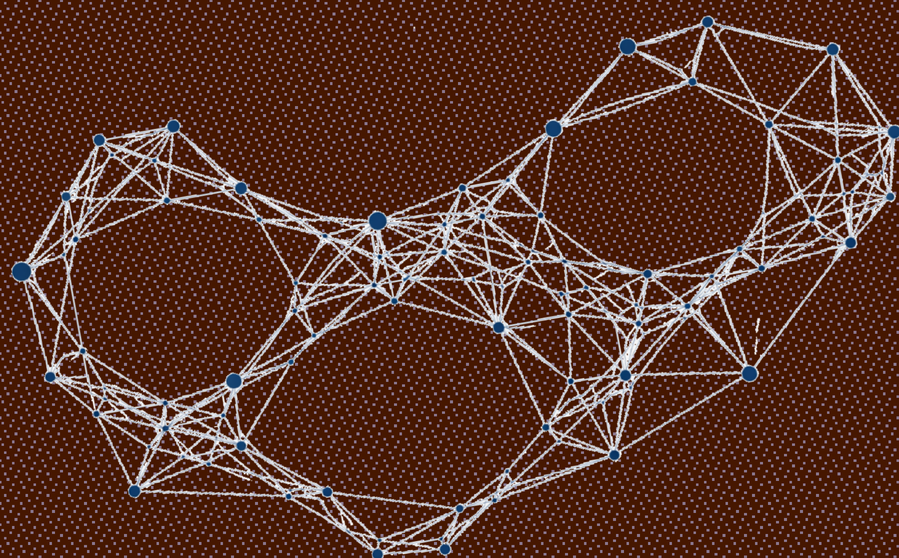
TAKAAKI FUJITA
FLORENTIN SMARANDACHE

HYPERGRAPH AND SUPERHYPERGRAPH THEORY WITH APPLICATIONS

I

FOUNDATIONS, DEFINITIONS, AND THEORETICAL MODELS



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HyperGraph and SuperHyperGraph Theory with Applications

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Abstract

Hypergraphs generalize this framework by allowing *hyperedges* that connect more than two vertices [1]. Superhypergraphs further enrich the model through iterated powerset constructions, capturing hierarchical and self-referential structures among hyperedges [2]. An (m, n)-SuperHyperGraph is a mathematical structure in which each vertex corresponds to an (m, n)-superhyperfunction defined on a base set, while the hyperedges group such functions together to represent higher-order relationships and contextual connections. Systematic research on SuperHyperGraphs is still relatively limited compared with the extensive literature on graphs and hypergraphs.

To help bridge this gap, this book presents a survey of fundamental and advanced concepts related to SuperHy-perGraphs. Our aim is twofold: (i) to increase the visibility and accessibility of SuperHyperGraph theory and thereby stimulate further research, and (ii) to deepen the mathematical understanding of their structures among researchers and practitioners who work with graph- and hypergraph-based models.

Keywords
SuperHyperGraph Theory, HyperGraph Theory, (m, n)-Superhypergraph Theory, Uncertain Graph Theory, Graph Applications

Chapter 1

Introduction

1.1 Graph, Hypergraph, and Superhypergraph

Network modeling often relies on graphs, where entities are represented by vertices and binary relations are represented by edges [3]. However, classical graphs can be inadequate for describing complex networks in which three or more entities interact simultaneously. Hypergraphs address this limitation by allowing each hyperedge to connect an arbitrary nonempty subset of vertices, thereby capturing higher-order interactions [4].

Despite their expressive power, hypergraphs may still be insufficient for representing layered, nested, and inherently hierarchical relationships that arise in many real-world systems. To bridge this gap, the notion of a *SuperHyperGraph* was introduced by F. Smarandache. A SuperHyperGraph employs iterative powerset-based constructions to encode nested connectivity patterns and multi-level relations [2,5], and has attracted substantial recent attention [6,7].

Graphs and hypergraphs provide intuitive visual metaphors for complex systems and support a wide range of applications in artificial intelligence, network science, data mining, informatics, chemistry, physics, and beyond [8–11]. By explicitly accommodating hierarchical and multi-level relationships, SuperHyperGraphs offer a robust framework for modeling and analyzing the intricate structures encountered in modern networked data (e.g.. [12–14, 14–16]).

Table 1.1 summarizes the key distinctions among graphs, hypergraphs, and superhypergraphs. Throughout this book, n is taken to be a natural number unless stated otherwise.

Table 1.1: Key distinctions among graph, hypergraph, and superhypergraph

Concept	**Notation**	**Edge Type**	**Extension Mechanism**
Graph [3]	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$	Standard edges connect exactly two vertices.
Hypergraph [17]	$H = (V, E)$	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Hyperedges may join any nonempty subset of vertices.
Superhypergraph [2]	$\mathrm{SHG}^{(n)} = (V_0, V, E)$	$V \subseteq \mathcal{P}^n(V_0),\ E \subseteq \mathcal{P}(V)$	Applies an n-fold powerset to capture nested structure.

Notation. $\mathcal{P}(X) = \{A \subseteq X\}$ and $\mathcal{P}^0(X) = X,\ \mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

Advantages of Using SuperHyperGraphs include several well-known benefits, such as:

- Naturally modeling hierarchical structures through iterated powersets.
- Representing multiway, multi-level relations within a unified framework.
- Containing graphs [3], multi-graphs [18,19], subset-vertex graphs [20–22], hypergraphs [17], supergraph [23], h-model [24], Quasi-SuperHyperGraphs [5], Powerset Graph [25], k-chain free sets [26], johnson [27], kneser graphs [28], and multi-hypergraphs [29] as special (flattened) instances.

- Supporting rich attribute systems (fuzzy, intuitionstic fuzzy, neutrosophic, plithogenic [2]) at every level.
- Providing a better fit for real hierarchical systems such as curricula or supply-chain networks.

1.2 Applications of Graph, HyperGraph, and SuperHyperGraph

Graph, HyperGraph, and SuperHyperGraph structures have been investigated in a wide range of application domains. Representative examples are summarized in Table 1.2 and 1.3. It should be noted that research on SuperHyperGraphs is still in its early stages, and most existing work remains theoretical at the time of writing. Consequently, future studies are expected to include more practical research supported by computational experiments, machine-learning techniques, and detailed case studies conducted by domain experts.

Table 1.2: Applied graph, hypergraph, and superhypergraph models

Application domain	Graph model	HyperGraph model	SuperHyperGraph model
Generic network	Graph	HyperGraph	SuperHyperGraph
Molecular structure	Molecular Graph [30]	Molecular HyperGraph [31]	Molecular SuperHyperGraph [32]
Competition / ecology	Competition Graph [33]	Competition HyperGraph [34]	Competition SuperHyperGraph [35]
Knowledge / semantics	Knowledge Graph [36]	Knowledge HyperGraph [37, 38]	Knowledge SuperHyperGraph [39]
Property-based system	Property Graph [40]	Property HyperGraph [41]	Property SuperHyperGraph [41]
Semi-structured network	SemiGraph [42]	SemiHyperGraph [43]	Semi-SuperHyperGraph [5]
Quantum system	Quantum Graph [44, 45]	Quantum HyperGraph [46]	Quantum SuperHyperGraph [47]
Semantic relations	Semantic Graph [48]	Semantic HyperGraph [49]	Semantic SuperHyperGraph [50]
Bonding	Bond Graph [51, 52]	Bond HyperGraph [53]	Bond SuperHyperGraph [53]
Chemical reactions	Chemical Graph [54]	Chemical HyperGraph [55, 56]	Chemical SuperHyperGraph [57, 58]
Legal citation network	Legal Citation Graph [59]	Legal Citation HyperGraph [60]	Legal Citation SuperHyperGraph [60]
Fuzzy uncertainty	Fuzzy Graph [61]	Fuzzy HyperGraph [62]	Fuzzy SuperHyperGraph [63]
Neutrosophic uncertainty	Neutrosophic Graph [64, 65]	Neutrosophic HyperGraph [66, 67]	Neutrosophic SuperHyperGraph [68]
Plithogenic uncertainty	Plithogenic Graph [69]	Plithogenic HyperGraph [70]	Plithogenic SuperHyperGraph [13, 71]
Soft-set based modeling	Soft Graph [72]	Soft HyperGraph [72]	Soft SuperHyperGraph [73]
Rough approximation	Rough Graph [74]	Rough HyperGraph [75]	Rough SuperHyperGraph [73]

Table 1.3: Additional applied graph, hypergraph, and superhypergraph models (Part 2)

Application domain	Graph model	HyperGraph model	SuperHyperGraph model
Higher-order containers	Graph Container	HyperGraph Container [76–78]	SuperHyperGraph Container [79]
Ecological food webs	Graph-based Food Web	Hypergraph-based Food Web [80]	SuperHyperGraph-Based Food Web [80]
Crystal and lattice structures	Crystal Graph [81–83]	Crystal HyperGraph [84]	Crystal SuperHyperGraph [85]
Neural architectures	Graph Neural Network [86–88]	Hypergraph Neural Network [4, 89, 90]	SuperHyperGraph Neural Network [91]
Social and behavioral modeling	Behavioral Graphs in Social Sciences [92, 93]	Behavioral HyperGraphs in Social Sciences [50]	Behavioral SuperHyperGraphs in Social Sciences [50]
Signal processing on networks	Graph Signal Processing [94, 95]	Hypergraph Signal Processing [96, 97]	SuperHyperGraph Signal Processing [53]
Brain connectivity	Brain Graphs [98, 99]	Brain HyperGraphs [100]	Brain SuperHyperGraphs [100]
River basin and watershed systems	River Network Graphs	River Network HyperGraphs [101]	River Network SuperHyperGraphs [101]
Transportation and logistics	Transportation Network Graphs	Transportation Network HyperGraphs [101]	Transportation Network SuperHyperGraphs [101]

1.3 Our Contributions

From the above discussion, it is clear that SuperHyperGraphs are highly important for modeling hierarchical and multiway structures. However, systematic research on SuperHyperGraphs remains relatively limited compared with the extensive literature on graphs and hypergraphs.

To help bridge this gap, this book provides a survey of fundamental and advanced concepts related to SuperHyperGraphs. Our aim is twofold: (i) to increase the visibility and accessibility of SuperHyperGraph theory and thereby stimulate further research, and (ii) to deepen the mathematical understanding of these structures among researchers and practitioners who work with graph- and hypergraph-based models.

This book primarily focuses on theoretical developments. We sincerely hope that further computational experiments and real-world case studies will be carried out by experts in the relevant domains.

Chapter 2

Preliminaries: Basic SuperHyperGraph Theory

We collect the basic terminology and notation used in what follows. Unless explicitly stated otherwise, all graphs considered are finite, undirected, and loopless; multiple edges are allowed only when this is specified.

2.1 Graphs and Hypergraphs

Graphs and hypergraphs are fundamental combinatorial models for discrete structures. A classical (undirected) graph can be viewed as a special case of a hypergraph in which every edge contains exactly two vertices [3]. In contrast, a classical hypergraph permits an edge to connect an arbitrary (finite) number of vertices, making it suitable for representing multiway relationships [1, 102, 103]. We briefly present the definitions along with the related concepts.

Definition 2.1.1 (Graph)**.** [3] A (simple) graph is an ordered pair $G = (V, E)$ where V is a nonempty finite set of *vertices* and

$$E \subseteq \big\{\{u, v\} \mid u, v \in V,\ u \neq v\big\}$$

is a set of unordered pairs of distinct vertices, called *edges*.

Definition 2.1.2 (Subgraph)**.** [3] Let $G = (V, E)$ be a graph. A graph $H = (V', E')$ is called a *subgraph* of G if

$$V' \subseteq V \quad \text{and} \quad E' \subseteq \big\{\{u, v\} \in E \mid u, v \in V'\big\}.$$

Definition 2.1.3 (Base set)**.** A *base (ground) set* is a fixed finite set S from which higher-level objects are generated:

$$S = \{\, x \mid x \text{ belongs to the chosen domain} \,\}.$$

All structures introduced below ultimately draw their elements from S.

Definition 2.1.4 (Powerset)**.** [104, 105] Given a set X, its powerset is

$$\mathcal{P}(X) := \{\, A \subseteq X \,\}.$$

We also use the *nonempty powerset*

$$\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}.$$

Definition 2.1.5 (Hypergraph [1, 17])**.** A *hypergraph* is a pair $H = (V(H), E(H))$ where

$$V(H) \neq \varnothing \quad \text{and} \quad E(H) \subseteq \mathcal{P}^*\big(V(H)\big).$$

Throughout this book both $V(H)$ and $E(H)$ are assumed to be finite. Elements of $V(H)$ are called *vertices*, and elements of $E(H)$ are called *hyperedges*.

We present below a concrete example of a HyperGraph.

Example 2.1.6 (A simple collaboration hypergraph)**.** Consider the finite set of researchers

$$V(H) := \{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dave}\}.$$

We define three research teams (hyperedges) by

$$e_1 := \{\text{Alice}, \text{Bob}, \text{Carol}\}, \qquad e_2 := \{\text{Bob}, \text{Dave}\}, \qquad e_3 := \{\text{Alice}, \text{Dave}\},$$

and set

$$E(H) := \{\, e_1, e_2, e_3 \,\}.$$

Each e_i is a nonempty subset of $V(H)$, so

$$E(H) \;\subseteq\; \mathcal{P}^*\big(V(H)\big),$$

and therefore

$$H := \big(V(H), E(H)\big)$$

is a hypergraph in the sense of the definition above.

Real-life interpretation.

- The vertices Alice, Bob, Carol, Dave represent individual researchers in a laboratory.
- The hyperedge e_1 represents a large joint project involving Alice, Bob, and Carol.
- The hyperedges e_2 and e_3 represent smaller projects: one between Bob and Dave, and one between Alice and Dave.

In this way, hyperedges encode multi-person collaborations, which cannot be captured by ordinary (pairwise) graph edges alone.

2.2 SuperHyperGraphs

A *SuperHyperGraph* carries this idea further by forming vertices and edges from iterated powersets of a base set; this viewpoint has appeared in several recent contexts [7, 106, 107]. Reported applications include, among others, molecular structure modeling, complex network analysis, and signal processing [108–112]. Throughout, the *level n* is a fixed nonnegative integer. We briefly present the definitions along with the related concepts.

Definition 2.2.1 (Iterated powerset)**.** [113–115] For $k \in \mathbb{N}_0$ define

$$\mathcal{P}^0(X) := X, \qquad \mathcal{P}^{k+1}(X) := \mathcal{P}\big(\mathcal{P}^k(X)\big).$$

For the nonempty version set

$$\big(\mathcal{P}^*\big)^0(X) := X, \qquad \big(\mathcal{P}^*\big)^{k+1}(X) := \mathcal{P}^*\big((\mathcal{P}^*)^k(X)\big).$$

Example 2.2.2 (Iterated powerset for a finite base set)**.** Let the base set be

$$X := \{a, b\}.$$

We compute the first few iterated powersets $\mathcal{P}^k(X)$ and their nonempty versions $\big(\mathcal{P}^*\big)^k(X)$.

Step 1: Level $k = 0$. By Definition 2.2.1,

$$\mathcal{P}^0(X) = X = \{a, b\}, \qquad \big(\mathcal{P}^*\big)^0(X) = X = \{a, b\}.$$

Step 2: Level $k = 1$ (ordinary powerset). The powerset of X is

$$\mathcal{P}^1(X) = \mathcal{P}(X) = \big\{\varnothing, \{a\}, \{b\}, \{a, b\}\big\}.$$

The nonempty powerset of X is

$$(\mathcal{P}^*)^1(X) = \mathcal{P}^*(X) = \{\{a\}, \{b\}, \{a, b\}\},$$

obtained by removing the empty set $\varnothing$.

Step 3: Level $k = 2$ (powerset of the powerset). Now $\mathcal{P}^1(X)$ has four elements, so its powerset has $2^4 = 16$ subsets:

$$\mathcal{P}^2(X) = \mathcal{P}(\mathcal{P}^1(X)) = \mathcal{P}(\{\varnothing, \{a\}, \{b\}, \{a, b\}\}).$$

Explicitly,

$$\mathcal{P}^2(X) = \{\varnothing,\ \{\varnothing\},\ \{\{a\}\},\ \{\{b\}\},\ \{\{a, b\}\},\ \{\varnothing, \{a\}\},\ \{\varnothing, \{b\}\},\ \{\varnothing, \{a, b\}\},$$
$$\{\{a\}, \{b\}\},\ \{\{a\}, \{a, b\}\},\ \{\{b\}, \{a, b\}\},\ \{\varnothing, \{a\}, \{b\}\},\ \{\varnothing, \{a\}, \{a, b\}\},$$
$$\{\varnothing, \{b\}, \{a, b\}\},\ \{\{a\}, \{b\}, \{a, b\}\},\ \{\varnothing, \{a\}, \{b\}, \{a, b\}\}\}.$$

The nonempty version at level 2 is obtained by removing the empty set:

$$(\mathcal{P}^*)^2(X) = \mathcal{P}^*((\mathcal{P}^*)^1(X)) = \mathcal{P}^*(\{\{a\}, \{b\}, \{a, b\}\}).$$

Since $(\mathcal{P}^*)^1(X)$ has three elements, its nonempty powerset has $2^3 - 1 = 7$ elements:

$$(\mathcal{P}^*)^2(X) = \{\{\{a\}\},\ \{\{b\}\},\ \{\{a, b\}\},\ \{\{a\}, \{b\}\},\ \{\{a\}, \{a, b\}\},$$
$$\{\{b\}, \{a, b\}\},\ \{\{a\}, \{b\}, \{a, b\}\}\}.$$

Interpretation.

- Level 0 ($\mathcal{P}^0(X)$): the original "atomic" elements a, b.
- Level 1 ($\mathcal{P}^1(X)$): all subsets of $\{a, b\}$, such as $\{a\}$ or $\{a, b\}$.
- Level 2 ($\mathcal{P}^2(X)$): sets whose elements are themselves subsets of $\{a, b\}$; for instance $\{\{a\}, \{a, b\}\}$ is one element of $\mathcal{P}^2(X)$.

Thus the iterated powerset construction builds higher and higher levels of "sets of sets", which is the basic combinatorial mechanism behind n-SuperHyperGraphs and related hierarchical structures.

We state below the definition of an n-SuperHyperGraph. Although several types of definitions exist for an n-SuperHyperGraph, we present one representative example below.

Definition 2.2.3 (n-SuperHyperGraph)**.** [2, 112, 116] Fix a finite base set V_0 and a level $n \in \mathbb{N}_0$. An *n-SuperHyperGraph over V_0* is a triple

$$\mathrm{SHG}^{(n)} = (V,\ E,\ \partial),$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of *n-supervertices*;
- $E \subseteq \mathcal{P}(V)$ is a finite set of *(super)edge identifiers*;
- $\partial : E \to \mathcal{P}^*(V)$ is an *incidence map* sending each edge to a nonempty finite subset of V.

For $e \in E$, the set $\partial(e) \subseteq V$ is called the *(super)edge incidence set*.

Table 2.1 re-presents the conceptual relationships among Graphs, HyperGraphs, and SuperHyperGraphs. SuperHyperGraphs are expected to provide a clear and expressive framework for representing hierarchical network structures that arise in real-world systems.

We present below concrete examples of SuperHyperGraphs.

Structure	Definition (Core Idea)	Relation to Other Structures
Graph	A graph $G = (V, E)$ where every edge connects exactly two vertices.	Special case of a hypergraph where all hyperedges have size 2.
HyperGraph	A hypergraph $H = (V, E)$ where each hyperedge $e \in E$ is any nonempty subset of V.	Generalizes graphs by allowing edges of arbitrary cardinality. Graphs embed as hypergraphs with all hyperedges of size 2.
SuperHyperGraph	An n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E, \partial)$ where $V \subseteq \mathcal{P}^n(V_0)$ are n-supervertices and $\partial : E \to \mathcal{P}^*(V)$ gives n-superedges.	Strict extension of hypergraphs. For $n = 0$ one recovers ordinary hypergraphs; for $n \geq 1$ vertices and edges are built from iterated powersets, enabling hierarchical and multi-level structures.

Table 2.1: Conceptual relationships among Graphs, HyperGraphs, and SuperHyperGraphs

Example 2.2.4 (University curriculum bundle network as a 2-SuperHyperGraph)**.** A university curriculum is a structured collection of courses organized into modules and programs to guide students' academic progression [117, 118]. We model a small part of a university curriculum in which courses are grouped into modules, and several modules are combined into degree-program patterns.

Step 1: Base set and iterated powersets. Let the finite base set of *atomic courses* be

$$V_0 := \{\text{Math101},\ \text{CS101},\ \text{AI201},\ \text{DS201}\}.$$

By Definition 2.2.1,

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0), \qquad \mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0)).$$

Elements of $\mathcal{P}^1(V_0)$ are *modules* (sets of courses), while elements of $\mathcal{P}^2(V_0)$ are *families of modules*.

For readability, define the following modules (elements of $\mathcal{P}^1(V_0)$):

$$M_{\text{core}} := \{\text{Math101}, \text{CS101}\}, \quad M_{\text{AI}} := \{\text{CS101}, \text{AI201}\}, \quad M_{\text{DS}} := \{\text{AI201}, \text{DS201}\}.$$

Step 2: 2-supervertices. We now form *program patterns* as subsets of $\mathcal{P}(V_0)$, hence elements of $\mathcal{P}^2(V_0)$:

$$v_{\text{AI-track}} := \{\, M_{\text{core}},\ M_{\text{AI}},\ M_{\text{DS}} \,\},$$

$$v_{\text{DS-track}} := \{\, M_{\text{core}},\ M_{\text{DS}} \,\},$$

$$v_{\text{foundation}} := \{\, M_{\text{core}} \,\}.$$

Each of $v_{\text{AI-track}}, v_{\text{DS-track}}, v_{\text{foundation}}$ is a subset of $\mathcal{P}(V_0)$, so

$$v_{\text{AI-track}},\ v_{\text{DS-track}},\ v_{\text{foundation}} \in \mathcal{P}(\mathcal{P}(V_0)) = \mathcal{P}^2(V_0).$$

We set the 2-supervertex set

$$V := \{v_{\text{AI-track}},\ v_{\text{DS-track}},\ v_{\text{foundation}}\} \subseteq \mathcal{P}^2(V_0).$$

Step 3: Superedges and incidence map. We introduce three superedges:

$$E := \{e_{\text{AI-only}},\ e_{\text{DS-only}},\ e_{\text{AI-DS-joint}}\},$$

and define the incidence map

$$\partial : E \longrightarrow \mathcal{P}^*(V)$$

by

$$\partial(e_{\text{AI-only}}) := \{\, v_{\text{foundation}},\ v_{\text{AI-track}} \,\},$$

$$\partial(e_{\text{DS-only}}) := \{\, v_{\text{foundation}},\ v_{\text{DS-track}} \,\},$$

$$\partial(e_{\text{AI-DS-joint}}) := \{\, v_{\text{AI-track}},\ v_{\text{DS-track}} \,\}.$$

Each $\partial(e)$ is a nonempty subset of V, so $\partial(e) \in \mathcal{P}^*(V)$. Thus

$$\mathrm{SHG}^{(2)} := \bigl(V,\ E,\ \partial\bigr)$$

is a valid 2-SuperHyperGraph in the sense of Definition 2.2.3.

Real-life interpretation.

- The base set V_0 collects individual courses.
- Each element of $\mathcal{P}(V_0)$ is a course module (e.g. "core mathematics and programming" or "artificial intelligence").
- Each 2-supervertex $v \in V$ is a *program pattern*: a finite family of modules that could be offered as a coherent track.
- Each superedge $e \in E$ bundles several such patterns that the university regards as mutually comparable or administratively linked (e.g. AI-only, DS-only, or joint AI-DS offering).

The double powerset level $n = 2$ is essential: vertices are not single modules, but families of modules, capturing the idea that program design operates on sets of module combinations rather than on individual courses alone.

Example 2.2.5 (Multi-hospital monitoring protocols as a 3-SuperHyperGraph)**.** Multi-hospital management has become increasingly necessary in recent years [119, 120]. We model how different hospitals organize multi-level vital-sign monitoring protocols.

Step 1: Base set and first-level combinations. Let the base set of *atomic vital signals* be

$$V_0 := \bigl\{\mathrm{HR},\ \mathrm{BP},\ \mathrm{SpO_2},\ \mathrm{Temp}\bigr\}.$$

Then

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0), \qquad \mathcal{P}^2(V_0) = \mathcal{P}\bigl(\mathcal{P}(V_0)\bigr), \qquad \mathcal{P}^3(V_0) = \mathcal{P}\bigl(\mathcal{P}^2(V_0)\bigr).$$

For clinical use, we define several *monitoring templates* (elements of $\mathcal{P}^1(V_0)$):

$$T_{\text{cardiac}} := \{\mathrm{HR}, \mathrm{BP}, \mathrm{SpO_2}\}, \qquad T_{\text{resp}} := \{\mathrm{SpO_2}, \mathrm{Temp}\},$$

$$T_{\text{basic}} := \{\mathrm{HR}, \mathrm{BP}\}.$$

Step 2: Second-level bundles (protocol families). We combine templates into *protocol families*, which are subsets of $\mathcal{P}(V_0)$ and hence elements of $\mathcal{P}^2(V_0)$:

$$Q_{\text{emerg}} := \{T_{\text{cardiac}}, T_{\text{resp}}\}, \qquad Q_{\text{routine}} := \{T_{\text{basic}}\}.$$

Since each $Q_\bullet$ is a subset of $\mathcal{P}(V_0)$, we have

$$Q_{\text{emerg}},\ Q_{\text{routine}} \ \in\ \mathcal{P}\bigl(\mathcal{P}(V_0)\bigr) \ =\ \mathcal{P}^2(V_0).$$

Step 3: Third-level vertices (hospital-specific portfolios). Each hospital chooses certain protocol families. Thus a *hospital portfolio* is a subset of $\mathcal{P}^2(V_0)$, i.e. an element of $\mathcal{P}^3(V_0)$.

Define

$$v_{\text{HospitalA}} := \{\, Q_{\text{emerg}}, Q_{\text{routine}} \,\}, \qquad v_{\text{HospitalB}} := \{\, Q_{\text{emerg}} \,\}.$$

Because

$$\{Q_{\text{emerg}}, Q_{\text{routine}}\} \subseteq \mathcal{P}^2(V_0), \qquad \{Q_{\text{emerg}}\} \subseteq \mathcal{P}^2(V_0),$$

we obtain

$$v_{\text{HospitalA}},\ v_{\text{HospitalB}} \ \in\ \mathcal{P}\bigl(\mathcal{P}^2(V_0)\bigr) \ =\ \mathcal{P}^3(V_0).$$

We set the 3-supervertex set

$$V := \{v_{\text{HospitalA}},\ v_{\text{HospitalB}}\} \ \subseteq\ \mathcal{P}^3(V_0).$$

Step 4: Superedges and incidence map. We describe national guidelines that relate these hospital portfolios. Introduce

$$E := \{e_{\text{national-minimum}},\ e_{\text{high-intensity}}\},$$

with incidence map $\partial : E \to \mathcal{P}^*(V)$ given by

$$\partial(e_{\text{national-minimum}}) := \{\ v_{\text{HospitalA}}, v_{\text{HospitalB}}\ \},$$

$$\partial(e_{\text{high-intensity}}) := \{\ v_{\text{HospitalA}}\ \}.$$

Again each $\partial(e)$ is a nonempty subset of V, so $\partial(e) \in \mathcal{P}^*(V)$. Hence

$$\text{SHG}^{(3)} := (V,\ E,\ \partial)$$

is a 3-SuperHyperGraph in the sense of Definition 2.2.3.

Real-life interpretation.

- Level 0 (V_0): individual vital signs (HR, BP, SpO_2, Temp).
- Level 1 ($\mathcal{P}^1(V_0)$): monitoring templates, each a concrete set of vital signs to measure together (e.g. cardiac or respiratory).
- Level 2 ($\mathcal{P}^2(V_0)$): protocol families combining templates for emergency or routine monitoring.
- Level 3 ($\mathcal{P}^3(V_0)$): hospital portfolios collecting protocol families actually implemented at each hospital.
- Superedges collect hospital portfolios subject to national or regional guidelines (e.g. "national minimum" vs. "high-intensity" monitoring requirements).

The triple powerset level $n = 3$ is crucial here: vertices are *portfolios of protocol families*, which captures the genuinely hierarchical nature of real-world clinical monitoring policies.

The theorem is stated as follows.

Theorem 2.2.6 (n-SuperHyperGraphs generalize hypergraphs)**.** *Every finite hypergraph can be realized as an n-SuperHyperGraph (in particular, as a 0-SuperHyperGraph). Consequently, the class of n-SuperHyperGraphs strictly generalizes the class of hypergraphs.*

Proof. Let $H = (V(H), E(H))$ be a finite hypergraph in the sense that

$$V(H) \neq \emptyset, \qquad E(H) \subseteq \mathcal{P}^*(V(H)) := \mathcal{P}(V(H)) \setminus \{\emptyset\},$$

so each $e \in E(H)$ is a nonempty subset of $V(H)$.

We construct a 0-SuperHyperGraph that reproduces H exactly.

Step 1: Base set and level. Take the base set

$$V_0 := V(H)$$

and the level

$$n := 0.$$

By Definition 2.2.1, we have

$$\mathcal{P}^0(V_0) = V_0.$$

Step 2: Supervertices. Set

$$V := V_0 = V(H).$$

Then, by $\mathcal{P}^0(V_0) = V_0$, we obtain

$$V \subseteq \mathcal{P}^0(V_0),$$

so V is an admissible set of 0-supervertices.

Step 3: Superedges and incidence map. Define the superedge set

$$E := E(H),$$

and the incidence map

$$\partial : E \longrightarrow \mathcal{P}^*(V)$$

by

$$\partial(e) := e \qquad \text{for all } e \in E.$$

Since $E(H) \subseteq \mathcal{P}^*(V(H))$ and $V = V(H)$, each $e \in E$ satisfies

$$\emptyset \neq e \subseteq V,$$

hence

$$\partial(e) = e \in \mathcal{P}^*(V) \qquad (\forall\, e \in E).$$

Therefore

$$\mathrm{SHG}^{(0)} := (V, E, \partial)$$

is a 0-SuperHyperGraph according to Definition 2.2.3.

Step 4: Identification with the original hypergraph. In the hypergraph H, the incidence relation is given by membership $v \in e \subseteq V(H)$. In the constructed 0-SuperHyperGraph $\mathrm{SHG}^{(0)}$, the incidence is given by $v \in \partial(e)$. But by construction

$$\partial(e) = e \quad \text{for all } e \in E,$$

so for all $v \in V$ and $e \in E$,

$$v \in e \quad \Longleftrightarrow \quad v \in \partial(e).$$

Hence the identity maps

$$V(H) \to V, \quad v \mapsto v, \qquad E(H) \to E, \quad e \mapsto e,$$

preserve both vertices, edges, and incidence. Thus H and $\mathrm{SHG}^{(0)}$ are isomorphic as incidence structures.

Consequently, every hypergraph is (up to isomorphism) a 0-SuperHyperGraph. Since n-SuperHyperGraphs are defined for all $n \in \mathbb{N}_0$ and allow vertices in $\mathcal{P}^n(V_0)$ for $n \geq 1$, they form a strictly larger class of objects, which contains all hypergraphs as the special case $n = 0$. □

2.3 Generalization Theorem for SuperHyperGraph

SuperHyperGraphs can generalize a wide variety of graphs and related mathematical structures. As an illustrative starting point, we explicitly examine how SuperHyperGraphs generalize several classical objects, namely abstract simplicial complexes, finite matroids, and balanced incomplete block designs (BIBDs).

Definition 2.3.1 (Abstract simplicial complex)**.** (cf. [121, 122]) Let V be a finite, nonempty set. A family $\Delta \subseteq \mathcal{P}(V)$ is called an *abstract simplicial complex* on V if

1. $\Delta \neq \emptyset$;
2. (downward closed) whenever $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.

The elements of Δ are called *simplices*; singletons $\{v\}$ with $v \in V$ are the *vertices* of the complex. We write $K = (V, \Delta)$ for an abstract simplicial complex.

Example 2.3.2 (Abstract simplicial complex of a filled triangle)**.** Let

$$V := \{1, 2, 3\}.$$

Define

$$\Delta := \big\{\varnothing,\ \{1\}, \{2\}, \{3\},\ \{1, 2\}, \{1, 3\}, \{2, 3\},\ \{1, 2, 3\}\big\}.$$

Then $\Delta \subseteq \mathcal{P}(V)$ is nonempty and downward closed: whenever $\sigma \in \Delta$ and $\tau \subseteq \sigma$, the face τ is also in Δ (for instance, $\{1, 2, 3\} \in \Delta$ implies that all of $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and singletons $\{1\}, \{2\}, \{3\}$ lie in Δ). Thus $K = (V, \Delta)$ is an abstract simplicial complex representing a filled triangle with vertices $1, 2, 3$.

Definition 2.3.3 (Finite matroid)**.** [123–125] Let E be a finite, nonempty set. A family $\mathcal{I} \subseteq \mathcal{P}(E)$ is called a system of *independent sets* on E if it satisfies:

1. $\varnothing \in \mathcal{I}$ (nonempty);
2. (hereditary) if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$;
3. (exchange axiom) if $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$.

A pair $M = (E, \mathcal{I})$ satisfying the above axioms is called a *finite matroid*.

Example 2.3.4 (Cycle matroid of a triangle graph)**.** Let G be the simple graph with vertex set

$$V(G) := \{a, b, c\}$$

and edge set

$$E := \{e_1, e_2, e_3\}$$

where

$$e_1 = ab, \quad e_2 = bc, \quad e_3 = ca.$$

Define

$$\mathcal{I} := \big\{I \subseteq E \ \big|\ I \text{ does not contain all three edges simultaneously}\big\}.$$

Explicitly,

$$\mathcal{I} = \big\{\varnothing,\ \{e_1\}, \{e_2\}, \{e_3\},\ \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\big\}.$$

Then $(E, \mathcal{I})$ satisfies the matroid axioms: $\varnothing \in \mathcal{I}$, it is hereditary under taking subsets, and the exchange axiom holds (any smaller independent set can be extended by an edge from a larger independent set while staying independent). Hence $M = (E, \mathcal{I})$ is a finite matroid, called the cycle matroid of the triangle graph G.

Definition 2.3.5 (Balanced incomplete block design (BIBD))**.** [126, 127] Let X be a finite set of *points* with $|X| = v$. A *balanced incomplete block design* with parameters (v, b, r, k, λ) is a pair

$$\mathcal{D} = (X, \mathcal{B}),$$

where $\mathcal{B}$ is a multiset of b *blocks* $B \subseteq X$ such that

1. (block size) each block has the same size k: $|B| = k$ for all $B \in \mathcal{B}$;
2. (replication) each point appears in exactly r blocks: for every $x \in X$,
$$\big|\{\, B \in \mathcal{B} \mid x \in B \,\}\big| = r;$$
3. (pair balance) every unordered pair of distinct points appears together in exactly λ blocks: for all $\{x, y\} \subseteq X, x \neq y$,
$$\big|\{\, B \in \mathcal{B} \mid \{x, y\} \subseteq B \,\}\big| = \lambda.$$

Example 2.3.6 (A $(3, 3, 2, 2, 1)$ balanced incomplete block design)**.** Let the point set be

$$X := \{1, 2, 3\}, \qquad v = |X| = 3.$$

Consider the multiset of blocks

$$\mathcal{B} := \big\{\{1, 2\},\ \{1, 3\},\ \{2, 3\}\big\},$$

so $b = |\mathcal{B}| = 3$. Each block has size $k = 2$.

Each point appears in exactly $r = 2$ blocks:

$$1 \text{ occurs in } \{1, 2\}, \{1, 3\}; \quad 2 \text{ occurs in } \{1, 2\}, \{2, 3\}; \quad 3 \text{ occurs in } \{1, 3\}, \{2, 3\}.$$

Every unordered pair of distinct points appears together in exactly $\lambda = 1$ block:

$$\{1, 2\} \text{ in } \{1, 2\}, \quad \{1, 3\} \text{ in } \{1, 3\}, \quad \{2, 3\} \text{ in } \{2, 3\}.$$

Thus

$$\mathcal{D} = (X, \mathcal{B})$$

is a balanced incomplete block design with parameters $(v, b, r, k, \lambda) = (3, 3, 2, 2, 1)$.

Theorem 2.3.7. *Every abstract simplicial complex, every finite matroid, and every balanced incomplete block design can be represented as a* 1*-SuperHyperGraph on a suitable base set. More precisely:*

1. *For each abstract simplicial complex $K = (V, \Delta)$ there exists a* 1*-SuperHyperGraph* $\mathrm{SHG}^{(1)}_{\mathrm{SC}} = (V_{\mathrm{SC}}, E_{\mathrm{SC}})$ *such that K is recovered from* $\mathrm{SHG}^{(1)}_{\mathrm{SC}}$.
2. *For each finite matroid $M = (E, \mathcal{I})$ there exists a* 1*-SuperHyperGraph* $\mathrm{SHG}^{(1)}_{\mathrm{M}} = (V_{\mathrm{M}}, E_{\mathrm{M}})$ *from which M is recovered.*
3. *For each BIBD $\mathcal{D} = (X, \mathcal{B})$ there exists a* 1*-SuperHyperGraph* $\mathrm{SHG}^{(1)}_{\mathrm{B}} = (V_{\mathrm{B}}, E_{\mathrm{B}})$ *from which $\mathcal{D}$ is recovered.*

Consequently, these three classes are special cases of 1*-SuperHyperGraphs, obtained by imposing additional axioms on the superedge family.*

Proof. We treat the three cases separately, always working with the same pattern: choose a base set V_0, construct a 1-SuperHyperGraph (V, E) with $V, E \subseteq \mathcal{P}(V_0)$, and then show that the original structure is uniquely recovered from (V, E).

(1) Abstract simplicial complexes. Let $K = (V, \Delta)$ be an abstract simplicial complex. Choose the base set

$$V_0 := V.$$

Define

$$V_{\mathrm{SC}} \ := \ \big\{ \{v\} \bigm| v \in V \big\} \ \subseteq \ \mathcal{P}(V_0),$$
$$E_{\mathrm{SC}} \ := \ \Delta \setminus \{\emptyset\} \ \subseteq \ \mathcal{P}(V_0).$$

Then both V_{SC} and E_{SC} are subsets of $\mathcal{P}(V_0)$, so

$$\mathrm{SHG}^{(1)}_{\mathrm{SC}} \ := \ (V_{\mathrm{SC}}, E_{\mathrm{SC}})$$

is a 1-SuperHyperGraph on V_0.

Conversely, from $(V_{\mathrm{SC}}, E_{\mathrm{SC}})$ we reconstruct K as follows. The vertex set is

$$V = \big\{v \in V_0 \bigm| \{v\} \in V_{\mathrm{SC}}\big\},$$

and the simplices form

$$\Delta := E_{\mathrm{SC}} \cup \{\emptyset\}.$$

Downward closedness of Δ comes exactly from the simplicial complex axioms; in the 1-SuperHyperGraph representation this appears as a closure property of E_{SC} inside $\mathcal{P}(V_0)$. Thus there is a one-to-one correspondence between abstract simplicial complexes on V and 1-SuperHyperGraphs of the above form on base $V_0 = V$, showing that 1-SuperHyperGraphs generalize simplicial complexes.

(2) Finite matroids. Let $M = (E, \mathcal{I})$ be a finite matroid. Take the base set

$$V_0 := E.$$

Define

$$V_{\mathrm{M}} := \{\{e\} \mid e \in E\} \subseteq \mathcal{P}(V_0),$$
$$E_{\mathrm{M}} := \mathcal{I} \setminus \{\emptyset\} \subseteq \mathcal{P}(V_0).$$

Again $V_{\mathrm{M}}, E_{\mathrm{M}} \subseteq \mathcal{P}(V_0)$, so

$$\mathrm{SHG}^{(1)}_{\mathrm{M}} := (V_{\mathrm{M}}, E_{\mathrm{M}})$$

is a 1-SuperHyperGraph on base V_0.

From $\mathrm{SHG}^{(1)}_{\mathrm{M}}$ we recover the matroid as follows. The ground set is

$$E = \{e \in V_0 \mid \{e\} \in V_{\mathrm{M}}\},$$

and the independent sets are

$$\mathcal{I} := E_{\mathrm{M}} \cup \{\emptyset\}.$$

The hereditary and exchange axioms of a matroid are now simply additional conditions on the superedge family E_{M}:

- hereditary: $I \in E_{\mathrm{M}}$, $J \subseteq I$ implies either $J \in E_{\mathrm{M}}$ or $J = \emptyset$;
- exchange: if $I, J \in E_{\mathrm{M}}$ and $|I| < |J|$, then there exists $e \in J \setminus I$ with $I \cup \{e\} \in E_{\mathrm{M}}$.

Thus every finite matroid can be seen as a 1-SuperHyperGraph satisfying specific incidence–regularity constraints on its superedges, and conversely any 1-SuperHyperGraph of this form determines a matroid. Hence finite matroids are special cases of 1-SuperHyperGraphs.

(3) Balanced incomplete block designs. Let $\mathcal{D} = (X, \mathcal{B})$ be a (v, b, r, k, λ) BIBD. Choose the base set

$$V_0 := X.$$

We represent the design as a 1-SuperHyperGraph by taking

$$V_{\mathrm{B}} := \{\{x\} \mid x \in X\} \subseteq \mathcal{P}(V_0),$$

$$E_{\mathrm{B}} := \{ B \subseteq X \mid B \in \mathcal{B} \} \subseteq \mathcal{P}(V_0).$$

Thus

$$\mathrm{SHG}^{(1)}_{\mathrm{B}} := (V_{\mathrm{B}}, E_{\mathrm{B}})$$

is a 1-SuperHyperGraph whose superedges are exactly the blocks of the design.

Conversely, given a 1-SuperHyperGraph $(V_{\mathrm{B}}, E_{\mathrm{B}})$ constructed in this way, we recover the design:

$$X = \{x \in V_0 \mid \{x\} \in V_{\mathrm{B}}\}, \qquad \mathcal{B} = E_{\mathrm{B}}$$

(viewing $\mathcal{B}$ as a multiset if some blocks repeat). The BIBD constraints (constant block size k, constant replication number r, and constant pair count λ) become numerical regularity conditions on the superedge family E_{B}:

- $|B| = k$ for all $B \in E_{\mathrm{B}}$;

- for each $x \in X$, the number of superedges containing x is r;
- for each unordered pair $\{x, y\} \subseteq X$ with $x \neq y$, the number of superedges containing $\{x, y\}$ is λ.

Hence BIBDs are 1-SuperHyperGraphs endowed with this specific incidence-regularity pattern.

In all three cases the construction is functorial and invertible at the level of underlying sets and incidence families: an abstract simplicial complex, a finite matroid, or a BIBD is completely encoded by a 1-SuperHyperGraph with appropriately constrained superedge family, and conversely such constrained 1-SuperHyperGraphs recover exactly these structures. Therefore abstract simplicial complexes, finite matroids, and balanced incomplete block designs are all generalized by the 1-SuperHyperGraph framework. □

From the above observations, we obtain Theorem 2.3.8 (cf. [128]).

Theorem 2.3.8 (Universality of SuperHyperGraphs). *Every one of the following structures can be faithfully represented as an n-SuperHyperGraph:*

- *Ordinary graphs;*
- *Hypergraphs [17];*
- *Multigraphs [18, 19];*
- *Supergraphs [23];*
- *Multihypergraphs [29];*
- *Quasi-SuperHyperGraphs [5];*
- *n-th power graphs [129];*
- *Subset-Vertex graphs [22];*
- *Subset-Vertex multigraphs [20];*
- *h-models [24];*
- *k-chain-free hypergraphs [26];*
- *Power set graphs [130];*
- *n-th power set graphs [128];*
- *Johnson graphs [131];*
- *Kneser graphs [132];*
- *Sets [105];*
- *Multisets [133];*
- *Iterated multisets [134];*
- *Powersets [130];*
- *n-th powersets [135];*
- *Abstract simplicial complexes [136, 137];*
- *Finite matroids [125];*
- *Balanced incomplete block designs (BIBDs) [127, 138].*

Chapter 3

Basic Definition for SuperHyperGraph

In the areas of graphs and hypergraphs, a wide variety of operations and graph structures have been defined and studied in terms of their properties. The same is true for SuperHyperGraphs. We present the core definitions and fundamental properties of SuperHyperGraphs.

3.1 Simple, Uniform, and Nonempty-tier SuperHyperGraph

A simple SuperHyperGraph forbids parallel superedges; the incidence map is injective, ensuring that each superedge has a unique incidence pattern. A k–uniform SuperHyperGraph is one in which every superedge is incident with exactly k supervertices. This extends the notion of uniformity from classical uniform hypergraphs to the multi-level setting of SuperHyperGraphs, modeling higher-order interactions of fixed arity [17, 139, 140]. A nonempty-tier SuperHyperGraph requires that every supervertex belongs to the iterated nonempty powersets at each hierarchical level, thereby excluding empty sets throughout the entire multi-tier construction.

Definition 3.1.1 (Simple n-SuperHyperGraph)**.** Let V_0 be a finite base set and let $n \in \mathbb{N}_0$. An n-SuperHyperGraph over V_0 is a triple

$$\mathrm{SHG}^{(n)} = (V, E, \partial),$$

where

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \text{ is a finite set of superedges},$$

and

$$\partial : E \longrightarrow \mathcal{P}^*(V)$$

is the incidence map that assigns to each superedge $e \in E$ a nonempty set $\partial(e) \subseteq V$ of n-supervertices.

The n-SuperHyperGraph $\mathrm{SHG}^{(n)}$ is called *simple* if it has no parallel superedges, that is, if the incidence map ∂ is injective:

$$\forall e_1, e_2 \in E: \quad e_1 \neq e_2 \implies \partial(e_1) \neq \partial(e_2).$$

Equivalently, in a simple n-SuperHyperGraph each superedge is uniquely determined by its incidence set.

Definition 3.1.2 (k-uniform n-SuperHyperGraph)**.** In the setting of Definition 3.1.1, fix an integer $k \in \mathbb{N}$ with $k \geq 1$. The n-SuperHyperGraph

$$\mathrm{SHG}^{(n)} = (V, E, \partial)$$

is called *k-uniform* if every superedge is incident with exactly k n-supervertices, i.e.,

$$\forall e \in E: \quad \left|\partial(e)\right| = k.$$

In particular, 1-uniform n-SuperHyperGraphs have superedges incident with a single n-supervertex, while 2-uniform n-SuperHyperGraphs may be viewed as superedge analogues of ordinary (hyper)edges between pairs of n-supervertices.

Example 3.1.3 (A 2-uniform 1-SuperHyperGraph: paired joint-degree programmes)**.** Fix the base set of *courses*

$$V_0 := \{\text{Math}, \text{Phys}, \text{CS}, \text{Econ}\}.$$

At level $n = 1$ we consider subsets of V_0, so

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0).$$

Define the following 1-supervertices, each representing a joint-degree programme built from a bundle of courses:

$$v_1 := \{\text{Math}, \text{Phys}\}, \quad v_2 := \{\text{Math}, \text{CS}\}, \quad v_3 := \{\text{CS}, \text{Econ}\}.$$

Set

$$V := \{v_1, v_2, v_3\} \subseteq \mathcal{P}^1(V_0).$$

We now describe how some pairs of programmes are jointly administered (for example, they share a common committee or accreditation process). Introduce two superedges

$$E := \{e_{12}, e_{23}\}$$

with incidence map

$$\partial : E \longrightarrow \mathcal{P}^*(V)$$

given by

$$\partial(e_{12}) := \{v_1, v_2\}, \qquad \partial(e_{23}) := \{v_2, v_3\}.$$

By construction,

$$\left|\partial(e_{12})\right| = \left|\partial(e_{23})\right| = 2,$$

so every superedge is incident with exactly two 1-supervertices. Hence

$$\text{SHG}^{(1)} := (V, E, \partial)$$

is a 2-uniform 1-SuperHyperGraph in the sense of Definition 3.1.2. Each superedge represents a pair of joint-degree programmes that are coordinated at the administrative level.

Example 3.1.4 (A 3-uniform 2-SuperHyperGraph: tri-regional emergency clusters)**.** Let the base set of *response resources* be

$$V_0 := \{r_1, r_2, r_3, r_4, r_5, r_6\},$$

where each r_i is, say, an ambulance or fire unit.

At level 1 we group resources into *local response cells*. For instance,

$$C_1 := \{r_1, r_2\}, \quad C_2 := \{r_3, r_4\}, \quad C_3 := \{r_5, r_6\} \in \mathcal{P}^1(V_0) = \mathcal{P}(V_0).$$

At level $n = 2$ we group these cells into *regional clusters*, so vertices live in

$$\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

Define three 2-supervertices

$$v_A := \{C_1, C_2\}, \quad v_B := \{C_2, C_3\}, \quad v_C := \{C_1, C_3\},$$

and set

$$V := \{v_A, v_B, v_C\} \subseteq \mathcal{P}^2(V_0).$$

Suppose a national emergency protocol specifies that certain training exercises always involve *exactly three* regional clusters working together. We model one such exercise by a single superedge

$$E := \{e_{ABC}\},$$

with incidence map

$$\partial : E \longrightarrow \mathcal{P}^*(V), \qquad \partial(e_{ABC}) := \{v_A, v_B, v_C\}.$$

Then

$$\left|\partial(e_{ABC})\right| = 3,$$

so every superedge (here, the unique one) is incident with exactly three 2-supervertices. Therefore

$$\mathrm{SHG}^{(2)} := (V, E, \partial)$$

is a 3-uniform 2-SuperHyperGraph.

In this model, each 2-supervertex represents a nested group of resources (local cells within a regional cluster), and the single superedge encodes a tri-regional emergency exercise involving exactly three such clusters.

Definition 3.1.5 (Nonempty-tier n-SuperHyperGraph)**.** Let V_0 be a nonempty base set and define the *nonempty iterated powersets* recursively by

$$(\mathcal{P}^*)^0(V_0) \ := \ V_0, \qquad (\mathcal{P}^*)^{k+1}(V_0) \ := \ \mathcal{P}^*\big((\mathcal{P}^*)^k(V_0)\big) \quad (k \in \mathbb{N}_0),$$

where $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}$ denotes the nonempty powerset of X.

An n-SuperHyperGraph

$$\mathrm{SHG}^{(n)} = (V, E, \partial)$$

over V_0 is called *nonempty-tier* if its vertex set is contained in the nonempty iterated powerset:

$$V \ \subseteq \ (\mathcal{P}^*)^n(V_0).$$

In this case, every n-supervertex is built recursively from nonempty sets at each tier, so that no empty set appears in the construction from V_0 up to level n. Together with the condition $\partial(e) \in \mathcal{P}^*(V)$ for all $e \in E$, this ensures that the entire hierarchy (base elements, intermediate tiers, vertices, and superedges) is free of empty components.

3.2 Matrix for SuperHyperGraph

A matrix is a rectangular array of numbers or symbols representing linear transformations, relationships, or structured data in mathematical applications [141, 142]. Derived notions such as fuzzy matrices [143, 144], bimatrices [145–147], and neutrosophic matrices [148, 149] are also well known.

A SuperHyperGraph matrix encodes the incidence between all tiered supervertices and superedges in the form of a binary table, providing a natural generalization of the classical incidence matrices used for graphs and hypergraphs. This construction extends the ideas behind the matrix for graphs [150, 151] and the matrix for hypergraphs [17, 152] to the multi-level setting of SuperHyperGraphs. The definition is given as follows.

Definition 3.2.1 (Hypergraph matrix (incidence matrix))**.** [17] Let $H = (V, E)$ be a finite hypergraph, where

$$V = \{v_1, \ldots, v_n\}, \qquad E = \{e_1, \ldots, e_m\}.$$

The *hypergraph matrix* (also called the *incidence matrix*) of H is the $n \times m$ matrix

$$M(H) = (a_{ij})_{1 \le i \le n,\ 1 \le j \le m}$$

with rows indexed by vertices and columns indexed by hyperedges, whose entries are defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \in e_j, \\ 0, & \text{if } v_i \notin e_j. \end{cases}$$

Equivalently, the j-th column of $M(H)$ is the characteristic column vector of the hyperedge $e_j \subseteq V$.

Definition 3.2.2 (Superhypergraph matrix)**.** Let $\mathcal{H}^{(n)} = (V_0, V_1, \ldots, V_n; E, \partial)$ be an n-superhypergraph, where

- V_0 is a nonempty base set of vertices,
- for each tier $i \in \{1, \dots, n\}$, the i-th tier vertex set V_i is a family of nonempty iterated subsets, for example $V_i \subseteq (\mathcal{P}^*)^i(V_0)$,
- $V := V_0 \cup V_1 \cup \cdots \cup V_n$ is the total vertex set, and E is a nonempty family of superedges,
- $\partial : E \to \mathcal{P}^*(V)$ is the boundary map that assigns to each superedge $e \in E$ the nonempty set of (possibly higher-tier) vertices incident with e.

Assume

$$V = \{w_1, \dots, w_N\}, \qquad E = \{e_1, \dots, e_M\}.$$

The *superhypergraph matrix* of $\mathcal{H}^{(n)}$ is the $N \times M$ matrix

$$M\big(\mathcal{H}^{(n)}\big) = (b_{pq})_{1 \le p \le N,\ 1 \le q \le M},$$

with rows indexed by all vertices $w_p \in V$ (from all tiers) and columns indexed by superedges $e_q \in E$, whose entries are defined by

$$b_{pq} = \begin{cases} 1, & \text{if } w_p \in \partial(e_q), \\ 0, & \text{if } w_p \notin \partial(e_q). \end{cases}$$

Equivalently, the q-th column of $M\big(\mathcal{H}^{(n)}\big)$ is the characteristic column vector of the incident vertex set $\partial(e_q) \subseteq V$. If the rows of $M\big(\mathcal{H}^{(n)}\big)$ are ordered so that

$$V_0, V_1, \dots, V_n$$

form consecutive blocks, then the matrix naturally decomposes into block rows corresponding to the different tiers:

$$M\big(\mathcal{H}^{(n)}\big) = \begin{pmatrix} M_0 \\ M_1 \\ \vdots \\ M_n \end{pmatrix},$$

where M_i has one row for each vertex in V_i and the same M columns indexed by the superedges in E. In the special case $n = 0$ and $V_0 = V$, the superhypergraph matrix $M\big(\mathcal{H}^{(0)}\big)$ reduces to the usual hypergraph matrix $M(H)$.

Example 3.2.3 (One-tier superhypergraph matrix for a small project structure)**.** Consider a 1-superhypergraph modelling two developers and one joint team.

Base vertices:

$$V_0 := \{\mathrm{Dev}_1, \mathrm{Dev}_2\}.$$

First-tier vertices (team-level group):

$$V_1 := \big\{T\big\}, \qquad T := \{\mathrm{Dev}_1, \mathrm{Dev}_2\}.$$

The total vertex set is

$$V := V_0 \cup V_1 = \{\mathrm{Dev}_1, \mathrm{Dev}_2, T\}.$$

We define two superedges:

$$E := \{e_1, e_2\},$$

with boundary map

$$\partial(e_1) := \{\mathrm{Dev}_1, T\}, \qquad \partial(e_2) := \{\mathrm{Dev}_2, T\}.$$

Here e_1 represents a task that involves developer 1 and the team as a whole, while e_2 involves developer 2 and the team.

Ordering the vertices as

$$w_1 = \mathrm{Dev}_1, \quad w_2 = \mathrm{Dev}_2, \quad w_3 = T,$$

and the superedges as e_1, e_2, the superhypergraph matrix is

$$M(\mathcal{H}^{(1)}) = (b_{pq}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$

where, for example, the entry $b_{31} = 1$ encodes that the team vertex T is incident with e_1, and $b_{12} = 0$ encodes that Dev_1 is not incident with e_2.

Example 3.2.4 (Two-tier superhypergraph matrix for a nested team hierarchy)**.** Consider a 2-superhypergraph describing employees, subteams, and a department.

Base vertices (employees):

$$V_0 := \{p, q, r\}.$$

First-tier vertices (subteams):

$$P := \{p, q\}, \qquad Q := \{q, r\}, \qquad V_1 := \{P, Q\}.$$

Second-tier vertex (department grouping subteams P and Q):

$$S := \{P, Q\}, \qquad V_2 := \{S\}.$$

Thus the total vertex set is

$$V := V_0 \cup V_1 \cup V_2 = \{p, q, r, P, Q, S\}.$$

We introduce three superedges that connect different tiers:

$$E := \{e_1, e_2, e_3\},$$

with boundary map

$$\partial(e_1) := \{p, P, S\}, \qquad \partial(e_2) := \{q, P, Q, S\}, \qquad \partial(e_3) := \{r, Q\}.$$

For instance, e_1 encodes an initiative involving employee p, subteam P, and the department S; e_2 involves q and both subteams plus the department; e_3 is specific to r and subteam Q.

Order the vertices as

$$w_1 = p,\ w_2 = q,\ w_3 = r,\ w_4 = P,\ w_5 = Q,\ w_6 = S,$$

and keep the superedge order e_1, e_2, e_3. The superhypergraph matrix is then

$$M(\mathcal{H}^{(2)}) = (b_{pq}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Each row block corresponds to a tier: rows 1–3 for employees V_0, rows 4–5 for subteams V_1, and row 6 for the department V_2. For example, $b_{46} = 0$ does not appear since there is no sixth column; instead, $b_{41} = 1$ records that subteam P is incident with e_1, and $b_{63} = 0$ shows that the department S is not incident with e_3.

The theorem is written as follows.

Theorem 3.2.5 (Superhypergraph matrix generalizes the hypergraph matrix)**.** *Let $H = (V(H), E(H))$ be a finite hypergraph with*

$$V(H) = \{v_1, \ldots, v_n\}, \qquad E(H) = \{e_1, \ldots, e_m\} \subseteq \mathcal{P}^*(V(H)).$$

Let $M(H) = (a_{ij})_{1\le i\le n,\ 1\le j\le m}$ be its hypergraph (incidence) matrix, defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \in e_j, \\ 0, & \text{if } v_i \notin e_j. \end{cases}$$

Consider the 0*-SuperHyperGraph*

$$\mathrm{SHG}^{(0)} := (V, E, \partial)$$

defined by

$$V := V(H), \qquad E := E(H), \qquad \partial(e) := e \quad (\forall e \in E).$$

Let $M(\mathrm{SHG}^{(0)}) = (b_{ij})_{1\le i\le n,\ 1\le j\le m}$ *be its superhypergraph matrix, defined by*

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \in \partial(e_j), \\ 0, & \text{if } v_i \notin \partial(e_j). \end{cases}$$

Then

$$M(\mathrm{SHG}^{(0)}) = M(H).$$

In particular, the notion of a superhypergraph matrix strictly generalizes the usual hypergraph matrix, obtained as the special case $n = 0$*.*

Proof. By construction we have, for every j,

$$\partial(e_j) = e_j \subseteq V = V(H).$$

Hence, for all indices i, j,

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \in \partial(e_j), \\ 0, & \text{if } v_i \notin \partial(e_j), \end{cases}$$

becomes

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \in e_j, \\ 0, & \text{if } v_i \notin e_j. \end{cases}$$

Therefore, by comparing with the definition of a_{ij},

$$b_{ij} = a_{ij} \qquad (\forall\, i, j),$$

so $M(\mathrm{SHG}^{(0)}) = M(H)$.

Since for $n \ge 1$ a superhypergraph allows vertices in higher iterated powersets and incidence with higher-tier objects, its matrix extends the usual hypergraph matrix beyond the case $n = 0$. Hence the superhypergraph matrix generalizes the hypergraph matrix. □

3.3 SuperHyperGraph Products

Graph products combine two graphs into a new one, with vertices paired and edges defined via specific product rules systematically [153–155]. Related concepts such as Directed Graph Products [156], Fuzzy Graph Products [157–159], and Neutrosophic Graph Products [160, 161] are also well established. A hypergraph product combines two hypergraphs into a new one, pairing vertices and constructing hyperedges according to a specified rule [162, 163]. A SuperHypergraph product forms an n-SuperHyperGraph from two factors, pairing supervertices and aggregating superedges through iterated powerset-based construction and connectivity. We recall that a (finite) hypergraph is a pair $H = (V(H), E(H))$ with $V(H) \neq \varnothing$ and $E(H) \subseteq \mathcal{P}^*(V(H))$.

Definition 3.3.1 (Hypergraph product)**.** [162, 163] Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. A *hypergraph product* is a binary operation $\boxtimes$ on hypergraphs such that

$$H_1 \boxtimes H_2 = (V_1 \times V_2,\ E(H_1 \boxtimes H_2)),$$

where the edge set $E(H_1 \boxtimes H_2)$ is specified by a rule that generalizes a chosen graph product. Typical examples are the Cartesian, direct, lexicographic, and square products defined below.

Definition 3.3.2 (Cartesian product of hypergraphs)**.** [163] Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. The *Cartesian product* $H = H_1 \Box H_2$ is the hypergraph with

$$V(H) = V_1 \times V_2,$$

and edge set

$$E(H) = \{\{x\} \times f \mid x \in V_1,\ f \in E_2\} \ \cup\ \{e \times \{y\} \mid e \in E_1,\ y \in V_2\}.$$

Each hyperedge of $H_1 \Box H_2$ is thus either "horizontal" ($\{x\} \times f$) or "vertical" ($e \times \{y\}$).

Definition 3.3.3 (Minimal rank–preserving direct product)**.** (cf. [164]) Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. For $e_i \in E_i$ put

$$r^-_{e_1,e_2} := \min\{|e_1|, |e_2|\}.$$

The *minimal rank–preserving direct product* $H = H_1 \dashv\!\!\ltimes H_2$ is the hypergraph with vertex set $V(H) = V_1 \times V_2$ and edge set

$$E(H) := \Big\{ e \subseteq V_1 \times V_2 \ \Big|\ \exists\, e_1 \in E_1,\ e_2 \in E_2 :\ |e| = r^-_{e_1,e_2},\ p_i(e) \subseteq e_i,\ |p_i(e)| = r^-_{e_1,e_2}\ (i = 1, 2) \Big\},$$

where p_1, p_2 denote the coordinate projections $V_1 \times V_2 \to V_1, V_2$. Equivalently, $e = \{(x_1, y_1), \dots, (x_r, y_r)\}$ is an edge of H if $r = r^-_{e_1,e_2}$ for some $e_1 \in E_1, e_2 \in E_2$ and

$$\{x_1, \dots, x_r\} \in \{e_1, \text{subset of } e_1\}, \quad \{y_1, \dots, y_r\} \in \{e_2, \text{subset of } e_2\},$$

with at least one of these sets equal to the corresponding e_i.

Definition 3.3.4 (Lexicographic product of hypergraphs)**.** (cf. [165,166]) Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. The *lexicographic product* $H = H_1 \circ H_2$ is the hypergraph with vertex set $V(H) = V_1 \times V_2$ and edge set

$$E(H) = \{ e \subseteq V(H) \mid p_1(e) \in E_1,\ |p_1(e)| = |e| \} \ \cup\ \{ \{x\} \times e_2 \mid x \in V_1,\ e_2 \in E_2 \},$$

where $p_1 : V_1 \times V_2 \to V_1$ is the projection onto the first coordinate. Thus an edge is either a "lift" of an edge of H_1, with all first coordinates distinct, or a copy of an edge of H_2 anchored at a fixed vertex $x \in V_1$.

Definition 3.3.5 (Square product of hypergraphs)**.** (cf. [162, 167]) Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be hypergraphs. The *square product* $H = H_1 \boxplus H_2$ is the hypergraph with vertex set $V(H) = V_1 \times V_2$ and edge set

$$E(H) = \{ e_1 \times e_2 \mid e_1 \in E_1,\ e_2 \in E_2 \}.$$

In particular, each edge of H is a full Cartesian product of an edge of H_1 and an edge of H_2.

Definition 3.3.6 (Product of n-SuperHyperGraphs induced by a hypergraph product)**.** Let $\boxtimes$ be a fixed hypergraph product as above, i.e. for hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ we have

$$H_1 \boxtimes H_2 = (V_1 \times V_2,\ E_1 \boxtimes E_2).$$

Let $\mathrm{SHG}^{(n)}_1 = (V_1, E_1)$ and $\mathrm{SHG}^{(n)}_2 = (V_2, E_2)$ be n-SuperHyperGraphs based on finite sets $V^{(1)}_0$ and $V^{(2)}_0$, respectively, so that

$$V_i,\ E_i \ \subseteq\ \mathcal{P}_n(V^{(i)}_0) \qquad (i = 1, 2).$$

Form the disjoint union

$$V_0 := V^{(1)}_0 \sqcup V^{(2)}_0,$$

and let

$$\iota_i : V^{(i)}_0 \hookrightarrow V_0 \qquad (i = 1, 2)$$

be the natural injections. Extend these injections levelwise to maps

$$\iota^{(k)}_i : \mathcal{P}_k(V^{(i)}_0) \longrightarrow \mathcal{P}_k(V_0) \quad (k \ge 0)$$

by recursion:

$$\iota^{(0)}_i := \iota_i, \qquad \iota^{(k+1)}_i(A) := \{ \iota^{(k)}_i(a) \mid a \in A \} \quad \text{for } A \in \mathcal{P}_{k+1}(V^{(i)}_0).$$

Define

$$\beta : V_1 \times V_2 \longrightarrow \mathcal{P}_n(V_0), \qquad \beta(v_1, v_2) := \iota_1^{(n)}(v_1) \cup \iota_2^{(n)}(v_2).$$

Let $H := H_1 \boxtimes H_2 = (V_1 \times V_2,\ E_1 \boxtimes E_2)$ be the hypergraph product of the underlying hypergraphs. We define the *SuperHyperGraph product*

$$\mathrm{SHG}_1^{(n)} \boxtimes \mathrm{SHG}_2^{(n)} := (V', E')$$

by

$$V' := \beta(V_1 \times V_2) \subseteq \mathcal{P}_n(V_0),$$

$$E' := \{ \beta[e] \mid e \in E_1 \boxtimes E_2 \} \subseteq \mathcal{P}_n(V_0),$$

where $\beta[e] := \{\beta(v) \mid v \in e\}$ is the pointwise image of e under β.

Then (V', E') is an n-SuperHyperGraph on the unified base set $V_0^{(1)} \sqcup V_0^{(2)}$; we call it the $\boxtimes$*-product of n-SuperHyperGraphs*.

Remark 3.3.7. Choosing $\boxtimes$ to be, for example, the Cartesian product $\square$, the minimal rank–preserving direct product, the lexicographic product, or the square product yields the corresponding *Cartesian*, *direct*, *lexicographic*, or *square* SuperHyperGraph products. When $n = 0$ (so that $\mathcal{P}_0(V_0) = V_0$), the above construction reduces to the usual hypergraph product on H_1 and H_2.

For reference, Table 3.1 presents a comparison of graph, hypergraph, and SuperHyperGraph products.

Table 3.1: Comparison of graph, hypergraph, and SuperHyperGraph products

Aspect	Graph products	HyperGraph products	SuperHyperGraph products
Input objects	$G_i = (V_i, E_i)$	$H_i = (V_i, E_i)$	$\mathrm{SHG}_i^{(n)} = (V_i, E_i)$ on bases $V_0^{(i)}$
Vertex set of the product	$V_1 \times V_2$	$V_1 \times V_2$	$\beta(V_1 \times V_2) \subseteq \mathcal{P}_n(V_0^{(1)} \sqcup V_0^{(2)})$
Edge / hyperedge / superedge type	Edges are 2-element subsets of $V_1 \times V_2$.	Hyperedges are arbitrary nonempty subsets of $V_1 \times V_2$.	Superedges are subsets of $\beta(V_1 \times V_2)$ (sets of n-supervertices).
Product rule	Edges determined by adjacencies in G_1, G_2 (Cartesian, direct, lexicographic, etc.).	Hyperedges built from hyperedges of H_1, H_2 by a chosen product rule (Cartesian, direct, lexicographic, square, etc.).	Superedges are images $\beta[e]$ of hyperedges $e \in E(H_1 \boxtimes H_2)$ for a fixed hypergraph product $H_1 \boxtimes H_2$.
Hierarchy level	Single-level (binary relations on V).	Single-level higher-order relations on V.	Multi-level relations on iterated powersets $\mathcal{P}_n(V_0)$; graphs and hypergraphs appear as the special case $n = 0$.

Several concrete examples are provided below.

Example 3.3.8 (Cartesian product of two small 1-SuperHyperGraphs)**.** We illustrate the construction for $n = 1$ using the Cartesian product of hypergraphs.

First component. Let the first base set be

$$V_0^{(1)} := \{a, b\}.$$

At level $n = 1$ the 1-supervertices are subsets of $V_0^{(1)}$. Set

$$V_1 := \{A_1, A_2\} \quad \text{with} \quad A_1 := \{a\},\ A_2 := \{a, b\}.$$

Define a single hyperedge

$$e_1 := \{A_1, A_2\} \subseteq V_1, \qquad E_1 := \{e_1\}.$$

Then

$$\mathrm{SHG}_1^{(1)} := (V_1, E_1)$$

is a 1-SuperHyperGraph (its underlying hypergraph has vertex set V_1 and edge set E_1).

Second component. Let the second base set be

$$V_0^{(2)} := \{x, y\},$$

and define 1-supervertices

$$V_2 := \{B_1, B_2\} \quad \text{with} \quad B_1 := \{x\},\ B_2 := \{x, y\}.$$

Again take a single hyperedge

$$f_1 := \{B_1, B_2\} \subseteq V_2, \qquad E_2 := \{f_1\},$$

so

$$\mathrm{SHG}_2^{(1)} := (V_2, E_2)$$

is a second 1-SuperHyperGraph.

Underlying Cartesian product of hypergraphs. Consider the underlying hypergraphs

$$H_1 := (V_1, E_1), \qquad H_2 := (V_2, E_2).$$

Their Cartesian product $H := H_1 \square H_2$ has vertex set

$$V(H) = V_1 \times V_2 = \{(A_1, B_1), (A_1, B_2), (A_2, B_1), (A_2, B_2)\},$$

and edge set

$$E(H) = \{\{A_i\} \times f_1 \mid i = 1, 2\} \cup \{e_1 \times \{B_j\} \mid j = 1, 2\}.$$

Explicitly,

$$\begin{aligned} e_{A_1}^{\mathrm{hor}} &= \{(A_1, B_1), (A_1, B_2)\}, \\ e_{A_2}^{\mathrm{hor}} &= \{(A_2, B_1), (A_2, B_2)\}, \\ e_{B_1}^{\mathrm{ver}} &= \{(A_1, B_1), (A_2, B_1)\}, \\ e_{B_2}^{\mathrm{ver}} &= \{(A_1, B_2), (A_2, B_2)\}, \end{aligned}$$

so $E(H) = \{e_{A_1}^{\mathrm{hor}}, e_{A_2}^{\mathrm{hor}}, e_{B_1}^{\mathrm{ver}}, e_{B_2}^{\mathrm{ver}}\}$.

Embedding into a product 1-SuperHyperGraph. Form the disjoint union of base sets

$$V_0 := V_0^{(1)} \sqcup V_0^{(2)} = \{a, b, x, y\}.$$

Since $n = 1$, each 1-supervertex is just a subset of the corresponding base set. We define the embedding

$$\beta : V_1 \times V_2 \longrightarrow \mathcal{P}(V_0)$$

by taking unions of the underlying subsets:

$$\beta(v_1, v_2) := v_1 \cup v_2 \quad \text{(now viewed as a subset of } V_0\text{)}.$$

Thus

$$\begin{aligned} \beta(A_1, B_1) &= \{a\} \cup \{x\} = \{a, x\}, \\ \beta(A_1, B_2) &= \{a\} \cup \{x, y\} = \{a, x, y\}, \\ \beta(A_2, B_1) &= \{a, b\} \cup \{x\} = \{a, b, x\}, \\ \beta(A_2, B_2) &= \{a, b\} \cup \{x, y\} = \{a, b, x, y\}. \end{aligned}$$

We take as the vertex set of the product 1-SuperHyperGraph

$$V' := \beta(V_1 \times V_2) = \{\{a, x\}, \{a, x, y\}, \{a, b, x\}, \{a, b, x, y\}\} \subseteq \mathcal{P}_1(V_0).$$

For each hyperedge $e \in E(H)$ we set

$$\beta[e] := \{\beta(v) \mid v \in e\} \subseteq V',$$

and define

$$E' := \{\beta[e] \mid e \in E(H)\}.$$

Concretely,

$$\beta[e^{\text{hor}}_{A_1}] = \big\{\{a, x\}, \{a, x, y\}\big\},$$
$$\beta[e^{\text{hor}}_{A_2}] = \big\{\{a, b, x\}, \{a, b, x, y\}\big\},$$
$$\beta[e^{\text{ver}}_{B_1}] = \big\{\{a, x\}, \{a, b, x\}\big\},$$
$$\beta[e^{\text{ver}}_{B_2}] = \big\{\{a, x, y\}, \{a, b, x, y\}\big\}.$$

Therefore

$$\mathrm{SHG}^{(1)}_1 \square \mathrm{SHG}^{(1)}_2 := (V', E')$$

is a 1-SuperHyperGraph whose vertices are 1-supervertices in $\mathcal{P}_1(V_0)$ and whose hyperedges are induced from the Cartesian product of the underlying hypergraphs.

Example 3.3.9 (Square product of two small 1-SuperHyperGraphs)**.** We now use the same level $n = 1$ but take the square product of hypergraphs.

Component SuperHyperGraphs. Let

$$V^{(1)}_0 := \{p, q\}, \qquad V^{(2)}_0 := \{r, s\}.$$

Define

$$V_1 := \{P_1, P_2\}, \quad P_1 := \{p\},\ P_2 := \{p, q\},$$

and

$$e_1 := \{P_1, P_2\} \subseteq V_1, \quad E_1 := \{e_1\},$$

so $\mathrm{SHG}^{(1)}_1 := (V_1, E_1)$ is a 1-SuperHyperGraph.

Similarly, set

$$V_2 := \{Q_1, Q_2\}, \quad Q_1 := \{r\},\ Q_2 := \{r, s\},$$

and

$$f_1 := \{Q_1, Q_2\} \subseteq V_2, \quad E_2 := \{f_1\},$$

so $\mathrm{SHG}^{(1)}_2 := (V_2, E_2)$ is another 1-SuperHyperGraph.

Underlying square product of hypergraphs. Consider the underlying hypergraphs

$$H_1 := (V_1, E_1), \qquad H_2 := (V_2, E_2).$$

Their square product $H := H_1 \boxplus H_2$ has vertex set

$$V(H) = V_1 \times V_2 = \{(P_1, Q_1), (P_1, Q_2), (P_2, Q_1), (P_2, Q_2)\},$$

and, by definition of the square product,

$$E(H) = \{e_1 \times f_1\},$$

i.e. a single hyperedge

$$e^* := e_1 \times f_1 = \{(P_1, Q_1), (P_1, Q_2), (P_2, Q_1), (P_2, Q_2)\}.$$

Embedding into a product 1-SuperHyperGraph. Form the combined base set

$$V_0 := V^{(1)}_0 \sqcup V^{(2)}_0 = \{p, q, r, s\}.$$

Again, at level $n = 1$ each 1-supervertex is a subset of V_0. Define

$$\beta : V_1 \times V_2 \longrightarrow \mathcal{P}(V_0), \qquad \beta(v_1, v_2) := v_1 \cup v_2.$$

Hence

$$\begin{aligned}\beta(P_1, Q_1) &= \{p\} \cup \{r\} = \{p, r\},\\ \beta(P_1, Q_2) &= \{p\} \cup \{r, s\} = \{p, r, s\},\\ \beta(P_2, Q_1) &= \{p, q\} \cup \{r\} = \{p, q, r\},\\ \beta(P_2, Q_2) &= \{p, q\} \cup \{r, s\} = \{p, q, r, s\}.\end{aligned}$$

Thus the vertex set of the product 1-SuperHyperGraph is

$$V'' := \beta(V_1 \times V_2) = \{\{p, r\}, \{p, r, s\}, \{p, q, r\}, \{p, q, r, s\}\} \subseteq \mathcal{P}_1(V_0).$$

The unique hyperedge $e^* \in E(H)$ induces

$$\beta[e^*] = \{\beta(P_1, Q_1), \beta(P_1, Q_2), \beta(P_2, Q_1), \beta(P_2, Q_2)\} = V'',$$

so we put

$$E'' := \{\beta[e^*]\}.$$

Therefore

$$\mathrm{SHG}_1^{(1)} \boxplus \mathrm{SHG}_2^{(1)} := (V'', E'')$$

is a 1-SuperHyperGraph whose single 1-superedge connects all four 1-supervertices. It is precisely the product of the two 1-SuperHyperGraphs induced by the square product of their underlying hypergraphs.

3.4 SuperHyperGraph Entropy

Entropy measures disorder or uncertainty in a system, quantifying missing information or the number of microscopic configurations accessible to it [168]. Related concepts such as Fuzzy Entropy [169, 170] and Neutrosophic Entropy [171, 172] are also well known.

SuperHyperGraph entropy quantifies the uncertainty inherent in weighted, multi-level hyperedge connectivity by employing minimal cut weights to evaluate how much information flows across external partitions of the supervertex hierarchy [173]. The formulation extends and adapts the principles of graph entropy [174, 175], and hypergraph entropy [176, 177] to the richer, iterated-powerset structure of SuperHyperGraphs.

Definition 3.4.1 (n-SuperHyperGraph Entropy)**.** [173] Let V_0 be a finite base set and, for each integer $k \geq 0$, define the iterated powerset by

$$\mathcal{P}_0(V_0) := V_0, \qquad \mathcal{P}_{k+1}(V_0) := \mathcal{P}(\mathcal{P}_k(V_0)),$$

where $P(\cdot)$ denotes the usual powerset.

Fix $n \in \mathbb{N}_0$ and an integer $m \geq 1$. A *weighted n-SuperHyperGraph* is a tuple

$$\mathrm{SHG}^{(n)} = (V, E, \partial, \omega, \partial V, \chi),$$

where

- $V \subseteq \mathcal{P}_n(V_0)$ is a finite set of n-supervertices,
- E is a finite set of n-superedges,
- $\partial : E \to P^*(V)$ is the incidence map, where $\mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\}$,
- $\omega : E \to \mathbb{R}_{>0}$ assigns a strictly positive weight to each n-superedge,
- $\partial V \subseteq V$ is the distinguished set of external n-supervertices,
- $\chi : \partial V \to [m] := \{1, 2, \ldots, m\}$ labels each external supervertex by one of m parties. For $I \subseteq [m]$ we set

$$\partial V_I := \chi^{-1}(I) = \{ v \in \partial V : \chi(v) \in I \}.$$

For any $W \subseteq V$ we define the *n-SuperHyperGraph cut*

$$C^{(n)}(W) := \{ e \in E \mid \partial(e) \cap W \neq \emptyset,\ \partial(e) \cap (V \setminus W) \neq \emptyset \},$$

and its total cut-weight

$$c^{(n)}(W) := \sum_{e \in C^{(n)}(W)} \omega(e).$$

For each nonempty index set $I \subseteq [m]$, the *n-SuperHyperGraph entropy* of the subsystem I is defined by

$$S^{(n)}(I) := \min_{\substack{W \subseteq V \\ W \cap \partial V = \partial V_I}} c^{(n)}(W),$$

provided that the set of admissible W is nonempty (otherwise one may set $S^{(n)}(I) := +\infty$).

The family

$$\{ S^{(n)}(I) \mid \emptyset \neq I \subseteq [m] \}$$

is called the *n-SuperHyperGraph entropy vector* of $\mathrm{SHG}^{(n)}$. When n is clear from the context, we simply write $S^{(n)}$ and refer to it as the *SuperHyperGraph Entropy*.

For reference, Table 3.2 presents a comparison of Graph Entropy, HyperGraph Entropy, and SuperHyperGraph Entropy.

Table 3.2: Comparison of Graph Entropy, HyperGraph Entropy, and SuperHyperGraph Entropy

Aspect	Graph Entropy	HyperGraph Entropy	SuperHyperGraph Entropy
Underlying structure	Finite graph $G = (V, E)$, where each edge joins exactly two vertices.	Finite hypergraph $H = (V, E)$, where each hyperedge $e \in E$ is a nonempty subset of V.	Weighted n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E, \partial, \omega, \partial V, \chi)$ with $V \subseteq \mathcal{P}_n(V_0)$.
Interaction level	Binary (pairwise) edges; only 2-vertex interactions are represented.	Higher-order interactions via hyperedges on arbitrary nonempty vertex subsets.	Multi-level and hierarchical interactions between supervertices formed from iterated powersets.
Entropy construction	Shannon-type entropy $H(p)$ of a probability distribution on V or E, typically induced by degrees, distances, or centrality measures.	Entropy of a probability distribution on vertices or hyperedges derived from incidence patterns or hyperedge weights, extending graph entropy to higher-order edges.	Cut-based entropy vector $S^{(n)}(I)$ defined by the minimal total weight of superedges crossing admissible cuts separating external parties $I \subseteq [m]$.
Main focus	Quantifies uncertainty or information content in ordinary network connectivity.	Quantifies uncertainty in higher-order group interactions on a single structural tier.	Quantifies uncertainty in weighted, multi-tier superedge connectivity across hierarchical partitions of the supervertex system.

Example 3.4.2 (A simple 1-SuperHyperGraph entropy)**.** Let $V_0 := \{a, b, c\}$ and $P_0(V_0) := V_0$, $P_1(V_0) := P(V_0)$. Consider the 1-SuperHyperGraph

$$\mathrm{SHG}^{(1)} = (V, E, \partial, \omega, \partial V, \chi)$$

with

$$V := \{\{a\}, \{b\}, \{a, b\}\} \subseteq P_1(V_0), \qquad E := \{e_1, e_2\},$$

incidence map

$$\partial(e_1) := \{\{a\}, \{a, b\}\}, \qquad \partial(e_2) := \{\{b\}, \{a, b\}\},$$

and positive weights

$$\omega(e_1) := 1, \qquad \omega(e_2) := 2.$$

Let the external supervertices and labels be

$$\partial V := \{\{a\}, \{b\}\}, \quad m := 2, \quad \chi(\{a\}) := 1, \; \chi(\{b\}) := 2.$$

For $I \subseteq [m] = \{1, 2\}$ we have

$$\partial V_{\{1\}} = \{\{a\}\}, \qquad \partial V_{\{2\}} = \{\{b\}\}.$$

We compute the entropy $S^{(1)}(\{1\})$. By definition,

$$S^{(1)}(\{1\}) := \min_{\substack{W \subseteq V \\ W \cap \partial V = \partial V_{\{1\}}}} c^{(1)}(W), \qquad c^{(1)}(W) := \sum_{e \in C^{(1)}(W)} \omega(e),$$

where

$$C^{(1)}(W) := \{\, e \in E \mid \partial(e) \cap W \neq \emptyset, \; \partial(e) \cap (V \setminus W) \neq \emptyset \,\}.$$

The constraint $W \cap \partial V = \partial V_{\{1\}} = \{\{a\}\}$ means that W must contain $\{a\}$ but not $\{b\}$. Thus the admissible subsets W are

$$W_1 := \{\{a\}\}, \qquad W_2 := \{\{a\}, \{a, b\}\}.$$

For W_1 we have $V \setminus W_1 = \{\{b\}, \{a, b\}\}$ and

$$C^{(1)}(W_1) = \{e_1\}, \qquad c^{(1)}(W_1) = \omega(e_1) = 1,$$

since $\partial(e_1)$ meets both W_1 and $V \setminus W_1$, whereas $\partial(e_2)$ does not intersect W_1.

For W_2 we have $V \setminus W_2 = \{\{b\}\}$ and

$$C^{(1)}(W_2) = \{e_2\}, \qquad c^{(1)}(W_2) = \omega(e_2) = 2,$$

since $\partial(e_2)$ meets both W_2 and $V \setminus W_2$, whereas $\partial(e_1)$ is entirely contained in W_2.

Hence

$$S^{(1)}(\{1\}) = \min\{c^{(1)}(W_1), c^{(1)}(W_2)\} = \min\{1, 2\} = 1.$$

This value $S^{(1)}(\{1\}) = 1$ is the 1-SuperHyperGraph entropy of the subsystem corresponding to party 1.

3.5 Similarity and Metric on SuperHyperGraphs

Similarity and metric on SuperHyperGraphs evaluate the closeness of multi-level incidence structures by comparing the patterns of their tiered superedges, producing normalized distance values and similarity scores for structured uncertain networks. This notion extends the established ideas of similarity and metric on graphs and on hypergraphs (cf. [17, 178, 179]) to the richer SuperHyperGraph framework.

Definition 3.5.1 (Similarity and metric on hypergraphs)**.** [17] Let $V = \{v_1, \ldots, v_p\}$ and $E = \{e_1, \ldots, e_q\}$ be finite sets. A (simple) hypergraph on (V, E) is given by a map

$$\partial : E \to \mathcal{P}^*(V), \qquad e \mapsto \partial(e),$$

where $\mathcal{P}^*(V)$ is the set of all nonempty subsets of V.

Its incidence matrix is the $(p \times q)$–matrix $M(H) = (m_{ij})$ with

$$m_{ij} := \begin{cases} 1, & v_i \in \partial(e_j), \\ 0, & v_i \notin \partial(e_j). \end{cases}$$

Fix $N := pq$. For two hypergraphs H_1, H_2 on (V, E) with incidence matrices $M(H_1) = (m_{ij}^{(1)})$ and $M(H_2) = (m_{ij}^{(2)})$, define the *hypergraph distance* and *similarity* by

$$d_H(H_1, H_2) := \frac{1}{N} \sum_{i=1}^{p} \sum_{j=1}^{q} \left| m_{ij}^{(1)} - m_{ij}^{(2)} \right|, \qquad s_H(H_1, H_2) := 1 - d_H(H_1, H_2).$$

Then d_H is a metric on the set of all hypergraphs on (V, E), and s_H takes values in $[0, 1]$.

Definition 3.5.2 (Similarity and metric on n-SuperHyperGraphs)**.** Fix a finite base set V_0 and $n \in \mathbb{N}_0$. Let $V \subseteq \mathcal{P}^n(V_0)$ and E be finite sets, and let

$$\partial : E \to \mathcal{P}^*(V)$$

be an incidence map. Then (V, E, ∂) is an n-SuperHyperGraph.

Label $V = \{v_1, \dots, v_p\}$ and $E = \{e_1, \dots, e_q\}$ and define the incidence matrix $M(\mathrm{SHG}^{(n)}) = (a_{ij})$ by

$$a_{ij} := \begin{cases} 1, & v_i \in \partial(e_j), \\ 0, & v_i \notin \partial(e_j). \end{cases}$$

For two n-SuperHyperGraphs $\mathsf{G}_1, \mathsf{G}_2$ on the same indexed sets (V, E) with incidence matrices $M(\mathsf{G}_1) = (a_{ij}^{(1)})$ and $M(\mathsf{G}_2) = (a_{ij}^{(2)})$, put $N := pq$ and define

$$d_{\mathrm{SH}}^{(n)}(\mathsf{G}_1, \mathsf{G}_2) := \frac{1}{N} \sum_{i=1}^{p} \sum_{j=1}^{q} \left| a_{ij}^{(1)} - a_{ij}^{(2)} \right|, \qquad s_{\mathrm{SH}}^{(n)}(\mathsf{G}_1, \mathsf{G}_2) := 1 - d_{\mathrm{SH}}^{(n)}(\mathsf{G}_1, \mathsf{G}_2).$$

Then $d_{\mathrm{SH}}^{(n)}$ is a metric on the set of all n-SuperHyperGraphs on (V, E), and $s_{\mathrm{SH}}^{(n)} \in [0, 1]$.

For reference, an overview of the comparison between Similarity and Metric on HyperGraphs and on SuperHyperGraphs is presented in Table 3.3.

Table 3.3: Comparison of Similarity and Metric on HyperGraphs and on SuperHyperGraphs

Aspect	Similarity / Metric on HyperGraphs	Similarity / Metric on SuperHyperGraphs
Underlying structure	Simple hypergraph $H = (V, E)$ with incidence map $\partial : E \to \mathcal{P}^*(V)$.	n-SuperHyperGraph (V, E, ∂) with $V \subseteq \mathcal{P}^n(V_0)$ and $\partial : E \to \mathcal{P}^*(V)$.
Incidence representation	Incidence matrix $M(H) = (m_{ij})$ with $m_{ij} = 1$ if $v_i \in \partial(e_j)$, and 0 otherwise.	Incidence matrix $M(\mathsf{G}^{(n)}) = (a_{ij})$ with $a_{ij} = 1$ if $v_i \in \partial(e_j)$, and 0 otherwise (same scheme, but vertices are n-supervertices).
Distance	Normalized Hamming distance on incidence matrices: $d_H(H_1, H_2) = \frac{1}{pq} \sum_{i,j} \lvert m_{ij}^{(1)} - m_{ij}^{(2)} \rvert$, a metric on all hypergraphs on (V, E).	Same normalized Hamming distance: $d_{\mathrm{SH}}^{(n)}(\mathsf{G}_1, \mathsf{G}_2) = \frac{1}{pq} \sum_{i,j} \lvert a_{ij}^{(1)} - a_{ij}^{(2)} \rvert$, a metric on all n-SuperHyperGraphs on (V, E).
Similarity	Similarity score $s_H(H_1, H_2) = 1 - d_H(H_1, H_2) \in [0, 1]$.	Similarity score $s_{\mathrm{SH}}^{(n)}(\mathsf{G}_1, \mathsf{G}_2) = 1 - d_{\mathrm{SH}}^{(n)}(\mathsf{G}_1, \mathsf{G}_2) \in [0, 1]$.
Generalization	Defined on a single-level hypergraph structure.	Extends the hypergraph case: when $n = 0$, similarity and metric reduce exactly to s_H and d_H. For $n \geq 1$, they compare multi-level superincidence patterns of SuperHyperGraphs.

The concrete example is presented as follows.

Example 3.5.3 (Concrete distance and similarity of two 1-SuperHyperGraphs)**.** Let the finite base set be

$$V_0 := \{a, b\},$$

and take $n = 1$, so $P_1(V_0) = P(V_0)$. Set

$$V := \big\{\{a\}, \{b\}\big\} \subseteq P_1(V_0), \qquad E := \{e_1\}.$$

Define two 1-SuperHyperGraphs on the same indexed sets (V, E):

$$\mathsf{G}_1^{(1)} := (V, E, \partial_1), \qquad \mathsf{G}_2^{(1)} := (V, E, \partial_2),$$

with incidences

$$\partial_1(e_1) := \{\{a\}\}, \qquad \partial_2(e_1) := \{\{a\}, \{b\}\}.$$

Index

$$V = \{v_1, v_2\} := \{\{a\}, \{b\}\}, \qquad E = \{e_1\},$$

so $p = 2$, $q = 1$, and $N = pq = 2$. The incidence matrices are

$$M(\mathsf{G}_1^{(1)}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad M(\mathsf{G}_2^{(1)}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By definition,

$$d_{\mathrm{SH}}^{(1)}(\mathsf{G}_1^{(1)}, \mathsf{G}_2^{(1)}) = \frac{1}{N} \sum_{i=1}^{2} \sum_{j=1}^{1} \left| a_{ij}^{(1)} - a_{ij}^{(2)} \right| = \frac{1}{2}\left(|1-1| + |0-1|\right) = \frac{1}{2},$$

and hence the similarity is

$$s_{\mathrm{SH}}^{(1)}(\mathsf{G}_1^{(1)}, \mathsf{G}_2^{(1)}) = 1 - d_{\mathrm{SH}}^{(1)}(\mathsf{G}_1^{(1)}, \mathsf{G}_2^{(1)}) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Thus these two 1-SuperHyperGraphs have distance $\frac{1}{2}$ and similarity $\frac{1}{2}$.

Theorem 3.5.4 (Generalization of hypergraph similarity and metric)**.** *Let V, E be fixed finite sets and let H_1, H_2 be two hypergraphs on (V, E). Regard each H_r as a 0-SuperHyperGraph*

$$\mathsf{G}_r^{(0)} := (V, E, \partial_r), \qquad r = 1, 2,$$

with the same incidence map ∂_r as H_r. Then

$$d_{\mathrm{SH}}^{(0)}(\mathsf{G}_1^{(0)}, \mathsf{G}_2^{(0)}) = d_H(H_1, H_2), \qquad s_{\mathrm{SH}}^{(0)}(\mathsf{G}_1^{(0)}, \mathsf{G}_2^{(0)}) = s_H(H_1, H_2).$$

In particular, similarity and metric on n-SuperHyperGraphs extend those on hypergraphs, which are recovered as the special case $n = 0$.

Proof. For $r = 1, 2$ the incidence matrix of H_r and of $\mathsf{G}_r^{(0)}$ have the same entries by definition:

$$m_{ij}^{(r)} = 1 \iff v_i \in \partial_r(e_j) \iff a_{ij}^{(r)} = 1.$$

Hence $M(H_r) = M(\mathsf{G}_r^{(0)})$ for $r = 1, 2$, and therefore the formulas defining d_H, s_H and $d_{\mathrm{SH}}^{(0)}, s_{\mathrm{SH}}^{(0)}$ coincide term by term. This proves the equalities in the statement. □

3.6 SuperHypergraph Morphism

SuperHyperGraph morphism maps supervertices and superedges between superhypergraphs, preserving incidence structure across all tiers and connections in a structure-preserving way. This is obtained by applying the concept of graph morphisms [180, 181] and hypergraph morphisms [17, 182, 183] to the framework of SuperHyperGraphs.

Definition 3.6.1 (Hypergraph morphism)**.** [17] Let $H = (V, E)$ and $H' = (V', E')$ be hypergraphs without repeated hyperedges. A map

$$f : V \longrightarrow V'$$

is called a *hypergraph morphism* from H to H' if for every hyperedge $e \in E$ the image

$$f[e] := \{ f(v) \mid v \in e \}$$

is a hyperedge of H', i.e. $f[e] \in E'$.

Definition 3.6.2 (SuperHypergraph morphism)**.** Let $\mathcal{H}_1 = (V_1, E_1, \partial_1)$ and $\mathcal{H}_2 = (V_2, E_2, \partial_2)$ be SuperHyperGraphs, where V_i is the (possibly tiered) supervertex set, E_i is the set of superedges, and

$$\partial_i : E_i \longrightarrow \mathcal{P}^*(V_i)$$

is the incidence map assigning to each superedge a nonempty set of supervertices, with $\mathcal{P}^*(V_i) := \mathcal{P}(V_i) \setminus \{\emptyset\}$.

A *SuperHyperGraph morphism*

$$F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

is a pair $F = (f, g)$ consisting of

$$f : V_1 \longrightarrow V_2, \qquad g : E_1 \longrightarrow E_2,$$

such that the following compatibility condition holds for every $e \in E_1$:

$$\partial_2\big(g(e)\big) \;=\; f_\#\big(\partial_1(e)\big),$$

where

$$f_\# : \mathcal{P}(V_1) \longrightarrow \mathcal{P}(V_2), \qquad f_\#(X) := \{\, f(v) \mid v \in X \,\},$$

is the direct image map on subsets.

Table 3.4 presents a comparison of Hypergraph morphisms and SuperHyperGraph morphisms.

Table 3.4: Comparison of Hypergraph morphisms and SuperHyperGraph morphisms

Aspect	Hypergraph morphism	SuperHyperGraph morphism
Underlying objects	Hypergraphs $H = (V, E)$ and $H' = (V', E')$ without repeated hyperedges.	SuperHyperGraphs $\mathcal{H}_1 = (V_1, E_1, \partial_1)$ and $\mathcal{H}_2 = (V_2, E_2, \partial_2)$, where vertices and edges may be tiered.
Maps	A single vertex map $f : V \to V'$.	A pair of maps $F = (f, g)$ with $f : V_1 \to V_2$ (on supervertices) and $g : E_1 \to E_2$ (on superedges).
Incidence preservation	For every $e \in E$, the image $f[e] = \{f(v) \mid v \in e\}$ is required to be a hyperedge of H', i.e. $f[e] \in E'$.	For every $e \in E_1$, the incidence condition $\partial_2\big(g(e)\big) = f_\#\big(\partial_1(e)\big)$ must hold, where $f_\#(X) = \{f(v) \mid v \in X\}$.
Level of structure	Preserves membership of vertices in hyperedges at a single level.	Preserves membership of supervertices in superedges across possibly multi-level superstructures via a compatible pair (f, g).
Reduction	Fundamental notion in classical hypergraph theory.	Extends the hypergraph case: when supervertices/edges collapse to ordinary vertices/edges, the notion reduces to a hypergraph morphism.

Example 3.6.3 (A simple SuperHyperGraph morphism)**.** Let $\mathcal{H}_1 = (V_1, E_1, \partial_1)$ and $\mathcal{H}_2 = (V_2, E_2, \partial_2)$ be SuperHyperGraphs defined by

$$V_1 := \{v_1, v_2, v_3\}, \quad E_1 := \{e_1\}, \quad \partial_1(e_1) := \{v_1, v_2\},$$

$$V_2 := \{w_1, w_2\}, \quad E_2 := \{f_1\}, \quad \partial_2(f_1) := \{w_1, w_2\}.$$

Define maps

$$f : V_1 \to V_2, \qquad f(v_1) := w_1,\ f(v_2) := w_2,\ f(v_3) := w_2,$$

$$g : E_1 \to E_2, \qquad g(e_1) := f_1.$$

The direct image of the incident set of e_1 is

$$f_\#\big(\partial_1(e_1)\big) = f_\#(\{v_1, v_2\}) = \{f(v_1), f(v_2)\} = \{w_1, w_2\}.$$

Hence

$$\partial_2\big(g(e_1)\big) = \partial_2(f_1) = \{w_1, w_2\} = f_\#\big(\partial_1(e_1)\big),$$

so the pair $F = (f, g)$ is a SuperHyperGraph morphism

$$F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2.$$

Theorem 3.6.4 (SuperHyperGraph morphisms generalize hypergraph morphisms)**.** *Let $H = (V, E)$ and $H' = (V', E')$ be hypergraphs without repeated hyperedges, and regard them as SuperHyperGraphs via*

$$\mathcal{H} = (V, E, \partial), \qquad \mathcal{H}' = (V', E', \partial'),$$

where

$$\partial(e) = e \subseteq V, \qquad \partial'(e') = e' \subseteq V' \qquad (e \in E,\ e' \in E').$$

Then a map $f : V \to V'$ is a hypergraph morphism $H \to H'$ if and only if there exists a map $g : E \to E'$ such that (f, g) is a SuperHyperGraph morphism $\mathcal{H} \to \mathcal{H}'$ in the sense of Definition 3.6.2.

Proof. ($\Rightarrow$) Suppose $f : V \to V'$ is a hypergraph morphism. For each $e \in E$ the image $f[e] = \{f(v) \mid v \in e\}$ is a hyperedge of H', so $f[e] \in E'$. Define $g : E \to E'$ by

$$g(e) := f[e] \in E'.$$

Then, for every $e \in E$,

$$\partial'\big(g(e)\big) = g(e) = f[e] = f_\#\big(\partial(e)\big),$$

because $\partial(e) = e$ and $\partial'(g(e)) = g(e)$ by construction. Hence (f, g) satisfies the compatibility condition and is a SuperHyperGraph morphism.

($\Leftarrow$) Conversely, assume there exists $g : E \to E'$ such that (f, g) is a SuperHyperGraph morphism. For each $e \in E$ we then have

$$\partial'\big(g(e)\big) = f_\#\big(\partial(e)\big).$$

Using $\partial(e) = e$ and $\partial'(g(e)) = g(e)$, this becomes

$$g(e) = f_\#(e) = \{f(v) \mid v \in e\}.$$

Thus $f[e] = g(e) \in E'$, so f sends every hyperedge of H to a hyperedge of H'. Therefore f is a hypergraph morphism.

Consequently, hypergraph morphisms are precisely those vertex maps that admit a completion to a SuperHyperGraph morphism between the associated SuperHyperGraphs, so the notion of SuperHyperGraph morphism is a genuine generalization of hypergraph morphism. □

3.7 SuperHyperGraph Partitioning

Graph partitioning divides a graph's vertices into disjoint blocks, minimizing the number (or weight) of edges crossing between different blocks [184]. Related concepts such as Fuzzy Graph Partitioning [185, 186] and Directed Graph Partitioning [187–189] are also well known. Graph partitioning is essential for scalable computation, reducing communication costs, enabling parallelism, improving clustering quality, and accelerating large-scale optimization and learning.

Hypergraph partitioning splits vertices into blocks, minimizing cut hyperedges while balancing block sizes, better capturing multiway relationships than standard graph partitioning [190–192]. SuperHyperGraph partitioning divides supervertices and superedges into balanced groups, minimizing cut weight while respecting hierarchical multi-level connectivity constraints and structure [193].

Definition 3.7.1 (k-way n-SuperHyperGraph partition)**.** [193] Let $H^{(n)} = (V, E)$ be an n-SuperHyperGraph, and let $k \in \mathbb{N}$, $k \geq 2$. A *k-way partition* of V is a family of subsets

$$\mathcal{V} := \{V_1, \dots, V_k\}$$

such that

$$V_i \subseteq V, \quad V_i \cap V_j = \varnothing \ (i \neq j), \quad \bigcup_{i=1}^{k} V_i = V.$$

Fix a balance parameter $c \geq 1$. The partition $\mathcal{V}$ is called *c-balanced* if

$$\frac{1}{c} \frac{|V|}{k} \ \leq \ |V_i| \ \leq \ c \, \frac{|V|}{k} \qquad \text{for all } i = 1, \dots, k.$$

For a given partition $\mathcal{V}$ and a superedge $e \in E$, define the number of parts spanned by e as

$$\mathrm{span}_{\mathcal{V}}(e) := \big|\{\, i \in \{1, \dots, k\} \mid e \cap V_i \neq \varnothing \,\}\big|.$$

The *superedge-cut cost* of $\mathcal{V}$ is

$$f_{\mathrm{cut}}(\mathcal{V}) := \sum_{e \in E} \big(\mathrm{span}_{\mathcal{V}}(e) - 1\big).$$

The *k-way n-SuperHyperGraph partitioning problem* is: given $H^{(n)} = (V, E)$, k, and $c \geq 1$, find a c-balanced k-way partition $\mathcal{V} = \{V_1, \dots, V_k\}$ of V that minimizes $f_{\mathrm{cut}}(\mathcal{V})$ (or another chosen objective such as the sum of external degrees).

Table 3.5 presents a Comparison of Graph, HyperGraph, and SuperHyperGraph Partitioning.

Table 3.5: Comparison of Graph, HyperGraph, and SuperHyperGraph Partitioning

Aspect	Graph Partitioning	HyperGraph Partitioning	SuperHyperGraph Partitioning
Underlying structure	Graph $G = (V, E)$ with edges joining two vertices.	HyperGraph $H = (V, E)$ with hyperedges $e \subseteq V$.	n-SuperHyperGraph $H^{(n)} = (V, E)$ with supervertices and superedges on iterated powersets.
Partition object	Vertex set V split into disjoint blocks.	Vertex set V split into disjoint blocks.	Supervertex set V split into disjoint blocks.
Cut notion	Number or total weight of edges crossing between blocks.	Number or total weight of hyperedges incident to more than one block.	Superedge-cut cost $f_{\mathrm{cut}}(\mathcal{V}) = \sum_{e \in E}(\mathrm{span}_{\mathcal{V}}(e) - 1)$, measuring how many blocks each superedge spans.
Balance constraint	Block sizes approximately equal (e.g., bounded ratio to $\lvert V\rvert/k$).	Block sizes approximately equal under a chosen balance parameter.	c-balanced: $\frac{1}{c}\frac{\lvert V\rvert}{k} \leq \lvert V_i\rvert \leq c\frac{\lvert V\rvert}{k}$ for all parts V_i.
Goal	Minimize edges crossing between blocks while maintaining balance.	Minimize cut hyperedges (and possibly their weights) while maintaining balance; better captures multiway relationships.	Minimize superedge-cut cost while respecting hierarchical, multi-level connectivity and structural constraints.

Hereafter, we present a concrete example.

Example 3.7.2 (2-way partition of a small 2-SuperHyperGraph)**.** We give a concrete 2-SuperHyperGraph and an explicit optimal 2-way partition with respect to the cut objective.

Base level (individuals). Let

$$V_0 := \{a, b, c, d\}$$

represent four individual employees.

Level 1 (teams). Consider the following nonempty subsets of V_0:

$$T_1 := \{a, b\}, \qquad T_2 := \{b, c\}, \qquad T_3 := \{c, d\}.$$

Level 2 (departments as sets of teams). Define three departments

$$D_1 := \{T_1, T_2\}, \qquad D_2 := \{T_2, T_3\}, \qquad D_3 := \{T_1, T_3\}.$$

Each D_i is a nonempty subset of $\{T_1, T_2, T_3\}$, so $D_i \in P\big(P(V_0)\big) = P_2(V_0)$.

We set the 2-supervertex set and 2-superedge set as

$$V := \{D_1, D_2, D_3\}, \qquad E := \Big\{e_1 := \{D_1, D_2\},\ e_2 := \{D_2, D_3\}\Big\}.$$

Then $H^{(2)} = (V, E)$ is a finite 2-SuperHyperGraph.

We now partition V into $k = 2$ parts. Let

$$V_1 := \{D_1, D_2\}, \qquad V_2 := \{D_3\}.$$

Clearly $V_1 \cap V_2 = \varnothing$ and $V_1 \cup V_2 = V$.

Balance check. Here $|V| = 3$ and $k = 2$, so the ideal size is $|V|/k = 3/2$. Choose $c = 2$. Then

$$\frac{1}{c}\,\frac{|V|}{k} = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}, \qquad c\,\frac{|V|}{k} = 2 \cdot \frac{3}{2} = 3.$$

We have $|V_1| = 2$ and $|V_2| = 1$, so

$$\frac{3}{4} \le 1 \le 3, \qquad \frac{3}{4} \le 2 \le 3,$$

hence the partition is 2-balanced.

Cut cost. We compute $\mathrm{span}_{\mathcal{V}}(e)$ for each superedge $e \in E$, where $\mathcal{V} = \{V_1, V_2\}$.

For $e_1 = \{D_1, D_2\}$ we have $e_1 \subseteq V_1$, so

$$\mathrm{span}_{\mathcal{V}}(e_1) = 1, \qquad \mathrm{span}_{\mathcal{V}}(e_1) - 1 = 0.$$

For $e_2 = \{D_2, D_3\}$ we have $D_2 \in V_1$ and $D_3 \in V_2$, hence

$$\mathrm{span}_{\mathcal{V}}(e_2) = 2, \qquad \mathrm{span}_{\mathcal{V}}(e_2) - 1 = 1.$$

Therefore the total cut cost is

$$f_{\mathrm{cut}}(\mathcal{V}) = \big(\mathrm{span}_{\mathcal{V}}(e_1) - 1\big) + \big(\mathrm{span}_{\mathcal{V}}(e_2) - 1\big) = 0 + 1 = 1.$$

If we instead consider the alternative partition

$$V_1' := \{D_1, D_3\}, \qquad V_2' := \{D_2\},$$

then both superedges span both parts:

$$\mathrm{span}_{\mathcal{V}'}(e_1) = 2, \qquad \mathrm{span}_{\mathcal{V}'}(e_2) = 2,$$

so

$$f_{\mathrm{cut}}(\mathcal{V}') = (2 - 1) + (2 - 1) = 2.$$

Hence the original partition $\mathcal{V} = \{V_1, V_2\}$ is strictly better with respect to the cut objective f_{cut}.

This example shows explicitly how a small 2-SuperHyperGraph can be partitioned into two balanced parts while minimizing the number of superedges that cross between parts.

3.8 SuperHyperGraph Coloring

Graph coloring assigns colors to vertices so adjacent vertices differ, modeling resource allocation, scheduling conflicts, or frequency assignment constraints [194–197]. Related concepts such as Fuzzy Graph Coloring [198, 199], Directed Graph Coloring [200–202], Edge Coloring [203, 204], Total coloring [205, 206], Face coloring [207, 208], and Neutrosophic Graph Coloring [194, 209] are also well known. HyperGraph coloring assigns colors to vertices so no hyperedge is monochromatic, extending classical coloring to multiway interactions and constraints [210–213]. SuperHyperGraph coloring assigns colors to supervertices across iterated powerset levels, preventing monochromatic superedges and capturing hierarchical multi-level conflict structures.

Definition 3.8.1 (Hypergraph coloring)**.** [214, 215] Let $H = (V(H), E(H))$ be a finite hypergraph with

$$\emptyset \neq V(H), \qquad \emptyset \neq E(H) \subseteq \mathcal{P}^*\big(V(H)\big),$$

where $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$.

Let $c \in \mathbb{N}$ with $c \geq 1$, and let

$$C \ := \ \{1, 2, \dots, c\}$$

be a set of c colors.

A *c-coloring* of H is a function

$$\varphi : V(H) \longrightarrow C.$$

Such a coloring φ is called *proper* if no hyperedge is monochromatic, that is,

$$\forall\, e \in E(H): \quad \{\varphi(v) \mid v \in e\} \neq \{i\} \quad \text{for every } i \in C.$$

Equivalently, every $e \in E(H)$ contains at least two vertices with distinct colors.

The *chromatic number* of H is

$$\chi(H) \ := \ \min\big\{\, c \in \mathbb{N} \mid H \text{ admits a proper } c\text{-coloring} \,\big\}.$$

Definition 3.8.2 (SuperHyperGraph coloring)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph as above. Fix $c \in \mathbb{N}$ with $c \geq 1$ and a color set

$$C \ := \ \{1, 2, \dots, c\}.$$

A *c-coloring* of the n-SuperHyperGraph $\mathrm{SHG}^{(n)}$ is a function

$$\psi : V_n \longrightarrow C.$$

Such a coloring ψ is called *proper* if no n-superedge is monochromatic, that is,

$$\forall\, e \in E_n: \quad \{\psi(v) \mid v \in e\} \neq \{i\} \quad \text{for every } i \in C.$$

The *SuperHyperGraph chromatic number* of $\mathrm{SHG}^{(n)}$ is

$$\chi\big(\mathrm{SHG}^{(n)}\big) \ := \ \min\big\{\, c \in \mathbb{N} \mid \mathrm{SHG}^{(n)} \text{ admits a proper } c\text{-coloring} \,\big\}.$$

Table 3.6 presents an overview of the comparison among Graph Coloring, HyperGraph Coloring, and Super-HyperGraph Coloring.

Hereafter, we present a concrete example.

Table 3.6: Comparison of Graph Coloring, HyperGraph Coloring, and SuperHyperGraph Coloring

Aspect	Graph Coloring	HyperGraph Coloring	SuperHyperGraph Coloring
Underlying structure	Graph $G = (V, E)$ with edges joining pairs of vertices.	HyperGraph $H = (V, E)$ with hyperedges $e \subseteq V$.	n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V_n, E_n)$ on iterated powersets.
Coloring object	Vertices V are colored.	Vertices V are colored.	Supervertices V_n (collections on higher levels) are colored.
Proper-coloring constraint	Adjacent vertices must receive different colors; no edge is monochromatic.	No hyperedge is monochromatic; every hyperedge contains at least two distinct colors.	No n-superedge is monochromatic; every superedge contains at least two distinct colors at the supervertex level.
Chromatic parameter	Chromatic number $\chi(G)$: minimum number of colors in a proper coloring.	Chromatic number $\chi(H)$: minimum number of colors in a proper hypergraph coloring.	SuperHyperGraph chromatic number $\chi(\mathrm{SHG}^{(n)})$: minimum number of colors in a proper n-SuperHyperGraph coloring.
Level of conflict modeled	Pairwise conflicts between individual vertices.	Multiway conflicts within vertex subsets (hyperedges).	Hierarchical multi-level conflicts between supervertices and their superedges.

