min(y,min(e_31,g_12))
h_4=min(y,min(f_31,h_12))
h_5=max(x,max(a_32,c_22))
h_6=max(x,max(b_32,d_22))
h_7=min(y,min(e_32,g_22))
h_8=min(y,min(f_32,h_22))
h_9=max(x,max(a_33,c_32))
h_10=max(x,max(b_33,d_32))
h_11=min(y,min(e_33,g_32))
h_12=min(y,min(f_33,h_32))
#creating cells
x_32=[max((max(h_1,h_5)),h_9),max(max(h_2,h_6),h_10)]
Y_32=[max((max(h_3,h_7)),h_11),max(max(h_4,h_8),h_12)]
print("C_32=",x_32)
print("D_32=",Y_32)
#creating variables for c_33
i_1=max(x,max(a_31,c_13))
i_2=max(x,max(b_31,d_13))
i_3=min(y,min(e_31,g_13))
i_4=min(y,min(f_31,h_13))
i_5=max(x,max(a_32,c_23))
```

```
i_6=max(x,max(b_32,d_23))
i_7=min(y,min(e_32,g_23))
i_8=min(y,min(f_32,h_23))
i_9=max(x,max(a_33,c_33))
i_10=max(x,max(b_33,d_33))
i_11=min(y,min(e_33,g_33))
i_12=min(y,min(f_33,h_33))
#creating cells
x_33=[max((max(i_1,i_5)),i_9),max(max(i_2,i_6),i_10)]
Y_33=[max((max(i_3,i_7)),i_11),max(max(i_4,i_8),i_12)]
print("C_33=",x_33)
print("D_33=",Y_33)
```

Output:

```
x=0.04
y=0.01
a_11=0.05
b_11=0.07
c_11=0.02
d_11=0.06
e_11=0.03
f_11=0.05
g_11=0.03
h_11=0.05
a_12=0.02
b_12=0.03
c_12=0.05
d_12=0.07
e_12=0.04
f_12=0.07
g_12=0.02
h_12=0.04
a_13=0.05
b_13=0.06
c_13=0.02
```

d_13=0.03

e_13=0.01

f_13=0.04

g_13=0.03

h_13=0.06

a_21=0.03

b_21=0.08

c_21=0.04

d_21=0.05

e_21=0.03

f_21=0.04

g_21=0.01

h_21=0.07

a_22=0.04

b_22=0.05

c_22=0.04

d_22=0.08

e_22=0.02

f_22=0.08

g_22=0.02

h_22=0.03

a_23=0.01

b_23=0.03

c_23=0.01

d_23=0.04

e_23=0.04

f_23=0.07

g_23=0.03

h_23=0.08

a_31=0.04

b_31=0.06

c_31=0.02

d_31=0.03

e_31=0.03

f_31=0.09

g_31=0.02

h_31=0.05

a_32=0.01

b_32=0.02

c_32=0.01

d_32=0.03

e_32=0.05

f_32=0.07

g_32=0.05

h_32=0.06

a_33=0.02

b_33=0.03

c_33=0.03

d_33=0.07

e_33=0.08

f_33=0.09

g_33=0.01

h_33=0.05

C_11= [0.05, 0.07]

D_11= [0.01, 0.01]

C_12= [0.05, 0.08]

D_12= [0.01, 0.01]

C_13= [0.05, 0.07]

D_13= [0.01, 0.01]

C_21= [0.04, 0.08]

D_21= [0.01, 0.01]

C_22= [0.04, 0.08]

D_22= [0.01, 0.01]

C_23= [0.04, 0.08]

D_23= [0.01, 0.01]

C_31= [0.04, 0.06]

D_31= [0.01, 0.01]

C_32= [0.04, 0.08]

D_32= [0.01, 0.01]

C_33= [0.04, 0.07]

D_33= [0.01, 0.01]

Conclusion:

In this chapter, we represent an interval- valued intuitionistic fuzzy matrix sets [IVIFMs] as the Cartesian product of its membership and non-membership matrices. We introduce " $\times_{4,}$ " $\times_5$ " of Cartesian product over interval- valued intuitionistic fuzzy matrix sets. A new interval- valued intuitionistic fuzzy matrix sets generated by the use of the Cartesian product of two interval- valued intuitionistic fuzzy matrix sets.

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PROPERTIES OF INTUITIONISTIC FUZZY SOFT SETS AND ITS APPLICATIONS

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Abstract

In this research work, we explain parameter fuzzy Intuitionistic soft sets and various characteristic rules. After we bring together parameter fuzzy Intuitionistic soft sets and some theorems, and also on selection taking intuitionistic fuzzy soft set followed by given some numerical examples.

Keywords: Fuzzy sets, fuzzy soft set, Intuitionistic soft set, Parameter Intuitionistic soft set, Selection taking.

1. Introduction:

Many fields deal with uncertain data that may not be successfully modeled by classical mathematics, probability theory,[26] L.A. Zadeh, provided the concept of Fuzzy sets , [23] Z. Pawlak gave the concept Rough sets. Molodtsov [21] in 1999 introduced new concept so soft set that is added Common Universe show off uncertainty and vagueness. After [18,19] Maji et al.make known to many operations of soft sets. [4,5] Ali etal. gave more or less different novel thinking like that , and extended for 2 soft fuzzy sets after [24] were developed the hypothetical part of the soft set procedures. And [19] present the idea of soft fuzzy set, well-stated a soft fuzzy set and all are delivered the uses of soft fuzzy set in selection taking difficult. In [19] Gave Soft structures of fuzzy algebra properties like rings, modules, fields, By putting on these senses, the uses of the set idea should have well-thought-out additional. Cagman et.al [10-13] thoughtful the soft set selection taking and also gave use of soft set in selection takes. Chen et al. [14], talk over the parameterization method. A suitable technique to soft set up on best making. The resolve of this work is to combine the intuitionistic fuzzy sets[6,8]. [15] A flexible method to soft set uses. This research work Intuitionistic fuzzy set and selection taking. [6] in 1986 introduce the notion of ,[2] M. Agarwal, K.K. Biswas presented the global intuitionistic fuzzy set with uses in selection taking,[5][1] U. Acar, gave the concept of soft groups, [25] Y. Yang, gave the idea of decision-making [16],[17] gave more application of fuzzy soft set,[20][22] gave more uses with decision making, [8] gave the rough guide of the soft group and [9] gave the application of well-adjusted result of a soft set constructed selection taking,[3] gave soft set and soft group based on this concept in this paper using parameter and its application.

2. Preliminary

In this portion, we define the elementary definition of a set soft theory fuzzy, intuitionistic fuzzy set theory.

Definition. 2.1. [21] Consider C is a Common set, and the power set is $P(C)$ over C and K is a parameter set. A soft set S along with C is a set stated by a mapping $g_S: K \to P(C)$, therefore this van be stated by $S = \{(r, g_S(r)): r \in K\}$. Were g_S be the come close to the value of the set S and $g_S(r)$ be the a come close to the value of $\in K$. Clearly that if $g_S(r) = \emptyset$,then the object $(r, g_S(r))$ is not looked in S.

Example. 2.2 [21]Consider $C = \{r_1, r_2, r_3, r_4\}$ be the four apartments under consideration in a broker and $K = \{t_1, t_2, t_3, t_4\}$ be the parameters set, where $e_j(j = 1,2,3,4)$ assigned for ''Security'', ''Expensive'', ''Average rate '', ''Costly'', respectively. A purchaser to choose a apartments from apartment Manager can build a soft set S that refer to the typical of apartments allowing to individual choose. Consider $g_S(t_1) = \{r_1, r_2, r_3\}$,$g_S(t_2) = \{r_2, r_4\}, g_S(t_3) = \emptyset, g_s(t_4) = C$ therefore the soft-set S can be written as $S = \{(t_1, \{r_1, r_2, r_3\}), (t_2, \{r_2, r_4\}), (t_4, C)\}$.

Definition.2.3.[26] let A be set and K be a common set then the fuzzy set A over K be a function stated as below

$A = \{(r, \mu_A(r)): r \in K$, where $\mu_A: K \to [0,1]$. Here a is membership value of A, also $\mu_A(r)$ is value of membership of $r \in K$,the grad represent them degree of $r \in A$.

Definition.2.4. [11] Let X be a set and K is a common set and an intuitionistic fuzzy set X on K can be stated as below `

$X = \{a, \mu_X(a), \eta_X(a): a \in K\}$

Here $\mu_X: K \to I$ and $\eta_X: K \to I$ such that $0 \le \mu_X(r) \le 1, 0 \le \eta_X(r) \le 1, r \in K$. Here,$\mu_X(r)$ and $\eta_X(r)$ is the value of membership and not a membership of the member r, respectively.

If X and Y be two IFS on K, then

(1) $X \subset Y \Leftrightarrow \mu_X(r) \le \eta_Y(r)$ and $\mu_X(r) \ge \eta_Y(r)$ for $\forall r \in K$
(2) $X = Y \Leftrightarrow \mu_X(r) = \eta_Y(r)$ and $\mu_X(r) = \eta_Y(r)$ for $\forall r \in K$
(3) $X^c = \{r, \mu_X(r), \eta_Y(r): r \in K\}$
(4) $X \coprod Y = \{r, \vee(\mu_X(r), \mu_Y(r)), \wedge(\eta_X(r), \eta_Y(r)): r \in K\}$,
(5) $X \Cap Y = \{r, \wedge(\mu_X(r), \mu_Y(r)), \vee(\eta_X(r), \eta_Y(r)) : r \in K\}$.

Definition .2.5.[11] The universe C and P^C be power set of C,and let parameter set K and A be a FS with K.then a parameter FS set (g_A, K) on the common set C is stated below

$$(g_A, K) = \{(\mu_A(r)/r, f_A(r)) : r \in K\}$$

Where $\mu_A: K \to I$ and $g_A: K \to P^C$ such that $g_A(r) = \emptyset$ if$\mu_A(r) = 0$. Now g_A is approximate value and μ_A is membership value of parameter fuzzy soft set.

Example: 2. 6[11] Consider $C = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9\}$ be a common set and $K = \{t_1, t_2, t_3, t_4\}$ be the parameters if $A = \left\{\left(t_1, \frac{7}{10}\right), \left(t_2, \frac{5}{10}\right), \left(t_3, \frac{6}{10}\right), \left(t_4, \frac{9}{10}\right)\right\}$ be the FS over K, therefore the parameter fuzzy soft set

$$(g_A, K) = \{\left(t_1, \frac{7}{10}\right), \{r_2, r_3, r_4, r_5, r_7\}), \left(r_2, \frac{5}{10}\right), \{r_4, r, r_7\}) , (t_3, \frac{6}{10}), \{r_1, r_2, r_3, r_4, r_9\}),$$
$$(t_4, \frac{9}{10}), \{r_1, r_2, r, r_8\})\}$$

3. Parameter fuzzy intuitionistic soft sets

In below section we have to see the procedures on parameter intuitionistic soft fuzzy sets.

Definition: 3.1. The Common set C and P^C be the PS, K is the parameter set and N be an intuitionistic fuzzy set over$K$. An Intuitionistic parameter fuzzy soft sets C_N over C is stated as follows

$$C_N = \{((r, \gamma_N(r), \delta_N(r)) , g_N(r): r \in K\}$$

Where$\gamma_N: K \to I, \delta_N: K \to [0,1]$ and $g_N: K \to [0,1]$ along with the property $g_N(r) = \emptyset$ and $\gamma_N(r) = 0$ and $\delta_N(r) = 1$, where $\gamma_N$and δ_N membership and not a membership $intuitionistic\ fuzzy$ soft set**.** The value $\gamma_N(r)$ and $\delta_N(r)$ is the value of significance and insignificance of the parameter $''r''$

Clearly, every usual parameter set can be written as below

$\coprod_N = \{(r, \gamma_N(r), 1-\gamma_N(r)), g_N(r): r \in K\}$, clearly $FFISS$ over C and it is symbolized by $FFISS(C)$.

Definition.3.2. Take$\coprod_N \in FFISS(C). \gamma_N(r) = 0$ And$\delta_N(r) = 1, \forall r \in K$, therefore $\coprod_N$ is said to be parameter fuzzy intuitionistic empty set and it is denoted $\coprod_\emptyset$.

Definition.3.3. Let$\coprod_N \in FFISS(C). \gamma_N(r) = 0$ And $\delta_N(r) = 1, \forall r \in K$ and $g_N(r) = C$ therefore $\coprod_N$ is said to be parameter fuzzy intuitionistic common set and it is denoted $\coprod_{\tilde{K}}$.

Example. 3. 4 Consider the common set$C = \{r_1, r_2, r_3, r_4, r_5\}$, and parameter set $K = \{t_2, t_2, t_3\}$,if $N = \left\{\left(t_1, \frac{2}{10}, \frac{5}{10}\right), \left(t_2, \frac{5}{10}, \frac{5}{10}\right), \left(t_3, \frac{6}{10}, \frac{3}{10}\right)\right\}$, And $g_N(r_1) = \{r_2, r_4\}, g_N(r_2) = \{\emptyset\}$, $g_N(r_3) = \{C\}$. Therefore the $FFISS$ $\coprod_N$ can be written as follows

$$\coprod_N = \{\left(\left(t_1, \frac{2}{10}, \frac{5}{10}\right), \{r_2, r_4\}\right), \left(t_2, \frac{5}{10}, \frac{5}{10}\right), \{\emptyset\}), \left(t_3, \frac{6}{10}, \frac{3}{10}\right), \{C\})\}$$

$A = \{(r_1, 0,1), (r_2, 0,1), (r_3, 0,1), (r_4, 0,1)\}$, then the $FFISS$ $\coprod_A$ is empty. If $B = \{(r_1, 0,1), (r_2, 0,1), (r_3, 0,1), (r_4, 0,1)\}$ and $g_B(r_1) = C, g_B(r_2) = C, g_B(r_3) = C, g_B(r_4) = C$ then the $FFIS$ common parameter fuzzy soft set.

Definition 3.5. Let$\coprod_A, \coprod_B \in FFISS(C)$. Then $\coprod_A$ parameter fuzzy intuitionistic soft subset of$\coprod_B$, and is denoted by $\coprod_B \supseteq \coprod_A$ iff $\Leftrightarrow$ if$\gamma_N(r) \leq \gamma_A(r)$,$\delta_N(r) \geq \delta_A(r)$ and $g_L(r) \supseteq g_N(r), \forall\ r \in K$.

Remark 3. 6. $\coprod_A \supseteq \coprod_N$ Which not given all members of $\coprod_N$ is a member of $\coprod_A$ as in the meaning of usual subset. Consider $C = \{r_1, r_2, r_3, r_4\}$ is a common set of items and $K = \{t_1, t_2, t_3\}$is a set of parameters. If $N = \left\{t_1, \frac{4}{10}, \frac{6}{10}\right\}$ and

$A = \left(\left\{t_1, \frac{5}{10}, \frac{5}{10}\right\}, \left\{t_3, \frac{4}{10}, \frac{5}{10}\right\}\right)$, and

$\coprod_N = \{\left(t_1, \frac{4}{10}, \frac{6}{10}\right), \{a_2, r_4\})\}$,

$$\coprod_A = \{\left(t_1, \frac{5}{10}, \frac{5}{10}\right)\{r_2, r_3, r_4\}), \left(t_3, \frac{4}{10}, \frac{5}{10}\right), \{r_1, r_5\})\}, \forall\ t \in K,$$

$\gamma_A(r) \geq \gamma_N(r), \delta_N(r) \leq \delta_A(r)$ And $\coprod_A(r) \supseteq \coprod_N(r)$ is suitable.Therfore$\coprod_A \tilde{\supseteq} \coprod_N$. It is understandable that $\left(\left(t_1, \frac{4}{10}, \frac{6}{10}\right), \{r_2, r_4\}\right) \in \coprod_N$ but $\left(\left(t_1, \frac{4}{10}, \frac{6}{10}\right), \{r_2, r_4\}\right) \notin \coprod_A$.

Theorem 3. 7. Let$\coprod_N, \coprod_{\tilde{K}} \in FFISS(C)$. Then

(a) $\coprod_N \subseteq \coprod_{\tilde{K}}$

(b) $\coprod_\emptyset \subseteq \coprod_N$

(c) $\coprod_N \subseteq \coprod_N$

Proof: The above properties of $\subseteq$ and above definition trivially true.

Definition3.8.$\coprod_N, \coprod_A \in FFISS(C)$. After $\coprod_N$ and $\coprod_A$ parameter fuzzy intuitionistic soft- equal, write by $\coprod_N = \coprod_A \Leftrightarrow \gamma_N(r) = \gamma_A(r), \delta_N(r) = \delta_A(r)$ and $g_N(r) = g_A(r), \forall r \in K$.

Theorem 3.2 Let $\coprod_N, \coprod_A, \coprod_F \in FFISS(C)$Then

(a) $\coprod_N = \coprod_A$ and $\coprod_A = \coprod_F$ iff $\coprod_N = \coprod_F$

(b) $\coprod_N \subseteq \coprod_A$ and $\coprod_A \subseteq \coprod_F$iff $\coprod_N = \coprod_A$

(c) $\coprod_N \subseteq \coprod_A$ and $\coprod_A \subseteq \coprod_F$ which implies$\coprod_N = \coprod_F$.

Proof: The above properties of $\equiv$ and $\subseteq$ is true form definition 3.4 and 3.5

Definition.3.9.$\coprod_N \in FFISS(C)$. Then parameter fuzzy intuitionistic soft complement set is stated as below

$$\coprod_N{}' = \{(a, \gamma_N(a), \delta_N(a)), g_N{}'(a)): a \in K\}.$$

Theorem 3.10 If $\coprod_N \in FFISS(C)$. Then

(a) $(\coprod_N{}')' = \coprod_N$

(b) $\coprod_\emptyset{}' = \coprod_{\tilde{K}}$

(c) $\coprod_{\tilde{K}}{}' = \coprod_\emptyset$

Proof: Let $\coprod_K = \{(r, 1,0), C)$:for all $r \in K$. by definition 3.6

$\coprod_{\tilde{K}}{}' = \{(r, 0,1), C)$: For all $r \in K\} = \coprod_\emptyset$ in same way we can prove (a) and (b)

Definition.3.11. If $\coprod_N, \coprod_A \in FFISS(C)$. Then parameter fuzzy intuitionistic soft union set is stated as below

$$\coprod_N \sqcup \coprod_A = \left\{\left(\left(r, \vee(\gamma_N(r)), \delta_A(r)\right), \wedge\left(\delta_N(r), \delta_A(r)\right)\right), g_{N\sqcup A}(a)\right): a \in K\right\}$$

Theorem.3.12 Let $\coprod_N, \coprod_A, \coprod_F \in FFISS(C)$. then

(a) $\coprod_N \sqcup \coprod_N = \coprod_N$

(b) $\coprod_N \sqcup \coprod_\emptyset = \coprod_N$

(c) $\coprod_N \sqcup \coprod_K = \coprod_K$

(d) $\coprod_N \sqcup \coprod_A = \coprod_A \sqcup \coprod_N$

(e) $\coprod_N \sqcup (\coprod_A \sqcup \coprod_F) = (\coprod_N \sqcup \coprod_A) \sqcup \coprod_F$.

Proof: Definition 3.2, 3.3, 3.5 and 3.7 help to see their proof of equality.

Definition.3.13. If $\coprod_N, \coprod_A \in FFISS(C)$. Then parameter fuzzy intuitionistic soft intersection set is stated as below

$$\coprod_N \sqcap \coprod_A = \left\{\left(\left(a, \wedge(\gamma_N(a)), \delta_A(a)\right), \vee\left(\delta_N(a), \delta_A(a)\right)\right), g_{N\sqcap A}(a)\right): a \in K\right\}$$

Theorem.3.12. Let $\coprod_N, \coprod_A, \coprod_F \in FFISS(C)$. then

(a) $\coprod_N \sqcap \coprod_N = \coprod_N$

(b) $\coprod_N \sqcap \coprod_\emptyset = \coprod_N$

(c) $\coprod_N \sqcap \coprod_K = \coprod_K$

(d) $\coprod_N \sqcap \coprod_A = \coprod_A \sqcap \coprod_N$

(e) $\coprod_N \sqcap (\coprod_A \sqcap \coprod_F) = (\coprod_N \sqcap \coprod_A) \sqcap \coprod_F$.

Proof: Definition 3.2, 3.3, 3.5 and 3.8 help to see their proof of equality.

Remark. 3.13 Let $\coprod_N \in FFISS(C)$. If $\coprod_N \neq \coprod_\emptyset$ or $\coprod_N \neq \coprod_K$, then $\coprod_N \sqcup \coprod_N{}' \neq \coprod_K$ and $\coprod_N \sqcap \coprod_K{}' \neq \coprod_\emptyset$. For instance, consider $C = \{r_1, r_2, r_3, r_4\}$ be the common set of item and $K = \{t_1, t_2\}$ be the set of parameters. If $N = \left\{\left(t_1, \frac{4}{10}, \frac{6}{10}\right), \left(t_2, \frac{5}{10}, \frac{5}{10}\right)\right\}$,and

$\coprod_N = \left\{\left(\left(t_1, \frac{4}{10}, \frac{6}{10}\right), \{r_2, r_4\}\right)\right\}, \left\{\left(\left(t_2, \frac{5}{10}, \frac{5}{10}\right), \{r_2, r_3, r_4\}\right)\right\}$, Therefore

$N' = \left\{\left(t_1, \frac{4}{10}, \frac{6}{10}\right), \left(t_2, \frac{5}{10}, \frac{5}{10}\right)\right\}$ And

$\coprod_N{}' = \left\{\left(\left(t_1, \frac{6}{10}, \frac{4}{10}\right), \{t_1, t_3\}\right)\right\}, \left\{\left(t_2, \frac{5}{10}, \frac{5}{10}\right)\{r_1\}\right\}$, since

$\coprod_N \sqcup \coprod_N{}' = \left\{\left(\left(t_1, \frac{6}{10}, \frac{4}{10}\right), \{C\}\right)\right\}, \left\{\left(t_2, \frac{5}{10}, \frac{5}{10}\right), \{C\}\right\} \neq \coprod_K$,

$\coprod_N \sqcap \coprod_N{}' = \left\{\left(\left(t_1, \frac{4}{10}, \frac{6}{10}\right), \{\emptyset\}\right)\right\}, \left\{\left(t_2, \frac{5}{10}, \frac{5}{10}\right), \{\emptyset\}\right\} \neq \coprod_\emptyset$.

Theorem.3.14 Let $\coprod_N, \coprod_A, \coprod_F \in FFISS(C)$. then

(a) $\coprod_N \tilde{\sqcup} (\coprod_A \tilde{\sqcup} \coprod_F) = (\coprod_N \tilde{\sqcup} \coprod_A) \tilde{\sqcap} (\coprod_N \tilde{\sqcup} \coprod_F)$

(b) $\coprod_N \tilde{\sqcap} (\coprod_A \tilde{\sqcup} \coprod_F) = (\coprod_N \tilde{\sqcap}\tilde{\sqcap} \coprod_A) \tilde{\sqcup} (\coprod_N \tilde{\sqcap} \coprod_F)$

Proof: Definition 3.7 and 3.8 we can easily made the proof.

Theorem 3.15. Let $\coprod_N, \coprod_A \in FFISS(C)$. then the De Morgan's laws are true

(a) $(\coprod_N \tilde{\sqcup} \coprod_A)' = \coprod_N{}' \tilde{\sqcap} \coprod_A{}'$

(b) $(\coprod_N \tilde{\sqcap} \coprod_A)' = \coprod_N{}' \tilde{\sqcup} \coprod_A{}'$

Proof: Definition 3.6, 3.7 and 3.8we can easily prove the proof.

Definition.3.16. Let $\coprod_N, \coprod_A \in FFISS(C)$. Therefore the max-sum of $\coprod_N$ and $\coprod_A$ and it is denoted by $\coprod_N \vee^{max} \coprod_A$, stated as $\coprod_N \vee^{max} \coprod_A = \left\{\left(\left(r, \gamma_N(r) \oplus \gamma_A(r) - \gamma_N(r)\gamma_A(r), \delta_N(r)\delta_A(r)\right), g_{N \sqcup A}(r)\right)\right\} : r \in K\}$ Where $g_{N \sqcup A}(r) = g_N(r) \sqcup g_A(r)$.

Definition.3.17. Let $\coprod_N, \coprod_A \in FFISS(C)$. Therefore the min-sum of $\coprod_N$ and $\coprod_A$ and it is denoted by $\coprod_N \wedge^{min} \coprod_A$, stated as $\coprod_N \wedge^{min} \coprod_A = \left\{\left(\left(r, \gamma_N(r) \oplus \gamma_A(r) \ominus \gamma_N(r)\gamma_A(r), \delta_N(r)\delta_A(r)\right), g_{N \sqcap A}(r)\right)\right\} : r \in K\}$ Where $g_{N \sqcap A}(r) = g_N(r) \sqcap g_A(r)$.

Theorem .3.18 Let $\coprod_N, \coprod_A, \coprod_F \in FFISS(C)$. Then

(a) $\coprod_N \vee^{max} \coprod_\emptyset = \coprod_N$

(b) $\coprod_N \vee^{max} \coprod_K = \coprod_K$

(c) $\coprod_N \vee^{max} \coprod_A = \coprod_A \vee^{max} \coprod_N$

(d) $\coprod_N \wedge^{min} \coprod_A = \coprod_N \wedge^{min} \coprod_A$

(e) $(\coprod_N \vee^{max} \coprod_A) \vee^{max} \coprod_F = \coprod_A \vee^{max} (\coprod_A \vee^{max} \coprod_F)$

(f) $(\coprod_N \wedge^{min} \coprod_A) \wedge^{min} \coprod_F = \coprod_A \wedge^{min} (\coprod_A \wedge^{min} \coprod_F)$.

Proof: Definition 3.2, 3.3 and 3.10 we can easily prove the proof.

Definition.3.19. Let $\coprod_N, \coprod_A \in FFISS(C)$. Then the max-product of $\coprod_N$ and $\coprod_A$, and it is denoted by $\coprod_N \wedge^{max} \coprod_A$ and is stated as

$$\coprod_N \vee^{max} \coprod_A = \left\{\left(\left(r, \gamma_N(r)\gamma_A(r), \delta_N(r) \oplus \delta_A(r) \ominus \delta_N(r)\delta_A(r)\right), g_{N \sqcup A}(r)\right)\right\} : r \in K\}$$

Where $g_{N\sqcup A}(r) = g_N(r) \sqcup g_A(r)$.

