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HYPERGRAPH AND SUPERHYPERGRAPH THEORY WITH APPLICATIONS (IX)

Revisiting SuperHyperGraph Concepts

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Preface

Graphs and hypergraphs have long served as the natural language for describing relationships between discrete objects. Yet as the systems we study grow richer, more hierarchical, and more deeply nested, the classical incidence structures of a single vertex set and a single edge set become increasingly restrictive. The SuperHyperGraph, introduced by Florentin Smarandache and built through iterated powerset operations, was proposed precisely to answer this need: a framework in which vertices may themselves be sets of sets, and in which edges may relate objects across many structural layers at once.

This book, the ninth installment of the HyperGraph and SuperHyperGraph Theory with Applications series, returns to a wide range of concepts that are already well established for ordinary graphs and hypergraphs, and asks how each one should be reformulated once nested supervertices and superhyperedges are taken seriously. Blocks, k-cores, girth, defensive alliances, zero-forcing processes, degree sequences, neural and tensor structures, semantic networks, and many further notions are each revisited in turn, with definitions and illustrative examples chosen to keep the exposition self-contained.

The material presented here is deliberately theoretical in character. It is intended less as a finished destination than as a set of foundations on which future work on structural parameters, algorithms, computational experiments, and applications to hierarchical and higher-order systems can be built. Readers already familiar with graph theory and hypergraph theory will recognize most of the underlying ideas; what changes is the setting in which they are cast, and the additional expressive power that this setting affords.

We are grateful to the colleagues, reviewers, and readers whose engagement with earlier volumes in this series has helped shape the present one, and to the Neutrosophic Science International Association Publishing House for making this work available to the wider research community. It is our hope that this book will encourage further exploration of SuperHyperGraphs, both as objects of independent mathematical interest and as tools for modeling the nested, multi-level systems that increasingly surround us.

Takaaki Fujita and Florentin Smarandache

HyperGraph and SuperHyperGraph Theory with Applications (IX): Revisiting SuperHyperGraph Concepts

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Abstract

Hypergraphs generalize ordinary graphs by allowing an edge to join an arbitrary nonempty subset of the vertex set. This broader incidence structure makes it possible to represent higher-order relationships that cannot be adequately captured by ordinary pairwise edges. A further extension is obtained by iterating the powerset construction, which gives rise to nested and higher-order vertex objects. This leads to the framework of finite *SuperHyperGraphs*, in which vertices and edges may be organized across multiple set-theoretic layers.

Although SuperHyperGraphs provide a highly expressive language for describing hierarchical, multi-level, and higher-order relational systems, their structural properties and graph-theoretic parameters have not yet been investigated as systematically as those of ordinary graphs and hypergraphs. In particular, many classical notions from graph theory and hypergraph theory can be reconsidered, extended, and reformulated within the SuperHyperGraph setting.

The purpose of this book is to discuss several new concepts related to SuperHyperGraphs and to present them in a mathematically consistent form. By revisiting familiar graph-theoretic and hypergraph-theoretic ideas from the viewpoint of nested supervertices and superhyperedges, this book aims to provide a foundation for further theoretical development and future applications of SuperHyperGraph theory.

Keywords: HyperGraph, SuperHyperGraph, higher-order structure, nested structure, graph theory, hypergraph theory.

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Chapter 1

Introduction

1.1 Graphs, HyperGraphs, and SuperHyperGraphs

Classical network structures are commonly modeled by *graphs*, in which objects are represented by vertices and binary relationships are represented by edges [1]. This framework is highly effective when interactions occur only between pairs of entities. Due to their broad applicability and theoretical importance, many variants and extensions of graphs have been studied, including Fuzzy Graphs [2–5], Intuitionistic Fuzzy Graphs [6–8], Hesitant Fuzzy Graphs [9–11], Weighted Graphs [12], Neutrosophic Graphs [8,13–15], Quadripartitioned Neutrosophic Graphs [16,17], Pentapartitioned Neutrosophic Graphs [18], Bipolar Neutrosophic Graphs [19–21], and Plithogenic Graphs [22,23].

However, in many real-world and theoretical settings, several entities may interact simultaneously. In such situations, ordinary graphs are often insufficient for describing the underlying relations in a faithful way. To overcome this limitation, *hypergraphs* extend the classical notion of a graph by allowing each hyperedge to connect an arbitrary nonempty collection of vertices. Thus, hypergraphs provide a direct and flexible way to represent higher-order interactions [24–27]. Due to their importance as objects of study, HyperGraphs have also been extended and investigated in various forms, including Fuzzy HyperGraphs [28–31], Knowledge HyperGraphs [32–34], Neutrosophic HyperGraphs [35–37], Soft HyperGraphs [38,39], and Directed HyperGraphs [40–42]. Moreover, in recent years, research on applications such as HyperGraph Neural Networks has also been carried out very actively [43–46].

Many natural, social, computational, and engineered systems are not only higher-order, but also *nested*, *multi-layered*, and *hierarchical*. To describe such rich incidence structures, F. Smarandache introduced the concept of a *SuperHyperGraph*. Roughly speaking, a SuperHyperGraph is constructed by means of iterative powerset operations. Hence, vertices may themselves be set-valued objects, often called *supervertices*, and edges may express incidence relations across multiple structural levels [47–49]. For this reason, SuperHyperGraphs have recently attracted increasing attention in both theoretical investigations and practical modeling studies [50–55].

Graphs and hypergraphs are also intuitive and visually transparent tools for describing complex systems. They have been widely applied in artificial intelligence, network science, data mining, informatics, chemistry, physics, and related disciplines [56–58]. By explicitly incorporating hierarchical and multi-level incidence, SuperHyperGraphs provide a broader and more adaptable framework for the study of complex structured data (e.g., [59–68]).

Table 1.1 summarizes the main distinctions among graphs, hypergraphs, and SuperHyperGraphs. Throughout this book, n denotes a nonnegative integer unless explicitly stated otherwise.

Table 1.1: Main differences among graphs, hypergraphs, and SuperHyperGraphs.

Concept	*Notation*	*Edge Family*	*Main Structural Feature*
Graph [1]	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$	Each edge represents a *binary* relation between two vertices.
Hypergraph [25]	$H = (V, \mathcal{E})$	$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Each hyperedge may connect an arbitrary nonempty subset of vertices, allowing direct modeling of higher-order interactions.
SuperHyperGraph [47]	$\mathrm{SHG}^{(n)} = (V, E)$	$V \subseteq \mathcal{P}^n(V_0)$, $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	An n-fold powerset construction allows the representation of *nested*, *hierarchical*, and *multi-level* incidence structures.

Notation. $\mathcal{P}(X) = \{A \mid A \subseteq X\}$, and $\mathcal{P}^0(X) = X,\ \mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

A more explicit side-by-side comparison of graphs, hypergraphs, and n-SuperHyperGraphs is given in Table 1.2.

Table 1.2: A concrete comparison of graphs, hypergraphs, and n-SuperHyperGraphs.

Aspect	*Graph* $G = (V, E)$	*Hypergraph* $H = (V, \mathcal{E})$	*n-SuperHyperGraph* $\mathrm{SHG}^{(n)} = (V_0, V, E)$
Vertices	$v \in V$	$v \in V$	$x \in V \subseteq \mathcal{P}^n(V_0)$, so vertices may themselves be nested objects
Edges	$E \subseteq \binom{V}{2}$	$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$
Incidence	$v \in e$	$v \in e$	$x \in \varepsilon$
Adjacency	$\{u, v\} \in E$	$\exists e \in \mathcal{E} : \{u, v\} \subseteq e$	$\exists \varepsilon \in E : \{x, y\} \subseteq \varepsilon$
Meaning of one edge	Pairwise relation	Multi-entity relation	Multi-entity relation among supervertices
Typical distance notion	Shortest-path distance	Berge distance or primal distance	Super-Berge distance or primal-type distance
Typical use	Binary connectivity	Higher-order group structure	Nested and hierarchical incidence structure

Notation. $\mathcal{P}^0(X) = X$ and $\mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

SuperHyperGraphs have been regarded as important objects of study, and therefore various extended frameworks have also been investigated, such as Fuzzy SuperHyperGraphs [69–73], Neutrosophic SuperHyperGraphs [35, 74], Directed SuperHyperGraphs [75, 76], and Meta SuperHyperGraphs.

1.2 Our Contributions

Since SuperHyperGraphs have only recently begun to be studied systematically, many of their structural concepts, graph-theoretic parameters, and higher-order extensions remain open to further development. This book revisits several notions from graph theory and hypergraph theory and reformulates them within the framework of SuperHyperGraphs. In particular, it aims to provide a coherent collection of definitions, examples, and basic structural results for concepts arising from nested supervertices, superhyperedges, and higher-order incidence relations.

The main contributions of this book may be summarized as follows. First, several classical graph-theoretic notions, such as blocks, k-cores, girth, defensive alliances, zero forcing, and degree sequences, are reconsidered in the setting of SuperHyperGraphs. Second, selected hypergraph-theoretic and uncertainty-based structures are extended to SuperHyperGraphs, including Tanner, Kneser, fuzzy, vague, hypersoft, and uncertain variants. Third, new higher-order frameworks, such as Multi-Axis SuperHyperGraphs, Edge-Iterated SuperHyperGraphs, Open-Incidence SuperHyperGraphs, and Ultra-SuperHyperGraphs, are presented. Fourth, network-oriented models, including Neural, Feedback, Tensor, Semantic, and Social SuperHyperNetworks, are introduced as possible foundations for future applications.

The present book is part of a broader series on HyperGraph and SuperHyperGraph theory. For related background and additional context, we refer the reader to [77–83]. Since the focus of this book is mainly theoretical, computational experiments, algorithmic implementations, and detailed practical analyses of the proposed concepts are left as important directions for future research.

For clarity, Table 1.3 summarizes the main categories of contributions presented in this book.

Table 1.3: Classification of the contributions in this book.

Type of contribution	**Representative concepts**
Reformulations of graph-theoretic notions	Blocks, k-cores, girth, defensive alliances, zero forcing, and degree sequences.
Extensions of hypergraph and uncertainty-based notions	Tanner, Kneser, fuzzy, vague, hypersoft, and uncertainty-based SuperHyperGraphs.
New higher-order frameworks	Multi-Axis SuperHyperGraphs, Edge-Iterated SuperHyperGraphs, Open-Incidence SuperHyperGraphs, and Ultra-SuperHyperGraphs.
Network-oriented models	Neural, Feedback, Tensor, Semantic, and Social SuperHyperNetworks.

Chapter 2

Preliminaries

This chapter fixes the notation and recalls the basic structures used throughout the book.

2.1 SuperHyperGraphs

Classical graph theory provides a standard language for describing systems built from *vertices* and *edges*. It also provides fundamental notions such as connectivity, graph invariants, and structural parameters, which arise in mathematics, computer science, and many applied domains [1]. When interactions are purely pairwise, graphs give a natural and effective model. However, many systems involve relations among more than two objects at the same time. In such cases, the ordinary graph model becomes too restrictive.

A *hypergraph* addresses this limitation by allowing a single edge, called a hyperedge, to connect an arbitrary nonempty collection of vertices. Thus, hypergraphs directly encode genuinely multiway relations and higher-order interactions [24, 84]. These higher-order structures now play an important role in modern modeling and learning frameworks, including neural architectures that explicitly use hypergraph-based incidence information [24, 85–88].

A further generalization is obtained by iterating the powerset construction. In this setting, vertices themselves may be set-valued and nested objects, leading to the notion of a finite *SuperHyperGraph*. This framework is well suited to the representation of hierarchical, multilevel, and nested relational systems [89, 90]. Such structures naturally arise in areas such as multiscale network analysis, molecular representation, and neural-network design [51, 60, 91–95]. Related variants have also been studied, including Directed SuperHyperGraphs [96, 97] and MetaSuperHyperGraphs [98].

Unless explicitly stated otherwise, n denotes a nonnegative integer when it appears in $\mathcal{P}^n(\cdot)$ or in the expression n-SuperHyperGraph.

Definition 2.1.1 (Base set)**.** A *base set* S is the underlying universe of admissible objects in the context under consideration. Formally, one may write

$$S = \{\, x \mid x \text{ is admissible in the given setting} \,\}.$$

All objects appearing in powerset constructions over S, such as $\mathcal{P}(S)$ and $\mathcal{P}^n(S)$, are ultimately generated from elements of S.

Definition 2.1.2 (Hypergraph)**.** [25,99] A *hypergraph* is an ordered pair
$$H = (V, E)$$
such that:

(i) V is a finite set of *vertices*;

(ii) E is a finite family of nonempty subsets of V, whose elements are called *hyperedges*.

Thus, unlike an ordinary graph, a hyperedge may contain more than two vertices. This allows hypergraphs to model multiway interactions directly.

Definition 2.1.3 (Iterated powerset)**.** [100,101] Let V_0 be a finite nonempty set. Define
$$\mathcal{P}^0(V_0) := V_0,$$
and, for $k \geq 0$,
$$\mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big).$$
An element of $\mathcal{P}^n(V_0)$ is called an *n-level object* over V_0.

Definition 2.1.4 (Flattening map)**.** [102] Let V_0 be a finite nonempty set. For each $k \geq 0$, define the *flattening map*
$$\mathrm{Flat}_k : \mathcal{P}^k(V_0) \longrightarrow \mathcal{P}(V_0)$$
recursively as follows. For $x \in V_0$, set
$$\mathrm{Flat}_0(x) := \{x\}.$$
For $k \geq 0$ and $X \in \mathcal{P}^{k+1}(V_0)$, set
$$\mathrm{Flat}_{k+1}(X) := \bigcup_{Y \in X} \mathrm{Flat}_k(Y).$$
Thus $\mathrm{Flat}_k(X)$ is the set of base-level elements of V_0 represented by the k-level object X. In particular, if $X = \emptyset$, then
$$\mathrm{Flat}_{k+1}(\emptyset) = \emptyset.$$

Definition 2.1.5 (n-SuperHyperGraph)**.** [47] Let V_0 be a finite nonempty base set and let $n \geq 0$. An *n-SuperHyperGraph on V_0* is a pair
$$\mathrm{SHG}^{(n)} = (V, E)$$
satisfying
$$V \subseteq \mathcal{P}^n(V_0)$$
and
$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$
The elements of V are called *n-supervertices*, and the elements of E are called *n-superedges*. Equivalently, each n-superedge is a nonempty subset of the set of n-supervertices.

Remark 2.1.6. When $n = 0$, one has
$$\mathcal{P}^0(V_0) = V_0.$$
Hence a 0-SuperHyperGraph is precisely an ordinary hypergraph on a vertex set contained in V_0. For $n \geq 1$, the vertices may be nested set-valued objects, and the corresponding superedges describe incidence relations among such nested objects.

Chapter 3

Revisiting Some Concepts

In this chapter, we introduce several new concepts that have recently been defined.

3.1 Conceptual SuperHyperGraph

A Conceptual SuperHyperGraph is a typed n-level relational structure whose nested supervertices and ordered superhyperedges encode concepts, individuals, and relations hierarchically consistently within layered incidence [103].

Definition 3.1.1 (Conceptual vocabulary)**.** A *conceptual vocabulary* is a tuple

$$\mathfrak{V} = (T_C, \leq_C, T_R, \mathrm{ar}, I)$$

where:

(i) T_C is a nonempty set of *concept types*;

(ii) $\leq_C$ is a preorder on T_C, interpreted as a type-subsumption relation;

(iii) T_R is a nonempty set of *relation types*;

(iv) $\mathrm{ar} : T_R \to \mathbb{N}_{\geq 1}$ assigns to each relation type its arity;

(v) I is a set of individual markers.

We also fix a distinguished generic marker

$$* \notin I,$$

which is used to denote an unspecified individual.

Definition 3.1.2 (Conceptual HyperGraph)**.** Let

$$\mathfrak{V} = (T_C, \leq_C, T_R, \mathrm{ar}, I)$$

be a conceptual vocabulary. A *Conceptual HyperGraph* over $\mathfrak{V}$ is a finite structure

$$\mathcal{CH} = (X, A, \lambda_X, \lambda_A, \mathrm{inc})$$

satisfying the following conditions:

(i) X is a finite set of *concept vertices*;

(ii) A is a finite set of *relation occurrences*, also called *conceptual hyperedges*;

(iii)
$$\lambda_X : X \to T_C \times (I \cup \{*\})$$
is a concept-labeling map. If
$$\lambda_X(x) = (\tau, \mu),$$
then $\tau \in T_C$ is the concept type of x, and $\mu \in I \cup \{*\}$ is its individual marker;

(iv)
$$\lambda_A : A \to T_R$$
is a relation-labeling map. For $a \in A$, the value $\lambda_A(a) \in T_R$ is the relation type represented by a;

(v) inc is an ordered incidence map such that, for every $a \in A$, if
$$k = \mathrm{ar}(\lambda_A(a)),$$
then
$$\mathrm{inc}(a) = (x_1, \ldots, x_k) \in X^k,$$
where $x_1, \ldots, x_k$ are pairwise distinct.

Thus each relation occurrence $a \in A$ represents an ordered instance of the k-ary relation type $\lambda_A(a)$, whose i-th argument is the concept vertex x_i.

Notation 3.1.3 (Underlying incidence set)**.** *Let*

$$\mathcal{CH} = (X, A, \lambda_X, \lambda_A, \mathrm{inc})$$

be a Conceptual HyperGraph. If
$$\mathrm{inc}(a) = (x_1, \ldots, x_k),$$
we define the underlying incidence set of a by

$$\mathrm{Inc}(a) := \{x_1, \ldots, x_k\} \subseteq X.$$

The underlying ordinary hypergraph of $\mathcal{CH}$ is

$$\mathrm{Sh}(\mathcal{CH}) = (X, E_{\mathcal{CH}}), \qquad E_{\mathcal{CH}} := \{\mathrm{Inc}(a) \mid a \in A\}.$$

Theorem 3.1.4 (Underlying hypergraph of a Conceptual HyperGraph)**.** *For every Conceptual HyperGraph*

$$\mathcal{CH} = (X, A, \lambda_X, \lambda_A, \mathrm{inc}),$$

the pair

$$\mathrm{Sh}(\mathcal{CH}) = (X, E_{\mathcal{CH}})$$

is a finite hypergraph.

Proof. Since X is finite by definition, it remains only to verify that every element of $E_{\mathcal{CH}}$ is a nonempty subset of X. Let $e \in E_{\mathcal{CH}}$. Then there exists $a \in A$ such that

$$e = \mathrm{Inc}(a).$$

Let $k = \mathrm{ar}(\lambda_A(a))$. Since $\mathrm{ar} : T_R \to \mathbb{N}_{\geq 1}$, we have $k \geq 1$. Moreover,

$$\mathrm{inc}(a) = (x_1, \ldots, x_k) \in X^k,$$

so

$$\mathrm{Inc}(a) = \{x_1, \ldots, x_k\}$$

is a nonempty subset of X. Therefore

$$E_{\mathcal{CH}} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}.$$

Hence $(X, E_{\mathcal{CH}})$ is a finite hypergraph. □

Definition 3.1.5 (Conceptual n-SuperHyperGraph)**.** Let

$$\mathfrak{V} = (T_C, \leq_C, T_R, \mathrm{ar}, I)$$

be a conceptual vocabulary, let V_0 be a finite nonempty base set, and let $n \geq 0$. A *Conceptual n-SuperHyperGraph* over $\mathfrak{V}$ on V_0 is a finite structure

$$\mathcal{CSH}^{(n)} = (V, A, \lambda_V, \lambda_A, \mathrm{inc})$$

satisfying the following conditions:

(i)
$$V \subseteq \mathcal{P}^n(V_0)$$
is a finite set of *n-supervertices*;

(ii) A is a finite set of *relation occurrences*, also called *conceptual n-superhyperedges*;

(iii)
$$\lambda_V : V \to T_C \times (I \cup \{*\})$$
labels each n-supervertex by a concept type and an individual marker;

(iv)
$$\lambda_A : A \to T_R$$
labels each relation occurrence by a relation type;

(v) inc is an ordered incidence map such that, for every $a \in A$, if

$$k = \mathrm{ar}(\lambda_A(a)),$$

then

$$\mathrm{inc}(a) = (v_1, \dots, v_k) \in V^k,$$

where $v_1, \dots, v_k$ are pairwise distinct n-supervertices.

Thus a Conceptual n-SuperHyperGraph is a typed ordered relational structure whose arguments are not merely ordinary vertices, but n-level objects in $\mathcal{P}^n(V_0)$.

Notation 3.1.6 (Underlying n-superedge)**.** *Let*

$$\mathcal{CSH}^{(n)} = (V, A, \lambda_V, \lambda_A, \mathrm{inc})$$

be a Conceptual n-SuperHyperGraph. If

$$\mathrm{inc}(a) = (v_1, \dots, v_k),$$

define

$$\mathrm{Inc}(a) := \{v_1, \dots, v_k\} \subseteq V.$$

The underlying n-SuperHyperGraph of $\mathcal{CSH}^{(n)}$ is

$$\mathrm{Sh}\big(\mathcal{CSH}^{(n)}\big) = (V, E_{\mathcal{CSH}}), \qquad E_{\mathcal{CSH}} := \{\mathrm{Inc}(a) \mid a \in A\}.$$

Theorem 3.1.7 (Underlying n-SuperHyperGraph)**.** *For every Conceptual n-SuperHyperGraph*

$$\mathcal{CSH}^{(n)} = (V, A, \lambda_V, \lambda_A, \mathrm{inc})$$

on V_0, the pair

$$\mathrm{Sh}\big(\mathcal{CSH}^{(n)}\big) = (V, E_{\mathcal{CSH}})$$

is an n-SuperHyperGraph on V_0.

Proof. By definition,

$$V \subseteq \mathcal{P}^n(V_0),$$

so the elements of V are n-supervertices over V_0. Let $e \in E_{\mathcal{CSH}}$. Then there exists $a \in A$ such that

$$e = \mathrm{Inc}(a).$$

Let

$$k = \mathrm{ar}(\lambda_A(a)).$$

Since $\mathrm{ar}(\lambda_A(a)) \in \mathbb{N}_{\geq 1}$, we have $k \geq 1$. Moreover,

$$\mathrm{inc}(a) = (v_1, \dots, v_k) \in V^k,$$

and hence

$$\mathrm{Inc}(a) = \{v_1, \dots, v_k\}$$

is a nonempty subset of V. Therefore

$$E_{\mathcal{CSH}} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Consequently,

$$(V, E_{\mathcal{CSH}})$$

satisfies the defining conditions of an n-SuperHyperGraph on V_0. $\square$

Theorem 3.1.8 (Reduction to Conceptual HyperGraphs). *A Conceptual* 0*-SuperHyperGraph is precisely a Conceptual HyperGraph whose vertex set is contained in the base set* V_0*.*

Proof. For $n = 0$, one has

$$\mathcal{P}^0(V_0) = V_0.$$

Therefore the condition

$$V \subseteq \mathcal{P}^0(V_0)$$

becomes simply

$$V \subseteq V_0.$$

All other components

$$A, \lambda_V, \lambda_A, \mathrm{inc}$$

are exactly the same as in the definition of a Conceptual HyperGraph. Hence a Conceptual 0-SuperHyperGraph is a Conceptual HyperGraph whose concept vertices are chosen from V_0. □

3.2 Multi-Axis SuperHyperGraph

A Multi Axis SuperHyperGraph represents higher order relations among vertices carrying multi indexed iterated powerset coordinates, with cross level inclusion and hierarchical multidimensional organization rules [104].

Definition 3.2.1 (Multi-indexed iterated powerset). [104] Let $d \in \mathbb{N}$, and let $\mathbf{U} = (U_1, \ldots, U_d)$ be a d-tuple of finite nonempty sets. For each axis $r \in \{1, \ldots, d\}$, define

$$\mathcal{P}^0(U_r) := U_r, \qquad \mathcal{P}^{n+1}(U_r) := \mathcal{P}(\mathcal{P}^n(U_r)) \quad (n \in \mathbb{N}_0).$$

For a multi-index $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$, define the $\mathbf{n}$-layer by

$$\mathbb{P}^{\mathbf{n}}(\mathbf{U}) := \prod_{r=1}^{d} \mathcal{P}^{n_r}(U_r).$$

An element $x \in \mathbb{P}^{\mathbf{n}}(\mathbf{U})$ is written as $x = (x_1, \ldots, x_d)$ with $x_r \in \mathcal{P}^{n_r}(U_r)$.

Definition 3.2.2 (Axis-wise singleton lift). Let $\mathbf{e}_r$ denote the r-th standard basis vector of $\mathbb{N}_0^d$. For $\mathbf{n} \in \mathbb{N}_0^d$, define

$$\Sigma_r^{\mathbf{n}} : \mathbb{P}^{\mathbf{n}}(\mathbf{U}) \to \mathbb{P}^{\mathbf{n}+\mathbf{e}_r}(\mathbf{U})$$

by

$$\Sigma_r^{\mathbf{n}}(x_1, \ldots, x_d) := (x_1, \ldots, x_{r-1}, \{x_r\}, x_{r+1}, \ldots, x_d).$$

If $\mathbf{n} \leq \mathbf{m}$ coordinatewise, define

$$\Sigma^{\mathbf{n}\to\mathbf{m}} : \mathbb{P}^{\mathbf{n}}(\mathbf{U}) \to \mathbb{P}^{\mathbf{m}}(\mathbf{U})$$

as the iterated composition of the axis-wise singleton lifts, applied $(m_r - n_r)$ times along axis r for each r.

Definition 3.2.3 (Typed coordinatewise inclusion)**.** Fix $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}_0^d$. For each axis r, define a relation $\preceq_r^{(n_r)}$ on $\mathcal{P}^{n_r}(U_r)$ by

$$a \preceq_r^{(n_r)} b \iff \begin{cases} a = b, & n_r = 0, \\ a \subseteq b, & n_r \geq 1. \end{cases}$$

Then define $\preceq_{\mathbf{n}}$ on $\mathbb{P}^{\mathbf{n}}(\mathbf{U})$ by

$$x \preceq_{\mathbf{n}} y \iff x_r \preceq_r^{(n_r)} y_r \quad (\forall r = 1, \dots, d).$$

Definition 3.2.4 (Cross-layer inclusion)**.** Let $x \in \mathbb{P}^{\mathbf{n}}(\mathbf{U})$ and $y \in \mathbb{P}^{\mathbf{m}}(\mathbf{U})$. If $\mathbf{n} \leq \mathbf{m}$, define

$$x \preceq y \iff \Sigma^{\mathbf{n} \to \mathbf{m}}(x) \preceq_{\mathbf{m}} y.$$

If $\mathbf{n} \not\leq \mathbf{m}$, we regard x and y as incomparable (with respect to this relation).

Definition 3.2.5 (Typed graded multi-axis vertex universe)**.** Let $\mathbf{U} = (U_1, \dots, U_d)$ be a d-axis base system, and let $I \subseteq \mathbb{N}_0^d$ be a finite index set of admissible multi-levels. Define

$$\mathfrak{V}_I(\mathbf{U}) := \bigsqcup_{\mathbf{n} \in I} \big(\{\mathbf{n}\} \times \mathbb{P}^{\mathbf{n}}(\mathbf{U})\big).$$

An element of $\mathfrak{V}_I(\mathbf{U})$ is written $(\mathbf{n}, x)$ with $x \in \mathbb{P}^{\mathbf{n}}(\mathbf{U})$.

Definition 3.2.6 (Multi-Axis SuperHyperGraph)**.** A *Multi-Axis SuperHyperGraph* (MASHG) on $(\mathbf{U}, I)$ is a pair

$$\mathcal{H} = (V, E),$$

where

$$V \subseteq \mathfrak{V}_I(\mathbf{U})$$

is a finite set of vertices (multi-axis supervertices), and

$$E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$$

is a finite set of hyperedges (multi-axis superhyperedges).

Definition 3.2.7 (Level map and content map)**.** Let $\mathcal{H} = (V, E)$ be a MASHG. For each vertex $v = (\mathbf{n}, x) \in V$, define

$$\lambda(v) := \mathbf{n}, \qquad \chi(v) := x.$$

Thus $\lambda : V \to I$ is the level map, and $\chi(v) \in \mathbb{P}^{\lambda(v)}(\mathbf{U})$ is the typed content of v.

Definition 3.2.8 (Multi-axis inclusion between vertices)**.** Let $u, v \in V$. We write $u \preceq_{\mathcal{H}} v$ if

$$\lambda(u) \leq \lambda(v) \quad \text{and} \quad \Sigma^{\lambda(u) \to \lambda(v)}(\chi(u)) \preceq_{\lambda(v)} \chi(v).$$

3.3 Edge-Iterated SuperHyperGraph

An Edge-Iterated SuperHyperGraph is a superhypergraph whose vertices and edges are both generated through iterated powerset constructions, allowing hierarchical incidence at both levels [104].

Definition 3.3.1 ((n, m)-SuperHyperGraph with iterated edge hierarchy)**.** Let $V_0 \neq \varnothing$ be a finite base set, and let $n, m \in \mathbb{N}_0$. An (n, m)-*SuperHyperGraph* on V_0 is a pair

$$\mathrm{SHG}^{(n,m)} = (V, E),$$

such that

$$V \subseteq \mathcal{P}^{*n}(V_0), \qquad E \subseteq \mathcal{P}^{*m}(V).$$

The elements of V are called n-*supervertices*, and the elements of E are called m-*superhyperedges*.

Definition 3.3.2 (Flattened support of an m-superhyperedge)**.** Let $V \neq \varnothing$ be a set. Define recursively the maps

$$\mathrm{Flat}_0 : V \to V, \qquad \mathrm{Flat}_0(v) := v,$$

$$\mathrm{Flat}_1 : \mathcal{P}^{*1}(V) \to \mathcal{P}^{*1}(V), \qquad \mathrm{Flat}_1(A) := A,$$

and for $k \geq 1$,

$$\mathrm{Flat}_{k+1} : \mathcal{P}^{*(k+1)}(V) \to \mathcal{P}^{*1}(V), \qquad \mathrm{Flat}_{k+1}(\mathcal{A}) := \bigcup_{B \in \mathcal{A}} \mathrm{Flat}_k(B).$$

If

$$\mathrm{SHG}^{(n,m)} = (V, E)$$

is an (n, m)-SuperHyperGraph and $\varepsilon \in E$, then

$$\mathrm{Supp}(\varepsilon) := \mathrm{Flat}_m(\varepsilon) \subseteq V$$

is called the *flattened support* (or simply the *support*) of ε. A supervertex $v \in V$ is said to be *incident* with ε if

$$v \in \mathrm{Supp}(\varepsilon).$$

Remark 3.3.3. If $m = 1$, then

$$E \subseteq \mathcal{P}^{*1}(V) = \mathcal{P}(V) \setminus \{\varnothing\},$$

so the above definition reduces to the usual set-based n-SuperHyperGraph. Hence this notion extends the standard superhypergraph framework by allowing not only the vertex domain, but also the edge domain, to be generated through iterated powerset constructions.

Remark 3.3.4. If $n = 0$, then $V \subseteq V_0$, so the vertices are ordinary base elements while the edge objects still carry an m-fold iterated powerset hierarchy. If $m = 0$, then $E \subseteq V$, so edges degenerate to distinguished supervertices. Therefore the most interesting cases are typically $n \geq 1$ and $m \geq 1$.

3.4 Tanner SuperHyperGraph

A Tanner graph is a bipartite graph representing a linear code, connecting variable nodes to check nodes according to parity-check matrix incidences for decoding analysis [105–107]. A Tanner hypergraph generalizes Tanner graphs by representing each parity-check constraint as a hyperedge incident with all involved variable nodes in coding systems and algorithms [108]. A Tanner SuperHyperGraph extends Tanner hypergraphs by allowing hierarchical supervertices and superhyperedges, modeling nested coding constraints, multilevel parity relations, and higher-order dependency structures in codes efficiently. To interpret a nested set as the collection of underlying variables it represents, define recursively:

$$\mathrm{flat}_0(x) := \{x\} \subseteq V_0 \quad (x \in V_0),$$

and for $k \geq 1$ and $A \in \mathcal{P}^k(V_0) = \mathcal{P}(\mathcal{P}^{k-1}(V_0))$,

$$\mathrm{flat}_k(A) := \bigcup_{B \in A} \mathrm{flat}_{k-1}(B) \subseteq V_0.$$

In particular, $\mathrm{flat}_k(A)$ is always a (possibly empty) subset of V_0 [104].