Example 3.8.3 (A 2-colorable SuperHyperGraph)**.** Let

$$V_0 := \{v_1, v_2, v_3\}, \qquad E_0 := \{e_1, e_2\}$$

with hyperedges

$$e_1 := \{v_1, v_2\}, \qquad e_2 := \{v_2, v_3\}.$$

Then $\mathrm{SHG}^{(0)} := (V_0, E_0)$ is a 0-SuperHyperGraph.

Take $c = 2$ and the color set

$$C := \{1, 2\}.$$

Define a coloring $\psi : V_0 \to C$ by

$$\psi(v_1) := 1, \qquad \psi(v_2) := 2, \qquad \psi(v_3) := 1.$$

For each $e \in E_0$ we have

$$\{\psi(v) \mid v \in e_1\} = \{\psi(v_1), \psi(v_2)\} = \{1, 2\},$$

$$\{\psi(v) \mid v \in e_2\} = \{\psi(v_2), \psi(v_3)\} = \{1, 2\},$$

so no edge is monochromatic and ψ is a proper 2-coloring.

If we tried $c = 1$ and colored all vertices with a single color, then each edge would be monochromatic, so no proper 1-coloring exists. Therefore

$$\chi(\mathrm{SHG}^{(0)}) = 2.$$

Example 3.8.4 (A 3-chromatic 1-SuperHyperGraph)**.** Let the base set be

$$V_0 := \{a, b, c\}.$$

Form three 1-supervertices

$$A := \{a, b\}, \qquad B := \{b, c\}, \qquad C := \{a, c\},$$

and set
$$V_1 := \{A, B, C\}.$$
Define 1-superedges
$$e_1 := \{A, B\}, \qquad e_2 := \{B, C\}, \qquad e_3 := \{A, C\},$$
and let
$$E_1 := \{e_1, e_2, e_3\}.$$
Then $\mathrm{SHG}^{(1)} := (V_1, E_1)$ is a 1-SuperHyperGraph whose underlying structure on the tier-1 vertices A, B, C is the complete graph K_3.

Take $c = 3$ with color set
$$C := \{1, 2, 3\},$$
and define $\psi : V_1 \to C$ by
$$\psi(A) := 1, \qquad \psi(B) := 2, \qquad \psi(C) := 3.$$
Then
$$\{\psi(v) \mid v \in e_1\} = \{\psi(A), \psi(B)\} = \{1, 2\},$$
$$\{\psi(v) \mid v \in e_2\} = \{\psi(B), \psi(C)\} = \{2, 3\},$$
$$\{\psi(v) \mid v \in e_3\} = \{\psi(A), \psi(C)\} = \{1, 3\},$$
so no 1-superedge is monochromatic and ψ is a proper 3-coloring.

On the other hand, any 2-coloring of A, B, C must assign the same color to some pair, say $\psi(A) = \psi(B)$. Since $\{A, B\}$ is a 1-superedge, this edge would then be monochromatic, contradicting properness. Hence no proper 2-coloring exists and
$$\chi\big(\mathrm{SHG}^{(1)}\big) = 3.$$

Theorem 3.8.5 (SuperHyperGraph coloring generalizes hypergraph coloring)**.** *Every hypergraph coloring problem is a special case of SuperHyperGraph coloring at level $n = 0$. More precisely, for every finite hypergraph $H = (V(H), E(H))$ there exists a level-0 SuperHyperGraph*

$$\mathrm{SHG}^{(0)} \; = \; (V_0, E_0)$$

such that:

1. *$V_0 = V(H)$ and $E_0 = E(H)$;*
2. *for every integer $c \geq 1$, the proper c-colorings of H are* exactly *the proper c-colorings of* $\mathrm{SHG}^{(0)}$ *(via the same vertex-coloring functions);*
3. *in particular,*
$$\chi(H) \; = \; \chi\big(\mathrm{SHG}^{(0)}\big).$$

Proof. Let $H = (V(H), E(H))$ be a finite hypergraph. Set

$$V_0 \; := \; V(H), \qquad E_0 \; := \; E(H).$$

Recall that by definition
$$\mathrm{P}_0(V_0) = V_0, \qquad \mathcal{P}^*(V_0) = \mathcal{P}(V_0) \setminus \{\emptyset\}.$$
Since $E(H) \subseteq \mathcal{P}^*\big(V(H)\big) = \mathcal{P}^*(V_0)$, the pair

$$\mathrm{SHG}^{(0)} \; := \; (V_0, E_0)$$

satisfies
$$\emptyset \neq V_0 \subseteq \mathrm{P}_0(V_0), \qquad \emptyset \neq E_0 \subseteq \mathcal{P}^*(V_0),$$
so it is a level-0 SuperHyperGraph on the base set V_0.

Fix $c \in \mathbb{N}$, $c \geq 1$, and let $C = \{1, \ldots, c\}$ be the set of colors.

(1) Suppose first that $\varphi : V(H) \to C$ is a proper hypergraph c-coloring of H. Define

$$\psi : V_0 \longrightarrow C, \qquad \psi(v) := \varphi(v) \quad (v \in V_0).$$

Since $V_0 = V(H)$ and $E_0 = E(H)$, for every edge $e \in E_0$ we have

$$\{\psi(v) \mid v \in e\} = \{\varphi(v) \mid v \in e\}.$$

Because φ is proper on H, no $e \in E(H) = E_0$ is monochromatic under φ, hence no $e \in E_0$ is monochromatic under ψ. Thus ψ is a proper c-coloring of the level-0 SuperHyperGraph $\mathrm{SHG}^{(0)}$.

(2) Conversely, suppose that $\psi : V_0 \to C$ is a proper c-coloring of $\mathrm{SHG}^{(0)}$. Define

$$\varphi : V(H) \longrightarrow C, \qquad \varphi(v) := \psi(v) \quad (v \in V(H) = V_0).$$

For every hyperedge $e \in E(H) = E_0$ we again have

$$\{\varphi(v) \mid v \in e\} = \{\psi(v) \mid v \in e\}.$$

Since ψ is proper on $\mathrm{SHG}^{(0)}$, no $e \in E_0$ is monochromatic under ψ, hence no $e \in E(H)$ is monochromatic under φ. Therefore φ is a proper c-coloring of the original hypergraph H.

The constructions in (1) and (2) are inverses of each other (they are both the identity map on the underlying set $V(H) = V_0$). Thus, for every $c \geq 1$, there is a natural bijection

$$\{\text{proper } c\text{-colorings of } H\} \;\cong\; \{\text{proper } c\text{-colorings of } \mathrm{SHG}^{(0)}\}.$$

Taking the minimum c for which these sets are nonempty on each side, we obtain

$$\chi(H) = \chi(\mathrm{SHG}^{(0)}).$$

Hence hypergraph coloring is realized exactly as the special case $n = 0$ of SuperHyperGraph coloring, which proves that SuperHyperGraph coloring is a proper generalization of hypergraph coloring. □

3.9 SuperHyperGraph Domination

Graph domination investigates subsets of vertices such that every vertex either belongs to the subset or is adjacent to at least one vertex in it, with a primary focus on minimizing the size of such subsets in graphs [216–218]. Related notions of domination and its variants have also been extensively studied in the settings of fuzzy graphs and neutrosophic graphs (e.g., [219–222]). In addition, several well-known variants of domination exist, including secure domination [223, 224], paired domination [225, 226], double domination [227, 228], roman domination [229–231], connected domination [232–234], star domination [235–237], and total domination [238].

Hypergraph domination extends graph domination by requiring that each vertex outside a dominating set shares at least one hyperedge with a dominating vertex, thereby modeling influence coverage in more complex multiway interaction systems [239–242]. SuperHyperGraph domination further generalizes hypergraph domination to multi-tier supervertices, by requiring that every supervertex is connected, via at least one superedge, to some dominating supervertex while respecting the hierarchical structure across the superhypergraph levels.

Definition 3.9.1 (Domination in a hypergraph)**.** (cf. [239–241, 243]) Let $H = (V, E)$ be a finite hypergraph with nonempty vertex set V and hyperedge family $E \subseteq \mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\}$.

A subset $D \subseteq V$ is called a *dominating set* of H if for every vertex $v \in V \setminus D$ there exists an edge $e \in E$ such that

$$v \in e \quad \text{and} \quad e \cap D \neq \emptyset.$$

Equivalently, every vertex outside D is contained in some edge that also contains at least one vertex of D.

The *domination number* of H is

$$\gamma(H) \;:=\; \min\{\, |D| \mid D \subseteq V \text{ is a dominating set of } H \,\}.$$

Example 3.9.2 (Domination in a small hypergraph)**.** Consider the hypergraph

$$H = (V, E)$$

with vertex set

$$V := \{1, 2, 3, 4\}$$

and hyperedge family

$$E := \{e_1, e_2, e_3\} := \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}.$$

Define

$$D := \{2, 3\} \subseteq V.$$

We check that D is a dominating set of H.

For each vertex $v \in V \setminus D = \{1, 4\}$:

- $v = 1$: we have $1 \in e_1 = \{1, 2\}$ and $e_1 \cap D = \{2\} \neq \emptyset$.
- $v = 4$: we have $4 \in e_3 = \{3, 4\}$ and $e_3 \cap D = \{3\} \neq \emptyset$.

Thus every vertex outside D lies in some hyperedge that also contains a vertex from D, so D is dominating.

No singleton $\{1\}$, $\{2\}$, $\{3\}$, or $\{4\}$ is dominating:

- $\{1\}$ fails to dominate 4,
- $\{2\}$ fails to dominate 4,
- $\{3\}$ fails to dominate 1,
- $\{4\}$ fails to dominate 1.

Hence no dominating set of size 1 exists, and the domination number is

$$\gamma(H) = 2.$$

Definition 3.9.3 (Domination in an n-SuperHyperGraph)**.** (cf. [112]) Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph on a finite base set V_0.

A subset $D \subseteq V$ is called a *dominating set* of $\mathrm{SHG}^{(n)}$ if for every n-supervertex $v \in V \setminus D$ there exists an n-superedge $e \in E$ such that

$$v \in e \quad \text{and} \quad e \cap D \neq \emptyset.$$

Equivalently, every n-supervertex outside D lies in some n-superedge that also contains at least one n-supervertex from D.

The *domination number* of $\mathrm{SHG}^{(n)}$ is

$$\gamma\big(\mathrm{SHG}^{(n)}\big) := \min\{\, |D| \mid D \subseteq V \text{ is a dominating set of } \mathrm{SHG}^{(n)}\,\}.$$

Table 3.7 provides an overview of the comparison between graph, hypergraph, and superhypergraph domination.

Hereafter, we present a concrete example.

Table 3.7: Comparison of graph, hypergraph, and superhypergraph domination

Framework	Underlying structure	Domination condition
Graph domination	Vertices joined by ordinary edges; adjacency is pairwise between vertices.	A set $D \subseteq V$ such that every $v \in V \setminus D$ is adjacent to at least one vertex in D.
HyperGraph domination	Vertices joined by hyperedges; each hyperedge is a nonempty subset of V.	A set $D \subseteq V$ such that for every $v \in V \setminus D$ there exists a hyperedge e with $v \in e$ and $e \cap D \neq \emptyset$.
SuperHyperGraph domination	n-supervertices $V \subseteq \mathcal{P}^n(V_0)$ joined by n-superedges (images of the incidence map).	A set $D \subseteq V$ such that for every $v \in V \setminus D$ there exists an n-superedge e with $v \in e$ and $e \cap D \neq \emptyset$, respecting the hierarchical supervertex structure.

Example 3.9.4 (Domination in a 1-SuperHyperGraph)**.** Let the base set be

$$V_0 := \{a, b, c\}.$$

Form three 1-supervertices

$$A := \{a, b\}, \qquad B := \{b, c\}, \qquad C := \{a, c\},$$

and set

$$V := \{A, B, C\}.$$

Define two 1-superedges

$$e_1 := \{A, B\}, \qquad e_2 := \{B, C\},$$

and let

$$E := \{e_1, e_2\}.$$

Then

$$\mathrm{SHG}^{(1)} := (V, E)$$

is a 1-SuperHyperGraph.

Consider the subset

$$D := \{B\} \subseteq V.$$

We verify that D is a dominating set of $\mathrm{SHG}^{(1)}$.

For each 1-supervertex $v \in V \setminus D = \{A, C\}$:

- $v = A$: we have $A \in e_1 = \{A, B\}$ and $e_1 \cap D = \{B\} \neq \emptyset$.
- $v = C$: we have $C \in e_2 = \{B, C\}$ and $e_2 \cap D = \{B\} \neq \emptyset$.

Thus every 1-supervertex outside D belongs to some 1-superedge that also contains an element of D, so D is dominating.

No dominating set of size 0 exists, so D is a minimum dominating set and the domination number is

$$\gamma\big(\mathrm{SHG}^{(1)}\big) = 1.$$

Theorem 3.9.5 (SuperHyperGraph domination generalizes hypergraph domination)**.** *Every finite hypergraph can be viewed as a* 0*-SuperHyperGraph in such a way that dominating sets (and hence the domination number) coincide. In particular,* hypergraph domination *is a special case of* n-SuperHyperGraph domination *(for* $n = 0$*).*

Proof. Let $H = (V, E)$ be an arbitrary finite hypergraph with $\emptyset \neq V$ and $\emptyset \neq E \subseteq \mathcal{P}^*(V)$.

Take the base set $V_0 := V$ and consider the 0-fold iterated powerset $\mathcal{P}^0(V_0) = V_0$. Define the 0-SuperHyperGraph

$$\mathrm{SHG}_H^{(0)} := (V^{(0)}, E^{(0)})$$

by

$$V^{(0)} := V, \qquad E^{(0)} := E.$$

Since $E^{(0)} \subseteq \mathcal{P}^*(V^{(0)})$ and $V^{(0)} \subseteq \mathcal{P}^0(V_0)$, this is a valid 0-SuperHyperGraph on the base set V_0 in the sense of the previous definition.

We now compare dominating sets.

Let $D \subseteq V$.

(i) Suppose D is a dominating set of H. By definition of hypergraph domination, for every vertex $v \in V \setminus D$ there exists an edge $e \in E$ such that

$$v \in e \quad \text{and} \quad e \cap D \neq \emptyset.$$

But $E^{(0)} = E$ and $V^{(0)} = V$, so the same condition reads: for every $v \in V^{(0)} \setminus D$ there exists $e \in E^{(0)}$ with $v \in e$ and $e \cap D \neq \emptyset$. This is exactly the condition that D is a dominating set of $\mathrm{SHG}_H^{(0)}$.

(ii) Conversely, suppose $D \subseteq V^{(0)} = V$ is a dominating set of $\mathrm{SHG}_H^{(0)}$. By definition of 0-SuperHyperGraph domination, for every $v \in V^{(0)} \setminus D$ there exists $e \in E^{(0)}$ such that

$$v \in e \quad \text{and} \quad e \cap D \neq \emptyset.$$

Since $E^{(0)} = E$, this is exactly the condition that D is a dominating set of the original hypergraph H.

Thus

$$\{ D \subseteq V \mid D \text{ dominates } H \} = \{ D \subseteq V^{(0)} \mid D \text{ dominates } \mathrm{SHG}_H^{(0)} \}.$$

Taking minima of the cardinalities of dominating sets on both sides, we obtain

$$\gamma(H) = \gamma\big(\mathrm{SHG}_H^{(0)}\big).$$

Therefore every hypergraph domination problem can be regarded as a special case of n-SuperHyperGraph domination (with $n = 0$), and SuperHyperGraph domination strictly generalizes hypergraph domination. □

3.10 Sombor index of SuperHypergraphs

Sombor index of a graph sums, over edges, the square root of squared endpoint degrees, capturing degree-based structural complexity information [244–246]. These concepts have been further extended and studied in various settings, including chemical graphs [247, 248], fuzzy graphs [249, 250], and neutrosophic graphs [251, 252]. Moreover, related concepts such as the modified Sombor index [253–255], the Zagreb index [256–258], the Hyper-Zagreb Index [259, 260], the ABC index [261, 262], and the GA index [263, 264] are also well known.

Sombor index of a hypergraph generalizes this by summing square-rooted degree squares over each hyperedge's incident vertices within complex interactions [265]. Sombor index of a superhypergraph extends further, aggregating degree-squared contributions over multi-tier superedges, reflecting hierarchical connectivity across nested structural levels.

Definition 3.10.1 (Sombor index of a hypergraph)**.** [265] Let $H = (V, E)$ be a finite hypergraph, where V is a nonempty finite vertex set and $E \subseteq \mathcal{P}^*(V)$ is a finite family of nonempty subsets of V (the hyperedges).

For each vertex $v \in V$, the *degree* of v in H is

$$d_H(v) := \big|\{ e \in E \mid v \in e \}\big|.$$

The *Sombor index* of the hypergraph H is defined by

$$SO(H) := \sum_{e \in E} \sqrt{\sum_{v \in e} d_H(v)^2}.$$

When H is 2-uniform (i.e., every hyperedge has size 2), this reduces to the classical Sombor index of a simple graph.

Example 3.10.2 (Sombor index of a small hypergraph)**.** Consider the hypergraph

$$H = (V, E),$$

where the vertex set and hyperedge family are

$$V := \{x, y, z\}, \qquad E := \{e_1, e_2, e_3\},$$

with

$$e_1 := \{x, y\}, \qquad e_2 := \{y, z\}, \qquad e_3 := \{x, y, z\}.$$

This is a non–2-uniform hypergraph (since $|e_3| = 3$).

Step 1: Vertex degrees. For each $v \in V$, the degree $d_H(v) := |\{e \in E \mid v \in e\}|$ is:

$$d_H(x) = 2 \quad \text{(appears in } e_1, e_3),$$

$$d_H(y) = 3 \quad \text{(appears in } e_1, e_2, e_3),$$

$$d_H(z) = 2 \quad \text{(appears in } e_2, e_3).$$

Step 2: Sombor index. By definition,

$$SO(H) := \sum_{e \in E} \sqrt{\sum_{v \in e} d_H(v)^2}.$$

We compute the contribution of each hyperedge.

For $e_1 = \{x, y\}$:

$$\sum_{v \in e_1} d_H(v)^2 = d_H(x)^2 + d_H(y)^2 = 2^2 + 3^2 = 4 + 9 = 13,$$

so the contribution of e_1 is

$$\sqrt{13}.$$

For $e_2 = \{y, z\}$:

$$\sum_{v \in e_2} d_H(v)^2 = d_H(y)^2 + d_H(z)^2 = 3^2 + 2^2 = 9 + 4 = 13,$$

so the contribution of e_2 is also

$$\sqrt{13}.$$

For $e_3 = \{x, y, z\}$:

$$\sum_{v \in e_3} d_H(v)^2 = d_H(x)^2 + d_H(y)^2 + d_H(z)^2 = 2^2 + 3^2 + 2^2 = 4 + 9 + 4 = 17,$$

so the contribution of e_3 is

$$\sqrt{17}.$$

Therefore, the Sombor index of H is

$$SO(H) = \sqrt{13} + \sqrt{13} + \sqrt{17} = 2\sqrt{13} + \sqrt{17}.$$

In this example, the presence of a 3-vertex hyperedge e_3 shows how the Sombor index naturally extends beyond the graph (2-uniform) case.

Definition 3.10.3 (Degree in an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be a level-n SuperHyperGraph. For each $v \in V$, the *degree* of v in $\mathrm{SHG}^{(n)}$ is

$$d_{\mathrm{SHG}^{(n)}}(v) \ := \ \big|\{\, e \in E \mid v \in \partial(e) \,\}\big|.$$

Definition 3.10.4 (Sombor index of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be a level-n SuperHyperGraph. The *Sombor index* of $\mathrm{SHG}^{(n)}$ is defined by

$$SO\big(\mathrm{SHG}^{(n)}\big) \ := \ \sum_{e \in E} \sqrt{\sum_{v \in \partial(e)} d_{\mathrm{SHG}^{(n)}}(v)^2} \ .$$

Remark 3.10.5. If we view a hypergraph $H = (V, E)$ in incidence form by taking $\partial(e) = e$ for all $e \in E$, then the above formula coincides with the Sombor index of a hypergraph.

Table 3.8 provides an overview of the comparison of the Sombor index for graphs, hypergraphs, and superhypergraphs.

Table 3.8: Comparison of Sombor index for graphs, hypergraphs, and superhypergraphs

Framework	Underlying structure	Sombor index
Graph	Simple graph $G = (V, E)$ with pairwise edges between vertices.	$SO(G) = \sum_{uv \in E} \sqrt{d_G(u)^2 + d_G(v)^2}.$
Hypergraph	Hypergraph $H = (V, E)$ where each hyperedge $e \in E$ is a nonempty subset of V.	$SO(H) = \sum_{e \in E} \sqrt{\sum_{v \in e} d_H(v)^2}.$
SuperHypergraph	Level-n SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E, \partial)$ with n-supervertices and n-superedges.	$SO\big(\mathrm{SHG}^{(n)}\big) = \sum_{e \in E} \sqrt{\sum_{v \in \partial(e)} d_{\mathrm{SHG}^{(n)}}(v)^2}.$

Example 3.10.6 (Sombor index of a 1-SuperHyperGraph with two tiers)**.** We now consider a simple 1-SuperHyperGraph that mixes base vertices and 1-supervertices.

Step 1: Vertex sets and supervertices. Let the base set be

$$V_0 := \{a, b, c\}.$$

Form two first-tier (super)vertices

$$P := \{a, b\}, \qquad Q := \{b, c\},$$

and set

$$V_1 := \{P, Q\}.$$

The total vertex set is

$$V := V_0 \cup V_1 = \{a, b, c, P, Q\}.$$

Step 2: Superedges and boundary map. Define three superedges

$$E := \{e_1, e_2, e_3\},$$

with boundary map $\partial : E \to \mathcal{P}^*(V)$ given by

$$\partial(e_1) := \{a, P\}, \qquad \partial(e_2) := \{b, P, Q\}, \qquad \partial(e_3) := \{c, Q\}.$$

Thus

$$\mathrm{SHG}^{(1)} := (V, E, \partial)$$

is a level-1 SuperHyperGraph: a, b, c are base vertices, P, Q are 1-supervertices, and each superedge connects a small subset of these across tiers.

Step 3: Vertex degrees in $\mathrm{SHG}^{(1)}$**.** For $v \in V$, the degree $d_{\mathrm{SHG}^{(1)}}(v) := \bigl|\{e \in E \mid v \in \partial(e)\}\bigr|$ is:

$$\begin{aligned} d_{\mathrm{SHG}^{(1)}}(a) &= 1 \quad \text{(only in } \partial(e_1)),\\ d_{\mathrm{SHG}^{(1)}}(b) &= 1 \quad \text{(only in } \partial(e_2)),\\ d_{\mathrm{SHG}^{(1)}}(c) &= 1 \quad \text{(only in } \partial(e_3)),\\ d_{\mathrm{SHG}^{(1)}}(P) &= 2 \quad \text{(in } \partial(e_1) \text{ and } \partial(e_2)),\\ d_{\mathrm{SHG}^{(1)}}(Q) &= 2 \quad \text{(in } \partial(e_2) \text{ and } \partial(e_3)). \end{aligned}$$

Step 4: Sombor index of the 1-SuperHyperGraph. By definition,

$$SO\bigl(\mathrm{SHG}^{(1)}\bigr) := \sum_{e \in E} \sqrt{\sum_{v \in \partial(e)} d_{\mathrm{SHG}^{(1)}}(v)^2}\,.$$

We compute each term.

For e_1 with $\partial(e_1) = \{a, P\}$:

$$\sum_{v \in \partial(e_1)} d_{\mathrm{SHG}^{(1)}}(v)^2 = d_{\mathrm{SHG}^{(1)}}(a)^2 + d_{\mathrm{SHG}^{(1)}}(P)^2$$

$$= 1^2 + 2^2 = 1 + 4 = 5,$$

so the contribution of e_1 is

$$\sqrt{5}.$$

For e_2 with $\partial(e_2) = \{b, P, Q\}$:

$$\sum_{v \in \partial(e_2)} d_{\mathrm{SHG}^{(1)}}(v)^2 = d_{\mathrm{SHG}^{(1)}}(b)^2 + d_{\mathrm{SHG}^{(1)}}(P)^2 + d_{\mathrm{SHG}^{(1)}}(Q)^2$$

$$= 1^2 + 2^2 + 2^2 = 1 + 4 + 4 = 9,$$

so the contribution of e_2 is

$$\sqrt{9} = 3.$$

For e_3 with $\partial(e_3) = \{c, Q\}$:

$$\sum_{v \in \partial(e_3)} d_{\mathrm{SHG}^{(1)}}(v)^2 = d_{\mathrm{SHG}^{(1)}}(c)^2 + d_{\mathrm{SHG}^{(1)}}(Q)^2$$

$$= 1^2 + 2^2 = 1 + 4 = 5,$$

so the contribution of e_3 is

$$\sqrt{5}.$$

Therefore, the Sombor index of the 1-SuperHyperGraph is

$$SO\bigl(\mathrm{SHG}^{(1)}\bigr) = \sqrt{5} + 3 + \sqrt{5} = 2\sqrt{5} + 3.$$

In this example, the Sombor index incorporates both base vertices (a, b, c) and higher-tier supervertices (P, Q), reflecting how hierarchical incidences in a SuperHyperGraph contribute jointly to the overall index.

Theorem 3.10.7 (SuperHyperGraph Sombor index generalises the hypergraph Sombor index)**.** *Let $H = (V, E)$ be a finite hypergraph. Define the associated level-0 SuperHyperGraph*

$$\mathrm{SHG}_H^{(0)} := (V, E, \partial_H), \qquad \partial_H(e) := e \quad (e \in E).$$

Then

$$SO\big(\mathrm{SHG}_H^{(0)}\big) = SO(H).$$

In particular, the Sombor index of an n-SuperHyperGraph is a strict extension of the Sombor index of a hypergraph (the case $n = 0$).

Proof. By construction, V is the vertex set of both H and $\mathrm{SHG}_H^{(0)}$, and E is the edge/superedge set in both structures. For every vertex $v \in V$ we have

$$d_{\mathrm{SHG}_H^{(0)}}(v) = \big|\{\, e \in E \mid v \in \partial_H(e) \,\}\big| = \big|\{\, e \in E \mid v \in e \,\}\big| = d_H(v).$$

Thus the degree of each vertex is identical in H and in $\mathrm{SHG}_H^{(0)}$.

Next, for every edge $e \in E$ we have $\partial_H(e) = e$, so the inner sum in the SuperHyperGraph Sombor index is

$$\sum_{v \in \partial_H(e)} d_{\mathrm{SHG}_H^{(0)}}(v)^2 = \sum_{v \in e} d_H(v)^2.$$

Therefore

$$SO\big(\mathrm{SHG}_H^{(0)}\big) = \sum_{e \in E} \sqrt{\sum_{v \in \partial_H(e)} d_{\mathrm{SHG}_H^{(0)}}(v)^2} = \sum_{e \in E} \sqrt{\sum_{v \in e} d_H(v)^2} = SO(H).$$

Hence the Sombor index of an n-SuperHyperGraph extends the Sombor index of a hypergraph, and the latter is recovered exactly when we restrict to level $n = 0$ with $\partial(e) = e$. □

3.11 SuperHyperGraph Labeling

Graph labeling assigns labels to a graph's vertices and/or edges so that specified constraints hold, typically driven by adjacency, incidence, or distances [266–268]. As related concepts in graph labeling, graceful labeling [269, 270], magic labeling [271, 272], harmonious labeling [273, 274], and lucky labeling [275, 276] are well known. Graph labeling has also been studied within the frameworks of fuzzy graphs, neutrosophic graphs, and related generalized graph models (e.g., [277–280]).

Hypergraph labeling assigns labels to vertices and/or hyperedges so that incidence-based or overlap-based constraints are satisfied across the hypergraph structure [281–283]. SuperHyperGraph labeling assigns labels to n-supervertices and/or n-superedges so that constraints respect the super-incidence structure induced at level n [284]. SuperHyperGraph multilabeling assigns several label components to each supervertex and/or superedge, requiring these components to satisfy coupled constraints and remain consistent [284].

Definition 3.11.1 (Graph Labeling)**.** [266–268] Let $G = (V, E)$ be a finite graph and let L_V and L_E be nonempty label alphabets. A (possibly partial) *graph labeling* on G consists of maps

$$\ell_V : V \to L_V \quad \text{and/or} \quad \ell_E : E \to L_E,$$

where either map may be omitted if not used. A *labeling schema* is a predicate $\Phi(G; \ell_V, \ell_E)$ built from adjacency/incidence in G, the graph distance dist_G on V, and fixed relations/operations on the alphabets (e.g. $=$, $\neq$, order, arithmetic, etc.). We call (ℓ_V, ℓ_E) *valid* (for Φ) if $\Phi(G; \ell_V, \ell_E)$ holds.

Example 3.11.2 (An $L(2, 1)$ vertex-labeling on the path P_5)**.** Let P_5 have vertices v_1, v_2, v_3, v_4, v_5 and edges $v_i v_{i+1}$ for $i = 1, 2, 3, 4$. Let $L_V = \mathbb{Z}$ and define a vertex labeling $\ell_V : V(P_5) \to \mathbb{Z}$ by

$$\ell_V(v_1) = 0,\ \ell_V(v_2) = 2,\ \ell_V(v_3) = 4,\ \ell_V(v_4) = 1,\ \ell_V(v_5) = 3.$$

Let Φ encode the $L(2,1)$ constraints:

$$\mathrm{dist}_{P_5}(u,v)=1 \Rightarrow |\ell_V(u)-\ell_V(v)|\geq 2, \qquad \mathrm{dist}_{P_5}(u,v)=2 \Rightarrow |\ell_V(u)-\ell_V(v)|\geq 1.$$

Verification (distance 1):

$$|0-2|=2,\ |2-4|=2,\ |4-1|=3,\ |1-3|=2.$$

Verification (distance 2):

$$|0-4|=4,\ |2-1|=1,\ |4-3|=1.$$

Hence $\Phi(P_5;\ell_V,\emptyset)$ holds, so ℓ_V is a valid $L(2,1)$ graph labeling.

Definition 3.11.3 (HyperGraph Labeling)**.** [281–283] Let $H=(V,E)$ be a finite hypergraph with $E\subseteq \mathcal{P}(V)\setminus\{\emptyset\}$, and let L_V, L_E be nonempty alphabets. A (possibly partial) *hypergraph labeling* consists of maps

$$\ell_V : V \to L_V \quad \text{and/or} \quad \ell_E : E \to L_E.$$

Write $G(H)$ for the *primal (2-section) graph* of H on vertex set V, where $\{u,v\}$ is an edge of $G(H)$ iff $u\neq v$ and $\exists e\in E$ with $\{u,v\}\subseteq e$. Let dist_H denote the usual graph distance in $G(H)$. A *labeling schema* is a predicate $\Phi(H;\ell_V,\ell_E)$ built from the incidence relation $v\in e$, the distance dist_H on V, and fixed relations/operations on the alphabets. We call (ℓ_V,ℓ_E) *valid* if $\Phi(H;\ell_V,\ell_E)$ holds.

Example 3.11.4 (Strong hyperedge coloring as a hypergraph labeling)**.** Let $V=\{1,2,3,4\}$ and let $E=\{e_1,e_2\}$ with

$$e_1=\{1,2,3\}, \qquad e_2=\{3,4\}.$$

Take $L_V=\{r,g,b\}$ (colors) and define ℓ_V by

$$\ell_V(1)=r, \quad \ell_V(2)=g, \quad \ell_V(3)=b, \quad \ell_V(4)=r.$$

Let Φ require *strong hyperedge coloring*: for every hyperedge $e\in E$, the set $\{\ell_V(v): v\in e\}$ is pairwise distinct. Then for e_1 we have $\{r,g,b\}$ (all distinct), and for e_2 we have $\{b,r\}$ (distinct). Hence $\Phi(H;\ell_V,\emptyset)$ holds, so ℓ_V is a valid hypergraph labeling.

Definition 3.11.5 (SuperHyperGraph Labeling)**.** [284] Fix $n\in\mathbb{N}_0$ and a finite base set V_0. Write $\mathcal{P}^0(V_0)=V_0$ and $\mathcal{P}^{k+1}(V_0)=\mathcal{P}(\mathcal{P}^k(V_0))$. An *n-SuperHyperGraph* is a pair $\mathrm{SHG}(n)=(V,E)$ where

$$V\subseteq \mathcal{P}^n(V_0), \qquad \emptyset\neq E\subseteq \mathcal{P}(V)\setminus\{\emptyset\}.$$

Its *primal (2-section) graph* is $G(\mathrm{SHG}(n))=(V,E')$ where $\{X,Y\}\in E'$ iff $X\neq Y$ and $\exists F\in E$ with $\{X,Y\}\subseteq F$. Let $\mathrm{dist}_{\mathrm{SHG}}$ be the graph distance in $G(\mathrm{SHG}(n))$.

Let L_V, L_E be nonempty alphabets. A (possibly partial) *SuperHyperGraph labeling* consists of maps

$$\ell_V : V \to L_V \quad \text{and/or} \quad \ell_E : E \to L_E.$$

A *labeling schema* is a predicate $\Phi(\mathrm{SHG}(n);\ell_V,\ell_E)$ built from: the incidence relation $X\in F$ (supervertex–superedge), the distance $\mathrm{dist}_{\mathrm{SHG}}$ on V, and the set-membership relations available inside $\mathcal{P}^k(V_0)$ for $0\leq k\leq n$ (e.g. $x\in X$ when $X\in\mathcal{P}(V_0)$). We call (ℓ_V,ℓ_E) *valid* if $\Phi(\mathrm{SHG}(n);\ell_V,\ell_E)$ holds.

Example 3.11.6 (An $L(2,1)$ vertex-labeling on a level-$n=1$ SuperHyperGraph)**.** Let $V_0=\{a,b,c,d\}$ and $n=1$. Define supervertices (elements of $\mathcal{P}(V_0)$):

$$A=\{a,b\}, \quad B=\{b,c\}, \quad C=\{c,d\}, \quad D=\{a,d\},$$

and let $V=\{A,B,C,D\}$. Define superedges (nonempty subsets of V):

$$F_1=\{A,B,C\}, \qquad F_2=\{A,D\},$$

and set $\mathrm{SHG}(1)=(V,E)$ with $E=\{F_1,F_2\}$.

Take $L_V=\mathbb{Z}$ and define $\ell_V : V\to\mathbb{Z}$ by

$$\ell_V(A)=0, \quad \ell_V(B)=2, \quad \ell_V(C)=4, \quad \ell_V(D)=3.$$

Let Φ encode $L(2,1)$ constraints using $\mathrm{dist}_{\mathrm{SHG}}$:

$$\mathrm{dist}_{\mathrm{SHG}}(X,Y)=1 \Rightarrow |\ell_V(X)-\ell_V(Y)|\ge 2, \qquad \mathrm{dist}_{\mathrm{SHG}}(X,Y)=2 \Rightarrow |\ell_V(X)-\ell_V(Y)|\ge 1.$$

In the primal graph, A is adjacent to B, C, D and B is adjacent to C (via F_1), so

$$|0-2|=2,\ |0-4|=4,\ |0-3|=3,\ |2-4|=2$$

for all distance-1 pairs, and

$$\mathrm{dist}_{\mathrm{SHG}}(D,B)=2,\ |3-2|=1; \qquad \mathrm{dist}_{\mathrm{SHG}}(D,C)=2,\ |3-4|=1.$$

Hence $\Phi(\mathrm{SHG}(1);\ell_V,\emptyset)$ holds, so ℓ_V is a valid SuperHyperGraph labeling.

Definition 3.11.7 (SuperHyperGraph MultiLabeling)**.** [284] Let $\mathrm{SHG}(n)=(V,E)$ be an n-SuperHyperGraph over a base set V_0. Define the *flattening* (base support) maps $\flat_n : \mathcal{P}^n(V_0)\to\mathcal{P}(V_0)$ recursively by

$$\flat_0(x)=x\ (x\in V_0), \qquad \flat_{k+1}(X)=\bigcup_{Y\in X}\flat_k(Y)\ (X\in\mathcal{P}^{k+1}(V_0)).$$

Thus $\flat_n(X)\subseteq V_0$ is the set of base elements that occur inside the n-level object X.

Fix nonnegative integers p,q and choose nonempty alphabets

$$L_V^{(1)},\dots,L_V^{(p)}, \qquad L_E^{(1)},\dots,L_E^{(q)}.$$

A *SuperHyperGraph MultiLabeling* on $\mathrm{SHG}(n)$ consists of coordinate maps

$$\ell_V^{(a)} : V\to L_V^{(a)}\ (1\le a\le p), \qquad \ell_E^{(b)} : E\to L_E^{(b)}\ (1\le b\le q),$$

equivalently $\ell_V : V\to\prod_{a=1}^{p}L_V^{(a)}$ and $\ell_E : E\to\prod_{b=1}^{q}L_E^{(b)}$.

A *multilabeling schema* is a predicate $\Phi(\mathrm{SHG}(n);\ell_V,\ell_E)$ built from the incidence relation $X\in F$, the primal distance $\mathrm{dist}_{\mathrm{SHG}}$ on V, the membership relations inside $\mathcal{P}^k(V_0)$ $(0\le k\le n)$, and the flattening operator $\flat_n$ (plus fixed relations/operations on the alphabets). We call (ℓ_V,ℓ_E) *valid* if $\Phi(\mathrm{SHG}(n);\ell_V,\ell_E)$ holds.

Example 3.11.8 (A distance-aware $(p,q)=(2,1)$ multilabel on a level-$n=1$ SuperHyperGraph)**.** Let $V_0=\{a,b,c,d\}$ and $n=1$. Define supervertices

$$X_1=\{a,b\},\quad X_2=\{b,c\},\quad X_3=\{c,d\},$$

set $V=\{X_1,X_2,X_3\}$, and define superedges

$$F_1=\{X_1,X_2\}, \qquad F_2=\{X_2,X_3\},$$

so $\mathrm{SHG}(1)=(V,E)$ with $E=\{F_1,F_2\}$. In the primal graph we have $\mathrm{dist}_{\mathrm{SHG}}(X_1,X_2)=1$, $\mathrm{dist}_{\mathrm{SHG}}(X_2,X_3)=1$, and $\mathrm{dist}_{\mathrm{SHG}}(X_1,X_3)=2$.

Take alphabets $L_V^{(1)}=\mathbb{Z}$, $L_V^{(2)}=\mathbb{N}$, and $L_E^{(1)}=\mathbb{N}$. Define the coordinate maps by

$$\ell_V^{(1)}(X_1)=0,\ \ell_V^{(1)}(X_2)=2,\ \ell_V^{(1)}(X_3)=4,$$
$$\ell_V^{(2)}(X_1)=2,\ \ell_V^{(2)}(X_2)=2,\ \ell_V^{(2)}(X_3)=2,$$
$$\ell_E^{(1)}(F_1)=1, \qquad \ell_E^{(1)}(F_2)=1.$$

Let Φ require the following three constraints:

$$\text{(L2,1)}\quad \mathrm{dist}_{\mathrm{SHG}}(X,Y)=1\Rightarrow|\ell_V^{(1)}(X)-\ell_V^{(1)}(Y)|\ge 2,\quad \mathrm{dist}_{\mathrm{SHG}}(X,Y)=2\Rightarrow|\ell_V^{(1)}(X)-\ell_V^{(1)}(Y)|\ge 1;$$

$$\text{(Sup)}\quad \ell_V^{(2)}(X)=|\flat_1(X)| \text{ for all } X\in V; \qquad \text{(Int)}\quad \ell_E^{(1)}(F)=\left|\bigcap_{X\in F}\flat_1(X)\right| \text{ for all } F\in E.$$

Verification:

$$|\ell_V^{(1)}(X_1)-\ell_V^{(1)}(X_2)|=|0-2|=2,\quad |\ell_V^{(1)}(X_2)-\ell_V^{(1)}(X_3)|=|2-4|=2,\quad |0-4|=4\ (\mathrm{dist}=2),$$

and

$$\flat_1(X_1)=\{a,b\},\ \flat_1(X_2)=\{b,c\},\ \flat_1(X_3)=\{c,d\},$$

so $|\flat_1(X_i)|=2$ for all i, and

$$|\flat_1(X_1)\cap\flat_1(X_2)|=|\{b\}|=1=\ell_E^{(1)}(F_1), \qquad |\flat_1(X_2)\cap\flat_1(X_3)|=|\{c\}|=1=\ell_E^{(1)}(F_2).$$

Hence $\Phi(\mathrm{SHG}(1);\ell_V,\ell_E)$ holds, so (ℓ_V,ℓ_E) is a valid SuperHyperGraph MultiLabeling.

3.12 SuperHyperGraph Grammar

A graph grammar rewrites labeled rank-two edges by replacement graphs, gluing two ordered attachment vertices, generating terminal graphs [285, 286]. Related concepts such as fuzzy graph grammars [287–289], digraph grammars [290,291], attributed graph grammars [292,293], and molecular graph grammars [294,295] are also known. A hypergraph grammar rewrites labeled hyperedges of arity r(A) by replacement hypergraphs, identifying ordered attachment tuples, generating terminals [296–298]. A superhypergraph grammar rewrites labeled superhyperedges by ranked n-superhypergraphs with port orderings, gluing external vertices, producing terminal superhypergraphs.

Definition 3.12.1 (Ranked hypergraphs in ordered attachment form)**.** Let Σ be a set of edge labels equipped with an arity map $r : \Sigma \to \mathbb{N}_{\geq 1}$. A *ranked* Σ*-hypergraph* is a tuple

$$H = (V, E, \mathrm{lab}, \mathrm{att})$$

where V is a finite vertex set, E is a finite edge set, $\mathrm{lab} : E \to \Sigma$, and

$$\mathrm{att}(e) = (v_1, \ldots, v_{r(\mathrm{lab}(e))}) \in V^{r(\mathrm{lab}(e))}$$

is the ordered attachment tuple of e.

Definition 3.12.2 (Graph grammar (rank-2 hyperedge replacement))**.** Let $\Sigma = N \cup T$ be a disjoint union of *nonterminals* N and *terminals* T, with arity map $r : \Sigma \to \mathbb{N}_{\geq 1}$ satisfying $r(a) = 2$ for all $a \in \Sigma$. A *graph grammar* (hyperedge-replacement, rank 2) is a tuple

$$\mathcal{G}_G = (N, T, S, P)$$

where $S \in N$ and each production is $A \Rightarrow R$ with $A \in N$ and R a ranked Σ-hypergraph together with an ordered list of external vertices

$$\mathrm{ext}(R) = (u_1, u_2) \in V(R)^2.$$

A derivation starts from the single-edge handle of S and repeatedly replaces a nonterminal edge labeled A by a fresh copy of R, identifying the two attachment vertices of the replaced edge with u_1, u_2, and deleting the replaced edge. The language $L(\mathcal{G}_G)$ is the set of terminal graphs obtained.

Definition 3.12.3 (Hypergraph grammar (hyperedge replacement))**.** Let $\Sigma = N \cup T$ be a disjoint union, with arity map $r : \Sigma \to \mathbb{N}_{\geq 1}$. A *hypergraph grammar* (hyperedge-replacement) is a tuple

$$\mathcal{G}_H = (N, T, S, P)$$

where $S \in N$ and each production is $A \Rightarrow R$ with $A \in N$ and R a ranked Σ-hypergraph equipped with an ordered list of external vertices

$$\mathrm{ext}(R) = (u_1, \ldots, u_{r(A)}) \in V(R)^{r(A)}.$$

A derivation starts from the single-edge handle of S and repeatedly replaces a nonterminal edge e labeled A (with attachment tuple $(x_1, \ldots, x_{r(A)})$) by a fresh copy of R, identifying u_i with x_i for all i, and deleting e. The language $L(\mathcal{G}_H)$ is the set of terminal hypergraphs obtained.

Definition 3.12.4 (Ranked n-superhypergraphs with ports)**.** Fix $n \in \mathbb{N}_0$, a base set V_0, a label set Σ and an arity map $r : \Sigma \to \mathbb{N}_{\geq 1}$. A *ranked n-superhypergraph* is a tuple

$$\mathsf{S} = (V, E, \mathrm{lab}, \partial, \mathrm{port})$$

such that (V, E, ∂) is an n-SuperHyperGraph over V_0, $\mathrm{lab} : E \to \Sigma$, and for each $e \in E$ we have a bijection (a port-ordering)

$$\mathrm{port}_e : [r(\mathrm{lab}(e))] \xrightarrow{\cong} \partial(e).$$

Equivalently, each edge has an ordered attachment tuple $\mathrm{att}(e) = (\mathrm{port}_e(1), \ldots, \mathrm{port}_e(r(\mathrm{lab}(e))))$.

Example 3.12.5 (A hypergraph grammar generating all finite loose paths (rank 2))**.** Let $\Sigma = N \cup T$ where $N = \{A\}$ and $T = \{t\}$, and let the arity map be

$$r(A) = 2, \qquad r(t) = 2.$$

We define a hyperedge-replacement hypergraph grammar

$$\mathcal{G}_H = (N, T, S, P) \quad \text{with} \quad S := A.$$

Intuitively, terminals t will form a *loose path* of 2-edges (i.e., an ordinary path graph seen as a 2-uniform hypergraph). A derivation starts from the handle consisting of one nonterminal edge labeled A.

Define two productions (both of rank 2):

(1) *Stop (produce one terminal edge).*

$$A \Rightarrow R_{\text{stop}},$$

where R_{stop} is the ranked hypergraph with vertex set

$$V(R_{\text{stop}}) = \{u_1, u_2\},$$

edge set

$$E(R_{\text{stop}}) = \{e\},$$

label $\text{lab}(e) = t$, attachment $\text{att}(e) = (u_1, u_2)$, and externals

$$\text{ext}(R_{\text{stop}}) = (u_1, u_2).$$

(2) *Extend (add one terminal edge and keep one nonterminal for further growth).*

$$A \Rightarrow R_{\text{ext}},$$

where R_{ext} has vertex set

$$V(R_{\text{ext}}) = \{u_1, u_2, w\},$$

edge set

$$E(R_{\text{ext}}) = \{e_1, e_2\},$$

labels

$$\text{lab}(e_1) = t, \qquad \text{lab}(e_2) = A,$$

attachments

$$\text{att}(e_1) = (u_1, w), \qquad \text{att}(e_2) = (w, u_2),$$

and externals

$$\text{ext}(R_{\text{ext}}) = (u_1, u_2).$$

Then $L(\mathcal{G}_H)$ is exactly the set of finite loose paths: each derivation applies R_{ext} some number of times and finally applies R_{stop}, yielding a terminal 2-uniform hypergraph whose hyperedges form a path.

Definition 3.12.6 (n-SuperHyperGraph grammar (superhyperedge replacement)). Let $n \in \mathbb{N}_0$, let $\Sigma = N \cup T$ be a disjoint union with arity map r, and fix a base set V_0. An *n-superhypergraph grammar* is a tuple

$$\mathcal{G}_S^{(n)} = (N, T, S, P; V_0)$$

where $S \in N$ and each production is $A \Rightarrow R$ with $A \in N$ and

$$R = (V_R, E_R, \text{lab}_R, \partial_R, \text{port}^R)$$

a ranked n-superhypergraph together with an ordered list of external vertices

$$\text{ext}(R) = (u_1, \dots, u_{r(A)}) \in V_R^{r(A)}.$$

A derivation starts from the single-edge handle of S (as a ranked n-superhypergraph) and repeatedly: choose a nonterminal edge e labeled A with ordered attachment tuple $(x_1, \dots, x_{r(A)})$, take a fresh copy of R, identify u_i with x_i for all $i \in [r(A)]$, and delete e. The language $L(\mathcal{G}_S^{(n)})$ is the set of terminal n-superhypergraphs obtained.

Example 3.12.7 (A 2-SuperHyperGraph grammar generating a chain of supervertices (rank 2))**.** Fix a base set

$$V_0 := \{a, b, c\}, \qquad n = 2.$$

Define the level-2 supervertex set (each element is a subset of $\mathcal{P}(V_0)$, hence lies in $\mathcal{P}^2(V_0)$)

$$p_a := \{a\}, \quad p_b := \{b\}, \quad p_c := \{c\},$$

$$v_L := \{p_a\}, \quad v_M := \{p_b\}, \quad v_R := \{p_c\}, \qquad V := \{v_L, v_M, v_R\} \subseteq \mathcal{P}^2(V_0).$$

Let $\Sigma = N \cup T$ with $N = \{A\}, T = \{t\}$, and arities

$$r(A) = 2, \qquad r(t) = 2.$$

We define a 2-superhypergraph grammar

$$\mathcal{G}_S^{(2)} = (N, T, S, P; V_0) \quad \text{with} \quad S := A.$$

We specify two productions $A \Rightarrow R_{\text{stop}}$ and $A \Rightarrow R_{\text{ext}}$. In both, the external list has length 2 and the port maps impose an order on each (super)edge incidence set.

(1) *Stop.* Let $R_{\text{stop}} = (V_R, E_R, \text{lab}_R, \partial_R, \text{port}^R)$ where

$$V_R = \{v_L, v_R\}, \qquad E_R = \{e\}, \qquad \text{lab}_R(e) = t,$$

$$\partial_R(e) = \{v_L, v_R\}, \qquad \text{port}_e^R(1) = v_L,\ \text{port}_e^R(2) = v_R,$$

and

$$\text{ext}(R_{\text{stop}}) = (v_L, v_R).$$

Thus R_{stop} produces one terminal superedge connecting the two external supervertices.

(2) *Extend.* Let $R_{\text{ext}} = (V_R, E_R, \text{lab}_R, \partial_R, \text{port}^R)$ where

$$V_R = \{v_L, v_M, v_R\}, \qquad E_R = \{e_1, e_2\}, \qquad \text{lab}_R(e_1) = t,\ \text{lab}_R(e_2) = A,$$

$$\partial_R(e_1) = \{v_L, v_M\}, \quad \partial_R(e_2) = \{v_M, v_R\},$$

$$\text{port}_{e_1}^R(1) = v_L,\ \text{port}_{e_1}^R(2) = v_M, \qquad \text{port}_{e_2}^R(1) = v_M,\ \text{port}_{e_2}^R(2) = v_R,$$

and

$$\text{ext}(R_{\text{ext}}) = (v_L, v_R).$$

Starting from the single-edge handle of $S = A$, repeated application of R_{ext} inserts a new intermediate supervertex (a fresh copy of v_M) and a terminal superedge to its left, while keeping a nonterminal superedge to the right. A final application of R_{stop} terminates the derivation. Hence $L(\mathcal{G}_S^{(2)})$ consists of terminal 2-superhypergraphs whose terminal superedges labeled t form a (super)edge-chain between the two original external supervertices.

Theorem 3.12.8 (n-superhypergraph grammars generalize hypergraph grammars)**.** *Let $\mathcal{G}_H = (N, T, S, P)$ be a hypergraph grammar over (Σ, r).*

(i) For $n = 0$, there exists a 0-superhypergraph grammar $\mathcal{G}_S^{(0)}$ such that (up to the notational identification of hypergraphs with 0-SuperHyperGraphs)

$$L(\mathcal{G}_S^{(0)}) \;=\; L(\mathcal{G}_H).$$

(ii) More generally, for any $n \in \mathbb{N}_0$ there exists an n-superhypergraph grammar $\mathcal{G}_S^{(n)}$ and an embedding ι_n from terminal hypergraphs to terminal n-superhypergraphs such that

$$L(\mathcal{G}_S^{(n)}) \;=\; \iota_n\big(L(\mathcal{G}_H)\big).$$

Proof. (i) View every ranked Σ-hypergraph $H = (V, E, \mathrm{lab}, \mathrm{att})$ as a ranked 0-superhypergraph $\mathsf{S}(H) = (V, E, \mathrm{lab}, \partial, \mathrm{port})$ over base set $V_0 := V$, defined by

$$\partial(e) \ := \ \{v_1, \dots, v_{r(\mathrm{lab}(e))}\} \quad \text{if } \mathrm{att}(e) = (v_1, \dots, v_{r(\mathrm{lab}(e))}),$$

and $\mathrm{port}_e(i) := v_i$ for each i. This is well-defined because $V \subseteq \mathcal{P}^0(V_0) = V_0$ and $\partial(e) \in \mathcal{P}^*(V)$. Now define $\mathcal{G}_S^{(0)}$ to have the same N, T, S, P, but interpret every right-hand side R of a rule $A \Rightarrow R$ as $\mathsf{S}(R)$ and keep the same external list. A single replacement step in $\mathcal{G}_H$ replaces an edge with attachment tuple $(x_1, \dots, x_{r(A)})$ by gluing the external vertices $(u_1, \dots, u_{r(A)})$ of R onto that tuple. Because $\mathsf{S}(\cdot)$ preserves the ordered attachment tuple via the port maps, the same gluing and deletion operation is exactly a replacement step in $\mathcal{G}_S^{(0)}$. Hence derivations correspond step-by-step, and terminal objects coincide (up to the identification above), so $L(\mathcal{G}_S^{(0)}) = L(\mathcal{G}_H)$.

(ii) Fix $n \in \mathbb{N}_0$ and define the n-fold singleton nesting map $s_n : V_0 \to \mathcal{P}^n(V_0)$ by

$$s_0(v) := v, \qquad s_{k+1}(v) := \{s_k(v)\} \ \ (k \geq 0).$$

Given a ranked Σ-hypergraph $H = (V, E, \mathrm{lab}, \mathrm{att})$, define

$$\iota_n(H) \ := \ \big(V', E, \mathrm{lab}, \partial', \mathrm{port}'\big)$$

where $V' := \{s_n(v) : v \in V\} \subseteq \mathcal{P}^n(V_0)$ (choose V_0 to contain all vertex names used), and for each edge e with $\mathrm{att}(e) = (v_1, \dots, v_{r(\mathrm{lab}(e))})$ set

$$\partial'(e) := \{s_n(v_1), \dots, s_n(v_{r(\mathrm{lab}(e))})\}, \qquad \mathrm{port}'_e(i) := s_n(v_i).$$

Thus $\iota_n(H)$ is a ranked n-superhypergraph and preserves attachment order.

Construct $\mathcal{G}_S^{(n)}$ from $\mathcal{G}_H$ by applying ι_n to every right-hand side R of a production $A \Rightarrow R$, and by applying s_n to the external vertex list: if $\mathrm{ext}(R) = (u_1, \dots, u_{r(A)})$, set $\mathrm{ext}(\iota_n(R)) = (s_n(u_1), \dots, s_n(u_{r(A)}))$. Now check one-step rewriting: replacing a nonterminal edge e in a sentential form identifies its attachment vertices $(x_1, \dots, x_{r(A)})$ with the external vertices of the chosen rule. Since ι_n maps each attachment vertex x_i to $s_n(x_i)$ and each external vertex u_i to $s_n(u_i)$, the identifications commute with ι_n:

$$s_n(u_i) \text{ identified with } s_n(x_i) \quad \Longleftrightarrow \quad u_i \text{ identified with } x_i.$$

Therefore, applying a rule in $\mathcal{G}_H$ and then embedding by ι_n yields the same result as first embedding the sentential form and then applying the corresponding rule in $\mathcal{G}_S^{(n)}$. By induction on the length of derivations, terminal derivations correspond, and terminal languages satisfy $L(\mathcal{G}_S^{(n)}) = \iota_n(L(\mathcal{G}_H))$. □

Chapter 4

Some Particular SuperHyperGraphs

In graph theory and hypergraph theory, many graph classes have been studied extensively. Here, a *graph class* is a family of graphs defined by shared properties or constraints, typically closed under specified operations, which enables systematic analysis and the design of algorithms [299]. Studying graph classes reveals structural patterns, yields efficient algorithms for restricted inputs, clarifies complexity boundaries, and enables transferable theorems across graphs sharing forbidden substructures. In this chapter, we present the description and formulation of several particular types of SuperHyperGraphs.

4.1 Directed SuperHyperGraph

As discussed above, graphs are widely applied across numerous domains. However, when modeling concepts that inherently involve directional information, the use of *directed graphs* becomes essential [300,301]. Several extensions of directed graphs are known, including fuzzy directed graphs [302], intuitionistic fuzzy directed graphs [303], rough directed graphs [304], soft directed graphs [305,306], and neutrosophic directed graphs [307–309]. These structures have been further extended to *directed hypergraphs* [310–312] and *directed superhypergraphs* [73,313,314], which have attracted growing research interest.

Definition 4.1.1 (Directed Hypergraph)**.** (cf. [315,316]) A *directed hypergraph* is a pair

$$H = (V, E),$$

where

- V is a finite set of *vertices*.
- E is a finite set of *hyperarcs*, each hyperarc $e \in E$ being an ordered pair

  $$e = \big(T(e), H(e)\big) \in \mathcal{P}(V) \times \mathcal{P}(V),$$

  with

  $$T(e) \subseteq V,\ T(e) \neq \emptyset, \quad H(e) \subseteq V,\ H(e) \neq \emptyset.$$

Intuitively, each $e = (T(e), H(e))$ carries "flow" from all vertices in $T(e)$ (the *tail*) to all vertices in $H(e)$ (the *head*).

Example 4.1.2 (A simple directed hypergraph: project dependency flow)**.** Consider the finite vertex set

$$V := \{\text{Spec, Design, Implement, Test}\},$$

representing the phases of a software project: Specification, Design, Implementation, and Testing.

Define the set of hyperarcs E by

$$E := \{e_1, e_2\},$$

where

$$e_1 := \bigl(T(e_1), H(e_1)\bigr) := \bigl(\{\text{Spec}\},\ \{\text{Design}, \text{Implement}\}\bigr),$$

$$e_2 := \bigl(T(e_2), H(e_2)\bigr) := \bigl(\{\text{Design}, \text{Implement}\},\ \{\text{Test}\}\bigr).$$

Then

$$H := (V, E)$$

is a directed hypergraph in the sense of Definition 4.1.1. The hyperarc e_1 models that once the specification is completed, both Design and Implementation can start, while e_2 models that Testing starts only after both Design and Implementation are available.

Definition 4.1.3 (Directed n-SuperHyperGraph)**.** (cf. [2,73]) Let S be a nonempty *base set* and let $n \geq 0$ be an integer. Define iterated powersets by

$$\mathcal{P}^0(S) = S, \quad \mathcal{P}^{k+1}(S) = \mathcal{P}\bigl(\mathcal{P}^k(S)\bigr) \quad (k \geq 0).$$

A *directed n-SuperHyperGraph* is a pair

$$\mathrm{DSHG}^{(n)} = (V,\ E),$$

where

$$V \subseteq \mathcal{P}^n(S), \quad E \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S),$$

and each directed n-superedge $e \in E$ is an ordered pair

$$e = \bigl(\mathrm{Tail}(e),\ \mathrm{Head}(e)\bigr), \quad \mathrm{Tail}(e),\ \mathrm{Head}(e) \subseteq \mathcal{P}^n(S),$$

typically both nonempty. Such an e carries "flow" from the entire set $\mathrm{Tail}(e)$ of n-supervertices into $\mathrm{Head}(e)$.

Example 4.1.4 (A simple directed 1-SuperHyperGraph: information flow between groups)**.** Let the base set of individuals be

$$S := \{u_1, u_2, u_3, u_4\},$$

so that

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^1(S) = \mathcal{P}(S).$$

We construct a directed 1-SuperHyperGraph, where each vertex is a group of individuals.

Define the set of 1-supervertices

$$V := \bigl\{A := \{u_1, u_2\},\ B := \{u_2, u_3, u_4\},\ C := \{u_3\}\bigr\} \subseteq \mathcal{P}^1(S),$$

and the set of directed 1-superedges

$$E := \{e_1, e_2\} \subseteq \mathcal{P}^1(S) \times \mathcal{P}^1(S),$$

where

$$e_1 := \bigl(\mathrm{Tail}(e_1), \mathrm{Head}(e_1)\bigr) := \bigl(\{A\},\ \{B\}\bigr),$$

$$e_2 := \bigl(\mathrm{Tail}(e_2), \mathrm{Head}(e_2)\bigr) := \bigl(\{B\},\ \{C\}\bigr).$$

Then

$$\mathrm{DSHG}^{(1)} := (V, E)$$

is a directed 1-SuperHyperGraph in the sense of Definition 4.1.3 with $n = 1$. Here, e_1 represents information flowing from the group $\{u_1, u_2\}$ to the larger group $\{u_2, u_3, u_4\}$, and e_2 represents subsequent forwarding from $\{u_2, u_3, u_4\}$ to the single individual $\{u_3\}$.

For reference, Table 4.1 provides information comparing directed graphs, directed hypergraphs, and directed n-SuperHyperGraphs.

Table 4.1: Comparison of directed graphs, directed hypergraphs, and directed n-SuperHyperGraphs

Framework	Vertices	Directed edge / arc structure
Directed graph	Finite vertex set V.	Arc set $A \subseteq V \times V$; each arc (u, v) carries direction from u (tail) to v (head).
Directed hypergraph	Finite vertex set V.	Hyperarc set E, each $e \in E$ is an ordered pair $(T(e), H(e))$ with nonempty $T(e), H(e) \subseteq V$; direction flows from all $T(e)$ to all $H(e)$.
Directed n-SuperHyperGraph	n-supervertex set $V \subseteq \mathcal{P}^n(S)$ over a base set S.	Directed n-superedge set $E \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S)$; each $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ carries flow from a family of n-supervertices to another, across hierarchical levels.

4.2 Bidirected SuperHyperGraph

One of the well-known extended notions of a directed graph is the bidirected graph. A bidirected graph assigns to each vertex–edge incidence a sign indicating whether the edge is locally directed toward or away from that vertex [317–320]. A bidirected hypergraph assigns such signs to vertex–hyperedge incidences, requiring that the signed values on each hyperedge sum to zero [321]. A bidirected superhypergraph assigns signs to supervertex–superedge incidences, again imposing that each superedge has total signed sum zero [5, 321].

Definition 4.2.1 (Bidirected Graph). [317] A *bidirected graph* (also called a *bigraph*) is a pair

$$B = (G, \tau),$$

where $G = (V, E)$ is a simple undirected graph (no loops and no parallel edges), and

$$\tau : V \times E \to \{-1, 0, 1\}$$

is a *bidirection function* such that for every vertex–edge pair (v, e):

1. $\tau(v, e) = 1$ means that the edge e is locally directed *towards* v;
2. $\tau(v, e) = -1$ means that the edge e is locally directed *away from* v;
3. $\tau(v, e) = 0$ means that v is not incident to e.

The graph G is called the *underlying graph* of B.

Example 4.2.2 (A concrete bidirected graph). Let

$$V = \{a, b, c\}, \qquad E = \{e_1, e_2\}, \qquad e_1 = \{a, b\},\ e_2 = \{b, c\}.$$

Define $\tau : V \times E \to \{-1, 0, 1\}$ by the table

$\tau(v, e)$	e_1	e_2
$v = a$	1	0
$v = b$	1	-1
$v = c$	0	1

Interpretation:

- On $e_1 = \{a, b\}$ we have $\tau(a, e_1) = 1$ and $\tau(b, e_1) = 1$, so e_1 is "towards" both a and b.
- On $e_2 = \{b, c\}$ we have $\tau(b, e_2) = -1$ (away from b) and $\tau(c, e_2) = 1$ (towards c).

All other pairs (v, e) have $\tau(v, e) = 0$ exactly when $v \notin e$.

Definition 4.2.3 (Bidirected Hypergraph)**.** [321] A *bidirected hypergraph* is a triple

$$H = (V, E, \tau),$$

where V is a nonempty set of vertices, E is a family of nonempty subsets of V (hyperedges), and

$$\tau : V \times E \to \{-1, 0, 1\}$$

is a bidirection function satisfying:

$$\tau(v, e) = 0 \iff v \notin e,$$

and, additionally, for each hyperedge $e \in E$ we impose the *balancing condition*

$$\sum_{v \in e} \tau(v, e) = 0.$$

Example 4.2.4 (A concrete bidirected hypergraph)**.** Let

$$V = \{1, 2, 3, 4\}, \qquad E = \{e, f\}, \qquad e = \{1, 2, 3, 4\},\ f = \{2, 3\}.$$

Define $\tau : V \times E \to \{-1, 0, 1\}$ by

$$\tau(1, e) = 1,\ \tau(2, e) = -1,\ \tau(3, e) = 1,\ \tau(4, e) = -1,$$

and

$$\tau(2, f) = 1,\ \tau(3, f) = -1,$$

and $\tau(v, e') = 0$ whenever $v \notin e'$.

Verification of the balancing condition:

$$\sum_{v \in e} \tau(v, e) = 1 + (-1) + 1 + (-1) = 0, \qquad \sum_{v \in f} \tau(v, f) = 1 + (-1) = 0.$$

Definition 4.2.5 (Bidirected Superhypergraph)**.** [321] A *bidirected superhypergraph* is a quadruple

$$\mathcal{H} = (V, S, E, \tau),$$

where:

1. V is a nonempty set of (base) vertices;
2. S is a set of nonempty subsets of V, called *supervertices*;
3. E is a family of *superedges*, where each $e \in E$ is a nonempty subset of S;
4. $\tau : S \times E \to \{-1, 0, 1\}$ is a bidirection function such that

$$\tau(s, e) = 0 \iff s \notin e,$$

and for each superedge $e \in E$ we impose the *balancing condition*

$$\sum_{s \in e} \tau(s, e) = 0.$$

Example 4.2.6 (A concrete bidirected superhypergraph)**.** Let the base vertex set be

$$V = \{a, b, c, d\}.$$

Define supervertices (nonempty subsets of V) by

$$s_1 = \{a, b\}, \quad s_2 = \{b, c\}, \quad s_3 = \{d\}, \quad s_4 = \{a, d\}, \qquad S = \{s_1, s_2, s_3, s_4\}.$$

Define superedges by

$$E = \{E_1, E_2\}, \qquad E_1 = \{s_1, s_2, s_3, s_4\}, \quad E_2 = \{s_1, s_3\}.$$

Framework	Objects	Incidence/sign rule (bidirection)
Bidirected graph	$B = (G, \tau)$, $G = (V, E)$	$\tau : V \times E \to \{-1, 0, 1\}$; $\tau(v, e) = 0 \Leftrightarrow v \notin e$. $\tau(v, e) = 1$ (towards v), -1 (away from v).
Bidirected hypergraph	$H = (V, E, \tau)$	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$; $\tau : V \times E \to \{-1, 0, 1\}$, $\tau(v, e) = 0 \Leftrightarrow v \notin e$, and *balance* on each hyperedge: $\sum_{v \in e} \tau(v, e) = 0$.
Bidirected SuperHyper-Graph	$\mathcal{H} = (V, S, E, \tau)$	$S \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ (supervertices), $E \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$ (superedges); $\tau : S \times E \to \{-1, 0, 1\}$, $\tau(s, e) = 0 \Leftrightarrow s \notin e$, and *balance* on each superedge: $\sum_{s \in e} \tau(s, e) = 0$.

Table 4.2: Concise overview of bidirected graphs, bidirected hypergraphs, and bidirected SuperHyperGraphs.

Define $\tau : S \times E \to \{-1, 0, 1\}$ by

$$\tau(s_1, E_1) = 1,\ \tau(s_2, E_1) = -1,\ \tau(s_3, E_1) = 1,\ \tau(s_4, E_1) = -1,$$

$$\tau(s_1, E_2) = 1,\ \tau(s_3, E_2) = -1,$$

and $\tau(s, e) = 0$ whenever $s \notin e$.

Verification of the balancing condition:

$$\sum_{s \in E_1} \tau(s, E_1) = 1 + (-1) + 1 + (-1) = 0, \qquad \sum_{s \in E_2} \tau(s, E_2) = 1 + (-1) = 0.$$

For reference, an overview of bidirected graphs, bidirected hypergraphs, and bidirected SuperHyperGraphs is presented in Table 4.2.

4.3 Multidirected SuperHyperGraph

Multidirected graph is a directed graph that permits multiple parallel directed edges between the same ordered pair of vertices, so edge multiplicity represents repeated or layered interactions [322, 323]. Multidirected hypergraph is a directed hypergraph in which each hyperedge has a tail set and a head set (or head vertex), and multiple identical directed hyperedges are allowed [314]. Multidirected SuperHyperGraph is a multidirected hypergraph whose vertices are level- n supervertices (iterated-powerset objects), with directed superhyperedges between supervertex sets and possible multiplicities [314].

Definition 4.3.1 (Multidirected Graph)**.** [322,323] A *multidirected graph* is a 5-tuple

$$G = (V, E, s, t, m),$$

where V is a finite set of vertices, E is a finite set of directed edges (allowing repetitions), $s : E \to V$ assigns the *source* of each edge, $t : E \to V$ assigns the *target* of each edge, and

$$m : V \times V \to \mathbb{N}_0$$

is a *multiplicity function* such that $m(u, v)$ counts how many edges are directed from u to v.

Example 4.3.2 (Parallel Data Channels)**.** Let $V = \{A, B, C\}$ (three servers). Take directed edges

$$E = \{f_{AB}^{(1)}, f_{AB}^{(2)}, f_{BC}, f_{CA}^{(1)}, f_{CA}^{(2)}, f_{CA}^{(3)}\},$$

with

$$s\Big(f_{AB}^{(i)}\Big) = A, \quad t\Big(f_{AB}^{(i)}\Big) = B\ (i = 1, 2), \qquad s(f_{BC}) = B,\ t(f_{BC}) = C,$$

$$s\Big(f_{CA}^{(j)}\Big) = C, \quad t\Big(f_{CA}^{(j)}\Big) = A\ (j = 1, 2, 3).$$

Define $m(A, B) = 2$, $m(B, C) = 1$, $m(C, A) = 3$, and $m(u, v) = 0$ otherwise. Then $G = (V, E, s, t, m)$ models the directed channels together with their parallel counts.

Type	Edge object	Keyword-style description
Undirected graph	$E \subseteq \{\{u, v\} : u, v \in V\}$	No orientation; adjacency is symmetric.
Directed graph (digraph)	$A \subseteq V \times V$	Arcs (u, v); orientation from tail u to head v.
Bidirected graph	$G = (V, E)$ with incidence signs $\tau : V \times E \to \{-1, 0, 1\}$	Local direction at each incidence (v, e); each edge has two signed ends.
Multidirected graph	Parallel arcs allowed (multiplicity $m(u, v) \in \mathbb{N}_0$)	Multiple directed edges between the same ordered pair; models repeated/layered flow.

Table 4.3: Concise overview of undirected, directed, bidirected, and multidirected graphs.

For reference, Table 4.3 presents an overview of undirected, directed, bidirected, and multidirected graphs.

Definition 4.3.3 (Multidirected Hypergraph)**.** [314] A *multidirected hypergraph* is a triple

$$H = (V, E, m),$$

where V is a finite vertex set, E is a finite set of directed hyperedges, and $m : E \to \mathbb{N}$ assigns a positive integer multiplicity to each hyperedge. Each hyperedge $e \in E$ is an ordered pair

$$e = \big(T(e), h(e)\big),$$

where $T(e) \subseteq V$ is a nonempty *tail* (a set of sources) and $h(e) \in V$ is a *head* (a single target). The value $m(e)$ records how many parallel instances of e occur.

Example 4.3.4 (Collaborative Report Workflow)**.** Let $V = \{\text{Hiroko}, \text{Shinya}, \text{Masahiro}, \text{Tae}\}$. Define two hyperedges (collaborations sending drafts to a manager)

$$e_1 = \big(\{\text{Hiroko}, \text{Shinya}\}, \text{Tae}\big), \qquad e_2 = \big(\{\text{Shinya}, \text{Masahiro}\}, \text{Tae}\big),$$

and set $m(e_1) = 2$ (two distinct monthly drafts) and $m(e_2) = 3$ (three distinct weekly drafts). Then $H = (V, \{e_1, e_2\}, m)$ encodes directed group-to-individual submissions with multiplicities.

Definition 4.3.5 (Multidirected n-SuperHyperGraph (Multidirected Superhypergraph))**.** [314] Fix an integer $n \geq 1$ and a finite base set V_0. Define iterated powersets by $\mathcal{P}^0(V_0) := V_0$ and $\mathcal{P}^{k+1}(V_0) := \mathcal{P}(\mathcal{P}^k(V_0))$. A *multidirected n-SuperHyperGraph* is a triple

$$SH = (V, E, m),$$

where

$$V \subseteq \mathcal{P}^n(V_0)$$

is a set of *n-supervertices*, and

$$E \subseteq \mathcal{P}(V) \times \mathcal{P}(V)$$

is a set of directed *n-superhyperedges*. Each $e \in E$ is an ordered pair

$$e = \big(T(e), H(e)\big)$$

with nonempty tail $T(e) \subseteq V$ and nonempty head $H(e) \subseteq V$. Finally, $m : E \to \mathbb{N}$ assigns a positive integer multiplicity to each directed n-superhyperedge.

Example 4.3.6 (Committees and Councils)**.** Let $V_0 = \{\text{Hiroko}, \text{Shinya}, \text{Tae}, \text{Masahiro}\}$. Form three committees in $\mathcal{P}^1(V_0)$:

$$C_1 = \{\text{Hiroko}, \text{Shinya}\}, \quad C_2 = \{\text{Shinya}, \text{Tae}\}, \quad C_3 = \{\text{Tae}, \text{Masahiro}\}.$$

Create two councils (2-supervertices) in $\mathcal{P}^2(V_0)$:

$$v_I = \{C_1, C_2\}, \qquad v_{II} = \{C_2, C_3\}, \qquad V = \{v_I, v_{II}\} \subseteq \mathcal{P}^2(V_0).$$

Define directed 2-superhyperedges

$$e_1 = \big(\{v_I\}, \{v_{II}\}\big), \qquad e_2 = \big(\{v_I, v_{II}\}, \{v_I\}\big),$$

with multiplicities $m(e_1) = 5$ and $m(e_2) = 2$. Then $SH = (V, \{e_1, e_2\}, m)$ captures hierarchical groupings (as supervertices) and repeated directed exchanges (as multiplicities).

For reference, an overview of multidirected graphs, multidirected hypergraphs, and multidirected n-SuperHyperGraphs is presented in Table 4.4.

Framework	Vertices	(Multi)directed edge structure
Multidirected Graph	Vertex set V	Directed edges with multiplicity; parallel arcs allowed between the same ordered pair $(u, v) \in V \times V$ (e.g. via a multiplicity map $m(u, v) \in \mathbb{N}_0$).
Multidirected HyperGraph	Vertex set V	Directed hyperedges with multiplicity; each hyperarc has a tail and head (e.g. $e = (T(e), H(e))$ with nonempty $T(e), H(e) \subseteq V$), and identical hyperarcs may occur multiple times.
Multidirected n-SuperHyperGraph	n-supervertex set $V \subseteq \mathcal{P}^n(V_0)$ over a base set V_0	Directed n-superhyperedges with multiplicity; each $e = (T(e), H(e))$ has nonempty $T(e), H(e) \subseteq V$ (families of n-supervertices), and parallel copies are recorded by $m(e) \in \mathbb{N}$.

Table 4.4: Concise overview of multidirected graphs, multidirected hypergraphs, and multidirected n-SuperHyperGraphs.

4.4 Mixed SuperHyperGraph

A mixed graph is a graph on a single vertex set that allows both undirected edges and directed arcs simultaneously [324, 325]. As related concepts, fuzzy mixed graphs [326–328] and neutrosophic mixed graphs [329] are also well known. A mixed hypergraph is a hypergraph that permits both undirected hyperedges and directed hyperedges (with tails and heads) in one model [314]. A mixed superhypergraph is a superhypergraph whose supervertices are nested-set objects and whose superedges may be undirected or directed [5].

Definition 4.4.1 (Mixed Graph)**.** [324, 325] A *mixed graph* is a pair

$$G = (V, E \cup A),$$

where $V \neq \emptyset$ is a set of vertices, $E \subseteq \{\{u, v\} : u, v \in V,\ u \neq v\}$ is a set of *undirected edges*, and $A \subseteq \{(u, v) \in V \times V : u \neq v\}$ is a set of *directed edges* (*arcs*). Thus, a mixed graph may contain both undirected and directed adjacencies.

Definition 4.4.2 (Mixed HyperGraph)**.** [5] A *mixed hypergraph* is a pair

$$H = (V, E \cup A),$$

where $V \neq \emptyset$ is a set of vertices, E is a set of *undirected hyperedges* with

$$E \subseteq \{e \subseteq V : e \neq \emptyset,\ |e| \geq 2\},$$

and A is a set of *directed hyperedges* (*dyperedges*) of the form

$$A \subseteq \{(Z, z) : Z \subseteq V \setminus \{z\},\ Z \neq \emptyset,\ z \in V\}.$$

Here (Z, z) represents a directed relation from the *tail set* Z to the *head* vertex z.

Definition 4.4.3 (Mixed SuperHyperGraph)**.** [5] A *mixed superhypergraph* (or *mixed SuperHyperGraph*) is a quadruple

$$\mathsf{H} = (V, S, E, A),$$

where:

1. $V \neq \emptyset$ is a set of (base) vertices,
2. $S \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a set of *supervertices* (each $s \in S$ is a nonempty subset of V),
3. $E \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$ is a set of *undirected superedges* (each $e \in E$ is a nonempty subset of S),
4. $A \subseteq \{(Z, z) : Z \subseteq S \setminus \{z\},\ Z \neq \emptyset,\ z \in S\}$ is a set of *directed superedges* (or *super-dyperedges*), so (Z, z) represents a directed relation from the tail supervertex-set Z to the head supervertex z.

Example 4.4.4 (A small mixed SuperHyperGraph)**.** Let the base vertex set be

$$V := \{1, 2, 3, 4\}.$$

Define the supervertex set

$$S := \{s_1, s_2, s_3\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}, \qquad s_1 := \{1, 2\}, \ \ s_2 := \{2, 3\}, \ \ s_3 := \{4\}.$$

Define one undirected superedge

$$E := \{e_1\} \subseteq \mathcal{P}(S) \setminus \{\emptyset\}, \qquad e_1 := \{s_1, s_2\}.$$

Define two directed superedges (super-dyperedges)

$$A := \{(Z_1, z_1), (Z_2, z_2)\},$$

where

$$(Z_1, z_1) := (\{s_1\}, s_3), \qquad (Z_2, z_2) := (\{s_2, s_3\}, s_1).$$

Then

$$\mathsf{H} = (V, S, E, A)$$

is a mixed SuperHyperGraph in the sense of Definition *(Mixed SuperHyperGraph)*: it contains an undirected superedge e_1 connecting s_1 and s_2, and directed superedges $\{s_1\} \to s_3$ and $\{s_2, s_3\} \to s_1$.

4.5 Multi-SuperHyperGraph

A Multi-Superhypergraph is a loopless n-SuperHyperGraph allowing parallel superedges, modeling multiset-style higher-order connections among repeated or weighted supervertex groups [5]. A Multi-SuperHyperGraph is an extension of both MultiGraphs [330, 331] and MultiHyperGraphs [332, 333]. Also, related concepts such as Fuzzy MultiGraphs [331, 334], Directed Multigraphs [19, 335–338], Soft multigraphs [339, 340], and Neutrosophic MultiGraphs [341, 342] are known.

Definition 4.5.1 (Undirected multigraph)**.** [330, 331] Let V be a nonempty set of vertices. Write

$$[V]^2 := \big\{ \{u, v\} \subseteq V \mid u \neq v \big\}, \qquad [V]^{\leq 2} := [V]^2 \cup \big\{ \{v\} \mid v \in V \big\}.$$

An *(undirected) multigraph* is a triple

$$G = (V, E, \partial),$$

where E is a finite set of edges and

$$\partial : E \longrightarrow [V]^{\leq 2}$$

is the *endpoint map*. For $e \in E$, the set $\partial(e)$ is the (unordered) set of endpoints of e.

- e is a *loop* if $|\partial(e)| = 1$.
- Distinct edges $e_1 \neq e_2$ are *parallel* if $\partial(e_1) = \partial(e_2)$.
- For $F \in [V]^{\leq 2}$, the *multiplicity* of F is

$$m_G(F) \ := \ \big|\{ e \in E \mid \partial(e) = F \}\big|.$$

Equivalently, a multigraph is specified by a function $\mu_G : [V]^{\leq 2} \to \mathbb{N}$ with finite support, where $\mu_G(F) = m_G(F)$.

Definition 4.5.2 (Loopless multigraph)**.** A multigraph $G = (V, E, \partial)$ is *loopless* if $|\partial(e)| = 2$ for every $e \in E$ (equivalently, $\mu_G(\{v\}) = 0$ for all $v \in V$).

Definition 4.5.3 (Multihypergraph)**.** [332, 333] Let V be a nonempty set. Put

$$\mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\}.$$

A *multihypergraph* is a triple

$$H = (V, E, \partial),$$

where E is a finite set of hyperedges and

$$\partial : E \longrightarrow \mathcal{P}^*(V)$$

is the *boundary map*. For $e \in E$, the set $\partial(e)$ is the vertex-set incident with e.

- e is a *loop hyperedge* if $|\partial(e)| = 1$.
- Distinct hyperedges $e_1 \neq e_2$ are *parallel* if $\partial(e_1) = \partial(e_2)$.
- For $F \in \mathcal{P}^*(V)$, the *multiplicity* of F is

$$m_H(F) := \big|\{ e \in E \mid \partial(e) = F \}\big|.$$

Equivalently, a multihypergraph is specified by a function $\mu_H : \mathcal{P}^*(V) \to \mathbb{N}$ with finite support, where $\mu_H(F) = m_H(F)$.

Definition 4.5.4 (Loopless multihypergraph)**.** A multihypergraph $H = (V, E, \partial)$ is *loopless* if $|\partial(e)| \geq 2$ for every $e \in E$ (equivalently, $\mu_H(\{v\}) = 0$ for all $v \in V$).

Definition 4.5.5 (Loops, parallel superedges, and multiplicity)**.** Let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be an n-SuperHyperGraph.

- A superedge $e \in E$ is called a *loop superedge* if $\big|\partial(e)\big| = 1$.
- Two distinct superedges $e_1, e_2 \in E$ are called *parallel* if

$$\partial(e_1) = \partial(e_2).$$

- For $F \in \mathcal{P}^*(V)$, the *multiplicity* of the incidence pattern F is

$$m(F) := \big|\{ e \in E \mid \partial(e) = F \}\big|.$$

Thus the family of superedges can be viewed as a finite multiset $\{\partial(e) \mid e \in E\}$ of nonempty subsets of V together with the multiplicity function m.

Definition 4.5.6 (Multi n-SuperHyperGraph)**.** [5] An n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E, \partial)$ is called a *Multi n-SuperHyperGraph* if

- it is *loopless*, i.e. $\big|\partial(e)\big| \geq 2$ for every $e \in E$;
- parallel superedges are allowed, i.e. the incidence map ∂ is not required to be injective, so some $F \in \mathcal{P}^*(V)$ may satisfy $m(F) \geq 2$.

Equivalently, a Multi n-SuperHyperGraph is an n-SuperHyperGraph whose edge family is an arbitrary finite multiset of nonempty subsets of V of size at least 2. For $n = 0$ this reduces to the usual notion of a loopless multihypergraph.

Example 4.5.7 (A simple Multi 1-SuperHyperGraph with parallel superedges)**.** We construct a loopless 1-SuperHyperGraph with parallel superedges, modeling repeated collaborations between the same employee–team pair.

Step 1: Vertex tiers. Let the base (tier 0) vertex set be

$$V_0 := \{p, q\},$$

where p and q represent two employees.

Define a tier 1 vertex

$$T := \{p, q\},$$

representing a team that consists of both employees p and q. Set

$$V_1 := \{T\}.$$

The total vertex set of the 1-SuperHyperGraph is

$$V := V_0 \cup V_1 = \{p, q, T\}.$$

Step 2: Superedges and boundary map. We introduce three superedges

$$E := \{e_1, e_2, e_3\}$$

and define the boundary map $\partial : E \to \mathcal{P}^*(V)$ by

$$\partial(e_1) := \{p, T\}, \qquad \partial(e_2) := \{p, T\}, \qquad \partial(e_3) := \{q, T\}.$$

Intuitively:

- e_1 and e_2 encode two different projects where employee p works with team T;
- e_3 encodes a project where employee q works with the same team T.

Thus

$$\mathrm{SHG}^{(1)} := (V, E, \partial)$$

is a level-1 SuperHyperGraph whose vertices come from two tiers (p, q at tier 0 and T at tier 1), and whose superedges connect these across tiers.

Step 3: Loopless property. A loop superedge is defined by the condition $\left|\partial(e)\right| = 1$. Here we have

$$\partial(e_1) = \{p, T\}, \quad \partial(e_2) = \{p, T\}, \quad \partial(e_3) = \{q, T\},$$

so

$$|\partial(e_1)| = |\partial(e_2)| = |\partial(e_3)| = 2 \geq 2.$$

Hence there are *no* loop superedges, and $\mathrm{SHG}^{(1)}$ is loopless.

Step 4: Parallel superedges and multiplicity. By definition, two distinct superedges $e_1, e_2 \in E$ are parallel if $\partial(e_1) = \partial(e_2)$.

In this example,

$$\partial(e_1) = \{p, T\} \quad \text{and} \quad \partial(e_2) = \{p, T\},$$

so e_1 and e_2 are parallel superedges.

For the incidence pattern

$$F := \{p, T\} \in \mathcal{P}^*(V),$$

its multiplicity is

$$m(F) := \left|\{\, e \in E \mid \partial(e) = F \,\}\right| = \left|\{e_1, e_2\}\right| = 2.$$

For the other incidence pattern $\{q, T\}$, we have

$$m(\{q, T\}) = 1,$$

corresponding to the single superedge e_3.

Therefore, the edge family of $\mathrm{SHG}^{(1)}$ can be viewed as the finite multiset

$$\{\partial(e) \mid e \in E\} = \big\{\{p, T\},\ \{p, T\},\ \{q, T\}\big\},$$

with multiplicities $m(\{p, T\}) = 2$ and $m(\{q, T\}) = 1$.

Conclusion. The structure

$$\mathrm{SHG}^{(1)} = (V, E, \partial)$$

is loopless (every superedge is incident with at least two vertices) and has parallel superedges (e_1 and e_2 share the same incidence pattern). Hence it is a *Multi* 1*-SuperHyperGraph* in the sense of the definition: its superedge family is an arbitrary finite multiset of nonempty subsets of V of size at least 2.

4.6 Semi-SuperHyperGraph

Semigraphs generalize graphs by using ordered vertex tuples as edges, permitting varying edge sizes and controlled pairwise intersections among them [42, 343, 344]. Semihypergraphs extend semigraph tuples to hyperedges, using ordered vertex sequences of arbitrary length, with restricted intersections and reversal equivalence criteria [43,345]. A Semi-Superhypergraph is an n-SuperHyperGraph without parallel superedges, permitting loops, thus representing unique incidence patterns with possible self-connected supervertices [5].

Definition 4.6.1 (SemiGraph (Semigraph))**.** [42, 343, 344] Let V be a nonempty set, and let E be a set of ordered tuples of distinct vertices from V, each of length at least 2. A *semigraph* is a pair $G = (V, E)$ satisfying:

(1) *Intersection condition:* any two edges in E have at most one vertex in common.

(2) *Reversal equivalence:* for edges $E_1 = (u_1, \ldots, u_m)$ and $E_2 = (v_1, \ldots, v_n)$, we consider $E_1 = E_2$ iff $m = n$ and either $u_i = v_i$ for all i, or $u_i = v_{n-i+1}$ for all i.

Definition 4.6.2 (SemiHyperGraph (Semihypergraph))**.** [43] A *semihypergraph* is a pair $H_s = (V, E_h)$ where V is a finite nonempty set of vertices and E_h is a set of ordered tuples of distinct vertices from V, each of length at least 2, such that:

(1) *Intersection condition:* any two hyperedges in E_h have at most one vertex in common.

(2) *Reversal equivalence:* for hyperedges $E^h_m = (u_1, \ldots, u_m)$ and $E^h_n = (v_1, \ldots, v_n)$, we consider $E^h_m = E^h_n$ iff $m = n$ and either $u_i = v_i$ for all i, or $u_i = v_{n-i+1}$ for all i.

Example 4.6.3 (A simple SemiHyperGraph (Semihypergraph))**.** Let

$$V := \{1, 2, 3, 4, 5, 6\}.$$

Define a family of ordered hyperedges (tuples of distinct vertices, each of length ≥ 2) by

$$E_h := \big\{\, e_1 := (1, 2, 3),\ e_2 := (3, 4),\ e_3 := (5, 6) \,\big\}.$$

Then $H_s = (V, E_h)$ is a semihypergraph:

(1) *Intersection condition.* We have

$$e_1 \cap e_2 = \{3\}, \qquad e_1 \cap e_3 = \emptyset, \qquad e_2 \cap e_3 = \emptyset,$$

so any two hyperedges share at most one vertex.

(2) *Reversal equivalence.* For instance, the tuple $(1, 2, 3)$ represents the same hyperedge as its reversal $(3, 2, 1)$, while it is different from $(1, 3, 2)$ because that is neither identical to $(1, 2, 3)$ nor its reversal.

Thus $H_s = (V, E_h)$ is a concrete example of a SemiHyperGraph.

Definition 4.6.4 (Semi n-SuperHyperGraph)**.** [5] An n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E, \partial)$ is called a *Semi n-SuperHyperGraph* if

- parallel superedges are forbidden, i.e. ∂ is injective:

$$\partial(e_1) = \partial(e_2) \implies e_1 = e_2 \quad \text{for all } e_1, e_2 \in E;$$

- loop superedges are allowed, i.e. we permit superedges $e \in E$ with $\left|\partial(e)\right| = 1$.

Thus a Semi n-SuperHyperGraph is an n-SuperHyperGraph with no parallel superedges but with possible loops. For $n = 0$ this is the natural analogue of a loop-allowing simple hypergraph (no parallel hyperedges).

Example 4.6.5 (A simple Semi 1-SuperHyperGraph)**.** Let the base set of elements be

$$S := \{a, b, c\}.$$

Then

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^1(S) = \mathcal{P}(S).$$

Define two 1-supervertices

$$A := \{a, b\}, \qquad B := \{b, c\},$$

and set

$$V := \{A, B\} \subseteq \mathcal{P}^1(S).$$

Next, take a set of superedges

$$E := \{e_1, e_2\},$$

and define the incidence map

$$\partial : E \longrightarrow \mathcal{P}^*(V)$$

by

$$\partial(e_1) := \{A, B\}, \qquad \partial(e_2) := \{A\}.$$

Then

$$\mathrm{SHG}^{(1)} := (V, E, \partial)$$

is a 1-SuperHyperGraph. We now check the Semi 1-SuperHyperGraph conditions:

- There are no parallel superedges, because $\partial(e_1) = \{A, B\} \neq \{A\} = \partial(e_2)$, so ∂ is injective on E.
- A loop superedge is present: e_2 satisfies $\left|\partial(e_2)\right| = |\{A\}| = 1$, so e_2 is a loop at the supervertex A.

Therefore, $\mathrm{SHG}^{(1)}$ is a Semi 1-SuperHyperGraph: it has no parallel superedges, but it does allow a loop on the supervertex A.

Table 4.5 presents an overview of SemiGraphs, SemiHyperGraphs, and Semi n-SuperHyperGraphs.

Model	Carrier objects	Edge / hyperedge / superedge rule (informal)
SemiGraph	Vertices V	Edges are ordered tuples $(u_1, \ldots, u_m)$ of distinct vertices ($m \geq 2$); any two edges intersect in at most one vertex; tuple reversal is identified.
SemiHyperGraph	Vertices V	Hyperedges are ordered tuples $(u_1, \ldots, u_m)$ of distinct vertices ($m \geq 2$); any two hyperedges intersect in at most one vertex; tuple reversal is identified.
Semi n-SuperHyperGraph	n-supervertices $V^{(n)} \subseteq \mathrm{P}_n(V_0)$	Superedges are subsets of $V^{(n)}$ (incidence patterns); no parallel superedges (incidence map injective); loop superedges $\lvert\partial(e)\rvert = 1$ are allowed.

Table 4.5: Overview of SemiGraphs, SemiHyperGraphs, and Semi n-SuperHyperGraphs

4.7 Pseudo-SuperHyperGraph

A pseudograph is a graph allowing loops and parallel edges, modeling repeated connections and self-interactions between vertices simultaneously, explicitly, often [346, 347]. Related concepts such as fuzzy pseudographs are also known [348–350]. A pseudo-hypergraph permits repeated hyperedges and loops, allowing identical vertex-subsets as distinct hyperedges for multiplicity in models of complex systems [351, 352]. A Pseudo-Superhypergraph is the most general n-SuperHyperGraph, allowing both loops and parallel superedges, capturing arbitrary nonempty incidence configurations [5].

Definition 4.7.1 (Pseudo n-SuperHyperGraph)**.** [5] An n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E, \partial)$ is called a *Pseudo n-SuperHyperGraph* if both loop superedges and parallel superedges are allowed. Equivalently, we impose only the basic conditions

$$\emptyset \neq \partial(e) \subseteq V \quad \text{for all } e \in E,$$

and place no restrictions on the cardinalities $\lvert\partial(e)\rvert$ or on the injectivity of ∂. Hence a Pseudo n-SuperHyperGraph is the most general (nonempty-incidence) form of n-SuperHyperGraph; Multi and Semi n-SuperHyperGraphs appear as special cases obtained by forbidding loops or parallel superedges, respectively. For $n = 0$ this coincides with the usual notion of a pseudohypergraph (loops and multiple hyperedges allowed).

Example 4.7.2 (A simple Pseudo 1-SuperHyperGraph)**.** Let the base set be

$$S := \{x, y\}.$$

Then

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^1(S) = \mathcal{P}(S).$$

Define two 1-supervertices

$$A := \{x\}, \qquad B := \{x, y\},$$

and set

$$V := \{A, B\} \subseteq \mathcal{P}^1(S).$$

Next, take a set of superedges

$$E := \{e_1, e_2, e_3\},$$

and define the incidence map

$$\partial : E \longrightarrow \mathcal{P}^*(V)$$

by

$$\partial(e_1) := \{A, B\}, \qquad \partial(e_2) := \{A, B\}, \qquad \partial(e_3) := \{B\}.$$

Then
$$\mathrm{SHG}^{(1)} := (V, E, \partial)$$
is a 1-SuperHyperGraph. It is a Pseudo 1-SuperHyperGraph because:

- e_1 and e_2 are parallel superedges, since $\partial(e_1) = \partial(e_2) = \{A, B\}$.
- e_3 is a loop superedge at the supervertex B, since $\left|\partial(e_3)\right| = |\{B\}| = 1$.
- No further restrictions are imposed on ∂, apart from $\emptyset \neq \partial(e) \subseteq V$ for all $e \in E$.

Thus (V, E, ∂) provides a concrete example of a Pseudo 1-SuperHyperGraph.

4.8 Directed Multi-SuperHyperGraph

A directed multigraph is a directed graph that permits parallel directed edges between the same ordered pair of vertices, thereby recording multiplicities of repeated interactions [353–355]. A directed multihypergraph is a directed hypergraph that permits parallel directed hyperedges (hyperarcs) from a tail set of vertices to a head set of vertices, so that one can model repeated higher-order directed relations. A directed multi-superhypergraph is a tiered directed multihypergraph whose vertex universe may include supervertices built via iterated powersets, and it permits parallel directed superhyperedges from a nonempty tail supervertex-set to a nonempty head supervertex-set.

Definition 4.8.1 (Directed multigraph)**.** (cf. [353–355]) A *directed multigraph* is a tuple
$$G = (V, A, s, t),$$
where $V \neq \emptyset$ is a set of vertices, A is a (finite or infinite) set of *arc identifiers*, and $s, t : A \to V$ are the *source* and *target* maps. An element $a \in A$ represents the directed arc $s(a) \to t(a)$.

Two distinct arcs $a_1 \neq a_2$ are *parallel* if $(s(a_1), t(a_1)) = (s(a_2), t(a_2))$. An arc a is a *loop* if $s(a) = t(a)$.

If A is finite, then G is called a *finite* directed multigraph.

Remark 4.8.2 (Multiplicity-function presentation)**.** Equivalently, a directed multigraph can be given by a multiplicity function
$$\mu : V \times V \to \mathbb{N}$$
(with $\mathbb{N} = \{0, 1, 2, \dots\}$), where $\mu(u, v)$ is the number of parallel arcs from u to v. This is equivalent to (V, A, s, t) whenever $\sum_{(u,v)\in V\times V} \mu(u, v) < \infty$ (finite support).

Definition 4.8.3 (Directed multihypergraph)**.** A *directed multihypergraph* is a directed hypergraph
$$H = (V, E, \partial^-, \partial^+)$$
in which *parallel hyperarcs are allowed*, i.e. (∂^-, ∂^+) is not required to be injective. Thus, distinct $e_1 \neq e_2$ may satisfy
$$\big(\partial^-(e_1), \partial^+(e_1)\big) = \big(\partial^-(e_2), \partial^+(e_2)\big),$$
and this equality encodes multiplicity.

A hyperarc e is a *loop-hyperarc* if $\partial^-(e) = \partial^+(e)$ (in particular, a *vertex-loop* occurs when $\partial^-(e) = \partial^+(e) = \{v\}$ for some $v \in V$).

If E is finite, then H is called a *finite* directed multihypergraph.

Remark 4.8.4 (Multiplicity-function presentation and reduction)**.** Equivalently, a directed multihypergraph can be represented by a multiplicity function
$$\mu : \mathcal{P}^*(V) \times \mathcal{P}^*(V) \to \mathbb{N},$$
where $\mu(T, H)$ counts how many directed hyperarcs have tail T and head H, typically assuming finite support for a finite structure.

A directed multigraph is recovered as the special case where every tail and head is a singleton:
$$\partial^-(e) = \{u\},\ \partial^+(e) = \{v\} \quad \Longleftrightarrow \quad e \text{ is an arc } u \to v \text{ (with possible multiplicity).}$$

Definition 4.8.5 (n-tier supervertex universe)**.** Fix an integer $n \geq 0$ and a nonempty base set V_0. For each $k \in \{1, \ldots, n\}$ choose a set

$$V_k \subseteq \mathcal{P}^k(V_0),$$

whose elements are called *level-k supervertices*. Define the total (tiered) supervertex universe by

$$\widehat{V}^{(n)} := \bigcup_{k=0}^{n} V_k.$$

Definition 4.8.6 (Directed Multi n-SuperHyperGraph)**.** Fix $n \geq 0$ and a tiered supervertex universe $\widehat{V}^{(n)} = \bigcup_{k=0}^{n} V_k$ as above. A *directed multi n-superhypergraph* is a tuple

$$\mathrm{DMSHG}^{(n)} := \big(V_0, (V_k)_{k=1}^{n}, E, \partial^-, \partial^+\big),$$

where E is a set of *directed superedge identifiers* and

$$\partial^-, \partial^+ : E \longrightarrow \mathcal{P}^*\big(\widehat{V}^{(n)}\big)$$

are the *tail* and *head* incidence maps. For each $e \in E$, the ordered pair

$$e : \partial^-(e) \longrightarrow \partial^+(e)$$

is a directed superedge from the (nonempty) tail set to the (nonempty) head set.

Two distinct directed superedges $e_1 \neq e_2$ are called *parallel* if

$$\partial^-(e_1) = \partial^-(e_2) \quad \text{and} \quad \partial^+(e_1) = \partial^+(e_2).$$

Definition 4.8.7 (Multiplicity of a directed incidence pattern)**.** Let $\mathrm{DMSHG}^{(n)}$ be a directed multi n-superhypergraph. For $(T, H) \in \mathcal{P}^*(\widehat{V}^{(n)}) \times \mathcal{P}^*(\widehat{V}^{(n)})$ define its *multiplicity* by

$$m(T, H) := \big|\{\, e \in E \mid \partial^-(e) = T,\ \partial^+(e) = H \,\}\big|.$$

Thus the directed superedge family may be viewed as a finite or infinite multiset of ordered pairs (T, H) with $T, H \neq \emptyset$.

Remark 4.8.8 (Loop edges and the loopless variant)**.** A directed superedge e may satisfy $\partial^-(e) = \partial^+(e)$; this is a (set-)loop. If one wishes to forbid such loops, one may additionally require $\partial^-(e) \neq \partial^+(e)$ for all $e \in E$; the present definition does not impose this.

Example 4.8.9 (A concrete directed multi 1-superhypergraph with parallel directed superedges)**.** We model a small workflow in which two analysts repeatedly submit reports to the same review committee, and the committee forwards the approved material to an archive.

Step 1: Tiered supervertex universe. Let the base (tier 0) vertex set be

$$V_0 := \{a, b, c\},$$

where a, b are two analysts and c is an archivist.

Define tier 1 supervertices (subsets of V_0):

$$R := \{a, b\}, \qquad C := \{a, b, c\}.$$

Set

$$V_1 := \{R, C\} \subseteq \mathcal{P}(V_0), \qquad \widehat{V}^{(1)} := V_0 \cup V_1 = \{a, b, c, R, C\}.$$

Step 2: Directed superedge identifiers. Let

$$E := \{e_1, e_2, e_3\}.$$

Intuitively, e_1 and e_2 represent two distinct submissions of the same type (hence parallel), while e_3 represents forwarding to the archive.

Step 3: Tail/head incidence maps. Define

$$\partial^-, \partial^+ : E \to \mathcal{P}^*\big(\widehat{V}^{(1)}\big)$$

by

$$\partial^-(e_1) = \{a, R\}, \quad \partial^+(e_1) = \{C\},$$
$$\partial^-(e_2) = \{a, R\}, \quad \partial^+(e_2) = \{C\},$$
$$\partial^-(e_3) = \{C\}, \quad \partial^+(e_3) = \{c\}.$$

Hence each $e \in E$ is a directed superedge

$$e : \partial^-(e) \longrightarrow \partial^+(e),$$

with nonempty tail and head sets.

Step 4: Parallelism and multiplicities. Since

$$\partial^-(e_1) = \partial^-(e_2) = \{a, R\} \quad \text{and} \quad \partial^+(e_1) = \partial^+(e_2) = \{C\},$$

the directed superedges e_1 and e_2 are *parallel*. Let

$$(T, H) := \big(\{a, R\}, \{C\}\big).$$

Then the multiplicity of this directed incidence pattern is

$$m(T, H) = \big|\{\, e \in E \mid \partial^-(e) = \{a, R\},\ \partial^+(e) = \{C\}\,\}\big| = \big|\{e_1, e_2\}\big| = 2.$$

Also, for

$$(T', H') := \big(\{C\}, \{c\}\big),$$

we have

$$m(T', H') = 1.$$

Conclusion. The structure

$$\mathrm{DMSHG}^{(1)} := \big(V_0, (V_1), E, \partial^-, \partial^+\big)$$

is a directed multi 1-superhypergraph. It is genuinely "multi" because the same directed incidence pattern $\{a, R\} \to \{C\}$ occurs with multiplicity 2 (via e_1 and e_2), and it is "superhyper" because it uses tier-1 supervertices $R, C \in \mathcal{P}(V_0)$.

Theorem 4.8.10 (Directed multihypergraphs are exactly the $n = 0$ case)**.** *Let $n = 0$ and let V_0 be a nonempty set. Then a directed multi* 0*-superhypergraph*

$$\mathrm{DMSHG}^{(0)} = \big(V_0,\ E,\ \partial^-,\ \partial^+\big)$$

is precisely a directed multihypergraph on V_0 in the standard sense:

$$\partial^-, \partial^+ : E \to \mathcal{P}^*(V_0).$$

Conversely, every directed multihypergraph $H = (V, E, \partial^-, \partial^+)$ is canonically a directed multi 0*-superhypergraph by taking $V_0 := V$ and $\widehat{V}^{(0)} = V_0$.*

Proof. When $n = 0$ we have $\widehat{V}^{(0)} = V_0$ and there are no higher tiers. By Definition, a directed multi 0-superhypergraph consists of a vertex set V_0, a set E of hyperarc identifiers, and maps $\partial^-, \partial^+ : E \to \mathcal{P}^*(V_0)$. This is exactly the usual definition of a directed multihypergraph: each $e \in E$ is a directed hyperarc from tail set $\partial^-(e)$ to head set $\partial^+(e)$, and multiplicity is encoded by allowing distinct identifiers $e_1 \neq e_2$ with the same ordered pair $(\partial^-(e), \partial^+(e))$.

Conversely, given a directed multihypergraph $H = (V, E, \partial^-, \partial^+)$, set $V_0 := V$ and observe that $\widehat{V}^{(0)} = V_0$. Then H matches the data of a directed multi 0-superhypergraph without any modification. □

Theorem 4.8.11 (Directed superhypergraphs are obtained by forbidding parallel edges)**.** *Let* $\mathrm{DMSHG}^{(n)} = \big(V_0, (V_k)_{k=1}^n, E, \partial^-, \partial^+\big)$ *be a directed multi n-superhypergraph. Assume that the pair map*

$$(\partial^-, \partial^+) : E \longrightarrow \mathcal{P}^*(\widehat{V}^{(n)}) \times \mathcal{P}^*(\widehat{V}^{(n)}), \qquad e \longmapsto \big(\partial^-(e), \partial^+(e)\big),$$

is injective (equivalently, there are no parallel directed superedges). Define

$$\mathcal{E} := \big\{\big(\partial^-(e), \partial^+(e)\big) \mid e \in E\big\} \subseteq \mathcal{P}^*(\widehat{V}^{(n)}) \times \mathcal{P}^*(\widehat{V}^{(n)}).$$

Then

$$\mathrm{DSHG}^{(n)} := \big(V_0, (V_k)_{k=1}^n, \mathcal{E}, \pi^-, \pi^+\big)$$

is a directed n-superhypergraph, and $\mathrm{DMSHG}^{(n)}$ *and* $\mathrm{DSHG}^{(n)}$ *carry the same directed incidence information (each* $e \in E$ *corresponds to exactly one element of* $\mathcal{E}$*).*

Proof. By construction, $\mathcal{E}$ is a set of ordered pairs of nonempty subsets of $\widehat{V}^{(n)}$, so $\mathcal{E} \subseteq \mathcal{P}^*(\widehat{V}^{(n)}) \times \mathcal{P}^*(\widehat{V}^{(n)})$. Hence $\mathrm{DSHG}^{(n)}$ satisfies Definition of a directed n-superhypergraph.

Injectivity of (∂^-, ∂^+) implies that the assignment

$$\Phi : E \longrightarrow \mathcal{E}, \qquad \Phi(e) := \big(\partial^-(e), \partial^+(e)\big),$$

is a bijection onto its image $\mathcal{E}$. Moreover, for every $e \in E$ we have

$$\pi^-\big(\Phi(e)\big) = \pi^-\big(\partial^-(e), \partial^+(e)\big) = \partial^-(e), \qquad \pi^+\big(\Phi(e)\big) = \partial^+(e).$$

Thus the directed incidence of e in $\mathrm{DMSHG}^{(n)}$ is exactly the directed incidence of $\Phi(e)$ in $\mathrm{DSHG}^{(n)}$. Therefore, forbidding parallel edges (injectivity) converts the multiset-of-edges presentation into a set-of-patterns presentation, i.e. a directed superhypergraph. □

4.9 Signed SuperHyperGraph

A Signed SuperHyperGraph assigns positive or negative incidence signs to multi-level superedges connecting nested supervertices, thereby extending the structural ideas of classical signed hypergraphs [356]. The concept of a Signed SuperHyperGraph applies and generalizes the principles of both Signed Graphs [357–359] and Signed HyperGraphs [360,361]. In addition, related concepts such as signed fuzzy graphs [362–365], weighted signed graphs [366,367], and signed neutrosophic graphs [368,369] are also well known.

Definition 4.9.1 (Signed HyperGraph)**.** [360, 361] Let $H = (V, E)$ be a finite hypergraph, where V is a nonempty set of vertices and

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$

is a finite family of nonempty hyperedges.

A *signed hypergraph* on H is a triple

$$H^{\pm} = (V, E, \varphi),$$

where $\varphi : V \times E \to \{-1, 0, +1\}$ is called the *incidence sign function* and satisfies

$$\varphi(v, e) = \begin{cases} +1, & \text{if } v \in e \text{ and the incidence of } v \text{ in } e \text{ is declared positive,} \\ -1, & \text{if } v \in e \text{ and the incidence of } v \text{ in } e \text{ is declared negative,} \\ 0, & \text{if } v \notin e. \end{cases}$$

For a vertex–hyperedge pair (v, e), we say that the incidence is *positive* if $\varphi(v, e) = +1$ and *negative* if $\varphi(v, e) = -1$. The underlying hypergraph of $H^{\pm}$ is the pair (V, E). When every hyperedge $e \in E$ has cardinality 2, this notion reduces to the usual concept of a signed graph.

Example 4.9.2 (A simple signed hypergraph)**.** Let the vertex set and hyperedge family be

$$V := \{a, b, c\}, \qquad E := \{e_1, e_2\},$$

with hyperedges

$$e_1 := \{a, b\}, \qquad e_2 := \{b, c\}.$$

Define the incidence sign function $\varphi : V \times E \to \{-1, 0, +1\}$ by

$$\varphi(v, e) = \begin{cases} +1, & \text{if } (v, e) \in \{(a, e_1),\ (b, e_2)\}, \\ -1, & \text{if } (v, e) \in \{(b, e_1),\ (c, e_2)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, the values of φ on incidences with $v \in e$ are

$$\varphi(a, e_1) = +1, \quad \varphi(b, e_1) = -1, \quad \varphi(b, e_2) = +1, \quad \varphi(c, e_2) = -1,$$

and $\varphi(v, e) = 0$ whenever $v \notin e$. Then $H^{\pm} = (V, E, \varphi)$ is a signed hypergraph: each hyperedge connects its incident vertices with specified positive or negative incidences.

Definition 4.9.3 (Signed n-SuperHyperGraph)**.** [356] Let V_0 be a finite base set, and for each integer $k \geq 0$ define the iterated powersets

$$\mathrm{P}_0(V_0) := V_0, \qquad \mathrm{P}_{k+1}(V_0) := \mathrm{P}\big(\mathrm{P}_k(V_0)\big),$$

where $\mathrm{P}(X)$ denotes the usual powerset of a set X. Fix $n \in \mathbb{N}_0$.

A *signed n-SuperHyperGraph* is a triple

$$\mathsf{SWSuHyG}(n) = (V, E, \varphi),$$

where

- $V \subseteq \mathrm{P}_n(V_0)$ is a finite set of n-supervertices;
- $E \subseteq \mathrm{P}_{n+1}(V_0)$ is a finite family of n-superedges, and each $e \in E$ is a nonempty subset of V (so $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ and, in particular, $E \subseteq \mathrm{P}_{n+1}(V_0)$);
- $\varphi : V \times E \to \{-1, 0, +1\}$ is the *incidence sign function*, defined by

$$\varphi(v, e) = \begin{cases} +1, & \text{if } v \in e \text{ and the incidence of } v \text{ in } e \text{ is positive,} \\ -1, & \text{if } v \in e \text{ and the incidence of } v \text{ in } e \text{ is negative,} \\ 0, & \text{if } v \notin e. \end{cases}$$

We call an incidence (v, e) *positive* if $\varphi(v, e) = +1$ and *negative* if $\varphi(v, e) = -1$.

When $n = 1$, interpreting the elements of V as vertices and the elements of E as hyperedges recovers a signed hypergraph in the above sense; if in addition every superedge $e \in E$ has exactly two supervertices, we obtain a signed graph.

Example 4.9.4 (A simple signed 1-SuperHyperGraph)**.** Let the finite base set be

$$V_0 := \{1, 2\}.$$

Then

$$\mathrm{P}_0(V_0) = V_0, \qquad \mathrm{P}_1(V_0) = \mathrm{P}(V_0) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\},$$

and

$$\mathrm{P}_2(V_0) = \mathrm{P}\big(\mathrm{P}_1(V_0)\big).$$

Choose the set of 1-supervertices

$$V := \big\{w_1, w_2\big\} := \big\{\{1\},\ \{1, 2\}\big\} \subseteq \mathrm{P}_1(V_0),$$

and define a single 1-superedge

$$e := \{w_1, w_2\}, \qquad E := \{e\}.$$

Since e is a nonempty subset of V and $V \subseteq P_1(V_0)$, we have $e \in \mathcal{P}(V) \setminus \{\emptyset\}$ and also $e \in P_2(V_0)$, so the pair (V, E) fits the structure of a 1-SuperHyperGraph.