Definition.3.20. Let $\coprod_N, \coprod_A \in FFISS(C)$. Then the min-product of $\coprod_N$ and$\coprod_A$, and it is denoted by $\coprod_N \wedge^{max} \coprod_A$ and is stated as

$$\coprod_N \vee^{min} \coprod_A = \left\{\left(\left(r, \gamma_N(r)\gamma_A(r), \delta_N(r) \oplus \delta_A(r) \ominus \delta_N(r)\delta_A(r)\right), g_{N\sqcap A}(r)\right)\right\}: r \in K\}$$

Where $g_{N\sqcup A}(r) = g_N(r) \sqcap g_A(r)$.

Theorem.3.21 Let $\coprod_N, \coprod_A, \coprod_F \in FFISS(C)$. Then

(a) $\coprod_N \wedge^{min} \coprod_\emptyset = \coprod_N$
(b) $\coprod_N \wedge^{min} \coprod_K = \coprod_K$
(c) $\coprod_N \wedge^{min} \coprod_A = \coprod_A \wedge^{min} \coprod_N$
(d) $\coprod_N \wedge^{min} \coprod_A = \coprod_N \wedge^{min} \coprod_A$
(e) $(\coprod_N \wedge^{min} \coprod_A) \wedge^{min} \coprod_F = \coprod_A \wedge^{min} (\coprod_A \wedge^{min} \coprod_F)$
(f) $(\coprod_N \vee^{max} \coprod_A) \vee^{max} \coprod_F = \coprod_A \vee^{max} (\coprod_A \vee^{max} \coprod_F)$.

Proof: Definition 3.2, 3.3 and 3.12 we can easily made the prove their inequality.

4. Selection taking Intuitionistic fuzzy soft set

In this division, stated a decreased parameter intuitionistic fuzzy soft set creates an intuitionistic fuzzy soft set from an intuitionistic fuzzy soft set. We then have stated a decreased fuzzy set of a parameter fuzzy intuitionistic fuzzy set that creates a fuzzy set from a parameter intuitionistic fuzzy soft set. These set nearby a flexible move toward intuitionistic fuzzy soft sets created on selection-taking problems.

Definition .4.1. Take $\coprod_N$ is a$FFISS$. Then a decreased IFS of $\coprod_N$, denoted by N^{difs},stated as below

$N^{difs} = \left\{\left(r, \gamma_N{}^{difs}(r), \delta_N{}^{difs}(r)\right): r \in C\right\}$, where

$\gamma_N{}^{difs}: C \to I, \gamma_N{}^{difs}(a) = \frac{1}{|C| \sum_{t\in K, r\in C} \quad \gamma_N(t)\psi_{g_N(t)}(r)}$

$\delta_N{}^{difs}: C \to I, \delta_N{}^{difs}(r) = \frac{1}{|C| \sum_{t\in K, r\in C} \quad \delta_N(t)\psi_{g_N(t)}(r)}$

Where $\gamma_N{}^{difs}$ and $\delta_N{}^{difs}$ are said to be decreased set-notation of N^{difs}.It is evident that N^{difs} is an IFS over C.

Definition.4.2. If $\coprod_N \in FFISS(C)$ and N^{difs} be decreased intuitionistic fuzzy of $\coprod_N$.Then a fuzzy decreased set of N^{difs} be a fuzzy set under C.Symbolized by N^{fd} and stated below

$N^{fd} = \{(r, \nu^{fd}(r): r \in C\}$, where $\nu^{fd}: C \to I, \nu^{fd}(r) = \gamma^{fd}(r)\left(1 - \delta^{fd}(r)\right)$.

Now, we make a $FFISS$ selection taking modal by the subsequent to produce a selection fuzzy set.

Selection taking Now, we make a selection-taking technique by the succeeding procedure to yield a selection set from an ordinary set of replacements. Therefore selection taker:

(a) Create a possible intuitionistic fuzzy sets over the parameter set based on a selection client who is a specialist.
(b) Creates a parameter fuzzy intuitionistic fuzzy set $\coprod_N$ over the different set C over on a selection taker.
(c) Find the decreases fuzzy intuitionistic set N^{difs} of N^{fd}
(d) Find the fuzzy decreases set N^{fd}ofN^{difs}.

(e) Select the parameter of N^{fd} that has highest membership value. Now, we can see the example.

Example: 4. 3 Assume that a firm fills a manager position. There are five applicants who assign in order.

Example. 4. 4. Assume that an administrative center requirements to positing a place. There are 5 applicants who fill in an application in order to apply officially for the place. There is a selection taker (ST) that is from the branch of HR. He wants to interview the applicants, but it is very hard to create all of them. Hence, by the fuzzy parameter Intuitionistic soft selection taker process, the total of applicants is decreased to a proper one. Think that the set of applicants $C = \{r_1, r_2, r_3, r_4, r_5\}$ which may be assign the parameter set $K = \{t_1, t_2, t_3, t_4\}$, which is $''t_1$ =knowledge ‘’t_2= mature ‘’t_3 =technical experimentation ‘’$t_4 =$ training person‘’.

Now we use the following steps

(a) Consider that selection taker constructs a feasible fuzzy intuitionistic subsets N along with parameter set K as;

$N = \left\{\left(t_1, \frac{7}{10}, \frac{3}{10}\right), \left(t_2, \frac{2}{10}, \frac{5}{10}\right), \left(t_3, \frac{5}{10}, \frac{5}{10}\right), \left(t_4, \frac{6}{10}, \frac{3}{10}\right)\right\}.$

(b) Selection taker constructs an parameter fuzzy intuitionistic soft set $\coprod_N$ along with alternatives set C

$$\coprod_N = \left\{\left(\left(t_1, \frac{7}{10}, \frac{3}{10}\right), \{r_1, r_2, r_4\}\right)\right\}, \left\{\left(t_2, \frac{2}{10}, \frac{5}{10}\right), \{C\}\right\}, \left\{\left(t_3, \frac{5}{10}, \frac{5}{10}\right)\{r_1, r_2, r_4\}\right\}, \left\{\left(t_4, \frac{6}{10}, \frac{3}{10}\right), \{r_2, r_3\}\right\}.$$

(c) Selection taker finds the decreased fuzzy intuitionistic set N^{difs} of $\coprod_N$

$$N^{difs} = \left\{\left(\left(r_1, \frac{28}{100}, \frac{26}{100}\right), \left(r_2, \frac{40}{100}, \frac{32}{100}\right), \left(r_3, \frac{16}{100}, \frac{16}{100}\right), \left(r_4, \frac{28}{100}, \frac{32}{100}\right), \left(r_5, \frac{4}{100}, \frac{10}{100}\right)\right)\right\}.$$

(d) Selection taker finds the decreased fuzzy set N^{fd} ofN^{difs}

$$N^{fd} = \left\{\left(r_1, \frac{2072}{10000}\right), \left(r_2, \frac{2720}{10000}\right), \left(r_3, \frac{1344}{10000}\right), \left(r_4, \frac{1904}{10000}\right), \left(r_4, \frac{360}{10000}\right)\right\}.$$

(e) In conclusion, selection chooses r_2 for the place from N^{fd} since it has the highest value $\frac{2720}{10000}$ along with the others.

5. Conclusion

In this work, we learn the Atanassov concept of intuitionistic fuzzy sets and we stated their choice of procedures and a few outcomes. Then, we presented the technique of selection taking on the parameter fuzzy intuitionistic fuzzy soft set theory. We also give an illustration that established the selection-taking methods. It can be applied to problems in so many areas.

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A CARTESIAN PRODUCT STRUCTURE ON SOCIAL STATES AND NEW RESOLUTIONS

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Abstract: This study is to introduce a Cartesian product structure into the social choice theoretical framework. We believe that a Cartesian product structure is a relevant way to describe individual rights in the social choice theory since it discriminates the personal attribute comprised in each social state. First, we define some conceptional and formal tools related to the Cartesian product structure. Then apply these notions to Gibbard's paradox and to Sen's impossibility of a Paretian liberal. Finally, we analyze the advantages of our approach to other solutions projected in the literary study for both impossibility theorems.

Keywords: Cartesian product, social choice, impossibility.

1 Introduction

In 1970, Sen introduced this concept into the social choice theoretical framework with a condition of liberalism based on the notion of decisiveness individuals must be decisive – their preferences must be acknowledged by society over some pairs of social states, which belong to their private sphere. Sen shows that this condition of liberalism and a weak Pareto principle lead to an impossibility of social choice: it is the impossibility of a Paretian liberal. But Sen's formal analysis does not need to distinguish between decisive pairs that enable an individual to take decisions that are "personal" to her and those that are not. He uses a Cartesian product structure to describe individual rights and points out the internal inconsistency caused by an extended condition of liberalism. This result is called Gibbard's paradox or Gibbard's First Libertarian Claim. Besides, Gibbard shows that his paradox arises only if individuals express conditional preferences. In other words, an individual expresses conditional preferences if her preferences depend on those of another individual. For example, Nikita is said to have conditional preferences if her desire is to wear a dress of the same color as Nisha's. On the contrary, if Nisha's desire is to differentiate from Nikita, it leads to Gibbard's paradox. Gibbard stresses that his paradox does not arise if unconditional preferences only are acknowledged by society.

This topic gave rise to many debates and attempts to develop new tools to take individual rights into account and to solve Gibbard's and Sen's para- doxes. This article is to introduce a Cartesian product structure on social states and to examine if new possibility results can be developed.

But a Cartesian product structure is inadequate in itself in order to deal with both impossibility results. It is necessary to determine a relevant way to take into account the implementation of these individual rights thus clarified.

2 Some conceptual and formal tools related to the Cartesian product structure

Let $M = \{1, 2, ..., n\}$ be the finite set of individuals, which is society ($n \geq 2$).With a Cartesian product structure on social states, each individual is a set X of personal features, this set being the same for all individuals. X is a finite set, where $|X| \geq 2$. A social state is a n-list $(x_1, x_2, ..., x_n)$ of personal features of the world, where $x_i \in X$, $\forall\, i \in M$. The set of all social states X^n is given by $X^n = X \times X \times ... \times X$. Each individual $i \in M$ has a binary relation $\geq_i$ on X^n, which is a linear ordering. A collective choice rule f specifies a social preference relation for each d : $\geq= f(d)$. If $\geq$ is a complete pre-ordering for all d in the domain, f is a "social welfare function" . Here, f is called a "social decision function". For any $i \in M$ and any $x = (x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n) \in X^n$, $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ...,x_n)$, where $x_{-i} \in X^n$. If $x_i \in X$ and $a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n) \in x^n{}_{-i}$, then $(x_i; a_{-i}) = (a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n)$. The personal sphere of individual i is the family of sets $\{D_i(a_{-i})\}_{a-i \in Xn}$ where $D_i(a_{-i})$ is defined as $D_i(a_{-i}) = \{x \in X^n \mid x_{-i} = a_{-i}\}$ As stated in our introduction, the difficulty we face with the problem of individual rights in the social choice theory is less analytical than conceptual. Consequently, it is crucial to find out first which values could be wished by the members of society and how they can be secured.

$-i$

Definition: 2.1 Strong and Light preferences For any $x, y \in X^n$, for any $j \in M$, if $x >_j y$ and if there exists at least one $z \in X^n$ such that $x >_j z$ and $z >_j y$, then individual j strongly prefers x toy: it will be denoted by $T [x >_j y] = S$. If $x >_j y$ but if such a social state z does not exist, individual j lightly prefers x to y: it is denoted by $T [x >_j y] = L$.For example, let us consider the following individual linear ordering $X^n = \{x, y, z, w\}$ and $x >_j w >_j z >_j y$. By transitivity, $x >_j y$. We then obtain $T [x >_j y] = S$. However, if $x >_j y >_j w >_j z$, $T [x >_j y] = L$.

Definition: 2.2 Set of Invasive Options For a given d, the set Y_j is com- posed of all social states for which the individual j has a preference which goes against a preference of another individual i /= j in her personal sphere is$Y_j(a_{-i})$=$\{y \in D_i(a_{-i}) | T[x >_j y] = S$ for at least one $x \in D_i(a_{-i})$ such that $y >_i x\}$ and Y_j=$\cup_{i \neq j} \cup_{a_{-i \in x^n}{}_{-i}} Y_j(a_{-i})$.For example, consider two individuals 1 and 2 and $X = \{p, q\}$. Thus, $X^n = \{(p, q), (q, p), (p, p), (q, q)\}$. Suppose that individual 1 has the following linear ordering $(p, p) >_1 (q, p) >_1 (q, q) >_1 (p, q)$. Suppose moreover that $(p, q) >_2 (p, p)$ and $(q, q) >_2 (q, p)$. Hence, $(p, q), (p, p) \in D_2(p)$

and $T [(p, p) >_1 (p, q)] = S$, $Y_1(p) = \{(p, q)\}$. And since $(q, q), (q, p) \in$

$D_2(q)$ and T [(q, p) $>_1$ (q, q)] = L, $Y_1(q) = \emptyset$. Finally, Y_1 = {(p, q)}. But, if (p, p) $>_1$ (q, p) $>_1$ (p, q) $>_1$ (q, q), all other things remaining equal, then Y_1 = {(p, q), (q, q)} since T [(q, p) $>_1$ (q, q)] = S.Sen's and Gibbard's theorems can now be presented and some possibility results be proposed thanks to the exclusion of invasive preferences.

Solution to Gibbard's paradox: It is based on an extended interpretation of the concept of personal sphere, every individual should be decisive over all pairs of social states, which differ only in her personal feature. Gibbard suggests the following claim **First Libertarian Claim:** For any x, y ∈ X^n, for any i ∈ N, for any $a_{-i} \in X^n$, if x, y ∈ $D_i(a_{-i})$ and x $>_i$ y, then x > y. Moreover, the collective choice rule f should respect the condition of unrestricted domain.

Second Unrestricted domain: The domain of f includes all logically possible n-lists of individual linear orderings.

Preference Modification Yk = ∅, ∀ t ∈ T where T ⊆ N and |T| ≥ n − 1.

Theorem 2.3 There exists a SDF satisfying conditions PM_1 and GL.

Proof The theorem is proved by constructing a SDF, which gives each person i an appropriate special voice on her feature. Let R be the relation between x and y, (∃i)[x, y ∈ $D_i(a_{-i})$ and x $>_i$ y]. Let ≥= f (d) be generated from Q in the following manner ∀ x, y ∈ X^n : x ≥ y ⟸⟹ ¬(yRx). Firstly, we prove that whenever yRx, y > x. Suppose that ¬(y> x). Hence x ≥ y and from the construction of f , we have ¬(yRx). Then, from ¬(y>x), it followed that ¬(yRx); therefore, if yRx, we obtain y > x as declared. Secondly, we show that f satisfies GL, then xRy. Therefore, x > y, and hence f satisfies GL. Now, consider the individual, which is responsible for the step x^1Rx^2 and call her individual j. Then, $x^1 >_j x^2$. Individual j is necessarily responsible for another step of the cycle $x^{\iota-1}Rx^{\iota}$, with ι = 4, ..., σ so that cycle 0 can exist. Hence, $x^{\iota-1} >_j x^{\iota}$. Suppose that $x^1 >_j x^{\iota-1}$. From x^{ι} to x^{σ} , steps originating from individuals i /= j or from individual j follow each other. In every case, we cannot obtain$x^{\sigma} >_j x^{\iota-1}$ so that Y_j can be empty. We necessarily have $x^{\iota-1} >_j x^{\sigma}$. Therefore, the step x^{σ} Qx^1 necessarily comes from an individual i /= j. Hence, T [$x^1 >_j x^{\sigma}$] = S and Y_j is nonempty. If $x^{\iota-1} >_j x^1$, we can prove that Y_j is nonempty according to the same line of reasoning.

Finally, we showed that if individual j is involved in cycle 0, her set of invasive options is nonempty. But the same conclusion remains for any individual involved in such a cycle.

Theorem 2.4 There exists a SDF satisfying conditions PM_2, P, and GL'.

Proof Let R be the relation between x and y, (∃j) x, y ∈ $D_j(a_{-j})$, x $>_j$ y and $x_jp_jy_j$ or (∀ i) x $>_i$ y, ∀ x, y ∈ X^n : x ≥ y ⟸⟹ ¬(yRx). From the way Q is defined, it is obvious that f satisfies conditions P and GL'. Next, we check that f is really a SDF, i.e., f is complete and acyclic. Since R is an asymmetric relation, f is necessarily complete. It remains to be shown that f is acyclic. Suppose there is a cycle 0, x^1Rx^2, ..., $x^{\sigma-1}Rx^{\sigma}$, x^{σ} Rx^1 where $x^1,\dots,x^{\sigma}$ belong to X^n (for the subscripts, we shall use mod τ arithmetic, so that 1 — 1 = σ and σ + 1 = 1. Variables ι and κ will range from 1 to σ). Hence at least two steps originating from condition GL' for two distinct individuals and one step initiating from condition P main to such a cycle. Now, consider $x^{\iota-1}Qx^{\iota}$. We get either (∀ i) $x^{\iota-1} >_i x^{\iota}$ or ¬(∀ i) $x^{\iota-1} >_i x^{\iota}$ and (∃ j) $x^{\iota-1}, x^{\iota} \in D_j(a_{-j})$, $x^{\iota-1} >_j x^{\iota}$ and $x^{\iota-1}p_jx^{\iota}$. We consider two individuals j and l: each of them is responsible for a step of cycle 0 proceeding from condition GL'. Then there is a ι such that $x^{\iota-1}, x^{\iota} \in D_j(a_{-j})$, $x^{\iota-1} >_j x^{\iota}$ and $x^{\iota-1}p_jx^{\iota}$ and such that (∀ i) $x^{\iota} >_i x^{\iota+1}$. Hence, we get: $x^{\iota-1} >_j x^{\iota}$ and $x^{\iota} >_j x^{\iota+1}$. In addition, individual l is involved in the cycle as well. There is a κ such that $x^{\kappa}, x^{\kappa+1} \in D_l(a_{-l})$, $x^{\kappa} >_l x^{\kappa+1}$. and $x^{\kappa}p_lx^{\kappa+1}$. Suppose that κ + 1 = ι-1, in other words, the step $x^{\kappa}Rx^{\iota-1}$ originates from condition GL'. It should be noted that this step can proceed from condition P, but this does not modify our proof. There is somewhere in the cycle a step originating from condition GL and from an individual different from j. For individual j, we have either $x^{\kappa} >_j x^{\iota-1}$ or $x^{\iota-1} >_j x^{\kappa}$ with T $[x^{\iota-1} >_j x^{\kappa}]$ = L so that Y_j can be empty. In both cases, T $[x^{\kappa} >_j x^{\iota+1}]$ = S. Then, in cycle 0, from $x^{\iota+1}$ to x^{κ}, steps necessarily come from condition GL'. Hence, the set Y_j is nonempty since j necessarily expresses at least one strong preference against a preference of another individual's protected sphere in the following subpart of cycle 0 $x^{\iota+1}Rx^{\iota+2}$, ..., $x^{\kappa}Rx^{\iota-1}$. The second stage of the proof requires to rely on the Cartesian product structure. Suppose that an individual m is involved in cycle 0 only in steps originating from condition P. According to (1), we get: $x^{\iota} >_m x^{\iota+1}$. For individual m, we could have $x^{\iota} >_m x^{\iota-1}$ and T $[x^{\iota} >_m x^{\iota-1}]$ = L, $x^{\iota-1} >_m x^{\kappa}$ and T $[x^{\iota-1} >_m x^{\kappa}]$ = L, if all steps from $x^{\iota+1}$ to x^{κ} proceed from condition GL'. In every other case, Y_m is nonempty. In order to complete this proof, we show that m's above preferences necessarily imply a non- empty set Y_m. For individual j, recall that $x^{\iota-1} >_j x^{\iota}$, $x^{\iota-1}, x^{\iota} \in D_j(a_{-j})$ and $x^{\iota-1}p_jx^{\iota}$. For individual l, $x^{\kappa} >_l x^{\iota-1}$, $x^{\kappa}, x^{\iota-1} \in D_l(a_{-l})$ and $x^{\kappa}p_lx^{\iota-1}$. Let $x^{\iota-1} = (x^1, ..., x^j, ..., x^l, ..., x^n)$, $x^{\iota} = (x^1, ..., x^{j*}, ..., x^l, ..., x^n)$ and $x^{\kappa} = (x^1, ..., x^j, ..., x^{l*}, ..., x^n)$. Since individuals j and l have to express uncondi- tional preferences, we obtain $(x^1, ..., x^j, ..., x^{l*}, ..., x^n) > (x^1, ..., x^{j*}, ..., x^{l*}, ..., x^n)$ and $(x^1, ..., x^{j*}, ..., x^{l*}, ..., x^n) > (x^1, ..., x^{j*}, ..., x^l, ..., x^n)$. But $(x^1, ..., x^j, ..., x^{l*}, ..., x^n) = x^{\kappa}$ and $(x^1, ..., x^{j*}, ..., x^l, ..., z) = x^{\iota}$. Let x^* be the social state $(x^1, ..., x^{j*}, ..., x^{l*}, ..., x^n)$. Therefore, Y_m is nonempty since individual m necessarily expresses at least one strong preference against a preference of individuals j or l in their protected sphere. Hence cycles cannot occur.

3.Conclusion

The aim of the article is to devise a reliable way of overcoming two impossibility results developed into a social choice theoretical context, which makes it possible to take individual rights into account properly. Some conceptual and formal tools are developed so that the private sphere can be protected from aggressive preferences. The Cartesian product structure matters since it provides improved results.

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CARTESIAN PRODUCT OVER TWO FUZZY MATRIX SETS

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Abstract

This paper, the notions of an operations and relations on the Cartesian product over two fuzzy matrices set are introduced and its some properties are explored. We prove some equality based on the operation and the relation over FSs. Finally, we introducing some Cartesian formulas x_4, x_5 in Cartesian product over two fuzzy matrixes sets.

Keyword: Fuzzy matrix, Cartesian product over fuzzy matrix.

1. Introduction: The theory of fuzzy sets (FS) introduction by Zadeh [42] has showed meaningful application in many fields of studies. A fuzzy matrix with element having values in closed interval [0, 1]. R.H. Kim and F.W. Roush [19] has introduced the concept of F.W. The concept of intuitionistic fuzzy sets proposed by Atanassov is a generalization of FS. He has introduced a non-membership grade, in addition to the membership grade, thus allowing an aspect of uncertainty in the membership grade [2]. The IFS (Intuitionistic Fuzzy Set) theory introduced by K.T. Atanassov [3] is interesting and useful to problem solving. The ideas of IFS were developed in later [4, 5]. Structures on Intuitionistic Fuzzy Relations, Fuzzy Set and System [7]. The IFS has captured much attention from researchers in various fields and many achievements have been made, such as entropy measure of IFS [8, 22, 30, 32, and 41]. Distance or similarly measure between IFSs [11, 20, 28, 37]. Some operations on intuitionistic Fuzzy sets, Fuzzy Set and System [29]. In recent years the IFS theory has been applied in medical diagnosis [9]. Using the concept of IFS, Im et al [14,15] studied Intuitionistic Fuzzy Matrix (IFM). The decision has been taken by measuring the smallest Euclidean distance between a person and a society. Many real-world decision-making problems such as academic career of the students, high school determination problem, medical problem, student performance determination of a course, career determination problem, career determinations etc. have been carried out by various researchers by using intuitionistic fuzzy Set [31]. intuitionistic fuzzy matrix, Notes on Intuitionistic Fuzzy Sets [26]. In research was carried out on how a transitive IFM decomposed into a sum of nilpotent IFM and symmetric IFM by Jeong et al [16]. Distance Measure between intuitionistic Fuzzy Sets [36]. Note on some operations on intuitionistic fuzzy sets, fuzzy sets and System [43]. Aggregation operations of IFS [34, 38, 40]. The concept of upper cut sets and lower cuts of IFSs are given by [21]. "On some properties of one Cartesian product over intuitionistic fuzzy sets", Notes on Intuitionistic Fuzzy Sets [6]. Intuitionistic Fuzzy Relations Equation, Advances in Fuzzy Mathematics [23]. He worked on IFSs and they also discussed the decomposition theorem, representation theorem of IFSs by using cut sets (α, α') -cut of IFMs [33]. Decomposition theorems of an Intuitionistic Fuzzy Sets, Notes on Intuitionistic Fuzzy Sets [17]. [39] studied intuitionistic Fuzzy Value and also IFMs. He defined intuitionistic fuzzy similarly relation and also utilizes it in clustering analysis, "intuitionistic Fuzzy Sets and its Application in Students Performance Determination of a Course via Normalized Euclidean Distance Method" [39]. Have contributed significantly for the development of cut sets [10, 13]. Representation and Decomposition of an intuitionistic Fuzzy Matrix using some (α, α') -Cuts [24]. Decomposed an IFMs into product of idempotent [25]." Application of intuitionistic Fuzzy Sets in the Academic Career of the Students" [18].

2. Preliminaries

Definition 2.1: Fuzzy sets: A fuzzy set is any set that allows its members to have different degree of membership function, having interval [0, 1].

Definition 2.2: Fuzzy matrix set: Fuzzy matrices play a vital role in scientific development. A Fuzzy matrix may be matrix that has its parts from [0, 1]. Consider a matrix $A = [a_{ij}]_{3\times 3}$ where $a_{ij} \in [0,1], 1 \leq j \leq n$. Then A is a Fuzzy Matrix [FM].

Definition 2.3: Fuzzy rectangular matrix: Let $A = [a_{ij}]_{m\times n}\,(m \neq n)$ where $a_{ij} \in [0,1], 1 \leq i \leq n, 1 \leq j \leq m$. Then A is a Fuzzy Rectangular Matrix.

Definition 2.4: Fuzzy square matrix

Let $A = \begin{bmatrix} a_{11} a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots \quad \vdots & & \vdots \quad \vdots & & \vdots \quad \vdots \\ a_{i1} a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots \quad \vdots & & \vdots \quad \vdots & & \vdots \quad \vdots \\ a_{n1} a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$ Where, $a_{ij} \in [0,1], 1 \leq i, j \leq n$. Then A is a fuzzy square matrix.

Definition 2.5: Fuzzy row matrix: Let A= $[a_1, a_2, a_3, \cdots, a_n]$ where $a_{ij} \in [0,1]$, $j = 1,2, \ldots, n$.Then A is called $1 \times n$ a fuzzy row matrix or row vector.