Definition 3.4.1 (Tanner n-superhypergraph)**.** Fix $n \in \mathbb{N}$. Let $V \subseteq \mathcal{P}^n(V_0)$ be a set of *n-supervertices*. For each check $i \in \{1, \ldots, m\}$ choose a nonempty *superhyperedge*

$$E_i \in \mathcal{P}(V) \setminus \{\emptyset\}.$$

Put $\mathcal{E} := \{E_1, \ldots, E_m\}$ and define

$$\mathrm{TSH}^{(n)}(H) := (V, \mathcal{E}).$$

We call $\mathrm{TSH}^{(n)}(H)$ a *Tanner n-superhypergraph* (for H) if for every i,

$$\bigcup_{v \in E_i} \mathrm{flat}_n(v) \;=\; \{\, x_j \mid j \in \mathrm{supp}(H_i) \,\}. \tag{T}$$

3.5 Kneser SuperHyperGraphs

A Kneser graph has vertices as k-subsets of an n-element set, with edges joining disjoint pairs, naturally encoding standard combinatorial disjointness relations in graph theory [109–111]. A Kneser hypergraph has vertices as k-subsets of an n-set, with hyperedges formed by mutually disjoint families, generalizing Kneser graphs to higher uniformity and dimension (cf. [112–115]). A Kneser superhypergraph has supervertices with base supports of size k; r-uniform superedges connect r vertices whose flattened supports are pairwise disjoint [104].

Definition 3.5.1 (Kneser (n, r, k)-SuperHyperGraph)**.** [104] Fix integers $N \in \mathbb{N}$, $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, and $r \in \mathbb{N}$ with $r \geq 2$. Let the base set be

$$V_0 = [N] := \{1, 2, \ldots, N\},$$

and let flat_n be the flattening map. Define the set of *n-supervertices of size k* by

$$V_{N,k}^{(n)} := \{\, v \in \mathcal{P}^n(V_0) \;\mid\; |\mathrm{flat}_n(v)| = k \,\}.$$

Let $V \subseteq V^{(n)}_{N,k}$ be a finite nonempty vertex set.

Define the set of *superedge identifiers* by

$$E := \left\{ e \subseteq V \;\middle|\; |e| = r \text{ and } \mathrm{flat}_n(u) \cap \mathrm{flat}_n(v) = \emptyset \text{ for all distinct } u, v \in e \right\}.$$

Define the incidence map $\partial : E \to \mathcal{P}(V) \setminus \{\emptyset\}$ by

$$\partial(e) := e \qquad (e \in E).$$

Then

$$\mathrm{KSHG}^{(n,r)}_{N,k}(V) \;:=\; (V, E, \partial)$$

is called the *Kneser* (n, r, k)*-SuperHyperGraph* (on V). When $V = V^{(n)}_{N,k}$, we write simply $\mathrm{KSHG}^{(n,r)}_{N,k}$.

3.6 Nested SuperHyperGraphs

A nested hypergraph allows hyperedges to contain other hyperedges as elements, with ranks enforcing well-founded nesting without cycles [116]. A nested superhypergraph uses supervertices from iterated powersets; superhyperedges may contain other superhyperedges, ranked to avoid cycles [116].

Definition 3.6.1 (Nested HyperGraph)**.** [116] Let V be a finite nonempty set (the vertex set). A *nested hypergraph* on V is a triple

$$H = (V, E, \rho),$$

where E is a finite set (the set of hyperedges) and

$$\rho : V \uplus E \to \mathbb{N}$$

is a *rank function* such that:

1. $\rho(v) = 0$ for all $v \in V$;
2. for every $e \in E$, the hyperedge e is a nonempty finite set satisfying

   $$e \subseteq V \uplus E,$$

   and for every $x \in e$ one has

   $$\rho(x) < \rho(e).$$

A hyperedge $e \in E$ is called a *nesting hyperedge* if $e \cap E \neq \emptyset$ (i.e., e contains at least one hyperedge as a member). We say that H is *(nontrivially) nested* if it has at least one nesting hyperedge.

Remark 3.6.2 (Well-foundedness)**.** The strict inequality $\rho(x) < \rho(e)$ for $x \in e$ excludes membership cycles among hyperedges, so the nesting relation is well-founded.

Definition 3.6.3 (Nested level-n SuperHyperGraph)**.** Fix a finite nonempty base set V_0 and an integer $n \geq 0$. Define iterated powersets recursively by

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0).$$

Let

$$V_n \subseteq \mathcal{P}^n(V_0)$$

be a finite set; its elements are called *n-supervertices*. A *nested level-n SuperHyperGraph* is a triple

$$H^{(n)} = (V_n, E_n, \rho),$$

where E_n is a finite set (of superhyperedges) and

$$\rho : V_n \uplus E_n \to \mathbb{N}$$

is a rank function such that:

1. $\rho(v) = 0$ for all $v \in V_n$;

2. for every $e \in E_n$, the superhyperedge e is a nonempty finite set satisfying

$$e \subseteq V_n \uplus E_n,$$

and for every $x \in e$ one has

$$\rho(x) < \rho(e).$$

A superhyperedge $e \in E_n$ is called *simple* if $e \subseteq V_n$, and it is called *nesting* if $e \cap E_n \neq \emptyset$ (i.e., e contains at least one superhyperedge as a member).

3.7 Social SuperHyperNetwork

A Social SuperHyperNetwork models hierarchical social actors and multi-group interactions using recursively nested social entities, weighted superconnections, and higher-order relational structures across complex community systems.

Definition 3.7.1 (Support of a Hierarchical Social Object)**.** Let V_0 be a finite nonempty set of *basic actors*. Define

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0).$$

For each $k \geq 0$, define the *actor-support map*

$$\sigma_k : \mathcal{P}^k(V_0) \longrightarrow \mathcal{P}(V_0)$$

recursively as follows:

$$\sigma_0(x) := \{x\} \qquad (x \in V_0),$$

and, for $X \in \mathcal{P}^{k+1}(V_0)$,

$$\sigma_{k+1}(X) := \bigcup_{Y \in X} \sigma_k(Y).$$

An element $X \in \mathcal{P}^k(V_0)$ is called *socially nonempty* if

$$\sigma_k(X) \neq \emptyset.$$

We denote the set of all socially nonempty k-level social objects by

$$\mathcal{P}^k_*(V_0) := \{\, X \in \mathcal{P}^k(V_0) \mid \sigma_k(X) \neq \emptyset \,\}.$$

Definition 3.7.2 (Social n-SuperHyperNetwork)**.** Let V_0 be a finite nonempty set of *basic actors*, such as individuals, organizations, households, institutions, or communities. Let $n \in \mathbb{N} \cup \{0\}$. A *Social n-SuperHyperNetwork* over V_0 is a triple

$$\mathcal{SSHN}^{(n)} = \big(V, \mathcal{E}, w\big)$$

satisfying the following conditions:

(i) V is a finite nonempty subset of $\mathcal{P}^n_*(V_0)$. Its elements are called *n-superactors*. Thus each $X \in V$ is a hierarchical social object ultimately supported by at least one basic actor in V_0.

(ii) $\mathcal{E}$ is a finite subset of

$$\mathcal{P}(V) \setminus \{\emptyset\}.$$

Its elements are called *n-superhyperconnections*. Hence each

$$e \in \mathcal{E}$$

is a nonempty finite family of n-superactors.

(iii) w is a weight function

$$w : \mathcal{E} \longrightarrow \mathbb{R}_{\geq 0}.$$

The value $w(e)$ may represent the strength, frequency, intensity, reliability, or importance of the social interaction represented by e.

When $w(e) = 1$ for all $e \in \mathcal{E}$, the structure is called an *unweighted Social n-SuperHyperNetwork*.

Remark 3.7.3 (Interpretation)**.** In a Social n-SuperHyperNetwork, the base set V_0 contains ordinary actors. For $n = 1$, an element of V may represent a group of actors, such as a team, family, or committee. For $n = 2$, an element of V may represent a family of groups, such as a consortium of committees. More generally, increasing n allows one to model social entities formed by iterated grouping. The set $\mathcal{E}$ then records interactions among such hierarchical social entities.

Theorem 3.7.4 (Well-definedness of Social n-SuperHyperNetworks)**.** *Let V_0 be a finite nonempty set and let $n \in \mathbb{N} \cup \{0\}$. If*

$$V \subseteq \mathcal{P}^n_*(V_0)$$

is finite and nonempty,

$$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

and

$$w : \mathcal{E} \to \mathbb{R}_{\geq 0}$$

is a function, then

$$\mathcal{SSHN}^{(n)} = (V, \mathcal{E}, w)$$

is a well-defined Social n-SuperHyperNetwork.

Proof. Since V_0 is finite, the iterated powerset $\mathcal{P}^n(V_0)$ is finite for every $n \geq 0$. Hence $\mathcal{P}^n_*(V_0) \subseteq \mathcal{P}^n(V_0)$ is also finite. Therefore every subset $V \subseteq \mathcal{P}^n_*(V_0)$ is finite.

By assumption, $V \neq \emptyset$, so the n-superactors are well-defined. Moreover, each element $X \in V$ is socially nonempty, since $X \in \mathcal{P}^n_*(V_0)$ implies $\sigma_n(X) \neq \emptyset$. Thus every superactor is ultimately supported by at least one basic actor in V_0.

Next, since

$$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

each $e \in \mathcal{E}$ is a nonempty subset of V. Hence each superhyperconnection e is a nonempty finite family of n-superactors. Finally, $w : \mathcal{E} \to \mathbb{R}_{\geq 0}$ assigns a unique nonnegative real number to every superhyperconnection. Therefore all components of $\mathcal{SSHN}^{(n)} = (V, \mathcal{E}, w)$ are well-defined. □

Theorem 3.7.5 (Underlying n-SuperHyperGraph)**.** *For every Social n-SuperHyperNetwork*

$$\mathcal{SSHN}^{(n)} = (V, \mathcal{E}, w),$$

the pair

$$(V, \mathcal{E})$$

is an n-SuperHyperGraph whose vertices are n-superactors and whose superhyperedges are social superhyperconnections.

Proof. By definition,

$$V \subseteq \mathcal{P}^n_*(V_0) \subseteq \mathcal{P}^n(V_0),$$

so the elements of V are n-level hierarchical objects over V_0. Also,

$$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

so each element of $\mathcal{E}$ is a nonempty subset of V. Therefore $(V, \mathcal{E})$ satisfies the usual incidence condition of an n-SuperHyperGraph. The function w adds weights to the superhyperedges, but it does not change the underlying incidence structure. □

Theorem 3.7.6 (Reduction to Social HyperNetworks)**.** *When $n = 0$, a Social 0-SuperHyperNetwork is exactly a weighted Social HyperNetwork.*

Proof. For $n = 0$, we have

$$\mathcal{P}^0(V_0) = V_0.$$

Moreover, for $x \in V_0$,

$$\sigma_0(x) = \{x\} \neq \emptyset.$$

Hence

$$\mathcal{P}^0_*(V_0) = V_0.$$

Therefore a Social 0-SuperHyperNetwork is a triple

$$(V, \mathcal{E}, w)$$

with

$$V \subseteq V_0, \qquad \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}, \qquad w : \mathcal{E} \to \mathbb{R}_{\geq 0}.$$

This is precisely a weighted Social HyperNetwork. □

Theorem 3.7.7 (Embedding of Ordinary Social Networks)**.** *Let*

$$\mathcal{S} = (V_0, E_0, w_0)$$

be an undirected weighted social network, where

$$E_0 \subseteq \{\{u, v\} \mid u, v \in V_0,\ u \neq v\}$$

and

$$w_0 : E_0 \to \mathbb{R}_{\geq 0}.$$

For every $n \geq 0$, $\mathcal{S}$ embeds into a Social n-SuperHyperNetwork.

Proof. Define the nested singleton map

$$\eta_0(v) := v, \qquad \eta_{k+1}(v) := \{\eta_k(v)\} \quad (k \geq 0).$$

Then

$$\eta_n(v) \in \mathcal{P}_*^n(V_0)$$

for every $v \in V_0$. Set

$$V^{(n)} := \{\eta_n(v) \mid v \in V_0\}$$

and

$$\mathcal{E}^{(n)} := \{\{\eta_n(u), \eta_n(v)\} \mid \{u, v\} \in E_0\}.$$

Since every element of $\mathcal{E}^{(n)}$ is a nonempty subset of $V^{(n)}$, we have

$$\mathcal{E}^{(n)} \subseteq \mathcal{P}(V^{(n)}) \setminus \{\emptyset\}.$$

Define

$$w^{(n)}\big(\{\eta_n(u), \eta_n(v)\}\big) := w_0(\{u, v\}).$$

Because the nested singleton map η_n is injective, this assignment is well-defined. Therefore

$$\mathcal{SSHN}_{\mathcal{S}}^{(n)} := \big(V^{(n)}, \mathcal{E}^{(n)}, w^{(n)}\big)$$

is a Social n-SuperHyperNetwork isomorphic to the original weighted social network $\mathcal{S}$. □

Theorem 3.7.8 (Flattening to a Social HyperNetwork)**.** *Let*

$$\mathcal{SSHN}^{(n)} = (V, \mathcal{E}, w)$$

be a Social n-SuperHyperNetwork. For each $e \in \mathcal{E}$, define its actor-level flattening by

$$\mathrm{flat}(e) := \bigcup_{X \in e} \sigma_n(X) \subseteq V_0.$$

Then

$$\mathcal{E}_{\mathrm{flat}} := \{\mathrm{flat}(e) \mid e \in \mathcal{E}\}$$

is a set of nonempty subsets of V_0. If several superhyperconnections have the same flattening, define

$$w_{\mathrm{flat}}(A) := \sum_{\substack{e \in \mathcal{E} \\ \mathrm{flat}(e) = A}} w(e) \qquad (A \in \mathcal{E}_{\mathrm{flat}}).$$

Then

$$\mathrm{Flat}\big(\mathcal{SSHN}^{(n)}\big) := \big(V_0, \mathcal{E}_{\mathrm{flat}}, w_{\mathrm{flat}}\big)$$

is a weighted Social HyperNetwork on the base actor set V_0.

Proof. Let $e \in \mathcal{E}$. Since $e \neq \emptyset$, there exists $X \in e$. Because $X \in V \subseteq \mathcal{P}_*^n(V_0)$, we have

$$\sigma_n(X) \neq \emptyset.$$

Thus

$$\mathrm{flat}(e) = \bigcup_{X \in e} \sigma_n(X)$$

is a nonempty subset of V_0. Hence

$$\mathcal{E}_{\mathrm{flat}} \subseteq \mathcal{P}(V_0) \setminus \{\emptyset\}.$$

Since $\mathcal{E}$ is finite, each sum defining w_{flat} is finite. Moreover, every summand is nonnegative, so

$$w_{\mathrm{flat}}(A) \in \mathbb{R}_{\geq 0}$$

for every $A \in \mathcal{E}_{\mathrm{flat}}$. Therefore

$$\big(V_0, \mathcal{E}_{\mathrm{flat}}, w_{\mathrm{flat}}\big)$$

is a weighted Social HyperNetwork. □

3.8 HyperSoft SuperHyperGraph

A hypersoft set assigns subsets of a universe to attribute-value tuples, enabling multi-attribute parameterized modeling of objects, alternatives, or states under structured uncertainty and context [117–121]. A hypersoft graph assigns vertex subsets and optionally edge subsets to attribute-value tuples, producing parameterized subgraphs of a fixed graph under multi-attribute conditions or scenarios [122–125]. A hypersoft superhypergraph assigns supervertex subsets and admissible superedge subsets to attribute-value tuples, producing parameterized sub-superhypergraphs of a fixed hierarchical superhypergraph under multi-attribute structured contexts [126].

Definition 3.8.1 (HyperSoft SuperHyperGraph)**.** [126] Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph over a finite nonempty base set V_0. Let $m \geq 1$, and let

$$A_1, A_2, \ldots, A_m$$

be nonempty attribute-value sets. Put

$$J := A_1 \times A_2 \times \cdots \times A_m.$$

A *HyperSoft SuperHyperGraph* on $\mathcal{S}^{(n)}$ with attribute domain J is a quadruple

$$\mathfrak{HSS}^{(n)} = \big(\mathcal{S}^{(n)}, J, F_V, F_E\big),$$

where

$$F_V : J \longrightarrow \mathcal{P}(V), \qquad F_E : J \longrightarrow \mathcal{P}(E),$$

such that, for every attribute tuple

$$\mathbf{a} = (a_1, \ldots, a_m) \in J,$$

the following admissibility condition holds:

$$F_E(\mathbf{a}) \subseteq E\big[F_V(\mathbf{a})\big] = \{\, \varepsilon \in E \mid \varepsilon \subseteq F_V(\mathbf{a}) \,\}.$$

For each $\mathbf{a} \in J$, the pair

$$\big(F_V(\mathbf{a}), F_E(\mathbf{a})\big)$$

is called the *HyperSoft slice* of $\mathfrak{HSS}^{(n)}$ at $\mathbf{a}$. By the admissibility condition, every slice is a sub-n-SuperHyperGraph of $\mathcal{S}^{(n)}$.

Remark 3.8.2. The adjective *HyperSoft* refers to the fact that the parameter domain is the Cartesian product

$$J = A_1 \times \cdots \times A_m$$

of several attribute-value sets. Thus each slice is indexed by a single attribute tuple $(a_1, \ldots, a_m)$.

Theorem 3.8.3 (Well-definedness of HyperSoft SuperHyperGraph slices)**.** *Let*

$$\mathfrak{H}\mathfrak{S}\mathcal{S}^{(n)} = \big(\mathcal{S}^{(n)}, J, F_V, F_E\big)$$

be a HyperSoft SuperHyperGraph. Then, for every $\mathbf{a} \in J$*,*

$$\big(F_V(\mathbf{a}), F_E(\mathbf{a})\big)$$

is a well-defined finite sub-n-SuperHyperGraph of $\mathcal{S}^{(n)}$*.*

Proof. Since $F_V(\mathbf{a}) \subseteq V$, every element of $F_V(\mathbf{a})$ is an n-supervertex. Also, by definition,

$$F_E(\mathbf{a}) \subseteq \{\, \varepsilon \in E \mid \varepsilon \subseteq F_V(\mathbf{a}) \,\}.$$

Hence every element of $F_E(\mathbf{a})$ is an n-superedge of $\mathcal{S}^{(n)}$ whose endpoints all lie in $F_V(\mathbf{a})$. Therefore

$$\big(F_V(\mathbf{a}), F_E(\mathbf{a})\big)$$

satisfies the defining incidence condition for an n-SuperHyperGraph. Since $\mathcal{S}^{(n)}$ is finite, the slice is finite. □

3.9 SuperHyperSoft SuperHyperGraph

A superhypersoft set assigns subsets of a universe to tuples of attribute-value subsets, enabling aggregated multi-valued parameterization of uncertain objects and contexts across complex systems [127–129]. Related concepts such as TreeSoft Sets [130–132], IndetermSoft Sets [133–135], and ContraSoft Sets [136, 137] are also known. A superhypersoft graph assigns vertex and edge subsets to tuples of attribute-value subsets, generating graph slices that aggregate multiple parameter choices simultaneously within host graphs [138]. A superhypersoft superhypergraph assigns supervertex and superedge subsets to tuples of attribute-value subsets, yielding aggregated sub-superhypergraph slices of a fixed superhypergraph under multi-valued selections simultaneously.

Definition 3.9.1 (SuperHyperSoft SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph over a finite nonempty base set V_0. Let $m \geq 1$, and let

$$A_1, A_2, \ldots, A_m$$

be nonempty attribute-value sets. Put

$$\mathfrak{A} := \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_m).$$

A *SuperHyperSoft SuperHyperGraph* on $\mathcal{S}^{(n)}$ with attribute-subset domain $\mathfrak{A}$ is a quadruple

$$\mathfrak{SHSS}^{(n)} = \big(\mathcal{S}^{(n)}, \mathfrak{A}, F_V, F_E\big),$$

where

$$F_V : \mathfrak{A} \longrightarrow \mathcal{P}(V), \qquad F_E : \mathfrak{A} \longrightarrow \mathcal{P}(E),$$

such that, for every attribute-subset tuple

$$\mathbf{A} = (A'_1, \dots, A'_m) \in \mathfrak{A}, \qquad A'_i \subseteq A_i,$$

the following admissibility condition holds:

$$F_E(\mathbf{A}) \subseteq E\big[F_V(\mathbf{A})\big] = \{\, \varepsilon \in E \mid \varepsilon \subseteq F_V(\mathbf{A}) \,\}.$$

For each $\mathbf{A} \in \mathfrak{A}$, the pair

$$\big(F_V(\mathbf{A}), F_E(\mathbf{A})\big)$$

is called the *SuperHyperSoft slice* of $\mathfrak{SHSS}^{(n)}$ at $\mathbf{A}$. By the admissibility condition, every slice is a sub-n-SuperHyperGraph of $\mathcal{S}^{(n)}$.

Remark 3.9.2. The adjective *SuperHyperSoft* refers to the fact that each attribute coordinate may be selected as a subset of possible values. Thus the parameter domain is

$$\mathfrak{A} = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_m),$$

rather than the ordinary HyperSoft domain

$$J = A_1 \times \cdots \times A_m.$$

If one wants to exclude empty attribute selections, one may replace $\mathcal{P}(A_i)$ by

$$\mathcal{P}_*(A_i) := \mathcal{P}(A_i) \setminus \{\emptyset\}.$$

Theorem 3.9.3 (Well-definedness of SuperHyperSoft SuperHyperGraph slices)**.** *Let*

$$\mathfrak{SHSS}^{(n)} = \big(\mathcal{S}^{(n)}, \mathfrak{A}, F_V, F_E\big)$$

be a SuperHyperSoft SuperHyperGraph. Then, for every $\mathbf{A} \in \mathfrak{A}$,

$$\big(F_V(\mathbf{A}), F_E(\mathbf{A})\big)$$

is a well-defined finite sub-n-SuperHyperGraph of $\mathcal{S}^{(n)}$.

Proof. Let

$$\mathbf{A} = (A'_1, \dots, A'_m) \in \mathfrak{A}.$$

By definition,

$$F_V(\mathbf{A}) \subseteq V.$$

Thus every element of $F_V(\mathbf{A})$ is an n-supervertex of $\mathcal{S}^{(n)}$. Also,

$$F_E(\mathbf{A}) \subseteq \{\, \varepsilon \in E \mid \varepsilon \subseteq F_V(\mathbf{A}) \,\}.$$

Therefore every selected n-superedge in $F_E(\mathbf{A})$ is incident only with n-supervertices lying in $F_V(\mathbf{A})$. Hence

$$\big(F_V(\mathbf{A}), F_E(\mathbf{A})\big)$$

satisfies the defining incidence condition for an n-SuperHyperGraph. Since $\mathcal{S}^{(n)}$ is finite, the slice is finite. $\square$

Theorem 3.9.4 (HyperSoft SuperHyperGraphs as restrictions of SuperHyperSoft SuperHyperGraphs)**.** *Let*

$$\mathfrak{SHSS}^{(n)} = \big(\mathcal{S}^{(n)}, \mathfrak{A}, F_V, F_E\big)$$

be a SuperHyperSoft SuperHyperGraph with

$$\mathfrak{A} = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_m).$$

Define the singleton embedding

$$\iota : A_1 \times \cdots \times A_m \longrightarrow \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_m)$$

by

$$\iota(a_1, \ldots, a_m) := (\{a_1\}, \ldots, \{a_m\}).$$

Then the restriction

$$F_V^{\iota}(\mathbf{a}) := F_V(\iota(\mathbf{a})), \qquad F_E^{\iota}(\mathbf{a}) := F_E(\iota(\mathbf{a}))$$

defines a HyperSoft SuperHyperGraph

$$\big(\mathcal{S}^{(n)}, A_1 \times \cdots \times A_m, F_V^{\iota}, F_E^{\iota}\big).$$

Proof. For every

$$\mathbf{a} = (a_1, \ldots, a_m) \in A_1 \times \cdots \times A_m,$$

the SuperHyperSoft admissibility gives

$$F_E(\iota(\mathbf{a})) \subseteq \{\, \varepsilon \in E \mid \varepsilon \subseteq F_V(\iota(\mathbf{a}))\,\}.$$

By the definitions of F_V^{ι} and F_E^{ι}, this is exactly

$$F_E^{\iota}(\mathbf{a}) \subseteq \{\, \varepsilon \in E \mid \varepsilon \subseteq F_V^{\iota}(\mathbf{a})\,\}.$$

Therefore the restricted structure satisfies the HyperSoft SuperHyperGraph admissibility condition. □

3.10 Fuzzy n-SuperHyperGraphs

A Fuzzy Graph assigns membership degrees to vertices and edges, modeling uncertain pairwise relationships, with each edge bounded above by the memberships of its endpoints [139–141]. A Fuzzy HyperGraph assigns membership degrees to vertices and hyperedges, representing uncertain multiway relationships, with each hyperedge bounded above by the memberships of participating vertices [31, 142, 143]. A Fuzzy SuperHyperGraph assigns membership degrees to supervertices and superhyperedges, modeling uncertain higher-order nested relations, with each superhyperedge bounded by incident supervertex memberships therein [77, 144].

Definition 3.10.1 (Fuzzy n-SuperHyperGraph)**.** (cf. [47, 70]) Let

$$\mathrm{SHG}^{(n)} = (V, E)$$

be an n-SuperHyperGraph. A *fuzzy n-SuperHyperGraph* is a quadruple

$$\widetilde{\mathrm{SHG}}^{(n)} = \big(V, E, \sigma, \mu\big),$$

where

$$\sigma : V \to [0,1] \qquad \text{and} \qquad \mu : E \to [0,1]$$

are the vertex-membership and edge-membership functions, respectively, satisfying

$$\mu(e) \le \min_{v \in e} \sigma(v) \qquad \text{for every } e \in E.$$

The function σ assigns to each n-supervertex $v \in V$ its degree of membership, while μ assigns to each n-superedge $e \in E$ its degree of membership.

3.11 Vague SuperHyperGraphs

A vague set assigns each element lower and upper membership evidence, representing uncertainty explicitly through truth-membership and false-membership bounds simultaneously [145–147]. A vague graph labels vertices and edges with vague-set membership bounds, modeling uncertain relationships while preserving endpoint-based consistency conditions structurally [148–150]. A *vague hypergraph* extends the hypergraph framework by assigning to each vertex and each hyperedge both a true-membership degree and a false-membership degree [147]. This concept is based on vague sets and vague graphs and is adapted here to the SuperHyperGraph setting.

Definition 3.11.1 (Vague Set)**.** [145,151] Let X be a finite nonempty set. A *vague set* A on X is a pair

$$A = (t_A, f_A),$$

where

$$t_A : X \to [0,1], \qquad f_A : X \to [0,1]$$

are the *true-membership* and *false-membership* functions, respectively, such that

$$t_A(x) + f_A(x) \leq 1 \qquad \text{for all } x \in X.$$

The associated *vague membership interval* of $x \in X$ is

$$V_A(x) := \big[t_A(x),\, 1 - f_A(x)\big].$$

Definition 3.11.2 (Vague Hypergraph)**.** [147] Let V be a finite nonempty crisp vertex set, and let

$$\mathcal{E} = \{E_1, E_2, \ldots, E_m\}$$

be a finite family of nontrivial vague subsets of V, where each

$$E_j = (t_{E_j}, f_{E_j})$$

is a vague set on V. Define the support of E_j by

$$\operatorname{supp}(E_j) := \{\, x \in V : (t_{E_j}(x), f_{E_j}(x)) \neq (0,0) \,\}.$$

If

$$\bigcup_{j=1}^{m} \operatorname{supp}(E_j) = V,$$

then the pair

$$H = (V, \mathcal{E})$$

is called a *vague hypergraph*, and the members of $\mathcal{E}$ are called its *vague hyperedges*.

We now extend this idea to n-SuperHyperGraphs.

Definition 3.11.3 (Vague n-SuperHyperGraph)**.** Let

$$\mathrm{SHG}^{(n)} = (V, E)$$

be an n-SuperHyperGraph. A *vague n-SuperHyperGraph* is a sextuple

$$\widehat{\mathrm{SHG}}^{(n)} = \big(V, E, t_V, f_V, t_E, f_E\big),$$

where:

(i)
$$t_V : V \to [0,1], \qquad f_V : V \to [0,1]$$
are the true-membership and false-membership functions on the n-supervertex set, satisfying
$$t_V(v) + f_V(v) \leq 1 \qquad \text{for all } v \in V;$$

(ii)
$$t_E : E \to [0,1], \qquad f_E : E \to [0,1]$$
are the true-membership and false-membership functions on the n-superedge set, satisfying
$$t_E(e) + f_E(e) \leq 1 \qquad \text{for all } e \in E;$$

(iii) for every n-superedge $e \in E$, the following appurtenance conditions hold:
$$t_E(e) \leq \min_{v \in e} t_V(v), \qquad f_E(e) \geq \max_{v \in e} f_V(v).$$

3.12 m-Polar Fuzzy SuperHyperGraphs

An m-polar fuzzy set assigns to each element of a universe an m-dimensional membership vector in $[0,1]^m$, thereby capturing multi-criteria fuzziness [152,153]. Special cases include bipolar fuzzy sets ($m = 2$) [154,155] and tripolar fuzzy sets ($m = 3$) [156,157]. In a similar spirit, an m-polar fuzzy graph assigns m-dimensional membership vectors to both vertices and edges [158,159], while an m-polar fuzzy hypergraph equips hyperedges with such vectors to encode multi-criteria interaction strengths [160–162]. Related notions also include bipolar fuzzy hypergraphs [163–165] and tripolar fuzzy hypergraphs.

Definition 3.12.1 (m-Polar Fuzzy Set)**.** [152,153] Let X be a nonempty crisp set. An *m-polar fuzzy set* A on X is a mapping
$$A : X \longrightarrow [0,1]^m.$$
For each $x \in X$, we write
$$A(x) = \big(P_1(A(x)), P_2(A(x)), \dots, P_m(A(x))\big),$$
where $P_i : [0,1]^m \to [0,1]$ denotes the projection onto the i-th coordinate.

The set $[0,1]^m$ is equipped with the pointwise order: for $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$ in $[0,1]^m$,
$$u \leq v \quad \Longleftrightarrow \quad u_i \leq v_i \quad \text{for all } i = 1, \dots, m.$$
We also write
$$1 = (1, 1, \dots, 1), \qquad 0 = (0, 0, \dots, 0),$$
for the top and bottom elements of this ordered set.

Definition 3.12.2 (m-Polar Fuzzy Hypergraph)**.** [160–162] Let X be a nonempty crisp set. An *m-polar fuzzy hypergraph* $H = (\mathcal{A}, \mathcal{B})$ on X consists of:

(i) a finite family
$$\mathcal{A} = \{\zeta_1, \zeta_2, \dots, \zeta_r\},$$
where each $\zeta_j : X \to [0,1]^m$ is an m-polar fuzzy subset of X, and
$$\operatorname{supp}(\zeta_j) = \{\, x \in X \mid \zeta_j(x) \neq 0 \,\}$$
denotes its crisp support;

(ii) an edge-membership mapping
$$\mathcal{B} : \{E_1, \dots, E_r\} \longrightarrow [0,1]^m,$$
where
$$E_j := \operatorname{supp}(\zeta_j) \qquad (j = 1, \dots, r),$$
such that for every $j = 1, \dots, r$ and every $i = 1, \dots, m$,
$$P_i\big(\mathcal{B}(E_j)\big) \le \min_{x \in E_j} P_i\big(\zeta_j(x)\big);$$

(iii) the covering condition
$$\bigcup_{j=1}^{r} \operatorname{supp}(\zeta_j) = X.$$

Each ζ_j is called an *m-polar fuzzy hypervertex membership*, and each $\mathcal{B}(E_j) \in [0,1]^m$ is called the *m-polar fuzzy membership vector of the hyperedge E_j*. The underlying crisp hypergraph is
$$H^* = \big(X, \{E_1, \dots, E_r\}\big).$$

Definition 3.12.3 (m-Polar Fuzzy n-SuperHyperGraph)**.** Let V_0 be a finite base set, and let $m \ge 1$ and $n \ge 0$ be integers. An *m-polar fuzzy n-SuperHyperGraph* is a quadruple
$$\widetilde{\mathrm{SHG}}_m^{(n)} = \big(V, E, \sigma, \mu\big),$$
where:

(i)
$$V \subseteq \mathcal{P}^n(V_0)$$
is a finite set of n-supervertices;

(ii)
$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$
is a family of n-superedges;

(iii)
$$\sigma : V \to [0,1]^m$$
assigns to each n-supervertex $v \in V$ an m-dimensional membership vector
$$\sigma(v) = \big(\sigma_1(v), \dots, \sigma_m(v)\big);$$

(iv)

$$\mu : E \to [0,1]^m$$

assigns to each n-superedge $e \in E$ an m-dimensional membership vector

$$\mu(e) = \big(\mu_1(e), \ldots, \mu_m(e)\big).$$

These data satisfy the appurtenance condition that for every $e \in E$ and each $i = 1, \ldots, m$,

$$\mu_i(e) \leq \min_{v \in e} \sigma_i(v).$$

Equivalently,

$$P_i\big(\mu(e)\big) \leq \min_{v \in e} P_i\big(\sigma(v)\big) \qquad (i = 1, \ldots, m).$$

3.13 q-Rung Orthopair Fuzzy n-SuperHyperGraphs

We now introduce q-rung orthopair fuzzy n-SuperHyperGraphs. A q-rung orthopair fuzzy set assigns to each element a truth-membership degree and a falsity-membership degree whose q-th powers sum to at most one [166–168]. When $q = 2$, one recovers a Pythagorean fuzzy set [169, 170], and when $q = 3$, one recovers a Fermatean fuzzy set [171, 172]. Analogously, a q-rung orthopair fuzzy hypergraph becomes a Pythagorean fuzzy hypergraph when $q = 2$ [143], and a Fermatean fuzzy hypergraph when $q = 3$ [173].

