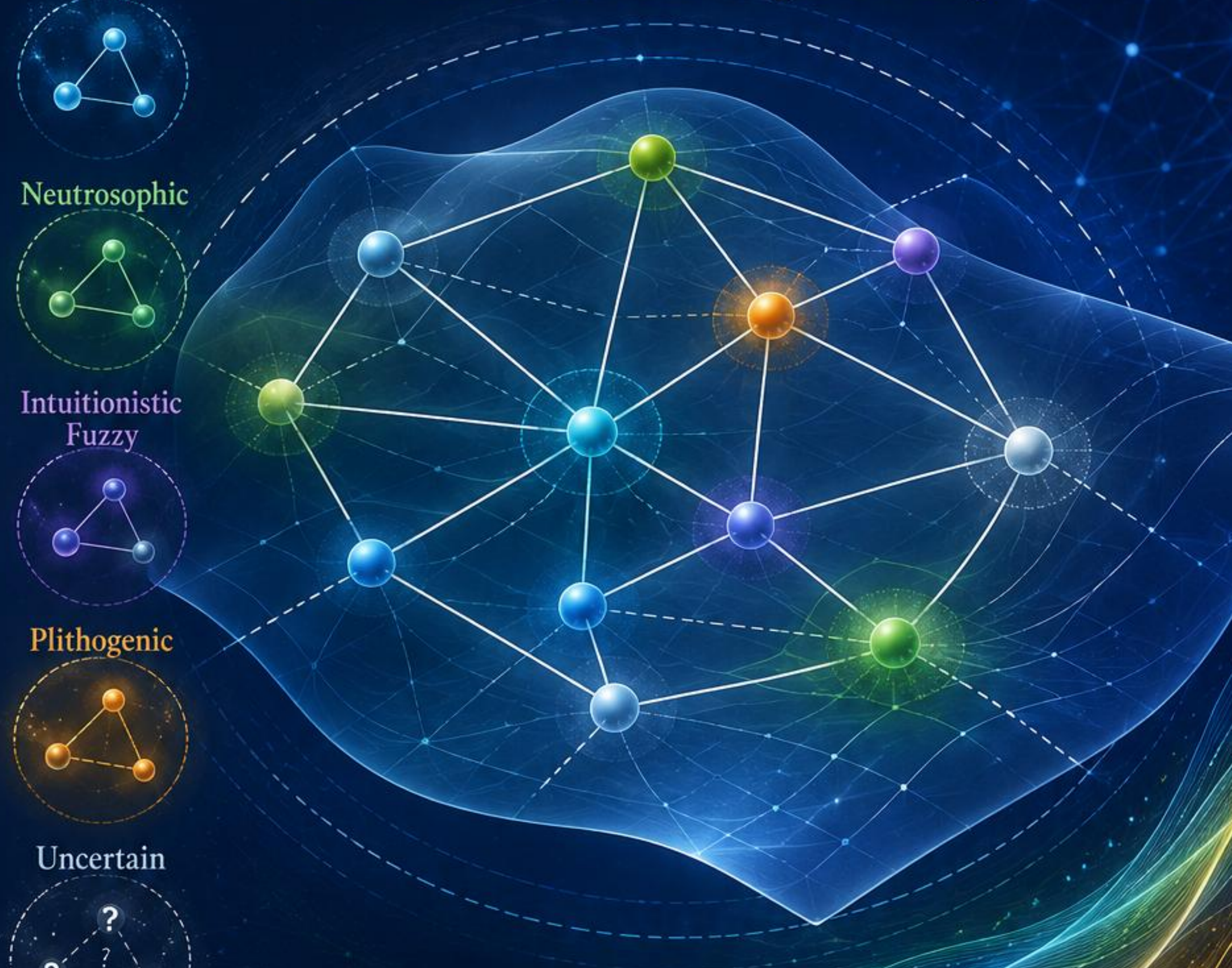
Fuzzy, Neutrosophic, and Uncertain Graph Theory: Properties and Applications

(II)

Uncertain Planar Graph Theory



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Neutrosophic Science International Association (NSIA)
Publishing House

Gallup - Guayaquil
United States of America – Ecuador
2026

Editor:

Neutrosophic Science International Association (NSIA)
Publishing House
https://fs.unm.edu/NSIA/

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ISBN 978-197250224-2

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Chapter 1

Introduction

1.1 Graph Theory

As is well known, graph theory is a fundamental branch of mathematics that studies networks consisting of vertices and edges, with particular attention to their paths, structures, and properties [2]. Graphs provide a clear and flexible way to represent relationships among real-world entities, which is one reason why graph theory has been extensively studied in a wide range of applications [3]. For example, in artificial intelligence, research on graph neural networks, which is grounded in graph-theoretic ideas, has become highly active [4].

One of the powerful aspects of graph theory is the classification of graphs into distinct graph classes according to shared properties or structural characteristics. A large number of graph classes have been introduced, each intended to capture particular behaviors or structural patterns. Such classes provide a foundation for designing efficient algorithms, simplifying problem solving, and obtaining deeper insight into computational complexity (cf. [5, 6]).

Examples of well-known graph classes include tree graphs [7], path graphs [8], complete graphs [9], outerplanar graphs [10], and total graphs [11]. The study of such graph classes enables researchers to identify common structural properties, develop more specialized and efficient algorithms, and apply these insights to practical problems. In line with this perspective, a growing body of research also investigates how closely a given graph or algorithm aligns with a target graph class or structural pattern (cf. [2]).

1.2 Planar Graph and Outerplanar Graph

In graph theory, many graph classes with diverse structural properties have been introduced and studied. Among them, this book focuses primarily on planar graphs and outerplanar graphs. A planar graph is a graph that can be drawn in the plane without any edge crossings, thereby preserving planarity [12,13]. A well-known example of a planar graph is the butterfly graph (cf. [14]).

An outerplanar graph is a planar graph that admits an embedding in which all vertices lie on the boundary of the outer face [15]. Another well-known graph class related to planar graphs is the apex graph. Because of the edge-crossing restrictions imposed by planarity, planar graphs are not only visually intuitive but also frequently used in the literature as mathematically tractable models.

Research on structural parameters of outerplanar graphs remains active. For example, every outerplanar graph has degeneracy at most 2 [16]. In addition, outerplanar graphs have boxicity at most 2 [17].

Other related graph classes include Halin graphs [18], upward planar graphs [19–21], convex planar graphs [22], 1-planar graphs [23, 24], and k-planar graphs [25], all of which arise in the broader study of planar structures.

One of the main motivations for studying these restricted graph classes is the development of efficient algorithms tailored to them. For example, like outerplanar graphs, Halin graphs have low treewidth, which makes many algorithmic problems more tractable than in unrestricted planar graphs [26]. Many other works have likewise investigated planar graphs and related graph classes, contributing substantially to this area of research [27, 28].

1.3 Fuzzy , Neutrosophic, Quadripartitioned Neutrosophic, and Plithogenic Graphs

Since many practical systems involve uncertainty not only in attributes but also in relations, several graph-theoretic frameworks have been developed to incorporate uncertainty directly into vertices, edges, and higher-level structural information. Among these, fuzzy graphs, neutrosophic graphs, quadripartitioned neutrosophic graphs, and plithogenic graphs form an important family of uncertainty-aware network models.

A fuzzy graph assigns to each vertex and each edge a membership degree in $[0, 1]$, thereby expressing the extent to which the corresponding object belongs to the modeled structure [29, 30]. In this sense, a fuzzy graph may be regarded as a graph-theoretic realization of fuzzy-set-based uncertainty [31, 32]. Because many real-world relationships are inherently imprecise, fuzzy graphs have been applied to problems in social networks, decision-making, transportation systems, and related areas [29, 30]. This broad applicability has led to the development of many variants and refinements, including intuitionistic fuzzy graphs [33], bipolar fuzzy graphs [34], fuzzy planar graphs [35], and complex hesitant fuzzy graphs [36].

More broadly, a wide variety of graph models have been proposed to represent uncertainty and enriched relational information. These include fuzzy graphs [29, 30], vague graphs [37], plithogenic graphs [38], soft graphs [39, 40], hypersoft graphs [41, 42], and rough graphs [43,44]. Taken together, these frameworks illustrate the breadth of approaches that have been developed to represent uncertainty, ambiguity, and enriched semantic structure in graph-based models.

In recent years, neutrosophic graphs [45, 46] and neutrosophic hypergraphs [47, 48] have attracted increasing attention within the broader development of neutrosophic set theory [49, 50]. The term *neutrosophic* refers to a framework in which truth, indeterminacy, and falsity are treated as distinct components. From a graph-theoretic perspective, this makes it possible to represent ambiguous or inconsistent relational information more flexibly than ordinary fuzzy graphs. Accordingly, many related classes have been introduced, including bipolar neutrosophic graphs [48, 51–53], neutrosophic incidence graphs [54–57], single-valued neutrosophic signed graphs [58], strong neutrosophic graphs [59], m-polar neutrosophic graphs [60–62], complex neutrosophic hypergraphs [47], and bipolar neutrosophic hypergraphs [48].

Plithogenic graphs extend uncertainty-aware graph models even further by describing vertices and edges through attribute values together with corresponding degrees of appurtenance, while also introducing a contradiction function that quantifies incompatibility among distinct attribute values [63–65]. They may therefore be viewed as graph-theoretic counterparts of plithogenic sets [66]. This richer structure supports context-dependent aggregation of heterogeneous and potentially conflicting information on networks, thereby refining classical fuzzy, intuitionistic fuzzy, and neutrosophic graph models [67, 68].

For convenience, Table 1.1 summarizes the canonical information attached to vertices and edges in several representative graph extensions.

Among graph types that incorporate such forms of uncertainty, especially those related to planar graphs, representative examples include fuzzy planar graphs [69,70], fuzzy outerplanar graphs [71], intuitionistic fuzzy planar graphs [72], and neutrosophic planar graphs [73].

1.4 Our Con tribution

This book develops a unified treatment of planar, outerplanar, apex, quasi-planar, and related graph classes under several uncertainty-based graph frameworks. We first review the corresponding notions for fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic graphs. We then formulate a model-independent notion of uncertain planar graph by imposing the relevant topological graph-class property on the underlying crisp graph. This approach clarifies which parts of the theory are purely structural and which parts depend on the chosen uncertainty model. The book also discusses related extensions to hypergraphs and SuperHyperGraphs, thereby providing a broader framework for future investigations of planarity under uncertainty. This book is a partially improved and revised version of [1].

Table 1.1: Representative graph extensions and the canonical information stored on vertices and/or edges.

Graph Type	**Canonical data attached to vertices/edges**
Fuzzy Graph	Vertex membership $\sigma : V \to [0,1]$ and edge membership $\mu : E \to [0,1]$ (typically with $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$).
Intuitionistic Fuzzy Graph	Vertex degrees $(\mu_A, \nu_A) : V \to [0,1]^2$ and edge degrees $(\mu_B, \nu_B) : E \to [0,1]^2$ with $\mu + \nu \leq 1$; the residual term represents hesitation.
Neutrosophic Graph	Vertex triple $(T_A, I_A, F_A) : V \to [0,1]^3$ and edge triple $(T_B, I_B, F_B) : E \to [0,1]^3$, representing truth, indeterminacy, and falsity.
Quadripartitioned Neutrosophic Graph	Vertex quadruple $(T, C, U, F) : V \to [0,1]^4$ and edge quadruple $(T, C, U, F) : E \to [0,1]^4$, typically representing truth, contradiction, unknown, and falsity.
Pentapartitioned Neutrosophic Graph	Vertex quintuple $(T, C, U, F, S) : V \to [0,1]^5$ and edge quintuple $(T, C, U, F, S) : E \to [0,1]^5$, that is, a five-component refinement of neutrosophic information.
Plithogenic Graph	Vertex structure $PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf})$ and edge structure $PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf})$, where $\mathrm{adf} : M \times M_\ell \to [0,1]^s$ and $\mathrm{bdf} : N \times N_m \to [0,1]^s$ encode s-dimensional appurtenance, while aCf and bCf are symmetric contradiction maps with values in $[0,1]^t$.

Chapter 2

Preliminaries

This chapter introduces the notation and foundational concepts used throughout this book. Unless explicitly stated otherwise, all sets considered here are finite.

2.1 Fuzzy Graph

A fuzzy set assigns to each element a value in the interval $[0,1]$, thereby expressing graded membership rather than crisp inclusion [74, 75]. A fuzzy graph incorporates this idea into graph theory by allowing both vertices and edges to have membership values, so that uncertainty and partial connectivity can be represented naturally [29].

Definition 2.1.1 (Fuzzy set). [74] Let Y be a nonempty universe. A *fuzzy set* τ on Y is a map

$$\tau : Y \longrightarrow [0,1],$$

where $\tau(y)$ denotes the membership degree of $y \in Y$.

A *fuzzy relation* on Y is a fuzzy subset δ of $Y \times Y$. If τ is a fuzzy set on Y, then δ is called a *fuzzy relation on* τ whenever

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\}, \qquad \forall\, y, z \in Y.$$

Definition 2.1.2 (Fuzzy graph). [29] A *fuzzy graph* on a vertex set V is an ordered pair

$$G = (\sigma, \mu),$$

where:

- $\sigma : V \to [0,1]$ is the vertex membership function, and $\sigma(x)$ represents the degree to which the vertex x belongs to the graph;

- $\mu : V \times V \to [0,1]$ is the edge membership function, regarded as a fuzzy relation on σ, satisfying
$$\mu(x,y) \le \sigma(x) \wedge \sigma(y), \qquad \forall\, x,y \in V,$$
where $\wedge$ denotes the minimum.

The associated *crisp graph* $G^* = (\sigma^*, \mu^*)$ is defined by
$$\sigma^* = \{x \in V : \sigma(x) > 0\}, \qquad \mu^* = \{(x,y) \in V \times V : \mu(x,y) > 0\}.$$

A *fuzzy subgraph* $H = (\sigma', \mu')$ of G is obtained by taking a subset $X \subseteq V$ together with

- a restricted vertex membership function $\sigma' : X \to [0,1]$,
- an edge membership function $\mu' : X \times X \to [0,1]$ such that
$$\mu'(x,y) \le \sigma'(x) \wedge \sigma'(y), \qquad \forall\, x,y \in X.$$

2.2 Intuitionistic Fuzzy Graph

An intuitionistic fuzzy set associates with each element both a membership degree and a non-membership degree, with the sum bounded above by 1, so that hesitation can also be represented explicitly [76,77]. An intuitionistic fuzzy graph transfers this framework to graph structures by assigning such degrees to vertices and edges, thus modeling uncertainty, partial adjacency, and incomplete information [78].

Definition 2.2.1 (Intuitionistic Fuzzy Graph)**.** Let V be a nonempty vertex set. An *intuitionistic fuzzy graph* on V is a pair
$$G = (A, B),$$
where
$$A = \{(v, \mu_A(v), \nu_A(v)) : v \in V\}$$
is an intuitionistic fuzzy set on V, and
$$B = \{((u,v), \mu_B(u,v), \nu_B(u,v)) : u,v \in V\}$$
is an intuitionistic fuzzy relation on V, such that
$$0 \le \mu_A(v) + \nu_A(v) \le 1 \qquad \text{for all } v \in V,$$
and
$$\mu_B(u,v) \le \min\{\mu_A(u), \mu_A(v)\}, \qquad \nu_B(u,v) \ge \max\{\nu_A(u), \nu_A(v)\},$$
for all $u,v \in V$, with
$$0 \le \mu_B(u,v) + \nu_B(u,v) \le 1.$$

Here, μ_A and μ_B denote the membership degrees of vertices and edges, respectively, whereas ν_A and ν_B denote the corresponding non-membership degrees.

2.3 Neutrosophic Graph

A single-valued neutrosophic graph enriches the classical graph model by attaching to each vertex and edge three independent components: truth, indeterminacy, and falsity [56,79–81].

Definition 2.3.1 (Single-Valued Neutrosophic Graph)**.** [81] Let $G^* = (V, E)$ be a crisp graph, where V is the vertex set and $E \subseteq V \times V$ is the edge set. A *single-valued neutrosophic graph* (SVNG) on G^* is a pair

$$G = (A, B),$$

where

- $$A = \{\langle v, T_A(v), I_A(v), F_A(v)\rangle : v \in V\}$$

  is the *single-valued neutrosophic vertex set*, with

  $$T_A, I_A, F_A : V \to [0, 1],$$

  representing the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively, and satisfying

  $$0 \le T_A(v) + I_A(v) + F_A(v) \le 3 \qquad \forall\, v \in V;$$

- $$B = \{\langle uv, T_B(uv), I_B(uv), F_B(uv)\rangle : uv \in E\}$$

  is the *single-valued neutrosophic edge set*, where

  $$T_B, I_B, F_B : E \to [0, 1],$$

  and for every edge $uv \in E$,

  $$T_B(uv) \le \min\{T_A(u), T_A(v)\}, \qquad I_B(uv) \le \min\{I_A(u), I_A(v)\},$$

  $$F_B(uv) \ge \max\{F_A(u), F_A(v)\}.$$

If B is symmetric, then $G = (A, B)$ is called an *undirected SVNG*; otherwise, it is called a *directed SVNG*.

2.4 Plithogenic Graph

A plithogenic set describes elements through attribute values, appurtenance degrees, and contradiction degrees, making it suitable for modeling multi-valued, context-dependent, and potentially inconsistent information [38,82]. A plithogenic graph extends this viewpoint to graph structures by assigning attribute-based appurtenance and contradiction information to both vertices and edges [63].

Definition 2.4.1 (Plithogenic Set)**.** [38, 82] Let S be a universal set and let $P \subseteq S$ be a nonempty subset. A *Plithogenic Set* is a quintuple

$$PS = (P,\ v,\ Pv,\ pdf,\ pCF),$$

where

- v is an attribute;
- Pv is the set of possible values of the attribute v;
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)*;[1]
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF).*

The DCF satisfies, for all $a, b \in Pv$,

$$pCF(a,a) = 0, \qquad pCF(a,b) = pCF(b,a).$$

Here $s \in \mathbb{N}$ denotes the appurtenance dimension and $t \in \mathbb{N}$ denotes the contradiction dimension.

Definition 2.4.2 (Plithogenic Graph)**.** (cf. [63, 84]) Let $G = (V, E)$ be a crisp simple undirected graph, where

$$E \subseteq \{\{x, y\} : x, y \in V,\ x \neq y\}.$$

A *plithogenic graph* is a pair

$$PG = (PM, PN),$$

whose vertex part and edge part are given as follows.

Vertex part.

$$PM = (M,\ \ell,\ ML,\ \text{adf},\ \text{aCf}),$$

where

- $M \subseteq V$ is a selected subset of vertices;
- ℓ is an attribute assigned to vertices;
- ML is the set of possible values of ℓ;
- $\text{adf} : M \times ML \to [0,1]^s$ is the vertex DAF;
- $\text{aCf} : ML \times ML \to [0,1]^t$ is the vertex DCF.

Edge part.

$$PN = (N,\ m,\ ML',\ \text{bdf},\ \text{bCf}),$$

where

[1] In the literature, DAF appears in several variants. Some formulations use powerset-valued structures, while others adopt the simpler codomain $[0,1]^s$. In this book, we use the latter standard form; see also [83].

- $N \subseteq E$ is a selected subset of edges;
- m is an attribute assigned to edges;
- ML' is the set of possible values of m;
- $\mathrm{bdf} : N \times ML' \to [0,1]^s$ is the edge DAF;
- $\mathrm{bCf} : ML' \times ML' \to [0,1]^t$ is the edge DCF.