Definition 2.6: Fuzzy column matrix: Let $A = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ where $a_i[0,1], i = 1,2, \ldots, n$.Then A is called $m \times 1$ a fuzzy column matrix.

Definition 2.7: Fuzzy diagonal matrix: A Fuzzy square matrix A= $[a_{ij}]_{m\times n}$ is said to fuzzy diagonal matrix. If $a_{ij} = 0$ when $i \neq j, a_{ij}[0,1], 1 \leq I$.

Definition 2.8: Fuzzy relation: A fuzzy relation is the Cartesian product of mathematical fuzzy sets. Two fuzzy sets are taken as input; the fuzzy relation is then equal to the cross product of the sets which is created by vector multiplication

Definition 2.9: Cartesian Products: Consider two sets A and B. The set of all ordered pairs {a, b} where aϵA & $b \epsilon B$ is called cartesian product. It is denoted by $A \times B$. $A \times B = \{(a, b) : a \epsilon A \text{ and } b \epsilon B\}$

Definition 2.10: Membership Function: The membership function of a fuzzy set A is denoted by μ_A, $\mu_A : E \rightarrow [0,1]$. The most commonly used range of value of membership function is the unit interval $[a, b]$.

Definition 2.11: Degree of membership function: Membership function for an intuitionistic fuzzy set A on the universe of discourse is defined as $\mu_A : X \rightarrow [0,1]$,Where each element X is mapped to a value between 0 and 1. The value $\mu_A(x), x \in X$ is called Membership value or degree of membership function. The most commonly used range of value of membership function is the unit interval $[a, b]$.

Definition 2.12: Degree of non-membership Function: Non-Membership function for an intuitionistic fuzzy set A on the universe of discourse is defined as $\vartheta_A : X \rightarrow [0,1]$,Where each element X is, mapped to a value between 0 and 1. The value $\vartheta_A(x)$, $x \epsilon X$ is called non-membership value or degree of non-membership function.

Definition 2.13: Intuitionistic fuzzy set: An Intuitionistic Fuzzy Set (IFs) A in E is defined as an object of the following form $A = \{< X, \mu_A(x), \vartheta_A(x) > \backslash x \in E\}$ Where the functions: $\mu_A : E \rightarrow [0,1]$ and $\vartheta_A : E \rightarrow [0,1]$.

Definition 2.14: Intuitionistic fuzzy matrix: An intuitionistic fuzzy matrix is a pair of fuzzy matrices, namely, a membership and non-membership function which represent positive and negative aspects. The concept of intuitionistic fuzzy matrices was introduced by pa le tal.

Definition 2.15: Operations on intuitionistic fuzzy sets: Let A and B be two intuitionistic fuzzy sets on the universe X. Where, A =$\{[\, x, \mu_A(x), \gamma_A(x)] | x \epsilon X\}$ and B= $\{[\, x, \mu_B(x), \gamma_B(x)] | x \epsilon X\}$.

Definition 2.16: The five Cartesian products of two IFSs A and B are defined as follows: Let A and B are two intuitionistic fuzzy sets of the universes A_E *and* B_F, then the Cartesian product of two IFSs is defined by

The Cartesian product " $\times_4$ " is defined by

A $\times_4 B = \{((x,y)\ ,\ min\big(\mu_A(x),\mu_B(y)\big)\ ,\ max(\lambda_A(x),\lambda_B(y))) : x\epsilon E_1, and\ y\epsilon E_2\}$.

The Cartesian product " $\times_5$ " is defined by

A $\times_5 B = \{((x,y)\ ,\ max\big(\mu_A(x),\mu_B(y)\big)\ ,\ min(\lambda_A(x),\lambda_B(y))) : x\epsilon E_1, and\ y\epsilon E_2\}$.

Theorem 3.1: If $\overline{A_E}$ and $\overline{B_F}$ are two intuitionistic fuzzy matrices set, then $\overline{A_E} \times_4 \overline{B_F}$ is also an intuitionistic fuzzy matrix set.

Proof: If $\overline{A_E} = \begin{pmatrix} [\overline{\mu_{E_{11}}}\ ,\ \overline{\lambda_{E_{11}}}] & [\overline{\mu_{E_{12}}}\ ,\ \overline{\lambda_{E_{12}}}] & [\overline{\mu_{E_{13}}}\ ,\ \overline{\lambda_{E_{13}}}] \\ [\overline{\mu_{E_{21}}}\ ,\ \overline{\lambda_{E_{21}}}] & [\overline{\mu_{E_{22}}}\ ,\ \overline{\lambda_{E_{22}}}] & [\overline{\mu_{E_{23}}}\ ,\ \overline{\lambda_{E_{23}}}] \\ [\overline{\mu_{E_{31}}}\ ,\ \overline{\lambda_{E_{31}}}] & [\overline{\mu_{E_{32}}}\ ,\ \overline{\lambda_{E_{32}}}] & [\overline{\mu_{E_{33}}}\ ,\ \overline{\lambda_{E_{33}}}] \end{pmatrix}$ and $\overline{B_F} = \begin{pmatrix} [\overline{\mu_{F_{11}}}\ ,\ \overline{\lambda_{F_{11}}}] & [\overline{\mu_{F_{12}}}\ ,\ \overline{\lambda_{F_{12}}}] & [\overline{\mu_{F_{13}}}\ ,\ \overline{\lambda_{F_{13}}}] \\ [\overline{\mu_{F_{21}}}\ ,\ \overline{\lambda_{F_{21}}}] & [\overline{\mu_{F_{22}}}\ ,\ \overline{\lambda_{F_{22}}}] & [\overline{\mu_{F_{23}}}\ ,\ \overline{\lambda_{F_{23}}}] \\ [\overline{\mu_{F_{31}}}\ ,\ \overline{\lambda_{F_{31}}}] & [\overline{\mu_{F_{32}}}\ ,\ \overline{\lambda_{F_{32}}}] & [\overline{\mu_{F_{33}}}\ ,\ \overline{\lambda_{F_{33}}}] \end{pmatrix}$ are two intuitionistic fuzzy matrix sets.Thus $\overline{A_E} \times_4 \overline{B_F}$,

$$\overline{A_E} \times_4 \overline{B_F} = \begin{pmatrix} [\overline{\mu_{E_{11}}}\ ,\ \overline{\lambda_{E_{11}}}] & [\overline{\mu_{E_{12}}}\ ,\ \overline{\lambda_{E_{12}}}] & [\overline{\mu_{E_{13}}}\ ,\ \overline{\lambda_{E_{13}}}] \\ [\overline{\mu_{E_{21}}}\ ,\ \overline{\lambda_{E_{21}}}] & [\overline{\mu_{E_{22}}}\ ,\ \overline{\lambda_{E_{22}}}] & [\overline{\mu_{E_{23}}}\ ,\ \overline{\lambda_{E_{23}}}] \\ [\overline{\mu_{E_{31}}}\ ,\ \overline{\lambda_{E_{31}}}] & [\overline{\mu_{E_{32}}}\ ,\ \overline{\lambda_{E_{32}}}] & [\overline{\mu_{E_{33}}}\ ,\ \overline{\lambda_{E_{33}}}] \end{pmatrix} \times_4 \begin{pmatrix} [\overline{\mu_{F_{11}}}\ ,\ \overline{\lambda_{F_{11}}}] & [\overline{\mu_{F_{12}}}\ ,\ \overline{\lambda_{F_{12}}}] & [\overline{\mu_{F_{13}}}\ ,\ \overline{\lambda_{F_{13}}}] \\ [\overline{\mu_{F_{21}}}\ ,\ \overline{\lambda_{F_{21}}}] & [\overline{\mu_{F_{22}}}\ ,\ \overline{\lambda_{F_{22}}}] & [\overline{\mu_{F_{23}}}\ ,\ \overline{\lambda_{F_{23}}}] \\ [\overline{\mu_{F_{31}}}\ ,\ \overline{\lambda_{F_{31}}}] & [\overline{\mu_{F_{32}}}\ ,\ \overline{\lambda_{F_{32}}}] & [\overline{\mu_{F_{33}}}\ ,\ \overline{\lambda_{F_{33}}}] \end{pmatrix}$$

$\overline{A_E} \times_4 \overline{B_F} = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$ Where,

$$X_{11} = \left([\overline{\mu_{E_{11}}},\overline{\lambda_{E_{11}}}] \quad [\overline{\mu_{E_{12}}},\overline{\lambda_{E_{12}}}] \quad [\overline{\mu_{E_{13}}},\overline{\lambda_{E_{13}}}]\right) \times_4 \begin{pmatrix} [\overline{\mu_{F_{11}}},\overline{\lambda_{F_{11}}}] \\ [\overline{\mu_{F_{21}}},\overline{\lambda_{F_{21}}}] \\ [\overline{\mu_{F_{31}}},\overline{\lambda_{F_{31}}}] \end{pmatrix}$$

$$X_{12} = \left([\overline{\mu_{E_{11}}},\overline{\lambda_{E_{11}}}] \quad [\overline{\mu_{E_{12}}},\overline{\lambda_{E_{12}}}] \quad [\overline{\mu_{E_{13}}},\overline{\lambda_{E_{13}}}]\right) \times_4 \begin{pmatrix} [\overline{\mu_{F_{12}}},\overline{\lambda_{F_{12}}}] \\ [\overline{\mu_{F_{22}}},\overline{\lambda_{F_{22}}}] \\ [\overline{\mu_{F_{32}}},\overline{\lambda_{F_{32}}}] \end{pmatrix}$$

$$X_{13} = \left([\overline{\mu_{E_{11}}},\overline{\lambda_{E_{11}}}] \quad [\overline{\mu_{E_{12}}},\overline{\lambda_{E_{12}}}] \quad [\overline{\mu_{E_{13}}},\overline{\lambda_{E_{13}}}]\right) \times_4 \begin{pmatrix} [\overline{\mu_{F_{13}}},\overline{\lambda_{F_{13}}}] \\ [\overline{\mu_{F_{23}}},\overline{\lambda_{F_{23}}}] \\ [\overline{\mu_{F_{33}}},\overline{\lambda_{F_{33}}}] \end{pmatrix}$$

$$X_{21} = \left([\overline{\mu_{E_{21}}},\overline{\lambda_{E_{21}}}] \quad [\overline{\mu_{E_{22}}},\overline{\lambda_{E_{22}}}] \quad [\overline{\mu_{E_{23}}},\overline{\lambda_{E_{23}}}]\right) \times_4 \begin{pmatrix} [\overline{\mu_{F_{11}}},\overline{\lambda_{F_{11}}}] \\ [\overline{\mu_{F_{21}}},\overline{\lambda_{F_{21}}}] \\ [\overline{\mu_{F_{31}}},\overline{\lambda_{F_{31}}}] \end{pmatrix}$$

$$X_{22} = \left([\overline{\mu_{E_{21}}},\overline{\lambda_{E_{21}}}] \quad [\overline{\mu_{E_{22}}},\overline{\lambda_{E_{22}}}] \quad [\overline{\mu_{E_{23}}},\overline{\lambda_{E_{23}}}]\right) \times_4 \begin{pmatrix} [\overline{\mu_{F_{12}}},\overline{\lambda_{F_{12}}}] \\ [\overline{\mu_{F_{22}}},\overline{\lambda_{F_{22}}}] \\ [\overline{\mu_{F_{32}}},\overline{\lambda_{F_{32}}}] \end{pmatrix}$$

$$X_{23} = \left([\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] \quad [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] \quad [\overline{\mu_{E_{23}}}, \overline{\lambda_{E_{23}}}]\right) \times_4 \begin{pmatrix} [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] \\ [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] \\ [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}] \end{pmatrix}$$

$$X_{31} = \left([\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \quad [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \quad [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}]\right) \times_4 \begin{pmatrix} [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] \\ [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] \\ [\overline{\mu_{F_{31}}}, \overline{\lambda_{F_{31}}}] \end{pmatrix}$$

$$X_{32} = \left([\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \quad [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \quad [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}]\right) \times_4 \begin{pmatrix} [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] \\ [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] \\ [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}] \end{pmatrix}$$

$$X_{33} = \left([\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \quad [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \quad [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}]\right) \times_4 \begin{pmatrix} [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] \\ [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] \\ [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}] \end{pmatrix}$$

$$X_{11} = [\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] \times_4 [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] + [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] \times_4 [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] + [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}] \times_4 [\overline{\mu_{F_{31}}}, \overline{\lambda_{F_{31}}}]$$

$$X_{12} = [\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] \times_4 [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] + [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] \times_4 [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] + [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}] \times_4 [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}]$$

$$X_{13} = [\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] \times_4 [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] + [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] \times_4 [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] + [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}] \times_4 [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}]$$

$$X_{21} = [\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] \times_4 [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] + [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] \times_4 [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] + [\overline{\mu_{E_{23}}}, \overline{\lambda_{E_{23}}}] \times_4 \left[\overline{\mu_{F_{31}, \overline{\lambda_{F_{31}}}}}\right]$$

$$X_{22} = [\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] \times_4 [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] + [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] \times_4 [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] + [\overline{\mu_{E_{23}}}, \overline{\lambda_{E_{23}}}] \times_4 [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}]$$

$$X_{23} = [\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] \times_4 [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] + [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] \times_4 [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] + [\overline{\mu_{23}}, \overline{\lambda_{E_{23}}}] \times_4 [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}]$$

$$X_{31} = [\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \times_4 [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] + [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \times_4 [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] + [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}] \times_4 [\overline{\mu_{F_{31}}}, \overline{\lambda_{F_{31}}}]$$

$$X_{32} = [\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \times_4 [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] + [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \times_4 [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] + [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}] \times_4 [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}]$$

$$X_{33} = [\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \times_4 [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] + [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \times_4 [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] + [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}] \times_4 [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}]$$

Now, by applying this A $\times_4 B = \{((x, y),\ min(\mu_A(x), \mu_B(y)),\ max(\lambda_A(x), \lambda_B(y))) : x \epsilon E_1, and\ y \epsilon E_2\}$.

$$X_{11} = [min(\overline{\mu_{E_{11}}}, \overline{\mu_{F_{11}}}), max(\overline{\lambda_{E_{11}}}, \overline{\lambda_{F_{11}}})] + [min(\overline{\mu_{E_{12}}}, \overline{\mu_{F_{21}}}), max(\overline{\lambda_{E_{12}}}, \overline{\lambda_{F_{21}}})]$$
$$+ [min(\overline{\mu_{E_{13}}}, \overline{\mu_{F_{31}}}), max(\overline{\lambda_{E_{13}}}, \overline{\lambda_{F_{31}}})]$$

$$X_{12} = [min(\overline{\mu_{E_{11}}}, \overline{\mu_{F_{12}}}), max(\overline{\lambda_{E_{11}}}, \overline{\lambda_{F_{12}}})] + [min(\overline{\mu_{E_{12}}}, \overline{\mu_{F_{22}}}), max(\overline{\lambda_{E_{12}}}, \overline{\lambda_{F_{22}}})]$$
$$+ [min(\overline{\mu_{E_{13}}}, \overline{\mu_{F_{32}}}), max(\overline{\lambda_{E_{13}}}, \overline{\lambda_{F_{32}}})]$$

$$X_{13} = [min(\overline{\mu_{E_{11}}}, \overline{\mu_{F_{13}}}), max(\overline{\lambda_{E_{11}}}, \overline{\lambda_{F_{13}}})] + [min(\overline{\mu_{E_{12}}}, \overline{\mu_{F_{23}}}), max(\overline{\lambda_{E_{12}}}, \overline{\lambda_{F_{23}}})]$$
$$+ [min(\overline{\mu_{E_{13}}}, \overline{\mu_{F_{33}}}), max(\overline{\lambda_{E_{13}}}, \overline{\lambda_{F_{33}}})]$$

$$X_{21} = [min(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{11}}}), max(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{11}}})] + [min(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{21}}}), max(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{21}}})]$$
$$+ [min(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{31}}}), max(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{31}}})]$$

$$X_{22} = [min(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{12}}}), max(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{12}}})] + [min(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{22}}}), max(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{22}}})] +$$
$$[min(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{32}}}), max(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{32}}})]$$

$X_{23} = \left[min\left(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{13}}}\right), max\left(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{13}}}\right)\right] + \left[min\left(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{23}}}\right), max\left(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{23}}}\right)\right] +$
$\left[min\left(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{33}}}\right), max\left(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{33}}}\right)\right]$

$X_{31} = \left[min\left(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{11}}}\right), max\left(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{11}}}\right)\right] + \left[min\left(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{21}}}\right), max\left(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{21}}}\right)\right] +$
$\left[min\left(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{31}}}\right), max\left(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{31}}}\right)\right]$

$X_{32} = \left[min\left(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{12}}}\right), max\left(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{12}}}\right)\right] + \left[min\left(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{22}}}\right), max\left(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{22}}}\right)\right]$
$+\left[min\left(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{32}}}\right), max\left(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{32}}}\right)\right]$

$X_{33} = \left[min\left(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{13}}}\right), max\left(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{13}}}\right)\right] + \left[min\left(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{23}}}\right), max\left(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{23}}}\right)\right] +$
$\left[min\left(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{33}}}\right), max\left(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{33}}}\right)\right]$

By using, $\mathrm{A} + \mathrm{B} + C = max(\mathrm{A}, \mathrm{B}) + C$

$X_{11} = max\left[\left[min\left(\overline{\mu_{E_{11}}}, \overline{\mu_{F_{11}}}\right), max\left(\overline{\lambda_{E_{11}}}, \overline{\lambda_{F_{11}}}\right)\right], \left[min\left(\overline{\mu_{E_{12}}}, \overline{\mu_{F_{21}}}\right), max\left(\overline{\lambda_{E_{12}}}, \overline{\lambda_{F_{21}}}\right)\right]\right]$
$+\left[min\left(\overline{\mu_{E_{13}}}, \overline{\mu_{F_{31}}}\right), max\left(\overline{\lambda_{E_{13}}}, \overline{\lambda_{F_{31}}}\right)\right]$

$X_{12} = max\left[\left[min\left(\overline{\mu_{E_{11}}}, \overline{\mu_{F_{12}}}\right), max\left(\overline{\lambda_{E_{11}}}, \overline{\lambda_{F_{12}}}\right)\right], \left[min\left(\overline{\mu_{E_{12}}}, \overline{\mu_{F_{22}}}\right), max\left(\overline{\lambda_{E_{12}}}, \overline{\lambda_{F_{22}}}\right)\right]\right] +$
$\left[min\left(\overline{\mu_{E_{13}}}, \overline{\mu_{F_{32}}}\right), max\left(\overline{\lambda_{E_{13}}}, \overline{\lambda_{F_{32}}}\right)\right]$

$X_{13} = max\left[\left[min\left(\overline{\mu_{E_{11}}}, \overline{\mu_{F_{13}}}\right), max\left(\overline{\lambda_{E_{11}}}, \overline{\lambda_{F_{13}}}\right)\right], \left[min\left(\overline{\mu_{E_{12}}}, \overline{\mu_{F_{23}}}\right), max\left(\overline{\lambda_{E_{12}}}, \overline{\lambda_{F_{23}}}\right)\right]\right] +$
$\left[min\left(\overline{\mu_{E_{13}}}, \overline{\mu_{F_{33}}}\right), max\left(\overline{\lambda_{E_{13}}}, \overline{\lambda_{F_{33}}}\right)\right]$

$X_{21} = max\left[\left[min\left(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{11}}}\right), max\left(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{11}}}\right)\right], \left[min\left(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{21}}}\right), max\left(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{21}}}\right)\right]\right] +$
$\left[min\left(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{31}}}\right), max\left(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{31}}}\right)\right]$

$X_{22} = max\left[\left[min\left(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{12}}}\right), max\left(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{12}}}\right)\right], \left[min\left(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{22}}}\right), max\left(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{22}}}\right)\right]\right]$
$+\left[min\left(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{32}}}\right), max\left(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{32}}}\right)\right]$

$X_{23} = max\left[\left[min\left(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{13}}}\right), max\left(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{13}}}\right)\right], \left[min\left(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{23}}}\right), max\left(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{23}}}\right)\right]\right]$
$+\left[min\left(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{33}}}\right), max\left(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{33}}}\right)\right]$

$X_{31} = max\left[\left[min\left(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{11}}}\right), max\left(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{11}}}\right)\right], \left[min\left(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{21}}}\right), max\left(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{21}}}\right)\right]\right]$
$+\left[min\left(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{31}}}\right), max\left(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{31}}}\right)\right]$

$X_{32} = max\left[\left[min\left(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{12}}}\right), max\left(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{12}}}\right)\right], \left[min\left(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{22}}}\right), max\left(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{22}}}\right)\right]\right]$
$+\left[min\left(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{32}}}\right), max\left(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{32}}}\right)\right]$

$X_{33} = max\left[\left[min\left(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{13}}}\right), max\left(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{13}}}\right)\right], \left[min\left(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{23}}}\right), max\left(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{23}}}\right)\right]\right]$
$+\left[min\left(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{33}}}\right), max\left(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{33}}}\right)\right]$

By using, $\mathrm{A} + \mathrm{B} + C = max\{max(\mathrm{A}, \mathrm{B}), C\}$

$$X_{11} = max\begin{Bmatrix} max\left[\left[min\left(\overline{\mu_{E_{11}}},\overline{\mu_{F_{11}}}\right), max\left(\overline{\lambda_{E_{11}}},\overline{\lambda_{F_{11}}}\right)\right],\left[min\left(\overline{\mu_{E_{12}}},\overline{\mu_{F_{21}}}\right), max\left(\overline{\lambda_{E_{12}}},\overline{\lambda_{F_{21}}}\right)\right]\right], \\ \left[min\left(\overline{\mu_{E_{13}}},\overline{\mu_{F_{31}}}\right), max\left(\overline{\lambda_{E_{13}}},\overline{\lambda_{F_{31}}}\right)\right]\end{Bmatrix}$$

$$X_{12} = max\begin{Bmatrix} max\left[\left[min\left(\overline{\mu_{E_{11}}},\overline{\mu_{F_{12}}}\right), max\left(\overline{\lambda_{E_{11}}},\overline{\lambda_{F_{12}}}\right)\right],\left[min\left(\overline{\mu_{E_{12}}},\overline{\mu_{F_{22}}}\right), max\left(\overline{\lambda_{E_{12}}},\overline{\lambda_{F_{22}}}\right)\right]\right], \\ \left[min\left(\overline{\mu_{E_{13}}},\overline{\mu_{F_{32}}}\right), max\left(\overline{\lambda_{E_{13}}},\overline{\lambda_{F_{32}}}\right)\right]\end{Bmatrix}$$

$$X_{13} = max\begin{Bmatrix} max\left[\left[min\left(\overline{\mu_{E_{11}}},\overline{\mu_{F_{13}}}\right), max\left(\overline{\lambda_{E_{11}}},\overline{\lambda_{F_{13}}}\right)\right],\left[min\left(\overline{\mu_{E_{12}}},\overline{\mu_{F_{23}}}\right), max\left(\overline{\lambda_{E_{12}}},\overline{\lambda_{F_{23}}}\right)\right]\right], \\ \left[min\left(\overline{\mu_{E_{13}}},\overline{\mu_{F_{33}}}\right), max\left(\overline{\lambda_{E_{13}}},\overline{\lambda_{F_{33}}}\right)\right]\end{Bmatrix}$$

$$X_{21} = max\begin{Bmatrix} max\left[\left[min\left(\overline{\mu_{E_{21}}},\overline{\mu_{F_{11}}}\right), max\left(\overline{\lambda_{E_{21}}},\overline{\lambda_{F_{11}}}\right)\right],\left[min\left(\overline{\mu_{E_{22}}},\overline{\mu_{F_{21}}}\right), max\left(\overline{\lambda_{E_{22}}},\overline{\lambda_{F_{21}}}\right)\right]\right], \\ \left[min\left(\overline{\mu_{E_{23}}},\overline{\mu_{F_{31}}}\right), max\left(\overline{\lambda_{E_{23}}},\overline{\lambda_{F_{31}}}\right)\right]\end{Bmatrix}$$

$$X_{22} = max\begin{Bmatrix} max\left[\left[min\left(\overline{\mu_{E_{21}}},\overline{\mu_{F_{12}}}\right), max\left(\overline{\lambda_{E_{21}}},\overline{\lambda_{F_{12}}}\right)\right],\left[min\left(\overline{\mu_{E_{22}}},\overline{\mu_{F_{22}}}\right), max\left(\overline{\lambda_{E_{22}}},\overline{\lambda_{F_{22}}}\right)\right]\right], \\ \left[min\left(\overline{\mu_{E_{23}}},\overline{\mu_{F_{32}}}\right), max\left(\overline{\lambda_{E_{23}}},\overline{\lambda_{F_{32}}}\right)\right]\end{Bmatrix}$$

$$X_{23} = max\begin{Bmatrix} max\left[\left[min\left(\overline{\mu_{E_{21}}},\overline{\mu_{F_{13}}}\right), max\left(\overline{\lambda_{E_{21}}},\overline{\lambda_{F_{13}}}\right)\right],\left[min\left(\overline{\mu_{E_{22}}},\overline{\mu_{F_{23}}}\right), max\left(\overline{\lambda_{E_{22}}},\overline{\lambda_{F_{23}}}\right)\right]\right], \\ \left[min\left(\overline{\mu_{E_{23}}},\overline{\mu_{F_{33}}}\right), max\left(\overline{\lambda_{E_{23}}},\overline{\lambda_{F_{33}}}\right)\right]\end{Bmatrix}$$

$$X_{31} = max\begin{Bmatrix} max\left[\left[min\left(\overline{\mu_{E_{31}}},\overline{\mu_{F_{11}}}\right), max\left(\overline{\lambda_{E_{31}}},\overline{\lambda_{F_{11}}}\right)\right],\left[min\left(\overline{\mu_{E_{32}}},\overline{\mu_{F_{21}}}\right), max\left(\overline{\lambda_{E_{32}}},\overline{\lambda_{F_{21}}}\right)\right]\right], \\ \left[min\left(\overline{\mu_{E_{33}}},\overline{\mu_{F_{31}}}\right), max\left(\overline{\lambda_{E_{33}}},\overline{\lambda_{F_{31}}}\right)\right]\end{Bmatrix}$$

$$X_{32} = max\begin{Bmatrix} max\left[\left[min\left(\overline{\mu_{E_{31}}},\overline{\mu_{F_{12}}}\right), max\left(\overline{\lambda_{E_{31}}},\overline{\lambda_{F_{12}}}\right)\right],\left[min\left(\overline{\mu_{E_{32}}},\overline{\mu_{F_{22}}}\right), max\left(\overline{\lambda_{E_{32}}},\overline{\lambda_{F_{22}}}\right)\right]\right], \\ \left[min\left(\overline{\mu_{E_{33}}},\overline{\mu_{F_{32}}}\right), max\left(\overline{\lambda_{E_{33}}},\overline{\lambda_{F_{32}}}\right)\right]\end{Bmatrix}$$

$$X_{33} = max\begin{Bmatrix} max\left[\left[min\left(\overline{\mu_{E_{31}}},\overline{\mu_{F_{13}}}\right), max\left(\overline{\lambda_{E_{31}}},\overline{\lambda_{F_{13}}}\right)\right],\left[min\left(\overline{\mu_{E_{32}}},\overline{\mu_{F_{23}}}\right), max\left(\overline{\lambda_{E_{32}}},\overline{\lambda_{F_{23}}}\right)\right]\right], \\ \left[min\left(\overline{\mu_{E_{33}}},\overline{\mu_{F_{33}}}\right), max\left(\overline{\lambda_{E_{33}}},\overline{\lambda_{F_{33}}}\right)\right]\end{Bmatrix}$$

$\overline{A_E} \times_4 \overline{B_F} = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$. Hence, $\overline{A_E} \times_4 \overline{B_F}$ is an intuitionistic fuzzy matrix set.