Definition 3.13.1 (q-Rung Orthopair Fuzzy Set)**.** [166–168] Let X be a nonempty finite universe. A *q-rung orthopair fuzzy set* (q-ROFS) Q on X is a mapping

$$Q : X \longrightarrow [0,1] \times [0,1], \qquad x \longmapsto \big(T_Q(x), F_Q(x)\big),$$

such that for each $x \in X$,

$$T_Q(x)^q + F_Q(x)^q \leq 1,$$

where $q \geq 1$ is fixed. Here $T_Q(x)$ is the truth-membership degree and $F_Q(x)$ is the falsity-membership degree of x. The corresponding indeterminacy degree is

$$\pi_Q(x) = \sqrt[q]{1 - T_Q(x)^q - F_Q(x)^q}\,.$$

Definition 3.13.2 (q-Rung Orthopair Fuzzy Relation)**.** (cf. [166–168]) Let X be a nonempty set, and let V be a q-ROFS on X. A *q-rung orthopair fuzzy relation* R on X is a mapping

$$R : X \times X \longrightarrow [0,1] \times [0,1], \qquad (x,y) \longmapsto \big(T_R(x,y), F_R(x,y)\big),$$

such that for all $x, y \in X$,

$$T_R(x,y)^q + F_R(x,y)^q \leq 1,$$

$$T_R(x,y) \leq \min\{T_V(x), T_V(y)\}, \qquad F_R(x,y) \geq \max\{F_V(x), F_V(y)\}.$$

If (x,y) is not an edge in the underlying crisp graph, then one sets

$$T_R(x,y) = 0, \qquad F_R(x,y) = 1.$$

Definition 3.13.3 (q-Rung Orthopair Fuzzy Graph)**.** [174,175] A *q-rung orthopair fuzzy graph* $G = (V, E)$ consists of:

- a q-ROFS V on the vertex set X, assigning $\big(T_V(x), F_V(x)\big)$ to each $x \in X$;
- a q-ROFR E on X, assigning $\big(T_E(x,y), F_E(x,y)\big)$ to each pair $(x,y) \in X \times X$, such that for all $x, y \in X$,

  $$T_E(x,y) \le \min\{T_V(x), T_V(y)\}, \qquad F_E(x,y) \ge \max\{F_V(x), F_V(y)\},$$

  and

  $$T_E(x,y)^q + F_E(x,y)^q \le 1.$$

Definition 3.13.4 (q-Rung Orthopair Fuzzy Hypergraph)**.** [176] Let X be a nonempty finite set. A *q-rung orthopair fuzzy hypergraph* (q-ROFH) $H = (\mathcal{Q}, \zeta)$ on X consists of:

(i) a finite family

$$\mathcal{Q} = \{Q_1, Q_2, \dots, Q_n\},$$

where each Q_i is a q-ROFS on X;

(ii) a mapping

$$\zeta : \{Q_1, \dots, Q_n\} \longrightarrow [0,1] \times [0,1], \qquad Q_i \longmapsto \big(T_\zeta(Q_i), F_\zeta(Q_i)\big),$$

such that

$$T_\zeta(Q_i)^q + F_\zeta(Q_i)^q \le 1,$$

and

$$T_\zeta(Q_i) \le \min_{x \in \operatorname{supp}(Q_i)} T_{Q_i}(x), \qquad F_\zeta(Q_i) \ge \max_{x \in \operatorname{supp}(Q_i)} F_{Q_i}(x);$$

(iii) the covering condition

$$\bigcup_{i=1}^{n} \operatorname{supp}(Q_i) = X,$$

where

$$\operatorname{supp}(Q_i) = \{\, x \in X \mid T_{Q_i}(x) > 0 \text{ or } F_{Q_i}(x) < 1 \,\}.$$

The underlying crisp hypergraph is

$$\big(X, \{\operatorname{supp}(Q_1), \dots, \operatorname{supp}(Q_n)\}\big).$$

Definition 3.13.5 (q-Rung Orthopair Fuzzy n-SuperHyperGraph)**.** Let V_0 be a finite base set, and let $n \ge 0$ and $q \ge 1$ be fixed. A *q-rung orthopair fuzzy n-SuperHyperGraph* is a quadruple

$$\mathbb{SHG}_q^{(n)} = \big(V, E, \sigma, \mu\big),$$

where:

(i)
$$V \subseteq \mathcal{P}^n(V_0)$$
is the set of n-supervertices;

(ii)
$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$
is the family of n-superedges;

(iii)
$$\sigma : V \to [0,1] \times [0,1], \qquad \sigma(v) = \big(T_\sigma(v), F_\sigma(v)\big),$$
with
$$T_\sigma(v)^q + F_\sigma(v)^q \leq 1 \qquad \text{for all } v \in V;$$

(iv)
$$\mu : E \to [0,1] \times [0,1], \qquad \mu(e) = \big(T_\mu(e), F_\mu(e)\big),$$
with
$$T_\mu(e)^q + F_\mu(e)^q \leq 1 \qquad \text{for all } e \in E.$$

These maps satisfy the appurtenance constraints
$$T_\mu(e) \leq \min_{v \in e} T_\sigma(v), \qquad F_\mu(e) \geq \max_{v \in e} F_\sigma(v) \qquad \text{for every } e \in E.$$
If desired, one may additionally impose the covering condition
$$\bigcup_{e \in E} e = V.$$

3.14 Fuzzy Tolerance n-SuperHyperGraph

A Fuzzy Tolerance Graph assigns fuzzy intervals and fuzzy tolerances to vertices, measuring pairwise overlap against tolerance thresholds to produce graded adjacency between vertices naturally [177, 178]. A Fuzzy Tolerance HyperGraph assigns fuzzy intervals and fuzzy tolerances to vertices, producing graded hyperedges whenever multiway overlap satisfies normalized tolerance conditions among vertices collectively [64]. A Fuzzy Tolerance n-SuperHyperGraph assigns fuzzy intervals and fuzzy tolerances to n-supervertices, producing graded superhyperedges from normalized multiway overlap within higher-order set-valued structures recursively therein [64].

Definition 3.14.1 (Fuzzy Tolerance n-SuperHyperGraph)**.** [64] Let $n \in \mathbb{N}_0$ and let $V \subseteq \mathcal{P}^n(V_0)$ be a finite nonempty set. For each supervertex $v \in V$, choose a fuzzy interval I_v and a fuzzy tolerance T_v.

For every nonempty $e \subseteq V$ with $|e| \geq 2$, define
$$L_c(e) := \ell\Big(\bigcap_{v \in e} c(I_v)\Big), \qquad L_s(e) := \ell\Big(\bigcap_{v \in e} s(I_v)\Big),$$

$$\tau_c(e) := \min_{v\in e} \ell\big(c(T_v)\big), \qquad \tau_s(e) := \min_{v\in e} \ell\big(s(T_v)\big),$$

and

$$\rho(x,a) := \begin{cases} \min\{1, x/a\}, & a > 0, \\ 0, & a = 0. \end{cases}$$

The tolerance overlap score of e is

$$\varphi(e) := \max\Big\{\rho\big(L_c(e), \tau_c(e)\big),\ \rho\big(L_s(e), \tau_s(e)\big)\Big\}.$$

Let

$$E := \{\, e \subseteq V :\ |e| \geq 2 \,\}.$$

Define

$$\sigma : V \to [0,1], \qquad \sigma(v) := \min\{h(I_v), h(T_v)\},$$

$$\mu : E \to [0,1], \qquad \mu(e) := \min\Big\{\varphi(e), \min_{v\in e} \sigma(v)\Big\},$$

and

$$\eta : V \times E \to [0,1], \qquad \eta(v,e) := \begin{cases} \mu(e), & v \in e, \\ 0, & v \notin e. \end{cases}$$

Then

$$S^{(n)}_{\mathrm{FT}} = (V, E; \sigma, \mu, \eta)$$

is called a *Fuzzy Tolerance n-SuperHyperGraph*.

3.15 Soft Expert SuperHyperGraph

A Soft Expert Set assigns subsets of a universe to parameter-expert-opinion triples, representing expert-dependent selections and uncertainty without requiring numerical membership values or probabilities directly [179,179,180]. A Soft Expert Graph assigns parameter-expert-opinion indexed vertex and edge subsets, so each triple determines an expert-approved subgraph that models structured uncertainty in graph-based relationships [181–183]. A Soft Expert SuperHyperGraph assigns parameter-expert-opinion indexed supervertex and superedge subsets, so each triple specifies an expert-approved sub-superhypergraph capturing higher-order uncertain relationships across multiple levels.

Definition 3.15.1 (Soft Expert n-SuperHyperGraph)**.** Let

$$\mathrm{SHG}^{(n)} = (V, E)$$

be an n-SuperHyperGraph, let Y be a nonempty set of parameters, let X be a nonempty set of experts, and let

$$S = \{0, 1\}$$

be the set of expert opinions. Define

$$Z := Y \times X \times S.$$

A *Soft Expert n-SuperHyperGraph* over $\mathrm{SHG}^{(n)}$ is a sextuple

$$\left(\mathrm{SHG}^{(n)},\, Y,\, X,\, S,\, A_V,\, A_E\right),$$

where

$$A_V : Z \to \mathcal{P}(V), \qquad A_E : Z \to \mathcal{P}(E),$$

such that for every $\alpha \in Z$,

$$A_E(\alpha) \subseteq \{\, e \in E : e \subseteq A_V(\alpha) \,\}.$$

For each $\alpha = (y, x, o) \in Z$, the pair

$$\big(A_V(\alpha),\, A_E(\alpha)\big)$$

is called the *soft expert sub-n-SuperHyperGraph* of $\mathrm{SHG}^{(n)}$ at α.

When n is fixed, this structure is also called simply a *Soft Expert SuperHyperGraph*.

3.16 Uncertain SuperHyperGraph

An Uncertain Set assigns to each element a degree from an uncertainty model, unifying fuzzy, intuitionistic, neutrosophic and plithogenic frameworks [184–187]. An Uncertain Graph is a graph where vertices or edges carry degrees in an uncertainty model, subsuming fuzzy, intuitionistic, neutrosophic [188]. We first recall the notion of an Uncertain Model, which provides the membership–degree domain.

Definition 3.16.1 (Uncertain Model)**.** [187] Let U denote the class of all *uncertain models*. Each $M \in U$ is specified by

- a nonempty set $\mathrm{Dom}(M) \subseteq [0,1]^k$ of *admissible degree tuples* for some fixed integer $k \geq 1$;
- model–specific algebraic or geometric constraints on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \leq 1$ in the intuitionistic fuzzy case, or $T + I + F \leq 3$ in the neutrosophic case).

Typical examples include:

- Fuzzy model: $\mathrm{Dom}(M) = [0,1]$;
- Intuitionistic fuzzy model: $\mathrm{Dom}(M) = \{(\mu, \nu) \in [0,1]^2 \mid \mu + \nu \leq 1\}$;
- Neutrosophic model: $\mathrm{Dom}(M) = \{(T, I, F) \in [0,1]^3 \mid 0 \leq T + I + F \leq 3\}$;
- Plithogenic model, and many other extensions.

Definition 3.16.2 (Uncertain Set (U-Set))**.** [187] Let X be a nonempty universe, and let M be a fixed uncertain model with degree–domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *Uncertain Set of type M* (or *U-Set* for short) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is called the *uncertainty–degree function* (or membership map) of $\mathcal{U}$.

For $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ encodes the degree(s) to which x belongs to the uncertain set, according to the model M.

Remark 3.16.3. Special cases:

- If M is the fuzzy model and $\mathrm{Dom}(M) = [0,1]$, then $\mu_M : X \to [0,1]$ is a usual fuzzy membership function and $\mathcal{U}$ is a fuzzy set.
- If M is neutrosophic, then $\mu_M(x) = (T(x), I(x), F(x))$ gives a neutrosophic set.
- Other choices of M recover intuitionistic fuzzy sets, picture fuzzy sets, plithogenic sets, and so on.

As noted in the remark, various generalizations are possible. For reference, Table 3.1 presents a catalogue of uncertainty-set families (U-Sets) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (cf. [189]).

The definitions and related concepts of Uncertain Graphs are presented below.

Definition 3.16.4 (Uncertain Graph)**.** Let $G = (V, E)$ be a (finite, undirected, loopless) graph and let M be an uncertain model with degree–domain $\mathrm{Dom}(M)$. An *Uncertain Graph of type M* is a triple

$$\mathcal{G}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns to each vertex $v \in V$ and each edge $e \in E$ an uncertainty degree $\mu_M(v)$ or $\mu_M(e)$ in $\mathrm{Dom}(M)$.

Optionally, one may impose model–specific consistency conditions between vertex and edge degrees (for instance, $\mu_M(e)$ bounded in terms of $\mu_M(u)$ and $\mu_M(v)$ for $e = \{u, v\}$ in fuzzy or intuitionistic fuzzy graph models), but these constraints are encoded in the choice of M and are not fixed at the level of this general definition.

Remark 3.16.5. Again, particular choices of M recover well–known graph models:

Table 3.1: A catalogue of uncertainty-set families (U-Sets) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [189].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [31,190]; N-Fuzzy Set [191–193] Shadowed Set [194–196]
2		Intuitionistic Fuzzy Set [197,198]; Vague Set [147,151]; Bipolar Fuzzy Set (two-component description) [154]; Variable Fuzzy Set [199–201]; Paraconsistent Fuzzy Set [202,203]; Bifuzzy Set [204,205]
3		Single-Valued Neutrosophic Set [206,207]; Picture Fuzzy Set [178,208]; Spherical Fuzzy Set [209,210]; Tripolar Fuzzy Set (three-component formalisms) [156,157,211]; Neutrosophic Vague Set [212,213]
4		Quadripartitioned Neutrosophic Set [214,215]; Double-Valued Neutrosophic Set [216,217]; Dual Hesitant Fuzzy Set [218,219]; Ambiguous Set [220–222]; Turiyam Neutrosophic Set [223–226]
5		Pentapartitioned Neutrosophic Set [227–229]; Triple-Valued Neutrosophic Set [230–232]
6		Hexapartitioned Neutrosophic Set; Quadruple-Valued Neutrosophic Set [231,233]
7		Heptapartitioned Neutrosophic Set; Quintuple-Valued Neutrosophic Set [231,234,235]
8		Octapartitioned Neutrosophic Set [236,237]
9		Nonapartitioned Neutrosophic Set [236,237]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [238]; MultiFuzzy Set [239]; n-Refined Fuzzy Set [240,241]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [241]; Multi-Intuitionistic Fuzzy Set [239]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [241]; Multi-Neutrosophic Set [239,242]

Reading guide. In the U-Set scheme [187], each model M is specified by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ and a membership map $\mu_M : X \to \mathrm{Dom}(M)$. The table groups representative families by the ambient dimension k (i.e., how many numerical components are stored per element).

(a) A widely cited viewpoint is that neutrosophic sets provide a unifying umbrella covering several earlier multi-component fuzzy models (and their generalizations); see [243].

(b) Ambiguous sets are commonly presented as subclasses of certain four-component neutrosophic families; see [214,215,222].

(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [244].

- Fuzzy graph (when M is fuzzy and $\mu_M : V \cup E \to [0,1]$);
- Intuitionistic fuzzy graph, neutrosophic graph, plithogenic graph, etc., for the corresponding models M.

As a reference, Table 3.2 presents a catalogue of uncertainty-graph families (Uncertain Graphs) organised by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

An Uncertain HyperGraph represents vertices and hyperedges whose existence, incidence, weights, or memberships are uncertain, enabling higher-order relations under incomplete or imprecise information mathematically.

Definition 3.16.6 (Uncertain HyperGraph)**.** Let $H = (V,E)$ be a hypergraph and let M be an uncertain model with degree–domain $\mathrm{Dom}(M)$. An *Uncertain HyperGraph of type M* is a triple

$$\mathcal{H}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

Table 3.2: A catalogue of uncertainty-graph families (Uncertain Graphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-graph type(s) $\mathcal{G}_M = (V, E, \mu_M)$ with $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$
1	Fuzzy graph; N-graph; shadowed-graph variants
2	Intuitionistic fuzzy graph [245, 246]; vague graph [247]; bipolar fuzzy graph [248, 249]; intuitionistic evidence graph; variable fuzzy graph; paraconsistent fuzzy graph; bifuzzy graph [250, 251]
3	Neutrosophic graph [13][(a)]; hesitant fuzzy graph [252]; tripolar fuzzy graph; three-way fuzzy graph; picture fuzzy graph [253, 254]; spherical fuzzy graph [209]; inconsistent intuitionistic fuzzy graph; ternary fuzzy / neutrosophic-fuzzy graph; neutrosophic vague graph
4	Quadripartitioned neutrosophic graph [255, 256]; double-valued neutrosophic graph [216, 257]; dual hesitant fuzzy graph [258]; ambiguous graph[(b)]; local-neutrosophic graph; support-neutrosophic graph; turiyam neutrosophic graph [259][(c)]
5	Pentapartitioned neutrosophic graph [260, 261]; triple-valued neutrosophic graph [262]
6	Hexapartitioned neutrosophic graph; quadruple-valued neutrosophic graph [262]
7	Heptapartitioned neutrosophic graph [263]; quintuple-valued neutrosophic graph [262]
8	Octapartitioned neutrosophic graph
9	Nonapartitioned neutrosophic graph
n	n-refined fuzzy graph; multi-valued (fuzzy) graphs; multi-fuzzy graphs [264]
$2n$	n-refined intuitionistic fuzzy graph; multi-intuitionistic fuzzy graphs
$3n$	n-refined neutrosophic graph; multi-neutrosophic graphs

(a) Neutrosophic graph models are often treated as broad frameworks that can specialize to many degree-based graph formalisms under suitable constraints.
(b) Ambiguous-graph models are commonly presented as subclasses of certain quadripartitioned and also double-valued neutrosophic graph models.
(c) Turiyam neutrosophic graphs are reported as subclasses of certain quadripartitioned neutrosophic graph models.

assigns an uncertainty degree to each vertex $v \in V$ and each hyperedge $e \in E$.

As in the graph case, possible relations between vertex and hyperedge degrees (for instance, bounds of $\mu_M(e)$ in terms of $\mu_M(v)$ for $v \in e$) are governed by the chosen model M and its constraints.

Remark 3.16.7. For suitable choices of M, this framework yields fuzzy hypergraphs, intuitionistic fuzzy hypergraphs, neutrosophic hypergraphs, plithogenic hypergraphs, and many further extensions. We present the catalogue of uncertainty-hypergraph families (Uncertain HyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ in Table 3.3.

Table 3.3: A catalogue of uncertainty-hypergraph families (Uncertain HyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-hypergraph family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy HyperGraph [265–267]:* $\mu_M : V \cup E \to [0,1]$.
2	*Intuitionistic-fuzzy HyperGraph [268–270]:* $\mu_M : V \cup E \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic HyperGraph [271–274]:* $\mu_M : V \cup E \to [0,1]^3$ (e.g., (T, I, F)).
4	*Quadripartitioned Neutrosophic / four-component uncertainty HyperGraph:* $\mu_M : V \cup E \to [0,1]^4$.
5	*Pentapartitioned Neutrosophic / five-component uncertainty HyperGraph:* $\mu_M : V \cup E \to [0,1]^5$.
k	*k-component uncertainty HyperGraph:* $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

An Uncertain n-SuperHyperGraph represents nested supervertices and superedges with uncertain existence, incidence, weights, or supports, modeling hierarchical higher-order relations under incomplete information within nested systems.

Definition 3.16.8 (Uncertain n-SuperHyperGraph)**.** [77] Let V_0 be a finite base set and let $n \in \mathbb{N}_0$. Assume that an n-SuperHyperGraph on V_0 is given by

$$\mathrm{SHG}^{(n)} = (V_n, E),$$

where

$$\emptyset \neq V_n \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad \emptyset \neq E \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\},$$

so that each n-superedge $e \in E$ is a nonempty subset of the n-supervertex set V_n.

Let M be a fixed uncertain model with degree–domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *Uncertain n-SuperHyperGraph of type M* is a triple

$$\mathcal{S}_M^{(n)} = (V_n, E, \mu_M),$$

where

$$\mu_M : V_n \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns to each n-supervertex $v \in V_n$ and each n-superedge $e \in E$ an uncertainty degree $\mu_M(v)$ or $\mu_M(e)$ in $\mathrm{Dom}(M)$.

Any additional relations between the degrees of n-superedges and the degrees of the n-supervertices they contain (for example, model- specific bounds or aggregations) are imposed by the chosen uncertain model M and are not fixed at the level of this general definition.

For $n = 0$ and $V_0 = V_n$, the above notion reduces to an Uncertain HyperGraph of type M.

Remark 3.16.9. Particular choices of the model M recover well–known uncertain SuperHyperGraph types:

- Fuzzy n-SuperHyperGraphs (when M is fuzzy);
- Intuitionistic fuzzy, neutrosophic, and plithogenic n-SuperHyperGraphs for the corresponding models M;
- More exotic variants (e.g. q-rung orthopair, picture fuzzy, refined neutrosophic) are obtained by choosing the appropriate degree–domain $\mathrm{Dom}(M)$.

Regarding the catalogue of uncertainty-superhypergraph families (Uncertain n-SuperHyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, we list them in Table 3.4.

Table 3.4: A catalogue of uncertainty-superhypergraph families (Uncertain n-SuperHyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-superhypergraph family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy n-SuperHyperGraph [275]:* $\mu_M : V_n \cup E \to [0,1]$.
2	*Intuitionistic-fuzzy n-SuperHyperGraph [275, 276]:* $\mu_M : V_n \cup E \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic n-SuperHyperGraph [93, 277, 278]:* $\mu_M : V_n \cup E \to [0,1]^3$ (e.g., (T, I, F)).
4	*Quadripartitioned / four-component uncertainty n-SuperHyperGraph:* $\mu_M : V_n \cup E \to [0,1]^4$.
k	*k-component uncertainty n-SuperHyperGraph:* $\mu_M : V_n \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

Chapter 4

New Concepts of a SuperHyperGraph

In this chapter, several new concepts of a SuperHyperGraph are introduced, and their properties are briefly examined. Some notions in this chapter are reformulations of existing graph-theoretic or hypergraph-theoretic concepts, while others are proposed as new SuperHyperGraph-based extensions. When a notion is a direct extension of an existing concept, this relationship is explicitly indicated by the corresponding reduction theorem.

4.1 Block of a SuperHyperGraph

A block of a graph is a maximal connected subgraph without cut-vertices, often representing a biconnected component or a bridge isolated vertex case in graphs (cf. [279]). A block of a HyperGraph is a connected maximal subhypergraph with no articulation vertex, formed through decomposition by vertices whose removal separates hypergraph path-connected components (cf. [280]). A block of a SuperHyperGraph is a connected maximal sub-superhypergraph without articulation supervertices, capturing inseparable incidence structure among nested supervertices and superedges within hierarchy levels.

Definition 4.1.1 (Induced and restricted subhypergraphs)**.** Let

$$H = (V, \mathcal{A})$$

be a finite hypergraph, where V is a finite vertex set and

$$\mathcal{A} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$

is a finite family of hyperedges.

For a subset $U \subseteq V$, the *induced hypergraph* of H on U is

$$H[U] := \big(U, \mathcal{A}[U]\big),$$

where

$$\mathcal{A}[U] := \{\, a \cap U \mid a \in \mathcal{A},\ |a \cap U| \geq 2 \,\}.$$

That is, empty intersections and singleton intersections are omitted.

The *restricted hypergraph* of H on U is

$$H|_U := \big(U, \mathcal{A}|_U\big),$$

where

$$\mathcal{A}|_U := \{\, a \in \mathcal{A} \mid a \subseteq U \,\}.$$

Definition 4.1.2 (Hypergraph path and connectedness)**.** Let

$$H = (V, \mathcal{A})$$

be a finite hypergraph. A v_0v_k*-path* in H is an alternating sequence

$$p: \quad v_0, a_1, v_1, a_2, \ldots, a_k, v_k$$

such that:

(i) $a_1, \ldots, a_k \in \mathcal{A}$;

(ii) $v_0 \in a_1$ and $v_k \in a_k$;

(iii) for every $i = 1, \ldots, k-1$,

$$v_i \in a_i \cap a_{i+1}.$$

Two vertices $v, w \in V$ are called *connected* in H if there exists a vw-path in H. The hypergraph H is called *connected* if every two vertices of V are connected.

Definition 4.1.3 (Connected components of a HyperGraph)**.** Let

$$H = (V, \mathcal{A})$$

be a finite hypergraph. The relation

$$v \sim_H w \quad \Longleftrightarrow \quad v = w \text{ or there exists a } vw\text{-path in } H$$

is an equivalence relation on V. The equivalence classes of $\sim_H$ are called the *connected components* of H. If

$$V = C_1 \dot\cup \cdots \dot\cup C_r$$

is the decomposition of V into connected components, then the connected components of H are the induced hypergraphs

$$H[C_1], \ldots, H[C_r].$$

Definition 4.1.4 (Articulation point of a HyperGraph)**.** Let

$$H = (V, \mathcal{A})$$

be a finite hypergraph and let $v \in V$. Consider the restricted hypergraph

$$H|_{V \setminus \{v\}}.$$

The connected components of $H|_{V \setminus \{v\}}$ are called the *parts of* v.

The vertex v is called an *articulation point* of H if

$$H|_{V \setminus \{v\}}$$

has more than one connected component.

Definition 4.1.5 (Block HyperGraph)**.** A finite hypergraph

$$H = (V, \mathcal{A})$$

is called a *block hypergraph*, or simply a *block*, if it satisfies the following two conditions:

(i) H is connected;

(ii) H has no articulation point.

Definition 4.1.6 (Blocks of a HyperGraph)**.** Let

$$H = (V, \mathcal{A})$$

be a finite hypergraph. The *blocks of* H are defined recursively as follows.

(i) If H is connected and has no articulation point, then H itself is a block.

(ii) If H is not connected, then the blocks of H are the blocks of its connected components.

(iii) If H is connected and has an articulation point v, let

$$W_1, \dots, W_k$$

be the vertex sets of the connected components of

$$H|_{V \setminus \{v\}}.$$

Then the blocks of H are the blocks of the induced hypergraphs

$$H[W_1 \cup \{v\}], \dots, H[W_k \cup \{v\}].$$

Each hypergraph obtained at the end of this recursive procedure is called a *block of* H.

Remark 4.1.7 (Non-uniqueness)**.** In contrast with the classical block decomposition of ordinary graphs, this recursive block decomposition of a hypergraph need not be unique. Different choices of articulation points during the recursion may produce different, but structurally related, block decompositions.

Theorem 4.1.8 (Well-definedness of HyperGraph blocks)**.** *Let*

$$H = (V, \mathcal{A})$$

be a finite hypergraph. Then every block obtained by the recursive procedure above is a finite connected hypergraph with no articulation point.

Proof. The recursive procedure only applies induced subhypergraphs of the form

$$H[U]$$

to subsets $U \subseteq V$. Since V is finite, every such induced hypergraph is finite. At each recursive step in which an articulation point v is chosen, the vertex sets

$$W_1 \cup \{v\}, \dots, W_k \cup \{v\}$$

are proper subsets of V unless the process has already reached a connected hypergraph without articulation points. Hence the recursion terminates after finitely many steps. By construction, the terminal objects are precisely connected hypergraphs with no articulation point. Therefore every terminal object is a well-defined block. □

Definition 4.1.9 (Induced and restricted sub-SuperHyperGraphs)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph over V_0. For a subset $U \subseteq V$, the *induced n-sub-SuperHyperGraph* on U is

$$\mathcal{S}^{(n)}[U] := \big(U, E[U]\big),$$

where

$$E[U] := \{\, \varepsilon \cap U \mid \varepsilon \in E,\ |\varepsilon \cap U| \geq 2 \,\}.$$

The *restricted n-sub-SuperHyperGraph* on U is

$$\mathcal{S}^{(n)}|_U := \big(U, E|_U\big),$$

where

$$E|_U := \{\, \varepsilon \in E \mid \varepsilon \subseteq U \,\}.$$

Definition 4.1.10 (SuperHyperGraph path and connectedness)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. A *XY-superpath* in $\mathcal{S}^{(n)}$ is an alternating sequence

$$P : \quad X_0, \varepsilon_1, X_1, \varepsilon_2, \dots, \varepsilon_k, X_k$$

such that:

(i) $X_0, \dots, X_k \in V$;

(ii) $\varepsilon_1, \dots, \varepsilon_k \in E$;

(iii) $X_0 \in \varepsilon_1$ and $X_k \in \varepsilon_k$;

(iv) for every $i = 1, \dots, k-1$,
$$X_i \in \varepsilon_i \cap \varepsilon_{i+1}.$$

Two n-supervertices $X, Y \in V$ are called *connected* if there exists an XY-superpath. The n-SuperHyperGraph $\mathcal{S}^{(n)}$ is called *connected* if every two n-supervertices in V are connected.

Definition 4.1.11 (Connected components of an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. Define a relation on V by

$$X \sim_{\mathcal{S}^{(n)}} Y \quad \Longleftrightarrow \quad X = Y \text{ or there exists an } XY\text{-superpath in } \mathcal{S}^{(n)}.$$

This is an equivalence relation on V. Its equivalence classes are called the *connected components* of $\mathcal{S}^{(n)}$. If

$$V = C_1 \dot\cup \cdots \dot\cup C_r$$

is the decomposition of V into connected components, then the connected components of $\mathcal{S}^{(n)}$ are

$$\mathcal{S}^{(n)}[C_1], \ldots, \mathcal{S}^{(n)}[C_r].$$

Definition 4.1.12 (Articulation n-supervertex)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph and let $X \in V$. Consider the restricted n-sub-SuperHyperGraph

$$\mathcal{S}^{(n)}|_{V \setminus \{X\}}.$$

The connected components of this restricted structure are called the *parts of X*.

The n-supervertex X is called an *articulation n-supervertex* if

$$\mathcal{S}^{(n)}|_{V \setminus \{X\}}$$

has more than one connected component.

Definition 4.1.13 (Block n-SuperHyperGraph)**.** A finite n-SuperHyperGraph

$$\mathcal{S}^{(n)} = (V, E)$$

is called a *block n-SuperHyperGraph*, or simply a *superblock*, if:

(i) $\mathcal{S}^{(n)}$ is connected;

(ii) $\mathcal{S}^{(n)}$ has no articulation n-supervertex.

Definition 4.1.14 (Blocks of an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. The *blocks of $\mathcal{S}^{(n)}$* are defined recursively as follows.

(i) If $\mathcal{S}^{(n)}$ is connected and has no articulation n-supervertex, then $\mathcal{S}^{(n)}$ itself is a block.

(ii) If $\mathcal{S}^{(n)}$ is not connected, then the blocks of $\mathcal{S}^{(n)}$ are the blocks of its connected components.