All inequalities in $[0,1]^k$ are understood componentwise, and $s, t \in \mathbb{N}$ are fixed. The following axioms are imposed.

(A1) **Edge–vertex compatibility.** For every $\{x, y\} \in N$ and all $a, b \in ML$,

$$\mathrm{bdf}\big(\{x, y\}, (a, b)\big) \le \min\big\{\mathrm{adf}(x, a),\ \mathrm{adf}(y, b)\big\}. \tag{2.1}$$

(A2) **Contradiction consistency.** For all $(a, b), (c, d) \in ML'$,

$$\mathrm{bCf}\big((a, b), (c, d)\big) \le \min\big\{\mathrm{aCf}(a, c),\ \mathrm{aCf}(b, d)\big\}. \tag{2.2}$$

(A3) **Reflexivity and symmetry of the DCFs.**

$$\mathrm{aCf}(u, u) = 0, \qquad \mathrm{aCf}(u, v) = \mathrm{aCf}(v, u) \qquad (\forall\, u, v \in ML),$$

$$\mathrm{bCf}(u, u) = 0, \qquad \mathrm{bCf}(u, v) = \mathrm{bCf}(v, u) \qquad (\forall\, u, v \in ML').$$

When $s = t = 1$, all of the above maps are scalar-valued on $[0, 1]$, and (2.1)–(2.2) reduce to ordinary scalar inequalities.

2.5 Uncertain Graph

An Uncertain Set assigns to each element a degree taken from a prescribed uncertainty model, thereby providing a common framework that includes fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic settings [85]. Likewise, an Uncertain Graph is a graph in which vertices and/or edges are equipped with degrees from such an uncertainty model, so that many well-known uncertainty-aware graph frameworks can be treated in a unified manner. We begin with the notion of an Uncertain Model, which specifies the domain of admissible membership degrees.

Definition 2.5.1 (Uncertain Model)**.** [85] Let U denote the class of all *uncertain models*. Each $M \in U$ is determined by the following data:

- a nonempty set $\mathrm{Dom}(M) \subseteq [0,1]^k$ of *admissible degree tuples*, for some fixed integer $k \ge 1$;

- a collection of model-dependent algebraic or geometric constraints imposed on elements of $\mathrm{Dom}(M)$ (for example, $\mu+\nu \le 1$ in the intuitionistic fuzzy setting, or $0 \le T+I+F \le 3$ in the neutrosophic setting).

Typical examples are as follows:

- Fuzzy model: $\mathrm{Dom}(M) = [0, 1]$;
- Intuitionistic fuzzy model:
$$\mathrm{Dom}(M) = \{(\mu, \nu) \in [0, 1]^2 : \mu + \nu \le 1\};$$
- Neutrosophic model:
$$\mathrm{Dom}(M) = \{(T, I, F) \in [0, 1]^3 : 0 \le T + I + F \le 3\};$$
- Plithogenic model, together with many further extensions.

Definition 2.5.2 (Uncertain Set (U-Set))**.** [85] Let X be a nonempty universe, and let M be a fixed uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$. An *Uncertain Set of type M* (briefly, a *U-Set*) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is called the *uncertainty-degree function* (or membership map) of $\mathcal{U}$.

For each $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ records the degree, or collection of degrees, with which x belongs to the uncertain set under the model M.

Remark 2.5.3. Several familiar structures appear as special cases:

- If M is the fuzzy model with $\mathrm{Dom}(M) = [0, 1]$, then $\mu_M : X \to [0, 1]$ is an ordinary fuzzy membership function, and $\mathcal{U}$ is a fuzzy set.
- If M is a neutrosophic model, then $\mu_M(x) = (T(x), I(x), F(x))$, and $\mathcal{U}$ becomes a neutrosophic set.
- Other choices of M recover intuitionistic fuzzy sets, picture fuzzy sets, plithogenic sets, and many related models.

As indicated above, this framework admits many different specializations. For reference, Table 2.1 lists representative families of uncertainty sets (U-Sets), organized according to the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$ (cf. [86]).

We now introduce the corresponding notion of an Uncertain Graph.

Table 2.1: A catalogue of uncertainty-set families (U-Sets) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [86].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [30, 74]; N-Fuzzy Set [87–89]; Shadowed Set [90–92]
2		Intuitionistic Fuzzy Set [77, 93]; Vague Set [94, 95]; Bipolar Fuzzy Set (two-component description) [96]; Variable Fuzzy Set [97–99]; Paraconsistent Fuzzy Set [100, 101]; Bifuzzy Set [102, 103]
3		Single-Valued Neutrosophic Set [104, 105]; Picture Fuzzy Set [106, 107]; Spherical Fuzzy Set [108, 109]; Tripolar Fuzzy Set (three-component formalisms) [110–112]; Neutrosophic Vague Set [49, 113]
4		Quadripartitioned Neutrosophic Set [114, 115]; Double-Valued Neutrosophic Set [116, 117]; Dual Hesitant Fuzzy Set [118, 119]; Ambiguous Set [120–122]; Turiyam Neutrosophic Set [123–126]
5		Pentapartitioned Neutrosophic Set [127–129]; Triple-Valued Neutrosophic Set [130–132]
6		Hexapartitioned Neutrosophic Set; Quadruple-Valued Neutrosophic Set [131, 133]
7		Heptapartitioned Neutrosophic Set; Quintuple-Valued Neutrosophic Set [131, 134, 135]
8		Octapartitioned Neutrosophic Set [136, 137]
9		Nonapartitioned Neutrosophic Set [136, 137]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [138]; MultiFuzzy Set [139]; n-Refined Fuzzy Set [140, 141]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [141]; Multi-Intuitionistic Fuzzy Set [139]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [141]; Multi-Neutrosophic Set [139, 142]

Reading guide. Within the U-Set framework [85], each model M is determined by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ together with a membership map $\mu_M : X \to \mathrm{Dom}(M)$. Accordingly, the table groups representative families by the ambient dimension k, that is, by the number of numerical components assigned to each element.

(a) A common viewpoint in the literature is that neutrosophic sets form a broad umbrella encompassing several earlier multi-component fuzzy models and their generalizations; see [143].

(b) Ambiguous sets are often described as subclasses of certain four-component neutrosophic families; see [114, 115, 122].

(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [144].

Definition 2.5.4 (Uncertain Graph)**.** Let $G = (V, E)$ be a finite undirected loopless graph, and let M be an uncertain model with degree-domain $\mathrm{Dom}(M)$. An *Uncertain Graph of type M* is a triple

$$\mathcal{G}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns to every vertex $v \in V$ and every edge $e \in E$ an uncertainty degree $\mu_M(v)$ or $\mu_M(e)$ belonging to $\mathrm{Dom}(M)$.

If desired, one may additionally impose model-dependent consistency conditions relating vertex and edge degrees. For example, in fuzzy or intuitionistic fuzzy settings, the degree of an edge $e = \{u, v\}$ is often bounded in terms of the degrees of its incident vertices. However, such conditions depend on the particular model M and are therefore not built into this general definition.

Remark 2.5.5. Well-known graph models are recovered by choosing M appropriately:

- fuzzy graphs arise when M is the fuzzy model and $\mu_M : V \cup E \to [0,1]$;

- intuitionistic fuzzy graphs, neutrosophic graphs, plithogenic graphs, and related models are obtained from the corresponding choices of M.

For reference, Table 2.2 presents a catalogue of uncertainty-graph families (Uncertain Graphs), again classified by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

Table 2.2: A catalogue of uncertainty-graph families (Uncertain Graphs) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-graph type(s) $\mathcal{G}_M = (V, E, \mu_M)$ with $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$
1	Fuzzy graph; N-graph; shadowed-graph variants
2	Intuitionistic fuzzy graph [145]; vague graph [146]; bipolar fuzzy graph [34]; intuitionistic evidence graph; variable fuzzy graph; paraconsistent fuzzy graph; bifuzzy graph [147,148]
3	Neutrosophic graph [81][(a)]; hesitant fuzzy graph [149]; tripolar fuzzy graph; three-way fuzzy graph; picture fuzzy graph [150, 151]; spherical fuzzy graph [108]; inconsistent intuitionistic fuzzy graph; ternary fuzzy / neutrosophic-fuzzy graph; neutrosophic vague graph
4	Quadripartitioned neutrosophic graph [152,153]; double-valued neutrosophic graph [116]; dual hesitant fuzzy graph [154]; ambiguous graph [(b)]; local-neutrosophic graph; support-neutrosophic graph; turiyam neutrosophic graph [155][(c)]
5	Pentapartitioned neutrosophic graph [156]; triple-valued neutrosophic graph
6	Hexapartitioned neutrosophic graph; quadruple-valued neutrosophic graph
7	Heptapartitioned neutrosophic graph [157]; quintuple-valued neutrosophic graph
8	Octapartitioned neutrosophic graph
9	Nonapartitioned neutrosophic graph
n	n-refined fuzzy graph; multi-valued (fuzzy) graphs; multi-fuzzy graphs [158]
$2n$	n-refined intuitionistic fuzzy graph; multi-intuitionistic fuzzy graphs
$3n$	n-refined neutrosophic graph; multi-neutrosophic graphs

[(a)] Neutrosophic graph models are often regarded as broad frameworks that can specialize to many degree-based graph formalisms under suitable additional constraints.
[(b)] Ambiguous graph models are commonly described as subclasses of certain quadripartitioned, and in some cases also double-valued, neutrosophic graph models.
[(c)] Turiyam neutrosophic graphs are reported as subclasses of certain quadripartitioned neutrosophic graph models.

2.6 Soft Graph

A soft graph is a parameter-dependent graph structure in which each parameter determines a corresponding subgraph. This framework is useful for modeling systems whose relationships change across conditions, viewpoints, or scenarios.

Definition 2.6.1 (Soft Graph)**.** Let $G^* = (V, E)$ be a simple graph, and let A be a nonempty set of parameters. A *soft graph* over G^* is a quadruple

$$G = (G^*, F, K, A),$$

where

$$F : A \to \mathcal{P}(V), \qquad K : A \to \mathcal{P}(E),$$

such that, for every $a \in A$, the pair

$$H(a) = \big(F(a), K(a)\big)$$

forms a subgraph of G^*.

2.7 Rough Graph

A rough graph describes a graph by means of lower and upper approximation graphs induced by an equivalence relation. In this way, it provides a graph-theoretic model for indiscernibility, vagueness, and boundary-region uncertainty [159, 160].

Definition 2.7.1 (Rough Graph)**.** Let $U = (V, E)$ be a universe graph, and let R be an equivalence relation on E. For each $e \in E$, denote by $[e]_R$ the R-equivalence class of e.

Let $T = (W, X)$ be a graph with $W \subseteq V$ and $X \subseteq E$. Define

$$\underline{R}(X) = \{\, e \in E : [e]_R \subseteq X \,\}, \qquad \overline{R}(X) = \{\, e \in E : [e]_R \cap X \neq \varnothing \,\}.$$

Then the pair

$$\big(\underline{R}(T), \overline{R}(T)\big) = \big((W, \underline{R}(X)), (W, \overline{R}(X))\big)$$

is called the *rough graph* associated with T.

If X is not a union of R-equivalence classes, then T is called an *R-rough graph*.

2.8 Sup erHyp ergraph

We begin with the set-theoretic operations underlying SuperHyperGraphs and then recall the relevant graph-theoretic definitions. A hypergraph is a natural generalization of an ordinary graph in which an edge, usually called a *hyperedge*, may join any number of vertices rather than only two. We begin with the standard definition [161].

Definition 2.8.1 (Hypergraph)**.** [161] A *hypergraph* is a pair

$$H = (V(H), E(H)),$$

where $V(H)$ is a nonempty set of vertices and $E(H)$ is a family of subsets of $V(H)$, called the *hyperedges* of H. In this book, all hypergraphs are assumed to be finite.

Hypergraphs are useful for modeling higher-order relationships in many areas, including computer science and biology [162, 163]. They are also actively studied in database theory [164, 165] and in hypergraph neural networks [166, 167].

A SuperHyperGraph is a higher-order structure whose supervertices belong to an iterated powerset of a base set, and whose superhyperedges are nonempty subsets thereof [168–170]. Moreover, related concepts such as Recursive SuperHyperGraphs [171, 172], Meta-SuperHyperGraphs [173, 174], and Hierarchical SuperHyperGraphs [173, 175, 176] are also known. Their definitions and related notions are given below.

Definition 2.8.2 (Base set)**.** A *base set* S is the underlying universe of admissible objects in the setting under consideration, i.e.,

$$S = \{\, x \mid x \text{ is an admissible object in the given context} \,\}.$$

Accordingly, every element of $\mathcal{P}(S)$—and, more generally, of any iterated powerset—is ultimately formed from elements of S.

Definition 2.8.3 (Powerset)**.** (see [177]) For a set S, its *powerset* is the family of all subsets of S:

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

In particular, $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$.

Definition 2.8.4 (Hypergraph)**.** [161, 178] A *hypergraph* is an ordered pair $H = (V, E)$ where

- V is a finite set of *vertices*, and
- E is a finite family of nonempty subsets of V, called *hyperedges*.

Thus a hyperedge may contain more than two vertices, allowing one to model genuinely multiway relations.

Definition 2.8.5 (n-fold iterated powerset)**.** [179, 180] Let X be a set. Define $\mathcal{P}^1(X) := \mathcal{P}(X)$ and, for $n \geq 1$, set recursively

$$\mathcal{P}^{n+1}(X) := \mathcal{P}\big(\mathcal{P}^n(X)\big).$$

When excluding the empty set, we write

$$\mathcal{P}_*^n(X) := \mathcal{P}^n(X) \setminus \{\emptyset\}.$$

Definition 2.8.6 (n-SuperHyperGraph)**.** (see [168, 181]) Let V_0 be a finite, nonempty base set, and define the iterated powersets by

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \in \mathbb{N} \cup \{0\}).$$

For $n \geq 0$, an *n-SuperHyperGraph on V_0* is a pair

$$\mathrm{SHG}^{(n)} = (V, E)$$

satisfying

$$V \subseteq \mathcal{P}^n(V_0) \qquad \text{and} \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Elements of V are called *n-supervertices*, and elements of E are called *superhyperedges*; equivalently, each superhyperedge is a nonempty subset of the supervertex set V.

Chapter 3

Classical Planar Graph and Related Concepts

In this chapter, we consider classical planar graphs and related concepts.

3.1 Planar Graph

Planar graphs and their related classes, such as co-planar graphs and k-planar graphs, have been studied extensively and have found applications in network analysis, graph drawing, VLSI design, and many other areas. We begin with the notion of an edge crossing.

Definition 3.1.1 (Edge Crossing)**.** (cf. [182, 183]) Let $G = (V, E)$ be a graph drawn in the plane. An *edge crossing* occurs when two edges $e_1 = (u, v)$ and $e_2 = (x, y)$ intersect at a point p other than a common endpoint. That is, p is not one of the vertices u, v, x, or y.

If a graph admits a drawing in the plane with no edge crossings, then it is called a *planar graph*.

Definition 3.1.2 (Planar Graph)**.** (cf. [184–186]) A graph $G = (V, E)$ is called *planar* if it can be embedded in the plane in such a way that no two edges intersect except possibly at a common endpoint. Equivalently, G is planar if it has a drawing in the plane with no edge crossings.

Example 3.1.3 (A Planar Graph)**.** A simple example of a planar graph is K_4 with one edge removed.

Let

$$V = \{1, 2, 3, 4\}, \qquad E = \big\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\big\}.$$

Then the graph $G = (V, E)$ has four vertices and five edges.

This graph can be drawn in the plane without any edge crossings. For instance, one may draw a triangle and place the fourth vertex inside it, connecting it appropriately so that no two edges cross. Hence, G is planar.

Definition 3.1.4 (Non-Planar Graph)**.** (cf. [187, 188]) A graph $G = (V, E)$ is called *non-planar* if it does not admit any drawing in the plane without edge crossings, except at shared endpoints. Equivalently, every planar drawing attempt of G necessarily produces at least one crossing between two edges at a point other than a vertex.

Example 3.1.5 (A Non-Planar Graph)**.** A classical example of a non-planar graph is the complete graph K_5.

Let
$$V = \{1, 2, 3, 4, 5\},$$
and let
$$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}.$$
This graph contains all possible edges among five vertices, and therefore has ten edges.

It is well known that K_5 cannot be drawn in the plane without edge crossings. Thus, K_5 is non-planar.

Among the graph classes related to planar graphs, one important example is the co-planar graph.

Definition 3.1.6 (Co-Planar Graph)**.** (cf. [189,190]) A graph $G = (V, E)$ is called *co-planar* if its complement graph $\overline{G}$ is planar. In other words, G is co-planar precisely when the graph on the same vertex set, whose edges are exactly the nonedges of G, can be drawn in the plane without edge crossings.

Example 3.1.7 (A Co-Planar Graph)**.** Consider the graph $G = (V, E)$ with
$$V = \{1, 2, 3, 4\},$$
and
$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$
Then the complement graph $\overline{G}$ has edge set
$$E(\overline{G}) = \{\{1, 4\}\}.$$
Since a graph with a single edge is clearly planar, $\overline{G}$ is planar. Therefore, G is co-planar.

We next recall the notion of a k-planar graph [191, 192], which generalizes the concept of planarity. Important special cases include 1-planar graphs [23, 24] and 2-planar graphs [193, 194].

Definition 3.1.8 (k-Planar Graph)**.** (cf. [191,192]) A graph $G = (V, E)$ is called k*-planar* if it admits a drawing in the plane such that each edge is crossed by at most k other edges.

More precisely, let D be a drawing of G in the plane, where each edge $e = (u, v) \in E$ is represented by a continuous curve joining u and v. Then G is k-planar if there exists such a drawing D for which
$$\text{cross}(e) \leq k \qquad \text{for every } e \in E,$$
where cross(e) denotes the number of crossings involving the edge e.

In particular:

- when $k = 0$, one recovers the class of planar graphs;
- when $k = 1$, one obtains the class of 1-planar graphs.

Planar graphs are structurally simple and often more manageable than general graphs because they avoid edge crossings. For this reason, they are useful in a variety of application domains. Some representative examples are listed below.

- **Urban street patterns.** Urban street patterns describe the arrangement and organization of streets in cities, including grid, radial, and irregular layouts [195, 196]. Such systems have also been studied from the viewpoint of planar graphs [197, 198].
- **Neural networks and graph learning.** Neural networks are computational models inspired by the human brain and are widely used for pattern recognition and data analysis [199, 200]. In graph-based machine learning, graph neural networks have become a particularly active topic [4, 201]. Related studies have also been developed for planar graph settings [202, 203].
- **Computer vision.** Computer vision is a field of artificial intelligence concerned with enabling computers to interpret visual information from images and videos [204, 205]. Planar graph methods have also been considered in this context [206, 207].
- **VLSI design.** VLSI (Very Large Scale Integration) concerns the design of integrated circuits containing very large numbers of transistors on a single chip [208, 209]. Within this field, floorplanning problems are often studied, since they involve the determination of module sizes, shapes, and placements, with direct impact on chip area, interconnections, and delay [210, 211]. Planar graph methods have also been investigated in this area [212, 213].
- **Graph drawing.** Graph drawing studies visual representations of graphs and supports the analysis of relationships arising in data science, networks, linguistics, bioinformatics, and related applications [214–216]. The drawing of planar graphs has, in particular, been a major topic of study [217, 218].