Python program for $\overline{A_E} \times_4 \overline{B_F}$

```
#input the values
a_11=float(input("a_11="))
b_11=float(input("b_11="))
c_11=float(input("c_11="))
d_11=float(input("d_11="))
a_12=float(input("a_12="))
b_12=float(input("b_12="))
c_12=float(input("c_12="))
d_12=float(input("d_12="))
a_13=float(input("a_13="))
b_13=float(input("b_13="))
```

```
c_13=float(input("c_13="))
d_13=float(input("d_13="))
a_21=float(input("a_21="))
b_21=float(input("b_21="))
c_21=float(input("c_21="))
d_21=float(input("d_21="))
a_22=float(input("a_22="))
b_22=float(input("b_22="))
c_22=float(input("c_22="))
d_22=float(input("d_22="))
a_23=float(input("a_23="))
b_23=float(input("b_23="))
c_23=float(input("c_23="))
d_23=float(input("d_23="))
a_31=float(input("a_31="))
b_31=float(input("b_31="))
c_31=float(input("c_31="))
d_31=float(input("d_31="))
a_32=float(input("a_32="))
b_32=float(input("b_32="))
c_32=float(input("c_32="))
d_32=float(input("d_32="))
a_33=float(input("a_33="))
b_33=float(input("b_33="))
c_33=float(input("c_33="))
d_33=float(input("d_33="))
#creating variables for c_11
a_1=min(a_11,c_11)
a_2=max(b_11,d_11)
a_3=min(a_12,c_21)
a_4=max(b_12,d_21)
a_5=min(a_13,c_31)
a_6=max(b_13,d_31)
#creating cells
```

```
x_11=[max((max(a_1,a_3)),a_5),max(max(a_2,a_4),a_6)]
print("c_11=",x_11)
#creating variables for c_12
b_1=min(a_11,c_12)
b_2=max(b_11,d_12)
b_3=min(a_12,c_22)
b_4=max(b_12,d_22)
b_5=min(a_13,c_32)
b_6=max(b_13,d_32)
#creating cells
x_12=[max((max(b_1,b_3)),b_5),max(max(b_2,b_4),b_6)]
print("c_12=",x_12)
#creating variables for c_13
c_1=min(a_11,c_13)
c_2=max(b_11,d_13)
c_3=min(a_12,c_23)
c_4=max(b_12,d_23)
c_5=min(a_13,c_33)
c_6=max(b_13,d_33)
#creating cells
x_13=[max((max(c_1,c_3)),c_5),max(max(c_2,c_4),c_6)]
print("c_13=",x_13)
#creating variables for c_21
d_1=min(a_21,c_11)
d_2=max(b_21,d_11)
d_3=min(a_22,c_21)
d_4=max(b_22,d_21)
d_5=min(a_23,c_31)
d_6=max(b_23,d_31)
#creating cells
x_21=[max((max(d_1,d_3)),d_5),max(max(d_2,d_4),d_6)]
print("c_21=",x_21)
#creating variables for c_22
e_1=min(a_21,c_12)
```

```
e_2=max(b_21,d_12)
e_3=min(a_22,c_22)
e_4=max(b_22,d_22)
e_5=min(a_23,c_32)
e_6=max(b_23,d_32)
#creating cells
x_22=[max((max(e_1,e_3)),e_5),max(max(e_2,e_4),e_6)]
print("c_22=",x_22)
#creating variables for c_23
f_1=min(a_21,c_13)
f_2=max(b_21,d_13)
f_3=min(a_22,c_23)
f_4=max(b_22,d_23)
f_5=min(a_23,c_33)
f_6=max(b_23,d_33)
#creating cells
x_23=[max((max(f_1,f_3)),f_5),max(max(f_2,f_4),f_6)]
print("c_23=",x_23)
#creating variables for c_31
g_1=min(a_31,c_11)
g_2=max(b_31,d_11)
g_3=min(a_32,c_21)
g_4=max(b_32,d_21)
g_5=min(a_33,c_31)
g_6=max(b_33,d_31)
#creating cells
x_31=[max((max(g_1,g_3)),g_5),max(max(g_2,g_4),g_6)]
print("c_31=",x_31)
#creating variables for c_32
h_1=min(a_31,c_12)
h_2=max(b_31,d_12)
h_3=min(a_32,c_22)
h_4=max(b_32,d_22)
h_5=min(a_33,c_32)
```

```
h_6=max(b_33,d_32)
#creating cells
x_32=[max((max(h_1,h_3)),h_5),max(max(h_2,h_4),h_6)]
print("c_32=",x_32)
#creating variables for c_33
i_1=min(a_31,c_13)
i_2=max(b_31,d_13)
i_3=min(a_32,c_23)
i_4=max(b_32,d_23)
i_5=min(a_33,c_33)
i_6=max(b_33,d_33)
#creating cells
x_33=[max((max(i_1,i_3)),i_5),max(max(i_2,i_4),i_6)]
print("c_33=",x_33)
```

Output:

```
a_11=0.4
b_11=0.6
c_11=0.3
d_11=0.6
a_12=0.4
b_12=0.5
c_12=0.3
d_12=0.7
a_13=0.3
b_13=0.6
c_13=0.2
d_13=0.4
a_21=0.2
b_21=0.8
c_21=0.4
d_21=0.6
a_22=0.3
b_22=0.7
c_22=0.4
```

d_22=0.5

a_23=0

b_23=1

c_23=0.2

d_23=0.7

a_31=0.2

b_31=0.4

c_31=0.2

d_31=0.8

a_32=0.3

b_32=0.4

c_32=0.2

d_32=0.4

a_33=0.2

b_33=0.7

c_33=0.3

d_33=0.4

c_11= [0.4, 0.8]

c_12= [0.4, 0.7]

c_13= [0.3, 0.7]

c_21= [0.3, 1.0]

c_22= [0.3, 1.0]

c_23= [0.2, 1.0]

c_31= [0.3, 0.8]

c_32= [0.3, 0.7]

c_33= [0.2, 0.7]

Theorem 3.2: If $\overline{A_E}$ and $\overline{B_F}$ are two intuitionistic fuzzy matrix set, then $\overline{A_E} \times_5 \overline{B_F}$ is also an intuitionistic fuzzy matrix set .

Proof: If $\overline{A_E} = \begin{pmatrix} [\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] & [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] & [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}] \\ [\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] & [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] & [\overline{\mu_{E_{23}}}, \overline{\lambda_{E_{23}}}] \\ [\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] & [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] & [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}] \end{pmatrix}$ and

$\overline{B_F} = \begin{pmatrix} [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] & [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] & [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] \\ [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] & [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] & [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] \\ [\overline{\mu_{F_{31}}}, \overline{\lambda_{F_{31}}}] & [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}] & [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}] \end{pmatrix}$ are two intuitionistic fuzzy matrix sets. Thus $\overline{A_E} \times_5 \overline{B_F}$,

$$\overline{A_E} \times_5 \overline{B_F} = \begin{pmatrix} [\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] & [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] & [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}] \\ [\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] & [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] & [\overline{\mu_{E_{23}}}, \overline{\lambda_{E_{23}}}] \\ [\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] & [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] & [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}] \end{pmatrix} \times_5 \begin{pmatrix} [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] & [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] & [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] \\ [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] & [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] & [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] \\ [\overline{\mu_{F_{31}}}, \overline{\lambda_{F_{31}}}] & [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}] & [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}] \end{pmatrix}$$

$\overline{A_E} \times_5 \overline{B_F} = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$ Where,

$$X_{11} = \left([\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] \quad [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] \quad [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}]\right) \times_5 \begin{pmatrix} [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] \\ [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] \\ [\overline{\mu_{F_{31}}}, \overline{\lambda_{F_{31}}}] \end{pmatrix}$$

$$X_{12} = \left([\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] \quad [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] \quad [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}]\right) \times_5 \begin{pmatrix} [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] \\ [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] \\ [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}] \end{pmatrix}$$

$$X_{13} = \left([\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] \quad [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] \quad [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}]\right) \times_5 \begin{pmatrix} [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] \\ [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] \\ [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}] \end{pmatrix}$$

$$X_{21} = \left([\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] \quad [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] \quad [\overline{\mu_{E_{23}}}, \overline{\lambda_{E_{23}}}]\right) \times_5 \begin{pmatrix} [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] \\ [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] \\ [\overline{\mu_{F_{31}}}, \overline{\lambda_{F_{31}}}] \end{pmatrix}$$

$$X_{22} = \left([\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] \quad [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] \quad [\overline{\mu_{E_{23}}}, \overline{\lambda_{E_{23}}}]\right) \times_5 \begin{pmatrix} [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] \\ [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] \\ [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}] \end{pmatrix}$$

$$X_{23} = \left([\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] \quad [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] \quad [\overline{\mu_{E_{23}}}, \overline{\lambda_{E_{23}}}]\right) \times_5 \begin{pmatrix} [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] \\ [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] \\ [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}] \end{pmatrix}$$

$$X_{31} = \left([\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \quad [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \quad [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}]\right) \times_5 \begin{pmatrix} [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] \\ [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] \\ [\overline{\mu_{F_{31}}}, \overline{\lambda_{F_{31}}}] \end{pmatrix}$$

$$X_{32} = \left([\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \quad [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \quad [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}]\right) \times_5 \begin{pmatrix} [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] \\ [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] \\ [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}] \end{pmatrix}$$

$$X_{33} = \left([\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \quad [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \quad [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}]\right) \times_5 \begin{pmatrix} [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] \\ [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] \\ [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}] \end{pmatrix}$$

$$X_{11} = [\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] \times_5 [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] + [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] \times_5 [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] + [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}] \times_5 [\overline{\mu_{F_{31}}}, \overline{\lambda_{F_{31}}}]$$

$$X_{12} = [\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] \times_5 [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] + [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] \times_5 [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] + [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}] \times_5 [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}]$$

$$X_{13} = [\overline{\mu_{E_{11}}}, \overline{\lambda_{E_{11}}}] \times_5 [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] + [\overline{\mu_{E_{12}}}, \overline{\lambda_{E_{12}}}] \times_5 [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] + [\overline{\mu_{E_{13}}}, \overline{\lambda_{E_{13}}}] \times_5 [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}]$$

$$X_{21} = [\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] \times_5 [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] + [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] \times_5 [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] + [\overline{\mu_{E_{23}}}, \overline{\lambda_{E_{23}}}] \times_5 \left[\overline{\mu_{F_{31},\overline{\lambda_{F_{31}}}}}\right]$$

$$X_{22} = [\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] \times_5 [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] + [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] \times_5 [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] + [\overline{\mu_{E_{23}}}, \overline{\lambda_{E_{23}}}] \times_5 [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}]$$

$$X_{23} = [\overline{\mu_{E_{21}}}, \overline{\lambda_{E_{21}}}] \times_5 [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] + [\overline{\mu_{E_{22}}}, \overline{\lambda_{E_{22}}}] \times_5 [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] + [\overline{\mu_{23}}, \overline{\lambda_{E_{23}}}] \times_5 [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}]$$

$$X_{31} = [\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \times_5 [\overline{\mu_{F_{11}}}, \overline{\lambda_{F_{11}}}] + [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \times_5 [\overline{\mu_{F_{21}}}, \overline{\lambda_{F_{21}}}] + [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}] \times_5 [\overline{\mu_{F_{31}}}, \overline{\lambda_{F_{31}}}]$$

$$X_{32} = [\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \times_5 [\overline{\mu_{F_{12}}}, \overline{\lambda_{F_{12}}}] + [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \times_5 [\overline{\mu_{F_{22}}}, \overline{\lambda_{F_{22}}}] + [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}] \times_5 [\overline{\mu_{F_{32}}}, \overline{\lambda_{F_{32}}}]$$

$$X_{33} = [\overline{\mu_{E_{31}}}, \overline{\lambda_{E_{31}}}] \times_5 [\overline{\mu_{F_{13}}}, \overline{\lambda_{F_{13}}}] + [\overline{\mu_{E_{32}}}, \overline{\lambda_{E_{32}}}] \times_5 [\overline{\mu_{F_{23}}}, \overline{\lambda_{F_{23}}}] + [\overline{\mu_{E_{33}}}, \overline{\lambda_{E_{33}}}] \times_5 [\overline{\mu_{F_{33}}}, \overline{\lambda_{F_{33}}}]$$

Now, by applying this A X_5B = $\{((x,y)\ ,\ max(\mu_A(x), \mu_B(y))\ ,\ min(\lambda_A(x), \lambda_B(y))) \colon x\epsilon E_1, and\ y\epsilon E_2\}$.

$$X_{11} = [max(\overline{\mu_{E_{11}}}, \overline{\mu_{F_{11}}}), min(\overline{\lambda_{E_{11}}}, \overline{\lambda_{F_{11}}})] + [max(\overline{\mu_{E_{12}}}, \overline{\mu_{F_{21}}}), min(\overline{\lambda_{E_{12}}}, \overline{\lambda_{F_{21}}})] + [max(\overline{\mu_{E_{13}}}, \overline{\mu_{F_{31}}}), min(\overline{\lambda_{E_{13}}}, \overline{\lambda_{F_{31}}})]$$

$$X_{12} = [max(\overline{\mu_{E_{11}}}, \overline{\mu_{F_{12}}}), min(\overline{\lambda_{E_{11}}}, \overline{\lambda_{F_{12}}})] + [max(\overline{\mu_{E_{12}}}, \overline{\mu_{F_{22}}}), min(\overline{\lambda_{E_{12}}}, \overline{\lambda_{F_{22}}})] + [max(\overline{\mu_{E_{13}}}, \overline{\mu_{F_{32}}}), min(\overline{\lambda_{E_{13}}}, \overline{\lambda_{F_{32}}})]$$

$$X_{13} = [max(\overline{\mu_{E_{11}}}, \overline{\mu_{F_{13}}}), min(\overline{\lambda_{E_{11}}}, \overline{\lambda_{F_{13}}})] + [max(\overline{\mu_{E_{12}}}, \overline{\mu_{F_{23}}}), min(\overline{\lambda_{E_{12}}}, \overline{\lambda_{F_{23}}})] + [max(\overline{\mu_{E_{13}}}, \overline{\mu_{F_{33}}}), min(\overline{\lambda_{E_{13}}}, \overline{\lambda_{F_{33}}})]$$

$$X_{21} = [max(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{11}}}), min(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{11}}})] + [max(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{21}}}), min(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{21}}})] + [max(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{31}}}), min(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{31}}})]$$

$$X_{22} = [max(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{12}}}), min(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{12}}})] + [max(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{22}}}), min(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{22}}})] + [max(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{32}}}), min(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{32}}})]$$

$$X_{23} = [max(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{13}}}), min(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{13}}})] + [max(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{23}}}), min(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{23}}})] + [max(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{33}}}), min(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{33}}})]$$

$$X_{31} = [max(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{11}}}), min(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{11}}})] + [max(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{21}}}), min(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{21}}})] + [max(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{31}}}), min(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{31}}})]$$

$$X_{32} = [max(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{12}}}), min(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{12}}})] + [max(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{22}}}), min(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{22}}})] + [max(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{32}}}), min(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{32}}})]$$

$$X_{33} = [max(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{13}}}), min(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{13}}})] + [max(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{23}}}), min(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{23}}})] + [max(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{33}}}), min(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{33}}})]$$

By using, $\mathrm{A}+\mathrm{B}+C=max(\mathrm{A},\mathrm{B})+C$

$X_{11}=max\left[\left[max\left(\overline{\mu_{E_{11}}},\overline{\mu_{F_{11}}}\right),min\left(\overline{\lambda_{E_{11}}},\overline{\lambda_{F_{11}}}\right)\right],\left[max\left(\overline{\mu_{E_{12}}},\overline{\mu_{F_{21}}}\right),min\left(\overline{\lambda_{E_{12}}},\overline{\lambda_{F_{21}}}\right)\right]\right]+$
$\left[max\left(\overline{\mu_{E_{13}}},\overline{\mu_{F_{31}}}\right),min\left(\overline{\lambda_{E_{13}}},\overline{\lambda_{F_{31}}}\right)\right]$

$X_{12}=max\left[\left[max\left(\overline{\mu_{E_{11}}},\overline{\mu_{F_{12}}}\right),min\left(\overline{\lambda_{E_{11}}},\overline{\lambda_{F_{12}}}\right)\right],\left[max\left(\overline{\mu_{E_{12}}},\overline{\mu_{F_{22}}}\right),min\left(\overline{\lambda_{E_{12}}},\overline{\lambda_{F_{22}}}\right)\right]\right]+$
$\left[max\left(\overline{\mu_{E_{13}}},\overline{\mu_{F_{32}}}\right),min\left(\overline{\lambda_{E_{13}}},\overline{\lambda_{F_{32}}}\right)\right]$

$X_{13}=max\left[\left[max\left(\overline{\mu_{E_{11}}},\overline{\mu_{F_{13}}}\right),min\left(\overline{\lambda_{E_{11}}},\overline{\lambda_{F_{13}}}\right)\right],\left[max\left(\overline{\mu_{E_{12}}},\overline{\mu_{F_{23}}}\right),min\left(\overline{\lambda_{E_{12}}},\overline{\lambda_{F_{23}}}\right)\right]\right]$
$+\left[max\left(\overline{\mu_{E_{13}}},\overline{\mu_{F_{33}}}\right),min\left(\overline{\lambda_{E_{13}}},\overline{\lambda_{F_{33}}}\right)\right]$

$X_{21}=max\left[\left[max\left(\overline{\mu_{E_{21}}},\overline{\mu_{F_{11}}}\right),min\left(\overline{\lambda_{E_{21}}},\overline{\lambda_{F_{11}}}\right)\right],\left[max\left(\overline{\mu_{E_{22}}},\overline{\mu_{F_{21}}}\right),min\left(\overline{\lambda_{E_{22}}},\overline{\lambda_{F_{21}}}\right)\right]\right]$
$+\left[max\left(\overline{\mu_{E_{23}}},\overline{\mu_{F_{31}}}\right),min\left(\overline{\lambda_{E_{23}}},\overline{\lambda_{F_{31}}}\right)\right]$

$X_{22}=max\left[\left[max\left(\overline{\mu_{E_{21}}},\overline{\mu_{F_{12}}}\right),min\left(\overline{\lambda_{E_{21}}},\overline{\lambda_{F_{12}}}\right)\right],\left[max\left(\overline{\mu_{E_{22}}},\overline{\mu_{F_{22}}}\right),min\left(\overline{\lambda_{E_{22}}},\overline{\lambda_{F_{22}}}\right)\right]\right]+$
$\left[max\left(\overline{\mu_{E_{23}}},\overline{\mu_{F_{32}}}\right),min\left(\overline{\lambda_{E_{23}}},\overline{\lambda_{F_{32}}}\right)\right]$

$X_{23}=max\left[\left[max\left(\overline{\mu_{E_{21}}},\overline{\mu_{F_{13}}}\right),min\left(\overline{\lambda_{E_{21}}},\overline{\lambda_{F_{13}}}\right)\right],\left[max\left(\overline{\mu_{E_{22}}},\overline{\mu_{F_{23}}}\right),min\left(\overline{\lambda_{E_{22}}},\overline{\lambda_{F_{23}}}\right)\right]\right]$
$+\left[max\left(\overline{\mu_{E_{23}}},\overline{\mu_{F_{33}}}\right),min\left(\overline{\lambda_{E_{23}}},\overline{\lambda_{F_{33}}}\right)\right]$

$X_{31}=max\left[\left[max\left(\overline{\mu_{E_{31}}},\overline{\mu_{F_{11}}}\right),min\left(\overline{\lambda_{E_{31}}},\overline{\lambda_{F_{11}}}\right)\right],\left[max\left(\overline{\mu_{E_{32}}},\overline{\mu_{F_{21}}}\right),min\left(\overline{\lambda_{E_{32}}},\overline{\lambda_{F_{21}}}\right)\right]\right]$
$+\left[max\left(\overline{\mu_{E_{33}}},\overline{\mu_{F_{31}}}\right),min\left(\overline{\lambda_{E_{33}}},\overline{\lambda_{F_{31}}}\right)\right]$

$X_{32}=max\left[\left[max\left(\overline{\mu_{E_{31}}},\overline{\mu_{F_{12}}}\right),min\left(\overline{\lambda_{E_{31}}},\overline{\lambda_{F_{12}}}\right)\right],\left[max\left(\overline{\mu_{E_{32}}},\overline{\mu_{F_{22}}}\right),min\left(\overline{\lambda_{E_{32}}},\overline{\lambda_{F_{22}}}\right)\right]\right]+$
$\left[max\left(\overline{\mu_{E_{33}}},\overline{\mu_{F_{32}}}\right),min\left(\overline{\lambda_{E_{33}}},\overline{\lambda_{F_{32}}}\right)\right]$

$X_{33}=max\left[\left[max\left(\overline{\mu_{E_{31}}},\overline{\mu_{F_{13}}}\right),min\left(\overline{\lambda_{E_{31}}},\overline{\lambda_{F_{13}}}\right)\right],\left[max\left(\overline{\mu_{E_{32}}},\overline{\mu_{F_{23}}}\right),min\left(\overline{\lambda_{E_{32}}},\overline{\lambda_{F_{23}}}\right)\right]\right]+$
$\left[max\left(\overline{\mu_{E_{33}}},\overline{\mu_{F_{33}}}\right),min\left(\overline{\lambda_{E_{33}}},\overline{\lambda_{F_{33}}}\right)\right]$

By using, $\mathrm{A}+\mathrm{B}+C=max\{max(\mathrm{A},\mathrm{B}),C\}$

$$X_{11}=max\left\{\begin{matrix}max\left[\left[max\left(\overline{\mu_{E_{11}}},\overline{\mu_{F_{11}}}\right),min\left(\overline{\lambda_{E_{11}}},\overline{\lambda_{F_{11}}}\right)\right],\left[max\left(\overline{\mu_{E_{12}}},\overline{\mu_{F_{21}}}\right),min\left(\overline{\lambda_{E_{12}}},\overline{\lambda_{F_{21}}}\right)\right]\right],\\ \left[max\left(\overline{\mu_{E_{13}}},\overline{\mu_{F_{31}}}\right),min\left(\overline{\lambda_{E_{13}}},\overline{\lambda_{F_{31}}}\right)\right]\end{matrix}\right\}$$

$$X_{12}=max\left\{\begin{matrix}max\left[\left[max\left(\overline{\mu_{E_{11}}},\overline{\mu_{F_{12}}}\right),min\left(\overline{\lambda_{E_{11}}},\overline{\lambda_{F_{12}}}\right)\right],\left[max\left(\overline{\mu_{E_{12}}},\overline{\mu_{F_{22}}}\right),min\left(\overline{\lambda_{E_{12}}},\overline{\lambda_{F_{22}}}\right)\right]\right],\\ \left[max\left(\overline{\mu_{E_{13}}},\overline{\mu_{F_{32}}}\right),min\left(\overline{\lambda_{E_{13}}},\overline{\lambda_{F_{32}}}\right)\right]\end{matrix}\right\}$$

$$X_{13}=max\left\{max\left[\left[max\left(\overline{\mu_{E_{11}}},\overline{\mu_{F_{13}}}\right),min\left(\overline{\lambda_{E_{11}}},\overline{\lambda_{F_{13}}}\right)\right],\left[max\left(\overline{\mu_{E_{12}}},\overline{\mu_{F_{23}}}\right),min\left(\overline{\lambda_{E_{12}}},\overline{\lambda_{F_{23}}}\right)\right]\right]\right\},$$
$$\left[max\left(\overline{\mu_{E_{13}}},\overline{\mu_{F_{33}}}\right),min\left(\overline{\lambda_{E_{13}}},\overline{\lambda_{F_{33}}}\right)\right]$$

$$X_{21}=max\left\{\begin{matrix}max\left[\left[max\left(\overline{\mu_{E_{21}}},\overline{\mu_{F_{11}}}\right),min\left(\overline{\lambda_{E_{21}}},\overline{\lambda_{F_{11}}}\right)\right],\left[max\left(\overline{\mu_{E_{22}}},\overline{\mu_{F_{21}}}\right),min\left(\overline{\lambda_{E_{22}}},\overline{\lambda_{F_{21}}}\right)\right]\right],\\ \left[max\left(\overline{\mu_{E_{23}}},\overline{\mu_{F_{31}}}\right),min\left(\overline{\lambda_{E_{23}}},\overline{\lambda_{F_{31}}}\right)\right]\end{matrix}\right\}$$

$$X_{22} = max\left\{\begin{matrix} max\left[\left[max\left(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{12}}}\right), min\left(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{12}}}\right)\right], \left[max\left(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{22}}}\right), min\left(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{22}}}\right)\right]\right], \\ \left[max\left(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{32}}}\right), min\left(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{32}}}\right)\right] \end{matrix}\right\}$$

$$X_{23} = max\left\{\begin{matrix} max\left[\left[max\left(\overline{\mu_{E_{21}}}, \overline{\mu_{F_{13}}}\right), min\left(\overline{\lambda_{E_{21}}}, \overline{\lambda_{F_{13}}}\right)\right], \left[max\left(\overline{\mu_{E_{22}}}, \overline{\mu_{F_{23}}}\right), min\left(\overline{\lambda_{E_{22}}}, \overline{\lambda_{F_{23}}}\right)\right]\right], \\ \left[max\left(\overline{\mu_{E_{23}}}, \overline{\mu_{F_{33}}}\right), min\left(\overline{\lambda_{E_{23}}}, \overline{\lambda_{F_{33}}}\right)\right] \end{matrix}\right\}$$

$$X_{31} = max\left\{\begin{matrix} max\left[\left[max\left(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{11}}}\right), min\left(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{11}}}\right)\right], \left[max\left(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{21}}}\right), min\left(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{21}}}\right)\right]\right], \\ \left[max\left(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{31}}}\right), min\left(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{31}}}\right)\right] \end{matrix}\right\}$$

$$X_{32} = max\left\{\begin{matrix} max\left[\left[max\left(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{12}}}\right), min\left(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{12}}}\right)\right], \left[max\left(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{22}}}\right), min\left(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{22}}}\right)\right]\right], \\ \left[max\left(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{32}}}\right), min\left(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{32}}}\right)\right] \end{matrix}\right\}$$

$$X_{33} = max\left\{\begin{matrix} max\left[\left[max\left(\overline{\mu_{E_{31}}}, \overline{\mu_{F_{13}}}\right), min\left(\overline{\lambda_{E_{31}}}, \overline{\lambda_{F_{13}}}\right)\right], \left[max\left(\overline{\mu_{E_{32}}}, \overline{\mu_{F_{23}}}\right), min\left(\overline{\lambda_{E_{32}}}, \overline{\lambda_{F_{23}}}\right)\right]\right], \\ \left[max\left(\overline{\mu_{E_{33}}}, \overline{\mu_{F_{33}}}\right), min\left(\overline{\lambda_{E_{33}}}, \overline{\lambda_{F_{33}}}\right)\right] \end{matrix}\right\}$$

$\overline{A_E} \times_5 \overline{B_F} = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix}$. Hence, $\overline{A_E} \times_5 \overline{B_F}$ is an intuitionistic fuzzy matrix set..