(iii) If $\mathcal{S}^{(n)}$ is connected and has an articulation n-supervertex X, let

$$W_1, \dots, W_k$$

be the vertex sets of the connected components of

$$\mathcal{S}^{(n)}|_{V \setminus \{X\}}.$$

Then the blocks of $\mathcal{S}^{(n)}$ are the blocks of

$$\mathcal{S}^{(n)}[W_1 \cup \{X\}], \dots, \mathcal{S}^{(n)}[W_k \cup \{X\}].$$

Each terminal n-SuperHyperGraph obtained in this recursion is called a *block of* $\mathcal{S}^{(n)}$.

Theorem 4.1.15 (Well-definedness of SuperHyperGraph blocks)**.** *Let*

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. Then every block obtained from the recursive procedure above is a finite connected n-SuperHyperGraph with no articulation n-supervertex.

Proof. The proof is identical in structure to the hypergraph case. Since V is finite, every induced n-sub-SuperHyperGraph

$$\mathcal{S}^{(n)}[U]$$

has finite supervertex set. Whenever an articulation n-supervertex X is chosen, the recursive step replaces the current structure by induced structures on the sets

$$W_1 \cup \{X\}, \dots, W_k \cup \{X\},$$

where $W_1, \dots, W_k$ are the connected components of the restricted structure

$$\mathcal{S}^{(n)}|_{V \setminus \{X\}}.$$

Thus the recursion cannot continue indefinitely. It terminates exactly when each remaining component is connected and has no articulation n-supervertex. Therefore each terminal object is a well-defined block n-SuperHyperGraph. □

Theorem 4.1.16 (Reduction to HyperGraph blocks)**.** *Let*

$$\mathcal{S}^{(0)} = (V, E)$$

be a 0-SuperHyperGraph over V_0. Since

$$\mathcal{P}^0(V_0) = V_0,$$

$\mathcal{S}^{(0)}$ is an ordinary hypergraph. Moreover, the blocks of $\mathcal{S}^{(0)}$ coincide with the blocks of the hypergraph

$$H = (V, E).$$

Proof. For $n = 0$, every 0-supervertex is an ordinary vertex. Hence the definitions of path, connected component, restricted substructure, induced substructure, and articulation point in $\mathcal{S}^{(0)}$ reduce exactly to the corresponding definitions for the hypergraph $H = (V, E)$. Therefore the recursive block decomposition is the same in both settings. □

4.2 k-Core of a SuperHyperGraph

A k-core of a graph is the maximal induced subgraph where every remaining vertex has degree at least k, obtained by iteratively deleting low-degree vertices [281–283]. Related concepts include the k-shell [284,285], k-cliques [286,287], and k-truss [288,289]. A k-core of a SuperHyperGraph is its maximal induced superhypergraph where every remaining supervertex belongs to at least k entirely retained superedges after iterative peeling.

Definition 4.2.1 (Induced subhypergraph for core theory)**.** Let

$$H = (V, E)$$

be a finite hypergraph, where $V \neq \emptyset$ and

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

For a subset $S \subseteq V$, the *core-induced subhypergraph* of H on S is

$$H[S] := (S, E_H[S]),$$

where

$$E_H[S] := \{\, e \in E \mid e \subseteq S \,\}.$$

For $u \in S$, the degree of u inside $H[S]$ is

$$\deg_{H[S]}(u) := |\{\, e \in E \mid u \in e \subseteq S \,\}|.$$

Equivalently,

$$\deg_{H[S]}(u) = |\{\, e \in E_H[S] \mid u \in e \,\}|.$$

Definition 4.2.2 (k-core-feasible set in a HyperGraph)**.** Let

$$H = (V, E)$$

be a finite hypergraph and let $k \in \mathbb{N}_0$. A subset $S \subseteq V$ is called *k-core-feasible* if

$$\deg_{H[S]}(u) \geq k \qquad \text{for every } u \in S.$$

The empty set is allowed to be k-core-feasible by convention.

Definition 4.2.3 (k-core of a HyperGraph)**.** Let

$$H = (V, E)$$

be a finite hypergraph and let $k \in \mathbb{N}_0$. The *k-core* of H, denoted by

$$\mathrm{Core}_k(H),$$

is the core-induced subhypergraph

$$\mathrm{Core}_k(H) := H[C_k],$$

where

$$C_k := \bigcup \{\, S \subseteq V \mid S \text{ is } k\text{-core-feasible} \,\}.$$

Equivalently, C_k is the unique largest subset of V such that

$$\deg_{H[C_k]}(u) \geq k \qquad \text{for every } u \in C_k.$$

If $C_k = \emptyset$, then the k-core is called empty.

Definition 4.2.4 (Connected k-core of a HyperGraph)**.** Let

$$H = (V, E)$$

be a finite hypergraph and let $k \in \mathbb{N}_0$. A *connected k-core* of H is a connected component of

$$\mathrm{Core}_k(H).$$

Equivalently, it is a maximal connected core-induced subhypergraph

$$H[S]$$

such that

$$\deg_{H[S]}(u) \geq k \qquad \text{for every } u \in S.$$

Definition 4.2.5 (Core value of a vertex in a HyperGraph)**.** Let

$$H = (V, E)$$

be a finite hypergraph. The *core value* of a vertex $v \in V$ is

$$c_H(v) := \max\{\, k \in \mathbb{N}_0 \mid v \in C_k \,\},$$

where C_k is the vertex set of $\mathrm{Core}_k(H)$. Equivalently,

$$c_H(v) = \max\{\, k \in \mathbb{N}_0 \mid v \text{ belongs to some induced subhypergraph of minimum degree at least } k \,\}.$$

Theorem 4.2.6 (Well-definedness of the HyperGraph k-core)**.** *Let*

$$H = (V, E)$$

be a finite hypergraph and let $k \in \mathbb{N}_0$. Then

$$\mathrm{Core}_k(H)$$

is well-defined and is the unique largest core-induced subhypergraph of H with minimum degree at least k.

Proof. Let

$$\mathcal{F}_k := \{\, S \subseteq V \mid S \text{ is } k\text{-core-feasible} \,\}.$$

Since V is finite, $\mathcal{F}_k$ is finite. Hence

$$C_k := \bigcup_{S \in \mathcal{F}_k} S$$

is a well-defined subset of V.

It remains to show that C_k itself is k-core-feasible. Let $u \in C_k$. Then $u \in S$ for some $S \in \mathcal{F}_k$. Since $S \subseteq C_k$, every hyperedge $e \in E$ satisfying

$$u \in e \subseteq S$$

also satisfies

$$u \in e \subseteq C_k.$$

Therefore

$$\deg_{H[C_k]}(u) \geq \deg_{H[S]}(u) \geq k.$$

Thus C_k is k-core-feasible.

By construction, every k-core-feasible subset $S \subseteq V$ is contained in C_k. Hence $H[C_k]$ is the unique largest core-induced subhypergraph of minimum degree at least k. Therefore $\mathrm{Core}_k(H)$ is well-defined. $\square$

Proposition 4.2.7 (Nestedness of HyperGraph k-cores)**.** *Let*

$$H = (V, E)$$

be a finite hypergraph. If $0 \leq k \leq \ell$*, then*

$$C_\ell \subseteq C_k.$$

Equivalently,

$$\mathrm{Core}_\ell(H)$$

is an induced subhypergraph of

$$\mathrm{Core}_k(H).$$

Proof. Every ℓ-core-feasible set is also k-core-feasible, because

$$\deg_{H[S]}(u) \geq \ell \geq k$$

for every $u \in S$. Hence the union of all ℓ-core-feasible sets is contained in the union of all k-core-feasible sets. Therefore

$$C_\ell \subseteq C_k.$$

□

Theorem 4.2.8 (Peeling characterization of the HyperGraph k-core)**.** *Let*

$$H = (V, E)$$

be a finite hypergraph and let $k \in \mathbb{N}_0$*. Start with*

$$S_0 := V.$$

Recursively define

$$S_{t+1} := S_t \setminus \{\, u \in S_t \mid \deg_{H[S_t]}(u) < k \,\}.$$

Since V *is finite, the sequence stabilizes. If*

$$S_\infty$$

denotes the stable set, then

$$H[S_\infty] = \mathrm{Core}_k(H).$$

Proof. The sequence

$$S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$$

is decreasing and V is finite, so it stabilizes after finitely many steps.

By construction, at the stable stage, no vertex of degree less than k remains. Therefore

$$\deg_{H[S_\infty]}(u) \geq k \qquad \text{for every } u \in S_\infty.$$

Thus S_∞ is k-core-feasible, so

$$S_\infty \subseteq C_k.$$

Conversely, let $S \subseteq V$ be any k-core-feasible set. We show by induction that

$$S \subseteq S_t$$

for every $t \geq 0$. This is clear for $t = 0$. Suppose $S \subseteq S_t$. If $u \in S$, then every hyperedge counted in $\deg_{H[S]}(u)$ is also contained in S_t. Hence

$$\deg_{H[S_t]}(u) \geq \deg_{H[S]}(u) \geq k.$$

Thus u is not removed in the step from S_t to S_{t+1}. Hence $S \subseteq S_{t+1}$. Therefore $S \subseteq S_\infty$. Taking the union over all k-core-feasible sets gives

$$C_k \subseteq S_\infty.$$

Consequently,

$$S_\infty = C_k,$$

and hence

$$H[S_\infty] = \mathrm{Core}_k(H).$$

□

Definition 4.2.9 (Core-induced sub-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. For $A \subseteq V$, define

$$E_{\mathcal{S}^{(n)}}[A] := \{\, \varepsilon \in E \mid \varepsilon \subseteq A \,\}.$$

The *core-induced n-sub-SuperHyperGraph* on A is

$$\mathcal{S}^{(n)}[A] := \big(A, E_{\mathcal{S}^{(n)}}[A]\big).$$

For $X \in A$, the degree of X inside $\mathcal{S}^{(n)}[A]$ is

$$\deg_{\mathcal{S}^{(n)}[A]}(X) := |\{\, \varepsilon \in E \mid X \in \varepsilon \subseteq A \,\}|.$$

Definition 4.2.10 (k-core-feasible set in an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph and let $k \in \mathbb{N}_0$. A subset $A \subseteq V$ is called *k-core-feasible* if

$$\deg_{\mathcal{S}^{(n)}[A]}(X) \geq k \qquad \text{for every } X \in A.$$

The empty set is allowed to be k-core-feasible by convention.

Definition 4.2.11 (k-core of an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph and let $k \in \mathbb{N}_0$. The *k-core* of $\mathcal{S}^{(n)}$, denoted by

$$\mathrm{Core}_k\big(\mathcal{S}^{(n)}\big),$$

is the core-induced n-sub-SuperHyperGraph

$$\mathrm{Core}_k\big(\mathcal{S}^{(n)}\big) := \mathcal{S}^{(n)}[C_k],$$

where

$$C_k := \bigcup \{\, A \subseteq V \mid A \text{ is } k\text{-core-feasible} \,\}.$$

Equivalently, C_k is the unique largest subset of V such that

$$\deg_{\mathcal{S}^{(n)}[C_k]}(X) \geq k \qquad \text{for every } X \in C_k.$$

Definition 4.2.12 (Connected k-core of an n-SuperHyperGraph)**.** A *connected k-core* of
$$\mathcal{S}^{(n)}$$
is a connected component of
$$\mathrm{Core}_k\big(\mathcal{S}^{(n)}\big).$$
Equivalently, it is a maximal connected core-induced n-sub-SuperHyperGraph
$$\mathcal{S}^{(n)}[A]$$
such that
$$\deg_{\mathcal{S}^{(n)}[A]}(X) \geq k \qquad \text{for every } X \in A.$$

Definition 4.2.13 (Core value of an n-supervertex)**.** Let
$$\mathcal{S}^{(n)} = (V, E)$$
be a finite n-SuperHyperGraph. The *core value* of an n-supervertex $X \in V$ is
$$c_{\mathcal{S}^{(n)}}(X) := \max\{\, k \in \mathbb{N}_0 \mid X \in C_k \,\},$$
where C_k is the n-supervertex set of
$$\mathrm{Core}_k\big(\mathcal{S}^{(n)}\big).$$
Equivalently,
$$c_{\mathcal{S}^{(n)}}(X) = \max\{\, k \in \mathbb{N}_0 \mid X \text{ belongs to an induced } n\text{-sub-SuperHyperGraph of minimum degree at least } k \,\}.$$

Theorem 4.2.14 (Well-definedness of the SuperHyperGraph k-core)**.** *Let*
$$\mathcal{S}^{(n)} = (V, E)$$
be a finite n-SuperHyperGraph and let $k \in \mathbb{N}_0$. Then
$$\mathrm{Core}_k\big(\mathcal{S}^{(n)}\big)$$
is well-defined and is the unique largest core-induced n-sub-SuperHyperGraph with minimum degree at least k.

Proof. Let
$$\mathcal{F}_k := \{\, A \subseteq V \mid A \text{ is } k\text{-core-feasible} \,\}.$$
Since V is finite, $\mathcal{F}_k$ is finite. Therefore
$$C_k := \bigcup_{A \in \mathcal{F}_k} A$$
is a well-defined subset of V.

Let $X \in C_k$. Then $X \in A$ for some $A \in \mathcal{F}_k$. Since $A \subseteq C_k$, every n-superedge ε satisfying
$$X \in \varepsilon \subseteq A$$
also satisfies
$$X \in \varepsilon \subseteq C_k.$$
Hence
$$\deg_{\mathcal{S}^{(n)}[C_k]}(X) \geq \deg_{\mathcal{S}^{(n)}[A]}(X) \geq k.$$
Thus C_k is k-core-feasible. Moreover, by definition, every k-core-feasible subset of V is contained in C_k. Therefore
$$\mathcal{S}^{(n)}[C_k]$$
is the unique largest core-induced n-sub-SuperHyperGraph with minimum degree at least k. □

Proposition 4.2.15 (Nestedness of SuperHyperGraph k-cores)**.** *Let*

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. If $0 \le k \le \ell$, then

$$C_\ell \subseteq C_k.$$

Equivalently,

$$\mathrm{Core}_\ell\big(\mathcal{S}^{(n)}\big)$$

is a core-induced n-sub-SuperHyperGraph of

$$\mathrm{Core}_k\big(\mathcal{S}^{(n)}\big).$$

Proof. Every ℓ-core-feasible subset is also k-core-feasible, because

$$\deg_{\mathcal{S}^{(n)}[A]}(X) \ge \ell \ge k$$

for every $X \in A$. Taking unions of all feasible subsets gives

$$C_\ell \subseteq C_k.$$

□

Theorem 4.2.16 (Peeling characterization of the SuperHyperGraph k-core)**.** *Let*

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph and let $k \in \mathbb{N}_0$. Define

$$A_0 := V$$

and, recursively,

$$A_{t+1} := A_t \setminus \{\, X \in A_t \mid \deg_{\mathcal{S}^{(n)}[A_t]}(X) < k \,\}.$$

Since V is finite, the sequence stabilizes. If A_∞ denotes the stable set, then

$$\mathcal{S}^{(n)}[A_\infty] = \mathrm{Core}_k\big(\mathcal{S}^{(n)}\big).$$

Proof. The sequence

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

is decreasing and V is finite, so it stabilizes after finitely many steps.

At the stable stage, every remaining n-supervertex has degree at least k inside the remaining induced structure. Hence A_∞ is k-core-feasible, and therefore

$$A_\infty \subseteq C_k.$$

Conversely, let $A \subseteq V$ be any k-core-feasible set. We prove by induction that

$$A \subseteq A_t$$

for all $t \geq 0$. This is true for $t = 0$. Suppose $A \subseteq A_t$. If $X \in A$, then every n-superedge counted in

$$\deg_{\mathcal{S}^{(n)}[A]}(X)$$

is also counted in

$$\deg_{\mathcal{S}^{(n)}[A_t]}(X).$$

Therefore

$$\deg_{\mathcal{S}^{(n)}[A_t]}(X) \geq \deg_{\mathcal{S}^{(n)}[A]}(X) \geq k.$$

Hence X is not removed from A_t when forming A_{t+1}. Thus $A \subseteq A_{t+1}$. Consequently,

$$A \subseteq A_\infty.$$

Taking the union over all k-core-feasible sets yields

$$C_k \subseteq A_\infty.$$

Thus

$$A_\infty = C_k,$$

and the result follows. □

Theorem 4.2.17 (Reduction to HyperGraph k-cores)**.** *Let*

$$\mathcal{S}^{(0)} = (V, E)$$

be a 0*-SuperHyperGraph over* V_0*. Since*

$$\mathcal{P}^0(V_0) = V_0,$$

$\mathcal{S}^{(0)}$ *is an ordinary hypergraph. Moreover,*

$$\mathrm{Core}_k\big(\mathcal{S}^{(0)}\big) = \mathrm{Core}_k(V, E)$$

for every $k \in \mathbb{N}_0$*.*

Proof. For $n = 0$, every 0-supervertex is an ordinary vertex of V_0. Hence

$$V \subseteq \mathcal{P}^0(V_0) = V_0.$$

The 0-superedges are precisely nonempty subsets of V, so

$$\mathcal{S}^{(0)} = (V, E)$$

is an ordinary hypergraph. For every $A \subseteq V$ and every $X \in A$,

$$\deg_{\mathcal{S}^{(0)}[A]}(X) = |\{\, \varepsilon \in E \mid X \in \varepsilon \subseteq A \,\}| = \deg_{H[A]}(X).$$

Thus the k-core-feasible subsets are identical in both definitions. Therefore their largest feasible subsets coincide, and so

$$\mathrm{Core}_k\big(\mathcal{S}^{(0)}\big) = \mathrm{Core}_k(H).$$

□

4.3 Girth of a SuperHyperGraph

The girth of a graph is the length of its shortest cycle; an acyclic graph has infinite girth by the standard convention in graph theory [290–292]. Related concepts such as odd girth [293, 294], even girth [295, 296], and directed girth [297, 298] are also known. Several studies have also investigated the girth of hypergraphs [299–301]. In this section, we consider an extension of this notion to the framework of SuperHyperGraphs. The girth of a SuperHyperGraph is defined as the minimum length of a Super-Berge cycle consisting of supervertices and superedges. If no such Super-Berge cycle exists, then the girth is taken to be infinite by convention.

Definition 4.3.1 (Berge cycle in a HyperGraph)**.** Let

$$H = (V, E)$$

be a finite hypergraph, where V is a finite set of vertices and

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

For an integer $\ell \geq 2$, a *Berge cycle of length* ℓ in H is a pair of sequences

$$(v_1, \ldots, v_\ell;\ e_1, \ldots, e_\ell)$$

such that:

(i) $v_1, \ldots, v_\ell \in V$ are pairwise distinct;

(ii) $e_1, \ldots, e_\ell \in E$ are pairwise distinct;

(iii) for every $i \in \{1, \ldots, \ell\}$,
$$\{v_i, v_{i+1}\} \subseteq e_i,$$
where indices are taken modulo ℓ, that is,
$$v_{\ell+1} = v_1.$$

The integer ℓ is called the *length* of the Berge cycle.

Definition 4.3.2 (Girth of a HyperGraph)**.** Let

$$H = (V, E)$$

be a finite hypergraph. The *girth* of H, denoted by

$$\mathrm{girth}(H),$$

is defined by

$$\mathrm{girth}(H) := \min\{\, \ell \in \mathbb{N}_{\geq 2} \mid H \text{ contains a Berge cycle of length } \ell \,\},$$

if H contains at least one Berge cycle. If H contains no Berge cycle, then we set

$$\mathrm{girth}(H) := \infty.$$

Remark 4.3.3. A Berge cycle of length 2 consists of two distinct hyperedges sharing at least two distinct vertices. Hence a hypergraph has girth at least 3 if and only if no two distinct hyperedges intersect in more than one vertex. In other words, girth at least 3 is equivalent to linearity.

Definition 4.3.4 (Large girth in a HyperGraph)**.** Let $g \geq 2$. A finite hypergraph H is said to have *girth at least* g if

$$\mathrm{girth}(H) \geq g.$$

Equivalently, H contains no Berge cycle of length

$$2, 3, \ldots, g-1.$$

Similarly, H is said to have *girth greater than* g if

$$\mathrm{girth}(H) > g,$$

equivalently, if H contains no Berge cycle of length at most g.

Theorem 4.3.5 (Well-definedness of HyperGraph girth)**.** *Let $H = (V, E)$ be a finite hypergraph. Then* $\mathrm{girth}(H)$ *is well-defined as an element of*

$$\mathbb{N}_{\geq 2} \cup \{\infty\}.$$

Proof. Since V and E are finite, there are only finitely many possible finite sequences

$$(v_1, \ldots, v_\ell;\ e_1, \ldots, e_\ell)$$

with pairwise distinct vertices and pairwise distinct hyperedges. Hence the set

$$\{\, \ell \in \mathbb{N}_{\geq 2} \mid H \text{ contains a Berge cycle of length } \ell \,\}$$

is finite. If this set is nonempty, it has a minimum, which defines $\mathrm{girth}(H)$. If the set is empty, the convention

$$\mathrm{girth}(H) = \infty$$

applies. Therefore $\mathrm{girth}(H)$ is well-defined. □

Definition 4.3.6 (Super-Berge cycle in an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be an n-SuperHyperGraph over V_0. For an integer $\ell \geq 2$, a *Super-Berge cycle of length* ℓ in $\mathcal{S}^{(n)}$ is a pair of sequences

$$(X_1, \ldots, X_\ell;\ \varepsilon_1, \ldots, \varepsilon_\ell)$$

such that:

(i) $X_1, \ldots, X_\ell \in V$ are pairwise distinct n-supervertices;

(ii) $\varepsilon_1, \ldots, \varepsilon_\ell \in E$ are pairwise distinct n-superedges;

(iii) for every $i \in \{1, \dots, \ell\}$,
$$\{X_i, X_{i+1}\} \subseteq \varepsilon_i,$$
where indices are taken modulo ℓ, that is,
$$X_{\ell+1} = X_1.$$

The integer ℓ is called the *length* of the Super-Berge cycle.

Definition 4.3.7 (Girth of an n-SuperHyperGraph)**.** Let
$$\mathcal{S}^{(n)} = (V, E)$$
be an n-SuperHyperGraph. The *girth* of $\mathcal{S}^{(n)}$, denoted by
$$\operatorname{girth}\big(\mathcal{S}^{(n)}\big),$$
is defined by
$$\operatorname{girth}\big(\mathcal{S}^{(n)}\big) := \min\{\, \ell \in \mathbb{N}_{\geq 2} \mid \mathcal{S}^{(n)} \text{ contains a Super-Berge cycle of length } \ell \,\},$$
if $\mathcal{S}^{(n)}$ contains at least one Super-Berge cycle. If $\mathcal{S}^{(n)}$ contains no Super-Berge cycle, then we set
$$\operatorname{girth}\big(\mathcal{S}^{(n)}\big) := \infty.$$

Remark 4.3.8 (Intrinsic interpretation)**.** The girth of an n-SuperHyperGraph is the Berge girth of its incidence structure on n-supervertices. Thus the internal set-theoretic nesting of an n-supervertex affects which objects are available as vertices, but a cycle is counted through incidence among the selected n-supervertices and n-superedges.

Definition 4.3.9 (Large girth in an n-SuperHyperGraph)**.** Let $g \geq 2$. An n-SuperHyperGraph $\mathcal{S}^{(n)}$ is said to have *girth at least* g if
$$\operatorname{girth}\big(\mathcal{S}^{(n)}\big) \geq g.$$
Equivalently, $\mathcal{S}^{(n)}$ contains no Super-Berge cycle of length
$$2, 3, \dots, g-1.$$
It is said to have *girth greater than* g if
$$\operatorname{girth}\big(\mathcal{S}^{(n)}\big) > g,$$
equivalently, if it contains no Super-Berge cycle of length at most g.

Theorem 4.3.10 (Well-definedness of SuperHyperGraph girth)**.** *Let*
$$\mathcal{S}^{(n)} = (V, E)$$
be a finite n-SuperHyperGraph. Then
$$\operatorname{girth}\big(\mathcal{S}^{(n)}\big)$$
is well-defined as an element of
$$\mathbb{N}_{\geq 2} \cup \{\infty\}.$$

Proof. Since $\mathcal{S}^{(n)}$ is finite, both V and E are finite. Hence there are only finitely many possible sequences

$$(X_1, \ldots, X_\ell;\ \varepsilon_1, \ldots, \varepsilon_\ell)$$

with pairwise distinct n-supervertices and pairwise distinct n-superedges. Therefore the set

$$\{\, \ell \in \mathbb{N}_{\geq 2} \mid \mathcal{S}^{(n)} \text{ contains a Super-Berge cycle of length } \ell \,\}$$

is finite. If this set is nonempty, it has a minimum. If it is empty, we define the girth to be ∞. Thus

$$\operatorname{girth}\bigl(\mathcal{S}^{(n)}\bigr)$$

is well-defined. □

Theorem 4.3.11 (Reduction to HyperGraph girth)**.** *Let*

$$\mathcal{S}^{(0)} = (V, E)$$

be a 0*-SuperHyperGraph over* V_0*. Since*

$$\mathcal{P}^0(V_0) = V_0,$$

the structure $\mathcal{S}^{(0)}$ *is an ordinary hypergraph. Moreover,*

$$\operatorname{girth}\bigl(\mathcal{S}^{(0)}\bigr) = \operatorname{girth}(V, E).$$

Proof. For $n = 0$, one has

$$V \subseteq \mathcal{P}^0(V_0) = V_0.$$

Therefore (V, E), with

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

is an ordinary finite hypergraph. The definition of a Super-Berge cycle in $\mathcal{S}^{(0)}$ is exactly the definition of a Berge cycle in the hypergraph (V, E). Hence the two sets of possible cycle lengths coincide, and so their minima coincide. Therefore

$$\operatorname{girth}\bigl(\mathcal{S}^{(0)}\bigr) = \operatorname{girth}(V, E).$$

□

Theorem 4.3.12 (Linearity criterion for SuperHyperGraph girth)**.** *Let*

$$\mathcal{S}^{(n)} = (V, E)$$

be an n*-SuperHyperGraph. Then*

$$\operatorname{girth}\bigl(\mathcal{S}^{(n)}\bigr) \geq 3$$

if and only if

$$|\varepsilon \cap \delta| \leq 1$$

for all distinct n*-superedges*

$$\varepsilon, \delta \in E.$$

Proof. Suppose first that there exist distinct n-superedges $\varepsilon, \delta \in E$ such that

$$|\varepsilon \cap \delta| \geq 2.$$

Choose two distinct n-supervertices

$$X_1, X_2 \in \varepsilon \cap \delta.$$

Then

$$(X_1, X_2;\ \varepsilon, \delta)$$

is a Super-Berge cycle of length 2. Hence

$$\operatorname{girth}\big(\mathcal{S}^{(n)}\big) = 2,$$

so the girth is not at least 3.

Conversely, suppose that $\mathcal{S}^{(n)}$ contains a Super-Berge cycle of length 2. Then there exist two distinct n-supervertices $X_1, X_2 \in V$ and two distinct n-superedges $\varepsilon_1, \varepsilon_2 \in E$ such that

$$\{X_1, X_2\} \subseteq \varepsilon_1 \quad \text{and} \quad \{X_1, X_2\} \subseteq \varepsilon_2.$$

Therefore

$$|\varepsilon_1 \cap \varepsilon_2| \geq 2.$$

Thus the absence of Super-Berge cycles of length 2 is equivalent to the condition that any two distinct n-superedges intersect in at most one n-supervertex. This proves the claim. □

4.4 Global Defensive Alliances in SuperHyperGraphs

Global defensive alliances are dominating vertex sets whose members have sufficient internal neighborhood support, resisting external neighbors through local defensive balance in finite graph-theoretic structures [302–304]. Related concepts such as strong defensive alliances [305,306] and offensive alliances [307,308] are also known. In this section, we consider an extension of this concept to the framework of SuperHyperGraphs.

Definition 4.4.1 (Vertex neighborhood in a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph, where V is a finite nonempty vertex set and

$$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$

is a finite family of hyperedges. For a vertex $v \in V$, its *open neighborhood* in H is defined by

$$N_H(v) := \{\, u \in V \setminus \{v\} \mid \exists e \in \mathcal{E} \text{ such that } \{u, v\} \subseteq e \,\}.$$

The *closed neighborhood* of v is

$$N_H[v] := N_H(v) \cup \{v\}.$$

For a subset $S \subseteq V$, its open and closed neighborhoods are defined by

$$N_H(S) := \bigcup_{v \in S} N_H(v), \qquad N_H[S] := N_H(S) \cup S.$$

Definition 4.4.2 (Dominating set in a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph. A subset $S \subseteq V$ is called a *dominating set* of H if

$$N_H[S] = V.$$

Equivalently, for every vertex $u \in V \setminus S$, there exists a vertex $v \in S$ and a hyperedge $e \in \mathcal{E}$ such that

$$\{u, v\} \subseteq e.$$

Definition 4.4.3 (Defensive alliance in a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph. A nonempty subset $S \subseteq V$ is called a *defensive alliance* in H if, for every $v \in S$,

$$|N_H[v] \cap S| \geq |N_H(v) \cap (V \setminus S)|.$$

Equivalently,

$$1 + |N_H(v) \cap S| \geq |N_H(v) \setminus S| \qquad \text{for every } v \in S.$$

The term 1 represents the vertex v itself, which is counted as part of the defending side.

Definition 4.4.4 (Strong defensive alliance in a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph. A nonempty subset $S \subseteq V$ is called a *strong defensive alliance* in H if, for every $v \in S$,

$$|N_H[v] \cap S| > |N_H(v) \cap (V \setminus S)|.$$

Equivalently,

$$1 + |N_H(v) \cap S| > |N_H(v) \setminus S| \qquad \text{for every } v \in S.$$

Definition 4.4.5 (Global defensive alliance in a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph. A subset $S \subseteq V$ is called a *global defensive alliance* in H if S satisfies both of the following conditions:

(i) S is a defensive alliance in H, that is,

$$|N_H[v] \cap S| \geq |N_H(v) \cap (V \setminus S)| \qquad \text{for every } v \in S;$$

(ii) S is a dominating set of H, that is,

$$N_H[S] = V.$$

Equivalently, S is a global defensive alliance if every vertex of S is defended by S, and every vertex outside S is adjacent, through at least one hyperedge, to some vertex of S.

Definition 4.4.6 (Global defensive alliance number of a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph. The *global defensive alliance number* of H is defined by

$$\gamma_a(H) := \min\{\, |S| \mid S \subseteq V,\ S \text{ is a global defensive alliance in } H \,\}.$$

Similarly, the *global strong defensive alliance number* of H is

$$\widehat{\gamma}_a(H) := \min\{\, |S| \mid S \subseteq V,\ S \text{ is a global strong defensive alliance in } H \,\}.$$

Since V itself is always a global defensive alliance and a global strong defensive alliance, both parameters are well-defined for every finite hypergraph.

Theorem 4.4.7 (Reduction to the graph case)**.** *Let $G = (V, E)$ be a finite simple graph, regarded as a hypergraph whose hyperedges are the two-element sets in E. Then a subset $S \subseteq V$ is a global defensive alliance in the hypergraph sense if and only if it is a global defensive alliance in the usual graph-theoretic sense.*

Proof. If G is regarded as a hypergraph, then for every vertex $v \in V$,

$$N_H(v) = \{\, u \in V \setminus \{v\} \mid \{u, v\} \in E \,\} = N_G(v).$$

Hence

$$N_H[v] = N_G[v].$$

Therefore the defensive condition

$$|N_H[v] \cap S| \geq |N_H(v) \cap (V \setminus S)|$$

is exactly the usual graph condition

$$|N_G[v] \cap S| \geq |N_G(v) \cap (V \setminus S)|.$$

Likewise,

$$N_H[S] = V$$

is equivalent to

$$N_G[S] = V.$$

Thus the two definitions coincide. □

Theorem 4.4.8 (Well-definedness)**.** *Let*

$$H = (V, \mathcal{E})$$

be a finite hypergraph. Then the notions of defensive alliance, strong defensive alliance, global defensive alliance, and global strong defensive alliance in H are well-defined.