Define the incidence sign function $\varphi : V \times E \to \{-1, 0, +1\}$ by

$$\varphi(v, e) = \begin{cases} +1, & \text{if } v = w_1, \\ -1, & \text{if } v = w_2, \\ 0, & \text{if } v \notin e. \end{cases}$$

Thus w_1 is positively incident with e, w_2 is negatively incident with e, and there are no other incidences.

The triple

$$\mathsf{SWSuHyG}(1) := (V, E, \varphi)$$

is a signed 1-SuperHyperGraph in the sense of the definition: V is a finite subset of $P_1(V_0)$, E is a finite family of 1-superedges (each a nonempty subset of V and hence an element of $P_2(V_0)$), and φ assigns ± 1 or 0 to each vertex–superedge pair according to membership.

For reference, Table 4.6 provides an overview of the comparison between signed graphs, signed hypergraphs, and signed n-SuperHyperGraphs.

Table 4.6: Comparison of signed graphs, signed hypergraphs, and signed n-SuperHyperGraphs

Framework	Underlying structure	Incidence / sign assignment
Signed graph	Simple graph $G = (V, E)$ with edges $uv \in E$ between vertex pairs.	Edge-sign function $\sigma : E \to \{-1, +1\}$, or equivalently signs on incidences (v, uv) indicating positive or negative edges.
Signed hypergraph	Hypergraph $H = (V, E)$ with hyperedges $e \in E \subseteq \mathcal{P}^*(V)$.	Incidence sign function $\varphi : V \times E \to \{-1, 0, +1\}$, with $\varphi(v, e) = 0$ if $v \notin e$ and ± 1 marking positive/negative incidences.
Signed n-SuperHyperGraph	n-SuperHyperGraph (V, E) with $V \subseteq P_n(V_0)$ and $E \subseteq \mathcal{P}^*(V)$ of n-superedges.	Incidence sign function $\varphi : V \times E \to \{-1, 0, +1\}$ on supervertex–superedge pairs, encoding positive/negative multi-level incidences across nested structures.

4.10 Weighted SuperHyperGraph

A weighted graph assigns numerical weights to edges or vertices, representing costs, capacities, distances, or strengths for optimization and analysis [370, 371]. Related notions are also known, such as HyperWeighted Graphs [372], Weighted Directed Graphs [325, 373, 374], Weighted Neutrosophic Graphs [375, 376], and Weighted Fuzzy Graphs [377–379]. A weighted hypergraph is a hypergraph whose vertices or hyperedges carry real-valued weights modeling capacities, costs, or strengths in applications [380–382]. A weighted SuperHypergraph is an n-level SuperHyperGraph assigning numerical weights to supervertices or superedges, encoding hierarchical importance or reliability scores [356]

Definition 4.10.1 (Weighted hypergraph)**.** [380–382] Let V be a nonempty finite set and let

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$

be a finite family of nonempty subsets of V. The pair

$$H := (V, E)$$

is called a *hypergraph*, and the elements of E are its *hyperedges*.

A *weighted hypergraph* is a triple

$$\text{W-HG} := (V, E, w),$$

where $H = (V, E)$ is a hypergraph and

$$w : E \longrightarrow \mathbb{R}_{>0}$$

is a *weight function* that assigns to each hyperedge $e \in E$ a strictly positive real weight $w(e)$ (for example, cost, capacity, strength of interaction, or frequency of co-occurrence). The underlying unweighted hypergraph is recovered by forgetting the function w.

Example 4.10.2 (Weighted hypergraph: project collaboration intensity)**.** Consider a small research lab with four researchers

$$V := \{r_1, r_2, r_3, r_4\}.$$

We model collaborative projects as hyperedges on V:

$$E := \{e_1, e_2, e_3\}, \quad e_1 := \{r_1, r_2\},\ e_2 := \{r_2, r_3, r_4\},\ e_3 := \{r_1, r_3\}.$$

Then

$$H := (V, E)$$

is a hypergraph in the sense of the above definition.

Suppose that each hyperedge $e \in E$ represents a multi-person project, and we assign a positive weight $w(e)$ equal to the number of joint publications produced by the researchers in that project:

$$w : E \to \mathbb{R}_{>0}, \qquad w(e_1) := 3, \quad w(e_2) := 5, \quad w(e_3) := 1.$$

The triple

$$\text{W-HG} := (V, E, w)$$

is then a weighted hypergraph. The hyperedge e_2 has the largest weight, indicating that the collaboration among $\{r_2, r_3, r_4\}$ is the most productive (in terms of joint publications), while e_3 is the weakest collaboration in this lab.

Definition 4.10.3 (Weighted SuperHyperGraph of depth n)**.** [356] Let V_0 be a nonempty finite base set and let $n \in \mathbb{N}_0$. Let

$$\mathrm{SHG}^{(n)} := (V, E)$$

be an n-SuperHyperGraph on V_0, so that

$$V \subseteq \mathrm{P}_n(V_0), \qquad E \subseteq \mathrm{P}_{n+1}(V_0).$$

A *weighted SuperHyperGraph of depth n* on $\mathrm{SHG}^{(n)}$ is a triple

$$\text{W-SHG}^{(n)} := (V, E, w),$$

where

$$w : E \longrightarrow \mathbb{R}_{>0}$$

is a weight function that assigns to each n-superedge $e \in E$ a strictly positive real weight $w(e)$. The value $w(e)$ may encode, for example, the intensity, capacity, cost, reliability, or frequency of the higher–order, hierarchical interaction represented by e.

In particular, when $n = 0$ and $V_0 = V$, the conditions

$$V \subseteq \mathrm{P}_0(V_0) = V_0, \qquad E \subseteq \mathrm{P}_1(V_0) = \mathcal{P}(V_0)$$

show that any weighted hypergraph (V, E, w) can be regarded as a weighted SuperHyperGraph of depth 0. Thus weighted SuperHyperGraphs strictly generalize weighted hypergraphs by allowing vertices and edges to inhabit iterated powerset layers of arbitrary depth.

Example 4.10.4 (Weighted SuperHyperGraph of depth 1: multi-team task forces)**.** Let the finite base set of *employees* be

$$V_0 := \{a, b, c, d\}.$$

We work at depth $n = 1$, so

$$P_1(V_0) = \mathcal{P}(V_0), \qquad P_2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

Choose three *teams* (subsets of employees) and treat them as 1-supervertices:

$$T_1 := \{a, b\}, \quad T_2 := \{b, c\}, \quad T_3 := \{c, d\},$$

and set

$$V := \{T_1, T_2, T_3\} \subseteq P_1(V_0).$$

Now define two *task-force groupings* of these teams as 1-superedges:

$$e_A := \{T_1, T_2\}, \qquad e_B := \{T_2, T_3\},$$

so that

$$E := \{e_A, e_B\} \subseteq P_2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

Each superedge is a nonempty set of 1-supervertices, hence (V, E) is a 1-SuperHyperGraph on the base set V_0.

Assume that each task force runs with a certain monthly budget (in arbitrary monetary units). We encode these budgets as a positive weight function

$$w : E \to \mathbb{R}_{>0}, \qquad w(e_A) := 10, \quad w(e_B) := 6.$$

Thus e_A (linking teams T_1 and T_2) has a higher weight and represents a larger, more expensive task force, while e_B (linking T_2 and T_3) is a smaller one.

The triple

$$\mathsf{W\text{-}SHG}^{(1)} := (V, E, w)$$

is a weighted SuperHyperGraph of depth 1. Vertices are teams (subsets of employees), superedges are task forces (groupings of teams), and weights quantify the intensity or cost of each multi-team collaboration in this hierarchical setting.

Table 4.7 provides an overview of the comparison of weighted graphs, weighted hypergraphs, and weighted SuperHyperGraphs.

Table 4.7: Comparison of weighted graphs, weighted hypergraphs, and weighted SuperHyperGraphs

Framework	Underlying structure	Weight assignment
Weighted graph	Simple graph $G = (V, E)$ with pairwise edges $uv \in E$.	A weight function $w : E \to \mathbb{R}_{>0}$ (and optionally on V) assigning a positive real weight to each edge.
Weighted hypergraph	Hypergraph $H = (V, E)$ with hyperedges $e \in E \subseteq \mathcal{P}^*(V)$.	A weight function $w : E \to \mathbb{R}_{>0}$ assigning a positive real weight to each hyperedge (e.g. cost, capacity, interaction strength).
Weighted n-SuperHyperGraph	n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ with $V \subseteq P_n(V_0)$ and $E \subseteq P_{n+1}(V_0)$.	A weight function $w : E \to \mathbb{R}_{>0}$ assigning a positive real weight to each n-superedge, encoding hierarchical importance or reliability.

4.11 SuperHyperTree and SuperHypertree Decomposition

A SuperHyperTree is an acyclic n-SuperHyperGraph whose superedges form a join-tree and satisfy the required connectedness conditions for every supervertex in the structure [116, 383, 384]. A SuperHyperTree generalizes the classical notion of a HyperTree [385–387].

A tree decomposition represents a graph by a tree of vertex-bags so every edge is inside some bag and bags intersect consistently [388, 389]. Tree decomposition is important because many hard graph problems become tractable on bounded treewidth, enabling dynamic programming, efficient algorithms, and structural insights in theory and practice. A SuperHypertree decomposition represents an n-SuperHyperGraph by means of a tree whose bags and guards guarantee both vertex and edge connectivity throughout the decomposition [383, 384, 390, 391]. SuperHypertree decompositions are closely related to tree-decompositions [389, 392] and Hypertree-decompositions [386, 387].

Definition 4.11.1 (n-SuperHyperTree)**.** [116] Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. An *n-SuperHyperGraph* is a pair

$$H^{(n)} = (V, E)$$

where

- $V \subseteq P_n(V_0)$ is a finite set of n-supervertices, and
- E is a finite family of nonempty subsets of V, whose elements are called n-superedges.

We say that $H^{(n)}$ is an *n-SuperHyperTree* if there exists a tree

$$J = (E, F)$$

whose vertex set is E (called a *join superhypertree* of $H^{(n)}$) such that, for every supervertex $v \in V$, the set

$$J_v := \{ e \in E \mid v \in e \} \subseteq V(J)$$

is nonempty and induces a connected subtree of J. In this case, the n-SuperHyperGraph $H^{(n)}$ is called *acyclic* (or *join-tree acyclic*) and J is a join superhypertree of $H^{(n)}$.

Example 4.11.2 (A simple 2-SuperHyperTree)**.** Let the finite base set be

$$V_0 := \{a, b, c\}.$$

Define

$$P_0(V_0) := V_0, \qquad P_1(V_0) := P(V_0), \qquad P_2(V_0) := P\big(P(V_0)\big),$$

and consider the following 2-supervertices (each is a nonempty subset of $P_1(V_0)$):

$$v_1 := \{\{a\}, \{a, b\}\}, \quad v_2 := \{\{b\}, \{b, c\}\}, \quad v_3 := \{\{c\}, \{a, c\}\}, \quad v_4 := \{\{a, b, c\}\}.$$

Put

$$V := \{v_1, v_2, v_3, v_4\} \ \subseteq \ P_2(V_0),$$

and define three 2-superedges

$$e_1 := \{v_1, v_2\}, \qquad e_2 := \{v_2, v_3\}, \qquad e_3 := \{v_3, v_4\},$$

so that

$$E := \{e_1, e_2, e_3\}.$$

Then

$$H^{(2)} := (V, E)$$

is a 2-SuperHyperGraph in the sense of the above definition.

Consider the graph

$$J := (E, F),$$

where the vertex set is $E = \{e_1, e_2, e_3\}$ and the edge set is

$$F := \{\{e_1, e_2\}, \{e_2, e_3\}\}.$$

Thus J is a path $e_1 - e_2 - e_3$, hence a tree.

For each 2-supervertex $v \in V$ we have:

$$J_{v_1} = \{e_1\}, \quad J_{v_2} = \{e_1, e_2\}, \quad J_{v_3} = \{e_2, e_3\}, \quad J_{v_4} = \{e_3\}.$$

Each J_{v_i} is nonempty and induces a connected subtree of J (a single vertex or a path of length one). Hence $H^{(2)}$ is a 2-SuperHyperTree, and J is a join superhypertree of $H^{(2)}$.

We describe below how to present the definitions of hypertree decompositions and superhypertree decompositions. Moreover, several concepts related to tree-width are known in the literature, including branch-width [393, 394], linear-width [395], path-distance-width [396], clique-width [397, 398], and sim-width [399]. Intuitively speaking, without fear of misunderstanding, these can be regarded as graph parameters that measure how close a given graph is to the desired structure. These are often referred to as graph width parameters, and relationships among different graph width parameters, as well as comparisons with other graph parameters, are frequently studied.

Definition 4.11.3 (Tree decomposition and treewidth)**.** [400,401] Let $G = (V, E)$ be a finite undirected graph. A *tree decomposition* of G is a pair (T, χ) where $T = (N, F)$ is a tree and χ is a mapping assigning to each node $p \in N$ a set (called a *bag*)

$$\chi(p) \subseteq V,$$

such that the following conditions hold:

1. **Vertex coverage.** For every vertex $v \in V$ there exists a node $p \in N$ with $v \in \chi(p)$.
2. **Edge coverage.** For every edge $\{u, v\} \in E$ there exists a node $p \in N$ with $\{u, v\} \subseteq \chi(p)$.
3. **Running intersection (connectedness).** For every vertex $v \in V$, the set of nodes

   $$N_v := \{ p \in N \mid v \in \chi(p) \}$$

   induces a connected subtree of T.

The *width* of (T, χ) is

$$\mathrm{width}(T, \chi) := \max_{p \in N} \big(|\chi(p)| - 1\big).$$

The *treewidth* of G is

$$\mathrm{tw}(G) := \min\{\mathrm{width}(T, \chi) \mid (T, \chi) \text{ is a tree decomposition of } G\}.$$

Definition 4.11.4 (Hypertree decomposition (HyperTree-Decomposition))**.** [386, 387, 402] Let $H = (V, E)$ be a finite hypergraph, where $V = \mathrm{var}(H)$ is the set of variables (vertices) and $E = \mathrm{edges}(H)$ is the set of hyperedges.

A *hypertree decomposition* of H is a triple

$$HD = \langle T, \chi, \lambda \rangle,$$

where T is a rooted tree, and χ and λ are labeling functions such that, for every node $p \in V(T)$,

$$\chi(p) \subseteq V \qquad \text{and} \qquad \lambda(p) \subseteq E.$$

For any $\mathcal{F} \subseteq E$, write

$$\mathrm{var}(\mathcal{F}) := \bigcup_{h \in \mathcal{F}} h \subseteq V.$$

For any node $p \in V(T)$, let T_p denote the subtree of T rooted at p, and set

$$\chi(T_p) := \bigcup_{q \in V(T_p)} \chi(q).$$

The triple $\langle T, \chi, \lambda \rangle$ is a hypertree decomposition of H if it satisfies all of the following conditions:

(1) (*Edge coverage*) For each hyperedge $h \in E$, there exists a node $p \in V(T)$ such that

$$h \subseteq \chi(p).$$

(2) (*Connectedness of variables*) For each variable $Y \in V$, the set

$$\{ p \in V(T) \mid Y \in \chi(p) \}$$

induces a connected subtree of T.

(3) (*Local covering by guards*) For each node $p \in V(T)$,

$$\chi(p) \subseteq \mathrm{var}(\lambda(p)).$$

(4) (*Special condition / descendant separation*) For each node $p \in V(T)$,

$$\mathrm{var}(\lambda(p)) \cap \chi(T_p) \subseteq \chi(p).$$

The *width* of HD is

$$\mathrm{width}(HD) := \max_{p \in V(T)} |\lambda(p)|.$$

Definition 4.11.5 (SuperHypertree decomposition of an n-SuperHyperGraph)**.** Let $H^{(n)} = (V, E)$ be a finite n-SuperHyperGraph. A *SuperHypertree decomposition* of $H^{(n)}$ is a triple

$$(T, \mathcal{B}, C),$$

where

- $T = (V_T, E_T)$ is a finite tree,
- $\mathcal{B} = \{ B_t \subseteq V \mid t \in V_T \}$ is a family of *bags*,
- $C = \{ C_t \subseteq E \mid t \in V_T \}$ is a family of *guards*,

such that the following conditions hold.

(1) **Vertex coverage.** Every supervertex appears in at least one bag:

$$V = \bigcup_{t \in V_T} B_t.$$

(2) **Superedge coverage.** For every superedge $e \in E$ there exists $t \in V_T$ with

$$e \subseteq B_t.$$

(3) **Vertex connectedness.** For every supervertex $v \in V$, the set

$$T_v := \{ t \in V_T \mid v \in B_t \}$$

is nonempty and induces a connected subtree of T.

(4) **Guard covering.** For each $t \in V_T$, the union of the superedges in C_t covers the bag B_t, that is

$$B_t \subseteq \bigcup_{e \in C_t} e.$$

(5) **Guard connectedness and consistency.** For every superedge $e \in E$, the set

$$T_e := \{ t \in V_T \mid e \in C_t \}$$

is nonempty and induces a connected subtree of T, and whenever $e \subseteq B_t$ holds for some $t \in V_T$, we also have $e \in C_t$.

In this situation we say that $(T, \mathcal{B}, C)$ is a *SuperHypertree decomposition* of the n-SuperHyperGraph $H^{(n)}$.

Example 4.11.6 (A SuperHypertree decomposition of a 2-SuperHyperGraph)**.** We continue with the 2-SuperHyperGraph

$$H^{(2)} = (V, E)$$

from the previous example, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3\}$ with

$$e_1 = \{v_1, v_2\}, \quad e_2 = \{v_2, v_3\}, \quad e_3 = \{v_3, v_4\}.$$

Define a tree

$$T = (V_T, E_T)$$

by

$$V_T := \{t_1, t_2, t_3\}, \qquad E_T := \big\{\{t_1, t_2\}, \{t_2, t_3\}\big\},$$

so T is again a path $t_1 - t_2 - t_3$.

Define the family of bags

$$\mathcal{B} = \{B_{t_1}, B_{t_2}, B_{t_3}\}$$

by

$$B_{t_1} := e_1 = \{v_1, v_2\}, \quad B_{t_2} := e_2 = \{v_2, v_3\}, \quad B_{t_3} := e_3 = \{v_3, v_4\},$$

and the family of guards

$$C = \{C_{t_1}, C_{t_2}, C_{t_3}\}$$

by

$$C_{t_1} := \{e_1\}, \quad C_{t_2} := \{e_2\}, \quad C_{t_3} := \{e_3\}.$$

We now verify the conditions of a SuperHypertree decomposition:

- Vertex coverage:
$$\bigcup_{t \in V_T} B_t = B_{t_1} \cup B_{t_2} \cup B_{t_3} = \{v_1, v_2, v_3, v_4\} = V.$$
- Superedge coverage: For each $e_i \in E$ we have $e_i = B_{t_i}$, hence $e_i \subseteq B_{t_i}$.
- Vertex connectedness: For each $v \in V$,
$$T_{v_1} = \{t_1\}, \quad T_{v_2} = \{t_1, t_2\}, \quad T_{v_3} = \{t_2, t_3\}, \quad T_{v_4} = \{t_3\},$$
each inducing a connected subtree of T.
- Guard covering: For each t_i,
$$\bigcup_{e \in C_{t_i}} e = e_i = B_{t_i},$$
so $B_{t_i} \subseteq \bigcup_{e \in C_{t_i}} e$ holds.
- Guard connectedness and consistency: For each superedge $e_i \in E$,
$$T_{e_i} := \{t \in V_T \mid e_i \in C_t\} = \{t_i\},$$
which is nonempty and connected. Moreover, whenever $e_i \subseteq B_t$, we have $t = t_i$ and thus $e_i \in C_{t_i}$ by definition.

Therefore, $(T, \mathcal{B}, C)$ is a SuperHypertree decomposition of the 2-SuperHyperGraph $H^{(2)}$.

Table 4.8 presents a concise overview of tree, hypertree, and superhypertree decompositions.

Decomposition	Input object	Structure and labels	Key constraints (informal)	Typical width
Tree decomposition [400,401]	Graph $G = (V, E)$	Tree T with bags $\chi(p) \subseteq V$ for $p \in V(T)$	(i) $\bigcup_p \chi(p) = V$; (ii) $\forall\{u, v\} \in E\ \exists p : \{u, v\} \subseteq \chi(p)$; (iii) $\{p \mid v \in \chi(p)\}$ is connected in T	$\max_p(\lvert\chi(p)\rvert - 1)$
Hypertree decomposition [386,387,402]	Hypergraph $H = (V, E)$	Rooted tree T with bags $\chi(p) \subseteq V$ and guards $\lambda(p) \subseteq E$	(i) $\forall h \in E\ \exists p : h \subseteq \chi(p)$; (ii) $\{p \mid v \in \chi(p)\}$ is connected; (iii) $\chi(p) \subseteq \mathrm{var}(\lambda(p))$; (iv) $\mathrm{var}(\lambda(p)) \cap \chi(T_p) \subseteq \chi(p)$	$\max_p \lvert\lambda(p)\rvert$
SuperHypertree decomposition [383,384,391]	n-SuperHyperGraph $H^{(n)} = (V, E)$	Tree T with bags $B_t \subseteq V$ and guards $C_t \subseteq E$	(i) $\bigcup_t B_t = V$; (ii) $\forall e \in E\ \exists t : e \subseteq B_t$; (iii) $\{t \mid v \in B_t\}$ is connected; (iv) $B_t \subseteq \bigcup_{e \in C_t} e$; (v) $\{t \mid e \in C_t\}$ connected and $(e \subseteq B_t \Rightarrow e \in C_t)$	$\max_t \lvert C_t\rvert$

Table 4.8: Concise overview of tree, hypertree, and superhypertree decompositions.

4.12 Complete n-SuperHyperGraph

A complete graph is a simple undirected graph in which every pair of distinct vertices is joined by exactly one edge [403–405]. As concepts related to complete graphs, notions such as complete digraphs [406, 407], bicomplete graphs [408], complete fuzzy graphs [409, 410], complete multipartite graph [411, 412], complete bipartite graph [413, 414], random complete graph [415], probe complete graphs [416, 417], and complete neutrosophic graphs [418, 419] are well known. Complete graphs serve as fundamental extremal objects: they often bound chromatic number [420], clique number [421], and density, underpin Ramsey theory, minors, and worst-case algorithmic complexity analyses.

A complete hypergraph is a hypergraph whose hyperedges consist of all nonempty vertex subsets, so every possible multiway connection appears [422–424]. A complete n-SuperHyperGraph is an n-SuperHyperGraph where every nonempty subset of n-supervertices forms an n-superedge, realizing all hierarchical relations.

Definition 4.12.1 (Complete hypergraph)**.** [422–424] Let V be a finite vertex set with $|V| = n$.

(1) The *complete k-uniform hypergraph* on V is the hypergraph

$$K_n^{(k)} := (V, E), \qquad E := \binom{V}{k} := \{\, e \subseteq V \mid |e| = k \,\}.$$

(2) The *(non-uniform) complete hypergraph* on V is the hypergraph

$$K_n := (V, E), \qquad E := \bigcup_{k=2}^{n} \binom{V}{k}.$$

Definition 4.12.2 (Complete n-SuperHyperGraph)**.** Let V_0 be a nonempty finite base set and let $n \in \mathbb{N}_0$. Recall that the iterated powersets are defined by

$$\mathrm{P}_0(V_0) := V_0, \qquad \mathrm{P}_{k+1}(V_0) := \mathrm{P}\big(\mathrm{P}_k(V_0)\big) \quad (k \geq 0),$$

where $\mathrm{P}(X)$ denotes the powerset of a set X.

A *level-n SuperHyperGraph* on V_0 is a pair

$$\mathrm{SHG}^{(n)} = (V_n, E),$$

where

$$\emptyset \neq V_n \subseteq \mathrm{P}_n(V_0) \quad \text{and} \quad \emptyset \neq E \subseteq \mathrm{P}(V_n) \setminus \{\emptyset\}.$$

The elements of V_n are called n-supervertices and the elements of E are called n-superedges.

The level-n SuperHyperGraph $\mathrm{SHG}^{(n)} = (V_n, E)$ is called a *complete n-SuperHyperGraph* if

$$E \;=\; \mathrm{P}(V_n) \setminus \{\emptyset\},$$

that is, every nonempty subset of V_n occurs as an n-superedge.

Equivalently, $\mathrm{SHG}^{(n)}$ is complete if and only if for every nonempty family $F \subseteq V_n$ there exists a unique $e \in E$ with $e = F$.

For $1 \le k \le |V_n|$, a complete n-SuperHyperGraph is called *k-uniform* if all its n-superedges have cardinality k; in this case

$$E = \{ F \subseteq V_n \mid |F| = k \}.$$

When $n = 0$, this definition reduces to the usual notions of complete hypergraph and complete k-uniform hypergraph on the vertex set V_0.

Example 4.12.3 (A complete 1-SuperHyperGraph on three supervertices)**.** Let the finite base set be

$$V_0 := \{a, b, c\}.$$

Then

$$\mathrm{P}_0(V_0) = V_0, \qquad \mathrm{P}_1(V_0) = \mathrm{P}(V_0).$$

Choose three 1-supervertices

$$v_1 := \{a\}, \qquad v_2 := \{b\}, \qquad v_3 := \{a, b\},$$

and set

$$V_1 := \{v_1, v_2, v_3\} \subseteq \mathrm{P}_1(V_0).$$

We now define the family of 1-superedges by taking *all* nonempty subsets of V_1:

$$E := \mathrm{P}(V_1) \setminus \{\emptyset\} = \big\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\big\}.$$

Then

$$\mathrm{SHG}^{(1)} := (V_1, E)$$

is a level-1 SuperHyperGraph on the base set V_0. Moreover, by construction we have

$$E = \mathrm{P}(V_1) \setminus \{\emptyset\},$$

so every nonempty subset of V_1 appears as a 1-superedge. Hence $\mathrm{SHG}^{(1)}$ is a complete 1-SuperHyperGraph in the sense of the above definition.

4.13 co-SuperHyperGraph

A co-graph is a graph obtained from K_1 by repeatedly applying disjoint union and complementation; equivalently, it is P_4-free [425–427]. A co-hypergraph is a pair $(X, \mathcal{A})$ where X is a nonempty vertex set and $\mathcal{A} \subseteq \mathcal{P}^*(X)$ is a family of nonempty subsets of X (co-edges), typically with $|A| \ge 2$ for all $A \in \mathcal{A}$ [428–430]. A co-n-superhypergraph is a tuple $(V, \mathcal{A}_1, \ldots, \mathcal{A}_n)$ with $\mathcal{A}_1 \subseteq \mathcal{P}^*(V)$ and $\mathcal{A}_i \subseteq \binom{\mathcal{A}_{i-1}}{2}$ for every $2 \le i \le n$, so level-i co-links connect pairs of level-$(i-1)$ objects.

Definition 4.13.1 (Graph complement)**.** Let $G = (V, E)$ be a finite simple undirected graph. Its *complement* is $\overline{G} = (V, \overline{E})$, where

$$\overline{E} := \big\{\{u, v\} \subseteq V \mid u \ne v,\ \{u, v\} \notin E\big\}.$$

Definition 4.13.2 (Co-graph (cograph))**.** A (finite simple) graph G is a *co-graph* (also called a *cograph*) if it belongs to the smallest class C of graphs satisfying:

1. $K_1 \in C$.
2. If $G, H \in C$, then their disjoint union $G \dot\cup H \in C$.
3. If $G \in C$, then $\overline{G} \in C$.

Equivalently, G is a cograph if and only if G has no induced path on four vertices (P_4-free).

Example 4.13.3 (A co-graph (cograph)). Let

$$V = \{1, 2, 3, 4\}.$$

Start from four isolated vertices K_1 and take the disjoint union

$$G_0 := K_1 \dot\cup K_1 \dot\cup K_1 \dot\cup K_1,$$

so $E(G_0) = \emptyset$. Now take the complement:

$$G := \overline{G_0} = K_4.$$

Since G is obtained from K_1 using disjoint union and complement, G is a co-graph. (Equivalently, K_4 contains no induced P_4.)

Definition 4.13.4 (Co-hypergraph (co-edge hypergraph)). Let X be a finite nonempty set and let $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$. A *co-hypergraph* is a pair

$$H = (X, \mathcal{A}),$$

where $\mathcal{A} \subseteq \mathcal{P}^*(X)$ is a family of nonempty subsets of X (typically one assumes $|A| \geq 2$ for all $A \in \mathcal{A}$). The members of $\mathcal{A}$ are called *co-edges*.

Example 4.13.5 (A co-graph (cograph)). Let

$$V = \{1, 2, 3, 4\}.$$

Start from four isolated vertices K_1 and take the disjoint union

$$G_0 := K_1 \dot\cup K_1 \dot\cup K_1 \dot\cup K_1,$$

so $E(G_0) = \emptyset$. Now take the complement:

$$G := \overline{G_0} = K_4.$$

Since G is obtained from K_1 using disjoint union and complement, G is a co-graph. (Equivalently, K_4 contains no induced P_4.)

Definition 4.13.6 (Co-n-superhypergraph). Fix an integer $n \geq 1$ and a finite nonempty vertex set V. A *co-n-superhypergraph* is a tuple

$$\mathcal{S}^{\mathrm{co}} = \big(V, \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\big)$$

such that

$$\mathcal{A}_1 \subseteq \mathcal{P}^*(V) \quad \text{and} \quad \mathcal{A}_i \subseteq \binom{\mathcal{A}_{i-1}}{2} \text{ for every } 2 \leq i \leq n.$$

Elements of $\mathcal{A}_1$ are *level-1 co-edges*, and elements of $\mathcal{A}_i$ ($i \geq 2$) are *level-i co-superlinks*, i.e., unordered links between two distinct level-$(i-1)$ objects.

Remark 4.13.7 (2-uniform case). If $\mathcal{A}_1 \subseteq \binom{V}{2}$, then $(V, \mathcal{A}_1)$ can be identified with the simple graph $G = (V, E)$ where $E = \mathcal{A}_1$.

Example 4.13.8 (A co-hypergraph). Let

$$X := \{a, b, c, d\}, \qquad \mathcal{A} := \big\{\{a, b\}, \{b, c, d\}\big\} \subseteq \mathcal{P}^*(X).$$

Then the pair

$$\mathcal{H} := (X, \mathcal{A})$$

is a co-hypergraph: its vertices are X and its co-edges are the nonempty subsets in $\mathcal{A}$ (and each has size at least 2, so there are no loops).

Theorem 4.13.9 (Co-n-superhypergraphs generalize co-hypergraphs and co-graphs). *Fix $n \geq 1$.*

1. *(Co-hypergraphs embed.) For every co-hypergraph* $H = (X, \mathcal{A})$*, the tuple*

$$\mathcal{S}^{\mathrm{co}}(H) := \big(X, \mathcal{A}, \varnothing, \ldots, \varnothing\big)$$

is a co-n-superhypergraph.

2. *(Co-graphs embed.) For every co-graph (cograph)* $G = (V, E)$*, the tuple*

$$\mathcal{S}^{\mathrm{co}}(G) := \big(V, E, \varnothing, \ldots, \varnothing\big)$$

is a co-n-superhypergraph (with $\mathcal{A}_1 = E \subseteq \binom{V}{2} \subseteq \mathcal{P}^*(V)$*).*

Hence the framework of co-n-superhypergraphs contains both co-hypergraphs and co-graphs as special cases.

Proof. (1) Let $H = (X, \mathcal{A})$ be a co-hypergraph. Set $\mathcal{A}_1 := \mathcal{A}$ and $\mathcal{A}_i := \varnothing$ for $2 \le i \le n$. Then $\mathcal{A}_1 \subseteq \mathcal{P}^*(X)$ holds by definition of co-hypergraph. For every $i \ge 2$, we have $\mathcal{A}_i = \varnothing \subseteq \binom{\mathcal{A}_{i-1}}{2}$ trivially. Therefore $\big(X, \mathcal{A}_1, \ldots, \mathcal{A}_n\big)$ is a co-n-superhypergraph.

(2) Let $G = (V, E)$ be a co-graph (cograph). Since G is a simple graph, every edge is a 2-element subset of V, i.e., $E \subseteq \binom{V}{2} \subseteq \mathcal{P}^*(V)$. Define $\mathcal{A}_1 := E$ and $\mathcal{A}_i := \varnothing$ for $2 \le i \le n$. As in (1), the level conditions for a co-n-superhypergraph hold: $\mathcal{A}_1 \subseteq \mathcal{P}^*(V)$ and $\mathcal{A}_i = \varnothing \subseteq \binom{\mathcal{A}_{i-1}}{2}$ for $i \ge 2$. Hence $\big(V, E, \varnothing, \ldots, \varnothing\big)$ is a co-n-superhypergraph.

This proves both embeddings, so co-n-superhypergraphs generalize co-hypergraphs and co-graphs in the sense of containing them as special cases. □

4.14 Perfect SuperHyperGraphs

A perfect graph is a graph in which every induced subgraph satisfies $\chi = \omega$, i.e., the chromatic number equals the clique number [431–434]. Related notions include superperfect graphs [435], perfect fuzzy graphs [436–438], strongly perfect graphs [431, 433], locally perfect graphs [439], and line perfect graphs [440, 441]. A perfect hypergraph is a hypergraph whose shadow (2-section) graph is perfect, so every induced subhypergraph's shadow satisfies $\chi = \omega$. A perfect superhypergraph is a superhypergraph such that all level interaction graphs of every induced substructure are perfect, hence $\chi = \omega$ at every level.

Definition 4.14.1 (Perfect graph)**.** [431–434] A (finite, simple) graph $G = (V, E)$ is *perfect* if for every induced subgraph $H = G[U]$ with $U \subseteq V$ we have

$$\chi(H) = \omega(H),$$

where $\chi(H)$ is the chromatic number of H, and $\omega(H)$ is the clique number of H.

Definition 4.14.2 (Hypergraph, induced subhypergraph, and shadow graph)**.** A (finite) hypergraph is a pair $H = (V, \mathcal{E})$ where V is a finite set and $\emptyset \notin \mathcal{E} \subseteq \mathcal{P}(V)$.

For $U \subseteq V$, the *induced subhypergraph* is

$$H[U] := \big(U,\ \mathcal{E}[U]\big), \qquad \mathcal{E}[U] := \{\, e \in \mathcal{E} \mid e \subseteq U \,\}.$$

The *shadow graph* (also called the *2-section*) of H is the simple graph

$$\partial(H) := \Big(V,\ \big\{\{u, v\} \subseteq V :\ u \neq v,\ \exists e \in \mathcal{E} \text{ with } \{u, v\} \subseteq e\big\}\Big).$$

Definition 4.14.3 (Perfect hypergraph (shadow-perfect))**.** A hypergraph H is *perfect* if its shadow graph $\partial(H)$ is a perfect graph. Equivalently, H is perfect if for every $U \subseteq V(H)$,

$$\chi\big(\partial(H[U])\big) = \omega\big(\partial(H[U])\big).$$

Example 4.14.4 (A perfect hypergraph (shadow-perfect))**.** Let

$$V := \{1,2,3,4\}, \qquad \mathcal{E} := \big\{\{1,2,3\},\{2,3,4\}\big\} \subseteq \mathcal{P}^*(V).$$

Define the hypergraph

$$H := (V,\mathcal{E}).$$

Its shadow graph $\partial(H)$ has an edge $\{u,v\}$ iff $u \neq v$ and $\{u,v\} \subseteq e$ for some $e \in \mathcal{E}$. Since $\{1,2,3\}$ induces K_3 on $\{1,2,3\}$ and $\{2,3,4\}$ induces K_3 on $\{2,3,4\}$, we obtain

$$E\big(\partial(H)\big) = \big\{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\big\}.$$

Thus $\partial(H)$ is exactly the graph K_4 with the single missing edge $\{1,4\}$.

We verify the perfectness condition:

$$\omega\big(\partial(H)\big) = 3 \quad \text{(for example, } \{1,2,3\} \text{ is a clique)},$$

and a proper 3-coloring is

$$c(2) = 1, \quad c(3) = 2, \quad c(1) = 3, \quad c(4) = 3,$$

so

$$\chi\big(\partial(H)\big) \leq 3.$$

Because $\chi \geq \omega$ holds for every graph, we get

$$\chi\big(\partial(H)\big) = \omega\big(\partial(H)\big) = 3.$$

Moreover, for every $U \subseteq V$, the induced subgraph $\partial(H)[U]$ is either a clique or a subgraph of this K_4-minus-one-edge, hence again satisfies $\chi = \omega$ (one checks directly for $|U| \leq 4$). Therefore H is a perfect hypergraph in the shadow-perfect sense.

Definition 4.14.5 (n-SuperHyperGraph via iterated super-links)**.** Fix an integer $n \geq 1$ and a finite base vertex set V. An *n-SuperHyperGraph* is a tuple

$$\mathcal{S} = \big(V,\mathcal{E}_1,\mathcal{E}_2,\ldots,\mathcal{E}_n\big)$$

such that

$$\mathcal{E}_1 \subseteq \mathcal{P}^*(V) \quad \text{and} \quad \mathcal{E}_i \subseteq \binom{\mathcal{E}_{i-1}}{2} \text{ for every } 2 \leq i \leq n,$$

where $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}$ and $\binom{X}{2} := \{\{a,b\} \subseteq X : a \neq b\}$. Elements of $\mathcal{E}_1$ are hyperedges, and elements of $\mathcal{E}_i$ ($i \geq 2$) are *level-i super-links* (unordered links between two distinct level-$(i-1)$ objects). For $n = 1$, $\mathcal{S}$ is exactly a hypergraph $(V,\mathcal{E}_1)$.

Definition 4.14.6 (Induced sub-n-SuperHyperGraph)**.** Let $\mathcal{S} = (V,\mathcal{E}_1,\ldots,\mathcal{E}_n)$ be an n-SuperHyperGraph and let $U \subseteq V$. Define recursively:

$$\mathcal{E}_1[U] := \{\, e \in \mathcal{E}_1 : e \subseteq U \,\}, \qquad \mathcal{E}_i[U] := \{\, \lambda \in \mathcal{E}_i : \lambda \subseteq \mathcal{E}_{i-1}[U] \,\}\ (2 \leq i \leq n).$$

Then

$$\mathcal{S}[U] := \big(U,\mathcal{E}_1[U],\ldots,\mathcal{E}_n[U]\big)$$

is called the *induced sub-n-SuperHyperGraph* on U.

Definition 4.14.7 (Level interaction graphs of an n-SuperHyperGraph)**.** Let $\mathcal{S} = (V,\mathcal{E}_1,\ldots,\mathcal{E}_n)$ be an n-SuperHyperGraph.

(0) The *level-0 interaction graph* is the shadow graph of $(V,\mathcal{E}_1)$:

$$G_0(\mathcal{S}) \;:=\; \partial\big((V,\mathcal{E}_1)\big).$$

(i) For each $1 \leq i \leq n-1$, the *level-i interaction graph* is the simple graph

$$G_i(\mathcal{S}) \;:=\; \big(\mathcal{E}_i,\; E_i^{\text{link}}\big), \qquad E_i^{\text{link}} := \big\{\{A,B\} \subseteq \mathcal{E}_i : \{A,B\} \in \mathcal{E}_{i+1}\big\}.$$

Definition 4.14.8 (Perfect n-SuperHyperGraph)**.** An n-SuperHyperGraph $\mathcal{S}$ is *perfect* if for every subset $U \subseteq V$ and every $i \in \{0, 1, \ldots, n-1\}$,

$$\chi\big(G_i(\mathcal{S}[U])\big) = \omega\big(G_i(\mathcal{S}[U])\big).$$

Example 4.14.9 (A perfect 2-SuperHyperGraph)**.** Let

$$V := \{1, 2, 3, 4\}.$$

Define level-1 hyperedges

$$\mathcal{E}_1 := \{e_1, e_2\}, \qquad e_1 := \{1, 2, 3\}, \;\; e_2 := \{2, 3, 4\}.$$

Define level-2 super-links by

$$\mathcal{E}_2 := \big\{\{e_1, e_2\}\big\} \subseteq \binom{\mathcal{E}_1}{2}.$$

Then

$$\mathcal{S} := (V, \mathcal{E}_1, \mathcal{E}_2)$$

is a 2-SuperHyperGraph.

We check perfectness of $\mathcal{S}$ under your definition.

Level 0 **interaction graph.** By definition,

$$G_0(\mathcal{S}) = \partial\big((V, \mathcal{E}_1)\big) = \partial(H),$$

where $H = (V, \mathcal{E}_1)$ is exactly the hypergraph from the previous example. Hence

$$\chi\big(G_0(\mathcal{S}[U])\big) = \omega\big(G_0(\mathcal{S}[U])\big) \quad \text{for all } U \subseteq V.$$

Level 1 **interaction graph.** Here the vertex set is $\mathcal{E}_1 = \{e_1, e_2\}$ and

$$E_1^{\text{link}} = \big\{\{A, B\} \subseteq \mathcal{E}_1 : \; \{A, B\} \in \mathcal{E}_2\big\} = \big\{\{e_1, e_2\}\big\}.$$

Thus

$$G_1(\mathcal{S}) \cong K_2,$$

so

$$\chi\big(G_1(\mathcal{S})\big) = 2, \qquad \omega\big(G_1(\mathcal{S})\big) = 2.$$

For an induced sub-2-SuperHyperGraph $\mathcal{S}[U]$, either:

- $\mathcal{E}_1[U] = \varnothing$ or $\{e_1\}$ or $\{e_2\}$, in which case $G_1(\mathcal{S}[U])$ has at most one vertex, hence $\chi = \omega = 0$ or 1; or
- $\mathcal{E}_1[U] = \{e_1, e_2\}$, which forces $U = V$, and we are back to K_2 with $\chi = \omega = 2$.

Therefore,

$$\chi\big(G_1(\mathcal{S}[U])\big) = \omega\big(G_1(\mathcal{S}[U])\big) \quad \text{for all } U \subseteq V.$$

We have verified that for every $U \subseteq V$ and each $i \in \{0, 1\}$,

$$\chi\big(G_i(\mathcal{S}[U])\big) = \omega\big(G_i(\mathcal{S}[U])\big).$$

Hence $\mathcal{S}$ is a perfect 2-SuperHyperGraph.

Lemma 4.14.10 (Induced compatibility of level graphs)**.** *Let $\mathcal{S}$ be an n-SuperHyperGraph and $U \subseteq V$. Then, for each $i \in \{0, 1, \ldots, n-1\}$, the graph $G_i(\mathcal{S}[U])$ is an induced subgraph of $G_i(\mathcal{S})$.*

Proof. For $i = 0$, by definition

$$G_0(\mathcal{S}[U]) = \partial\big((U, \mathcal{E}_1[U])\big),$$

so its vertex set is U and its edges are exactly those pairs $\{u, v\} \subseteq U$ that lie in some $e \in \mathcal{E}_1$ with $e \subseteq U$. This is precisely the induced subgraph of $\partial((V, \mathcal{E}_1))$ on U, i.e. $G_0(\mathcal{S})[U]$.

For $1 \leq i \leq n-1$, the vertex set of $G_i(\mathcal{S}[U])$ is $\mathcal{E}_i[U] \subseteq \mathcal{E}_i$. Moreover,

$$\{A, B\} \in E_i^{\text{link}}(\mathcal{S}[U]) \iff \{A, B\} \in \mathcal{E}_{i+1}[U] \iff \{A, B\} \in \mathcal{E}_{i+1},$$

with $A, B \in \mathcal{E}_i[U]$. Hence adjacency in $G_i(\mathcal{S}[U])$ is exactly adjacency in $G_i(\mathcal{S})$ restricted to $\mathcal{E}_i[U]$, i.e. $G_i(\mathcal{S}[U])$ is an induced subgraph of $G_i(\mathcal{S})$. □

Theorem 4.14.11 (Perfect superhypergraphs generalize perfect hypergraphs and perfect graphs)**.** *Fix $n \geq 1$.*

1. *(Hypergraph case) Let $H = (V, \mathcal{E})$ be a hypergraph. Define an n-SuperHyperGraph*

$$\mathcal{S}_H := \big(V, \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\big) \quad by \quad \mathcal{E}_1 := \mathcal{E},\ \mathcal{E}_2 = \cdots = \mathcal{E}_n := \varnothing.$$

   Then H is a perfect hypergraph if and only if $\mathcal{S}_H$ is a perfect n-SuperHyperGraph.

2. *(Graph case) Let $G = (V, E)$ be a (finite, simple) graph and define the 2-uniform hypergraph*

$$H_G := (V, \mathcal{E}), \qquad \mathcal{E} := \{\{u, v\} : \{u, v\} \in E\}.$$

   Form $\mathcal{S}_{H_G}$ as in (1). Then G is a perfect graph if and only if $\mathcal{S}_{H_G}$ is a perfect n-SuperHyperGraph.

Proof. (1) By construction, $\mathcal{S}_H$ has no super-links at levels ≥ 2. Hence for every $U \subseteq V$ we have

$$G_0(\mathcal{S}_H[U]) = \partial\big(H[U]\big),$$

and for each $1 \leq i \leq n-1$ the level graph $G_i(\mathcal{S}_H[U])$ has vertex set $\mathcal{E}_i[U] = \varnothing$, hence is an empty graph and satisfies $\chi = \omega = 0$.

Therefore, the definition of perfect n-SuperHyperGraph for $\mathcal{S}_H$ reduces to

$$\forall U \subseteq V :\ \chi\big(\partial(H[U])\big) = \omega\big(\partial(H[U])\big),$$

which is exactly the definition of a perfect hypergraph (shadow-perfect). Thus H is perfect if and only if $\mathcal{S}_H$ is perfect.

(2) For the 2-uniform hypergraph H_G, its shadow graph satisfies

$$\partial(H_G) = G,$$

because a pair $\{u, v\}$ is an edge of $\partial(H_G)$ exactly when $\{u, v\}$ is contained in some hyperedge of H_G, and the hyperedges of H_G are precisely the edges of G. More generally, for every $U \subseteq V$,

$$\partial\big(H_G[U]\big) = G[U].$$

Applying (1) to H_G yields that $\mathcal{S}_{H_G}$ is perfect if and only if

$$\forall U \subseteq V :\ \chi\big(G[U]\big) = \omega\big(G[U]\big),$$

which is exactly the definition that G is perfect. □

4.15 Line SuperHyperGraphs

Line graph represents each edge of an original graph as a vertex, connecting vertices whenever corresponding edges share an endpoint [442–444]. Related notions include the fuzzy line graph [443, 445], the neutrosophic line graph [446, 447], the weighted line graph [448, 449], and the plithogenic line graph [444]. Also, active research has been conducted on related topics such as total graphs [450,451], line digraphs [452,453], pancyclic line graphs [454], quasi-line graphs [455, 456], and iterated line graphs [457, 458]. Line hypergraph uses each hyperedge as a vertex, adding hyperedges that connect hyperedges sharing at least one common original vertex [459, 460]. Line SuperHypergraph takes each n-superedge as vertex, forming superedges from stars: sets of superedges incident to same n-supervertex within structure [461].

Definition 4.15.1 (Line hypergraph)**.** [459,460] Let $H = (V, E)$ be a finite hypergraph. For each $v \in V$, define the *star*

$$\mathrm{Star}_H(v) := \{\, e \in E \mid v \in e \,\} \subseteq E.$$

The *line hypergraph* of H is the hypergraph

$$L(H) := \big(E,\ \{\, \mathrm{Star}_H(v) \mid v \in V,\ \mathrm{Star}_H(v) \neq \emptyset \,\}\big).$$

Thus, vertices of $L(H)$ are the hyperedges of H, and each $v \in V$ contributes a hyperedge of $L(H)$ collecting all hyperedges of H incident with v.

Definition 4.15.2 (Multi-valued line hypergraph (Tyshkevich–Zverovich))**.** Let $H = (V, E)$ be a hypergraph without isolated vertices, and list $V = \{v_1, \ldots, v_n\}$. Let $\deg(v)$ denote the number of hyperedges containing v. Define integer vectors

$$\mathbf{1}_H := \big(\deg(v_1), \ldots, \deg(v_n)\big), \qquad \mathbf{0}_H := \big(0_{v_1}, \ldots, 0_{v_n}\big),$$

where

$$0_{v_i} := \begin{cases} 0, & \deg(v_i) = 1, \\ 2, & \deg(v_i) \geq 2. \end{cases}$$

Let

$$D_H := \{\, D = (d_{v_i})_{i=1}^{n} \mid \mathbf{0}_H \leq D \leq \mathbf{1}_H \text{ componentwise} \,\}.$$

For $v \in V$, write $E(v) := \mathrm{Star}_H(v) = \{e \in E \mid v \in e\}$. For $D \in D_H$ and $v \in V$, let F_v be the *clique of rank* d_v on the vertex set $E(v)$, i.e.,

$$F_v := \big(E(v),\ \{\, S \subseteq E(v) \mid |S| = d_v \,\}\big).$$

Define

$$L_D(H) \ := \ \bigcup_{v \in V} F_v.$$

The *(multi-valued) line hypergraph* of H is the set

$$\mathcal{L}(H) := \{\, L_D(H) \mid D \in D_H \,\}.$$

Definition 4.15.3 ((Recall) Iterated powerset and level-n SuperHyperGraph)**.** Let V_0 be a nonempty finite base set. Define the iterated powersets

$$\mathrm{P}_0(V_0) := V_0, \qquad \mathrm{P}_{k+1}(V_0) := \mathrm{P}\big(\mathrm{P}_k(V_0)\big) \quad (k \geq 0),$$

where $\mathrm{P}(X)$ denotes the usual powerset of a set X.

Fix $n \in \mathbb{N}_0$. A *level-n SuperHyperGraph* is a pair

$$H^{(n)} = (V_n, E),$$

where

$$V_n \subseteq \mathrm{P}_n(V_0)$$

is a finite set of *n-supervertices*, and

$$\emptyset \neq E \subseteq \mathrm{P}(V_n) \setminus \{\emptyset\}$$

is a finite family of nonempty subsets of V_n, whose elements are called *n-superedges*.

For $v \in V_n$, the *star* of v in $H^{(n)}$ is

$$\mathrm{Star}_{H^{(n)}}(v) := \{\, F \in E : v \in F \,\} \subseteq E.$$

Definition 4.15.4 (Line SuperHyperGraph)**.** [461] Let $H^{(n)} = (V_n, E)$ be a level-n SuperHyperGraph on the finite base set V_0. The *line SuperHyperGraph* of $H^{(n)}$ is the pair

$$L^{(n)}\big(H^{(n)}\big) := \big(V'_{n+1}, E'_{n+1}\big),$$

defined as follows:

- the vertex set is the set of superedges of $H^{(n)}$,

$$V'_{n+1} := E;$$

- the superedge family is the collection of all nonempty stars,

$$E'_{n+1} := \big\{ \mathrm{Star}_{H^{(n)}}(v) \subseteq E \ \big|\ v \in V_n,\ \mathrm{Star}_{H^{(n)}}(v) \neq \emptyset \big\}.$$

Since $V_n \subseteq \mathrm{P}_n(V_0)$, we have

$$V'_{n+1} = E \subseteq \mathrm{P}(V_n) \subseteq \mathrm{P}\big(\mathrm{P}_n(V_0)\big) = \mathrm{P}_{n+1}(V_0),$$

and each element of E'_{n+1} is a nonempty subset of V'_{n+1}. Hence $L^{(n)}\big(H^{(n)}\big)$ is a level-$(n+1)$ SuperHyperGraph.

Intuitively, $L^{(n)}\big(H^{(n)}\big)$ has one vertex for each n-superedge of $H^{(n)}$, and for every n-supervertex $v \in V_n$ it adds a (super)hyperedge collecting all superedges that contain v.

Example 4.15.5 (A simple line SuperHyperGraph)**.** Let $V_0 := \{1, 2, 3\}$ and take $n = 0$. Then $\mathrm{P}_0(V_0) = V_0$, and we consider the level-0 SuperHyperGraph

$$H^{(0)} = (V_0, E),$$

where

$$V_0 = \{1, 2, 3\}, \qquad E = \big\{e_1, e_2\big\} = \big\{\{1, 2\},\ \{2, 3\}\big\}.$$

Thus $e_1 = \{1, 2\}$ and $e_2 = \{2, 3\}$ are the 0-superedges.

For each vertex $v \in V_0$ the star in $H^{(0)}$ is

$$\mathrm{Star}_{H^{(0)}}(1) = \{e_1\}, \quad \mathrm{Star}_{H^{(0)}}(2) = \{e_1, e_2\}, \quad \mathrm{Star}_{H^{(0)}}(3) = \{e_2\}.$$

The line SuperHyperGraph of $H^{(0)}$ is

$$L^{(0)}\big(H^{(0)}\big) = \big(V'_1, E'_1\big),$$

where the vertex set is the set of superedges of $H^{(0)}$,

$$V'_1 := E = \{e_1, e_2\},$$

and the superedge family is the collection of all nonempty stars,

$$E'_1 := \big\{\{e_1\},\ \{e_1, e_2\},\ \{e_2\}\big\}.$$

Since $V'_1 \subseteq \mathrm{P}_1(V_0) = \mathrm{P}(V_0)$ and each element of E'_1 is a nonempty subset of V'_1, this shows that $L^{(0)}\big(H^{(0)}\big)$ is a level-1 SuperHyperGraph obtained as the line SuperHyperGraph of $H^{(0)}$.

Definition 4.15.6 (Multi-valued line SuperHyperGraph)**.** Let V_0 be a finite nonempty base set, let $n \in \mathbb{N}_0$, and let

$$H^{(n)} = (V_n, E)$$

be a level-n SuperHyperGraph on V_0, where $V_n \subseteq \mathrm{P}_n(V_0)$ is the finite set of n-supervertices and $E \subseteq \mathrm{P}(V_n) \setminus \{\emptyset\}$ is the finite family of nonempty n-superedges.

For each $v \in V_n$, define its *star* and *degree* by

$$E(v) := \mathrm{Star}_{H^{(n)}}(v) := \{\, F \in E \mid v \in F \,\} \subseteq E, \qquad \deg(v) := |E(v)|.$$

Assume that $H^{(n)}$ has *no isolated n-supervertices*, i.e. $\deg(v) \geq 1$ for all $v \in V_n$. Fix an enumeration $V_n = \{v_1, \ldots, v_m\}$ and define integer vectors

$$\mathbf{1}_{H^{(n)}} := \big(\deg(v_1), \ldots, \deg(v_m)\big), \qquad \mathbf{0}_{H^{(n)}} := \big(0_{v_1}, \ldots, 0_{v_m}\big),$$

where

$$0_{v_i} := \begin{cases} 0, & \deg(v_i) = 1, \\ 2, & \deg(v_i) \geq 2. \end{cases}$$

Let

$$D_{H^{(n)}} := \Big\{ D = (d_{v_i})_{i=1}^{m} \;\Big|\; \mathbf{0}_{H^{(n)}} \leq D \leq \mathbf{1}_{H^{(n)}} \text{ componentwise} \Big\}.$$

For a finite set X and an integer $r \geq 0$, define the *rank-r clique hypergraph* on X by

$$\mathrm{Clique}_r(X) := \begin{cases} \big(X,\ \{\, S \subseteq X \mid |S| = r \,\}\big), & r \geq 1, \\ \big(X,\ \emptyset\big), & r = 0. \end{cases}$$

For $D \in D_{H^{(n)}}$ and $v \in V_n$, set

$$F_{v,D} := \mathrm{Clique}_{d_v}\big(E(v)\big),$$

and define the *D-line SuperHyperGraph* of $H^{(n)}$ as

$$L_D^{(n)}\big(H^{(n)}\big) := \big(E,\ E_D\big), \qquad E_D := \bigcup_{v \in V_n} E\big(F_{v,D}\big).$$

Finally, the *multi-valued line SuperHyperGraph* of $H^{(n)}$ is the family

$$\mathcal{L}^{(n)}\big(H^{(n)}\big) := \Big\{ L_D^{(n)}\big(H^{(n)}\big) \;\Big|\; D \in D_{H^{(n)}} \Big\}.$$

Example 4.15.7 (A concrete multi-valued line SuperHyperGraph)**.** Let the finite base set be

$$V_0 := \{a, b\},$$

and take $n = 1$, so $\mathrm{P}_1(V_0) = \mathrm{P}(V_0) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Choose the following 1-supervertices (nonempty subsets of V_0):

$$v_1 := \{a\}, \qquad v_2 := \{b\}, \qquad v_3 := \{a, b\}.$$

Thus

$$V_1 := \{v_1, v_2, v_3\} \subseteq \mathrm{P}_1(V_0).$$

Define a level-1 SuperHyperGraph

$$H^{(1)} = (V_1, E)$$

with three nonempty 1-superedges

$$e_1 := \{v_1, v_3\}, \qquad e_2 := \{v_2, v_3\}, \qquad e_3 := \{v_1, v_2\},$$

and hence

$$E := \{e_1, e_2, e_3\} \subseteq \mathrm{P}(V_1) \setminus \{\emptyset\}.$$

Every supervertex has positive degree, so $H^{(1)}$ has no isolated 1-supervertices.

Step 1: stars and degrees. Compute the stars:

$$E(v_1) = \{e_1, e_3\}, \qquad E(v_2) = \{e_2, e_3\}, \qquad E(v_3) = \{e_1, e_2\},$$

so

$$\deg(v_1) = \deg(v_2) = \deg(v_3) = 2.$$

Step 2: admissible vectors D. Enumerate $V_1 = \{v_1, v_2, v_3\}$. Then

$$\mathbf{1}_{H^{(1)}} = (2, 2, 2), \qquad \mathbf{0}_{H^{(1)}} = (2, 2, 2)$$

because each degree is ≥ 2 and hence each $0_{v_i} = 2$. Therefore the admissible set is a singleton:

$$D_{H^{(1)}} = \{(2, 2, 2)\}.$$

Let $D = (2, 2, 2)$.

Step 3: build the D-line SuperHyperGraph. For each v_i, since $d_{v_i} = 2$ and $|E(v_i)| = 2$, the rank-2 clique hypergraph $\mathrm{Clique}_2(E(v_i))$ contributes exactly one hyperedge, namely $E(v_i)$ itself:

$$E\big(F_{v_1,D}\big) = \big\{\{e_1, e_3\}\big\}, \quad E\big(F_{v_2,D}\big) = \big\{\{e_2, e_3\}\big\}, \quad E\big(F_{v_3,D}\big) = \big\{\{e_1, e_2\}\big\}.$$

Hence

$$E_D = \bigcup_{v \in V_1} E\big(F_{v,D}\big) = \Big\{\{e_1, e_3\}, \{e_2, e_3\}, \{e_1, e_2\}\Big\}.$$

Thus the (unique) D-line SuperHyperGraph is

$$L_D^{(1)}\big(H^{(1)}\big) = \big(E, E_D\big),$$

where the vertex set is $E = \{e_1, e_2, e_3\}$ and the hyperedge family is $E_D = \{\{e_1, e_3\}, \{e_2, e_3\}, \{e_1, e_2\}\}$.

Step 4: the multi-valued line SuperHyperGraph. Since $D_{H^{(1)}} = \{D\}$, the multi-valued line SuperHyperGraph is the singleton family

$$\mathcal{L}^{(1)}\big(H^{(1)}\big) = \Big\{ L_D^{(1)}\big(H^{(1)}\big)\Big\}.$$

Vertices of $L_D^{(1)}(H^{(1)})$ correspond to the original 1-superedges e_1, e_2, e_3. Each original supervertex v_i produces a hyperedge on $\{e \in E \mid v_i \in e\}$, i.e. on its star.

Lemma 4.15.8. *For every $D \in D_{H^{(n)}}$, the object $L_D^{(n)}\big(H^{(n)}\big) = \big(E, E_D\big)$ is a level-$(n+1)$ SuperHyperGraph on the same base set V_0.*

Proof. Since $E \subseteq \mathrm{P}(V_n)$ and $V_n \subseteq \mathrm{P}_n(V_0)$, we have

$$E \subseteq \mathrm{P}(V_n) \subseteq \mathrm{P}\big(\mathrm{P}_n(V_0)\big) = \mathrm{P}_{n+1}(V_0),$$

so the vertex set E of $L_D^{(n)}(H^{(n)})$ is a finite subset of $\mathrm{P}_{n+1}(V_0)$. Moreover, each hyperedge in E_D is a subset of E (because $E(F_{v,D}) \subseteq \mathrm{P}(E(v)) \subseteq \mathrm{P}(E)$), and by construction every hyperedge in $E(F_{v,D})$ has cardinality d_v when $d_v \geq 1$, hence is nonempty. Therefore $E_D \subseteq \mathrm{P}(E) \setminus \{\emptyset\}$ is a finite family of nonempty subsets of E. This is exactly the definition of a level-$(n+1)$ SuperHyperGraph. □

Theorem 4.15.9 (Generalization properties)**.** *Let $H^{(n)} = (V_n, E)$ be a level-n SuperHyperGraph without isolated n-supervertices, and let $\mathcal{L}^{(n)}(H^{(n)})$ be its multi-valued line SuperHyperGraph as in Definition 4.15.6.*

(i) (Generalizes the multi-valued line hypergraph.) If $n = 0$, then $H^{(0)} = (V_0, E)$ is an ordinary hypergraph (with no isolated vertices), and the family $\mathcal{L}^{(0)}(H^{(0)})$ coincides with the Tyshkevich–Zverovich multi-valued line hypergraph construction $\mathcal{L}(H)$.

(ii) (Generalizes the line SuperHyperGraph.) Let $D^{\max} \in D_{H^{(n)}}$ be defined by $d_v^{\max} = \deg(v)$ for every $v \in V_n$ (i.e. $D^{\max} = \mathbf{1}_{H^{(n)}}$). Then

$$L_{D^{\max}}^{(n)}\big(H^{(n)}\big) \;=\; L^{(n)}\big(H^{(n)}\big),$$

where $L^{(n)}(H^{(n)})$ is the (single-valued) line SuperHyperGraph defined via stars.

Proof. (i) Assume $n = 0$. Then $\mathrm{P}_0(V_0) = V_0$ and V_0 is precisely the vertex set of the hypergraph $H^{(0)} = (V_0, E)$. For each $v \in V_0$, the star

$$E(v) = \{ h \in E \mid v \in h \}$$

is exactly the usual star used in the definition of the (multi-valued) line hypergraph. The degree $\deg(v) = |E(v)|$ also matches the usual hypergraph degree. Hence the vectors $\mathbf{0}_{H^{(0)}}, \mathbf{1}_{H^{(0)}}$ and the admissible set $D_{H^{(0)}}$ agree with those in the Tyshkevich–Zverovich construction. For each $D \in D_{H^{(0)}}$, the hyperedges contributed by v are precisely the d_v-subsets of $E(v)$ when $d_v \geq 1$ (and none when $d_v = 0$), i.e. the rank-d_v clique on $E(v)$. Therefore the resulting hypergraph on vertex set E is exactly $L_D(H)$, and so

$$\mathcal{L}^{(0)}(H^{(0)}) = \{ L_D^{(0)}(H^{(0)}) \mid D \in D_{H^{(0)}} \} = \{ L_D(H) \mid D \in D_H \} = \mathcal{L}(H).$$

(ii) Fix $n \geq 0$ and let $D^{\max} = \mathbf{1}_{H^{(n)}}$, so that $d_v^{\max} = \deg(v) = |E(v)|$. For any $v \in V_n$, the rank-$\deg(v)$ clique on $E(v)$ has exactly one hyperedge, namely $E(v)$ itself: indeed,

$$\{ S \subseteq E(v) \mid |S| = \deg(v) \} = \{ E(v) \}.$$

Thus

$$E\big(F_{v,D^{\max}}\big) = \{ E(v) \} = \{ \mathrm{Star}_{H^{(n)}}(v) \}.$$

Taking the union over all $v \in V_n$, we obtain

$$E_{D^{\max}} = \bigcup_{v \in V_n} E\big(F_{v,D^{\max}}\big) = \bigcup_{v \in V_n} \{ \mathrm{Star}_{H^{(n)}}(v) \} = \{ \mathrm{Star}_{H^{(n)}}(v) \mid v \in V_n \},$$

where duplicates (if any) are ignored because we are working with a set of hyperedges. Therefore

$$L^{(n)}_{D^{\max}}\big(H^{(n)}\big) = \big(E,\ \{ \mathrm{Star}_{H^{(n)}}(v) \mid v \in V_n \}\big) = L^{(n)}\big(H^{(n)}\big),$$

which is exactly the star-based line SuperHyperGraph. □

4.16 Total Superhypergraph

Total graph of G has vertices $V(G) \cup E(G)$; adjacency means adjacent vertices, adjacent edges, or incidence between vertex and edge [450, 462, 463]. Total hypergraph $T(H)$ uses $V \uplus E$ as vertices; hyperedges encode original edges, vertex-stars, and vertex–incident edge sets [461]. Total n-superhypergraph of $H^{(n)}$ uses $\iota(V_n) \uplus E_n$; superedges encode incidences, stars, and lifted original superedges together [461].

Definition 4.16.1 (Total HyperGraph). [461] Let $H = (V, E)$ be a finite hypergraph, i.e., $V \neq \emptyset$ and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Define the (typed) disjoint union

$$U := V \uplus E,$$

with canonical injections $\iota_V : V \to U$ and $\iota_E : E \to U$. For each $v \in V$, define the star (incidence family)

$$\mathrm{Star}_H(v) := \{ e \in E : v \in e \} \subseteq E.$$

Define three families of hyperedges on U by

$$A := \Big\{ \{\iota_V(x) : x \in e\} : e \in E \Big\},$$

$$B := \Big\{ \{\iota_E(f) : f \in \mathrm{Star}_H(v)\} : v \in V,\ |\mathrm{Star}_H(v)| \geq 2 \Big\},$$

$$C := \Big\{ \{\iota_V(v)\} \cup \{\iota_E(f) : f \in \mathrm{Star}_H(v)\} : v \in V \Big\}.$$

The *total hypergraph* of H is the hypergraph

$$T(H) := (U,\ A \cup B \cup C).$$

A concrete example is given below.

Example 4.16.2 (A concrete total hypergraph)**.** Let

$$V = \{1, 2, 3\}, \qquad E = \{e_1, e_2\}, \qquad e_1 = \{1, 2\}, \ \ e_2 = \{2, 3\}.$$

Then $H = (V, E)$ is a finite hypergraph. The stars are

$$\mathrm{Star}_H(1) = \{e_1\}, \qquad \mathrm{Star}_H(2) = \{e_1, e_2\}, \qquad \mathrm{Star}_H(3) = \{e_2\}.$$

Form the typed disjoint union

$$U := V \uplus E = \{1, 2, 3, e_1, e_2\},$$

where elements from V and E are regarded as different types.

The three hyperedge families in Definition 4.16.1 are:

$$A = \big\{\{\iota_V(1), \iota_V(2)\}, \ \{\iota_V(2), \iota_V(3)\}\big\},$$

$$B = \big\{\{\iota_E(e_1), \iota_E(e_2)\}\big\} \quad (\text{only } v = 2 \text{ has } |\,\mathrm{Star}_H(v)| \geq 2),$$

$$C = \big\{\{\iota_V(1), \iota_E(e_1)\}, \{\iota_V(2), \iota_E(e_1), \iota_E(e_2)\}, \{\iota_V(3), \iota_E(e_2)\}\big\}.$$

Hence the total hypergraph is

$$T(H) = (U, \ A \cup B \cup C).$$

Definition 4.16.3 (Total n-SuperHyperGraph (Total SuperHyperGraph))**.** [461] Fix a finite base set V_0 and $n \in \mathbb{N}_0$. Let $H^{(n)} = (V_n, E_n)$ be a level-n SuperHyperGraph over V_0, meaning that $V_n \subseteq \mathcal{P}^n(V_0)$ is finite and $E_n \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}$ is finite. Set

$$U_{n+1} \ := \ \iota(V_n) \uplus E_n,$$

where $\iota : V_n \to \mathcal{P}(V_n)$ is the singleton embedding $\iota(v) = \{v\}$, and $\uplus$ indicates a typed disjoint union. For each $v \in V_n$, define the star

$$\mathrm{Star}_{H^{(n)}}(v) \ := \ \{\, E \in E_n : \ v \in E \,\} \ \subseteq \ E_n.$$

Define three families of (super)hyperedges on U_{n+1} by

$$A \ := \ \big\{\, \{\iota(u) : u \in E\} \ : \ E \in E_n \,\big\},$$

$$B \ := \ \big\{\, \mathrm{Star}_{H^{(n)}}(v) \ : \ v \in V_n, \ |\,\mathrm{Star}_{H^{(n)}}(v)| \geq 2 \,\big\},$$

$$C \ := \ \big\{\, \{\iota(v)\} \cup \mathrm{Star}_{H^{(n)}}(v) \ : \ v \in V_n \,\big\}.$$

The *total n-superhypergraph* (or *total superhypergraph*) of $H^{(n)}$ is

$$T\Big(H^{(n)}\Big) \ := \ \big(U_{n+1}, \ A \cup B \cup C\big).$$

A concrete example is given below.

Example 4.16.4 (A concrete total n-superhypergraph (take $n = 1$))**.** Let the base set be

$$V_0 = \{a, b, c\},$$

and take $n = 1$, so $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$. Define the level-1 superhypergraph

$$H^{(1)} = (V_1, E_1)$$

by

$$V_1 = \big\{\{a\}, \{b\}, \{c\}\big\} \subseteq \mathcal{P}(V_0), \qquad E_1 = \{E^*, E^\dagger\},$$

$$E^* = \big\{\{a\}, \{b\}\big\}, \qquad E^\dagger = \big\{\{b\}, \{c\}\big\}.$$

For each $v \in V_1$, its star is

$$\mathrm{Star}_{H^{(1)}}(\{a\}) = \{E^*\}, \qquad \mathrm{Star}_{H^{(1)}}(\{b\}) = \{E^*, E^\dagger\}, \qquad \mathrm{Star}_{H^{(1)}}(\{c\}) = \{E^\dagger\}.$$

The singleton embedding is $\iota(v) = \{v\}$, so

$$\iota(V_1) = \big\{\{\{a\}\}, \{\{b\}\}, \{\{c\}\}\big\}.$$

Define the typed disjoint union

$$U_2 := \iota(V_1) \uplus E_1 = \big\{\{\{a\}\}, \{\{b\}\}, \{\{c\}\}, E^*, E^\dagger\big\}.$$

Now compute the three families in Definition 4.16.3:

$$A = \big\{\{\iota(\{a\}), \iota(\{b\})\}, \{\iota(\{b\}), \iota(\{c\})\}\big\} = \big\{\{\{\{a\}\}, \{\{b\}\}\}, \{\{\{b\}\}, \{\{c\}\}\}\big\},$$

$$B = \big\{\mathrm{Star}_{H^{(1)}}(\{b\})\big\} = \big\{\{E^*, E^\dagger\}\big\},$$

$$C = \big\{\{\iota(\{a\})\} \cup \mathrm{Star}_{H^{(1)}}(\{a\}), \{\iota(\{b\})\} \cup \mathrm{Star}_{H^{(1)}}(\{b\}), \{\iota(\{c\})\} \cup \mathrm{Star}_{H^{(1)}}(\{c\})\big\}$$
$$= \big\{\{\{\{a\}\}, E^*\}, \{\{\{b\}\}, E^*, E^\dagger\}, \{\{\{c\}\}, E^\dagger\}\big\}.$$

Therefore the total 1-superhypergraph is

$$T\Big(H^{(1)}\Big) = (U_2,\ A \cup B \cup C).$$

4.17 Interval SuperHypergraphs

An interval graph represents each vertex as a real line interval, adding edges whenever the corresponding intervals intersect in order [464, 465]. Related concepts include proper interval graphs [466, 467], interval bigraphs [468, 469], co-interval graphs [470, 471], and fuzzy interval graphs [472, 473], among others. An interval hypergraph assigns each hyperedge to a contiguous block of ordered vertices, so every hyperedge equals one underlying interval [474, 475]. An interval SuperHypergraph organizes level supervertices over an ordered base set, requiring each superedge's flattened support to form one interval.

Definition 4.17.1 (Interval hypergraph)**.** (cf. [17, 474, 475]) Let V be a nonempty finite set equipped with a total order $\le$. For $x, y \in V$ with $x \le y$, define the (closed) interval

$$[x, y]_\le \ :=\ \{\, z \in V \mid x \le z \le y \,\}.$$

A hypergraph $H = (V, E)$ (without loops) is called an *interval hypergraph* if there exists a total order $\le$ on V such that for every hyperedge $e \in E$ there are vertices $x, y \in V$ with $x \le y$ and

$$e \ =\ [x, y]_\le.$$

A concrete example is given below.

Example 4.17.2 (Interval hypergraph)**.** Let

$$V := \{1, 2, 3, 4\}$$

equipped with the natural total order

$$1 \ <\ 2\ <\ 3\ <\ 4.$$

For $x, y \in V$ with $x \le y$, the interval is

$$[x, y]_\le := \{\, z \in V \mid x \le z \le y \,\}.$$

Define the hyperedges

$$e_1 := [1, 3]_\le = \{1, 2, 3\}, \qquad e_2 := [2, 4]_\le = \{2, 3, 4\},$$

and let

$$E := \{e_1, e_2\}.$$

Then

$$H := (V, E)$$

is a hypergraph in which every hyperedge is exactly an interval of $(V, \le)$. Hence H is an interval hypergraph.

Definition 4.17.3 ((Recall) Iterated powerset and flattening)**.** Let V_0 be a finite base set. For each integer $k \geq 0$ define the iterated powerset by

$$P_0(V_0) := V_0, \qquad P_{k+1}(V_0) := P\big(P_k(V_0)\big),$$

where $P(\cdot)$ denotes the usual powerset.

Assume V_0 is equipped with a total order $\leq$. For each $k \geq 0$ define the *flattening map*

$$\mathrm{flat}_k : P_k(V_0) \longrightarrow P(V_0)$$

recursively by

$$\mathrm{flat}_0(x) := \{x\} \quad (x \in V_0),$$

and, for $k \geq 0$,

$$\mathrm{flat}_{k+1}(X) := \bigcup_{Y \in X} \mathrm{flat}_k(Y) \quad \text{for } X \in P_{k+1}(V_0) = P\big(P_k(V_0)\big).$$

Definition 4.17.4 (Interval n-SuperHyperGraph)**.** Let V_0 be a finite set with a total order $\leq$, and let $H^{(n)} = (V, E)$ be an n-SuperHyperGraph with $V \subseteq P_n(V_0)$ and $E \subseteq P^*(V)$.

For $x, y \in V_0$ with $x \leq y$, write

$$[x, y]_{\leq} := \{\, z \in V_0 \mid x \leq z \leq y \,\}.$$

We say that $H^{(n)}$ is an *interval n-SuperHyperGraph* if there exists a total order $\leq$ on V_0 such that for every superedge $e \in E$ there are vertices $x, y \in V_0$ with $x \leq y$ and

$$\mathrm{supp}(e) \;=\; [x, y]_{\leq}.$$

A concrete example is given below.

Example 4.17.5 (Interval 2-SuperHyperGraph)**.** Let

$$V_0 := \{1, 2, 3, 4\}$$

equipped with the natural order $1 < 2 < 3 < 4$. Recall that

$$\mathrm{P}_0(V_0) = V_0, \quad \mathrm{P}_1(V_0) = \mathrm{P}(V_0), \quad \mathrm{P}_2(V_0) = \mathrm{P}\big(\mathrm{P}(V_0)\big).$$

Consider the following 2-supervertices:

$$v_1 := \big\{\{1, 2\}, \{2, 3\}\big\}, \qquad v_2 := \big\{\{2, 3\}, \{3, 4\}\big\},$$

so that $v_1, v_2 \in \mathrm{P}_2(V_0)$. Put

$$V := \{v_1, v_2\} \;\subseteq\; \mathrm{P}_2(V_0),$$

and define a single 2-superedge

$$e := \{v_1, v_2\}, \qquad E := \{e\} \;\subseteq\; \mathrm{P}^*(V).$$

Then

$$H^{(2)} := (V, E)$$

is a 2-SuperHyperGraph.

Let $\mathrm{flat}_2 : \mathrm{P}_2(V_0) \to \mathrm{P}(V_0)$ be the flattening map (from the recalled definition), so that for a 2-element $Y \in \mathrm{P}_2(V_0)$,

$$\mathrm{flat}_2(Y) = \bigcup_{X \in Y} X.$$

By definition of the support of a superedge,

$$\mathrm{supp}(e) = \mathrm{flat}_2\Big(\bigcup_{v \in e} v\Big) = \mathrm{flat}_2\big(v_1 \cup v_2\big).$$

Since

$$v_1 \cup v_2 = \{\{1,2\},\{2,3\},\{3,4\}\},$$

we obtain

$$\mathrm{supp}(e) = \{1,2\} \cup \{2,3\} \cup \{3,4\} = \{1,2,3,4\} = [1,4]_{\le}.$$

Thus $\mathrm{supp}(e)$ is exactly an interval of $(V_0, \le)$, so $H^{(2)}$ is an interval 2-SuperHyperGraph (with respect to the total order $1 < 2 < 3 < 4$).

The overview of the comparison of interval graphs, interval hypergraphs, and interval n-SuperHyperGraphs is presented in Table 4.9.

Table 4.9: Comparison of interval graphs, interval hypergraphs, and interval n-SuperHyperGraphs

Framework	Underlying objects	Interval condition
Interval graph	Simple graph $G = (V, E)$; each vertex $v \in V$ is represented by an interval I_v on the real line.	There is an edge $uv \in E$ if and only if $I_u \cap I_v \neq \emptyset$, i.e. edges arise exactly from intersecting intervals.
Interval hypergraph	Hypergraph $H = (V, E)$ with a total order $\le$ on V; hyperedges $e \in E \subseteq \mathcal{P}^*(V)$.	For each $e \in E$ there exist $x, y \in V$ with $x \le y$ such that $e = [x, y]_{\le} = \{z \in V \mid x \le z \le y\}$.
Interval n-SuperHyperGraph	n-SuperHyperGraph $H^{(n)} = (V, E)$ with $V \subseteq P_n(V_0)$ over an ordered base set $(V_0, \le)$.	For each superedge $e \in E$ there exist $x, y \in V_0$ with $x \le y$ such that the flattened support $\mathrm{supp}(e)$ equals the interval $[x, y]_{\le}$.

Theorem 4.17.6 (Interval n-SuperHyperGraphs generalize interval hypergraphs)**.** *Let V_0 be a finite set with a total order $\le$. Consider the following two structures on V_0:*

1. *A hypergraph $H = (V_0, E)$.*
2. *A 0-SuperHyperGraph*

$$H^{(0)} = (V, E) \quad \text{with } V := V_0,\ E := E.$$

Then H is an interval hypergraph (in the sense of the above definition) if and only if $H^{(0)}$ is an interval 0-SuperHyperGraph. Hence, interval n-SuperHyperGraphs (for general n) extend the notion of interval hypergraphs, which are exactly the case $n = 0$.

Proof. First note that for $n = 0$ we have $P_0(V_0) = V_0$ and, by definition of the flattening map,

$$\mathrm{flat}_0(x) = \{x\} \quad \text{for all } x \in V_0.$$

Thus, for a 0-supervertex $v \in V = V_0$,

$$\mathrm{supp}(v) = \mathrm{flat}_0(v) = \{v\},$$

and for a superedge $e \in E \subseteq P^*(V_0)$ we obtain

$$\mathrm{supp}(e) = \bigcup_{v \in e} \mathrm{supp}(v) = \bigcup_{v \in e} \{v\} = e.$$

($\Rightarrow$) Assume $H = (V_0, E)$ is an interval hypergraph. Then there exists a total order $\le$ on V_0 such that for each $e \in E$ there are $x, y \in V_0$ with $x \le y$ and

$$e = [x, y]_{\le}.$$

Define $H^{(0)} = (V, E)$ with $V := V_0$ as above. For every $e \in E$ we have seen that $\mathrm{supp}(e) = e$, so

$$\mathrm{supp}(e) = e = [x, y]_{\le}.$$

Hence $H^{(0)}$ is an interval 0-SuperHyperGraph with respect to the same order $\leq$.

($\Leftarrow$) Conversely, suppose $H^{(0)} = (V, E)$ with $V = V_0$ is an interval 0-SuperHyperGraph. Then there exists a total order $\leq$ on V_0 such that for every $e \in E$ there are $x, y \in V_0$ with $x \leq y$ and

$$\mathrm{supp}(e) = [x, y]_{\leq}.$$

But for $n = 0$ we have $\mathrm{supp}(e) = e$, so

$$e = [x, y]_{\leq}$$

for all $e \in E$. Therefore $H = (V_0, E)$ is an interval hypergraph in the usual sense.

This proves the equivalence, and shows that when $n = 0$, interval n-SuperHyperGraphs coincide exactly with interval hypergraphs, so the former constitute a genuine generalization of the latter. □

4.18 Unimodular SuperHypergraphs

A unimodular function is a function whose values have absolute value one, typically complex-valued, e.g., $|f(x)| = 1$ everywhere. A unimodular graph has a totally unimodular incidence matrix, so related linear programs have integral optimal solutions (cf. [476, 477]). An unimodular hypergraph has a totally unimodular incidence matrix, so all subdeterminants are restricted to minus one, zero, or one [17, 478–480]. Unimodular n-SuperHyperGraphs have totally unimodular incidence matrices, ensuring integer solutions in associated linear programming and combinatorial optimization problems and algorithms.

Definition 4.18.1 (Totally unimodular matrix)**.** Let A be a real matrix. We say that A is *totally unimodular* if every square submatrix of A has determinant in $\{-1, 0, 1\}$.

Example 4.18.2 (Totally unimodular matrix)**.** Consider the 3×3 identity matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Every square submatrix of A is either a smaller identity matrix (or one of its permutations) or a matrix containing a zero row or zero column. Hence every such determinant is 1, 0, or -1. Therefore A is a totally unimodular matrix.

Definition 4.18.3 (Unimodular hypergraph)**.** [17] Let $H = (V, E)$ be a (finite) hypergraph, where $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$, and let $A(H) = (a_{ij})$ be its incidence matrix defined by

$$a_{ij} := \begin{cases} 1, & \text{if } v_i \in e_j, \\ 0, & \text{otherwise.} \end{cases}$$

The hypergraph H is called *unimodular* if its incidence matrix $A(H)$ is totally unimodular.

A concrete example is given below.

Example 4.18.4 (Unimodular hypergraph)**.** Let

$$V := \{v_1, v_2\}, \qquad E := \{e_1, e_2\},$$

with hyperedges

$$e_1 := \{v_1\}, \qquad e_2 := \{v_2\}.$$

The incidence matrix $A(H)$ of the hypergraph $H = (V, E)$ is

$$A(H) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is the 2×2 identity matrix, which is totally unimodular (since all its square subdeterminants are 0 or 1). Hence H is a unimodular hypergraph.

Definition 4.18.5 (n-SuperHyperGraph incidence matrix)**.** Let V_0 be a finite base set and $n \in \mathbb{N}_0$. An *n-SuperHyperGraph* is a triple

$$\mathcal{H}^{(n)} = (V, E, \partial),$$

where

- $V \subseteq P_n(V_0)$ is a finite set of n-supervertices,
- E is a finite set of n-superedges,
- $\partial : E \to \mathcal{P}^*(V)$ is the incidence map, with $\mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\}$.

Enumerate $V = \{w_1, \dots, w_N\}$ and $E = \{f_1, \dots, f_M\}$. The *incidence matrix* of $\mathcal{H}^{(n)}$ is the $N \times M$ matrix $A(\mathcal{H}^{(n)}) = (b_{ij})$ defined by

$$b_{ij} := \begin{cases} 1, & \text{if } w_i \in \partial(f_j), \\ 0, & \text{otherwise.} \end{cases}$$

A concrete example is given below.

Example 4.18.6 (1-SuperHyperGraph incidence matrix)**.** Let the base set be

$$V_0 := \{a, b, c\},$$

so that the first powerset is $\mathrm{P}_1(V_0) = \mathrm{P}(V_0)$. Define the set of 1-supervertices

$$V := \{w_1, w_2\} := \{\{a, b\}, \{b, c\}\} \subseteq \mathrm{P}_1(V_0),$$

and the set of 1-superedges

$$E := \{f_1, f_2\},$$

with incidence map $\partial : E \to \mathcal{P}^*(V)$ given by

$$\partial(f_1) := \{w_1, w_2\}, \qquad \partial(f_2) := \{w_2\}.$$

Enumerating $V = \{w_1, w_2\}$ and $E = \{f_1, f_2\}$, the incidence matrix $A(\mathcal{H}^{(1)}) = (b_{ij})$ of the 1-SuperHyperGraph $\mathcal{H}^{(1)} = (V, E, \partial)$ is

$$A(\mathcal{H}^{(1)}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

because $w_1 \in \partial(f_1)$, $w_1 \notin \partial(f_2)$, and $w_2 \in \partial(f_1) \cap \partial(f_2)$.

Definition 4.18.7 (Unimodular n-SuperHyperGraph)**.** Let $\mathcal{H}^{(n)} = (V, E, \partial)$ be an n-SuperHyperGraph with incidence matrix $A(\mathcal{H}^{(n)})$ as above. We say that $\mathcal{H}^{(n)}$ is *unimodular* if $A(\mathcal{H}^{(n)})$ is totally unimodular.

Example 4.18.8 (Unimodular 1-SuperHyperGraph)**.** Let the base set be

$$V_0 := \{a, b\},$$

so that $\mathrm{P}_1(V_0) = \mathrm{P}(V_0)$. Define the 1-supervertex set

$$V := \{w_1, w_2\} := \{\{a\}, \{b\}\} \subseteq \mathrm{P}_1(V_0),$$

and the 1-superedge set

$$E := \{f_1, f_2\},$$

with incidence map

$$\partial(f_1) := \{w_1\}, \qquad \partial(f_2) := \{w_2\}.$$

Enumerating $V = \{w_1, w_2\}$ and $E = \{f_1, f_2\}$, the incidence matrix of the 1-SuperHyperGraph $\mathcal{H}^{(1)} = (V, E, \partial)$ is

$$A(\mathcal{H}^{(1)}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is the 2×2 identity matrix, hence it is totally unimodular. Therefore $\mathcal{H}^{(1)}$ is a unimodular 1-SuperHyperGraph.

Feature	Unimodular hypergraph $H = (V, E)$ [17]	Unimodular n-SuperHyperGraph $\mathcal{H}^{(n)} = (V, E, \partial)$
Underlying universe	A finite vertex set V	A finite base set V_0 and iterated powerset level $V \subseteq P_n(V_0)$
Vertices	Ordinary vertices $v \in V$	n-supervertices $w \in V \subseteq P_n(V_0)$
Edges / superedges	Hyperedges $e \in E$ with $e \subseteq V$	n-superedges $f \in E$ with incidence map $\partial(f) \in \mathcal{P}(V) \setminus \{\emptyset\}$
Incidence matrix	$A(H) = (a_{ij})$, $a_{ij} = 1 \Leftrightarrow v_i \in e_j$	$A(\mathcal{H}^{(n)}) = (b_{ij})$, $b_{ij} = 1 \Leftrightarrow w_i \in \partial(f_j)$
Unimodularity condition	H is unimodular iff $A(H)$ is totally unimodular	$\mathcal{H}^{(n)}$ is unimodular iff $A(\mathcal{H}^{(n)})$ is totally unimodular
Meaning of total unimodularity	Every square subdeterminant of $A(H)$ lies in $\{-1, 0, 1\}$	Every square subdeterminant of $A(\mathcal{H}^{(n)})$ lies in $\{-1, 0, 1\}$
Reduction / generalization	Base notion (level 0)	Extends unimodular hypergraphs: $n = 0$ (with $\partial(e) = e$) recovers $A(\mathcal{H}^{(0)}) = A(H)$
Typical implication (informal)	LP relaxations associated with incidence constraints often have integral optima	Same integrality benefit, now for higher-level (supervertex/superedge) incidence constraints

Table 4.10: Concise comparison of unimodular hypergraphs and unimodular n-SuperHyperGraphs.

Theorem 4.18.9 (Unimodular n-SuperHyperGraphs generalize unimodular hypergraphs)**.** *Let $H = (V, E)$ be a finite hypergraph with incidence matrix $A(H)$. Regard H as a 0-SuperHyperGraph*

$$\mathcal{H}^{(0)} := (V, E, \partial),$$

where $\partial(e) = e$ for each $e \in E$. Then

$$H \text{ is a unimodular hypergraph} \iff \mathcal{H}^{(0)} \text{ is a unimodular 0-SuperHyperGraph}.$$

In particular, the notion of unimodularity for n-SuperHyperGraphs extends the usual notion of unimodular hypergraphs.

Proof. By construction, the incidence matrix $A(\mathcal{H}^{(0)})$ of the 0-SuperHyperGraph $\mathcal{H}^{(0)}$ coincides with the incidence matrix $A(H)$ of the hypergraph H, entry by entry: for all i, j,

$$A(\mathcal{H}^{(0)})_{ij} = 1 \iff v_i \in \partial(e_j) = e_j \iff A(H)_{ij} = 1,$$

and similarly for 0. Therefore $A(H)$ is totally unimodular if and only if $A(\mathcal{H}^{(0)})$ is totally unimodular. By the definitions of unimodular hypergraph and unimodular n-SuperHyperGraph, this is equivalent to the desired statement. □

A comparison of unimodular hypergraphs and unimodular n-SuperHyperGraphs is presented in Table 4.10.

4.19 Probabilistic Superhypergraphs

A probabilistic graph assigns probabilities to vertices or edges, representing uncertain existence or interactions and enabling stochastic inference and analysis [481–483]. Moreover, as a related concept to probabilistic graphs, the probabilistic directed graph [484, 485] is well known. A probabilistic hypergraph models hyperedges with vertex-level connection probabilities, capturing uncertain multiway interactions, weights, and affinity-based structure in complex networks [76, 486–488]. A probabilistic superhypergraph extends hypergraphs to hierarchical supervertices, assigning probabilities to multi-level superedges, modeling uncertain interactions across nested structures robustly [489]. The relevant definitions and related notions are presented below.

Definition 4.19.1 (Probabilistic Hypergraph)**.** [76, 486–488] Let V be a finite, nonempty set and let $\mathcal{P}(V)$ denote its powerset. A *probabilistic hypergraph* is a triplet

$$H = (V, E, A),$$

where

- $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a finite family of nonempty hyperedges;
- $A : V \times V \to [0,1]$ is an *affinity* (or probability) function, where $A(u,v)$ encodes the probability or similarity of a connection between u and v.

For each hyperedge $e \in E$ we choose a *centroid* vertex $c(e) \in e$ according to a specified optimality criterion, for example

$$c(e) \ := \ \arg\max_{w \in e} \sum_{u \in e} A(w,u).$$

The *incidence matrix* of H is the $|V| \times |E|$ matrix

$$\mathbf{H} : V \times E \longrightarrow [0,1]$$

defined by

$$\mathbf{H}(v,e) := \begin{cases} A\big(c(e),v\big), & \text{if } v \in e, \\ 0, & \text{if } v \notin e. \end{cases}$$

The *weight* of a hyperedge $e \in E$ is

$$w(e) \ := \ \sum_{v \in e} A\big(c(e),v\big),$$

the *degree* of a vertex $v \in V$ is

$$d(v) \ := \ \sum_{e \in E} w(e)\,\mathbf{H}(v,e),$$

and the *degree* of a hyperedge $e \in E$ is

$$\delta(e) \ := \ \sum_{v \in e} \mathbf{H}(v,e).$$

Example 4.19.2 (A small probabilistic hypergraph)**.** Let the finite vertex set be

$$V := \{a,b,c\},$$

and define the family of hyperedges by

$$E := \{e_1, e_2\}, \qquad e_1 := \{a,b\}, \quad e_2 := \{b,c\}.$$

Thus $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$.

Define the affinity (probability) function

$$A : V \times V \longrightarrow [0,1]$$

by the table

$\big(A(u,v)\big)_{u,v \in V} =$

	a	b	c
a	1.0	0.8	0.2
b	0.5	1.0	0.9
c	0.4	0.3	1.0

.

For each hyperedge $e \in E$ we choose a centroid

$$c(e) := \arg\max_{w \in e} \sum_{u \in e} A(w,u).$$

For $e_1 = \{a,b\}$ we compute

$$\sum_{u \in e_1} A(a,u) = A(a,a) + A(a,b) = 1.0 + 0.8 = 1.8,$$

$$\sum_{u \in e_1} A(b,u) = A(b,a) + A(b,b) = 0.5 + 1.0 = 1.5,$$

so $c(e_1) = a$.

For $e_2 = \{b, c\}$ we compute

$$\sum_{u \in e_2} A(b,u) = A(b,b) + A(b,c) = 1.0 + 0.9 = 1.9,$$

$$\sum_{u \in e_2} A(c,u) = A(c,b) + A(c,c) = 0.3 + 1.0 = 1.3,$$

hence $c(e_2) = b$.

The incidence matrix $\mathbf{H} : V \times E \to [0,1]$ is

$$\mathbf{H}(v,e) := \begin{cases} A\big(c(e), v\big), & \text{if } v \in e, \\ 0, & \text{if } v \notin e, \end{cases}$$

so explicitly

$$\mathbf{H} = \begin{array}{c|cc} & e_1 & e_2 \\ \hline a & A(c(e_1),a) & 0 \\ b & A(c(e_1),b) & A(c(e_2),b) \\ c & 0 & A(c(e_2),c) \end{array} = \begin{pmatrix} 1.0 & 0 \\ 0.8 & 1.0 \\ 0 & 0.9 \end{pmatrix}.$$

The hyperedge weights are

$$w(e_1) = \sum_{v \in e_1} A\big(c(e_1), v\big) = A(a,a) + A(a,b) = 1.0 + 0.8 = 1.8,$$

$$w(e_2) = \sum_{v \in e_2} A\big(c(e_2), v\big) = A(b,b) + A(b,c) = 1.0 + 0.9 = 1.9.$$

The degrees of vertices (in the sense of the definition) are

$$d(a) = w(e_1)\,\mathbf{H}(a, e_1) = 1.8 \cdot 1.0 = 1.8,$$

$$d(b) = w(e_1)\,\mathbf{H}(b, e_1) + w(e_2)\,\mathbf{H}(b, e_2) = 1.8 \cdot 0.8 + 1.9 \cdot 1.0 = 1.44 + 1.9 = 3.34,$$

$$d(c) = w(e_2)\,\mathbf{H}(c, e_2) = 1.9 \cdot 0.9 = 1.71.$$

Thus $H = (V, E, A)$ is a concrete probabilistic hypergraph in the sense of the definition.

Definition 4.19.3 (Probabilistic n-SuperHyperGraph)**.** [489] Let V_0 be a finite, nonempty *base set*. For each integer $k \geq 0$, define the iterated powersets

$$\mathrm{P}_0(V_0) := V_0, \qquad \mathrm{P}_{k+1}(V_0) := \mathrm{P}\big(\mathrm{P}_k(V_0)\big),$$

where $\mathrm{P}(X)$ denotes the powerset of a set X.

Fix a level $n \in \mathbb{N}_0$. A *probabilistic n-SuperHyperGraph* over V_0 is a triplet

$$G = (V, E, A),$$

where

- $V \subseteq \mathrm{P}_n(V_0)$ is a finite set of *n-supervertices*;
- $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a finite family of nonempty *n-superedges*, each $e \in E$ being a nonempty subset of V;
- $A : V \times V \to [0,1]$ is an *affinity* (or probability) function on pairs of n-supervertices, where $A(u,v)$ encodes the probability or similarity of an interaction between u and v.

For each n-superedge $e \in E$ we choose a *centroid n-supervertex* $c(e) \in e$ according to

$$c(e) := \arg\max_{w \in e} \sum_{u \in e} A(w,u).$$

The *incidence matrix* of G is the $|V| \times |E|$ matrix

$$\mathbf{H} : V \times E \longrightarrow [0,1]$$

defined by

$$\mathbf{H}(v,e) := \begin{cases} A\big(c(e), v\big), & \text{if } v \in e, \\ 0, & \text{if } v \notin e. \end{cases}$$

The *weight* of an n-superedge $e \in E$ is

$$w(e) := \sum_{v \in e} A\big(c(e), v\big),$$

the *degree* of an n-supervertex $v \in V$ is

$$d(v) := \sum_{e \in E} w(e)\, \mathbf{H}(v,e),$$

and the *degree* of an n-superedge $e \in E$ is

$$\delta(e) := \sum_{v \in e} \mathbf{H}(v,e).$$

When $n = 0$, this definition reduces to that of a probabilistic hypergraph with vertex set V_0.

Example 4.19.4 (A probabilistic 2-SuperHyperGraph)**.** Let the finite base set be

$$V_0 := \{x_1, x_2, x_3\}.$$

Then

$$\mathrm{P}_1(V_0) = \mathrm{P}(V_0), \qquad \mathrm{P}_2(V_0) = \mathrm{P}\big(\mathrm{P}(V_0)\big).$$

Define two 2-supervertices

$$v_1 := \big\{\{x_1, x_2\}\big\}, \qquad v_2 := \big\{\{x_2, x_3\}\big\}.$$

Since $\{x_1, x_2\}, \{x_2, x_3\} \in \mathrm{P}_1(V_0)$, both v_1 and v_2 are subsets of $\mathrm{P}_1(V_0)$, hence

$$v_1, v_2 \in \mathrm{P}\big(\mathrm{P}_1(V_0)\big) = \mathrm{P}_2(V_0).$$

Set the 2-supervertex set

$$V := \{v_1, v_2\} \subseteq \mathrm{P}_2(V_0).$$

Introduce a single 2-superedge

$$e_1 := \{v_1, v_2\}, \qquad E := \{e_1\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Define the affinity function

$$A : V \times V \longrightarrow [0,1]$$

by

$$\big(A(u,v)\big)_{u,v \in V} = \begin{array}{c|cc} & v_1 & v_2 \\ \hline v_1 & 1.0 & 0.6 \\ v_2 & 0.3 & 1.0 \end{array}.$$

For the unique 2-superedge $e_1 = \{v_1, v_2\}$ we choose the centroid

$$c(e_1) := \arg\max_{w \in e_1} \sum_{u \in e_1} A(w, u).$$

We compute

$$\sum_{u \in e_1} A(v_1, u) = A(v_1, v_1) + A(v_1, v_2) = 1.0 + 0.6 = 1.6,$$

$$\sum_{u \in e_1} A(v_2, u) = A(v_2, v_1) + A(v_2, v_2) = 0.3 + 1.0 = 1.3,$$

so $c(e_1) = v_1$.

The incidence matrix $\mathbf{H} : V \times E \to [0, 1]$ is given by

$$\mathbf{H}(v, e) := \begin{cases} A\big(c(e), v\big), & \text{if } v \in e, \\ 0, & \text{if } v \notin e, \end{cases}$$

hence

$$\mathbf{H} = \begin{array}{c|c} & e_1 \\ \hline v_1 & A(c(e_1), v_1) \\ v_2 & A(c(e_1), v_2) \end{array} = \begin{pmatrix} 1.0 \\ 0.6 \end{pmatrix}.$$

The weight of the 2-superedge e_1 is

$$w(e_1) = \sum_{v \in e_1} A\big(c(e_1), v\big) = A(v_1, v_1) + A(v_1, v_2) = 1.0 + 0.6 = 1.6.$$

The degrees of the 2-supervertices are

$$d(v_1) = w(e_1)\, \mathbf{H}(v_1, e_1) = 1.6 \cdot 1.0 = 1.6,$$

$$d(v_2) = w(e_1)\, \mathbf{H}(v_2, e_1) = 1.6 \cdot 0.6 = 0.96.$$

Thus

$$G := (V, E, A)$$

is a concrete probabilistic 2-SuperHyperGraph over the base set V_0, in the sense of the given definition.

We include in Table 4.11 a comparison of probabilistic graphs, probabilistic hypergraphs, and probabilistic n-SuperHyperGraphs.

4.20 Balanced SuperHypergraphs

A balanced n-SuperHyperGraph is one whose incidence matrix contains no odd-order square submatrix in which every row and column has exactly two entries equal to 1. This condition guarantees even-cycle parity and ensures structural stability across its multi-level superedge connections. A balanced n-SuperHyperGraph is an applied generalization of the classical notions of balanced graphs and balanced hypergraphs (cf. [17,490–492]). Related concepts are also known, such as balanced fuzzy graphs [493, 494], balanced intuitinistic fuzzy graphs [495, 496], balanced directed graphs [497, 498], balanced picture fuzzy graphs [499], and balanced neutrosophic graphs [500]. The relevant definitions and related notions are presented below.

Definition 4.20.1 (Balanced $\{0, 1\}$–matrix)**.** [17] A $\{0, 1\}$–matrix A is called *balanced* if it does not contain, as a square submatrix, any odd–order matrix B such that every row and every column of B has exactly two entries equal to 1. Equivalently, for every square submatrix B of A with exactly two 1's in each row and each column, the order of B must be even.

Table 4.11: Comparison of probabilistic graphs, probabilistic hypergraphs, and probabilistic n-SuperHyperGraphs

Framework	Underlying structure	Probabilistic / affinity modeling
Probabilistic graph	Simple graph $G = (V, E)$ (or potential edges on $V \times V$).	A probability or affinity function $P : V \times V \to [0, 1]$ assigns to each vertex pair the likelihood or strength of an edge.
Probabilistic hypergraph	Hypergraph $H = (V, E)$ with hyperedges $e \in E \subseteq \mathcal{P}^*(V)$.	An affinity function $A : V \times V \to [0, 1]$ encodes pairwise probabilities inside hyperedges; centroids, incidence matrix, and hyperedge weights are derived from A.
Probabilistic n-SuperHyperGraph	n-SuperHyperGraph $G = (V, E)$ with $V \subseteq \mathrm{P}_n(V_0)$ and $E \subseteq \mathcal{P}^*(V)$.	An affinity function $A : V \times V \to [0, 1]$ on n-supervertices induces probabilistic incidence, superedge weights, and degrees across hierarchical, multi-level interactions.

Example 4.20.2 (Balanced $\{0, 1\}$–matrix)**.** Consider the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Any square submatrix of A is either a (smaller) identity matrix or a matrix with at most one entry equal to 1 in each row or column. In particular, no odd–order square submatrix of A has exactly two entries equal to 1 in every row and every column. Hence A is a balanced $\{0, 1\}$–matrix.

Definition 4.20.3 (Balanced hypergraph)**.** [17] Let $H = (V, E)$ be a finite hypergraph and let $A(H)$ denote its incidence matrix (as defined previously). The hypergraph H is called *balanced* if the matrix $A(H)$ is balanced in the sense above.

Example 4.20.4 (Balanced hypergraph)**.** Let

$$V := \{v_1, v_2, v_3\}, \qquad E := \{e_1, e_2\},$$

with hyperedges

$$e_1 := \{v_1, v_2\}, \qquad e_2 := \{v_2, v_3\}.$$

The incidence matrix of the hypergraph $H = (V, E)$ is

$$A(H) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Every 1×1 or 2×2 square submatrix of $A(H)$ has at most two entries equal to 1 in total, so none of them has exactly two 1's in each row and each column. Therefore $A(H)$ does not contain any odd–order square submatrix with two 1's in every row and column. Thus $A(H)$ is balanced, and the hypergraph H is a balanced hypergraph.

Definition 4.20.5 (Balanced n-SuperHyperGraph)**.** Let $\mathcal{H}^{(n)} = (V, E, \partial)$ be an n-SuperHyperGraph with incidence matrix $A(\mathcal{H}^{(n)})$ (as defined previously). We say that $\mathcal{H}^{(n)}$ is a *balanced n-SuperHyperGraph* if $A(\mathcal{H}^{(n)})$ is a balanced $\{0, 1\}$–matrix.

Example 4.20.6 (Balanced 1-SuperHyperGraph)**.** Let the base set be

$$V_0 := \{a, b\},$$

so that $\mathrm{P}_1(V_0) = \mathrm{P}(V_0)$. Define the set of 1–supervertices

$$V := \{w_1, w_2\} := \{\{a\}, \{b\}\} \subseteq \mathrm{P}_1(V_0),$$

Feature	Balanced hypergraph $H = (V, E)$ [17, 490–492]	Balanced n-SuperHyperGraph $\mathcal{H}^{(n)} = (V, E, \partial)$
Underlying universe	Finite vertex set V	Finite base set V_0 and level-n supervertex set $V \subseteq P_n(V_0)$
Vertices	Ordinary vertices $v \in V$	n-supervertices $w \in V \subseteq P_n(V_0)$
Edges / superedges	Hyperedges $e \in E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	n-superedges $f \in E$ with incidence map $\partial(f) \in \mathcal{P}(V) \setminus \{\emptyset\}$
Incidence matrix	$A(H) = (a_{ij}),\ a_{ij} = 1 \Leftrightarrow v_i \in e_j$	$A(\mathcal{H}^{(n)}) = (b_{ij}),\ b_{ij} = 1 \Leftrightarrow w_i \in \partial(f_j)$
Balancedness criterion (matrix form)	$A(H)$ has no odd-order square submatrix with exactly two 1's in each row and column	$A(\mathcal{H}^{(n)})$ has no odd-order square submatrix with exactly two 1's in each row and column
Combinatorial intuition (informal)	Excludes "odd cycles" in the incidence structure encoded by $A(H)$	Enforces the same even-parity restriction across multi-level incidences of supervertices and superedges
Reduction / generalization	Base notion (level 0)	Extends balanced hypergraphs: for $n = 0$ with $\partial(e) = e$, one has $A(\mathcal{H}^{(0)}) = A(H)$

Table 4.12: Concise comparison of balanced hypergraphs and balanced n-SuperHyperGraphs.

and the set of 1–superedges

$$E := \{f_1, f_2\},$$

with incidence map

$$\partial(f_1) := \{w_1\}, \qquad \partial(f_2) := \{w_2\}.$$

Enumerating $V = \{w_1, w_2\}$ and $E = \{f_1, f_2\}$, the incidence matrix of the 1–SuperHyperGraph $\mathcal{H}^{(1)} = (V, E, \partial)$ is

$$A\big(\mathcal{H}^{(1)}\big) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is the 2×2 identity matrix. As in the first example, no odd–order square submatrix of $A\big(\mathcal{H}^{(1)}\big)$ has two 1's in every row and every column. Hence $A\big(\mathcal{H}^{(1)}\big)$ is a balanced $\{0, 1\}$–matrix, and $\mathcal{H}^{(1)}$ is a balanced 1–SuperHyperGraph.

Theorem 4.20.7 (Balanced n-SuperHyperGraphs generalize balanced hypergraphs)**.** *Let $H = (V, E)$ be a finite hypergraph. Regard H as a 0-SuperHyperGraph*

$$\mathcal{H}^{(0)} := (V, E, \partial),$$

where $\partial(e) = e$ for all $e \in E$. Then

$$H \text{ is a balanced hypergraph} \iff \mathcal{H}^{(0)} \text{ is a balanced 0-SuperHyperGraph}.$$

In particular, the notion of a balanced n-SuperHyperGraph extends the classical notion of a balanced hypergraph, which is recovered by taking $n = 0$.

Proof. By construction, the incidence matrix of the 0-SuperHyperGraph $\mathcal{H}^{(0)}$ coincides with the incidence matrix of H:

$$A(\mathcal{H}^{(0)}) = A(H),$$

entrywise. Therefore $A(H)$ is balanced if and only if $A(\mathcal{H}^{(0)})$ is balanced. By the definitions above, this is equivalent to H being a balanced hypergraph and $\mathcal{H}^{(0)}$ being a balanced 0-SuperHyperGraph, respectively. □

A comparison of balanced hypergraphs and balanced n-SuperHyperGraphs is presented in Table 4.12.

4.21 Spatial Superhypergraphs

Spatial hypergraph assigns each vertex a distinct point in Euclidean space, modeling multiway relationships constrained by geometric or geographic structure [501–504]. Spatial SuperHypergraph embeds base elements in space, while higher-level supervertices and superedges capture hierarchical, multi-scale relationships across locations and regions [504]. The relevant definitions and related notions are presented below.

Definition 4.21.1 (Spatial hypergraph)**.** [504] Let $d \in \mathbb{N}$ and let V be a finite, nonempty set. A *hypergraph* on V is a pair

$$H := (V, E), \qquad E \subseteq \mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\},$$

whose elements $e \in E$ are called *hyperedges*.

A *d-dimensional spatial hypergraph* is a triple

$$\mathsf{SpHG} := (V, E, \lambda),$$

where

- (V, E) is a hypergraph as above;
- $\lambda : V \to \mathbb{R}^d$ is an *embedding map* that assigns to each vertex $v \in V$ a point $\lambda(v)$ in Euclidean space, and is injective (no two vertices share the same location).

The pair (V, E) captures the combinatorial structure, while λ endows the hypergraph with a fixed spatial geometry.

Example 4.21.2 (Spatial hypergraph: wireless sensors in the plane)**.** Let $d = 2$ and consider three wireless sensors located in the plane. Set the vertex set

$$V := \{s_1, s_2, s_3\}.$$

Define the hyperedge family

$$E := \{e_1, e_2\}, \qquad e_1 := \{s_1, s_2\},\ e_2 := \{s_2, s_3\},$$

so that

$$H := (V, E)$$

is a hypergraph in the above sense.

We now fix the physical positions of the sensors in $\mathbb{R}^2$ by an injective embedding map

$$\lambda : V \longrightarrow \mathbb{R}^2,$$

given by

$$\lambda(s_1) := (0, 0), \quad \lambda(s_2) := (1, 0), \quad \lambda(s_3) := (1, 1).$$

The triple

$$\mathsf{SpHG} := (V, E, \lambda)$$

is a 2-dimensional spatial hypergraph. The combinatorial structure of group communication is encoded by E, while the map λ records the spatial locations of the sensors.

Definition 4.21.3 (Spatial n-SuperHyperGraph)**.** [504] Let $d \in \mathbb{N}$, let V_0 be a finite base set, and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph on V_0, that is

$$V, E \subseteq \mathcal{P}^n(V_0).$$

A *d-dimensional spatial n-SuperHyperGraph* is a quadruple

$$\mathsf{SpSHG}^{(n)} := (V_0, V, E, \lambda),$$

where

- (V, E) is an n-SuperHyperGraph on V_0;
- $\lambda : V_0 \to \mathbb{R}^d$ is an injective *embedding map* that assigns to each base element $x \in V_0$ a fixed position $\lambda(x)$ in Euclidean space.

Thus the combinatorial hierarchy is encoded by $(V, E) \subseteq \mathcal{P}^n(V_0)$, while the map λ equips all n-supervertices and n-superedges (via their elements in V_0) with a concrete spatial realization.

For $n = 1$ and $V = \{\{v\} : v \in V_0\}$, the structure $\mathsf{SpSHG}^{(1)}$ reduces to a spatial hypergraph (V_0, E, λ) in the sense of the previous definition.

Example 4.21.4 (Spatial 2-SuperHyperGraph: grouped facilities on a map)**.** Let $d = 2$ and let the finite base set of facilities be

$$V_0 := \{A, B, C\}.$$

We interpret these as three physical sites in the plane and fix their coordinates by an injective embedding map

$$\lambda : V_0 \longrightarrow \mathbb{R}^2,$$

for example

$$\lambda(A) := (0, 0), \quad \lambda(B) := (2, 0), \quad \lambda(C) := (2, 2).$$

Recall that

$$\mathrm{P}_0(V_0) = V_0, \qquad \mathrm{P}_1(V_0) = \mathrm{P}(V_0), \qquad \mathrm{P}_2(V_0) = \mathrm{P}\big(\mathrm{P}(V_0)\big).$$

Define two 2-supervertices by

$$v_1 := \big\{\{A, B\}\big\}, \qquad v_2 := \big\{\{B, C\}\big\},$$

so $v_1, v_2 \in \mathrm{P}_2(V_0)$. Set

$$V := \{v_1, v_2\} \ \subseteq \ \mathrm{P}_2(V_0),$$

and define a single 2-superedge

$$e := \{v_1, v_2\}, \qquad E := \{e\} \ \subseteq \ \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then

$$\mathrm{SHG}^{(2)} := (V, E)$$

is a 2-SuperHyperGraph on the base set V_0. The quadruple

$$\mathsf{SpSHG}^{(2)} := (V_0, V, E, \lambda)$$

is a 2-dimensional spatial 2-SuperHyperGraph.