3.2 Outerplanar Graph

Outerplanar graphs, like planar graphs, have been studied extensively. Related parameters include outerthickness [219], outercoarseness [219], outerplanarity [220], and outer-2-planarity [221, 222]. Moreover, as in the case of planar graphs, outerplanar graphs and their variants have been investigated in connection with networks and many other applications. We now recall the definitions of outerplanar graphs and two related generalizations, namely outer-k-planar graphs and k-outerplanar graphs.

Definition 3.2.1 (Outerplanar Graph)**.** (cf. [15, 223]) An *outerplanar graph* is an undirected graph $G = (V, E)$ that admits a plane embedding in which all vertices lie on the boundary of the outer face. Equivalently, G can be drawn in the plane so that

1. no two edges cross except possibly at a common endpoint, and

2. every vertex lies on the boundary of the unbounded face.

Definition 3.2.2 (Outer-k-Planar Graph)**.** (cf. [224]) A graph $G = (V, E)$ is called *outer-k-planar* if it has a drawing in the plane such that all vertices lie on the boundary of the outer face and each edge is crossed by at most k other edges.

More precisely, G is outer-k-planar if there exists a drawing D of G in which

- every vertex $v \in V$ lies on the boundary of the unbounded face, and
- for every edge $e \in E$,
$$\mathrm{cross}(e) \leq k,$$
where $\mathrm{cross}(e)$ denotes the number of crossings involving e in the drawing D.

Definition 3.2.3 (k-Outerplanar Graph)**.** (cf. [225–228]) A graph $G = (V, E)$ is called *k-outerplanar* if it admits a planar embedding whose vertices can be arranged into at most k layers relative to the outer face.

More precisely, let D be a planar embedding of G. The layers are defined recursively as follows:

1. The first layer consists of all vertices lying on the outer face of D.
2. After removing all vertices belonging to layers $1, \ldots, i-1$, the vertices that lie on the outer face of the remaining graph form the i-th layer.
3. The process continues until all vertices have been assigned to layers.

If all vertices are assigned within at most k layers, then G is called k-outerplanar.

The following basic fact is immediate.

Theorem 3.2.4. *(cf. [5, 229]) Every outerplanar graph is planar.*

Many graph classes are closely related to planar and outerplanar graphs. Their abundance further illustrates the importance of studying planar-type structures.

Notation 3.2.5. *In this book, the term* related graph class *refers to a graph class that extends, restricts, or otherwise modifies a given graph class in a structurally meaningful way.*

Remark 3.2.6 (Examples of related graph classes)**.** Examples of graph classes related to planar or outerplanar graphs include, among many others, the following:

- Halin graphs [18]
- Fan-planar graphs [230]

- Fan-crossing-free graphs [231]
- IC-planar graphs [232]
- Upward planar graphs [19–21]
- Convex planar graphs [22]
- 1-planar graphs [23, 24]
- 2-planar graphs [193, 194]
- 3-planar graphs [233]
- Interval planar graphs [234]
- k-planar graphs [25]
- Outer-1-planar graphs [235–237]
- Outer-k-planar graphs [220]
- Quasi-planar graphs [238, 239]
- k-quasi-planar graphs [240–242]
- Apex graphs [243–246] [1]
- Pseudo-outerplanar graphs [248]
- Biconnected outerplanar graphs [249, 250]
- Strongly connected outerplanar graphs [251]
- Biconnected planar graphs [252]
- Gap-planar graphs [253]
- Non-separating planar graphs [254]
- Apex-outerplanar graphs [255]
- Universal outerplanar graphs [256]
- Outer-projective-planar graphs [257]
- Planar Ramsey graphs [258]
- Cubic outerplanar graphs [259]
- Outerplanar partial cubes [260]
- Directed planar graphs [261–263]
- Planar partial cubes [264, 265]
- Planar median graphs [266]
- Planar straight-line graphs [267]
- Directed outerplanar graphs [268]
- Maximum planar subgraph [269]
- Weighted planar graphs [270, 271]
- RAC graphs [272–274]
- Planarly connected graphs [275]
- 2-connected planar graphs [276]
- Bipartite IC-planar graphs [274]

[1] The term *apex* is also used more broadly in graph theory to measure how close a graph is to a desired class; see [247].

- Bipartite NIC-planar graphs [274]
- Bipartite RAC graphs [274]
- Cubic planar graphs [277]

3.3 Ap ex Graph

An apex graph is a graph that becomes planar after the deletion of a single vertex. This notion has been studied extensively in connection with planar graph theory [243, 244], and it is naturally related to vertex-deletion operations in graphs (cf. [278, 279]). We first recall the standard definition.

Definition 3.3.1 (Apex Graph)**.** (cf. [243,244]) A graph $G = (V, E)$ is called an *apex graph* if there exists a vertex $v \in V$ such that the graph

$$G[V \setminus \{v\}]$$

is planar. Such a vertex v is called an *apex* of G.

Some immediate observations are as follows.

- Every planar graph is trivially an apex graph.
- A graph may have more than one apex vertex.
- The class of apex graphs may be viewed as a one-vertex extension of the class of planar graphs.

A natural generalization is obtained by allowing the removal of more than one vertex.

Definition 3.3.2 (k-Apex Graph)**.** (cf. [280–282]) Let $k \geq 0$. A graph $G = (V, E)$ is called a *k-apex graph* if there exists a subset

$$S \subseteq V, \qquad |S| \leq k,$$

such that the induced subgraph

$$G[V \setminus S]$$

is planar.

The following special cases are immediate.

- If $k = 0$, then a k-apex graph is simply a planar graph.
- If $k = 1$, then one recovers the usual notion of an apex graph.

- In general, there may be several different vertex subsets whose removal yields a planar graph.

Example 3.3.3 (A concrete example of a k-apex graph)**.** Let $k \geq 1$ be an integer. For each $i \in \{1, \ldots, k\}$, let

$$H_i \cong K_5,$$

and assume that the graphs $H_1, \ldots, H_k$ are pairwise vertex-disjoint. Define

$$G := \bigsqcup_{i=1}^{k} H_i,$$

that is, G is the disjoint union of k copies of the complete graph K_5.

Then G is a k-apex graph.

Indeed, for each i, choose a vertex $v_i \in V(H_i)$, and let

$$X := \{v_1, \ldots, v_k\}.$$

Since

$$H_i - v_i \cong K_4$$

for every i, we obtain

$$G - X \cong \bigsqcup_{i=1}^{k} K_4.$$

Because K_4 is planar and a disjoint union of planar graphs is planar, it follows that $G - X$ is planar. Hence there exists a set $X \subseteq V(G)$ with $|X| = k$ such that $G - X$ is planar, so G is k-apex.

Moreover, G is not $(k-1)$-apex. In fact, if fewer than k vertices are deleted from G, then at least one component $H_j \cong K_5$ remains untouched. Since K_5 is non-planar, the resulting graph is still non-planar. Therefore, G is precisely a k-apex graph.

3.4 Ap ex Outerplanar Graph

In an analogous manner, one may consider graphs that become outerplanar after the deletion of a single vertex. Such graphs have also been studied in the literature [283, 284].

Definition 3.4.1 (Apex Outerplanar Graph)**.** [283, 284] A graph $G = (V, E)$ is called an *apex outerplanar graph* if there exists a vertex $v \in V$ such that

$$G[V \setminus \{v\}]$$

is outerplanar. Such a vertex v is called an *apex* of G with respect to outerplanarity.

Again, several basic remarks follow.

- Every outerplanar graph is trivially an apex outerplanar graph.
- A graph may admit more than one apex whose deletion yields an outerplanar graph.
- This class may be regarded as a one-vertex extension of the class of outerplanar graphs.

This leads naturally to the following generalization.

Definition 3.4.2 (k-Apex Outerplanar Graph)**.** Let $k \geq 0$. A graph $G = (V, E)$ is called a *k-apex outerplanar graph* if there exists a subset

$$S \subseteq V, \qquad |S| \leq k,$$

such that

$$G[V \setminus S]$$

is outerplanar.

In particular:

- if $k = 0$, then the graph is already outerplanar;
- if $k = 1$, then the graph is an apex outerplanar graph;
- for general k, different choices of S may produce an outerplanar graph.

3.5 Quasi-Planar Graph

A quasi-planar graph is a graph that can be drawn in the plane in such a way that no three edges are pairwise crossing. This class has been studied extensively together with planar and near-planar graph classes [238, 285, 286].

Definition 3.5.1 (Quasi-Planar Graph)**.** [238, 239] A graph $G = (V, E)$ is called *quasi-planar* if it admits a drawing in the plane in which no three edges are pairwise crossing. Equivalently, there is no set of three edges each pair of which crosses in the drawing.

A few immediate observations are worth noting.

- Every planar graph is quasi-planar.
- Quasi-planar graphs form a broader class than planar graphs, since crossings are allowed as long as no three edges cross pairwise.

A more general version is obtained by forbidding k pairwise crossing edges.

Definition 3.5.2 (k-Quasi-Planar Graph)**.** [240, 287] Let $k \geq 2$. A graph $G = (V, E)$ is called *k-quasi-planar* if it can be drawn in the plane so that no k edges are pairwise crossing. In other words, there is no set of k edges such that every two of them cross.

Some special cases are immediate.

- When $k = 2$, the condition forbids any crossing at all, so the graph is planar.
- When $k = 3$, one recovers the usual notion of a quasi-planar graph.
- Thus, k-quasi-planar graphs provide a hierarchy of increasingly permissive crossing constraints.

3.6 Outer- k-Quasi-Planar Graph

An outer-k-quasi-planar graph is a graph drawable with all vertices on the outer face and no k edges pairwise crossing. Motivated by the classes above, we introduce an outer analogue of k-quasi-planarity.

Definition 3.6.1 (Outer-k-Quasi-Planar Graph)**.** A graph $G = (V, E)$ is called *outer-k-quasi-planar* if it admits a drawing in the plane such that

- all vertices lie on the boundary of the outer face, and
- no k edges are pairwise crossing.

Equivalently, G has an outer-face drawing in which every set of k edges fails to be pairwise crossing.

This notion combines the outer-face condition of outerplanar-type graphs with the crossing restriction of k-quasi-planar graphs.

3.7 Gap-Planar Graph and Fan-Planar Graph

Among the graph classes related to planar graphs, gap-planar graphs [253, 288] and fan-planar graphs [230, 289–291] have attracted considerable attention. A gap-planar graph is a graph drawable so that each crossing is assigned to one incident edge without assignment conflicts. A fan-planar graph is a graph drawable so that crossed edges intersect only within a single fan around one vertex. We first recall their definitions.

Definition 3.7.1 (k-Gap-Planar Graph)**.** [253] A graph $G = (V, E)$ is called *k-gap-planar* if it admits a drawing in the plane such that each crossing is assigned to exactly one of the two edges involved, and each edge is assigned at most k crossings.

Equivalently, G is k-gap-planar if there exists a drawing in which the crossings can be distributed among the incident edges so that no edge receives more than k assigned crossings.

Some immediate remarks are as follows.

- When $k = 0$, a k-gap-planar graph is simply a planar graph.
- The notion of k-gap-planarity measures, in a controlled way, how far a graph may deviate from planarity.

Definition 3.7.2 (Fan-Planar Graph)**.** (cf. [230, 289–291]) A graph $G = (V, E)$ is called *fan-planar* if it has a drawing in the plane such that whenever an edge is crossed, all edges crossing it are pairwise adjacent, that is, they share a common endpoint and form a fan-like configuration.

Thus, fan-planarity does not forbid crossings altogether, but restricts them to a highly structured pattern. This distinguishes fan-planar graphs from arbitrary non-planar graphs.

Graphs related to outerplanar graphs include, in particular, outer-fan-planar graphs [231, 292–294]. We record the corresponding definition below.

Definition 3.7.3 (Outer-Fan-Planar Graph)**.** (cf. [231, 292–294]) A graph $G = (V, E)$ is called *outer-fan-planar* if it admits a drawing in the plane such that

- all vertices lie on the boundary of the outer face, and
- whenever an edge is crossed, all edges crossing it are pairwise adjacent, forming a fan.

Equivalently, one may view such a drawing as an outer-face drawing in which the allowed crossings satisfy the fan-planar condition.

Several useful remarks from the literature may be kept in mind.

- In outer-fan-planar drawings, the vertices are often placed on a common circle representing the outer face.
- In the study of maximal outer-fan-planar graphs, biconnectivity plays an important role.
- It has been shown that an outer-fan-planar graph with n vertices has at most $5n - 10$ edges.

3.8 Inner Planar Graph

An inner planar graph is a planar graph admitting an embedding with internal vertices not lying on the outer boundary. More recently, the notion of an inner planar graph has also been considered. We recall the definition below [295].

Definition 3.8.1 (k-Inner Planar Graph)**.** [295] A graph $G = (V, E)$ is called *k-inner planar* if it admits a planar embedding in which at most k vertices do not lie on the boundary of the outer face. Such vertices are called *inner vertices*.

This definition yields the following immediate special cases.

- When $k = 0$, the graph is outerplanar.
- For $k > 0$, one allows up to k vertices to lie away from the outer face while preserving planarity.

3.9 Almost Planar Graph

Whereas apex-type graph classes are defined by deleting vertices, there are also related classes defined by deleting edges. Among these are almost planar graphs [296–298] and almost outerplanar graphs [299, 300]. These are also closely connected with edge-deletion operations (cf. [301–304]). We now introduce the relevant definitions.

Definition 3.9.1 (Almost Planar Graph)**.** [296–298] A graph G is called *almost planar* if there exists an edge $e \in E(G)$ such that

$$G - e$$

is planar.

Definition 3.9.2 (k-Almost Planar Graph)**.** (cf. [305, 306]) A graph $G = (V, E)$ is called *k-almost planar* if there exists a subset

$$F \subseteq E(G), \qquad |F| \leq k,$$

such that

$$G - F$$

is planar. In other words, the graph can be made planar by deleting at most k edges.

Definition 3.9.3 (Almost Outerplanar Graph)**.** [299, 300] A graph G is called *almost outerplanar* if there exists an edge $e \in E(G)$ such that

$$G - e$$

is outerplanar.

Definition 3.9.4 (k-Almost Outerplanar Graph)**.** A graph $G = (V, E)$ is called *k-almost outerplanar* if there exists a subset

$$F \subseteq E(G), \qquad |F| \leq k,$$

such that

$$G - F$$

is outerplanar. Equivalently, the graph becomes outerplanar after the deletion of at most k edges.

Almost planar and almost outerplanar graphs are closely related to the notion of *skewness-k* in graph drawing [307, 308]. A graph is said to have *skewness k* if k edge deletions are sufficient to make it planar [308]. In addition to the classes introduced above, other related notions are also well known, including (k, ℓ)-grid-free drawings [309] and k-fan-crossing-free drawings [310]. Taken together, these developments further illustrate the richness and importance of research on planar and near-planar graph classes.

Chapter 4

Fuzzy Planar Graph and Fuzzy Outerplanar Graph

In recent years, motivated by the classical theories of planar and outerplanar graphs, the notions of fuzzy planar graphs and fuzzy outerplanar graphs have also been introduced (cf. [70, 311–314]). To prepare for these concepts, we first recall the definition of a fuzzy graph. A fuzzy graph extends ordinary graph theory by incorporating the basic idea of fuzzy sets [74, 315]. Extensive work has been carried out on fuzzy graphs [29, 316] as well as on fuzzy planar graphs [69, 317, 318].

Definition 4.0.1 (Fuzzy Graph)**.** [29, 30] A *fuzzy graph* is a triple

$$\psi = (V, \sigma, \mu),$$

where

- V is a set of vertices;
- $\sigma : V \to [0, 1]$ assigns to each vertex $v \in V$ its membership degree;
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation satisfying

$$\mu(u, v) \le \min\{\sigma(u), \sigma(v)\} \qquad \text{for all } u, v \in V.$$

Typically, μ is assumed to be symmetric, that is,

$$\mu(u, v) = \mu(v, u) \qquad \text{for all } u, v \in V,$$

and loopless in the sense that

$$\mu(v, v) = 0 \qquad \text{for all } v \in V.$$

Thus, a fuzzy graph represents uncertainty not only in the presence of vertices but also in the strength of adjacency relations.

Definition 4.0.2 (Fuzzy Planar Graph)**.** (cf. [69, 70, 319–321]) A fuzzy graph

$$\psi = (V, \sigma, \mu, E)$$

is called a *fuzzy planar graph* if it admits a planar representation compatible with its fuzzy structure, where:

- V is the vertex set;
- $\sigma : V \to [0, 1]$ is the vertex-membership function;
- $\mu : V \times V \to [0, 1]$ is the fuzzy adjacency relation satisfying

$$\mu(u, v) \le \min\{\sigma(u), \sigma(v)\} \qquad \text{for all } u, v \in V;$$

- $E \subseteq V \times V$ is the underlying edge set, with each edge $(u, v) \in E$ carrying membership value $\mu(u, v)$.

If two fuzzy edges (a, b) and (c, d) intersect at a point p, then the corresponding intersection strength is defined by

$$I_p = \frac{I(a, b) + I(c, d)}{2},$$

where

$$I(a, b) = \frac{\mu(a, b)}{\sigma(a) \wedge \sigma(b)}.$$

The *fuzzy planarity value* of the graph is then given by

$$f = \frac{1}{1 + \sum_{i=1}^{n} I_{p_i}},$$

where $p_1, \ldots, p_n$ are the intersection points in the drawing. In particular, $f = 1$ for a completely planar fuzzy graph.

Proposition 4.0.3. *Every fuzzy planar graph is, in particular, a fuzzy graph.*

Proof. This follows immediately from the definition, since a fuzzy planar graph is defined as a fuzzy graph together with an additional planarity condition. □

Remark 4.0.4 (Examples of related graph classes)**.** Examples of graph classes related to fuzzy planar graphs include:

- interval-valued fuzzy planar graphs [319],
- m-polar fuzzy planar graphs [322, 323],
- complex Pythagorean fuzzy planar graphs [324],
- complex q-rung orthopair fuzzy planar graphs [325],
- picture fuzzy planar graphs [326],
- inverse fuzzy mixed planar graphs [327],
- bipolar fuzzy planar graphs [328].

4.1 Fuzzy Outerplanar Graph

A fuzzy outerplanar graph is a fuzzy graph admitting an outerplanar embedding with all vertices on the outer face boundary. We next recall the notion of a fuzzy outerplanar graph [71], which incorporates the outer-face condition into the fuzzy setting.

Definition 4.1.1 (Fuzzy Outerplanar Graph)**.** [71] A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy outerplanar graph* if it can be embedded in the plane in such a way that every vertex lies on the boundary of the outer face and no fuzzy edge intersections occur.

Equivalently, a fuzzy outerplanar graph is a fuzzy graph satisfying the following conditions:

- $\sigma : V \to [0, 1]$ gives the membership degree of each vertex;
- $\mu : V \times V \to [0, 1]$ gives the membership degree of each fuzzy edge, with

$$\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\};$$

- all vertices lie on the boundary of the exterior region in some embedding;
- no fuzzy edge crossings occur in that embedding.

If $i(\psi)$ denotes the number of vertices not lying on the boundary of the outer face, then ψ is fuzzy outerplanar precisely when

$$i(\psi) = 0.$$

If $i(\psi) \neq 0$, then the graph is regarded as fuzzy non-outerplanar.