Proof. For every $v \in V$, the open neighborhood $N_H(v)$ is a subset of the finite set V. Hence the quantities

$$|N_H[v] \cap S| \quad \text{and} \quad |N_H(v) \cap (V \setminus S)|$$

are finite nonnegative integers. Therefore the inequalities defining defensive and strong defensive alliances are meaningful. Also, $N_H[S]$ is a subset of the finite set V, so the domination condition

$$N_H[S] = V$$

is well-defined. Finally, since V itself satisfies

$$N_H[V] = V$$

and, for every $v \in V$,

$$|N_H[v] \cap V| = |N_H[v]| \geq 0 = |N_H(v) \cap (V \setminus V)|,$$

the family of global defensive alliances is nonempty. Similarly, V is a global strong defensive alliance because

$$|N_H[v] \cap V| = |N_H[v]| \geq 1 > 0 = |N_H(v) \cap (V \setminus V)|.$$

Hence the corresponding minimum cardinalities are well-defined. □

Definition 4.4.9 (Supervertex neighborhood)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be an n-SuperHyperGraph. For an n-supervertex $X \in V$, its *open super-neighborhood* is defined by

$$N_{\mathcal{S}^{(n)}}(X) := \{\, Y \in V \setminus \{X\} \mid \exists \varepsilon \in E \text{ such that } \{X, Y\} \subseteq \varepsilon \,\}.$$

The *closed super-neighborhood* of X is

$$N_{\mathcal{S}^{(n)}}[X] := N_{\mathcal{S}^{(n)}}(X) \cup \{X\}.$$

For a subset $A \subseteq V$, define

$$N_{\mathcal{S}^{(n)}}(A) := \bigcup_{X \in A} N_{\mathcal{S}^{(n)}}(X),$$

and

$$N_{\mathcal{S}^{(n)}}[A] := N_{\mathcal{S}^{(n)}}(A) \cup A.$$

Definition 4.4.10 (Dominating set in an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be an n-SuperHyperGraph. A subset $A \subseteq V$ is called a *dominating set* of $\mathcal{S}^{(n)}$ if

$$N_{\mathcal{S}^{(n)}}[A] = V.$$

Equivalently, for every $Y \in V \setminus A$, there exists $X \in A$ and there exists $\varepsilon \in E$ such that

$$\{X, Y\} \subseteq \varepsilon.$$

Definition 4.4.11 (Defensive alliance in an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be an n-SuperHyperGraph. A nonempty subset $A \subseteq V$ is called a *defensive alliance* in $\mathcal{S}^{(n)}$ if, for every $X \in A$,

$$\bigl|N_{\mathcal{S}^{(n)}}[X] \cap A\bigr| \geq \bigl|N_{\mathcal{S}^{(n)}}(X) \cap (V \setminus A)\bigr|.$$

Equivalently,

$$1 + \bigl|N_{\mathcal{S}^{(n)}}(X) \cap A\bigr| \geq \bigl|N_{\mathcal{S}^{(n)}}(X) \setminus A\bigr| \qquad \text{for every } X \in A.$$

Definition 4.4.12 (Strong defensive alliance in an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be an n-SuperHyperGraph. A nonempty subset $A \subseteq V$ is called a *strong defensive alliance* in $\mathcal{S}^{(n)}$ if, for every $X \in A$,

$$\big|N_{\mathcal{S}^{(n)}}[X] \cap A\big| > \big|N_{\mathcal{S}^{(n)}}(X) \cap (V \setminus A)\big|.$$

Equivalently,

$$1 + \big|N_{\mathcal{S}^{(n)}}(X) \cap A\big| > \big|N_{\mathcal{S}^{(n)}}(X) \setminus A\big| \qquad \text{for every } X \in A.$$

Definition 4.4.13 (Global defensive alliance in an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be an n-SuperHyperGraph. A subset $A \subseteq V$ is called a *global defensive alliance* in $\mathcal{S}^{(n)}$ if A satisfies both of the following conditions:

(i) A is a defensive alliance in $\mathcal{S}^{(n)}$, that is,

$$\big|N_{\mathcal{S}^{(n)}}[X] \cap A\big| \geq \big|N_{\mathcal{S}^{(n)}}(X) \cap (V \setminus A)\big| \qquad \text{for every } X \in A;$$

(ii) A is a dominating set of $\mathcal{S}^{(n)}$, that is,

$$N_{\mathcal{S}^{(n)}}[A] = V.$$

Thus every selected n-supervertex is defended by the alliance, and every unselected n-supervertex is adjacent to at least one selected n-supervertex through some n-superedge.

Definition 4.4.14 (Global defensive alliance number of an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. The *global defensive alliance number* of $\mathcal{S}^{(n)}$ is

$$\gamma_a\big(\mathcal{S}^{(n)}\big) := \min\{\, |A| \mid A \subseteq V,\ A \text{ is a global defensive alliance in } \mathcal{S}^{(n)} \,\}.$$

The *global strong defensive alliance number* of $\mathcal{S}^{(n)}$ is

$$\widehat{\gamma}_a\big(\mathcal{S}^{(n)}\big) := \min\{\, |A| \mid A \subseteq V,\ A \text{ is a global strong defensive alliance in } \mathcal{S}^{(n)} \,\}.$$

Theorem 4.4.15 (Well-definedness for n-SuperHyperGraphs)**.** *Let*

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. Then the notions of defensive alliance, strong defensive alliance, global defensive alliance, and global strong defensive alliance in $\mathcal{S}^{(n)}$ are well-defined.

Proof. Since V is finite, for every $X \in V$ the sets

$$N_{\mathcal{S}^{(n)}}(X), \qquad N_{\mathcal{S}^{(n)}}[X], \qquad N_{\mathcal{S}^{(n)}}(X) \cap A, \qquad N_{\mathcal{S}^{(n)}}(X) \cap (V \setminus A)$$

are finite. Therefore the cardinalities appearing in the defensive and strong defensive inequalities are finite nonnegative integers.

The domination condition

$$N_{\mathcal{S}^{(n)}}[A] = V$$

is also well-defined, because both sides are subsets of the finite set V. Finally, the whole supervertex set V is always a global defensive alliance: for every $X \in V$,

$$N_{\mathcal{S}^{(n)}}[X] \cap V = N_{\mathcal{S}^{(n)}}[X],$$

while

$$N_{\mathcal{S}^{(n)}}(X) \cap (V \setminus V) = \emptyset.$$

Hence

$$\left|N_{\mathcal{S}^{(n)}}[X] \cap V\right| \geq 0.$$

Moreover, $N_{\mathcal{S}^{(n)}}[V] = V$. Thus the family of global defensive alliances is nonempty. The same argument, with the strict inequality

$$\left|N_{\mathcal{S}^{(n)}}[X] \cap V\right| \geq 1 > 0,$$

shows that V is also a global strong defensive alliance. Therefore the minimum cardinality parameters are well-defined. □

Theorem 4.4.16 (Reduction to HyperGraphs)**.** *Let*

$$\mathcal{S}^{(0)} = (V, E)$$

be a 0*-SuperHyperGraph over* V_0*. Since*

$$\mathcal{P}^0(V_0) = V_0,$$

$\mathcal{S}^{(0)}$ *is an ordinary hypergraph. A subset* $A \subseteq V$ *is a global defensive alliance in* $\mathcal{S}^{(0)}$ *if and only if it is a global defensive alliance in the corresponding hypergraph* (V, E)*.*

Proof. For $n = 0$, every 0-supervertex is an ordinary element of V_0, and the superedge family satisfies

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Thus the super-neighborhood

$$N_{\mathcal{S}^{(0)}}(X)$$

is exactly the hypergraph neighborhood

$$N_H(X).$$

Consequently, the defensive inequality and the domination condition coincide with those of the corresponding hypergraph. Hence the two notions of global defensive alliance are identical. □

4.5 Zero forcing process in a SuperHyperGraph

Zero forcing in a graph is a coloring process where blue vertices repeatedly force their unique white neighbors blue until no further forcing moves remain [309–312]. Related concepts such as skew zero forcing [313,314] and signed zero forcing [315] are also known. A zero forcing process in a HyperGraph starts with blue vertices; a blue vertex forces its unique white hypergraph-neighbor blue, iterating until stabilization finally occurs (cf. [316,317]). A zero forcing process in a SuperHyperGraph starts with blue supervertices; a blue supervertex forces its unique white super-neighbor blue, iterating until stabilization finally occurs.

Definition 4.5.1 (Primal graph of a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph, where $V \neq \emptyset$ and

$$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

The *primal graph* or *2-section* of H is the simple graph

$$\partial_2 H = (V, E_2)$$

defined by

$$E_2 := \big\{\{u, v\} \subseteq V \mid u \neq v,\ \exists e \in \mathcal{E} \text{ such that } \{u, v\} \subseteq e\big\}.$$

Thus two distinct vertices are adjacent in $\partial_2 H$ if and only if they occur together in at least one hyperedge of H.

Definition 4.5.2 (Hypergraph neighborhood)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph. For $v \in V$, the *hypergraph neighborhood* of v is

$$N_H(v) := \{\, u \in V \setminus \{v\} \mid \exists e \in \mathcal{E} \text{ such that } \{u, v\} \subseteq e \,\}.$$

Equivalently,

$$N_H(v) = N_{\partial_2 H}(v).$$

Definition 4.5.3 (Zero forcing process in a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph. A *blue-white coloring* of H is specified by a set

$$B \subseteq V,$$

whose elements are called *blue vertices*; the vertices in $V \setminus B$ are called *white vertices*.

The *hypergraph zero forcing color-change rule* is the following rule: a blue vertex $v \in B$ *forces* a white vertex $w \in V \setminus B$, written

$$v \to w,$$

if

$$N_H(v) \setminus B = \{w\}.$$

That is, v has exactly one white hypergraph-neighbor, and that unique white neighbor is w.

Definition 4.5.4 (Zero forcing closure in a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph and let $B \subseteq V$. Define a sequence of subsets

$$B_0, B_1, B_2, \ldots$$

by

$$B_0 := B$$

and

$$B_{t+1} := B_t \cup \{\, w \in V \setminus B_t \mid \exists v \in B_t \text{ such that } N_H(v) \setminus B_t = \{w\} \,\}.$$

Since V is finite, this increasing sequence stabilizes. The stable set is called the *zero forcing closure* of B in H, and is denoted by

$$\mathrm{cl}_Z^H(B).$$

Thus

$$\mathrm{cl}_Z^H(B) := \bigcup_{t \geq 0} B_t.$$

Definition 4.5.5 (Zero forcing set and zero forcing number of a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph. A subset $B \subseteq V$ is called a *zero forcing set* of H if

$$\mathrm{cl}_Z^H(B) = V.$$

The *zero forcing number* of H is

$$Z(H) := \min\{\, |B| \mid B \subseteq V,\ \mathrm{cl}_Z^H(B) = V \,\}.$$

Theorem 4.5.6 (Well-definedness of HyperGraph zero forcing)**.** *Let*

$$H = (V, \mathcal{E})$$

be a finite hypergraph. Then the zero forcing closure

$$\mathrm{cl}_Z^H(B)$$

is well-defined for every $B \subseteq V$, and $Z(H)$ is well-defined.

Proof. For every $B_t \subseteq V$, the set B_{t+1} is uniquely determined by B_t and the neighborhoods $N_H(v)$. Hence the sequence

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$$

is deterministic and increasing.

Since V is finite, no strictly increasing chain of subsets of V has length greater than $|V|$. Therefore the sequence stabilizes after finitely many steps. Hence

$$\mathrm{cl}_Z^H(B) = \bigcup_{t \geq 0} B_t$$

is well-defined.

Moreover, V itself is always a zero forcing set, since

$$\mathrm{cl}_Z^H(V) = V.$$

Thus the family of zero forcing sets is nonempty, and the minimum defining $Z(H)$ exists. □

Theorem 4.5.7 (Primal-graph characterization)**.** *Let*

$$H = (V, \mathcal{E})$$

be a finite hypergraph. A subset $B \subseteq V$ is a zero forcing set of H if and only if B is a zero forcing set of the primal graph

$$\partial_2 H.$$

Consequently,

$$Z(H) = Z(\partial_2 H).$$

Proof. For every $v \in V$, the definition of the primal graph gives

$$N_H(v) = N_{\partial_2 H}(v).$$

Therefore, for every $B \subseteq V$,

$$N_H(v) \setminus B = \{w\}$$

if and only if

$$N_{\partial_2 H}(v) \setminus B = \{w\}.$$

Hence the available forcing moves in H and in $\partial_2 H$ are identical at every stage of the process. Thus the two closure processes produce the same stable set:

$$\mathrm{cl}_Z^H(B) = \mathrm{cl}_Z^{\partial_2 H}(B).$$

Therefore B is a zero forcing set of H if and only if it is a zero forcing set of $\partial_2 H$, and the equality

$$Z(H) = Z(\partial_2 H)$$

follows. □

Theorem 4.5.8 (Reduction to graph zero forcing)**.** *Let*

$$G = (V, E)$$

be a finite simple graph, and regard G as the 2-uniform hypergraph

$$H_G = (V, \mathcal{E}_G), \qquad \mathcal{E}_G := \{\{u, v\} \mid uv \in E\}.$$

Then

$$Z(H_G) = Z(G).$$

Proof. For the 2-uniform hypergraph H_G, two vertices $u, v \in V$ occur together in a hyperedge if and only if they are adjacent in G. Hence

$$\partial_2 H_G = G.$$

By the primal-graph characterization,

$$Z(H_G) = Z(\partial_2 H_G) = Z(G).$$

□

Definition 4.5.9 (Primal graph of an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be an n-SuperHyperGraph. The *primal graph* or *2-section* of $\mathcal{S}^{(n)}$ is the simple graph

$$\partial_2 \mathcal{S}^{(n)} = (V, E_2)$$

where

$$E_2 := \big\{ \{X, Y\} \subseteq V \mid X \neq Y,\ \exists \varepsilon \in E \text{ such that } \{X, Y\} \subseteq \varepsilon \big\}.$$

Thus two distinct n-supervertices are adjacent if and only if they occur together in some n-superedge.

Definition 4.5.10 (Super-neighborhood)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be an n-SuperHyperGraph. For an n-supervertex $X \in V$, define its *super-neighborhood* by

$$N_{\mathcal{S}^{(n)}}(X) := \{\, Y \in V \setminus \{X\} \mid \exists \varepsilon \in E \text{ such that } \{X, Y\} \subseteq \varepsilon \,\}.$$

Equivalently,

$$N_{\mathcal{S}^{(n)}}(X) = N_{\partial_2 \mathcal{S}^{(n)}}(X).$$

Definition 4.5.11 (Zero forcing process in an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. A *blue-white coloring* of $\mathcal{S}^{(n)}$ is specified by a set

$$B \subseteq V,$$

whose elements are called *blue n-supervertices*; the elements of $V \setminus B$ are called *white n-supervertices.*

The *zero forcing color-change rule* is the following rule: a blue n-supervertex $X \in B$ *forces* a white n-supervertex $Y \in V \setminus B$, written

$$X \to Y,$$

if

$$N_{\mathcal{S}^{(n)}}(X) \setminus B = \{Y\}.$$

That is, X has exactly one white super-neighbor, and that unique white super-neighbor is Y.

Definition 4.5.12 (Zero forcing closure in an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph and let $B \subseteq V$. Define

$$B_0 := B$$

and, for $t \geq 0$,

$$B_{t+1} := B_t \cup \{\, Y \in V \setminus B_t \mid \exists X \in B_t \text{ such that } N_{\mathcal{S}^{(n)}}(X) \setminus B_t = \{Y\} \,\}.$$

The *zero forcing closure* of B in $\mathcal{S}^{(n)}$ is

$$\mathrm{cl}_Z^{\mathcal{S}^{(n)}}(B) := \bigcup_{t \geq 0} B_t.$$

Definition 4.5.13 (Zero forcing set and zero forcing number of an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. A subset $B \subseteq V$ is called a *zero forcing set* of $\mathcal{S}^{(n)}$ if

$$\mathrm{cl}_Z^{\mathcal{S}^{(n)}}(B) = V.$$

The *zero forcing number* of $\mathcal{S}^{(n)}$ is

$$Z(\mathcal{S}^{(n)}) := \min\{\, |B| \mid B \subseteq V,\ \mathrm{cl}_Z^{\mathcal{S}^{(n)}}(B) = V \,\}.$$

Theorem 4.5.14 (Well-definedness of SuperHyperGraph zero forcing)**.** *Let*

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. Then

$$\mathrm{cl}_Z^{\mathcal{S}^{(n)}}(B)$$

is well-defined for every $B \subseteq V$, and

$$Z(\mathcal{S}^{(n)})$$

is well-defined.

Proof. For every $B_t \subseteq V$, the next set B_{t+1} is uniquely determined by B_t and the super-neighborhoods

$$N_{\mathcal{S}^{(n)}}(X).$$

Hence

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$$

is an increasing deterministic sequence of subsets of V.

Since V is finite, this increasing sequence stabilizes after finitely many steps. Thus the closure

$$\mathrm{cl}_Z^{\mathcal{S}^{(n)}}(B) = \bigcup_{t \geq 0} B_t$$

is well-defined.

Moreover,

$$\mathrm{cl}_Z^{\mathcal{S}^{(n)}}(V) = V,$$

so V itself is a zero forcing set. Therefore the set of zero forcing sets is nonempty, and the minimum defining

$$Z(\mathcal{S}^{(n)})$$

exists. □

Theorem 4.5.15 (Primal-graph characterization for n-SuperHyperGraphs)**.** *Let*

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. A subset $B \subseteq V$ is a zero forcing set of $\mathcal{S}^{(n)}$ if and only if B is a zero forcing set of

$$\partial_2 \mathcal{S}^{(n)}.$$

Consequently,

$$Z(\mathcal{S}^{(n)}) = Z(\partial_2 \mathcal{S}^{(n)}).$$

Proof. For every $X \in V$, the definition of the primal graph gives

$$N_{\mathcal{S}^{(n)}}(X) = N_{\partial_2 \mathcal{S}^{(n)}}(X).$$

Therefore, at each stage of the zero forcing process, the available forcing moves in $\mathcal{S}^{(n)}$ and in $\partial_2 \mathcal{S}^{(n)}$ coincide. Thus, for every $B \subseteq V$,

$$\mathrm{cl}_Z^{\mathcal{S}^{(n)}}(B) = \mathrm{cl}_Z^{\partial_2 \mathcal{S}^{(n)}}(B).$$

Hence B is a zero forcing set of $\mathcal{S}^{(n)}$ if and only if it is a zero forcing set of $\partial_2 \mathcal{S}^{(n)}$. Taking the minimum cardinality over all zero forcing sets gives

$$Z\big(\mathcal{S}^{(n)}\big) = Z\big(\partial_2 \mathcal{S}^{(n)}\big).$$

□

Theorem 4.5.16 (Reduction to HyperGraph zero forcing)**.** *Let*

$$\mathcal{S}^{(0)} = (V, E)$$

be a 0*-SuperHyperGraph over* V_0*. Since*

$$\mathcal{P}^0(V_0) = V_0,$$

$\mathcal{S}^{(0)}$ *is an ordinary hypergraph. Moreover,*

$$Z\big(\mathcal{S}^{(0)}\big) = Z(V, E).$$

Proof. For $n = 0$, every 0-supervertex is simply an element of V_0. Hence the definition of super-neighborhood becomes exactly the hypergraph neighborhood:

$$N_{\mathcal{S}^{(0)}}(X) = N_H(X).$$

Therefore the zero forcing color-change rule, the closure process, and the notion of a zero forcing set are identical for $\mathcal{S}^{(0)}$ and for the ordinary hypergraph (V, E). Consequently,

$$Z\big(\mathcal{S}^{(0)}\big) = Z(V, E).$$

□

4.6 Superhypergraph Degree Sequences

A HyperGraph degree sequence lists vertex degrees, where each degree counts how many hyperedges contain the corresponding vertex, usually sorted nonincreasingly or recorded as multiset (cf. [318–320]). A SuperHyperGraph degree sequence lists supervertex degrees, where each degree counts how many superedges contain the corresponding nested supervertex, usually presented nonincreasingly as a multiset.

Definition 4.6.1 (Degree of a vertex in a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph, where V is a finite nonempty set of vertices and

$$\mathcal{E}$$

is a finite family of nonempty subsets of V. For a vertex $v \in V$, the *degree* of v in H is

$$d_H(v) := |\{\, e \in \mathcal{E} \mid v \in e \,\}|.$$

Equivalently, if the incidence matrix of H is

$$M_H = (m_{v,e})_{v \in V,\ e \in \mathcal{E}}, \qquad m_{v,e} = \begin{cases} 1, & v \in e, \\ 0, & v \notin e, \end{cases}$$

then

$$d_H(v) = \sum_{e \in \mathcal{E}} m_{v,e}.$$

Definition 4.6.2 (Degree vector and degree sequence of a HyperGraph)**.** Let

$$H = (V, \mathcal{E})$$

be a finite hypergraph with

$$V = \{v_1, \dots, v_n\}.$$

The *labeled degree vector* of H with respect to the ordering $(v_1, \dots, v_n)$ is

$$\mathbf{d}(H) := \big(d_H(v_1), \dots, d_H(v_n)\big) \in \mathbb{N}_0^n.$$

The *degree sequence* of H is the nonincreasing rearrangement of this vector:

$$d_1 \geq d_2 \geq \cdots \geq d_n,$$

where

$$(d_1, \dots, d_n)$$

is a permutation of

$$\big(d_H(v_1), \dots, d_H(v_n)\big).$$

Equivalently, the degree sequence may be regarded as the multiset

$$\{\, d_H(v) \mid v \in V \,\}.$$

Definition 4.6.3 (Hypergraphic sequence)**.** Let $n \geq 1$. A finite sequence

$$\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}_0^n$$

is called *hypergraphic* if there exists a finite simple hypergraph

$$H = (V, \mathcal{E})$$

with

$$|V| = n$$

such that the degree sequence of H is

$$\mathbf{d}.$$

If, in addition, all hyperedges of H have cardinality r, then $\mathbf{d}$ is called *r-uniform hypergraphic.*

Proposition 4.6.4 (Handshaking identity for HyperGraphs)**.** *Let*

$$H = (V, \mathcal{E})$$

be a finite hypergraph. Then

$$\sum_{v \in V} d_H(v) = \sum_{e \in \mathcal{E}} |e|.$$

In particular, if H is r-uniform, then

$$\sum_{v \in V} d_H(v) = r|\mathcal{E}|.$$

Proof. By definition,

$$d_H(v) = |\{\, e \in \mathcal{E} \mid v \in e \,\}|.$$

Therefore

$$\sum_{v \in V} d_H(v) = \sum_{v \in V} \sum_{e \in \mathcal{E}} m_{v,e} = \sum_{e \in \mathcal{E}} \sum_{v \in V} m_{v,e} = \sum_{e \in \mathcal{E}} |e|.$$

If H is r-uniform, then $|e| = r$ for every $e \in \mathcal{E}$, and hence

$$\sum_{e \in \mathcal{E}} |e| = \sum_{e \in \mathcal{E}} r = r|\mathcal{E}|.$$

□

Remark 4.6.5 (Simple and non-simple cases)**.** If $\mathcal{E}$ is a set, then H is a simple hypergraph and repeated hyperedges are not allowed. If $\mathcal{E}$ is a multiset, then repeated hyperedges are allowed, and the degree $d_H(v)$ counts hyperedges with multiplicity.

Definition 4.6.6 (Degree of an n-supervertex)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. For an n-supervertex $X \in V$, the *degree* of X in $\mathcal{S}^{(n)}$ is

$$d_{\mathcal{S}^{(n)}}(X) := |\{\, \varepsilon \in E \mid X \in \varepsilon \,\}|.$$

Equivalently, if the incidence matrix of $\mathcal{S}^{(n)}$ is

$$M_{\mathcal{S}^{(n)}} = (m_{X,\varepsilon})_{X \in V,\ \varepsilon \in E}, \qquad m_{X,\varepsilon} = \begin{cases} 1, & X \in \varepsilon, \\ 0, & X \notin \varepsilon, \end{cases}$$

then

$$d_{\mathcal{S}^{(n)}}(X) = \sum_{\varepsilon \in E} m_{X,\varepsilon}.$$

Definition 4.6.7 (Degree vector and degree sequence of an n-SuperHyperGraph)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph with

$$V = \{X_1, \ldots, X_N\}.$$

The *labeled degree vector* of $\mathcal{S}^{(n)}$, with respect to the ordering

$$(X_1, \dots, X_N),$$

is

$$\mathbf{d}\big(\mathcal{S}^{(n)}\big) := \big(d_{\mathcal{S}^{(n)}}(X_1), \dots, d_{\mathcal{S}^{(n)}}(X_N)\big) \in \mathbb{N}_0^N.$$

The *degree sequence* of $\mathcal{S}^{(n)}$ is the nonincreasing rearrangement

$$d_1 \geq d_2 \geq \cdots \geq d_N$$

of the labeled degree vector. Equivalently, it is the multiset

$$\{\, d_{\mathcal{S}^{(n)}}(X) \mid X \in V \,\}.$$

Definition 4.6.8 (Superhypergraphic sequence)**.** Let V_0 be a finite nonempty base set and let $n \geq 0$. A finite sequence

$$\mathbf{d} = (d_1, \dots, d_N) \in \mathbb{N}_0^N$$

is called *n-superhypergraphic over* V_0 if there exists a finite n-SuperHyperGraph

$$\mathcal{S}^{(n)} = (V, E)$$

over V_0 such that

$$|V| = N$$

and the degree sequence of $\mathcal{S}^{(n)}$ is

$$\mathbf{d}.$$

If every n-superedge $\varepsilon \in E$ has the same cardinality r, then $\mathbf{d}$ is called *r-uniform n-superhypergraphic.*

Proposition 4.6.9 (Handshaking identity for n-SuperHyperGraphs)**.** *Let*

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph. Then

$$\sum_{X \in V} d_{\mathcal{S}^{(n)}}(X) = \sum_{\varepsilon \in E} |\varepsilon|.$$

In particular, if $\mathcal{S}^{(n)}$ is r-uniform, then

$$\sum_{X \in V} d_{\mathcal{S}^{(n)}}(X) = r|E|.$$

Proof. By definition,

$$d_{\mathcal{S}^{(n)}}(X) = |\{\, \varepsilon \in E \mid X \in \varepsilon \,\}|.$$

Hence

$$\sum_{X \in V} d_{\mathcal{S}^{(n)}}(X) = \sum_{X \in V} \sum_{\varepsilon \in E} m_{X,\varepsilon} = \sum_{\varepsilon \in E} \sum_{X \in V} m_{X,\varepsilon} = \sum_{\varepsilon \in E} |\varepsilon|.$$

If every n-superedge has cardinality r, then

$$\sum_{\varepsilon \in E} |\varepsilon| = \sum_{\varepsilon \in E} r = r|E|.$$

$\square$

Theorem 4.6.10 (Reduction to HyperGraph degree sequences)**.** *Let*

$$\mathcal{S}^{(0)} = (V, E)$$

be a 0*-SuperHyperGraph over* V_0*. Since*

$$\mathcal{P}^0(V_0) = V_0,$$

$\mathcal{S}^{(0)}$ *is an ordinary hypergraph. Moreover, for every* $X \in V$*,*

$$d_{\mathcal{S}^{(0)}}(X) = d_H(X),$$

where $H = (V, E)$*. Consequently, the degree sequence of* $\mathcal{S}^{(0)}$ *coincides with the ordinary hypergraph degree sequence of* H*.*

Proof. For $n = 0$, the condition

$$V \subseteq \mathcal{P}^0(V_0)$$

becomes

$$V \subseteq V_0.$$

Thus the elements of V are ordinary vertices, and the elements of

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$

are ordinary hyperedges. For any $X \in V$,

$$d_{\mathcal{S}^{(0)}}(X) = |\{\, \varepsilon \in E \mid X \in \varepsilon \,\}| = d_H(X).$$

Therefore the labeled degree vectors, and hence the nonincreasing degree sequences, are identical. $\square$

4.7 Neural HyperNetwork and Neural SuperHyperNetwork

A neural network is a computational model of interconnected artificial neurons that adaptively learns patterns from data to perform prediction, classification, decision-making, and representation tasks [321–324]. Fuzzy neural networks [325–328] and neutrosophic neural networks [322, 329–333] have also been studied in recent years. In this section, we define a Neural HyperNetwork and a Neural SuperHyperNetwork in a feedforward form. The main point is that a hyperedge, or a superhyperedge, is not only an input group but also has a specified target neuron in the next layer. This target information is encoded by a directed incidence relation.

Definition 4.7.1 (Neural HyperNetwork)**.** Let $L \in \mathbb{N}_{\geq 1}$, and let

$$n_0, n_1, \ldots, n_L \in \mathbb{N}_{\geq 1}.$$

For each layer $\ell = 0, 1, \ldots, L$, put

$$N_\ell := \{1, 2, \ldots, n_\ell\}.$$

A *feedforward Neural HyperNetwork* is a tuple

$$\mathcal{NH} = \Big(L, \{N_\ell\}_{\ell=0}^{L}, \{\mathcal{E}^{(\ell)}\}_{\ell=1}^{L}, \{\mathcal{A}^{(\ell)}\}_{\ell=1}^{L}, \{w^{(\ell)}\}_{\ell=1}^{L}, \{b^{(\ell)}\}_{\ell=1}^{L}, \{\sigma^{(\ell)}\}_{\ell=1}^{L}\Big)$$

satisfying the following conditions:

(i) For each $\ell = 1, \dots, L$,
$$\mathcal{E}^{(\ell)} \subseteq \mathcal{P}(N_{\ell-1}) \setminus \{\emptyset\}$$
is a finite set of input hyperedges from layer $\ell - 1$.

(ii) For each $\ell = 1, \dots, L$,
$$\mathcal{A}^{(\ell)} \subseteq \mathcal{E}^{(\ell)} \times N_\ell$$
is a finite directed incidence relation. If
$$(e, i) \in \mathcal{A}^{(\ell)},$$
then the hyperedge $e \subseteq N_{\ell-1}$ contributes to the i-th neuron of layer ℓ.

(iii) For each $\ell = 1, \dots, L$,
$$w^{(\ell)} : \mathcal{A}^{(\ell)} \longrightarrow \mathbb{R}$$
is a weight function. The value $w^{(\ell)}(e, i)$ is the weight assigned to the directed hyperconnection from e to i.

(iv) For each $\ell = 1, \dots, L$,
$$b^{(\ell)} = (b_1^{(\ell)}, \dots, b_{n_\ell}^{(\ell)}) \in \mathbb{R}^{n_\ell}$$
is a bias vector.

(v) For each $\ell = 1, \dots, L$,
$$\sigma^{(\ell)} : \mathbb{R} \longrightarrow \mathbb{R}$$
is an activation function.

For an input vector
$$a^{(0)} = x \in \mathbb{R}^{n_0},$$
the feedforward propagation is defined as follows. Suppose that
$$a^{(\ell-1)} = (a_1^{(\ell-1)}, \dots, a_{n_{\ell-1}}^{(\ell-1)}) \in \mathbb{R}^{n_{\ell-1}}$$
has already been defined. For every $e \in \mathcal{E}^{(\ell)}$, define its hyperedge value by
$$\mathrm{Val}_\ell(e; a^{(\ell-1)}) := \prod_{j \in e} a_j^{(\ell-1)}.$$
The product is finite because $e \subseteq N_{\ell-1}$ and $N_{\ell-1}$ is finite. For each $i \in N_\ell$, define
$$z_i^{(\ell)} := b_i^{(\ell)} + \sum_{\substack{e \in \mathcal{E}^{(\ell)} \\ (e,i) \in \mathcal{A}^{(\ell)}}} w^{(\ell)}(e, i)\ \mathrm{Val}_\ell(e; a^{(\ell-1)}),$$
where an empty sum is interpreted as 0. Then set
$$a_i^{(\ell)} := \sigma^{(\ell)}\big(z_i^{(\ell)}\big).$$
Thus
$$a^{(\ell)} = (a_1^{(\ell)}, \dots, a_{n_\ell}^{(\ell)}) \in \mathbb{R}^{n_\ell}.$$
The function represented by $\mathcal{NH}$ is
$$F_{\mathcal{NH}} : \mathbb{R}^{n_0} \longrightarrow \mathbb{R}^{n_L}, \qquad F_{\mathcal{NH}}(x) := a^{(L)}.$$

Theorem 4.7.2 (Well-definedness of a Neural HyperNetwork)**.** *Every feedforward Neural HyperNetwork*

$$\mathcal{NH}$$

defines a unique function

$$F_{\mathcal{NH}} : \mathbb{R}^{n_0} \longrightarrow \mathbb{R}^{n_L}.$$

Proof. Let $x \in \mathbb{R}^{n_0}$ be arbitrary and set $a^{(0)} := x$. We prove by induction on ℓ that $a^{(\ell)} \in \mathbb{R}^{n_\ell}$ is uniquely determined.