Here:

- V_0 are individual facilities with concrete positions $\lambda(A), \lambda(B), \lambda(C)$ in the plane;
- each 2-supervertex (such as v_1) is a small cluster of facility groups (e.g. a service bundle $\{A, B\}$);
- the 2-superedge e links the clusters v_1 and v_2, representing a higher-level relation (for example, a regional planning zone that coordinates the two bundles).

Thus (V, E) encodes the hierarchical combinatorial structure, while λ provides a spatial realization at the base level V_0.

4.22 Planar SuperHypergraphs

A planar graph is a finite graph drawable on the plane without edge crossings, using nonintersecting straight or curved edges [505–507]. As concepts related to planar graphs, notions such as quasi-planar graphs [508–510], planar digraphs [511, 512], co-planar graphs [513, 514], fuzzy planar graphs [515, 516], biplanar graphs [517, 518], and neutrosophic planar graphs [342, 519–521] are well known. Planar graphs are often not structurally complex, and because their edges do not cross, they offer high visual clarity for human interpretation. For this reason, the notion of planarity has been widely applied in many research papers.

A planar hypergraph is a finite hypergraph whose incidence graph admits a planar embedding without edge crossings in the plane [17, 522]. A planar SuperHyperGraph is a finite SuperHyperGraph whose incidence graph can be embedded in the plane without any edge crossings. The relevant definitions and related notions are presented below.

Definition 4.22.1 (Incidence graph of a hypergraph)**.** Let $H = (V, E)$ be a finite hypergraph with incidence map $\partial : E \to \mathcal{P}^*(V)$. The *incidence graph* of H is the bipartite graph

$$B(H) := (V \cup E,\ F),$$

where

$$F := \big\{ \{v, e\} \subseteq V \cup E \mid e \in E,\ v \in \partial(e) \big\}.$$

Example 4.22.2 (Incidence graph of a hypergraph)**.** Let

$$V := \{v_1, v_2, v_3\}, \qquad E := \{e_1, e_2\},$$

with hyperedges

$$e_1 := \{v_1, v_2\}, \qquad e_2 := \{v_2, v_3\}.$$

Define the hypergraph $H = (V, E)$ with the natural incidence map $\partial : E \to \mathcal{P}^*(V)$ given by $\partial(e_i) = e_i$.

The incidence graph $B(H) = (V \cup E, F)$ has vertex set

$$V \cup E = \{v_1, v_2, v_3, e_1, e_2\},$$

and edge set

$$F = \big\{\{v_1, e_1\},\ \{v_2, e_1\},\ \{v_2, e_2\},\ \{v_3, e_2\}\big\},$$

since each pair $\{v, e\}$ is an edge of $B(H)$ exactly when $v \in \partial(e)$.

Definition 4.22.3 (Planar hypergraph)**.** [17, 522] A hypergraph $H = (V, E)$ is called *planar* if its incidence graph $B(H)$ is a planar graph, i.e. it admits a drawing in the plane with no edge crossings.

Example 4.22.4 (Planar hypergraph)**.** Consider again the hypergraph

$$H = (V, E)$$

from the previous example, with

$$V = \{v_1, v_2, v_3\}, \quad E = \{e_1, e_2\}, \quad e_1 = \{v_1, v_2\}, \quad e_2 = \{v_2, v_3\}.$$

Its incidence graph $B(H)$ is the bipartite graph with vertex set $\{v_1, v_2, v_3, e_1, e_2\}$ and edges $\{v_1, e_1\}, \{v_2, e_1\}, \{v_2, e_2\}, \{v_3, e_2\}$.

This graph can be drawn in the plane without edge crossings, for instance by placing v_1, v_2, v_3 on a horizontal line and e_1, e_2 above them and drawing straight-line edges. Hence $B(H)$ is planar and therefore H is a planar hypergraph.

Definition 4.22.5 (Incidence graph of an n-SuperHyperGraph)**.** Let $\mathsf{H}^{(n)} = (V, E, \partial)$ be an n-SuperHyperGraph, where $\partial : E \to \mathcal{P}^*(V)$ is the incidence map (as defined earlier in the text). The *incidence graph* of $\mathsf{H}^{(n)}$ is the bipartite graph

$$B\big(\mathsf{H}^{(n)}\big) := (V \cup E,\ F^{(n)}),$$

where

$$F^{(n)} := \big\{ \{v, e\} \subseteq V \cup E \mid e \in E,\ v \in \partial(e) \big\}.$$

Example 4.22.6 (Incidence graph of a 1-SuperHyperGraph)**.** Let the base set be

$$V_0 := \{a, b\},$$

and consider

$$V := \{w_1, w_2\} := \{\{a\}, \{b\}\} \subseteq \mathrm{P}_1(V_0),$$

as the set of 1-supervertices. Let

$$E := \{f_1\}$$

with incidence map

$$\partial(f_1) := \{w_1, w_2\}.$$

Then $\mathsf{H}^{(1)} = (V, E, \partial)$ is a 1-SuperHyperGraph.

Its incidence graph

$$B\big(\mathsf{H}^{(1)}\big) = (V \cup E, F^{(1)})$$

has vertex set

$$V \cup E = \{w_1, w_2, f_1\}$$

and edge set

$$F^{(1)} = \big\{\{w_1, f_1\},\ \{w_2, f_1\}\big\},$$

because $w_i \in \partial(f_1)$ for $i = 1, 2$. Thus $B\big(\mathsf{H}^{(1)}\big)$ is a small star-shaped bipartite graph.

Definition 4.22.7 (Planar n-SuperHyperGraph)**.** An n-SuperHyperGraph $\mathsf{H}^{(n)} = (V, E, \partial)$ is called *planar* if its incidence graph $B\big(\mathsf{H}^{(n)}\big)$ is a planar graph.

Example 4.22.8 (Planar 1-SuperHyperGraph)**.** Consider the 1-SuperHyperGraph

$$\mathsf{H}^{(1)} = (V, E, \partial)$$

from the previous example, with

$$V = \{w_1, w_2\}, \quad E = \{f_1\}, \quad \partial(f_1) = \{w_1, w_2\}.$$

Its incidence graph $B\big(\mathsf{H}^{(1)}\big)$ has vertices $\{w_1, w_2, f_1\}$ and edges $\{w_1, f_1\}$ and $\{w_2, f_1\}$, which is a path of length 2.

Such a graph is clearly planar (it can be drawn as two line segments meeting at f_1 without crossings). Hence $\mathsf{H}^{(1)}$ is a planar 1-SuperHyperGraph.

Theorem 4.22.9 (Planar n-SuperHyperGraphs generalize planar hypergraphs)**.** *Let $H = (V, E)$ be a finite hypergraph. Regard H as a* 0*-SuperHyperGraph*

$$\mathsf{H}^{(0)} := (V, E, \partial),$$

with the same incidence map ∂. Then

$$H \text{ is planar} \quad \Longleftrightarrow \quad \mathsf{H}^{(0)} \text{ is a planar } 0\text{-SuperHyperGraph}.$$

Consequently, the notion of a planar n-SuperHyperGraph is a genuine extension of the classical notion of a planar hypergraph, which is recovered when $n = 0$.

Proof. By construction, the incidence graph of H and that of $\mathsf{H}^{(0)}$ coincide:

$$B(H) \ = \ B\big(\mathsf{H}^{(0)}\big)$$

as graphs, since both have vertex set $V \cup E$ and edges $\{v, e\}$ precisely when $v \in \partial(e)$. Therefore $B(H)$ is planar if and only if $B\big(\mathsf{H}^{(0)}\big)$ is planar. By the preceding definitions, this is equivalent to H being a planar hypergraph and $\mathsf{H}^{(0)}$ being a planar 0-SuperHyperGraph, respectively. □

A concise comparison of planar graphs, planar hypergraphs, and planar n-SuperHyperGraphs is presented in Table 4.13.

Feature	Planar graph $G = (V, E)$ [505–507]	Planar hypergraph $H = (V, E)$ [17, 522]	Planar n-SuperHyperGr (V, E, ∂)
Underlying objects	Vertices and edges (2-uniform incidence)	Vertices and hyperedges (k-ary incidence)	n-supervertices ($V \subseteq P_n$ superedges with inciden
Planarity criterion	G admits a plane drawing with no edge crossings	Incidence graph $B(H)$ is planar	Incidence graph $B(\mathsf{H}^{(n)}$
Incidence graph	Not needed (can be used via subdivision)	Bipartite graph on $V \cup E$ with edges $\{v, e\}$ iff $v \in e$	Bipartite graph on $V \cup$ $\{v, e\}$ iff $v \in \partial(e)$
What "embeds in the plane"	The graph itself	The bipartite incidence graph	The bipartite incidence
Reduction / generalization	Special case of hypergraph (2-uniform)	Recovered from SuperHyperGraphs at level $n = 0$	Generalizes planar hype $n = 0$, $B(\mathsf{H}^{(0)}) = B(H)$

Table 4.13: Concise comparison of planar graphs, planar hypergraphs, and planar n-SuperHyperGraphs.

4.23 Outerplanar SuperHypergraph

An outerplanar graph is a planar graph that admits an embedding in which every vertex lies on the boundary of the outer face [523, 524]. Outerplanar graphs generalize the class of planar graphs, and several related variants have been studied, including fuzzy outerplanar graphs [525–529], neutrosophic outerplanar graphs [520], and outerplanar directed graphs [530–532]. Outerplanar graphs form a tractable planar subclass with strong structural characterizations, enabling efficient algorithms, clear embeddings, and useful bounds for width parameters and graph drawing applications (cf. [533, 534]). The relevant definitions and related notions are presented below.

An *outerplanar hypergraph* is defined as a hypergraph whose incidence bipartite graph is outerplanar, so that all vertices can be placed on the outer face in some planar embedding [535, 536]. An *outerplanar superhypergraph* is a superhypergraph whose extended incidence graph—obtained by adding auxiliary links between hyperedges—admits an outerplanar embedding. In other words, the enriched incidence structure must remain outerplanar when drawn in the plane.

Definition 4.23.1 (Outerplanar graph)**.** [523, 524] A (finite, simple) graph $G = (V, E)$ is *outerplanar* if there exists a plane embedding of G in which every vertex of G lies on the boundary of the unbounded (outer) face.

Definition 4.23.2 (Incidence (bipartite) representation of a hypergraph)**.** Let $H = (V, \mathcal{E})$ be a (finite) hypergraph, where $\emptyset \notin \mathcal{E} \subseteq \mathcal{P}(V)$. Its *incidence graph* (or *bipartite representation*) is the bipartite graph

$$B(H) := \big(V \,\dot\cup\, \mathcal{E},\ F\big), \qquad F := \big\{\{v, e\} : v \in V,\ e \in \mathcal{E},\ v \in e\big\}.$$

Definition 4.23.3 (Outerplanar hypergraph)**.** [535, 536] A hypergraph H is *outerplanar* if its incidence graph $B(H)$ is an outerplanar graph.

Example 4.23.4 (An outerplanar hypergraph (via the incidence graph))**.** Let

$$V := \{v_1, v_2, v_3, v_4\}, \qquad \mathcal{E} := \{e_1, e_2\},$$

where

$$e_1 := \{v_1, v_2, v_3\}, \qquad e_2 := \{v_3, v_4\}.$$

Define the hypergraph

$$H := (V, \mathcal{E}).$$

Its incidence graph is

$$B(H) = \big(V \dot\cup \mathcal{E},\ F\big), \qquad F = \big\{\{v_1, e_1\}, \{v_2, e_1\}, \{v_3, e_1\}, \{v_3, e_2\}, \{v_4, e_2\}\big\}.$$

Observe that $B(H)$ is a tree (it has no cycles). Since every tree is outerplanar (one can embed it in the plane with all vertices on the boundary of the outer face), it follows that $B(H)$ is outerplanar. Hence H is an outerplanar hypergraph in the sense of Definition (Outerplanar hypergraph).

Definition 4.23.5 (Shadow of a hypergraph)**.** Let $H = (V, \mathcal{E})$ be a hypergraph. Its *shadow* (or *2-section*) is the graph

$$\partial(H) := \big(V,\ \{\{u, v\} : u \neq v,\ \exists e \in \mathcal{E} \text{ with } \{u, v\} \subseteq e\}\big).$$

Definition 4.23.6 (Outerplanar 3-uniform hypergraph (Zykov-type))**.** Let $H = (V, \mathcal{E})$ be 3-uniform (i.e., $|e| = 3$ for all $e \in \mathcal{E}$). We say that H is *outerplanar* if $\partial(H)$ has an outerplanar embedding such that, for every hyperedge $e = \{a, b, c\} \in \mathcal{E}$, the vertices a, b, c bound an interior triangular face of that embedding.

Definition 4.23.7 (n-SuperHyperGraph via iterated super-links)**.** Fix an integer $n \geq 1$. An *n-superhypergraph* is a tuple

$$\mathcal{S} = \big(V, \mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n\big)$$

such that

$$\mathcal{E}_1 \subseteq \mathcal{P}^*(V) \quad \text{and} \quad \mathcal{E}_i \subseteq \binom{\mathcal{E}_{i-1}}{2} \text{ for every } 2 \leq i \leq n,$$

where $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}$. Elements of $\mathcal{E}_1$ are (hyper)edges, and elements of $\mathcal{E}_i$ ($i \geq 2$) are called *level-i super-links* (links between level-($i-1$) objects).

For $n = 1$ this is exactly a hypergraph $(V, \mathcal{E}_1)$. For $n = 2$ this is exactly a superhypergraph $(V, \mathcal{E}_1, \Lambda)$ with $\Lambda = \mathcal{E}_2 \subseteq \binom{\mathcal{E}_1}{2}$.

Definition 4.23.8 (Extended bipartite representation of an n-superhypergraph)**.** Let $\mathcal{S} = (V, \mathcal{E}_1, \dots, \mathcal{E}_n)$ be an n-superhypergraph. Its *extended bipartite representation* is the (simple) graph

$$B^{(n)}_{\mathrm{ext}}(\mathcal{S}) := \big(V_{\mathrm{ext}},\ F_{\mathrm{ext}}\big),$$

where the vertex set is the tagged (disjoint) union

$$V_{\mathrm{ext}} := V \,\dot\cup\, \mathcal{E}_1 \,\dot\cup\, \mathcal{E}_2 \,\dot\cup\, \cdots \,\dot\cup\, \mathcal{E}_n,$$

and the edge set is the union

$$F_{\mathrm{ext}} := F_1 \cup F_2 \cup \cdots \cup F_n,$$

with

$$F_1 := \big\{\{v, e\} \ :\ v \in V,\ e \in \mathcal{E}_1,\ v \in e\big\},$$

and for every $2 \leq i \leq n$,

$$F_i := \big\{\{x, \lambda\} \ :\ \lambda \in \mathcal{E}_i,\ x \in \mathcal{E}_{i-1},\ x \in \lambda\big\}.$$

Definition 4.23.9 (Outerplanar n-SuperHyperGraph)**.** An n-superhypergraph $\mathcal{S}$ is *outerplanar* if

$$B^{(n)}_{\mathrm{ext}}(\mathcal{S})$$

is an outerplanar graph.

Example 4.23.10 (An outerplanar 2-superhypergraph (outerplanar n-superhypergraph with $n = 2$))**.** Let

$$V := \{1, 2, 3\}, \qquad \mathcal{E}_1 := \{e_{12}, e_{23}\},$$

where

$$e_{12} := \{1, 2\}, \qquad e_{23} := \{2, 3\}.$$

Define the level-2 super-links by

$$\mathcal{E}_2 := \{\lambda\}, \qquad \lambda := \{e_{12}, e_{23}\} \ \in\ \binom{\mathcal{E}_1}{2}.$$

Then

$$\mathcal{S} := (V, \mathcal{E}_1, \mathcal{E}_2)$$

is a 2-superhypergraph.

Its extended bipartite representation $B^{(2)}_{\mathrm{ext}}(\mathcal{S}) = (V_{\mathrm{ext}}, F_{\mathrm{ext}})$ has

$$V_{\mathrm{ext}} = V \,\dot\cup\, \mathcal{E}_1 \,\dot\cup\, \mathcal{E}_2 = \{1, 2, 3\} \dot\cup \{e_{12}, e_{23}\} \dot\cup \{\lambda\},$$

and

$$F_{\mathrm{ext}} = F_1 \cup F_2,$$

where

$$F_1 = \big\{\{1, e_{12}\}, \{2, e_{12}\}, \{2, e_{23}\}, \{3, e_{23}\}\big\}, \qquad F_2 = \big\{\{e_{12}, \lambda\}, \{e_{23}, \lambda\}\big\}.$$

In $B^{(2)}_{\mathrm{ext}}(\mathcal{S})$ the vertices

$$2,\ e_{12},\ \lambda,\ e_{23}$$

form a 4-cycle

$$2 - e_{12} - \lambda - e_{23} - 2,$$

and the remaining vertices 1 and 3 are leaves attached to e_{12} and e_{23}, respectively. This graph admits an outerplanar embedding by placing the 4-cycle on the boundary of the outer face and attaching the leaves 1 and 3 outside the cycle. Therefore

$$B^{(2)}_{\mathrm{ext}}(\mathcal{S})$$

is outerplanar, and $\mathcal{S}$ is an outerplanar 2-superhypergraph (i.e., an outerplanar n-superhypergraph with $n = 2$).

Proposition 4.23.11 (Outerplanar n-superhypergraphs generalize outerplanar hypergraphs)**.** *Let $H = (V, \mathcal{E})$ be a hypergraph. Define the associated n-superhypergraph*

$$\mathcal{S}^{(n)}_H := \big(V, \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n\big) \quad by \quad \mathcal{E}_1 := \mathcal{E} \ \ and \ \ \mathcal{E}_i := \varnothing \ (2 \le i \le n).$$

If H is outerplanar (i.e., its incidence graph $B(H)$ is outerplanar), then $\mathcal{S}^{(n)}_H$ is an outerplanar n-superhypergraph.

Proof. Let $H = (V, \mathcal{E})$ and $\mathcal{S}^{(n)}_H = (V, \mathcal{E}_1, \ldots, \mathcal{E}_n)$ be as defined, so $\mathcal{E}_1 = \mathcal{E}$ and $\mathcal{E}_2 = \cdots = \mathcal{E}_n = \varnothing$.

By the definition of $B^{(n)}_{\mathrm{ext}}(\mathcal{S})$, we have

$$V_{\mathrm{ext}} = V \,\dot\cup\, \mathcal{E}_1 \,\dot\cup\, \underbrace{\mathcal{E}_2 \,\dot\cup\, \cdots \,\dot\cup\, \mathcal{E}_n}_{=\varnothing} = V \,\dot\cup\, \mathcal{E}.$$

For the edge set,

$$F_{\mathrm{ext}} = F_1 \cup F_2 \cup \cdots \cup F_n.$$

Because $\mathcal{E}_i = \varnothing$ for every $i \ge 2$, it follows from

$$F_i = \big\{\{x, \lambda\} : \lambda \in \mathcal{E}_i,\ x \in \mathcal{E}_{i-1},\ x \in \lambda\big\}$$

that

$$F_i = \varnothing \quad \text{for all } i \ge 2.$$

Hence

$$F_{\mathrm{ext}} = F_1.$$

But F_1 is exactly the incidence edge set of the usual incidence graph $B(H)$:

$$F_1 = \big\{\{v, e\} : v \in V,\ e \in \mathcal{E},\ v \in e\big\}.$$

Therefore,

$$B^{(n)}_{\mathrm{ext}}(\mathcal{S}^{(n)}_H) = \big(V \dot\cup \mathcal{E},\ F_1\big) = B(H).$$

If H is outerplanar, then $B(H)$ is outerplanar by definition, hence $B^{(n)}_{\mathrm{ext}}(\mathcal{S}^{(n)}_H)$ is outerplanar, and consequently $\mathcal{S}^{(n)}_H$ is an outerplanar n-superhypergraph. □

4.24 Multimodal Superhypergraphs

A Multimodal n-SuperHyperGraph represents one vertex set with multiple labeled superedge layers, each encoding a distinct interaction modality type assignment [537]. Multimodal Superhypergraphs are known to generalize both multimodal graphs [538–540] and multimodal hypergraphs [541–543]. Moreover, as a related concept, fuzzy multimodal graphs [544, 545] are also well known. The relevant definitions and related notions are presented below.

Definition 4.24.1 (Multimodal Hypergraph). [541–543] Let V be a finite set of vertices and let $M \in \mathbb{N}$ be the number of modalities. For each modality $m \in \{1, 2, \dots, M\}$, let

$$G_m \; = \; (V, E_m, W_m)$$

be a (weighted) hypergraph on the common vertex set V, where:

1. E_m is a set of hyperedges, and each hyperedge $e \in E_m$ is a nonempty subset of V (typically $|e| \geq 2$);
2. $W_m : E_m \to \mathbb{R}_{>0}$ assigns a positive weight to each hyperedge $e \in E_m$.

Let $\{\alpha_m\}_{m=1}^{M}$ be combination weights satisfying

$$\alpha_m \geq 0 \; (m = 1, \dots, M), \qquad \sum_{m=1}^{M} \alpha_m = 1.$$

A *multimodal hypergraph* is the tuple

$$G \; := \; \Big(V, \; \{E_m\}_{m=1}^{M}, \; \{W_m\}_{m=1}^{M}, \; \{\alpha_m\}_{m=1}^{M}\Big),$$

which integrates the modality-specific hypergraphs by weighting each G_m by α_m.

Remark 4.24.2 (Optional combined Laplacian). In applications where each G_m induces a Laplacian matrix L_m, one often forms a unified operator by the convex combination

$$L \; := \; \sum_{m=1}^{M} \alpha_m \, L_m.$$

Example 4.24.3 (Real-life multimodal hypergraph: smart-city multimodal commuting analytics). Let a city analyze *commuter groups* using multiple data modalities to plan signal timing and public-transport capacity.

Vertex set. Let

$$V := \{v_1, \dots, v_N\}$$

be a finite set of commuters (or anonymized commuter IDs).

Modalities. Fix $M = 3$ modalities:

$m = 1$: mobile-phone GPS traces, $\qquad m = 2$: transit smart-card taps, $\qquad m = 3$: road-sensor / traffic-counter detections.

For each modality $m \in \{1, 2, 3\}$, define a weighted hypergraph

$$G_m = (V, E_m, W_m),$$

where each hyperedge groups commuters who *jointly exhibit* a commuting pattern under that modality.

Hyperedges and weights (concrete meaning).

1. *GPS modality* $m = 1$. For each weekday time window (e.g., 7:00–9:00) and corridor C (a sequence of road segments), let
$$e_C^{(1)} := \{ \, v_i \in V : \; v_i \text{ traverses corridor } C \text{ in the window} \, \} \in E_1.$$
A typical weight is
$$W_1\big(e_C^{(1)}\big) := \frac{1}{\varepsilon + \mathrm{Var}\big(T_C^{(1)}\big)},$$
where $T_C^{(1)}$ is the set of observed travel-times of members of $e_C^{(1)}$ and $\varepsilon > 0$ is small; thus more consistent group travel implies larger weight.

2. *Transit modality* $m = 2$. For a transit line ℓ and time window τ, let

$$e^{(2)}_{\ell,\tau} := \{\, v_i \in V : \ v_i \text{ taps-in on line } \ell \text{ during } \tau \,\} \in E_2,$$

and define

$$W_2\big(e^{(2)}_{\ell,\tau}\big) := \frac{\#(\text{co-occurring tap sequences within } \Delta \text{ minutes})}{|e^{(2)}_{\ell,\tau}|},$$

so groups with frequent co-occurrence receive higher weight.

3. *Road-sensor modality* $m = 3$. For an intersection cluster I and time window τ, let

$$e^{(3)}_{I,\tau} := \{\, v_i \in V : \ v_i \text{ is detected near } I \text{ during } \tau \,\} \in E_3,$$

and set

$$W_3\big(e^{(3)}_{I,\tau}\big) := \text{average detected flow intensity in } (I, \tau),$$

so heavily loaded junction-period groups are emphasized.

Integration weights. Choose convex combination weights, for example,

$$(\alpha_1, \alpha_2, \alpha_3) = (0.4, 0.4, 0.2), \qquad \alpha_m \geq 0, \ \sum_{m=1}^{3} \alpha_m = 1,$$

reflecting that GPS and smart-card data are equally trusted, while sensors are noisier.

Multimodal hypergraph. The resulting real-life multimodal hypergraph is

$$G = \Big(V, \{E_m\}_{m=1}^{3}, \{W_m\}_{m=1}^{3}, \{\alpha_m\}_{m=1}^{3}\Big),$$

which jointly encodes *group commuting relations* across GPS, transit, and sensor modalities, and supports downstream tasks such as multimodal clustering of commuting communities or identifying critical multimodal bottlenecks.

Definition 4.24.4 (Multimodal n-SuperHyperGraph). [537] Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph and let M be a nonempty finite set of *modes* (or *modalities*).

A *Multimodal n-SuperHyperGraph* on $\mathrm{SHG}^{(n)}$ is a triple

$$\textsf{MM-SHG}^{(n)} := (V, E, \lambda),$$

where $\lambda : E \to M$ assigns to each n-superedge a mode label.

For $m \in M$ we set

$$E_m := \{\, e \in E \mid \lambda(e) = m \,\},$$

so that $\{E_m\}_{m \in M}$ is the family of *modal layers* of $\textsf{MM-SHG}^{(n)}$ on the common n-supervertex set V.

Example 4.24.5 (Multimodal 1-SuperHyperGraph: road vs. rail connections). Let the finite base set of *atomic districts* be

$$V_0 := \{\text{North}, \text{Center}, \text{South}\}.$$

Then

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$$

is the collection of all subsets of districts.

Define two 1-supervertices

$$v_1 := \{\text{North}, \text{Center}\}, \qquad v_2 := \{\text{Center}, \text{South}\},$$

so that $v_1, v_2 \in \mathcal{P}^1(V_0)$ and

$$V := \{v_1, v_2\} \subseteq \mathcal{P}^1(V_0).$$

Consider the 1-SuperHyperGraph

$$\mathsf{SHG}^{(1)} := (V, E),$$

where the 1-superedge family is

$$E := \{e_{\text{road}}, e_{\text{rail}}\}, \qquad e_{\text{road}} := \{v_1, v_2\}, \quad e_{\text{rail}} := \{v_1, v_2\}.$$

Both superedges connect the same pair of 1-supervertices but will be distinguished by their *mode*.

Let the set of modes be

$$M := \{\text{road}, \text{rail}\},$$

and define

$$\lambda : E \longrightarrow M, \qquad \lambda(e_{\text{road}}) := \text{road}, \quad \lambda(e_{\text{rail}}) := \text{rail}.$$

For each $m \in M$ we obtain the modal layers

$$E_{\text{road}} = \{e_{\text{road}}\}, \qquad E_{\text{rail}} = \{e_{\text{rail}}\}.$$

Thus

$$\mathsf{MM\text{-}SHG}^{(1)} := (V, E, \lambda)$$

is a Multimodal 1-SuperHyperGraph. The 1-supervertices represent district pairs (regional clusters), while e_{road} and e_{rail} encode the same structural connection realized through two distinct modalities: a road corridor and a rail line.

4.25 Lattice Superhypergraphs

A lattice is a partially ordered set where any two elements have a unique join (supremum) and meet (infimum) [546, 547]. A Lattice n-SuperHyperGraph assigns each supervertex and superedge a lattice-valued label, enforcing edge values beneath incident-vertex meets in the lattice [537]. Lattice SuperHyperGraphs are known to generalize lattice hypergraphs [548, 549]. The relevant definitions and related notions are presented below.

Definition 4.25.1 (Lattice n-SuperHyperGraph)**.** [537] Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph and let $(L, \leq)$ be a complete lattice with meet $\wedge$.

A *Lattice n-SuperHyperGraph* on $\mathrm{SHG}^{(n)}$ with value lattice L is a quintuple

$$\mathsf{Lat\text{-}SHG}^{(n)} := (V, E, L, \sigma, \mu),$$

where

- $\sigma : V \to L$ assigns to each n-supervertex $v \in V$ a lattice value $\sigma(v)$,
- $\mu : E \to L$ assigns to each n-superedge $e \in E$ a lattice value $\mu(e)$,

subject to the *lattice admissibility constraint*

$$\mu(e) \;\leq\; \bigwedge_{v \in e} \sigma(v) \qquad \text{for every } e \in E,$$

where the meet $\bigwedge_{v \in e} \sigma(v)$ is taken in the lattice $(L, \leq)$.

Example 4.25.2 (Lattice 2-SuperHyperGraph: discrete risk levels in a hierarchical grid)**.** Let the base set of *components* be

$$V_0 := \{\text{SubstationA}, \text{SubstationB}, \text{SubstationC}\}.$$

The first iterated powerset is

$$\mathrm{P}_1(V_0) := \mathrm{P}(V_0),$$

whose elements are ordinary subsets of substations.

Define the following first-level *region subsets* (elements of $\mathrm{P}_1(V_0)$):

$$R_{AB} := \{\text{SubstationA}, \text{SubstationB}\},$$

$$R_{BC} := \{\text{SubstationB}, \text{SubstationC}\},$$

$$R_{\text{all}} := \{\text{SubstationA}, \text{SubstationB}, \text{SubstationC}\}.$$

The second iterated powerset is

$$\mathrm{P}_2(V_0) := \mathrm{P}\big(\mathrm{P}_1(V_0)\big),$$

whose elements are sets of such regions. Define two 2-supervertices

$$v_1 := \{R_{AB}, R_{\text{all}}\}, \qquad v_2 := \{R_{BC}, R_{\text{all}}\},$$

so that $v_1, v_2 \in \mathrm{P}_2(V_0)$. Set

$$V := \{v_1, v_2\} \subseteq \mathrm{P}_2(V_0).$$

Introduce a single 2-superedge

$$e_{\text{grid}} := \{v_1, v_2\}, \qquad E := \{e_{\text{grid}}\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then

$$\mathrm{SHG}^{(2)} := (V, E)$$

is a level-2 SuperHyperGraph over the base set V_0.

Let $(L, \leq)$ be the finite chain

$$L := \{0, 1, 2\}, \qquad 0 \leq 1 \leq 2,$$

with meet $\wedge$ given by the minimum in this order, interpreted as "low", "medium", and "high" risk levels.

Define the lattice-valued vertex and edge labels

$$\sigma : V \to L, \qquad \sigma(v_1) := 2, \quad \sigma(v_2) := 1,$$

$$\mu : E \to L, \qquad \mu(e_{\text{grid}}) := 1.$$

For the unique 2-superedge e_{grid} we check the lattice admissibility constraint

$$\mu(e_{\text{grid}}) \ \leq \ \bigwedge_{v \in e_{\text{grid}}} \sigma(v).$$

Indeed,

$$\bigwedge_{v \in e_{\text{grid}}} \sigma(v) = \sigma(v_1) \wedge \sigma(v_2) = 2 \wedge 1 = 1,$$

and $\mu(e_{\text{grid}}) = 1 \leq 1$ holds.

Thus

$$\text{Lat-SHG}^{(2)} := (V, E, L, \sigma, \mu)$$

is a Lattice 2-SuperHyperGraph. Each 2-supervertex represents a hierarchical cluster of substations (a set of regional groupings), and the 2-superedge e_{grid} encodes a joint operational constraint whose lattice-valued risk does not exceed the meet of the incident superclusters' risk levels.

4.26 Hyperbolic Superhypergraphs

A Hyperbolic n-SuperHyperGraph equips its vertex set with a Gromov-hyperbolic graph metric compatible with superedge-induced geodesic paths respecting SuperHyperGraph connectivity [537]. Hyperbolic SuperHypergraphs generalize both *hyperbolic graphs* [550–552] and *hyperbolic hypergraphs* [38, 553, 554], extending their geometric behavior to iterated-powerset hierarchies and multi-level relational structures. The relevant definitions and related notions are presented below.

Definition 4.26.1 (Hyperbolic n-SuperHyperGraph)**.** [537] Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A function

$$d : V \times V \longrightarrow [0, \infty)$$

is called a *graph metric* on V (for $\mathrm{SHG}^{(n)}$) if (V, d) is a metric space and any two vertices of V can be joined by a finite d–geodesic path whose successive vertices are contained in a common n-superedge of E.

The pair

$$\textsf{Hyp-SHG}^{(n)} := \big(\mathrm{SHG}^{(n)}, d\big)$$

is called a *Hyperbolic n-SuperHyperGraph* if there exists $\delta \geq 0$ such that the metric space (V, d) is δ–hyperbolic in the sense of Gromov (i.e. every geodesic triangle in (V, d) is δ–thin).

Example 4.26.2 (Hyperbolic 2-SuperHyperGraph: tree-shaped hierarchy of router clusters)**.** Let the base set of routers be

$$V_0 := \{r_1, r_2, r_3, r_4\}.$$

The first iterated powerset

$$\mathrm{P}_1(V_0) := \mathrm{P}(V_0)$$

contains all subsets of routers. Define the following first-level *link groups*:

$$G_{12} := \{r_1, r_2\}, \quad G_{23} := \{r_2, r_3\}, \quad G_{34} := \{r_3, r_4\},$$

all elements of $\mathrm{P}_1(V_0)$.

The second iterated powerset

$$\mathrm{P}_2(V_0) := \mathrm{P}\big(\mathrm{P}_1(V_0)\big)$$

consists of sets of such link groups. Define three 2-supervertices

$$v_1 := \{G_{12}, G_{23}\}, \qquad v_2 := \{G_{23}, G_{34}\}, \qquad v_3 := \{G_{34}\},$$

so that $v_1, v_2, v_3 \in \mathrm{P}_2(V_0)$. Set

$$V := \{v_1, v_2, v_3\} \subseteq \mathrm{P}_2(V_0).$$

Define the 2-superedges

$$e_{12} := \{v_1, v_2\}, \qquad e_{23} := \{v_2, v_3\},$$

and put

$$E := \{e_{12}, e_{23}\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then

$$\mathrm{SHG}^{(2)} := (V, E)$$

is a level-2 SuperHyperGraph whose 2-superedges connect router clusters in a path-like (tree) fashion.

Define $d : V \times V \to [0, \infty)$ to be the usual *graph distance* on the underlying simple graph with vertex set V and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$. Explicitly,

$$d(v_1, v_2) = d(v_2, v_3) = 1, \qquad d(v_1, v_3) = 2, \qquad d(v_i, v_i) = 0 \text{ for } i = 1, 2, 3.$$

Then (V, d) is a tree metric, hence 0-hyperbolic in the sense of Gromov: every geodesic triangle in (V, d) is 0-thin. Moreover, any d–geodesic path between two 2-supervertices has successive vertices contained in a common 2-superedge of E (for example, the path v_1–v_2–v_3 uses e_{12} and e_{23}).

Therefore

$$\textsf{Hyp-SHG}^{(2)} := \big(\mathrm{SHG}^{(2)}, d\big)$$

is a Hyperbolic 2-SuperHyperGraph. It models a hierarchical backbone where 2-supervertices are clusters of router-link groups and the tree-shaped connectivity induces a Gromov-hyperbolic graph metric on the space of such clusters.

4.27 Directed Acyclic Superhypergraphs (dash)

A directed acyclic graph is a directed graph containing no directed cycles, admitting a topological ordering respecting all edges globally [555–557]. Directed acyclic graphs model causal, temporal, and dependency structures; they enable topological ordering, efficient scheduling and compilation, and underpin Bayesian networks, workflows, and version-control build systems (cf. [558–560]).

A directed acyclic hypergraph is a directed hypergraph whose hyperedges create no directed cycles, allowing hierarchical topological ordering constraints overall [561, 562]. A directed acyclic SuperHypergraph is a directed n-SuperHyperGraph without directed superhyperedge cycles, supporting level-wise topological ordering across all nested structures [563]. The relevant definitions and related notions are presented below.

Definition 4.27.1 (Directed cycle in a directed n-SuperHyperGraph)**.** [563] Let $\mathrm{DSH}_n = (V, E)$ be a directed n-SuperHyperGraph. A *directed cycle* in DSH_n is a sequence of distinct n-supervertices

$$v_1, v_2, \dots, v_k \in V \qquad (k \geq 2),$$

together with directed n-superhyperedges

$$e_1, e_2, \dots, e_k \in E$$

such that for each $i = 1, \dots, k$ we have

$$v_i \in T(e_i) \quad \text{and} \quad v_{i+1} \in H(e_i),$$

where the indices are taken cyclically, i.e. $v_{k+1} := v_1$.

Example 4.27.2 (Directed cycle in a directed n-SuperHyperGraph)**.** Let

$$V := \{v_1, v_2, v_3\}$$

be a set of three distinct n-supervertices. Define three directed n-superhyperedges

$$E := \{e_1, e_2, e_3\}$$

with tail and head sets

$$T(e_1) := \{v_1\}, \quad H(e_1) := \{v_2\},$$
$$T(e_2) := \{v_2\}, \quad H(e_2) := \{v_3\},$$
$$T(e_3) := \{v_3\}, \quad H(e_3) := \{v_1\}.$$

Then

$$\mathrm{DSH}_n := (V, E)$$

is a directed n-SuperHyperGraph.

Consider the sequence of distinct n-supervertices

$$v_1, v_2, v_3$$

together with the sequence of directed n-superhyperedges

$$e_1, e_2, e_3.$$

We have

$$v_1 \in T(e_1), \quad v_2 \in H(e_1),$$
$$v_2 \in T(e_2), \quad v_3 \in H(e_2),$$
$$v_3 \in T(e_3), \quad v_1 \in H(e_3),$$

and by convention $v_{3+1} := v_1$. Hence v_1, v_2, v_3 with e_1, e_2, e_3 form a directed cycle in DSH_n in the sense of Definition 4.27.1.

Definition 4.27.3 (Directed Acyclic SuperHyperGraph (DASH))**.** A directed n-SuperHyperGraph

$$\mathrm{DSH}_n = (V, E)$$

is called a *Directed Acyclic SuperHyperGraph* (abbreviated *DASH*) if it contains no directed cycle in the sense of Definition 4.27.1; that is, there is no sequence of distinct n-supervertices $v_1, \dots, v_k$ and directed n-superhyperedges $e_1, \dots, e_k \in E$ with $k \geq 2$ such that

$$v_i \in T(e_i), \quad v_{i+1} \in H(e_i) \quad (i = 1, \dots, k), \qquad v_{k+1} := v_1.$$

Equivalently, a directed n-SuperHyperGraph is a DASH if and only if its supervertices can be arranged in a topological order compatible with all directed n-superhyperedges.

Example 4.27.4 (A simple DASH (Directed Acyclic SuperHyperGraph))**.** Let

$$V := \{u_1, u_2, u_3\}$$

be a set of three distinct n-supervertices. Define two directed n-superhyperedges

$$E := \{f_1, f_2\}$$

with tails and heads

$$T(f_1) := \{u_1\}, \quad H(f_1) := \{u_2\},$$
$$T(f_2) := \{u_2\}, \quad H(f_2) := \{u_3\}.$$

Then

$$\mathrm{DSH}_n := (V, E)$$

is a directed n-SuperHyperGraph.

In this structure, every directed n-superhyperedge points "forward" from u_1 to u_2 and from u_2 to u_3. There is no sequence of distinct n-supervertices $v_1, \dots, v_k$ with $k \geq 2$ and directed n-superhyperedges $e_1, \dots, e_k \in E$ such that

$$v_i \in T(e_i), \quad v_{i+1} \in H(e_i) \quad (i = 1, \dots, k), \qquad v_{k+1} := v_1,$$

because there is no way to return from u_3 back to u_1 or u_2. Thus no directed cycle exists.

Equivalently, the ordering

$$u_1 \prec u_2 \prec u_3$$

is a topological order of the supervertices compatible with all directed n-superhyperedges, so DSH_n is a Directed Acyclic SuperHyperGraph (DASH) in the sense of Definition 4.27.3.

4.28 Meta-SuperHyperGraph

A Meta-Graph is a graph whose vertices are graphs, and whose edges represent specified relations between those graphs [48, 564–566]. Meta-Graph research is important because it models relations between whole graphs, enabling higher-level reasoning, transfer, compression, and learning across multiple networks, domains, and scales. A Meta-HyperGraph is a hypergraph whose vertices are hypergraphs, and whose hyperedges group several hypergraphs whenever a chosen higher-level relation holds [567]. A Meta-SuperHyperGraph is a hypergraph whose vertices are n-SuperHyperGraphs, and whose hyperedges connect multiple such structures according to a prescribed multi-way relation [567]. The relevant definitions and related notions are presented below.

Definition 4.28.1 (MetaGraph (graph of graphs))**.** [567] Fix a nonempty universe $\mathcal{G}$ of finite graphs and a nonempty family of binary relations $\mathcal{R} \subseteq \mathcal{P}(\mathcal{G} \times \mathcal{G})$. A *MetaGraph over* $(\mathcal{G}, \mathcal{R})$ is a directed labelled multigraph

$$M = (V, E, s, t, \lambda)$$

such that $V \subseteq \mathcal{G}$, $s, t : E \to V$, and $\lambda : E \to \mathcal{R}$, satisfying the incidence constraint

$$\forall e \in E : \ (\, s(e), t(e)\,) \in \lambda(e).$$

Vertices are graphs (meta-vertices), and each meta-edge e is justified by the relation-label $\lambda(e)$.

Example 4.28.2 (MetaGraph: cross-citing departments)**.** Let $\mathcal{G}$ be the class of finite directed acyclic citation graphs (vertices = papers, arcs = citations). Consider the three graphs:

$$G_{\mathrm{CS}}: \ V = \{c_1, c_2, c_3\}, \ E = \{c_2 \to c_1, \ c_3 \to c_2\},$$

$$G_{\mathrm{Bio}}: \ V = \{b_1, b_2\}, \ E = \{b_2 \to b_1\}, \qquad G_{\mathrm{Math}}: \ V = \{m_1, m_2\}, \ E = \{m_2 \to m_1\}.$$

Let the observed cross-department citations be

$$X = \{c_3 \to b_1, \ c_1 \to m_1, \ b_2 \to c_2, \ m_2 \to c_1\}.$$

For a threshold $\tau \in \mathbb{N}$, define a relation R_τ on $\mathcal{G}$ by:

$$c(G, H) := \Big|\{(p, q) \in V(G) \times V(H) \ : \ p \to q \in X\}\Big|, \qquad (G, H) \in R_\tau \iff c(G, H) \geq \tau.$$

With $\tau = 1$, we have

$$c(G_{\mathrm{CS}}, G_{\mathrm{Bio}}) = 1, \ c(G_{\mathrm{CS}}, G_{\mathrm{Math}}) = 1, \ c(G_{\mathrm{Bio}}, G_{\mathrm{CS}}) = 1, \ c(G_{\mathrm{Math}}, G_{\mathrm{CS}}) = 1,$$

and all other cross-counts are 0. Hence the MetaGraph over $(\mathcal{G}, \{R_1\})$ can be taken as

$$V = \{G_{\mathrm{CS}}, G_{\mathrm{Bio}}, G_{\mathrm{Math}}\}, \quad E = \{e_1, e_2, e_3, e_4\}, \quad \lambda(e_i) = R_1 \ (i = 1, 2, 3, 4),$$

with sources/targets

$$s(e_1) = G_{\mathrm{CS}}, \ t(e_1) = G_{\mathrm{Bio}}, \qquad s(e_2) = G_{\mathrm{CS}}, \ t(e_2) = G_{\mathrm{Math}},$$

$$s(e_3) = G_{\mathrm{Bio}}, \ t(e_3) = G_{\mathrm{CS}}, \qquad s(e_4) = G_{\mathrm{Math}}, \ t(e_4) = G_{\mathrm{CS}}.$$

Each incidence condition $(s(e_i), t(e_i)) \in R_1$ holds by the computed counts.

Definition 4.28.3 (MetaHyperGraph (HyperGraph of HyperGraphs))**.** [567] Let U be a nonempty universe of objects and let

$$\mathcal{R} \subseteq \mathcal{P}\big(\mathcal{P}_{\mathrm{fin}}(U) \times \mathcal{P}_{\mathrm{fin}}(U)\big)$$

be a nonempty family of admissible set-relations. A *MetaHyperGraph over* $(U, \mathcal{R})$ is a labelled directed hypergraph

$$M = (V, E, T, Hd, \lambda)$$

with $V \subseteq U$, tail/head maps $T, Hd : E \to \mathcal{P}_{\mathrm{fin}}(V)$, and label map $\lambda : E \to \mathcal{R}$, such that

$$\forall e \in E : \ \big(T(e), Hd(e)\big) \in \lambda(e).$$

If U is chosen as the class of finite hypergraphs, then M is literally a "hypergraph whose vertices are hypergraphs."

Example 4.28.4 (MetaHyperGraph: hospital departments sharing patients)**.** Let each meta-vertex be a finite (undirected) hypergraph whose vertices are patient IDs and whose hyperedges are procedure sessions. Consider three departmental hypergraphs:

$$H_{\mathrm{Rad}}: \ V = \{p_1, p_2, p_3\}, \ E = \big\{\{p_1, p_2\}, \{p_2, p_3\}\big\},$$

$$H_{\mathrm{Card}}: \ V = \{p_2, p_4\}, \ E = \big\{\{p_2\}, \{p_2, p_4\}\big\}, \qquad H_{\mathrm{Onc}}: \ V = \{p_1, p_2, p_5\}, \ E = \big\{\{p_1, p_2\}, \{p_2, p_5\}\big\}.$$

Define a set-relation R_{share} on finite families of departmental hypergraphs by:

$$(S, T) \in R_{\mathrm{share}} \iff \exists x \text{ (patient ID) such that } \forall H \in S \cup T, \ \exists e \in E(H) \text{ with } x \in e.$$

Form a MetaHyperGraph $M = (V, E, T, Hd, \lambda)$ over $(\{H_{\mathrm{Rad}}, H_{\mathrm{Card}}, H_{\mathrm{Onc}}\}, \{R_{\mathrm{share}}\})$ by

$$V = \{H_{\mathrm{Rad}}, H_{\mathrm{Card}}, H_{\mathrm{Onc}}\}, \quad E = \{e_1\}, \quad T(e_1) = \{H_{\mathrm{Rad}}, H_{\mathrm{Card}}\}, \quad Hd(e_1) = \{H_{\mathrm{Onc}}\}, \quad \lambda(e_1) = R_{\mathrm{share}}.$$

Incidence check: choose $x = p_2$. Then

$$p_2 \in \{p_1, p_2\} \in E(H_{\mathrm{Rad}}), \quad p_2 \in \{p_2\} \in E(H_{\mathrm{Card}}), \quad p_2 \in \{p_1, p_2\} \in E(H_{\mathrm{Onc}}),$$

so $\big(T(e_1), Hd(e_1)\big) \in R_{\mathrm{share}} = \lambda(e_1)$.

Definition 4.28.5 (MetaSuperHyperGraph (SuperHyperGraph of SuperHyperGraphs))**.** [567] Fix $n \in \mathbb{N}_0$ and let S_n denote the class of all finite directed n-SuperHyperGraphs (over arbitrary base sets). Let

$$\mathcal{R} \subseteq \mathcal{P}\big(\mathcal{P}_{\text{fin}}(\mathsf{S}_n) \times \mathcal{P}_{\text{fin}}(\mathsf{S}_n)\big)$$

be a nonempty family of admissible set-relations on n-SuperHyperGraphs. A *MetaSuperHyperGraph over* $(\mathsf{S}_n, \mathcal{R})$ is a labelled directed hypergraph

$$M = (V, E, T, Hd, \lambda)$$

such that $V \subseteq \mathsf{S}_n$, $T, Hd : E \to \mathcal{P}_{\text{fin}}(V)$, $\lambda : E \to \mathcal{R}$, and

$$\forall e \in E : \ \big(T(e), Hd(e)\big) \in \lambda(e).$$

Example 4.28.6 (MetaSuperHyperGraph: multi-department clinical cohorts (depth $n = 1$))**.** Let the patient-ID universe be $\Omega = \{u_1, u_2, u_3, u_4, u_5\}$ and consider depth-1 SuperHyperGraphs (so their vertices are subsets of Ω). Define three 1-SuperHyperGraphs:

Department A (oncology):

$$H_A = (V_A, E_A, T_A, Hd_A), \quad V_A = \big\{\{u_1, u_2\}, \{u_2, u_3\}\big\}, \quad E_A = \{e_A\},$$
$$T_A(e_A) = \big\{\{u_1, u_2\}\big\}, \qquad Hd_A(e_A) = \big\{\{u_2, u_3\}\big\}.$$

Department B (radiology):

$$H_B = (V_B, E_B, T_B, Hd_B), \quad V_B = \big\{\{u_2, u_4\}\big\}, \quad E_B = \emptyset.$$

Department C (cardiology):

$$H_C = (V_C, E_C, T_C, Hd_C), \quad V_C = \big\{\{u_1, u_2, u_5\}\big\}, \quad E_C = \emptyset.$$

Define a set-relation $R_{\text{share}V}$ on finite families of 1-SuperHyperGraphs by

$$(S, T) \in R_{\text{share}V} \iff \exists x \in \Omega \text{ such that } \forall H \in S \cup T, \ \exists A_H \in V(H) \text{ with } x \in A_H.$$

Now form a MetaSuperHyperGraph $M = (V, E, T, Hd, \lambda)$ over $(\{H_A, H_B, H_C\}, \{R_{\text{share}V}\})$ by

$$V = \{H_A, H_B, H_C\}, \quad E = \{e^\star\}, \quad T(e^\star) = \{H_A, H_B\}, \quad Hd(e^\star) = \{H_C\}, \quad \lambda(e^\star) = R_{\text{share}V}.$$

Incidence check: pick $x = u_2$. Then

$$u_2 \in \{u_1, u_2\} \in V_A, \qquad u_2 \in \{u_2, u_4\} \in V_B, \qquad u_2 \in \{u_1, u_2, u_5\} \in V_C,$$

so $\big(T(e^\star), Hd(e^\star)\big) \in R_{\text{share}V} = \lambda(e^\star)$.

An overview of Meta-Graph, Meta-HyperGraph, and Meta-SuperHyperGraph is presented in Table 4.14.

4.29 Regular SuperHyperGraph

A regular graph has each vertex incident to exactly k edges, so all vertices share the same degree [444, 568]. Related concepts include regular fuzzy graphs [569, 570], irregular graphs [571–573], regular intuitionistic fuzzy graphs [574, 575], strongly regular graphs [576, 577], edge-regular graphs [578–580], regular neutrosophic graphs [581–583], as well as biregular [584, 585] and triregular graphs [586]. Regular graph research is important because uniform degrees yield clean theory, sharp extremal bounds, efficient algorithms, and realistic models for symmetric networks, designs, codes, and expanders. The relevant definitions and related notions are presented below.

A regular hypergraph has each vertex contained in exactly r hyperedges, so every vertex has the same hyperedge-degree [587–589]. A regular SuperHyperGraph has each supervertex incident with exactly r superedges via ∂, so all supervertices have identical degree.

Object	Meta-vertices	Meta-edges / hyperedges (relation layer)
Meta-Graph	Graphs	Edges encode a chosen binary relation between graphs (e.g., "shares a motif", "cites", "is similar").
Meta-HyperGraph	HyperGraphs	Hyperedges group multiple hypergraphs when a multi-way relation holds (e.g., "share a common vertex-set pattern").
Meta-SuperHyperGraph	n-SuperHyperGraphs	Hyperedges connect several n-SuperHyperGraphs according to a prescribed multi-way relation on super-level objects.

Table 4.14: Concise overview of Meta-Graph, Meta-HyperGraph, and Meta-SuperHyperGraph.

Definition 4.29.1 (Regular graph)**.** [590] Let $G = (V(G), E(G))$ be a finite undirected loopless graph, where

$$E(G) \subseteq \{\{u, v\} \mid u, v \in V(G),\ u \neq v\}.$$

For $v \in V(G)$, the (vertex) degree of v is

$$\deg_G(v) := |\{ e \in E(G) \mid v \in e \}|.$$

For an integer $k \in \mathbb{N}_0$, the graph G is called *k-regular* if

$$\deg_G(v) = k \qquad (\forall v \in V(G)).$$

Definition 4.29.2 (Regular hypergraph)**.** [587–589] Let $H = (V(H), E(H))$ be a finite hypergraph, where

$$V(H) \neq \emptyset, \qquad E(H) \subseteq \mathcal{P}^*(V(H)) := \mathcal{P}(V(H)) \setminus \{\emptyset\}.$$

For $v \in V(H)$, the (vertex) degree of v is

$$\deg_H(v) := |\{ e \in E(H) \mid v \in e \}|.$$

For an integer $r \in \mathbb{N}_0$, the hypergraph H is called *r-regular* if

$$\deg_H(v) = r \qquad (\forall v \in V(H)).$$

Example 4.29.3 (A 2-regular hypergraph)**.** Let

$$V(H) := \{1, 2, 3\}, \qquad E(H) := \{ e_{12}, e_{23}, e_{31} \},$$

where

$$e_{12} := \{1, 2\}, \quad e_{23} := \{2, 3\}, \quad e_{31} := \{3, 1\}.$$

For each vertex $v \in V(H)$,

$$\deg_H(v) = |\{ e \in E(H) \mid v \in e \}|.$$

Hence

$$\deg_H(1) = |\{e_{12}, e_{31}\}| = 2, \qquad \deg_H(2) = |\{e_{12}, e_{23}\}| = 2, \qquad \deg_H(3) = |\{e_{23}, e_{31}\}| = 2.$$

Therefore H is 2-regular.

Definition 4.29.4 (Regular n-SuperHyperGraph)**.** Let $n \in \mathbb{N}_0$ and let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be an n-SuperHyperGraph, where $V \subseteq \mathcal{P}^n(V_0)$ is finite, E is finite, and

$$\partial : E \to \mathcal{P}^*(V)$$

is the incidence map. For a supervertex $v \in V$, define its (vertex) degree by

$$\deg_{\mathrm{SHG}}(v) := |\{ e \in E \mid v \in \partial(e) \}|.$$

For an integer $r \in \mathbb{N}_0$, the n-SuperHyperGraph $\mathrm{SHG}^{(n)}$ is called *r-regular* if

$$\deg_{\mathrm{SHG}}(v) = r \qquad (\forall v \in V).$$

Example 4.29.5 (A 2-regular 2-SuperHyperGraph)**.** Fix the base set

$$V_0 := \{a, b, c\}.$$

Define 2-supervertices (note $V \subseteq \mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$) by

$$V := \{v_a, v_b, v_c\}, \qquad v_a := \{\{a\}\},\ \ v_b := \{\{b\}\},\ \ v_c := \{\{c\}\}.$$

Let

$$E := \{e_{ab}, e_{bc}, e_{ca}\}, \qquad e_{ab} := \{v_a, v_b\},\ \ e_{bc} := \{v_b, v_c\},\ \ e_{ca} := \{v_c, v_a\},$$

and define the incidence map

$$\partial : E \to \mathcal{P}^*(V), \qquad \partial(e) := e \quad (e \in E).$$

For each supervertex $v \in V$, define

$$\deg_{\mathrm{SHG}}(v) = \bigl|\{\, e \in E \mid v \in \partial(e) \,\}\bigr|.$$

Then

$$\deg_{\mathrm{SHG}}(v_a) = |\{e_{ab}, e_{ca}\}| = 2, \quad \deg_{\mathrm{SHG}}(v_b) = |\{e_{ab}, e_{bc}\}| = 2, \quad \deg_{\mathrm{SHG}}(v_c) = |\{e_{bc}, e_{ca}\}| = 2.$$

Therefore $\mathrm{SHG}^{(2)} = (V, E, \partial)$ is a 2-regular 2-SuperHyperGraph.

Theorem 4.29.6 (Regular SuperHyperGraphs generalize regular hypergraphs)**.** *For every finite r-regular hypergraph $H = (V(H), E(H))$, there exists a finite r-regular 0-SuperHyperGraph $\mathrm{SHG}^{(0)} = (V, E, \partial)$ whose incidence structure is identical to that of H (hence H is recovered from $\mathrm{SHG}^{(0)}$ by forgetting the superscript 0).*

Equivalently, the class of r-regular hypergraphs embeds into the class of r-regular SuperHyperGraphs (already at level $n = 0$).

Proof. Let $H = (V(H), E(H))$ be a finite r-regular hypergraph. Thus

$$E(H) \subseteq \mathcal{P}^*(V(H)) \quad \text{and} \quad \deg_H(v) = r \ \ (\forall\, v \in V(H)),$$

where

$$\deg_H(v) = \bigl|\{\, e \in E(H) \mid v \in e \,\}\bigr|.$$

We construct a 0-SuperHyperGraph $\mathrm{SHG}^{(0)}$ that preserves degrees.

Step 1: Choose base set and level. Set

$$V_0 := V(H), \qquad n := 0.$$

Then $\mathcal{P}^0(V_0) = V_0$.

Step 2: Define supervertices. Let

$$V := V_0 = V(H).$$

Hence $V \subseteq \mathcal{P}^0(V_0)$ holds.

Step 3: Define superedges and incidence map. Let

$$E := E(H), \qquad \partial : E \to \mathcal{P}^*(V) \text{ by } \partial(e) := e \ \ (\forall\, e \in E).$$

This is well-defined because each $e \in E(H)$ is a nonempty subset of $V(H) = V$, so

$$\partial(e) = e \in \mathcal{P}^*(V).$$

Therefore

$$\mathrm{SHG}^{(0)} := (V, E, \partial)$$

is a valid 0-SuperHyperGraph.

Step 4: Verify preservation of regularity (degree equality). Fix any $v \in V = V(H)$. By definition of $\deg_{\mathrm{SHG}}$ and $\partial(e) = e$,

$$\deg_{\mathrm{SHG}}(v) = \left|\{ e \in E \mid v \in \partial(e) \}\right| = \left|\{ e \in E(H) \mid v \in e \}\right| = \deg_H(v).$$

Since H is r-regular, $\deg_H(v) = r$ for all $v \in V(H)$, hence

$$\deg_{\mathrm{SHG}}(v) = r \qquad (\forall\, v \in V).$$

Thus $\mathrm{SHG}^{(0)}$ is r-regular.

Finally, the incidence relation in H is "$v \in e$", while in $\mathrm{SHG}^{(0)}$ it is "$v \in \partial(e)$"; but $\partial(e) = e$, so these coincide. Hence H is recovered exactly from $\mathrm{SHG}^{(0)}$, and regular hypergraphs are included as special cases of regular SuperHyperGraphs (level 0). □

4.30 Intersection SuperHyperGraph

Intersection graph represents sets as vertices and connects two vertices exactly when the corresponding sets share at least one element [591–595]. As related concepts to intersection graphs, Fuzzy Intersection Graphs [596, 597], Neutrosophic Intersection Graphs [598], and Intersection Directed Graphs [599–601] are also known. In addition, Appendix A provides an overview of Multi-Intersection Graphs. Intersection graph research is important because it models overlap constraints in geometry and systems, enabling structural theorems, recognition algorithms, complexity insights, and applications in scheduling and VLSI (cf. [602,603]).

Intersection hypergraph represents sets as vertices and assigns each hyperedge from a witness set B, collecting all vertices whose sets intersect B [604–606]. Intersection superhypergraph represents sets as supervertices and uses a superhyperedge for any subfamily with nonempty common intersection, capturing higher-order overlaps. The relevant definitions and related notions are presented below.

Definition 4.30.1 (Intersection graph). [607] Let X be a set and let $\mathcal{S} = \{S_1, \dots, S_m\} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. The *intersection graph* of $\mathcal{S}$ is the (simple, undirected) graph

$$\mathrm{IG}(\mathcal{S}) = (V, E), \qquad V := \{1, \dots, m\}, \qquad \{i, j\} \in E \iff i \neq j \text{ and } S_i \cap S_j \neq \emptyset.$$

Equivalently, one may take $V := \mathcal{S}$ and join S_i, S_j iff $S_i \cap S_j \neq \emptyset$.

Definition 4.30.2 (Intersection hypergraph (with respect to a second family)). [608] Let X be a set and let

$$\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}, \qquad \mathcal{B} = \{B_1, \dots, B_r\} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}.$$

The *intersection hypergraph of $\mathcal{A}$ with respect to $\mathcal{B}$* is the hypergraph

$$\mathrm{IH}(\mathcal{A} \mid \mathcal{B}) = (V, \mathcal{E}), \qquad V := \{1, \dots, m\}, \qquad \mathcal{E} := \{e_1, \dots, e_r\},$$

where, for each $j \in \{1, \dots, r\}$,

$$e_j := \{\, i \in V : A_i \cap B_j \neq \emptyset \,\}.$$

If one wants a *simple* hypergraph, discard hyperedges of size ≤ 1 and remove duplicates.

Example 4.30.3 (Intersection hypergraph $\mathrm{IH}(\mathcal{A} \mid \mathcal{B})$). Let $X := \{1, 2, 3, 4\}$ and

$$\mathcal{A} = \{A_1, A_2, A_3\} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}, \qquad A_1 = \{1, 2\},\ A_2 = \{2, 3\},\ A_3 = \{4\},$$

$$\mathcal{B} = \{B_1, B_2, B_3\} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}, \qquad B_1 = \{2\},\ B_2 = \{3, 4\},\ B_3 = \{1, 4\}.$$

Then $V = \{1, 2, 3\}$ and the hyperedges are

$$e_1 = \{\, i \in V : A_i \cap B_1 \neq \emptyset \,\} = \{1, 2\},$$

$$e_2 = \{\, i \in V : A_i \cap B_2 \neq \emptyset \,\} = \{2, 3\},$$

$$e_3 = \{\, i \in V : A_i \cap B_3 \neq \emptyset \,\} = \{1, 3\}.$$

Hence

$$\mathrm{IH}(\mathcal{A} \mid \mathcal{B}) = (\{1, 2, 3\},\ \{\,\{1, 2\}, \{2, 3\}, \{1, 3\}\,\}).$$

In particular, each e_j collects exactly those A_i that intersect the witness set B_j.

Definition 4.30.4 (Intersection n-superhypergraph (common-intersection superhyperedges))**.** Fix an integer $n \geq 1$. Let X be a set and let

$$\mathcal{S} = \{S_1, \ldots, S_m\} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}.$$

Define the ground set

$$V_X := \bigcup_{i=1}^{m} S_i,$$

and define the *supervertex family* (level-1 vertices) by

$$\mathcal{V} := \{S_1, \ldots, S_m\} \subseteq \mathcal{P}(V_X) \setminus \{\emptyset\}.$$

For each $T \subseteq \mathcal{V}$ with $|T| \geq 2$, define the associated *level-n superhyperedge* by

$$e_T^{(n)} \; := \; \widehat{T}^{(n-1)} \; \in \; \mathcal{P}^n(\mathcal{V}),$$

where the lifting $\widehat{\cdot}^{(n-1)}$ is defined recursively by

$$\widehat{A}^{(0)} := A, \qquad \widehat{A}^{(t+1)} := \left\{\widehat{a}^{(t)} : a \in A\right\} \quad (t \geq 0),$$

and, for an individual element a, we set $\widehat{a}^{(0)} := a$ and $\widehat{a}^{(t+1)} := \{\widehat{a}^{(t)}\}$.

The *intersection n-superhypergraph* of $\mathcal{S}$ is the set-based n-superhypergraph

$$\mathrm{ISH}^{(n)}(\mathcal{S}) \; := \; \left(V_X, \; \mathcal{V}, \; \mathcal{SE}^{(n)}\right),$$

whose *level-n superhyperedge family* $\mathcal{SE}^{(n)} \subseteq \mathcal{P}^n(\mathcal{V})$ is

$$\mathcal{SE}^{(n)} \; := \; \left\{e_T^{(n)} \;\middle|\; T \subseteq \mathcal{V}, \; |T| \geq 2, \; \bigcap_{U \in T} U \neq \emptyset\right\}.$$

Thus a level-n superhyperedge is (the $(n-1)$-fold lift of) a subfamily of $\mathcal{S}$ having a nonempty *common* intersection.

Example 4.30.5 (Intersection n-superhypergraph $\mathrm{ISH}^{(n)}(\mathcal{S})$)**.** Let $X := \{1, 2, 3, 4\}$ and

$$\mathcal{S} = \{S_1, S_2, S_3, S_4\} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}, \qquad S_1 = \{1, 2\}, \; S_2 = \{2, 3\}, \; S_3 = \{2, 4\}, \; S_4 = \{3, 4\}.$$

Then $V_X = \bigcup_{i=1}^{4} S_i = X$ and $\mathcal{V} = \{S_1, S_2, S_3, S_4\}$.

A subset $T \subseteq \mathcal{V}$ with $|T| \geq 2$ yields a level-n superhyperedge $e_T^{(n)} = \widehat{T}^{(n-1)} \in \mathcal{P}^n(\mathcal{V})$ if and only if $\bigcap_{U \in T} U \neq \emptyset$.

Pairwise intersections:

$$S_1 \cap S_2 = \{2\} \neq \emptyset, \quad S_1 \cap S_3 = \{2\} \neq \emptyset, \quad S_1 \cap S_4 = \emptyset,$$

$$S_2 \cap S_3 = \{2\} \neq \emptyset, \quad S_2 \cap S_4 = \{3\} \neq \emptyset, \quad S_3 \cap S_4 = \{4\} \neq \emptyset.$$

Hence the level-n superhyperedges coming from 2-subfamilies are exactly

$$e^{(n)}_{\{S_1,S_2\}}, \; e^{(n)}_{\{S_1,S_3\}}, \; e^{(n)}_{\{S_2,S_3\}}, \; e^{(n)}_{\{S_2,S_4\}}, \; e^{(n)}_{\{S_3,S_4\}}.$$

A genuine higher-order overlap occurs for the triple:

$$S_1 \cap S_2 \cap S_3 = \{2\} \neq \emptyset,$$

so the corresponding level-n superhyperedge

$$e^{(n)}_{\{S_1,S_2,S_3\}} = \widehat{\{S_1, S_2, S_3\}}^{(n-1)}$$

belongs to $\mathcal{SE}^{(n)}$.

But

$$S_1 \cap S_2 \cap S_4 = \emptyset, \quad S_1 \cap S_3 \cap S_4 = \emptyset, \quad S_2 \cap S_3 \cap S_4 = \emptyset,$$

and also

$$S_1 \cap S_2 \cap S_3 \cap S_4 = \emptyset.$$

Therefore, in this example,

$$\mathcal{SE}^{(n)} = \left\{ e^{(n)}_{\{S_1,S_2\}}, e^{(n)}_{\{S_1,S_3\}}, e^{(n)}_{\{S_2,S_3\}}, e^{(n)}_{\{S_2,S_4\}}, e^{(n)}_{\{S_3,S_4\}}, e^{(n)}_{\{S_1,S_2,S_3\}} \right\}.$$

This shows that $\mathrm{ISH}^{(n)}(\mathcal{S})$ records not only pairwise intersections, but also common intersections of larger subfamilies, lifted to level n.

4.31 Bipartite SuperHyperGraph

A bipartite graph is a graph whose vertices split into two disjoint parts, and every edge has endpoints in different parts only [609–611]. As related concepts, bipartite fuzzy graphs [612], bipartite neutrosophic graphs [613, 614], tripartite graphs [615–617], nearly bipartite graphs [618], bipartable graphs [619], and bipartite directed graphs [620, 621] are well known. In addition, as graph classes formed by bipartite + an additional property, several families are known, including bipartite tolerance graphs [622,623], circular convex bipartite graphs [624, 625], chordal bipartite graphs [626, 627], and bipartite permutation graphs [628, 629]. Bipartite graphs admit efficient matchings and colorings, model two-type relations naturally, and avoid odd cycles, simplifying many algorithms.

A bipartite hypergraph is a hypergraph with a vertex partition $A \sqcup B$ such that each hyperedge intersects both parts, $e \cap A \neq \emptyset$ and $e \cap B \neq \emptyset$ [630–632]. A bipartite superhypergraph is a superhypergraph with a supervertex partition $\mathcal{V}_1 \sqcup \mathcal{V}_2$ such that every superhyperedge meets both parts nontrivially. The relevant definitions and related notions are presented below.

Definition 4.31.1 (Bipartite Graph)**.** [609–611] A graph is a pair $G = (V, E)$ where V is a (finite) set and $E \subseteq \{\{u, v\} \subseteq V : u \neq v\}$. The graph G is *bipartite* if there exist disjoint sets $V_1, V_2 \subseteq V$ such that

$$V = V_1 \sqcup V_2 \quad \text{and} \quad \forall \{u, v\} \in E : \ (u \in V_1 \ \&\ v \in V_2) \text{ or } (u \in V_2 \ \&\ v \in V_1).$$

Equivalently, no edge has both endpoints in the same part.

Definition 4.31.2 (Bipartite Hypergraph)**.** [630–632] A hypergraph is a pair $H = (V, \mathcal{E})$ where V is a (finite) set and $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a family of nonempty subsets of V (hyperedges). The hypergraph H is *bipartite* if there exist disjoint sets $A, B \subseteq V$ such that

$$V = A \sqcup B \quad \text{and} \quad \forall e \in \mathcal{E} : \ e \cap A \neq \emptyset \text{ and } e \cap B \neq \emptyset.$$

(In particular, no hyperedge is contained entirely in one part.)

Example 4.31.3 (Bipartite hypergraph)**.** Let

$$V := \{a, b, c, d, e\}, \qquad A := \{a, b\}, \qquad B := \{c, d, e\}, \qquad V = A \sqcup B.$$

Define the hyperedge family

$$\mathcal{E} := \{e_1, e_2, e_3\}, \qquad e_1 := \{a, c\}, \quad e_2 := \{b, d, e\}, \quad e_3 := \{a, b, c, d\}.$$

Then each hyperedge meets both parts:

$$e_1 \cap A = \{a\} \neq \emptyset, \ \ e_1 \cap B = \{c\} \neq \emptyset,$$

$$e_2 \cap A = \{b\} \neq \emptyset, \ \ e_2 \cap B = \{d, e\} \neq \emptyset,$$

$$e_3 \cap A = \{a, b\} \neq \emptyset, \ \ e_3 \cap B = \{c, d\} \neq \emptyset.$$

Hence $H := (V, \mathcal{E})$ is a bipartite hypergraph with bipartition (A, B).

Definition 4.31.4 (Bipartite n-superhypergraph)**.** Fix an integer $n \geq 1$. An *n-superhypergraph* is a pair

$$\mathrm{SHG}^{(n)} = (\mathcal{V}, \mathcal{F}^{(n)})$$

where $\mathcal{V}$ is a nonempty set (of *supervertices*) and

$$\mathcal{F}^{(n)} \subseteq \mathcal{P}^n(\mathcal{V}) \setminus \{\emptyset\}$$

is a family of *level-n superhyperedges*.

We say that $\mathrm{SHG}^{(n)}$ is *bipartite* if there exist disjoint sets $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$ such that

$$\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2 \quad \text{and} \quad \forall F \in \mathcal{F}^{(n)} : \ \mathrm{Flat}(F) \cap \mathcal{V}_1 \neq \varnothing \text{ and } \mathrm{Flat}(F) \cap \mathcal{V}_2 \neq \varnothing,$$

where $\mathrm{Flat}(F) \subseteq \mathcal{V}$ denotes the *vertex-level flattening* of F, defined recursively by

$$\mathrm{Flat}_0(U) := U \ \ (U \subseteq \mathcal{V}), \qquad \mathrm{Flat}_{t+1}(A) := \bigcup_{x \in A} \mathrm{Flat}_t(x),$$

and $\mathrm{Flat}(F) := \mathrm{Flat}_n(F)$ for $F \in \mathcal{P}^n(\mathcal{V})$. Thus each level-$n$ superhyperedge, when flattened down to the vertex level, meets both sides of the bipartition.

Example 4.31.5 (Bipartite n-superhypergraph)**.** Fix $n \geq 1$ and let

$$\mathcal{V} := \{U_1, U_2, U_3, U_4, U_5\}, \qquad \mathcal{V}_1 := \{U_1, U_2\}, \qquad \mathcal{V}_2 := \{U_3, U_4, U_5\}, \qquad \mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2.$$

Define three vertex-level supports

$$T_1 := \{U_1, U_3\}, \qquad T_2 := \{U_2, U_4, U_5\}, \qquad T_3 := \{U_1, U_2, U_4\}.$$

Lift each T_i to a level-n superhyperedge by

$$F_i^{(n)} := \widehat{T_i}^{(n-1)} \in \mathcal{P}^n(\mathcal{V}),$$

where $\widehat{\cdot}^{(n-1)}$ is the $(n-1)$-fold singleton-lift:

$$\widehat{A}^{(0)} := A, \qquad \widehat{A}^{(t+1)} := \{\widehat{a}^{(t)} : a \in A\} \quad (t \geq 0),$$

and for $a \in \mathcal{V}$ we set $\widehat{a}^{(0)} := a$, $\widehat{a}^{(t+1)} := \{\widehat{a}^{(t)}\}$. Put

$$\mathcal{F}^{(n)} := \{F_1^{(n)}, F_2^{(n)}, F_3^{(n)}\}, \qquad \mathrm{SHG}^{(n)} := (\mathcal{V}, \mathcal{F}^{(n)}).$$

Then $\mathrm{Flat}(F_i^{(n)}) = T_i$ for each i by construction, and hence each flattened edge meets both parts:

$$\mathrm{Flat}(F_1^{(n)}) \cap \mathcal{V}_1 = \{U_1\} \neq \emptyset, \ \ \mathrm{Flat}(F_1^{(n)}) \cap \mathcal{V}_2 = \{U_3\} \neq \emptyset,$$

$$\mathrm{Flat}(F_2^{(n)}) \cap \mathcal{V}_1 = \{U_2\} \neq \emptyset, \ \ \mathrm{Flat}(F_2^{(n)}) \cap \mathcal{V}_2 = \{U_4, U_5\} \neq \emptyset,$$

$$\mathrm{Flat}(F_3^{(n)}) \cap \mathcal{V}_1 = \{U_1, U_2\} \neq \emptyset, \ \ \mathrm{Flat}(F_3^{(n)}) \cap \mathcal{V}_2 = \{U_4\} \neq \emptyset.$$

Therefore $\mathrm{SHG}^{(n)}$ is a bipartite n-superhypergraph with bipartition $(\mathcal{V}_1, \mathcal{V}_2)$.

Theorem 4.31.6 (Bipartite n-superhypergraphs generalize bipartite hypergraphs)**.** *Fix $n \geq 1$. For every bipartite hypergraph $H = (V, \mathcal{E})$ there exists a bipartite n-superhypergraph*

$$\widehat{H}^{(n)} = \left(\widehat{\mathcal{V}}, \widehat{\mathcal{F}}^{(n)}\right)$$

such that H is obtained from $\widehat{H}^{(n)}$ by identifying each supervertex with a singleton $\{v\}$ and flattening each level-n superhyperedge to a hyperedge of V. In particular, the class of bipartite n-superhypergraphs contains all bipartite hypergraphs as special cases.

Proof. Let $H = (V, \mathcal{E})$ be a bipartite hypergraph. Fix a bipartition $V = A \sqcup B$ such that

$$\forall e \in \mathcal{E} : \ e \cap A \neq \varnothing \text{ and } e \cap B \neq \varnothing.$$

Define the supervertex set by singleton embedding:

$$\widehat{\mathcal{V}} := \{\{v\} : v \in V\}.$$

Define the vertex-level encoding map $f : \widehat{\mathcal{V}} \to V$ by $f(\{v\}) = v$.

For each hyperedge $e \in \mathcal{E}$ define its lifted level-n superhyperedge by

$$T_e := \{\{v\} : v \in e\} \subseteq \widehat{\mathcal{V}}, \qquad \widehat{e}^{(n)} := \widehat{T_e}^{(n-1)} \in \mathcal{P}^n(\widehat{\mathcal{V}}),$$

and set

$$\widehat{\mathcal{F}}^{(n)} := \{\widehat{e}^{(n)} : e \in \mathcal{E}\}.$$

Then $\widehat{H}^{(n)} := (\widehat{\mathcal{V}}, \widehat{\mathcal{F}}^{(n)})$ is an n-superhypergraph.

Now define the induced bipartition of supervertices:

$$\widehat{\mathcal{V}}_A := \{\{a\} : a \in A\}, \qquad \widehat{\mathcal{V}}_B := \{\{b\} : b \in B\}.$$

Clearly $\widehat{\mathcal{V}} = \widehat{\mathcal{V}}_A \sqcup \widehat{\mathcal{V}}_B$.

Take any $\widehat{e}^{(n)} \in \widehat{\mathcal{F}}^{(n)}$, so $\widehat{e}^{(n)} = \widehat{T_e}^{(n-1)}$ for some $e \in \mathcal{E}$. By construction of the lift and the recursive flattening,

$$\mathrm{Flat}(\widehat{e}^{(n)}) = T_e = \{\{v\} : v \in e\}.$$

Hence

$$\mathrm{Flat}(\widehat{e}^{(n)}) \cap \widehat{\mathcal{V}}_A = \{\{a\} : a \in e \cap A\} \neq \varnothing$$

because $e \cap A \neq \varnothing$, and similarly

$$\mathrm{Flat}(\widehat{e}^{(n)}) \cap \widehat{\mathcal{V}}_B = \{\{b\} : b \in e \cap B\} \neq \varnothing$$

because $e \cap B \neq \varnothing$. Therefore every level-$n$ superhyperedge meets both sides after flattening, so $\widehat{H}^{(n)}$ is bipartite.

Finally, identify each supervertex $\{v\} \in \widehat{\mathcal{V}}$ with $v \in V$ via f. For any $e \in \mathcal{E}$ we have

$$f[\mathrm{Flat}(\widehat{e}^{(n)})] = f[T_e] = \{ f(\{v\}) : v \in e \} = e.$$

Thus flattening $\widehat{H}^{(n)}$ and then applying f recovers exactly the hypergraph H. This proves the claimed generalization. □

4.32 Threshold SuperHyperGraph

A threshold graph assigns a nonnegative weight to each vertex and fixes a threshold τ; two vertices are adjacent exactly when their weights sum to at least τ. Related concepts such as fuzzy threshold graphs [633,634], mock threshold graphs [635], neutrosophic threshold graphs [636,637], bithreshold graphs [638], quasi-threshold graphs [639–641], and threshold directed graphs [642] are also known. Threshold graphs have simple structure, admit linear-time recognition, and enable fast solutions for many NP-hard problems on general graphs.

A threshold hypergraph assigns nonnegative weights to vertices and fixes a threshold τ; a nonempty vertex subset X is a hyperedge exactly when $\sum_{v \in X} w(v) \geq \tau$ [643–645]. A threshold superhypergraph assigns nonnegative weights to supervertices and fixes a threshold τ; a nonempty family X of supervertices is a superhyperedge exactly when $\sum_{v \in X} w(v) \geq \tau$. The relevant definitions and related notions are presented below.

Definition 4.32.1 (Threshold graph)**.** [646] A (simple) graph $G = (V, E)$ is a *threshold graph* if there exist a weight function $w : V \to \mathbb{R}_{\geq 0}$ and a threshold $\tau \in \mathbb{R}_{\geq 0}$ such that, for all distinct $u, v \in V$,

$$\{u, v\} \in E \iff w(u) + w(v) \geq \tau.$$

Definition 4.32.2 (Threshold hypergraph)**.** A hypergraph $H = (V, E)$ with $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a *threshold hypergraph* if there exist a weight function $w : V \to \mathbb{R}_{\geq 0}$ and a threshold $\tau \in \mathbb{R}_{\geq 0}$ such that, for every nonempty subset $X \subseteq V$,

$$X \in E \iff \sum_{v \in X} w(v) \geq \tau.$$

Example 4.32.3 (A threshold hypergraph)**.** Let

$$V := \{a, b, c\}, \qquad w(a) = 2,\ w(b) = 1,\ w(c) = 1, \qquad \tau := 3.$$

Define a hypergraph $H = (V, E)$ by

$$E := \Big\{ X \subseteq V \setminus \{\emptyset\} \;\Big|\; \sum_{v \in X} w(v) \geq \tau \Big\}.$$

Then the nonempty subsets $X \subseteq V$ satisfying the threshold condition are exactly

$$\{a, b\}\ (w = 3), \quad \{a, c\}\ (w = 3), \quad \{a, b, c\}\ (w = 4),$$

while $\{a\}$ has weight $2 < 3$, $\{b\}$ has weight $1 < 3$, $\{c\}$ has weight $1 < 3$, and $\{b, c\}$ has weight $2 < 3$. Hence

$$E = \big\{ \{a, b\}, \{a, c\}, \{a, b, c\} \big\},$$

and H is a threshold hypergraph witnessed by (w, τ).

Definition 4.32.4 (Threshold n-SuperHyperGraph (threshold superhypergraph))**.** Let V_0 be a finite base set and define iterated powersets by

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0).$$

Fix an integer $n \geq 1$. An *n-SuperHyperGraph* is a pair $S^{(n)} = (V_n, E_n)$ such that

$$V_n \subseteq \mathcal{P}^n(V_0), \qquad E_n \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}.$$

It is called a *threshold n-SuperHyperGraph* if there exist a weight function $w : V_n \to \mathbb{R}_{\geq 0}$ and a threshold $\tau \in \mathbb{R}_{\geq 0}$ such that, for every nonempty $X \subseteq V_n$,

$$X \in E_n \iff \sum_{v \in X} w(v) \geq \tau.$$

(When $n = 1$, this is exactly a threshold hypergraph on vertex set V_1.)

Example 4.32.5 (A threshold 2-SuperHyperGraph)**.** Let the base set be

$$V_0 := \{1, 2, 3\},$$

so $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$. Define three 2-supervertices (elements of $\mathcal{P}^2(V_0)$) by

$$s_1 := \big\{\{1\}\big\}, \qquad s_2 := \big\{\{2\}\big\}, \qquad s_3 := \big\{\{1, 2\}\big\},$$

and set

$$V_2 := \{s_1, s_2, s_3\} \subseteq \mathcal{P}^2(V_0).$$

Assign weights and a threshold by

$$w(s_1) = 2, \quad w(s_2) = 1, \quad w(s_3) = 2, \qquad \tau := 3.$$

Define

$$E_2 := \Big\{ X \subseteq V_2 \setminus \{\emptyset\} \;\Big|\; \sum_{s \in X} w(s) \geq \tau \Big\}.$$

Then

$$w(\{s_1, s_2\}) = 3, \quad w(\{s_1, s_3\}) = 4, \quad w(\{s_2, s_3\}) = 3, \quad w(\{s_1, s_2, s_3\}) = 5,$$

so these four subsets are superhyperedges, while the singletons have weights $2, 1, 2 < 3$ and are not superhyperedges. Therefore

$$E_2 = \big\{ \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_2, s_3\} \big\},$$

and $S^{(2)} = (V_2, E_2)$ is a threshold 2-SuperHyperGraph witnessed by (w, τ).

4.33 Fractional SuperHyperGraph

A fractional graph is an equivalence class of finite graphs under fractional (doubly stochastic) relabelings that transform adjacency consistently (cf. [647–649]). A fractional hypergraph is an equivalence class of finite hypergraphs under fractional vertex and hyperedge matchings that transform incidence consistently (cf. [650, 651]). A fractional SuperHyperGraph is an equivalence class of finite n-SuperHyperGraphs under fractional supervertex and superedge matchings that transform superincidence consistently. The relevant definitions and related notions are presented below.

Definition 4.33.1 (Doubly stochastic matrix)**.** Let $n \in \mathbb{N}$. A matrix $S \in \mathbb{R}^{n\times n}$ is called *doubly stochastic* if

$$S_{ij} \geq 0 \ \ (1 \leq i, j \leq n), \qquad S\mathbf{1} = \mathbf{1}, \qquad S^{\mathsf{T}}\mathbf{1} = \mathbf{1},$$

where $\mathbf{1}$ is the all-ones column vector in $\mathbb{R}^n$.

Definition 4.33.2 (Fractional graph (fractional-isomorphism viewpoint))**.** (cf. [647–649]) Let G and H be finite directed or undirected graphs on the same number n of vertices, and let $A_G, A_H \in \mathbb{R}^{n\times n}$ be their adjacency matrices. We say that G and H are *fractionally isomorphic* (and write $G \cong_f H$) if there exists a doubly stochastic matrix $S \in \mathbb{R}^{n\times n}$ such that

$$A_G S = S A_H.$$

A *fractional graph* can be regarded as an equivalence class

$$[G]_f := \{\, H \mid H \cong_f G \,\}$$

under the relation $\cong_f$.

Definition 4.33.3 ((Recall) Incidence matrix of a hypergraph)**.** Let $G = (V, E)$ be a finite hypergraph with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. Its *vertex–hyperedge incidence matrix* is the matrix $M_G \in \{0, 1\}^{n\times m}$ defined by

$$(M_G)_{ij} := \begin{cases} 1, & v_i \in e_j, \\ 0, & v_i \notin e_j. \end{cases}$$

Definition 4.33.4 (Fractional hypergraph (fractional-isomorphism viewpoint))**.** Let G and H be finite hypergraphs. Write $M_G \in \{0, 1\}^{n\times m}$ and $M_H \in \{0, 1\}^{n\times m}$ for their incidence matrices (after fixing vertex and hyperedge orderings). We say that G and H are *fractionally isomorphic* (and write $G \equiv H$) if either

(i) G and H have the same number of vertices and no hyperedges; or

(ii) there exist doubly stochastic matrices $S_1 \in \mathbb{R}^{n\times n}$ and $S_2 \in \mathbb{R}^{m\times m}$ such that

$$S_1 M_G = M_H S_2^{\mathsf{T}} \qquad \text{and} \qquad M_G S_2 = S_1^{\mathsf{T}} M_H.$$

A *fractional hypergraph* can be regarded as an equivalence class

$$[G]_{\equiv} := \{\, H \mid H \equiv G \,\}$$

under the relation $\equiv$.

Example 4.33.5 (A concrete fractional hypergraph)**.** Let $G = (V, E)$ and $H = (V, F)$ be 3-uniform hypergraphs on the same vertex set

$$V = \{1, 2, 3, 4, 5, 6\}.$$

Define

$$E = \{e_1, e_2, e_3, e_4\}, \qquad e_1 = \{1, 2, 3\},\ e_2 = \{1, 4, 5\},\ e_3 = \{2, 4, 6\},\ e_4 = \{3, 5, 6\},$$

and

$$F = \{f_1, f_2, f_3, f_4\}, \qquad f_1 = \{1, 2, 3\},\ f_2 = \{1, 2, 4\},\ f_3 = \{3, 5, 6\},\ f_4 = \{4, 5, 6\}.$$

With vertex order $(1, 2, 3, 4, 5, 6)$ and edge orders (e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4), the incidence matrices are

$$M_G = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad M_H = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Both are (r, s)-biregular with $r = 2$ (each vertex is in exactly two hyperedges) and $s = 3$ (each hyperedge has size three), i.e.

$$M_G \mathbf{1}_4 = 2\mathbf{1}_6, \quad \mathbf{1}_6^\top M_G = 3\mathbf{1}_4^\top, \qquad M_H \mathbf{1}_4 = 2\mathbf{1}_6, \quad \mathbf{1}_6^\top M_H = 3\mathbf{1}_4^\top.$$

Let J_k denote the $k \times k$ all-ones matrix and set the doubly stochastic matrices

$$S_1 := \frac{1}{6} J_6, \qquad S_2 := \frac{1}{4} J_4.$$

Then, using $J_6 = \mathbf{1}_6 \mathbf{1}_6^\top$ and $J_4 = \mathbf{1}_4 \mathbf{1}_4^\top$,

$$S_1 M_G = \frac{1}{6} J_6 M_G = \frac{1}{6} \mathbf{1}_6 \big(\mathbf{1}_6^\top M_G\big) = \frac{1}{6} \mathbf{1}_6 (3\mathbf{1}_4^\top) = \frac{1}{2} J_{6\times 4},$$

and

$$M_H S_2^\top = M_H \frac{1}{4} J_4 = \frac{1}{4} \big(M_H \mathbf{1}_4\big) \mathbf{1}_4^\top = \frac{1}{4} (2\mathbf{1}_6) \mathbf{1}_4^\top = \frac{1}{2} J_{6\times 4}.$$

Similarly,

$$M_G S_2 = \frac{1}{4} \big(M_G \mathbf{1}_4\big) \mathbf{1}_4^\top = \frac{1}{2} J_{6\times 4} \qquad \text{and} \qquad S_1^\top M_H = \frac{1}{6} \mathbf{1}_6 \big(\mathbf{1}_6^\top M_H\big) = \frac{1}{2} J_{6\times 4}.$$

Hence $S_1 M_G = M_H S_2^\top$ and $M_G S_2 = S_1^\top M_H$, so $G \equiv H$. Therefore the fractional hypergraph $[G]_\equiv$ is a concrete example, and it contains H.

Definition 4.33.6 ((Recall) Incidence matrix of an n-SuperHyperGraph)**.** Let $\mathcal{H}^{(n)} = (V^{(n)}, E^{(n)})$ be finite with

$$V^{(n)} = \{v_1, \ldots, v_p\}, \qquad E^{(n)} = \{e_1, \ldots, e_q\}.$$

Its *incidence matrix* is $M_{\mathcal{H}^{(n)}} \in \{0, 1\}^{p\times q}$ defined by

$$\big(M_{\mathcal{H}^{(n)}}\big)_{ij} := \begin{cases} 1, & v_i \in e_j, \\ 0, & v_i \notin e_j. \end{cases}$$

Definition 4.33.7 (Fractional isomorphism of n-SuperHyperGraphs)**.** Let $\mathcal{H}^{(n)} = (V^{(n)}, E^{(n)})$ and $\mathcal{K}^{(n)} = (W^{(n)}, F^{(n)})$ be finite n-SuperHyperGraphs with

$$|V^{(n)}| = |W^{(n)}| = p, \qquad |E^{(n)}| = |F^{(n)}| = q,$$

and incidence matrices $M_{\mathcal{H}^{(n)}}, M_{\mathcal{K}^{(n)}} \in \{0, 1\}^{p\times q}$ after fixing orderings of supervertices and superedges. We say that $\mathcal{H}^{(n)}$ and $\mathcal{K}^{(n)}$ are *fractionally isomorphic* (written $\mathcal{H}^{(n)} \equiv_n \mathcal{K}^{(n)}$) if there exist doubly stochastic matrices

$$S_V \in \mathbb{R}^{p\times p}, \qquad S_E \in \mathbb{R}^{q\times q}$$

such that

$$S_V\, M_{\mathcal{H}^{(n)}} \;=\; M_{\mathcal{K}^{(n)}}\, S_E^\top \qquad \text{and} \qquad M_{\mathcal{H}^{(n)}}\, S_E \;=\; S_V^\top\, M_{\mathcal{K}^{(n)}}.$$

Definition 4.33.8 (Fractional n-SuperHyperGraph)**.** A *Fractional n-SuperHyperGraph* is an equivalence class

$$[\mathcal{H}^{(n)}]_{\equiv_n} \;:=\; \{\, \mathcal{K}^{(n)} \mid \mathcal{K}^{(n)} \equiv_n \mathcal{H}^{(n)} \,\}$$

under the relation $\equiv_n$ of Definition 4.33.7. When n is understood, we also call it a *Fractional SuperHyperGraph*.

Example 4.33.9 (A concrete fractional SuperHyperGraph (level $n = 1$))**.** Let the base set be

$$V_0 = \{a, b, c, d, e, f\},$$

and define six 1-supervertices (nonempty subsets of V_0) by

$$s_1 = \{a, b\},\ s_2 = \{b, c\},\ s_3 = \{c, d\},\ s_4 = \{d, e\},\ s_5 = \{e, f\},\ s_6 = \{f, a\}.$$

Consider two 1-SuperHyperGraphs

$$\mathcal{H}^{(1)} = (V^{(1)}, E^{(1)}), \qquad \mathcal{K}^{(1)} = (V^{(1)}, F^{(1)}), \quad V^{(1)} = \{s_1, s_2, s_3, s_4, s_5, s_6\},$$

with 1-superedges

$$E^{(1)} = \{E_1, E_2, E_3, E_4\}, \quad E_1 = \{s_1, s_2, s_3\},\ E_2 = \{s_1, s_4, s_5\},\ E_3 = \{s_2, s_4, s_6\},\ E_4 = \{s_3, s_5, s_6\},$$

and

$$F^{(1)} = \{F_1, F_2, F_3, F_4\}, \quad F_1 = \{s_1, s_2, s_3\},\ F_2 = \{s_1, s_2, s_4\},\ F_3 = \{s_3, s_5, s_6\},\ F_4 = \{s_4, s_5, s_6\}.$$

Using the supervertex order $(s_1, \dots, s_6)$ and the superedge orders $(E_1, \dots, E_4)$ and $(F_1, \dots, F_4)$, their incidence matrices are exactly the matrices M_G and M_H from the previous example. In particular, each supervertex is incident with exactly two superedges and each superedge contains exactly three supervertices.

Let

$$S_V := \frac{1}{6} J_6, \qquad S_E := \frac{1}{4} J_4,$$

which are doubly stochastic. The same calculation yields

$$S_V\, M_{\mathcal{H}^{(1)}} = M_{\mathcal{K}^{(1)}}\, S_E^{\top} \qquad \text{and} \qquad M_{\mathcal{H}^{(1)}}\, S_E = S_V^{\top}\, M_{\mathcal{K}^{(1)}}.$$

Hence $\mathcal{H}^{(1)} \equiv_1 \mathcal{K}^{(1)}$, so $[\mathcal{H}^{(1)}]_{\equiv_1}$ is a concrete Fractional 1-SuperHyperGraph (Fractional SuperHyperGraph), and it contains $\mathcal{K}^{(1)}$.

Theorem 4.33.10 (SuperHyperGraph representation and generalization of fractional hypergraphs)**.**

(i) Every Fractional n-SuperHyperGraph $[\mathcal{H}^{(n)}]_{\equiv_n}$ has an underlying n-SuperHyperGraph representative $\mathcal{H}^{(n)}$ in the sense that

$$\mathcal{H}^{(n)} \in [\mathcal{H}^{(n)}]_{\equiv_n}.$$

In particular, the notion is built from (and represented by) n-SuperHyperGraphs.

(ii) For $n = 0$, the notion of Fractional 0-SuperHyperGraph coincides with the notion of Fractional HyperGraph (defined via doubly stochastic matrices acting on the usual vertex–hyperedge incidence matrix). Concretely, if $H = (V, E)$ is a hypergraph, then viewing it as a 0-SuperHyperGraph $\mathcal{H}^{(0)} := (V, E)$ yields

$$[H]_{\equiv} = [\mathcal{H}^{(0)}]_{\equiv_0}.$$

Proof.

(i) By Definition 4.33.8, a Fractional n-SuperHyperGraph is, by construction, an equivalence class of n-SuperHyperGraphs under $\equiv_n$. Since $\equiv_n$ is an equivalence relation, it is reflexive; hence

$$\mathcal{H}^{(n)} \equiv_n \mathcal{H}^{(n)}.$$

Taking $S_V = I_p$ and $S_E = I_q$ (both are doubly stochastic), we have

$$S_V M_{\mathcal{H}^{(n)}} = I_p M_{\mathcal{H}^{(n)}} = M_{\mathcal{H}^{(n)}}, \qquad M_{\mathcal{H}^{(n)}} S_E^{\top} = M_{\mathcal{H}^{(n)}} I_q = M_{\mathcal{H}^{(n)}},$$

and similarly

$$M_{\mathcal{H}^{(n)}} S_E = M_{\mathcal{H}^{(n)}} I_q = M_{\mathcal{H}^{(n)}}, \qquad S_V^{\top} M_{\mathcal{H}^{(n)}} = I_p M_{\mathcal{H}^{(n)}} = M_{\mathcal{H}^{(n)}}.$$

Thus the defining equalities of Definition 4.33.7 hold, so $\mathcal{H}^{(n)} \in [\mathcal{H}^{(n)}]_{\equiv_n}$.

(ii) Let $n = 0$. Then $P_0(V_0) = V_0$, so an 0-SuperHyperGraph $\mathcal{H}^{(0)} = (V^{(0)}, E^{(0)})$ is exactly a (finite) hypergraph on vertex set $V^{(0)}$ with hyperedges $E^{(0)} \subseteq \mathcal{P}(V^{(0)}) \setminus \{\emptyset\}$. Moreover, the incidence matrix in Definition 4.33.6 is the usual vertex–hyperedge incidence matrix of a hypergraph.

Now take two hypergraphs $H = (V, E)$ and $H' = (V', E')$ with $|V| = |V'| = p$ and $|E| = |E'| = q$. Identify them with 0-SuperHyperGraphs

$$\mathcal{H}^{(0)} := (V, E), \qquad \mathcal{K}^{(0)} := (V', E').$$

Let M_H and $M_{H'}$ denote their usual incidence matrices. Under this identification,

$$M_{\mathcal{H}^{(0)}} = M_H, \qquad M_{\mathcal{K}^{(0)}} = M_{H'}.$$

Therefore, the existence of doubly stochastic matrices $S_V \in \mathbb{R}^{p\times p}$ and $S_E \in \mathbb{R}^{q\times q}$ satisfying the fractional-hypergraph equalities

$$S_V M_H = M_{H'} S_E^{\top} \qquad \text{and} \qquad M_H S_E = S_V^{\top} M_{H'}$$

is *equivalent* (by literal substitution of $M_{\mathcal{H}^{(0)}}$ and $M_{\mathcal{K}^{(0)}}$) to the equalities

$$S_V M_{\mathcal{H}^{(0)}} = M_{\mathcal{K}^{(0)}} S_E^{\top} \qquad \text{and} \qquad M_{\mathcal{H}^{(0)}} S_E = S_V^{\top} M_{\mathcal{K}^{(0)}},$$

i.e. to $\mathcal{H}^{(0)} \equiv_0 \mathcal{K}^{(0)}$ from Definition 4.33.7.

Hence the equivalence classes defined by the two notions coincide: the class of all H' fractionally isomorphic to H is exactly the class of all $\mathcal{K}^{(0)}$ fractionally isomorphic to $\mathcal{H}^{(0)}$. This proves

$$[H]_{\equiv} = [\mathcal{H}^{(0)}]_{\equiv_0}.$$

□

4.34 Cycle SuperHyperGraph

A cycle graph is a simple graph where vertices form one closed loop, each vertex adjacent to exactly two vertices [652–654]. Several related concepts are also known, such as fuzzy cycle graphs [655, 656], directed cycle graphs [657], and neutrosophic cycle graphs [658]. A cycle hypergraph is a hypergraph admitting a Berge cycle: alternating distinct vertices and hyperedges, each hyperedge containing consecutive vertices [659–661]. A cycle superhypergraph is an n-superhypergraph whose underlying hypergraph contains a Berge cycle, via supervertices and incidence-defined superhyperedges. The relevant definitions and related notions are presented below.

Definition 4.34.1 (Cycle graph)**.** (cf. [652–654]) Let $\ell \in \mathbb{N}$ with $\ell \geq 3$. The *cycle graph of length* ℓ is the graph

$$C_\ell := (V_\ell, E_\ell),$$

where

$$V_\ell := \{v_1, \dots, v_\ell\}, \qquad E_\ell := \big\{\{v_i, v_{i+1}\} \mid i = 1, \dots, \ell - 1\big\} \cup \big\{\{v_\ell, v_1\}\big\}.$$

(Here indices are taken modulo ℓ in the obvious way, i.e., $v_{\ell+1} := v_1$.)

Definition 4.34.2 (Berge cycle and cycle hypergraph)**.** Let $H = (V(H), E(H))$ be a finite hypergraph, i.e., $V(H) \neq \emptyset$ and $E(H) \subseteq \mathcal{P}^*(V(H))$.

A *Berge cycle of length* $\ell \geq 3$ *in* H is an alternating sequence

$$(v_1, e_1, v_2, e_2, \dots, v_\ell, e_\ell, v_1)$$

such that

$$v_1, \dots, v_\ell \in V(H) \text{ are pairwise distinct}, \qquad e_1, \dots, e_\ell \in E(H) \text{ are pairwise distinct},$$

and for every $i \in \{1, \dots, \ell\}$,

$$\{v_i, v_{i+1}\} \subseteq e_i, \quad \text{where } v_{\ell+1} := v_1.$$

A hypergraph H is called a *cycle hypergraph of length* ℓ (in the Berge sense) if H has a Berge cycle $(v_1, e_1, \dots, v_\ell, e_\ell, v_1)$ of length ℓ and moreover

$$V(H) = \{v_1, \dots, v_\ell\}, \qquad E(H) = \{e_1, \dots, e_\ell\}.$$

Definition 4.34.3 (Underlying hypergraph of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be an n-SuperHyperGraph (with incidence map $\partial : E \to \mathcal{P}^*(V)$). Its *underlying hypergraph* is

$$U\big(\mathrm{SHG}^{(n)}\big) := \big(V, \{\partial(e) \mid e \in E\}\big),$$

which is a hypergraph on vertex set V because $\partial(e) \in \mathcal{P}^*(V)$ for all $e \in E$.

Definition 4.34.4 (Cycle n-SuperHyperGraph)**.** An n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E, \partial)$ is called a *cycle n-SuperHyperGraph of length ℓ* (Berge type) if its underlying hypergraph $U(\mathrm{SHG}^{(n)})$ is a cycle hypergraph of length ℓ in the sense of Definition 4.34.2. Equivalently, there exist pairwise distinct supervertices $u_1, \dots, u_\ell \in V$ and pairwise distinct edges $f_1, \dots, f_\ell \in E$ such that

$$V = \{u_1, \dots, u_\ell\}, \qquad \{\partial(f_1), \dots, \partial(f_\ell)\} = \{F_1, \dots, F_\ell\},$$

and for each i,

$$\{u_i, u_{i+1}\} \subseteq \partial(f_i), \quad \text{with } u_{\ell+1} := u_1.$$

Lemma 4.34.5 (Iterated singleton embedding)**.** *Let V_0 be a set and let $n \in \mathbb{N}_0$. Define a map $\iota_n : V_0 \to \mathcal{P}^n(V_0)$ by*

$$\iota_0(x) := x, \qquad \iota_{k+1}(x) := \{\iota_k(x)\} \quad (k \ge 0).$$

Then:

(i) For every $x \in V_0$, one has $\iota_n(x) \in \mathcal{P}^n(V_0)$.

(ii) The map ι_n is injective.

Proof. (i) We prove by induction on n. If $n = 0$, then $\iota_0(x) = x \in V_0 = \mathcal{P}^0(V_0)$. Assume $\iota_n(x) \in \mathcal{P}^n(V_0)$. Then

$$\iota_{n+1}(x) = \{\iota_n(x)\} \subseteq \mathcal{P}^n(V_0),$$

hence $\iota_{n+1}(x) \in \mathcal{P}\big(\mathcal{P}^n(V_0)\big) = \mathcal{P}^{n+1}(V_0)$.

(ii) We prove by induction on n. If $n = 0$, then $\iota_0 = \mathrm{id}_{V_0}$ is injective. Assume ι_n is injective. If $\iota_{n+1}(x) = \iota_{n+1}(y)$, then

$$\{\iota_n(x)\} = \{\iota_n(y)\} \implies \iota_n(x) = \iota_n(y) \implies x = y,$$

using injectivity of ι_n. Thus ι_{n+1} is injective. □

Theorem 4.34.6 (Cycle n-SuperHyperGraphs generalize cycle hypergraphs)**.** *Let $H = (V(H), E(H))$ be a cycle hypergraph of length ℓ in the sense of Definition 4.34.2. Then for every $n \in \mathbb{N}_0$ there exists a cycle n-SuperHyperGraph $\mathrm{CSHG}^{(n)}$ whose underlying hypergraph is isomorphic to H. In particular, when $n = 0$ one recovers H exactly (as a 0-level cycle SuperHyperGraph).*

Proof. Let $H = (V(H), E(H))$ be a cycle hypergraph of length ℓ. By Definition 4.34.2, there exist pairwise distinct vertices $v_1, \dots, v_\ell$ and pairwise distinct hyperedges $e_1, \dots, e_\ell$ such that

$$V(H) = \{v_1, \dots, v_\ell\}, \qquad E(H) = \{e_1, \dots, e_\ell\}, \qquad \{v_i, v_{i+1}\} \subseteq e_i \ (1 \le i \le \ell),$$

where $v_{\ell+1} := v_1$.

Step 1 (Base set and embedded supervertices). Set the base set $V_0 := V(H)$ and define $\iota_n : V_0 \to \mathcal{P}^n(V_0)$ as in Lemma 4.34.5. Define

$$V^{(n)} := \iota_n\big(V(H)\big) = \{\iota_n(v) \mid v \in V(H)\}.$$

By Lemma 4.34.5(i), $V^{(n)} \subseteq \mathcal{P}^n(V_0)$.

Step 2 (Lifted superedges). For each hyperedge $e \in E(H)$ define its lift

$$\widehat{e}^{(n)} := \{\iota_n(v) \mid v \in e\}.$$

Since $e \neq \emptyset$ and ι_n is a function, we have $\widehat{e}^{(n)} \neq \emptyset$. Also $\widehat{e}^{(n)} \subseteq V^{(n)}$, hence $\widehat{e}^{(n)} \in \mathcal{P}^*(V^{(n)})$. Set

$$E^{(n)} := \{\widehat{e}^{(n)} \mid e \in E(H)\} \subseteq \mathcal{P}^*(V^{(n)}).$$

Define the incidence map $\partial^{(n)} : E^{(n)} \to \mathcal{P}^*(V^{(n)})$ by

$$\partial^{(n)}(f) := f \qquad (\forall f \in E^{(n)}),$$

i.e., $\partial^{(n)}$ is the identity on the (lifted) edge-set.

Thus

$$\mathrm{CSHG}^{(n)} := \big(V^{(n)}, E^{(n)}, \partial^{(n)}\big)$$

is an n-SuperHyperGraph over V_0 (it satisfies $V^{(n)} \subseteq \mathcal{P}^n(V_0)$ and $\partial^{(n)}(f) \in \mathcal{P}^*(V^{(n)})$ for all $f \in E^{(n)}$).

Step 3 (Cycle property at level n). Define $u_i := \iota_n(v_i) \in V^{(n)}$ and $f_i := \widehat{e}_i^{(n)} \in E^{(n)}$. Then for each i,

$$\{u_i, u_{i+1}\} = \{\iota_n(v_i), \iota_n(v_{i+1})\} \subseteq \{\iota_n(v) \mid v \in e_i\} = \widehat{e}_i^{(n)} = \partial^{(n)}(f_i),$$

because $\{v_i, v_{i+1}\} \subseteq e_i$ in H. The vertices $u_1, \ldots, u_\ell$ are pairwise distinct by Lemma 4.34.5(ii). Also the lifted edges $f_1, \ldots, f_\ell$ are pairwise distinct: if $\widehat{e}_i^{(n)} = \widehat{e}_j^{(n)}$, then for every $v \in e_i$ we have $\iota_n(v) \in \widehat{e}_j^{(n)}$, so $\iota_n(v) = \iota_n(w)$ for some $w \in e_j$; injectivity of ι_n yields $v = w$, hence $e_i \subseteq e_j$, and similarly $e_j \subseteq e_i$, so $e_i = e_j$. Therefore $U(\mathrm{CSHG}^{(n)}) = (V^{(n)}, E^{(n)})$ is a cycle hypergraph of length ℓ, so $\mathrm{CSHG}^{(n)}$ is a cycle n-SuperHyperGraph by Definition 4.34.4.

Step 4 (Isomorphism back to the original cycle hypergraph). The map $\iota_n : V(H) \to V^{(n)}$ is a bijection onto its image by injectivity, and by construction it sends each hyperedge $e \in E(H)$ to the lifted hyperedge $\widehat{e}^{(n)} \in E^{(n)}$. Hence H is isomorphic to the underlying hypergraph $U(\mathrm{CSHG}^{(n)})$.

Finally, if $n = 0$, then $\iota_0 = \mathrm{id}_{V(H)}$, so

$$V^{(0)} = V(H), \qquad E^{(0)} = \{\widehat{e}^{(0)} \mid e \in E(H)\} = \{e \mid e \in E(H)\} = E(H),$$

and $\partial^{(0)}$ is the identity. Thus $\mathrm{CSHG}^{(0)}$ reproduces H exactly. □

4.35 Friendship SuperHyperGraphs

A friendship graph is a graph where every vertex pair has exactly one common neighbor, forcing triangle−based "friend" structure [662]. A friendship r−hypergraph is r-uniform: each r-set has a unique external vertex completing all $(r - 1)$-subsets into edges [663–665]. A friendship r−superhypergraph is an $(r + 1)$-uniform block system whose r-subset flattening is friendship, with unique block containment. The relevant definitions and related notions are presented below.

Definition 4.35.1 (Friendship graph)**.** [662, 666] A (finite, simple) graph $G = (V, E)$ is a *friendship graph* if for every two distinct vertices $u, v \in V$ there exists a *unique* vertex $w \in V \setminus \{u, v\}$ such that $\{u, w\} \in E$ and $\{v, w\} \in E$. Equivalently, every pair of vertices has a unique common neighbour.

Definition 4.35.2 (Friendship r-hypergraph)**.** Fix an integer $r \geq 2$. An *r-uniform hypergraph* is a pair $H = (V, E)$ where V is a finite set and $E \subseteq \binom{V}{r}$. We call H a *friendship r-hypergraph* if for every $R \in \binom{V}{r}$ there exists a *unique* vertex $x \in V \setminus R$ such that

$$A \cup \{x\} \in E \quad \text{for every } A \in \binom{R}{r-1}.$$

Such an x is called the *friend* of R.

Remark 4.35.3. For $r = 2$, a friendship 2-hypergraph is exactly a friendship graph. For $r = 3$, the condition reads: for every triple $\{x, y, z\}$ there is a unique w such that $\{x, y, w\}, \{x, z, w\}, \{y, z, w\}$ are hyperedges.

Definition 4.35.4 (K_r^{r+1} on r+1 vertices)**.** Let $q \subseteq V$ with $|q| = r + 1$. We say $H[q]$ is a *copy of* K_r^{r+1} if

$$\binom{q}{r} \subseteq E,$$

i.e., every r-subset of q is a hyperedge.

Definition 4.35.5 (Friendship r-superhypergraph and flattening)**.** Fix $r \geq 2$. A *(uniform) r-superhypergraph* is a pair

$$\mathcal{S} = (V, Q) \quad \text{with} \quad Q \subseteq \binom{V}{r+1},$$

whose elements are called *superhyperedges* (or *blocks*). Its *flattening* is the r-uniform hypergraph

$$\mathrm{Flat}(\mathcal{S}) := (V, E_{\mathrm{Flat}}), \qquad E_{\mathrm{Flat}} := \bigcup_{q \in Q} \binom{q}{r}.$$

We call $\mathcal{S}$ a *friendship r-superhypergraph* if $\mathrm{Flat}(\mathcal{S})$ is a friendship r-hypergraph and, moreover, every hyperedge $e \in E_{\mathrm{Flat}}$ is contained in a *unique* block $q \in Q$.

