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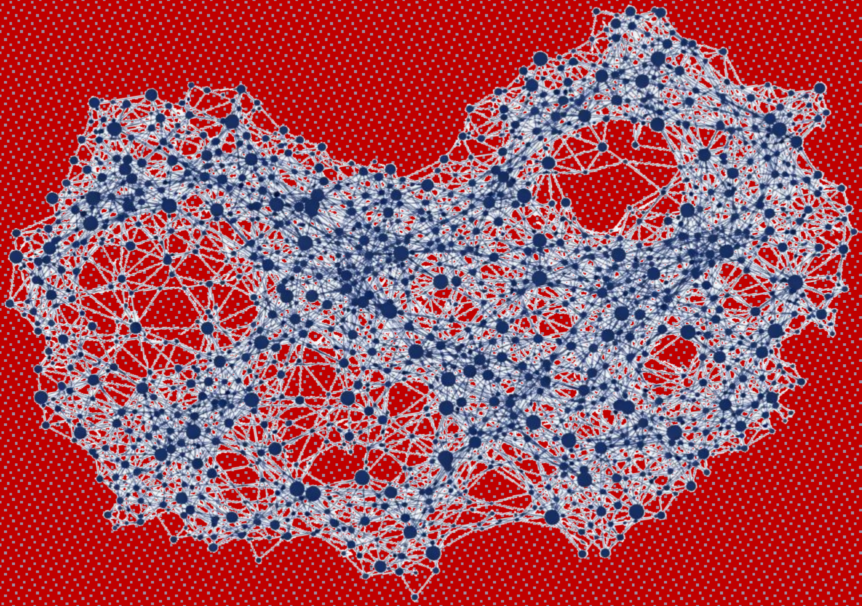
HYPERGRAPH AND SUPERHYPERGRAPH THEORY WITH APPLICATIONS II

GRAPH PROPERTY AND PARAMETER



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II

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HyperGraph and SuperHyperGraph Theory with Applications (II): Graph Property and Parameter

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Abstract

A *hypergraph* extends an ordinary graph by permitting each edge to join an arbitrary nonempty subset of the vertex set. By iterating the powerset construction one additional level, one obtains nested (higher-order) vertex objects and, consequently, finite *SuperHyperGraphs* whose vertices and edges can themselves be set-valued across multiple layers. tudies specifically devoted to *SuperHyperGraph* properties and *SuperHyperGraph* parameters have not progressed substantially so far. To help increase the visibility of SuperHyperGraphs, this book investigates various graph properties and graph parameters in the SuperHyperGraph setting. In particular, we place special emphasis on well-studied graph parameters such as domination and topological indices. This book is a sequel to the work cited in [1].

Keywords: SuperHyperGraph, HyperGraph, Graph Property, Graph Parameter

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Table of Contents

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Chapter 1

Introduction

1.1 Graph, HyperGraph, and SuperHyperGraph

Network modeling is often based on graphs, where entities are represented by vertices and binary relations by edges [2]. Such graphs are applied across a wide range of areas—including machine learning and neural networks—and constitute a highly important research concept. However, classical graphs can be inadequate for describing complex networks in which three or more entities interact simultaneously. Hypergraphs overcome this limitation by allowing each hyperedge to connect an arbitrary nonempty subset of vertices, thereby capturing higher-order interactions [3].

Despite their expressive power, hypergraphs may still be insufficient for modeling layered, nested, and intrinsically hierarchical relationships that occur in many real-world systems. To address this gap, F. Smarandache introduced the notion of a *SuperHyperGraph* [4, 5]. A SuperHyperGraph uses iterative powerset-based constructions to encode nested connectivity patterns and multilevel relations [4, 6], and it has received substantial attention in recent years [7–10].

Table 1.1 highlights the main distinctions among graphs, hypergraphs, and superhypergraphs. Throughout this book, n denotes a natural number unless stated otherwise (cf. [1]).

Table 1.1.: Salient differences among graphs, hypergraphs, and superhypergraphs.

Concept	*Notation*	*Edge Type*	*Extension Mechanism*
Graph [2]	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$	Standard edges encode relations between exactly two vertices.
Hypergraph [11]	$H = (V, E)$	$E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$	Hyperedges may join any nonempty subset of vertices.
Superhypergraph [4]	$\mathrm{SHG}^{(n)} = (V_0, V, E)$	$V \subseteq \mathrm{PS}^n(V_0),\ E \subseteq \mathrm{PS}(V)$	An n-fold powerset construction is used to capture nested structure.