For $\ell = 0$, this is true by assumption, since

$$a^{(0)} = x \in \mathbb{R}^{n_0}.$$

Assume that $a^{(\ell-1)} \in \mathbb{R}^{n_{\ell-1}}$ is uniquely determined. Let $e \in \mathcal{E}^{(\ell)}$. Since

$$e \subseteq N_{\ell-1} = \{1, \ldots, n_{\ell-1}\},$$

each coordinate $a_j^{(\ell-1)}$ appearing in

$$\prod_{j \in e} a_j^{(\ell-1)}$$

is defined. Since e is finite and nonempty,

$$\mathrm{Val}_\ell(e; a^{(\ell-1)}) \in \mathbb{R}$$

is uniquely determined.

For each $i \in N_\ell$, the set

$$\{\, e \in \mathcal{E}^{(\ell)} \mid (e, i) \in \mathcal{A}^{(\ell)} \,\}$$

is finite because $\mathcal{E}^{(\ell)}$ is finite. Therefore

$$z_i^{(\ell)} = b_i^{(\ell)} + \sum_{\substack{e \in \mathcal{E}^{(\ell)} \\ (e,i) \in \mathcal{A}^{(\ell)}}} w^{(\ell)}(e, i)\ \mathrm{Val}_\ell(e; a^{(\ell-1)})$$

is a finite sum of real numbers and hence is a uniquely determined real number. Since

$$\sigma^{(\ell)} : \mathbb{R} \to \mathbb{R}$$

is a function, the value

$$a_i^{(\ell)} = \sigma^{(\ell)}(z_i^{(\ell)})$$

is also uniquely determined and belongs to $\mathbb{R}$. This holds for every $i \in N_\ell$, so

$$a^{(\ell)} \in \mathbb{R}^{n_\ell}$$

is uniquely determined.

By induction, $a^{(L)} \in \mathbb{R}^{n_L}$ is uniquely determined for every input $x \in \mathbb{R}^{n_0}$. Hence the assignment

$$F_{\mathcal{NH}}(x) := a^{(L)}$$

defines a unique function

$$F_{\mathcal{NH}} : \mathbb{R}^{n_0} \to \mathbb{R}^{n_L}.$$

□

Definition 4.7.3 (Recursive Value of a Nested Neuron Object)**.** Let $X = N_{\ell-1}$ and let

$$a^{(\ell-1)} \in \mathbb{R}^{n_{\ell-1}}.$$

For $k \geq 0$, define

$$\mathrm{Val}_\ell^{(k)}(\,\cdot\,; a^{(\ell-1)}) : \mathcal{P}_*^k(N_{\ell-1}) \longrightarrow \mathbb{R}$$

recursively as follows.

For $j \in \mathcal{P}_*^0(N_{\ell-1}) = N_{\ell-1}$, set

$$\mathrm{Val}_\ell^{(0)}(j; a^{(\ell-1)}) := a_j^{(\ell-1)}.$$

For $k \geq 0$ and

$$S \in \mathcal{P}_*^{k+1}(N_{\ell-1}),$$

set

$$\mathrm{Val}_\ell^{(k+1)}(S; a^{(\ell-1)}) := \prod_{T \in S} \mathrm{Val}_\ell^{(k)}(T; a^{(\ell-1)}).$$

The product is finite and nonempty because

$$S \in \mathcal{P}\big(\mathcal{P}_*^k(N_{\ell-1})\big) \setminus \{\emptyset\}.$$

Definition 4.7.4 (Neural n-SuperHyperNetwork)**.** Let $n \in \mathbb{N}_{\geq 1}$. A *feedforward Neural n-SuperHyperNetwork* is a tuple

$$\mathcal{NSH}^{(n)} = \Big(L, \{N_\ell\}_{\ell=0}^L, \{\mathcal{E}^{(\ell,n)}\}_{\ell=1}^L, \{\mathcal{A}^{(\ell,n)}\}_{\ell=1}^L, \{w^{(\ell,n)}\}_{\ell=1}^L, \{b^{(\ell)}\}_{\ell=1}^L, \{\sigma^{(\ell)}\}_{\ell=1}^L\Big)$$

satisfying the following conditions:

(i) $L \in \mathbb{N}_{\geq 1}$, and

$$N_\ell = \{1, 2, \ldots, n_\ell\}$$

for some $n_\ell \in \mathbb{N}_{\geq 1}$, for every $\ell = 0, 1, \ldots, L$.

(ii) For each $\ell = 1, \ldots, L$,

$$\mathcal{E}^{(\ell,n)} \subseteq \mathcal{P}_*^n(N_{\ell-1})$$

is a finite set of n-level superhyperedges from layer $\ell - 1$.

(iii) For each $\ell = 1, \ldots, L$,

$$\mathcal{A}^{(\ell,n)} \subseteq \mathcal{E}^{(\ell,n)} \times N_\ell$$

is a finite directed incidence relation. If

$$(S, i) \in \mathcal{A}^{(\ell,n)},$$

then the n-level superhyperedge S contributes to the i-th neuron of layer ℓ.

(iv) For each $\ell = 1, \ldots, L$,

$$w^{(\ell,n)} : \mathcal{A}^{(\ell,n)} \longrightarrow \mathbb{R}$$

is a weight function.

(v) For each $\ell = 1, \dots, L$,

$$b^{(\ell)} \in \mathbb{R}^{n_\ell}$$

is a bias vector.

(vi) For each $\ell = 1, \dots, L$,

$$\sigma^{(\ell)} : \mathbb{R} \longrightarrow \mathbb{R}$$

is an activation function.

For an input vector

$$a^{(0)} = x \in \mathbb{R}^{n_0},$$

the feedforward propagation is defined recursively. Suppose that

$$a^{(\ell-1)} \in \mathbb{R}^{n_{\ell-1}}$$

has already been defined. For each $S \in \mathcal{E}^{(\ell,n)}$, its nested superhyperedge value is

$$\mathrm{Val}_\ell^{(n)}(S; a^{(\ell-1)}),$$

as defined by the recursive value construction above. For each $i \in N_\ell$, set

$$z_i^{(\ell)} := b_i^{(\ell)} + \sum_{\substack{S \in \mathcal{E}^{(\ell,n)} \\ (S,i) \in \mathcal{A}^{(\ell,n)}}} w^{(\ell,n)}(S,i)\ \mathrm{Val}_\ell^{(n)}(S; a^{(\ell-1)}),$$

where an empty sum is interpreted as 0. Then define

$$a_i^{(\ell)} := \sigma^{(\ell)}\big(z_i^{(\ell)}\big).$$

The function represented by $\mathcal{NSH}^{(n)}$ is

$$F_{\mathcal{NSH}^{(n)}} : \mathbb{R}^{n_0} \longrightarrow \mathbb{R}^{n_L}, \qquad F_{\mathcal{NSH}^{(n)}}(x) := a^{(L)}.$$

Theorem 4.7.5 (Well-definedness of the recursive nested value)**.** *Let $n \in \mathbb{N}_{\geq 1}$, let $N_{\ell-1}$ be finite and nonempty, and let*

$$a^{(\ell-1)} \in \mathbb{R}^{n_{\ell-1}}.$$

Then, for every $k = 0, 1, \dots, n$, the map

$$\mathrm{Val}_\ell^{(k)}(\,\cdot\,; a^{(\ell-1)}) : \mathcal{P}_*^k(N_{\ell-1}) \longrightarrow \mathbb{R}$$

is well-defined.

Proof. We prove the assertion by induction on k.

For $k = 0$, one has

$$\mathcal{P}_*^0(N_{\ell-1}) = N_{\ell-1}.$$

For every $j \in N_{\ell-1}$, the coordinate $a_j^{(\ell-1)}$ is a real number. Hence

$$\mathrm{Val}_\ell^{(0)}(j; a^{(\ell-1)}) = a_j^{(\ell-1)}$$

is well-defined.

Assume that

$$\mathrm{Val}_\ell^{(k)}(\,\cdot\,; a^{(\ell-1)}) : \mathcal{P}_*^k(N_{\ell-1}) \to \mathbb{R}$$

is well-defined. Let

$$S \in \mathcal{P}_*^{k+1}(N_{\ell-1}).$$

By definition,

$$S \in \mathcal{P}\big(\mathcal{P}_*^k(N_{\ell-1})\big) \setminus \{\emptyset\}.$$

Thus S is a finite nonempty set whose elements belong to $\mathcal{P}_*^k(N_{\ell-1})$. By the induction hypothesis, for every $T \in S$,

$$\mathrm{Val}_\ell^{(k)}(T; a^{(\ell-1)}) \in \mathbb{R}$$

is uniquely determined. Since S is finite and nonempty, the product

$$\prod_{T \in S} \mathrm{Val}_\ell^{(k)}(T; a^{(\ell-1)})$$

is a uniquely determined real number. Therefore

$$\mathrm{Val}_\ell^{(k+1)}(S; a^{(\ell-1)})$$

is well-defined.

By induction, the statement holds for every $k = 0, 1, \ldots, n$. $\square$

Theorem 4.7.6 (Well-definedness of a Neural n-SuperHyperNetwork)**.** *Every feedforward Neural n-SuperHyperNetwork*

$$\mathcal{NSH}^{(n)}$$

defines a unique function

$$F_{\mathcal{NSH}^{(n)}} : \mathbb{R}^{n_0} \longrightarrow \mathbb{R}^{n_L}.$$

Proof. Let $x \in \mathbb{R}^{n_0}$ be arbitrary and set

$$a^{(0)} := x.$$

We prove by induction on ℓ that

$$a^{(\ell)} \in \mathbb{R}^{n_\ell}$$

is uniquely determined.

The case $\ell = 0$ follows from the choice

$$a^{(0)} = x \in \mathbb{R}^{n_0}.$$

Assume that

$$a^{(\ell-1)} \in \mathbb{R}^{n_{\ell-1}}$$

has been uniquely determined. By the well-definedness of the recursive nested value, for every

$$S \in \mathcal{E}^{(\ell,n)} \subseteq \mathcal{P}_*^n(N_{\ell-1}),$$

the quantity

$$\mathrm{Val}_\ell^{(n)}(S; a^{(\ell-1)})$$

is a uniquely determined real number.

For each $i \in N_\ell$, the set

$$\{ S \in \mathcal{E}^{(\ell,n)} \mid (S,i) \in \mathcal{A}^{(\ell,n)} \}$$

is finite because $\mathcal{E}^{(\ell,n)}$ is finite. Hence

$$z_i^{(\ell)} = b_i^{(\ell)} + \sum_{\substack{S \in \mathcal{E}^{(\ell,n)} \\ (S,i) \in \mathcal{A}^{(\ell,n)}}} w^{(\ell,n)}(S,i)\ \mathrm{Val}_\ell^{(n)}(S; a^{(\ell-1)})$$

is a finite sum of real numbers and is therefore a uniquely determined real number. Since

$$\sigma^{(\ell)} : \mathbb{R} \to \mathbb{R}$$

is a function,

$$a_i^{(\ell)} = \sigma^{(\ell)}(z_i^{(\ell)})$$

is uniquely determined and belongs to $\mathbb{R}$. This holds for every $i \in N_\ell$, so

$$a^{(\ell)} = (a_1^{(\ell)}, \ldots, a_{n_\ell}^{(\ell)}) \in \mathbb{R}^{n_\ell}$$

is uniquely determined.

By induction, $a^{(L)} \in \mathbb{R}^{n_L}$ is uniquely determined for every input $x \in \mathbb{R}^{n_0}$. Therefore

$$F_{\mathcal{NSH}^{(n)}}(x) := a^{(L)}$$

defines a unique function

$$F_{\mathcal{NSH}^{(n)}} : \mathbb{R}^{n_0} \to \mathbb{R}^{n_L}.$$

□

Theorem 4.7.7 (Reduction to Neural HyperNetworks)**.** *A feedforward Neural* 1*-SuperHyperNetwork is precisely a feedforward Neural HyperNetwork with the same directed incidence, weights, biases, and activation functions.*

Proof. For $n = 1$, one has

$$\mathcal{P}_*^1(N_{\ell-1}) = \mathcal{P}(N_{\ell-1}) \setminus \{\emptyset\}.$$

Thus each 1-level superhyperedge

$$S \in \mathcal{P}_*^1(N_{\ell-1})$$

is exactly an ordinary nonempty hyperedge

$$S \subseteq N_{\ell-1}.$$

Moreover, by the recursive definition of the nested value,

$$\mathrm{Val}_\ell^{(1)}(S; a^{(\ell-1)}) = \prod_{j \in S} \mathrm{Val}_\ell^{(0)}(j; a^{(\ell-1)}) = \prod_{j \in S} a_j^{(\ell-1)}.$$

This is exactly the hyperedge value used in the Neural HyperNetwork definition. Therefore the pre-activation formula

$$z_i^{(\ell)} = b_i^{(\ell)} + \sum_{\substack{S \in \mathcal{E}^{(\ell,1)} \\ (S,i) \in \mathcal{A}^{(\ell,1)}}} w^{(\ell,1)}(S,i)\ \mathrm{Val}_\ell^{(1)}(S; a^{(\ell-1)})$$

becomes

$$z_i^{(\ell)} = b_i^{(\ell)} + \sum_{\substack{e \in \mathcal{E}^{(\ell)} \\ (e,i) \in \mathcal{A}^{(\ell)}}} w^{(\ell)}(e,i) \prod_{j \in e} a_j^{(\ell-1)},$$

which is the Neural HyperNetwork feedforward rule. Hence the 1-SuperHyperNetwork case coincides with the Neural HyperNetwork case. □

4.8 MultiMeta Graph, MultiMeta HyperGraph, and MultiMeta SuperHyperGraph

A Meta Graph is a higher-level graph whose vertices represent graphs or graph-like structures, while edges encode relations between those structured objects abstractly and systematically [334,335]. Throughout this section, the notions of graph, hypergraph, n-SuperHyperGraph, and the iterated powerset $\mathcal{P}^q(\cdot)$ are assumed to have been fixed earlier. We do not redefine them here.

Notation 4.8.1 (Finite nonempty families)**.** *For a set X, let*

$$\mathcal{P}_{\mathrm{fin}}(X) := \{\, A \subseteq X \mid A \text{ is finite}\,\}, \qquad \mathcal{P}^*_{\mathrm{fin}}(X) := \mathcal{P}_{\mathrm{fin}}(X) \setminus \{\emptyset\}.$$

For $k \in \mathbb{N}_{\geq 1}$, define

$$X^{\underline{k}} := \{(x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \text{ whenever } i \neq j\}.$$

Thus $X^{\underline{k}}$ is the set of ordered k-tuples of pairwise distinct elements of X.

Notation 4.8.2 (Finitary relation schemes)**.** *For a set X, define*

$$\mathrm{Rel}_{\mathrm{fin}}(X) := \bigcup_{k \geq 1} \big(\{k\} \times \mathcal{P}(X^k)\big).$$

An element

$$\rho = (k, R) \in \mathrm{Rel}_{\mathrm{fin}}(X)$$

is called a finitary relation scheme on X. Its arity and extension are defined by

$$\mathrm{ar}(\rho) := k, \qquad \mathrm{rel}(\rho) := R \subseteq X^k.$$

Definition 4.8.3 (MultiMeta Graph)**.** Let $\mathfrak{G}$ be a nonempty set of finite graphs, and put

$$\mathfrak{U}_{\mathrm{G}} := \mathcal{P}^*_{\mathrm{fin}}(\mathfrak{G}).$$

Let

$$\mathfrak{R}_{\mathrm{G}} \subseteq \mathcal{P}(\mathfrak{U}_{\mathrm{G}} \times \mathfrak{U}_{\mathrm{G}})$$

be a nonempty set of admissible binary relations on $\mathfrak{U}_{\mathrm{G}}$.

A *MultiMeta Graph* over $(\mathfrak{G}, \mathfrak{R}_{\mathrm{G}})$ is a finite directed labelled multigraph

$$\mathcal{MMG} = (V, E, s, t, \lambda)$$

satisfying the following conditions:

(i)
$$\emptyset \neq V \subseteq \mathfrak{U}_{\mathrm{G}}.$$
Each $v \in V$ is called a *multi-meta vertex*. Thus each v is a finite nonempty family of graphs.

(ii) E is a finite set of edge identifiers.

(iii)
$$s, t : E \to V$$
are the source and target maps.

(iv)
$$\lambda : E \to \mathfrak{R}_{\mathrm{G}}$$
is an edge-labeling map.

(v) For every $e \in E$,
$$(s(e), t(e)) \in \lambda(e).$$

If $\lambda(e) = R$, then the edge e may be written as
$$s(e) \xrightarrow{R} t(e).$$

Theorem 4.8.4 (Well-definedness of MultiMeta Graphs)**.** *Every MultiMeta Graph*
$$\mathcal{MMG} = (V, E, s, t, \lambda)$$
is a well-defined finite directed labelled multigraph whose vertices are finite nonempty families of graphs.

Proof. By definition,
$$V \subseteq \mathcal{P}^*_{\mathrm{fin}}(\mathfrak{G}).$$
Hence every $v \in V$ is a finite nonempty family of elements of $\mathfrak{G}$, and each element of $\mathfrak{G}$ is a finite graph.

Since E is finite and
$$s, t : E \to V$$
are functions, each edge identifier $e \in E$ has a uniquely determined source $s(e) \in V$ and a uniquely determined target $t(e) \in V$. Therefore
$$e \mapsto (s(e), t(e))$$
is a well-defined directed incidence assignment.

Since
$$\lambda : E \to \mathfrak{R}_{\mathrm{G}}$$
is a function, each edge $e \in E$ has a uniquely determined admissible binary relation label $\lambda(e)$. The compatibility condition
$$(s(e), t(e)) \in \lambda(e)$$
ensures that the labelled directed edge is admissible with respect to its own relation label.

Thus all vertices, edges, sources, targets, and labels are uniquely specified, and the structure is a well-defined finite directed labelled multigraph whose vertices are finite nonempty families of graphs. □

Definition 4.8.5 (MultiMeta HyperGraph)**.** Let $\mathfrak{H}$ be a nonempty set of finite hypergraphs, and put

$$\mathfrak{U}_{\mathrm{H}} := \mathcal{P}^{*}_{\mathrm{fin}}(\mathfrak{H}).$$

Let

$$\mathfrak{R}_{\mathrm{H}} \subseteq \mathrm{Rel}_{\mathrm{fin}}(\mathfrak{U}_{\mathrm{H}})$$

be a nonempty set of admissible finitary relation schemes on $\mathfrak{U}_{\mathrm{H}}$.

A *MultiMeta HyperGraph* over $(\mathfrak{H}, \mathfrak{R}_{\mathrm{H}})$ is a finite structure

$$\mathcal{MMH} = (V, A, \iota, \lambda)$$

satisfying the following conditions:

(i)
$$\emptyset \neq V \subseteq \mathfrak{U}_{\mathrm{H}}.$$
Each $v \in V$ is called a *multi-meta hypervertex*. Thus each v is a finite nonempty family of hypergraphs.

(ii) A is a finite set of hyperedge identifiers.

(iii)
$$\lambda : A \to \mathfrak{R}_{\mathrm{H}}$$
assigns an admissible finitary relation scheme to each hyperedge identifier.

(iv) If $a \in A$ and
$$\mathrm{ar}(\lambda(a)) = k,$$
then
$$\iota(a) = (v_1, \dots, v_k) \in V^{\underline{k}}.$$

(v) For every $a \in A$,
$$\iota(a) \in \mathrm{rel}(\lambda(a)).$$

If

$$\iota(a) = (v_1, \dots, v_k),$$

then the underlying unordered incidence set of a is denoted by

$$\partial(a) := \{v_1, \dots, v_k\} \subseteq V.$$

Theorem 4.8.6 (Well-definedness of MultiMeta HyperGraphs)**.** *Every MultiMeta HyperGraph*

$$\mathcal{MMH} = (V, A, \iota, \lambda)$$

determines a well-defined finite hypergraph

$$\mathrm{Und}(\mathcal{MMH}) = (V, E_{\mathcal{MMH}})$$

where

$$E_{\mathcal{MMH}} := \{\, \partial(a) \mid a \in A \,\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Proof. Since

$$V \subseteq \mathcal{P}^*_{\mathrm{fin}}(\mathfrak{H}),$$

every element of V is a finite nonempty family of hypergraphs. Hence the vertex set V is a well-defined finite set of multi-meta hypervertices.

Let $a \in A$. By definition,

$$\lambda(a) \in \mathfrak{R}_{\mathrm{H}} \subseteq \mathrm{Rel}_{\mathrm{fin}}(\mathfrak{U}_{\mathrm{H}}).$$

Thus $\lambda(a)$ has a uniquely determined arity

$$k = \mathrm{ar}(\lambda(a)) \in \mathbb{N}_{\geq 1}.$$

The incidence condition gives

$$\iota(a) = (v_1, \ldots, v_k) \in V^{\underline{k}}.$$

Therefore $v_1, \ldots, v_k$ are pairwise distinct elements of V, and

$$\partial(a) := \{v_1, \ldots, v_k\}$$

is a nonempty finite subset of V. Hence

$$\partial(a) \in \mathcal{P}(V) \setminus \{\emptyset\}.$$

Since A is finite, the family

$$E_{\mathcal{MMH}} = \{\, \partial(a) \mid a \in A \,\}$$

is finite. Therefore

$$E_{\mathcal{MMH}} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Consequently,

$$\mathrm{Und}(\mathcal{MMH}) = (V, E_{\mathcal{MMH}})$$

is a well-defined finite hypergraph. □

Definition 4.8.7 (MultiMeta q-SuperHyperGraph over n-SuperHyperGraphs)**.** Let $n, q \in \mathbb{N}_0$. Let $\mathfrak{S}^{(n)}$ be a nonempty set of finite n-SuperHyperGraphs, and put

$$\mathfrak{U}^{(n)}_{\mathrm{SH}} := \mathcal{P}^*_{\mathrm{fin}}(\mathfrak{S}^{(n)}).$$

Let

$$\mathfrak{B}^{(q,n)} := \mathcal{P}^q\big(\mathfrak{U}^{(n)}_{\mathrm{SH}}\big).$$

Let

$$\mathfrak{R}^{(q,n)}_{\mathrm{SH}} \subseteq \mathrm{Rel}_{\mathrm{fin}}\big(\mathfrak{B}^{(q,n)}\big)$$

be a nonempty set of admissible finitary relation schemes on $\mathfrak{B}^{(q,n)}$.

A *MultiMeta q-SuperHyperGraph over n-SuperHyperGraphs* is a finite structure

$$\mathcal{MMSH}^{(q,n)} = (V, A, \iota, \lambda)$$

satisfying the following conditions:

(i)
$$V \subseteq \mathcal{P}^q\big(\mathfrak{U}^{(n)}_{\mathrm{SH}}\big) = \mathfrak{B}^{(q,n)}$$
is a finite set. Its elements are called *multi-meta q-supervertices*. At level $q = 0$, each such vertex is a finite nonempty family of n-SuperHyperGraphs.

(ii) A is a finite set of superhyperedge identifiers.

(iii)
$$\lambda : A \to \mathfrak{R}^{(q,n)}_{\mathrm{SH}}$$
assigns an admissible finitary relation scheme to each superhyperedge identifier.

(iv) If $a \in A$ and
$$\mathrm{ar}(\lambda(a)) = k,$$
then
$$\iota(a) = (X_1, \dots, X_k) \in V^{\underline{k}}.$$

(v) For every $a \in A$,
$$\iota(a) \in \mathrm{rel}(\lambda(a)).$$

If
$$\iota(a) = (X_1, \dots, X_k),$$
then the underlying unordered superincidence set of a is denoted by
$$\partial(a) := \{X_1, \dots, X_k\} \subseteq V.$$

Theorem 4.8.8 (Well-definedness of MultiMeta SuperHyperGraphs)**.** *Every MultiMeta q-SuperHyperGraph over n-SuperHyperGraphs*
$$\mathcal{MMSH}^{(q,n)} = (V, A, \iota, \lambda)$$
determines a well-defined finite q-SuperHyperGraph
$$\mathrm{Und}\big(\mathcal{MMSH}^{(q,n)}\big) = (V, E_{\mathcal{MMSH}})$$
over the base set
$$\mathfrak{U}^{(n)}_{\mathrm{SH}} = \mathcal{P}^*_{\mathrm{fin}}(\mathfrak{S}^{(n)}),$$
where
$$E_{\mathcal{MMSH}} := \{\, \partial(a) \mid a \in A \,\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Proof. By definition,
$$V \subseteq \mathcal{P}^q\big(\mathfrak{U}^{(n)}_{\mathrm{SH}}\big).$$
Thus every element of V is a q-level object over the base set
$$\mathfrak{U}^{(n)}_{\mathrm{SH}} = \mathcal{P}^*_{\mathrm{fin}}(\mathfrak{S}^{(n)}).$$
Hence V is a legitimate finite set of q-supervertices over $\mathfrak{U}^{(n)}_{\mathrm{SH}}$.

Let $a \in A$. Since

$$\lambda(a) \in \mathfrak{R}_{\mathrm{SH}}^{(q,n)} \subseteq \mathrm{Rel}_{\mathrm{fin}}\big(\mathfrak{B}^{(q,n)}\big),$$

the arity

$$k = \mathrm{ar}(\lambda(a))$$

is a uniquely determined positive integer. By the incidence condition,

$$\iota(a) = (X_1, \dots, X_k) \in V^{\underline{k}}.$$

Therefore $X_1, \dots, X_k$ are pairwise distinct elements of V, and

$$\partial(a) := \{X_1, \dots, X_k\}$$

is a nonempty finite subset of V. Hence

$$\partial(a) \in \mathcal{P}(V) \setminus \{\emptyset\}.$$

Since A is finite, the family

$$E_{\mathcal{MMSH}} = \{\, \partial(a) \mid a \in A \,\}$$

is finite. Thus

$$E_{\mathcal{MMSH}} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Combining this with

$$V \subseteq \mathcal{P}^q\big(\mathfrak{U}_{\mathrm{SH}}^{(n)}\big),$$

we conclude that

$$\mathrm{Und}\big(\mathcal{MMSH}^{(q,n)}\big) = (V, E_{\mathcal{MMSH}})$$

satisfies the defining conditions of a finite q-SuperHyperGraph over the base set $\mathfrak{U}_{\mathrm{SH}}^{(n)}$. Therefore it is well-defined. □

4.9 Feedback HyperNetworks and Feedback SuperHyperNetworks

A feedback network is a system where outputs influence future inputs, enabling adaptation, regulation, learning, stability, or amplification through recurrent interactions among interconnected components dynamically (cf. [336]). Throughout this section, the notions of Feedback Network, HyperNetwork, n-SuperHyperGraph, and iterated powerset are assumed to have been fixed earlier. We do not redefine them here. All node sets and hyperarc sets are assumed to be finite. The empty indexed product is understood to be a one-point set.

Definition 4.9.1 (Feedback HyperNetwork)**.** A *Feedback HyperNetwork* is a finite directed hypernetwork equipped with state spaces and feedback update maps. It is a tuple

$$\mathcal{FHN} = \big(V, A, T, H, \{\mathcal{X}_v\}_{v \in V}, \{\mathcal{Y}_{a,v}\}_{a \in A,\ v \in H(a)}, \{\alpha_{a,v}\}_{a \in A,\ v \in H(a)}, \{\varphi_v\}_{v \in V}\big)$$

satisfying the following conditions:

(i) V is a finite nonempty set of nodes.

(ii) A is a finite set of directed hyperarc identifiers.

(iii)
$$T, H : A \longrightarrow \mathcal{P}(V) \setminus \{\emptyset\}$$
are the tail and head maps. For $a \in A$, the set $T(a)$ is the input group of a, and $H(a)$ is the output group of a.

(iv) For every $v \in V$,
$$\mathcal{X}_v \neq \emptyset$$
is the state space of node v.

(v) For every $a \in A$ and every $v \in H(a)$,
$$\mathcal{Y}_{a,v} \neq \emptyset$$
is the message space from the hyperarc a to the head node v.

(vi) For every $a \in A$ and every $v \in H(a)$,
$$\alpha_{a,v} : \prod_{u \in T(a)} \mathcal{X}_u \longrightarrow \mathcal{Y}_{a,v}$$
is a hyperarc aggregation map.

(vii) For every $v \in V$, define the incoming hyperarc set by
$$\mathrm{In}(v) := \{\, a \in A \mid v \in H(a) \,\}.$$
The local feedback update map at v is a function
$$\varphi_v : \mathcal{X}_v \times \prod_{a \in \mathrm{In}(v)} \mathcal{Y}_{a,v} \longrightarrow \mathcal{X}_v.$$

The global state space is
$$\mathcal{X} := \prod_{v \in V} \mathcal{X}_v.$$
For a global state
$$x = (x_v)_{v \in V} \in \mathcal{X},$$
the one-step feedback update
$$F_{\mathcal{FHN}} : \mathcal{X} \longrightarrow \mathcal{X}$$
is defined coordinatewise by
$$\big(F_{\mathcal{FHN}}(x)\big)_v := \varphi_v\Big(x_v,\ \big(\alpha_{a,v}\big((x_u)_{u \in T(a)}\big)\big)_{a \in \mathrm{In}(v)}\Big), \qquad v \in V.$$
The associated discrete-time feedback dynamics is
$$x(t+1) = F_{\mathcal{FHN}}(x(t)), \qquad t = 0, 1, 2, \ldots.$$
No acyclicity condition is imposed on the directed hyperarc structure; directed cycles and self-feedback are permitted.

Theorem 4.9.2 (Well-definedness of Feedback HyperNetworks)**.** *Every Feedback HyperNetwork*

$$\mathcal{FHN}$$

defines a unique global update map

$$F_{\mathcal{FHN}} : \mathcal{X} \longrightarrow \mathcal{X}.$$

Consequently, for every initial state

$$x(0) \in \mathcal{X},$$

the discrete-time trajectory

$$x(t+1) = F_{\mathcal{FHN}}(x(t))$$

is uniquely determined for all $t \in \mathbb{N}_0$*.*

Proof. Let

$$x = (x_v)_{v \in V} \in \mathcal{X}$$

be arbitrary. By the definition of the product state space, for every $u \in V$,

$$x_u \in \mathcal{X}_u.$$

Fix $v \in V$. For every $a \in \mathrm{In}(v)$, one has $v \in H(a)$. Hence the message space $\mathcal{Y}_{a,v}$ and the aggregation map

$$\alpha_{a,v} : \prod_{u \in T(a)} \mathcal{X}_u \longrightarrow \mathcal{Y}_{a,v}$$

are defined. Since

$$T(a) \subseteq V$$

and $x_u \in \mathcal{X}_u$ for every $u \in T(a)$, the restricted tuple

$$(x_u)_{u \in T(a)}$$

belongs to

$$\prod_{u \in T(a)} \mathcal{X}_u.$$

Therefore

$$\alpha_{a,v}\big((x_u)_{u \in T(a)}\big) \in \mathcal{Y}_{a,v}$$

is uniquely determined.

Since A is finite, the incoming set

$$\mathrm{In}(v) = \{\, a \in A \mid v \in H(a) \,\}$$

is finite. Thus the indexed family

$$\big(\alpha_{a,v}\big((x_u)_{u \in T(a)}\big)\big)_{a \in \mathrm{In}(v)}$$

is an element of

$$\prod_{a \in \mathrm{In}(v)} \mathcal{Y}_{a,v}.$$

Also $x_v \in \mathcal{X}_v$. Hence the argument of

$$\varphi_v : \mathcal{X}_v \times \prod_{a \in \mathrm{In}(v)} \mathcal{Y}_{a,v} \longrightarrow \mathcal{X}_v$$

is well-defined, and therefore

$$\big(F_{\mathcal{FHN}}(x)\big)_v \in \mathcal{X}_v$$

is uniquely determined.

Since this holds for every $v \in V$, the family

$$\big((F_{\mathcal{FHN}}(x))_v\big)_{v\in V}$$

belongs to

$$\prod_{v\in V} \mathcal{X}_v = \mathcal{X}.$$

Thus $F_{\mathcal{FHN}}(x) \in \mathcal{X}$ is uniquely determined for every $x \in \mathcal{X}$. Therefore

$$F_{\mathcal{FHN}} : \mathcal{X} \to \mathcal{X}$$

is a well-defined function.