Python program for $\overline{A_E} \times_5 \overline{B_F}$

```
#input the values
a_11=float(input("a_11="))
b_11=float(input("b_11="))
c_11=float(input("c_11="))
d_11=float(input("d_11="))
a_12=float(input("a_12="))
b_12=float(input("b_12="))
c_12=float(input("c_12="))
d_12=float(input("d_12="))
a_13=float(input("a_13="))
b_13=float(input("b_13="))
c_13=float(input("c_13="))
d_13=float(input("d_13="))
a_21=float(input("a_21="))
b_21=float(input("b_21="))
c_21=float(input("c_21="))
d_21=float(input("d_21="))
a_22=float(input("a_22="))
b_22=float(input("b_22="))
c_22=float(input("c_22="))
```

```
d_22=float(input("d_22="))
a_23=float(input("a_23="))
b_23=float(input("b_23="))
c_23=float(input("c_23="))
d_23=float(input("d_23="))
a_31=float(input("a_31="))
b_31=float(input("b_31="))
c_31=float(input("c_31="))
d_31=float(input("d_31="))
a_32=float(input("a_32="))
b_32=float(input("b_32="))
c_32=float(input("c_32="))
d_32=float(input("d_32="))
a_33=float(input("a_33="))
b_33=float(input("b_33="))
c_33=float(input("c_33="))
d_33=float(input("d_33="))
#creating variables for c_11
a_1=2*((a_11*c_11)/(a_11+c_11))
a_2=2*((b_11*d_11)/(b_11+d_11))
a_3=2*((a_12*c_21)/(a_12+c_21))
a_4=2*((b_12*d_21)/(b_12+d_21))
a_5=2*((a_13*c_31)/(a_13+c_31))
a_6=2*((b_13*d_31)/(b_13+d_31))
#creating cells
x_11=[max((max(a_1, a_3)),a_5),max(max(a_2,a_4),a_6)]
print("c_11=", x_11)
#creating variables for c_12
b_1=2*((a_11*c_12)/(a_11+c_12))
b_2=2*((b_11*d_12)/(b_11+d_12))
b_3=2*((a_12*c_22)/(a_12+c_22))
b_4=2*((b_12*d_22)/(b_12+d_22))
b_5=2*((a_13*c_32)/(a_13+c_32))
b_6=2*((b_13*d_32)/(b_13+d_32))
```

```
#creating cells
x_12=[max((max(b_1,b_3)),b_5),max(max(b_2,b_4),b_6)]
print("c_12=",x_12)
#creating variables for c_13
c_1=2*((a_11*c_13)/(a_11+c_13))
c_2=2*((b_11*d_13)/(b_11+d_13))
c_3=2*((a_12*c_23)/(a_12+c_23))
c_4=2*((b_12*d_23)/(b_12+d_23))
c_5=2*((a_13*c_33)/(a_13+c_33))
c_6=2*((b_13*d_33)/(b_13+d_33))
#creating cells
x_13=[max((max(c_1,c_3)),c_5),max(max(c_2,c_4),c_6)]
print("c_13=",x_13)
#creating variables for c_21
d_1=2*((a_21*c_11)/(a_21+c_11))
d_2=2*((b_21*d_11)/(b_21+d_11))
d_3=2*((a_22*c_21)/(a_22+c_21))
d_4=2*((b_22*d_21)/(b_22+d_21))
d_5=2*((a_23*c_31)/(a_23+c_31))
d_6=2*((b_23*d_31)/(b_23+d_31))
#creating cells
x_21=[max((max(d_1,d_3)),d_5),max(max(d_2,d_4),d_6)]
print("c_21=",x_21)
#creating variables for c_22
e_1=2*((a_21*c_12)/(a_21+c_12))
e_2=2*((b_21*d_12)/(b_21+d_12))
e_3=2*((a_22*c_22)/(a_22+c_22))
e_4=2*((b_22*d_22)/(b_22+d_22))
e_5=2*((a_23*c_32)/(a_23+c_32))
e_6=2*((b_23*d_32)/(b_23+d_32))
#creating cells
x_22=[max((max(e_1,e_3)),e_5),max(max(e_2,e_4),e_6)]
print("c_22=",x_22)
#creating variables for c_23
```

```
f_1=2*((a_21*c_13)/(a_21+c_13))
f_2=2*((b_21*d_13)/(b_21+d_13))
f_3=2*((a_22*c_23)/(a_22+c_23))
f_4=2*((b_22*d_23)/(b_22+d_23))
f_5=2*((a_23*c_33)/(a_23+c_33))
f_6=2*((b_23*d_33)/(b_23+d_33))
#creating cells
x_23=[max((max(f_1,f_3)),f_5),max(max(f_2,f_4),f_6)]
print("c_23=",x_23)
#creating variables for c_31
g_1=2*((a_31*c_11)/(a_31+c_11))
g_2=2*((b_31*d_11)/(b_31+d_11))
g_3=2*((a_32*c_21)/(a_32+c_21))
g_4=2*((b_32*d_21)/(b_32+d_21))
g_5=2*((a_33*c_31)/(a_33+c_31))
g_6=2*((b_33*d_31)/(b_33+d_31))
#creating cells
x_31=[max((max(g_1,g_3)),g_5),max(max(g_2,g_4),g_6)]
print("c_31=",x_31)
#creating variables for c_32
h_1=2*((a_31*c_12)/(a_31+c_12))
h_2=2*((b_31*d_12)/(b_31+d_12))
h_3=2*((a_32*c_22)/(a_32+c_22))
h_4=2*((b_32*d_22)/(b_32+d_22))
h_5=2*((a_33*c_32)/(a_33+c_32))
h_6=2*((b_33*d_32)/(b_33+d_32))
#creating cells
x_32=[max((max(h_1,h_3)),h_5),max(max(h_2,h_4),h_6)]
print("c_32=",x_32)
#creating variables for c_33
i_1=2*((a_31*c_13)/(a_31+c_13))
i_2=2*((b_31*d_13)/(b_31+d_13))
i_3=2*((a_32*c_23)/(a_32+c_23))
i_4=2*((b_32*d_23)/(b_32+d_23))
```

```
i_5=2*((a_33*c_33)/(a_33+c_33))
i_6=2*((b_33*d_33)/(b_33+d_33))
#creating cells
x_33=[max((max(i_1,i_3)),i_5),max(max(i_2,i_4),i_6)]
print("c_33=",x_33)
```

Output:

```
a_11=0.4
b_11=0.6
c_11=0.3
d_11=0.6
a_12=0.4
b_12=0.5
c_12=0.3
d_12=0.7
a_13=0.3
b_13=0.6
c_13=0.2
d_13=0.4
a_21=0.2
b_21=0.8
c_21=0.4
d_21=0.6
a_22=0.3
b_22=0.7
c_22=0.4
d_22=0.5
a_23=0
b_23=1
c_23=0.2
d_23=0.7
a_31=0.2
b_31=0.4
c_31=0.2
d_31=0.8
```

a_32=0.3

b_32=0.4

c_32=0.2

d_32=0.4

a_33=0.2

b_33=0.7

c_33=0.3

d_33=0.4

c_11= [0.4, 0.6]

c_12= [0.4, 0.6]

c_13= [0.4, 0.5]

c_21= [0.4, 0.8]

c_22= [0.4, 0.7]

c_23= [0.3, 0.7]

c_31= [0.4, 0.7]

c_32= [0.4, 0.4]

c_33= [0.3, 0.4]

Conclusion: In this paper, we represent an intuitionistic fuzzy matrix set (IFMs) as the Cartesian product of its membership and non-membership matrices. We introduce "$\times_1$","$\times_2$","$\times_3$","$\times_4$","$\times_5$" of Cartesian product over intuitionistic fuzzy matrix sets. A new intuitionistic fuzzy matrix sets can be generated by the use of the Cartesian product of two intuitionistic fuzzy matrix sets.

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QUADRIPARTITIONED NEUTROSOPHIC TOPOLOGICAL GROUP ON d-ALGEBRAS

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Abstract

In this paper we derive the notion of quadripartitioned neutrosophic topological group on d-algebra and using d-algebra as a tool we present the features of quadripartitioned neutrosophic BCK-ideal, d-algebra and quick ideals of d-algebra and its topological group structure.

Keywords : Fuzzy Neutrosophic set, Fuzzy Neutrosophic topological space, Quadripartitioned Neutrosophic set and Fuzzy Neutrosophic product space.

1 Introduction

The concept of neutrosophic set was introduced by Smarnandache [28, 29]. The traditional neutrosophic sets is characterized by the truth value, indeterminate value and false value. Neutrosophic set is a mathematically tool for handling problems involving imprecise, indeterminacy inconsistent data and inconsistent information which exits in belief system. The concept of neutrosophic set which overcomes the inherent difficulties that existed in fuzzy sets and intuitionistic fuzzy sets.

2 Preliminary Notes

Definition 2.1. [1] A Fuzzy neutrosophic set A over the non-empty set X is said to be empty fuzzy neutrosophic set if $T_A(x) = 0$, $I_A(x) = 0, F_A(x) = 1, \forall x \in X$. It is denoted by 0_N.

A Fuzzy neutrosophic set A over the non-empty set X is said to be universe fuzzy neutrosophic set if $T_A(x) = 1$, $I_A(x) = 1, F_A(x) = 0, \forall x \in X$. It is de- noted by 1_N.

3. QUADRIPARTITIONED NEUTROSOPHIC TOPOLOGICAL GROUP ON d-ALGEBRAS

In this section we derive the notion of quadripartitioned neutrosophic sets on d-algebra and using d-algebra as a tool we present the featured of quadripartitioned neutrosophic BCK-ideal, d-ideal and quick ideals of d-algebra and its topological group structures.

Definition 3.1. Let X be a d-algebra. An quadripartitioned neutrosophic set D=<x, T_D, C_D, U_D, F_D > in X is called an quadripartitioned neutrosophic d-algebra if it satisfies $T_D(x*y) \geq \min(T_D(x), T_D(y)), C_D(x,y) \geq \min(C_D(x), C_D(y)), U_D(x*y) \leq \max(U_D(x), U_D(y))$ and $F_D(x*y) \leq \max(F_D(x), F_D(y))$ for all x,y ∈ X.

Example 3.2. consider a d-algebra X={0,a,b,c} with the following Cayley table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Let D=<x, T_D, C_D, U_D, F_D> be an quadripartitoned neutrosophic set in X defined by

$T_D(0) = T_D(a) = 0.8, T_D(b) = T_D(c) = 0.3$

$C_D(0) = C_D(a) = 0.75, C_D(b) = C_D(c) = 0.15$

$U_D(0) = U_D(a) = 0.03, U_D(b) = U_D(c) = 0.08$

$F_D(0) = F_D(a) = 0.03, F_D(b) = F_D(c) = 0.08$

Then D=<x, T_D, C_D, U_D, F_D> be an quadripartitoned neutrosophic d-algebra.

Example 3.3. Consider a d-algebra X={0,a,b,c} with the following Cayley table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Let D=<x, T_D, C_D, U_D, F_D> be an quadripartitoned neutrosophic set in X defined by

$T_D(0) = T_D(a) = T_D(c) = \alpha_1, T_D(b) = \alpha_2$

$C_D(0) = C_D(a) = C_D(c) = \beta_1, C_D(b) = \beta_2$

$U_D(0) = U_D(a) = U_D(c) = \gamma_1, U_D(b) = \gamma_2$

$F_D(0) = F_D(a) = F_D(c) = \delta_1 F_D(b) = \delta_2$

Where $\alpha_1 > \alpha_2$, $\beta_1 > \beta_2$, $\gamma_1 < \gamma_2$ and $\delta_1 < \delta_2$ and $\alpha_i + \beta_i + \gamma_i + \delta_i \in [0,4]$ for i=1,2. Then D=<x, T_D, C_D, U_D, F_D> be an quadripartitoned neutrosophic d-algebra.

Proposition 3.4. If a quadripartitioned neutrosophic set D= < x, T_D, C_D, U_D, F_D> in X a quadripartitioned neutrosophic d-algebra of X, then $T_D(0) \geq T_D(x), C_D(0) \geq C_D(x), U_D(0) \leq U_D(x)$ and $F_D(0) \leq F_D(x)$, for all x,y ∈ X.

Proof: Let x∈ X. Then $T_D(0) = T_D(x*y) \geq \min\left(T_D(x), T_D(x)\right) = T_D(x)$

$$C_D(0) = C_D(x*y) \geq \min\left(C_D(x), C_D(x)\right) = C_D(x),$$

$$U_D(0) = U_D(x*y) \leq \max\left(U_D(x), U_D(y)\right) = U_D(x)$$

$$F_D(0) = F_D(x*y) \leq \max(F_D(x), F_D(y)) = F_D(x)$$

Theorem 3.5. If $\{D_k/k \in K\}$ is an arbitrary family of quadripartitioned neutrosophic d-algebra of X, then $\cap D_k$ is a quadripartitioned neutrosophic d-algebra of X where $\cap D_k =< x, \wedge T_{D_k}, \wedge C_{D_k}, \vee U_{D_k}, \vee F_{D_k} >/x \in$ X.

Proof : Let x,y∈X. Then

$\wedge T_{D_k}(x * y) \geq \wedge (\min\{T_{D_k}(x), T_{D_k}(y)\} = (\min\{\wedge T_{D_k}(x), \wedge T_{D_k}(y)\}$
$\wedge C_{D_k}(x * y) \geq \wedge (\min\{C_{D_k}(x), C_{D_k}(y)\} = (\min\{\wedge C_{D_k}(x), \wedge C_{D_k}(y)\}$
$\vee U_{D_k}(x * y) \leq \vee (\max\{U_{D_k}(x), U_{D_k}(y)\} = (\max\{\vee U_{D_k}(x), \vee U_{D_k}(y)\}$
$\vee F_{D_k}(x * y) \leq \vee (\max\{F_{D_k}(x), F_{D_k}(y)\} = (\max\{\vee F_{D_k}(x), \vee F_{D_k}(y)\}$
Hence $\cap D_k = < x, \wedge T_{D_k}, \wedge C_{D_k}, \vee U_{D_k}, \vee F_{D_k} > / x \in X$ is a quadripartitiomed neutrosophic d-algebra of X.

Theorem 3.6. If a quadripartitioned neutrosophic set D= $< x, T_D, C_D, U_D, F_D>$ in X a quadripartitioned neutrosophic d-algebra of X, then the sets $X_T = \{x \in X/T_D(x) = T_D(0)\}$, $X_C = \{x \in X/C_D(x) = C_D(0)\}$, $X_U = \{x \in X/U_D(x) = U_D(0)\}$ and $X_F = \{x \in X/F_D(x) = F_D(0)\}$ are d-subalgebras of X.
Proof: Let x,y$\in X_T$. Then $T_D(x) = T_D(0) = T_D(y)$ and $T_D(x * y) \geq \min(T_D(x), T_D(y)) = T_D(0)$. By using the proposition 6.4.4, we have $T_D(x * y) = T_D(0)$ implies $x * y \in X_T$. similarly we can prove for X_C, X_U and X_F.

Definition 3.7. Let D=<x, T_D, C_D, U_D, F_D> be an quadripartitoned neutrosophic set in X and let $\alpha, \beta, \gamma, \delta \in [0,1]$. Then the set $L(T_D, \alpha) = \{x \in X/T_D(x) \geq \alpha\}$,
$M(C_D, \beta) = \{x \in X/C_D(x) \geq \beta\}, N(U_D, \gamma) = \{x \in X/U_D(x) \leq \gamma\}, P(F_D, \delta) = \{x \in X/F_D(x) \leq \delta\}$ are called T-level α −cut ,C-level β −cut,U-level γ −cut and F-level δ −cut respectively of D.

Theorem 3.8. If a quadripartitioned neutrosophic set D= $< x, T_D, C_D, U_D, F_D>$ in X a quadripartitioned neutrosophic d-algebra of X, then the T-level α −cut ,C-level β −cut,U-level γ −cut and F-level δ −cut are d-algebra of X for every $\alpha, \beta, \gamma, \delta \in [0,1]$
Proof: Let x,y$\in$ $L(T_D, \alpha)$. Then $T_D(x) \geq \alpha$ and$T_D(y) \geq \alpha$. It follows that $T_D(x * y) \geq \min(T_D(x), T_D(y)) = \alpha$ so that $x * y \in L(T_D, \alpha)$. Hence $L(T_D, \alpha)$ is ad-algebra of X. Similarly we can prove that $M(C_D, \beta)$, $N(U_D, \gamma)$ and $P(F_D, \delta)$ is a d- algebra of X.

Theorem 3.9. Let D=<x, T_D, C_D, U_D, F_D> be an quadripartitoned neutrosophic set in X such that sets $L(T_D, \alpha), M(C_D, \beta)$, $N(U_D, \gamma)$ and $P(F_D, \delta)$ are d-algebra of X. Then D $=< x, T_D, C_D, U_D, F_D>$ is a quadripartitoned neutrosophic d-algebra of X.
Proof: Assume that there exist $x_0, y_0 \in X$ such that $T_D(x_0 * y_0) < \min(T_D(x_0), T_D(y_0))$
Let $\alpha_0 = \frac{1}{2}[T_D(x_0 * y_0) + \min(T_D(x_0), T_D(y_0))]$ then
$T_D(x_0 * y_0) < \alpha_0 < \min(T_D(x_0), T_D(y_0))$ and so $x_0 * y_0 \notin U(T_D, \alpha_0)$ but $x_0 * y_0 \in U(T_D, \alpha_0)$. This is a contradiction and therefore $T_D(x * y) \geq \min(T_D(x), T_D(y))$. Similarly we prove $C_D(x * y) \geq \min(C_D(x), C_D(x))$. Now suppose that $F_D(x_0 * y_0) > max\,(F_D(x_0), F_D(y_0))$ Let $\delta_0 = \frac{1}{2}[F_D(x_0 * y_0) + \max(F_D(x_0), F_D(y_0))]$ then $\max(F_D(x_0), F_D(y_0)) < \delta_0 < F_D(x_0 * y_0)$ and so $x_0 * y_0 \notin U(F_D, \delta_0)$ but $x_0 * y_0 \in U(F_D, \delta_0)$. This is a contradiction and therefore $F_D(x * y) \leq \max(F_D(x), F_D(y))$ for all x,y$\in$X , Hence the proof.

Theorem 3.10. Any d-algebra of X can be realized as T-level α −cut ,C-level β −cut,U-level γ −cut and F-level δ −cut d-algebra of some quadripartitioned neutrosophic d-algebra of X.
Proof: Let S be a d-algebra of X. Let T_D, C_D, U_D and F_D in X are defined as $T_D(x) = \begin{cases} \alpha, & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}$, $C_D(x) = \begin{cases} \beta, & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}$, $U_D(x) = \begin{cases} \gamma, & \text{if } x \in S \\ 1, & \text{otherwise} \end{cases}$ and $F_D(x) = \begin{cases} \delta, & \text{if } x \in S \\ 1, & \text{otherwise} \end{cases}$, for all

$x \in X$ Where α, β, γ and δ are fixed numbers in (0,1) such that $\alpha + \beta + \gamma + \delta < 4$. Let x,y∈ X. If x,y∈ S , then $x * y \in S$. Hence $T_D(x * y) = T_D(x) = T_D(y) = \alpha$ and $T_D(x * y) \geq \min\{T_D(x), T_D(y)\}$,$C_D(x * y) = C_D(x) = C_D(y) = \beta$, $C_D(x * y) \geq \min\{C_D(x), C_D(y)\}$and $U_D(x * y) = U_D(x) = U_D(y) = \gamma$ and $U_D(x * y) \leq \min\{U_D(x), U_D(y)\}$, and $F_D(x * y) = F_D(x) = F_D(y) = \delta$ and $F_D(x * y) \leq \min\{F_D(x), F_D(y)\}$.
If x,y∉ S then $T_D(x) = T_D(y) = 0, C_D(x) = C_D(y) = 0, U_D(x) = U_D(y) = 0$ and $F_D(x) = F_D(y) = 0$. Then $T_D(x * y) \geq \min\{T_D(x), T_D(y)\} = 0$, $C_D(x * y) \geq \min\{C_D(x), C_D(y)\} = 0, U_D(x * y) \leq \max\{U_D(x), U_D(y)\} = 0$ and$F_D(x * y) \leq \max\{F_D(x), F_D(y)\} = 0$. If almost one of x,y∈ S then atleast one of $T_D(x)$ and $T_D(y)$ is equal to one. Therefore $T_D(x * y) \geq 0 = \min\{T_D(x), T_D(y)\}, C_D(x * y) \geq 0 = \min\{C_D(x), C_D(y)\}$,$U_D(x * y) \leq 1 = \max\{U_D(x), U_D(y)\}$ and $F_D(x * y) \leq 1 = \max\{F_D(x), F_D(y)\}$. hence D=<x, $T_D, C_D, U_{D,} F_D$> is a quadripartitioned neutrosophic d-algebra of X.

Theorem 3.11. Let f be a d-homomorphism of ad-algebra X into a d-algebra Y and D a quadripartitioned neutrosophic d-algebra of Y. Then $f^{-1}(D)$ is a quadripartitioned neutrosophic d-algebra of X.

Proof: For any x,y∈ X we have $f^{-1}(T_D(x * y) = T_D(f(x * y)) = T_D[f(X) * f(y)] \geq \min[T_D(f(x)), T_D f((y))] = min[f^{-1}(T_D(x)), f^{-1}(T_D(y))]$. Similarly we can show that
$f^{-1}(C_D(x * y) \geq min[f^{-1}(C_D(x)), f^{-1}(C_D(y))]$
$f^{-1}(U_D(x * y) \leq max\ [f^{-1}(U_D(x)), f^{-1}(U_D(y))]$ and
$f^{-1}(F_D(x * y) \leq max[f^{-1}(F_D(x)), f^{-1}(F_D(y))]$. Hence $f^{-1}(D)$ is a quadripartitioned neutrosophic d-algebra of X.

Definition 3.12. Let X be ad-algebra. A quadripartitioned neutrosophic set
D $=< x, T_D, C_D, U_{D,} F_D$> in X is called a quadripartiitoned neutrosophic BCK-ideal of X if it satisfies

i. For all $x \in X$, $T_D(0) \geq T_D(x), C_D(0) \geq C_D(x), U_D(0) \leq U_D(x), F_D(0) \leq F_D(x)$
ii. For all $x \in X$, $T_D(x) \geq \min\{T_D(x * y), T_D(y)\}$, $C_D(x) \geq \min\{C_D(x * y), C_D(y)\}, U_D(x) \leq max\ \{U_D(x * y), U_D(y)\}$ and $F_D(x) \leq max\ \{F_D(x * y), F_D(y)\}$.

A quadripartitioned neutrosophic set D $=< x, T_D, C_D, U_{D,} F_D$> in X is called a quadripartiitoned neutrosophic d-ideal of X if it satisfies (ii) and

iii. For all $x \in X$, $T_D(x * y) \geq T_D(x), C_D(x * y) \geq C_D(x), U_D(x * y) \leq U_D(x)$ and $F_D(x * y) \leq F_D(x)$.
A quadripartitioned neutrosophic set D $=< x, T_D, C_D, U_{D,} F_D$> in X is called a quadripartiitoned neutrosophic d-ideal of X if it satisfies (i) and
iv. For all $x \in X$, with $x * y \neq 0, \min\{T_D(x), T_D(y)\} \geq C_D(x * y), \min\{C_D(x), C_D(y)\} \geq C_D(x * y), \max\{U_D(x), U_D(y)\} \leq U_D(x * y)$ and $\max\{F_D(x), F_D(y)\} \leq F_D(x * y)$

Example 3.13. Let X={0,a,b,c,d} be a d-algebra which is not BCK-algebra with the following Cayley table.

∗	0	a	b	c	*d*
0	0	0	0	0	0
a	a	0	0	a	0
b	b	b	0	0	*b*
c	c	c	c	0	0
d	*d*	*c*	*c*	*a*	0

Let D $=< x, T_D, C_D, U_{D,} F_D$> be a quadripartitioned neutrosophic set in X defined by $T_D(c) = 0.03, T_D(x) = 0.5, C_D(c) = 0.12, C_D(x) = 0.6, U_D(c) = 0.4, U_D(x) = 0.22$ and $F_D(c) = 0.3, F_D(x) = 0.24$, for x≠c.

Then $D =< x, T_D, C_D, U_{D,} F_D >$ is a quadripartitioned neutrosophic BCK-ideal of X which is not a quadripatitioned neutrosophic d-ideal of X.
Since $T_D(d * c) = T_D(c) = 0.03 \leq T_D(d) = 0.5, C_D(d * c) = C_D(c) = 0.12 \leq T_D(d) = 0.6, U_D(d * c) = U(c) = 0.4 \geq U_D(d) = 0.22$ and $F_D(d * c) = F(c) = 0.3 \geq F_D(d) = 0.24$.