Proposition 4.1.2. *Every fuzzy outerplanar graph is a fuzzy graph.*

Proof. The statement is immediate from the definition. □

Proposition 4.1.3. *Let $\psi = (V, \sigma, \mu)$ be a fuzzy outerplanar graph. Then the support graph obtained by retaining the vertices and edges of positive membership is a classical outerplanar graph.*

Proof. Consider the support graph

$$G^* = (V^*, E^*),$$

where

$$V^* = \{v \in V : \sigma(v) > 0\}, \qquad E^* = \{\{u, v\} : \mu(u, v) > 0\}.$$

Since ψ admits an embedding in which all vertices lie on the boundary of the outer face and no fuzzy edge crossings occur, the same embedding induces an outerplanar drawing of G^*. Hence G^* is a classical outerplanar graph. □

4.2 Fuzzy Ap ex Graph

In ordinary graph theory, apex graphs form a well-known extension of planar graphs. Motivated by this idea, one may introduce their fuzzy counterpart. A fuzzy apex graph is a fuzzy graph becoming planar after deleting one vertex while retaining memberships on vertices and edges.

Definition 4.2.1 (Fuzzy Apex Graph)**.** (cf. [243, 329]) A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy apex graph* if there exists a vertex $v \in V$ such that the induced fuzzy subgraph

$$\psi - v = (V \setminus \{v\}, \sigma', \mu')$$

is a fuzzy planar graph, where σ' and μ' are the restrictions of σ and μ to the remaining vertices. Such a vertex v is called an *apex* of ψ.

Remark 4.2.2. A fuzzy apex graph preserves the ordinary fuzzy graph structure:

- $\sigma : V \to [0, 1]$ assigns vertex-membership degrees;
- $\mu : V \times V \to [0, 1]$ assigns edge-membership degrees with

$$\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\};$$

- if ψ is already fuzzy planar, then every vertex may be regarded as an apex;
- a fuzzy apex graph may have more than one apex vertex.

Proposition 4.2.3. *Every fuzzy apex graph can be reduced to a fuzzy planar graph by deleting a suitable vertex.*

Proof. This is exactly the defining property of a fuzzy apex graph. □

Definition 4.2.4 (Fuzzy k-Apex Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy k-apex graph* if it can be transformed into a fuzzy planar graph by deleting at most k vertices. Equivalently, there exists a subset

$$S \subseteq V, \qquad |S| \leq k,$$

such that the induced fuzzy subgraph on $V \setminus S$ is fuzzy planar.

In particular, when $k = 0$, the graph is itself fuzzy planar, and when $k = 1$, one recovers the notion of a fuzzy apex graph.

Example 4.2.5 (A Fuzzy Apex Outerplanar Graph)**.** Let

$$V = \{v_1, v_2, v_3, v_4\}.$$

Define the vertex-membership function $\sigma : V \to [0,1]$ by

$$\sigma(v_1) = 0.9, \qquad \sigma(v_2) = 0.8, \qquad \sigma(v_3) = 0.7, \qquad \sigma(v_4) = 0.6.$$

Let the edge-membership function $\mu : V \times V \to [0,1]$ be given by

$$\mu(v_i, v_j) = \begin{cases} 0.6, & \{v_i, v_j\} \in \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\big\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\psi = (V, \sigma, \mu)$ is a fuzzy graph, since for every edge $\{v_i, v_j\}$,

$$\mu(v_i, v_j) = 0.6 \le \min\{\sigma(v_i), \sigma(v_j)\}.$$

Observe that the underlying crisp graph of ψ is K_4. Now remove the vertex v_4. Then

$$\psi - v_4 = (V \setminus \{v_4\}, \sigma', \mu')$$

has vertex set

$$\{v_1, v_2, v_3\},$$

and its underlying crisp graph is the triangle K_3, which is outerplanar.

Therefore, $\psi - v_4$ is a fuzzy outerplanar graph. Hence ψ is a *fuzzy apex outerplanar graph*, with apex vertex v_4.

Proposition 4.2.6. *Every fuzzy k-apex graph can be transformed into a fuzzy planar graph.*

Proof. This follows directly from the definition. □

Proposition 4.2.7. *Every fuzzy apex graph is a fuzzy k-apex graph for every $k \geq 1$.*

Proof. If one vertex is sufficient, then certainly at most k vertices are sufficient for every $k \geq 1$. □

4.3 Fuzzy Apex Outerplanar Graph

A fuzzy apex outerplanar graph becomes outerplanar after deleting a vertex while preserving fuzzy memberships on remaining vertices and edges. By replacing fuzzy planarity with fuzzy outerplanarity, one obtains the following analogous class.

Definition 4.3.1 (Fuzzy Apex Outerplanar Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy apex outerplanar graph* if there exists a vertex $v \in V$ such that the induced fuzzy subgraph

$$\psi - v = (V \setminus \{v\}, \sigma', \mu')$$

is a fuzzy outerplanar graph, where σ' and μ' are the restrictions of σ and μ to the remaining vertices.

Remark 4.3.2. For a fuzzy apex outerplanar graph:

- $\sigma : V \to [0, 1]$ gives the vertex-membership degrees;
- $\mu : V \times V \to [0, 1]$ gives the fuzzy adjacency values and satisfies

$$\mu(u, v) \le \min\{\sigma(u), \sigma(v)\};$$

- if the graph is already fuzzy outerplanar, then every vertex may serve as an apex.

Proposition 4.3.3. *Every fuzzy apex outerplanar graph can be transformed into a fuzzy outerplanar graph.*

Proof. This is immediate from the definition. □

Definition 4.3.4 (Fuzzy k-Apex Outerplanar Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy k-apex outerplanar graph* if there exists a subset

$$S \subseteq V, \qquad |S| \le k,$$

such that the induced fuzzy subgraph

$$\psi' = (V \setminus S, \sigma', \mu')$$

is fuzzy outerplanar, where σ' and μ' are the restrictions of σ and μ.

Remark 4.3.5. For a fuzzy k-apex outerplanar graph:

- if $k = 0$, then the graph is already fuzzy outerplanar;
- if $k = 1$, then one obtains a fuzzy apex outerplanar graph;
- for general k, more than one deletion set may yield a fuzzy outerplanar graph.

Theorem 4.3.6. *Every fuzzy k-apex outerplanar graph can be transformed into a fuzzy outerplanar graph.*

Proof. The conclusion is exactly the content of the definition. □

Proposition 4.3.7. *Every fuzzy apex outerplanar graph is a fuzzy k-apex outerplanar graph for every $k \geq 1$.*

Proof. The claim is immediate. □

4.4 Fuzzy Quasi-Planar Graph

A fuzzy quasi-planar graph is a fuzzy graph drawable so that no specified number of edges pairwise cross in interiors. A fuzzy quasi-planar graph is the fuzzy analogue of a classical quasi-planar graph.

Definition 4.4.1 (Fuzzy Quasi-Planar Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy quasi-planar graph* if it admits a drawing in the plane in which no three fuzzy edges are pairwise crossing.

More explicitly:

- $\sigma : V \to [0, 1]$ assigns membership degrees to vertices;
- $\mu : V \times V \to [0, 1]$ assigns membership degrees to edges, with

$$\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\};$$

- in some drawing of ψ, there is no set of three fuzzy edges all of which intersect pairwise in their interiors.

Definition 4.4.2 (Fuzzy k-Quasi-Planar Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy k-quasi-planar graph* if it can be drawn in the plane so that no set of k fuzzy edges is pairwise crossing.

Equivalently, there exists a drawing of ψ in which no k fuzzy edges mutually intersect pairwise in their interiors.

Example 4.4.3 (A Fuzzy 3-Quasi-Planar Graph)**.** Let

$$V = \{v_1, v_2, v_3, v_4\}.$$

Define the vertex-membership function $\sigma : V \to [0, 1]$ by

$$\sigma(v_1) = 0.9, \qquad \sigma(v_2) = 0.8, \qquad \sigma(v_3) = 0.8, \qquad \sigma(v_4) = 0.9.$$

Let the edge-membership function $\mu : V \times V \to [0, 1]$ be defined by

$$\mu(v_i, v_j) = \begin{cases} 0.8, & \{v_i, v_j\} \in E, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_1\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_3, v_4\}\big\}.$$

Thus, the underlying crisp graph is K_4.

Since

$$0.8 \le \min\{\sigma(v_i), \sigma(v_j)\} \qquad \text{for every } \{v_i, v_j\} \in E,$$

the pair

$$\psi = (V, \sigma, \mu)$$

is a fuzzy graph.

Now observe that K_4 is planar. Hence there exists a drawing of the underlying crisp graph in the plane with no edge crossings. Using this planar drawing for ψ, no two fuzzy edges cross, and therefore there cannot exist any set of three fuzzy edges that are pairwise crossing.

Consequently, ψ is a fuzzy 3-quasi-planar graph.

Theorem 4.4.4. *Every fuzzy quasi-planar graph is a fuzzy k-quasi-planar graph for every $k \ge 3$.*

Proof. If no three fuzzy edges are pairwise crossing, then certainly no larger set of $k \ge 3$ fuzzy edges can be pairwise crossing. □

4.5 Fuzzy Quasi-Outerplanar Graph

We next consider the outer-face counterpart of fuzzy quasi-planarity. A fuzzy quasi-outerplanar graph is a fuzzy graph drawable outerplanarly with all vertices outermost and no prescribed pairwise edge crossings.

Definition 4.5.1 (Fuzzy Quasi-Outerplanar Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy quasi-outerplanar graph* if it admits a drawing in the plane such that

- all vertices lie on the boundary of the outer face, and
- no three fuzzy edges are pairwise crossing.

Definition 4.5.2 (Fuzzy Outer-k-Quasi-Planar Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy outer-k-quasi-planar graph* if it admits a drawing in the plane such that

- all vertices lie on the boundary of the outer face, and
- no set of k fuzzy edges is pairwise crossing.

Theorem 4.5.3. *Every fuzzy quasi-outerplanar graph is a fuzzy outer-k-quasi-planar graph for every $k \geq 3$.*

Proof. The statement follows in the same way as in the quasi-planar case. □

4.6 Fuzzy Quasi Ap ex Graph

We now combine bounded vertex deletion with fuzzy quasi-planarity.

Definition 4.6.1 (Fuzzy k-Quasi Apex Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy k-quasi apex graph* if there exists a subset

$$S \subseteq V, \qquad |S| \leq k,$$

such that the induced fuzzy subgraph

$$\psi' = (V \setminus S, \sigma', \mu')$$

is a fuzzy quasi-planar graph.

Remark 4.6.2. In particular:

- if $k = 0$, then the graph is already fuzzy quasi-planar;
- if $k = 1$, then one obtains the one-vertex-deletion version of fuzzy quasi-planarity.

Example 4.6.3 (A Fuzzy 1-Quasi Apex Graph)**.** Let

$$V = \{v_1, v_2, v_3, v_4, v_5\}.$$

Define the vertex-membership function $\sigma : V \to [0, 1]$ by

$$\sigma(v_1) = 0.9, \qquad \sigma(v_2) = 0.8, \qquad \sigma(v_3) = 0.8, \qquad \sigma(v_4) = 0.9, \qquad \sigma(v_5) = 0.7.$$

Let the edge-membership function $\mu : V \times V \to [0,1]$ be given by

$$\mu(v_i, v_j) = \begin{cases} 0.7, & \{v_i, v_j\} \in E, \\ 0, & \text{otherwise,} \end{cases}$$

where

$E = \{\{v_1,v_2\},\{v_1,v_3\},\{v_1,v_4\},\{v_1,v_5\},\{v_2,v_3\},\{v_2,v_4\},\{v_2,v_5\},\{v_3,v_4\},\{v_3,v_5\},\{v_4,v_5\}\}$.

Thus, the underlying crisp graph of $\psi = (V, \sigma, \mu)$ is the complete graph K_5.

Since

$$0.7 \le \min\{\sigma(v_i), \sigma(v_j)\} \qquad \text{for every } \{v_i, v_j\} \in E,$$

the pair

$$\psi = (V, \sigma, \mu)$$

is a fuzzy graph.

Now take

$$S = \{v_5\}.$$

Then $|S| = 1$, and the induced fuzzy subgraph

$$\psi' = (V \setminus S, \sigma', \mu')$$

has underlying crisp graph K_4, which is planar. Hence ψ' is a fuzzy quasi-planar graph.

Therefore, there exists a subset $S \subseteq V$ with $|S| \le 1$ such that the induced fuzzy subgraph ψ' is fuzzy quasi-planar. Thus, ψ is a *fuzzy 1-quasi apex graph.*

Theorem 4.6.4. *Every fuzzy k-quasi apex graph can be transformed into a fuzzy quasi-planar graph.*

Proof. This is immediate from the definition. □

4.7 Fuzzy Quasi Ap ex Outerplanar Graph

A fuzzy graph that becomes fuzzy outerplanar after removing an apex vertex set while preserving fuzzy vertex and edge memberships. The analogous outerplanar version is defined as follows.

Definition 4.7.1 (Fuzzy k-Quasi Apex Outerplanar Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy k-quasi apex outerplanar graph* if there exists a subset

$$S \subseteq V, \qquad |S| \le k,$$

such that the induced fuzzy subgraph

$$\psi' = (V \setminus S, \sigma', \mu')$$

is a fuzzy quasi-outerplanar graph.

Remark 4.7.2. In particular:

- if $k = 0$, then the graph is already fuzzy quasi-outerplanar;
- if $k = 1$, then one obtains the one-vertex-deletion version of fuzzy quasi-outerplanarity.

Theorem 4.7.3. *Every fuzzy k-quasi apex outerplanar graph can be transformed into a fuzzy quasi-outerplanar graph.*

Proof. The statement is immediate. □

4.8 Fuzzy Almost Planar Graph

A fuzzy graph drawable in the plane with at most one crossing, while retaining fuzzy values on vertices and edges. Finally, one may extend almost planar and almost outerplanar graphs to the fuzzy setting.

Definition 4.8.1 (Fuzzy Almost Planar Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy almost planar graph* if there exists an edge $e \in E(\psi)$ such that the fuzzy graph

$$\psi - e$$

is a fuzzy planar graph.

Definition 4.8.2 (Fuzzy Almost Outerplanar Graph)**.** A fuzzy graph

$$\psi = (V, \sigma, \mu)$$

is called a *fuzzy almost outerplanar graph* if there exists an edge $e \in E(\psi)$ such that the fuzzy graph

$$\psi - e$$

is a fuzzy outerplanar graph.

Proposition 4.8.3. *Every fuzzy almost planar graph becomes fuzzy planar after deleting a suitable edge.*

Proof. This is exactly the defining property. □

Proposition 4.8.4. *Every fuzzy almost outerplanar graph becomes fuzzy outerplanar after deleting a suitable edge.*

Proof. Again, this follows directly from the definition. □

Chapter 5

Intuitionistic Fuzzy Planar Graphs

Intuitionistic fuzzy graphs may be viewed as a natural extension of fuzzy graphs. They enrich the usual membership description by adding a non-membership degree and, implicitly, a hesitancy degree [145]. Intuitionistic fuzzy planar graphs [72] extend fuzzy planar graphs in the same direction by incorporating membership and non-membership information into planar graph representations. These notions are also closely related to the general theory of intuitionistic fuzzy sets [93].

In this chapter, we review intuitionistic fuzzy planar graphs and introduce several related classes, including intuitionistic fuzzy outerplanar graphs, apex graphs, quasi-planar graphs, quasi-outerplanar graphs, quasi-apex graphs, and almost planar graphs.

Definition 5.0.1 (Intuitionistic Fuzzy Graph)**.** [33] Let

$$G^* = (V, E)$$

be a classical graph, where V is the vertex set and E is the edge set. An *intuitionistic fuzzy graph* on G^* is a structure

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B),$$

where

$$\mu_A, \nu_A : V \to [0, 1]$$

are the vertex membership and vertex non-membership functions, and

$$\mu_B, \nu_B : E \to [0, 1]$$

are the edge membership and edge non-membership functions.

These functions satisfy the following conditions:

$$\mu_A(x) + \nu_A(x) \leq 1 \qquad \text{for all } x \in V,$$

and

$$\mu_B(e) + \nu_B(e) \leq 1 \qquad \text{for all } e \in E.$$

Moreover, for every edge $e = \{x, y\} \in E$, one has

$$\mu_B(e) \leq \min\{\mu_A(x), \mu_A(y)\},$$

and

$$\nu_B(e) \geq \max\{\nu_A(x), \nu_A(y)\}.$$

For each vertex $x \in V$, the number

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

is called the *vertex hesitancy degree*. Similarly, for each edge $e \in E$, the number

$$\pi_B(e) = 1 - \mu_B(e) - \nu_B(e)$$

is called the *edge hesitancy degree*.

If

$$\nu_A(x) = 0 \quad \text{for all } x \in V,$$

and

$$\nu_B(e) = 0 \quad \text{for all } e \in E,$$

then the intuitionistic fuzzy graph reduces to an ordinary fuzzy graph.

5.1 Intuitionistic Fuzzy Planar Graph

Intuitionistic fuzzy planar graphs extend fuzzy planar graphs by assigning intuitionistic fuzzy information to vertices and edges while retaining the geometric notion of planar drawing. Since edge crossings depend on the chosen drawing, it is useful to distinguish the planarity value of a fixed drawing from the planarity of the graph itself.

Definition 5.1.1 (Intersection Value in an Intuitionistic Fuzzy Drawing)**.** Let

$$G_{\text{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

be an intuitionistic fuzzy graph, and let D be a drawing of G_{IF} in the plane. Suppose that two distinct edges

$$e, h \in E$$

intersect at a point P which is not a common endpoint of the two edges. The *intuitionistic fuzzy intersection value* at P is defined by

$$S_P = (M_P, N_P),$$

where

$$M_P = \frac{\mu_B(e) + \mu_B(h)}{2}, \qquad N_P = \frac{\nu_B(e) + \nu_B(h)}{2}.$$

Here M_P measures the membership contribution of the crossing, while N_P measures the non-membership contribution of the crossing.

Definition 5.1.2 (Intuitionistic Fuzzy Planarity Value of a Drawing)**.** Let

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

be an intuitionistic fuzzy graph, and let D be a drawing of G_{IF} in the plane. Denote by

$$\mathcal{C}(D)$$

the set of all edge-crossing points in D, where crossings at common endpoints are not counted.

For each crossing point $P \in \mathcal{C}(D)$, let

$$S_P = (M_P, N_P)$$

be the intuitionistic fuzzy intersection value at P. Define

$$C_M(D) = \sum_{P \in \mathcal{C}(D)} M_P, \qquad C_N(D) = \sum_{P \in \mathcal{C}(D)} N_P.$$

The *intuitionistic fuzzy planarity value* of the drawing D is the ordered pair

$$f_D = (f_M(D), f_N(D)),$$

where

$$f_M(D) = \frac{1}{1 + C_M(D)}, \qquad f_N(D) = \frac{1}{1 + C_N(D)}.$$

Thus

$$0 < f_M(D) \leq 1, \qquad 0 < f_N(D) \leq 1.$$

In particular,

$$f_D = (1, 1)$$

whenever D has no edge crossings.

Definition 5.1.3 (Intuitionistic Fuzzy Planar Graph)**.** [72,78] An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy planar graph* if there exists a drawing D of its underlying crisp graph

$$G^* = (V, E)$$

in the plane such that no two edges cross except possibly at a common endpoint.

Equivalently, G_{IF} is intuitionistic fuzzy planar if there exists a drawing D such that

$$f_D = (1, 1),$$

where f_D is the intuitionistic fuzzy planarity value of D.

Remark 5.1.4. The value f_D is attached to a fixed drawing D, whereas intuitionistic fuzzy planarity is a graph-level property. Thus two different drawings of the same intuitionistic fuzzy graph may have different planarity values, but the graph is called intuitionistic fuzzy planar if at least one crossing-free drawing exists.