Notation. $\mathrm{PS}(X) = \{A \subseteq X\}$ and $\mathrm{PS}^0(X) = X,\ \mathrm{PS}^{k+1}(X) = \mathrm{PS}(\mathrm{PS}^k(X))$.

A more concrete side-by-side comparison of graphs, hypergraphs, and n-superhypergraphs is given in Table 1.2.

Table 1.2.: A concrete comparison of graphs, hypergraphs, and n-superhypergraphs.

Aspect	*Graph* $G=(V,E)$	*Hypergraph* $H=(V,\mathcal{E})$	*n-SuperHyperGraph* $\mathrm{SHG}^{(n)}=(V,E)$
Vertices	$v\in V$	$v\in V$	$x\in V\subseteq \mathrm{PS}^n(V_0)$ (nesting allowed)
Edges	$E\subseteq\binom{V}{2}$	$\mathcal{E}\subseteq \mathrm{PS}(V)\setminus\{\emptyset\}$	$E\subseteq \mathrm{PS}(V)\setminus\{\emptyset\}$
Incidence	$v\in e$	$v\in e$	$x\in\varepsilon$
Adjacency	$\{u,v\}\in E$	$\exists e\in\mathcal{E}:\{u,v\}\subseteq e$	$\exists\varepsilon\in E:\{x,y\}\subseteq\varepsilon$
One edge encodes	pairwise relation	multiway relation	multiway relation among supervertices
Distance (typ.)	shortest-path	Berge / primal	super-Berge / primal
Use (typ.)	binary links	higher-order groups	hierarchical/nested incidence

Notation. $\mathrm{PS}^0(X)=X$ and $\mathrm{PS}^{k+1}(X)=\mathrm{PS}(\mathrm{PS}^k(X))$.

1.2 Graph Property and Graph Parameters

A *graph property* is a structural condition imposed on graphs—typically invariant under graph isomorphism—that specifies whether a given graph satisfies a prescribed constraint. A *graph parameter* is a numerical invariant associated with a graph (usually isomorphism-invariant) that measures some structural feature, such as size, sparsity, or combinatorial complexity. Several representative parameters are listed in Table 1.3.

Table 1.3.: Examples of parameters for graphs, hypergraphs, and n-SuperHyperGraphs.

Family	Graph G	Hypergraph H	n-SHG $\mathrm{SHG}^{(n)}$
Distance/diameter	d_G, $\mathrm{diam}(G)$	Berge d_H, $\mathrm{diam}(H)$	super-Berge $d_{\mathrm{SHG}^{(n)}}$, $\mathrm{diam}(\mathrm{SHG}^{(n)})$
Girth	$\mathrm{girth}(G)$	Berge $\mathrm{girth}(H)$	super-Berge $\mathrm{girth}(\mathrm{SHG}^{(n)})$
Degree sequence	$(\deg_G(v))$	$(d_H(v))$	$(\deg_{\mathrm{SHG}^{(n)}}(x))$
Cyclomatic number	$\mathrm{cyc}(G)=\lvert E\rvert-\lvert V\rvert+\mathrm{cc}(G)$	$\mathrm{cyc}(\mathrm{Inc}(H))$	$\mathrm{cyc}(\mathrm{Inc}(\mathrm{SHG}^{(n)}))$
Width-type	$\mathrm{tw}(G)$	$\mathrm{hw}(H)$	$\mathrm{shtw}(\mathrm{SHG}^{(n)})$
Domination	$\gamma(G)$	$\gamma(H)$	$\gamma(\mathrm{SHG}^{(n)})$
Total domination	$\gamma_t(G)$	$\gamma_t(H)$	$\gamma_t(\mathrm{SHG}^{(n)})$
Signed domination	$\gamma_s(G)$	$\gamma_s(H)$	$\gamma_s(\mathrm{SHG}^{(n)})$
Double domination	$\gamma_{\times 2}(G)$	$\gamma_{\times 2}(H)$	$\gamma_{\times 2}(\mathrm{SHG}^{(n)})$
Wiener index	$W(G)$	$W(H)$	$W(\mathrm{SHG}^{(n)})$
Zagreb indices	M_1, M_2	Zagreb-type	M_1, M_2
Randić index	R_α	Randić-type	R_α
Hosoya index	$Z(G)$	matching-type	$Z(\mathrm{SHG}^{(n)})$
Sombor index	$\mathrm{SO}(G)$	$\mathrm{SO}(H)$	$\mathrm{SO}(\mathrm{SHG}^{(n)})$
Skewness	$\mathrm{skew}(G)$	$\mathrm{skew}(\mathrm{Pr}(H))$	$\mathrm{skew}(\mathrm{Pr}(\mathrm{SHG}^{(n)}))$

Note. Hypergraph (and SHG) parameters often admit several non-equivalent extensions; common choices are based on Berge, primal, or incidence constructions.