Lemma 4.35.6 (Canonical $(r$+1$)$-block containing a given hyperedge)**.** *Let $H = (V, E)$ be a friendship r-hypergraph and let $e \in E$. Let x be the unique friend of the r-set e. Set*

$$q(e) := e \cup \{x\},$$

so $|q(e)| = r + 1$. Then:

$$\binom{q(e)}{r} \subseteq E,$$

i.e., $H[q(e)]$ is a copy of K_r^{r+1}. Moreover, $q(e)$ is the unique *$(r$+1$)$-subset $q \supseteq e$ with $\binom{q}{r} \subseteq E$.*

Proof. Write $e = \{v_1, \ldots, v_r\}$ and let x be the unique friend of e. By the friendship property applied to $R = e$, for every $(r-1)$-subset $A \in \binom{e}{r-1}$ we have $A \cup \{x\} \in E$. But the r-subsets of $q(e) = e \cup \{x\}$ are exactly:

$$e \quad \text{and} \quad (e \setminus \{v_i\}) \cup \{x\} \;\; (i = 1, \ldots, r).$$

The set e is a hyperedge by assumption, and each $(e \setminus \{v_i\}) \cup \{x\}$ is a hyperedge by the friendship property, hence $\binom{q(e)}{r} \subseteq E$.

For uniqueness, suppose $q \supseteq e$ with $|q| = r + 1$ and $\binom{q}{r} \subseteq E$. Then $q = e \cup \{y\}$ for some $y \notin e$. Since every $(r-1)$-subset $A \subset e$ satisfies $A \cup \{y\} \in E$, the vertex y is a friend of e. By uniqueness of the friend of e, we get $y = x$, hence $q = q(e)$. □

Theorem 4.35.7 (Friendship superhypergraphs generalise friendship hypergraphs)**.** *Let $H = (V, E)$ be a friendship r-hypergraph. Define*

$$Q(H) := \{\, q(e) \,:\, e \in E \,\} \subseteq \binom{V}{r+1}, \qquad \mathcal{S}(H) := (V, Q(H)).$$

Then $\mathcal{S}(H)$ is a friendship r-superhypergraph and

$$\mathrm{Flat}\big(\mathcal{S}(H)\big) = H.$$

In particular, every friendship r-hypergraph is (canonically) obtained as the flattening of a friendship r-superhypergraph.

Proof. By the lemma, for each $e \in E$ we have $\binom{q(e)}{r} \subseteq E$; thus, for every $q \in Q(H)$,

$$\binom{q}{r} \subseteq E.$$

Let $E_{\mathrm{Flat}} := \bigcup_{q \in Q(H)} \binom{q}{r}$ be the edge set of $\mathrm{Flat}(\mathcal{S}(H))$.

First, $E \subseteq E_{\text{Flat}}$: take any $e \in E$; then $e \subseteq q(e) \in Q(H)$, hence $e \in \binom{q(e)}{r} \subseteq E_{\text{Flat}}$.

Second, $E_{\text{Flat}} \subseteq E$: take any $f \in E_{\text{Flat}}$; then $f \in \binom{q}{r}$ for some $q \in Q(H)$, and we already know $\binom{q}{r} \subseteq E$, hence $f \in E$.

Therefore $E_{\text{Flat}} = E$, so $\text{Flat}(\mathcal{S}(H)) = (V, E) = H$.

Finally, the lemma also shows that each $e \in E$ is contained in exactly one block $q(e) \in Q(H)$, so $\mathcal{S}(H)$ satisfies the defining uniqueness condition of a friendship r-superhypergraph. □

4.36 Wheel SuperHyperGraph

A wheel graph has one hub vertex joined to all cycle vertices, plus the cycle edges connecting consecutive rim vertices [667–669]. Related notions such as fuzzy wheel graphs [670–673], Double-Wheel graphs [674–676], and their variants are also known. A wheel hypergraph has rim hyperedges on consecutive rim vertices and spoke hyperedges containing the hub with consecutive rim vertices (cf. [677]). A wheel n-superhypergraph replaces each vertex by an $(n-1)$-fold singleton lift and lifts each wheel hyperedge accordingly. The relevant definitions and related notions are presented below.

Definition 4.36.1 (Wheel graph)**.** [667–669] Let $N \geq 4$. Put

$$V_0 := \{h\} \cup \{v_1, v_2, \dots, v_{N-1}\}.$$

The *wheel graph* W_N is the simple graph (V_0, E) with

$$E := \big\{\{v_i, v_{i+1}\} : 1 \leq i \leq N-1\big\} \ \cup \ \big\{\{h, v_i\} : 1 \leq i \leq N-1\big\},$$

where indices on rim vertices are taken modulo $N-1$ (so $v_N := v_1$).

Definition 4.36.2 (Wheel r-uniform hypergraph)**.** Let $r \geq 2$ and $N \geq r+1$. With the same vertex set

$$V_0 := \{h\} \cup \{v_1, \dots, v_{N-1}\},$$

define the *wheel r-uniform hypergraph* $W_N^{(r)} = (V_0, \mathcal{E}_N^{(r)})$ by

$$\mathcal{E}_N^{(r)} := \{R_i, S_i : 1 \leq i \leq N-1\},$$

where, using indices modulo $N-1$ (so $v_{N-1+j} := v_j$),

$$R_i := \{v_i, v_{i+1}, \dots, v_{i+r-1}\}, \qquad S_i := \{h, v_i, v_{i+1}, \dots, v_{i+r-2}\}.$$

Thus every hyperedge has size r, and R_i are rim hyperedges while S_i are spoke hyperedges.

Definition 4.36.3 (Singleton lifting)**.** Fix V_0 and an integer $t \geq 0$. For each $x \in V_0$ define recursively

$$\widehat{x}^{(0)} := x, \qquad \widehat{x}^{(t+1)} := \{\widehat{x}^{(t)}\}.$$

Then $\widehat{x}^{(t)} \in \mathcal{P}^t(V_0)$ for every $t \geq 0$.

Definition 4.36.4 (Wheel (n,r)-superhypergraph)**.** Let $r \geq 2$, $N \geq r+1$, and $n \geq 1$. Let $W_N^{(r)} = (V_0, \mathcal{E}_N^{(r)})$ be the wheel r-uniform hypergraph.

Define the $(n-1)$-level lifted vertex family

$$\mathcal{V}_N^{(n-1)} := \{\widehat{x}^{(n-1)} : x \in V_0\} \subseteq \mathcal{P}^{n-1}(V_0).$$

For each hyperedge $e \in \mathcal{E}_N^{(r)}$ define its n-lift by

$$\widehat{e}^{(n)} := \{\widehat{x}^{(n-1)} : x \in e\} \subseteq \mathcal{V}_N^{(n-1)}.$$

Set

$$\mathcal{E}_{N,r}^{(n)} := \{\widehat{e}^{(n)} : e \in \mathcal{E}_N^{(r)}\}.$$

The *wheel (n,r)-superhypergraph* is

$$\mathbb{W}_{N,r}^{(n)} := (V_0, \mathcal{E}_{N,r}^{(n)}).$$

Example 4.36.5 (A concrete wheel (n, r)-superhypergraph)**.** Take

$$n = 2, \qquad r = 3, \qquad N = 5.$$

Let the base vertex set be

$$V_0 = \{h, 1, 2, 3, 4\},$$

where h is the hub and $1, 2, 3, 4$ are the rim vertices (with indices taken modulo 4).

Define the wheel 3-uniform hypergraph

$$W_5^{(3)} = (V_0, \mathcal{E}_5^{(3)})$$

by specifying its hyperedge family as the union of rim-triangles and spoke-triangles:

$$\mathcal{E}_5^{(3)} := \Big\{\{i, i+1, i+2\} : i \in \{1, 2, 3, 4\}\Big\} \ \cup \ \Big\{\{h, i, i+1\} : i \in \{1, 2, 3, 4\}\Big\},$$

where arithmetic on $\{1, 2, 3, 4\}$ is modulo 4 (so $4 + 1 = 1$, etc.). Concretely,

$$\mathcal{E}_5^{(3)} = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 1\}, \{4, 1, 2\}\} \cup \{\{h, 1, 2\}, \{h, 2, 3\}, \{h, 3, 4\}, \{h, 4, 1\}\}.$$

Since $n = 2$, the $(n - 1)$-level lifted vertex family is the collection of singletons:

$$\mathcal{V}_5^{(1)} = \big\{\widehat{x}^{(1)} : x \in V_0\big\} = \big\{\{h\}, \{1\}, \{2\}, \{3\}, \{4\}\big\} \subseteq \mathcal{P}(V_0).$$

For each $e \in \mathcal{E}_5^{(3)}$ its 2-lift is

$$\widehat{e}^{(2)} = \big\{\widehat{x}^{(1)} : x \in e\big\} = \big\{\{x\} : x \in e\big\} \subseteq \mathcal{V}_5^{(1)}.$$

Hence the lifted hyperedge family is

$$\mathcal{E}_{5,3}^{(2)} = \Big\{\widehat{e}^{(2)} : e \in \mathcal{E}_5^{(3)}\Big\},$$

for instance

$$\widehat{\{1, 2, 3\}}^{(2)} = \{\{1\}, \{2\}, \{3\}\}, \qquad \widehat{\{h, 3, 4\}}^{(2)} = \{\{h\}, \{3\}, \{4\}\}.$$

Therefore the wheel $(2, 3)$-superhypergraph on $N = 5$ vertices is

$$\mathbb{W}_{5,3}^{(2)} = \big(V_0, \mathcal{E}_{5,3}^{(2)}\big),$$

whose superhyperedges are exactly the 2-lifts of the eight 3-hyperedges listed above.

Theorem 4.36.6 (Well-definedness: $\mathbb{W}_{N,r}^{(n)}$ is an n-SuperHyperGraph)**.** *For $r \geq 2$, $N \geq r + 1$, and $n \geq 1$, the pair $\mathbb{W}_{N,r}^{(n)} = (V_0, \mathcal{E}_{N,r}^{(n)})$ is an n-superhypergraph on V_0.*

Proof. By construction, $\mathcal{V}_N^{(n-1)} \subseteq \mathcal{P}^{n-1}(V_0)$.

Take any $\widehat{e}^{(n)} \in \mathcal{E}_{N,r}^{(n)}$. Then $\widehat{e}^{(n)} = \{\widehat{x}^{(n-1)} : x \in e\}$ for some $e \in \mathcal{E}_N^{(r)}$. Since $e \neq \emptyset$, the set $\widehat{e}^{(n)}$ is nonempty. Also, by definition, every element of $\widehat{e}^{(n)}$ lies in $\mathcal{P}^{n-1}(V_0)$. Hence

$$\widehat{e}^{(n)} \in \mathcal{P}^*\big(\mathcal{P}^{n-1}(V_0)\big).$$

Therefore

$$\mathcal{E}_{N,r}^{(n)} \subseteq \mathcal{P}^*\big(\mathcal{P}^{n-1}(V_0)\big),$$

so $\mathbb{W}_{N,r}^{(n)}$ is an n-superhypergraph on V_0. □

4.37 Submodular SuperHypergraph

A submodular graph assigns each edge a submodular cut function, producing a global cut cost via weighted summation always consistently [678–680]. A submodular hypergraph equips every hyperedge with a submodular splitting penalty, so cut evaluations aggregate weighted within-hyperedge variations for subsets [681]. A submodular superhypergraph uses supervertices from iterated powersets and superedges with submodular cut functions, extending hypergraphs hierarchically across abstraction levels. The relevant definitions and related notions are presented below.

Definition 4.37.1 (Submodular set function)**.** [682–684] Let U be a finite set. A function $f : 2^U \to \mathbb{R}$ is *submodular* if for all $A, B \subseteq U$,

$$f(A) + f(B) \ \geq \ f(A \cap B) + f(A \cup B).$$

Definition 4.37.2 (Submodular cut function on a hyperedge)**.** Let h be a finite set. A function $\delta_h : 2^h \to [0,1]$ is called a *submodular cut function* if (i) δ_h is submodular (as a set function on 2^h), and (ii) $\delta_h(\emptyset) = \delta_h(h) = 0$.

Definition 4.37.3 (Submodular graph)**.** [678–680] A *submodular graph* is a tuple

$$G = (V, E, w, \{\delta_e\}_{e \in E})$$

such that

$$E \subseteq \binom{V}{2}, \qquad w : E \to \mathbb{R}_{>0},$$

and for every edge $e = \{u, v\} \in E$, the function $\delta_e : 2^e \to [0,1]$ is a submodular cut function.

For any $S \subseteq V$, the induced cut-cost is

$$\delta_G(S) \ := \ \sum_{e \in E} w(e)\, \delta_e(S \cap e).$$

(Example: the standard undirected edge-cut is $\delta_{\{u,v\}}(S) = |\mathbf{1}_S(u) - \mathbf{1}_S(v)|$.)

Example 4.37.4 (A concrete submodular graph)**.** Let

$$V = \{1, 2, 3\}, \qquad E = \big\{\{1, 2\}, \{2, 3\}\big\}.$$

Assign positive weights

$$w(\{1, 2\}) = 2, \qquad w(\{2, 3\}) = 1.$$

For each edge $e = \{u, v\}$ define the (normalized) cut function

$$\delta_e(T) := \begin{cases} 0, & T = \emptyset \text{ or } T = e, \\ 1, & T = \{u\} \text{ or } T = \{v\}, \end{cases} \qquad (T \subseteq e).$$

This δ_e is submodular on the ground set e (it is the usual edge-cut on $\{u, v\}$).

Hence, for any $S \subseteq V$,

$$\delta_G(S) = 2\, \delta_{\{1,2\}}(S \cap \{1, 2\}) + 1\, \delta_{\{2,3\}}(S \cap \{2, 3\}).$$

For example, if $S = \{1\}$ then

$$S \cap \{1, 2\} = \{1\} \Rightarrow \delta_{\{1,2\}} = 1, \qquad S \cap \{2, 3\} = \emptyset \Rightarrow \delta_{\{2,3\}} = 0,$$

so $\delta_G(\{1\}) = 2 \cdot 1 + 1 \cdot 0 = 2$.

Definition 4.37.5 (Submodular hypergraph)**.** A *submodular hypergraph* is a tuple

$$H = (V, E, w, \{\delta_h\}_{h \in E})$$

such that

$$E \subseteq 2^V \setminus \{\emptyset\}, \qquad w : E \to \mathbb{R}_{>0},$$

and for every hyperedge $h \in E$, the function $\delta_h : 2^h \to [0,1]$ is a submodular cut function.

For any $S \subseteq V$, define

$$\delta_H(S) \ := \ \sum_{h \in E} w(h)\, \delta_h(S \cap h).$$

(Example: the standard hypergraph cut function is $\delta_h^{\mathrm{cut}}(T) = \min\{1, |T|, |h \setminus T|\}$ for $T \subseteq h$.)

Example 4.37.6 (A concrete submodular hypergraph)**.** Let

$$V = \{a, b, c, d\}, \qquad E = \{h_1, h_2\}, \quad h_1 = \{a, b, c\},\ h_2 = \{b, c, d\}.$$

Assign weights

$$w(h_1) = 3, \qquad w(h_2) = 1.$$

Define for each hyperedge h the standard (normalized) hypergraph cut function

$$\delta_h(T) := \min\{1,\ |T|,\ |h \setminus T|\}, \qquad (T \subseteq h).$$

It is submodular on 2^h.

Thus, for any $S \subseteq V$,

$$\delta_H(S) = 3\,\delta_{h_1}(S \cap h_1) + 1\,\delta_{h_2}(S \cap h_2).$$

For example, take $S = \{a, d\}$. Then

$$S \cap h_1 = \{a\} \ \Rightarrow\ \delta_{h_1}(S \cap h_1) = \min\{1, 1, 2\} = 1,$$

$$S \cap h_2 = \{d\} \ \Rightarrow\ \delta_{h_2}(S \cap h_2) = \min\{1, 1, 2\} = 1,$$

so

$$\delta_H(\{a, d\}) = 3 \cdot 1 + 1 \cdot 1 = 4.$$

Definition 4.37.7 (Submodular n-SuperHyperGraph)**.** Fix a finite base set V_0 and an integer $n \geq 0$. A *(vertex-)level n-SuperHyperGraph* is a pair

$$\mathcal{H} = (\mathcal{V}, \mathcal{E})$$

where

$$\mathcal{V} \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\} \quad \text{(supervertices)}, \qquad \mathcal{E} \subseteq 2^{\mathcal{V}} \setminus \{\emptyset\} \quad \text{(superedges)}.$$

A *submodular n-SuperHyperGraph* is a tuple

$$\mathcal{H} = (\mathcal{V}, \mathcal{E}, w, \{\delta_e\}_{e \in \mathcal{E}})$$

with $w : \mathcal{E} \to \mathbb{R}_{>0}$ and, for each superedge $e \in \mathcal{E}$, a submodular cut function $\delta_e : 2^e \to [0, 1]$.

For any $\mathcal{S} \subseteq \mathcal{V}$, define the cut-cost

$$\delta_{\mathcal{H}}(\mathcal{S}) \ :=\ \sum_{e \in \mathcal{E}} w(e)\,\delta_e(\mathcal{S} \cap e).$$

If $n = 0$ and we identify each base vertex $v \in V_0$ with the singleton $\{v\} \in \mathcal{P}(V_0)$, then $\mathcal{V} \subseteq V_0$ and this definition reduces to a submodular hypergraph; furthermore, if all edges have size 2, it reduces to a submodular graph.

Example 4.37.8 (A concrete submodular n-SuperHyperGraph (take $n = 1$))**.** Let the base set be

$$V_0 = \{1, 2, 3\}, \qquad n = 1.$$

Define the supervertex family (nonempty subsets of V_0)

$$\mathcal{V} := \big\{\{1\}, \{2\}, \{3\}, \{1, 2\}\big\} \subseteq \mathcal{P}^1(V_0) \setminus \{\emptyset\}.$$

Define two superedges (each is a nonempty subset of $\mathcal{V}$):

$$e_1 := \big\{\{1\}, \{2\}, \{1, 2\}\big\}, \qquad e_2 := \big\{\{2\}, \{3\}\big\}, \qquad \mathcal{E} := \{e_1, e_2\}.$$

Assign weights

$$w(e_1) = 2, \qquad w(e_2) = 5.$$

Define submodular cut functions as follows.

(1) For e_2 (a 2-element ground set), use the standard edge-cut:

$$\delta_{e_2}(T) := \begin{cases} 0, & T = \emptyset \text{ or } T = e_2, \\ 1, & \text{otherwise,} \end{cases} \qquad (T \subseteq e_2).$$

(2) For e_1 (a 3-element ground set), use the normalized cardinality cut

$$\delta_{e_1}(T) := \min\{1,\ |T|,\ |e_1 \setminus T|\}, \qquad (T \subseteq e_1).$$

This is submodular on 2^{e_1}.

Therefore, for any $\mathcal{S} \subseteq \mathcal{V}$,

$$\delta_{\mathcal{H}}(\mathcal{S}) = 2\,\delta_{e_1}(\mathcal{S} \cap e_1) + 5\,\delta_{e_2}(\mathcal{S} \cap e_2).$$

Example evaluation: let

$$\mathcal{S} = \big\{\{1\}, \{3\}\big\} \subseteq \mathcal{V}.$$

Then

$$\mathcal{S} \cap e_1 = \big\{\{1\}\big\} \Rightarrow \delta_{e_1}(\mathcal{S} \cap e_1) = \min\{1, 1, 2\} = 1,$$

$$\mathcal{S} \cap e_2 = \big\{\{3\}\big\} \Rightarrow \delta_{e_2}(\mathcal{S} \cap e_2) = 1,$$

so

$$\delta_{\mathcal{H}}(\mathcal{S}) = 2 \cdot 1 + 5 \cdot 1 = 7.$$

4.38 Multipartite SuperHypergraph

A multipartite graph partitions vertices into k disjoint parts, and every edge connects two vertices from different parts only [685–688]. Multipartite graphs model multi-group interactions, forbid within-group edges, simplify constraints, and enable efficient colorings, matchings, and partition-based optimizations.

A multipartite hypergraph partitions vertices into k disjoint parts, and each hyperedge contains at most one vertex from each part [689–692]. A multipartite superhypergraph partitions n-supervertices into k disjoint classes, and each superhyperedge selects one supervertex from each class. The relevant definitions and related notions are presented below.

Definition 4.38.1 (k-partite graph; multipartite graph)**.** [693–695] Let $k \geq 2$. A (simple) graph is a pair $G = (V, E)$ where V is a set of vertices and $E \subseteq \binom{V}{2}$ is a set of (2-element) edges. We say that G is *k-partite* if there exist pairwise disjoint sets $V_1, \ldots, V_k$ such that

$$V = V_1 \,\dot\cup\, \cdots \,\dot\cup\, V_k$$

and for every edge $\{u, v\} \in E$ there exist indices $i \neq j$ with $u \in V_i$ and $v \in V_j$. If G is k-partite for some $k \geq 2$, then G is called *multipartite*.

Definition 4.38.2 (k-uniform multipartite hypergraph)**.** Let $k \geq 2$. A *k-uniform multipartite hypergraph* is a hypergraph $H = (V, \mathcal{E})$ for which there exist pairwise disjoint vertex classes $V_1, \ldots, V_k$ such that

$$V = V_1 \,\dot\cup\, \cdots \,\dot\cup\, V_k,$$

and the hyperedge family $\mathcal{E}$ can be represented as a set of k-tuples

$$\mathcal{E} \subseteq V_1 \times \cdots \times V_k,$$

where each hyperedge $e = (v_1, \ldots, v_k) \in \mathcal{E}$ corresponds to the k-element subset $\{v_1, \ldots, v_k\} \subseteq V$, equivalently satisfying $|\{v_1, \ldots, v_k\} \cap V_i| = 1$ for all $i = 1, \ldots, k$.

Example 4.38.3 (A 3-uniform multipartite hypergraph)**.** Let $k = 3$ and let the vertex classes be

$$V_1 := \{a_1, a_2\}, \qquad V_2 := \{b_1, b_2\}, \qquad V_3 := \{c_1, c_2\}, \qquad V := V_1 \dot{\cup} V_2 \dot{\cup} V_3.$$

Define the hyperedge family by the set of 3-tuples

$$\mathcal{E} \subseteq V_1 \times V_2 \times V_3, \qquad \mathcal{E} := \{(a_1, b_1, c_1),\ (a_2, b_1, c_2),\ (a_2, b_2, c_1)\}.$$

Equivalently, the three hyperedges (as 3-element subsets of V) are

$$\{a_1, b_1, c_1\}, \qquad \{a_2, b_1, c_2\}, \qquad \{a_2, b_2, c_1\}.$$

Each hyperedge meets every class V_i in exactly one vertex, so $H = (V, \mathcal{E})$ is 3-uniform multipartite.

Definition 4.38.4 (k-uniform multipartite n-superhypergraph)**.** Let $n \geq 1$ and $k \geq 2$. Let $V_{0,1}, \ldots, V_{0,k}$ be pairwise disjoint nonempty base sets, and define the k vertex classes by

$$V_i \ := \ \mathcal{P}^{n-1}(V_{0,i}) \qquad (i = 1, \ldots, k).$$

A *k-uniform multipartite n-superhypergraph* is a hypergraph $H = (V, \mathcal{E})$ such that

$$V \ = \ V_1 \dot{\cup} \cdots \dot{\cup} V_k, \qquad \mathcal{E} \subseteq V_1 \times \cdots \times V_k,$$

and each hyperedge $e = (v_1, \ldots, v_k) \in \mathcal{E}$ selects exactly one (super)vertex $v_i \in V_i$ from each class.

In particular, when $n = 1$ we have $V_i = \mathcal{P}^0(V_{0,i}) = V_{0,i}$, so the above definition reduces to the usual k-uniform multipartite hypergraph on the base vertex classes.

Example 4.38.5 (A 3-uniform multipartite n-superhypergraph (take $n = 2$))**.** Let $n = 2$ and $k = 3$. Take pairwise disjoint base sets

$$V_{0,1} := \{1, 2\}, \qquad V_{0,2} := \{3, 4\}, \qquad V_{0,3} := \{5, 6\}.$$

Then

$$V_i = \mathcal{P}^{n-1}(V_{0,i}) = \mathcal{P}(V_{0,i}) \qquad (i = 1, 2, 3),$$

so explicitly

$$V_1 = \big\{\{1\}, \{2\}, \{1, 2\}\big\}, \quad V_2 = \big\{\{3\}, \{4\}, \{3, 4\}\big\}, \quad V_3 = \big\{\{5\}, \{6\}, \{5, 6\}\big\},$$

and

$$V := V_1 \dot{\cup} V_2 \dot{\cup} V_3.$$

Define the hyperedge family by a set of triples

$$\mathcal{E} \subseteq V_1 \times V_2 \times V_3, \qquad \mathcal{E} := \Big\{(\{1, 2\}, \{3\}, \{5\}),\ (\{2\}, \{3, 4\}, \{6\})\Big\}.$$

Equivalently, the hyperedges (as 3-element subsets of V) are

$$\big\{\{1, 2\}, \{3\}, \{5\}\big\}, \qquad \big\{\{2\}, \{3, 4\}, \{6\}\big\}.$$

Each hyperedge chooses exactly one supervertex from each class V_1, V_2, V_3, hence $H = (V, \mathcal{E})$ is a 3-uniform multipartite 2-superhypergraph in the stated sense.

4.39 Annotated HyperGraph and SuperHyperGraph

Annotated HyperGraph is a hypergraph equipped with additional labels or metadata on vertices, hyperedges, or incidences to represent roles, attributes, or contextual information beyond connectivity (cf. [696, 697]). Annotated SuperHyperGraph is a higher-order hypergraph built via iterated powerset-based objects, where vertices and superhyperedges also carry annotations describing hierarchical roles, attributes, and context [698]. The relevant definitions and related notions are presented below.

Definition 4.39.1 (Annotated HyperGraph)**.** [696, 697] Let V be a finite set (vertices). Let E be a finite *hyperedge multiset* such that every $e \in E$ is a nonempty subset of V (parallel hyperedges are allowed). Let X be a finite set (role labels). An *Annotated HyperGraph* is a quadruple

$$H = (V, E, X, \varphi),$$

where the *role-labeling function* is

$$\varphi : \{(v, e) \in V \times E \mid v \in e\} \longrightarrow X.$$

For each incidence (v, e) with $v \in e$, the value $\varphi(v, e) \in X$ specifies the role of v inside the hyperedge e.

Example 4.39.2 (Annotated HyperGraph: Project-Team Roles)**.** Let

$$V = \{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dave}\}, \qquad X = \{\text{Manager}, \text{Developer}, \text{Tester}\}.$$

Let the hyperedge multiset be $E = \{P_1, P_2\}$ with

$$P_1 = \{\text{Alice}, \text{Bob}, \text{Carol}\}, \qquad P_2 = \{\text{Bob}, \text{Carol}, \text{Dave}\}.$$

Define φ on incidences by

$$\varphi(\text{Alice}, P_1) = \text{Manager}, \;\; \varphi(\text{Bob}, P_1) = \text{Developer}, \;\; \varphi(\text{Carol}, P_1) = \text{Tester},$$

$$\varphi(\text{Bob}, P_2) = \text{Manager}, \;\; \varphi(\text{Carol}, P_2) = \text{Developer}, \;\; \varphi(\text{Dave}, P_2) = \text{Tester}.$$

Then $H = (V, E, X, \varphi)$ is an annotated hypergraph in which each member has an explicit role in each project-hyperedge.

Definition 4.39.3 (Annotated n-SuperHyperGraph)**.** [698] Let S be a nonempty *base set* and let $n \geq 0$ be an integer. Define iterated powersets recursively by

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^{k+1}(S) = \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \geq 0).$$

An *Annotated n-SuperHyperGraph* is a quadruple

$$H = (V, E, X, \varphi),$$

where:

(1) $V \subseteq \mathcal{P}^n(S)$ is a finite set of *n-supervertices*;

(2) E is a finite *n-superedge multiset* with $E \subseteq \mathcal{P}(V)$, and each $e \in E$ is a nonempty subset of V (parallel superedges are allowed);

(3) X is a finite set of role labels;

(4) φ is the role-labeling function

$$\varphi : \{(v, e) \in V \times E \mid v \in e\} \longrightarrow X.$$

Thus $\varphi(v, e) = x$ means that the n-supervertex v plays role x in the n-superedge e.

Example 4.39.4 (Annotated 2-SuperHyperGraph: Cross-Department Projects)**.** Let the base set of employees be

$$S = \{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dave}, \text{Eve}, \text{Frank}\}.$$

Choose first-level teams (elements of $\mathcal{P}^1(S) = \mathcal{P}(S)$):

$$T_1 = \{\text{Alice}, \text{Bob}\}, \quad T_2 = \{\text{Carol}, \text{Dave}\}, \quad T_3 = \{\text{Eve}, \text{Frank}\}.$$

Define second-level departments (elements of $\mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S))$):

$$D_1 = \{T_1, T_2\}, \qquad D_2 = \{T_2, T_3\}.$$

Let the set of 2-supervertices be $V = \{D_1, D_2\} \subseteq \mathcal{P}^2(S)$ and let

$$X = \{\text{Coordinator}, \text{Contributor}\}.$$

Define two cross-department projects (superedges) by

$$e_A = \{D_1, D_2\}, \qquad e_B = \{D_2\}, \qquad E = \{e_A, e_B\} \subseteq \mathcal{P}(V).$$

Define the role-labeling function φ on incidences by

$$\varphi(D_1, e_A) = \text{Coordinator}, \quad \varphi(D_2, e_A) = \text{Contributor}, \quad \varphi(D_2, e_B) = \text{Coordinator}.$$

Then $H = (V, E, X, \varphi)$ is an annotated 2-superhypergraph, where each department plays an explicit role in each cross-department project.

4.40 Chordal n-SuperHyperGraph

A chordal graph is a graph in which every cycle of length at least four contains a chord, equivalently, it has no induced cycles of length at least four [699–701]. In addition, for chordal graphs, related concepts such as fuzzy chordal graphs [702–704] , strongly chordal graphs [705, 706], proper chordal graphs [707], co-chordal graphs [708,709], dually chordal graphs [710,711], locally chordal graphs [712], and directed chordal graphs [713] are also known. Chordal graphs admit perfect elimination orderings and clique trees, enabling efficient algorithms for coloring, maximum clique, and many NP-hard problems on general graphs.

A chordal hypergraph is a hypergraph whose two-section (primal) graph is chordal, meaning that whenever two vertices lie in some common hyperedge, they are adjacent in a chordal graph [714, 715]. A chordal superhypergraph is an n-SuperHyperGraph whose two-section graph on supervertices is chordal, so the induced-cycle obstruction is excluded at the supervertex level. The relevant definitions and related notions are presented below.

Definition 4.40.1 (Chordal graph)**.** [699–701] Let $G = (V, E)$ be a finite simple undirected graph. A *chord* of a cycle $C = v_1 v_2 \cdots v_k v_1$ $(k \geq 4)$ is an edge $\{v_i, v_j\} \in E$ with $|i - j| \not\equiv 1 \pmod{k}$. The graph G is *chordal* if every cycle of length at least 4 has a chord. Equivalently, G has no induced cycle of length ≥ 4.

Definition 4.40.2 (Two-section (shadow) graph of a hypergraph)**.** Let $H = (V, \mathcal{E})$ be a finite hypergraph, i.e. $\varnothing \neq V$ and $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$. The *two-section graph* (or *shadow graph*) of H is the graph

$$\Gamma(H) = (V, E_2), \qquad \{u, v\} \in E_2 \iff u \neq v \text{ and } \exists\, e \in \mathcal{E} \text{ with } \{u, v\} \subseteq e.$$

Definition 4.40.3 (Chordal hypergraph)**.** A finite hypergraph H is *chordal* if its two-section graph $\Gamma(H)$ is chordal in the sense of Definition 4.40.1.

Example 4.40.4 (A chordal hypergraph)**.** Let

$$V := \{1, 2, 3, 4\} \quad \text{and} \quad \mathcal{E} := \big\{\{1, 2, 3\}, \{1, 3, 4\}\big\}.$$

Define the hypergraph $H := (V, \mathcal{E})$.

Its two-section graph $\Gamma(H)$ has vertex set V and edges joining any two vertices that appear together in a hyperedge. Hence

$$E\big(\Gamma(H)\big) = \big\{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\big\}.$$

Equivalently, $\Gamma(H)$ is obtained from the 4-cycle $(2, 1, 4, 3, 2)$ by adding the chord $\{1, 3\}$. Therefore every cycle of length at least 4 in $\Gamma(H)$ has a chord, so $\Gamma(H)$ is chordal. By Definition 4.40.3, the hypergraph H is chordal.

Definition 4.40.5 (Two-section graph of an n-SuperHyperGraph)**.** Let $n \in \mathbb{N}_0$ and let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be a finite n-SuperHyperGraph (with incidence map $\partial : E \to \mathcal{P}^*(V)$). Its *two-section graph* is

$$\Gamma\big(\mathrm{SHG}^{(n)}\big) = (V, E_2), \qquad \{U, W\} \in E_2 \iff U \neq W \text{ and } \exists\, e \in E \text{ with } \{U, W\} \subseteq \partial(e).$$

Example 4.40.6 (Two-section graph of a 1-SuperHyperGraph)**.** Let the base set be $V_0 := \{1, 2, 3, 4\}$ and define the 1-level supervertex set

$$V := \big\{\{1\}, \{2\}, \{3\}, \{4\}\big\} \subseteq \mathcal{P}(V_0) \setminus \{\emptyset\}.$$

Let the edge-identifier set be $E := \{a, b\}$ and define the incidence map $\partial : E \to \mathcal{P}^*(V)$ by

$$\partial(a) := \big\{\{1\}, \{2\}, \{3\}\big\}, \qquad \partial(b) := \big\{\{1\}, \{3\}, \{4\}\big\}.$$

Then $\mathrm{SHG}^{(1)} := (V, E, \partial)$ is a finite 1-SuperHyperGraph.

By Definition 4.40.5, its two-section graph $\Gamma(\mathrm{SHG}^{(1)}) = (V, E_2)$ satisfies

$$\{U, W\} \in E_2 \iff U \neq W \text{ and } \exists e \in \{a, b\} \text{ with } \{U, W\} \subseteq \partial(e).$$

Thus the edges are exactly the pairs of supervertices that co-occur in some incidence set:

$$E_2 = \big\{\{\{1\}, \{2\}\}, \{\{2\}, \{3\}\}, \{\{1\}, \{3\}\}, \{\{1\}, \{4\}\}, \{\{3\}, \{4\}\}\big\}.$$

So $\Gamma(\mathrm{SHG}^{(1)})$ is the graph on $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ whose adjacency is induced by the two incidence triples $\partial(a)$ and $\partial(b)$.

Definition 4.40.7 (Chordal n-SuperHyperGraph)**.** An n-SuperHyperGraph $\mathrm{SHG}^{(n)}$ is *chordal* if $\Gamma\big(\mathrm{SHG}^{(n)}\big)$ is chordal as a graph (Definitions 4.40.5 and 4.40.1).

Example 4.40.8 (A chordal 1-SuperHyperGraph)**.** Consider the same 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E, \partial)$ as in Example 4.40.6. We claim that $\mathrm{SHG}^{(1)}$ is chordal in the sense of Definition 4.40.7.

Indeed, Example 4.40.6 computed $\Gamma(\mathrm{SHG}^{(1)})$ explicitly and showed that it is the graph on four vertices obtained from the 4-cycle $(\{2\}, \{1\}, \{4\}, \{3\}, \{2\})$ by adding the chord $\{\{1\}, \{3\}\}$. Hence $\Gamma(\mathrm{SHG}^{(1)})$ is chordal as a graph. Therefore, by Definition 4.40.7, $\mathrm{SHG}^{(1)}$ is a chordal 1-SuperHyperGraph.

Theorem 4.40.9 (Chordal n-SuperHyperGraphs generalize chordal hypergraphs)**.** *Let $H = (V, \mathcal{E})$ be a finite hypergraph. Define the associated* 0*-SuperHyperGraph*

$$\mathrm{SHG}^{(0)}(H) := (V, E, \partial)$$

by taking $E := \mathcal{E}$ and $\partial(e) := e$ for all $e \in E$. Then

$$H \text{ is chordal (Definition 4.40.3)} \iff \mathrm{SHG}^{(0)}(H) \text{ is chordal (Definition 4.40.7).}$$

Consequently, the notion of chordal n-SuperHyperGraph extends (contains) the notion of chordal hypergraph as the special case $n = 0$.

Proof. Let $H = (V, \mathcal{E})$ be given and form $\mathrm{SHG}^{(0)}(H) = (V, E, \partial)$ with $E = \mathcal{E}$ and $\partial(e) = e$ for all $e \in E$.

By Definition 4.40.2,

$$\{u, v\} \in E\big(\Gamma(H)\big) \iff u \neq v \text{ and } \exists\, e \in \mathcal{E} \text{ with } \{u, v\} \subseteq e.$$

Since $E = \mathcal{E}$ and $\partial(e) = e$, Definition 4.40.5 gives

$$\{u, v\} \in E\Big(\Gamma\big(\mathrm{SHG}^{(0)}(H)\big)\Big) \iff u \neq v \text{ and } \exists\, e \in E \text{ with } \{u, v\} \subseteq \partial(e) \iff u \neq v \text{ and } \exists\, e \in \mathcal{E} \text{ with } \{u, v\} \subseteq e.$$

Hence the edge sets coincide:

$$\Gamma(H) = \Gamma\big(\mathrm{SHG}^{(0)}(H)\big).$$

Therefore $\Gamma(H)$ is chordal if and only if $\Gamma(\mathrm{SHG}^{(0)}(H))$ is chordal, i.e., H is chordal (Definition 4.40.3) if and only if $\mathrm{SHG}^{(0)}(H)$ is chordal (Definition 4.40.7). □

4.41 Kneser SuperHypergraph

A Kneser graph connects two k-subsets of $[n]$ by an edge exactly when the subsets are disjoint [132, 716, 717]. Related concepts such as the Bipartite Kneser graph [718, 719] and the generalized Kneser graph [720, 721] are also known.

A Kneser hypergraph uses k-subsets of $[n]$ as vertices, and forms r-uniform hyperedges from r pairwise disjoint vertices [722, 723]. A Kneser superhypergraph uses k-subsets as vertices, takes blocks of $r+1$ pairwise disjoint vertices, and flattens blocks into r-hyperedges. The relevant definitions and related notions are presented below.

Notation 4.41.1. *For $n \in \mathbb{N}$ write $[n] := \{1, 2, \ldots, n\}$. For a finite set X and integer $k \geq 0$, write*

$$\binom{X}{k} := \{A \subseteq X : |A| = k\}.$$

A family $\mathcal{F}$ of sets is pairwise disjoint *if*

$$A \cap B = \varnothing \quad \textit{for all distinct } A, B \in \mathcal{F}.$$

Definition 4.41.2 (Kneser graph $\mathrm{KG}(n, k)$)**.** [132, 716, 717] Let n, k be integers with $n \geq 2k$. The *Kneser graph* $\mathrm{KG}(n, k)$ is the simple graph with

$$V\big(\mathrm{KG}(n,k)\big) := \binom{[n]}{k}, \qquad E\big(\mathrm{KG}(n,k)\big) := \Big\{\{A, B\} : A, B \in \binom{[n]}{k},\ A \cap B = \varnothing\Big\}.$$

Equivalently, two k-subsets are adjacent iff they are disjoint.

Definition 4.41.3 (Kneser r-uniform hypergraph $\mathrm{KG}^{(r)}(n, k)$)**.** Fix integers $r \geq 2$ and n, k with $n \geq rk$. The *Kneser r-uniform hypergraph* (also called the *Kneser r-graph*) $\mathrm{KG}^{(r)}(n, k)$ is the r-uniform hypergraph with vertex set

$$V\big(\mathrm{KG}^{(r)}(n,k)\big) := \binom{[n]}{k},$$

and hyperedge set

$$E\big(\mathrm{KG}^{(r)}(n,k)\big) := \Big\{e \in \binom{\binom{[n]}{k}}{r} \ :\ e \text{ is a pairwise disjoint family of } k\text{-subsets}\Big\}.$$

Thus $e = \{A_1, \ldots, A_r\}$ is a hyperedge iff $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Remark 4.41.4. For $r = 2$, the hypergraph $\mathrm{KG}^{(2)}(n, k)$ is exactly the graph $\mathrm{KG}(n, k)$.

Definition 4.41.5 (Uniform $(r+1)$-block n-superhypergraph and flattening)**.** Fix integers $r \geq 2$ and $n \geq 1$. A *uniform $(r+1)$-block n-superhypergraph* is a pair

$$\mathbb{S} = (V, Q^{(n)})$$

such that, with the $(n-1)$-level lifted vertex family

$$\mathcal{V}^{(n-1)} := \{\widehat{x}^{(n-1)} : x \in V\} \subseteq \mathcal{P}^{n-1}(V),$$

the block family satisfies

$$Q^{(n)} \subseteq \binom{\mathcal{V}^{(n-1)}}{r+1}.$$

Its *flattening* is the r-uniform hypergraph on the lifted vertex set

$$\mathrm{Flat}(\mathbb{S}) := (\mathcal{V}^{(n-1)}, E^{(n)}_{\mathrm{Flat}}), \qquad E^{(n)}_{\mathrm{Flat}} := \bigcup_{q \in Q^{(n)}} \binom{q}{r}.$$

Definition 4.41.6 (Kneser $(r+1)$-block n-superhypergraph)**.** Fix integers $r \geq 2$, $n \geq 1$, $N \geq 1$, and $k \geq 1$ with

$$N \geq (r+1)k$$

(so that $(r+1)$ pairwise disjoint k-subsets of $[N]$ exist). Let the base set be

$$V := V_{N,k} = \binom{[N]}{k}.$$

Define the lifted vertex family at level $(n-1)$ by

$$\mathcal{V}_{N,k}^{(n-1)} := \{\widehat{A}^{(n-1)} : A \in \binom{[N]}{k}\} \subseteq \mathcal{P}^{n-1}(V).$$

For every pairwise disjoint family

$$q_0 = \{A_0, A_1, \ldots, A_r\} \in \binom{V}{r+1},$$

define its n-level block (a subset of $\mathcal{P}^{n-1}(V)$) by

$$\widehat{q_0}^{(n)} := \{\widehat{A_0}^{(n-1)}, \widehat{A_1}^{(n-1)}, \ldots, \widehat{A_r}^{(n-1)}\} \in \binom{\mathcal{V}_{N,k}^{(n-1)}}{r+1}.$$

Define the block family

$$\mathcal{Q}_{N,k,r}^{(n)} := \Big\{ \widehat{q_0}^{(n)} \;\Big|\; q_0 \in \binom{V}{r+1} \text{ and } q_0 \text{ is pairwise disjoint} \Big\}.$$

The *Kneser* $(r+1)$*-block n-superhypergraph* is

$$\mathbb{KG}_{N,k,r}^{(n)} := \big(V, \mathcal{Q}_{N,k,r}^{(n)}\big).$$

Theorem 4.41.7 (Well-definedness as an n-superhypergraph)**.** *For all parameters as in Definition 4.41.6, the structure $\mathbb{KG}_{N,k,r}^{(n)}$ is a well-defined uniform $(r+1)$-block n-superhypergraph in the sense of Definition 4.41.5.*

Proof. Let $V = \binom{[N]}{k}$.

Step 1 (lifted vertices lie in $\mathcal{P}^{n-1}(V)$). For each $A \in V$ we have $\widehat{A}^{(n-1)} \in \mathcal{P}^{n-1}(V)$. Hence

$$\mathcal{V}_{N,k}^{(n-1)} = \{\widehat{A}^{(n-1)} : A \in V\} \subseteq \mathcal{P}^{n-1}(V).$$

Step 2 (each block has size $r+1$ and lies in $\binom{\mathcal{V}_{N,k}^{(n-1)}}{r+1}$). Take any pairwise disjoint family $q_0 = \{A_0, \ldots, A_r\} \in \binom{V}{r+1}$. Then $\widehat{A_i}^{(n-1)} \in \mathcal{V}_{N,k}^{(n-1)}$ for each i, so

$$\widehat{q_0}^{(n)} = \{\widehat{A_0}^{(n-1)}, \ldots, \widehat{A_r}^{(n-1)}\} \subseteq \mathcal{V}_{N,k}^{(n-1)}.$$

Also $|q_0| = r+1$ implies $|\widehat{q_0}^{(n)}| = r+1$, hence

$$\widehat{q_0}^{(n)} \in \binom{\mathcal{V}_{N,k}^{(n-1)}}{r+1}.$$

By Step 2, every element of $\mathcal{Q}_{N,k,r}^{(n)}$ is an $(r+1)$-subset of $\mathcal{V}_{N,k}^{(n-1)}$. Therefore

$$\mathcal{Q}_{N,k,r}^{(n)} \subseteq \binom{\mathcal{V}_{N,k}^{(n-1)}}{r+1},$$

so $\mathbb{KG}_{N,k,r}^{(n)} = (V, \mathcal{Q}_{N,k,r}^{(n)})$ is a uniform $(r+1)$-block n-superhypergraph as required. □

Theorem 4.41.8 (Flattening and projection recover the Kneser hypergraph)**.** *Let $\mathbb{KG}^{(n)}_{N,k,r}$ be as in Definition 4.41.6, and form its flattening*

$$\mathrm{Flat}\big(\mathbb{KG}^{(n)}_{N,k,r}\big) = \big(\mathcal{V}^{(n-1)}_{N,k}, E^{(n)}_{\mathrm{Flat}}\big).$$

Define the projection map

$$\mathrm{Proj} : \mathcal{V}^{(n-1)}_{N,k} \to V_{N,k}, \qquad \mathrm{Proj}\big(\widehat{A}^{(n-1)}\big) := A.$$

Let the projected r-uniform hypergraph be

$$\mathrm{Proj}_*\Big(\mathrm{Flat}\big(\mathbb{KG}^{(n)}_{N,k,r}\big)\Big) := \big(V_{N,k},\ \{\mathrm{Proj}[e] : e \in E^{(n)}_{\mathrm{Flat}}\}\big), \qquad \mathrm{Proj}[e] := \{\mathrm{Proj}(U) : U \in e\}.$$

Then

$$\mathrm{Proj}_*\Big(\mathrm{Flat}\big(\mathbb{KG}^{(n)}_{N,k,r}\big)\Big) = \mathrm{KG}^{(r)}(N,k).$$

In particular, $\mathbb{KG}^{(n)}_{N,k,r}$ determines the classical Kneser r-uniform hypergraph via flattening and projection.

Proof. Let $V := V_{N,k} = \binom{[N]}{k}$ and $\mathcal{V}^{(n-1)} := \mathcal{V}^{(n-1)}_{N,k}$.

Write

$$E^{(r)} := E\big(\mathrm{KG}^{(r)}(N,k)\big) = \Big\{e \in \binom{V}{r} : \text{the } k\text{-sets in } e \text{ are pairwise disjoint}\Big\},$$

and

$$E^{(n)}_{\mathrm{Flat}} = \bigcup_{q \in Q^{(n)}_{N,k,r}} \binom{q}{r}.$$

We prove the two inclusions

$$\{\mathrm{Proj}[e] : e \in E^{(n)}_{\mathrm{Flat}}\} \subseteq E^{(r)} \quad \text{and} \quad E^{(r)} \subseteq \{\mathrm{Proj}[e] : e \in E^{(n)}_{\mathrm{Flat}}\}.$$

Step 1 (first inclusion). Take any $e \in E^{(n)}_{\mathrm{Flat}}$. Then $e \in \binom{q}{r}$ for some block $q \in Q^{(n)}_{N,k,r}$. By definition of $Q^{(n)}_{N,k,r}$, there exists a pairwise disjoint family

$$q_0 = \{A_0, \ldots, A_r\} \in \binom{V}{r+1}$$

such that

$$q = \widehat{q_0}^{(n)} = \{\widehat{A_0}^{(n-1)}, \ldots, \widehat{A_r}^{(n-1)}\}.$$

Since $e \subseteq q$, we may write

$$e = \{\widehat{A_{i_1}}^{(n-1)}, \ldots, \widehat{A_{i_r}}^{(n-1)}\}$$

for distinct indices $i_1, \ldots, i_r$. Applying $\mathrm{Proj}(\widehat{A}^{(n-1)}) = A$ yields

$$\mathrm{Proj}[e] = \{A_{i_1}, \ldots, A_{i_r}\}.$$

Because the A_j are pairwise disjoint, the subfamily $\{A_{i_1}, \ldots, A_{i_r}\}$ is also pairwise disjoint. Hence $\mathrm{Proj}[e] \in E^{(r)}$. This proves

$$\{\mathrm{Proj}[e] : e \in E^{(n)}_{\mathrm{Flat}}\} \subseteq E^{(r)}.$$

Step 2 (second inclusion). Let $f = \{A_1, \ldots, A_r\} \in E^{(r)}$. Then $A_1, \ldots, A_r$ are pairwise disjoint k-subsets of $[N]$. Therefore

$$\Big|\bigcup_{i=1}^{r} A_i\Big| = \sum_{i=1}^{r} |A_i| = rk.$$

Since $N \geq (r+1)k$, we have

$$|[N] \setminus \textstyle\bigcup_{i=1}^{r} A_i| = N - rk \geq k,$$

so we can choose a k-subset

$$B \in \binom{[N] \setminus \bigcup_{i=1}^{r} A_i}{k}.$$

Then B is disjoint from each A_i, so

$$q_0 := \{A_1, \dots, A_r, B\} \in \binom{V}{r+1}$$

is a pairwise disjoint family. Hence its lift

$$q := \widehat{q_0}^{(n)} = \{\widehat{A_1}^{(n-1)}, \dots, \widehat{A_r}^{(n-1)}, \widehat{B}^{(n-1)}\}$$

lies in $Q^{(n)}_{N,k,r}$. Now set

$$e := \{\widehat{A_1}^{(n-1)}, \dots, \widehat{A_r}^{(n-1)}\} \in \binom{q}{r} \subseteq E^{(n)}_{\mathrm{Flat}}.$$

Applying Proj gives

$$\mathrm{Proj}[e] = \{A_1, \dots, A_r\} = f.$$

Thus $f \in \{\mathrm{Proj}[e] : e \in E^{(n)}_{\mathrm{Flat}}\}$, proving

$$E^{(r)} \subseteq \{\mathrm{Proj}[e] : e \in E^{(n)}_{\mathrm{Flat}}\}.$$

Steps 1–2 give equality of the r-edge families after projection, and the vertex set is $V_{N,k}$ on both sides. Therefore

$$\mathrm{Proj}_*\Big(\mathrm{Flat}\big(\mathbb{KG}^{(n)}_{N,k,r}\big)\Big) = \mathrm{KG}^{(r)}(N, k).$$

□

4.42 Turán SuperHyperGraph

A Turán graph $T(n, k)$ is the complete balanced k-partite graph on n vertices, maximizing edges among all K_{k+1}-free graphs [724–726]. A Turán r-hypergraph $T_r(n, k)$ is the complete k-partite r-uniform hypergraph on n vertices, maximizing hyperedges under forbidding $K^{(r)}_{k+1}$. A Turán r-superhypergraph is a block system whose r-flattening equals the Turán r-hypergraph, encoding hierarchical constraints via superhyperedges. The relevant definitions and related notions are presented below.

Definition 4.42.1 (Balanced k-partition)**.** Let V be a finite set with $|V| = n$ and let $k \geq 2$. A family $\mathcal{V} = \{V_1, \dots, V_k\}$ is a *balanced k-partition* of V if $V = \bigsqcup_{i=1}^{k} V_i$ and $||V_i| - |V_j|| \leq 1$ for all i, j.

Definition 4.42.2 (Turán graph)**.** [724–726] Let $n \geq 1$ and $k \geq 2$. Fix a balanced k-partition $\mathcal{V} = \{V_1, \dots, V_k\}$ of a vertex set V with $|V| = n$. The *Turán graph* $T_2(n, k)$ is the complete k-partite simple graph on V:

$$E\big(T_2(n, k)\big) := \big\{\{u, v\} \subseteq V : u \in V_i,\ v \in V_j,\ i \neq j\big\}.$$

Definition 4.42.3 (Turán r-hypergraph)**.** Let $n \geq 1$, $k \geq 2$, and $r \geq 2$. Fix a balanced k-partition $\mathcal{V} = \{V_1, \dots, V_k\}$ of V with $|V| = n$. The *Turán r-hypergraph* $T_r(n, k)$ is the r-uniform hypergraph (V, E) where

$$E\big(T_r(n, k)\big) := \Big\{e \in \binom{V}{r} : |e \cap V_i| \leq 1 \text{ for every } i \in \{1, \dots, k\}\Big\}.$$

(Equivalently, the hyperedges are exactly the r-sets that choose vertices from r distinct parts.)

Definition 4.42.4 (Balanced k-partition)**.** Let V_0 be a finite set with $|V_0| = N \geq 1$ and let $k \geq 2$. A family $\mathcal{V} = \{U_1, \dots, U_k\}$ is called a *balanced k-partition* of V_0 if

$$V_0 = \bigsqcup_{i=1}^{k} U_i \quad \text{and} \quad \big||U_i| - |U_j|\big| \leq 1 \ \text{ for all } i, j.$$

Definition 4.42.5 (Iterated singleton embedding)**.** Let $n \in \mathbb{N}_0$ and let V_0 be a set. Define $\iota_n : V_0 \to \mathcal{P}^n(V_0)$ recursively by

$$\iota_0(x) := x, \qquad \iota_{t+1}(x) := \{\iota_t(x)\} \ \ (t \geq 0).$$

For a subset $S \subseteq V_0$ we write

$$\iota_n(S) := \{\iota_n(x) \mid x \in S\} \subseteq \mathcal{P}^n(V_0).$$

Definition 4.42.6 (Turán r-superhypergraph as an n-SuperHyperGraph)**.** Let $N \geq 1$, $k \geq 2$, $r \geq 2$, and $n \in \mathbb{N}_0$. Fix a balanced k-partition $\mathcal{V} = \{U_1, \dots, U_k\}$ of a base set V_0 with $|V_0| = N$.

Define the n-supervertex set

$$V^{(n)} := \iota_n(V_0) \subseteq \mathcal{P}^n(V_0), \qquad V_i^{(n)} := \iota_n(U_i) \ \ (1 \leq i \leq k).$$

Define the block family Q by

$$Q := \begin{cases} \Big\{\iota_n(S) \in \binom{V^{(n)}}{r} \ : \ S \in \binom{V_0}{r},\ |S \cap U_i| \leq 1 \text{ for all } i\Big\}, & \text{if } k = r, \\ \Big\{\iota_n(S) \in \binom{V^{(n)}}{r+1} \ : \ S \in \binom{V_0}{r+1},\ |S \cap U_i| \leq 1 \text{ for all } i\Big\}, & \text{if } k \geq r+1, \\ \emptyset, & \text{if } k < r. \end{cases}$$

Now define the *Turán r-superhypergraph of level n* by the triple

$$\mathcal{T}_r^{(n)}(N, k, r) \ := \ \big(V^{(n)},\, E^{(n)},\, \partial\big),$$

where $E^{(n)} := Q \subseteq \mathcal{P}(V^{(n)})$ and the incidence map is

$$\partial : E^{(n)} \to \mathcal{P}^*(V^{(n)}), \qquad \partial(q) := q \ \text{ for all } q \in E^{(n)}.$$

Theorem 4.42.7 (Well-definedness: $\mathcal{T}_r^{(n)}(N, k, r)$ is an n-SuperHyperGraph)**.** *The structure $\mathcal{T}_r^{(n)}(N, k, r)$ in Definition 4.42.6 is an n-SuperHyperGraph over the base set V_0.*

Proof. We verify the three required conditions.

Step 1: $V^{(n)} \subseteq \mathcal{P}^n(V_0)$. By definition, each element of $V^{(n)}$ has the form $\iota_n(x)$ for some $x \in V_0$. The map ι_n was defined so that $\iota_n(x) \in \mathcal{P}^n(V_0)$ for every $x \in V_0$. Hence

$$V^{(n)} = \iota_n(V_0) \subseteq \mathcal{P}^n(V_0).$$

Step 2: $E^{(n)} \subseteq \mathcal{P}(V^{(n)})$. By construction, every block in Q is of the form $\iota_n(S)$ for some $S \subseteq V_0$, so $\iota_n(S) \subseteq \iota_n(V_0) = V^{(n)}$. Therefore each $q \in Q$ is a subset of $V^{(n)}$, and hence

$$E^{(n)} = Q \subseteq \mathcal{P}(V^{(n)}).$$

Step 3: $\partial : E^{(n)} \to \mathcal{P}^*(V^{(n)})$ is well-defined. If $q \in E^{(n)}$, then $q \neq \emptyset$ because $q = \iota_n(S)$ with $|S| = r$ or $|S| = r + 1$. Also $q \subseteq V^{(n)}$ by Step 2. Thus $q \in \mathcal{P}^*(V^{(n)})$, and since $\partial(q) = q$ we obtain

$$\partial(q) \in \mathcal{P}^*(V^{(n)}) \quad \text{for all } q \in E^{(n)}.$$

This proves that $\mathcal{T}_r^{(n)}(N, k, r) = (V^{(n)}, E^{(n)}, \partial)$ satisfies the definition of an n-SuperHyperGraph over V_0. □

Definition 4.42.8 (Flattening to an r-uniform hypergraph)**.** Let $\mathcal{T}_r^{(n)}(N,k,r) = (V^{(n)}, E^{(n)}, \partial)$ be as in Definition 4.42.6. Define the projection $\pi : V^{(n)} \to V_0$ by

$$\pi(\iota_n(x)) := x \quad (x \in V_0),$$

and for $X \subseteq V^{(n)}$ define $\pi(X) := \{\pi(v) \mid v \in X\} \subseteq V_0$.

The r*-flattening* of $\mathcal{T}_r^{(n)}(N,k,r)$ is the r-uniform hypergraph

$$\mathrm{Flat}\big(\mathcal{T}_r^{(n)}(N,k,r)\big) := (V_0, E_{\mathrm{Flat}}), \qquad E_{\mathrm{Flat}} := \bigcup_{q \in E^{(n)}} \binom{\pi(\partial(q))}{r}.$$

Theorem 4.42.9 (Turán superhypergraphs generalize Turán hypergraphs via flattening)**.** *Let $N \geq 1$, $k \geq 2$, $r \geq 2$, and $n \in \mathbb{N}_0$. With the same balanced k-partition $\mathcal{V}$ of V_0,*

$$\mathrm{Flat}\big(\mathcal{T}_r^{(n)}(N,k,r)\big) = T_r(N,k,r).$$

Proof. Let $\mathcal{V} = \{U_1, \dots, U_k\}$ be fixed. Write E_T for the edge set and E_{Flat} for the edge set in Definition 4.42.8.

Case 1: $k < r$. Then no r-subset can meet each part in at most one vertex because there are fewer than r parts, so $E_T = \varnothing$. Also Definition 4.42.6 gives $E^{(n)} = \varnothing$, hence $E_{\mathrm{Flat}} = \varnothing$. Thus $E_{\mathrm{Flat}} = E_T$.

Case 2: $k = r$. Here each block $q \in E^{(n)}$ has size r and equals $\iota_n(S)$ for a unique $S \in \binom{V_0}{r}$ satisfying $|S \cap U_i| \leq 1$ for all i. Moreover $\pi(q) = S$, and $\partial(q) = q$, hence

$$\binom{\pi(\partial(q))}{r} = \binom{\pi(q)}{r} = \binom{S}{r} = \{S\}.$$

Therefore

$$E_{\mathrm{Flat}} = \bigcup_{q \in E^{(n)}} \binom{\pi(\partial(q))}{r} = \bigcup_{S \in E_T} \{S\} = E_T.$$

Case 3: $k \geq r+1$. In this case each block $q \in E^{(n)}$ has size $r+1$ and equals $\iota_n(S)$ for some $S \in \binom{V_0}{r+1}$ with $|S \cap U_i| \leq 1$ for all i. We prove the two inclusions $E_{\mathrm{Flat}} \subseteq E_T$ and $E_T \subseteq E_{\mathrm{Flat}}$.

$\underline{E_{\mathrm{Flat}} \subseteq E_T}$. Take any $e \in E_{\mathrm{Flat}}$. Then $e \in \binom{\pi(\partial(q))}{r}$ for some $q \in E^{(n)}$. Since $\partial(q) = q$ and $q = \iota_n(S)$ for some S meeting each U_i in at most one vertex, we have $\pi(q) = S$ and hence $e \subseteq S$ with $|e| = r$. Because S meets each part U_i in at most one vertex, every subset $e \subseteq S$ also satisfies

$$|e \cap U_i| \leq 1 \quad \text{for all } i,$$

so $e \in E_T$. This proves $E_{\mathrm{Flat}} \subseteq E_T$.

$\underline{E_T \subseteq E_{\mathrm{Flat}}}$. Take any $e \in E_T$. Then $e \in \binom{V_0}{r}$ and $|e \cap U_i| \leq 1$ for all i, so e uses vertices from exactly r distinct parts. Since $k \geq r+1$, there exists an index $j \in \{1, \dots, k\}$ such that $e \cap U_j = \varnothing$. Choose any vertex $x \in U_j$ and set $S := e \cup \{x\}$. Then $|S| = r+1$ and still $|S \cap U_i| \leq 1$ for all i, hence $q := \iota_n(S) \in E^{(n)}$. Moreover $\pi(\partial(q)) = \pi(q) = S$, so

$$e \in \binom{S}{r} = \binom{\pi(\partial(q))}{r} \subseteq E_{\mathrm{Flat}}.$$

Thus $E_T \subseteq E_{\mathrm{Flat}}$.

Combining the two inclusions yields $E_{\mathrm{Flat}} = E_T$, i.e. $\mathrm{Flat}(\mathcal{T}_r^{(n)}(N,k,r)) = T_r(N,k,r)$. □

4.43 Book SuperHyperGraph

Book graph is a graph of p triangles sharing one common edge, with p additional vertices each adjacent to both endpoints [727, 728]. Book hypergraph is a 3-uniform hypergraph whose hyperedges are $\{s_0, s_1, w_i\}$, so all pages share the spine $\{s_0, s_1\}$ (cf. [729]). Book superhypergraph is an n-superhypergraph whose superhyperedges are lifted triples $\{\hat{s}_0, \hat{s}_1, \hat{w}_i\}$, flattening to the book hypergraph. The relevant definitions and related notions are presented below.

Definition 4.43.1 (Triangular book graph)**.** Let $p \in \mathbb{Z}_{\geq 1}$. The *p-page triangular book graph* (briefly, the *book graph*) is the simple graph

$$B_p := (V(B_p), E(B_p))$$

defined by

$$V(B_p) := \{s_0, s_1\} \cup \{w_1, \ldots, w_p\},$$

and

$$E(B_p) := \{\{s_0, s_1\}\} \cup \{\{s_0, w_i\}, \{s_1, w_i\} : 1 \leq i \leq p\}.$$

Equivalently, $B_p \cong K_{1,1,p}$, and for each i the triple $\{s_0, s_1, w_i\}$ induces a triangle, with all p triangles sharing the common edge $\{s_0, s_1\}$ (the *spine*).

Definition 4.43.2 (Shadow (2-section) graph of a hypergraph)**.** Let $H = (V, E)$ be a hypergraph, where E is a family of nonempty subsets of V. The *shadow graph* (or *2-section*) of H is the graph

$$\partial(H) := (V, E_\partial)$$

where

$$E_\partial := \{\{x, y\} \subseteq V : x \neq y \text{ and } \exists e \in E \text{ with } \{x, y\} \subseteq e\}.$$

Definition 4.43.3 (Book hypergraph)**.** Let $p \in \mathbb{Z}_{\geq 1}$. The *p-page book hypergraph* is the 3-uniform hypergraph

$$\mathcal{B}_p := (V(\mathcal{B}_p), E(\mathcal{B}_p))$$

defined by

$$V(\mathcal{B}_p) := \{s_0, s_1\} \cup \{w_1, \ldots, w_p\}, \qquad E(\mathcal{B}_p) := \{\{s_0, s_1, w_i\} : 1 \leq i \leq p\}.$$

Each hyperedge $\{s_0, s_1, w_i\}$ is called a *page* and the pair $\{s_0, s_1\}$ is the *spine*.

Theorem 4.43.4 (Book hypergraph is a hypergraph)**.** *For every $p \geq 1$, $\mathcal{B}_p$ is a hypergraph.*

Proof. By definition, $V(\mathcal{B}_p)$ is a nonempty set. For each $i \in \{1, \ldots, p\}$, the set $e_i := \{s_0, s_1, w_i\}$ satisfies

$$e_i \subseteq V(\mathcal{B}_p), \qquad e_i \neq \emptyset.$$

Hence $E(\mathcal{B}_p) = \{e_i : 1 \leq i \leq p\}$ is a family of nonempty subsets of $V(\mathcal{B}_p)$, so $\mathcal{B}_p = (V(\mathcal{B}_p), E(\mathcal{B}_p))$ is a hypergraph. □

Theorem 4.43.5 (Book hypergraph generalizes the book graph)**.** *For every $p \geq 1$, the shadow graph of the book hypergraph equals the book graph:*

$$\partial(\mathcal{B}_p) = B_p.$$

Proof. Let $V := V(\mathcal{B}_p) = V(B_p) = \{s_0, s_1\} \cup \{w_1, \ldots, w_p\}$.

First, we prove $E(B_p) \subseteq E_\partial$, where $E_\partial := E(\partial(\mathcal{B}_p))$. Take an arbitrary edge of B_p.

(i) If the edge is $\{s_0, s_1\}$, then for any i we have $\{s_0, s_1\} \subseteq \{s_0, s_1, w_i\} \in E(\mathcal{B}_p)$, so $\{s_0, s_1\} \in E_\partial$.

(ii) If the edge is $\{s_0, w_i\}$ for some i, then $\{s_0, w_i\} \subseteq \{s_0, s_1, w_i\} \in E(\mathcal{B}_p)$, so $\{s_0, w_i\} \in E_\partial$.

(iii) If the edge is $\{s_1, w_i\}$ for some i, similarly $\{s_1, w_i\} \subseteq \{s_0, s_1, w_i\} \in E(\mathcal{B}_p)$, hence $\{s_1, w_i\} \in E_\partial$.

Thus every edge of B_p is an edge of $\partial(\mathcal{B}_p)$, so $E(B_p) \subseteq E_\partial$.

Second, we prove $E_\partial \subseteq E(B_p)$. Let $\{x, y\} \in E_\partial$. Then there exists i with

$$\{x, y\} \subseteq \{s_0, s_1, w_i\}.$$

Hence $\{x, y\}$ is one of the three pairs

$$\{s_0, s_1\}, \quad \{s_0, w_i\}, \quad \{s_1, w_i\},$$

all of which belong to $E(B_p)$ by definition. Therefore $E_\partial \subseteq E(B_p)$.

Combining both inclusions yields $E_\partial = E(B_p)$, and the vertex sets coincide, so $\partial(\mathcal{B}_p) = B_p$. □

Definition 4.43.6 (Singleton embeddings and vertex lifting)**.** For each $k \geq 0$, define the singleton embedding

$$\iota_k : \mathcal{P}^k(V) \to \mathcal{P}^{k+1}(V), \qquad \iota_k(x) := \{x\}.$$

For $n \geq 1$ and $v \in V$, define the $(n-1)$-*lift* of v by

$$\widehat{v}^{(n-1)} \; := \; (\iota_{n-2} \circ \cdots \circ \iota_0)(v) \in \mathcal{P}^{n-1}(V),$$

with the convention $\widehat{v}^{(0)} = v$ when $n = 1$.

Definition 4.43.7 (Book n-superhypergraph)**.** Fix $p \geq 1$ and $n \geq 1$. Let

$$V := \{s_0, s_1\} \cup \{w_1, \ldots, w_p\}.$$

The *p-page book n-superhypergraph* is

$$\mathbb{B}_p^{(n)} := (V, \mathcal{E}_p^{(n)})$$

where

$$\mathcal{E}_p^{(n)} := \left\{ e_i^{(n)} \;:\; 1 \leq i \leq p \right\}, \qquad e_i^{(n)} := \left\{ \widehat{s_0}^{(n-1)},\ \widehat{s_1}^{(n-1)},\ \widehat{w_i}^{(n-1)} \right\}.$$

Thus each $e_i^{(n)}$ is a 3-element subset of $\mathcal{P}^{n-1}(V)$, hence an element of $\mathcal{P}^n(V)$.

Example 4.43.8 (A concrete 2-page book 2-superhypergraph)**.** Let $p = 2$ and $n = 2$. Set

$$V := \{s_0, s_1, w_1, w_2\}.$$

Recall that for $n = 2$ we have $(n-1) = 1$, so

$$\widehat{x}^{(1)} = \{x\} \in \mathcal{P}(V) \qquad (x \in V).$$

Hence the 2-page book 2-superhypergraph

$$\mathbb{B}_2^{(2)} = (V, \mathcal{E}_2^{(2)})$$

has superhyperedge family

$$\mathcal{E}_2^{(2)} = \{e_1^{(2)}, e_2^{(2)}\},$$

where, explicitly,

$$e_1^{(2)} = \left\{ \{s_0\}, \{s_1\}, \{w_1\} \right\} \in \mathcal{P}^2(V), \qquad e_2^{(2)} = \left\{ \{s_0\}, \{s_1\}, \{w_2\} \right\} \in \mathcal{P}^2(V).$$

Thus $\mathbb{B}_2^{(2)}$ consists of two "pages" (the two superhyperedges) sharing the common spine $\{\{s_0\}, \{s_1\}\}$, and differing by the third supervertex $\{w_1\}$ versus $\{w_2\}$.

Theorem 4.43.9 (Book n-superhypergraph is an n-SuperHyperGraph)**.** *For every $p \geq 1$ and $n \geq 1$, $\mathbb{B}_p^{(n)}$ is an n-SuperHyperGraph.*

Proof. Let $V = \{s_0, s_1\} \cup \{w_1, \ldots, w_p\}$. For each i, we have

$$\widehat{s_0}^{(n-1)},\ \widehat{s_1}^{(n-1)},\ \widehat{w_i}^{(n-1)} \in \mathcal{P}^{n-1}(V),$$

so

$$e_i^{(n)} = \{\widehat{s_0}^{(n-1)}, \widehat{s_1}^{(n-1)}, \widehat{w_i}^{(n-1)}\} \subseteq \mathcal{P}^{n-1}(V), \qquad e_i^{(n)} \neq \emptyset.$$

Hence $e_i^{(n)} \in \mathcal{P}(\mathcal{P}^{n-1}(V)) = \mathcal{P}^n(V)$ and $e_i^{(n)} \neq \emptyset$. Therefore

$$\mathcal{E}_p^{(n)} \subseteq \mathcal{P}^n(V) \setminus \{\emptyset\},$$

so $\mathbb{B}_p^{(n)} = (V, \mathcal{E}_p^{(n)})$ is an n-SuperHyperGraph by definition. □

Definition 4.43.10 (Flattening on lifted vertices)**.** Let $n \geq 1$ and let V be fixed. Define the flattening map on lifted vertices by

$$\flat_{n-1} : \{\widehat{v}^{(n-1)} : v \in V\} \to V, \qquad \flat_{n-1}(\widehat{v}^{(n-1)}) := v.$$

Extend it to $e \subseteq \{\widehat{v}^{(n-1)} : v \in V\}$ by

$$\flat_n(e) := \{\flat_{n-1}(x) : x \in e\} \subseteq V.$$

Theorem 4.43.11 (Book n-superhypergraph generalizes the book graph)**.** *For every $p \geq 1$ and $n \geq 1$, if we flatten $\mathbb{B}_p^{(n)}$ level-by-level, we recover the book hypergraph, and hence the book graph:*

$$\big(V, \{\flat_n(e) : e \in \mathcal{E}_p^{(n)}\}\big) = \mathcal{B}_p, \qquad \textit{and thus} \qquad \partial\Big(V, \{\flat_n(e) : e \in \mathcal{E}_p^{(n)}\}\Big) = B_p.$$

Proof. Fix $i \in \{1, \ldots, p\}$. By definition,

$$e_i^{(n)} = \{\widehat{s_0}^{(n-1)}, \widehat{s_1}^{(n-1)}, \widehat{w_i}^{(n-1)}\}.$$

Applying $\flat_n$ gives

$$\flat_n(e_i^{(n)}) = \{\flat_{n-1}(\widehat{s_0}^{(n-1)}), \flat_{n-1}(\widehat{s_1}^{(n-1)}), \flat_{n-1}(\widehat{w_i}^{(n-1)})\} = \{s_0, s_1, w_i\}.$$

Therefore

$$\{\flat_n(e) : e \in \mathcal{E}_p^{(n)}\} = \big\{\{s_0, s_1, w_i\} : 1 \leq i \leq p\big\} = E(\mathcal{B}_p).$$

Hence the flattened hypergraph equals $\mathcal{B}_p$.

By the previous theorem "Book hypergraph generalizes the book graph", we already proved $\partial(\mathcal{B}_p) = B_p$. Substituting $\mathcal{B}_p$ by the flattened hypergraph yields the desired identity. □

4.44 Pancake SuperHyperGraph

The pancake graph is a Cayley graph [730] on $\mathfrak{S}_n$ where edges connect permutations differing by one prefix reversal [731, 732]. The pancake hypergraph is the 2-uniform hypergraph on $\mathfrak{S}_n$ whose hyperedges are exactly prefix-reversal pairs of permutations. The pancake n-SuperHyperGraph is a level-n superhypergraph lifting $\mathfrak{S}_n$ to n-supervertices, whose flattening recovers pancake hyperedges. The relevant definitions and related notions are presented below.

Definition 4.44.1 (Prefix reversal on $\mathfrak{S}_n$)**.** Fix an integer $n \geq 2$. For each $k \in \{2, 3, \ldots, n\}$ and each permutation $\pi \in \mathfrak{S}_n$, define $\rho_k(\pi) \in \mathfrak{S}_n$ by

$$\rho_k(\pi)(i) := \begin{cases} \pi(k+1-i), & 1 \leq i \leq k, \\ \pi(i), & k < i \leq n. \end{cases}$$

Equivalently, in one-line notation $\pi = (\pi_1\ \pi_2\ \cdots\ \pi_n)$,

$$\rho_k(\pi) = (\pi_k\ \pi_{k-1}\ \cdots\ \pi_1\ \pi_{k+1}\ \cdots\ \pi_n).$$

Definition 4.44.2 (Pancake graph)**.** [731, 732] For $n \geq 2$, the *pancake graph P_n* is the graph

$$P_n = (\mathfrak{S}_n, E(P_n)), \qquad E(P_n) := \Big\{\{\pi, \rho_k(\pi)\} \;\Big|\; \pi \in \mathfrak{S}_n,\ 2 \leq k \leq n\Big\}.$$

Definition 4.44.3 (Pancake hypergraph)**.** For $n \geq 2$, the *pancake hypergraph $\mathcal{H}_n$* is the 2-uniform hypergraph

$$\mathcal{H}_n = (\mathfrak{S}_n, E(\mathcal{H}_n)), \qquad E(\mathcal{H}_n) := E(P_n).$$

(Thus every hyperedge has size 2, and $\mathcal{H}_n$ encodes exactly the same adjacencies as P_n.)

Example 4.44.4 (A concrete pancake hypergraph on $\mathfrak{S}_3$)**.** A *hypergraph* is a pair $H = (V, E)$ where V is a finite set and $E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$.

Let $V := \mathfrak{S}_3$, written in one-line notation:

$$\mathfrak{S}_3 = \{[123], [132], [213], [231], [312], [321]\}.$$

For $k \in \{2, 3\}$, define the prefix-reversal map $\rho_k : \mathfrak{S}_3 \to \mathfrak{S}_3$ by

$$\rho_2([a\,b\,c]) := [b\,a\,c], \qquad \rho_3([a\,b\,c]) := [c\,b\,a].$$

Define the *pancake hypergraph* (which is 2-uniform) by

$$\mathcal{H}_3 := \big(\mathfrak{S}_3,\ E(\mathcal{H}_3)\big), \qquad E(\mathcal{H}_3) := \big\{\{\pi, \rho_k(\pi)\} \ \big|\ \pi \in \mathfrak{S}_3,\ k \in \{2, 3\}\big\}.$$

Computing explicitly, the distinct hyperedges are

$$E(\mathcal{H}_3) = \big\{\{[123], [213]\}, \{[123], [321]\}, \{[132], [231]\}, \{[132], [312]\}, \{[213], [312]\}, \{[231], [321]\}\big\}.$$

Thus $\mathcal{H}_3$ is a concrete pancake hypergraph.

Definition 4.44.5 (Iterated singleton lift)**.** Let X be a set and $x \in X$. Define $\widehat{x}^{(t)}$ recursively by

$$\widehat{x}^{(0)} := x, \qquad \widehat{x}^{(t+1)} := \{\widehat{x}^{(t)}\} \quad (t \in \mathbb{N}_0).$$

Lemma 4.44.6. *Let X be a set and $x \in X$. Then for every $t \in \mathbb{N}_0$,*

$$\widehat{x}^{(t)} \in \mathcal{P}^t(X),$$

where $\mathcal{P}^0(X) := X$ and $\mathcal{P}^{t+1}(X) := \mathcal{P}(\mathcal{P}^t(X))$.

Proof. We prove the claim by induction on t.

Base case $t = 0$. By definition, $\widehat{x}^{(0)} = x \in X = \mathcal{P}^0(X)$.

Inductive step. Assume $\widehat{x}^{(t)} \in \mathcal{P}^t(X)$ for some $t \geq 0$. Then $\{\widehat{x}^{(t)}\} \subseteq \mathcal{P}^t(X)$, hence

$$\widehat{x}^{(t+1)} = \{\widehat{x}^{(t)}\} \in \mathcal{P}\big(\mathcal{P}^t(X)\big) = \mathcal{P}^{t+1}(X).$$

This completes the induction. □

Definition 4.44.7 (Pancake n-superhypergraph)**.** Fix an integer $n \geq 2$ and let the base set be

$$V_0 := \mathfrak{S}_n.$$

Define the set of n-supervertices by

$$V^{(n)} := \Big\{\widehat{\pi}^{(n)} \ \Big|\ \pi \in \mathfrak{S}_n\Big\} \ \subseteq\ \mathcal{P}^n(V_0).$$

Define the set of (super)edge identifiers by

$$E^{(n)} := \mathfrak{S}_n \times \{2, 3, \ldots, n\}.$$

Define the incidence map $\partial^{(n)} : E^{(n)} \to \mathcal{P}^*(V^{(n)})$ by

$$\partial^{(n)}(\pi, k) := \Big\{\widehat{\pi}^{(n)},\ \widehat{\rho_k(\pi)}^{(n)}\Big\}.$$

The triple

$$\mathcal{S}^{(n)}_{\text{pan}} := \big(V^{(n)},\ E^{(n)},\ \partial^{(n)}\big)$$

is called the *pancake n-superhypergraph*.

Example 4.44.8 (A concrete pancake 2-superhypergraph over $\mathfrak{S}_3$)**.** A *level-2 superhypergraph over* V_0 is a triple $\mathcal{S} = (V, E, \partial)$ such that

$$V \subseteq \mathcal{P}^2(V_0), \qquad E \text{ is a finite set}, \qquad \partial : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Let the base set be $V_0 := \mathfrak{S}_3$. Define iterated singleton-lifts for $x \in V_0$ by

$$\widehat{x}^{(0)} := x, \qquad \widehat{x}^{(1)} := \{x\} \in \mathcal{P}(V_0), \qquad \widehat{x}^{(2)} := \{\{x\}\} \in \mathcal{P}^2(V_0).$$

Define the set of level-2 supervertices by

$$V := \{\widehat{\pi}^{(2)} \mid \pi \in \mathfrak{S}_3\} \subseteq \mathcal{P}^2(\mathfrak{S}_3).$$

Let the edge-identifier set be

$$E := \mathfrak{S}_3 \times \{2, 3\}.$$

Using the same prefix reversals ρ_2, ρ_3 as in the previous example, define the incidence map

$$\partial(\pi, k) := \{\widehat{\pi}^{(2)},\ \widehat{\rho_k(\pi)}^{(2)}\} \in \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then

$$\mathcal{S}_3^{(2)} := (V, E, \partial)$$

is a concrete *pancake* 2-*superhypergraph* over $\mathfrak{S}_3$. For instance,

$$\partial([123], 2) = \{\widehat{[123]}^{(2)}, \widehat{[213]}^{(2)}\} = \big\{\{\{[123]\}\}, \{\{[213]\}\}\big\},$$

and

$$\partial([123], 3) = \{\widehat{[123]}^{(2)}, \widehat{[321]}^{(2)}\} = \big\{\{\{[123]\}\}, \{\{[321]\}\}\big\}.$$

Theorem 4.44.9 (The pancake n-superhypergraph is an n-SuperHyperGraph)**.** *For every integer $n \geq 2$, the structure*

$$\mathcal{S}_{\text{pan}}^{(n)} = \big(V^{(n)}, E^{(n)}, \partial^{(n)}\big)$$

is a well-defined level-n SuperHyperGraph over the base set $V_0 = \mathfrak{S}_n$.

Proof. We verify the defining requirements explicitly.

Step 1 ($V^{(n)} \subseteq \mathcal{P}^n(V_0)$). Take any $v \in V^{(n)}$. Then $v = \widehat{\pi}^{(n)}$ for some $\pi \in \mathfrak{S}_n = V_0$. By Lemma 4.44.6 with $X = V_0$ and $t = n$,

$$\widehat{\pi}^{(n)} \in \mathcal{P}^n(V_0).$$

Hence $V^{(n)} \subseteq \mathcal{P}^n(V_0)$.

Step 2 (the incidence map lands in $\mathcal{P}^*(V^{(n)})$). Take any $(\pi, k) \in E^{(n)} = \mathfrak{S}_n \times \{2, \ldots, n\}$. Then $\pi \in \mathfrak{S}_n$ and, by Definition 4.44.1, $\rho_k(\pi) \in \mathfrak{S}_n$. Therefore $\widehat{\pi}^{(n)} \in V^{(n)}$ and $\widehat{\rho_k(\pi)}^{(n)} \in V^{(n)}$, so

$$\partial^{(n)}(\pi, k) = \left\{\widehat{\pi}^{(n)}, \widehat{\rho_k(\pi)}^{(n)}\right\} \subseteq V^{(n)}.$$

Moreover, $\partial^{(n)}(\pi, k)$ contains (at least) the element $\widehat{\pi}^{(n)}$, hence it is nonempty:

$$\partial^{(n)}(\pi, k) \neq \emptyset.$$

Thus $\partial^{(n)}(\pi, k) \in \mathcal{P}^*(V^{(n)})$.

Step 3 (conclusion). Steps 1–2 show that $\mathcal{S}_{\text{pan}}^{(n)}$ satisfies the requirements of a level-n SuperHyperGraph over $V_0 = \mathfrak{S}_n$. □

Definition 4.44.10 (Flattening map and flattening)**.** Define $f_n : V^{(n)} \to \mathfrak{S}_n$ by

$$f_n(\widehat{\pi}^{(n)}) := \pi.$$

Define the *flattening* of $\mathcal{S}_{\mathrm{pan}}^{(n)}$ to be the hypergraph

$$\mathrm{Flat}(\mathcal{S}_{\mathrm{pan}}^{(n)}) := (\mathfrak{S}_n,\, E_{\mathrm{Flat}}), \qquad E_{\mathrm{Flat}} := \Big\{ f_n[\partial^{(n)}(e)] \;\Big|\; e \in E^{(n)} \Big\},$$

where $f_n[\partial^{(n)}(e)] := \{ f_n(x) \mid x \in \partial^{(n)}(e) \}$.

Theorem 4.44.11 (Flattening recovers the pancake hypergraph (and hence the pancake graph))**.** *For every $n \geq 2$,*

$$\mathrm{Flat}(\mathcal{S}_{\mathrm{pan}}^{(n)}) = \mathcal{H}_n.$$

Consequently, viewing each 2*-element hyperedge of* $\mathrm{Flat}(\mathcal{S}_{\mathrm{pan}}^{(n)})$ *as a graph edge yields exactly P_n.*

Proof. Let $e = (\pi, k) \in E^{(n)} = \mathfrak{S}_n \times \{2, \ldots, n\}$ be arbitrary. By Definition 4.44.7,

$$\partial^{(n)}(e) = \partial^{(n)}(\pi, k) = \Big\{ \widehat{\pi}^{(n)}, \widehat{\rho_k(\pi)}^{(n)} \Big\}.$$

Apply $f_n(\widehat{\sigma}^{(n)}) = \sigma$ to obtain

$$f_n[\partial^{(n)}(\pi, k)] = \Big\{ f_n(\widehat{\pi}^{(n)}),\; f_n(\widehat{\rho_k(\pi)}^{(n)}) \Big\} = \{\pi, \rho_k(\pi)\}.$$

Therefore

$$E_{\mathrm{Flat}} = \Big\{ \{\pi, \rho_k(\pi)\} \;\Big|\; \pi \in \mathfrak{S}_n,\; 2 \leq k \leq n \Big\} = E(\mathcal{H}_n)$$

by Definition 4.44.3. Since the vertex sets are both $\mathfrak{S}_n$, we have $\mathrm{Flat}(\mathcal{S}_{\mathrm{pan}}^{(n)}) = \mathcal{H}_n$.

Finally, $\mathcal{H}_n$ is 2-uniform, so interpreting each hyperedge $\{\pi, \rho_k(\pi)\}$ as an edge produces precisely the pancake graph P_n from Definition 4.44.2. □

4.45 Connected n-SuperHyperGraph

A connected graph has a path between every two vertices, so the whole vertex set forms one component [733]. Related notions include Fuzzy Connected Graphs [734], Biconnected Graphs [735], and MultiConnected Graphs [736]. A connected hypergraph has a Berge path between any two vertices, alternating vertices and hyperedges containing consecutive vertices [737,738]. A connected n-superhypergraph has a super-Berge path between any two supervertices, alternating supervertices and superedges via incidence. The relevant definitions and related notions are presented below.

Definition 4.45.1 (Connected graph)**.** [733] Let $G = (V, E)$ be a finite undirected graph. A *(vertex-)path* in G from u to v is a sequence of vertices

$$u = v_0, v_1, \ldots, v_\ell = v$$

such that $\{v_{i-1}, v_i\} \in E$ for every $i = 1, \ldots, \ell$. The graph G is called *connected* if for every two vertices $u, v \in V$ there exists a path in G from u to v.

Definition 4.45.2 (Connected hypergraph (Berge connectivity))**.** [737, 738] Let $H = (V, E)$ be a finite hypergraph, i.e.,

$$V \neq \varnothing, \qquad E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}.$$

A *Berge path* in H from u to v is an alternating sequence

$$u = v_0,\; e_1,\; v_1,\; e_2,\; \ldots,\; e_\ell,\; v_\ell = v$$

such that for each $i = 1, \ldots, \ell$ one has

$$v_{i-1} \in e_i \quad \text{and} \quad v_i \in e_i.$$

The hypergraph H is called *connected* if for every two vertices $u, v \in V$ there exists a Berge path from u to v.

Definition 4.45.3 ((Recall) Incidence graph (of a hypergraph))**.** Let $H = (V, E)$ be a hypergraph. The *incidence graph* (or *Levi graph*) of H is the bipartite graph

$$I(H) := \big(V \cup E,\ F\big), \qquad F := \big\{\{x, e\} \mid x \in V,\ e \in E,\ x \in e\big\}.$$

Equivalently, H is connected (in the Berge sense) if and only if $I(H)$ is connected.

Definition 4.45.4 (Connected n-SuperHyperGraph)**.** Fix a finite base set V_0 and an integer $n \in \mathbb{N}_0$. An *n-SuperHyperGraph* is a triple

$$\mathcal{S} = \big(V,\ E,\ \partial\big)$$

where $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of n-supervertices, E is a finite set of superedge identifiers, and

$$\partial : E \to \mathcal{P}(V) \setminus \{\varnothing\}$$

is an incidence map.

A *super-Berge path* in $\mathcal{S}$ from u to v (where $u, v \in V$) is an alternating sequence

$$u = v_0,\ e_1,\ v_1,\ e_2,\ \ldots,\ e_\ell,\ v_\ell = v$$

such that for every $i = 1, \ldots, \ell$ one has

$$v_{i-1} \in \partial(e_i) \quad \text{and} \quad v_i \in \partial(e_i).$$

The n-SuperHyperGraph $\mathcal{S}$ is called *connected* if for every two supervertices $u, v \in V$ there exists a super-Berge path from u to v.

Example 4.45.5 (A connected n-SuperHyperGraph)**.** Fix an integer $n \geq 1$ and take the base set

$$V_0 := \{a, b, c\}.$$

Define the n-supervertex set by n-lifting singletons:

$$V := \left\{ \widehat{\{a\}}^{(n)},\ \widehat{\{b\}}^{(n)},\ \widehat{\{c\}}^{(n)} \right\} \subseteq \mathcal{P}^n(V_0), \qquad \widehat{X}^{(0)} := X,\ \ \widehat{X}^{(t+1)} := \{\widehat{X}^{(t)}\}.$$

Let the superedge-identifier set be

$$E := \{e_{ab},\ e_{bc}\},$$

and define the incidence map $\partial : E \to \mathcal{P}(V) \setminus \{\varnothing\}$ by

$$\partial(e_{ab}) := \big\{\widehat{\{a\}}^{(n)},\ \widehat{\{b\}}^{(n)}\big\}, \qquad \partial(e_{bc}) := \big\{\widehat{\{b\}}^{(n)},\ \widehat{\{c\}}^{(n)}\big\}.$$

Then

$$\mathcal{S} := (V, E, \partial)$$

is an n-SuperHyperGraph, and it is connected.

Indeed, for any two supervertices $u, v \in V$:

- if $\{u, v\} = \{\widehat{\{a\}}^{(n)}, \widehat{\{b\}}^{(n)}\}$, then u, e_{ab}, v is a super-Berge path;
- if $\{u, v\} = \{\widehat{\{b\}}^{(n)}, \widehat{\{c\}}^{(n)}\}$, then u, e_{bc}, v is a super-Berge path;
- if $\{u, v\} = \{\widehat{\{a\}}^{(n)}, \widehat{\{c\}}^{(n)}\}$, then

  $$\widehat{\{a\}}^{(n)},\ e_{ab},\ \widehat{\{b\}}^{(n)},\ e_{bc},\ \widehat{\{c\}}^{(n)}$$

  is a super-Berge path because $\widehat{\{a\}}^{(n)}, \widehat{\{b\}}^{(n)} \in \partial(e_{ab})$ and $\widehat{\{b\}}^{(n)}, \widehat{\{c\}}^{(n)} \in \partial(e_{bc})$.

Hence $\mathcal{S}$ is connected.

Definition 4.45.6 ((Recall) Incidence graph (of an n-SuperHyperGraph))**.** Let $\mathcal{S} = (V, E, \partial)$ be an n-SuperHyperGraph. Its *incidence graph* is the bipartite graph

$$I(\mathcal{S}) := \big(V \cup E,\ F\big), \qquad F := \big\{\{x, e\} \mid x \in V,\ e \in E,\ x \in \partial(e)\big\}.$$

Equivalently, $\mathcal{S}$ is connected (in the super-Berge sense) if and only if $I(\mathcal{S})$ is connected.

Theorem 4.45.7 (Connected n-SuperHyperGraph generalizes connected hypergraph)**.** *Let $H = (V, E_H)$ be a (finite) hypergraph, i.e., $V \neq \emptyset$ and $E_H \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Define the associated* 0*-SuperHyperGraph*

$$\mathcal{S}_H \ := \ (V,\ E_H,\ \partial_H), \qquad \partial_H(e) \ := \ e \quad (e \in E_H).$$

Then H is connected (in the Berge sense) if and only if $\mathcal{S}_H$ is connected (in the super-Berge sense of Definition 3.?(Connected n-SuperHyperGraph) with $n = 0$). Consequently, the notion of a connected n-SuperHyperGraph is a genuine extension of Berge-connectedness of hypergraphs (obtained by setting $n = 0$ and $\partial = \mathrm{id}$).

Proof. Since $n = 0$, the 0-supervertex set is just the vertex set:

$$V \subseteq \mathcal{P}^0(V) = V,$$

so the supervertices of $\mathcal{S}_H$ are exactly the vertices of H.

Fix $u, v \in V$. A Berge path in H from u to v is an alternating sequence

$$u = v_0,\ e_1,\ v_1,\ e_2,\ \ldots,\ e_\ell,\ v_\ell = v$$

such that for each $i = 1, \ldots, \ell$ one has $v_{i-1} \in e_i$ and $v_i \in e_i$. In $\mathcal{S}_H$, a super-Berge path from u to v is an alternating sequence with the same form, but the membership conditions are written using the incidence map:

$$v_{i-1} \in \partial_H(e_i) \quad \text{and} \quad v_i \in \partial_H(e_i).$$

By definition of $\partial_H(e) = e$, these conditions are identical to $v_{i-1} \in e_i$ and $v_i \in e_i$.

Hence, for every pair $u, v \in V$, there exists a Berge path from u to v in H if and only if there exists a super-Berge path from u to v in $\mathcal{S}_H$. Therefore H is connected if and only if $\mathcal{S}_H$ is connected. □

Theorem 4.45.8 (Incidence graph generalizes the hypergraph incidence (Levi) graph)**.** *Let $H = (V, E_H)$ be a hypergraph and let $\mathcal{S}_H = (V, E_H, \partial_H)$ be the associated* 0*-SuperHyperGraph defined in Theorem 4.45.7 with $\partial_H(e) = e$. Then the incidence graph of $\mathcal{S}_H$ coincides with the incidence graph (Levi graph) of H:*

$$I(\mathcal{S}_H) \ = \ I(H) \quad \textit{as bipartite graphs on the vertex set } V \cup E_H.$$

In particular, the construction $I(\mathcal{S})$ for n-SuperHyperGraphs extends the classical incidence graph $I(H)$ of hypergraphs (obtained by setting $n = 0$ and $\partial = \mathrm{id}$).

Proof. By definition, the incidence graph of the hypergraph H is

$$I(H) \ = \ \big(V \cup E_H,\ F_H\big), \qquad F_H \ := \ \big\{\{x, e\} \mid x \in V,\ e \in E_H,\ x \in e\big\}.$$

On the other hand, the incidence graph of the 0-SuperHyperGraph $\mathcal{S}_H$ is

$$I(\mathcal{S}_H) \ = \ \big(V \cup E_H,\ F_{\mathcal{S}}\big), \qquad F_{\mathcal{S}} \ := \ \big\{\{x, e\} \mid x \in V,\ e \in E_H,\ x \in \partial_H(e)\big\}.$$

Since $\partial_H(e) = e$ for all $e \in E_H$, we have $x \in \partial_H(e)$ if and only if $x \in e$. Therefore $F_{\mathcal{S}} = F_H$, and the two bipartite graphs are identical: $I(\mathcal{S}_H) = I(H)$. □

Chapter 5

Uncertain SuperHyperGraph

In this chapter, we investigate the notion of Uncertain SuperHyperGraphs. Here, an Uncertain Set refers to any uncertainty-handling framework such as Fuzzy Sets, Intuitionistic Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets. Moreover, within graph theory, numerous extended concepts related to these notions have been proposed. For additional details, the reader may consult [739, 740] as needed.

Each of graph models—fuzzy and neutrosophic—can be lifted to the hypergraph setting, giving rise to fuzzy hypergraphs [62, 741] and neutrosophic hypergraphs [742, 743]. These ideas can be pushed further into "recursive nested" architectures, producing fuzzy superhypergraphs and neutrosophic superhypergraphs (see, e.g., [2, 744]). By capturing increasingly intricate and uncertain network structures, these generalized frameworks play a vital role not only in graph theory but also across computational science, informatics, network theory, decision science, engineering, and applied mathematics [7, 63, 112, 745]. Table 5.1, 5.2, and 5.3 presents an overview of uncertain hypergraph and superhypergraph models.

Model	**Description**	**Membership relations**
Classical Graph	A pair (V, E) where E is a set of unordered vertex pairs, representing crisp binary adjacency.	Edge indicator $\chi_E : V \times V \to \{0, 1\} \subset [0, 1]$; recovered as the $\{0, 1\}$–valued restriction of all uncertain models.
Fuzzy Graph [61]	Each vertex and edge has a fuzzy degree of presence in $[0, 1]$, modeling gradual adjacency.	$\mu_V : V \to [0, 1]$, $\mu_E : E \to [0, 1]$; classical graph when $\mu_V, \mu_E \in \{0, 1\}$.
Intuitionistic Fuzzy Graph	Vertices and edges have membership and non-membership with hesitation, encoding partial belief in adjacency.	$(\mu, \nu) : V \cup E \to [0, 1]^2$ with $\mu + \nu \leq 1$; fuzzy graph when $\nu = 1 - \mu$.
Neutrosophic Graph	Vertices and edges carry truth, indeterminacy, and falsity degrees, handling incomplete and inconsistent link information.	$(T, I, F) : V \cup E \to [0, 1]^3$; intuitionistic fuzzy graph when $T = \mu$, $F = \nu$, $I = 1 - \mu - \nu$.
Plithogenic Graph	Membership of vertices and edges depends on attribute values and their contradiction, unifying all above graph models.	$pdf : (V \cup E) \times Pv \to [0, 1]^s$, $pCF : Pv \times Pv \to [0, 1]^t$; suitable choices give classical, fuzzy, intuitionistic, neutrosophic.

Table 5.1: Classical and uncertain Graph models with membership relations

5.1 Fuzzy n-SuperHyperGraphs

We first address the notion of Fuzzy n-SuperHyperGraphs. A fuzzy n-SuperHyperGraph is a higher-level network representation in which supervertices and superedges carry membership values for modeling complex interactions.

Model	Description	Membership relations
Classical Hypergraph	A pair (V, E) where each hyperedge is a nonempty subset of V, encoding arbitrary multiway relations.	Incidence map $\chi : E \to \{0, 1\} \subset [0, 1]$; crisp limit of all uncertain hypergraph models.
Fuzzy Hypergraph	Each vertex and hyperedge is assigned a fuzzy degree in $[0, 1]$, representing uncertain multiway connections.	$\mu_V : V \to [0, 1]$, $\mu_E : E \to [0, 1]$; classical hypergraph when all values lie in $\{0, 1\}$.
Intuitionistic Fuzzy Hypergraph	Vertices and hyperedges have membership and non-membership with hesitation, for higher-arity relations.	$(\mu, \nu) : V \cup E \to [0, 1]^2$, $\mu + \nu \leq 1$; fuzzy hypergraph when $\nu = 1 - \mu$.
Neutrosophic Hypergraph	Vertices and hyperedges carry truth, indeterminacy, and falsity degrees for multiway uncertain information.	$(T, I, F) : V \cup E \to [0, 1]^3$; intuitionistic fuzzy hypergraph when $(T, F, I) = (\mu, \nu, 1 - \mu - \nu)$.
Plithogenic Hypergraph	Hyperedges and vertices use attribute-based membership and contradiction to model multi-criteria hyper-relations.	$pdf : (V \cup E) \times Pv \to [0, 1]^s$, $pCF : Pv \times Pv \to [0, 1]^t$; all previous hypergraph models obtained as special cases.

Table 5.2: Classical and uncertain Hypergraph models with membership relations

5.1.1 Fuzzy Graph and Fuzzy HyperGraph

A fuzzy set assigns to each element a membership degree in $[0, 1]$ [746, 747]. Fuzzy sets play a major role in diverse domains such as control theory [748], decision-making [749], graph theory [61], topology [750], signal processing [751], and engineering. Fuzzy graphs and fuzzy hypergraphs extend this notion by assigning membership degrees to vertices and to (hyper)edges [61, 409, 752]. These structures have been extensively studied, particularly for applications in decision–making and other uncertainty–driven tasks.

Definition 5.1.1 (Fuzzy Set)**.** [746] Let X be a nonempty universe of discourse. A *fuzzy set* A on X is specified by a membership function

$$\mu_A : X \longrightarrow [0, 1],$$

where $\mu_A(x)$ represents the degree to which $x \in X$ belongs to A. Equivalently, one may write

$$A = \{ (x, \mu_A(x)) \mid x \in X \}.$$

A classical (crisp) subset $C \subseteq X$ is recovered by restricting μ_A to $\{0, 1\}$.

Definition 5.1.2 (Fuzzy graph)**.** [61] A *fuzzy graph* is a triple $G = (V, \sigma, \mu)$ where V is a finite nonempty vertex set, $\sigma : V \to [0, 1]$ assigns vertex-membership degrees, and $\mu : V \times V \to [0, 1]$ assigns edge-membership degrees subject to

$$\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\} \qquad (\forall\, u, v \in V).$$

We write uv for $\{u, v\}$ and abbreviate $\mu(uv) := \mu(u, v)$. The (crisp) underlying graph of G has vertex set V and edge set $E^* := \{ uv : \mu(uv) > 0 \}$.

Example 5.1.3 (Fuzzy graph: trust network on three people)**.** Consider a small social network of three people

$$V := \{u_1, u_2, u_3\},$$

where $\sigma(u_i)$ encodes how strongly each person belongs to the “core” of the group, and $\mu(u_i, u_j)$ encodes the strength of the mutual trust between u_i and u_j.

Define the vertex-membership function $\sigma : V \to [0, 1]$ by

$$\sigma(u_1) = 1.0, \qquad \sigma(u_2) = 0.8, \qquad \sigma(u_3) = 0.6.$$

Define the edge-membership function $\mu : V \times V \to [0, 1]$ (symmetric) by

$$\mu(u_1, u_2) = 0.8, \qquad \mu(u_2, u_3) = 0.5, \qquad \mu(u_1, u_3) = 0.4,$$

Model	**Description**	**Membership relations**
Classical n-SuperHyperGraph	Vertices and superedges lie in iterated powersets of a base set, representing hierarchical multi-level relations.	Crisp incidence $\chi : V_n \cup E \to \{0,1\} \subset [0,1]$; obtained from uncertain models by enforcing $\{0,1\}$ values.
Fuzzy n-SuperHyperGraph	Each n-supervertex and n-superedge has a fuzzy membership in $[0,1]$ across hierarchy levels.	$\mu_V : V_n \to [0,1]$, $\mu_E : E \to [0,1]$; classical case when $\mu_V, \mu_E \in \{0,1\}$.
Intuitionistic Fuzzy n-SuperHyperGraph	Membership and non-membership (with hesitation) are assigned to n-supervertices and n-superedges.	$(\mu,\nu) : V_n \cup E \to [0,1]^2$, $\mu+\nu \leq 1$; fuzzy n-SuperHyperGraph when $\nu = 1-\mu$.
Neutrosophic n-SuperHyperGraph	Each n-supervertex and n-superedge has truth, indeterminacy, and falsity degrees.	$(T,I,F) : V_n \cup E \to [0,1]^3$; intuitionistic fuzzy n-SuperHyperGraph when $(T,F,I) = (\mu,\nu,1-\mu-\nu)$.
Plithogenic n-SuperHyperGraph	Supervertices and superedges use attribute-based membership vectors and contradiction degrees, at all levels.	$pdf : (V_n \cup E) \times Pv \to [0,1]^s$, $pCF : Pv \times Pv \to [0,1]^t$; specializes to all previous n-SuperHyperGraph models.

Table 5.3: Classical and uncertain n-SuperHyperGraph models with membership relations

and $\mu(u_i,u_i) = 0$ and $\mu(u_i,u_j) = 0$ for any unordered pair not listed above.

Then for every pair $u_i, u_j \in V$ we have

$$\mu(u_i,u_j) \;\leq\; \min\{\sigma(u_i),\sigma(u_j)\},$$

for example

$$\mu(u_2,u_3) = 0.5 \;\leq\; \min\{\sigma(u_2),\sigma(u_3)\} = \min\{0.8, 0.6\} = 0.6.$$

Thus $G = (V,\sigma,\mu)$ is a fuzzy graph. The underlying crisp graph has edges u_1u_2, u_2u_3, u_1u_3, forming a triangle whose edges are equipped with fuzzy trust strengths.

As examples of extensions of fuzzy graphs, concepts such as those listed in Table 5.4 are well known.

The definition of a Fuzzy HyperGraph is presented below.

Definition 5.1.4 (Fuzzy hypergraph)**.** (cf. [62, 773]) Let $H^* = (V,E,\partial)$ be a crisp hypergraph. A *fuzzy hypergraph* on H^* is a sextuple

$$\mathcal{H} \;=\; \big(V,E,\partial;\;\sigma,\mu,\eta\big),$$

with maps

$$\sigma : V \to [0,1], \qquad \mu : E \to [0,1], \qquad \eta : V \times E \to [0,1],$$

such that for all $v \in V$ and $e \in E$,

$$\textbf{(support)}\quad \big[v \in \partial(e)\big] \iff \eta(v,e) > 0, \tag{5.1}$$

$$\textbf{(incidence bound)}\quad \eta(v,e) \;\leq\; \min\{\sigma(v),\mu(e)\}, \tag{5.2}$$

$$\textbf{(edge–vertex bound)}\quad \mu(e) \;\leq\; \min_{u \in \partial(e)} \sigma(u). \tag{5.3}$$

Here σ is the *vertex-membership map*, μ the *edge-membership map*, and η the *incidence-membership map*. The underlying crisp hypergraph is (V,E,∂), recoverable via (5.1).

Table 5.4: Overview of some extensions of fuzzy graphs

Type of graph	Brief description
Hesitant Fuzzy Graph [753, 754]	Vertices and edges carry hesitant fuzzy sets with several possible membership degrees.
Picture Fuzzy Graph [755, 755, 756]	Vertices and edges have picture fuzzy labels (positive, neutral, negative, refusal).
Bipolar Fuzzy Graph [757, 758]	Each vertex and edge has both positive and negative membership degrees.
Spherical Fuzzy Graph [749, 759]	Membership, non-membership, and hesitancy degrees satisfy a spherical constraint.
Complex Fuzzy Graph [760–762]	Membership degrees are complex numbers encoding magnitude and phase-like information.
Interval-Valued Fuzzy Graph [763–765]	Vertices and edges are assigned intervals of membership degrees instead of single values.
m-Polar Fuzzy Graph [443, 766]	Each vertex and edge has an m-tuple of membership degrees capturing multiple polar attitudes.
Fuzzy Soft Graph [218, 420, 767]	Combines fuzzy graphs with parameterized soft sets on vertices and edges.
Fuzzy Rough Graph [768–770]	Uses fuzzy lower and upper approximations on vertices and edges.
Linear Diophantine Fuzzy Graph [771, 772]	Membership is defined via linear Diophantine fuzzy information on integer pairs.

Example 5.1.5 (Fuzzy hypergraph: project teams and participation)**.** Let $V = \{v_1, v_2, v_3, v_4\}$ be a set of employees. Suppose there are two project teams:

$$e_1 := \{v_1, v_2, v_3\}, \qquad e_2 := \{v_2, v_4\}.$$

Define a crisp hypergraph

$$H^* = (V, E, \partial), \qquad E := \{e_1, e_2\}, \quad \partial(e_i) := e_i.$$

Interpret $\sigma(v)$ as how fully v is assigned to the department, $\mu(e)$ as how firmly the project team e is established, and $\eta(v, e)$ as the degree of participation of v in team e.

Set the vertex-membership function $\sigma : V \to [0, 1]$ as

$$\sigma(v_1) = 0.9, \quad \sigma(v_2) = 0.8, \quad \sigma(v_3) = 0.7, \quad \sigma(v_4) = 0.6.$$

Define the edge-membership function $\mu : E \to [0, 1]$ by

$$\mu(e_1) = 0.7, \qquad \mu(e_2) = 0.6.$$

Note that

$$\mu(e_1) = 0.7 = \min\{\sigma(v_1), \sigma(v_2), \sigma(v_3)\}, \qquad \mu(e_2) = 0.6 = \min\{\sigma(v_2), \sigma(v_4)\},$$

so the edge–vertex bound (5.3) is satisfied.

Define the incidence-membership map $\eta : V \times E \to [0, 1]$ by

$$\eta(v_1, e_1) = \eta(v_2, e_1) = \eta(v_3, e_1) = 0.7,$$

$$\eta(v_2, e_2) = \eta(v_4, e_2) = 0.6,$$

and set

$$\eta(v, e) = 0 \quad \text{whenever } v \notin \partial(e).$$

Then for each $v \in V$ and $e \in E$ we have

$$v \in \partial(e) \iff \eta(v, e) > 0,$$

and

$$\eta(v, e) \ \leq \ \min\{\sigma(v), \mu(e)\},$$

for instance

$$\eta(v_2, e_2) = 0.6 \ \leq \ \min\{\sigma(v_2), \mu(e_2)\} = \min\{0.8, 0.6\} = 0.6.$$

Thus

$$\mathcal{H} := (V, E, \partial; \sigma, \mu, \eta)$$

is a fuzzy hypergraph in the sense of Definition 5.1.4, modeling two fuzzy project teams with graded participation of employees.

As examples of extensions of fuzzy hypergraphs, concepts such as those listed in Table 5.5 are well known.

Table 5.5: Overview of some extensions of fuzzy hypergraphs

Type of hypergraph	Brief description
Hesitant Fuzzy HyperGraph [774]	Vertices and hyperedges carry hesitant fuzzy sets with several possible membership degrees.
Picture Fuzzy HyperGraph [775]	Vertices and hyperedges have picture fuzzy labels (positive, neutral, negative, refusal).
Bipolar Fuzzy HyperGraph [776, 777]	Each vertex and hyperedge has both positive and negative membership degrees.
Spherical Fuzzy HyperGraph	Membership, non-membership, and hesitancy degrees on vertices and hyperedges satisfy a spherical constraint.
Complex Fuzzy HyperGraph [778]	Membership degrees on vertices and hyperedges are complex numbers encoding magnitude and phase-like information.
Interval-Valued Fuzzy HyperGraph [773, 779]	Vertices and hyperedges are assigned intervals of membership degrees instead of single values.
m-Polar Fuzzy HyperGraph [780, 781]	Each vertex and hyperedge has an m-tuple of membership degrees capturing multiple polar attitudes.
Fuzzy Soft HyperGraph [72, 782]	Combines fuzzy hypergraphs with parameterized soft sets on vertices and hyperedges.
Fuzzy Rough HyperGraph	Uses fuzzy lower and upper approximations on vertices and hyperedges induced by an indiscernibility or similarity relation.

5.1.2 Fuzzy n-SuperHyperGraph

A fuzzy n-SuperHyperGraph is a higher-level network representation in which supervertices and superedges carry membership values for modeling complex interactions (cf. [2, 783]).

Definition 5.1.6 (Fuzzy n-SuperHyperGraph)**.** (cf. [2]) Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A *fuzzy n-SuperHyperGraph* is a quadruple

$$(V, E, \sigma, \mu),$$

where $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ obey the *admissibility constraint*

$$\mu(e) \ \leq \ \min_{v \in e} \sigma(v) \qquad \text{for every } e \in E.$$

Example 5.1.7 (Fuzzy 1-SuperHyperGraph: overlapping research groups)**.** Let $V_0 = \{a, b, c\}$ be a set of researchers. Consider the first iterated powerset

$$\mathrm{P}_1(V_0) = \mathcal{P}(V_0) = \big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\big\}.$$

Define two 1-supervertices

$$v_A := \{a, b\}, \qquad v_B := \{b, c\},$$

and set

$$V := \{v_A, v_B\} \ \subseteq \ \mathrm{P}_1(V_0).$$

Table 5.6: Compact comparison: fuzzy graph vs. fuzzy hypergraph vs. fuzzy n-superhypergraph.