Example 3.15. Let X={0,a,b,c} be a d-algebra with the following Cayley table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	c
b	b	b	0	0
c	c	0	b	0

Let $D =< x, T_D, C_D, U_{D,} F_D >$ be a quadripartitioned neutrosophic set in X defined by $T_D(b) = 0.4, T_D(x) = 0.7, C_D(b) = 0.23, C_D(x) = 0.6, U_D(b) = 0.6, U_D(x) = 0.14$ *and* $F_D(b) = 0.2, F_D(x) = 0.12$, for x≠b. Then $D =< x, T_D, C_D, U_{D,} F_D >$ is a quadripartitioned neutrosophic quick -ideal of X which is not a quadripatitioned neutrosophic BCK-ideal of X.
Since $T_D(b) = 0.4 \leq \min\{T_D(b * c) = T_D(0), T_D(c)\} = 0.7, C_D(b) = 0.23 \leq \min\{C_D(b * c) = C_D(0), C_D(c)\} = 0.6$, $U_D(b) = 0.6 \geq \max\{U_D(b * c) = U_D(0), U_D(c)\} = 0.14$ and $F_D(b) = 0.2 \geq \max\{F_D(b * c) = F_D(0), F_D(c)\} = 0.12$. Also D is a quadripariititioned neutrosophic d-algebra.

Example 3.16. Let X={0,1,2,3} be ad-algebra with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	2
2	2	2	0	0
3	3	3	3	0

Let $D =< x, T_D, C_D, U_{D,} F_D >$ be a quadripartitioned neutrosophic set in X defined by $T_D(3) = 0.2, T_D(x) = 0.8, C_D(3) = 0.3, C_D(x) = 0.7, U_D(b) = 0.06, U_D(x) = 0.02$ *and* $F_D(b) = 0.08, F_D(x) = 0.01$, for x≠3. Then $D =< x, T_D, C_D, U_{D,} F_D >$ is a quadripartitioned neutrosophic BCK -ideal of X which is not a quadripatitioned neutrosophic quick-ideal of X.
Since $1 * 3 = 2 \neq 0$, $\min\{T_D(1), T_D(3)\} = 0.2 < T_D(1 * 3) = T_D(2) = 0.8$, $\min\{C_D(1), C_D(3)\} = 0.3 < C_D(1 * 3) = C_D(2) = 0.7, \max\{U_D(1), U_D(3)\} = 0.06 > U_D(1 * 3) = U_D(2) = 0.02$ and $\max\{F_D(1), F_D(3)\} = 0.08 > F_D(1 * 3) = F_D(2) = 0.01$.

Example 3.17. Let X={0,1,2,3} be ad-algebra with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	2
2	2	2	0	0
3	3	0	2	0

Let $D =< x, T_D, C_D, U_{D,} F_D >$ be a quadripartitioned neutrosophic set in X defined by $T_D(2) = 0.4, T_D(x) = 0.7, C_D(2) = 0.3, C_D(x) = 0.8, U_D(b) = 0.06, U_D(x) = 0.01$ *and* $F_D(b) = 0.07, F_D(x) = 0.02$, for x≠2. Then
$D =< x, T_D, C_D, U_{D,} F_D >$ is a quadripartitioned neutrosophic quick -ideal of X but not a quadripatitioned neutrosophic d-algebra of X.
Since $T_D(1 * 3) = T_D(2) = 0.4 < \min\{T_D(1), T_D(3)\} = 0.7$, $C_D(1 * 3) = C_D(2) = 0.3 < \min\{C_D(1), C_D(3)\} = 0.8, U_D(1 * 3) = U_D(2) = 0.01 > \max\{U_D(1), U_D(3)\} = 0.06$ and $F_D(1 * 3) =$

$\mathrm{F_D}(2) = 0.07 > \max\{\mathrm{F_D}(1), \mathrm{F_D}(3)\} = 0.02$. Also D $=< x, \mathrm{T_D}, \mathrm{C_D}, \mathrm{U_D}, \mathrm{F_D} >$ is not a quadripartitioned neutrosophic BCK-ideal of X. Since $\mathrm{T_D}(2) = 0.4 < \min\{\mathrm{T_D}(2*3) = \mathrm{T_D}(0), \mathrm{T_D}(3)\} = 0.7$, $\mathrm{C_D}(2) = 0.2 < \min\{\mathrm{C_D}(2*3) = \mathrm{C_D}(0), \mathrm{C_D}(3)\} = 0.8$, $\mathrm{U_D}(2) = 0.06 > \max\{\mathrm{U_D}(2*3) = \mathrm{U_D}(0), \mathrm{U_D}(3)\} = 0.01$ and $\mathrm{F_D}(2) = 0.07 > \max\{\mathrm{F_D}(2*3) = \mathrm{F_D}(0), \mathrm{F_D}(3)\} = 0.02$.

Theorem 3.18. Let D $=< x, \mathrm{T_D}, \mathrm{C_D}, \mathrm{U_D}, \mathrm{F_D} >$ by a quadripartitioned neutrosophic BCK -ideal of a d-algebra X. Then

i. $\mathrm{T_D}, \mathrm{C_D}$ are order reversing.
ii. $x * y \leq z$ implies $\mathrm{T_D}(x*y) \geq \mathrm{T_D}(z), \mathrm{C_D}(x*y) \geq \mathrm{C_D}(z), \mathrm{U_D}(x*y) \leq \mathrm{U_D}(z)$ and $\mathrm{F_D}(x*y) \leq \mathrm{F_D}(z)$ for all x,y,z∈X.

Proof:

i. Let x,y∈ X with $x \leq y$, then $x * y$=0. Hence $\mathrm{T_D}(x) \geq \min\{\mathrm{T_D}(x*y), \mathrm{T_D}(y)\} = min\{\mathrm{T_D}(0), \mathrm{T_D}(y)\} = \mathrm{T_D}(y)$. Thus $\mathrm{T_D}(x) \geq \mathrm{T_D}(y)$. Similarly we can prove that $\mathrm{C_D}$ is order reversing.
ii. Proof is obvious.

Theorem 3.19. Let D $=< x, \mathrm{T_D}, \mathrm{C_D}, \mathrm{U_D}, \mathrm{F_D} >$ by a quadripartitioned neutrosophic BCK -ideal of a d-algebra X. Then $x * y \leq z$ implies $\mathrm{T_D}(x) \geq \min\{\mathrm{T_D}(y), \mathrm{T_D}(z)\}, \mathrm{C_D}(x) \geq \min\{\mathrm{C_D}(y), \mathrm{C_D}(z)\}, \mathrm{U_D}(x) \leq \max\{\mathrm{U_D}(y), \mathrm{U_D}(z)\}$ and $\mathrm{F_D}(x) \leq \max\{\mathrm{F_D}(y), \mathrm{F_D}(z)\}$ for x,y,z∈X.

Proof: Let x,y,z∈X such that $x * y \leq z$. Then $(x * y) * z = 0$. Hence $\mathrm{T_D}(x*y) \geq \min\{\mathrm{T_D}((x*y)*z), \mathrm{T_D}(z)\} = min\{\mathrm{T_D}(0), \mathrm{T_D}(z)\} = \mathrm{T_D}(z)$. Therefore we have $\mathrm{T_D}(x) \geq min\{\mathrm{T_D}((x*y), \mathrm{T_D}(y)\} \geq \min\{\mathrm{T_D}(z), \mathrm{T_D}(y)\}$. Thus $\mathrm{T_D}(x) \geq \min\{\mathrm{T_D}(y), \mathrm{T_D}(z)\}$. in the similar way we obtain $\mathrm{C_D}(x) \geq min\{\mathrm{C_D}(y), \mathrm{C_D}(z)\}$,$\mathrm{U_D}(x) \leq \max\{\mathrm{U_D}(y), \mathrm{U_D}(z)\} and\ \mathrm{F_D}(x) \leq \max\{\mathrm{F_D}(y), \mathrm{F_D}(z)\}$.

Theorem 3.20. If $\{D_i / i \in \wedge\}$ be an arbitrary family of quadripartiitoned neutrosophic quick ideal of a d-algebra X, then $\cap D_i$ is a quadripartitioned neutrosophic quick ideal of X where $\cap D_i = \{< x, \wedge \mathrm{T}_{\mathrm{D}_i}(x), \wedge \mathrm{C}_{\mathrm{D}_i}(x), \wedge \mathrm{U}_{\mathrm{D}_i}(x), \wedge \mathrm{F}_{\mathrm{D}_i}(x) > / x \in X\}$.

Theorem 3.21. If a quadripatitioned neutrosophic set D $=< x, \mathrm{T_D}, \mathrm{C_D}, \mathrm{U_D}, \mathrm{F_D} >$ in X is a quadripartitioned neutrosophic quick ideal of a d-algebra X, then the T-level α −cut ,C-level β −cut,U-level γ −cut and F-level δ −cut of D are quick ideal of X for each $\alpha, \beta, \gamma, \delta \in [0,1]$

Theorem 3.22. Let D $=< x, \mathrm{T_D}, \mathrm{C_D}, \mathrm{U_D}, \mathrm{F_D} >$ be a quadripartitioned neutrosophic set in X such that the sets $L(\mathrm{T_D}, \alpha), M(\mathrm{C_D}, \beta), N(\mathrm{U_D}, \gamma)$ and $P(\mathrm{F_D}, \delta)$ are quick ideals of X. Then D $=< x, \mathrm{T_D}, \mathrm{C_D}, \mathrm{U_D}, \mathrm{F_D} >$ is a quadripartitioned neutrosophic quick ideal of X.

Theorem 3.23. Any quick ideal of a d-algebra X can be realized as T-level α −cut ,C-level β −cut,U-level γ −cut and F-level δ −cut d-algebras of some quadripartitioned neutrosophic quick ideal of X.

Proof: Let S be quick ideal of X. Let $\mathrm{T_D}, \mathrm{C_D}, \mathrm{U_D}\ and\ \mathrm{F_D}$ in X are defined as

$$\mathrm{T_D}(x) = \begin{cases} \alpha_1, & \text{if } x \in S \\ \alpha_2, & \text{otherwise} \end{cases}, \quad \mathrm{C_D}(x) = \begin{cases} \beta_1, & \text{if } x \in S \\ \beta_2, & \text{otherwise} \end{cases},$$

$\mathrm{U_D}(x) = \begin{cases} \gamma_1, & \text{if } x \in S \\ \gamma_2, & \text{otherwise} \end{cases}$, $\mathrm{F_D}(x) = \begin{cases} \delta_1, & \text{if } x \in S \\ \delta_2, & \text{otherwise} \end{cases}$, for all x $\in$ X where $\alpha_i + \beta_i + \gamma_i + \delta_i < 4$ for i=1,2.

Let $0 \in \mathrm{S}$, we have $\mathrm{T_D}(0) = \alpha_1, \mathrm{C_D}(0) = \beta_1, \mathrm{U_D}(0) = \gamma_1$ and $\mathrm{F_D}(0) = \delta_1$. Hence $\mathrm{T_D}(0) \geq \mathrm{T_D}(x), \mathrm{C_D}(0) \geq \mathrm{C_D}(x), \mathrm{U_D}(0) \leq \mathrm{U_D}(x)$ and $\mathrm{F_D}(0) \leq \mathrm{F_D}(x)$, for all x $\in$ X.

Let $\mathrm{x, y} \in \mathrm{X}$ with $\mathrm{x} * \mathrm{y} \neq 0$. If $\mathrm{x} * \mathrm{y} \in \mathrm{S}$ then $\mathrm{x, y} \in \mathrm{S}$.

Hence $\mathrm{T_D}(\mathrm{x} * \mathrm{y}) = min\{\mathrm{T_D}(x), \mathrm{T_D}(y)\}, \mathrm{C_D}(\mathrm{x} * \mathrm{y}) = min\{\mathrm{C_D}(x), \mathrm{C_D}(y)\}, \mathrm{U_D}(\mathrm{x} * \mathrm{y}) = max\{\mathrm{U_D}(x), \mathrm{U_D}(y)\}$ *and* $\mathrm{F_D}(\mathrm{x} * \mathrm{y}) = max\{\mathrm{F_D}(x), \mathrm{F_D}(y)\}$. If $\mathrm{x} * \mathrm{y} \notin \mathrm{S}$, then $\mathrm{T_D}(\mathrm{x} * \mathrm{y}) = \alpha_2, \mathrm{C_D}(\mathrm{x} * \mathrm{y}) = \beta_2, \mathrm{U_D}(\mathrm{x} * \mathrm{y}) = \gamma_2$ and $\mathrm{F_D}(\mathrm{x} * \mathrm{y}) = \delta_2$.

Hence $\alpha_2 = \mathrm{T_D}(\mathrm{x} * \mathrm{y}) \leq min\{\mathrm{T_D}(x), \mathrm{T_D}(y)\}, \beta_2 = \mathrm{C_D}(\mathrm{x} * \mathrm{y}) \leq min\{\mathrm{T_D}(x), \mathrm{T_D}(y)\}, \gamma_2 = \mathrm{U_D}(\mathrm{x} * \mathrm{y}) \geq max\{\mathrm{U_D}(x), \mathrm{U_D}(y)\}$ and $\delta_2 = \mathrm{F_D}(\mathrm{x} * \mathrm{y}) \geq max\{\mathrm{U_D}(x), \mathrm{U_D}(y)\}$. Hence D $=< x, \mathrm{T_D}, \mathrm{C_D}, \mathrm{U_D}, \mathrm{F_D} >$ is a quadripartitioned neutrosophic quick ideal of X.

Proposition 3.24. Let (D,τ_D) and (B, σ_D) be quadripartitioned neutrosophic subspace of quadripatitioned neutrosophic topological spaces (X,τ) and (Y,σ) respectively and let f be a quadripartitioned neutrosophic continuous mapping of X into Y such that f(D)$\subset$ B.

Proof: Let V_B be a quadripartitioned neutrosophic set in σ_B.then there exist V$\in \sigma$ such that $V_B = V \cap B$. since f is quadripartitioned neutrosophic continuous it follows that f^{-1}(V) is a quadripartitioned neutrosophic set in τ. Hence $f^{-1}(V_B) \cap D = f^{-1}(V \cap B) \cap D = f^{-1}(V) \cap f^{-1}(B) \cap D = f^{-1}(V) \cap D$ is a quadripartitioned neutrosophic set in τ_D. Hence the proof.

Note: for any d-algebra X and any element the right translation of X is defined by $R_a(x) = \mathrm{x} * \mathrm{a}$ for all $\mathrm{x} \in \mathrm{X}$. and$R_x(0) = 0 = R_x(x)$.

Definition 3.25. Let X be a d-algebra, τ is quadripartitioned neutrosophic topology on X and Y a quadripartitioned neutrosophic d-algebra with IFNT τ_D. Then D is called a quadripartitioned neutrosophic topological d- algebra if for each $a \in X$, the mapping $R_a: (D, \tau_D) \to (D, \tau_D)$ defined be $R_a(x) = \mathrm{x} * \mathrm{a}$ for all $\mathrm{x} \in \mathrm{X}$, is relatively quadripartitioned neutrosophic continuous.

Theorem 3.26. Given d-algebra X and Y and a d-homomorphism f:X→Y, let τ and σ be the quadripartitioned neutrosophic topologies on X and Y respectively such that $\tau = f^{-1}(\sigma)$. If B is a quadripartitioned neutrosophic topological d-algebra in Y, then $f^{-1}(B)$ is a quadripartitioned neutrsophic topological d-algebra in X.

Proof: Let $a \in X$ and let U be a quadripartitioned neutrosophic set in $\tau_{f^{-1}(B)}$. Since f is a quadripartitioned neutrosophic mapping of (X,τ) into (Y,σ), By proposition 3.24 f is relatively quadripartitioned neutrosophic continuous mapping of $(f^{-1}(B), \tau_{f^{-1}(B)})$ into (B, τ_B). Note that there exist a quadripartitioned neutrosophic set V in τ_B such that $f^{-1}(V) = U$. Then $\mathrm{T}_{R_a^{-1}(U)}(x) = \mathrm{T_U}(\mathrm{x} * \mathrm{a}) = \mathrm{T}_{f^{-1}(V)}(\mathrm{x} * \mathrm{a}) = \mathrm{T_V}(f(\mathrm{x} * \mathrm{a})) = \mathrm{T_V}(f(\mathrm{x}) * \mathrm{f(a)})$. Similarly we obtain $\mathrm{C}_{f^{-1}(V)}(\mathrm{x} * \mathrm{a}) = \mathrm{C_V}(f(\mathrm{x}) * \mathrm{f(a)})$, $\mathrm{U}_{f^{-1}(V)}(\mathrm{x} * \mathrm{a}) = \mathrm{U_V}(f(\mathrm{x}) * \mathrm{f(a)})$ and $\mathrm{F}_{f^{-1}(V)}(\mathrm{x} * \mathrm{a}) = \mathrm{F_V}(f(\mathrm{x}) * \mathrm{f(a)})$. Since B is a quadripartitioned neutrosophic topological d-algebra in Y, then the mapping $R_a = (B, \sigma_B) \to (B, \sigma_B)$ is relatively quadripartitioned neutrosophic continuous for each $b \in Y$. Hence $\mathrm{T}_{R_a^{-1}(U)}(x) = \mathrm{T_V}(f(\mathrm{x}) * \mathrm{f(a)}) = \mathrm{T_V}(R_{f(a)}(\mathrm{f(x)}) = \mathrm{T}_{R_{f(a)(V)}^{-1}}(f(x)) = \mathrm{T}_{f^{-1}(R_a^{-1})(V)}(x)$ similarly we get $\mathrm{C}_{R_a^{-1}(U)}(x) = \mathrm{C}_{f^{-1}(R_a^{-1})(V)}(x)$, $\mathrm{U}_{R_a^{-1}(U)}(x) = \mathrm{U}_{f^{-1}(R_a^{-1})(V)}(x)$and $\mathrm{F}_{R_a^{-1}(U)}(x) = \mathrm{F}_{f^{-1}(R_a^{-1})(V)}(x)$ which implies

${R_a}^{-1}(U) = f^{-1}({R_a}^{-1})(V))$ so that ${R_a}^{-1}(U) \cap f^{-1}(B) = f^{-1}[{R_{f(a)}}^{-1}(V)] \cap f^{-1}(B)$ is a quadripartitioned neutrosophic set in $\tau_{f^{-1}(B)}$. Hence the proof.

Theorem 3.27. Given d-algebras X and Y and a d-isomorphism f of X onto Y, Let τ and σ be the quadrioartitioned neutrosophic topologies on X and Y respectively such that f(τ) = σ. If D is a quadripartitioned neutrosophic topological d-algebra in X, then f(D) is a quadripartitioned neutrosophic topological d-algebra in Y.

Proof: Consider the mapping $R_b\colon (f(D), \sigma_{f(D)}) \to\colon (f(D), \sigma_{f(D)})$. Then we can prove that it is relatively quadripartitioned neutrosophic continuous for each $b \in Y$. Let $\mathrm{U_D}$ be a quadripartitioned neutrosophic set in τ_D, then there exist a quadripartitioned neutrosophic set U in τ such that $\mathrm{U_D} = U \cap D$. Since f is one-one it follows that f($\mathrm{U_D}$)=f($U \cap D$)=f(U) $\cap$ f(D) which is a quadripartitioned neutrosophic set in $\sigma_{f(D)}$. This shows that f is relatively quadripartitioned neutrosophic open. Let $\mathrm{V}_{f(D)}$ be a quadripartitioned neutrosophic set in $\sigma_{f(D)}$. Since f is onto, for each $b \in Y$ there exist $a \in X$ such that b = f(a). Hence $\mathrm{T}_{f^{-1}({R_b}^{-1}(\mathrm{V}_{f(D)}))}(x) = \mathrm{T}_{f^{-1}({R_{f(a)}}^{-1}(\mathrm{V}_{f(D)}))}(x) = \mathrm{T}_{{R_{f(a)}}^{-1}\mathrm{V}_{f(D)}}(f(x)) = \mathrm{T}_{\mathrm{V}_{f(D)}}(R_{f(a)}f(x)) =$ $\mathrm{T}_{\mathrm{V}_{f(D)}}(f(x) * \mathrm{f(a)}) = \mathrm{T}_{\mathrm{V}_{f(D)}}(f(x * \mathrm{a})) = \mathrm{T}_{f^{-1}(\mathrm{V}_{f(D)})}(x * \mathrm{a}) = \mathrm{T}_{f^{-1}(\mathrm{V}_{f(D)})}(R_a(x)) =$ $\mathrm{T}_{{R_a}^{-1}(f^{-1}(\mathrm{V}_{f(D)}))}(x)$. similarly we prove that $\mathrm{C}_{f^{-1}({R_b}^{-1}(\mathrm{V}_{f(D)}))}(x) = \mathrm{C}_{{R_a}^{-1}(f^{-1}(\mathrm{V}_{f(D)}))}(x)$, $\mathrm{U}_{f^{-1}({R_b}^{-1}(\mathrm{V}_{f(D)}))}(x) = \mathrm{U}_{{R_a}^{-1}(f^{-1}(\mathrm{V}_{f(D)}))}(x)$ and $\mathrm{F}_{f^{-1}({R_b}^{-1}(\mathrm{V}_{f(D)}))}(x)= \mathrm{F}_{{R_a}^{-1}(f^{-1}(\mathrm{V}_{f(D)}))}(x)$. By hypothesis R_a is a relatively quadripartitioned neutrosophic continuous mapping from $(D, \tau_D)\ to\ (D, \tau_D)$ and f is a relatively quadripartitioned neutrosophic continuous mapping from $(D, \tau_D)\ to (f(D), \sigma_{f(D)})$. Hence $f^{-1}({R_b}^{-1}(\mathrm{V}_{f(D)}) \cap D = {R_b}^{-1}(\mathrm{V}_{f(D)}) \cap D$ is open in τ_D. Since f is relatively quadripartitioned neutrosophic open, $f(f^{-1}({R_b}^{-1}(\mathrm{V}_{f(D)}) \cap D) = {R_b}^{-1}(\mathrm{V}_{f(D)}) \cap f(D)$ is a open in $\sigma_{f(D)}$. Hence the proof.

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NEW SIMILARITY AND ENTROPY MEASURES OF NEUTROSOPHIC VAGUE SETS WITH MULTI-ATTRIBUTE DECISION-MAKING

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Abstract:. The main purpose of this paper is to study the similarity and entropy of Neutrosophic Vague sets with multi-attribute decision-making. We propose the axiomatic definitions of the similarity measure and entropy of the neutrosophic vague set (NVS).Finally we apply these measures in a Multi-Attribute decision making problem.

Keywords: *Neutrosophic Vague sets; Inclusion relation in NVS; similarity measure; entropy.*

1.Introduction: Zadeh [9] put forward the theory of fuzzy sets in 1965, which is an effective method to deal with fuzzy information, but only limited to the truth-membership function. In actual decision-making, because of the fuzziness of people's thinking and the complexity of objective things, it is difficult for decision-makers to evaluate only through truth-membership function. On this basis, Atanassov [2] proposed an intuitionistic fuzzy set, and added a falsity-membership function to the fuzzy set to represent uncertain information. That is to say, the intuitionistic fuzzy concentration has both truth-membership function $T_A(x)$ and falsity-membership function $F_A(x)$, and $T_A(x)$, $F_A(x) \in [0, 1]$, $0 \leq T_A(x) + F_A(x) \leq 1$.. The correlation coefficients and weighted correlation coefficients of single-valued neutrosophic sets are proposed by Ye [11]. It is proved that the cosine similarity under singular concentration is a special case of the correlation coefficients. Furthermore, a single-valued neutrosophic cross-entropy measurement method is proposed and applied to multi-attribute decision-making in single-valued neutrosophic environment. Chi and Liu [3] applied a TOPSIS (The Order Performance technique based on Similarity to Ideal Solution) method to classify interval neutrosophic multi-attribute decision-making problems to alternative levels. further proposed the comparison rules on the basis of truth-membership function, et al. Garg developed an entropy measure under IVIFSs and used the proposed measure in solving MCDM with unknown attribute weights., Smarandache proposed a neutrosopic set (NS) which is the three components of truth, indeterminacy, and falsity degrees and that can be denoted as T,I,F respectively. NS is characterized independently and the ranges of functions T,I,F are in form of real standard and the nonstandard interval $]0^-,1^+[$ which cannot be used in real applications. Therefore, Wang et al. [6] proposed single valued neutrosophic set (SVNS) where the truth membership degree, indeterminacy-membership degree, and falsity-membership degree in form of real standard interval. Alkhazaleh [10], introduced a neutrosophic vague set (NV) by incorporating the features of SVNS and vague set . Besides that, he also defined several operators for NV and proved related properties. NV has played a significant role in the uncertain information system. In certain NV sets, the degree of truth, falsity, and indeterminacy of a given statement cannot be strictly described in real-world contexts, but it is instead denoted by several possible interval values. In 1972, De Luca and Termin gave the axiomatization definition of fuzzy entropy to characterize the degree of uncertainty [11]. Similarity is mainly used to estimate the degree of similarity between two objects. Wang [3] proposes the definition of similarity based on distance.

The main purpose of this paper is to study the similarity and entropy of Neutrosophic Vague sets with multi-attribute decision-making. We propose the axiomatic definitions of the similarity measure and entropy of the neutrosophic vague set (NVS) Based on the Hamming distance, cosine function and cotangent function, some new similarity measures and entropies of NVS are constructed.

2. Preliminaries:

In this section, we recall some fundamental notions and properties related to an Neutrosophic vague set.

Definition 2.1. [5] Let X be an object set and x be an element in the object set X. A neutrosophic set A of X can be expressed as $A = \{[x,(T_A(x), I_A(x), F_A(x))]|x \in X\}$, where ,$(T_A(x), I_A(x), F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$ which represent truth-membership, indeterminacy-membership, and falsity-membership respectively, $0^- \le TA(x) + IA(x) + FA(x) \le 3^+$.

Definition 2.2 : [1] A neutrosophic vague set A_{NV} (NVS in short) on the universe of discourse X be written as $A_{NV} = \{\langle x, T_{A_{NV}}(x), I_{A_{NV}}(x), F_{A_{NV}}(x)\rangle, x \in \mathbb{X}\}$, whose truth-membership, indeterminacy-membership and falsity-membership function is defined as

$T_{A_{NV}}(x) = [\, T^-(x), T^+(x)\,]$, $I_{A_{NV}}(x) = [\, I^-(x), I^+(x)\,]$,]and $F_{A_{NV}}(x) = [\, F^-(x), F^+(x)\,]$,

where$T^+(x) = 1 - F^-(x)$, $F^+(x) = 1 - T^-(x)$ and $0 \le T^-(x)) + I^-(x) + F^-(x) \le 2$.