Finally, given $x(0) \in \mathcal{X}$, define recursively

$$x(t+1) := F_{\mathcal{FHN}}(x(t)).$$

Since $F_{\mathcal{FHN}}$ is a function from $\mathcal{X}$ to itself, a straightforward induction on t shows that $x(t) \in \mathcal{X}$ is uniquely determined for every $t \in \mathbb{N}_0$. □

Definition 4.9.3 (Feedback n-SuperHyperNetwork)**.** Let $n \in \mathbb{N}_0$, and let V_0 be a finite nonempty base set. A *Feedback n-SuperHyperNetwork* over V_0 is a feedback dynamical structure

$$\mathcal{FSHN}^{(n)} = \big(V, A, T, H, \{\mathcal{X}_X\}_{X\in V}, \{\mathcal{Y}_{a,X}\}_{a\in A,\ X\in H(a)}, \{\alpha_{a,X}\}_{a\in A,\ X\in H(a)}, \{\varphi_X\}_{X\in V}\big)$$

satisfying the following conditions:

(i)
$$V \subseteq \mathcal{P}^n(V_0)$$
is a finite nonempty set of n-supernodes.

(ii) A is a finite set of directed superhyperarc identifiers.

(iii)
$$T, H : A \longrightarrow \mathcal{P}(V) \setminus \{\emptyset\}$$
are the supertail and superhead maps. Thus, for each $a \in A$, both $T(a)$ and $H(a)$ are nonempty finite sets of n-supernodes.

(iv) For every $X \in V$,
$$\mathcal{X}_X \neq \emptyset$$
is the state space of the n-supernode X.

(v) For every $a \in A$ and every $X \in H(a)$,

$$\mathcal{Y}_{a,X} \neq \emptyset$$

is the message space from the directed superhyperarc a to the superhead node X.

(vi) For every $a \in A$ and every $X \in H(a)$,

$$\alpha_{a,X} : \prod_{Y \in T(a)} \mathcal{X}_Y \longrightarrow \mathcal{Y}_{a,X}$$

is a superhyperarc aggregation map.

(vii) For every $X \in V$, define

$$\mathrm{In}(X) := \{\, a \in A \mid X \in H(a) \,\}.$$

The local feedback update map at X is a function

$$\varphi_X : \mathcal{X}_X \times \prod_{a \in \mathrm{In}(X)} \mathcal{Y}_{a,X} \longrightarrow \mathcal{X}_X.$$

The global state space is

$$\mathcal{X}^{(n)} := \prod_{X \in V} \mathcal{X}_X.$$

For

$$x = (x_X)_{X \in V} \in \mathcal{X}^{(n)},$$

the one-step feedback update

$$F_{\mathcal{FSHN}^{(n)}} : \mathcal{X}^{(n)} \longrightarrow \mathcal{X}^{(n)}$$

is defined by

$$\big(F_{\mathcal{FSHN}^{(n)}}(x)\big)_X := \varphi_X\left(x_X,\ \big(\alpha_{a,X}\big((x_Y)_{Y \in T(a)}\big)\big)_{a \in \mathrm{In}(X)}\right), \qquad X \in V.$$

The associated discrete-time feedback dynamics is

$$x(t+1) = F_{\mathcal{FSHN}^{(n)}}(x(t)), \qquad t = 0, 1, 2, \ldots.$$

No acyclicity condition is imposed; feedback loops among supernodes and superhyperarcs are allowed.

Theorem 4.9.4 (Underlying directed n-SuperHyperGraph)**.** *Let*

$$\mathcal{FSHN}^{(n)}$$

be a Feedback n-SuperHyperNetwork over V_0. Define

$$\partial(a) := T(a) \cup H(a) \qquad (a \in A),$$

and set

$$E_{\mathcal{FSHN}^{(n)}} := \{\, \partial(a) \mid a \in A \,\}.$$

Then

$$\mathrm{Und}\big(\mathcal{FSHN}^{(n)}\big) := \big(V, E_{\mathcal{FSHN}^{(n)}}\big)$$

is a well-defined n-SuperHyperGraph over V_0.

Proof. By definition of a Feedback n-SuperHyperNetwork,

$$V \subseteq \mathcal{P}^n(V_0).$$

Thus V is a finite set of n-supernodes over V_0.

Let $a \in A$. Since

$$T(a), H(a) \in \mathcal{P}(V) \setminus \{\emptyset\},$$

both $T(a)$ and $H(a)$ are nonempty subsets of V. Hence

$$\partial(a) := T(a) \cup H(a)$$

is also a nonempty subset of V. Therefore

$$\partial(a) \in \mathcal{P}(V) \setminus \{\emptyset\}.$$

Since A is finite,

$$E_{\mathcal{FSHN}^{(n)}} = \{\, \partial(a) \mid a \in A \,\}$$

is finite and satisfies

$$E_{\mathcal{FSHN}^{(n)}} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Consequently,

$$\big(V, E_{\mathcal{FSHN}^{(n)}}\big)$$

satisfies the defining conditions of an n-SuperHyperGraph over V_0. □

Theorem 4.9.5 (Well-definedness of Feedback n-SuperHyperNetworks)**.** *Every Feedback n-SuperHyperNetwork*

$$\mathcal{FSHN}^{(n)}$$

defines a unique global update map

$$F_{\mathcal{FSHN}^{(n)}} : \mathcal{X}^{(n)} \longrightarrow \mathcal{X}^{(n)}.$$

Consequently, for every initial state

$$x(0) \in \mathcal{X}^{(n)},$$

the discrete-time trajectory

$$x(t+1) = F_{\mathcal{FSHN}^{(n)}}(x(t))$$

is uniquely determined for all $t \in \mathbb{N}_0$.

Proof. Let

$$x = (x_X)_{X \in V} \in \mathcal{X}^{(n)}$$

be arbitrary. Then, for every $Y \in V$,

$$x_Y \in \mathcal{X}_Y.$$

Fix $X \in V$. For every $a \in \mathrm{In}(X)$, one has

$$X \in H(a).$$

Thus the message space $\mathcal{Y}_{a,X}$ and the aggregation map

$$\alpha_{a,X} : \prod_{Y \in T(a)} \mathcal{X}_Y \longrightarrow \mathcal{Y}_{a,X}$$

are defined. Since

$$T(a) \subseteq V$$

and $x_Y \in \mathcal{X}_Y$ for every $Y \in T(a)$, the restricted family

$$(x_Y)_{Y \in T(a)}$$

belongs to

$$\prod_{Y \in T(a)} \mathcal{X}_Y.$$

Therefore

$$\alpha_{a,X}\big((x_Y)_{Y \in T(a)}\big) \in \mathcal{Y}_{a,X}$$

is uniquely determined.

Because A is finite,

$$\mathrm{In}(X) = \{\, a \in A \mid X \in H(a) \,\}$$

is finite. Hence the family

$$\big(\alpha_{a,X}\big((x_Y)_{Y \in T(a)}\big)\big)_{a \in \mathrm{In}(X)}$$

is an element of

$$\prod_{a \in \mathrm{In}(X)} \mathcal{Y}_{a,X}.$$

Also $x_X \in \mathcal{X}_X$. Therefore the local update

$$\varphi_X\left(x_X,\ \big(\alpha_{a,X}\big((x_Y)_{Y \in T(a)}\big)\big)_{a \in \mathrm{In}(X)}\right)$$

is well-defined and belongs to $\mathcal{X}_X$.

Thus, for each $X \in V$,

$$\big(F_{\mathcal{FSHN}^{(n)}}(x)\big)_X \in \mathcal{X}_X$$

is uniquely determined. Consequently,

$$F_{\mathcal{FSHN}^{(n)}}(x) = \big((F_{\mathcal{FSHN}^{(n)}}(x))_X\big)_{X \in V}$$

belongs to

$$\prod_{X \in V} \mathcal{X}_X = \mathcal{X}^{(n)}.$$

Hence

$$F_{\mathcal{FSHN}^{(n)}} : \mathcal{X}^{(n)} \to \mathcal{X}^{(n)}$$

is a well-defined function.

Finally, for any initial state

$$x(0) \in \mathcal{X}^{(n)},$$

the recursion

$$x(t+1) = F_{\mathcal{FSHN}^{(n)}}(x(t))$$

defines $x(t) \in \mathcal{X}^{(n)}$ uniquely for every $t \in \mathbb{N}_0$ by induction on t. □

Theorem 4.9.6 (Reduction to Feedback HyperNetworks)**.** *Every Feedback* 0*-SuperHyperNetwork is a Feedback HyperNetwork whose node set is contained in the base set* V_0*.*

Proof. For $n = 0$, one has
$$\mathcal{P}^0(V_0) = V_0.$$
Hence the condition
$$V \subseteq \mathcal{P}^0(V_0)$$
becomes
$$V \subseteq V_0.$$
Thus the supernodes are ordinary nodes. The supertail and superhead maps
$$T, H : A \to \mathcal{P}(V) \setminus \{\emptyset\}$$
are exactly directed hyperarc tail and head maps on the ordinary node set V. The state spaces, message spaces, aggregation maps, and local update maps have the same domains and codomains as in a Feedback HyperNetwork. Therefore the coordinate update formula for
$$\mathcal{FSHN}^{(0)}$$
is exactly the coordinate update formula for a Feedback HyperNetwork. □

4.10 Tensor HyperGraphs and Tensor SuperHyperGraphs

Throughout this section, the notions of HyperGraph, n-SuperHyperGraph, and iterated powerset are assumed to have been fixed earlier. We do not redefine them here.

Notation 4.10.1. *For $N \in \mathbb{N}_{\geq 1}$, write*
$$[N] := \{1, 2, \ldots, N\}.$$
If V is a finite nonempty set with $|V| = N$, an indexing *of V is a bijection*
$$\iota : V \longrightarrow [N].$$
For a nonempty subset $e \subseteq V$, define
$$\iota(e) := \{\iota(v) \mid v \in e\} \subseteq [N].$$

Notation 4.10.2 (Tensor normalization factor)**.** *Let $M \in \mathbb{N}_{\geq 1}$ and let $c \in \{1, \ldots, M\}$. Define*
$$\alpha_{M,c} := \sum_{\substack{r_1,\ldots,r_c \geq 1 \\ r_1+\cdots+r_c=M}} \binom{M}{r_1, r_2, \ldots, r_c}.$$
Equivalently, $\alpha_{M,c}$ is the number of ordered M-tuples whose set of distinct symbols is a prescribed c-element set and in which each of the c symbols appears at least once.

Lemma 4.10.3 (Well-definedness of the normalization factor)**.** *For every $M \in \mathbb{N}_{\geq 1}$ and every $c \in \{1, \ldots, M\}$, the number $\alpha_{M,c}$ is a well-defined positive integer.*

Proof. The set of integer c-tuples

$$(r_1, \dots, r_c) \in \mathbb{N}_{\geq 1}^c \quad \text{with} \quad r_1 + \cdots + r_c = M$$

is finite. For each such tuple, the multinomial coefficient

$$\binom{M}{r_1, r_2, \dots, r_c} = \frac{M!}{r_1! r_2! \cdots r_c!}$$

is a positive integer. Hence $\alpha_{M,c}$ is a finite sum of positive integers.

Moreover, since $c \leq M$, at least one such tuple exists; for instance,

$$(r_1, \dots, r_c) = (M - c + 1, 1, \dots, 1)$$

is admissible. Therefore $\alpha_{M,c}$ is positive. □

Definition 4.10.4 (Tensor HyperGraph)**.** Let

$$H = (V, E)$$

be a finite HyperGraph with $V \neq \emptyset$. Let

$$M \in \mathbb{N}_{\geq 1}$$

be such that

$$1 \leq |e| \leq M \qquad (e \in E).$$

Let

$$\iota : V \longrightarrow [N], \qquad N := |V|,$$

be an indexing of V, and let

$$w : E \longrightarrow \mathbb{R}_{\geq 0}$$

be a nonnegative weight function. In the unweighted case, one may take $w(e) = 1$ for all $e \in E$.

The *order-M Tensor HyperGraph* associated with (H, ι, w) is the tuple

$$\mathcal{TH}^{(M)} = (H, \iota, w, A),$$

where

$$A = (a_{i_1 \dots i_M}) \in \mathbb{R}_{\geq 0}^{N \times \cdots \times N}$$

is defined by

$$a_{i_1 \dots i_M} := \sum_{e \in E} \frac{w(e)}{\alpha_{M,|e|}} \mathbf{1}[\{i_1, \dots, i_M\} = \iota(e)], \qquad (i_1, \dots, i_M) \in [N]^M.$$

Here $\mathbf{1}[P]$ denotes 1 if the proposition P is true and 0 otherwise.

The tensor A is called the *adjacency tensor* of $\mathcal{TH}^{(M)}$.

Theorem 4.10.5 (Well-definedness of Tensor HyperGraphs)**.** *Let*

$$H = (V, E)$$

be a finite HyperGraph with $V \neq \emptyset$, let $M \in \mathbb{N}_{\geq 1}$ satisfy

$$1 \leq |e| \leq M \qquad (e \in E),$$

let $\iota : V \to [N]$ be an indexing, and let

$$w : E \to \mathbb{R}_{\geq 0}$$

be a nonnegative weight function. Then

$$\mathcal{TH}^{(M)} = (H, \iota, w, A)$$

is well-defined. More precisely, A is a uniquely determined symmetric nonnegative M-th order tensor of dimension $N = |V|$.

Proof. Since V is finite and nonempty, $N = |V|$ is a positive integer and an indexing

$$\iota : V \to [N]$$

assigns a unique tensor index to each vertex of V.

Let

$$(i_1, \ldots, i_M) \in [N]^M$$

be arbitrary. For each $e \in E$, one has

$$1 \leq |e| \leq M.$$

By the previous lemma,

$$\alpha_{M,|e|}$$

is a well-defined positive integer. Hence

$$\frac{w(e)}{\alpha_{M,|e|}}$$

is a well-defined nonnegative real number. The condition

$$\{i_1, \ldots, i_M\} = \iota(e)$$

is a well-defined statement about two subsets of $[N]$, so the indicator

$$\mathbf{1}[\{i_1, \ldots, i_M\} = \iota(e)]$$

is well-defined.

Since E is finite, the sum

$$a_{i_1 \ldots i_M} = \sum_{e \in E} \frac{w(e)}{\alpha_{M,|e|}} \mathbf{1}[\{i_1, \ldots, i_M\} = \iota(e)]$$

is a finite sum of nonnegative real numbers. Therefore

$$a_{i_1 \ldots i_M} \in \mathbb{R}_{\geq 0}$$

is uniquely determined for every $(i_1, \ldots, i_M) \in [N]^M$.

Thus

$$A = (a_{i_1 \dots i_M}) \in \mathbb{R}_{\geq 0}^{N \times \cdots \times N}$$

is a uniquely determined nonnegative M-th order tensor.

Finally, let π be any permutation of $\{1, \dots, M\}$. Since

$$\{i_{\pi(1)}, \dots, i_{\pi(M)}\} = \{i_1, \dots, i_M\},$$

the defining formula gives

$$a_{i_{\pi(1)} \dots i_{\pi(M)}} = a_{i_1 \dots i_M}.$$

Hence A is symmetric. Therefore the Tensor HyperGraph

$$\mathcal{TH}^{(M)} = (H, \iota, w, A)$$

is well-defined. □

Definition 4.10.6 (Tensor n-SuperHyperGraph)**.** Let V_0 be a finite nonempty base set, and let

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph over V_0, with $V \neq \emptyset$. Let

$$M \in \mathbb{N}_{\geq 1}$$

be such that

$$1 \leq |\varepsilon| \leq M \qquad (\varepsilon \in E).$$

Let

$$\iota : V \longrightarrow [N], \qquad N := |V|,$$

be an indexing of the n-supervertices, and let

$$w : E \longrightarrow \mathbb{R}_{\geq 0}$$

be a nonnegative superedge weight function.

The *order-M Tensor n-SuperHyperGraph* associated with $(\mathcal{S}^{(n)}, \iota, w)$ is the tuple

$$\mathcal{TSH}^{(n,M)} = (V_0, \mathcal{S}^{(n)}, \iota, w, A),$$

where

$$A = (a_{i_1 \dots i_M}) \in \mathbb{R}_{\geq 0}^{N \times \cdots \times N}$$

is defined by

$$a_{i_1 \dots i_M} := \sum_{\varepsilon \in E} \frac{w(\varepsilon)}{\alpha_{M,|\varepsilon|}} \mathbf{1}[\{i_1, \dots, i_M\} = \iota(\varepsilon)], \qquad (i_1, \dots, i_M) \in [N]^M.$$

Here

$$\iota(\varepsilon) := \{\iota(X) \mid X \in \varepsilon\} \subseteq [N].$$

The tensor A is called the *adjacency tensor* of $\mathcal{TSH}^{(n,M)}$.

Theorem 4.10.7 (Well-definedness of Tensor n-SuperHyperGraphs)**.** *Let*

$$\mathcal{S}^{(n)} = (V, E)$$

be a finite n-SuperHyperGraph over a finite nonempty base set V_0, with $V \neq \emptyset$. Let $M \in \mathbb{N}_{\geq 1}$ satisfy

$$1 \leq |\varepsilon| \leq M \qquad (\varepsilon \in E),$$

let

$$\iota : V \to [N], \qquad N := |V|,$$

be an indexing of the n-supervertices, and let

$$w : E \to \mathbb{R}_{\geq 0}$$

be a nonnegative superedge weight function. Then

$$\mathcal{TSH}^{(n,M)} = (V_0, \mathcal{S}^{(n)}, \iota, w, A)$$

is well-defined. More precisely, A is a uniquely determined symmetric nonnegative M-th order tensor of dimension $N = |V|$.

Proof. Because $\mathcal{S}^{(n)} = (V, E)$ is an n-SuperHyperGraph over V_0, the set V is a finite set of n-supervertices and every $\varepsilon \in E$ is a nonempty subset of V. Since $V \neq \emptyset$, the integer

$$N = |V|$$

is positive, and the indexing

$$\iota : V \to [N]$$

assigns a unique tensor index to each n-supervertex.

Let

$$(i_1, \ldots, i_M) \in [N]^M$$

be arbitrary. For every $\varepsilon \in E$, one has

$$1 \leq |\varepsilon| \leq M.$$

Hence, by the normalization lemma,

$$\alpha_{M,|\varepsilon|}$$

is a well-defined positive integer. Therefore

$$\frac{w(\varepsilon)}{\alpha_{M,|\varepsilon|}}$$

is a well-defined nonnegative real number.

The set

$$\iota(\varepsilon) = \{\iota(X) \mid X \in \varepsilon\}$$

is a well-defined nonempty subset of $[N]$, and the condition

$$\{i_1, \ldots, i_M\} = \iota(\varepsilon)$$

is therefore a well-defined proposition. Consequently, each summand

$$\frac{w(\varepsilon)}{\alpha_{M,|\varepsilon|}} \mathbf{1}[\{i_1, \ldots, i_M\} = \iota(\varepsilon)]$$

is a well-defined nonnegative real number.

Since E is finite, the sum

$$a_{i_1 \dots i_M} = \sum_{\varepsilon \in E} \frac{w(\varepsilon)}{\alpha_{M,|\varepsilon|}} \mathbf{1}[\{i_1, \dots, i_M\} = \iota(\varepsilon)]$$

is finite and uniquely determined. Thus

$$A = (a_{i_1 \dots i_M}) \in \mathbb{R}_{\geq 0}^{N \times \dots \times N}$$

is a well-defined nonnegative M-th order tensor.

To prove symmetry, let π be a permutation of $\{1, \dots, M\}$. Since

$$\{i_{\pi(1)}, \dots, i_{\pi(M)}\} = \{i_1, \dots, i_M\},$$

the defining indicator has the same value before and after the permutation of indices. Hence

$$a_{i_{\pi(1)} \dots i_{\pi(M)}} = a_{i_1 \dots i_M}.$$

Therefore A is symmetric.

Thus all components of

$$\mathcal{TSH}^{(n,M)} = (V_0, \mathcal{S}^{(n)}, \iota, w, A)$$

are well-defined, and the adjacency tensor is uniquely determined. □

Theorem 4.10.8 (Reduction to Tensor HyperGraphs)**.** *A Tensor* 0*-SuperHyperGraph is precisely a Tensor HyperGraph whose vertex set is contained in the base set* V_0*.*

Proof. For $n = 0$, one has

$$\mathcal{P}^0(V_0) = V_0.$$

Hence the condition that

$$\mathcal{S}^{(0)} = (V, E)$$

is a 0-SuperHyperGraph over V_0 means precisely that

$$V \subseteq V_0$$

and

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Thus $\mathcal{S}^{(0)}$ is an ordinary HyperGraph on the vertex set V.

The indexing

$$\iota : V \to [N],$$

the weight function

$$w : E \to \mathbb{R}_{\geq 0},$$

and the tensor formula

$$a_{i_1 \dots i_M} = \sum_{e \in E} \frac{w(e)}{\alpha_{M,|e|}} \mathbf{1}[\{i_1, \dots, i_M\} = \iota(e)]$$

are exactly the same as in the definition of a Tensor HyperGraph. Therefore a Tensor 0-SuperHyperGraph is precisely a Tensor HyperGraph whose vertex set is contained in V_0. □

4.11 Semantic HyperNetworks and Semantic SuperHyperNetworks

A semantic network represents knowledge as nodes and labeled links, capturing concepts, relationships, inheritance, and reasoning structure in machine-readable form (cf. [337–340]). Throughout this section, the notions of Semantic Network, HyperNetwork, n-SuperHyperGraph, and iterated powerset are assumed to have been fixed earlier. We do not redefine them here.

Notation 4.11.1. *For a set X and $k \in \mathbb{N}_{\geq 1}$, write*

$$X^{\underline{k}} := \{(x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \text{ whenever } i \neq j\}.$$

Thus $X^{\underline{k}}$ is the set of ordered k-tuples of pairwise distinct elements of X.

Definition 4.11.2 (Semantic type frame)**.** A *semantic type frame* is a tuple

$$\mathfrak{S} = (\mathcal{T}, \mathcal{R}, \mathrm{ar}, \mathrm{sig}, \Omega, \mathcal{I}_{\mathcal{T}}, \mathcal{I}_{\mathcal{R}})$$

satisfying the following conditions:

(i) $\mathcal{T}$ is a nonempty set of semantic types.

(ii) $\mathcal{R}$ is a nonempty set of relation symbols.

(iii)
$$\mathrm{ar} : \mathcal{R} \to \mathbb{N}_{\geq 1}$$
assigns an arity to each relation symbol.

(iv) For every $r \in \mathcal{R}$, if
$$\mathrm{ar}(r) = k,$$
then
$$\mathrm{sig}(r) = (\tau_1, \ldots, \tau_k) \in \mathcal{T}^k$$
is the semantic type signature of r.

(v) $\Omega \neq \emptyset$ is a universe of meanings.

(vi)
$$\mathcal{I}_{\mathcal{T}} : \mathcal{T} \to \mathcal{P}(\Omega) \setminus \{\emptyset\}$$
assigns to each semantic type a nonempty set of admissible meanings.

(vii) For every $r \in \mathcal{R}$, if
$$\mathrm{sig}(r) = (\tau_1, \ldots, \tau_k),$$
then
$$\mathcal{I}_{\mathcal{R}}(r) \subseteq \mathcal{I}_{\mathcal{T}}(\tau_1) \times \cdots \times \mathcal{I}_{\mathcal{T}}(\tau_k)$$
is the semantic interpretation of r.

Definition 4.11.3 (Semantic HyperNetwork)**.** Let

$$\mathfrak{S} = (\mathcal{T}, \mathcal{R}, \mathrm{ar}, \mathrm{sig}, \Omega, \mathcal{I}_{\mathcal{T}}, \mathcal{I}_{\mathcal{R}})$$

be a semantic type frame. A *Semantic HyperNetwork* over $\mathfrak{S}$ is a finite structure

$$\mathcal{SHN} = (V, A, \tau, \mu, \lambda, \iota)$$

satisfying the following conditions:

(i) V is a finite nonempty set of semantic nodes.

(ii) A is a finite set of semantic hyperrelation identifiers.

(iii)
$$\tau : V \to \mathcal{T}$$
assigns a semantic type to each node.

(iv)
$$\mu : V \to \Omega$$
assigns a meaning to each node, and it satisfies
$$\mu(v) \in \mathcal{I}_{\mathcal{T}}(\tau(v)) \qquad (v \in V).$$

(v)
$$\lambda : A \to \mathcal{R}$$
assigns a relation symbol to each semantic hyperrelation identifier.

(vi) For every $a \in A$, if
$$\mathrm{ar}(\lambda(a)) = k \quad \text{and} \quad \mathrm{sig}(\lambda(a)) = (\tau_1, \ldots, \tau_k),$$
then
$$\iota(a) = (v_1, \ldots, v_k) \in V^{\underline{k}}$$
and
$$\tau(v_j) = \tau_j \qquad (j = 1, \ldots, k).$$

(vii) For every $a \in A$, if
$$\iota(a) = (v_1, \ldots, v_k),$$
then the semantic compatibility condition
$$(\mu(v_1), \ldots, \mu(v_k)) \in \mathcal{I}_{\mathcal{R}}(\lambda(a))$$
holds.

For $a \in A$ with
$$\iota(a) = (v_1, \dots, v_k),$$
define its underlying unordered hyperincidence set by
$$\partial(a) := \{v_1, \dots, v_k\} \subseteq V.$$

Theorem 4.11.4 (Well-definedness of Semantic HyperNetworks)**.** *Every Semantic HyperNetwork*
$$\mathcal{SHN} = (V, A, \tau, \mu, \lambda, \iota)$$
over a semantic type frame $\mathfrak{S}$ determines a well-defined finite semantic hypernetwork. In particular, its underlying incidence structure
$$\mathrm{Und}(\mathcal{SHN}) := (V, E_{\mathcal{SHN}})$$
with
$$E_{\mathcal{SHN}} := \{\, \partial(a) \mid a \in A \,\}$$
is a finite HyperNetwork incidence structure, namely
$$E_{\mathcal{SHN}} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Proof. Let $a \in A$. Since
$$\lambda : A \to \mathcal{R}$$
is a function, $\lambda(a) \in \mathcal{R}$ is uniquely determined. Hence
$$k := \mathrm{ar}(\lambda(a)) \in \mathbb{N}_{\geq 1}$$
is uniquely determined. By definition of a Semantic HyperNetwork,
$$\iota(a) = (v_1, \dots, v_k) \in V^{\underline{k}}.$$
Thus $v_1, \dots, v_k$ are pairwise distinct elements of V. Since $k \geq 1$, the set
$$\partial(a) = \{v_1, \dots, v_k\}$$
is a nonempty subset of V. Therefore
$$\partial(a) \in \mathcal{P}(V) \setminus \{\emptyset\}.$$

Since A is finite, the family
$$E_{\mathcal{SHN}} = \{\, \partial(a) \mid a \in A \,\}$$
is finite. Hence
$$E_{\mathcal{SHN}} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$
Thus the underlying incidence structure is a finite HyperNetwork incidence structure.

It remains to verify that the semantic data are well-defined. For each $v \in V$, the type $\tau(v) \in \mathcal{T}$ and the meaning $\mu(v) \in \Omega$ are uniquely determined because τ and μ are functions. The condition
$$\mu(v) \in \mathcal{I}_{\mathcal{T}}(\tau(v))$$
ensures that every node meaning belongs to the interpretation of its assigned type.

For each $a \in A$, the relation symbol $\lambda(a)$, its arity, and its type signature are uniquely determined. If

$$\iota(a) = (v_1, \dots, v_k),$$

then the type compatibility condition gives

$$\tau(v_j) = \tau_j \qquad (j = 1, \dots, k),$$

where

$$\mathrm{sig}(\lambda(a)) = (\tau_1, \dots, \tau_k).$$

Therefore

$$(\mu(v_1), \dots, \mu(v_k)) \in \mathcal{I}_{\mathcal{T}}(\tau_1) \times \cdots \times \mathcal{I}_{\mathcal{T}}(\tau_k).$$

The additional semantic compatibility condition states precisely that

$$(\mu(v_1), \dots, \mu(v_k)) \in \mathcal{I}_{\mathcal{R}}(\lambda(a)).$$

Thus every semantic hyperrelation occurrence has a well-defined arity, well-defined ordered arguments, well-defined node meanings, and a well-defined relation interpretation.

Consequently, all structural and semantic components of

$$\mathcal{SHN} = (V, A, \tau, \mu, \lambda, \iota)$$

are well-defined. □

Definition 4.11.5 (Semantic n-SuperHyperNetwork)**.** Let

$$\mathfrak{S} = (\mathcal{T}, \mathcal{R}, \mathrm{ar}, \mathrm{sig}, \Omega, \mathcal{I}_{\mathcal{T}}, \mathcal{I}_{\mathcal{R}})$$

be a semantic type frame. Let V_0 be a finite nonempty base set, and let $n \in \mathbb{N}_0$. A *Semantic n-SuperHyperNetwork* over $(V_0, \mathfrak{S})$ is a finite structure

$$\mathcal{SSHN}^{(n)} = (V, A, \tau, \mu, \lambda, \iota)$$

satisfying the following conditions:

(i)
$$\emptyset \neq V \subseteq \mathcal{P}^n(V_0).$$
The elements of V are called semantic n-supernodes.

(ii) A is a finite set of semantic superhyperrelation identifiers.

(iii)
$$\tau : V \to \mathcal{T}$$
assigns a semantic type to each semantic n-supernode.

(iv)
$$\mu : V \to \Omega$$
assigns a meaning to each semantic n-supernode, and it satisfies
$$\mu(X) \in \mathcal{I}_{\mathcal{T}}(\tau(X)) \qquad (X \in V).$$

(v)
$$\lambda : A \to \mathcal{R}$$
assigns a relation symbol to each semantic superhyperrelation identifier.

(vi) For every $a \in A$, if
$$\mathrm{ar}(\lambda(a)) = k \quad \text{and} \quad \mathrm{sig}(\lambda(a)) = (\tau_1, \ldots, \tau_k),$$
then
$$\iota(a) = (X_1, \ldots, X_k) \in V^{\underline{k}}$$
and
$$\tau(X_j) = \tau_j \qquad (j = 1, \ldots, k).$$

(vii) For every $a \in A$, if
$$\iota(a) = (X_1, \ldots, X_k),$$
then
$$(\mu(X_1), \ldots, \mu(X_k)) \in \mathcal{I}_{\mathcal{R}}(\lambda(a)).$$

For $a \in A$ with
$$\iota(a) = (X_1, \ldots, X_k),$$
define its underlying unordered superhyperincidence set by
$$\partial(a) := \{X_1, \ldots, X_k\} \subseteq V.$$

Theorem 4.11.6 (Well-definedness of Semantic n-SuperHyperNetworks)**.** *Every Semantic n-SuperHyperNetwork*
$$\mathcal{SSHN}^{(n)} = (V, A, \tau, \mu, \lambda, \iota)$$
over $(V_0, \mathfrak{S})$ determines a well-defined finite semantic superhypernetwork. In particular, its underlying incidence structure
$$\mathrm{Und}\big(\mathcal{SSHN}^{(n)}\big) := (V, E_{\mathcal{SSHN}^{(n)}})$$
with
$$E_{\mathcal{SSHN}^{(n)}} := \{\, \partial(a) \mid a \in A \,\}$$
is a finite n-SuperHyperGraph over V_0.

Proof. By definition,
$$V \subseteq \mathcal{P}^n(V_0).$$
Thus every element of V is an n-level object over the base set V_0. Hence V is a well-defined finite set of semantic n-supernodes.

Let $a \in A$. Since λ is a function, the relation symbol
$$\lambda(a) \in \mathcal{R}$$

is uniquely determined. Therefore

$$k := \mathrm{ar}(\lambda(a)) \in \mathbb{N}_{\geq 1}$$

is uniquely determined. By the incidence condition,

$$\iota(a) = (X_1, \ldots, X_k) \in V^{\underline{k}}.$$

Thus $X_1, \ldots, X_k$ are pairwise distinct elements of V. Since $k \geq 1$, the set

$$\partial(a) = \{X_1, \ldots, X_k\}$$

is a nonempty subset of V. Hence

$$\partial(a) \in \mathcal{P}(V) \setminus \{\emptyset\}.$$

Since A is finite,

$$E_{\mathcal{SSHN}^{(n)}} = \{\, \partial(a) \mid a \in A \,\}$$

is finite. Therefore

$$E_{\mathcal{SSHN}^{(n)}} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Combining this with

$$V \subseteq \mathcal{P}^n(V_0),$$

we obtain that

$$(V, E_{\mathcal{SSHN}^{(n)}})$$

satisfies the incidence conditions of a finite n-SuperHyperGraph over V_0.