Item	Fuzzy Graph	Fuzzy HyperGraph	Fuzzy n-SuperHyperGraph
Underlying carrier	Finite vertex set V	Finite V and hyperedge set E with incidence $\partial : E \to \mathcal{P}(V)\setminus\{\emptyset\}$	Base set V_0; supervertices $V \subseteq \mathcal{P}^n(V_0)$; superedges E with $\partial : E \to \mathcal{P}(V)\setminus\{\emptyset\}$
Membership maps	$\sigma : V \to [0,1], \mu : V\times V \to [0,1]$	$\sigma : V \to [0,1]$, $\mu : E \to [0,1], \eta : V \times E \to [0,1]$	$\sigma : V \to [0,1]$, $\mu : E \to [0,1]$ (optional: incidence degree $\eta : V \times E \to [0,1]$)
Admissibility / bounds	$\mu(u,v) \le \min\{\sigma(u), \sigma(v)\}$	$\eta(v,e) \le \min\{\sigma(v), \mu(e)\}$ and $\mu(e) \le \min_{u\in\partial(e)} \sigma(u)$	$\mu(e) \le \min_{v\in\partial(e)} \sigma(v)$ (and if η used: $\eta(v,e) \le \min\{\sigma(v), \mu(e)\}$)
Interaction pattern	Pairwise edges (binary)	Higher-order hyperedges (sets of vertices)	Multi-level groups as vertices (nested sets) and higher-order superedges among groups
Typical viewpoint	Uncertain strength of pairwise relations	Uncertain multiway relations plus graded incidence	Uncertain relations between higher-level entities (clusters/teams/modules), possibly across multiple abstraction levels

Interpret v_A as the research group jointly led by a and b, and v_B as the group jointly led by b and c.

Let the 1-superedge set be

$$E := \{e_1\}, \qquad e_1 := \{v_A, v_B\} \subseteq V,$$

representing a higher-level collaboration project that involves both groups v_A and v_B.

Define the vertex-membership function $\sigma : V \to [0,1]$ and superedge-membership function $\mu : E \to [0,1]$ by

$$\sigma(v_A) = 0.9, \qquad \sigma(v_B) = 0.7,$$

$$\mu(e_1) = 0.7.$$

Then

$$\mu(e_1) = 0.7 \ \le \ \min\{\sigma(v_A), \sigma(v_B)\} = \min\{0.9, 0.7\} = 0.7,$$

so the admissibility constraint

$$\mu(e) \le \min_{v\in e} \sigma(v) \qquad (e \in E)$$

is satisfied. Therefore

$$(V, E, \sigma, \mu)$$

is a fuzzy 1-SuperHyperGraph, modeling two overlapping research groups (as supervertices) and their joint collaborative project (as a superedge) with graded membership strengths.

For reference, a compact comparison of fuzzy graphs, fuzzy hypergraphs, and fuzzy n-superhypergraphs is provided in Table 5.6.

5.2 Intuitionistic Fuzzy SuperHyperGraph

An intuitionistic fuzzy set assigns each element degrees of membership and nonmembership whose sum is at most one, capturing hesitation [784,785]. An intuitionistic fuzzy graph equips each vertex and edge with membership and nonmembership degrees, modeling uncertain relationships and hesitant connectivity [786,787]. An intuitionistic fuzzy hypergraph extends intuitionistic fuzzy graphs by assigning such degrees to vertices and hyperedges representing multiway uncertain interactions [788–790]. An intuitionistic fuzzy SuperHyperGraph labels multi-level supervertices and superedges with intuitionistic membership, nonmembership, and hesitation degrees, capturing hierarchical uncertainty precisely (cf. [783,791,792]).

Definition 5.2.1 (Intuitionistic Fuzzy Set)**.** [793] Let X be a nonempty universe. An *intuitionistic fuzzy set* (in the sense of Atanassov) A on X is given by a pair of functions

$$\mu_A,\ \nu_A : X \longrightarrow [0,1],$$

where for every $x \in X$,

$$0 \ \le\ \mu_A(x) + \nu_A(x) \ \le\ 1.$$

Here $\mu_A(x)$ is the degree of membership of x in A, $\nu_A(x)$ is the degree of non–membership, and the *hesitation* (or indeterminacy) degree is

$$\pi_A(x) := 1 - \mu_A(x) - \nu_A(x) \ \in\ [0,1].$$

We write

$$A \ =\ \{\, \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \,\}.$$

Definition 5.2.2 (Intuitionistic Fuzzy Hypergraph)**.** (cf. [794–796]) Let V be a nonempty finite set of *vertices*. An *intuitionistic fuzzy hyperedge* on V is an ordered pair

$$E = (\mu_E, \nu_E),$$

where

$$\mu_E, \nu_E \colon V \longrightarrow [0,1]$$

such that

$$0 \ \le\ \mu_E(v) + \nu_E(v) \ \le\ 1, \quad \forall\, v \in V.$$

Its *support* is defined by

$$\operatorname{supp}(E) \ =\ \bigl\{ v \in V \bigm| \mu_E(v) > 0 \text{ or } \nu_E(v) < 1 \bigr\}.$$

An *intuitionistic fuzzy hypergraph* is a pair

$$H = \bigl(V, \mathcal{E}\bigr),$$

where $\mathcal{E} = \{E_1, \ldots, E_m\}$ is a finite family of intuitionistic fuzzy hyperedges on V satisfying the covering condition

$$\bigcup_{j=1}^{m} \operatorname{supp}(E_j) \ =\ V.$$

The elements of V are called *vertices*, and each $E_j \in \mathcal{E}$ is called an *intuitionistic fuzzy hyperedge*. The *order* of H is $|V|$, and the number of hyperedges is $|\mathcal{E}|$.

Definition 5.2.3 (Intuitionistic fuzzy n-SuperHyperGraph)**.** Let

$$\mathrm{SHG}^{(n)} = (V, E, \partial)$$

be an n-SuperHyperGraph, where $\partial : E \to \mathcal{P}^*(V)$ is the incidence map.

An *intuitionistic fuzzy n-SuperHyperGraph* on $\mathrm{SHG}^{(n)}$ is a tuple

$$\mathcal{H}^{(n)}_{\mathrm{IF}} = (V, E, \partial, \mu_V, \nu_V, \mu_E, \nu_E),$$

where

- $\mu_V, \nu_V : V \to [0,1]$ are the *vertex membership* and *vertex nonmembership* functions, satisfying the Atanassov condition

  $$0 \ \le\ \mu_V(v) + \nu_V(v) \ \le\ 1 \quad \text{for all } v \in V.$$

  For each $v \in V$ the *vertex indeterminacy degree* is

  $$\pi_V(v) := 1 - \mu_V(v) - \nu_V(v) \in [0,1].$$

- $\mu_E, \nu_E : E \times V \to [0,1]$ are the *edge–vertex membership* and *edge–vertex nonmembership* functions, such that
$$\mu_E(e,v) = \nu_E(e,v) = 0 \quad \text{whenever } v \notin \partial(e),$$
and for every $e \in E$ and $v \in \partial(e)$ we have
$$0 \le \mu_E(e,v) + \nu_E(e,v) \le 1.$$
The corresponding *edge–vertex indeterminacy degree* is
$$\pi_E(e,v) := 1 - \mu_E(e,v) - \nu_E(e,v) \in [0,1].$$
- These functions satisfy the *edge–vertex appurtenance constraints*
$$\mu_E(e,v) \le \mu_V(v), \qquad \nu_E(e,v) \le \nu_V(v),$$
for all $e \in E$ and all $v \in \partial(e)$.

The structure $\mathcal{H}_{\mathrm{IF}}^{(n)}$ is called an *intuitionistic fuzzy n-SuperHyperGraph* on the underlying n-SuperHyperGraph $\mathrm{SHG}^{(n)}$.

Example 5.2.4 (Intuitionistic fuzzy 1-SuperHyperGraph: collaboration between two project teams)**.** We model how two cross–functional project teams in a company are jointly assigned to an "AI–Analytics" initiative, with degrees of suitability, opposition, and hesitation.

Step 1: Underlying 1-SuperHyperGraph. Let the base set of employees be
$$V_0 := \{\text{Alice}, \text{Bob}, \text{Carol}\}.$$
At level $n = 1$ the 1-supervertices are subsets of V_0:
$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0).$$
Define two project teams (subsets of employees)
$$v_1 := \{\text{Alice}, \text{Bob}\}, \qquad v_2 := \{\text{Bob}, \text{Carol}\},$$
and set
$$V := \{v_1, v_2\} \subseteq \mathcal{P}(V_0).$$
We consider a single 1-superedge
$$e_{\mathrm{AI}} := \{v_1, v_2\}, \qquad E := \{e_{\mathrm{AI}}\},$$
representing the joint "AI–Analytics" project linking the two teams. The incidence map
$$\partial : E \to \mathcal{P}^*(V)$$
is defined by
$$\partial(e_{\mathrm{AI}}) := \{v_1, v_2\}.$$
Thus
$$\mathrm{SHG}^{(1)} := (V, E, \partial)$$
is a 1-SuperHyperGraph as in Definition 2.2.3.

Step 2: Intuitionistic fuzzy vertex suitability. We now assign to each 1-supervertex $v \in V$ an intuitionistic fuzzy degree of suitability for company–wide strategic projects:
$$\mu_V, \nu_V : V \to [0,1].$$
Let
$$\mu_V(v_1) := 0.90, \quad \nu_V(v_1) := 0.05,$$

$$\mu_V(v_2) := 0.70, \quad \nu_V(v_2) := 0.20.$$

For each $v \in V$ we have

$$0 \leq \mu_V(v) + \nu_V(v) \leq 1,$$

since

$$\mu_V(v_1) + \nu_V(v_1) = 0.90 + 0.05 = 0.95 \leq 1,$$
$$\mu_V(v_2) + \nu_V(v_2) = 0.70 + 0.20 = 0.90 \leq 1.$$

Hence the vertex indeterminacy degrees are

$$\pi_V(v_1) := 1 - \mu_V(v_1) - \nu_V(v_1) = 0.05,$$
$$\pi_V(v_2) := 1 - \mu_V(v_2) - \nu_V(v_2) = 0.10.$$

Here $\mu_V(v_i)$ expresses how suitable team v_i is for strategic projects, $\nu_V(v_i)$ how unsuitable it is perceived to be, and $\pi_V(v_i)$ quantifies remaining hesitation.

Step 3: Intuitionistic fuzzy edge–vertex incidence for the AI project. We next specify intuitionistic fuzzy appurtenance of each team to the particular joint AI project e_{AI}:

$$\mu_E, \nu_E : E \times V \to [0, 1].$$

By definition we set

$$\mu_E(e_{\mathrm{AI}}, v) = \nu_E(e_{\mathrm{AI}}, v) = 0 \quad \text{whenever } v \notin \partial(e_{\mathrm{AI}}),$$

but here $\partial(e_{\mathrm{AI}}) = \{v_1, v_2\}$ so we only need to define values for (e_{AI}, v_1) and (e_{AI}, v_2).

Choose

$$\mu_E(e_{\mathrm{AI}}, v_1) := 0.80, \qquad \nu_E(e_{\mathrm{AI}}, v_1) := 0.03,$$
$$\mu_E(e_{\mathrm{AI}}, v_2) := 0.60, \qquad \nu_E(e_{\mathrm{AI}}, v_2) := 0.10.$$

Then for each incident pair (e_{AI}, v_i) we have

$$0 \leq \mu_E(e_{\mathrm{AI}}, v_i) + \nu_E(e_{\mathrm{AI}}, v_i) \leq 1,$$

indeed

$$\mu_E(e_{\mathrm{AI}}, v_1) + \nu_E(e_{\mathrm{AI}}, v_1) = 0.80 + 0.03 = 0.83 \leq 1,$$
$$\mu_E(e_{\mathrm{AI}}, v_2) + \nu_E(e_{\mathrm{AI}}, v_2) = 0.60 + 0.10 = 0.70 \leq 1.$$

The edge–vertex indeterminacy degrees are therefore

$$\pi_E(e_{\mathrm{AI}}, v_1) := 1 - \mu_E(e_{\mathrm{AI}}, v_1) - \nu_E(e_{\mathrm{AI}}, v_1) = 0.17,$$
$$\pi_E(e_{\mathrm{AI}}, v_2) := 1 - \mu_E(e_{\mathrm{AI}}, v_2) - \nu_E(e_{\mathrm{AI}}, v_2) = 0.30.$$

Finally, the edge–vertex appurtenance constraints hold:

$$\mu_E(e_{\mathrm{AI}}, v_1) = 0.80 \leq \mu_V(v_1) = 0.90, \qquad \nu_E(e_{\mathrm{AI}}, v_1) = 0.03 \leq \nu_V(v_1) = 0.05,$$
$$\mu_E(e_{\mathrm{AI}}, v_2) = 0.60 \leq \mu_V(v_2) = 0.70, \qquad \nu_E(e_{\mathrm{AI}}, v_2) = 0.10 \leq \nu_V(v_2) = 0.20.$$

Real-life interpretation.

- $v_1 = \{\mathrm{Alice}, \mathrm{Bob}\}$ is a highly suitable AI–engineering team with small opposition and mild hesitation.
- $v_2 = \{\mathrm{Bob}, \mathrm{Carol}\}$ is reasonably suitable but has larger non–membership and hesitation (e.g. less experience).
- The superedge e_{AI} represents the joint AI–Analytics project; its intuitionistic fuzzy incidence values μ_E, ν_E, π_E reflect how strongly each team is actually committed to this specific initiative, constrained by their overall suitability.

Thus the tuple

$$\mathcal{H}^{(1)}_{\mathrm{IF}} := (V, E, \partial, \mu_V, \nu_V, \mu_E, \nu_E)$$

is an intuitionistic fuzzy 1-SuperHyperGraph in the sense of Definition 5.2.3, modeling a real-world collaborative project between two teams with graded support, opposition, and uncertainty.

5.3 Neutrosophic SuperHyperGraph

A Neutrosophic Set assigns independent truth, indeterminacy, and falsity degrees to each element, allowing explicit modeling of incomplete, inconsistent information [797–800]. Moreover, as generalizations of the Neutrosophic Set, concepts such as the Quadripartitioned Neutrosophic Set and the Pentapartitioned Neutrosophic Set are also well known. A *Single-valued Neutrosophic n-Superhypergraph* [801] is a concept that generalizes both the Single-valued Neutrosophic graph [802–804] and the Single-valued Neutrosophic hypergraph [805, 806]. It also extends the notion of a Fuzzy n-Superhypergraph. The formal definition and a representative example are given below(cf. [109]).

Definition 5.3.1 (Single–valued Neutrosophic Set)**.** [64, 800] Let X be a nonempty universe. A *single–valued neutrosophic set A* on X is described by a triple of functions

$$T_A,\ I_A,\ F_A : X \longrightarrow [0,1],$$

such that for every $x \in X$,

$$0 \ \le\ T_A(x) + I_A(x) + F_A(x) \ \le\ 3.$$

Here $T_A(x), I_A(x)$, and $F_A(x)$ denote, respectively, the degrees of truth–membership, indeterminacy–membership, and falsity–membership of x with respect to A. We write

$$A \ =\ \{\, \langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X \,\}.$$

A fuzzy set is recovered when $I_A(x) = 0$ and $F_A(x) = 1 - T_A(x)$ for all x.

Definition 5.3.2 (Single-Valued Neutrosophic Graph)**.** [64] Let $G^* = (V, E)$ be a crisp (classical) graph, where V is the vertex set and $E \subseteq V \times V$ the edge set. A *single-valued neutrosophic graph* (SVNG) on G^* is defined as a pair

$$G \ =\ (A, B),$$

where

- $A = \{\langle v, T_A(v), I_A(v), F_A(v)\rangle : v \in V\}$ is the *single-valued neutrosophic vertex set*, with

  $$T_A, I_A, F_A : V \to [0,1],$$

  denoting respectively the *truth-membership*, *indeterminacy-membership*, and *falsity-membership* functions of vertices, such that for every $v \in V$,

  $$0 \ \le\ T_A(v) + I_A(v) + F_A(v) \ \le\ 3.$$

- $B = \{\langle uv, T_B(uv), I_B(uv), F_B(uv)\rangle : uv \in E\}$ is the *single-valued neutrosophic edge set*, with

  $$T_B, I_B, F_B : E \to [0,1],$$

  satisfying for all $u, v \in V$ with $uv \in E$:

  $$T_B(uv) \ \le\ \min\{T_A(u), T_A(v)\}, \qquad I_B(uv) \ \le\ \min\{I_A(u), I_A(v)\}, \qquad F_B(uv) \ \ge\ \max\{F_A(u), F_A(v)\}.$$

If B is symmetric, $G = (A, B)$ is called an *undirected SVNG*; otherwise, it is a *directed SVNG*.

For reference, a compact comparison of a Fuzzy Graph and a Single-Valued Neutrosophic Graph (SVNG) is provided in Table 5.7. As the table illustrates, neutrosophic structures handle ambiguity and uncertainty in a more granular manner, and therefore the study of Neutrosophic Graphs is just as important as that of Fuzzy Graphs.

In what follows, we continue the discussion by introducing the definition of a Single-Valued Neutrosophic Hypergraph.

Table 5.7: Compact comparison: Fuzzy Graph vs. Single-Valued Neutrosophic Graph (SVNG).

Aspect	Fuzzy Graph	Single-Valued Neutrosophic Graph (SVNG) [64]
Underlying crisp object	A crisp graph $G^* = (V, E)$ (finite, undirected in the usual fuzzy-graph setting).	A crisp graph $G^* = (V, E)$ (directed or undirected; undirected when the edge set B is symmetric).
Vertex labeling	One membership degree $\sigma : V \to [0, 1]$.	Three membership degrees $T_A, I_A, F_A : V \to [0, 1]$ (truth, indeterminacy, falsity) with $0 \leq T_A(v) + I_A(v) + F_A(v) \leq 3$.
Edge labeling	One membership degree $\mu : V \times V \to [0, 1]$ (often written $\mu(uv)$ for $uv \in E$).	Three membership degrees $T_B, I_B, F_B : E \to [0, 1]$ attached to each $uv \in E$.
Admissibility constraints	For all $u, v \in V$, $$\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}.$$	For all $uv \in E$, $$T_B(uv) \leq \min\{T_A(u), T_A(v)\}, \quad I_B(uv) \leq \min\{I_A(u), I_A(v)\},$$ $$F_B(uv) \geq \max\{F_A(u), F_A(v)\}.$$
Uncertainty representation	Uncertainty is encoded by a single grade (degree of membership).	Uncertainty is decomposed into three independent grades (truth/indeterminacy/falsity).
Special case relation	–	If one sets $I_A \equiv 0, F_A \equiv 0$ and identifies $\sigma := T_A$ (and similarly $\mu := T_B$), then the SVNG constraints reduce to the fuzzy-graph constraint.
Typical use	Graded presence/strength of vertices and edges (e.g., similarity, trust, reliability).	Modeling truth with explicit indeterminacy and falsity (richer uncertainty in decision models).

Definition 5.3.3 (Single-Valued Neutrosophic Hypergraph)**.** [742, 743, 806, 807] Let $V = \{v_1, \ldots, v_N\}$ be a finite vertex set, and let $\{E_i\}_{i=1}^{M}$ be a collection of non-empty neutrosophic subsets of V such that $V = \bigcup_{i=1}^{M} \text{supp}(E_i)$. Each hyperedge E_i is specified by three membership functions

$$T_{E_i},\ I_{E_i},\ F_{E_i} : V \to [0, 1],$$

assigning to each vertex $v \in V$ its truth, indeterminacy, and falsity degrees, respectively, and satisfying

$$0 \ \leq \ T_{E_i}(v) + I_{E_i}(v) + F_{E_i}(v) \ \leq \ 3 \quad \forall\, v \in V.$$

We represent E_i as the set

$$E_i \ = \ \big\{(v,\, T_{E_i}(v),\, I_{E_i}(v),\, F_{E_i}(v)) : v \in V\big\}.$$

The pair $H = (V, \{E_i\})$ is called a *single-valued neutrosophic hypergraph*.

Definition 5.3.4 (Neutrosophic n-Superhypergraph)**.** (cf. [109, 801]) Let V_0 be a finite *base set* of vertices, and for each integer $k \geq 0$ define

$$\mathcal{P}^0(V_0) = V_0,$$
$$\mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big),$$

where $\mathcal{P}(\cdot)$ denotes the usual powerset. An *n-Superhypergraph* is a pair

$$\text{SHG}^{(n)} = (V,\, E), \quad V \ \subseteq \ \mathcal{P}^n(V_0), \ \ E \ \subseteq \ \mathcal{P}^n(V_0).$$

A *Neutrosophic n-Superhypergraph* is then the tuple

$$\big(V,\, E,\, T_V,\, I_V,\, F_V,\, T_E,\, I_E,\, F_E\big),$$

where

- $T_V,\, I_V,\, F_V : V \to [0, 1]$ assign to each n-supervertex $v \in V$ its truth-, indeterminacy-, and falsity-membership degrees, respectively, subject to

$$0 \ \leq \ T_V(v) + I_V(v) + F_V(v) \ \leq \ 3,$$

$$\forall v \in V.$$

- $T_E, I_E, F_E : E \times V \to [0,1]$ assign to each n-superedge $e \in E$ and vertex $v \in e$ its truth-, indeterminacy-, and falsity-membership degrees, respectively, subject to

$$0 \le T_E(e,v) + I_E(e,v) + F_E(e,v) \le 3,$$
$$\forall e \in E, \ \forall v \in e.$$

These functions satisfy the *edge-appurtenance constraints*:

$$T_E(e,v) \le T_V(v),$$
$$I_E(e,v) \le I_V(v),$$
$$F_E(e,v) \le F_V(v),$$
$$\forall e \in E, \ \forall v \in e.$$

Example 5.3.5 (Neutrosophic 2-Superhypergraph: uncertain adoption of program bundles)**.** We model uncertain adoption of multi–course program bundles across two university campuses by a Neutrosophic 2-Superhypergraph.

Step 1: Base set and 2-supervertices. Let the base set of atomic courses be

$$V_0 := \{\text{Math101}, \text{CS101}, \text{AI201}, \text{DS201}\}.$$

Then

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^1(V_0) = \mathcal{P}(V_0), \qquad \mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

Interpret elements of $\mathcal{P}^1(V_0)$ as *modules* (sets of courses). Define three modules

$$M_{\text{core}} := \{\text{Math101}, \text{CS101}\}, \quad M_{\text{AI}} := \{\text{CS101}, \text{AI201}\}, \quad M_{\text{DS}} := \{\text{AI201}, \text{DS201}\},$$

so $M_{\text{core}}, M_{\text{AI}}, M_{\text{DS}} \in \mathcal{P}^1(V_0)$.

Now form *program bundles* as subsets of $\mathcal{P}(V_0)$, hence elements of $\mathcal{P}^2(V_0)$:

$$v_{\text{AI-track}} := \{M_{\text{core}}, M_{\text{AI}}, M_{\text{DS}}\}, \qquad v_{\text{DS-track}} := \{M_{\text{core}}, M_{\text{DS}}\}.$$

Both $v_{\text{AI-track}}$ and $v_{\text{DS-track}}$ are subsets of $\mathcal{P}(V_0)$, so

$$v_{\text{AI-track}}, v_{\text{DS-track}} \in \mathcal{P}\big(\mathcal{P}(V_0)\big) = \mathcal{P}^2(V_0).$$

Set the 2-supervertex set

$$V := \{v_{\text{AI-track}}, v_{\text{DS-track}}\} \subseteq \mathcal{P}^2(V_0).$$

Step 2: Underlying 2-Superhypergraph edges. We consider two campuses:

- Campus A offers only the AI track bundle.
- Campus B offers both AI and Data Science (DS) track bundles.

Let

$$E := \{e_{\text{A}}, e_{\text{B}}\},$$

where e_{A} and e_{B} are *edge identifiers* for “Campus A offering” and “Campus B offering”, respectively. We take the incidence map

$$\partial : E \to \mathcal{P}^*(V)$$

to be

$$\partial(e_{\text{A}}) := \{v_{\text{AI-track}}\}, \qquad \partial(e_{\text{B}}) := \{v_{\text{AI-track}}, v_{\text{DS-track}}\}.$$

Then

$$\text{SHG}^{(2)} := (V, E, \partial)$$

is a 2-Superhypergraph in the sense of Definition 2.2.3.

Step 3: Neutrosophic degrees on 2-supervertices. We now equip $\text{SHG}^{(2)}$ with Neutrosophic degrees (T_V, I_V, F_V) as in Definition 5.3.4. Interpret:

- $T_V(v)$ = degree that bundle v is *truly* an institutional "core strategic program";
- $I_V(v)$ = degree of *indeterminacy* (ongoing negotiation);
- $F_V(v)$ = degree that v is *rejected* as strategic.

Define

$$T_V(v_{\text{AI-track}}) := 0.90, \quad I_V(v_{\text{AI-track}}) := 0.05, \quad F_V(v_{\text{AI-track}}) := 0.05,$$
$$T_V(v_{\text{DS-track}}) := 0.70, \quad I_V(v_{\text{DS-track}}) := 0.20, \quad F_V(v_{\text{DS-track}}) := 0.10.$$

For each $v \in V$,

$$0 \le T_V(v) + I_V(v) + F_V(v) \le 3,$$

indeed

$$T_V(v_{\text{AI-track}}) + I_V(v_{\text{AI-track}}) + F_V(v_{\text{AI-track}}) = 0.90 + 0.05 + 0.05 = 1,$$
$$T_V(v_{\text{DS-track}}) + I_V(v_{\text{DS-track}}) + F_V(v_{\text{DS-track}}) = 0.70 + 0.20 + 0.10 = 1.$$

Step 4: Neutrosophic edge–vertex degrees. Next we define $(T_E, I_E, F_E) : E \times V \to [0, 1]$. We set these functions to 0 whenever $v \notin \partial(e)$ and specify values only for incident pairs (e, v).

Campus A. Campus A strongly adopts the AI track:

$$T_E(e_{\text{A}}, v_{\text{AI-track}}) := 0.85, \quad I_E(e_{\text{A}}, v_{\text{AI-track}}) := 0.04, \quad F_E(e_{\text{A}}, v_{\text{AI-track}}) := 0.01.$$

Then

$$T_E(e_{\text{A}}, v_{\text{AI-track}}) + I_E(e_{\text{A}}, v_{\text{AI-track}}) + F_E(e_{\text{A}}, v_{\text{AI-track}}) = 0.85 + 0.04 + 0.01 = 0.90 \le 3,$$

and the appurtenance constraints hold:

$$T_E(e_{\text{A}}, v_{\text{AI-track}}) = 0.85 \le T_V(v_{\text{AI-track}}) = 0.90,$$
$$I_E(e_{\text{A}}, v_{\text{AI-track}}) = 0.04 \le I_V(v_{\text{AI-track}}) = 0.05,$$
$$F_E(e_{\text{A}}, v_{\text{AI-track}}) = 0.01 \le F_V(v_{\text{AI-track}}) = 0.05.$$

For $v_{\text{DS-track}} \notin \partial(e_{\text{A}})$ we set

$$T_E(e_{\text{A}}, v_{\text{DS-track}}) = I_E(e_{\text{A}}, v_{\text{DS-track}}) = F_E(e_{\text{A}}, v_{\text{DS-track}}) := 0.$$

Campus B. Campus B offers both tracks but with slightly lower certainty for the DS track:

$$T_E(e_{\text{B}}, v_{\text{AI-track}}) := 0.80, \quad I_E(e_{\text{B}}, v_{\text{AI-track}}) := 0.05, \quad F_E(e_{\text{B}}, v_{\text{AI-track}}) := 0.05,$$
$$T_E(e_{\text{B}}, v_{\text{DS-track}}) := 0.60, \quad I_E(e_{\text{B}}, v_{\text{DS-track}}) := 0.15, \quad F_E(e_{\text{B}}, v_{\text{DS-track}}) := 0.05.$$

Again, for each incident pair (e_{B}, v),

$$0 \le T_E(e_{\text{B}}, v) + I_E(e_{\text{B}}, v) + F_E(e_{\text{B}}, v) \le 3,$$

since

$$0.80 + 0.05 + 0.05 = 0.90, \qquad 0.60 + 0.15 + 0.05 = 0.80.$$

The appurtenance constraints also hold:

$$T_E(e_{\text{B}}, v_{\text{AI-track}}) = 0.80 \le T_V(v_{\text{AI-track}}) = 0.90,$$
$$I_E(e_{\text{B}}, v_{\text{AI-track}}) = 0.05 \le I_V(v_{\text{AI-track}}) = 0.05,$$
$$F_E(e_{\text{B}}, v_{\text{AI-track}}) = 0.05 \le F_V(v_{\text{AI-track}}) = 0.05,$$
$$T_E(e_{\text{B}}, v_{\text{DS-track}}) = 0.60 \le T_V(v_{\text{DS-track}}) = 0.70,$$
$$I_E(e_{\text{B}}, v_{\text{DS-track}}) = 0.15 \le I_V(v_{\text{DS-track}}) = 0.20,$$
$$F_E(e_{\text{B}}, v_{\text{DS-track}}) = 0.05 \le F_V(v_{\text{DS-track}}) = 0.10.$$

Interpretation.

- Level 0 (V_0): individual courses.
- Level 1 ($\mathcal{P}^1(V_0)$): course modules (core, AI, DS).
- Level 2 ($\mathcal{P}^2(V_0)$): program bundles, each a finite family of modules; these are the 2-supervertices $v_{\text{AI-track}}$ and $v_{\text{DS-track}}$.
- Superedges $e_{\text{A}}, e_{\text{B}}$ encode which bundles each campus offers. Their Neutrosophic degrees (T_E, I_E, F_E) describe, for each campus and bundle, the truth, indeterminacy, and falsity of the claim "this bundle is effectively implemented here", bounded by the campus–independent degrees (T_V, I_V, F_V) attached to each bundle.

Thus

$$\big(V, E, T_V, I_V, F_V, T_E, I_E, F_E\big)$$

is a Neutrosophic 2-Superhypergraph, modelling real-world uncertainty about the adoption of multi–course program bundles across two campuses.

5.4 Plithogenic SuperHyperGraph

Plithogenic set assigns multi-criteria membership vectors to elements, modulated by contradiction degrees between attribute values and dominance levels interactions globally [808–811]. Plithogenic graph attaches plithogenic sets to vertices and edges, modeling networks where attribute contradictions influence weighted connectivity patterns dynamically significantly [69, 740, 812]. Plithogenic hypergraph generalizes plithogenic graphs, assigning plithogenic memberships to hyperedges connecting arbitrary vertex subsets under contradictory attributes and uncertainties simultaneously [70]. Plithogenic SuperHyperGraph equips multi-level supervertices and superedges with plithogenic attribute degrees, capturing hierarchical contradictions across nested interaction structures faithfully everywhere [2, 13, 14, 16].

Definition 5.4.1 (Plithogenic Set)**.** [808, 809] Let P be a nonempty universe of discourse, and let v be a (fixed) attribute whose possible values form a nonempty set Pv. Fix dimensions $s, t \in \mathbb{N}$.

A *plithogenic set* on (P, v, Pv) is a quintuple

$$PS = (P, v, Pv, pdf, pCF),$$

where

- $pdf : P \times Pv \longrightarrow [0,1]^s$ is the *degree of appurtenance function* (DAF); for $x \in P$ and $a \in Pv$, $pdf(x, a)$ is the (possibly vector–valued) membership degree of x corresponding to the attribute value a;
- $pCF : Pv \times Pv \longrightarrow [0,1]^t$ is the *degree of contradiction function* (DCF), satisfying

$$pCF(a, a) = 0, \qquad pCF(a, b) = pCF(b, a) \quad \text{for all } a, b \in Pv.$$

In plithogenic theory, a (typically fixed) *dominant attribute value* $a^* \in Pv$ is chosen, and set–theoretic operations (such as union and intersection) are defined by combining the appurtenance degrees pdf with the contradiction degrees $pCF(\cdot, a^*)$ in order to model interaction and opposition between different attribute values.

Example 5.4.2 (Plithogenic set: smartphone choice by reliability)**.** Let $P := \{p_1, p_2, p_3\}$ be three smartphone models and let the attribute be v = "reliability". Take

$$Pv := \{\text{High}, \text{Medium}, \text{Low}\}, \qquad s = t = 1,$$

and choose the dominant attribute value $a^* := \text{High}$.

Define the degree of appurtenance function $pdf : P \times Pv \to [0,1]$ by

$pdf(x, a)$	High	Medium	Low
p_1	0.85	0.20	0.05
p_2	0.55	0.45	0.10
p_3	0.25	0.40	0.70

and define the degree of contradiction function $pCF : Pv \times Pv \to [0,1]$ by

$$pCF(a,a) = 0 \ (a \in Pv), \qquad pCF(\text{High}, \text{Medium}) = 0.3, \quad pCF(\text{High}, \text{Low}) = 0.9, \quad pCF(\text{Medium}, \text{Low}) = 0.6,$$

extended to all pairs by symmetry. Then

$$PS := (P, v, Pv, pdf, pCF)$$

is a plithogenic set on (P, v, Pv): each p_i has a membership degree for each attribute value, and $pCF(\cdot,\cdot)$ quantifies the opposition between attribute values relative to the dominant value High.

Definition 5.4.3 (Plithogenic Graph)**.** (cf. [69, 813]) Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
   - $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.
2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
   - $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x,a), (y,b) \in M \times Ml$:

   $$bdf\,((xy),(a,b)) \le \min\{adf(x,a), adf(y,b)\}$$

   where $xy \in N$ is an edge between vertices x and y, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.
2. *Contradiction Function Constraint*: For all $(a,b), (c,d) \in Nm \times Nm$:

   $$bCf\,((a,b),(c,d)) \le \min\{aCf(a,c), aCf(b,d)\}$$

3. *Reflexivity and Symmetry of Contradiction Functions*:

   $$\begin{aligned} aCf(a,a) &= 0, && \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), && \forall a,b \in Ml \\ bCf(a,a) &= 0, && \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

Example 5.4.4 (Plithogenic set: smartphone choice by reliability)**.** Let $P := \{p_1, p_2, p_3\}$ be three smartphone models and let the attribute be v = "reliability". Take

$$Pv := \{\text{High}, \text{Medium}, \text{Low}\}, \qquad s = t = 1,$$

and choose the dominant attribute value $a^* :=$ High.

Define the degree of appurtenance function $pdf : P \times Pv \to [0, 1]$ by

$pdf(x, a)$	High	Medium	Low
p_1	0.85	0.20	0.05
p_2	0.55	0.45	0.10
p_3	0.25	0.40	0.70

and define the degree of contradiction function $pCF : Pv \times Pv \to [0, 1]$ by

$$pCF(a, a) = 0\ (a \in Pv), \qquad pCF(\text{High}, \text{Medium}) = 0.3, \quad pCF(\text{High}, \text{Low}) = 0.9, \quad pCF(\text{Medium}, \text{Low}) = 0.6,$$

extended to all pairs by symmetry. Then

$$PS := (P, v, Pv, pdf, pCF)$$

is a plithogenic set on (P, v, Pv): each p_i has a membership degree for each attribute value, and $pCF(\cdot, \cdot)$ quantifies the opposition between attribute values relative to the dominant value High.

Definition 5.4.5 (Plithogenic Hypergraph)**.** (cf. [70]) Let V be a finite set of vertices and $E \subseteq \mathcal{P}(V)$ a family of hyperedges. A *plithogenic vertex system* is a tuple

$$PM = (V, \ell, M_\ell, \text{adf}, \text{aCf}),$$

where

- ℓ is a vertex-attribute,
- M_ℓ is the finite set of possible attribute values,
- adf : $V \times M_\ell \to [0, 1]^s$ is the degree-of-appurtenance function,
- aCf : $M_\ell \times M_\ell \to [0, 1]^t$ is the degree-of-contradiction function,

satisfying $\text{aCf}(a, a) = 0$ and symmetry. Similarly, a *plithogenic hyperedge system* is

$$PN = (E, m, N_m, \text{bdf}, \text{bCf}),$$

with

- m an edge-attribute,
- N_m its value set,
- bdf : $E \times N_m \to [0, 1]^s$ the edge-appurtenance function,
- bCf : $N_m \times N_m \to [0, 1]^t$ the edge-contradiction function,

satisfying analogous reflexivity and symmetry. The tuple

$$H = (PM, PN)$$

is called a *plithogenic hypergraph* if for every $e = \{x, y, \dots\} \in E$ and every attribute-value combination $(a, b, \dots) \in M_\ell \times N_m$ the following hold:

$$\text{bdf}(e, (a, b, \dots)) \le \min\{\text{adf}(x, a), \text{adf}(y, b), \dots\},$$
$$\text{bCf}(\alpha, \beta) \le \min\{\text{aCf}(\alpha), \text{aCf}(\beta)\}.$$

Example 5.4.6 (Plithogenic hypergraph: project teams with reliability labels)**.** Let

$$V := \{a, b, c, d\}, \qquad E := \big\{\{a, b, c\}, \{b, d\}\big\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Take the vertex-attribute ℓ = "availability" with

$$M_\ell := \{\text{High}, \text{Low}\}, \qquad s = t = 1,$$

and define adf : $V \times M_\ell \to [0, 1]$ by

$$\text{adf}(a, \text{High}) = 0.8,\ \text{adf}(a, \text{Low}) = 0.2, \quad \text{adf}(b, \text{High}) = 0.6,\ \text{adf}(b, \text{Low}) = 0.4,$$

$$\text{adf}(c, \text{High}) = 0.5,\ \text{adf}(c, \text{Low}) = 0.5, \quad \text{adf}(d, \text{High}) = 0.3,\ \text{adf}(d, \text{Low}) = 0.7.$$

Let aCf satisfy $\text{aCf}(u, u) = 0$ and $\text{aCf}(\text{High}, \text{Low}) = 0.9$ (symmetric).

Let the hyperedge-attribute be m = "team stability" with

$$N_m := \{\text{Stable}, \text{Unstable}\}.$$

Define bdf : $E \times N_m \to [0, 1]$ by

$$\text{bdf}(\{a, b, c\}, \text{Stable}) = 0.5, \qquad \text{bdf}(\{b, d\}, \text{Unstable}) = 0.4,$$

and define bCf with $\text{bCf}(u, u) = 0$ and $\text{bCf}(\text{Stable}, \text{Unstable}) = 0.8$ (symmetric).

For the hyperedge $e = \{a, b, c\}$, choosing the attribute-values (High, High, High) for (a, b, c) gives

$$\min\{\text{adf}(a, \text{High}), \text{adf}(b, \text{High}), \text{adf}(c, \text{High})\} = \min\{0.8, 0.6, 0.5\} = 0.5,$$

hence $\text{bdf}(e, \text{Stable}) = 0.5$ satisfies the plithogenic appurtenance bound. For $e' = \{b, d\}$, choosing (Low, Low) yields

$$\min\{\text{adf}(b, \text{Low}), \text{adf}(d, \text{Low})\} = \min\{0.4, 0.7\} = 0.4,$$

hence $\text{bdf}(e', \text{Unstable}) = 0.4$ is admissible. Thus $H = (PM, PN)$ is a plithogenic hypergraph in the sense of the definition.

Definition 5.4.7 (Plithogenic n-SuperHyperGraph)**.** [2] Let V_0 be a finite base set and let $n \in \mathbb{N}_0$. Consider an n-SuperHyperGraph over V_0 in the sense of Definition 2.2.3, that is,

$$\text{SHG}^{(n)} \;=\; (V, E, \partial),$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of n-supervertices;
- E is a finite set of (super)edge identifiers;
- $\partial : E \to \mathcal{P}^*(V)$ is the incidence map, so that for each $e \in E$, the set $\partial(e) \subseteq V$ is the nonempty incidence set of e.

Fix the same dimensions $s, t \in \mathbb{N}$ as above.

A *plithogenic vertex system* on V is a tuple

$$PM^{(n)} \;=\; \big(V, \ell, M_\ell, \text{adf}^{(n)}, \text{aCF}\big),$$

where

- ℓ is a vertex attribute;
- M_ℓ is a nonempty finite set of possible attribute values for vertices;

- $\mathrm{adf}^{(n)} : V \times M_\ell \to [0,1]^s$ is the (vertex) degree–of–appurtenance function; for $v \in V$ and $a \in M_\ell$, $\mathrm{adf}^{(n)}(v,a)$ encodes the (possibly vector–valued) membership degree of v having attribute value a;
- $\mathrm{aCF} : M_\ell \times M_\ell \to [0,1]^t$ is the (vertex) degree–of–contradiction function, satisfying

$$\mathrm{aCF}(a,a) = 0, \qquad \mathrm{aCF}(a,b) = \mathrm{aCF}(b,a) \quad \text{for all } a,b \in M_\ell.$$

A *plithogenic superedge system* on E is a tuple

$$PN^{(n)} = \big(E, m, N_m, \mathrm{bdf}^{(n)}, \mathrm{bCF}\big),$$

where

- m is a superedge attribute;
- N_m is a nonempty finite set of possible attribute values for superedges;
- $\mathrm{bdf}^{(n)} : E \times N_m \to [0,1]^s$ is the (superedge) degree–of–appurtenance function; for $e \in E$ and $u \in N_m$, $\mathrm{bdf}^{(n)}(e,u)$ encodes the membership degree of e having attribute value u;
- $\mathrm{bCF} : N_m \times N_m \to [0,1]^t$ is the (superedge) degree–of–contradiction function, satisfying

$$\mathrm{bCF}(u,u) = 0, \qquad \mathrm{bCF}(u,v) = \mathrm{bCF}(v,u) \quad \text{for all } u,v \in N_m.$$

For each n-superedge $e \in E$ we assume a prescribed *plithogenic aggregation rule*

$$\beta_e : M_\ell^{\partial(e)} \longrightarrow N_m,$$

which assigns to every family of vertex–attribute values

$$\alpha = (\alpha_v)_{v \in \partial(e)} \in M_\ell^{\partial(e)}$$

a superedge–attribute value $\beta_e(\alpha) \in N_m$.

The triple

$$\textsf{Plith-SHG}^{(n)} := \big(\mathrm{SHG}^{(n)},\ PM^{(n)},\ PN^{(n)}\big)$$

is called a *Plithogenic n-SuperHyperGraph* if, for every $e \in E$ and every family $\alpha = (\alpha_v)_{v\in\partial(e)} \in M_\ell^{\partial(e)}$, the following *plithogenic appurtenance–compatibility* condition holds, componentwise in $[0,1]^s$:

$$\mathrm{bdf}^{(n)}\big(e, \beta_e(\alpha)\big) \le \min_{v \in \partial(e)} \mathrm{adf}^{(n)}\big(v, \alpha_v\big).$$

Here the minimum is taken pointwise in $\mathbb{R}^s$, that is, for each coordinate $j \in \{1,\ldots,s\}$,

$$\mathrm{bdf}^{(n)}\big(e, \beta_e(\alpha)\big)_j \le \min_{v \in \partial(e)} \mathrm{adf}^{(n)}\big(v, \alpha_v\big)_j.$$

We call $\textsf{Plith-SHG}^{(n)}$ a *Plithogenic n-SuperHyperGraph*. For $n = 1$ and $V \subseteq V_0$, E corresponding to nonempty subsets of V_0, this construction reduces to a Plithogenic hypergraph, while for $n = 0$ and $|E| = 1$ it recovers a single plithogenic set on the vertex universe.

Example 5.4.8 (Plithogenic 2-SuperHyperGraph: nested teams and repeated joint tasks)**.** Let the base set be

$$V_0 := \{1,2,3,4\}.$$

Consider two 2-supervertices in $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$:

$$v_1 := \big\{\{1,2\},\{2,3\}\big\}, \qquad v_2 := \big\{\{2,3\},\{3,4\}\big\},$$

and let

$$V := \{v_1, v_2\} \subseteq \mathcal{P}^2(V_0).$$

Let $E := \{e\}$ be a singleton superedge set and define the incidence map

$$\partial(e) := \{v_1, v_2\} \in \mathcal{P}^*(V).$$

Hence $\mathrm{SHG}^{(2)} = (V, E, \partial)$ is a 2-SuperHyperGraph.

Take $s = t = 1$. Let the vertex-attribute be ℓ = "reliability class" with

$$M_\ell := \{\mathrm{High}, \mathrm{Low}\}.$$

Define $\mathrm{adf}^{(2)} : V \times M_\ell \to [0, 1]$ by

$$\mathrm{adf}^{(2)}(v_1, \mathrm{High}) = 0.7,\ \mathrm{adf}^{(2)}(v_1, \mathrm{Low}) = 0.3, \qquad \mathrm{adf}^{(2)}(v_2, \mathrm{High}) = 0.6,\ \mathrm{adf}^{(2)}(v_2, \mathrm{Low}) = 0.4.$$

Let aCF satisfy $\mathrm{aCF}(u, u) = 0$ and $\mathrm{aCF}(\mathrm{High}, \mathrm{Low}) = 0.8$ (symmetric).

Let the superedge-attribute be m = "task criticality" with

$$N_m := \{\mathrm{Critical}, \mathrm{Routine}\}.$$

Define $\mathrm{bdf}^{(2)} : E \times N_m \to [0, 1]$ by

$$\mathrm{bdf}^{(2)}(e, \mathrm{Critical}) = 0.6, \qquad \mathrm{bdf}^{(2)}(e, \mathrm{Routine}) = 0.4.$$

Let bCF satisfy $\mathrm{bCF}(u, u) = 0$ and $\mathrm{bCF}(\mathrm{Critical}, \mathrm{Routine}) = 0.5$ (symmetric).

Define an aggregation rule $\beta_e : M_\ell^{\partial(e)} \to N_m$ by

$$\beta_e(\alpha_{v_1}, \alpha_{v_2}) = \begin{cases} \mathrm{Critical}, & \text{if } \alpha_{v_1} = \alpha_{v_2} = \mathrm{High}, \\ \mathrm{Routine}, & \text{otherwise.} \end{cases}$$

Now take $\alpha_{v_1} = \alpha_{v_2} = \mathrm{High}$. Then

$$\beta_e(\alpha) = \mathrm{Critical}, \qquad \min_{v \in \partial(e)} \mathrm{adf}^{(2)}(v, \alpha_v) = \min\{0.7, 0.6\} = 0.6,$$

so the required compatibility holds:

$$\mathrm{bdf}^{(2)}\big(e, \beta_e(\alpha)\big) = \mathrm{bdf}^{(2)}(e, \mathrm{Critical}) = 0.6 \le 0.6.$$

Therefore

$$\textsf{Plith-SHG}^{(2)} = \big(\mathrm{SHG}^{(2)}, PM^{(2)}, PN^{(2)}\big)$$

with the above data is a concrete Plithogenic 2-SuperHyperGraph.

5.5 Uncertain SuperHyperGraph

An Uncertain Set assigns to each element a degree from an uncertainty model, unifying fuzzy, intuitionistic, neutrosophic and plithogenic frameworks [814]. An Uncertain Graph is a graph where vertices or edges carry degrees in an uncertainty model, subsuming fuzzy, intuitionistic, neutrosophic. An Uncertain HyperGraph assigns uncertainty-model degrees to vertices and hyperedges in a hypergraph, modeling complex higher-order connections under incomplete information. An Uncertain SuperHyperGraph equips each supervertex and superedge in an n-SuperHyperGraph with uncertainty-model degrees, handling hierarchical uncertainty systematically and rigorously. We first recall the notion of an Uncertain Model, which provides the membership–degree domain.

Definition 5.5.1 (Uncertain Model)**.** [814] Let U denote the class of all *uncertain models*. Each $M \in U$ is specified by

- a nonempty set $\mathrm{Dom}(M) \subseteq [0, 1]^k$ of *admissible degree tuples* for some fixed integer $k \geq 1$;

- model–specific algebraic or geometric constraints on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \leq 1$ in the intuitionistic fuzzy case, or $T + I + F \leq 3$ in the neutrosophic case).

Typical examples include:

- Fuzzy model: $\mathrm{Dom}(M) = [0, 1]$;
- Intuitionistic fuzzy model: $\mathrm{Dom}(M) = \{(\mu, \nu) \in [0, 1]^2 \mid \mu + \nu \leq 1\}$;
- Neutrosophic model: $\mathrm{Dom}(M) = \{(T, I, F) \in [0, 1]^3 \mid 0 \leq T + I + F \leq 3\}$;
- Plithogenic model, and many other extensions.

Definition 5.5.2 (Uncertain Set (U-Set))**.** [814] Let X be a nonempty universe, and let M be a fixed uncertain model with degree–domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$. An *Uncertain Set of type M* (or *U-Set* for short) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is called the *uncertainty–degree function* (or membership map) of $\mathcal{U}$.

For $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ encodes the degree(s) to which x belongs to the uncertain set, according to the model M.

Remark 5.5.3. Special cases:

- If M is the fuzzy model and $\mathrm{Dom}(M) = [0, 1]$, then $\mu_M : X \to [0, 1]$ is a usual fuzzy membership function and $\mathcal{U}$ is a fuzzy set.
- If M is neutrosophic, then $\mu_M(x) = (T(x), I(x), F(x))$ gives a neutrosophic set.
- Other choices of M recover intuitionistic fuzzy sets, picture fuzzy sets, plithogenic sets, and so on.

As noted in the remark, various generalizations are possible. For reference, Table 5.8 presents a catalogue of uncertainty-set families (U-Sets) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$ (cf. [739]).

The definitions and related concepts of Uncertain Graphs are presented below.

Definition 5.5.4 (Uncertain Graph)**.** Let $G = (V, E)$ be a (finite, undirected, loopless) graph and let M be an uncertain model with degree–domain $\mathrm{Dom}(M)$. An *Uncertain Graph of type M* is a triple

$$\mathcal{G}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns to each vertex $v \in V$ and each edge $e \in E$ an uncertainty degree $\mu_M(v)$ or $\mu_M(e)$ in $\mathrm{Dom}(M)$.

Optionally, one may impose model–specific consistency conditions between vertex and edge degrees (for instance, $\mu_M(e)$ bounded in terms of $\mu_M(u)$ and $\mu_M(v)$ for $e = \{u, v\}$ in fuzzy or intuitionistic fuzzy graph models), but these constraints are encoded in the choice of M and are not fixed at the level of this general definition.

Remark 5.5.5. Again, particular choices of M recover well–known graph models:

- Fuzzy graph (when M is fuzzy and $\mu_M : V \cup E \to [0, 1]$);
- Intuitionistic fuzzy graph, neutrosophic graph, plithogenic graph, etc., for the corresponding models M.

Table 5.8: A catalogue of uncertainty-set families (U-Sets) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [739].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [62, 746]; N-Fuzzy Set [815–817] Shadowed Set [818–820]
2		Intuitionistic Fuzzy Set [784, 821]; Vague Set [102, 822]; Bipolar Fuzzy Set (two-component description) [823]; Variable Fuzzy Set [824–826]; Paraconsistent Fuzzy Set [827, 828]; Bifuzzy Set [829, 830]
3		Single-Valued Neutrosophic Set [800, 831]; Picture Fuzzy Set [832, 833]; Spherical Fuzzy Set [749, 834]; Tripolar Fuzzy Set (three-component formalisms) [835–837]; Neutrosophic Vague Set [838, 839]
4		Quadripartitioned Neutrosophic Set [840, 841]; Double-Valued Neutrosophic Set [842, 843]; Dual Hesitant Fuzzy Set [844, 845]; Ambiguous Set [846–848]; Turiyam Neutrosophic Set [849–852]
5		Pentapartitioned Neutrosophic Set [853–855]; Triple-Valued Neutrosophic Set [856–858]
6		Hexapartitioned Neutrosophic Set; Quadruple-Valued Neutrosophic Set [857, 859]
7		Heptapartitioned Neutrosophic Set; Quintuple-Valued Neutrosophic Set [857, 860, 861]
8		Octapartitioned Neutrosophic Set [862]
9		Nonapartitioned Neutrosophic Set [862]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [863]; MultiFuzzy Set [864]; n-Refined Fuzzy Set [865, 866]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [866]; Multi-Intuitionistic Fuzzy Set [864]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [866]; Multi-Neutrosophic Set [864, 867]

Reading guide. In the U-Set scheme [814], each model M is specified by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ and a membership map $\mu_M : X \to \mathrm{Dom}(M)$. The table groups representative families by the ambient dimension k (i.e., how many numerical components are stored per element).
(a) A widely cited viewpoint is that neutrosophic sets provide a unifying umbrella covering several earlier multi-component fuzzy models (and their generalizations); see [868].
(b) Ambiguous sets are commonly presented as subclasses of certain four-component neutrosophic families; see [840, 841, 848].
(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [869].

As a reference, Table 5.9 presents a catalogue of uncertainty-graph families (Uncertain Graphs) organised by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

Definition 5.5.6 (Uncertain HyperGraph)**.** Let $H = (V, E)$ be a hypergraph and let M be an uncertain model with degree–domain $\mathrm{Dom}(M)$. An *Uncertain HyperGraph of type M* is a triple

$$\mathcal{H}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns an uncertainty degree to each vertex $v \in V$ and each hyperedge $e \in E$.

As in the graph case, possible relations between vertex and hyperedge degrees (for instance, bounds of $\mu_M(e)$ in terms of $\mu_M(v)$ for $v \in e$) are governed by the chosen model M and its constraints.

Remark 5.5.7. For suitable choices of M, this framework yields fuzzy hypergraphs, intuitionistic fuzzy hypergraphs, neutrosophic hypergraphs, plithogenic hypergraphs, and many further extensions. We present the catalogue of uncertainty-hypergraph families (Uncertain HyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ in Table 5.10.

Definition 5.5.8 (Uncertain n-SuperHyperGraph)**.** Let V_0 be a finite base set and let $n \in \mathbb{N}_0$. Assume that an n-SuperHyperGraph on V_0 is given by

$$\mathrm{SHG}^{(n)} = (V_n, E),$$

where

$$\emptyset \neq V_n \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad \emptyset \neq E \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\},$$

so that each n-superedge $e \in E$ is a nonempty subset of the n-supervertex set V_n.

Let M be a fixed uncertain model with degree–domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *Uncertain n-SuperHyperGraph of type M* is a triple

$$\mathcal{S}_M^{(n)} = (V_n, E, \mu_M),$$

Table 5.9: A catalogue of uncertainty-graph families (Uncertain Graphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-graph type(s) $\mathcal{G}_M = (V, E, \mu_M)$ with $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$
1	Fuzzy graph; N-graph; shadowed-graph variants
2	Intuitionistic fuzzy graph [786]; vague graph [870]; bipolar fuzzy graph [758]; intuitionistic evidence graph; variable fuzzy graph; paraconsistent fuzzy graph; bifuzzy graph [871, 872]
3	Neutrosophic graph [64][(a)]; hesitant fuzzy graph [873]; tripolar fuzzy graph; three-way fuzzy graph; picture fuzzy graph [499, 874]; spherical fuzzy graph [749]; inconsistent intuitionistic fuzzy graph; ternary fuzzy / neutrosophic-fuzzy graph; neutrosophic vague graph
4	Quadripartitioned neutrosophic graph [875, 876]; double-valued neutrosophic graph [842]; dual hesitant fuzzy graph [877]; ambiguous graph[(b)]; local-neutrosophic graph; support-neutrosophic graph; turiyam neutrosophic graph [878][(c)]
5	Pentapartitioned neutrosophic graph [879]; triple-valued neutrosophic graph
6	Hexapartitioned neutrosophic graph; quadruple-valued neutrosophic graph
7	Heptapartitioned neutrosophic graph [880]; quintuple-valued neutrosophic graph
8	Octapartitioned neutrosophic graph
9	Nonapartitioned neutrosophic graph
n	n-refined fuzzy graph; multi-valued (fuzzy) graphs; multi-fuzzy graphs [881]
$2n$	n-refined intuitionistic fuzzy graph; multi-intuitionistic fuzzy graphs
$3n$	n-refined neutrosophic graph; multi-neutrosophic graphs

[(a)] Neutrosophic graph models are often treated as broad frameworks that can specialize to many degree-based graph formalisms under suitable constraints.
[(b)] Ambiguous-graph models are commonly presented as subclasses of certain quadripartitioned and also double-valued neutrosophic graph models.
[(c)] Turiyam neutrosophic graphs are reported as subclasses of certain quadripartitioned neutrosophic graph models.

Table 5.10: A catalogue of uncertainty-hypergraph families (Uncertain HyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-hypergraph family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy HyperGraph [882–884]:* $\mu_M : V \cup E \to [0,1]$.
2	*Intuitionistic-fuzzy HyperGraph [885–887]:* $\mu_M : V \cup E \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic HyperGraph [66, 67, 888, 889]:* $\mu_M : V \cup E \to [0,1]^3$ (e.g., (T, I, F)).
4	*Quadripartitioned Neutrosophic / four-component uncertainty HyperGraph:* $\mu_M : V \cup E \to [0,1]^4$.
5	*Pentapartitioned Neutrosophic / five-component uncertainty HyperGraph:* $\mu_M : V \cup E \to [0,1]^5$.
k	*k-component uncertainty HyperGraph:* $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

where

$$\mu_M : V_n \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns to each n-supervertex $v \in V_n$ and each n-superedge $e \in E$ an uncertainty degree $\mu_M(v)$ or $\mu_M(e)$ in $\mathrm{Dom}(M)$.

Any additional relations between the degrees of n-superedges and the degrees of the n-supervertices they contain (for example, model- specific bounds or aggregations) are imposed by the chosen uncertain model M and are not fixed at the level of this general definition.

For $n = 0$ and $V_0 = V_n$, the above notion reduces to an Uncertain HyperGraph of type M.

Remark 5.5.9. Particular choices of the model M recover well–known uncertain SuperHyperGraph types:

- Fuzzy n-SuperHyperGraphs (when M is fuzzy);
- Intuitionistic fuzzy, neutrosophic, and plithogenic n-SuperHyperGraphs for the corresponding models M;
- More exotic variants (e.g. q-rung orthopair, picture fuzzy, refined neutrosophic) are obtained by choosing the appropriate degree–domain $\mathrm{Dom}(M)$.

Regarding the catalogue of uncertainty-superhypergraph families (Uncertain n-SuperHyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, we list them in Table 5.11.

Table 5.11: A catalogue of uncertainty-superhypergraph families (Uncertain n-SuperHyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-superhypergraph family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy n-SuperHyperGraph [783]:* $\mu_M : V_n \cup E \to [0,1]$.
2	*Intuitionistic-fuzzy n-SuperHyperGraph [783]:* $\mu_M : V_n \cup E \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic n-SuperHyperGraph [68, 109, 801]:* $\mu_M : V_n \cup E \to [0,1]^3$ (e.g., (T, I, F)).
4	*Quadripartitioned / four-component uncertainty n-SuperHyperGraph:* $\mu_M : V_n \cup E \to [0,1]^4$.
k	*k-component uncertainty n-SuperHyperGraph:* $\mu_M : V_n \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

5.6 Functorial SuperHyperGraph

A Functorial Set is a functor assigning each object a set and pushing elements along structure-preserving morphisms in a category [814]. A Functorial Graph functorially assigns each object a graph and maps graph homomorphisms along morphisms, preserving composition and identities everywhere. A Functorial HyperGraph assigns each object a hypergraph and transports hyperedges via hypergraph homomorphisms induced by morphisms, respecting categorical composition. A Functorial SuperHyperGraph associates each object with a superhypergraph and sends morphisms to homomorphisms preserving supervertices, superedges, and hierarchical structure.

Definition 5.6.1 (Functorial Set)**.** [814] Let C be a category and let

$$F : C \longrightarrow \mathbf{Set}$$

be a covariant functor.

We call F a *Functorial Set* on C. For each object $X \in \mathrm{Ob}(C)$, the set

$$F(X)$$

is interpreted as the collection of "F–sets over X", and every element $s \in F(X)$ is an individual F–set based at X.

Every morphism $f : X \to Y$ in C induces a *pushforward*

$$F(f) : F(X) \longrightarrow F(Y), \qquad s \longmapsto F(f)(s),$$

and the usual functoriality conditions

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \qquad F(g \circ f) = F(g) \circ F(f)$$

hold for all composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Definition 5.6.2 (Functorial Graph)**.** Let **Graph** denote the category whose objects are finite, simple, undirected graphs $G = (V, E)$ with

$$E \subseteq \big\{\{u, v\} \mid u, v \in V,\ u \neq v\big\},$$

and whose morphisms $\varphi : G \to G'$ are *graph homomorphisms*, i.e. vertex maps $\varphi : V \to V'$ such that

$$\{u, v\} \in E \implies \{\varphi(u), \varphi(v)\} \in E'.$$

Let C be a category. A *Functorial Graph* on C is a covariant functor

$$\mathsf{G} : C \longrightarrow \mathbf{Graph}.$$

Equivalently, to each object $X \in \mathrm{Ob}(C)$ it assigns a graph

$$\mathsf{G}(X) = (V_X, E_X),$$

and to each morphism $f : X \to Y$ in C it assigns a graph homomorphism

$$\mathsf{G}(f) : \mathsf{G}(X) \longrightarrow \mathsf{G}(Y)$$

such that

$$\mathsf{G}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{G}(X)}, \qquad \mathsf{G}(g \circ f) = \mathsf{G}(g) \circ \mathsf{G}(f)$$

for all composable f, g in C.

Definition 5.6.3 (Functorial HyperGraph)**.** Let **HGraph** denote the category whose objects are finite hypergraphs $H = (V, E)$ with

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

and whose morphisms $\psi : H \to H'$ are *hypergraph homomorphisms*, i.e. vertex maps $\psi : V \to V'$ satisfying

$$\forall e \in E : \psi[e] \in E',$$

where $\psi[e] := \{\psi(v) \mid v \in e\}$ is the image of the hyperedge e.

Let C be a category. A *Functorial HyperGraph* on C is a covariant functor

$$\mathsf{H} : C \longrightarrow \textbf{HGraph}.$$

Equivalently, for each object $X \in \mathrm{Ob}(C)$ it assigns a hypergraph

$$\mathsf{H}(X) = (V_X, E_X),$$

and for each morphism $f : X \to Y$ a hypergraph homomorphism

$$\mathsf{H}(f) : \mathsf{H}(X) \longrightarrow \mathsf{H}(Y),$$

such that

$$\mathsf{H}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{H}(X)}, \qquad \mathsf{H}(g \circ f) = \mathsf{H}(g) \circ \mathsf{H}(f)$$

for all composable f, g in C.

Definition 5.6.4 (Functorial SuperHyperGraph)**.** Fix an integer $n \geq 1$. Let $\textbf{SHGraph}_n$ denote the category whose objects are finite level-n SuperHyperGraphs. Concretely, an object is a triple

$$\mathsf{SH} = (V_0, V, E),$$

where

- V_0 is a finite base set;
- $V \subseteq \mathcal{P}^n(V_0)$ is a nonempty set of n-supervertices;
- $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a nonempty family of n-superedges, each superedge being a nonempty subset of V.

Thus the supervertices live at the n-th iterated powerset level, while the superedges are ordinary (nonempty) subsets of the supervertex set V.

A morphism

$$\Phi : (V_0, V, E) \longrightarrow (V_0', V', E')$$

in $\textbf{SHGraph}_n$ is a *superhypergraph homomorphism*, i.e. a base map $\varphi_0 : V_0 \to V_0'$ such that the induced map on the n-th iterated powerset

$$\varphi_n := \mathcal{P}^n(\varphi_0) : \mathcal{P}^n(V_0) \longrightarrow \mathcal{P}^n(V_0')$$

satisfies

$$\varphi_n(V) \subseteq V' \quad \text{and} \quad \varphi_n[e] := \{\varphi_n(v) \mid v \in e\} \in E' \quad \text{for all } e \in E.$$

Let C be a category. A *Functorial SuperHyperGraph of level n* on C is a covariant functor

$$\mathsf{SH} : C \longrightarrow \textbf{SHGraph}_n.$$

For each object $X \in \mathrm{Ob}(C)$, the value

$$\mathsf{SH}(X) = (V_0^X, V_X, E_X)$$

is a level-n SuperHyperGraph, and for each morphism $f : X \to Y$ in C, the arrow

$$\mathsf{SH}(f) : \mathsf{SH}(X) \longrightarrow \mathsf{SH}(Y)$$

is a superhypergraph homomorphism in the above sense, satisfying

$$\mathsf{SH}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{SH}(X)}, \qquad \mathsf{SH}(g \circ f) = \mathsf{SH}(g) \circ \mathsf{SH}(f)$$

for all composable f, g in C.

In particular, when $n = 0$ and $V = V_0$, a Functorial SuperHyperGraph reduces to a Functorial HyperGraph.

The overview of Functorial Graphs, Functorial HyperGraphs, and Functorial SuperHyperGraphs is presented in Table 5.12.

Table 5.12: A concise overview of Functorial Graphs, Functorial HyperGraphs, and Functorial SuperHyperGraphs.

Concept	**Object assigned to each** $X \in \mathrm{Ob}(C)$	**Morphisms / functoriality requirement**
Functorial Graph	A graph $\mathsf{G}(X) = (V_X, E_X)$ in **Graph**.	Each $f : X \to Y$ induces a graph homomorphism $\mathsf{G}(f) : \mathsf{G}(X) \to \mathsf{G}(Y)$, preserving edges. Moreover, $\mathsf{G}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{G}(X)}$ and $\mathsf{G}(g \circ f) = \mathsf{G}(g) \circ \mathsf{G}(f)$.
Functorial HyperGraph	A hypergraph $\mathsf{H}(X) = (V_X, E_X)$ in **HGraph** with $E_X \subseteq \mathcal{P}(V_X) \setminus \{\emptyset\}$.	Each $f : X \to Y$ induces a hypergraph homomorphism $\mathsf{H}(f) : \mathsf{H}(X) \to \mathsf{H}(Y)$, sending every hyperedge $e \in E_X$ to $\mathsf{H}(f)[e] \in E_Y$. Moreover, $\mathsf{H}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{H}(X)}$ and $\mathsf{H}(g \circ f) = \mathsf{H}(g) \circ \mathsf{H}(f)$.
Functorial SuperHyperGraph (level n)	A level-n SuperHyperGraph $\mathsf{SH}(X) = (V_0^X, V_X, E_X)$ in **SHGraph**$_n$, with $V_X \subseteq \mathcal{P}^n(V_0^X)$ and $E_X \subseteq \mathcal{P}(V_X) \setminus \{\emptyset\}$.	Each $f : X \to Y$ induces a superhypergraph homomorphism $\mathsf{SH}(f) : \mathsf{SH}(X) \to \mathsf{SH}(Y)$ arising from a base map $\varphi_0 : V_0^X \to V_0^Y$ and its lift $\varphi_n = \mathcal{P}^n(\varphi_0)$, preserving supervertices and superedges. Moreover, $\mathsf{SH}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{SH}(X)}$ and $\mathsf{SH}(g \circ f) = \mathsf{SH}(g) \circ \mathsf{SH}(f)$.

5.7 Soft SuperHyperGraph

A soft set is a parameterized family of subsets representing approximate descriptions of objects under flexible, context-dependent attributes or conditions [890, 891]. A soft graph assigns to each parameter a subgraph, modeling uncertain relationships varying with expert choices, scenarios, or contexts over [420, 841, 892]. Fuzzy Soft Graphs [420, 893], HyperSoft Graph [894, 895], Soft Expert Graph [896], Neutrosophic Soft Graphs [897, 898], and other related concepts are also well established in the literature.

A soft hypergraph maps parameters to subhypergraphs, capturing uncertain higher-order interactions among vertex groups under varying criteria, preferences, or environments [72, 805, 899, 900]. Fuzzy soft hypergraph [72, 901, 902], Neutrosophic Soft Hypergraphs [805], and other related concepts are also well established in the literature.

A soft superhypergraph associates each parameter with a sub-superhypergraph, modeling multilevel uncertain connections between nested structures under requirements or viewpoints [801, 903, 904].

Definition 5.7.1 (Soft set [890])**.** Let U be a universe of discourse and E a set of parameters. Let $A \subseteq E$ and

$$F : A \longrightarrow \mathcal{P}(U)$$

be a mapping that assigns to each parameter $e \in A$ a subset $F(e) \subseteq U$ of objects possessing the property e. The pair (F, A) is called a *soft set* over U (with parameter subset A).

Definition 5.7.2 (Soft Graph)**.** [905] Let $G^* = (V, E)$ be a simple graph and A a nonempty set of parameters. Let $S \subseteq A$ be nonempty, and let

$$F : S \to \mathcal{P}(V), \qquad K : S \to \mathcal{P}(E).$$

The triple (F, K, S) is called a *soft graph over* G^* if, for every $a \in S$ and every edge $e = \{u, v\} \in K(a)$, one has $u, v \in F(a)$. Equivalently, for each $a \in S$, the pair

$$G_a := (F(a), K(a))$$

is a (crisp) subgraph of G^* and is called the a-section of the soft graph.

Definition 5.7.3 (Soft Hypergraph)**.** [899, 900] Let $H = (V, E)$ be a hypergraph with $E \subseteq \mathrm{PSET}(V)$, and let C be a nonempty set of parameters. A *soft hypergraph* over H with parameters C is a quadruple

$$\big(H,\ C,\ A,\ B\big),$$

where

$$A : C \longrightarrow \mathrm{PSET}(V), \qquad B : C \longrightarrow \mathrm{PSET}(E),$$

and for each $c \in C$,

$$B(c)\ \subseteq\ \{\, e \in E \mid e \subseteq A(c)\}.$$

The pair $\big(A(c), B(c)\big)$ is called the *soft subhypergraph* of H at parameter c.

Definition 5.7.4 (Soft n-SuperHyperGraph)**.** [904] Let $\mathrm{SHG}(n) = (V, E)$ be an n-SuperHyperGraph and let C be a nonempty set of parameters. A *soft n-SuperHyperGraph* (or soft SuperHyperGraph) over $\mathrm{SHG}(n)$ with parameter set C consists of maps

$$A : C \longrightarrow \mathrm{P}(V), \qquad B : C \longrightarrow \mathrm{P}(E),$$

such that for every parameter $c \in C$ the pair

$$\big(A(c),\ B(c)\big)$$

is a sub-n-SuperHyperGraph of $\mathrm{SHG}(n)$; that is,

$$A(c) \subseteq V, \qquad B(c) \subseteq \{\, e \in E \mid e \subseteq A(c)\,\}.$$

For each $c \in C$, the pair $\big(A(c), B(c)\big)$ is called the *soft slice* (or soft sub-n-SuperHyperGraph) of $\mathrm{SHG}(n)$ at parameter c.

Example 5.7.5 (Soft 2-SuperHyperGraph: public–transport demand scenarios)**.** Consider a small public–transport network.

Let the base set of stops be

$$V_0 := \{s_1, s_2, s_3\},$$

where s_1 is a residential stop, s_2 a business–district stop, and s_3 a shopping–area stop.

On the first level, take the following (crisp) routes as subsets of V_0:

$$r_1 := \{s_1, s_2\}, \qquad r_2 := \{s_2, s_3\}.$$

These are elements of $P(V_0)$. Define two level–2 supervertices by

$$v_A := \{r_1, r_2\}, \qquad v_B := \{r_2\},$$

so that

$$V := \{v_A, v_B\} \subseteq P(P(V_0)) = P_2(V_0).$$

Interpretation: v_A represents a *daily commuting pattern* combining both routes r_1 and r_2, while v_B represents a *shopping–oriented pattern* concentrated on r_2.

Let

$$e_1 := \{v_A, v_B\} \subseteq V,$$

and put $E := \{e_1\}$. Then

$$\mathrm{SHG}(2) := (V, E)$$

is a 2-SuperHyperGraph: the single superedge e_1 models that these two demand patterns interact (e.g. they share vehicles or drivers).

Now let the parameter set be

$$C := \{\text{weekday}, \ \text{weekend}\}.$$

Define

$$A : C \to P(V), \qquad B : C \to P(E)$$

by

$$A(\text{weekday}) := \{v_A, v_B\}, \quad B(\text{weekday}) := \{e_1\},$$
$$A(\text{weekend}) := \{v_B\}, \quad B(\text{weekend}) := \emptyset.$$

For the weekday parameter, the soft slice

$$\big(A(\text{weekday}), B(\text{weekday})\big) = \big(\{v_A, v_B\}, \{e_1\}\big)$$

is a sub–2-SuperHyperGraph of SHG(2) because $A(\text{weekday}) \subseteq V$ and $e_1 \subseteq A(\text{weekday})$. For the weekend parameter, the slice

$$\big(A(\text{weekend}), B(\text{weekend})\big) = \big(\{v_B\}, \emptyset\big)$$

is also a sub–2-SuperHyperGraph: we keep only the shopping–oriented pattern v_B and drop the superedge (no strong interaction constraint is enforced in the weekend timetable).

Thus (A, B) defines a soft 2-SuperHyperGraph that models different service configurations for weekday and weekend demand in a small public–transport system.

5.8 Rough SuperHyperGraph

A rough set represents a subset using lower and upper approximations induced by an equivalence relation modeling indiscernibility among elements [906, 907]. A rough graph describes a graph whose vertex and edge sets are approximated via equivalence classes, capturing uncertainty in connectivity [74, 768, 908, 909]. Fuzzy Rough Graphs [769, 910], Neutrosophic Rough Graphs [911], HyperSoft Rough Graphs [909], and Soft Rough Graphs [74] are also known as related concepts. A rough hypergraph extends rough sets to hypergraphs, approximating vertex and hyperedge families with lower and upper bounds under indiscernibility [167, 912]. A rough superhypergraph applies rough approximations to multilevel supervertices and superedges, using equivalence relations to bound uncertain hierarchical interactions robustly [904].

Definition 5.8.1 (Rough set [913])**.** Let U be a nonempty finite set (universe) and let $R \subseteq U \times U$ be an equivalence relation. For any $X \subseteq U$, the *lower* and *upper* R–approximations of X are defined by

$$R_*(X) := \{\, x \in U \mid [x]_R \subseteq X \,\}, \qquad R^*(X) := \{\, x \in U \mid [x]_R \cap X \neq \emptyset \,\},$$

where $[x]_R := \{\, y \in U \mid (x, y) \in R \,\}$ is the R–equivalence class of x. The pair $\big(R_*(X), R^*(X)\big)$ is called the *rough set* determined by X in the approximation space (U, R).

Definition 5.8.2 (Rough Graph)**.** [768] Let $G = (V, E)$ be a simple graph and let R_E be an equivalence relation on the edge set E. For any $X \subseteq E$, define

$$\underline{R_E}(X) = \{\, e \in E : [e]_{R_E} \subseteq X\}, \qquad \overline{R_E}(X) = \{\, e \in E : [e]_{R_E} \cap X \neq \emptyset\}.$$

The pair $\big(\underline{G}, \overline{G}\big)$, where $\underline{G} = (V, \underline{R_E}(X))$ and $\overline{G} = (V, \overline{R_E}(X))$, is called the *rough graph* approximation of the subedge-set $X \subseteq E$. If X is not a union of equivalence classes under R_E, then G is said to be a *rough graph* with respect to R_E.

Definition 5.8.3 (Rough Hypergraph)**.** [167] Let $H = (V, E)$ be a hypergraph, R_V an equivalence relation on V, and R_E an equivalence relation on E. For any $A \subseteq V$ and $D \subseteq E$, define

$$\underline{R_V}(A) = \{ v \in V : [v]_{R_V} \subseteq A \}, \quad \overline{R_V}(A) = \{ v \in V : [v]_{R_V} \cap A \neq \emptyset \},$$

$$\underline{R_E}(D) = \{ e \in E : [e]_{R_E} \subseteq D \}, \quad \overline{R_E}(D) = \{ e \in E : [e]_{R_E} \cap D \neq \emptyset \}.$$

The tuple

$$\big(V,\ E,\ \underline{R_V},\ \overline{R_V},\ \underline{R_E},\ \overline{R_E}\big)$$

is called a *rough hypergraph*, capturing uncertainty on both vertices and hyperedges via rough set approximations.

Definition 5.8.4 (Rough n-SuperHyperGraph)**.** [904] Let $\mathrm{SHG}(n) = (V, E)$ be an n-SuperHyperGraph. Let φ be an equivalence relation on the n-supervertex set V, and let ψ be an equivalence relation on the n-superedge set E.

For any subset $X \subseteq V$, the *lower* and *upper* φ-approximations of X are defined by

$$\varphi_*(X) := \{ v \in V \mid [v]_\varphi \subseteq X \}, \qquad \varphi^*(X) := \{ v \in V \mid [v]_\varphi \cap X \neq \emptyset \},$$

where $[v]_\varphi := \{ u \in V \mid (u, v) \in \varphi \}$ is the φ-equivalence class of v.

Similarly, for any subset $D \subseteq E$, the *lower* and *upper* ψ-approximations of D are

$$\psi_*(D) := \{ e \in E \mid [e]_\psi \subseteq D \}, \qquad \psi^*(D) := \{ e \in E \mid [e]_\psi \cap D \neq \emptyset \},$$

where $[e]_\psi := \{ f \in E \mid (f, e) \in \psi \}$ is the ψ-equivalence class of e.

The sextuple

$$\big(V,\ E,\ \varphi_*,\ \varphi^*,\ \psi_*,\ \psi^*\big)$$

is called a *rough n-SuperHyperGraph* (or rough SuperHyperGraph) on $\mathrm{SHG}(n)$. It models uncertainty on the n-supervertices and n-superedges of $\mathrm{SHG}(n)$ via the rough approximations induced by φ and ψ.

Example 5.8.5 (Rough 2-SuperHyperGraph: city–logistics service areas)**.** Consider a city–logistics planning problem.

Let the base set of delivery addresses be

$$V_0 := \{a, b, c\},$$

corresponding to three small zones in a city. On the first level, define delivery tours as

$$t_1 := \{a, b\}, \qquad t_2 := \{b, c\}, \qquad t_3 := \{a, c\} \in P(V_0).$$

Define level–2 supervertices

$$v_A := \{t_1, t_2\}, \quad v_B := \{t_2, t_3\}, \quad v_C := \{t_1, t_3\},$$

so that

$$V := \{v_A, v_B, v_C\} \subseteq P(P(V_0)) = P_2(V_0).$$

Interpretation: each v_* is a *delivery pattern*, i.e. a collection of tours serving certain combinations of zones.

Let the superedges be

$$e_1 := \{v_A, v_B\}, \qquad e_2 := \{v_B, v_C\},$$

and put $E := \{e_1, e_2\}$. Then

$$\mathrm{SHG}(2) := (V, E)$$

is a 2-SuperHyperGraph: each superedge groups delivery patterns that share vehicles or depots.

We now introduce roughness on both vertices and superedges to reflect imprecise information.

On V define an equivalence relation φ by the classes

$$[v_A]_\varphi := \{v_A\}, \qquad [v_B]_\varphi := [v_C]_\varphi := \{v_B, v_C\}.$$

Intuitively, v_A is a *central* delivery pattern, while v_B and v_C are grouped together as *peripheral/mixed* patterns: in coarse data, they are indistinguishable.

On E define an equivalence relation ψ by the single class

$$[e_1]_\psi := [e_2]_\psi := \{e_1, e_2\},$$

meaning the two superedges are regarded as one coarse type of *evening delivery coordination*.

Consider the subset

$$X := \{v_B\} \subseteq V,$$

representing "delivery patterns believed to mainly serve the peripheral zones." The φ–approximations of X are

$$\varphi_*(X) = \{\, v \in V \mid [v]_\varphi \subseteq X \,\} = \emptyset,$$

because none of the equivalence classes is entirely contained in X, and

$$\varphi^*(X) = \{\, v \in V \mid [v]_\varphi \cap X \neq \emptyset \,\} = \{v_B, v_C\}.$$

Thus, only the rough upper approximation can capture the intended peripheral behavior: v_C is forced into $\varphi^*(X)$ since it shares the same coarse profile as v_B.

Similarly, for the subset of superedges

$$D := \{e_1\} \subseteq E,$$

representing "clearly observed evening coordination," we obtain

$$\psi_*(D) = \emptyset, \qquad \psi^*(D) = \{e_1, e_2\},$$

because the only equivalence class $\{e_1, e_2\}$ is not contained in D but intersects D nontrivially.

The sextuple

$$\bigl(V,\, E,\, \varphi_*,\, \varphi^*,\, \psi_*,\, \psi^*\bigr)$$

therefore forms a rough 2-SuperHyperGraph modeling uncertain knowledge about which delivery patterns and coordination links are central versus peripheral in the city–logistics network.

5.9 Fuzzy Directed n-Superhypergraph

A fuzzy directed graph orients edges and assigns each vertex and arc a membership degree expressing directional connection uncertainty levels [914–916]. A fuzzy directed hypergraph generalizes directed graphs, using hyperarcs between vertex sets with fuzzy membership degrees modelling uncertain multiway influence [311, 917, 918]. We define the *Fuzzy Directed n-Superhypergraph* as an extension of the classical *Fuzzy Directed Hypergraph* (cf. [311, 917, 918]) by incorporating the hierarchical structure of n-Superhypergraphs (cf. [12]).

Definition 5.9.1 (Fuzzy Directed Hypergraph)**.** (cf. [311, 917, 919, 920]) Let V be a nonempty set of *vertices*. A *fuzzy directed hypergraph* is a quadruple

$$H = \bigl(V,\ E,\ \sigma,\ \mu\bigr),$$

where

- E is a finite set of *directed hyperarcs*. Each $e \in E$ is an ordered pair $\bigl(T(e),\ H(e)\bigr)$ with

  $$T(e)\ \subseteq\ V, \quad T(e) \neq \emptyset, \qquad H(e)\ \subseteq\ V \setminus T(e)$$

  called its *tail* and *head*, respectively.

- $\sigma : V \to [0,1]$ assigns to each vertex $v \in V$ a membership degree $\sigma(v)$.
- $\mu : E \to [0,1]$ assigns to each hyperarc $e \in E$ a membership degree $\mu(e)$.

These functions satisfy the *consistency constraint*

$$\mu(e) \le \min_{x \in T(e) \cup H(e)} \sigma(x), \quad \forall e \in E.$$

Definition 5.9.2 (Fuzzy Directed n-Superhypergraph)**.** Let S be a nonempty *base set* and let $n \ge 0$ be an integer. Define iterated powersets by

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^{k+1}(S) = \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \ge 0).$$

A *directed n-Superhypergraph* is a pair $\mathrm{DSHG}^{(n)} = (V, E)$ with

$$V \subseteq \mathcal{P}^n(S), \qquad E \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S),$$

where each $e \in E$ is of the form $\big(\mathrm{Tail}(e), \mathrm{Head}(e)\big)$. A *fuzzy directed n-Superhypergraph* is then the quadruple

$$\big(V,\ E,\ \sigma,\ \mu\big),$$

where

- $\sigma : V \to [0,1]$ assigns to each n-supervertex v a membership degree $\sigma(v)$.
- $\mu : E \to [0,1]$ assigns to each directed n-superedge e a membership degree $\mu(e)$.

These satisfy the *edge-appurtenance constraint*

$$\mu(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x),$$

$$\forall e \in E.$$

Example 5.9.3 (Fuzzy directed 1-SuperHyperGraph: emergency alert routing)**.** Consider a small city with three emergency sensor stations

$$S := \{s_1, s_2, s_3\},$$

for example "downtown" (s_1), "riverside" (s_2), and "industrial area" (s_3). We take $n = 1$, so $\mathcal{P}^1(S) = \mathcal{P}(S)$, and define the set of 1-supervertices by

$$v_1 := \{s_1, s_2\}, \quad v_2 := \{s_2, s_3\}, \quad v_3 := \{s_1, s_3\},$$

representing overlapping monitoring zones. Then

$$V := \{v_1, v_2, v_3\} \subseteq \mathcal{P}^1(S).$$

We define two directed 1-superedges:

$$e_1 := \big(\{v_1\},\ \{v_2, v_3\}\big), \qquad e_2 := \big(\{v_2\},\ \{v_3\}\big),$$

where e_1 models a possible alert propagation from the "downtown" zone v_1 to the other zones v_2 and v_3, and e_2 models an alert from v_2 to v_3. Put

$$E := \{e_1, e_2\} \subseteq \mathcal{P}^1(S) \times \mathcal{P}^1(S).$$

Next, we assign fuzzy membership degrees to supervertices and superedges. Let

$$\sigma : V \to [0,1], \qquad \sigma(v_1) = 0.9,\ \sigma(v_2) = 0.7,\ \sigma(v_3) = 0.8,$$

where $\sigma(v_i)$ expresses the reliability/health of the monitoring zone v_i (e.g., sensor uptime and communication quality).

For the directed superedges, define

$$\mu : E \to [0,1], \qquad \mu(e_1) = 0.7, \quad \mu(e_2) = 0.6.$$

We check the edge–appurtenance constraint

$$\mu(e) \;\le\; \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x) \quad \text{for all } e \in E.$$

Indeed,

$$\mathrm{Tail}(e_1) \cup \mathrm{Head}(e_1) = \{v_1, v_2, v_3\}, \quad \min \sigma = \min\{0.9, 0.7, 0.8\} = 0.7 \ge \mu(e_1),$$

and

$$\mathrm{Tail}(e_2) \cup \mathrm{Head}(e_2) = \{v_2, v_3\}, \quad \min \sigma = \min\{0.7, 0.8\} = 0.7 \ge \mu(e_2).$$

Thus (V, E, σ, μ) is a fuzzy directed 1-Superhypergraph in the sense of Definition 5.9.2.

Intuitively, the supervertices model overlapping sensor zones in the city, while the directed superedges model possible multi-zone alert propagation patterns. The vertex degrees σ quantify how reliable each zone is, and the edge degrees μ quantify the confidence that an emergency alert successfully flows along each multi-zone route.

5.10 Single-valued Neutrosophic Directed n-Superhypergraph

A single-valued neutrosophic directed graph orients edges and assigns each vertex and arc triple degrees of truth, indeterminacy, falsity levels [807, 921]. A single-valued neutrosophic directed hypergraph uses oriented hyperarcs between vertex sets with neutrosophic truth, indeterminacy, falsity degrees modeling uncertainty precisely [807]. We define the *Single-valued Neutrosophic Directed n-Superhypergraph* as an extension of the classical *Single-valued Neutrosophic Directed Hypergraph* by incorporating the hierarchical structure of n-Superhypergraphs [314].

Definition 5.10.1 (Single-valued Neutrosophic Directed Hypergraph)**.** (cf. [921]) A *single-valued neutrosophic directed hypergraph* on a nonempty set X is an ordered pair

$$G' \;=\; \bigl(G,\; \{F_j\}_{j=1}^{n}\bigr),$$

where

$$G \;=\; \{\,G_j\}_{j=1}^{n}, \quad G_j \;=\; \bigl(T(G_j),\, H(G_j)\bigr)$$

is a family of nontrivial single-valued neutrosophic subsets of X, with

$$T(G_j) \;=\; \{\,\bigl(v, \alpha_G(v), \beta_G(v), \gamma_G(v)\bigr) \mid v \in X\},$$

$$H(G_j) \;=\; \{\,\bigl(v', \alpha_G(v'), \beta_G(v'), \gamma_G(v')\bigr) \mid v' \in X\},$$

and each neutrosophic *hyperarc* is

$$F_j\bigl(T(G_j), H(G_j)\bigr) = (\alpha_{F_j},\, \beta_{F_j},\, \gamma_{F_j})$$

satisfying, for all j,

$$\alpha_{F_j} \;\le\; \bigwedge_{v \in T(G_j),\, v' \in H(G_j)} \bigl(\alpha_G(v) \wedge \alpha_G(v')\bigr),$$

$$\beta_{F_j} \;\le\; \bigwedge_{v \in T(G_j),\, v' \in H(G_j)} \bigl(\beta_G(v) \wedge \beta_G(v')\bigr),$$

$$\gamma_{F_j} \;\le\; \bigvee_{v \in T(G_j),\, v' \in H(G_j)} \bigl(\gamma_G(v) \wedge \gamma_G(v')\bigr),$$

and

$$X \;=\; \bigcup_{j=1}^{n} \mathrm{supp}(G_j).$$

Definition 5.10.2 (Neutrosophic Directed Superhypergraph)**.** Let S be a nonempty *base set* and $n \geq 0$ an integer. Define

$$\mathcal{P}^0(S) = S,$$
$$\mathcal{P}^{k+1}(S) = \mathcal{P}(\mathcal{P}^k(S)) \quad (k \geq 0).$$

A *directed n-Superhypergraph* is a pair $\mathrm{DSHG}^{(n)} = (V, E)$ with

$$V \subseteq \mathcal{P}^n(S),$$

$$E \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S),$$

where each $e \in E$ is $\big(\mathrm{Tail}(e), \mathrm{Head}(e)\big)$. A *single-valued neutrosophic directed n-Superhypergraph* is the septuple

$$\big(V,\, E,\, T_V,\, I_V,\, F_V,\, T_E,\, I_E,\, F_E\big),$$

where

$$\begin{aligned}
&T_V,\, I_V,\, F_V : V \to [0,1],\\
&T_V(v) + I_V(v) + F_V(v) \leq 3,\ \forall v \in V,\\
&T_E,\, I_E,\, F_E : E \to [0,1],\\
&\quad T_E(e) \leq \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} T_V(x),\\
&\quad I_E(e) \leq \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} I_V(x),\\
&\quad F_E(e) \leq \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} F_V(x), \quad \forall e \in E.
\end{aligned}$$

Example 5.10.3 (Single-valued neutrosophic directed 1-Superhypergraph: parcel delivery network)**.** Let the base set of local depots be

$$S := \{d_1, d_2, d_3\},$$

where d_1 is an "airport hub", d_2 a "city depot", and d_3 a "suburban depot". Take $n = 1$, so that $\mathcal{P}^1(S) = \mathcal{P}(S)$.

Define three 1-supervertices (each representing a delivery zone covered jointly by several depots):

$$v_1 := \{d_1, d_2\}, \quad v_2 := \{d_2, d_3\}, \quad v_3 := \{d_1, d_3\},$$

and set

$$V := \{v_1, v_2, v_3\} \subseteq \mathcal{P}^1(S).$$

We introduce two directed 1-superedges:

$$e_1 := \big(\{v_1\},\, \{v_2\}\big), \qquad e_2 := \big(\{v_1\},\, \{v_3\}\big),$$

so that e_1 models a planned flow of parcels from the zone $\{d_1, d_2\}$ toward $\{d_2, d_3\}$ (airport → city+suburb), and e_2 models a flow from $\{d_1, d_2\}$ toward $\{d_1, d_3\}$ (airport+city → airport+suburb). Put

$$E := \{e_1, e_2\} \subseteq \mathcal{P}^1(S) \times \mathcal{P}^1(S).$$

We now assign single-valued neutrosophic degrees to supervertices and superedges. For vertices, let

$$T_V, I_V, F_V : V \to [0,1]$$

be given by

$$(T_V(v_1), I_V(v_1), F_V(v_1)) := (0.9,\ 0.1,\ 0.0),$$
$$(T_V(v_2), I_V(v_2), F_V(v_2)) := (0.8,\ 0.15,\ 0.05),$$
$$(T_V(v_3), I_V(v_3), F_V(v_3)) := (0.7,\ 0.2,\ 0.1).$$

Here $T_V(v)$ measures how strongly the zone v is considered operationally reliable, $I_V(v)$ measures indeterminacy (e.g. unknown traffic or staffing conditions), and $F_V(v)$ measures the degree to which the zone is considered unreliable or unavailable. In each case $T_V(v) + I_V(v) + F_V(v) \leq 3$ is trivially satisfied since each term lies in $[0,1]$.

For edges, define

$$T_E, I_E, F_E : E \to [0,1]$$

by

$$(T_E(e_1), I_E(e_1), F_E(e_1)) := (0.75,\ 0.08,\ 0.00),$$
$$(T_E(e_2), I_E(e_2), F_E(e_2)) := (0.70,\ 0.10,\ 0.05).$$

We verify the constraints from Definition 5.10.2. For e_1 we have

$$\mathrm{Tail}(e_1) \cup \mathrm{Head}(e_1) = \{v_1, v_2\},$$

so

$$\min_{x \in \mathrm{Tail}(e_1)\cup\mathrm{Head}(e_1)} T_V(x) = \min\{0.9, 0.8\} = 0.8 \geq T_E(e_1),$$
$$\min_{x \in \mathrm{Tail}(e_1)\cup\mathrm{Head}(e_1)} I_V(x) = \min\{0.1, 0.15\} = 0.1 \geq I_E(e_1),$$
$$\min_{x \in \mathrm{Tail}(e_1)\cup\mathrm{Head}(e_1)} F_V(x) = \min\{0.0, 0.05\} = 0.0 \geq F_E(e_1).$$

Similarly, for e_2 we have

$$\mathrm{Tail}(e_2) \cup \mathrm{Head}(e_2) = \{v_1, v_3\},$$

so

$$\min T_V = \min\{0.9, 0.7\} = 0.7 \geq T_E(e_2), \quad \min I_V = \min\{0.1, 0.2\} = 0.1 \geq I_E(e_2),$$
$$\min F_V = \min\{0.0, 0.1\} = 0.0 \geq F_E(e_2).$$

Thus all neutrosophic edge-constraints are satisfied, and

$$\big(V,\ E,\ T_V,\ I_V,\ F_V,\ T_E,\ I_E,\ F_E\big)$$

is a single-valued neutrosophic directed 1-Superhypergraph in the sense of Definition 5.10.2.

Operationally, this structure models a parcel delivery network where each 1-supervertex is an overlapping group of depots, each directed 1-superedge represents a multi-depot shipping route, and the neutrosophic triples quantify, for both zones and routes, the degrees of reliable operation (truth), uncertainty (indeterminacy), and anticipated failure or disruption (falsity).

5.11 Fuzzy Tolerance SuperHyperGraph

A Fuzzy Tolerance Graph assigns fuzzy intervals and tolerances to vertices; edge degrees measure normalized overlap, expressing uncertain pairwise compatibility [833, 922–924]. A Fuzzy Tolerance HyperGraph assigns fuzzy intervals and tolerances to vertices; hyperedge degrees measure multiway normalized overlap, expressing uncertain group feasibility [925]. A Fuzzy Tolerance SuperHyperGraph lifts fuzzy tolerance hypergraphs to n-supervertices; superhyperedge degrees measure multiway overlap between hierarchical groups under uncertainty [925].

Definition 5.11.1 (Fuzzy Tolerance Graph)**.** [833, 922–924] Let V be a finite nonempty set. For each $v \in V$, let I_v be a fuzzy interval on $\mathbb{R}$ (with core $c(I_v)$ and support $s(I_v)$, both compact real intervals), and let T_v be a fuzzy tolerance (a fuzzy number) whose core and support have positive lengths. Write $\ell(J)$ for the (Lebesgue) length of a real interval J (and $\ell(\emptyset) = 0$).

Define the normalization

$$\rho(x, a) := \begin{cases} \min\{1,\ x/a\}, & a > 0, \\ 0, & a = 0. \end{cases}$$

Define vertex-membership by

$$\sigma(v) := h(I_v) \in [0,1] \quad (\text{typically } h(I_v) = 1 \text{ for normal fuzzy intervals}).$$

For distinct $u, v \in V$, define edge-membership

$$\mu(u,v) := \max\Big\{\rho\big(\ell(c(I_u)\cap c(I_v)),\ \min\{\ell(c(T_u)), \ell(c(T_v))\}\big),\ \rho\big(\ell(s(I_u)\cap s(I_v)),\ \min\{\ell(s(T_u)), \ell(s(T_v))\}\big)\Big\} \in [0,1],$$

and set $\mu(v,v) := \sigma(v)$. Then $\Xi = (V, \sigma, \mu)$ is called a *fuzzy tolerance graph.*

Example 5.11.2 (Fuzzy tolerance graph: uncertain call coordination)**.** Let $V = \{X, Y, Z\}$ be three consultants. Their fuzzy availability intervals and fuzzy tolerances are summarized only by core/support intervals and core/support lengths (hours):

$$\begin{aligned}
&c(I_X) = [9, 12],\ s(I_X) = [8.5, 12.5], && \ell(c(T_X)) = 1.5,\ \ell(s(T_X)) = 3.0;\\
&c(I_Y) = [10, 13],\ s(I_Y) = [9.5, 13.5], && \ell(c(T_Y)) = 1.0,\ \ell(s(T_Y)) = 2.0;\\
&c(I_Z) = [12, 14],\ s(I_Z) = [10.5, 14.5], && \ell(c(T_Z)) = 1.25,\ \ell(s(T_Z)) = 2.5.
\end{aligned}$$

Take $\sigma(X) = \sigma(Y) = \sigma(Z) = 1$ (normal fuzzy intervals).

Compute $\mu(X, Y)$:

$$\ell(c(I_X) \cap c(I_Y)) = \ell([10, 12]) = 2, \quad \min\{\ell(c(T_X)), \ell(c(T_Y))\} = \min\{1.5, 1.0\} = 1.0,$$

$$\rho(2, 1.0) = \min\{1, 2/1\} = 1.$$

Also

$$\ell(s(I_X) \cap s(I_Y)) = \ell([9.5, 12.5]) = 3, \quad \min\{\ell(s(T_X)), \ell(s(T_Y))\} = \min\{3.0, 2.0\} = 2.0,$$

$$\rho(3, 2.0) = \min\{1, 3/2\} = 1.$$

Hence $\mu(X, Y) = \max\{1, 1\} = 1$.

Compute $\mu(X, Z)$:

$$\ell(c(I_X) \cap c(I_Z)) = \ell([9, 12] \cap [12, 14]) = \ell(\{12\}) = 0, \quad \min\{1.5, 1.25\} = 1.25, \quad \rho(0, 1.25) = 0,$$

$$\ell(s(I_X) \cap s(I_Z)) = \ell([8.5, 12.5] \cap [10.5, 14.5]) = \ell([10.5, 12.5]) = 2, \quad \min\{3.0, 2.5\} = 2.5, \quad \rho(2, 2.5) = \min\{1, 2/2.5\} = 0.8.$$

Hence $\mu(X, Z) = \max\{0, 0.8\} = 0.8$.

Compute $\mu(Y, Z)$:

$$\ell(c(I_Y) \cap c(I_Z)) = \ell([10, 13] \cap [12, 14]) = \ell([12, 13]) = 1, \quad \min\{1.0, 1.25\} = 1.0, \quad \rho(1, 1.0) = 1,$$

$$\ell(s(I_Y) \cap s(I_Z)) = \ell([9.5, 13.5] \cap [10.5, 14.5]) = \ell([10.5, 13.5]) = 3, \quad \min\{2.0, 2.5\} = 2.0, \quad \rho(3, 2.0) = 1.$$

Hence $\mu(Y, Z) = 1$. Therefore the fuzzy tolerance graph has full-feasibility edges XY, YZ and partial-feasibility edge XZ with degree 0.8.

Definition 5.11.3 (Fuzzy Tolerance Hypergraph)**.** [925] Let V be a finite nonempty vertex set. For each $v \in V$, fix a fuzzy interval I_v and a fuzzy tolerance T_v as above. For any nonempty $e \subseteq V$, define

$$L_c(e) := \ell\Big(\bigcap_{v \in e} c(I_v)\Big), \qquad L_s(e) := \ell\Big(\bigcap_{v \in e} s(I_v)\Big),$$

$$\tau_c(e) := \min_{v \in e} \ell(c(T_v)), \qquad \tau_s(e) := \min_{v \in e} \ell(s(T_v)).$$

Let ρ be as in the previous definition, and define the overlap score

$$\varphi(e) := \max\{\rho(L_c(e), \tau_c(e)),\ \rho(L_s(e), \tau_s(e))\} \in [0, 1].$$

Define the (crisp) edge universe

$$E := \{\, e \subseteq V :\ |e| \geq 2,\ \varphi(e) > 0 \,\}.$$

Define fuzzy memberships

$$\sigma(v) := \min\{h(I_v), h(T_v)\} \in [0, 1], \qquad \mu(e) := \min\Big\{\varphi(e),\ \min_{v \in e} \sigma(v)\Big\} \in [0, 1],$$

and the incidence membership

$$\eta(v, e) := \begin{cases} \mu(e), & v \in e, \\ 0, & v \notin e. \end{cases}$$

Then $H = (V, E; \sigma, \mu, \eta)$ is called a *fuzzy tolerance hypergraph*.

Example 5.11.4 (Fuzzy tolerance hypergraph: joint specialist consultation)**.** Let $V = \{C, N, O\}$ denote Cardiology, Neurology, Orthopedics. Use (hours)

$$c(I_C) = [9, 12],\ s(I_C) = [8.5, 12.5],\ \ell(c(T_C)) = 1.0,\ \ell(s(T_C)) = 1.5,$$

$$c(I_N) = [10, 13],\ s(I_N) = [9.5, 13.5],\ \ell(c(T_N)) = 1.2,\ \ell(s(T_N)) = 2.0,$$

$$c(I_O) = [11, 14],\ s(I_O) = [10.5, 14.5],\ \ell(c(T_O)) = 1.5,\ \ell(s(T_O)) = 2.5.$$

Assume heights $(h(I_C), h(T_C)) = (0.95, 0.90)$, $(h(I_N), h(T_N)) = (0.90, 0.95)$, $(h(I_O), h(T_O)) = (0.85, 0.80)$, hence

$$\sigma(C) = \min\{0.95, 0.90\} = 0.90, \quad \sigma(N) = \min\{0.90, 0.95\} = 0.90, \quad \sigma(O) = \min\{0.85, 0.80\} = 0.80.$$

Compute $\varphi(\{C, N, O\})$:

$$L_c(\{C, N, O\}) = \ell([9, 12] \cap [10, 13] \cap [11, 14]) = \ell([11, 12]) = 1,$$

$$\tau_c(\{C, N, O\}) = \min\{1.0, 1.2, 1.5\} = 1.0, \quad \rho(L_c, \tau_c) = \rho(1, 1) = 1.$$

Also

$$L_s(\{C, N, O\}) = \ell([8.5, 12.5] \cap [9.5, 13.5] \cap [10.5, 14.5]) = \ell([10.5, 12.5]) = 2,$$

$$\tau_s(\{C, N, O\}) = \min\{1.5, 2.0, 2.5\} = 1.5, \quad \rho(L_s, \tau_s) = \rho(2, 1.5) = \min\{1, 2/1.5\} = 1.$$

Hence $\varphi(\{C, N, O\}) = \max\{1, 1\} = 1$ and

$$\mu(\{C, N, O\}) = \min\{1, \min(\sigma(C), \sigma(N), \sigma(O))\} = \min\{1, 0.80\} = 0.80.$$

Similarly, for the pairs:

$$\mu(\{C, N\}) = \min\{1, \min(0.90, 0.90)\} = 0.90, \quad \mu(\{C, O\}) = \min\{1, \min(0.90, 0.80)\} = 0.80, \quad \mu(\{N, O\}) = \min\{1, \min(0.9$$

and $\eta(v, e) = \mu(e)$ for $v \in e$ (otherwise 0). This quantifies feasibility of multi-clinic sessions under uncertainty.

Definition 5.11.5 (Fuzzy Tolerance SuperHyperGraph)**.** [925] Fix $n \in \mathbb{N}_0$ and a finite base set V_0. Let $V \subseteq \mathcal{P}^n(V_0)$ be a finite nonempty set of n-supervertices. For each $v \in V$, choose a fuzzy interval I_v and a fuzzy tolerance T_v.

For any nonempty $e \subseteq V$ define

$$L_c(e) := \ell\Big(\bigcap_{v \in e} c(I_v)\Big), \quad L_s(e) := \ell\Big(\bigcap_{v \in e} s(I_v)\Big), \quad \tau_c(e) := \min_{v \in e} \ell(c(T_v)), \quad \tau_s(e) := \min_{v \in e} \ell(s(T_v)),$$

$$\varphi(e) := \max\{\rho(L_c(e), \tau_c(e)), \rho(L_s(e), \tau_s(e))\} \in [0, 1], \qquad \sigma(v) := \min\{h(I_v), h(T_v)\} \in [0, 1].$$

Let the (crisp) superedge universe be

$$E := \{\, e \subseteq V : \ |e| \geq 2 \,\}.$$

Define superedge-membership and incidence-membership by

$$\mu(e) := \min\Big\{\varphi(e),\ \min_{v \in e} \sigma(v)\Big\}, \qquad \eta(v, e) := \begin{cases} \mu(e), & v \in e, \\ 0, & v \notin e. \end{cases}$$

Then $S^{(n)}_{FT} = (V, E; \sigma, \mu, \eta)$ is called a *fuzzy tolerance n-SuperHyperGraph.*

Example 5.11.6 (Fuzzy tolerance superhypergraph: workshops among teams (level $n = 1$))**.** Let $V_0 = \{a, b, c, d, e, f\}$ be employees and take $n = 1$. Define three team supervertices

$$T_1 = \{a, b, c\}, \quad T_2 = \{c, d\}, \quad T_3 = \{d, e, f\}, \qquad V = \{T_1, T_2, T_3\} \subseteq \mathcal{P}(V_0).$$

Assign fuzzy availability intervals and fuzzy tolerances via cores/supports (hours):

$$c(I_{T_1}) = [9, 12],\ s(I_{T_1}) = [8.5, 12.5],\ \ell(c(T_{T_1})) = 1.5,\ \ell(s(T_{T_1})) = 2.5,$$

$$c(I_{T_2}) = [10, 14],\ s(I_{T_2}) = [9.5, 14.5],\ \ell(c(T_{T_2})) = 1.0,\ \ell(s(T_{T_2})) = 2.0,$$
$$c(I_{T_3}) = [11, 13],\ s(I_{T_3}) = [10.5, 13.5],\ \ell(c(T_{T_3})) = 1.0,\ \ell(s(T_{T_3})) = 1.5.$$

Heights: $(h(I_{T_1}), h(T_{T_1})) = (0.95, 0.90)$, $(h(I_{T_2}), h(T_{T_2})) = (0.90, 0.90)$, $(h(I_{T_3}), h(T_{T_3})) = (0.90, 0.85)$, so

$$\sigma(T_1) = \min\{0.95, 0.90\} = 0.90, \quad \sigma(T_2) = \min\{0.90, 0.90\} = 0.90, \quad \sigma(T_3) = \min\{0.90, 0.85\} = 0.85.$$

Check the triple overlap score $\varphi(\{T_1, T_2, T_3\})$:

$L_c = \ell([9, 12] \cap [10, 14] \cap [11, 13]) = \ell([11, 12]) = 1, \quad \tau_c = \min\{1.5, 1.0, 1.0\} = 1.0, \quad \rho(L_c, \tau_c) = \rho(1, 1) = 1,$

$L_s = \ell([8.5, 12.5] \cap [9.5, 14.5] \cap [10.5, 13.5]) = \ell([10.5, 12.5]) = 2, \quad \tau_s = \min\{2.5, 2.0, 1.5\} = 1.5, \quad \rho(L_s, \tau_s) = \rho(2, 1.5) = 1.$

Hence $\varphi(\{T_1, T_2, T_3\}) = 1$ and thus

$$\mu(\{T_1, T_2, T_3\}) = \min\{1, \min(\sigma(T_1), \sigma(T_2), \sigma(T_3))\} = \min\{1, 0.85\} = 0.85.$$

For the pairs, the same overlap computation gives $\varphi(\{T_i, T_j\}) = 1$, so

$\mu(\{T_1, T_2\}) = \min\{1, \min(0.90, 0.90)\} = 0.90, \quad \mu(\{T_1, T_3\}) = \min\{1, \min(0.90, 0.85)\} = 0.85, \quad \mu(\{T_2, T_3\}) = \min\{1, \min(0.90,$

Finally, $\eta(T_i, e) = \mu(e)$ if $T_i \in e$ (else 0). This yields a fuzzy tolerance superhypergraph quantifying feasibility of joint workshops among teams under uncertainty.

5.12 Neutrosophic HyperEdgeWeighted n-SuperHyperGraph

A Neutrosophic HyperEdgeWeighted n-SuperHyperGraph assigns each n-superedge a neutrosophic weight (T, I, F), capturing uncertain edge strength at level n [926, 927].

Definition 5.12.1 (Single-valued neutrosophic weight domain)**.** [927] Define the (single-valued) neutrosophic weight domain by

$$\mathbb{N}_{SV} := \Big\{(T, I, F) \in [0, 1]^3 \ \Big|\ 0 \le T + I + F \le 3\Big\}.$$

Definition 5.12.2 (Neutrosophic HyperEdgeWeighted n-SuperHyperGraph)**.** [927] Let $\mathcal{H}^{(n)} = (V, E)$ be an n-SuperHyperGraph over V_0. A *Neutrosophic HyperEdgeWeighted n-SuperHyperGraph* (briefly, *NHEW-n-SHG*) is a triple

$$\Omega = (V, E, W)$$

where W is a *neutrosophic hyperedge-weight function*

$$W : E \longrightarrow \mathbb{N}_{SV}, \qquad e \longmapsto W(e) = (T_e, I_e, F_e).$$

Here T_e is the truth-component, I_e the indeterminacy-component, and F_e the falsity-component of the superhyperedge e.

Example 5.12.3 (A concrete Neutrosophic HyperEdgeWeighted 1-SuperHyperGraph)**.** Let the finite base vertex set be

$$V_0 = \{a, b, c\}, \qquad n = 1, \qquad \mathcal{P}_1(V_0) = \mathcal{P}(V_0).$$

Define the 1-supervertex set (each element is a nonempty subset of V_0) by

$$V = \big\{ v_1, v_2, v_3, v_4 \big\} := \big\{ \{a, b\}, \{b, c\}, \{a\}, \{c\} \big\} \subseteq \mathcal{P}(V_0) \setminus \{\emptyset\}.$$

Define two 1-superhyperedges (each is a nonempty set of supervertices) by

$$E = \{e_1, e_2\}, \qquad e_1 = \{v_1, v_2, v_4\} = \big\{\{a, b\}, \{b, c\}, \{c\}\big\}, \qquad e_2 = \{v_1, v_3\} = \big\{\{a, b\}, \{a\}\big\}.$$

Thus $\mathcal{H}^{(1)} = (V, E)$ is a 1-SuperHyperGraph.

Now assign neutrosophic hyperedge-weights by a map

$$W : E \to [0, 1]^3, \qquad W(e) = (T_e, I_e, F_e),$$

for instance

$$W(e_1) = (0.85, 0.10, 0.25), \qquad W(e_2) = (0.40, 0.55, 0.30).$$

Constraint check:

$$0 \le 0.85 + 0.10 + 0.25 = 1.20 \le 3, \qquad 0 \le 0.40 + 0.55 + 0.30 = 1.25 \le 3.$$

Hence

$$\Omega = (V, E, W)$$

is a concrete *Neutrosophic HyperEdgeWeighted* 1*-SuperHyperGraph.*

Theorem 5.12.4 (Neutrosophic HyperEdgeWeighted n-SuperHyperGraphs generalize n-SuperHyperGraphs)**.** *Let $n \ge 0$. For every (finite) n-SuperHyperGraph*

$$\mathcal{H}^{(n)} = (V^{(n)}, E^{(n)}),$$

there exists a Neutrosophic HyperEdgeWeighted n-SuperHyperGraph

$$\Omega^{(n)} = \big(V^{(n)}, E^{(n)}, W\big)$$

whose underlying n-SuperHyperGraph is exactly $\mathcal{H}^{(n)}$. Equivalently, the forgetful map

$$U : (V^{(n)}, E^{(n)}, W) \longmapsto (V^{(n)}, E^{(n)})$$

from Neutrosophic HyperEdgeWeighted n-SuperHyperGraphs to n-SuperHyperGraphs is surjective on objects.

Proof. Fix an arbitrary (finite) n-SuperHyperGraph $\mathcal{H}^{(n)} = (V^{(n)}, E^{(n)})$. Define a hyperedge-weight map

$$W : E^{(n)} \to [0, 1]^3$$

by choosing the constant neutrosophic triple

$$W(e) := (1, 0, 0) \qquad \text{for every } e \in E^{(n)}.$$

This is well-defined because $(1, 0, 0) \in [0, 1]^3$, and it satisfies the usual basic feasibility constraint used for neutrosophic weights:

$$0 \le 1 + 0 + 0 = 1 \le 3.$$

Now set

$$\Omega^{(n)} := \big(V^{(n)}, E^{(n)}, W\big).$$

By construction, $\Omega^{(n)}$ has the same n-supervertex set and the same n-superedge family as $\mathcal{H}^{(n)}$, and W assigns a valid neutrosophic weight to each n-superedge. Hence $\Omega^{(n)}$ is a Neutrosophic HyperEdgeWeighted n-SuperHyperGraph.

Finally, forgetting the weights yields

$$U\big(\Omega^{(n)}\big) = (V^{(n)}, E^{(n)}) = \mathcal{H}^{(n)}.$$

Therefore every n-SuperHyperGraph arises as the underlying structure of some Neutrosophic HyperEdgeWeighted n-SuperHyperGraph, proving the claim. □

Note that, by following the construction of Neutrosophic HyperEdgeWeighted n-SuperHyperGraphs, one can define a Fuzzy HyperEdgeWeighted n-SuperHyperGraph in the following manner.