Definition 2.3: [1] The complement of NVS A_{NV} is denoted by A^C_{NV} and it is given by

- $T^C_{A_{NV}} = [\, 1 - T^+(x), 1 - T^-(x)]$
- $I^C_{A_{NV}} = [\, 1 - I^+(x), 1 - I^-(x)]$
- $T^C_{A_{NV}} = [\, 1 - F^+(x), 1 - F^-(x)]$,

3. Similarity and Entropy of Neutrosophic Vague Sets

Let $D* = \{x|x = ([x_1^-, x_1^+], ([x_2^-, x_2^+], ([x_3^-, x_3^+]$ be the set of Neutrosophic vague values.

Definition 3.1:

Letting S: D* × D* → [0, 1], the real function S is a similarity interval Neutrosophic vague values x and y, if S satisfies the following conditions:

(P1) $0 \le S(x, y) \le 1$;

(P2) S(x, y) = 1 if and only if x = y;

(P3) S(x, y) = S(y, x);

(P4) For all x, y, z ∈ D* , if $x \le y \le z$, then $S(x, z) \le S(x, y)$, $S(x, z) \le S(y, z)$.

Definition 3.2: Let A and B be two Neutrosophic Vague Sets in the universe X, if A ⊆ B if and only if x ∈ X, ($T_A(x) < T_B(x)$, $F_A(x) < F_B(x)$), or ($T_A(x) = T_B(x)$, $F_A(x) \geq F_B(x)$), or ($T_A(x) = T_B(x)$, $F_A(x) = F_B(x)$ and $I_A(x) > I_B(x)$).

Definition 3.3: Let x =($[x_1^-, x_1^+]$, ($[x_2^-, x_2^+]$, ($[x_3^-, x_3^+]$]), y = ($[y_1^-, y_1^+]$, ($[y_2^-, y_2^+]$, $(y_3^-, y_3^+]$] be the neutrosophic vague values. x ≤y if and only if one of the following three conditions is true:

(1) $[x_1^-, x_1^+] < [y_1^-, y_1^+]$ and $[x_3^-, x_3^+] \geq [y_3^-, y_3^+]$

(2) $[x_1^-, x_1^+] = [y_1^-, y_1^+]$ and $[x_3^-, x_3^+]$] > $[y_3^-, y_3^+]$

(3) $[x_1^-, x_1^+] = [y_1^-, y_1^+]$] and $[x_3^-, x_3^+]$ $[y_3^-, y_3^+]$ and $[x_2^-, x_2^+] \geq [y_2^-, y_2^+]$.

Let A, B be the two Neutrosophic vague sets, A ⊆ B if and only if one of the following three conditions is true:

(1) $[T_A^-(x), T_A^+(x)] < [T_B^-(x), T_B^+(x)]$ and $[F_A^-(x), F_A^+(x)] \geq [F_B^-(x), F_B^+(x)]$;

(2) $[T_A^-(x), T_A^+(x)] = [T_B^-(x), T_B^+(x)]$ and $[F_A^-(x), F_A^+(x)] > [F_B^-(x), F_B^+(x)]$;

(3) $[T_A^-(x), T_A^+(x)] = [T_B^-(x), T_B^+(x)]$ and $[F_A^-(x), F_A^+(x)] = F_B^-(x), F_B^+(x)]$; and $[I_A^-(x), I_A^+(x)] \geq [I_B^-(x), I_B^+(x)]$;

Definition 3.4: Let x =($[x_1^-, x_1^+]$, ($[x_2^-, x_2^+]$, ($[x_3^-, x_3^+]$, y = ($[y_1^-, y_1^+]$, ($[y_2^-, y_2^+]$, ($[y_3^-, y_3^+]$

We define the following similarity

S(x,y)

$$=\begin{cases} 1-\frac{|x_2^- - y_2^-|+|x_2^+ - y_2^+|}{4} & [x_1^-, x_1^+] = [y_1^-, y_1^+], [x_3^-, x_3^+] = [y_3^-, y_3^+] \\ \frac{4-|x_1^- - y_1^-|-|x_1^+ - y_1^+|-|x_3^- - y_3^-|-|x_3^+ - y_3^+|}{8} & else \end{cases} \rightarrow (1)$$

Theorem 3.5. S(x, y) defined in formula (1) is a similarity between x and y.

Proof.

Let x =($[x_1^-, x_1^+]$, ($[x_2^-, x_2^+]$, ($[x_3^-, x_3^+]$ ∈ D∗ , y =($[y_1^-, y_1^+]$, ($[y_2^-, y_2^+]$, ($[y_3^-, y_3^+]$ ∈ D∗ ,

if $[x_1^-, x_1^+] = [y_1^-, y_1^+]$ and $[x_3^-, x_3^+] = [y_3^-, y_3^+]$,then S(x,y)= $1-\frac{|x_2^- - y_2^-|+|x_2^+ - y_2^+|}{4}$, so 0.5≤S(x,y) ≤1;

If $[x_1^-, x_1^+] \neq [y_1^-, y_1^+]$ and $[x_3^-, x_3^+] \neq [y_3^-, y_3^+]$,then S(x,y)= $\frac{4-|x_1^- - y_1^-|-|x_1^+ - y_1^+|-|x_3^- - y_3^-|-|x_3^+ - y_3^+|}{8}$

So 0 ≤ S(x,y) ≤0.5

(P1) Obviously, 0 ≤ S(x, y) ≤ 1.

(P2) S(x, y) = 1, if and only if S(x, y) = $1-\frac{|x_2^- - y_2^-|+|x_2^+ - y_2^+|}{4}$ if and only if $[x_1^-, x_1^+] = [y_1^-, y_1^+]$ and $[x_3^-, x_3^+] = [y_3^-, y_3^+]$ and $[x_2^-, x_2^+] = [y_2^-, y_2^+]$.

(P3) Obviously, S(x, y) = S(y, x).

(P4) Let X=($[x_1^-, x_1^+]$, $[x_2^-, x_2^+]$, $[x_3^-, x_3^+]$) ,Y=($[y_1^-, y_1^+]$, $[y_2^-, y_2^+]$, $[y_3^-, y_3^+]$) , Z=($[z_1^-, z_1^+]$,$[z_2^-, z_2^+]$, ($[z_3^-, z_3^+]$). and x ≤ y ≤ z, then

1) If $[x_1^-, x_1^+] < [y_1^-, y_1^+]$, $[x_3^-, x_3^+] \geq [y_3^-, y_3^+]$ and $[y_1^-, y_1^+] < [z_1^-, z_1^+]$, $[y_3^-, y_3^+] \geq [z_3^-, z_3^+]$,

so $S(x,y) = \frac{4-|x_1^- - y_1^-|-|x_1^+ - y_1^+|-|x_3^- - y_3^-|-|x_3^+ - y_3^+|}{8}$, $S(y,z) = \frac{4-|y_1^- - z_1^-|-|y_1^+ - z_1^+|-|y_3^- - z_3^-|-|y_3^+ - z_3^+|}{8}$

$S(x,z) = \frac{4-|x_1^- - z_1^-|-|x_1^+ - z_1^+|-|x_3^- - z_3^-|-|x_3^+ - z_3^+|}{8}$, Also because $[x_1^-, x_1^+] < [y_1^-, y_1^+] < [z_1^-, z_1^+]$, $[x_3^-, x_3^+] \geq [y_3^-, y_3^+] \geq [z_3^-, z_3^+]$, so $S(x, z) \leq S(x, y)$, $S(x, z) \leq S(y, z)$

2) If $[x_1^-, x_1^+] < [y_1^-, y_1^+]$, $[x_3^-, x_3^+] \geq [y_3^-, y_3^+]$ and $[y_1^-, y_1^+] = [z_1^-, z_1^+]$, $[y_3^-, y_3^+] > [z_3^-, z_3^+]$, so $S(x,y) = \frac{4-|x_1^- - y_1^-|-|x_1^+ - y_1^+|-|x_3^- - y_3^-|-|x_3^+ - y_3^+|}{8}$, $S(y,z) = \frac{4-|y_1^- - z_1^-|-|y_1^+ - z_1^+|-|y_3^- - z_3^-|-|y_3^+ - z_3^+|}{8}$,

$S(x,z) = \frac{4-|x_1^- - z_1^-|-|x_1^+ - z_1^+|-|x_3^- - z_3^-|-|x_3^+ - z_3^+|}{8}$, Also because $[x_1^-, x_1^+] < [y_1^-, y_1^+] = [z_1^-, z_1^+]$, $[x_3^-, x_3^+] \geq [y_3^-, y_3^+] > [z_3^-, z_3^+]$, so $S(x, z) \leq S(x, y)$, $S(x, z) \leq S(y, z)$.

3) If $[x_1^-, x_1^+] < [y_1^-, y_1^+]$, $[x_3^-, x_3^+] \geq [y_3^-, y_3^+]$ and $[y_1^-, y_1^+] = [z_1^-, z_1^+]$, $[y_3^-, y_3^+] = [z_3^-, z_3^+]$, $[y_2^-, y_2^+] \geq [z_2^-, z_2^+]$, so $S(x,y) = \frac{4-|x_1^- - y_1^-|-|x_1^+ - y_1^+|-|x_3^- - y_3^-|-|x_3^+ - y_3^+|}{8}$,

$S(y,z) = 1 - \frac{|y_2^- - z_2^-|+|y_2^+ - z_2^+|}{4}$, $S(x,z) = \frac{4-|x_1^- - z_1^-|-|x_1^+ - z_1^+|-|x_3^- - z_3^-|-|x_3^+ - z_3^+|}{8}$

Also because $[x_1^-, x_1^+] < [y_1^-, y_1^+] = [z_1^-, z_1^+]$, $[x_3^-, x_3^+] \geq [y_3^-, y_3^+] = [z_3^-, z_3^+]$, $[y_2^-, y_2^+] \geq [z_2^-, z_2^+]$

so $S(x, z) \leq S(x, y)$, $S(x, z) < 0.5 \leq S(y, z)$.

4) If $[x_1^-, x_1^+] = [y_1^-, y_1^+]$, $[x_3^-, x_3^+] > [y_3^-, y_3^+]$ and $[y_1^-, y_1^+] < [z_1^-, z_1^+]$, $[y_3^-, y_3^+] \geq [z_3^-, z_3^+]$,

so $S(x,y) = \frac{4-|x_1^- - y_1^-|-|x_1^+ - y_1^+|-|x_3^- - y_3^-|-|x_3^+ - y_3^+|}{8}$,

$S(y,z) = \frac{4-|y_1^- - z_1^-|-|y_1^+ - z_1^+|-|y_3^- - z_3^-|-|y_3^+ - z_3^+|}{8}$, $S(x,z) = \frac{4-|x_1^- - z_1^-|-|x_1^+ - z_1^+|-|x_3^- - z_3^-|-|x_3^+ - z_3^+|}{8}$,

Also because $[x_1^-, x_1^+] = [y_1^-, y_1^+] < [z_1^-, z_1^+]$, $[x_3^-, x_3^+] > [y_3^-, y_3^+] \geq [z_3^-, z_3^+]$, so $S(x, z) \leq S(x, y)$, $S(x, z) \leq S(y, z)$.

5) If $[x_1^-, x_1^+] = [y_1^-, y_1^+]$, $[x_3^-, x_3^+] > [y_3^-, y_3^+]$ and $[y_1^-, y_1^+] = [z_1^-, z_1^+]$, $[y_3^-, y_3^+] > [z_3^-, z_3^+]$,

so $S(x,y) = \frac{4-|x_1^- - y_1^-|-|x_1^+ - y_1^+|-|x_3^- - y_3^-|-|x_3^+ - y_3^+|}{8}$, $S(y,z) = \frac{4-|y_1^- - z_1^-|-|y_1^+ - z_1^+|-|y_3^- - z_3^-|-|y_3^+ - z_3^+|}{8}$,

$S(x,z) = \frac{4-|x_1^- - z_1^-|-|x_1^+ - z_1^+|-|x_3^- - z_3^-|-|x_3^+ - z_3^+|}{8}$, Also because $[x_1^-, x_1^+] = [y_1^-, y_1^+] = [z_1^-, z_1^+]$,

$[x_3^-, x_3^+] > [y_3^-, y_3^+] > [z_3^-, z_3^+]$, so $S(x, z) \leq S(x, y)$, $S(x, z) \leq S(y, z)$.

6) If $[x_1^-, x_1^+] = [y_1^-, y_1^+]$, $[x_3^-, x_3^+] > [y_3^-, y_3^+]$ and $[y_1^-, y_1^+] = [z_1^-, z_1^+]$, $[y_3^-, y_3^+] =$

$[z_3^-, z_3^+]$, $[y_2^-, y_2^+] \geq [z_2^-, z_2^+]$, so $S(x,y) = \frac{4-|x_1^- - y_1^-|-|x_1^+ - y_1^+|-|x_3^- - y_3^-|-|x_3^+ - y_3^+|}{8}$,

$S(y,z) = 1 - \frac{|y_2^- - z_2^-|+|y_2^+ - z_2^+|}{4}$, $S(x,z) = \frac{4-|x_1^- - z_1^-|-|x_1^+ - z_1^+|-|x_3^- - z_3^-|-|x_3^+ - z_3^+|}{8}$,

Also because $[x_1^-, x_1^+] = [y_1^-, y_1^+] = [z_1^-, z_1^+]$, $[x_3^-, x_3^+] > [y_3^-, y_3^+] = [z_3^-, z_3^+]$, $[y_2^-, y_2^+]$

$\geq$ $[z_2^-, z_2^+]$ so, S(x, z) ≤ S(x, y), S(x, z)<0.5 ≤ S(y, z).

7) If $[x_1^-, x_1^+]= [y_1^-, y_1^+], [x_3^-, x_3^+] = [y_3^-, y_3^+],, [x_2^-, x_2^+] \geq [y_2^-, y_2^+]$ and

$[y_1^-, y_1^+]<[z_1^-, z_1^+], [y_3^-, y_3^+] > [z_3^-, z_3^+]$,so S(x, y)= $1-\frac{|x_2^- - y_2^-|+|x_2^+ - y_2^+|}{4}$,
S(y,z)= $\frac{4-|y_1^- - z_1^-|-|y_1^+ - z_1^+|-|y_3^- - z_3^-|-|y_3^+ - z_3^+|}{8}$, S(x ,z)= $\frac{4-|x_1^- - z_1^-|-|x_1^+ - z_1^+|-|x_3^- - z_3^-|-|x_3^+ - z_3^+|}{8}$

Also because $[x_1^-, x_1^+]=[y_1^-, y_1^+]< [z_1^-, z_1^+], [x_3^-, x_3^+]=[y_3^-, y_3^+] > [z_3^-, z_3^+], [x_2^-, x_2^+] > [y_2^-, y_2^+]$. So S(x, z) <0.5≤S(x, y), S(x, z) ≤ S(y, z).

8) If $[x_1^-, x_1^+]= [y_1^-, y_1^+], [x_3^-, x_3^+] = [y_3^-, y_3^+],, [x_2^-, x_2^+] \geq [y_2^-, y_2^+]$ and

$[y_1^-, y_1^+]=[z_1^-, z_1^+], [y_3^-, y_3^+] > [z_3^-, z_3^+]$, so S(x, y)= $1-\frac{|x_2^- - y_2^-|+|x_2^+ - y_2^+|}{4}$
S(y,z)= $\frac{4-|y_1^- - z_1^-|-|y_1^+ - z_1^+|-|y_3^- - z_3^-|-|y_3^+ - z_3^+|}{8}$, S(x ,z)= $\frac{4-|x_1^- - z_1^-|-|x_1^+ - z_1^+|-|x_3^- - z_3^-|-|x_3^+ - z_3^+|}{8}$,
Also because $[x_1^-, x_1^+]=[y_1^-, y_1^+]= [z_1^-, z_1^+], [x_3^-, x_3^+]=[y_3^-, y_3^+] > [z_3^-, z_3^+], [x_2^-, x_2^+] \geq [y_2^-, y_2^+]$. So S(x, z) <0.5≤S(x, y), S(x, z) ≤ S(y, z).

9) If $[x_1^-, x_1^+]= [y_1^-, y_1^+], [x_3^-, x_3^+] = [y_3^-, y_3^+],, [x_2^-, x_2^+] \geq [y_2^-, y_2^+]$ and

$[y_1^-, y_1^+]=[z_1^-, z_1^+], [y_3^-, y_3^+] = [z_3^-, z_3^+][y_2^-, y_2^+] \geq [z_2^-, z_2^+]$,
So S(x, y)= $1-\frac{|x_2^- - y_2^-|+|x_2^+ - y_2^+|}{4}$, $S(y,z) = 1-\frac{|y_2^- - z_2^-|+|y_2^+ - z_2^+|}{4}$
$S(x,z) = 1-\frac{|x_2^- - z_2^-|+|x_2^+ - z_2^+|}{4}$, Also because $[x_1^-, x_1^+]=[y_1^-, y_1^+]= [z_1^-, z_1^+]$,
$[x_3^-, x_3^+]=[y_3^-, y_3^+] = [z_3^-, z_3^+], [x_2^-, x_2^+] \geq [y_2^-, y_2^+] \geq [z_2^-, z_2^+]$
so S(x, z) <0.5≤S(x, y), S(x, z) ≤ S(y, z).

Therefore, defined in formula (1) a similarity between x and y is defined.

Entropy of Neutrosophic vague Value

Since entropy is also an important means in the analysis of uncertainty information, we give the concept of entropy of Neutrosophic vague value.

Definition 3.6: Letting E: D* → [0, 1], the real function E is an entropy of neutrosophic vague value, if E satisfies the following conditions:

(N1) E(x) = 0 if and only if $[x_1^-, x_1^+]$ = [0, 0] or [1, 1] and$[x_3^-, x_3^+]$ = [0, 0] or [1, 1];
(N2) E(x) = 1 if and only if $[[x_1^-, x_1^+] = [x_2^-, x_2^+] = [x_3^-, x_3^+]$ = [0.5, 0.5];
(N3) E(x) = E(x^c);
(N4) Let x =$([x_1^-, x_1^+], ([x_2^-, x_2^+], ([x_3^-, x_3^+] \in D^*$, y=$[y_1^-, y_1^+],[y_2^-, y_2^+], [y_3^-, y_3^+] \in D^*$then
y^c= $([y_3^-, y_3^+], [1-y_2^+, 1-y_2^-], [y_1^-, y_1^+]$, E(x)≤ E(y), that is more ambiguous than y,
if x ≤ y, when y ≤y^c , or if y ≤ x, when y^c ≤ y, because Entropy is usually calculated by the similarity of x and x^c , so we define the following entropy

$$E(x)=S(x,x^c)=\begin{cases} 1-\frac{|2x_2^- -1|+|2x_2^+ -1|}{4} & [x_1^-,x_1^+]=[x_3^-,x_3^+]=[0.5,0.5] \\ \frac{4-2|x_1^- - x_3^-|-2|x_1^+ - x_3^+|}{8} & else \end{cases} \rightarrow (2)$$

Theorem 3. 7 E(x) defined as (2) is an entropy of x

Proof: If $[x_1^-,x_1^+]=[x_3^-,x_3^+]=[0.5,0.5]$ $, then\, E(x)=1-\frac{|2x_2^- -1|+|2x_2^+ -1|}{4}$

so $0.5 \le E(x) \le 1$; otherwise, $E(x)=\frac{4-2|x_1^- - x_3^-|-2|x_1^+ - x_3^+|}{8}$, so $0 \le E(x) \le 0.5$.

(N1) E(x) = 0 if and only if$[[x_1^-,x_1^+]$ = 1and $[x_3^-,x_3^+]$ =1,also because $[x_1^-,x_1^+][0.1]$ *and* $[x_3^-,x_3^+]\in[0.1]$, so $[x_1^-,x_1^+]$ = [0, 0] or [1, 1] and$[x_1^-,x_1^+]$ = [1,1]or [0,0];

(N2) Obviously E(x) = 1 if and only if $[[x_1^-,x_1^+]=[x_2^-,x_2^+]=[x_3^-,x_3^+]$ = [0.5, 0.5];

(N3) Obviously E(x) = E(x^c);

(N4) Let x =($[x_1^-,x_1^+]$, ($[x_2^-,x_2^+]$, ($[x_3^-,x_3^+]\in D^*$, ,y=$[y_1^-,y_1^+]$, $[y_2^-,y_2^+]$, ($[y_3^-,y_3^+]\in D^*$, then y= ($[y_3^-,y_3^+]$, $[1-y_2^+, 1-y_2^-]$, $[y_1^-,y_1^+]$,

if x ≤ y, when y ≤ y^c, because

$$E(x)=\begin{cases} 1-\frac{|2x_2^- -1|+|2x_2^+ -1|}{4} & [x_1^-,x_1^+]=[x_3^-,x_3^+]=[0.5,0.5] \\ \frac{4-2|x_1^- - x_3^-|-2|x_1^+ - x_3^+|}{8} & else \end{cases}$$

$$E(y)=\begin{cases} 1-\frac{|2y_2^- -1|+|2y_2^+ -1|}{4} & [x_1^-,x_1^+]=[x_3^-,x_3^+]=[0.5,0.5] \\ \frac{4-2|y_1^- - y_3^-|-2|y_1^+ - y_3^+|}{8} & else \end{cases}$$

1) If $[y_1^-,y_1^+]<[y_3^-,y_3^+][,y_3^-,y_3^+]\ge[y_1^-,y_1^+]$ and $[x_1^-,x_1^+]<[y_1^-,y_1^+]$,$[x_3^-,x_3^+]\ge$ $[y_3^-,y_3^+]$, *so* $[x_1^-,x_1^+]<[y_1^-,y_1^+]<[y_3^-,y_3^+]\le[x_3^-,x_3^+]$, therefore $[x_1^-,x_3^-]\ge$ $[y_1^-,y_3^-]$, $[x_1^+,x_3^+]\ge$ $[y_1^+,y_3^+]$ also because ,$E(x)=\frac{4-2|x_1^- - x_3^-|-2|x_1^+ - x_3^+|}{8}$

$E(y)=\frac{4-2|y_1^- - y_3^-|-2|y_1^+ - y_3^+|}{8}$, *so* $E(x)\le E(y)$.

2) If $[y_1^-,y_1^+]<[y_3^-,y_3^+][,y_3^-,y_3^+]\ge[y_1^-,y_1^+]$ and $[x_1^-,x_1^+]=[y_1^-,y_1^+]$,$[x_3^-,x_3^+]>$ $[y_3^-,y_3^+]$, *so* $[x_1^-,x_1^+]=[y_1^-,y_1^+]<[y_3^-,y_3^+]\le[x_3^-,x_3^+]$, therefore $[x_1^-,x_3^-]\ge[y_1^-,y_3^-]$, $[x_1^+,x_3^+]\ge[y_1^+,y_3^+]$ also because $E(x)=\frac{4-2|x_1^- - x_3^-|-2|x_1^+ - x_3^+|}{8}$, $E(y)=\frac{4-2|y_1^- - y_3^-|-2|y_1^+ - y_3^+|}{8}$ *so* $E(x)\le E(y)$.

3) If $[y_1^-,y_1^+]<[y_3^-,y_3^+]$,$[y_3^-,y_3^+]\ge[y_1^-,y_1^+]$ and $[x_1^-,x_1^+]=[y_1^-,y_1^+]$,$[x_3^-,x_3^+]=$ $[y_3^-,y_3^+]$, $[x_2^-,x_2^+]=[y_2^-,y_2^+]$ *so* $[x_1^-,x_1^+]=[y_1^-,y_1^+]<[y_3^-,y_3^+]=$ $[x_3^-,x_3^+]$,therefore $[x_1^-,x_3^-]\ge[y_1^-,y_3^-]$, $[x_1^+,x_3^+]\ge[y_1^+,y_3^+]$ also because $E(x)=\frac{4-2|x_1^- - x_3^-|-2|x_1^+ - x_3^+|}{8}$, $E(y)=\frac{4-2|y_1^- - y_3^-|-2|y_1^+ - y_3^+|}{8}$, *so* $E(x)\le E(y)$.

4) If $[y_1^-, y_1^+] = [y_3^-, y_3^+][, y_3^-, y_3^+] > [y_1^-, y_1^+]$ contradiction

5) If $[y_1^-, y_1^+] = [y_3^-, y_3^+], [y_2^-, y_2^+] \geq [1 - y_2^+, 1 - y_1^-]$ and $[x_1^-, x_1^+] < [y_1^-, y_1^+], [x_3^-, x_3^+] \geq [y_3^-, y_3^+]$, so $[x_1^-, x_1^+] < [y_1^-, y_1^+] = [y_3^-, y_3^+] \leq [x_3^-, x_3^+]$,

If $[y_1^-, y_1^+] = [y_3^-, y_3^+] = [0.5,0.5]$, then $E(x) = \frac{4-2|x_1^- - x_3^-| - 2|x_1^+ - x_3^+|}{8}$,

$E(y) = 1 - \frac{|2y_2^- - 1| + |2y_2^+ - 1|}{4}$, so $E(x) < 0.5 \leq E(y)$.

If $[y_1^-, y_1^+] = [y_3^-, y_3^+] \neq [0.5,0.5]$, then $E(x) = \frac{4-2|x_1^- - x_3^-| - 2|x_1^+ - x_3^+|}{8}$, $E(y) = \frac{4-2|y_1^- - y_3^-| - 2|y_1^+ - y_3^+|}{8}$, then $[x_1^-, x_3^-] \geq [y_1^-, y_3^-] = 0$, $[x_1^+, x_3^+] \geq [y_1^+, y_3^+] = 0$, so $E(x) \leq E(y)$

6) If $[y_1^-, y_1^+] = [y_3^-, y_3^+], [y_2^-, y_2^+] \geq [1 - y_2^+, 1 - y_1^-]$ and $[x_1^-, x_1^+] = [y_1^-, y_1^+], [x_3^-, x_3^+] > [y_3^-, y_3^+]$, so $[x_1^-, x_1^+] = [y_1^-, y_1^+] = [y_3^-, y_3^+] < [x_3^-, x_3^+]$,

If $[y_1^-, y_1^+] = [y_3^-, y_3^+] = [0.5,0.5]$, then $E(x) = \frac{4-2|x_1^- - x_3^-| - 2|x_1^+ - x_3^+|}{8}$,

$E(y) = 1 - \frac{|2y_2^- - 1| + |2y_2^+ - 1|}{4}$, so $E(x) < 0.5 \leq E(y)$.