Proposition 5.1.5. *Every intuitionistic fuzzy planar graph is an intuitionistic fuzzy graph.*

Proof. This follows directly from the definition. Intuitionistic fuzzy planarity is defined only after an underlying intuitionistic fuzzy graph has already been specified. □

Proposition 5.1.6. *Every intuitionistic fuzzy planar graph induces a fuzzy planar graph by discarding the non-membership data.*

Proof. Let

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

be an intuitionistic fuzzy planar graph. Define a fuzzy graph

$$G_F = (V, E, \sigma, \mu)$$

by

$$\sigma(x) = \mu_A(x) \quad \text{for all } x \in V,$$

and

$$\mu(e) = \mu_B(e) \quad \text{for all } e \in E.$$

For every edge $e = \{x, y\} \in E$, the intuitionistic fuzzy graph condition gives

$$\mu_B(e) \le \min\{\mu_A(x), \mu_A(y)\}.$$

Therefore,

$$\mu(e) \le \min\{\sigma(x), \sigma(y)\},$$

so G_F is a well-defined fuzzy graph.

Since G_{IF} is intuitionistic fuzzy planar, its underlying crisp graph (V, E) has a crossing-free drawing in the plane. The induced fuzzy graph G_F has the same underlying crisp graph and therefore admits the same crossing-free drawing. Hence G_F is a fuzzy planar graph. □

5.2 Intuitionistic Fuzzy Outerplanar Graph

We next introduce intuitionistic fuzzy outerplanar graphs. They are obtained by imposing the classical outerplanar embedding condition on the underlying crisp graph while preserving the intuitionistic fuzzy vertex and edge data.

Definition 5.2.1 (Intuitionistic Fuzzy Outerplanar Graph)**.** An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy outerplanar graph* if its underlying crisp graph

$$G^* = (V, E)$$

admits a plane embedding such that:

- no two edges cross except possibly at a common endpoint; and
- every vertex lies on the boundary of the outer face.

Proposition 5.2.2. *Every intuitionistic fuzzy outerplanar graph is an intuitionistic fuzzy planar graph.*

Proof. Every outerplanar embedding is, in particular, a planar embedding. Therefore, if an intuitionistic fuzzy graph admits an outerplanar embedding, then it also admits a planar embedding. Hence every intuitionistic fuzzy outerplanar graph is intuitionistic fuzzy planar. □

Proposition 5.2.3. *Every intuitionistic fuzzy outerplanar graph induces a fuzzy outerplanar graph by discarding the non-membership data.*

Proof. Let

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

be an intuitionistic fuzzy outerplanar graph. Retain only the membership functions μ_A and μ_B. This produces a fuzzy graph on the same underlying crisp graph (V, E). Since the outerplanar embedding is unchanged, the resulting fuzzy graph is fuzzy outerplanar. □

5.3 Intuitionistic Fuzzy Apex Graph

An intuitionistic fuzzy apex graph is an intuitionistic fuzzy graph that becomes intuitionistic fuzzy planar after the deletion of one suitable vertex.

Definition 5.3.1 (Intuitionistic Fuzzy Apex Graph)**.** An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy apex graph* if there exists a vertex

$$v \in V$$

such that the induced intuitionistic fuzzy subgraph

$$G_{\mathrm{IF}} - v$$

on the vertex set $V \setminus \{v\}$ is an intuitionistic fuzzy planar graph. Such a vertex v is called an *apex vertex* of G_{IF}.

Remark 5.3.2. The following observations are immediate.

- An intuitionistic fuzzy apex graph may have more than one apex vertex.
- If G_{IF} is intuitionistic fuzzy planar and $V \neq \emptyset$, then every vertex of G_{IF} may be regarded as an apex vertex.
- The definition requires the existence of an apex vertex. Therefore, when the empty graph is considered, it is not apex in this exact sense unless a separate convention allowing deletion of at most one vertex is adopted.

Proposition 5.3.3. *Every intuitionistic fuzzy apex graph induces a fuzzy apex graph by discarding the non-membership data.*

Proof. Let G_{IF} be an intuitionistic fuzzy apex graph. Then there exists a vertex $v \in V$ such that

$$G_{\mathrm{IF}} - v$$

is intuitionistic fuzzy planar. By retaining only the membership data, one obtains a fuzzy graph on the same underlying crisp graph. After deleting the same vertex v, the resulting fuzzy graph is fuzzy planar. Hence the membership projection of G_{IF} is a fuzzy apex graph. □

Proposition 5.3.4. *Deleting an apex vertex from an intuitionistic fuzzy apex graph yields an intuitionistic fuzzy planar graph.*

Proof. This is precisely the defining property of an intuitionistic fuzzy apex graph. □

5.4 Intuitionistic Fuzzy Quasi-Planar Graph

A quasi-planar graph is a graph that admits a drawing with no three pairwise crossing edges. The following definition transfers this condition to the intuitionistic fuzzy setting by applying the drawing condition to the underlying crisp graph.

Definition 5.4.1 (Intuitionistic Fuzzy Quasi-Planar Graph)**.** An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy quasi-planar graph* if its underlying crisp graph

$$G^* = (V, E)$$

admits a drawing in the plane such that no three edges are pairwise crossing. Equivalently, there is no set of three edges in which every pair of edges crosses in the chosen drawing.

Definition 5.4.2 (Intuitionistic Fuzzy k-Quasi-Planar Graph)**.** Let $k \geq 3$ be an integer. An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy k-quasi-planar graph* if its underlying crisp graph

$$G^* = (V, E)$$

admits a drawing in the plane such that no k edges are pairwise crossing.

Proposition 5.4.3. *Every intuitionistic fuzzy quasi-planar graph is an intuitionistic fuzzy k-quasi-planar graph for every integer $k \geq 3$.*

Proof. If a drawing contains no three pairwise crossing edges, then it cannot contain a larger family of $k \geq 3$ pairwise crossing edges. Therefore every intuitionistic fuzzy quasi-planar graph is intuitionistic fuzzy k-quasi-planar for every $k \geq 3$. □

Proposition 5.4.4. *Every intuitionistic fuzzy k-quasi-planar graph induces a fuzzy k-quasi-planar graph by discarding the non-membership data.*

Proof. Discard the functions ν_A and ν_B, and retain only the membership functions μ_A and μ_B. The underlying crisp graph and the chosen drawing are unchanged. Hence the condition forbidding k pairwise crossing edges remains valid, and the induced fuzzy graph is fuzzy k-quasi-planar. □

5.5 Intuitionistic Fuzzy Quasi-Outerplanar Graph

The quasi-outerplanar version combines the outer-face condition with the absence of three pairwise crossing edges.

Definition 5.5.1 (Intuitionistic Fuzzy Quasi-Outerplanar Graph)**.** An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy quasi-outerplanar graph* if its underlying crisp graph

$$G^* = (V, E)$$

admits a drawing in the plane such that:

- every vertex lies on the boundary of the outer face; and
- no three edges are pairwise crossing.

Definition 5.5.2 (Intuitionistic Fuzzy Outer-k-Quasi-Planar Graph)**.** Let $k \geq 3$ be an integer. An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy outer-k-quasi-planar graph* if its underlying crisp graph

$$G^* = (V, E)$$

admits a drawing in the plane such that:

- every vertex lies on the boundary of the outer face; and
- no k edges are pairwise crossing.

Proposition 5.5.3. *Every intuitionistic fuzzy quasi-outerplanar graph is an intuitionistic fuzzy outer-k-quasi-planar graph for every integer $k \geq 3$.*

Proof. If a drawing has no three pairwise crossing edges, then it has no k pairwise crossing edges for any $k \geq 3$. Since the same drawing also places all vertices on the boundary of the outer face, the assertion follows. □

Proposition 5.5.4. *Every intuitionistic fuzzy outer-k-quasi-planar graph induces a fuzzy outer-k-quasi-planar graph by discarding the non-membership data.*

Proof. The membership projection preserves the underlying crisp graph and the chosen drawing. Therefore the outer-face condition and the prohibition of k pairwise crossing edges remain valid. Hence the induced fuzzy graph is fuzzy outer-k-quasi-planar. □

Proposition 5.5.5. *Every intuitionistic fuzzy quasi-outerplanar graph induces a fuzzy quasi-outerplanar graph by discarding the non-membership data.*

Proof. This is the special case $k = 3$ of the preceding argument. Retaining only the membership data preserves the same drawing, the same outer-face condition, and the same absence of three pairwise crossing edges. □

5.6 Intuitionistic Fuzzy Quasi Ap ex Graph

We now combine bounded vertex deletion with intuitionistic fuzzy quasi-planarity.

Definition 5.6.1 (Intuitionistic Fuzzy k-Quasi Apex Graph)**.** Let $k \geq 0$ be an integer. An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy k-quasi apex graph* if there exists a subset

$$S \subseteq V$$

with

$$|S| \leq k$$

such that the induced intuitionistic fuzzy subgraph

$$G_{\mathrm{IF}} - S$$

on $V \setminus S$ is an intuitionistic fuzzy quasi-planar graph.

Proposition 5.6.2. *Every intuitionistic fuzzy k-quasi apex graph induces a fuzzy k-quasi apex graph by discarding the non-membership data.*

Proof. Let G_{IF} be an intuitionistic fuzzy k-quasi apex graph. Then there exists $S \subseteq V$ with $|S| \leq k$ such that

$$G_{\mathrm{IF}} - S$$

is intuitionistic fuzzy quasi-planar. After discarding the non-membership functions, the same vertex deletion produces a fuzzy quasi-planar graph. Therefore the induced fuzzy graph is fuzzy k-quasi apex. □

5.7 Intuitionistic Fuzzy Quasi Apex Outerplanar Graph

The corresponding outerplanar version is obtained by replacing quasi-planarity with quasi-outerplanarity after bounded vertex deletion.

Definition 5.7.1 (Intuitionistic Fuzzy k-Quasi Apex Outerplanar Graph)**.** Let $k \geq 0$ be an integer. An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy k-quasi apex outerplanar graph* if there exists a subset

$$S \subseteq V$$

with

$$|S| \leq k$$

such that the induced intuitionistic fuzzy subgraph

$$G_{\mathrm{IF}} - S$$

on $V \setminus S$ is an intuitionistic fuzzy quasi-outerplanar graph.

Proposition 5.7.2. *Every intuitionistic fuzzy k-quasi apex outerplanar graph induces a fuzzy k-quasi apex outerplanar graph by discarding the non-membership data.*

Proof. Let G_{IF} be an intuitionistic fuzzy k-quasi apex outerplanar graph. Then there exists $S \subseteq V$ with $|S| \leq k$ such that

$$G_{\mathrm{IF}} - S$$

is intuitionistic fuzzy quasi-outerplanar. Retaining only the membership data does not change the underlying crisp graph or the relevant drawing condition. Thus the induced fuzzy graph is fuzzy k-quasi apex outerplanar. □

5.8 Intuitionistic Fuzzy Almost Planar Graph

Finally, almost planar and almost outerplanar graphs may also be extended to the intuitionistic fuzzy setting. In these definitions, one deletes a single edge rather than a vertex.

Definition 5.8.1 (Intuitionistic Fuzzy Almost Planar Graph)**.** An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy almost planar graph* if there exists an edge

$$e \in E$$

such that the intuitionistic fuzzy graph

$$G_{\mathrm{IF}} - e$$

is intuitionistic fuzzy planar.

Definition 5.8.2 (Intuitionistic Fuzzy Almost Outerplanar Graph)**.** An intuitionistic fuzzy graph

$$G_{\mathrm{IF}} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is called an *intuitionistic fuzzy almost outerplanar graph* if there exists an edge

$$e \in E$$

such that the intuitionistic fuzzy graph

$$G_{\mathrm{IF}} - e$$

is intuitionistic fuzzy outerplanar.

Proposition 5.8.3. *Every intuitionistic fuzzy almost planar graph induces a fuzzy almost planar graph by discarding the non-membership data.*

Proof. Let G_{IF} be an intuitionistic fuzzy almost planar graph. Then there exists an edge $e \in E$ such that

$$G_{\mathrm{IF}} - e$$

is intuitionistic fuzzy planar. After deleting the same edge and retaining only the membership data, one obtains a fuzzy planar graph. Therefore the membership projection of G_{IF} is fuzzy almost planar. □

Proposition 5.8.4. *Every intuitionistic fuzzy almost outerplanar graph induces a fuzzy almost outerplanar graph by discarding the non-membership data.*

Proof. The proof is identical to the previous one, with "planar" replaced by "outerplanar". If deleting an edge e yields an intuitionistic fuzzy outerplanar graph, then deleting the same edge after discarding the non-membership data yields a fuzzy outerplanar graph. Hence the induced fuzzy graph is fuzzy almost outerplanar. □

Chapter 6

Neutrosophic Graph and Planar Graph

We begin with the notion of a neutrosophic graph. Like fuzzy graphs, neutrosophic graphs have been studied extensively in the literature [45, 46, 330–332]. They are also closely connected with neutrosophic sets [105, 333–335], which provide the underlying set-theoretic framework.

Definition 6.0.1 (Neutrosophic Graph)**.** [46] A *neutrosophic graph* is a structure

$$NTG = (V, E, \sigma, \mu),$$

where

$$\sigma = (\sigma_T, \sigma_I, \sigma_F) : V \to [0,1]^3$$

is the neutrosophic vertex assignment and

$$\mu = (\mu_T, \mu_I, \mu_F) : E \to [0,1]^3$$

is the neutrosophic edge assignment.

For each vertex $v \in V$, the triple

$$\sigma(v) = \big(\sigma_T(v), \sigma_I(v), \sigma_F(v)\big)$$

represents the truth, indeterminacy, and falsity degrees of v, and similarly, for each edge $e \in E$,

$$\mu(e) = \big(\mu_T(e), \mu_I(e), \mu_F(e)\big)$$

represents the corresponding neutrosophic edge values.

Typically, the edge values are required to be compatible with the endpoint values. In particular, for an edge $v_i v_j \in E$,

$$\mu_T(v_i v_j) \le \min\{\sigma_T(v_i), \sigma_T(v_j)\},$$

$$\mu_I(v_i v_j) \le \min\{\sigma_I(v_i), \sigma_I(v_j)\},$$

$$\mu_F(v_i v_j) \ge \max\{\sigma_F(v_i), \sigma_F(v_j)\}.$$

Moreover:

1. σ is called the *neutrosophic vertex set*;
2. μ is called the *neutrosophic edge set*;
3. $|V|$ is called the *order* of NTG, denoted by $O(NTG)$;
4. $\sum_{v\in V}\sigma(v)$ is called the *neutrosophic order* of NTG, denoted by $On(NTG)$;
5. $|E|$ is called the *size* of NTG, denoted by $S(NTG)$;
6. $\sum_{e\in E}\mu(e)$ is called the *neutrosophic size* of NTG, denoted by $Sn(NTG)$.

6.1 Neutrosophic Planar Graph

A neutrosophic planar graph may be regarded as a neutrosophic counterpart of fuzzy planar graphs and intuitionistic fuzzy planar graphs [48, 73, 336, 337]. In this section, we distinguish between the planarity value of a fixed drawing and the planarity of the graph itself. This distinction is important because edge crossings are properties of a particular geometric representation, whereas planarity is usually defined by the existence of a crossing-free representation.

Definition 6.1.1 (Intersection Value)**.** [338] Let

$$G_N = (V, E, \sigma, \mu)$$

be a neutrosophic graph, where

$$\sigma(v) = \big(T_A(v), I_A(v), F_A(v)\big) \quad (v \in V),$$

and

$$\mu(e) = \big(T_B(e), I_B(e), F_B(e)\big) \quad (e \in E).$$

Let D be a drawing of G_N in the plane. Suppose that two distinct edges

$$e = \{a, b\}, \qquad h = \{c, d\}$$

intersect at a point P which is not a common endpoint of the two edges. The *neutrosophic intersection value* at P is defined by

$$S_P = \big((S_T)_P, (S_I)_P, (S_F)_P\big),$$

where

$$(S_T)_P = \frac{T_B(e) + T_B(h)}{2}, \qquad (S_I)_P = \frac{I_B(e) + I_B(h)}{2}, \qquad (S_F)_P = \frac{F_B(e) + F_B(h)}{2}.$$

Equivalently,

$$S_P = \left(\frac{T_B(e) + T_B(h)}{2}, \frac{I_B(e) + I_B(h)}{2}, \frac{F_B(e) + F_B(h)}{2}\right).$$

The value S_P measures the neutrosophic weight of the crossing at P. Larger accumulated intersection values indicate a lower degree of planarity for the given drawing.

Definition 6.1.2 (Neutrosophic Planarity Value of a Drawing)**.** Let

$$G_N = (V, E, \sigma, \mu)$$

be a neutrosophic graph, and let D be a drawing of G_N in the plane. Denote by

$$\mathcal{C}(D)$$

the set of all edge-crossing points of D, where crossings at common endpoints are not counted. For each $P \in \mathcal{C}(D)$, let

$$S_P = \big((S_T)_P, (S_I)_P, (S_F)_P\big)$$

be the neutrosophic intersection value at P, as in Definition 6.1.1.

Define the accumulated neutrosophic crossing values of D by

$$C_T(D) = \sum_{P\in\mathcal{C}(D)} (S_T)_P, \qquad C_I(D) = \sum_{P\in\mathcal{C}(D)} (S_I)_P, \qquad C_F(D) = \sum_{P\in\mathcal{C}(D)} (S_F)_P.$$

Let

$$c(D) = |\mathcal{C}(D)|$$

be the number of crossing points in D. The *neutrosophic planarity value* of the drawing D is the triple

$$f_D = \big(f_T(D), f_I(D), f_F(D)\big),$$

where

$$f_T(D) = \frac{1}{1 + c(D) + C_T(D)}, \qquad f_I(D) = \frac{1}{1 + c(D) + C_I(D)}, \qquad f_F(D) = \frac{1}{1 + c(D) + C_F(D)}.$$

Thus

$$0 < f_T(D), f_I(D), f_F(D) \leq 1.$$

Moreover,

$$f_D = (1, 1, 1)$$

if and only if D has no edge crossings. Hence the value f_D gives a drawing-dependent neutrosophic measure of planarity, while preserving the classical crossing-free case as the maximal value.

Definition 6.1.3 (Neutrosophic Planar Graph)**.** Let

$$G_N = (V, E, \sigma, \mu)$$

be a neutrosophic graph. We say that G_N is a *neutrosophic planar graph* if there exists a drawing D of G_N in the plane such that

$$f_D = (1, 1, 1),$$

where f_D is the neutrosophic planarity value of D.

Equivalently, G_N is neutrosophic planar if its underlying crisp graph

$$G^* = (V, E)$$

admits a crossing-free drawing in the plane. In this sense, neutrosophic planarity is a structural property of the underlying graph, whereas the neutrosophic degrees are retained as vertex and edge information.

Remark 6.1.4. Definition 6.1.2 is drawing-dependent: different drawings of the same neutrosophic graph may have different neutrosophic planarity values. By contrast, Definition 6.1.3 is graph-dependent: it requires the existence of at least one drawing whose planarity value is maximal.