Definition 5.12.5 (Fuzzy HyperEdgeWeighted n-SuperHyperGraph)**.** Let V_0 be a nonempty finite base set and let

$$\mathrm{P}_0(V_0) := V_0, \qquad \mathrm{P}_{k+1}(V_0) := \mathcal{P}(\mathrm{P}_k(V_0)) \quad (k \ge 0).$$

Fix an integer $n \ge 0$. An *n-SuperHyperGraph* is a pair

$$\mathcal{H}^{(n)} = (V^{(n)}, E^{(n)}),$$

where $V^{(n)} \subseteq \mathrm{P}_n(V_0)$ is a finite set of n-supervertices and

$$E^{(n)} \subseteq \mathcal{P}(V^{(n)}) \setminus \{\emptyset\}$$

is a finite family of n-superedges.

A *Fuzzy HyperEdgeWeighted n-SuperHyperGraph* is a triple

$$\mathcal{H}_F^{(n)} := \big(V^{(n)}, E^{(n)}, w_F\big),$$

where $\big(V^{(n)}, E^{(n)}\big)$ is an n-SuperHyperGraph and

$$w_F : E^{(n)} \longrightarrow [0,1]$$

is a *fuzzy hyperedge-weight function*. For each $e \in E^{(n)}$, the value $w_F(e)$ is interpreted as the (fuzzy) strength/reliability/capacity of the n-superedge e.

Theorem 5.12.6 (Neutrosophic HyperEdgeWeighted n-SuperHyperGraphs generalize fuzzy ones)**.** *Every Fuzzy HyperEdgeWeighted n-SuperHyperGraph is obtained as a special case of a Neutrosophic HyperEdgeWeighted n-SuperHyperGraph. More precisely, there is an embedding*

$$\Phi : \Big\{\textit{Fuzzy HyperEdgeWeighted n-SuperHyperGraphs}\Big\} \hookrightarrow \Big\{\textit{Neutrosophic HyperEdgeWeighted n-SuperHyperGraphs}\Big\}$$

defined by keeping $(V^{(n)}, E^{(n)})$ and mapping fuzzy weights w_F to neutrosophic weights w_N via

$$w_N(e) := \big(w_F(e),\, 0,\, 1 - w_F(e)\big) \qquad (e \in E^{(n)}).$$

Proof. Let $\mathcal{H}_F^{(n)} = (V^{(n)}, E^{(n)}, w_F)$ be any Fuzzy HyperEdgeWeighted n-SuperHyperGraph. Define a map $w_N : E^{(n)} \to [0,1]^3$ by

$$w_N(e) := \big(T(e), I(e), F(e)\big) := \big(w_F(e),\, 0,\, 1 - w_F(e)\big) \qquad (e \in E^{(n)}).$$

Since $w_F(e) \in [0,1]$, we have $0 \le w_F(e) \le 1$, hence

$$0 \le T(e) = w_F(e) \le 1, \qquad I(e) = 0 \in [0,1], \qquad 0 \le F(e) = 1 - w_F(e) \le 1,$$

so indeed $w_N(e) \in [0,1]^3$ for every $e \in E^{(n)}$. Therefore

$$\mathcal{H}_N^{(n)} := \big(V^{(n)}, E^{(n)}, w_N\big)$$

is a Neutrosophic HyperEdgeWeighted n-SuperHyperGraph, and it has the same underlying n-SuperHyperGraph $(V^{(n)}, E^{(n)})$ as $\mathcal{H}_F^{(n)}$.

Moreover, the construction is injective on weights: if w_F and w'_F satisfy

$$\big(w_F(e), 0, 1 - w_F(e)\big) = \big(w'_F(e), 0, 1 - w'_F(e)\big) \quad \text{for all } e,$$

then comparing first coordinates gives $w_F(e) = w'_F(e)$ for all e, hence $w_F = w'_F$. Thus the mapping $\Phi(\mathcal{H}_F^{(n)}) = \mathcal{H}_N^{(n)}$ is an embedding, showing that Neutrosophic HyperEdgeWeighted n-SuperHyperGraphs generalize Fuzzy HyperEdgeWeighted n-SuperHyperGraphs. □

Chapter 6

Applications of SuperHyperGraph

This chapter presents several applications of SuperHyperGraphs. Classical Graphs and HyperGraphs are widely used in many fields—such as chemistry, physics, engineering, and informatics—because of their conceptual flexibility and modeling power. SuperHyperGraphs inherit and extend these advantages, and therefore offer promising applications in chemistry, physics, engineering, and other domains where multi-level and multi-structure relationships naturally arise.

6.1 Molecular SuperHyperGraphs

A molecular graph represents a molecule by treating atoms as vertices and covalent bonds as labeled edges in a simple graph structure [30, 928, 929]. A molecular hypergraph extends this idea by allowing hyperedges that connect several atoms at once, thereby capturing functional groups, aromatic rings, delocalized electron systems, and other multi-atom chemical interactions [31, 298, 930, 931]. A molecular SuperHyperGraph further organizes atoms, bonds, fragments, and whole molecular units across iterated powerset layers, offering a hierarchical representation of complex chemical structures, multi-scale motifs, and overlapping functional contexts [32, 932]. Related notions include Chemical Graphs [750, 933, 934], Chemical HyperGraphs [55, 56, 935], Chemical SuperHyperGraphs [57, 58], Goal-directed molecular graph [936, 937], and Chemical Reaction Networks [938–940], which appear in computational chemistry, cheminformatics, and reaction-mechanism modeling.

Definition 6.1.1 (Molecular Graph)**.** (cf. [30]) A *molecular graph* is a labeled simple graph

$$G = (V, E, \ell_V, \ell_E),$$

where

- V is a finite set of *atoms*;
- $E \subseteq \big\{\{u, v\} \mid u, v \in V,\ u \neq v\big\}$ is the set of *covalent bonds*;
- $\ell_V : V \to \mathcal{L}_V$ assigns to each vertex $v \in V$ its *atomic label* (e.g. element symbol such as C, H, O);
- $\ell_E : E \to \mathcal{L}_E$ assigns to each edge $e \in E$ its *bond label* (e.g. single, double, triple).

Thus vertices represent atoms, edges represent bonds, and the labeling functions encode atom types and bond types.

Definition 6.1.2 (Molecular n-SuperHyperGraph)**.** [32, 932] Let V_0 be a finite set of *bond identifiers* of a molecule. Define the iterated powersets by

$$P_0(V_0) := V_0, \qquad P_{k+1}(V_0) := P\big(P_k(V_0)\big) \quad (k \geq 0),$$

where $P(\cdot)$ denotes the usual powerset.

Fix an integer $n \geq 0$. A *molecular n-SuperHyperGraph* on the base set V_0 is a quintuple

$$H^{(n)} = \left(V_H, E_H, \partial, \ell_V^H, \ell_E^H\right),$$

where

- $V_H \subseteq \mathrm{P}_n(V_0)$ is a finite set of *n-supervertices* (each $v \in V_H$ represents a possibly nested collection of bonds up to level n);
- E_H is a finite set of *n-superedges*;
- $\partial : E_H \to \mathcal{P}^*(V_H)$ is the incidence map, with $\mathcal{P}^*(V_H) := \mathcal{P}(V_H) \setminus \{\emptyset\}$; for $e \in E_H$, the set $\partial(e) \subseteq V_H$ is the family of n-supervertices incident with the n-superedge e;
- $\ell_V^H : V_H \to \mathcal{L}_V$ assigns to each n-supervertex v a *vertex label* (for example, a bond-pattern type, functional-group name, or moiety type);
- $\ell_E^H : E_H \to \mathcal{L}_E$ assigns to each n-superedge e an *edge label* (for example, an atom symbol, a fragment name, or a whole-molecule/functional-unit identifier).

The underlying n-SuperHyperGraph of $H^{(n)}$ is the triple (V_H, E_H, ∂). For $n = 0$, the condition $V_H \subseteq \mathrm{P}_0(V_0) = V_0$ implies that each vertex corresponds to a single bond identifier, and the structure reduces to a labeled molecular hypergraph on V_0.

Example 6.1.3 (Molecular 1-SuperHyperGraph for a benzene core)**.** Let V_0 be the set of bond identifiers of a benzene ring [941],

$$V_0 = \{b_1, b_2, b_3, b_4, b_5, b_6\},$$

where each b_i denotes the C–C bond between C_i and C_{i+1} (with $C_7 := C_1$). Then

$$\mathrm{P}_0(V_0) := V_0, \qquad \mathrm{P}_1(V_0) := \mathcal{P}(V_0).$$

Fix $n = 1$. Define the set of 1-supervertices by

$$V_H := \left\{v_{\text{ring}}, v_{\text{ortho}}, v_{\text{meta}}, v_{\text{para}}\right\} \subseteq \mathrm{P}_1(V_0),$$

where

$$\begin{aligned} v_{\text{ring}} &:= \{b_1, b_2, b_3, b_4, b_5, b_6\} \quad \text{(entire aromatic ring)}, \\ v_{\text{ortho}} &:= \{b_1, b_2\} \quad \text{(two adjacent bonds: an ortho-like fragment)}, \\ v_{\text{meta}} &:= \{b_1, b_3\} \quad \text{(two bonds at meta separation)}, \\ v_{\text{para}} &:= \{b_1, b_4\} \quad \text{(two bonds at para separation)}. \end{aligned}$$

Let $E_H := \{e_{\text{benzene}}\}$ consist of a single 1-superedge, with incidence map

$$\partial(e_{\text{benzene}}) := \{v_{\text{ring}}, v_{\text{ortho}}, v_{\text{meta}}, v_{\text{para}}\} \subseteq V_H.$$

Interpret e_{benzene} as the "benzene core context" that simultaneously relates the whole aromatic ring and its standard substitution patterns.

Choose label sets

$$\mathcal{L}_V := \{\text{"ring"}, \text{"ortho"}, \text{"meta"}, \text{"para"}\}, \qquad \mathcal{L}_E := \{\text{"benzene\_core"}\},$$

and define

$$\ell_V^H(v_{\text{ring}}) = \text{"ring"}, \quad \ell_V^H(v_{\text{ortho}}) = \text{"ortho"}, \quad \ell_V^H(v_{\text{meta}}) = \text{"meta"}, \quad \ell_V^H(v_{\text{para}}) = \text{"para"},$$
$$\ell_E^H(e_{\text{benzene}}) = \text{"benzene\_core"}.$$

Then

$$H^{(1)} := \left(V_H, E_H, \partial, \ell_V^H, \ell_E^H\right)$$

is a molecular 1-SuperHyperGraph: each 1-supervertex is a bond-based fragment (subset of V_0), and the single 1-superedge encodes the benzene core that ties these fragments together.

Example 6.1.4 (Molecular 2-SuperHyperGraph for an aspirin-like molecule)**.** Consider an aspirin-like aromatic molecule [942] with a ring, a carboxyl group, and an ester group. Let V_0 be the set of selected bond identifiers

$$V_0 := \{b_{\mathrm{ring},1}, b_{\mathrm{ring},2}, b_{\mathrm{ring},3}, b_{\mathrm{COOH},1}, b_{\mathrm{COOH},2}, b_{\mathrm{ester},1}, b_{\mathrm{ester},2}\},$$

where, for example, $b_{\mathrm{ring},i}$ are representative C–C bonds in the aromatic ring, $b_{\mathrm{COOH},1}, b_{\mathrm{COOH},2}$ are bonds in the carboxyl group, and $b_{\mathrm{ester},1}, b_{\mathrm{ester},2}$ are bonds in the ester group.

Define

$$\mathrm{P}_0(V_0) := V_0, \qquad \mathrm{P}_1(V_0) := \mathcal{P}(V_0), \qquad \mathrm{P}_2(V_0) := \mathcal{P}\big(\mathrm{P}_1(V_0)\big).$$

At level 1, consider the following bond-based fragments

$$\begin{aligned} X_{\mathrm{ring}} &:= \{b_{\mathrm{ring},1}, b_{\mathrm{ring},2}, b_{\mathrm{ring},3}\} \in \mathrm{P}_1(V_0), \\ X_{\mathrm{COOH}} &:= \{b_{\mathrm{COOH},1}, b_{\mathrm{COOH},2}\} \in \mathrm{P}_1(V_0), \\ X_{\mathrm{ester}} &:= \{b_{\mathrm{ester},1}, b_{\mathrm{ester},2}\} \in \mathrm{P}_1(V_0). \end{aligned}$$

Fix $n = 2$. Define the 2-supervertex set $V_H \subseteq \mathrm{P}_2(V_0)$ by

$$\begin{aligned} v_{\mathrm{local}} &:= \{X_{\mathrm{ring}}, X_{\mathrm{COOH}}\} \in \mathrm{P}_2(V_0) \quad \text{(aromatic ring + carboxyl fragment)}, \\ v_{\mathrm{global}} &:= \{X_{\mathrm{ring}}, X_{\mathrm{COOH}}, X_{\mathrm{ester}}\} \in \mathrm{P}_2(V_0) \quad \text{(ring + carboxyl + ester = full aspirin motif)}, \\ V_H &:= \{v_{\mathrm{local}}, v_{\mathrm{global}}\}. \end{aligned}$$

Let

$$E_H := \{e_{\mathrm{local}}, e_{\mathrm{aspirin}}\},$$

with incidence map

$$\partial(e_{\mathrm{local}}) := \{v_{\mathrm{local}}\}, \qquad \partial(e_{\mathrm{aspirin}}) := \{v_{\mathrm{global}}\}.$$

Here e_{local} encodes a “salicylic-acid-like” local motif (ring plus carboxyl group), while e_{aspirin} encodes the full aspirin pharmacophore (ring, carboxyl, and ester).

Choose label sets

$$\mathcal{L}_V := \{\text{“local\_fragment”}, \text{“aspirin\_motif”}\}, \qquad \mathcal{L}_E := \{\text{“salicylic\_core”}, \text{“aspirin\_pharmacophore”}\},$$

and define

$$\ell_V^H(v_{\mathrm{local}}) = \text{“local\_fragment”}, \qquad \ell_V^H(v_{\mathrm{global}}) = \text{“aspirin\_motif”},$$

$$\ell_E^H(e_{\mathrm{local}}) = \text{“salicylic\_core”}, \qquad \ell_E^H(e_{\mathrm{aspirin}}) = \text{“aspirin\_pharmacophore”}.$$

Then

$$H^{(2)} := \big(V_H,\, E_H,\, \partial,\, \ell_V^H,\, \ell_E^H\big)$$

is a molecular 2-SuperHyperGraph. Each 2-supervertex is a set of level-1 bond fragments (elements of $\mathrm{P}_1(V_0)$), so elements of V_H belong to $\mathrm{P}_2(V_0)$, and superedges distinguish different chemically meaningful groupings: a local aromatic–carboxyl core and the full aspirin-like pharmacophore.

For reference, an overview of molecular graph, molecular hypergraph, and molecular n-SuperHyperGraph viewpoints is presented in Table 6.1.

Table 6.1: Concise overview of molecular graph, molecular hypergraph, and molecular n-SuperHyperGraph viewpoints.

Aspect	Molecular Graph	Molecular HyperGraph	Molecular n-SuperHyperGraph
Carrier objects	Atoms (vertices)	Atoms (vertices)	n-supervertices (nested fragments over a base carrier)
Relation objects	Bonds (edges)	Multi-atom interactions (hyperedges)	Hierarchical relations among fragments (superhyperedges)
What it captures	Pairwise connectivity, bond types	Functional groups, rings, multi-body constraints	Multi-scale motifs, overlapping fragments, context across levels
Typical labels/weights	Atom type, bond order	Hyperedge type (group/ring), interaction strength	Fragment type, role, reliability/importance across levels
Use cases (keywords)	QSAR, cheminformatics, descriptors	Hypergraph neural nets, group-level chemistry	Multi-level representation, coarse-to-fine reasoning

6.2 Competition SuperHyperGraphs

Competition graphs connect two vertices when they share a common out-neighbor in a digraph, modeling competition for the same resource [943–945]. Related concepts such as fuzzy competition graphs [946–948] and neutrosophic competition graphs [878, 949] are also known. A competition hypergraph has vertices of a digraph and hyperedges grouping vertices sharing the same in-neighborhood prey target vertex node [950–952]. A competition Superhypergraph lifts this construction to n-supervertices, forming superhyperedges connecting supervertices whose flattened elements share predator relationships in digraphs [35].

Definition 6.2.1 (Competition hypergraph)**.** [950–952] Let $D = (V, A)$ be a finite directed graph, where V is the vertex set and $A \subseteq V \times V$ is the arc set. For each vertex $v \in V$, its in–neighborhood is

$$N^-(v) := \{\, u \in V \mid (u, v) \in A \,\}.$$

The *competition hypergraph* of D is the hypergraph

$$\mathrm{CH}(D) := (V, E),$$

where the hyperedge family E is defined by

$$E := \left\{ N^-(v) \subseteq V \;\middle|\; v \in V,\ |N^-(v)| \geq 2 \right\}.$$

Thus each hyperedge of $\mathrm{CH}(D)$ groups all vertices that compete for the same target v (i.e. all predators of v), provided there are at least two such vertices.

Example 6.2.2 (Competition hypergraph $\mathrm{CH}(D)$)**.** Let $D = (V, A)$ be the directed graph with

$$V = \{a, b, c, d\}, \qquad A = \{(a, d), (b, d), (c, d), (a, c)\}.$$

Compute in-neighborhoods:

$$N^-(a) = \emptyset, \quad N^-(b) = \emptyset, \quad N^-(c) = \{a\}, \quad N^-(d) = \{a, b, c\}.$$

By definition, the hyperedge family of $\mathrm{CH}(D)$ is

$$E = \left\{ N^-(v) \mid v \in V,\ |N^-(v)| \geq 2 \right\} = \left\{\{a, b, c\}\right\}.$$

Hence

$$\mathrm{CH}(D) = (V, E)$$

has the single hyperedge $\{a, b, c\}$, representing that a, b, c compete for the same target d.

Definition 6.2.3 ((Recall) Iterated powerset and flattening)**.** Let V_0 be a finite base set. Define the iterated powersets by

$$P_0(V_0) := V_0, \qquad P_{k+1}(V_0) := P\big(P_k(V_0)\big) \quad (k \geq 0),$$

where $P(\cdot)$ denotes the usual powerset.

For each $k \geq 0$ define the *flattening map*

$$\mathrm{flat}_k : P_k(V_0) \longrightarrow P(V_0)$$

recursively by

$$\mathrm{flat}_0(x) := \{x\} \quad (x \in V_0),$$

and, for $k \geq 0$,

$$\mathrm{flat}_{k+1}(X) := \bigcup_{Y \in X} \mathrm{flat}_k(Y) \quad \text{for } X \in P_{k+1}(V_0) = P\big(P_k(V_0)\big).$$

Definition 6.2.4 (Competition n-SuperHyperGraph)**.** [35] Let $D = (V_0, A)$ be a finite directed graph, where V_0 is the base vertex set and $A \subseteq V_0 \times V_0$ is the arc set. For a fixed integer $n \geq 0$, set

$$V_n := P_n(V_0),$$

so that each element $S \in V_n$ is an n–supervertex (an n–fold iterated subset of V_0).

For $S \in V_n$, define its *n–level in–neighborhood* by

$$N_n^-(S) := \Big\{ T \in V_n \;\Big|\; \exists\, u \in \mathrm{flat}_n(T),\ \exists\, v \in \mathrm{flat}_n(S) \text{ with } (u, v) \in A \Big\}.$$

The *competition n–SuperHyperGraph* of D is the pair

$$\mathrm{CompSuHG}^{(n)}(D) := \big(V_n,\ E_n^{\mathrm{comp}}\big),$$

where

$$E_n^{\mathrm{comp}} := \big\{ N_n^-(S) \subseteq V_n \;\big|\; S \in V_n,\ |N_n^-(S)| \geq 2 \big\}.$$

Thus each (n–level) competition superhyperedge $N_n^-(S)$ collects all n–supervertices T whose flattened elements contain some predator u of some prey v in the flattened elements of S. For $n = 0$ we have V_0 itself and $\mathrm{CompSuHG}^{(0)}(D)$ reduces to the classical competition hypergraph $\mathrm{CH}(D)$.

Example 6.2.5 (Competition 1-SuperHyperGraph $\mathrm{CompSuHG}^{(1)}(D)$)**.** Let the base digraph $D = (V_0, A)$ be given by

$$V_0 = \{a, b, c, d\}, \qquad A = \{(a, d), (b, d), (c, d), (a, c)\}.$$

Fix $n = 1$. Then

$$V_1 = P_1(V_0) = \mathcal{P}(V_0),$$

and for $T \in V_1$ we have $\mathrm{flat}_1(T) = T$.

Choose the 1-supervertex (a subset of V_0)

$$S := \{c, d\} \in V_1.$$

We compute its 1-level in-neighborhood:

$$N_1^-(S) = \Big\{ T \in V_1 \;\Big|\; \exists u \in T,\ \exists v \in S \text{ with } (u, v) \in A \Big\}.$$

Since $S = \{c, d\}$ and A contains arcs into c from a and into d from a, b, c, we get

$$\{u \in V_0 : \exists v \in S \text{ with } (u, v) \in A\} = \{a, b, c\}.$$

Therefore $T \in N_1^-(S)$ holds exactly when T contains at least one element of $\{a, b, c\}$. Equivalently,

$$N_1^-(S) = \{\, T \subseteq V_0 \mid T \cap \{a, b, c\} \neq \emptyset \,\}.$$

In particular,

$$| N_1^-(S) | = 2^4 - 2^1 = 16 - 2 = 14 \geq 2$$

(the only subsets of V_0 not in $N_1^-(S)$ are $\emptyset$ and $\{d\}$). Hence $N_1^-(S)$ is a competition 1-superhyperedge, and

$$\mathrm{CompSuHG}^{(1)}(D) = (V_1, E_1^{\mathrm{comp}})$$

has (at least) the superhyperedge

$$N_1^-(\{c,d\}) = \{\, T \subseteq V_0 \mid T \cap \{a,b,c\} \neq \emptyset \,\}.$$

This superhyperedge groups all 1-supervertices (subsets) that contain some predator of c or d.

As reference information, an overview of competition graphs, competition hypergraphs, and competition n-SuperHyperGraphs is provided in Table 6.2.

Model	Objects	Competition relation (edge / hyperedge / superhyperedge)
Competition graph	Digraph $D = (V, A)$	Undirected edge $\{x, y\}$ exists iff $\exists\, v \in V$ such that $(x, v) \in A$ and $(y, v) \in A$ (two vertices share a common out-neighbor).
Competition hypergraph [950–952]	Digraph $D = (V, A)$	Hyperedge is the in-neighborhood $N^-(v) = \{u \in V \mid (u, v) \in A\}$ for some v, included only when $\lvert N^-(v)\rvert \geq 2$ (all predators competing for the same target v).
Competition n-SuperHyperGraph [35]	Digraph $D = (V_0, A)$ and level-n supervertices $V_n = \mathrm{P}_n(V_0)$	Superhyperedge is $N_n^-(S) = \{T \in V_n \mid \exists u \in \mathrm{flat}_n(T), \exists v \in \mathrm{flat}_n(S) : (u, v) \in A\}$, included only when $\lvert N_n^-(S)\rvert \geq 2$.

Table 6.2: Overview of competition graphs, competition hypergraphs, and competition n-SuperHyperGraphs.

6.3 Property SuperHyperGraphs

Property graphs represent entities as vertices and relationships as edges, both carrying key–value properties for flexible, attributed graph modeling (cf. [953–955]). A Property HyperGraph annotates vertices and hyperedges with attribute values, enabling constraint checking, classification, and reasoning over multiway relational structures [41]. A Property n-SuperHyperGraph extends this labeling to n-supervertices and n-superedges, supporting hierarchical attributes, constraints, and analyses across powerset levels effectively [41].

Definition 6.3.1 (Property HyperGraph)**.** [41] Fix three (possibly infinite) sets

Σ (hyperedge–label alphabet), K (property keys), S (property values),

and let $\perp \notin S$ be a distinguished symbol.

A *Property HyperGraph* is a quadruple

$$H = (V, E, \lambda, \mu)$$

satisfying:

- V is a finite (or at most countable) set of vertices.
- E is a finite family of nonempty subsets of V, i.e. $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$; the elements of E are called hyperedges.
- $\lambda : E \longrightarrow \Sigma$ assigns a label to each hyperedge.
- $\mu : (V \cup E) \times K \longrightarrow S \cup \{\perp\}$ is the property map; for $x \in V \cup E$ and $k \in K$ the value $\mu(x, k) = s \in S$ means that x has property key k with value s, while $\mu(x, k) = \perp$ means that x carries no value for key k.

For each $x \in V \cup E$ we define the keyset and value notation

$$\mathrm{keyset}(x) \;:=\; \{\, k \in K \mid \mu(x,k) \neq \perp \,\}, \qquad \mathrm{val}(x,k) \;:=\; \mu(x,k) \quad (k \in \mathrm{keyset}(x)).$$

Definition 6.3.2 (Property n-SuperHyperGraph (Property SuperHyperGraph))**.** [41] Let V_0 be a finite, nonempty base set and define the iterated powersets

$$\mathrm{P}_0(V_0) := V_0, \qquad \mathrm{P}_{k+1}(V_0) := \mathcal{P}\big(\mathrm{P}_k(V_0)\big) \quad (k \geq 0),$$

where $\mathcal{P}(\cdot)$ denotes the usual powerset. Fix an integer $n \geq 0$.

As above, fix sets

$$\Sigma \text{ (superedge–label alphabet)}, \qquad K \text{ (property keys)}, \qquad S \text{ (property values)},$$

and a distinguished symbol $\perp \notin S$.

A *Property n-SuperHyperGraph on* V_0 (or *Property SuperHyperGraph of level* n) is a quadruple

$$H^{(n)} \;=\; \big(V^{(n)}, E^{(n)}, \lambda, \mu\big)$$

with:

- $V^{(n)} \subseteq \mathrm{P}_n(V_0)$, whose elements are called n-supervertices;
- $E^{(n)} \subseteq \mathcal{P}\big(V^{(n)}\big) \setminus \{\emptyset\}$, whose elements are called n-superedges;
- $\lambda : E^{(n)} \to \Sigma$ is a superedge–labelling map;
- $\mu : \big(D^{(n)} \times K\big) \to S \cup \{\perp\}$ is a property map, where

$$D^{(n)} \;:=\; \Big(\bigcup_{k=0}^{n} \mathrm{P}_k(V_0)\Big) \;\cup\; E^{(n)}.$$

  For $x \in D^{(n)}$ and $k \in K$, the value $\mu(x,k) = s \in S$ means that x carries property key k with value s, while $\mu(x,k) = \perp$ means that x has no value for key k.

For each $x \in D^{(n)}$ we write

$$\mathrm{keyset}(x) \;:=\; \{\, k \in K \mid \mu(x,k) \neq \perp \,\}, \qquad \mathrm{val}(x,k) \;:=\; \mu(x,k) \quad (k \in \mathrm{keyset}(x)).$$

When $n = 1$ and only the level–1 carriers and their superedges are used, $H^{(1)}$ reduces to a Property HyperGraph; when Σ is a singleton and $\mu \equiv \perp$, $H^{(n)}$ reduces to an ordinary n-SuperHyperGraph.

Example 6.3.3 (Property SuperHyperGraph for a hospital patient-care pathway (real-life))**.** A hospital patient-care pathway is an ordered sequence of clinical steps, departments, and decisions guiding diagnosis, treatment, monitoring, and discharge for patients (cf. [956]). Let the base set V_0 be a finite set of atomic care-events:

$$V_0 := \{\mathrm{Triage}, \mathrm{CT}, \mathrm{Lab}, \mathrm{Consult}, \mathrm{Surgery}, \mathrm{ICU}, \mathrm{Discharge}\}.$$

Fix level $n = 1$, so $\mathrm{P}_1(V_0) = \mathcal{P}(V_0)$.

Define 1-supervertices (departments / care-units as bundles of events):

$$v_{\mathrm{ER}} := \{\mathrm{Triage}, \mathrm{Lab}\}, \qquad v_{\mathrm{Imaging}} := \{\mathrm{CT}\}, \qquad v_{\mathrm{OR}} := \{\mathrm{Surgery}\}, \qquad v_{\mathrm{Ward}} := \{\mathrm{ICU}, \mathrm{Discharge}\},$$

$$V^{(1)} := \{v_{\mathrm{ER}}, v_{\mathrm{Imaging}}, v_{\mathrm{OR}}, v_{\mathrm{Ward}}\} \subseteq \mathrm{P}_1(V_0).$$

Define 1-superedges (multiway episodes linking several units):

$$e_{\mathrm{workup}} := \{v_{\mathrm{ER}}, v_{\mathrm{Imaging}}\}, \qquad e_{\mathrm{operative}} := \{v_{\mathrm{ER}}, v_{\mathrm{OR}}, v_{\mathrm{Ward}}\},$$

$$E^{(1)} := \{e_{\text{workup}}, e_{\text{operative}}\} \subseteq \mathcal{P}(V^{(1)}) \setminus \{\emptyset\}.$$

Choose a label alphabet and property keys/values:

$\Sigma := \{\text{diagnostic}, \text{treatment}\}$, $K := \{\text{capacity}, \text{risk}, \text{priority}, \text{SLA}\}$, $S := \mathbb{R}_{\geq 0} \cup \{\text{low}, \text{medium}, \text{high}\}$.

Let $\bot \notin S$ be the "undefined" symbol.

Define the superedge labels $\lambda : E^{(1)} \to \Sigma$ by

$$\lambda(e_{\text{workup}}) = \text{diagnostic}, \qquad \lambda(e_{\text{operative}}) = \text{treatment}.$$

Define the property map $\mu : (D^{(1)} \times K) \to S \cup \{\bot\}$ on

$$D^{(1)} = \big(V_0 \cup \mathrm{P}_1(V_0)\big) \cup E^{(1)}$$

by specifying representative values (all unspecified pairs map to $\bot$):

$$\mu(v_{\text{ER}}, \text{capacity}) = 40, \quad \mu(v_{\text{Imaging}}, \text{capacity}) = 12, \quad \mu(v_{\text{OR}}, \text{capacity}) = 6, \quad \mu(v_{\text{Ward}}, \text{capacity}) = 20,$$

$$\mu(e_{\text{workup}}, \text{SLA}) = 2, \quad \mu(e_{\text{operative}}, \text{SLA}) = 6,$$

$$\mu(e_{\text{workup}}, \text{risk}) = \text{medium}, \qquad \mu(e_{\text{operative}}, \text{risk}) = \text{high}.$$

Then

$$H^{(1)} = (V^{(1)}, E^{(1)}, \lambda, \mu)$$

is a *Property* 1*-SuperHyperGraph*. It models hospital care at two levels: atomic events (V_0), grouped units (supervertices in $V^{(1)}$), and multiway episodes (superedges in $E^{(1)}$), while μ attaches operational attributes (capacity, risk, SLA/priority) enabling constraint checks such as

$$\text{val}(e_{\text{operative}}, \text{risk}) = \text{high} \;\Rightarrow\; \text{val}(v_{\text{OR}}, \text{capacity}) \geq 1 \text{ and } \text{val}(e_{\text{operative}}, \text{SLA}) \leq 8.$$

6.4 Knowledge SuperHyperGraphs

Knowledge graphs are structured semantic networks representing entities and relationships, enabling integration, reasoning, and querying across heterogeneous data sources [957–960]. Knowledge graphs are being actively studied and applied across a wide range of fields, including chemistry, computer science, and medicine (cf. [961–964]). Fuzzy knowledge graphs [965–967] and directed knowledge graphs [968, 969] are known as related concepts. Moreover, in machine–learning domains, extensive research has also been conducted on knowledge graph embeddings [970–972] and related representation–learning methods.

A knowledge hypergraph encodes entities as vertices and relational facts as hyperedges, representing structured, queryable knowledge bases in various domains [37,38,973,974]. A knowledge n-SuperHyperGraph lifts entities to iterated powerset supervertices, linking hierarchical superfacts as superhyperedges across abstraction levels in complex domains [975, 976].

Definition 6.4.1 (Knowledge hypergraph)**.** (cf. [977]) Let E be a finite set of *entities* and R a finite set of *relation symbols*. For each $r \in R$ let $|r| \in \mathbb{N}$ denote its arity (the number of arguments of r).

The set of all (ground) candidate facts over (E, R) is

$$\tau(E, R) \;:=\; \Big\{ r(e_1, \ldots, e_{|r|}) \;\Big|\; r \in R,\; e_i \in E \; (i = 1, \ldots, |r|) \Big\}.$$

A *world* on (E, R) is a subset $\tau_0 \subseteq \tau(E, R)$, whose elements are interpreted as true facts.

A *knowledge hypergraph* is a triple

$$H \;=\; (E, R, \tau_0),$$

where $\tau_0 \subseteq \tau(E, R)$ is a world on (E, R). Each fact $r(e_1, \ldots, e_{|r|}) \in \tau_0$ is viewed as a (labelled) hyperedge that simultaneously links the entities $e_1, \ldots, e_{|r|}$. The underlying (unlabelled) hypergraph associated with H has vertex set E and hyperedge set

$$E_H \;:=\; \Big\{ \{e_1, \ldots, e_{|r|}\} \;\Big|\; r(e_1, \ldots, e_{|r|}) \in \tau_0 \Big\}.$$

Definition 6.4.2 (Knowledge n-SuperHyperGraph)**.** [975,976] Let E_0 be a finite *base set of entities* and let R be a finite set of *relation symbols* with arity map

$$r \longmapsto |r| \in \mathbb{N} \qquad (r \in R).$$

For $n \in \mathbb{N}_0$ define the iterated powersets of E_0 by

$$\mathrm{P}_0(E_0) := E_0, \qquad \mathrm{P}_{k+1}(E_0) := \mathrm{P}\big(\mathrm{P}_k(E_0)\big) \quad (k \geq 0),$$

where $\mathrm{P}(\cdot)$ denotes the usual powerset.

A *knowledge n-SuperHyperGraph* (or knowledge SuperHyperGraph of level n) over (E_0, R) is a triple

$$\mathsf{KH}^{(n)} = \big(V,\ R,\ \tau_0^{(n)}\big),$$

where

- $V \subseteq \mathrm{P}_n(E_0)$ is a finite set of *n-superentities* (or n-supervertices); each $v \in V$ is a nested group of entities obtained by applying the powerset operator n times to E_0;
- $\tau_0^{(n)}$ is a set of *true n-superfacts* of the form

$$\tau_0^{(n)} \subseteq \Big\{ r(v_1, \ldots, v_{|r|}) \ \Big|\ r \in R,\ v_i \in V\ (i = 1, \ldots, |r|) \Big\}.$$

Each fact $r(v_1, \ldots, v_{|r|}) \in \tau_0^{(n)}$ is interpreted as a higher–order, recursively nested relation between the superentities $v_1, \ldots, v_{|r|}$.

The underlying (unlabelled) level-n SuperHyperGraph associated with $\mathsf{KH}^{(n)}$ has vertex set V and hyperedge set

$$E_H^{(n)} := \Big\{ \{v_1, \ldots, v_{|r|}\} \ \Big|\ r(v_1, \ldots, v_{|r|}) \in \tau_0^{(n)} \Big\} \subseteq \mathrm{P}(V).$$

For $n = 0$ and $V = E_0$ this reduces to a (labelled) knowledge hypergraph in the sense of the previous definition. For $n \geq 1$ the construction yields a hierarchical, multi–level generalization based on the iterated powerset of the base entity set E_0.

Example 6.4.3 (Knowledge 1-SuperHyperGraph: Data Science Curriculum)**.** Consider a small curriculum in data science (cf. [978]). Let the base entity set be

$$E_0 = \{\,\mathsf{LinAlg}, \mathsf{Calc}, \mathsf{Prob}, \mathsf{ML}, \mathsf{DL}\,\},$$

where

- LinAlg = Linear Algebra,
- Calc = Calculus,
- Prob = Probability,
- ML = Machine Learning,
- DL = Deep Learning.

For $n = 1$ we have

$$\mathrm{P}_0(E_0) = E_0, \qquad \mathrm{P}_1(E_0) = \mathrm{P}(E_0).$$

Define the set of 1-superentities (supervertices) by

$$V := \big\{ \{\mathsf{LinAlg}\}, \{\mathsf{Calc}\}, \{\mathsf{Prob}\}, \{\mathsf{ML}\}, \{\mathsf{DL}\}, \{\mathsf{LinAlg}, \mathsf{Calc}, \mathsf{Prob}\}, \{\mathsf{ML}, \mathsf{DL}\} \big\} \subseteq \mathrm{P}_1(E_0).$$

Thus each $v \in V$ is a set of base courses, viewed as a *superentity*; for instance:

$$v_{\text{found}} := \{\mathsf{LinAlg}, \mathsf{Calc}, \mathsf{Prob}\}$$

is a "foundational knowledge cluster", and

$$v_{\text{adv}} := \{\mathsf{ML}, \mathsf{DL}\}$$

is an "advanced modeling cluster".

Let R be a set of relation symbols containing a single binary symbol

$$r_{\text{prereq}} \in R, \qquad |r_{\text{prereq}}| = 2,$$

to be read as "is a prerequisite cluster for".

We now specify the set of true 1-superfacts $\tau_0^{(1)}$ by

$$\tau_0^{(1)} := \Big\{ r_{\text{prereq}}\big(\{\mathsf{LinAlg}, \mathsf{Calc}, \mathsf{Prob}\}, \{\mathsf{ML}\}\big),\ r_{\text{prereq}}\big(\{\mathsf{ML}\}, \{\mathsf{DL}\}\big),\ r_{\text{prereq}}\big(\{\mathsf{LinAlg}, \mathsf{Calc}, \mathsf{Prob}\}, \{\mathsf{ML}, \mathsf{DL}\}\big) \Big\}.$$

These facts encode the following higher–order statements:

- the foundational cluster $\{\mathsf{LinAlg}, \mathsf{Calc}, \mathsf{Prob}\}$ is a prerequisite for the course ML;
- the course ML is a prerequisite for DL;
- the same foundational cluster is also a prerequisite for the advanced cluster $\{\mathsf{ML}, \mathsf{DL}\}$ as a whole.

The triple

$$\mathsf{KH}^{(1)} := \big(V,\ R,\ \tau_0^{(1)}\big)$$

is therefore a *knowledge* 1*-SuperHyperGraph* over (E_0, R) in the sense of the definition above: vertices are nested groups of courses, and hyperedges (encoded by the facts in $\tau_0^{(1)}$) represent higher–order prerequisite relations between course clusters.

The underlying (unlabelled) level-1 SuperHyperGraph $E_H^{(1)}$ has vertex set V and hyperedge set

$$E_H^{(1)} = \Big\{ \{\{\mathsf{LinAlg}, \mathsf{Calc}, \mathsf{Prob}\}, \{\mathsf{ML}\}\},\ \{\{\mathsf{ML}\}, \{\mathsf{DL}\}\},\ \{\{\mathsf{LinAlg}, \mathsf{Calc}, \mathsf{Prob}\}, \{\mathsf{ML}, \mathsf{DL}\}\} \Big\},$$

obtained by forgetting the label r_{prereq}. This 1-SuperHyperGraph compactly encodes multi-course, multi-level prerequisite structures in the curriculum.

6.5 Quantum Superhypergraph

Quantum theory describes physical systems using wavefunctions, operators, and probability amplitudes, explaining superposition, entanglement, measurement outcomes, and quantized energies (cf. [979, 980]). Quantum graphs model wave or quantum dynamics on metric graph edges using differential operators, spectra, and boundary conditions at vertices [44, 45, 981]. Related concepts, such as quantum directed graphs [982, 983], periodic quantum graphs [984, 985], and infinite quantum graphs [986, 987] are also known. A Quantum HyperGraph assigns qubits to hypergraph vertices and controlled-phase operations to hyperedges, defining an entangled hypergraph state in Hilbert-space [46, 988–990]. A Quantum SuperHyperGraph equips each supervertex of an n-SuperHyperGraph with a qubit, using multi-qubit phase gates to encode entanglement structure [47].

Definition 6.5.1 (Quantum Graph)**.** [44, 45, 981] A *quantum graph* is a *metric graph* $G = (V, E, (L_e)_{e\in E})$ (each edge e is identified with an interval $[0, L_e]$) together with a self-adjoint differential operator H on the Hilbert space

$$\mathcal{H} := \bigoplus_{e\in E} L^2(0, L_e),$$

such that on each edge e the operator acts as a one-dimensional Schrödinger operator

$$(H\psi)_e(x) = -\frac{d^2}{dx^2}\psi_e(x) + V_e(x)\psi_e(x) \qquad (x \in (0, L_e)),$$

and the edge-components are coupled at vertices by self-adjoint vertex boundary conditions (e.g. continuity at each vertex and a Kirchhoff/δ-type condition $\sum_{e\sim v} \partial_v\psi_e(v) = \alpha_v\, \psi(v)$). The spectrum and eigenfunctions of H are called the spectrum and eigenstates of the quantum graph.

Definition 6.5.2 (Quantum HyperGraph)**.** [46, 988–990] Let $H = (V, E)$ be a finite hypergraph with vertex set $V = \{v_1, \dots, v_n\}$. Associate to each vertex $v_i \in V$ a qubit with Hilbert space $\mathcal{H}_i \cong \mathbb{C}^2$, and set

$$\mathcal{H} \ := \ \bigotimes_{i=1}^{n} \mathcal{H}_i.$$

Define the single–qubit state

$$|+\rangle \ := \ \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big), \qquad |+\rangle^{\otimes n} \ := \ \bigotimes_{i=1}^{n} |+\rangle_i.$$

For each hyperedge $e \in E$, define the projector

$$P_e \ := \ \bigotimes_{i=1}^{n} M_i^{(e)}, \qquad M_i^{(e)} \ := \ \begin{cases} |1\rangle\langle 1|_i, & v_i \in e, \\ I_i, & v_i \notin e, \end{cases}$$

and the associated controlled–phase operator

$$CZ_e \ := \ I_{\mathcal{H}} - 2P_e,$$

where $I_{\mathcal{H}}$ is the identity on $\mathcal{H}$. Then CZ_e multiplies by -1 exactly those computational basis vectors in which all qubits indexed by vertices of e are in state $|1\rangle$.

The *quantum hypergraph state* associated with H is

$$|H\rangle \ := \ \Big(\prod_{e\in E} CZ_e\Big)\, |+\rangle^{\otimes n},$$

where the product may be taken in any fixed order (all CZ_e are diagonal and therefore commute).

A *Quantum HyperGraph* is the pair

$$\mathsf{QH} \ := \ (H, |H\rangle),$$

consisting of the finite hypergraph H and its associated quantum hypergraph state $|H\rangle \in \mathcal{H}$.

Example 6.5.3 (Quantum HyperGraph for a three-qubit phase-entangled state)**.** Consider the finite hypergraph

$$H = (V, E), \qquad V = \{v_1, v_2, v_3\}, \qquad E = \big\{\, \{v_1, v_2, v_3\} \,\big\}.$$

Thus there is a single hyperedge joining all three vertices.

Associate to each vertex v_i a qubit with Hilbert space $\mathcal{H}_i \cong \mathbb{C}^2$, and set

$$\mathcal{H} \ := \ \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \ \cong \ \mathbb{C}^8.$$

Let

$$|+\rangle \ := \ \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big), \qquad |+\rangle^{\otimes 3} \ := \ |+\rangle_1 \otimes |+\rangle_2 \otimes |+\rangle_3.$$

For the unique hyperedge $e = \{v_1, v_2, v_3\}$, define

$$P_e \;:=\; |1\rangle\langle 1|_1 \otimes |1\rangle\langle 1|_2 \otimes |1\rangle\langle 1|_3, \qquad CZ_e \;:=\; I_{\mathcal{H}} - 2P_e.$$

The quantum hypergraph state associated with H is then

$$|H\rangle \;:=\; CZ_e\,|+\rangle^{\otimes 3}.$$

In words, we begin with three independent qubits in the equal superposition state $|+\rangle^{\otimes 3}$ and apply a three-qubit controlled phase gate that flips the sign of the basis state $|111\rangle$. The pair

$$\mathsf{QH} \;:=\; (H, |H\rangle)$$

is a concrete *Quantum HyperGraph*: the hypergraph H encodes the three-body interaction pattern, and $|H\rangle$ is the resulting entangled quantum state.

Definition 6.5.4 (Quantum SuperHyperGraph)**.** [47] Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph with finite n-supervertex set $V = \{v_1, \ldots, v_m\}$ and n-superedge set E. Associate to each n-supervertex $v \in V$ a qubit with Hilbert space $\mathcal{H}_v \cong \mathbb{C}^2$ and define the total Hilbert space

$$\mathcal{H} \;:=\; \bigotimes_{v \in V} \mathcal{H}_v.$$

Set

$$|+\rangle \;:=\; \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big), \qquad |+\rangle^{\otimes m} \;:=\; \bigotimes_{v \in V} |+\rangle_v.$$

For each n-superedge $e \in E$, define the projector

$$P_e \;:=\; \bigotimes_{v \in V} M_v^{(e)}, \qquad M_v^{(e)} \;:=\; \begin{cases} |1\rangle\langle 1|_v, & v \in e, \\ I_v, & v \notin e, \end{cases}$$

and the generalized controlled–phase operator

$$CZ_e \;:=\; I_{\mathcal{H}} - 2P_e.$$

Thus CZ_e multiplies by -1 exactly those computational basis vectors in which all qubits indexed by supervertices in e are in state $|1\rangle$.

The *quantum n-SuperHyperGraph state* associated with $\mathrm{SHG}^{(n)}$ is

$$\big|\mathrm{SHG}^{(n)}\big\rangle \;:=\; \Big(\prod_{e \in E} CZ_e\Big)\,|+\rangle^{\otimes m}.$$

A *Quantum SuperHyperGraph of level n* is the pair

$$\mathsf{QSH}^{(n)} \;:=\; \big(\mathrm{SHG}^{(n)}, \big|\mathrm{SHG}^{(n)}\big\rangle\big),$$

consisting of the n-SuperHyperGraph $\mathrm{SHG}^{(n)}$ and its associated quantum SuperHyperGraph state in $\mathcal{H}$. For $n = 1$ and when V is a set of ordinary vertices, this construction reduces to a Quantum HyperGraph.

Example 6.5.5 (Quantum SuperHyperGraph for two logical qubits built from four physical qubits)**.** Let the base set of physical qubits be

$$V_0 = \{q_1, q_2, q_3, q_4\}.$$

We group them into two *logical modules*

$$v_A := \{q_1, q_2\}, \qquad v_B := \{q_3, q_4\},$$

and define the level-1 supervertex set

$$V \;:=\; \{v_A, v_B\} \;\subseteq\; \mathcal{P}_1(V_0) = \mathcal{P}(V_0).$$

Model	Underlying object	State space / dynamics	Typical data encoded
Quantum Graph [44, 45, 981]	Metric graph $(V, E, (L_e))$ (edges are intervals)	Hilbert space $\bigoplus_{e \in E} L^2(0, L_e)$; self-adjoint operator on edges + vertex boundary conditions	Spectrum, eigenfunctions, wave/quantum transport on networks
Quantum HyperGraph [46, 988–990]	Finite hypergraph (V, E) (hyperedges are subsets of vertices)	$\lvert V\rvert$ qubits; hyperedge-controlled phase gates CZ_e produce a hypergraph state $\lvert H\rangle$	Multi-body entanglement pattern specified by hyperedges
Quantum n-SuperHyperGraph [47]	n-SuperHyperGraph (V, E) (vertices are n-level supervertices)	One qubit per n-supervertex; multi-qubit phase gates per n-superedge yield $\lvert \mathrm{SHG}^{(n)}\rangle$	Entanglement among hierarchical/coarse-grained subsystems (supervertices)

Table 6.3: Concise overview of Quantum Graphs, Quantum HyperGraphs, and Quantum n-SuperHyperGraphs.

Consider the 1-SuperHyperGraph

$$\mathrm{SHG}^{(1)} = (V, E), \qquad E := \{ e \}, \quad e := \{v_A, v_B\},$$

so there is a single superedge e connecting the two supervertices v_A and v_B. This models a higher-level interaction between the two logical modules $\{q_1, q_2\}$ and $\{q_3, q_4\}$.

We now construct a Quantum SuperHyperGraph of level 1 based on $\mathrm{SHG}^{(1)}$. Assign to each supervertex $v \in V$ a qubit with Hilbert space $\mathcal{H}_v \cong \mathbb{C}^2$ and define

$$\mathcal{H} \ := \ \mathcal{H}_{v_A} \otimes \mathcal{H}_{v_B} \ \cong \ \mathbb{C}^4.$$

Let

$$|+\rangle \ := \ \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big), \qquad |+\rangle^{\otimes 2} \ := \ |+\rangle_{v_A} \otimes |+\rangle_{v_B}.$$

For the superedge $e = \{v_A, v_B\}$, define

$$P_e \ := \ |1\rangle\langle 1|_{v_A} \otimes |1\rangle\langle 1|_{v_B}, \qquad CZ_e \ := \ I_{\mathcal{H}} - 2P_e,$$

so that CZ_e is a controlled-phase gate acting on the two *logical* qubits corresponding to the supervertices v_A and v_B.

The associated Quantum SuperHyperGraph state is

$$\big|\mathrm{SHG}^{(1)}\big\rangle \ := \ CZ_e \, |+\rangle^{\otimes 2}.$$

The pair

$$\mathsf{QSH}^{(1)} \ := \ \big(\mathrm{SHG}^{(1)}, \ \big|\mathrm{SHG}^{(1)}\big\rangle\big)$$

is a concrete *Quantum SuperHyperGraph*: the level-1 SuperHyperGraph $\mathrm{SHG}^{(1)}$ records an interaction between two modules (each module being a subset of physical qubits in V_0), and the quantum state $\big|\mathrm{SHG}^{(1)}\big\rangle$ is the entangled state of the corresponding two logical qubits. In this way, Quantum SuperHyperGraphs naturally model entanglement *between coarse-grained subsystems* rather than only between individual physical qubits.

The overview of Quantum Graphs, Quantum HyperGraphs, and Quantum n-SuperHyperGraphs is presented in Table 6.3.

6.6 SuperHyperGraph Containter

A hypergraph container is a selected vertex subset system capturing all independent sets while greatly reducing combinatorial search complexity space(cf. [76, 78, 991, 992]). A SuperHyperGraph container is a family of n-supervertex subsets covering all independent supervertex configurations within hierarchical SuperHyperGraph structures and dynamics [992].

Definition 6.6.1 ((Recall) r-uniform n-SuperHyperGraph)**.** Let V_0 be a nonempty finite base set and let

$$P_0(V_0) := V_0, \qquad P_{k+1}(V_0) := P\big(P_k(V_0)\big) \quad (k \geq 0),$$

where $P(\cdot)$ denotes the usual powerset.

A *level-n SuperHyperGraph* on V_0 is a pair

$$\mathrm{SHG}^{(n)} := (V, E),$$

where

$$\emptyset \neq V \subseteq P_n(V_0), \qquad \emptyset \neq E \subseteq P^*(V) := P(V) \setminus \{\emptyset\}.$$

Elements of V are called n-supervertices and elements of E are called n-superedges.

For $r \in \mathbb{N}$, the level-n SuperHyperGraph $\mathrm{SHG}^{(n)}$ is called *r-uniform* if

$$E \;\subseteq\; \{\, e \subseteq V \mid |e| = r \,\},$$

that is, every n-superedge contains exactly r n-supervertices.

Definition 6.6.2 (Degree, average degree, and degree measure)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an r-uniform n-SuperHyperGraph with $|V| = N$ and $|E| \geq 1$.

For $v \in V$, the *degree* of v is

$$d(v) \;:=\; \big|\{\, e \in E \mid v \in e \,\}\big|.$$

The *average degree* of $\mathrm{SHG}^{(n)}$ is

$$d \;:=\; \frac{1}{N} \sum_{v \in V} d(v).$$

For any subset $S \subseteq V$, the *degree measure* of S is

$$\mu(S) \;:=\; \begin{cases} 0, & S = \emptyset, \\ \dfrac{1}{Nd} \displaystyle\sum_{v \in S} d(v), & S \neq \emptyset. \end{cases}$$

Note that $0 \leq \mu(S) \leq 1$ for all $S \subseteq V$.

Definition 6.6.3 (Induced sub-n-SuperHyperGraph and edge count)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an r-uniform n-SuperHyperGraph. For any $S \subseteq V$, the *induced sub-n-SuperHyperGraph* on S is

$$\mathrm{SHG}^{(n)}[S] := \big(S, E[S]\big), \qquad E[S] := \{\, e \in E \mid e \subseteq S \,\}.$$

We write

$$e(S) \;:=\; |E[S]|$$

for the number of n-superedges entirely contained in S.

Definition 6.6.4 (Independent set in an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an r-uniform n-SuperHyperGraph. A set $I \subseteq V$ is called *independent* if it spans no n-superedges, i.e.

$$E[I] = \emptyset \quad \Longleftrightarrow \quad e(I) = 0.$$

Definition 6.6.5 (SuperHyperGraph container family)**.** [992] Let $\mathrm{SHG}^{(n)} = (V, E)$ be an r-uniform n-SuperHyperGraph with average degree d and degree measure μ as in Definition 6.6.2. A family

$$\mathcal{C} \;\subseteq\; P(V)$$

is called a *container family* (or a family of *n-SuperHyperGraph containers*) for $\mathrm{SHG}^{(n)}$ if the following two conditions hold.

- **Covering of independent sets.** Every independent set $I \subseteq V$ satisfies

$$\exists\, C \in \mathcal{C} \quad \text{such that} \quad I \subseteq C.$$

- **Smallness of containers.** There exist fixed parameters

$$\varepsilon \in (0,1), \qquad \alpha \in (0,1),$$

such that for every $C \in \mathcal{C}$ one has both

$$\mu(C) \le \alpha \quad \text{and} \quad e(C) \le (1-\varepsilon)\,|E|.$$

In other words, every independent set of the r-uniform n-SuperHyperGraph is contained in at least one container $C \in \mathcal{C}$, and each container is "small" both in degree measure (it occupies at most an α-fraction of the total degree mass) and in the number of n-superedges it spans (it contains at most a $(1-\varepsilon)$-fraction of all n-superedges).

For $n = 0$ and $V \subseteq V_0$ this notion reduces to the usual hypergraph container family on an r-uniform hypergraph.

Example 6.6.6 (SuperHyperGraph container: curriculum track design)**.** We now describe a container family for a 2-SuperHyperGraph modeling university programme tracks built from course modules.

Let V_0 be a finite base set of courses and let

$$P_1(V_0) := P(V_0), \qquad P_2(V_0) := P\big(P(V_0)\big)$$

be the first and second iterated powersets. Consider three 2-supervertices

$$v_{\text{found}},\ v_{\text{AI}},\ v_{\text{DS}} \in P_2(V_0),$$

defined informally as follows.

- v_{found} is a "foundation" track, consisting of a single core module (e.g. mathematics plus introductory programming).
- v_{AI} is an AI-oriented track, containing the foundation module together with AI-related and data-science modules.
- v_{DS} is a data-science track, combining the foundation module with data-science oriented modules.

We form the 2-SuperHyperGraph

$$\text{SHG}^{(2)} := (V, E),$$

where

$$V := \{v_{\text{found}}, v_{\text{AI}}, v_{\text{DS}}\} \subseteq P_2(V_0),$$

and the 2-superedges are

$$e_1 := \{v_{\text{found}}, v_{\text{AI}}\}, \quad e_2 := \{v_{\text{found}}, v_{\text{DS}}\}, \quad e_3 := \{v_{\text{AI}}, v_{\text{DS}}\},$$

so that

$$E := \{e_1, e_2, e_3\}.$$

This is a 2-uniform 2-SuperHyperGraph, since each e_i contains exactly two 2-supervertices.

Degree, average degree, and degree measure. Each 2-supervertex lies in exactly two 2-superedges, so

$$d\big(v_{\text{found}}\big) = d\big(v_{\text{AI}}\big) = d\big(v_{\text{DS}}\big) = 2.$$

Hence

$$N := |V| = 3, \qquad \sum_{v \in V} d(v) = 2 + 2 + 2 = 6,$$

and the average degree is

$$d = \frac{1}{N} \sum_{v \in V} d(v) = \frac{6}{3} = 2.$$

Thus

$$Nd = 3 \cdot 2 = 6,$$

and for any nonempty $S \subseteq V$ the degree measure is

$$\mu(S) \;=\; \frac{1}{Nd}\sum_{v\in S} d(v) \;=\; \frac{1}{6}\sum_{v\in S} d(v).$$

Independent sets. A subset $I \subseteq V$ is independent if it spans no 2-superedge, i.e. if $e(I) = 0$. Since every pair of distinct 2-supervertices forms a 2-superedge (the structure is a "triangle" on V), any independent set I can contain at most one vertex. Therefore the independent sets are exactly

$$I \in \big\{\emptyset, \{v_{\text{found}}\}, \{v_{\text{AI}}\}, \{v_{\text{DS}}\}\big\}.$$

Container family. Define

$$C_{\text{found}} := \{v_{\text{found}}\}, \quad C_{\text{AI}} := \{v_{\text{AI}}\}, \quad C_{\text{DS}} := \{v_{\text{DS}}\},$$

and set

$$\mathcal{C} := \{C_{\text{found}}, C_{\text{AI}}, C_{\text{DS}}\} \;\subseteq\; P(V).$$

We verify that $\mathcal{C}$ is a container family for $\text{SHG}^{(2)}$ in the sense of Definition 6.6.5.

(1) *Covering of independent sets.* Each nonempty independent set is equal to one of the three singletons above, and the empty set $\emptyset$ is contained in every container. Hence for every independent I there exists $C \in \mathcal{C}$ with $I \subseteq C$.

(2) *Smallness of containers.* For each container we have

$$\mu\big(C_{\text{found}}\big) = \frac{1}{6}\, d\big(v_{\text{found}}\big) = \frac{1}{6}\cdot 2 = \frac{2}{6} = \frac{1}{3},$$

and similarly

$$\mu\big(C_{\text{AI}}\big) = \mu\big(C_{\text{DS}}\big) = \frac{1}{3}.$$

Moreover, each container is a singleton, so it cannot contain a 2-superedge as a subset. Thus

$$e\big(C_{\text{found}}\big) = e\big(C_{\text{AI}}\big) = e\big(C_{\text{DS}}\big) = 0.$$

Since $|E| = 3$, we choose

$$\varepsilon := \frac{1}{2}, \qquad \alpha := \frac{1}{2}.$$

Then

$$(1-\varepsilon)|E| \;=\; \Big(1 - \frac{1}{2}\Big)\cdot 3 \;=\; \frac{3}{2},$$

and we have

$$e(C) = 0 \le \frac{3}{2} = (1-\varepsilon)|E| \quad \text{for all } C \in \mathcal{C}.$$

Furthermore,

$$\mu(C) = \frac{1}{3} \;\le\; \frac{1}{2} = \alpha \quad \text{for all } C \in \mathcal{C}.$$

Hence $\mathcal{C}$ satisfies the smallness conditions.

The 2-supervertices represent hierarchical curriculum patterns: each $v \in V$ is a family of course modules forming a programme track (e.g. foundation, AI-focused, or data-science focused). The 2-superedges encode administrative couplings between tracks (for example, shared committees or accreditation constraints). Independent sets are collections of tracks that are not jointly constrained, so at most one track can be chosen freely at a time. The three containers $\{v_{\text{found}}\}$, $\{v_{\text{AI}}\}$, and $\{v_{\text{DS}}\}$ summarize all independent choices of higher-level programme configurations in this toy model.

6.7 SuperHyperGraph-Based Food Web

A food web is an ecological network showing who eats whom among species, capturing energy flow and trophic interactions dynamically [943, 993, 994]. A graph-based food web models species as vertices and predator–prey relations as directed edges in a digraph for network analysis. A SuperHyperGraph-based food web uses n-supervertices from iterated powersets of species, encoding hierarchical, overlapping predator–prey groupings as superedges and contexts [80].

Definition 6.7.1 (SuperHyperGraph-Based Food Web (Food n-SuperHyperWeb))**.** [80] Let $G = (V_0, E_0)$ be a Food Web, that is, a finite directed graph whose vertices are *species* and whose arcs encode predator–prey relations:

$$(u, v) \in E_0 \quad \Longleftrightarrow \quad \text{species } u \text{ preys upon species } v,$$

with no self-loops and no parallel arcs.

For each predator $u \in V_0$ define its *prey set*

$$P(u) \ := \ \{ v \in V_0 \mid (u, v) \in E_0 \}.$$

Fix an integer $n \in \mathbb{N}_0$ and let $\mathcal{P}^n(V_0)$ denote the n-fold iterated powerset of V_0 as in Definition 2.2.1. Set

$$V_n \ := \ \mathcal{P}^n(V_0),$$

and for each $u \in V_0$ with $P(u) \neq \emptyset$ define the *n-superedge*

$$e_u^{(n)} \ := \ \mathcal{P}^n\big(P(u)\big) \ \subseteq \ \mathcal{P}^n(V_0) \ = \ V_n.$$

Thus each $e_u^{(n)}$ is a nonempty subset of V_n, hence $e_u^{(n)} \in \mathcal{P}^*(V_n)$, where $\mathcal{P}^*(V_n) := \mathcal{P}(V_n) \setminus \{\varnothing\}$. Define

$$E_n \ := \ \left\{ e_u^{(n)} \ \middle|\ u \in V_0,\ P(u) \neq \emptyset \right\} \ \subseteq \ \mathcal{P}^*(V_n).$$

The pair

$$F^{(n)} \ := \ (V_n, E_n)$$

is called the *Food n-SuperHyperWeb* or *SuperHyperGraph-Based Food Web of level n* associated with the Food Web G.

By construction,

$$\varnothing \neq V_n \subseteq \mathcal{P}^n(V_0), \qquad \varnothing \neq E_n \subseteq \mathcal{P}^*(V_n),$$

so $F^{(n)}$ is an n-SuperHyperGraph on the base set V_0 in the sense used throughout this work.

Moreover, let flat_n be the full-flattening operator on $\mathcal{P}^n(V_0)$ (as in the definition of the competition n-SuperHyperGraph). Applying flat_n elementwise to each n-superedge $e_u^{(n)}$ and taking the union yields exactly the prey set $P(u)$:

$$\bigcup_{X \in e_u^{(n)}} \mathrm{flat}_n(X) \ = \ P(u).$$

In this sense, the usual Food Hypergraph (with hyperedges $P(u)$, $u \in V_0$) and the original directed Food Web G appear as flattened shadows of the SuperHyperGraph-based model $F^{(n)}$.

Example 6.7.2 (Coastal bay ecosystem as a Food 1-SuperHyperWeb)**.** Consider a simplified coastal bay with the following five species:

$$V_0 := \{\text{Grass}, \text{Shrimp}, \text{Herring}, \text{Seal}, \text{Shark}\},$$

where Grass is a primary producer (seagrass or microalgae), Shrimp is a small benthic consumer, Herring is a small pelagic fish, Seal is a marine mammal, and Shark is a top predator.

The predator–prey relations in the underlying Food Web $G = (V_0, E_0)$ are:

$$E_0 := \{(\text{Shrimp}, \text{Grass}),\ (\text{Herring}, \text{Shrimp}),\ (\text{Seal}, \text{Herring}),\ (\text{Shark}, \text{Seal}),\ (\text{Shark}, \text{Herring})\}.$$

For instance, $(\mathrm{Shrimp}, \mathrm{Grass}) \in E_0$ means that shrimp graze on seagrass, and $(\mathrm{Shark}, \mathrm{Seal}) \in E_0$ means that sharks prey on seals.

For each predator $u \in V_0$, its prey set $P(u) \subseteq V_0$ is

$$P(\mathrm{Shrimp}) = \{\mathrm{Grass}\}, \qquad P(\mathrm{Herring}) = \{\mathrm{Shrimp}\},$$

$$P(\mathrm{Seal}) = \{\mathrm{Herring}\}, \qquad P(\mathrm{Shark}) = \{\mathrm{Seal}, \mathrm{Herring}\},$$

and $P(\mathrm{Grass}) = \emptyset$ since Grass is a primary producer.

We now construct the Food 1-SuperHyperWeb $F^{(1)} = (V_1, E_1)$ associated with this Food Web, using Definition 2.2.1 and the SuperHyperGraph-Based Food Web construction.

First, the level-1 vertex set is the powerset

$$V_1 := \mathcal{P}^1(V_0) = \mathcal{P}(V_0),$$

so a 1-supervertex is any subset of species, for example $\{\mathrm{Grass}, \mathrm{Shrimp}\}$ or $\{\mathrm{Seal}, \mathrm{Shark}\}$. Intuitively, such a 1-supervertex represents a *group* of species that we consider jointly (e.g. a group of prey or a combined foraging guild).

For each predator $u \in V_0$ with nonempty prey set $P(u)$, the 1-superedge is the powerset of its prey:

$$e_u^{(1)} := \mathcal{P}^1(P(u)) = \mathcal{P}(P(u)) \subseteq \mathcal{P}(V_0) = V_1.$$

We now compute these 1-superedges explicitly.

(1) For Shrimp we have $P(\mathrm{Shrimp}) = \{\mathrm{Grass}\}$, so

$$e_{\mathrm{Shrimp}}^{(1)} = \mathcal{P}\big(\{\mathrm{Grass}\}\big) = \big\{\emptyset,\ \{\mathrm{Grass}\}\big\}.$$

(2) For Herring we have $P(\mathrm{Herring}) = \{\mathrm{Shrimp}\}$, hence

$$e_{\mathrm{Herring}}^{(1)} = \mathcal{P}\big(\{\mathrm{Shrimp}\}\big) = \big\{\emptyset,\ \{\mathrm{Shrimp}\}\big\}.$$

(3) For Seal we have $P(\mathrm{Seal}) = \{\mathrm{Herring}\}$, hence

$$e_{\mathrm{Seal}}^{(1)} = \mathcal{P}\big(\{\mathrm{Herring}\}\big) = \big\{\emptyset,\ \{\mathrm{Herring}\}\big\}.$$

(4) For Shark we have $P(\mathrm{Shark}) = \{\mathrm{Seal}, \mathrm{Herring}\}$, so

$$e_{\mathrm{Shark}}^{(1)} = \mathcal{P}\big(\{\mathrm{Seal}, \mathrm{Herring}\}\big)$$

$$= \big\{\emptyset,\ \{\mathrm{Seal}\},\ \{\mathrm{Herring}\},\ \{\mathrm{Seal}, \mathrm{Herring}\}\big\}.$$

Collecting all nonempty 1-superedges gives the edge set

$$E_1 := \big\{e_{\mathrm{Shrimp}}^{(1)}, e_{\mathrm{Herring}}^{(1)}, e_{\mathrm{Seal}}^{(1)}, e_{\mathrm{Shark}}^{(1)}\big\} \subseteq \mathcal{P}^*(V_1),$$

and the Food 1-SuperHyperWeb is

$$F^{(1)} := (V_1, E_1).$$

In this coastal-bay interpretation, each 1-supervertex $X \in V_1$ is a *prey group* (a set of species viewed together), while each 1-superedge $e_u^{(1)}$ represents all possible prey-group configurations formed from the predator u's prey list. For example, $e_{\mathrm{Shark}}^{(1)}$ contains:

- {Seal, Herring}: shark foraging on both seals and herring together,
- {Seal} or {Herring}: shark feeding predominantly on just one prey type,
- $\emptyset$: a "no-catch" or inactive-foraging configuration.

If we denote by flat_1 the full-flattening operator on $\mathcal{P}(V_0)$ (so that $\mathrm{flat}_1(X) = X$ for any $X \subseteq V_0$), then for each predator u we can recover its original prey set as

$$\bigcup_{X \in e_u^{(1)}} \mathrm{flat}_1(X) = \bigcup_{X \in \mathcal{P}(P(u))} X = P(u).$$

Thus the usual Food Hypergraph (whose hyperedges are the prey sets $P(u)$) and the directed Food Web G appear as flattened shadows of the richer SuperHyperGraph-based food-web model $F^{(1)}$, which captures not only "who eats whom" but also how an apex predator such as Shark may dynamically combine or switch among different prey groups in a realistic coastal ecosystem.

6.8 Crystal SuperHyperGraph in material sciences

A crystal graph represents atoms as vertices and bonds as edges in a periodic lattice, capturing coordination connectivity for analysis [83, 995, 996]. A crystal SuperHyperGraph organizes atoms and motifs into iterated powerset supervertices and superedges, modeling hierarchical coordination patterns and overlapping environments [85].

Definition 6.8.1 (Crystal n-SuperHyperGraph)**.** [85] Let C be a periodic crystal structure with atom set

$$V_0 = \{v_1, \ldots, v_N\}$$

in a unit cell and lattice Λ. For each integer $k \geq 0$, define the iterated powerset of V_0 by

$$\mathcal{P}_0(V_0) := V_0, \qquad \mathcal{P}_{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}_k(V_0)\big),$$

where $\mathcal{P}(X)$ denotes the powerset of X.

Fix $n \in \mathbb{N}_0$. A *Crystal n-SuperHyperGraph* is a pair

$$\mathsf{CSHT}(n) := \big(V^{(n)}, E^{(n)}\big),$$

where

$$\emptyset \neq V^{(n)} \subseteq \mathcal{P}_n(V_0) \quad \text{and} \quad \emptyset \neq E^{(n)} \subseteq \mathcal{P}(V^{(n)}) \setminus \{\emptyset\}.$$

The elements of $V^{(n)}$ are called *n-supervertices*, and the elements of $E^{(n)}$ are called *n-superedges*. The incidence relation between n-supervertices and n-superedges is given by set membership:

$$(v, e) \text{ is incident} \quad \Longleftrightarrow \quad v \in V^{(n)},\ e \in E^{(n)},\ v \in e.$$

Each n-supervertex $v \in V^{(n)}$ represents a (possibly nested) cluster of atoms or lower-level motifs of the crystal C across n powerset layers, and each n-superedge $e \in E^{(n)}$ collects a finite nonempty family of such clusters that interact structurally (e.g., overlapping coordination environments).

In particular:

- for $n = 0$, $\mathsf{CSHT}(0)$ reduces to the usual *Crystal Graph* on the atom set V_0;
- for $n = 1$, $\mathsf{CSHT}(1)$ recovers the standard *Crystal HyperGraph*, whose hyperedges encode local coordination motifs in the crystal lattice.

Thus every Crystal n-SuperHyperGraph is an n-SuperHyperGraph built over the base atom set V_0 of the crystal C.

Example 6.8.2 (Crystal 2-SuperHyperGraph for a perovskite-like unit)**.** Consider a simplified perovskite-type unit cell (cf. [997]) with one A-site cation, one B-site cation, and three oxygen atoms. Let the base atom set be

$$V_0 \ = \ \{A_1,\, B_1,\, O_1,\, O_2,\, O_3\}.$$

The first iterated powerset is

$$\mathcal{P}_1(V_0) \ = \ \mathcal{P}(V_0),$$

and the second iterated powerset is

$$\mathcal{P}_2(V_0) \ = \ \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

Within $\mathcal{P}_1(V_0)$ we single out two local coordination motifs:

$$M_A := \{A_1, O_1, O_2, O_3\} \qquad \text{(the A-centered coordination polyhedron),}$$
$$M_B := \{B_1, O_1, O_2, O_3\} \qquad \text{(the B-centered coordination polyhedron).}$$

Both M_A and M_B are ordinary subsets of V_0 and hence elements of $\mathcal{P}_1(V_0)$.

We now pass to the second powerset $\mathcal{P}_2(V_0)$. Define three 2-supervertices by

$$v_1 := \{M_A\}, \qquad v_2 := \{M_B\}, \qquad v_3 := \{M_A, M_B\},$$

so that $v_1, v_2, v_3 \in \mathcal{P}_2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$. Set

$$V^{(2)} \ := \ \{v_1, v_2, v_3\} \ \subseteq \ \mathcal{P}_2(V_0).$$

We interpret v_1 as the A-centered motif, v_2 as the B-centered motif, and v_3 as a "layer" or combined structural unit that simultaneously contains both motifs (for example, an ABO_3 sheet in a layered crystal).

Next, define two 2-superedges as

$$e_1 := \{v_1, v_3\}, \qquad e_2 := \{v_2, v_3\}.$$

Clearly $e_1, e_2 \in \mathcal{P}(V^{(2)}) \setminus \{\emptyset\}$, so if we set

$$E^{(2)} \ := \ \{e_1, e_2\},$$

then

$$\mathsf{CSHT}(2) \ := \ \big(V^{(2)}, E^{(2)}\big)$$

is a Crystal 2-SuperHyperGraph in the sense of Definition 6.8.1.

The incidence is given by set membership:

$$v_1 \in e_1, \quad v_3 \in e_1, \qquad v_2 \in e_2, \quad v_3 \in e_2.$$

Chemically, this structure can be read as follows.

- The 2-supervertices v_1 and v_2 encode the A- and B-centered coordination polyhedra as first-level motifs.
- The 2-supervertex v_3 encodes a higher-level structural unit (a local ABO_3 layer) that contains *both* motifs.
- The 2-superedges e_1 and e_2 record that the combined layer v_3 is structurally coupled with each of its component motifs (v_1 and v_2), representing, for instance, shared oxygen atoms and cooperative distortions.

Thus $\mathsf{CSHT}(2)$ provides a simple Crystal 2-SuperHyperGraph which organizes atoms, coordination polyhedra, and their composite layer into two iterated powerset levels.

6.9 SuperHyperGraph Neural Networks

AI has recently become a highly prominent concept, exerting a substantial and positive influence on many aspects of human society. Machine learning and neural-network research have attracted significant attention in modern artificial-intelligence studies [998–1003]. A Graph Neural Network learns vector representations of graph nodes or edges by aggregating and transforming information from neighborhood structures [1004, 1005]. Related variants of Graph Neural Networks are also well known, including Fuzzy Graph Neural Networks [1006,1007], Neutrosophic Graph Neural Networks [1005], Molecular Graph Neural Networks [1008–1010], and Directed Graph Neural Networks [1011, 1012].

A HyperGraph Neural Network generalizes GNNs by propagating features through hyperedges, capturing higher-order relationships among multiple nodes simultaneously within hypergraphs [4, 1013–1018]. A SuperHyperGraph Neural Network extends HGNNs to n-SuperHyperGraphs, learning representations over iterated powerset vertices and hierarchical superedges across multiple levels [12, 91, 1019–1021].

Definition 6.9.1 (SuperHyperGraph Neural Network (SHGNN))**.** [91] Let V_0 be a finite base vertex set. Define the iterated powersets by

$$P_0(V_0) := V_0, \qquad P_{k+1}(V_0) := P\big(P_k(V_0)\big) \quad (k \geq 0),$$

where $P(\cdot)$ denotes the usual powerset, and $P^*(X) := P(X) \setminus \{\emptyset\}$.

An *n-SuperHyperGraph* on V_0 is a pair

$$\mathrm{SHG}^{(n)} = \big(V^{(n)}, E^{(n)}\big),$$

with

$$\emptyset \neq V^{(n)} \subseteq P_n(V_0), \qquad \emptyset \neq E^{(n)} \subseteq P^*\big(V^{(n)}\big),$$

whose elements are called n-supervertices and n-superedges, respectively.

For each $k \geq 0$ the *flattening map*

$$\mathrm{flat}_k : P_k(V_0) \longrightarrow P(V_0)$$

is defined recursively by

$$\mathrm{flat}_0(x) := \{x\} \quad (x \in V_0),$$

and, for $k \geq 0$,

$$\mathrm{flat}_{k+1}(X) := \bigcup_{Y \in X} \mathrm{flat}_k(Y) \quad \text{for } X \in P_{k+1}(V_0) = P\big(P_k(V_0)\big).$$

Given the n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V^{(n)}, E^{(n)})$, we flatten each n-supervertex $v \in V^{(n)}$ to a base-level vertex set $\mathrm{flat}_n(v) \subseteq V_0$, and each n-superedge $e \in E^{(n)}$, which is a nonempty subset of $V^{(n)}$, to

$$\mathrm{Flat}_n(e) := \bigcup_{v \in e} \mathrm{flat}_n(v) \subseteq V_0.$$

The *expanded hypergraph* associated with $\mathrm{SHG}^{(n)}$ is

$$H' := (V_0, E'), \qquad E' := \Big\{\mathrm{Flat}_n(e) \;\Big|\; e \in E^{(n)},\ \mathrm{Flat}_n(e) \neq \emptyset\Big\}.$$

Example 6.9.2 (SuperHyperGraph Neural Network for curriculum recommendation)**.** A curriculum recommendation suggests personalized courses and learning sequences based on goals, prerequisites, skills gaps, constraints, and predicted learning outcomes (cf. [1022, 1023]). Consider a small curriculum with the following undergraduate courses as base vertices:

$$V_0 := \{\mathsf{Alg}, \mathsf{Lin}, \mathsf{DL}, \mathsf{DB}\},$$

where Alg = Algorithms, Lin = Linear Algebra, DL = Deep Learning, DB = Databases.

We form a 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V^{(1)}, E^{(1)})$ on V_0 as follows. The 1-supervertices are *modules*, each being a subset of courses:

$$V^{(1)} := \{v_1, v_2, v_3\}, \qquad v_1 := \{\mathsf{Alg}, \mathsf{Lin}\},\ v_2 := \{\mathsf{Lin}, \mathsf{DB}\},\ v_3 := \{\mathsf{Alg}, \mathsf{DL}, \mathsf{DB}\}.$$

The 1-superedges are *study programmes* grouping related modules:

$$E^{(1)} := \{e_1, e_2\}, \qquad e_1 := \{v_1, v_3\},\ e_2 := \{v_2\}.$$

Thus e_1 represents an "AI & Systems" programme (mixing $\mathsf{Alg}, \mathsf{Lin}, \mathsf{DL}, \mathsf{DB}$ via v_1, v_3), while e_2 is a "Data Systems" programme.

To apply a SuperHyperGraph Neural Network, we first flatten the 1-superedges to obtain a hypergraph on the base courses. For $v \in V^{(1)}$ we set

$$\mathrm{flat}_1(v) = v \subseteq V_0, \qquad \mathrm{Flat}_1(e) := \bigcup_{v \in e} \mathrm{flat}_1(v) \subseteq V_0.$$

Hence

$$\mathrm{Flat}_1(e_1) = v_1 \cup v_3 = \{\mathsf{Alg}, \mathsf{Lin}, \mathsf{DL}, \mathsf{DB}\}, \qquad \mathrm{Flat}_1(e_2) = v_2 = \{\mathsf{Lin}, \mathsf{DB}\}.$$

The expanded hypergraph is

$$H' = (V_0, E'), \qquad E' := \{\mathrm{Flat}_1(e_1), \mathrm{Flat}_1(e_2)\}.$$

Let $m := |V_0| = 4$ and $p := |E'| = 2$. We index $V_0 = \{v_1^{(0)}, \dots, v_4^{(0)}\} = \{\mathsf{Alg}, \mathsf{Lin}, \mathsf{DL}, \mathsf{DB}\}$ and $E' = \{e'_1, e'_2\} = \{\mathrm{Flat}_1(e_1), \mathrm{Flat}_1(e_2)\}$. The incidence matrix of H' is

$$H' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix},$$

where row i corresponds to course $v_i^{(0)}$ and column j to hyperedge e'_j.

Suppose each course has a feature vector in $\mathbb{R}^d$ (e.g. difficulty level, pass rate, credit weight, topical category). Collecting them row-wise gives the input feature matrix

$$X \in \mathbb{R}^{4 \times d}, \qquad X = \begin{pmatrix} x_{\mathsf{Alg}}^{\top} \\ x_{\mathsf{Lin}}^{\top} \\ x_{\mathsf{DL}}^{\top} \\ x_{\mathsf{DB}}^{\top} \end{pmatrix}.$$

Choose a positive weight function $w : E' \to \mathbb{R}_{>0}$, e.g. $w(e'_1) = 1.0$, $w(e'_2) = 0.8$, and construct the degree matrices D_V, D_E as in the SHGNN definition. For a learnable hyperedge-weight matrix $W \in \mathbb{R}^{2 \times 2}$, parameter matrix $\Theta \in \mathbb{R}^{d \times c}$, and activation σ, a single SHGNN layer produces

$$Y := \sigma\Big(D_V^{-1/2}\, H'\, W\, D_E^{-1}\, H'^{\top}\, D_V^{-1/2}\, X\, \Theta\Big) \in \mathbb{R}^{4 \times c},$$

where the i-th row of Y is the learned representation of course $v_i^{(0)}$ informed by all multi-course programmes that contain it.

In a practical system, these SHGNN embeddings can be used for tasks such as course recommendation, prerequisite prediction, or early-warning classification of at-risk students, while the 1-SuperHyperGraph explicitly encodes the hierarchical structure (course $\subset$ module $\subset$ programme) guiding the message passing.

6.10 Semantic SuperHyperGraphs in Psychology

A semantic graph represents entities as vertices and labeled edges as typed relations, enabling meaning-aware queries and inference [48, 1024, 1025]. A semantic hypergraph uses hyperedges linking several entities within one relation instance, capturing multi-entity interactions and contextual constraints [49, 1026, 1027]. A semantic n-SuperHyperGraph organizes entities into iterated powerset levels, with superedges connecting nested semantic groups across abstraction layers [50].

Definition 6.10.1 (Semantic n-SuperHyperGraph)**.** [50] Let

$$G = (V, E, L, \ell, w)$$

be a Semantic Graph, where V is a finite concept set, $E \subseteq V \times V$ is the directed edge set, L is a finite set of relation labels, $\ell : E \to L$ assigns each edge its semantic relation type, and $w : E \to [0, 1]$ assigns each edge a relation strength or confidence.

Let

$$H = (V, E_H, L_H, \ell_H, w_H)$$

be the induced Semantic HyperGraph, where $E_H \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is the family of semantic hyperedges, $L_H = \mathcal{P}(L) \setminus \{\emptyset\}$ is the hyperedge–label set, $\ell_H : E_H \to L_H$ aggregates edge labels on each hyperedge, and $w_H : E_H \to [0, 1]$ assigns each hyperedge a weight (e.g. maximal edge weight inside the hyperedge).

For an integer $n \geq 1$, the *Semantic n-SuperHyperGraph* (or *Semantic SuperHyperGraph of level n*) associated with G is the tuple

$$\mathsf{SNHG}^{(n)} := \big(V_n, E_n, L_n, \ell^{(n)}, w^{(n)}\big),$$

where

- the level-n supervertex set is
$$V_n := \mathcal{P}^n(V);$$
- the level-n superedge family is
$$E_n := \big\{\mathcal{P}^{n-1}(e) \setminus \{\emptyset\} \;\big|\; e \in E_H\big\} \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\};$$
- the level-n label set is
$$L_n := \mathcal{P}(L_H) \setminus \{\emptyset\};$$
- the lifted labeling and weight functions $\ell^{(n)} : E_n \to L_n$, $w^{(n)} : E_n \to [0, 1]$ are defined by
$$\ell^{(n)}\big(\mathcal{P}^{n-1}(e) \setminus \{\emptyset\}\big) := \ell_H(e), \qquad w^{(n)}\big(\mathcal{P}^{n-1}(e) \setminus \{\emptyset\}\big) := w_H(e)$$
for all $e \in E_H$.

Again $V_n \subseteq \mathcal{P}^n(V)$ and $E_n \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}$, so $\mathsf{SNHG}^{(n)}$ is an n-SuperHyperGraph encoding hierarchical, multi-scale semantic relations.

Example 6.10.2 (Semantic 2-SuperHyperGraph for a small health–risk ontology)**.** We illustrate a Semantic 2-SuperHyperGraph that organizes health-related concepts and their relations at multiple semantic levels.

Step 1: Semantic graph. Let the concept set be

$$V := \{\mathsf{Smoking}, \mathsf{Obesity}, \mathsf{Hypertension}, \mathsf{HeartDisease}\}.$$

Let the set of relation labels be

$$L := \{\mathsf{risk\_factor\_for},\ \mathsf{co\_morbid\_with}\}.$$

We define a directed Semantic Graph

$$G = (V, E, L, \ell, w),$$

where the edge set $E \subseteq V \times V$ consists of

$$(\mathsf{Smoking}, \mathsf{HeartDisease}), \quad (\mathsf{Obesity}, \mathsf{HeartDisease}),$$

$$(\mathsf{Hypertension}, \mathsf{HeartDisease}), \quad (\mathsf{Obesity}, \mathsf{Hypertension}),$$

and the labeling and weight functions are given by

$$\ell(\textsf{Smoking}, \textsf{HeartDisease}) = \ell(\textsf{Obesity}, \textsf{HeartDisease}) = \ell(\textsf{Hypertension}, \textsf{HeartDisease}) = \textsf{risk\_factor\_for},$$
$$\ell(\textsf{Obesity}, \textsf{Hypertension}) = \textsf{co\_morbid\_with},$$
$$w(\textsf{Smoking}, \textsf{HeartDisease}) = 0.90, \quad w(\textsf{Obesity}, \textsf{HeartDisease}) = 0.80,$$
$$w(\textsf{Hypertension}, \textsf{HeartDisease}) = 0.85, \quad w(\textsf{Obesity}, \textsf{Hypertension}) = 0.70.$$

Step 2: Semantic hypergraph. From G we build a Semantic HyperGraph

$$H = (V, E_H, L_H, \ell_H, w_H).$$

Consider the hyperedge

$$e_1 := \{\textsf{Smoking}, \textsf{Obesity}, \textsf{Hypertension}, \textsf{HeartDisease}\},$$

representing the joint semantic pattern "*cluster of risk factors and outcome*". We set

$$E_H := \{e_1\}, \qquad L_H := \mathcal{P}(L) \setminus \{\emptyset\},$$

and define

$$\ell_H(e_1) := \{\textsf{risk\_factor\_for}, \textsf{co\_morbid\_with}\},$$
$$w_H(e_1) := \max\{w(e) \mid e \in E,\ e \text{ uses only vertices in } e_1\} = 0.90.$$

Thus H captures, in one hyperedge, the semantic fact that Smoking, Obesity, and Hypertension jointly relate to HeartDisease as correlated risk factors and comorbid conditions.

Step 3: Semantic 2-SuperHyperGraph. We now form the associated Semantic 2-SuperHyperGraph

$$\textsf{SNHG}^{(2)} := (V_2, E_2, L_2, \ell^{(2)}, w^{(2)})$$

as in the definition of Semantic n-SuperHyperGraph.

First,

$$V_2 := \mathcal{P}^2(V) = \mathcal{P}(\mathcal{P}(V))$$

is the set of all subsets of $\mathcal{P}(V)$. For example, the following are 2-supervertices:

$$v_1^{(2)} := \big\{\{\textsf{Smoking}, \textsf{HeartDisease}\}, \{\textsf{Obesity}, \textsf{HeartDisease}\}\big\},$$
$$v_2^{(2)} := \big\{\{\textsf{Obesity}, \textsf{Hypertension}\}\big\},$$
$$v_3^{(2)} := \big\{\{\textsf{Smoking}, \textsf{Obesity}, \textsf{Hypertension}, \textsf{HeartDisease}\}\big\}.$$

Intuitively, $v_1^{(2)}$ groups "two risk-factor relations to HeartDisease", $v_2^{(2)}$ represents a "co-morbidity relation", and $v_3^{(2)}$ is the whole hyperedge e_1 seen as a single nested semantic unit.

Next, for $n = 2$ the level-2 superedge family is

$$E_2 := \Big\{\mathcal{P}^1(e) \setminus \{\emptyset\} \ \Big|\ e \in E_H\Big\} \ \subseteq\ \mathcal{P}(V_2) \setminus \{\emptyset\}.$$

Since $E_H = \{e_1\}$ with $e_1 \subseteq V$, we have

$$\mathcal{P}^1(e_1) \setminus \{\emptyset\} = \mathcal{P}(e_1) \setminus \{\emptyset\},$$

so a typical 2-superedge is

$$E_1^{(2)} := \big\{\{\textsf{Smoking}, \textsf{HeartDisease}\}, \{\textsf{Obesity}, \textsf{HeartDisease}\}, \{\textsf{Hypertension}, \textsf{HeartDisease}\}\big\},$$

which is an element of V_2 (a subset of $\mathcal{P}(V)$), and thus an element of E_2 when viewed as a member of $\mathcal{P}(V_2)$. This 2-superedge collects all pairwise risk-factor relations to HeartDisease inside the cluster e_1.

The label and weight of $E_1^{(2)}$ are inherited from e_1:

$$\ell^{(2)}(E_1^{(2)}) := \ell_H(e_1) = \{\textsf{risk\_factor\_for}, \textsf{co\_morbid\_with}\}, \qquad w^{(2)}(E_1^{(2)}) := w_H(e_1) = 0.90.$$

Therefore $\textsf{SNHG}^{(2)}$ is a Semantic 2-SuperHyperGraph that encodes not only basic semantic relations between single concepts, but also higher-order *patterns of relations* (e.g. clusters of risk factors and co-morbidities) as supervertices and superedges at level 2. Such a structure can be used, for instance, in medical decision support or knowledge-based risk analysis, where hierarchical semantic patterns are crucial.

6.11 Behavioral SuperHyperGraphs in Social Sciences

A behavioral graph represents discrete behavioral states as vertices and observed transitions as directed edges, supporting Markovian or temporal analysis (cf. [1028, 1029]). A behavioral SuperHyperGraph organizes behavioral states into iterated powerset supervertices and superedges, modeling hierarchical routines, co-occurring behaviors, and multi-level transitions [50].

Definition 6.11.1 (Behavior n-SuperHyperGraph)**.** [50] Let

$$G = (V, E_G, f, p)$$

be a Behavior Graph, where V is a finite set of behavioral states, $E_G \subseteq V \times V$ is the transition set, $f : E_G \to \mathbb{N}$ is a frequency function, and $p : E_G \to [0, 1]$ is a transition–probability function.

Let

$$H = (V, E_H, F, P)$$

be the induced Behavior HyperGraph on V, where $E_H \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is the family of hyperedges (co-occurring state sets), $F : E_H \to \mathbb{N}$ is an aggregate frequency, and $P : E_H \to [0, 1]$ is a normalized weight.

For an integer $n \geq 1$, the *Behavior n-SuperHyperGraph* (or *Behavior SuperHyperGraph of level n*) associated with G is the tuple

$$\mathsf{BHG}^{(n)} := \big(V_n, E_n, F^{(n)}, P^{(n)}\big),$$

where

- the level-n supervertex set is
$$V_n := \mathcal{P}^n(V),$$
i.e. the n-fold iterated powerset of V;
- the level-n superedge family is
$$E_n := \big\{\mathcal{P}^{n-1}(e) \setminus \{\emptyset\} \;\big|\; e \in E_H\big\} \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\};$$
- the lifted frequency and weight functions $F^{(n)} : E_n \to \mathbb{N}$, $P^{(n)} : E_n \to [0, 1]$ are defined by
$$F^{(n)}\big(\mathcal{P}^{n-1}(e) \setminus \{\emptyset\}\big) := F(e), \qquad P^{(n)}\big(\mathcal{P}^{n-1}(e) \setminus \{\emptyset\}\big) := P(e)$$
for all $e \in E_H$.

By construction $V_n \subseteq \mathcal{P}^n(V)$ and $E_n \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}$, so $\mathsf{BHG}^{(n)}$ is an n-SuperHyperGraph in the usual sense.

Example 6.11.2 (Behavior 2-SuperHyperGraph for smartphone usage patterns)**.** Smartphone usage patterns describe recurring sequences and co-occurrences of smartphone actions (unlocking, app use, notifications) across time, contexts, and users (cf. [1030, 1031]). We describe a simple real-world Behavioral 2-SuperHyperGraph that models daily smartphone usage routines of a user.

Step 1: Behavior graph. Let the set of behavioral states be

$$V := \{\mathsf{Unlock},\ \mathsf{CheckNews},\ \mathsf{OpenSNS},\ \mathsf{WatchVideo},\ \mathsf{Lock}\}.$$

We consider the Behavior Graph

$$G = (V, E_G, f, p),$$

where the directed transitions are

$$E_G := \{(\mathsf{Unlock}, \mathsf{CheckNews}), (\mathsf{CheckNews}, \mathsf{OpenSNS}), (\mathsf{OpenSNS}, \mathsf{WatchVideo}),$$
$$(\mathsf{WatchVideo}, \mathsf{Lock}), (\mathsf{Unlock}, \mathsf{OpenSNS}), (\mathsf{OpenSNS}, \mathsf{Lock})\}.$$

The frequency function $f : E_G \to \mathbb{N}$ counts how many times each transition occurs in logged data. For instance, suppose

$$f(\mathsf{Unlock}, \mathsf{CheckNews}) = 60, \quad f(\mathsf{CheckNews}, \mathsf{OpenSNS}) = 50, \quad f(\mathsf{OpenSNS}, \mathsf{WatchVideo}) = 40,$$
$$f(\mathsf{WatchVideo}, \mathsf{Lock}) = 40, \quad f(\mathsf{Unlock}, \mathsf{OpenSNS}) = 20, \quad f(\mathsf{OpenSNS}, \mathsf{Lock}) = 15.$$

The transition probabilities $p : E_G \to [0, 1]$ are obtained by normalizing frequencies outgoing from each state (e.g. by dividing by the sum of frequencies from that source).

Step 2: Behavior hypergraph. From G we build the Behavior HyperGraph

$$H = (V, E_H, F, P).$$

Assume empirical logs reveal two frequently co-occurring interaction patterns ("sessions"):

$$e_1 := \{\mathsf{Unlock}, \mathsf{CheckNews}, \mathsf{OpenSNS}, \mathsf{Lock}\},$$
$$e_2 := \{\mathsf{Unlock}, \mathsf{OpenSNS}, \mathsf{WatchVideo}, \mathsf{Lock}\}.$$

We set

$$E_H := \{e_1, e_2\}.$$

The aggregate frequency $F : E_H \to \mathbb{N}$ and normalized weight $P : E_H \to [0, 1]$ summarize how often each session pattern occurs. For example,

$$F(e_1) = 80, \quad F(e_2) = 50, \qquad P(e_1) = \frac{80}{80 + 50}, \quad P(e_2) = \frac{50}{80 + 50}.$$

Step 3: Behavior 2-SuperHyperGraph. We now form the Behavior 2-SuperHyperGraph

$$\mathsf{BHG}^{(2)} := \left(V_2, E_2, F^{(2)}, P^{(2)}\right)$$

as in the definition of Behavior n-SuperHyperGraph.

The level-2 supervertex set is the second iterated powerset

$$V_2 := \mathcal{P}^2(V) = \mathcal{P}\big(\mathcal{P}(V)\big),$$

whose elements are sets of subsets of V. For instance, the following are 2-supervertices:

$$v_1^{(2)} := \big\{\{\mathsf{Unlock}, \mathsf{CheckNews}\}, \{\mathsf{CheckNews}, \mathsf{OpenSNS}\}\big\},$$
$$v_2^{(2)} := \big\{\{\mathsf{OpenSNS}, \mathsf{WatchVideo}\}, \{\mathsf{WatchVideo}, \mathsf{Lock}\}\big\}.$$

Intuitively, $v_1^{(2)}$ encodes the "news-checking" part of a session, while $v_2^{(2)}$ encodes the "video-watching" part.

For $n = 2$, each level-2 superedge is of the form

$$\mathcal{P}^1(e) \setminus \{\emptyset\} = \mathcal{P}(e) \setminus \{\emptyset\} \subseteq \mathcal{P}(V), \qquad e \in E_H.$$

For the pattern e_1, the corresponding 2-superedge is

$$E_1^{(2)} := \mathcal{P}(e_1) \setminus \{\emptyset\} \in V_2,$$

which contains all nonempty subsets of $\{\mathsf{Unlock}, \mathsf{CheckNews}, \mathsf{OpenSNS}, \mathsf{Lock}\}$. Analogously,

$$E_2^{(2)} := \mathcal{P}(e_2) \setminus \{\emptyset\}$$

is the second 2-superedge induced by e_2. Collecting these, we set

$$E_2 := \{E_1^{(2)}, E_2^{(2)}\} \subseteq \mathcal{P}(V_2) \setminus \{\emptyset\}.$$

The lifted frequency and weight functions $F^{(2)} : E_2 \to \mathbb{N}$ and $P^{(2)} : E_2 \to [0, 1]$ are defined by

$$F^{(2)}\big(E_1^{(2)}\big) := F(e_1) = 80, \qquad F^{(2)}\big(E_2^{(2)}\big) := F(e_2) = 50,$$
$$P^{(2)}\big(E_1^{(2)}\big) := P(e_1), \qquad P^{(2)}\big(E_2^{(2)}\big) := P(e_2).$$

In this way, $\mathsf{BHG}^{(2)}$ captures not only single transitions between behavioral states, but also higher-order *session patterns* and their internal structure as 2-supervertices and 2-superedges. Such a Behavioral 2-SuperHyperGraph can be used in social and behavioral sciences to analyze habitual smartphone usage routines, identify clusters of actions, and study how complex behaviors evolve over time.

6.12 SuperHyperGraph Signal Processing

Signal processing studies acquisition, representation, transformation, and analysis of signals to extract information, reduce noise, or enable communication in systems [1032, 1033]. Graph signal processing extends classical signal processing to data on graph vertices, using graph structure for filtering, sampling, and transforms [1034–1036]. Related concepts in graph signal processing include the Graph Fourier Transform (GFT) [1037–1039], Graph Image Processing [1040], graph filtering (in spectral or vertex-domain forms) [1041], and the Graph Laplacian [1042, 1043].

Hypergraph signal processing generalizes graph methods to hyperedges connecting many vertices, modeling multiway interactions and higher-order dependencies in data analysis [96, 97, 1044, 1045]. SuperHyperGraph signal processing extends hypergraph approaches to iterated powerset supervertices, enabling hierarchical, multi-level filtering, transforms, and spectral analysis of signals [53].

Definition 6.12.1 (n-SuperHyperGraph Signal Processing)**.** [53] Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. Let

$$\mathrm{SHG}^{(n)} = (V, E)$$

be an n-SuperHyperGraph on V_0, in the sense that

$$\emptyset \neq V \subseteq \mathrm{P}_n(V_0), \qquad \emptyset \neq E \subseteq \mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\},$$

where $\mathrm{P}_k(V_0)$ denotes the k-fold iterated powerset of V_0 and $\mathcal{P}(\cdot)$ is the usual powerset.

Write $V = \{v_1, \dots, v_{N_n}\}$ and set

$$M \ := \ \max_{e \in E} |e|,$$

the maximum cardinality of an n-superedge.

The *adjacency tensor* of $\mathrm{SHG}^{(n)}$ is the order-M tensor

$$A \ = \ (A_{i_1 \cdots i_M})_{1 \le i_1, \dots, i_M \le N_n} \ \in \ \mathbb{R}^{N_n \times \cdots \times N_n} \quad (M \text{ factors}),$$

defined as follows. For each superedge

$$e_\ell = \{v_{\ell_1}, \dots, v_{\ell_c}\} \in E, \qquad 1 \le c \le M,$$

we set

$$A_{i_1 \cdots i_M} \ := \ \begin{cases} c \displaystyle\sum_{\substack{k_1, \dots, k_c \ge 1 \\ k_1 + \cdots + k_c = M}} \left(\frac{M!}{k_1! \cdots k_c!} \right)^{-1}, & \text{if the multiset } \{v_{i_1}, \dots, v_{i_M}\} \text{ is obtained from } e_\ell \\ & \text{by repeating } v_{\ell_j} \text{ exactly } k_j \text{ times for } j = 1, \dots, c, \text{ for some } (k_1, \dots, k_c), \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, $A_{i_1 \cdots i_M} > 0$ precisely when all indices $v_{i_1}, \dots, v_{i_M}$ belong to the same superedge $e_\ell \in E$, and 0 otherwise.

A (real-valued) *signal* on $\mathrm{SHG}^{(n)}$ is a function $s : V \to \mathbb{R}$, which we identify with the column vector

$$s \ = \ \big(s(v_1), \dots, s(v_{N_n})\big)^\top \in \mathbb{R}^{N_n}.$$

From s we form the $(M-1)$-th order *signal tensor*

$$S \ := \ \underbrace{s \circ s \circ \cdots \circ s}_{M-1 \text{ times}} \ \in \ \mathbb{R}^{N_n \times \cdots \times N_n} \quad (M-1 \text{ factors}),$$

where $\circ$ denotes the outer product, so that in coordinates

$$S_{i_1 \cdots i_{M-1}} \ = \ s_{i_1} \cdots s_{i_{M-1}}.$$

The *shifted (or filtered) signal tensor* S' is obtained by the mode-M tensor–vector product

$$S' \;:=\; A \times_M s,$$

that is,

$$S'_{i_1\cdots i_{M-1}} \;:=\; \sum_{j=1}^{N_n} A_{i_1\cdots i_{M-1}j}\, s_j, \qquad 1 \le i_1, \ldots, i_{M-1} \le N_n.$$

Assume that the adjacency tensor A admits an orthogonal CANDECOMP/PARAFAC decomposition

$$A \;=\; \sum_{r=1}^{R} \lambda_r \, \underbrace{f_r \circ \cdots \circ f_r}_{M \text{ times}}, \qquad f_r \in \mathbb{R}^{N_n},\; \langle f_r, f_s\rangle = \delta_{rs},$$

where $\lambda_r \in \mathbb{R}$ and δ_{rs} is the Kronecker delta. Then the rank-one tensors

$$f_r^{\circ(M-1)} \;:=\; \underbrace{f_r \circ \cdots \circ f_r}_{M-1 \text{ times}}, \qquad r = 1, \ldots, R,$$

form an orthonormal basis of the $(M-1)$-th order signal-tensor space (with respect to the Frobenius inner product).

The *n-SuperHyperGraph Fourier transform* of the signal s (equivalently, of S) is the coefficient vector

$$\widehat{S} \;=\; \big(\widehat{S}_1, \ldots, \widehat{S}_R\big)^{\top} \in \mathbb{R}^R,$$

defined by

$$\widehat{S}_r \;:=\; \big\langle S,\, f_r^{\circ(M-1)} \big\rangle, \qquad r = 1, \ldots, R,$$

where $\langle \cdot, \cdot \rangle$ denotes the Frobenius inner product on $(M-1)$-th order tensors.

The data consisting of the adjacency tensor A, the shift operation $S' = A \times_M s$, and the Fourier transform $\widehat{S}$ is called the *n-SuperHyperGraph Signal Processing structure* on $\mathrm{SHG}^{(n)}$. For $n = 0$ this construction reduces to hypergraph signal processing on the underlying hypergraph.

Example 6.12.2 (SuperHyperGraph Signal Processing in a smart building)**.** We present a concrete real-world instance of n-SuperHyperGraph Signal Processing for temperature monitoring in a smart building.

Step 1: Building layout as an n-SuperHyperGraph. Consider a small office floor with four rooms:

$$V_0 := \{\mathsf{R}_1, \mathsf{R}_2, \mathsf{R}_3, \mathsf{R}_4\}.$$

We form a level-1 SuperHyperGraph (i.e. $n = 1$) whose 1-supervertices represent *zones* (unions of rooms):

$$V := \{\, v_1, v_2, v_3 \,\} \subseteq \mathrm{P}_1(V_0) = \mathcal{P}(V_0),$$

where

$$v_1 := \{\mathsf{R}_1, \mathsf{R}_2\}, \quad v_2 := \{\mathsf{R}_2, \mathsf{R}_3\}, \quad v_3 := \{\mathsf{R}_3, \mathsf{R}_4\}.$$

Each v_i is a 1-supervertex (a subset of rooms) corresponding to a controllable HVAC zone.

We define 1-superedges as *clusters of zones* that share walls, ducts, or strong thermal coupling:

$$e_1 := \{v_1, v_2\}, \qquad e_2 := \{v_2, v_3\}.$$

Thus the superedge family is

$$E := \{e_1, e_2\} \;\subseteq\; \mathcal{P}^*(V) = \mathcal{P}(V) \setminus \{\emptyset\}.$$

The pair

$$\mathrm{SHG}^{(1)} := (V, E)$$

is a 1-SuperHyperGraph on the base set V_0.

Step 2: Adjacency tensor of the SuperHyperGraph. We index the 1-supervertices as

$$V = \{v_1, v_2, v_3\}, \quad N_1 = 3.$$

The maximum superedge size is

$$M := \max_{e \in E} |e| = 2,$$

so the adjacency tensor A has order $M = 2$ and can be viewed as a 3×3 matrix

$$A = (A_{ij})_{1 \le i,j \le 3} \in \mathbb{R}^{3\times 3}.$$

Specializing the general definition to $M = 2$, we set

$$A_{ij} := \begin{cases} \alpha, & \text{if } \{v_i, v_j\} \subseteq e_\ell \text{ for some } e_\ell \in E, \\ 0, & \text{otherwise,} \end{cases}$$

for some fixed positive constant $\alpha > 0$ (for example, $\alpha = 1$).

Because $e_1 = \{v_1, v_2\}$ and $e_2 = \{v_2, v_3\}$, the nonzero entries are

$$A_{12} = A_{21} = \alpha, \qquad A_{23} = A_{32} = \alpha,$$

and all other off-diagonal entries are zero. (Diagonal entries may be chosen as 0 or as self-loop weights, depending on the modeling choice.)

Step 3: Temperature signal on the SuperHyperGraph. A temperature control system assigns to each zone v_i a real-valued temperature (for example, deviation from the desired setpoint in degrees Celsius). This defines a signal

$$s : V \to \mathbb{R}, \qquad s(v_i) = \text{temperature deviation in zone } v_i,$$

which we write as a column vector

$$s := \begin{pmatrix} s(v_1) \\ s(v_2) \\ s(v_3) \end{pmatrix} \in \mathbb{R}^3.$$

Since $M = 2$, the signal tensor reduces to the vector s itself; the shifted (or filtered) signal s' is obtained by the mode-2 tensor–vector product, which here is just matrix multiplication:

$$s' := As, \qquad s'_i = \sum_{j=1}^{3} A_{ij}\, s_j, \quad i = 1, 2, 3.$$

Concretely,

$$s' = \begin{pmatrix} \alpha s(v_2) \\ \alpha s(v_1) + \alpha s(v_3) \\ \alpha s(v_2) \end{pmatrix}.$$

Thus $s'(v_1)$ and $s'(v_3)$ depend on the temperature in the shared neighbor zone v_2, while $s'(v_2)$ averages (up to the factor α) the deviations in v_1 and v_3. This operation implements a simple *SuperHyperGraph filter* that smooths temperature differences along the zone clusters encoded by the 1-SuperHyperGraph.

Step 4: SuperHyperGraph Fourier analysis. Suppose the adjacency matrix A is diagonalizable with an orthonormal basis of eigenvectors $\{f_1, f_2, f_3\}$:

$$A = \sum_{r=1}^{3} \lambda_r\, f_r f_r^\top, \qquad \langle f_r, f_s \rangle = \delta_{rs}.$$

Then the 1-SuperHyperGraph Fourier transform of s is the coefficient vector

$$\widehat{s} := \begin{pmatrix} \widehat{s}_1 \\ \widehat{s}_2 \\ \widehat{s}_3 \end{pmatrix}, \qquad \widehat{s}_r := \langle s, f_r \rangle,$$

which decomposes the temperature signal into "frequency" modes on the SuperHyperGraph. Low-frequency components correspond to slowly varying temperature deviations across strongly coupled zones, while high-frequency components capture abrupt differences between adjacent zones.

In practice, building engineers can design filters that attenuate high-frequency components (reducing abrupt temperature jumps) while preserving low-frequency patterns (overall heating/cooling level). This yields smoother and more energy-efficient control strategies that explicitly exploit the hierarchical zone structure encoded by the SuperHyperGraph.

6.13 Bond SuperHyperGraph

Bond graphs represent physical systems as elements and junctions connected by bonds carrying power variables, unifying multi-domain dynamics modeling (cf. [51, 52, 1046, 1047]). Bond SuperHyperGraphs extend bond graphs using supervertices and superedges over iterated powersets, capturing hierarchical subnetworks and higher-order multi-port coupling patterns [53].

Definition 6.13.1 (Bond n-SuperHyperGraph)**.** [53] Let $G = (V_{\text{elem}} \dot{\cup} V_{\text{junc}}, E_G)$ be a (finite) bond graph, where V_{elem} is the set of element nodes (resistors, capacitors, inductors, sources, etc.), V_{junc} is the set of junction nodes, and

$$E_G \subseteq V_{\text{elem}} \times V_{\text{junc}}$$

encodes the incidence relation between element nodes and junctions.

The associated Bond HyperGraph is the hypergraph

$$H = (V_{\text{elem}}, E_H),$$

where for each junction $j \in V_{\text{junc}}$ the hyperedge

$$e_j := \{ u \in V_{\text{elem}} \mid (u, j) \in E_G \}$$

collects exactly the element nodes incident on j, and

$$E_H := \{ e_j \subseteq V_{\text{elem}} \mid j \in V_{\text{junc}} \}.$$

Let $V_0 := V_{\text{elem}}$ be the base set of element nodes, and recall the iterated powersets

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0),$$

as in Definition 2.2.1. Fix $n \in \mathbb{N}$.

A *Bond n-SuperHyperGraph* on the base bond graph G is a pair

$$\text{BnSHT}(n) := (V^{(n)}, E^{(n)}),$$

where

- $V^{(n)} \subseteq \mathcal{P}^n(V_0)$ is a nonempty finite set of *n-supervertices*. Each $v \in V^{(n)}$ is a nested collection (up to depth n) of element nodes and/or Bond HyperGraph hyperedges, representing a junction subnetwork of G;
- $E^{(n)} \subseteq \mathcal{P}\big(V^{(n)}\big) \setminus \{\emptyset\}$ is a nonempty finite family of *n-superedges*. Each $e \in E^{(n)}$ is a finite set of n-supervertices, interpreted as a higher-level "meta-junction" grouping together those subnetworks that are coupled in the underlying bond graph.

The canonical incidence relation is given by membership: an n-supervertex $v \in V^{(n)}$ is incident to an n-superedge $e \in E^{(n)}$ precisely when $v \in e$.

When $n = 1$ and we choose

$$V^{(1)} = \big\{ \{v\} \mid v \in V_0 \big\}, \qquad E^{(1)} = \big\{ e_j \mid j \in V_{\text{junc}} \big\},$$

the Bond 1-SuperHyperGraph BnSHT(1) coincides with the Bond HyperGraph H. If, in addition, every hyperedge e_j has cardinality two, then H further reduces to the classical bond graph G. Thus Bond n-SuperHyperGraphs strictly extend Bond HyperGraphs and ordinary bond graphs by encoding hierarchical junction groupings via iterated powersets.

Example 6.13.2 (Real-world Bond 1-SuperHyperGraph for a DC motor drive)**.** Consider a simple DC motor drive circuit (cf. [1048]) with the following physical components:

$$V_{\text{elem}} := \{\mathsf{Se}, \mathsf{R}, \mathsf{L}, \mathsf{M}\},$$

where

- Se is a DC voltage source,
- R is a series resistor (wiring and losses),
- L is an inductor (armature inductance),
- M is the motor element (electromechanical converter).

These element nodes are connected via three junction nodes

$$V_{\text{junc}} := \{\mathsf{J}_{\mathsf{s}}, \mathsf{J}_1, \mathsf{J}_2\},$$

where:

- J_{s} is the source junction (connection between the source and the series resistor),
- J_1 is the electrical intermediate junction (between resistor and inductor),
- J_2 is the electromechanical junction (between inductor and motor element).