It remains to check semantic well-definedness. For each $X \in V$, the values

$$\tau(X) \in \mathcal{T} \quad \text{and} \quad \mu(X) \in \Omega$$

are uniquely determined because τ and μ are functions. The condition

$$\mu(X) \in \mathcal{I}_{\mathcal{T}}(\tau(X))$$

ensures that every semantic n-supernode has an admissible meaning for its assigned type.

For each $a \in A$, the relation symbol $\lambda(a)$, its arity, and its type signature are uniquely determined. If

$$\mathrm{sig}(\lambda(a)) = (\tau_1, \ldots, \tau_k) \quad \text{and} \quad \iota(a) = (X_1, \ldots, X_k),$$

then the type compatibility condition gives

$$\tau(X_j) = \tau_j \qquad (j = 1, \ldots, k).$$

Therefore

$$(\mu(X_1), \ldots, \mu(X_k)) \in \mathcal{I}_{\mathcal{T}}(\tau_1) \times \cdots \times \mathcal{I}_{\mathcal{T}}(\tau_k).$$

The semantic compatibility condition further gives

$$(\mu(X_1), \ldots, \mu(X_k)) \in \mathcal{I}_{\mathcal{R}}(\lambda(a)).$$

Thus every semantic superhyperrelation occurrence has a well-defined ordered argument tuple, a well-defined type signature, and a well-defined semantic interpretation.

Consequently,

$$\mathcal{SSHN}^{(n)} = (V, A, \tau, \mu, \lambda, \iota)$$

is a well-defined semantic n-SuperHyperNetwork. □

Theorem 4.11.7 (Reduction to Semantic HyperNetworks)**.** *A Semantic* 0*-SuperHyperNetwork over* $(V_0, \mathfrak{S})$ *is exactly a Semantic HyperNetwork whose semantic node set is contained in* V_0*.*

Proof. For $n = 0$, one has

$$\mathcal{P}^0(V_0) = V_0.$$

Therefore the condition

$$V \subseteq \mathcal{P}^0(V_0)$$

is precisely

$$V \subseteq V_0.$$

Thus the semantic 0-supernodes are ordinary semantic nodes selected from the base set V_0.

All remaining components

$$A, \tau, \mu, \lambda, \iota$$

have exactly the same domains, codomains, and compatibility conditions as in the definition of a Semantic HyperNetwork. Moreover, for

$$\iota(a) = (v_1, \ldots, v_k),$$

the underlying incidence set

$$\partial(a) = \{v_1, \ldots, v_k\}$$

is an ordinary hyperedge on V. Hence the semantic and incidence structures coincide with those of a Semantic HyperNetwork whose node set is contained in V_0. □

4.12 Open-Incidence n-SuperHyperGraphs

An open-incidence n-SuperHyperGraph represents n-supervertices and superhyperedges using incidence pins whose endpoints may remain temporarily unresolved during construction or analysis. A Vertex-Revealing Edge-Growth n-SuperHyperGraph evolves through time by resolving open incidence pins into existing or newly created n-supervertices progressively and dynamically. Throughout this section, the notions of n-SuperHyperGraph and iterated powerset are assumed to have been fixed earlier. We do not redefine them here.

Definition 4.12.1 (Open-incidence n-SuperHyperGraph)**.** Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. Put

$$U_n := \mathcal{P}^n(V_0).$$

Let

$$\perp \notin U_n$$

be a distinguished symbol, interpreted as an unresolved incidence endpoint.

An *open-incidence n-SuperHyperGraph* on V_0 is a quintuple

$$\mathcal{S} = (V, E, P, \pi_E, \pi_V)$$

satisfying the following conditions:

(i)
$$V \subseteq U_n$$
is a finite set of n-supervertices.

(ii) E is a finite set of superhyperedge identifiers.

(iii) P is a finite set of *pins*, also called incidence tokens.

(iv)
$$\pi_E : P \to E$$
assigns each pin to the superhyperedge to which it belongs.

(v)
$$\pi_V : P \to V \sqcup \{\perp\}$$
assigns each pin either to a resolved n-supervertex in V or to the unresolved symbol $\perp$.

(vi) For every $e \in E$, its pin-set
$$P_e := \pi_E^{-1}(e)$$
is finite and nonempty.

(vii) For every $e \in E$, the restriction
$$\pi_V\big|_{P_e \setminus \pi_V^{-1}(\perp)} : P_e \setminus \pi_V^{-1}(\perp) \longrightarrow V$$
is injective.

The last condition means that a resolved n-supervertex cannot occur twice inside the same superhyperedge, while several unresolved pins may coexist.

Definition 4.12.2 (Realized incidence and status of a superhyperedge)**.** Let
$$\mathcal{S} = (V, E, P, \pi_E, \pi_V)$$
be an open-incidence n-SuperHyperGraph on V_0. For $e \in E$, define the *realized incidence* of e by
$$\mathrm{Inc}_{\mathcal{S}}(e) := \{\, \pi_V(p) \in V \mid p \in P_e,\ \pi_V(p) \neq \perp \,\} \subseteq V.$$
The superhyperedge e is called:

(a) *closed* if
$$\pi_V(p) \in V \qquad \text{for every } p \in P_e;$$

(b) *partially open* if
$$\pi_V(p) = \perp \qquad \text{for at least one } p \in P_e.$$

The whole open-incidence n-SuperHyperGraph is called *closed* if every superhyperedge is closed.

Remark 4.12.3 (Closed realization)**.** If

$$\mathcal{S} = (V, E, P, \pi_E, \pi_V)$$

is closed, then

$$\mathrm{Inc}_{\mathcal{S}}(e) \in \mathcal{P}(V) \setminus \{\emptyset\} \qquad (e \in E).$$

Hence

$$\mathrm{Cl}(\mathcal{S}) := \big(V,\ \{\mathrm{Inc}_{\mathcal{S}}(e) \mid e \in E\}\big)$$

is an n-SuperHyperGraph on V_0. Thus an open-incidence n-SuperHyperGraph is an incidence-token presentation of an n-SuperHyperGraph in which some incidences may remain unresolved.

Definition 4.12.4 (Vertex-revealing edge-growth n-SuperHyperGraph)**.** Let V_0 be a finite nonempty base set, let $n \in \mathbb{N}_0$, and put

$$U_n := \mathcal{P}^n(V_0).$$

Let T be either $\mathbb{N}_0$ or a linearly ordered time set. A *vertex-revealing edge-growth n-SuperHyperGraph* on V_0 is a time-indexed family

$$\mathbb{S} = \big(\mathcal{S}_t\big)_{t\in T}, \qquad \mathcal{S}_t = (V_t, E_t, P_t, \pi_{E,t}, \pi_{V,t}),$$

such that each $\mathcal{S}_t$ is an open-incidence n-SuperHyperGraph on V_0, and each transition

$$\mathcal{S}_t \longrightarrow \mathcal{S}_{t^+}$$

is obtained by a finite composition of the following elementary operations.

(A) Sprouting a partially open superhyperedge. Choose a nonempty finite set

$$S \subseteq V_t$$

and an integer $r \in \mathbb{N}_0$. Introduce a new superhyperedge identifier

$$e \notin E_t$$

and new pins

$$\{p_X \mid X \in S\} \cup \{q_1, \dots, q_r\}$$

disjoint from P_t. Define

$$E_{t^+} := E_t \cup \{e\},$$
$$P_{t^+} := P_t \cup \{p_X \mid X \in S\} \cup \{q_1, \dots, q_r\},$$

and set

$$\pi_{E,t^+}(p_X) = e, \qquad \pi_{V,t^+}(p_X) = X \qquad (X \in S),$$
$$\pi_{E,t^+}(q_j) = e, \qquad \pi_{V,t^+}(q_j) = \perp \qquad (j = 1, \dots, r).$$

All previously existing vertices, superhyperedges, pins, and map values are kept unchanged.

(B) Growing a superhyperedge by adding an unresolved pin. Choose an existing superhyperedge

$$e \in E_t$$

and introduce a new pin

$$q \notin P_t.$$

Define

$$V_{t^+} := V_t, \qquad E_{t^+} := E_t, \qquad P_{t^+} := P_t \cup \{q\},$$

and set

$$\pi_{E,t^+}(q) = e, \qquad \pi_{V,t^+}(q) = \perp.$$

All previous map values are kept unchanged.

(C) Resolving an unresolved incidence. Choose a pin

$$p \in P_t \qquad \text{with} \qquad \pi_{V,t}(p) = \perp,$$

and put

$$e := \pi_{E,t}(p).$$

The pin p is resolved by one of the following two operations.

(C1) Revealing a new n-supervertex. Choose

$$X_{\mathrm{new}} \in U_n \setminus V_t.$$

Set

$$V_{t^+} := V_t \cup \{X_{\mathrm{new}}\}, \qquad \pi_{V,t^+}(p) := X_{\mathrm{new}}.$$

All other components and map values are kept unchanged.

(C2) Attaching to an existing n-supervertex. Choose

$$X \in V_t \quad \text{such that} \quad X \notin \mathrm{Inc}_{\mathcal{S}_t}(e).$$

Set

$$\pi_{V,t^+}(p) := X.$$

All other components and map values are kept unchanged.

In every case, the condition

$$V_{t^+} \subseteq U_n$$

must be preserved. The operation is called *vertex-revealing* because an unresolved incidence may be resolved either by attaching it to an existing n-supervertex or by creating a new n-supervertex from the ambient universe U_n.

4.13 HyperGraph-Encoded Maps and SuperHyperGraph-Encoded Maps

Throughout this section, the notions of Graph-Encoded Map, HyperGraph, n-SuperHyperGraph, and iterated powerset are assumed to have been fixed earlier. We do not redefine them here.

Notation 4.13.1 (Support of a hyperedge family)**.** *Let*

$$H = (V, E)$$

be a HyperGraph. Define the support *of H by*

$$\mathrm{Supp}(H) := \bigcup_{e \in E} e \subseteq V.$$

If $\mathrm{Supp}(H) = V$*, then H is called* edge-covering.

Definition 4.13.2 (HyperGraph-Encoded Map)**.** Let

$$H = (V, E) \quad \text{and} \quad K = (W, F)$$

be HyperGraphs. A *HyperGraph-Encoded Map* from H to K is a tuple

$$\mathcal{M} = \big(H, K, \theta, \{\varphi_e\}_{e \in E}\big)$$

satisfying the following conditions:

(i)
$$\theta : E \to F$$
assigns to each source hyperedge $e \in E$ a target hyperedge $\theta(e) \in F$.

(ii) For each $e \in E$,
$$\varphi_e : e \to \theta(e)$$
is a local map from the vertices incident with e to the vertices incident with the target hyperedge $\theta(e)$.

(iii) The local maps are overlap-compatible: for all $e, e' \in E$ and all $v \in e \cap e'$,
$$\varphi_e(v) = \varphi_{e'}(v).$$

The map θ is called the *edge-encoding map*, and the family

$$\{\varphi_e : e \to \theta(e)\}_{e \in E}$$

is called the *local vertex-encoding family.*

Theorem 4.13.3 (Well-definedness of HyperGraph-Encoded Maps)**.** *Let*

$$\mathcal{M} = \big(H, K, \theta, \{\varphi_e\}_{e \in E}\big)$$

be a HyperGraph-Encoded Map from

$$H = (V, E) \quad \textit{to} \quad K = (W, F).$$

Then there exists a unique map

$$\Phi_{\mathcal{M}} : \mathrm{Supp}(H) \to W$$

such that

$$\Phi_{\mathcal{M}}(v) = \varphi_e(v)$$

whenever $e \in E$ and $v \in e$. Moreover, for every $e \in E$,

$$\Phi_{\mathcal{M}}(e) \subseteq \theta(e).$$

In particular, if H is edge-covering, then $\Phi_{\mathcal{M}}$ is a well-defined map

$$\Phi_{\mathcal{M}} : V \to W.$$

Proof. Let

$$v \in \mathrm{Supp}(H).$$

By definition of support, there exists at least one hyperedge $e \in E$ such that

$$v \in e.$$

Define

$$\Phi_{\mathcal{M}}(v) := \varphi_e(v).$$

We must show that this definition is independent of the choice of e.

Suppose that $e, e' \in E$ both contain v. Then

$$v \in e \cap e'.$$

By the overlap-compatibility condition,

$$\varphi_e(v) = \varphi_{e'}(v).$$

Hence the value $\Phi_{\mathcal{M}}(v)$ is independent of the chosen hyperedge. Therefore

$$\Phi_{\mathcal{M}} : \mathrm{Supp}(H) \to W$$

is well-defined.

For each $e \in E$ and each $v \in e$, the local map satisfies

$$\varphi_e(v) \in \theta(e).$$

Since

$$\Phi_{\mathcal{M}}(v) = \varphi_e(v),$$

we obtain

$$\Phi_{\mathcal{M}}(v) \in \theta(e) \qquad (v \in e).$$

Thus

$$\Phi_{\mathcal{M}}(e) = \{\Phi_{\mathcal{M}}(v) \mid v \in e\} \subseteq \theta(e).$$

Uniqueness follows immediately. If

$$\Psi : \mathrm{Supp}(H) \to W$$

also satisfies

$$\Psi(v) = \varphi_e(v)$$

for every $e \in E$ and $v \in e$, then for every $v \in \mathrm{Supp}(H)$ one may choose $e \in E$ with $v \in e$, and obtain

$$\Psi(v) = \varphi_e(v) = \Phi_{\mathcal{M}}(v).$$

Therefore $\Psi = \Phi_{\mathcal{M}}$.

Finally, if H is edge-covering, then

$$\mathrm{Supp}(H) = V,$$

so the same construction gives a well-defined map

$$\Phi_{\mathcal{M}} : V \to W.$$

□

Notation 4.13.4 (Support of a superhyperedge family)**.** *Let*

$$\mathcal{S}^{(n)} = (V, E)$$

be an n-SuperHyperGraph. Define

$$\mathrm{Supp}\big(\mathcal{S}^{(n)}\big) := \bigcup_{\varepsilon \in E} \varepsilon \subseteq V.$$

If

$$\mathrm{Supp}\big(\mathcal{S}^{(n)}\big) = V,$$

then $\mathcal{S}^{(n)}$ is called superedge-covering.

Definition 4.13.5 (SuperHyperGraph-Encoded Map)**.** Let

$$\mathcal{S}^{(n)} = (V, E)$$

be an n-SuperHyperGraph over a finite nonempty base set V_0, and let

$$\mathcal{T}^{(m)} = (W, F)$$

be an m-SuperHyperGraph over a finite nonempty base set W_0. A *SuperHyperGraph-Encoded Map* from $\mathcal{S}^{(n)}$ to $\mathcal{T}^{(m)}$ is a tuple

$$\mathcal{N} = \big(\mathcal{S}^{(n)}, \mathcal{T}^{(m)}, \Theta, \{\psi_\varepsilon\}_{\varepsilon \in E}\big)$$

satisfying the following conditions:

(i)

$$\Theta : E \to F$$

assigns to each source superhyperedge $\varepsilon \in E$ a target superhyperedge $\Theta(\varepsilon) \in F$.

(ii) For each $\varepsilon \in E$,

$$\psi_\varepsilon : \varepsilon \to \Theta(\varepsilon)$$

is a local map from the n-supervertices incident with ε to the m-supervertices incident with $\Theta(\varepsilon)$.

(iii) The local maps are overlap-compatible: for all $\varepsilon, \varepsilon' \in E$ and all

$$X \in \varepsilon \cap \varepsilon',$$

one has

$$\psi_\varepsilon(X) = \psi_{\varepsilon'}(X).$$

The map Θ is called the *superedge-encoding map*, and the family

$$\{\psi_\varepsilon : \varepsilon \to \Theta(\varepsilon)\}_{\varepsilon \in E}$$

is called the *local supervertex-encoding family*.

Theorem 4.13.6 (Well-definedness of SuperHyperGraph-Encoded Maps)**.** *Let*

$$\mathcal{N} = \big(\mathcal{S}^{(n)}, \mathcal{T}^{(m)}, \Theta, \{\psi_\varepsilon\}_{\varepsilon \in E}\big)$$

be a SuperHyperGraph-Encoded Map from

$$\mathcal{S}^{(n)} = (V, E)$$

to

$$\mathcal{T}^{(m)} = (W, F).$$

Then there exists a unique map

$$\Psi_{\mathcal{N}} : \mathrm{Supp}\big(\mathcal{S}^{(n)}\big) \to W$$

such that

$$\Psi_{\mathcal{N}}(X) = \psi_\varepsilon(X)$$

whenever $\varepsilon \in E$ and $X \in \varepsilon$. Moreover, for every $\varepsilon \in E$,

$$\Psi_{\mathcal{N}}(\varepsilon) \subseteq \Theta(\varepsilon).$$

In particular, if $\mathcal{S}^{(n)}$ is superedge-covering, then

$$\Psi_{\mathcal{N}} : V \to W$$

is a well-defined map from the n-supervertex set of $\mathcal{S}^{(n)}$ to the m-supervertex set of $\mathcal{T}^{(m)}$.

Proof. Let

$$X \in \mathrm{Supp}\big(\mathcal{S}^{(n)}\big).$$

Then there exists at least one superhyperedge $\varepsilon \in E$ such that

$$X \in \varepsilon.$$

Define

$$\Psi_{\mathcal{N}}(X) := \psi_\varepsilon(X).$$

We prove that this value is independent of the chosen superhyperedge.

Suppose that

$$X \in \varepsilon \cap \varepsilon'$$

for some $\varepsilon, \varepsilon' \in E$. By overlap-compatibility,

$$\psi_\varepsilon(X) = \psi_{\varepsilon'}(X).$$

Hence $\Psi_{\mathcal{N}}(X)$ is independent of the choice of ε. Therefore

$$\Psi_{\mathcal{N}} : \mathrm{Supp}\big(\mathcal{S}^{(n)}\big) \to W$$

is well-defined.

For every $\varepsilon \in E$, the local map has codomain

$$\Theta(\varepsilon) \subseteq W.$$

Thus for every $X \in \varepsilon$,

$$\psi_\varepsilon(X) \in \Theta(\varepsilon).$$

Since

$$\Psi_{\mathcal{N}}(X) = \psi_\varepsilon(X),$$

we have

$$\Psi_{\mathcal{N}}(X) \in \Theta(\varepsilon) \qquad (X \in \varepsilon).$$

Consequently,

$$\Psi_{\mathcal{N}}(\varepsilon) = \{\Psi_{\mathcal{N}}(X) \mid X \in \varepsilon\} \subseteq \Theta(\varepsilon).$$

Uniqueness is immediate. Let

$$\Psi' : \mathrm{Supp}\big(\mathcal{S}^{(n)}\big) \to W$$

be another map satisfying

$$\Psi'(X) = \psi_\varepsilon(X)$$

whenever $X \in \varepsilon$. For any

$$X \in \mathrm{Supp}\big(\mathcal{S}^{(n)}\big),$$

choose $\varepsilon \in E$ such that $X \in \varepsilon$. Then

$$\Psi'(X) = \psi_\varepsilon(X) = \Psi_{\mathcal{N}}(X).$$

Hence

$$\Psi' = \Psi_{\mathcal{N}}.$$

If $\mathcal{S}^{(n)}$ is superedge-covering, then

$$\mathrm{Supp}\big(\mathcal{S}^{(n)}\big) = V.$$

Therefore the induced map is a well-defined map

$$\Psi_{\mathcal{N}} : V \to W.$$

□

Theorem 4.13.7 (Reduction to HyperGraph-Encoded Maps)**.** *If $n = m = 0$, then a SuperHyperGraph-Encoded Map from*

$$\mathcal{S}^{(0)} = (V, E)$$

to

$$\mathcal{T}^{(0)} = (W, F)$$

is exactly a HyperGraph-Encoded Map between the corresponding HyperGraphs.

Proof. For $n = 0$ and $m = 0$, one has

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^0(W_0) = W_0.$$

Hence the elements of V and W are ordinary vertices, while the elements of E and F are ordinary hyperedges. The superedge-encoding map

$$\Theta : E \to F$$

is therefore an ordinary hyperedge-encoding map, and each local map

$$\psi_\varepsilon : \varepsilon \to \Theta(\varepsilon)$$

is a local map between ordinary hyperedges.

The overlap-compatibility condition

$$\psi_\varepsilon(X) = \psi_{\varepsilon'}(X) \qquad (X \in \varepsilon \cap \varepsilon')$$

is precisely the compatibility condition in the definition of a HyperGraph-Encoded Map. Therefore the two notions coincide when $n = m = 0$. □

4.14 Ultra-SuperHyperGraph

An Ultra-SuperHyperGraph is a finite well-founded ranked incidence structure over a cumulative supervertex hierarchy. Unlike ordinary n-SuperHyperGraphs, whose superedges are incident only with rank-homogeneous supervertices, an Ultra-SuperHyperGraph allows a higher-rank superedge to be incident with lower-rank supervertices and lower-rank superedges themselves. The strict rank-decreasing incidence condition preserves well-foundedness and prevents circular edge-to-edge incidence. Thus ordinary n-SuperHyperGraphs are recovered as the rank-homogeneous vertex-only special case, while Ultra-SuperHyperGraphs retain non-flattened cross-rank and edge-to-edge incidence information.

Definition 4.14.1 (Cumulative supervertex hierarchy)**.** Let V_0 be a nonempty finite base set. Define

$$\mathbb{U}_0(V_0) := \{(0, x) \mid x \in V_0\}.$$

For $r \geq 0$, define recursively

$$\mathbb{U}_{\leq r}(V_0) := \bigcup_{i=0}^{r} \mathbb{U}_i(V_0),$$

and

$$\mathbb{U}_{r+1}(V_0) := \{(r + 1, A) \mid A \subseteq_{\text{fin}} \mathbb{U}_{\leq r}(V_0)\}.$$

An element of $\mathbb{U}_r(V_0)$ is called a rank-r cumulative supervertex over V_0.

Definition 4.14.2 (Ultra-SuperHyperGraph)**.** Let V_0 be a nonempty finite base set and let $R \in \mathbb{N}_0$. An R-rank Ultra-SuperHyperGraph over V_0 is a tuple

$$\mathfrak{U} = \big(V_0, (\mathcal{V}_r)_{0 \leq r \leq R}, (\mathcal{E}_s)_{1 \leq s \leq R+1}, \partial\big),$$

satisfying the following conditions.

1. For each $0 \le r \le R$,
$$\mathcal{V}_r \subseteq \mathbb{U}_r(V_0)$$
is a finite set. Its elements are called rank-r ultra-supervertices.

2. For each $1 \le s \le R+1$, $\mathcal{E}_s$ is a finite set. Its elements are called rank-s ultra-superedges.

3. The sets
$$\mathcal{V}_0, \dots, \mathcal{V}_R, \mathcal{E}_1, \dots, \mathcal{E}_{R+1}$$
are pairwise disjoint.

4. Put
$$\mathcal{V}_{<s} := \bigcup_{0 \le r < s} \mathcal{V}_r, \qquad \mathcal{E}_{<s} := \bigcup_{1 \le t < s} \mathcal{E}_t.$$
For each $1 \le s \le R+1$, the incidence map is a function
$$\partial_s : \mathcal{E}_s \longrightarrow \mathcal{P}^*_{\mathrm{fin}}\big(\mathcal{V}_{<s} \dot{\cup} \mathcal{E}_{<s}\big),$$
where $\dot{\cup}$ denotes disjoint union and
$$\mathcal{P}^*_{\mathrm{fin}}(X) := \{A \subseteq X \mid 0 < |A| < \infty\}.$$

5. Equivalently, every ultra-superedge is incident only with strictly lower-rank objects. That is, if $e \in \mathcal{E}_s$ and $z \in \partial_s(e)$, then
$$\rho(z) < s,$$
where
$$\rho(z) = \begin{cases} r, & z \in \mathcal{V}_r, \\ t, & z \in \mathcal{E}_t. \end{cases}$$

The total ultra-supervertex set and total ultra-superedge set are
$$\mathcal{V}(\mathfrak{U}) := \bigcup_{r=0}^{R} \mathcal{V}_r, \qquad \mathcal{E}(\mathfrak{U}) := \bigcup_{s=1}^{R+1} \mathcal{E}_s.$$

Theorem 4.14.3 (Well-definedness)**.** *Let V_0 be a nonempty finite set and let $R \in \mathbb{N}_0$. Every R-rank Ultra-SuperHyperGraph*
$$\mathfrak{U} = \big(V_0, (\mathcal{V}_r)_{0 \le r \le R}, (\mathcal{E}_s)_{1 \le s \le R+1}, \partial\big)$$
is a well-defined finite ranked incidence structure. Moreover, its edge-to-edge incidence relation is acyclic.

Proof. Since V_0 is finite, $\mathbb{U}_0(V_0)$ is finite. Suppose that $\mathbb{U}_{\le r}(V_0)$ is finite. Then
$$\mathbb{U}_{r+1}(V_0) = \{(r+1, A) \mid A \subseteq_{\mathrm{fin}} \mathbb{U}_{\le r}(V_0)\}$$
is finite, because the finite powerset of a finite set is finite. Therefore, by induction, every $\mathbb{U}_r(V_0)$ with $0 \le r \le R$ is finite.

Each $\mathcal{V}_r \subseteq \mathbb{U}_r(V_0)$ is finite by definition, and each $\mathcal{E}_s$ is finite by assumption. Hence

$$\mathcal{V}(\mathfrak{U})\dot{\cup}\mathcal{E}(\mathfrak{U})$$

is finite.

For each $e \in \mathcal{E}_s$, the incidence set

$$\partial_s(e) \in \mathcal{P}^*_{\mathrm{fin}}\big(\mathcal{V}_{<s}\dot{\cup}\mathcal{E}_{<s}\big)$$

is a nonempty finite set of lower-rank objects. Thus the incidence relation is well defined.

Finally, if $e' \in \partial_s(e) \cap \mathcal{E}_t$, then $t < s$. Therefore, along any edge-to-edge incidence chain

$$e_0 \to e_1 \to e_2 \to \cdots ,$$

the edge rank strictly decreases:

$$\rho(e_0) > \rho(e_1) > \rho(e_2) > \cdots .$$

Since the ranks lie in the finite set $\{1, \ldots, R+1\}$, such a chain cannot be infinite and cannot contain a cycle. Hence the edge-to-edge incidence relation is acyclic. □

Theorem 4.14.4. *Every finite n-SuperHyperGraph is representable as an Ultra-SuperHyperGraph.*

Proof. Let

$$SHG^{(n)} = (V, E, \partial)$$

be a finite n-SuperHyperGraph over V_0, where

$$V \subseteq \mathcal{P}^n(V_0), \qquad \partial : E \to \mathcal{P}^*(V).$$

Define maps

$$J_k : \mathcal{P}^k(V_0) \to \mathbb{U}_k(V_0)$$

recursively as follows:

$$J_0(x) := (0, x) \qquad (x \in V_0),$$

and

$$J_{k+1}(A) := \big(k+1, \{J_k(a) \mid a \in A\}\big) \qquad (A \subseteq \mathcal{P}^k(V_0)).$$

Now define an Ultra-SuperHyperGraph $\mathfrak{U}$ by

$$\mathcal{V}_n := J_n[V], \qquad \mathcal{V}_r := \emptyset \quad (r \neq n),$$

and introduce a tagged copy

$$\mathcal{E}_{n+1} := \{\widehat{e} \mid e \in E\}, \qquad \mathcal{E}_s := \emptyset \quad (s \neq n+1).$$

For each $\widehat{e} \in \mathcal{E}_{n+1}$, define

$$\partial_{n+1}(\widehat{e}) := \{J_n(v) \mid v \in \partial(e)\}.$$

Since $\partial(e) \neq \emptyset$, the set $\partial_{n+1}(\widehat{e})$ is nonempty. Moreover every element of $\partial_{n+1}(\widehat{e})$ has rank n, while $\widehat{e}$ has rank $n+1$. Hence the rank-decreasing condition is satisfied.

The incidence relation is preserved because

$$v \in \partial(e) \quad \Longleftrightarrow \quad J_n(v) \in \partial_{n+1}(\widehat{e}).$$

Thus the given n-SuperHyperGraph is recovered as the rank-homogeneous, vertex-only special case of an Ultra-SuperHyperGraph. □

Example 4.14.5 (Edge-to-edge and mixed-rank incidence)**.** Let

$$V_0 = \{a, b\}.$$

Put

$$u = (0, a), \qquad v = (0, b),$$

and

$$w = (1, \{u, v\}).$$

Set

$$\mathcal{V}_0 = \{u, v\}, \qquad \mathcal{V}_1 = \{w\}.$$

Let

$$\mathcal{E}_1 = \{e_1\}, \qquad \mathcal{E}_2 = \{e_2\}.$$

Define

$$\partial_1(e_1) = \{u, v\},$$

and

$$\partial_2(e_2) = \{e_1, w, u\}.$$

Then e_1 is a rank-1 ultra-superedge incident with two atomic vertices, while e_2 is a rank-2 ultra-superedge incident with the lower-rank edge e_1, the rank-1 supervertex w, and the rank-0 vertex u.

Chapter 5

Conclusions

This book has revisited several notions from graph theory and hypergraph theory and reformulated them within the framework of SuperHyperGraphs. In particular, it has introduced and discussed a variety of concepts related to SuperHyperGraphs, with the aim of presenting them in a mathematically coherent and structurally consistent form.

By extending familiar graph-theoretic and hypergraph-theoretic ideas to settings involving nested supervertices, superhyperedges, and higher-order incidence relations, this book contributes to the further development of SuperHyperGraph theory. The results and definitions presented here are mainly theoretical, but they may provide a useful foundation for future work on structural parameters, algorithms, computational experiments, and practical modeling applications.

Since many aspects of SuperHyperGraph theory remain open, further investigation is still needed. Possible directions include deeper studies of connectivity, cycles, domination-type parameters, width parameters, optimization problems, and applications to hierarchical networks, complex systems, data science, and higher-order relational modeling. It is hoped that the concepts discussed in this book will stimulate additional theoretical, computational, and applied research on SuperHyperGraphs and their extensions.

We also hope that further research will be conducted on the relationships with related structures such as HyperStructures [341–343], Hv-Structures [344–346], higher-order networks [108, 347, 348], HyperFuzzy Sets [349–351], SuperHyperFuzzy Sets [**?**, **?**, 352–354], SuperHyperStructures [102, 355], and SHv-Structures [356, 357].

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Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this book.

Code Availability

No code or software was developed for this study.

Use of Generative AI and AI-Assisted Tools

The authors used generative AI and AI-assisted tools only for limited editorial purposes, such as English grammar checking, stylistic improvement, and LaTeX error checking. These tools were not used to generate mathematical results, fabricate data, manipulate images, or perform any activity that would violate ethical or scholarly standards.

Disclaimer (Others)

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

Appendix (List of Tables)

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HYPERGRAPH AND SUPERHYPERGRAPH THEORY WITH APPLICATIONS (IX)

Revisiting SuperHyperGraph Concepts

Hypergraphs extend ordinary graphs by allowing an edge to join any nonempty subset of vertices, making it possible to represent higher-order relationships that pairwise edges cannot capture. Iterating the powerset construction over such structures gives rise to the SuperHyperGraph, a framework in which vertices and edges may be organized across multiple nested, set-theoretic layers.

This book revisits a broad collection of classical graph- and hypergraph-theoretic notions — including blocks, k-cores, girth, defensive alliances, zero-forcing, degree sequences, neural and tensor networks, and semantic hypernetworks — and reformulates each one within the SuperHyperGraph setting. The result is a mathematically consistent foundation intended to support further theoretical development and future applications of SuperHyperGraph theory to hierarchical, multi-level, and higher-order relational systems.

Takaaki Fujita is an independent researcher based in Tokyo, Japan, working on hypergraph and SuperHyperGraph theory. Florentin Smarandache is a professor of mathematics at the University of New Mexico, Gallup Campus, and the founder of neutrosophic theory and SuperHyperGraph theory.

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