Proposition 6.1.5. *Every neutrosophic planar graph induces a fuzzy planar graph by retaining only the truth-membership component.*

Proof. Let

$$G_N = (V, E, \sigma, \mu)$$

be a neutrosophic planar graph, where

$$\sigma(v) = \big(T_A(v), I_A(v), F_A(v)\big) \quad (v \in V),$$

and

$$\mu(e) = \big(T_B(e), I_B(e), F_B(e)\big) \quad (e \in E).$$

Define a fuzzy graph

$$G_F = (V, E, \sigma_F, \mu_F)$$

by

$$\sigma_F(v) = T_A(v) \quad (v \in V),$$

and

$$\mu_F(e) = T_B(e) \quad (e \in E).$$

Since G_N is a neutrosophic graph, its truth-membership component satisfies the usual edge–vertex compatibility condition

$$T_B(\{u, v\}) \leq \min\{T_A(u), T_A(v)\}$$

for every edge $\{u, v\} \in E$. Hence

$$\mu_F(\{u, v\}) \leq \min\{\sigma_F(u), \sigma_F(v)\},$$

so G_F is a well-defined fuzzy graph.

Because G_N is neutrosophic planar, there exists a drawing D of its underlying crisp graph (V, E) such that

$$f_D = (1, 1, 1).$$

By Definition 6.1.2, this means that D has no edge crossings. The fuzzy projection G_F has the same underlying crisp graph and the same drawing D. Therefore G_F is a fuzzy planar graph. □

Proposition 6.1.6. *Let*

$$G_N = (V, E, \sigma, \mu)$$

be a neutrosophic planar graph. Suppose that

$$T_A(v) + F_A(v) \leq 1 \quad \textit{for all } v \in V,$$

and

$$T_B(e) + F_B(e) \leq 1 \quad \textit{for all } e \in E.$$

Then G_N induces an intuitionistic fuzzy planar graph by deleting the indeterminacy component.

Proof. For each vertex $v \in V$, define

$$\mu_A(v) = T_A(v), \qquad \nu_A(v) = F_A(v).$$

For each edge $e \in E$, define

$$\mu_B(e) = T_B(e), \qquad \nu_B(e) = F_B(e).$$

By assumption,

$$\mu_A(v) + \nu_A(v) = T_A(v) + F_A(v) \le 1$$

for every $v \in V$, and

$$\mu_B(e) + \nu_B(e) = T_B(e) + F_B(e) \le 1$$

for every $e \in E$. Thus the membership and non-membership degrees satisfy the basic intuitionistic fuzzy constraints.

Moreover, since G_N is a neutrosophic graph, for every edge $e = \{u, v\} \in E$, we have

$$T_B(e) \le \min\{T_A(u), T_A(v)\},$$

and

$$F_B(e) \ge \max\{F_A(u), F_A(v)\}.$$

Therefore,

$$\mu_B(e) \le \min\{\mu_A(u), \mu_A(v)\},$$

and

$$\nu_B(e) \ge \max\{\nu_A(u), \nu_A(v)\}.$$

Hence the induced structure

$$G_{IF} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$$

is a well-defined intuitionistic fuzzy graph.

Finally, since G_N is neutrosophic planar, its underlying crisp graph (V, E) admits a crossing-free drawing in the plane. The induced intuitionistic fuzzy graph G_{IF} has the same underlying crisp graph and therefore admits the same crossing-free drawing. Consequently, G_{IF} is an intuitionistic fuzzy planar graph. □

6.2 Neutrosophic Outerplanar Graph

We next introduce the neutrosophic version of an outerplanar graph. This notion extends both fuzzy outerplanar graphs and intuitionistic fuzzy outerplanar graphs.

Definition 6.2.1 (Neutrosophic Outerplanar Graph)**.** A *neutrosophic outerplanar graph* is a neutrosophic multigraph

$$G = (A, B, \sigma, \mu),$$

where

$$\sigma : A \to [0, 1]^3, \qquad \mu : B \to [0, 1]^3,$$

such that G admits an embedding in the plane satisfying:

1. no two edges intersect except possibly at common endpoints;
2. every vertex lies on the boundary of the outer face.

If the drawing contains intersection points $P_1, \ldots, P_z$, then the *neutrosophic outerplanarity value* is defined by

$$f = \left(\frac{1}{1 + \sum_{i=1}^{z} (S_T)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_I)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_F)_{P_i}} \right),$$

where

$$S_{P_i} = \big((S_T)_{P_i}, (S_I)_{P_i}, (S_F)_{P_i} \big)$$

is the intersection value at P_i. If no intersections occur, then

$$f = (1, 1, 1).$$

Definition 6.2.2 (Maximal Neutrosophic Outerplanar Graph)**.** A neutrosophic outerplanar graph ψ is called *maximal neutrosophic outerplanar* if no additional edge can be added without destroying outerplanarity.

Definition 6.2.3 (Minimally Neutrosophic Non-Outerplanar Graph)**.** A neutrosophic planar graph ψ is called *minimally neutrosophic non-outerplanar* if $i(\psi) \neq 0$, where $i(\psi)$ denotes the number of vertices not on the outer face, and this non-outerplanarity is minimal in the sense specified by the chosen embedding.

Proposition 6.2.4. *Every neutrosophic outerplanar graph induces a fuzzy outerplanar graph by retaining only the truth-membership component.*

Proof. Let

$$G = (A, B, \sigma, \mu)$$

be a neutrosophic outerplanar graph. Define a fuzzy graph $\psi = (V, \sigma', \mu')$ by

$$V = A, \qquad \sigma'(v) = \sigma_T(v), \qquad \mu'(e) = \mu_T(e).$$

The original embedding places all vertices on the outer face and has no forbidden crossings. Hence the induced fuzzy graph is outerplanar. □

6.3 Neutrosophic Ap ex Graph

A neutrosophic apex graph is a neutrosophic graph that becomes planar after deleting a vertex, while preserving truth, indeterminacy, and falsity on vertices and edges. We now define the neutrosophic analogue of an apex graph.

Definition 6.3.1 (Neutrosophic Apex Graph)**.** A *neutrosophic apex graph* is a neutrosophic graph

$$\psi = (V, E, \sigma, \mu)$$

such that there exists a vertex $v \in V$ for which the induced subgraph

$$\psi' = (V \setminus \{v\}, E', \sigma', \mu')$$

is neutrosophic planar. The removed vertex v is called an *apex* of ψ.

Remark 6.3.2. If ψ is itself neutrosophic planar, then every vertex may be regarded as an apex. As in the classical setting, multiple apex vertices may exist.

Definition 6.3.3 (Neutrosophic k-Apex Graph)**.** A *neutrosophic k-apex graph* is a neutrosophic graph that can be made neutrosophic planar by deleting at most k vertices.

In particular:

- when $k = 0$, the graph is already neutrosophic planar;
- when $k = 1$, one recovers the notion of a neutrosophic apex graph.

6.4 Neutrosophic Ap ex Outerplanar Graph

The outerplanar analogue is defined in the same spirit.

Definition 6.4.1 (Neutrosophic Apex Outerplanar Graph)**.** A *neutrosophic apex outerplanar graph* is a neutrosophic graph

$$\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$$

such that there exists a vertex $v \in V$ for which the induced neutrosophic subgraph obtained by deleting v is neutrosophic outerplanar.

6.5 Neutrosophic Quasi-Planar Graph

We next consider the neutrosophic version of quasi-planarity. A neutrosophic quasi-planar graph is a neutrosophic graph drawable so that no prescribed number of edges pairwise cross, while preserving truth, indeterminacy, and falsity information.

Definition 6.5.1 (Neutrosophic Quasi-Planar Graph)**.** A *neutrosophic quasi-planar graph* is a neutrosophic graph

$$\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$$

that admits a drawing in the plane such that no three neutrosophic edges are pairwise crossing.

Definition 6.5.2 (Neutrosophic k-Quasi-Planar Graph)**.** A *neutrosophic k-quasi-planar graph* is a neutrosophic graph

$$\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$$

that can be drawn in the plane in such a way that no k edges are pairwise crossing.

6.6 Neutrosophic Quasi-Outerplanar Graph

We now add the outer-face condition. A neutrosophic quasi-outerplanar graph is a neutrosophic graph drawable with all vertices on the outer face and no prescribed number of edges pairwise crossing simultaneously.

Definition 6.6.1 (Neutrosophic Quasi-Outerplanar Graph). A *neutrosophic quasi-outerplanar graph* is a neutrosophic graph

$$\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$$

that admits a drawing in the plane such that:

- all vertices lie on the boundary of the outer face; and
- no three neutrosophic edges are pairwise crossing.

Definition 6.6.2 (Neutrosophic Outer-k-Quasi-Planar Graph). A *neutrosophic outer-k-quasi-planar graph* is a neutrosophic graph

$$\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$$

that admits a drawing in the plane such that:

- all vertices lie on the boundary of the outer face; and
- no k neutrosophic edges are pairwise crossing.

6.7 Neutrosophic Quasi Apex Graph

A neutrosophic quasi apex graph is a neutrosophic graph that becomes quasi-planar after deleting a vertex subset while preserving truth, indeterminacy, and falsity information throughout. The corresponding deletion-based generalization is as follows.

Definition 6.7.1 (Neutrosophic k-Quasi Apex Graph). A *neutrosophic k-quasi apex graph* is a neutrosophic graph

$$\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$$

for which there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced neutrosophic subgraph on $V \setminus S$ is neutrosophic quasi-planar.

6.8 Neutrosophic Quasi Apex Outerplanar Graph

A neutrosophic quasi apex outerplanar graph is a neutrosophic graph that becomes quasi-outerplanar after deleting a vertex subset while preserving truth, indeterminacy, and falsity information. Likewise, one may define the outerplanar version.

Definition 6.8.1 (Neutrosophic k-Quasi Apex Outerplanar Graph). A *neutrosophic k-quasi apex outerplanar graph* is a neutrosophic graph

$$\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$$

for which there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced neutrosophic subgraph on $V \setminus S$ is neutrosophic quasi-outerplanar.

6.9 Neutrosophic Almost Planar Graph

Finally, the notions of almost planar and almost outerplanar graphs extend naturally to the neutrosophic setting. A neutrosophic almost planar graph is a neutrosophic graph drawable in the plane with at most one crossing while preserving truth, indeterminacy, and falsity values.

Definition 6.9.1 (Neutrosophic Almost Planar Graph)**.** A *neutrosophic almost planar graph* is a neutrosophic graph

$$NTG = (V, E, \sigma, \mu)$$

for which there exists an edge $e \in E(NTG)$ such that

$$NTG - e$$

is a neutrosophic planar graph.

Definition 6.9.2 (Neutrosophic Almost Outerplanar Graph)**.** A *neutrosophic almost outerplanar graph* is a neutrosophic graph

$$NTG = (V, E, \sigma, \mu)$$

for which there exists an edge $e \in E(NTG)$ such that

$$NTG - e$$

is a neutrosophic outerplanar graph.

Proposition 6.9.3. *Let NTG be a neutrosophic almost planar graph. If the induced truth–falsity pairs satisfy the intuitionistic fuzzy constraints after deleting the indeterminacy component, then NTG induces an intuitionistic fuzzy almost planar graph.*

Proof. Delete the same edge e, discard the indeterminacy component, and interpret truth and falsity as membership and non-membership, respectively. Under the stated constraints, the resulting graph is intuitionistic fuzzy almost planar. □

Proposition 6.9.4. *Let NTG be a neutrosophic almost outerplanar graph. If the induced truth–falsity pairs satisfy the intuitionistic fuzzy constraints after deleting the indeterminacy component, then NTG induces an intuitionistic fuzzy almost outerplanar graph.*

Proof. The argument is identical to the previous one, replacing planar by outerplanar. □

Chapter 7

Plithogenic Graphs and Planar Graphs

In recent years, plithogenic graphs have been introduced as a broad extension of fuzzy and neutrosophic graph models, and also as graph-theoretic counterparts of plithogenic sets [339]. This framework has attracted growing attention in recent research [63]. We begin with the basic definition.

Definition 7.0.1 (Plithogenic Graph)**.** [63] Let

$$G = (V, E)$$

be a crisp simple undirected graph, where V is the vertex set and

$$E \subseteq \big\{\{x, y\} : x, y \in V,\ x \neq y\big\}$$

is the edge set. A *plithogenic graph* associated with G is a pair

$$PG = (PM, PN),$$

where PM is the plithogenic vertex component and PN is the plithogenic edge component, defined as follows.

(i) **Plithogenic vertex component.**

$$PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf}),$$

where:

- $M \subseteq V$ is the set of selected vertices;
- ℓ is an attribute assigned to vertices;
- M_ℓ is the set of possible values of the vertex attribute ℓ;
- $$\mathrm{adf} : M \times M_\ell \longrightarrow [0, 1]^s$$

  is the *vertex degree of appurtenance function*;

- $$\mathrm{aCf} : M_\ell \times M_\ell \longrightarrow [0,1]^t$$
  is the *vertex degree of contradiction function.*

(ii) **Plithogenic edge component.**

$$PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf}),$$

where:

- $$N \subseteq \big\{\{x,y\} \in E : x,y \in M\big\}$$
  is the set of selected edges;
- m is an attribute assigned to edges;
- $$N_m \subseteq M_\ell \times M_\ell$$
  is the set of admissible edge attribute values. Thus an element $(a,b) \in N_m$ represents an attribute-value pair assigned to an edge $\{x,y\}$, where a is associated with x and b is associated with y;
- $$\mathrm{bdf} : N \times N_m \longrightarrow [0,1]^s$$
  is the *edge degree of appurtenance function*;
- $$\mathrm{bCf} : N_m \times N_m \longrightarrow [0,1]^t$$
  is the *edge degree of contradiction function.*

All inequalities between elements of $[0,1]^r$ are understood componentwise. The plithogenic graph $PG = (PM, PN)$ is required to satisfy the following conditions.

(P1) **Edge–vertex appurtenance compatibility.** For every edge $\{x,y\} \in N$ and every admissible edge attribute value $(a,b) \in N_m$,

$$\mathrm{bdf}\big(\{x,y\},(a,b)\big) \le \min\big\{\mathrm{adf}(x,a), \mathrm{adf}(y,b)\big\}.$$

Here the minimum is taken componentwise in $[0,1]^s$.

(P2) **Contradiction consistency.** For all $(a,b),(c,d) \in N_m$,

$$\mathrm{bCf}\big((a,b),(c,d)\big) \le \min\big\{\mathrm{aCf}(a,c), \mathrm{aCf}(b,d)\big\}.$$

Here the minimum is taken componentwise in $[0,1]^t$.

(P3) **Reflexivity and symmetry of contradiction functions.** The vertex and edge contradiction functions satisfy

$$\mathrm{aCf}(a,a) = 0, \qquad \mathrm{aCf}(a,b) = \mathrm{aCf}(b,a) \quad \text{for all } a,b \in M_\ell,$$

and

$$\mathrm{bCf}(u,u) = 0, \qquad \mathrm{bCf}(u,v) = \mathrm{bCf}(v,u) \quad \text{for all } u,v \in N_m.$$

The integers $s, t \in \mathbb{N}$ represent the dimensions of appurtenance degrees and contradiction degrees, respectively. In the scalar case $s = t = 1$, all appurtenance and contradiction values belong to the interval $[0, 1]$.

Example 7.0.2 (A Simple Plithogenic Graph)**.** Let

$$G = (V, E)$$

be the crisp graph with

$$V = \{x, y, z\}$$

and

$$E = \{\{x, y\}, \{y, z\}\}.$$

We construct a scalar plithogenic graph, so let $s = t = 1$.

Let

$$M = V, \qquad N = E.$$

Suppose that the vertex attribute ℓ is called "reliability type", and let

$$M_\ell = \{H, L\},$$

where H denotes high reliability and L denotes low reliability.

Define the vertex appurtenance function

$$\mathrm{adf} : M \times M_\ell \to [0, 1]$$

by the following table:

	H	L
x	0.8	0.2
y	0.6	0.4
z	0.3	0.7

Define the vertex contradiction function by

$$\mathrm{aCf}(H, H) = \mathrm{aCf}(L, L) = 0,$$

and

$$\mathrm{aCf}(H, L) = \mathrm{aCf}(L, H) = 0.7.$$

The admissible edge attribute values are chosen as

$$N_m = M_\ell \times M_\ell = \{(H, H), (H, L), (L, H), (L, L)\}.$$

Define the edge appurtenance function

$$\mathrm{bdf} : N \times N_m \to [0, 1]$$

by

	(H, H)	(H, L)	(L, H)	(L, L)
$\{x, y\}$	0.5	0.3	0.2	0.1
$\{y, z\}$	0.25	0.5	0.2	0.35

We now define the edge contradiction function by

$$\mathrm{bCf}\big((a,b),(c,d)\big) = \min\big\{\mathrm{aCf}(a,c), \mathrm{aCf}(b,d)\big\}$$

for all $(a,b),(c,d) \in N_m$.

Then aCf and bCf are reflexive and symmetric. Moreover, the edge–vertex appurtenance compatibility condition is satisfied. For instance,

$$\mathrm{bdf}\big(\{x,y\},(H,H)\big) = 0.5 \le \min\{\mathrm{adf}(x,H), \mathrm{adf}(y,H)\} = \min\{0.8, 0.6\} = 0.6,$$

and

$$\mathrm{bdf}\big(\{y,z\},(H,L)\big) = 0.5 \le \min\{\mathrm{adf}(y,H), \mathrm{adf}(z,L)\} = \min\{0.6, 0.7\} = 0.6.$$

The remaining cases are checked in the same way.

Therefore,

$$PG = (PM, PN)$$

with

$$PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf})$$

and

$$PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf})$$

is a well-defined plithogenic graph.

7.1 Plithogenic Planar Graph

A plithogenic planar graph extends the ordinary notion of a planar graph to the plithogenic setting.

Definition 7.1.1 (Plithogenic Planar Graph)**.** A *plithogenic planar graph* $PG = (PM, PN)$ is a plithogenic graph that admits an embedding in the plane with no edge intersections other than common endpoints.

More explicitly, let

$$PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf}) \quad \text{and} \quad PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf}).$$

If two plithogenic edges e_1 and e_2 intersect at a point P, define the intersection value by

$$S_P = \left(\frac{\mathrm{bdf}(e_1) + \mathrm{bdf}(e_2)}{2}, \frac{\mathrm{bCf}(e_1, e_2) + \mathrm{bCf}(e_2, e_1)}{2} \right).$$

Then PG is called plithogenic planar when

$$S_P = (0,0) \qquad \text{for every intersection point } P.$$

Equivalently, the graph has a planar embedding with no genuine edge crossings.

Example 7.1.2 (A Plithogenic Planar Graph)**.** Let the underlying crisp graph be the triangle

$$G = (V, E), \qquad V = \{v_1, v_2, v_3\}, \qquad E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\big\}.$$

We define a plithogenic graph

$$PG = (PM, PN)$$

with scalar-valued appurtenance and contradiction functions ($s = t = 1$) as follows.