The (directed) incidence relation between element nodes and junctions is encoded by the edge set

$$E_G := \big\{(\mathsf{Se}, \mathsf{J}_{\mathsf{s}}), (\mathsf{R}, \mathsf{J}_{\mathsf{s}}), (\mathsf{R}, \mathsf{J}_1), (\mathsf{L}, \mathsf{J}_1), (\mathsf{L}, \mathsf{J}_2), (\mathsf{M}, \mathsf{J}_2)\big\} \subseteq V_{\text{elem}} \times V_{\text{junc}}.$$

Thus

$$G := (V_{\text{elem}} \,\dot{\cup}\, V_{\text{junc}}, E_G)$$

is a (finite) bond graph.

Step 1: Bond HyperGraph. For each junction $j \in V_{\text{junc}}$ we form the hyperedge

$$e_j := \{\, u \in V_{\text{elem}} \mid (u, j) \in E_G \,\},$$

giving

$$e_{\mathsf{J}_{\mathsf{s}}} = \{\mathsf{Se}, \mathsf{R}\}, \quad e_{\mathsf{J}_1} = \{\mathsf{R}, \mathsf{L}\}, \quad e_{\mathsf{J}_2} = \{\mathsf{L}, \mathsf{M}\}.$$

The Bond HyperGraph is then

$$H := (V_{\text{elem}}, E_H), \qquad E_H := \{e_{\mathsf{J}_{\mathsf{s}}}, e_{\mathsf{J}_1}, e_{\mathsf{J}_2}\}.$$

Step 2: Bond 1-SuperHyperGraph. Let the base set be

$$V_0 := V_{\text{elem}} = \{\mathsf{Se}, \mathsf{R}, \mathsf{L}, \mathsf{M}\},$$

and fix $n = 1$. Recall that

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^1(V_0) = \mathcal{P}(V_0),$$

so a 1-supervertex is simply a subset of element nodes.

We now group elements into *functional subnetworks (modules)*:

$$v_{\text{sup}} := \{\mathsf{Se}, \mathsf{R}\} \quad \text{(supply and series resistance)},$$

$$v_{\text{line}} := \{\mathsf{R}, \mathsf{L}\} \quad \text{(line and armature inductance)},$$

$$v_{\text{load}} := \{\mathsf{L}, \mathsf{M}\} \quad \text{(inductor and motor load)}.$$

Each of these is an element of $\mathcal{P}(V_0)$, so we set

$$V^{(1)} := \{v_{\text{sup}}, v_{\text{line}}, v_{\text{load}}\} \subseteq \mathcal{P}^1(V_0).$$

The 1-supervertices represent higher-level subnetworks built from the original element nodes.

Next, we describe how these subnetworks are coupled at a meta-level. Define the 1-superedges

$$e^{(1)}_{\text{el}} := \{v_{\text{sup}}, v_{\text{line}}\}, \qquad e^{(1)}_{\text{mech}} := \{v_{\text{line}}, v_{\text{load}}\},$$

and optionally a global coupling

$$e^{(1)}_{\text{sys}} := \{v_{\text{sup}}, v_{\text{line}}, v_{\text{load}}\}.$$

We then set

$$E^{(1)} := \{e^{(1)}_{\text{el}}, e^{(1)}_{\text{mech}}, e^{(1)}_{\text{sys}}\} \subseteq \mathcal{P}(V^{(1)}) \setminus \{\emptyset\}.$$

The pair

$$\text{BnSHT}(1) := (V^{(1)}, E^{(1)})$$

is a Bond 1-SuperHyperGraph on the base bond graph G in the sense of the general definition: $V^{(1)} \subseteq \mathcal{P}^1(V_0)$ is a finite nonempty set of 1-supervertices, and $E^{(1)} \subseteq \mathcal{P}(V^{(1)}) \setminus \{\emptyset\}$ is a finite nonempty family of 1-superedges.

The Bond SuperHyperGraph BnSHT(1) encodes a two-level view of the DC motor drive:

- At the base level V_0 we see individual physical components (source, resistor, inductor, motor) and their junction-based couplings (Bond HyperGraph H).
- At the supervertex level $V^{(1)}$ we group these components into functional subnetworks (supply, line, load), while the superedges $E^{(1)}$ capture how these subnetworks are interconnected (electrical coupling and electromechanical coupling, plus an optional system-wide superedge).

Thus this example shows how a real DC motor drive can be modeled as a Bond SuperHyperGraph, revealing hierarchical structure and modular interactions beyond the ordinary bond graph representation.

6.14 Brain Hypergraphs in Neuroscience

Brain graphs represent brain regions as nodes and pairwise connections as edges, modeling structural or functional connectivity patterns between regions [1049–1051]. Brain SuperHyperGraphs use iterated powersets to form supervertices and superedges, encoding hierarchical, overlapping brain modules and higher-order functional interactions explicitly [1052].

Definition 6.14.1 (Brain n-SuperHyperGraph)**.** [1052] Let $V_0 = \{r_1, \dots, r_N\}$ be a finite *base set of brain regions* (Regions of Interest, ROIs). For each integer $k \geq 0$ define the iterated powersets by

$$\mathrm{P}_0(V_0) := V_0, \qquad \mathrm{P}_{k+1}(V_0) := \mathrm{P}(\mathrm{P}_k(V_0)),$$

where $\mathrm{P}(\cdot)$ denotes the usual powerset.

Fix $n \in \mathbb{N}_0$. A *Brain n-SuperHyperGraph* (or Brain SuperHyperGraph of level n) on V_0 is a pair

$$\mathsf{BSHT}^{(n)} = (V^{(n)}, E^{(n)}),$$

where

$$\varnothing \neq V^{(n)} \subseteq \mathrm{P}_n(V_0), \qquad \varnothing \neq E^{(n)} \subseteq \mathcal{P}(V^{(n)}) \setminus \{\varnothing\}.$$

The elements of $V^{(n)}$ are called *n-supervertices* and represent nested clusters (modules, subnetworks, or meta-modules) of brain regions obtained by applying the powerset operator n times to V_0. The elements of $E^{(n)}$ are called *n-superedges* and are nonempty sets of n-supervertices, encoding higher–order, hierarchical interactions between such nested brain modules.

The incidence relation is

$$I^{(n)} := \{ (v,e) \in V^{(n)} \times E^{(n)} \mid v \in e \},$$

with canonical projections

$$\pi_V : I^{(n)} \to V^{(n)},\ (v,e) \mapsto v, \qquad \pi_E : I^{(n)} \to E^{(n)},\ (v,e) \mapsto e.$$

When $n = 0$ and $V^{(0)} = V_0$, the conditions

$$V^{(0)} \subseteq \mathrm{P}_0(V_0) = V_0, \qquad E^{(0)} \subseteq \mathcal{P}(V^{(0)}) \setminus \{\emptyset\}$$

show that $\mathsf{BSHT}^{(0)}$ is exactly a Brain Hypergraph on the vertex set V_0. For $n \geq 1$, the construction yields a genuine hierarchical generalization in which both nodes and higher–order connections live on the n-th iterated powerset of the base set of brain regions.

Example 6.14.2 (Brain 2-SuperHyperGraph for large-scale functional networks)**.** Consider a toy fMRI study with four regions of interest (ROIs)

$$V_0 = \{r_1, r_2, r_3, r_4\},$$

where r_1 is a primary visual area, r_2 an auditory area, r_3 a dorsolateral prefrontal cortex region, and r_4 a hippocampal region.

First form the usual powerset

$$\mathrm{P}_1(V_0) = \mathcal{P}(V_0).$$

Among all subsets in $\mathrm{P}_1(V_0)$, suppose empirical connectivity analysis identifies the following task-related *functional modules*:

$$\begin{aligned} M_1 &:= \{r_1, r_3\} && \text{(visual–prefrontal attention module)},\\ M_2 &:= \{r_2, r_3\} && \text{(auditory–prefrontal control module)},\\ M_3 &:= \{r_3, r_4\} && \text{(prefrontal–hippocampal memory module)}. \end{aligned}$$

Each M_i is an element of $\mathrm{P}_1(V_0)$ and represents a hyperedge-level functional subnetwork at the "first" hierarchical layer.

Next, form the second iterated powerset

$$\mathrm{P}_2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

Inside $\mathrm{P}_2(V_0)$ consider the following two 2-supervertices:

$$\begin{aligned} v_A &:= \{M_1, M_2\} && \in \mathrm{P}_2(V_0) && \text{(task-positive fronto-sensory network)},\\ v_B &:= \{M_3\} && \in \mathrm{P}_2(V_0) && \text{(fronto-hippocampal memory network)}. \end{aligned}$$

Set

$$V^{(2)} := \{v_A, v_B\} \subseteq \mathrm{P}_2(V_0).$$

Each element of $V^{(2)}$ is a nested collection of ROIs, obtained by applying the powerset operator twice to V_0, and encodes a large-scale functional network composed of several lower-level modules.

During a demanding working-memory task, suppose neuroimaging analysis indicates that these two large-scale networks interact as a unit. This is captured by the 2-superedge

$$e_1 := \{v_A, v_B\} \subseteq V^{(2)},$$

which represents the higher-order, task-related coupling between the task-positive fronto-sensory network and the memory network. For completeness we may also include a self-network superedge

$$e_2 := \{v_A\},$$

modeling within-network integration of the task-positive system.

Define

$$E^{(2)} := \{e_1, e_2\} \subseteq \mathcal{P}\big(V^{(2)}\big) \setminus \{\varnothing\}.$$

Then the pair

$$\mathsf{BSHT}^{(2)} := \big(V^{(2)}, E^{(2)}\big)$$

is a Brain 2-SuperHyperGraph in the sense of the preceding definition. The 2-supervertices v_A and v_B encode nested functional modules (hierarchical brain networks), while the 2-superedges e_1 and e_2 encode higher-order interactions between these networks during the cognitive task. If we flatten $V^{(2)}$ and $E^{(2)}$ down to V_0, we recover the usual hypergraph representation of functional connectivity between ROIs.

6.15 Legal Citation SuperHyperGraphs

Legal citation is referencing statutes, cases, or regulations within legal writing to support authority, trace precedent, and justify arguments [1053, 1054]. A Legal Citation Graph represents legal provisions as nodes and citations as directed edges, modeling pairwise authority, precedent, or dependence [59, 1055]. A Legal Citation SuperHyperGraph groups provisions into nested supervertices and superedges, capturing higher-order, multi-document citation patterns and interpretive contexts explicitly [60].

Definition 6.15.1 (Legal Citation n-SuperHyperGraph)**.** [60] Let V_0 be a finite, nonempty set of legal provisions (statutes, cases, regulations, guidance, etc.). For $k \in \mathbb{N}_0$ define the iterated powersets by

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big),$$

where $\mathcal{P}(\cdot)$ denotes the usual powerset.

Fix an integer $n \ge 0$ and let L be a finite set of citation–type labels (e.g. *Authority*, *Exception*, *InterpretiveGuidance*).

A *Legal Citation n-SuperHyperGraph* on the base set V_0 is a triple

$$\mathsf{LCSHG}^{(n)} := \big(V^{(n)}, E^{(n)}, \ell^{(n)}\big),$$

where

- $V^{(n)} \subseteq \mathcal{P}^n(V_0)$ is a finite set of *n-supervertices*; each $u \in V^{(n)}$ is a (possibly nested) group of provisions obtained by applying the powerset operator n times to V_0.
- $E^{(n)} \subseteq V^{(n)} \times \big(\mathcal{P}(V^{(n)}) \setminus \{\emptyset\}\big)$ is a finite set of *oriented n-superedges*. An element $e \in E^{(n)}$ is a pair

  $$e = (u, T), \qquad u \in V^{(n)},\ \emptyset \ne T \subseteq V^{(n)},$$

  with the intended meaning that the citing supervertex u cites every target supervertex $v \in T$. By convention one may impose $u \notin T$ to exclude self–citation (this restriction can be relaxed if desired).
- $\ell^{(n)} : E^{(n)} \to L$ is a labeling function assigning to each n-superedge $e = (u, T)$ a citation label $\ell^{(n)}(e) \in L$ describing the semantic type of the citation (for example, whether it is used as an authority, an exception, or interpretive guidance).

For $n = 0$, if we identify $V^{(0)} = V_0$ and restrict every 0-superedge to pairs of the form $(u, \{v\})$, then $\mathsf{LCSHG}^{(0)}$ reduces to a (labeled) Legal Citation Graph. For $n = 1$, if we take $V^{(1)} \subseteq \mathcal{P}(V_0)$ and view each 1-superedge (u, T) as a citation event in which one provision group u cites all provision groups in T, then $\mathsf{LCSHG}^{(1)}$ recovers a Legal Citation Hypergraph with oriented hyperedges. Thus Legal Citation n-SuperHyperGraphs strictly generalize both Legal Citation Graphs and Legal Citation Hypergraphs by allowing nested, higher–order citation structures.

Example 6.15.2 (Legal Citation 1-SuperHyperGraph for a data-protection regime)**.** A data-protection regime is the legal and institutional framework governing collection, use, sharing, and security of personal data, protecting rights (cf. [1056, 1057]). Consider a simplified legal ecosystem for data protection. Let the base set V_0 consist of specific provisions drawn from real-world sources:

$$\begin{aligned} V_0 := \{a &:= \text{“GDPR Article 6(1)”}, \\ b &:= \text{“GDPR Recital 47”}, \\ c &:= \text{“EU Data Protection Directive 95/46/EC Article 7”}, \\ d &:= \text{“National Data Protection Act Section 15”}, \\ e &:= \text{“Supreme Court Case X vs. Y (Lawful Interest)”}\}. \end{aligned}$$

Here a, b, c, d are statutory or regulatory provisions and e is a judicial decision interpreting "legitimate interest" as a lawful basis for processing.

Form the powerset

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0),$$

and define the following *provision groups* (first-level supervertices):

$$\begin{aligned} u_1 &:= \{a, d\} && \text{(domestic implementation of lawful-basis rules),} \\ u_2 &:= \{b, c\} && \text{(EU-level interpretive context for legitimate interest),} \\ u_3 &:= \{e\} && \text{(leading case on legitimate-interest balancing test).} \end{aligned}$$

Set

$$V^{(1)} := \{u_1, u_2, u_3\} \subseteq \mathcal{P}^1(V_0).$$

Each u_i is a 1-supervertex, i.e. a nonempty subset of V_0 grouping related provisions.

Let the label set of citation types be

$$L := \{\mathit{Authority},\ \mathit{InterpretiveGuidance},\ \mathit{Implementation}\}.$$

We now define oriented 1-superedges in the sense of the definition of Legal Citation n-SuperHyperGraphs. Intuitively:

- the domestic implementation block u_1 cites the EU interpretive context u_2 as *Authority*;
- the case u_3 cites both u_1 and u_2 as *InterpretiveGuidance* for its reasoning.

Formally, set

$$E^{(1)} := \{e_A, e_I\},$$

where

$$e_A := (u_1, \{u_2\}), \qquad e_I := (u_3, \{u_1, u_2\})$$

are oriented 1-superedges of the form $(u, T) \in V^{(1)} \times \big(\mathcal{P}(V^{(1)}) \setminus \{\emptyset\}\big)$. Define the citation-type labeling function

$$\ell^{(1)} : E^{(1)} \to L$$

by

$$\ell^{(1)}(e_A) := \mathit{Authority}, \qquad \ell^{(1)}(e_I) := \mathit{InterpretiveGuidance}.$$

Thus the Legal Citation 1-SuperHyperGraph is

$$\mathsf{LCSHG}^{(1)} := \big(V^{(1)}, E^{(1)}, \ell^{(1)}\big),$$

where:

- each 1-supervertex u_i is a group of concrete provisions (statutory sections or a case) from V_0;
- the superedge e_A encodes that the domestic implementation block u_1 cites the EU interpretive block u_2 as authoritative legal basis;
- the superedge e_I encodes that the case u_3 simultaneously cites both u_1 and u_2 as interpretive guidance, capturing a higher-order citation event involving multiple provision groups.

If we "flatten" the supervertices by replacing each u_i with its underlying set of atomic provisions, e_A and e_I reduce to ordinary multi-target citation relations on V_0, so $\mathsf{LCSHG}^{(1)}$ extends a standard legal citation graph/hypergraph by grouping legal clauses into higher-level supervertices and modeling citation events at that grouped level.

6.16 River Network SuperHyperGraphs in Geoscientific and Civil Applications

A River Network Graph models river channels as directed edges between confluences and outlets, annotated with discharge, slope, and length [101]. A River Network SuperHyperGraph organizes channels, tributaries, and basins into iterated powerset levels, capturing hierarchical catchments, confluences, and multi-scale flow [101].

Definition 6.16.1 (River Network n-SuperHyperGraph)**.** [101] Let

$$G = (V, E, \varphi, \ell)$$

be a River Network Graph, where V is the finite set of river nodes, $E \subseteq V \times V$ the set of directed channels, $\varphi : E \to \mathbb{R}_{>0}$ the discharge capacity, and $\ell : E \to \mathbb{R}_{>0}$ the channel length.

Let

$$H = \big(V, E^{(1)}, \varphi_H, \ell_H\big)$$

be the associated River Network HyperGraph of G, whose hyperedge set $E^{(1)}$ consists of

- binary hyperedges $\{u, v\}$ for each directed channel $(u, v) \in E$,
- junction hyperedges $U_j \cup \{j\}$ for each confluence node j with at least two distinct upstream neighbours U_j,

and where $\varphi_H, \ell_H : E^{(1)} \to \mathbb{R}_{>0}$ are defined as in the River Network HyperGraph construction (capacities aggregated, lengths chosen as maximum upstream reach).

Let $\mathcal{P}^k(V)$ denote the k-fold iterated powerset of V as in Definition 2.2.1. Fix $n \in \mathbb{N}$. We define:

$$V_n := \mathcal{P}^n(V),$$
$$E_n := \big\{\mathcal{P}^{n-1}(e) \setminus \{\varnothing\} \;\big|\; e \in E^{(1)}\big\} \subseteq \mathcal{P}^n(V).$$

Each element of V_n is called an *n-supervertex*, and each element of E_n an *n-superedge*. Thus both V_n and E_n are subsets of the same iterated powerset $\mathcal{P}^n(V)$.

The level-n capacity and length labelings are the maps

$$\varphi^{(n)}, \ell^{(n)} : E_n \longrightarrow \mathbb{R}_{>0}$$

given on generators by

$$\varphi^{(n)}\big(\mathcal{P}^{n-1}(e) \setminus \{\varnothing\}\big) := \varphi_H(e), \qquad \ell^{(n)}\big(\mathcal{P}^{n-1}(e) \setminus \{\varnothing\}\big) := \ell_H(e) \quad (e \in E^{(1)}).$$

The *River Network n-SuperHyperGraph* associated with G is the quadruple

$$\mathrm{RNHG}^{(n)} := \big(V_n, E_n, \varphi^{(n)}, \ell^{(n)}\big).$$

For $n = 1$ one has $V_1 = \mathcal{P}(V)$ and $E_1 = \{e \mid e \in E^{(1)}\} \subseteq V_1$, and the labels $\varphi^{(1)}, \ell^{(1)}$ coincide with φ_H, ℓ_H on $E^{(1)}$. Hence the family $\big(\mathrm{RNHG}^{(n)}\big)_{n\geq 1}$ forms a hierarchical River Network SuperHyperGraph built on top of the River Network HyperGraph.

Example 6.16.2 (River Network 2-SuperHyperGraph for a mountainous catchment)**.** Consider a small mountainous catchment (cf. [1058]) used in a civil–engineering flood–risk study. The main river has three tributaries that join upstream of a town protected by levees.

Let the base set of river nodes be

$$V := \{A, B, C, J_1, J_2, O\},$$

where

- A, B, C are outlet points of three headwater subcatchments (upper tributaries),
- J_1 is the confluence where tributaries A and B meet,
- J_2 is the downstream confluence where J_1 and tributary C meet,
- O is the outlet node at the town.

The directed river–channel graph is

$$G = (V, E, \varphi, \ell),$$

with arcs

$$E := \{(A, J_1),\ (B, J_1),\ (J_1, J_2),\ (C, J_2),\ (J_2, O)\}.$$

Here $\varphi : E \to \mathbb{R}_{>0}$ gives bankfull discharge capacity (e.g. in m^3/s) and $\ell : E \to \mathbb{R}_{>0}$ gives channel length (e.g. in km). For instance,

$$\varphi(A, J_1) = 25, \quad \varphi(B, J_1) = 20, \quad \varphi(C, J_2) = 30, \quad \varphi(J_2, O) = 90,$$

and similarly for ℓ.

The associated River Network HyperGraph

$$H = \left(V, E^{(1)}, \varphi_H, \ell_H\right)$$

has hyperedges

$$\begin{aligned} e_{A,J_1} &:= \{A, J_1\}, & e_{B,J_1} &:= \{B, J_1\}, & e_{J_1,J_2} &:= \{J_1, J_2\}, \\ e_{C,J_2} &:= \{C, J_2\}, & e_{J_2,O} &:= \{J_2, O\}, & h_{J_1} &:= \{A, B, J_1\}, \\ h_{J_2} &:= \{J_1, C, J_2\}, \end{aligned}$$

where h_{J_1} and h_{J_2} are *junction hyperedges* representing multi–tributary confluences at J_1 and J_2. The hyperedge capacities and lengths φ_H, ℓ_H are obtained from φ, ℓ (e.g. $\varphi_H(h_{J_2})$ is the sum of upstream capacities entering J_2, and $\ell_H(h_{J_2})$ is the longest upstream path length into J_2).

We now build a River Network 2-SuperHyperGraph. First, the 2-fold iterated powerset of V is

$$V_2 := \mathcal{P}^2(V) = \mathcal{P}(\mathcal{P}(V)).$$

Each element of V_2 is a nonempty family of vertex subsets and is interpreted as a *cluster of river substructures*. For example

$$\begin{aligned} U_1 &:= \big\{\{A, J_1\}, \{B, J_1\}, \{A, B, J_1\}\big\} \in \mathcal{P}(\mathcal{P}(V)) = V_2, \\ U_2 &:= \big\{\{J_1, J_2\}, \{C, J_2\}, \{J_1, C, J_2\}\big\} \in V_2, \end{aligned}$$

represent, respectively, the local upstream confluence system around J_1 and the downstream confluence system around J_2.

According to the general construction, for $n = 2$ the level-2 superedge set is

$$E_2 := \left\{\mathcal{P}(e) \setminus \{\emptyset\} \;\middle|\; e \in E^{(1)}\right\} \subseteq \mathcal{P}^2(V) = V_2.$$

For instance, for the junction hyperedge $h_{J_2} = \{J_1, C, J_2\}$ we have

$$\mathcal{P}(h_{J_2}) \setminus \{\emptyset\} = \big\{\{J_1\}, \{C\}, \{J_2\}, \{J_1, C\}, \{J_1, J_2\}, \{C, J_2\}, \{J_1, C, J_2\}\big\} \in E_2,$$

which is a 2-superedge whose elements are all nonempty vertex subsets lying inside the local confluence structure h_{J_2}.

The lifted capacity and length maps

$$\varphi^{(2)},\ \ell^{(2)} : E_2 \to \mathbb{R}_{>0}$$

are defined by

$$\varphi^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) := \varphi_H(e), \qquad \ell^{(2)}\big(\mathcal{P}(e) \setminus \{\emptyset\}\big) := \ell_H(e), \quad e \in E^{(1)}.$$

The River Network 2-SuperHyperGraph

$$\mathrm{RNHG}^{(2)} := \big(V_2, E_2, \varphi^{(2)}, \ell^{(2)}\big)$$

thus simultaneously encodes:

- individual channels and confluences (via the original hyperedges in $E^{(1)}$),
- all internal groupings of nodes within each channel or confluence (via 2-superedges such as $\mathcal{P}(h_{J_2}) \setminus \{\emptyset\}$),
- clusters of such groupings (via 2-supervertices such as U_1 and U_2).

In practice, hydrologists and civil engineers can use $\mathrm{RNHG}^{(2)}$ to design multi–scale flood–mitigation strategies: for example, they may assign different control policies (retention basins, levee upgrades, temporary diversions) to distinct 2-supervertices representing upstream, midstream, and downstream confluence systems, while still keeping track of the detailed channel structure inside each system via the associated 2-superedges.

6.17 Transportation Network SuperHyperGraphs in Geoscientific and Civil Applications

A Transportation Network Graph represents intersections, stations as nodes and edges as links, annotated with travel time, distance, and capacity. A Transportation Network SuperHyperGraph lifts routes and junctions into iterated powersets, modeling hierarchical corridors, multimodal hubs, and higher-order flow interactions [101].

Definition 6.17.1 (Transportation Network n-SuperHyperGraph)**.** [101] Let

$$G = (V, E, \tau, \ell, \kappa)$$

be a Transportation Network Graph, where V is the finite set of transportation nodes (intersections, stations, terminals), $E \subseteq V \times V$ the set of directed links, and

$$\tau, \ell, \kappa : E \longrightarrow \mathbb{R}_{>0}$$

give, respectively, the typical travel time, physical length, and flow capacity of each directed edge.

Let

$$H = \big(V, E^{(1)}, \tau_H, \ell_H, \kappa_H\big)$$

be the associated Transportation Network HyperGraph of G, whose hyperedge set $E^{(1)}$ consists of

- binary hyperedges $\{u, v\}$ for each directed link $(u, v) \in E$,
- outgoing junction hyperedges $\{i\} \cup \{j : (i, j) \in E\}$ at nodes i with at least two outgoing links,
- incoming junction hyperedges $\{i\} \cup \{j : (j, i) \in E\}$ at nodes i with at least two incoming links,

with $(\tau_H, \ell_H, \kappa_H)$ defined by aggregating the corresponding edge weights in the usual way (e.g. maximum travel time and length, summed capacities).

Let $\mathcal{P}^k(V)$ again denote the k-fold iterated powerset of V, and fix $n \in \mathbb{N}$. We define:

$$V_n := \mathcal{P}^n(V),$$
$$E_n := \{\mathcal{P}^{n-1}(e) \setminus \{\varnothing\} \mid e \in E^{(1)}\} \subseteq \mathcal{P}^n(V).$$

Elements of V_n are called *n-supervertices* and elements of E_n are *n-superedges*.

The level-n travel-time, length, and capacity labelings are the maps

$$\tau^{(n)}, \ell^{(n)}, \kappa^{(n)} : E_n \longrightarrow \mathbb{R}_{>0}$$

defined on generators by

$$\tau^{(n)}\big(\mathcal{P}^{n-1}(e) \setminus \{\varnothing\}\big) := \tau_H(e),$$
$$\ell^{(n)}\big(\mathcal{P}^{n-1}(e) \setminus \{\varnothing\}\big) := \ell_H(e),$$
$$\kappa^{(n)}\big(\mathcal{P}^{n-1}(e) \setminus \{\varnothing\}\big) := \kappa_H(e) \qquad (e \in E^{(1)}).$$

The *Transportation Network n-SuperHyperGraph* associated with G is the quintuple

$$\mathrm{TNHG}^{(n)} := \big(V_n, E_n, \tau^{(n)}, \ell^{(n)}, \kappa^{(n)}\big).$$

For $n = 1$ the hyperedge labels $(\tau^{(1)}, \ell^{(1)}, \kappa^{(1)})$ coincide with $(\tau_H, \ell_H, \kappa_H)$ on $E^{(1)}$, so $\mathrm{TNHG}^{(1)}$ recovers the Transportation Network HyperGraph (up to the canonical identification of vertices with subsets of V). The family $\big(\mathrm{TNHG}^{(n)}\big)_{n\geq 1}$ is called the Transportation Network SuperHyperGraphs of G.

Example 6.17.2 (Transportation Network 2-SuperHyperGraph for an urban hub)**.** Consider a small urban transportation network with one central railway station and three surrounding terminals:

$$V := \{c, s, a, t\},$$

where c = Central Station, s = Suburban Station, a = Airport, t = Bus Terminal.

Directed links connect the central hub c to each terminal in both directions:

$$E := \{(c, s), (s, c), (c, a), (a, c), (c, t), (t, c)\}.$$

For each edge $e \in E$ we specify

$$\tau(e) \text{ (typical travel time)}, \quad \ell(e) \text{ (physical length)}, \quad \kappa(e) \text{ (flow capacity)},$$

for example

$$\tau(c, s) = 10,\ \tau(c, a) = 40,\ \tau(c, t) = 15,$$
$$\ell(c, s) = 8,\ \ell(c, a) = 30,\ \ell(c, t) = 10,$$
$$\kappa(c, s) = 500,\ \kappa(c, a) = 800,\ \kappa(c, t) = 600,$$

with similar values assigned to the reverse directions.

From this we build the Transportation Network HyperGraph

$$H = \big(V, E^{(1)}, \tau_H, \ell_H, \kappa_H\big)$$

as in the definition. The hyperedge set $E^{(1)}$ contains:

- binary hyperedges $\{u, v\}$ for each $(u, v) \in E$, e.g. $\{c, s\}$, $\{c, a\}$, $\{c, t\}$;
- an outgoing junction hyperedge
$$e_{\text{out}} := \{c, s, a, t\}$$
at the central hub c (three outgoing links);

- an incoming junction hyperedge
$$e_{\mathrm{in}} := \{c, s, a, t\}$$
at the same hub (three incoming links).

The aggregated labels $(\tau_H, \ell_H, \kappa_H)$ can be chosen, for instance, as

$$\tau_H(e_{\mathrm{out}}) := \max\{\tau(c,s), \tau(c,a), \tau(c,t)\}, \quad \kappa_H(e_{\mathrm{out}}) := \kappa(c,s) + \kappa(c,a) + \kappa(c,t),$$

with similar conventions for e_{in} and the binary hyperedges.

Now let $n = 2$ and consider the 2-fold iterated powerset $\mathcal{P}^2(V) = \mathcal{P}(\mathcal{P}(V))$. Define the level-2 supervertex set and superedge family by

$$V_2 := \mathcal{P}^2(V),$$
$$E_2 := \{\mathcal{P}(e) \setminus \{\varnothing\} \mid e \in E^{(1)}\} \subseteq \mathcal{P}^2(V).$$

Each element $U \in V_2$ is a 2-*supervertex*, i.e. a set of subsets of $\{c, s, a, t\}$. For example, the supervertex

$$U_{\mathrm{hub}} := \{\{c,s,a,t\}, \{c,s\}, \{c,a\}, \{c,t\}\} \in V_2$$

represents the central hub together with all direct corridors from c to the terminals.

For the outgoing junction hyperedge $e_{\mathrm{out}} = \{c, s, a, t\}$, the associated 2-superedge is

$$E^{(2)}_{\mathrm{out}} := \mathcal{P}(e_{\mathrm{out}}) \setminus \{\varnothing\} \in E_2,$$

which consists of all nonempty subsets of $\{c, s, a, t\}$ and thus encodes every possible group of nodes sharing the central hub. Similarly, each binary hyperedge $e = \{u, v\}$ yields a 2-superedge

$$E^{(2)}_{u,v} := \mathcal{P}(\{u,v\}) \setminus \{\varnothing\} = \{\{u\}, \{v\}, \{u,v\}\},$$

capturing the local corridor between u and v with all its sub-combinations.

The level-2 labels
$$\tau^{(2)},\ \ell^{(2)},\ \kappa^{(2)} : E_2 \to \mathbb{R}_{>0}$$
are defined by

$$\tau^{(2)}(\mathcal{P}(e) \setminus \{\varnothing\}) := \tau_H(e), \quad \ell^{(2)}(\mathcal{P}(e) \setminus \{\varnothing\}) := \ell_H(e), \quad \kappa^{(2)}(\mathcal{P}(e) \setminus \{\varnothing\}) := \kappa_H(e).$$

The resulting Transportation Network 2-SuperHyperGraph

$$\mathrm{TNHG}^{(2)} := (V_2, E_2, \tau^{(2)}, \ell^{(2)}, \kappa^{(2)})$$

provides a hierarchical model of the urban hub: supervertices encode collections of routes and transfer patterns, while 2-superedges capture all higher-order groupings of those collections relevant for capacity planning, multimodal scheduling, and robustness analysis.

6.18 SuperHyperGraph Learning

Graph learning predicts vertex labels or embeddings using edge weights, fitting labeled nodes while enforcing smoothness across adjacent vertices [1059–1061]. As related concepts, fuzzy graph learning [1062–1064], digraph learning [1065–1067], and molecular graph learning [1068–1070] are also well known.

Hypergraph learning predicts vertex labels using hyperedge weights, penalizing within-hyperedge disagreement by pulling each vertex toward its hyperedge mean [1071–1074]. Superhypergraph learning predicts labels on supervertices via superedge incidences, enforcing within-incidence smoothness over nested vertex sets and relations.

Definition 6.18.1 (Weighted graph)**.** A *weighted (undirected) graph* is a triple $G = (V, E, w)$ where V is a finite set, $E \subseteq \big\{\{u, v\} \subseteq V : u \neq v\big\}$, and $w : E \to \mathbb{R}_{>0}$ assigns a positive weight to each edge.

Definition 6.18.2 (Graph learning (Laplacian-regularized transductive learning))**.** Let $G = (V, E, w)$ be a weighted graph, let $L \subseteq V$ be a labeled vertex set, let Y be a label space, and let $\ell : \mathbb{R}^c \times Y \to \mathbb{R}_{\geq 0}$ be a loss. A *graph learning problem* is to find a prediction function $f : V \to \mathbb{R}^c$ minimizing

$$\min_{f:V\to\mathbb{R}^c} \sum_{v\in L} \ell\big(f(v), y_v\big) \; + \; \lambda \sum_{\{u,v\}\in E} w(\{u,v\}) \, \|f(u) - f(v)\|_2^2, \quad \lambda > 0.$$

Example 6.18.3 (Graph learning: toy semi-supervised node classification)**.** Let $V = \{1, 2, 3, 4\}$ and $E = \big\{\{1, 2\}, \{2, 3\}, \{3, 4\}\big\}$ with weights $w(\{1, 2\}) = w(\{2, 3\}) = w(\{3, 4\}) = 1$. Let the labeled set be $L = \{1, 4\}$ with binary labels $y_1 = 0$ and $y_4 = 1$. Graph learning seeks $f : V \to \mathbb{R}$ minimizing

$$\ell\big(f(1), 0\big) + \ell\big(f(4), 1\big) + \lambda \sum_{\{u,v\}\in E} \big(f(u) - f(v)\big)^2,$$

so $f(2), f(3)$ are encouraged to interpolate smoothly between the labeled endpoints.

Definition 6.18.4 (Weighted hypergraph)**.** A *weighted hypergraph* is a triple $H = (V, \mathcal{E}, w)$ where V is a finite set of vertices, $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a finite family of hyperedges, and $w : \mathcal{E} \to \mathbb{R}_{>0}$ assigns a positive weight to each hyperedge.

Definition 6.18.5 (Hypergraph learning (incidence-regularized learning))**.** Let $H = (V, \mathcal{E}, w)$ be a weighted hypergraph. For $e \in \mathcal{E}$ define the (hyperedge) mean

$$\bar{f}_e \; := \; \frac{1}{|e|} \sum_{v\in e} f(v) \in \mathbb{R}^c.$$

A *hypergraph learning problem* (semi-supervised transductive) is to find $f : V \to \mathbb{R}^c$ minimizing

$$\min_{f:V\to\mathbb{R}^c} \sum_{v\in L} \ell\big(f(v), y_v\big) \; + \; \lambda \sum_{e\in\mathcal{E}} w(e) \sum_{v\in e} \|f(v) - \bar{f}_e\|_2^2, \quad \lambda > 0.$$

This regularizer penalizes label/representation variation *inside each hyperedge*.

Example 6.18.6 (Hypergraph learning: group-consistency on overlapping teams)**.** Let $V = \{a, b, c, d\}$ and hyperedges

$$\mathcal{E} = \{e_1, e_2\}, \qquad e_1 = \{a, b, c\}, \quad e_2 = \{b, c, d\},$$

with $w(e_1) = w(e_2) = 1$. Let $L = \{a, d\}$ with labels $y_a = 0$ and $y_d = 1$. Hypergraph learning seeks $f : V \to \mathbb{R}$ minimizing

$$\ell\big(f(a), 0\big) + \ell\big(f(d), 1\big) + \lambda \sum_{e\in\mathcal{E}} \sum_{v\in e} \big\|f(v) - \bar{f}_e\big\|_2^2,$$

where $\bar{f}_{e_1} = \frac{1}{3}(f(a) + f(b) + f(c))$ and $\bar{f}_{e_2} = \frac{1}{3}(f(b) + f(c) + f(d))$. Thus (b, c) are influenced by both groups, enforcing consistent predictions inside each hyperedge.

Definition 6.18.7 (Weighted n-superhypergraph)**.** A *weighted n-superhypergraph* is a quadruple (V, E, ∂, w) where (V, E, ∂) is an n-SuperHyperGraph and $w : E \to \mathbb{R}_{>0}$ assigns a positive weight to each edge-identifier.

Definition 6.18.8 (n-SuperHyperGraph learning)**.** Let (V, E, ∂, w) be a weighted n-superhypergraph. For each $e \in E$ and each $f : V \to \mathbb{R}^c$, define the incidence-mean

$$\bar{f}_{\partial(e)} \; := \; \frac{1}{|\partial(e)|} \sum_{U\in\partial(e)} f(U) \in \mathbb{R}^c.$$

An *n-SuperHyperGraph learning problem* is to find $f : V \to \mathbb{R}^c$ minimizing

$$\min_{f:V\to\mathbb{R}^c} \sum_{U\in L} \ell\big(f(U), y_U\big) \; + \; \lambda \sum_{e\in E} w(e) \sum_{U\in\partial(e)} \|f(U) - \bar{f}_{\partial(e)}\|_2^2, \quad \lambda > 0,$$

where $L \subseteq V$ is a labeled set of n-supervertices. When $n \geq 1$, this is also called *superhypergraph learning*.

Example 6.18.9 (SuperHypergraph learning: learning on nested supervertices ($n = 2$))**.** Let $V_0 = \{x, y, z\}$ and define 2-supervertices

$$U_1 = \{\{x\}\}, \qquad U_2 = \{\{y\}\}, \qquad U_3 = \{\{x\}, \{y\}\}, \qquad U_4 = \{\{z\}\}.$$

Set $V = \{U_1, U_2, U_3, U_4\}$ and let $E = \{s_1, s_2\}$ with incidence map

$$\partial(s_1) = \{U_1, U_3\}, \qquad \partial(s_2) = \{U_2, U_3, U_4\},$$

and weights $w(s_1) = w(s_2) = 1$. Let the labeled set be $L = \{U_1, U_4\}$ with $y_{U_1} = 0$ and $y_{U_4} = 1$. Superhypergraph learning seeks $f : V \to \mathbb{R}$ minimizing

$$\ell\big(f(U_1), 0\big) + \ell\big(f(U_4), 1\big) + \lambda \sum_{e \in E} \sum_{U \in \partial(e)} \big\| f(U) - \bar{f}_{\partial(e)} \big\|_2^2,$$

where $\bar{f}_{\partial(s_1)} = \frac{1}{2}\big(f(U_1) + f(U_3)\big)$ and $\bar{f}_{\partial(s_2)} = \frac{1}{3}\big(f(U_2) + f(U_3) + f(U_4)\big)$. Hence U_3 (a nested set-of-sets) couples both incidences and mediates information between labeled supervertices.

Theorem 6.18.10 (n-SuperHyperGraph learning generalizes hypergraph learning)**.** *Fix $n \in \mathbb{N}_0$. For every weighted hypergraph $H = (V_0, \mathcal{E}, w)$ and every hypergraph learning instance as in Definition 6.18.5, there exists a weighted n-superhypergraph $(V^{(n)}, E^{(n)}, \partial^{(n)}, w^{(n)})$ and an n-superhypergraph learning instance as in Definition 6.18.8 such that:*

1. *the vertex sets are in bijection and labels/loss terms correspond exactly; and*
2. *the regularization terms are equal under this bijection; hence minimizers correspond.*

Therefore, n-SuperHyperGraph learning is a strict extension of hypergraph learning (at least in the sense of containing it as a special case).

Proof. Define the nested-singleton injection $\iota_n : V_0 \to \mathcal{P}^n(V_0)$ by

$$\iota_0(v) := v, \qquad \iota_{k+1}(v) := \{\iota_k(v)\} \quad (k \geq 0).$$

Let

$$V^{(n)} \;:=\; \iota_n(V_0) \subseteq \mathcal{P}^n(V_0).$$

Let the edge-identifier set be $E^{(n)} := \mathcal{E}$ (reuse each hyperedge as an identifier), define weights $w^{(n)}(e) := w(e)$ for $e \in \mathcal{E}$, and define incidence by

$$\partial^{(n)}(e) \;:=\; \{\iota_n(v) : v \in e\} \subseteq V^{(n)} \qquad (e \in \mathcal{E}).$$

Then $(V^{(n)}, E^{(n)}, \partial^{(n)})$ is an n-SuperHyperGraph over V_0, hence $(V^{(n)}, E^{(n)}, \partial^{(n)}, w^{(n)})$ is a weighted n-superhypergraph.

Now transfer any prediction function $f_0 : V_0 \to \mathbb{R}^c$ to $f_n : V^{(n)} \to \mathbb{R}^c$ by

$$f_n(\iota_n(v)) \;:=\; f_0(v) \qquad (v \in V_0).$$

This correspondence is bijective because ι_n is injective and $V^{(n)} = \iota_n(V_0)$.

Fix $e \in \mathcal{E}$. The incidence set $\partial^{(n)}(e)$ is in bijection with e via $v \mapsto \iota_n(v)$, hence

$$\bar{f}_{\partial^{(n)}(e)} = \frac{1}{|\partial^{(n)}(e)|} \sum_{U \in \partial^{(n)}(e)} f_n(U) = \frac{1}{|e|} \sum_{v \in e} f_n(\iota_n(v)) = \frac{1}{|e|} \sum_{v \in e} f_0(v) = \bar{f}_e.$$

Therefore,

$$\sum_{U \in \partial^{(n)}(e)} \|f_n(U) - \bar{f}_{\partial^{(n)}(e)}\|_2^2 = \sum_{v \in e} \|f_n(\iota_n(v)) - \bar{f}_e\|_2^2 = \sum_{v \in e} \|f_0(v) - \bar{f}_e\|_2^2.$$

Multiplying by the weights $w^{(n)}(e) = w(e)$ and summing over $e \in \mathcal{E}$ yields equality of the full regularizers:

$$\sum_{e\in E^{(n)}} w^{(n)}(e) \sum_{U\in\partial^{(n)}(e)} \|f_n(U) - \bar{f}_{\partial^{(n)}(e)}\|_2^2 = \sum_{e\in\mathcal{E}} w(e) \sum_{v\in e} \|f_0(v) - \bar{f}_e\|_2^2.$$

Finally, if the labeled set in the hypergraph instance is $L_0 \subseteq V_0$, define $L_n := \iota_n(L_0) \subseteq V^{(n)}$ and set $y_{\iota_n(v)} := y_v$. Then

$$\sum_{U\in L_n} \ell\big(f_n(U), y_U\big) = \sum_{v\in L_0} \ell\big(f_0(v), y_v\big),$$

so the full objectives coincide under $f_n(\iota_n(v)) = f_0(v)$. Hence minimizers correspond bijectively, and hypergraph learning is realized as a special case of n-superhypergraph learning. □

6.19 SuperHyperGraph Attention Networks

Graph Attention Networks (GATs) learn node embeddings on graphs by attention-weighted neighbor aggregation, emphasizing important adjacent nodes per layer. As related concepts, Fuzzy Graph Attention Networks [1075, 1076] and Directed Graph Attention Networks [1077, 1078] are also well known.

HyperGraph Attention Networks (HGATs) extend GATs to hypergraphs, attentively aggregating messages between vertices and hyperedges via incidence relations [1079–1082]. SuperHyperGraph Attention Networks (SuHGATs) generalize HGATs to n-superhypergraphs, applying attention-based message passing between supervertices and superedges [1083].

Definition 6.19.1 (HyperGraph Attention Network (HGAT))**.** [1079–1082] Let $G = (V, E, \omega)$ be a (weighted) hypergraph with $|V| = N$ vertices and $|E| = M$ hyperedges. Let $A \in \{0, 1\}^{N\times M}$ be the incidence matrix, where

$$A_{ij} = 1 \iff v_i \in e_j.$$

At layer ℓ, let $H^{(\ell)} \in \mathbb{R}^{N\times d}$ be the vertex-feature matrix and $E^{(\ell)} \in \mathbb{R}^{M\times d}$ the hyperedge-feature matrix.

A single HGAT layer is the composition of two attentive aggregations:

(1) *Vertex → Hyperedge attention.* Choose a learnable projection $W \in \mathbb{R}^{d\times d'}$ and an attention kernel $a : \mathbb{R}^{d'} \times \mathbb{R}^{d'} \to \mathbb{R}$. For each incident pair (i, j) with $A_{ij} = 1$, compute a raw score

$$s_{ij}^{(\ell)} = a\big(h_i^{(\ell)} W,\ e_j^{(\ell)} W\big),$$

then row-normalize over hyperedges incident to i (masking non-incidences) to obtain attention weights:

$$\alpha_{ij}^{(\ell)} = \frac{\exp(s_{ij}^{(\ell)})}{\sum\limits_{j':\,A_{ij'}=1} \exp(s_{ij'}^{(\ell)})} \quad \text{and} \quad \alpha_{ij}^{(\ell)} = 0 \text{ if } A_{ij} = 0.$$

Equivalently, in masked matrix form one may write

$$\mathbf{A}^{(\ell)} \;=\; A \odot \operatorname{softmax}\Big(\operatorname{LeakyReLU}\big(H^{(\ell)} W \,(E^{(\ell)} W)^{\top}\big)\Big) \in [0, 1]^{N\times M}.$$

Update hyperedge features by attentive aggregation:

$$E^{(\ell+1)} \;=\; \sigma\big((\mathbf{A}^{(\ell)})^{\top} H^{(\ell)}\big).$$

(2) *Hyperedge → Vertex attention.* Choose another projection $W_1 \in \mathbb{R}^{d\times d'}$ and define

$$\mathbf{B}^{(\ell)} \;=\; A^{\top} \odot \operatorname{softmax}\Big(\operatorname{LeakyReLU}\big(E^{(\ell)} W_1 \,(H^{(\ell)} W_1)^{\top}\big)\Big) \in [0, 1]^{M\times N}.$$

Update vertex features by

$$H^{(\ell+1)} \;=\; \sigma\big((\mathbf{B}^{(\ell)})^{\top} E^{(\ell)}\big).$$

Stacking L such layers yields final vertex embeddings $H^{(L)}$ (and optionally hyperedge embeddings $E^{(L)}$). Multi-head variants replace (W, a) by $(W^{(k)}, a^{(k)})$ and concatenate/average head outputs.

Example 6.19.2 (A concrete HGAT example on a tiny hypergraph)**.** Consider the hypergraph $G = (V, E)$ with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{e_1, e_2\}, \qquad e_1 = \{v_1, v_2\}, \;\; e_2 = \{v_2, v_3\}.$$

Its incidence matrix $A \in \{0, 1\}^{3\times 2}$ (rows v_i, columns e_j) is

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Take one-dimensional features ($d = 1$):

$$H^{(0)} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{(vertex features)}, \qquad E^{(0)} = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix} \quad \text{(hyperedge features)}.$$

Choose $W = 1$ and an attention kernel $a(x, y) = xy$ (dot-product in $d = 1$), and take σ as the identity.

Vertex→Hyperedge attention. For each incidence (i, j) with $A_{ij} = 1$, the raw score is $s_{ij} = h_i^{(0)} e_j^{(0)}$:

$$s_{11} = 1 \cdot 0.5 = 0.5, \quad s_{21} = 2 \cdot 0.5 = 1, \quad s_{22} = 2 \cdot 1.5 = 3, \quad s_{32} = 3 \cdot 1.5 = 4.5.$$

Row-normalize by softmax over hyperedges incident to each vertex:

$$\alpha_{11} = 1, \; \alpha_{12} = 0, \qquad \alpha_{21} = \frac{e^1}{e^1 + e^3} = \frac{2.7182818}{2.7182818 + 20.0855369} = 0.1192029,$$

$$\alpha_{22} = \frac{e^3}{e^1 + e^3} = \frac{20.0855369}{2.7182818 + 20.0855369} = 0.8807971, \qquad \alpha_{31} = 0, \; \alpha_{32} = 1.$$

Hence

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0.1192029 & 0.8807971 \\ 0 & 1 \end{pmatrix}.$$

Update hyperedge features by $E^{(1)} = \alpha^{\mathsf{T}} H^{(0)}$:

$$E_1^{(1)} = 1 \cdot 1 + 0.1192029 \cdot 2 + 0 \cdot 3 = 1 + 0.2384058 = 1.2384058,$$

$$E_2^{(1)} = 0 \cdot 1 + 0.8807971 \cdot 2 + 1 \cdot 3 = 1.7615942 + 3 = 4.7615942,$$

so

$$E^{(1)} = \begin{pmatrix} 1.2384058 \\ 4.7615942 \end{pmatrix}.$$

Hyperedge→Vertex attention (using $E^{(1)}$). For each incidence (j, i) with $A_{ij} = 1$, set $t_{ji} = E_j^{(1)} h_i^{(0)}$ and softmax over vertices in each hyperedge. For e_1 (incident to v_1, v_2):

$$t_{1,1} = 1.2384058 \cdot 1 = 1.2384058, \qquad t_{1,2} = 1.2384058 \cdot 2 = 2.4768116,$$

$$\beta_{1,1} = \frac{e^{1.2384058}}{e^{1.2384058} + e^{2.4768116}} = \frac{3.4500998}{3.4500998 + 11.9032595} = 0.2242645, \quad \beta_{1,2} = 0.7757355.$$

For e_2 (incident to v_2, v_3):

$$t_{2,2} = 4.7615942 \cdot 2 = 9.5231884, \qquad t_{2,3} = 4.7615942 \cdot 3 = 14.2847826,$$

$$\beta_{2,2} = \frac{e^{9.5231884}}{e^{9.5231884} + e^{14.2847826}} = 0.008479449, \quad \beta_{2,3} = 0.991520551.$$

Thus (rows e_1, e_2; columns v_1, v_2, v_3)

$$\beta = \begin{pmatrix} 0.2242645 & 0.7757355 & 0 \\ 0 & 0.008479449 & 0.991520551 \end{pmatrix}.$$

Update vertex features by $H^{(1)} = \beta^{\top} E^{(1)}$:

$$h_1^{(1)} = 0.2242645 \cdot 1.2384058 = 0.277730456,$$

$$h_2^{(1)} = 0.7757355 \cdot 1.2384058 + 0.008479449 \cdot 4.7615942 = 0.9606891 + 0.0403619 = 1.0010510,$$

$$h_3^{(1)} = 0.991520551 \cdot 4.7615942 = 4.7212185,$$

so

$$H^{(1)} = \begin{pmatrix} 0.277730456 \\ 1.001051040 \\ 4.721218505 \end{pmatrix}.$$

This is a fully specified, numerical HGAT layer computation on a concrete hypergraph.

Definition 6.19.3 (n-SuperHyperGraph Attention Network (n-SuHGAT))**.** [1083] Fix a finite base set V_0 and an integer $n \geq 1$. Let $\mathcal{P}^n(V_0)$ denote the n-th iterated powerset. An n-SuperHyperGraph is a pair

$$\mathrm{SuHG}(n) = (V^{(n)}, E^{(n)}), \qquad V^{(n)}, E^{(n)} \subseteq \mathcal{P}^n(V_0),$$

whose elements are called n-supervertices and n-superedges, respectively. Let $N = |V^{(n)}|$, $M = |E^{(n)}|$, and let the incidence matrix be

$$A^{(n)} \in \{0,1\}^{N \times M}, \qquad A_{uv}^{(n)} = 1 \iff n\text{-supervertex } u \text{ is incident to } n\text{-superedge } v.$$

At layer ℓ, let $H^{(\ell)} \in \mathbb{R}^{N \times d}$ be the n-supervertex features and $E^{(\ell)} \in \mathbb{R}^{M \times d}$ the n-superedge features.

A single n-SuHGAT layer is defined by the same two-phase attentive message passing, but using $A^{(n)}$:

(1) *Supervertex → Superedge attention.* With $W \in \mathbb{R}^{d \times d'}$,

$$\mathbf{A}^{(\ell,n)} = A^{(n)} \odot \operatorname{softmax}\Big(\mathrm{LeakyReLU}\big(H^{(\ell)} W \, (E^{(\ell)} W)^{\top}\big)\Big) \in [0,1]^{N \times M},$$

$$E^{(\ell+1)} = \sigma\big((\mathbf{A}^{(\ell,n)})^{\top} H^{(\ell)}\big).$$

(2) *Superedge → Supervertex attention.* With $W_1 \in \mathbb{R}^{d \times d'}$,

$$\mathbf{B}^{(\ell,n)} = (A^{(n)})^{\top} \odot \operatorname{softmax}\Big(\mathrm{LeakyReLU}\big(E^{(\ell)} W_1 \, (H^{(\ell)} W_1)^{\top}\big)\Big) \in [0,1]^{M \times N},$$

$$H^{(\ell+1)} = \sigma\big((\mathbf{B}^{(\ell,n)})^{\top} E^{(\ell)}\big).$$

Stacking L layers defines the n-SuHGAT and yields embeddings $H^{(L)}$ on n-supervertices (and optionally $E^{(L)}$ on n-superedges). Again, multi-head variants are obtained by running multiple attention heads in parallel and concatenating/averaging.

Example 6.19.4 (A concrete SuperHyperGraph Attention example (level $n = 2$) on nested supervertices)**.** Let the base set be $V_0 = \{a, b, c\}$ and take $n = 2$. Define three 2-supervertices (elements of $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$) by

$$u_1 = \{\{a\}\}, \qquad u_2 = \{\{b\}\}, \qquad u_3 = \{\{a\}, \{b\}\}.$$

Let $V^{(2)} = \{u_1, u_2, u_3\}$ and let $E^{(2)} = \{s_1, s_2\}$ with incidence map

$$\partial(s_1) = \{u_1, u_3\}, \qquad \partial(s_2) = \{u_2, u_3\}.$$

Equivalently, the incidence matrix $A^{(2)} \in \{0,1\}^{3 \times 2}$ (rows u_i, columns s_j) is

$$A^{(2)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which has the same pattern as the previous hypergraph example, but now the "vertices" are nested objects.

Assign one-dimensional initial features to supervertices and superedges:

$$H^{(0)} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{(2-supervertex features for } u_1, u_2, u_3\text{)}, \qquad E^{(0)} = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix} \quad \text{(2-superedge features for } s_1, s_2\text{)}.$$

Using exactly the same two-phase attentive message passing (with $A^{(2)}$ in place of A), and the same choices $W = 1$, $a(x, y) = xy$, and $\sigma = \text{id}$, we obtain the same numerical attentions

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0.1192029 & 0.8807971 \\ 0 & 1 \end{pmatrix}, \qquad E^{(1)} = \alpha^{\mathsf{T}} H^{(0)} = \begin{pmatrix} 1.2384058 \\ 4.7615942 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 0.2242645 & 0.7757355 & 0 \\ 0 & 0.008479449 & 0.991520551 \end{pmatrix}, \qquad H^{(1)} = \beta^{\mathsf{T}} E^{(1)} = \begin{pmatrix} 0.277730456 \\ 1.001051040 \\ 4.721218505 \end{pmatrix}.$$

Interpretation: $u_3 = \{\{a\}, \{b\}\}$ is a "higher" supervertex sharing incidence with both superedges, so its updated embedding becomes large after the second attention phase, reflecting strong multi-incidence influence.

Chapter 7

Extensional Definitions: (m, n)-SuperHyperGraph

In this chapter, we define the notion of an (m, n)-SuperHyperGraph as an extension of the n-SuperHyperGraph and examine its fundamental properties.

7.1 (m, n)-SuperHyperGraph

A (m, n)-SuperHyperGraph is a mathematical structure in which each vertex corresponds to an (m, n)-superhyperfunction defined on a base set, while the hyperedges group such functions together to represent higher-order relationships and contextual connections. An (h, k)-ary (m, n)-SuperHyperGraph further generalizes this idea by taking vertices as (h, k)-ary (m, n)-superhyperfunctions [1084].

Notation 7.1.1. *For a nonempty base set S define*

$$\mathcal{P}_0(S) := S, \qquad \mathcal{P}_{m+1}(S) := \mathcal{P}\big(\mathcal{P}_m(S)\big) \quad (m \in \mathbb{N}_0),$$

so $\mathcal{P}_1(S) = \mathcal{P}(S)$, $\mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S))$, etc. We also use the Cartesian power $X^h := \underbrace{X \times \cdots \times X}_{h \text{ copies}}$ for $h \in \mathbb{N}$.

Definition 7.1.2 (*(m, n)-superhyperfunction*)**.** [115,1085] Let $m, n \in \mathbb{N}$ and $S \neq \emptyset$. An *(m, n)-superhyperfunction on S* is a map

$$f : \mathcal{P}_m(S) \longrightarrow \mathcal{P}_n(S).$$

Equivalently, $f \in \mathrm{Hom}\big(\mathcal{P}_m(S), \mathcal{P}_n(S)\big)$ as functions of sets.

Definition 7.1.3 (*(m, n)-SuperHyperGraph*)**.** [1084] Fix $m, n \in \mathbb{N}$ and a nonempty base set S. Let

$$\mathfrak{F}_{m,n}(S) := \Big\{ f : \mathcal{P}_m(S) \to \mathcal{P}_n(S) \Big\}.$$

An *(m, n)-SuperHyperGraph* is a pair

$$\mathsf{SHG}^{(m,n)} := (V, \mathcal{E}),$$

where $V \subseteq \mathfrak{F}_{m,n}(S)$ is a nonempty set of vertices (each vertex is a concrete (m, n)-superhyperfunction) and

$$\emptyset \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$

is a nonempty family of nonempty *hyperedges*. Each hyperedge $E \in \mathcal{E}$ groups a finite, nonempty set of superhyperfunctions to encode higher-order relations/constraints among them.

For reference, a comparison between an n-SuperHyperGraph and an (m, n)-SuperHyperGraph is summarized in Table 7.1.

Definition 7.1.4 (*(h, k)-ary (m, n)-superhyperfunction*)**.** [115] Let $h, k \in \mathbb{N}$ and $m, n \in \mathbb{N}$. An *(h, k)-ary (m, n)-superhyperfunction on S* is a map

$$F : \big(\mathcal{P}_m(S)\big)^h \longrightarrow \big(\mathcal{P}_n(S)\big)^k.$$

Writing $F(A_1, \ldots, A_h) = (B_1, \ldots, B_k)$, each component $B_j \in \mathcal{P}_n(S)$ is an n–level object determined by h many m–level inputs.

Table 7.1: Compact comparison: n-SuperHyperGraph vs. (m, n)-SuperHyperGraph.

Item	n-SuperHyperGraph	(m, n)-SuperHyperGraph
Base data	A finite base set V_0 (universe for iterated powersets).	A nonempty base set S (universe for iterated powersets).
Vertex type	Supervertices are n-level objects: $$V \subseteq \mathcal{P}^n(V_0).$$	Vertices are concrete (m, n)-superhyperfunctions: $$V \subseteq \mathfrak{F}_{m,n}(S) := \{f : \mathcal{P}_m(S) \to \mathcal{P}_n(S)\}.$$
Edge / incidence model	Identifiers E with an incidence map $$\partial : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$	Hyperedges are subsets of V: $$\emptyset \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$ (Equivalently, one may encode each $E \in \mathcal{E}$ by an identifier plus an incidence map.)
What is "higher level"?	Higher level is in the *objects* (vertices live in $\mathcal{P}^n(V_0)$).	Higher level is in the *morphisms* (vertices are functions $\mathcal{P}_m(S) \to \mathcal{P}_n(S)$).
Typical reading	Each supervertex represents a grouped/aggregated entity; each superedge links multiple supervertices via incidence.	Each vertex is a set-to-set transformer (rule, policy, operator); each hyperedge groups several transformers to encode joint constraints or contexts.
Specialization / relation	Recovers hypergraphs by restricting $n = 0$ and taking $V \subseteq V_0$.	Extends the n-SuperHyperGraph viewpoint by shifting vertices from n-level sets to (m, n)-level set-valued functions (Definition 7.1.3).

Definition 7.1.5 (*(h, k)-ary (m, n)-SuperHyperGraph*)**.** [1084] Fix $m, n, h, k \in \mathbb{N}$ and $S \neq \emptyset$. Let

$$\mathfrak{F}^{h,k}_{m,n}(S) \;:=\; \Big\{ F : \big(\mathcal{P}_m(S)\big)^h \to \big(\mathcal{P}_n(S)\big)^k \Big\}.$$

An *(h, k)-ary (m, n)-SuperHyperGraph* is a pair

$$\mathsf{SHG}^{(h,k)}_{(m,n)} \;:=\; (V,\ \mathcal{E}), \qquad \emptyset \neq V \subseteq \mathfrak{F}^{h,k}_{m,n}(S), \quad \emptyset \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

For reference, the comparison between an (m, n)-SuperHyperGraph and an (h, k)-ary (m, n)-SuperHyperGraph is provided in Table 7.2.

Example 7.1.6 (Collaborative task planning: $(h, k, m, n) = (2, 2, 1, 1)$)**.** Let the base set of tasks be

$$S := \{\text{Doc},\ \text{Code},\ \text{Test},\ \text{Deploy}\}.$$

Here $\mathcal{P}_1(S) = \mathcal{P}(S)$ and outputs also lie in $\mathcal{P}_1(S)$. Each vertex is a concrete (h, k)-ary (m, n)-superhyperfunction

$$F :\ \big(\mathcal{P}_1(S)\big)^2 \;\longrightarrow\; \big(\mathcal{P}_1(S)\big)^2.$$

We interpret the inputs as two teams' current selections $A, B \subseteq S$, and the two outputs as *Must_Do* and *Plan* sets.

Define two vertices (set transformers):

$$F_{\text{ui}}(A, B) := \big(A \cap B,\ A \cup B\big) \quad \text{(consensus vs. union plan)},$$
$$F_{\text{prio}}(A, B) := \big(\,(A \cup B) \cap C^\star,\ (A \cup B) \setminus C^\star\,\big) \quad \text{with } C^\star := \{\text{Test}, \text{Deploy}\}.$$

Set

$$V := \{F_{\text{ui}},\ F_{\text{prio}}\}, \qquad \mathcal{E} := \big\{\{F_{\text{ui}}\},\ \{F_{\text{prio}}\},\ \{F_{\text{ui}}, F_{\text{prio}}\}\big\}.$$

Hyperedges encode whether one deploys only the union/intersection policy, only the priority split, or both together.

Concrete input–output calculation. For team selections

$$A = \{\text{Doc}, \text{Test}\}, \qquad B = \{\text{Test}, \text{Code}\},$$

Table 7.2: Compact comparison: (m,n)-SuperHyperGraph vs. (h,k)-ary (m,n)-SuperHyperGraph.

Category	(m,n)-SuperHyperGraph $\mathsf{SHG}^{(m,n)} = (V,\mathcal{E})$	(h,k)-ary (m,n)-SuperHyperGraph $\mathsf{SHG}^{(h,k)}_{(m,n)} = (V,\mathcal{E})$
Underlying base data	A nonempty base set S and two levels $m,n \in \mathbb{N}$.	A nonempty base set S, levels $m,n \in \mathbb{N}$, and arities $h,k \in \mathbb{N}$.
Vertex meaning	Each vertex is a concrete (m,n)-superhyperfunction $f : \mathcal{P}_m(S) \to \mathcal{P}_n(S)$.	Each vertex is a concrete (h,k)-ary (m,n)-superhyperfunction $F : (\mathcal{P}_m(S))^h \to (\mathcal{P}_n(S))^k$.
Input-output interface	Single m-level input set $A \in \mathcal{P}_m(S)$ mapped to one n-level output set $f(A) \in \mathcal{P}_n(S)$.	h many m-level input sets $(A_1,\ldots,A_h)$ mapped to a k-tuple of n-level outputs $F(A_1,\ldots,A_h) = (B_1,\ldots,B_k)$ with $B_j \in \mathcal{P}_n(S)$.
Interpretation	Unary rule/transformer on m-level objects producing one n-level object.	Multi-input, multi-output rule/transformer: fuses h sources (or contexts) and returns k output channels.
Hyperedges	$\varnothing \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$; each hyperedge groups a finite nonempty set of vertices (functions) to encode higher-order relations/constraints.	Same hyperedge formalism $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$, but the grouped vertices are multi-ary, multi-output transformers.
Specialization	Recovered as the case $(h,k) = (1,1)$ of the arity-extended model.	Strictly generalizes (m,n)-SuperHyperGraphs by allowing $h \neq 1$ and/or $k \neq 1$.

we obtain

$$F_{\mathrm{ui}}(A,B) = \big(\{\mathrm{Test}\},\ \{\mathrm{Doc},\mathrm{Test},\mathrm{Code}\}\big),$$

$$F_{\mathrm{prio}}(A,B) = \big(\{\mathrm{Test}\},\ \{\mathrm{Doc},\mathrm{Code}\}\big).$$

Thus $\mathsf{SHG}^{(h,k)}_{(m,n)} = (V,\mathcal{E})$ models two cooperative planning rules whose joint use is represented by the hyperedge $\{F_{\mathrm{ui}}, F_{\mathrm{prio}}\}$.

Example 7.1.7 (Urban traffic response with detour bundles: $(h,k,m,n) = (3,1,1,2)$)**.** Let the base set of road segments be

$$S := \{\mathrm{R1},\mathrm{R2},\mathrm{R3},\mathrm{R4},\mathrm{R5}\}.$$

Inputs are three incident reports $X_{\mathrm{cam}}, X_{\mathrm{nav}}, X_{\mathrm{pol}} \subseteq S$ from cameras, navigation apps, and police, respectively, so the domain is $(\mathcal{P}_1(S))^3$. Outputs live in $\mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S))$, i.e., *detour bundles* (sets of candidate detour sets).

Define two vertices:

$$F_{\mathrm{cons}}(X_{\mathrm{cam}}, X_{\mathrm{nav}}, X_{\mathrm{pol}}) := \big\{\, C,\ (U \setminus C) \,\big\} \setminus \{\varnothing\},$$
$$\text{where } U := X_{\mathrm{cam}} \cup X_{\mathrm{nav}} \cup X_{\mathrm{pol}},\ C := (X_{\mathrm{cam}} \cap X_{\mathrm{nav}}) \ \cup\ (X_{\mathrm{cam}} \cap X_{\mathrm{pol}}) \ \cup\ (X_{\mathrm{nav}} \cap X_{\mathrm{pol}}),$$

so C aggregates segments supported by at least two sources (majority congestion) and $U \setminus C$ are alternates; and

$$F_{\mathrm{pairs}}(X_{\mathrm{cam}}, X_{\mathrm{nav}}, X_{\mathrm{pol}})$$

$$:= \big\{\ X_{\mathrm{cam}} \cap X_{\mathrm{nav}},\ X_{\mathrm{cam}} \cap X_{\mathrm{pol}},\ X_{\mathrm{nav}} \cap X_{\mathrm{pol}}\ \big\} \setminus \{\varnothing\}.$$

Set

$$V := \{F_{\mathrm{cons}},\ F_{\mathrm{pairs}}\}, \qquad \mathcal{E} := \big\{\{F_{\mathrm{cons}}\},\ \{F_{\mathrm{pairs}}\},\ \{F_{\mathrm{cons}}, F_{\mathrm{pairs}}\}\big\}.$$

Hyperedges represent admissible joint deployment of consensus detouring and pairwise corroboration.

Concrete input–output calculation. For reports

$$X_{\mathrm{cam}} = \{\mathrm{R1},\mathrm{R2}\}, \quad X_{\mathrm{nav}} = \{\mathrm{R2},\mathrm{R3}\}, \quad X_{\mathrm{pol}} = \{\mathrm{R2},\mathrm{R4}\},$$

we compute $U = \{\mathrm{R1},\mathrm{R2},\mathrm{R3},\mathrm{R4}\}$ and

$$X_{\mathrm{cam}} \cap X_{\mathrm{nav}} = \{\mathrm{R2}\}, \quad X_{\mathrm{cam}} \cap X_{\mathrm{pol}} = \{\mathrm{R2}\}, \quad X_{\mathrm{nav}} \cap X_{\mathrm{pol}} = \{\mathrm{R2}\},$$

so $C = \{\mathrm{R2}\}$ and $U \setminus C = \{\mathrm{R1}, \mathrm{R3}, \mathrm{R4}\}$. Hence

$$F_{\mathrm{cons}}(X_{\mathrm{cam}}, X_{\mathrm{nav}}, X_{\mathrm{pol}}) = \big\{ \{\mathrm{R2}\},\ \{\mathrm{R1}, \mathrm{R3}, \mathrm{R4}\} \big\},$$

$$F_{\mathrm{pairs}}(X_{\mathrm{cam}}, X_{\mathrm{nav}}, X_{\mathrm{pol}}) = \big\{ \{\mathrm{R2}\} \big\}.$$

Therefore $\mathsf{SHG}^{(h,k)}_{(m,n)} = (V, \mathcal{E})$ concretely captures multi-source incident fusion with outputs as detour bundles in $\mathcal{P}_2(S)$; their combined use is encoded by the hyperedge $\{F_{\mathrm{cons}}, F_{\mathrm{pairs}}\}$.

7.2 Fuzzy (m, n)-SuperHyperGraph

In this section we give a mathematically precise definition of a *Fuzzy (m, n)-SuperHyperGraph* and prove that it simultaneously generalizes (i) a fuzzy n-SuperHyperGraph and (ii) a (crisp) (m, n)-SuperHyperGraph (Definition 7.1.3). Throughout, $S \neq \varnothing$ is a fixed base set and $m, n \in \mathbb{N}$ are fixed levels. We use the iterated powerset notation

$$\mathcal{P}_0(S) := S, \qquad \mathcal{P}_{t+1}(S) := \mathcal{P}\big(\mathcal{P}_t(S)\big) \quad (t \in \mathbb{N}_0).$$

Vertices of a (m, n)-SuperHyperGraph are (m, n)-superhyperfunctions $f : \mathcal{P}_m(S) \to \mathcal{P}_n(S)$ (Definition 7.1.2).

Definition 7.2.1 (Fuzzy (m, n)-SuperHyperGraph)**.** Let $\mathsf{SHG}^{(m,n)} = (V, \mathcal{E})$ be a (crisp) (m, n)-SuperHyperGraph (Definition 7.1.3) with nonempty vertex set $V \subseteq \mathrm{Hom}\big(\mathcal{P}_m(S), \mathcal{P}_n(S)\big)$ and nonempty family of nonempty hyperedges $\varnothing \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$. A *Fuzzy (m, n)-SuperHyperGraph on* $(V, \mathcal{E})$ is a quadruple

$$\mathfrak{F} \ := \ (V,\ \mathcal{E};\ \sigma,\ \mu),$$

where

$$\sigma : V \to [0, 1] \quad \text{(vertex-membership)}, \qquad \mu : \mathcal{E} \to [0, 1] \quad \text{(hyperedge-membership)}$$

satisfy the *admissibility (edge–vertex) constraint*

$$\forall\, E \in \mathcal{E} : \qquad \mu(E) \ \leq \ \min_{v \in E} \ \sigma(v). \tag{7.1}$$

Equivalently, $\mu(E) \leq \sigma(v)$ for every incident pair (v, E) with $v \in E$.

Optionally, one may introduce an *incidence-membership* map $\eta : V \times \mathcal{E} \to [0, 1]$ by the canonical choice

$$\eta(v, E) \ := \ \begin{cases} \mu(E), & v \in E, \\ 0, & v \notin E, \end{cases} \tag{7.2}$$

which then satisfies the support equivalence $[\, v \in E\,] \iff \eta(v, E) > 0$ and the bounds $\eta(v, E) \leq \min\{\sigma(v), \mu(E)\}$ together with (7.1).

Remark 7.2.2 (Simple/uniform restrictions)**.** A Fuzzy (m, n)-SuperHyperGraph is called *simple* if $\mathcal{E}$ has no parallel hyperedges (i.e. distinct $E, F \in \mathcal{E}$ have $E \neq F$), and *k-uniform* if $|E| = k$ for every $E \in \mathcal{E}$. Both restrictions are orthogonal to fuzziness and can be assumed when convenient.

For reference, a comparison between the (m, n)-SuperHyperGraph and the Fuzzy (m, n)-SuperHyperGraph is presented in Table 7.3.

Several illustrative examples are presented below.

Example 7.2.3 (E–commerce recommendation as a Fuzzy (m, n)-SuperHyperGraph with $(m, n) = (1, 1)$)**.** Let the base (item) set be

$$S := \{\text{Laptop},\ \text{Phone},\ \text{Headphones},\ \text{Charger},\ \text{Case}\}.$$

Here $\mathcal{P}_1(S) = \mathcal{P}(S)$ and $\mathcal{P}_n(S) = \mathcal{P}(S)$ for $n = 1$. A vertex $f \in V$ is a concrete recommender

$$f :\ \mathcal{P}(S) \longrightarrow \mathcal{P}(S).$$

Table 7.3: Compact comparison: (m, n)-SuperHyperGraph vs. Fuzzy (m, n)-SuperHyperGraph.

Aspect	(m, n)-SuperHyperGraph (crisp)	Fuzzy (m, n)-SuperHyperGraph
Base data	A pair $\mathsf{SHG}^{(m,n)} = (V, \mathcal{E})$ with $V \subseteq \mathrm{Hom}(\mathcal{P}_m(S), \mathcal{P}_n(S))$ and $\varnothing \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$.	A quadruple $\mathfrak{F} = (V, \mathcal{E}; \sigma, \mu)$ on a fixed $(V, \mathcal{E})$, where $\sigma : V \to [0, 1]$ and $\mu : \mathcal{E} \to [0, 1]$.
Vertices	Elements $v \in V$ are concrete (m, n)-superhyperfunctions $v : \mathcal{P}_m(S) \to \mathcal{P}_n(S)$.	Same vertex set V, but each vertex has a membership degree $\sigma(v) \in [0, 1]$.
Hyperedges	Each hyperedge is a nonempty set $E \in \mathcal{E}$ with $E \subseteq V$.	Same hyperedge family $\mathcal{E}$, but each hyperedge has a membership degree $\mu(E) \in [0, 1]$.
Constraints	No membership constraints (crisp incidence is $v \in E$).	Admissibility (edge–vertex) constraint: $\mu(E) \leq \min_{v \in E} \sigma(v)$ for all $E \in \mathcal{E}$ (Definition 7.2.1).
Incidence viewpoint	Incidence is boolean: v is incident to E iff $v \in E$.	Incidence can be encoded by the canonical map $\eta(v, E) = \mu(E)$ if $v \in E$, and 0 otherwise (optional), consistent with the admissibility bound.
Crisp reduction	Already crisp.	If σ, μ take only values in $\{0, 1\}$ (and $\mu(E) = 1$ implies $\sigma(v) = 1$ for all $v \in E$), then $\mathfrak{F}$ reduces to a crisp (m, n)-SuperHyperGraph on the support.
Modeling intent	Encodes higher-order relations among concrete superhyperfunctions without uncertainty.	Encodes graded reliability/importance of superhyperfunctions and of their grouped relations via (σ, μ).

Define two recommendation functions (vertices):

$$f_{\text{top}}(X) := \{\text{Phone}, \text{Headphones}\} \setminus X \quad \text{(bestseller filter, excluding already owned)},$$

$$f_{\text{cb}}(X) := \Big(X \cup \bigcup_{x \in X} \text{sim}(x) \Big) \setminus X \quad \text{(content–based expansion, then exclude } X\text{)},$$

where the similarity dictionary is fixed as

$$\text{sim}(\text{Laptop}) = \{\text{Charger}\}, \ \text{sim}(\text{Phone}) = \{\text{Charger}, \text{Case}\}, \ \text{sim}(\text{Headphones}) = \{\text{Case}\},$$

$$\text{sim}(\text{Charger}) = \{\text{Laptop}, \text{Phone}\}, \ \text{sim}(\text{Case}) = \{\text{Phone}, \text{Headphones}\}.$$

Set the vertex set and hyperedge family

$$V := \{f_{\text{top}}, \ f_{\text{cb}}\}, \qquad \mathcal{E} := \big\{\{f_{\text{top}}\}, \ \{f_{\text{cb}}\}, \ \{f_{\text{top}}, f_{\text{cb}}\}\big\}.$$

Assign fuzzy memberships (model credibilities)

$$\sigma(f_{\text{top}}) = 0.86, \qquad \sigma(f_{\text{cb}}) = 0.74,$$

and hyperedge-memberships

$$\mu\big(\{f_{\text{top}}\}\big) = 0.86, \quad \mu\big(\{f_{\text{cb}}\}\big) = 0.74, \quad \mu\big(\{f_{\text{top}}, f_{\text{cb}}\}\big) = 0.74.$$

Admissibility (edge–vertex) constraints (7.1) hold numerically:

$$\mu(\{f_{\text{top}}\}) = 0.86 \leq \min\{0.86\} = 0.86, \quad \mu(\{f_{\text{cb}}\}) = 0.74 \leq \min\{0.74\} = 0.74,$$

$$\mu(\{f_{\text{top}}, f_{\text{cb}}\}) = 0.74 \leq \min\{0.86, 0.74\} = 0.74.$$

Concrete input–output illustration. For $X = \{\text{Laptop}\}$,

$$f_{\text{top}}(X) = \{\text{Phone}, \text{Headphones}\}, \qquad f_{\text{cb}}(X) = \{\text{Charger}\}.$$

With the canonical incidence-membership (7.2), we have, e.g., $\eta\big(f_{\text{top}}, \{f_{\text{top}}, f_{\text{cb}}\}\big) = 0.74$ and $\eta\big(f_{\text{cb}}, \{f_{\text{top}}\}\big) = 0$. Thus $\mathfrak{F} = (V, \mathcal{E}; \sigma, \mu)$ is a valid Fuzzy $(1, 1)$-SuperHyperGraph for retail recommendations.

Example 7.2.4 (Smart-grid demand response as a Fuzzy (m, n)-SuperHyperGraph with $(m, n) = (1, 2)$)**.** Let the base (device) set be

$$S := \{\text{HVAC}, \ \text{EV}, \ \text{Washer}, \ \text{Dryer}\}.$$

Here $\mathcal{P}_1(S) = \mathcal{P}(S)$ is the set of currently active devices, and $\mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S))$ is the set of *curtailment group candidates* (each element is a subset of devices). A vertex $f \in V$ maps an active-device set $X \subseteq S$ to a family of candidate curtailment groups $f(X) \in \mathcal{P}_2(S)$:

$$f : \mathcal{P}(S) \longrightarrow \mathcal{P}(\mathcal{P}(S)).$$

Define two operational policies (vertices):

$$f_{\text{peak}}(X) := \{ \{\text{HVAC}\}, \{\text{EV}\}, X \cap \{\text{HVAC}, \text{EV}\} \} \setminus \{\varnothing\},$$
$$f_{\text{balance}}(X) := \{ \{\text{Washer}, \text{Dryer}\}, X \cap \{\text{HVAC}, \text{Washer}\}, \{\text{EV}, \text{Washer}\} \} \setminus \{\varnothing\}.$$

Take

$$V := \{f_{\text{peak}}, f_{\text{balance}}\}, \qquad \mathcal{E} := \{\{f_{\text{peak}}\}, \{f_{\text{balance}}\}, \{f_{\text{peak}}, f_{\text{balance}}\}\}.$$

Assign memberships (policy reliabilities)

$$\sigma(f_{\text{peak}}) = 0.83, \qquad \sigma(f_{\text{balance}}) = 0.68,$$

and hyperedge-memberships

$$\mu(\{f_{\text{peak}}\}) = 0.83, \quad \mu(\{f_{\text{balance}}\}) = 0.68, \quad \mu(\{f_{\text{peak}}, f_{\text{balance}}\}) = 0.68.$$

Admissibility checks:

$$\mu(\{f_{\text{peak}}\}) = 0.83 \le \min\{0.83\} = 0.83, \qquad \mu(\{f_{\text{balance}}\}) = 0.68 \le \min\{0.68\} = 0.68,$$
$$\mu(\{f_{\text{peak}}, f_{\text{balance}}\}) = 0.68 \le \min\{0.83, 0.68\} = 0.68.$$

Concrete input–output illustration. For the active set $X = \{\text{HVAC}, \text{Washer}\}$,

$$f_{\text{peak}}(X) = \{ \{\text{HVAC}\}, \{\text{EV}\}, \{\text{HVAC}\} \} = \{ \{\text{HVAC}\}, \{\text{EV}\} \},$$
$$f_{\text{balance}}(X) = \{ \{\text{Washer}, \text{Dryer}\}, \{\text{HVAC}, \text{Washer}\}, \{\text{EV}, \text{Washer}\} \}.$$

With the canonical incidence-membership (7.2), $\eta(f_{\text{peak}}, \{f_{\text{peak}}, f_{\text{balance}}\}) = 0.68$ and $\eta(f_{\text{balance}}, \{f_{\text{peak}}\}) = 0$, matching support. Hence $\mathfrak{F} = (V, \mathcal{E}; \sigma, \mu)$ is a Fuzzy $(1, 2)$-SuperHyperGraph modeling demand-response policies with quantified confidence.

Theorem 7.2.5 (Embedding of fuzzy n-SuperHyperGraphs)**.** *Fix $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Every fuzzy n-SuperHyperGraph $(W, \mathcal{F}; \sigma_n, \mu_n)$ is (isomorphic to) a Fuzzy (m, n)-SuperHyperGraph as in Definition 7.2.1. Concretely, define the* constant superhyperfunction embedding

$$\iota : W \hookrightarrow \mathrm{Hom}(\mathcal{P}_m(S), \mathcal{P}_n(S)), \qquad \iota(A)(X) := A \quad (A \in W, X \in \mathcal{P}_m(S)).$$

Set

$$V' := \iota(W), \qquad \mathcal{E}' := \{ \iota[F] \mid F \in \mathcal{F} \} \subseteq \mathcal{P}(V') \setminus \{\varnothing\},$$

and define $\sigma' : V' \to [0, 1]$, $\mu' : \mathcal{E}' \to [0, 1]$ by pullback:

$$\sigma'(\iota(A)) := \sigma_n(A), \qquad \mu'(\iota[F]) := \mu_n(F).$$

Then $\mathfrak{F}' := (V', \mathcal{E}'; \sigma', \mu')$ is a Fuzzy (m, n)-SuperHyperGraph and the map ι extends to an isomorphism of fuzzy structures between $(W, \mathcal{F}; \sigma_n, \mu_n)$ and $\mathfrak{F}'$.

Proof. The map ι is injective by definition of constants. For $E' = \iota[F] \in \mathcal{E}'$ we have

$$\min_{v' \in E'} \sigma'(v') = \min_{\iota(A) \in \iota[F]} \sigma'(\iota(A)) = \min_{A \in F} \sigma_n(A).$$

Hence the admissibility constraint for $\mathfrak{F}'$ is

$$\mu'(E') = \mu_n(F) \le \min_{A \in F} \sigma_n(A) = \min_{v' \in E'} \sigma'(v'),$$

. Thus $\mathfrak{F}'$ is a Fuzzy (m, n)-SuperHyperGraph. The assignments $W \ni A \mapsto \iota(A) \in V'$ and $\mathcal{F} \ni F \mapsto \iota[F] \in \mathcal{E}'$ yield a bijection preserving incidence, and the pullback definitions of σ', μ' ensure preservation of membership values. Therefore the two fuzzy structures are isomorphic. □

We now show that the crisp structure is obtained as the $\{0,1\}$-valued special case.

Theorem 7.2.6 (Crisp (m,n)-SuperHyperGraphs are $\{0,1\}$-valued fuzzy ones)**.** *Let* $\mathsf{SHG}^{(m,n)} = (V,\mathcal{E})$ *be a (crisp)* (m,n)*-SuperHyperGraph. Define*

$$\sigma(v) := 1 \quad (\forall v \in V), \qquad \mu(E) := 1 \quad (\forall E \in \mathcal{E}).$$

Then $\mathfrak{F} := (V,\mathcal{E};\sigma,\mu)$ *is a Fuzzy* (m,n)*-SuperHyperGraph. Moreover, if one also defines* η *by* (7.2)*, then*

$$\eta(v,E) \;=\; \begin{cases} 1, & v \in E, \\ 0, & v \notin E, \end{cases}$$

so the fuzzy incidence reduces to the crisp incidence relation.

Proof. For any $E \in \mathcal{E}$ we have $\min_{v\in E}\sigma(v) = \min_{v\in E} 1 = 1$, hence (7.1) becomes $\mu(E) \leq 1$, which is satisfied since $\mu(E) = 1$. The formula for η is immediate from (7.2). □

Chapter 8

SuperHyperStructure

Hyper- and SuperHyper-based viewpoints are not confined to graph models; they can be transferred to many mathematical and real-world formalisms. In this chapter, we formalize the notions of *HyperStructure* and *SuperHyperStructure* and outline several basic properties that motivate their use. Both *HyperStructure* and *SuperHyperStructure* can be transformed into graph-based models, such as HyperGraphs (including directed hypergraphs, bidirected hypergraphs, etc.) and SuperHyperGraphs (including directed SuperHyperGraphs, bidirected SuperHyperGraphs, etc.). Table 8.1 presents an overview of Structure, Hyperstructure, n-SuperHyperstructure, and (m, n)-SuperHyperStructure.

Moreover, for reference, Tables 8.2, 8.3, and 8.4 present a comparison of classical, hyper-, and superhyper-viewpoints across common domains. Research on HyperStructures and SuperHyperStructures is important in that it has the potential to clarify the mathematical characteristics of such conceptual extensions and their interrelationships.

8.1 HyperStructure

We use the phrase *classical structure* for a standard mathematical (or application-driven) framework arising, for example, in logic, probability, statistics, algebra, geometry, graph theory, automata, or game theory [1200,1205,1206]. A *hyperstructure* broadens such a framework by replacing a base set S with its powerset $\mathcal{P}(S)$ and by permitting *hyperoperations* whose outputs are subsets rather than single elements. This replacement makes it possible to encode multi-valued and higher-order interactions among elements [1207–1209]. We now state precise definitions.

Definition 8.1.1 (Classical Structure)**.** [814] A *Classical Structure* is a structured object from a standard domain such as set theory, logic, probability, statistics, algebra, geometry, graph theory, automata theory, or game theory. Formally, it is a pair

$$C = \big(H, \{\#^{(m)}\}_{m \in \mathcal{I}}\big),$$

where:

- H is a nonempty *carrier* (underlying) set;
- for each $m \in \mathcal{I} \subseteq \mathbb{Z}_{>0}$, there is an m-ary operation $\#^{(m)} : H^m \to H$ satisfying the axioms required by the intended context (e.g., associativity, commutativity, identity elements, inverses, etc.).

We say that C is of *type* $\{\#^{(m)} : m \in \mathcal{I}\}$. Representative examples include:

- *Set*: $(S, \varnothing)$, where S may additionally carry designated elements, relations, or constants [105].
- *Logic*: $(L, \wedge, \vee, \neg)$ with binary connectives $\wedge, \vee$ and unary negation $\neg$, satisfying the usual logical laws [1210].

Table 8.1: Concise overview of Structure, Hyperstructure, n-SuperHyperstructure, and (m,n)-SuperHyperStructure.

Model	Carrier	Operation(s) and type	Main viewpoint / typical use
Structure	Nonempty set H.	m-ary operations $\#^{(m)} : H^m \to H$ ($m \in \mathcal{I} \subseteq \mathbb{Z}_{>0}$). Single-valued outputs.	Base-level single-valued interactions (algebra, logic, probability, graphs, automata, games, etc.).
Hyperstructure	Powerset $\mathcal{P}(S)$ (often nonempty subsets).	Set-valued operation, e.g. $\circ : S \times S \to \mathcal{P}(S)$, or an operation on subsets $\circ : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$.	Multi-valued interactions: outputs are subsets rather than single elements.
n-SuperHyperstructure	Iterated powerset $\mathcal{P}^n(S)$ ($n \geq 1$).	Operation on the n-level layer, e.g. $\circ : (\mathcal{P}^n(S))^s \to \mathcal{P}^n(S)$ ($s \geq 1$).	Nested (hierarchical) collections up to depth n; layered relations or hierarchical uncertainty.
(m,n)-SuperHyperStructure	Domain layer $\mathcal{P}^m(S)$ and codomain layer $\mathcal{P}^n(S)$.	Superhyper-operation (arity s): $\odot^{(m,n)} : (\mathcal{P}^m(S))^s \to \mathcal{P}^n(S)$ ($m,n \geq 0$). Special cases: $(0,0)$ classical; $(0,1)$ hyper; (m,m) within-layer.	Unified viewpoint: classical/hyper/iterated layers and cross-level mappings between them.

- *Probability*: $(\Omega, \mathcal{F}, P)$, where $P : \mathcal{F} \to [0,1]$ is a probability measure on a σ-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ [1211].
- *Statistics*: $(X, \mathcal{A}, \theta)$, where θ associates observed data (or statistics) to parameters of interest [1212].
- *Algebra*:
  - *Group* $(G, *)$ with binary operation $* : G \times G \to G$ satisfying the group axioms [1213, 1214];
  - *Ring* $(R, +, \times)$ with two binary operations satisfying the ring axioms [1196, 1215];
  - *Vector space* $(V, +, \cdot)$ over a field $\mathbb{F}$ with scalar multiplication $\cdot : \mathbb{F} \times V \to V$ [1216, 1217].
- *Geometry*: (X, dist) with a metric $\mathrm{dist} : X \times X \to \mathbb{R}$ [1218, 1219].
- *Graph*: (V, E) with $E \subseteq \{\{u,v\} : u, v \in V\}$ in the undirected case, or $E \subseteq V \times V$ in the directed case; adjacency and incidence are defined in the usual way [1, 17].
- *Automaton*: $(Q, \Sigma, \delta, q_0, F)$, where Q is a finite state set, Σ is the input alphabet, $\delta : Q \times \Sigma \to Q$ is the transition function, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of accepting states [1220, 1221].
- *Game*: $(N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N})$, where N is the player set, A_i is the action set of player i, and $u_i : \prod_{j \in N} A_j \to \mathbb{R}$ is the payoff function of player i [1222, 1223].

Definition 8.1.2 (Hyperoperation)**.** *(cf. [1224, 1225])* Let S be a set. A *hyperoperation* on S is a binary rule whose output is a subset of S (possibly a singleton), namely a map

$$\circ : S \times S \longrightarrow \mathcal{P}(S).$$

Thus for $(x,y) \in S \times S$, the value $x \circ y$ is a (possibly multi-element) subset of S.

Definition 8.1.3 (Hyperstructure)**.** *(cf. [1198, 1226, 1227])* A *Hyperstructure* is obtained by lifting a classical structure to the powerset level. Concretely, it can be presented in the form

$$\mathcal{H} = \big(\mathcal{P}(S), \circ\big),$$

where $\circ$ is a hyperoperation on S (or, more generally, an operation acting on subsets). In this setting, the basic operations take collections of elements as inputs and return collections as outputs, thereby encoding multi-valued interactions.

Example 8.1.4 (Hyperstructure induced by a group operation)**.** Let $(G, *)$ be a (classical) group with carrier set $G \neq \emptyset$ and binary operation $* : G \times G \to G$. Define a hyperoperation

$$\circ : \mathcal{P}(G) \times \mathcal{P}(G) \longrightarrow \mathcal{P}(G)$$

Table 8.2: Keyword-style comparison of classical, hyper-, and superhyper- viewpoints across common domains (Part 1).

Domain	Classical	Hyper-	SuperHyper-
Graph theory	Graph [3]	Hypergraph [17, 1081, 1086]	n-SuperHyperGraph [2]
Directed graphs	Digraph [468]	DiHypergraph [1087, 1088]	DiSuperHyperGraph [801]
Functions	Function [1089, 1090]	Hyperfunction [115]	SuperHyperfunction [115, 1091]
Algebra	Algebra [1092, 1093]	Hyperalgebra [1094, 1095]	SuperHyperalgebra [1096–1098]
Topology	Topology [1099, 1100]	Hypertopology [1101]	SuperHypertopology [1102, 1103]
Groups	Group [1104, 1105]	Hypergroup [1106, 1107]	SuperHypergroup [1108]
Games	Game [1109, 1110]	Hypergame [1111]	SuperHypergame [1111]
Probability	Probability [1112, 1113]	Hyperprobability [1114]	SuperHyperprobability [1114]
Entropy	Entropy [1115, 1116]	Hyperentropy [1117]	SuperHyperentropy [1117]
Chemistry	Chemical structure [1118, 1119]	Chemical hyperstructure [1120–1122]	Chemical superhyperstructure [1123]
Automata	Automaton [1124, 1125]	Hyperautomaton [1126, 1127]	Superhyperautomaton [1126, 1127]

by the *setwise product*

$$A \circ B \ := \ \{\, a * b : \ a \in A, \ b \in B \,\} \subseteq G \qquad (A, B \subseteq G).$$

Then $\mathcal{H} = \big(\mathcal{P}(G), \circ\big)$ is a hyperstructure in the sense of Definition *(Hyperstructure)*.

Indeed, the carrier is the powerset $\mathcal{P}(G)$, and the operation $\circ$ takes two collections of group elements and returns the collection of all possible products formed by choosing one element from each input set. For instance, if $A = \{g_1, g_2\}$ and $B = \{h_1\}$, then

$$A \circ B = \{g_1 * h_1, \ g_2 * h_1\}.$$

Thus the operation encodes a multi-valued interaction: a single pair of sets (A, B) produces (in general) many outcomes in G.

Example 8.1.5 (Reachability hyperstructure on a directed graph)**.** Let $D = (V, E)$ be a directed graph with vertex set $V \neq \emptyset$ and edge set $E \subseteq V \times V$. For $u \in V$, write

$$N^+(u) := \{\, v \in V : \ (u, v) \in E \,\}$$

for the (out-)neighborhood of u. Define a hyperoperation

$$\circ : \mathcal{P}(V) \times \mathcal{P}(V) \longrightarrow \mathcal{P}(V)$$

by

$$A \circ B \ := \ \{\, v \in V : \ \exists u \in A \text{ with } v \in N^+(u) \,\} \ \cup \ B \qquad (A, B \subseteq V).$$

Equivalently, $A \circ B$ consists of all vertices reachable from A by one directed edge, together with all vertices already listed in B.

Then $\mathcal{H} = \big(\mathcal{P}(V), \circ\big)$ is a hyperstructure: the carrier is $\mathcal{P}(V)$, and $\circ$ maps a pair of vertex-collections to a new vertex-collection. Concretely, if $A = \{u_1, u_2\}$ and $B = \{w\}$, then

$$A \circ B = N^+(u_1) \cup N^+(u_2) \cup \{w\},$$

which aggregates multiple possible next-step vertices at once. Hence $\circ$ captures multi-valued dynamics (many possible next states) in a single set-valued operation.

Table 8.3: Keyword-style comparison across additional domains (Part 2).

Domain	Classical	Hyper-	SuperHyper-
Language	Language structure [1128, 1129]	Hyperlanguage structure [1130, 1131]	SuperHyperlanguage structure [1130, 1131]
Geometry [1132, 1133]	Geometric structure	Hypergeometric structure [1134]	SuperHypergeometric structure [1135, 1136]
Meta	Meta-structure [1137]	Meta-hyperstructure [1137]	Meta-superhyperstructure [1137]
Medicine [1138]	Medical structure	Medical hyperstructure [39]	Medical Superhyperstructure [39]
Randomness [1139]	Random structure	Hyperrandom structure [1140]	SuperHyperrandom structure [1140]
Decision-making [1141, 1142]	Decision structure [1143]	Hyperdecision structure [1144]	SuperHyperdecision structure [1144]
Category theory [1145]	Category [1146, 1147]	Hypercategory [1148]	SuperHypercategory [1148]
Variables	Variable	Hypervariable [1140]	SuperHypervariable [1140]
Integral	Integral structure [1149, 1150]	Hyperintegral structure [1151]	SuperHyperintegral structure [1151]
Space	Space	Hyperspace [1151]	SuperHyperspace [1151]

8.2 SuperHyperStructure

A *SuperHyperStructure* strengthens the above idea by iterating the powerset construction a prescribed number of times. One then works with nested families of subsets, and operations act across these nested tiers, which naturally models hierarchical and multi-layer interactions [114, 1228, 1229]. This viewpoint supports notions such as SuperHyperAlgebra [1230, 1231], SuperHyperGraph [16, 107, 1232], and other super-level algebraic and combinatorial systems.

Definition 8.2.1 (SuperHyperOperations)**.** *(cf. [1198])* Let $H \neq \emptyset$, and let $\mathcal{P}^k(H)$ denote the k-fold iterated powerset of H (as defined in the preliminaries). An (m, n)*-SuperHyperOperation* is an m-ary mapping

$$\circ^{(m,n)} : H^m \longrightarrow \mathcal{P}^n_*(H),$$

where $\mathcal{P}^n_*(H)$ denotes either the full n-th iterated powerset $\mathcal{P}^n(H)$ or the corresponding nonempty subfamily $\mathcal{P}^n(H)\setminus\{\emptyset\}$. If $\emptyset$ is excluded we call the operation *classical-type*, whereas allowing $\emptyset$ yields a *neutrosophic-type* SuperHyperOperation.

Definition 8.2.2 (n-Superhyperstructure)**.** *(cf. [114, 1198, 1233, 1234])* Let $S \neq \emptyset$ and fix $n \in \mathbb{N}$. An *n-Superhyperstructure* on S is a hyperstructure whose carrier is the n-fold iterated powerset:

$$\mathcal{SH}_n = \big(\mathcal{P}^n(S), \circ\big),$$

where $\circ$ is an operation (possibly multi-valued) acting on $\mathcal{P}^n(S)$. Equivalently, $\mathcal{SH}_n$ records interactions among nested collections of subsets up to depth n.

Definition 8.2.3 (SuperHyperStructure of order (m, n))**.** *(cf. [115, 1229])* Let $S \neq \emptyset$ and let $m, n \geq 0$. A (m, n)*-SuperHyperStructure* (of arity s) is specified by a map

$$\odot^{(m,n)} : \big(\mathcal{P}^m(S)\big)^s \longrightarrow \mathcal{P}^n(S).$$

This unifies several standard situations:

- $m = n = 0$ recovers ordinary s-ary operations on S;
- $m = 0$ and $n = 1$ yields (set-valued) hyperoperations on S;
- $s = 1$ gives a (super)operation acting on a single input from $\mathcal{P}^m(S)$.

Table 8.4: Keyword-style comparison across uncertainty/set- and algebra-oriented domains (Part 3).

Domain	Classical	Hyper-	SuperHyper-
Fuzzy set	Fuzzy set [746, 1152]	Hyperfuzzy set [1153–1155]	SuperHyperfuzzy set [1156–1158]
Weighted set	Weighted set	Hyperweighted set [372, 1159]	SuperHyperweighted set [372, 1159]
Vague set	Vague set [1160, 1161]	Hypervague set [1162]	SuperHypervague set [1162]
Neutrosophic set	Neutrosophic set [800, 831, 1163]	Hyperneutrosophic set [1164, 1165]	SuperHyperneutrosophic set [1158, 1166]
Plithogenic set	Plithogenic set [809, 1167]	Hyperplithogenic set [1168–1170]	SuperHyperplithogenic set [1158, 1166]
Uncertain set	Uncertain set [814]	Hyperuncertain set [814, 1158]	SuperHyperuncertain set [814, 1158]
Rough sets	Rough set [906, 1171]	Hyperrough set [1166, 1172, 1173]	SuperHyperrough set [1166, 1174]
Soft sets	Soft set [890, 891]	Hypersoft set [1166, 1175]	SuperHypersoft set [1166, 1176–1178]
Z-number	Z-number [1179–1181]	Hyper Z-number [1182]	SuperHyper Z-number [1182]
Partition	Partition	Hyperpartition [1151]	SuperHyperpartition [1151]
Matrix	Matrix [1183–1185]	Hypermatrix [1140, 1186]	SuperHypermatrix [1140, 1187]
Cognitive map	Cognitive map [1188–1190]	Cognitive Hypermap [1140]	Cognitive SuperHyper-map [1140]
Floorplan	Floorplan [1191, 1192]	Hyperfloorplan [1193]	SuperHyperfloorplan [1193]
Code	Code	Hypercode [1193]	SuperHypercode [1193]
Ring	Ring [1194, 1195]	Hyperring [1196, 1197]	SuperHyperring [1198]
Field	Field [1199]	HyperField [1197, 1200, 1201]	SuperHyperField [1140]
Lattice	Lattice [546, 547, 1202]	Hyperlattice [1140,1203,1204]	SuperHyperlattice [1140]

Hence (m, n)-SuperHyperStructures provide a common language for classical, hyper-, and multi-level super-hyper constructions.

Example 8.2.4 (A $(0, 2)$-SuperHyperStructure: "two-level recommendation bundle")**.** Let S be a nonempty set of items (e.g., products in an online catalog). We define a $(0, 2)$-SuperHyperStructure of arity $s = 2$ by the mapping

$$\odot^{(0,2)} : S^2 \longrightarrow \mathcal{P}^2(S), \qquad (x, y) \longmapsto \big\{\{x\}, \{y\}, \{x, y\}\big\}.$$

Here $\mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S))$ is the set of *families* of subsets of S. For an ordered pair (x, y), the output

$$\odot^{(0,2)}(x, y) = \big\{\{x\}, \{y\}, \{x, y\}\big\} \in \mathcal{P}(\mathcal{P}(S))$$

is a two-level object: it is a *set whose elements are subsets of* S. In words, the operation returns a nested bundle consisting of the singleton recommendations $\{x\}$ and $\{y\}$ and the joint bundle $\{x, y\}$. Therefore $\odot^{(0,2)}$ is a concrete example of a SuperHyperStructure map of order $(m, n) = (0, 2)$.

(If one wishes to exclude the empty set at every level, one may instead regard the codomain as $\mathcal{P}^2(S) \setminus \{\emptyset\}$; the above output is already nonempty.)

Example 8.2.5 (A $(1, 1)$-SuperHyperStructure: "closure under pairwise fusion")**.** Let S be a nonempty set, and consider the arity $s = 2$ mapping

$$\odot^{(1,1)} : \big(\mathcal{P}(S)\big)^2 \longrightarrow \mathcal{P}(S), \qquad (A, B) \longmapsto A \cup B \cup (A * B),$$

where $*$ is an auxiliary operation on subsets defined by

$$A * B := \{\, a *_S b : \ a \in A, \ b \in B \,\} \subseteq S,$$

and $*_S : S \times S \to S$ is any fixed binary operation on S (for instance, addition on a group, multiplication on a ring, or concatenation on a free monoid).

Then, for any $A, B \subseteq S$, the value $\odot^{(1,1)}(A, B)$ is a subset of S, hence an element of $\mathcal{P}(S)$. The three components have the following meanings:

$$A \quad \text{(keep the first input)}, \qquad B \quad \text{(keep the second input)}, \qquad A * B \quad \text{(all pairwise fusions under } *_S \text{)}.$$

Thus $\odot^{(1,1)}$ takes *two first-level collections* (elements of $\mathcal{P}(S)$) and returns a *first-level collection* (again in $\mathcal{P}(S)$) obtained by closing under pairwise fusion and union. This is a concrete SuperHyperStructure of order $(m, n) = (1, 1)$, and it illustrates that (m, n)-SuperHyperStructures can model operations acting directly on sets of objects, not only on individual objects.

Chapter 9

Conclusion

In this book, we extended the framework by introducing *Fuzzy* (m, n)*-SuperHyperGraphs*, where fuzzy memberships are incorporated into both vertices and hyperedges. For future work, we aim to explore further generalizations using other fuzzy and uncertainty-based graph models, such as Intuitionistic Fuzzy Graphs [1235, 1236], Hesitant Fuzzy Graphs [754, 1237], Bipolar Fuzzy Graphs [758, 1238], Neutrosophic Graphs [64, 65, 1239], and Plithogenic Graphs [69, 1240]. These directions hold promise for developing richer models capable of handling diverse forms of uncertainty and contradiction in real-world applications.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

Appendix A

Appendix: Multi-Intersection Graph

A multi-intersection graph is the intersection graph of a multiset of sets, allowing repeated identical sets as distinct vertices [1241].

Definition A.0.1 (Intersection graph of a finite family)**.** Let S be a set and let $\mathcal{F}$ be a finite family of subsets of S. The *intersection graph* of $\mathcal{F}$, denoted by $(\mathcal{F})$, is the (simple) graph with vertex set $V\big((\mathcal{F})\big) = \mathcal{F}$, and with an edge AB between two distinct vertices $A, B \in \mathcal{F}$ if and only if $A \cap B \neq \emptyset$.

Definition A.0.2 (Intersection graph of a finite multiset)**.** Let S be a set and let F be a finite *multiset* [1242] of subsets of S. Formally, write F as a finite list

$$F = \{A_1, \dots, A_N\} \quad (A_i \subseteq S),$$

allowing repetitions. The *intersection graph* (F) is the simple graph with vertex set

$$V\big((F)\big) = \{1, \dots, N\},$$

and an edge ij for $i \neq j$ if and only if $A_i \cap A_j \neq \emptyset$. (Thus, if $A_i = A_j \neq \emptyset$ with $i \neq j$, then ij is an edge.)

Definition A.0.3 (Reduction of a multiset)**.** Let F be a finite multiset of subsets of a set S. The *reduction* of F, denoted by $[F]$, is the set obtained from F by keeping exactly one representative of each distinct subset occurring in F. Equivalently, $[F]$ is the set of distinct elements appearing in the multiset F.

Definition A.0.4 (Multi-intersection graph classes P_m and P^*)**.** [1241] Fix a hereditary class $P = P_1$ of intersection graphs, described by a hereditary predicate on finite set families (equivalently: P is closed under induced subgraphs). Let F be a finite multiset of subsets of some ground set.

1. P^* is the class of all graphs (F) such that $[F]$ belongs to the underlying family domain and the reduced family satisfies the predicate of P, i.e. $P([F]) = 1$.
2. For an integer $m \geq 1$, P_m is the class of all graphs (F) such that every subset appears in F with multiplicity at most m, and $P([F]) = 1$.

Graphs in P^* (resp. P_m) are called *multi-intersection graphs* (resp. *m-multi-intersection graphs*) with respect to the base class P.

Remark A.0.5 (Equivalent "vertex duplication" view)**.** Let P be a hereditary class of graphs. A graph lies in P_m exactly when it can be obtained from some $G \in P$ by replacing each vertex $v \in V(G)$ by a clique of size $m(v)$ with $1 \leq m(v) \leq m$, and keeping adjacency between cliques according to adjacency of the original vertices. For P^* one allows arbitrary finite clique sizes.

As one example, we extend string graphs and interval graphs to multi-string graphs and multi-interval graphs. A string graph is the intersection graph of a family of curves in the plane: each vertex represents one curve, and two vertices are adjacent exactly when the corresponding curves intersect [509, 1243–1245]. As superclasses of string graphs, families such as VPG [1246] and k-SEG graphs [1247] are known. As subclasses of string graphs, planar graphs and circular permutation graphs [1248] are known. An interval graph is the intersection graph of a family of intervals on the real line: each vertex represents one interval, and two vertices are adjacent exactly when the corresponding intervals overlap (have nonempty intersection).

Definition A.0.6 (String and string graph)**.** [509, 1243–1245] A *string* is a continuous map $\gamma : [0,1] \to \mathbb{R}^2$; we identify it with its image $\gamma([0,1]) \subseteq \mathbb{R}^2$. A simple graph $G = (V, E)$ is a *string graph* if there exists a family of strings $\{\gamma_v\}_{v \in V}$ in the plane such that for all distinct $u, v \in V$,

$$uv \in E \iff \gamma_u([0,1]) \cap \gamma_v([0,1]) \neq \emptyset.$$

Definition A.0.7 (MultiString graph)**.** A simple graph G is a *multi-string graph* if there exists a finite multiset F of strings in the plane such that $G \cong (F)$. Equivalently, G is the intersection graph of a multiset of planar strings, where multiple vertices are allowed to correspond to identical strings.

Definition A.0.8 (Interval graph)**.** A simple graph $G = (V, E)$ is an *interval graph* if there exists an assignment $v \mapsto I_v$ of a nonempty real interval $I_v \subseteq \mathbb{R}$ to each vertex $v \in V$ such that for all distinct $u, v \in V$,

$$uv \in E \iff I_u \cap I_v \neq \emptyset.$$

Definition A.0.9 (MultiInterval graph)**.** A simple graph G is a *multi-interval graph* if there exists a finite multiset F of nonempty real intervals such that $G \cong (F)$. Equivalently, G is the intersection graph of a multiset of intervals, allowing repeated use of the same interval.

We state the theorem below.

Theorem A.0.10 (MultiInterval graphs are multi-intersection graphs)**.** *Let P be the hereditary class of all interval graphs. Then every multi-interval graph belongs to the multi-intersection class P^*. More precisely, if G is the intersection graph of a multiset of intervals, then $G \in P^*$ with respect to P.*

Proof. Let $G \cong (F)$ where $F = \{I_1, \ldots, I_N\}$ is a finite multiset of nonempty real intervals. Let $[F] = \{J_1, \ldots, J_t\}$ be the set of distinct intervals appearing in F.

Consider the intersection graph $H := ([F])$ of the *set* $[F]$. By definition of interval graphs, H is an interval graph: it is realized by the interval family $\{J_1, \ldots, J_t\}$, so $H \in P$.

Now compare (F) and $([F])$: for each distinct interval J_ℓ occurring in F, let $m_\ell \geq 1$ be its multiplicity. In (F), the vertices corresponding to the m_ℓ copies of J_ℓ form a clique (because $J_\ell \cap J_\ell = J_\ell \neq \emptyset$), and they have identical adjacency to vertices coming from other intervals (because intersection depends only on the underlying interval). Hence (F) is obtained from $H = ([F])$ by replacing each vertex J_ℓ by a clique of size m_ℓ and preserving adjacencies between cliques.

Therefore $G \cong (F) \in P^*$ by the definition of P^* (equivalently, by the vertex-duplication characterization of multi-intersection classes). □

Theorem A.0.11 (MultiString graphs are multi-intersection graphs)**.** *Let P be the hereditary class of all string graphs. Then every multi-string graph belongs to the multi-intersection class P^*.*

Proof. Let $G \cong (F)$ where $F = \{\gamma_1, \ldots, \gamma_N\}$ is a finite multiset of planar strings. Let $[F] = \{\eta_1, \ldots, \eta_t\}$ be the set of distinct strings appearing in F, and put $H := ([F])$.

By definition of string graphs, H is a string graph, since it is realized by the string family $\{\eta_1, \ldots, \eta_t\}$; hence $H \in P$.

As in the interval case, duplicates behave as vertex duplications: if a string η_ℓ appears with multiplicity $m_\ell \geq 1$ in F, then the corresponding m_ℓ vertices form a clique in (F) whenever $\eta_\ell([0,1]) \neq \emptyset$ (always true), and they have identical neighborhoods outside the clique because intersections with any other string depend only on the underlying set $\eta_\ell([0,1])$. Thus (F) is obtained from H by replacing each vertex by a clique (of size equal to its multiplicity) and preserving adjacencies between cliques.

Hence $G \in P^*$ by the definition of multi-intersection classes. □

Appendix (List of Tables)

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Hypergraphs generalize this framework by allowing hyperedges that connect more than two vertices. Superhypergraphs further enrich the model through iterated powerset constructions, capturing hierarchical and self-referential structures among hyperedges.

An (m, n)-SuperHyperGraph is a mathematical structure in which each vertex corresponds to an (m, n)-superhyperfunction defined on a base set, while the hyperedges group such functions together to represent higher-order relationships and contextual connections.

Systematic research on SuperHyperGraphs is still relatively limited compared with the extensive literature on graphs and hypergraphs. To help bridge this gap, this book presents a survey of fundamental and advanced concepts related to SuperHyperGraphs.

Our aim is twofold:

(i) to increase the visibility and accessibility of SuperHyperGraph theory and thereby stimulate further research, and

(ii) to deepen the mathematical understanding of their structures among researchers and practitioners who work with graph- and hypergraph-based models.

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