If $[y_1^-, y_1^+] = [y_3^-, y_3^+] \neq [0.5,0.5]$, then $E(x) = \frac{4-2|x_1^- - x_3^-| - 2|x_1^+ - x_3^+|}{8}$, $E(y) = \frac{4-2|y_1^- - y_3^-| - 2|y_1^+ - y_3^+|}{8}$

then $[x_1^-, x_3^-] \geq [y_1^-, y_3^-] = 0$, $[x_1^+, x_3^+] \geq [y_1^+, y_3^+] = 0$, so $E(x) \leq E(y)$.

7) If $[y_1^-, y_1^+] = [y_3^-, y_3^+], [y_2^-, y_2^+] \geq [1 - y_2^+, 1 - y_1^-]$ and $[x_1^-, x_1^+] = [y_1^-, y_1^+], [x_3^-, x_3^+] = [y_3^-, y_3^+], [x_2^-, x_2^+] \geq [y_2^-, y_2^+]$, so if $[x_1^-, x_1^+] = [y_1^-, y_1^+] = [y_3^-, y_3^+] = [x_3^-, x_3^+] = [0.5,0.5]$, then

$E(x) = 1 - \frac{|2x_2^- - 1| + |2x_2^+ - 1|}{4}$, , $E(y) = 1 - \frac{|2y_2^- - 1| + |2y_2^+ - 1|}{4}$, also because $[x_2^-, x_2^+] \geq [y_2^-, y_2^+]$,

so $E(x) < 0.5 \leq E(y)$.

If $[x_1^-, x_1^+] = [y_1^-, y_1^+] = [y_3^-, y_3^+] = [x_3^-, x_3^+] \neq [0.5,0.5]$ then

$E(x) = \frac{4-2|x_1^- - x_3^-| - 2|x_1^+ - x_3^+|}{8}$, $E(y) = \frac{4-2|y_1^- - y_3^-| - 2|y_1^+ - y_3^+|}{8}$, then $[x_1^-, x_3^-] = [y_1^-, y_3^-] = 0$, $[x_1^+, x_3^+] = [y_1^+, y_3^+] = 0$, so $E(x) \leq E(y)$.

As the same reason, we can easily get the conclusion that if $y \leq x$, when $y^c \leq y$, then $E(x) \leq E(y)$. Therefore, as defined in formula (2), an entropy is defined.

4. New Similarity and Entropy of Neutrosophic vague Sets

Definition 4.1 : Let A, B be the two Neutrosophic vague sets, the real function S is a similarity between Neutrosophic vague sets A and B, if S satisfies the following conditions:

(P1) $0 \leq S(A, B) \leq 1$;

(P2) $S(A, B) = 1$ if and only if $A = B$;

(P3) $S(A, B) = S(B, A)$;

(P4) For all A, B, C ∈ NVSs, if $A \subseteq B \subseteq C$, then $S(A, C) \leq S(A, B)$, $S(A, C) \leq S(B, C)$.

Definition 4.2: Let A be the an Neutrosophic vague set, the real function E is the entropy of Neutrosophic vague sets, if E satisfies the following conditions:

(N1) $E(A) = 0$ if and only if $[T_A^-, T_A^+] = [0, 0]$or$[1, 1]$, $[F_A^-, F_A^+] = [0, 0]$or$[1, 1]$;

(N2) $E(A) = 1$ if and only if $[T_A^-, T_A^+] = [I_A^-, I_A^+] = [F_A^-, F_A^+] = [0.5, 0.5]$;

(N3) E(A) = E(A^C);
(N4) Let A, B be the two neutrosophic vague sets, E(A) ≤ E(B), that is, B is more ambiguous than A, if A ⊆ B, when B ⊆ B c , or B ⊆A, when B^c ⊆B.

By aggregating the similarities and entropies of Neutrosophic vague values, we have the following similarity and entropy of Neutrosophic vague sets.

Theorem 4.3. Let X = {x_1, x_2, ...x_n} be an Neutrosophic vague set, s : $D^* \times D^* \to [0, 1]$ is the similarity of neutrosophic vague sets, ∀A, B ⊆ X, the similarity S of A and B is defined as follows:

$$S(A, B) = \frac{1}{n}\sum_{i=1}^{n} s(A(x_i), B(x_i)). \qquad \ldots\ldots..(3)$$

Theorem 4.4. Let X = {x_1, x_2, ...x_n} be an Neutrosophic vague set, e : $D^* \to [0, 1]$ is the entropy of neutrosophic vague sets, ∀A,B ⊆ X, the entropy E of A is defined as follows:

$$E(A) = \frac{1}{n}\sum_{i=1}^{n} e(A(xi)$$

If the weights W = (w_1, w_2, ..., w_n) is added, $w_i \in [0, 1]$ and $\sum_{i=1}^{n} W_i = 1$, then the similarities of A and B and the entropy of A are defined as follows:

$$S(A, B)= \frac{1}{n}\sum_{i=1}^{n} w_{i.} s(A(x_i), B(x_i)).$$

$$E(A) = \frac{1}{n}\sum_{i=1}^{n} w_i . e(A(xi)).$$

5. Multi-Attributes Decision Making Based on a New Similarity Measure

Suppose that there is a group with four possible alternatives to invest: (1) A_1 is a food company; (2) A_2 is a car company; (3) A_3 is a weapons company; (4) A_4 is a computer company. Investment companies must make decisions based on three criteria: (1) C_1 is growth analysis; (2) C_2 is risk analysis; and (3) C_3 is a environmental impact analysis. By using Neutrosphic vague information, decision makers evaluated four possible alternatives based on the above three criteria and the evaluation are expressed as three neutrosophic vague sets (Table 1)

Table 1: The evaluation of Alternative

	C_1	**C_2**	**C_3**
A_1	([0.2,0.7], [0.2,0.4], [0.3,0.8])	([0.0,0.2], [0.8,0.9], [0.8,1])	([0.3,0.4], [0.6,0.8], [0.6,0.7])
A_2	([0.1,0.5], [0.5,0.7], [0.5,0.9])	([0.4,0.5], [0.3,0.7], [0.5,0.6])	([0.5,1], [0.3,0.5], [0.0,0.5])
A_3	([0.1,0.3], [0.2,0.4],[0.7,0.9])	([0.3,0.9], [0.5,0.8], [0.1,0.7])	([0.5,0.8], [0.5,0.8], [0.2,0.5])
A_4	([0.4,0.6], [0.1,0.4],[0.4,0.6])	([0.2,0.8], [0.7,0.9], [0.2,0.8])	([0.1,0.7], [0.7,0.9], [0.3,0.9])

We use the newly proposed similarity and entropy to get the best alternative. The best choice is $A = ([1, 1], [1, 1], [0, 0])$. For convenience, we use A_{ij} that indicates the Neutrosophic Vague value in line i column j. It is available from (1),

$S(A_{11}, A) = 0.225$, $S(A_{12}, A) = 0.05$, $S(A_{13}, A) = 0.175$,

$S(A_{21}, A) = 0.15$, $S(A_{22}, A) = 0.225$, $S(A_{23}, A) = 0.0.375$,

$S(A_{31}, A) = 0.1$, $S(A_{32}, A) = 0.3$, $S(A_{33}, A) = 0.325$,

$S(A_{41}, A) = 0.25$, $S(A_{42}, A) = 0.3$, $S(A_{43}, A) = 0.575$.

Thus, by (3), we can obtain that $S(A_1, A) \approx 0.15$, for the same reason, we can obtain that $S(A_2, A) \approx 0.25$, $S(A_3, A) \approx 0.241$, $S(A_4, A) \approx 0.375$. Therefore, $S(A_4, A) > S(A_2, A) > S(A_3, A) > S(A_1, A)$, so A4 is the best choice.

6. Conclusions

In this paper, we have introduced novel similarity and entropy measures for Neutrosophic Vague Sets (NVS) and applied them to multi-attribute decision-making problems. The newly proposed measures effectively handle the uncertainty and ambiguity present in real-world decision-making environments, making them more suitable for complex systems where decision attributes are not crisply defined**.**

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A NEUTROSOPHIC VAGUE APPROACH TO THE TRANSPORTATION PROBLEM USING TRAPEZOIDAL NEUTROSOPHIC VAGUE NUMBERS

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Abstract

This paper presents a Neutrosophic Vague approach to solving transportation problem using Trapezoidal Neutrosophic Vague Numbers (NVTNVN). By incorporating degrees of truth, indeterminacy, and falsity, NVTNVN effectively model uncertainty in transportation data. We develop an algorithm to optimize transportation costs under vague conditions, demonstrated through practical examples and we find the transportation cost using Modified distribution method (MODI), also we compare the initial basic feasible solution using the method of Vogel's approximation method, least cost method and North west method.

Keywords: Neutrosophic Vague approach, Trapezoidal Neutrosophic vague numbers, Neutrosophic vague transportation problem, Modified distribution method.

1.Introduction:

Fuzzy set was introduced by Zadeh in the year 1965 [11], According to its definition, it is a set where each member is represented along with a membership grade, which is represented by a real integer in a closed interval ranging from 0 to 1.Neutosopic sets, which consider truth, falsity, and indeterminacy as three distinct components, are a generalization of both fuzzy and intuitionistic sets and it was presented by F. Smarandache in 1998 [10].Each components can take values from the real interval [0, 1] .In a Neutrosophic set, where they are not necessarily dependent on one another and may differ independently. The Transportation problem was introduced by Frank L. Hitchcock developed it in 1941[4]. It is a special case of linear programming problem that involves determining the best cost-effective way to distribute a product from multiple suppliers to multiple consumers while reducing total transportation costs. A vague set is a generalization of the traditional fuzzy set, introduced to handle uncertainty in a more flexible way. These numbers are used to handle incomplete information when we don't have complete confidence. They represent the degree to which an element is definitely true and definitely false, respectively. Vague sets were introduced by Gau and Buehrer in the late 1990s as an extension of fuzzy sets. A numerical representation obtained from fuzzy sets is called a vague number. It is described as a number that has two bounds, an upper bound and a lower bound, that indicate the range that the true value is most likely to fall inside. The fuzzy number provides a more understandable means of expressing uncertainty in computational and practical applications, and is particularly helpful when working with incorrect or incomplete numerical data. A Neutrosophic vague number (NVN) is a concept in Neutrosophic logic and set theory, which deals with uncertainty, vagueness, and indeterminacy in decision-making processes.

In this paper, we solved the transportation problem using trapezoidal Neutrosophic vague numbers, and we focus to solve the transportation problem using the method of Modified distribution method, The Modified distribution method also known as the Modified minimum cost method. It is an efficient method for finding the optimal solution for minimizing the cost of transportation goods from multiple sources to multiple destinations, and we compare the initial basic feasible solution using the method of Vogel's approximation method, north west method, and least cost method. Our goal is to identify the strategy that will reduce the total cost of transportation among the three approaches, and we compare these strategies to see which is the least expensive and find the optimality.

2.Preliminaries:

Definition 2.1[8]: Let X be the universe. Neutrosophic set A in X is characterized by a membership grade T_A, indeterminancy grade I_A and non-membership grade F_A, where, T_A:X→[0,1], I_A:X→[0,1], F_A:X→[0,1]. It can be written as A={(x,T_A(x),I_A(x),F_A(x)/xϵX, satisfying 0≤ T_A(x) +I_A(x)+ F_A(x)≤3.

Definition 2.2[5]: A Vague set A on a non empty set X is a pair (T_A,F_A), where T_A:X→ $[0,1]$ and F_A:X→ $[0,1]$ are true membership and false membership functions, respectively, such that 0≤ $T_A(x) + F_A(x)$ ≤1 for any x∈ X. Let X and Y be non -empty sets. A Vague relation R of X to Y is a Vague set R on X×Y That is R= (T_A,F_A),where T_R:X×Y→ $[0,1]$,F_R: X×Y→ $[0,1]$ and satisfy the condition:0≤ T_R(x,y) +F_R(x,y)≤1 for any x,y∈X.

Definiton2.3[5]: A Neutrosophic Vague set A_{NV} on the universe of discourse X be written as

$$A_{NV} =\{\langle x, \hat{T}_{A_{NV}}(x), \hat{I}_{A_{NV}}(x), \hat{F}_{A_{NV}}(x), x \in X\rangle\},$$

whose truth-membership, indeterminacy-membership, falsity-membership function is defined as

$\hat{T}_{A_{NV}}(x) = [T^-(x), T^+(x)], \hat{I}_{A_{NV}}(x) = [I^-(x), I^+(x)]$ and , $\hat{F}_{A_{NV}}(x) = [F^-(x), F^+(x)]$,

Where, $T^+(x) = 1 - F^-(x), F^+(x) = 1 - T^-(x)$,and0≤ $T^-(x) + I^-(x) + F^-(x) \leq 23$**:**

3. Trapezoidal Neutrosophic Vague Number: Let N be a trapezoidal Neutrosophic vague number in the set of real numbers with the truth, indeterminacy and falsity membership functions are defined by

$$T_{N^L}(X) = \begin{cases} \frac{(x-a)t_N}{b-a}, & a \le x \le b \\ t_N, & b \le x \le c \\ \frac{(d-x)t_N}{d-c}, & c \le x \le d \\ 0, & otherwise \end{cases} \qquad T_{N^U}(X) = \begin{cases} \frac{(x-a)t_N}{b-a}, & a \le x \le b \\ t_N, & b \le x \le c \\ \frac{(d-x)t_N}{d-c}, & c \le x \le d \\ 0, & otherwise \end{cases}$$

$$I_{N^L}(X) = \begin{cases} \frac{b-x+(x-a)i_N}{b-a}, & a \le x \le b \\ i_N, & b \le x \le c \\ \frac{x-c+(d-x)i_N}{d-c}, & c \le x \le d \\ 0, & otherwise \end{cases} \qquad I_{N^U}(X) = \begin{cases} \frac{b-x+(x-a)i_N}{b-a}, & a \le x \le b \\ i_N, & b \le x \le c \\ \frac{x-c+(d-x)i_N}{d-c}, & c \le x \le d \\ 0, & otherwise \end{cases}$$

$$F_{N^L}(X) = \begin{cases} \frac{b-x+(x-a)f_N}{b-a}, & a \le x \le b \\ f_N, & b \le x \le c \\ \frac{x-c+(d-x)f_N}{d-c}, & c \le x \le d \\ 0, & otherwise \end{cases} \qquad F_{N^U}(X) = \begin{cases} \frac{b-x+(x-a)f_N}{b-a}, & a \le x \le b \\ f_N, & b \le x \le c \\ \frac{x-c+(d-x)f_N}{d-c}, & c \le x \le d \\ 0, & otherwise \end{cases}$$

4.The Transportation Problem:

The Transportation Problem is mathematically formulated as follows.

Minimize

$$Z=\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij},$$

Where,

$\sum_{j=1}^{n} x_{ij}= a_i$; i=1,2, 3,……n,

$\sum_{i=1}^{m} x_{ij} = b_j$; j=1,2, 3,.m , $x_{ij} \geq 0$ for all i and j.

If $\sum_{j=1}^{n} x_{ij}=a_i = \sum_{i=1}^{m} x_{ij} = b_j$; where i=1, 2, m; j=1, 2,….n, then the transportation issue is a balanced Neutrosophic one. if it isn't balanced. Use a dummy row or dummy column to balance it. The goal of a Neutrosophic transportation challenge is to move goods with uncertain transported units from source to destination the lowest possible cost.

The notation of the Transportation Problem is

m is the total number of supplies(sources)

n is the total number of demands(destinations)

a_i is the amount of supply at source i

$b_j$is the amount of demand at destination j

$c_{ij}$is the transpotation cost from supply i to demand j

x_{ij} is the amount to be shipped to be from source i to destination j.

4.1: Algorithms:

Vogel's approximation method:

Step 1: Calculate Row and Column Penalties: For each row and column, find the two lowest transportation costs. Compute the penalty as the difference between them.

Step 2: Identify the Highest Penalty: Select the row or column with the largest penalty, as this represents the area where ignoring cost would be most expensive.

Step 3: Allocate as Much as Possible: In the chosen row/column, allocate as much as possible to the lowest-cost cell (subject to supply and demand constraints). Adjust supply and demand by subtracting the allocated amount

Step 4: Eliminate the Satisfied Row/Column: If supply or demand is fulfilled (becomes zero), eliminate that row or column from further consideration.

Step 5: Repeat Steps 1-4 until all supplies and demands are allocated.

North west corner method

Step 1: first we select the north-west corner cell of the transportation matrix and find the lowest value of supply or demand.

Step 2: Subtract the minimal value from each row and column. If the supply is zero, strike that row and go to the next cell. If the demand is zero, strike that column and go to the next cell. If both supply and demand are zero, strike both rows and columns before proceeding diagonally to the next cell.

Step 3: Repeat the process until all supply and demand variables are zero.

Least cost method:

Step 1: Select the cell with the lowest unit cost and assign as much as possible.

Step 2: Subtract the maximum value from supply and demand. If the supply is zero, cross (strike) that row; if the demand is zero, cross that column. If the minimum unit cost cell is not unique, choose the cell with the highest potential allocation.

Step 3: Repeat these steps for all uncrossed (unstriked) rows and columns until all supply and demand numbers equal 0.

Modified distribution method:

Step 1: Determine an initial basic feasible solution using one of the three methods: NWCM, LCM, or VAM.

Step 2: Determine u_i and v_j for rows and columns. assign 0 to u_i or v_j,where the maximum number of allocations in a row or column, respectively. For all occupied cells, use the formula $c_{ij} = u_i + v_j$.

Step 3: For each unoccupied cell, calculate $d_{ij} = c_{ij} - (u_i + v_j)$.

Step 4: Check the symbol of d_{ij}. a. If $d_{ij} > 0$, the current basic viable solution is optimal, and the method should be stopped. If $d_{ij} = 0$, an alternative solution exists with a different set allocation but the same transportation cost. Now, stop this procedure. b. If $d_{ij} < 0$, then the given solution is not an optimal solution.

Step 5: select the unoccupied cell with the greatest negative value of d_{ij}.

Step 6: Create a closed path (or loop) from the unoccupied cell. The right angle turn in this path is allowed only at occupied cells and the original unoccupied cell. Mark the (+) and (-) signs alternately in each corner, beginning with the original unoccupied cell.

Step-7: 1. Choose the minimum value from the cells marked with (-) sign of the closed path. Assign this value to the selected unoccupied cell. Enter this value into the other occupied cells marked with a (+) sign. Subtract this value from the other occupied cells marked with the (-) symbol.

Step 8: Repeat steps 2-7 until the ideal solution is reached. This process ends when all $d_{ij} \geq 0$ for unoccupied cells.

5.Application:

A corporation has three warehouses (A, B and C) that sell goods to four retail locations (1,2,3 and 4). The corporation wants to minimize the total transportation cost and the monthly supply capacity of each warehouse and demand for each retail store are as follows:

	1	2	3	4	Supply
A	(10,12,14,16) [0.4,0.5] [0.1,0.2] [0.5,0.6]	(3,4,5,6) [0.6,0.8] [0.5,0.7] [0.2,0.4]	(8,9,10,11) [0.5,0.8] [0.1,0.3] [0.2,0.5]	(1,2,3,4) [0.5,0.7] [0.1,0.3] [0.3,0.5]	26
B	(6,7,8,9) [0.5,0.6] [0.2,0.5] [0.4,0.5]	(10,11,12,13) [0.5,0.7] [0.1,0.2] [0.3,0.5]	(11,13,15,17) [0.6,0.8] [0.1,0.8] [0.2,0.4]	(9,10,11,12) [0.6,0.8] [0.1,0.2] [0.2,0.4]	28
C	(5,7,9,11) [0.1,0.9] [0.4,0.5] [0.1,0.9]	(2,4,6,8) [0.4,0.7] [0.2,0.3] [0.3,0.6]	(4,6,8,10) [0.6,0.7] [0.1,0.8] [0.3,0.4]	(7,8,9,10) [0.6,0.7] [0,0.2] [0.3,0.4]	22
Demand	10	29	16	21	Balanced

Conversion for the A Neutrosophic vague transportation problem into its crisp transportation problem

Here we use the score function for Trapezoidal Neutrosophic Vague Numbers. Also, for solving A transportation problem using Trapezoidal Neutrosophic Vague Numbers.

Let A_N= $\{(a_1, a_2, a_3, a_4; [T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U]\}; a_1 \leq a_2 \leq a_3 \leq a_4$ be trapezoidal Neutrosophic vague number. Then the centre of gravity (COG) in R is

$$\text{COG(R)} = \begin{cases} a \; if \; a_1 = a_2 = a_3 = a_4 \\ \left(\frac{1}{4}\right) [a_1 + a_2 + a_3 + a_4], otherwise \end{cases}$$

Score function S (A_N) = COG(R)× $\left(\frac{1}{4}\right) [2 + T^u + T^l - 2I^U - 2I^l - F^u - F^l] \longrightarrow (1)$

$= \left(\frac{1}{4}\right)[10+12+14+16] = 13$

$= 13 \times \left(\frac{1}{4}\right)[2 + 0.4 + 0.5 - 2(0.2) - 2(0.1) - 0.5 - 0.6]$

$= 3.9$

which is approximately equal to 4.

we convert the Neutrosophic vague transportation problem into its crisp model by using (1):

Crisp Transportation Table

Optimal solution:

Modified distribution method

The Transportation problem can be solved in two steps. The first phase involves determining the initial basic feasible solution. There are three ways for determining an initial basic workable solution:

1. Northwest Corner Method
2. Least Cost Method
3. Vogel's Approximation Method

and the second phase involves optimization of the initial basic feasible solution.

In this problem we compare the initial feasible solution from each method in terms of total transportation cost and optimality.

Phase 1:

	D_1	D_2	D_3	D_4	Supply
S_1	4	0	4	1	26
S_2	2	5	4	6	28
S_3	0	2	1	5	22
Demand	10	29	16	21	Balanced

Initial basic feasible solution:

1.Northwest Corner Method

	D_1	D_2	D_3	D_4	Supply
S_1	**10** 4	**16** 0	4	1	26
S_2	2	**13** 5	**15** 4	6	28
S_3	0	2	**1** 1	**21** 5	22
Demand	10	29	16	21	Balanced

The minimum total transportation cost = 4×10 + 0× 16 + 5×13 + 4×15 + 1×1 + 5×21 = 271.

2. Least Cost Method

	D_1	D_2	D_3	D_4	Supply
S_1	4	**26** 0	4	1	26
S_2	2	**3** 5	**4** 4	**21** 6	28
S_3	**10** 0	2	**12** 1	5	22
Demand	10	29	16	21	Balanced

The minimum total transportation cost = 0×26 + 5× 3 + 4×4 + 6×21 + 0× 10 + 1× 12 =169

3. Vogel's Approximation Method

The minimum total transportation cost = 0×5 + 1×21 + 2×10 + 5×18 + 2×6 + 1×16 = 159

The number of allocated cells = 6 is equal to m + n -1 = 3+4-1 = 6

∴ This solution is non – degenerate.

Phase 2:

	D_1	D_2	D_3	D_4	Supply
S_1	4	$\boxed{\mathbf{5}}$ 0	4	$\boxed{\mathbf{21}}$ 1	26
S_2	$\boxed{\mathbf{10}}$ 2	5	4	6	28
S_3	0	$\boxed{\mathbf{6}}$ 2	$\boxed{\mathbf{16}}$ 1	5	22
Demand	10	29	16	21	Balanced

optimality test:

Iteration-1

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	$\boxed{\mathbf{7}}$ 4	$\boxed{\mathbf{5}}$ 0	$\boxed{\mathbf{5}}$ 4	$\boxed{\mathbf{21}}$ 1	26	u_1 = -2
S_2	$\boxed{\mathbf{10}}$ 2	$\boxed{\mathbf{18}}$ 5	$\boxed{\mathbf{0}}$ 4	$\boxed{\mathbf{0}}$ 6	28	u_2 = 3
S_3	$\boxed{\mathbf{1}}$ 0	$\boxed{\mathbf{6}}$ 2	$\boxed{\mathbf{16}}$ 1	$\boxed{\mathbf{2}}$ 5	22	u_3 = 0
Demand	10	29	16	21		
v_j	v_1 = -1	v2 = 2	v_3 = 1	v_4 = 3		

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	4	**5** 0	4	**21** 1	26	u_1 = -2
S_2	**10** 2	**18** 5	4	6	28	u_2 = 3
S_3	0	**6** 2	**16** 1	5	22	u_3 = 0
Demand	10	29	16	21		
v_j	v_1 = -1	v2 = 2	v_3 = 1	v_4 =3		

Iteration-2:

Since all $d_{ij} \geq 0$.

So, the final optimal solution is arrived.

Optimal solution

	D_1	D_2	D_3	D_4	Supply
S_1	4	**5** 0	4	**21** 1	26
S_2	**10** 2	**18** 5	4	6	28
S_3	0	**6** 2	**16** 1	5	22
Demand	10	29	16	21	Balanced

The minimum total transportation cost = 0×5 + 1×21 + 2×10 + 5×18 + 2×6 + 1×16 = 159

6.Comparision:

As a result, the Vogel's Approximation Method (VAM) offers an initial solution that is more effective and economical than the North-West Corner and Least Cost approaches. Usually, VAM reduces transportation costs and eliminates the need for additional optimization by taking into account both cost and supply-demand penalties. Its balanced approach gives it to minimize overall transportation costs better, providing a more ideal solution for supply chain management and logistics.

6.Conclusion:

In operations research, the transportation problem is a specific kind of optimization problem where the goal is to find the most economical way to distribute a product from multiple suppliers (or sources) to multiple consumers (or destinations) while minimizing the overall cost of transportation. This paper addresses a transportation problem model under trapezoidal Neutrosophic Vague Numbers, and stepwise numerical applications are used to explain and prove the performance of the transportation problem. These kinds of new findings will help to get the best optimal decision for the transportation problem using a Neutrosophic Vague approach.

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This book explores the development and importance of these mathematical approaches in managing uncertainty and imprecision in decision-making, and focuses on the practical applications of these theories in decision-making, demonstrating how they improve uncertainty modeling and facilitate more effective decision-making across various fields. By leveraging these frameworks, decision-makers can better navigate complexity, leading to more accurate and dependable outcomes. Ultimately, it highlights the progression from fuzzy to intuitionistic fuzzy to neutrosophic set theories as a significant advancement in capturing and analyzing uncertainty within decision-making contexts.

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