Vertex part. Let

$$PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf}),$$

where

$$M = V, \qquad M_\ell = \{a, b\}.$$

Define the vertex appurtenance function $\mathrm{adf} : M \times M_\ell \to [0, 1]$ by

$$\mathrm{adf}(v_1, a) = 0.9, \qquad \mathrm{adf}(v_1, b) = 0.2,$$

$$\mathrm{adf}(v_2, a) = 0.8, \qquad \mathrm{adf}(v_2, b) = 0.3,$$

$$\mathrm{adf}(v_3, a) = 0.2, \qquad \mathrm{adf}(v_3, b) = 0.9.$$

Define the vertex contradiction function $\mathrm{aCf} : M_\ell \times M_\ell \to [0, 1]$ by

$$\mathrm{aCf}(a, a) = 0, \qquad \mathrm{aCf}(b, b) = 0, \qquad \mathrm{aCf}(a, b) = \mathrm{aCf}(b, a) = 0.4.$$

Edge part. Let

$$PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf}),$$

where

$$N = E, \qquad N_m = M_\ell \times M_\ell = \{(a, a), (a, b), (b, a), (b, b)\}.$$

For each edge $\{x, y\} \in N$ and each $(p, q) \in N_m$, define

$$\mathrm{bdf}\big(\{x, y\}, (p, q)\big) := \min\{\mathrm{adf}(x, p), \mathrm{adf}(y, q)\}.$$

Hence, for example,

$$\mathrm{bdf}\big(\{v_1, v_2\}, (a, a)\big) = \min\{0.9, 0.8\} = 0.8,$$
$$\mathrm{bdf}\big(\{v_2, v_3\}, (a, b)\big) = \min\{0.8, 0.9\} = 0.8,$$
$$\mathrm{bdf}\big(\{v_1, v_3\}, (b, b)\big) = \min\{0.2, 0.9\} = 0.2.$$

Define the edge contradiction function $\mathrm{bCf} : N_m \times N_m \to [0, 1]$ by

$$\mathrm{bCf}\big((p, q), (r, s)\big) := \min\{\mathrm{aCf}(p, r), \mathrm{aCf}(q, s)\}.$$

Then bCf is symmetric and satisfies

$$\mathrm{bCf}(u, u) = 0 \qquad \text{for all } u \in N_m.$$

Moreover, by construction,

$$\mathrm{bdf}\big(\{x, y\}, (p, q)\big) \le \min\{\mathrm{adf}(x, p), \mathrm{adf}(y, q)\},$$

and

$$\mathrm{bCf}\big((p,q),(r,s)\big) \le \min\{\mathrm{aCf}(p,r),\mathrm{aCf}(q,s)\},$$

so the compatibility conditions of a plithogenic graph are satisfied.

Finally, the underlying crisp graph is K_3, which admits a planar embedding as an ordinary triangle in the plane. In this drawing, no two edges intersect except at common endpoints. Therefore, there is no genuine intersection point P of two plithogenic edges, and thus the planar condition is satisfied.

Hence $PG = (PM, PN)$ is a *plithogenic planar graph.*

Theorem 7.1.3. *A plithogenic planar graph specializes to a neutrosophic planar graph when the appurtenance data are interpreted in three components, and it specializes to a fuzzy planar graph in the scalar-valued case.*

Proof. Let $PG = (PM, PN)$ be a plithogenic planar graph.

If the appurtenance functions are interpreted in three components, then the vertex and edge data can be read as truth, indeterminacy, and falsity values. This yields a neutrosophic graph representation. Since the original plithogenic graph is planar, the induced neutrosophic graph inherits a planar embedding with no edge intersections.

If the appurtenance functions are scalar-valued, then the plithogenic data reduce to ordinary membership values for vertices and edges. In this case, the graph becomes a fuzzy graph, and the same embedding shows that it is fuzzy planar.

Hence the plithogenic planar framework contains both neutrosophic planar graphs and fuzzy planar graphs as special cases under suitable specializations of the associated data. □

7.2 Plithogenic Outerplanar Graph

We next consider the outerplanar version. A plithogenic outerplanar graph is a plithogenic graph admitting an outerplanar embedding, with all vertices on the outer face and attribute-based appurtenance and contradiction information.

Definition 7.2.1 (Plithogenic Outerplanar Graph)**.** A *plithogenic outerplanar graph* $PG = (PM, PN)$ is a plithogenic graph that admits an embedding in the plane such that all vertices lie on the boundary of the outer face and no edge intersections occur except at common endpoints.

Equivalently, if

$$PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf}) \quad \text{and} \quad PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf}),$$

then every vertex of M lies on the outer boundary in the chosen embedding, and every intersection value satisfies

$$S_P = (0, 0).$$

Example 7.2.2 (A Plithogenic Outerplanar Graph)**.** Let the underlying crisp graph be the cycle

$$G = (V, E), \qquad V = \{v_1, v_2, v_3, v_4\},$$
$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\big\}.$$

Thus, the underlying graph is C_4.

We define a plithogenic graph

$$PG = (PM, PN)$$

with scalar-valued appurtenance and contradiction functions ($s = t = 1$) as follows.

Vertex part. Let

$$PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf}),$$

where

$$M = V, \qquad M_\ell = \{a, b\}.$$

Define the vertex appurtenance function $\mathrm{adf} : M \times M_\ell \to [0, 1]$ by

$$\mathrm{adf}(v_1, a) = 0.9, \qquad \mathrm{adf}(v_1, b) = 0.2,$$
$$\mathrm{adf}(v_2, a) = 0.8, \qquad \mathrm{adf}(v_2, b) = 0.3,$$
$$\mathrm{adf}(v_3, a) = 0.3, \qquad \mathrm{adf}(v_3, b) = 0.8,$$
$$\mathrm{adf}(v_4, a) = 0.2, \qquad \mathrm{adf}(v_4, b) = 0.9.$$

Define the vertex contradiction function $\mathrm{aCf} : M_\ell \times M_\ell \to [0, 1]$ by

$$\mathrm{aCf}(a, a) = 0, \qquad \mathrm{aCf}(b, b) = 0, \qquad \mathrm{aCf}(a, b) = \mathrm{aCf}(b, a) = 0.4.$$

Edge part. Let

$$PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf}),$$

where

$$N = E, \qquad N_m = M_\ell \times M_\ell = \{(a, a), (a, b), (b, a), (b, b)\}.$$

For each edge $\{x, y\} \in N$ and each $(p, q) \in N_m$, define

$$\mathrm{bdf}\big(\{x, y\}, (p, q)\big) := \min\{\mathrm{adf}(x, p), \mathrm{adf}(y, q)\}.$$

For example,

$$\mathrm{bdf}\big(\{v_1, v_2\}, (a, a)\big) = \min\{0.9, 0.8\} = 0.8,$$
$$\mathrm{bdf}\big(\{v_2, v_3\}, (a, b)\big) = \min\{0.8, 0.8\} = 0.8,$$
$$\mathrm{bdf}\big(\{v_4, v_1\}, (b, a)\big) = \min\{0.9, 0.9\} = 0.9.$$

Define the edge contradiction function $\mathrm{bCf} : N_m \times N_m \to [0, 1]$ by

$$\mathrm{bCf}\big((p, q), (r, s)\big) := \min\{\mathrm{aCf}(p, r), \mathrm{aCf}(q, s)\}.$$

Then bCf is symmetric and satisfies

$$\mathrm{bCf}(u, u) = 0 \qquad \text{for all } u \in N_m.$$

Moreover, by construction,

$$\mathrm{bdf}\big(\{x,y\},(p,q)\big) \le \min\{\mathrm{adf}(x,p),\mathrm{adf}(y,q)\},$$

and

$$\mathrm{bCf}\big((p,q),(r,s)\big) \le \min\{\mathrm{aCf}(p,r),\mathrm{aCf}(q,s)\},$$

so the plithogenic compatibility conditions are satisfied.

Now draw the graph C_4 as a square in the plane, with vertices v_1, v_2, v_3, v_4 appearing in cyclic order on the boundary of the outer face. In this embedding, all vertices lie on the boundary of the outer face, and no two edges intersect except at common endpoints. Hence there is no genuine crossing point P, so the condition

$$S_P = (0,0)$$

is satisfied for every intersection point P.

Therefore, $PG = (PM, PN)$ is a *plithogenic outerplanar graph.*

Theorem 7.2.3. *A plithogenic outerplanar graph specializes to a neutrosophic outerplanar graph in the three-component case, and to a fuzzy outerplanar graph in the scalar-valued case.*

Proof. Let $PG = (PM, PN)$ be a plithogenic outerplanar graph.

When the appurtenance data are interpreted as three-component values, the graph induces a neutrosophic graph. Since the original embedding is outerplanar, all vertices remain on the boundary of the outer face, and no crossings are introduced. Hence the induced graph is neutrosophic outerplanar.

When the appurtenance data are scalar-valued, the plithogenic structure reduces to a fuzzy graph. The same embedding then yields a fuzzy outerplanar graph.

Therefore, plithogenic outerplanar graphs include both neutrosophic outerplanar graphs and fuzzy outerplanar graphs as special cases. □

7.3 Plithogenic Ap ex Graph

The apex version is defined in the natural way. A plithogenic apex graph is a plithogenic graph that becomes planar after deleting one vertex while preserving appurtenance and contradiction information on vertices and edges.

Definition 7.3.1 (Plithogenic Apex Planar Graph)**.** A *plithogenic apex planar graph* $PG = (PM, PN)$ is a plithogenic graph that becomes plithogenic planar after the deletion of a single vertex $v \in M$.

Formally, PG is plithogenic apex planar if there exists a vertex $v \in M$ such that the induced subgraph

$$PG' = (PM', PN')$$

with

$$PM' = (M \setminus \{v\}, \ell', M'_\ell, \mathrm{adf}', \mathrm{aCf}')$$

and

$$PN' = (N', m', N'_m, \mathrm{bdf}', \mathrm{bCf}')$$

is a plithogenic planar graph, where the primed functions are the natural restrictions of the original data.

7.4 Plithogenic Apex Outerplanar Graph

Similarly, one may define the outerplanar counterpart. A plithogenic apex outerplanar graph is a plithogenic graph that becomes outerplanar after deleting one vertex while preserving appurtenance-contradiction information on vertices and edges.

Definition 7.4.1 (Plithogenic Apex Outerplanar Graph)**.** A *plithogenic apex outerplanar graph* $PG = (PM, PN)$ is a plithogenic graph that becomes plithogenic outerplanar after the deletion of a single vertex $v \in M$.

Formally, PG is plithogenic apex outerplanar if there exists a vertex $v \in M$ such that the induced subgraph

$$PG' = (PM', PN')$$

with

$$PM' = (M \setminus \{v\}, \ell', M'_\ell, \mathrm{adf}', \mathrm{aCf}')$$

and

$$PN' = (N', m', N'_m, \mathrm{bdf}', \mathrm{bCf}')$$

is a plithogenic outerplanar graph, where the primed functions are the restrictions of the original functions to the remaining vertices and edges.

Theorem 7.4.2. *Under a suitable neutrosophic specialization of the plithogenic data, a plithogenic apex planar graph induces a neutrosophic apex planar graph, and a plithogenic apex outerplanar graph induces a neutrosophic apex outerplanar graph.*

Proof. We treat the planar and outerplanar cases separately.

Case 1: Apex planar case. Let $PG = (PM, PN)$ be a plithogenic apex planar graph. By definition, there exists a vertex $v \in M$ such that deleting v yields a plithogenic planar graph PG'. If the plithogenic vertex and edge data are specialized to neutrosophic triples, then PG induces a neutrosophic graph. Since the deletion of the same vertex v yields a planar plithogenic graph, the induced neutrosophic graph becomes neutrosophic planar after deleting v. Hence the induced graph is a neutrosophic apex planar graph.

Case 2: Apex outerplanar case. Let $PG = (PM, PN)$ be a plithogenic apex outerplanar graph. Then there exists a vertex $v \in M$ such that deleting v yields a plithogenic outerplanar graph PG'. After the same neutrosophic specialization, the induced graph becomes neutrosophic outerplanar upon deleting v. Therefore the resulting graph is a neutrosophic apex outerplanar graph.

In both cases, the apex property is preserved under the specialization from the plithogenic framework to the neutrosophic framework. □

Chapter 8

Planar Hyp ergraphs

In this chapter, we discuss planar hypergraphs and SuperHyperGraphs.

8.1 Planar Hyp ergraphs

A planar hypergraph may be viewed as the hypergraph analogue of a planar graph. Indeed, an ordinary planar graph is a special case of a planar hypergraph in which every hyperedge has exactly two vertices (cf. [340–344]).

Definition 8.1.1 (Planar Hypergraph)**.** (cf. [340,344]) Let $H = (V, E)$ be a hypergraph. Its *bipartite representation*, denoted by $B(H)$, is the bipartite graph with vertex set

$$V \cup E,$$

where a vertex $v \in V$ is adjacent to a vertex $e \in E$ if and only if

$$v \in e$$

in the hypergraph H.

The hypergraph H is called a *planar hypergraph* if its bipartite representation $B(H)$ is planar. Equivalently, H is planar if $B(H)$ can be drawn in the plane without edge crossings.

Theorem 8.1.2. *Every planar hypergraph $H = (V, E)$ canonically determines a planar graph via its bipartite representation $B(H)$.*

Proof. By definition, H is planar precisely when $B(H)$ is a planar graph. Since $B(H)$ is constructed from the incidence relation between vertices and hyperedges, it is a graph associated canonically with H. Hence a planar hypergraph yields a planar graph through its bipartite representation. □

8.2 Outerplanar Hyp ergraph

An outerplanar hypergraph is a hypergraph admitting a planar embedding with all vertices on the outer face, so no hyperedge crossings occur beyond shared incidences.

Definition 8.2.1 (Outerplanar Hypergraph)**.** Let $H = (V, E)$ be a hypergraph, and let $B(H)$ be its bipartite representation. Then H is called an *outerplanar hypergraph* if $B(H)$ is an outerplanar graph. Equivalently, H is outerplanar if its bipartite representation admits an embedding in the plane such that all vertices lie on the boundary of the outer face and no two edges cross except at common endpoints.

Theorem 8.2.2. *Every outerplanar hypergraph $H = (V, E)$ canonically determines an outerplanar graph via its bipartite representation $B(H)$.*

Proof. If H is outerplanar, then by definition its bipartite representation $B(H)$ is outerplanar. Therefore $B(H)$ is an outerplanar graph associated with H. □

8.3 Ap ex Hyp ergraph

An apex hypergraph is a hypergraph that becomes planar after deleting one vertex, so the remaining incidence structure admits a planar embedding.

Definition 8.3.1 (Apex Hypergraph)**.** A hypergraph $H = (V, E)$ is called an *apex hypergraph* if there exists a vertex $v \in V$ such that the hypergraph obtained by deleting v becomes planar.

More precisely, let

$$H - v$$

denote the hypergraph obtained from H by removing v and deleting v from every hyperedge. Then H is an apex hypergraph if there exists $v \in V$ such that the bipartite representation

$$B(H - v)$$

is planar.

Remark 8.3.2. If H is already planar, then every vertex may be regarded as an apex.

Theorem 8.3.3. *If $H = (V, E)$ is an apex hypergraph, then there exists a vertex $v \in V$ such that $H - v$ is a planar hypergraph.*

Proof. This is exactly the defining property of an apex hypergraph. By assumption, there exists a vertex $v \in V$ such that $B(H - v)$ is planar. Hence $H - v$ is a planar hypergraph. □

Theorem 8.3.4. *If $H = (V, E)$ is an apex hypergraph, then its bipartite representation $B(H)$ is an apex graph.*

Proof. Since H is an apex hypergraph, there exists a vertex $v \in V$ such that $H - v$ is planar. Now $B(H)$ contains the vertices of V and the hyperedge-vertices of E. Deleting the vertex v from $B(H)$ yields the bipartite representation of the incidence structure after removing v. Because $B(H - v)$ is planar, the graph $B(H)$ becomes planar after deletion of the single vertex v. Therefore $B(H)$ is an apex graph. □

8.4 Apex Outerplanar Hypergraph

An apex outerplanar hypergraph is a hypergraph that becomes outerplanar after deleting one vertex, so the remaining incidence structure admits an embedding with vertices outermost.

Definition 8.4.1 (Apex Outerplanar Hypergraph)**.** A hypergraph $H = (V, E)$ is called an *apex outerplanar hypergraph* if there exists a vertex $v \in V$ such that $H - v$ is outerplanar.

Equivalently, H is apex outerplanar if there exists $v \in V$ such that

$$B(H - v)$$

is an outerplanar graph.

Remark 8.4.2. If H is already outerplanar, then every vertex may be regarded as an apex.

Theorem 8.4.3. *If $H = (V, E)$ is an apex outerplanar hypergraph, then there exists a vertex $v \in V$ such that $H - v$ is an outerplanar hypergraph.*

Proof. By definition, there exists $v \in V$ such that $B(H - v)$ is outerplanar. Hence $H - v$ is an outerplanar hypergraph. □

Theorem 8.4.4. *If $H = (V, E)$ is an apex outerplanar hypergraph, then its bipartite representation $B(H)$ is an apex outerplanar graph.*

Proof. Let $v \in V$ be a vertex such that $H - v$ is outerplanar. Then $B(H - v)$ is outerplanar by definition. Deleting the same vertex v from $B(H)$ yields the bipartite incidence graph corresponding to the reduced hypergraph. Therefore $B(H)$ becomes outerplanar after deletion of one vertex, so it is an apex outerplanar graph. □

8.5 Planar and Outerplanar SuperHyperGraphs

In this section, we consider planar, outerplanar, apex, and apex outerplanar SuperHyperGraphs. As noted earlier in this book, SuperHyperGraphs [345] may be regarded as higher-order structures that extend ordinary graphs and hypergraphs. Because of their importance, SuperHyperGraphs have attracted considerable attention in recent years [346–349].

Definition 8.5.1 (Extended Bipartite Representation of a SuperHyperGraph)**.** Let SHG be a SuperHyperGraph. Its *extended bipartite representation*, denoted by $B(SHG)$, is a bipartite graph (U, F) defined as follows:

- $$U = V \cup E,$$
  where hyperedges are also treated as vertices;
- an edge $(u, e) \in F$ is included whenever:

– $u \in V$ and $u \in e$ in SHG; or
– $u \in E$ and there exists a higher-order relation between the hyperedges u and e.

A Planar SuperHyperGraph is a SuperHyperGraph admitting a plane embedding of its incidence structure without crossings, preserving higher-order relations among supervertices and superhyperedges in visualization. An Outerplanar SuperHyperGraph is a planar SuperHyperGraph whose supervertices can be arranged entirely on the outer boundary, allowing a crossing-free embedding of higher-order incidence relations.

Definition 8.5.2 (Planar SuperHyperGraph)**.** A *planar SuperHyperGraph* is a SuperHyperGraph whose extended bipartite representation $B(SHG)$ is planar.

Example 8.5.3 (A Planar SuperHyperGraph)**.** Let the base set be

$$V_0 = \{a, b, c, d\},$$

and consider the 0-SuperHyperGraph

$$SHG = (V, E),$$

where

$$V = \{a, b, c, d\},$$

and

$$E = \big\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\big\}.$$

Thus, SHG is the ordinary complete graph K_4, viewed as a SuperHyperGraph.

Its extended bipartite representation $B(SHG)$ has bipartition

$$\{a, b, c, d\} \quad \text{and} \quad \big\{e_{ab}, e_{ac}, e_{ad}, e_{bc}, e_{bd}, e_{cd}\big\},$$

where each e_{xy} corresponds to the superhyperedge $\{x, y\}$, and

$$x \sim e_{xy}, \qquad y \sim e_{xy}.$$

Hence $B(SHG)$ is exactly the subdivision graph of K_4, obtained by subdividing each edge of K_4 once.

Since K_4 is planar, its subdivision graph is also planar. Therefore, $B(SHG)$ is planar, and so SHG is a *planar SuperHyperGraph*.

Definition 8.5.4 (Outerplanar SuperHyperGraph)**.** An *outerplanar SuperHyperGraph* is a SuperHyperGraph whose extended bipartite representation $B(SHG)$ is outerplanar.

Example 8.5.5 (An Outerplanar SuperHyperGraph)**.** Let the base set be

$$V_0 = \{a, b, c, d\},$$

and consider the 0-SuperHyperGraph

$$SHG = (V, E),$$

where

$$V = \{a, b, c, d\},$$

and

$$E = \big\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\big\}.$$

Thus, SHG is the cycle graph C_4, viewed as a SuperHyperGraph.

Its extended bipartite representation $B(SHG)$ has bipartition

$$\{a, b, c, d\} \quad \text{and} \quad \{e_{ab}, e_{bc}, e_{cd}, e_{da}\},$$

where each e_{xy} corresponds to the superhyperedge $\{x, y\}$, and the incidences are

$$a \sim e_{ab}, \quad b \sim e_{ab}, \quad b \sim e_{bc}, \quad c \sim e_{bc},$$

$$c \sim e_{cd}, \quad d \sim e_{cd}, \quad d \sim e_{da}, \quad a \sim e_{da}.$$

Therefore, $B(SHG)$ is a cycle of length 8, namely C_8.

Since every cycle admits an outerplanar embedding, $B(SHG)$ is outerplanar. Therefore, SHG is an *outerplanar SuperHyperGraph.*

Theorem 8.5.6. *Planar SuperHyperGraphs and outerplanar SuperHyperGraphs generalize planar hypergraphs and outerplanar hypergraphs, respectively.*

Proof. We verify the planar and outerplanar cases separately.

First, every hypergraph may be regarded as a special case of a SuperHyperGraph in which no higher-order relations among hyperedges are added. If $H = (V, E)$ is a planar hypergraph, then its bipartite representation $B(H)$ is planar. Viewing H as a SuperHyperGraph SHG without additional higher-order structure, the extended bipartite representation $B(SHG)$ coincides with $B(H)$. Hence SHG is a planar SuperHyperGraph.

Conversely, planar SuperHyperGraphs allow additional relations among hyperedges that do not occur in ordinary hypergraphs. Thus they strictly extend the class of planar hypergraphs.

The same argument applies to outerplanarity. If H is an outerplanar hypergraph, then $B(H)$ is outerplanar. When H is viewed as a SuperHyperGraph with no extra higher-order relations, its extended bipartite representation is unchanged. Hence every outerplanar hypergraph is an outerplanar SuperHyperGraph.

Therefore planar SuperHyperGraphs generalize planar hypergraphs, and outerplanar SuperHyperGraphs generalize outerplanar hypergraphs. □

Chapter 9

Uncertain Planar Graph

In this chapter, we explain Uncertain Planar Graphs and related concepts.

9.1 Uncertain Planar Graph

In order to define planarity in a mathematically clean and model-independent way, we separate the topological graph structure from the uncertainty data. The uncertainty values are treated as labels attached to vertices and edges, whereas planarity is imposed on the underlying ordinary graph.

Definition 9.1.1 (Underlying Crisp Graph)**.** Let M be an uncertain model, and let

$$\mathcal{G}_M = (V, E, \mu_M)$$

be an uncertain graph of type M, where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M).$$

The graph

$$\underline{\mathcal{G}_M} := (V, E)$$

is called the *underlying crisp graph* of $\mathcal{G}_M$.

The following notion should be understood as a structural, or label-invariant, notion of uncertain planarity. The uncertainty degrees are preserved as labels, but the planarity condition is imposed only on the underlying crisp graph.

Definition 9.1.2 (Uncertain Planar Graph)**.** Let M be an uncertain model, and let

$$\mathcal{G}_M = (V, E, \mu_M)$$

be an uncertain graph of type M, where (V, E) is a finite undirected loopless graph.

Then $\mathcal{G}_M$ is called an *Uncertain Planar Graph* if its underlying crisp graph

$$\underline{\mathcal{G}_M} = (V, E)$$

is planar.

Equivalently, $\mathcal{G}_M$ is an Uncertain Planar Graph if there exists an embedding of $\underline{\mathcal{G}_M}$ in the plane such that no two distinct edges intersect except possibly at a common endpoint.

Remark 9.1.3. This definition is intentionally model-independent. It does not require any thresholding, support extraction, or additional comparison rule on the degree-domain $\mathrm{Dom}(M)$. Hence it applies uniformly to fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and other uncertainty models.

Definition 9.1.4 (Support-Based Uncertain Planarity)**.** Let M be an uncertain model and let

$$\mathcal{G}_M = (V, E, \mu_M)$$

be an uncertain graph. Suppose that a support predicate

$$\mathrm{Supp}_M : \mathrm{Dom}(M) \to \{0, 1\}$$

is fixed. The support graph of $\mathcal{G}_M$ is the crisp graph

$$\mathrm{Supp}(\mathcal{G}_M) = (V_M, E_M),$$

where

$$V_M = \{v \in V : \mathrm{Supp}_M(\mu_M(v)) = 1\},$$

and

$$E_M = \{\{u, v\} \in E : u, v \in V_M,\ \mathrm{Supp}_M(\mu_M(\{u, v\})) = 1\}.$$

The uncertain graph $\mathcal{G}_M$ is called *support-planar* if $\mathrm{Supp}(\mathcal{G}_M)$ is planar.

Proposition 9.1.5. *Let $\mathcal{G}_M = (V, E, \mu_M)$ be an uncertain graph, and let $\mathrm{Supp}_M : \mathrm{Dom}(M) \to \{0, 1\}$ be a support predicate. If $\mathcal{G}_M$ is an Uncertain Planar Graph, then $\mathcal{G}_M$ is support-planar.*

Proof. If $\mathcal{G}_M$ is an Uncertain Planar Graph, then its underlying crisp graph (V, E) is planar. The support graph

$$\mathrm{Supp}(\mathcal{G}_M) = (V_M, E_M)$$

is a subgraph of (V, E). Since every subgraph of a planar graph is planar, $\mathrm{Supp}(\mathcal{G}_M)$ is planar. Hence $\mathcal{G}_M$ is support-planar. □

Example 9.1.6. Let the underlying crisp graph be K_5, which is non-planar. Suppose that the support predicate selects no edge, for example because all edge uncertainty degrees are below the chosen threshold. Then the support graph is edgeless and therefore planar. Hence the uncertain graph is support-planar, but it is not an Uncertain Planar Graph in the structural sense.

Theorem 9.1.7 (Well-Definedness of Uncertain Planarity)**.** *Let M be an uncertain model, and let*

$$\mathcal{G}_M = (V, E, \mu_M)$$

be an uncertain graph of type M. Then the statement

"$\mathcal{G}_M$ is an Uncertain Planar Graph"

is well-defined.

More precisely:

1. *it is independent of the particular drawing used to test planarity;*

2. *it depends only on the underlying crisp graph* $\underline{\mathcal{G}_M} = (V, E)$*;*

3. *it is independent of the specific uncertainty assignment* μ_M*.*

Proof. By definition, $\mathcal{G}_M$ is an Uncertain Planar Graph if and only if its underlying crisp graph

$$\underline{\mathcal{G}_M} = (V, E)$$

is planar.

First, the graph (V, E) is uniquely determined by the first two components of $\mathcal{G}_M = (V, E, \mu_M)$. The map $\mu_M : V \cup E \to \mathrm{Dom}(M)$ assigns uncertainty data to already existing vertices and edges, but it does not alter the vertex set, the edge set, the incidence relation, or the adjacency structure.

Second, planarity is a standard property of an ordinary finite undirected loopless graph: a graph is planar precisely when there exists at least one embedding in the plane with no edge crossings except at common endpoints. This property is independent of which particular drawing is chosen, because the definition is existential: it requires the existence of a crossing-free embedding, not that every drawing be crossing-free.

Therefore, the truth value of the statement

$$\text{“}\mathcal{G}_M \text{ is an Uncertain Planar Graph”}$$

is completely determined by the ordinary graph (V, E), and not by the uncertainty labels μ_M. Hence the notion is well-defined. □

Corollary 9.1.8. *Let*

$$\mathcal{G}_M = (V, E, \mu_M) \qquad \textit{and} \qquad \mathcal{H}_M = (V', E', \nu_M)$$

be uncertain graphs of the same type M*. If their underlying crisp graphs* (V, E) *and* (V', E') *are isomorphic, then*

$$\mathcal{G}_M \textit{ is an Uncertain Planar Graph} \iff \mathcal{H}_M \textit{ is an Uncertain Planar Graph.}$$

Proof. Planarity is invariant under graph isomorphism. Since uncertain planarity is defined entirely by the planarity of the underlying crisp graph, the conclusion follows immediately. □

9.2 Uncertain Outerplanar Graph, Uncertain Apex Graph, and Uncertain Apex Outerplanar Graph

We continue the same model-independent philosophy used for uncertain planar graphs: the uncertainty data are treated as labels, while the topological graph-class property is imposed on the underlying crisp graph.

Definition 9.2.1 (Vertex Deletion in an Uncertain Graph)**.** Let M be an uncertain model, and let

$$\mathcal{G}_M = (V, E, \mu_M)$$

be an uncertain graph of type M, where $E \subseteq \binom{V}{2}$.

For a vertex $v \in V$, the *vertex-deleted uncertain graph*

$$\mathcal{G}_M - v$$

is defined by

$$\mathcal{G}_M - v := \big(V \setminus \{v\},\ E_v,\ \mu_M|_{(V \setminus \{v\}) \cup E_v}\big),$$

where

$$E_v := \{\, e \in E : v \notin e \,\}.$$

Thus $\mathcal{G}_M - v$ is obtained by removing the vertex v, removing all edges incident with v, and restricting the uncertainty map to the remaining vertices and edges.

Definition 9.2.2 (Uncertain Outerplanar Graph)**.** Let M be an uncertain model, and let

$$\mathcal{G}_M = (V, E, \mu_M)$$

be an uncertain graph of type M, where (V, E) is a finite undirected loopless graph.

Then $\mathcal{G}_M$ is called an *Uncertain Outerplanar Graph* if its underlying crisp graph

$$\underline{\mathcal{G}_M} = (V, E)$$

is outerplanar.

Equivalently, $\mathcal{G}_M$ is an Uncertain Outerplanar Graph if there exists a plane embedding of $\underline{\mathcal{G}_M}$ such that all vertices lie on the boundary of the outer face.

Definition 9.2.3 (Uncertain Apex Graph)**.** Let M be an uncertain model, and let

$$\mathcal{G}_M = (V, E, \mu_M)$$

be an uncertain graph of type M, where (V, E) is a finite undirected loopless graph.

Then $\mathcal{G}_M$ is called an *Uncertain Apex Graph* if there exists a vertex $v \in V$ such that

$$\mathcal{G}_M - v$$

is an Uncertain Planar Graph.

Equivalently, $\mathcal{G}_M$ is an Uncertain Apex Graph if there exists a vertex $v \in V$ such that

$$\underline{\mathcal{G}_M - v} = (V \setminus \{v\}, E_v)$$

is planar.

Such a vertex v is called an *apex* of $\mathcal{G}_M$ with respect to planarity.

Definition 9.2.4 (Uncertain Apex Outerplanar Graph)**.** Let M be an uncertain model, and let

$$\mathcal{G}_M = (V, E, \mu_M)$$

be an uncertain graph of type M, where (V, E) is a finite undirected loopless graph.

Then $\mathcal{G}_M$ is called an *Uncertain Apex Outerplanar Graph* if there exists a vertex $v \in V$ such that

$$\mathcal{G}_M - v$$

is an Uncertain Outerplanar Graph.

Equivalently, $\mathcal{G}_M$ is an Uncertain Apex Outerplanar Graph if there exists a vertex $v \in V$ such that

$$\underline{\mathcal{G}_M - v} = (V \setminus \{v\}, E_v)$$

is outerplanar.

Such a vertex v is called an *apex* of $\mathcal{G}_M$ with respect to outerplanarity.

Remark 9.2.5. These definitions are model-independent. No thresholding, support extraction, ranking rule, or additional order structure on $\mathrm{Dom}(M)$ is required. Hence they apply uniformly to fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and other uncertainty models.

Theorem 9.2.6 (Well-Definedness)**.** *Let M be an uncertain model, and let*

$$\mathcal{G}_M = (V, E, \mu_M)$$

be an uncertain graph of type M. Then each of the following statements is well-defined:

1. *$\mathcal{G}_M$ is an Uncertain Outerplanar Graph;*
2. *$\mathcal{G}_M$ is an Uncertain Apex Graph;*
3. *$\mathcal{G}_M$ is an Uncertain Apex Outerplanar Graph.*

More precisely, each property depends only on the underlying crisp graph (V, E) and is independent of the particular uncertainty assignment μ_M.

Proof. We prove the three cases separately.

(1) Uncertain Outerplanar Graph. By definition, $\mathcal{G}_M$ is an Uncertain Outerplanar Graph if and only if

$$\underline{\mathcal{G}_M} = (V, E)$$

is outerplanar. The graph (V, E) is uniquely determined by the first two components of $\mathcal{G}_M$, and outerplanarity is a standard graph-theoretic property of ordinary finite undirected loopless graphs. Hence this notion is independent of μ_M, and therefore well-defined.

(2) Uncertain Apex Graph. For each $v \in V$, the deleted object

$$\mathcal{G}_M - v = \big(V \setminus \{v\}, E_v, \mu_M|_{(V\setminus\{v\})\cup E_v}\big)$$

is uniquely determined by $\mathcal{G}_M$ and v. Moreover,

$$\underline{\mathcal{G}_M - v} = (V \setminus \{v\}, E_v)$$

is precisely the ordinary vertex-deleted graph of (V, E). Thus the statement

"there exists $v \in V$ such that $\mathcal{G}_M - v$ is an Uncertain Planar Graph"

is equivalent to

"there exists $v \in V$ such that $(V \setminus \{v\}, E_v)$ is planar."

The latter is an ordinary graph-theoretic statement depending only on (V, E). Hence the notion of Uncertain Apex Graph is well-defined.

(3) Uncertain Apex Outerplanar Graph. The argument is completely analogous. By definition, $\mathcal{G}_M$ is an Uncertain Apex Outerplanar Graph if and only if there exists a vertex $v \in V$ such that

$$\mathcal{G}_M - v$$

is an Uncertain Outerplanar Graph. By part (1), this is equivalent to requiring that

$$\underline{\mathcal{G}_M - v} = (V \setminus \{v\}, E_v)$$

be outerplanar. Again, this depends only on the ordinary graph structure of (V, E), not on the uncertainty map μ_M. Therefore, the notion is well-defined.

Combining the three cases completes the proof. □

Proposition 9.2.7. *Every Uncertain Outerplanar Graph is an Uncertain Planar Graph.*

Proof. If $\mathcal{G}_M$ is uncertain outerplanar, then its underlying crisp graph is outerplanar. Every outerplanar graph is planar. Hence the underlying crisp graph of $\mathcal{G}_M$ is planar, so $\mathcal{G}_M$ is an Uncertain Planar Graph. □

Proposition 9.2.8. *Every Uncertain Planar Graph is an Uncertain Apex Graph.*

Proof. Let $\mathcal{G}_M = (V, E, \mu_M)$ be an Uncertain Planar Graph with $V \neq \emptyset$. Then $\mathcal{G}_M$ is an Uncertain Apex Graph. Choose any vertex $v \in V$. Since every subgraph of a planar graph is planar, the graph

$$(V \setminus \{v\}, E_v)$$

is planar. Therefore $\mathcal{G}_M - v$ is an Uncertain Planar Graph, and hence $\mathcal{G}_M$ is an Uncertain Apex Graph. □

Proposition 9.2.9. *Every Uncertain Outerplanar Graph is an Uncertain Apex Outerplanar Graph.*

Proof. Let $\mathcal{G}_M = (V, E, \mu_M)$ be an Uncertain Outerplanar Graph. Then its underlying crisp graph (V, E) is outerplanar. Choose any vertex $v \in V$. Since every subgraph of an outerplanar graph is outerplanar, the graph

$$(V \setminus \{v\}, E_v)$$

is outerplanar. Hence $\mathcal{G}_M - v$ is an Uncertain Outerplanar Graph, and therefore $\mathcal{G}_M$ is an Uncertain Apex Outerplanar Graph. □

Corollary 9.2.10. *Let*

$$\mathcal{G}_M = (V, E, \mu_M) \qquad \textit{and} \qquad \mathcal{H}_M = (V', E', \nu_M)$$

be uncertain graphs of type M. If the underlying crisp graphs (V, E) and (V', E') are isomorphic, then the following are equivalent:

1. *$\mathcal{G}_M$ is an Uncertain Outerplanar Graph;*
2. *$\mathcal{G}_M$ is an Uncertain Apex Graph;*
3. *$\mathcal{G}_M$ is an Uncertain Apex Outerplanar Graph;*

for $\mathcal{G}_M$ if and only if the corresponding statement holds for $\mathcal{H}_M$.

Proof. Outerplanarity, planarity after one-vertex deletion, and outerplanarity after one-vertex deletion are all invariant under graph isomorphism. Since the uncertain versions are defined entirely through the underlying crisp graphs, the conclusion follows. □

Chapter 10

Conclusion

In this book, we investigated outerplanar graphs and aim to clarify the relationships between planar graphs, fuzzy graphs, and neutrosophic graphs. Given the extensive and continuously expanding body of literature on fuzzy mathematics, it is inevitable that similar concepts may emerge independently across different journals and periods. Nevertheless, we believe that efforts to unify these concepts are essential and will greatly contribute to advancing the field. We aim to conduct further investigations in this direction in the future.

The model-independent notions introduced in this book are structural in nature: the planar-type property is imposed on the underlying crisp graph, while the uncertainty degrees are preserved as vertex and edge labels. Therefore, these definitions should be distinguished from degree-sensitive notions, such as support-based or threshold-based uncertain planarity. Developing such degree-sensitive variants systematically remains a natural direction for future research.

Disclaimer

Funding

This study was conducted without any financial support from external organizations or grants.

Acknowledgments

We would like to express our sincere gratitude to everyone who provided valuable insights, support, and encouragement throughout this research. We also extend our thanks to the readers for their interest and to the authors of the referenced works, whose scholarly contributions have greatly influenced this study. Lastly, we are deeply grateful to the publishers and reviewers who facilitated the dissemination of this work.

Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this book.

Code Availability

No code or software was developed for this study.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer (Others)

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

App endix (List of Tables)

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Fuzzy, Neutrosophic, and Uncertain Graph Theory (II): Uncertain Planar Graph Theory

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Abstract

Graph theory is a fundamental branch of mathematics concerned with networks formed by vertices and edges, together with their paths, embeddings, structures, and invariants. A planar graph is one that can be drawn on a plane without any edges intersecting, ensuring planarity. Outerplanar graphs, a subset of planar graphs, have all their vertices located on the boundary of the outer face in their planar embedding. In recent years, outerplanar graphs have been formally defined within the context of fuzzy graphs. To capture uncertain parameters and concepts, various graphs such as fuzzy, neutrosophic, Turiyam, and plithogenic graphs have been studied. In this book, we investigate planar graphs, outerplanar graphs, apex graphs, apex graphs, quasi-planar graphs, almost planar graphs, and related planar-type graph classes within the frameworks of neutrosophic graphs, fuzzy graphs, and plithogenic graphs. This book is a partially improved and revised version of [1].

Keywords: Planar Graph, Outerplanar Graph, Apex Graph, Fuzzy Graph, Intuitionistic Fuzzy Graph, Neutrosophic Graph, Plithogenic Graph, Uncertain Graph.

This book presents a comprehensive study of fuzzy, neutrosophic, and uncertain graph models. It extends classical notions of planarity to fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and uncertain environments, and investigates a wide range of related structures including outerplanar, apex, quasi-planar, almost planar, and other graph classes. By integrating uncertainty into graph theoretical frameworks, this work provides new perspectives, properties, and applications for modeling complex systems where information is incomplete, imprecise, or indeterminate.

ISBN 978-197250224-2

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