1.3 Our Contributions

In view of the above, research on SuperHyperGraphs—and on graph properties and graph parameters more broadly—is highly important. Nevertheless, work devoted specifically to *SuperHy-*

perGraph properties and *SuperHyperGraph* parameters has not advanced substantially to date. To enhance the visibility of SuperHyperGraphs, this book studies a range of graph properties and graph parameters within the SuperHyperGraph framework. In particular, we emphasize well-established parameters such as domination and topological indices. This book serves as a sequel to the work in [1].

Chapter 2

Preliminaries

This section fixes notation and recalls the basic objects.

2.1 SuperHyperGraphs

Graph theory investigates systems of *vertices* connected by *edges*, with an emphasis on connectivity, structural properties, and algorithmic questions arising in mathematics, computer science, and numerous applications [2]. A *hypergraph* extends an ordinary graph by permitting an edge to join an arbitrary nonempty subset of the vertex set, and therefore provides a natural formalism for genuinely multiway interactions [3, 12]. Such higher-arity relations have become increasingly relevant in modern learning pipelines, including recent neural-network architectures that exploit hypergraph structure [3, 13–16].

If one iterates the powerset construction further, then vertices themselves may be *nested* set-valued objects. This leads to finite *SuperHyperGraphs*, whose vertex set and edge set can both live at multiple "levels" of set nesting [17,18]. These hierarchical representations appear naturally in settings where relations are layered or multiscale, for example in molecular design, complex-network analysis, and neural-network modeling, among other applications [9, 19–24]. Related variants have also been studied, including Directed SuperHyperGraphs [25, 26] and MetaSuperHyperGraphs [27].

Throughout, the index n in $\mathrm{PS}^n(\cdot)$ and in an n-SuperHyperGraph is always a nonnegative integer.

Definition 2.1.1 (Base set)**.** A *base set* S is the underlying universe of discourse:

$$S = \{\, x \mid x \text{ is an admissible object in the context under consideration} \,\}.$$

All sets appearing in $\mathrm{PS}(S)$ and in the iterated powersets $\mathrm{PS}^n(S)$ are ultimately built from elements of S.

Definition 2.1.2 (Powerset)**.** (see [28]) For a set S, the *powerset* of S is

$$\mathrm{PS}(S) \;=\; \{\, A \mid A \subseteq S \,\}.$$

In particular, $\emptyset \in \mathrm{PS}(S)$ and $S \in \mathrm{PS}(S)$.

Definition 2.1.3 (Hypergraph)**.** [11,29] A *hypergraph* is a pair $H = (V, E)$ such that:

- V is a finite set (the *vertices*), and
- E is a finite family of nonempty subsets of V (the *hyperedges*).

Thus, a hyperedge may involve more than two vertices, capturing genuinely multiway relations.

Example 2.1.4 (A simple hypergraph)**.** Let $V = \{1, 2, 3, 4\}$. Define a family of nonempty subsets of V by

$$E = \{\{1, 2\},\ \{2, 3, 4\},\ \{1, 3\}\}.$$

Then $H = (V, E)$ is a hypergraph in the sense of Definition 2.1.3. For instance, the hyperedge $\{2, 3, 4\}$ simultaneously joins three vertices, so it cannot be represented by a single edge in an ordinary graph.

Definition 2.1.5 (Iterated powerset and flattening)**.** [30] Let V_0 be a finite nonempty set. Define $\mathrm{PS}^0(V_0) := V_0$ and

$$\mathrm{PS}^{k+1}(V_0) := \mathrm{PS}\big(\mathrm{PS}^k(V_0)\big) \qquad (k \geq 0).$$

For each $k \geq 0$, define the flattening map

$$\mathrm{Flat}_k : \mathrm{PS}^k(V_0) \setminus \{\emptyset\} \longrightarrow \mathrm{PS}(V_0) \setminus \{\emptyset\}$$

recursively by

$$\mathrm{Flat}_0(x) := \{x\} \quad (x \in V_0), \qquad \mathrm{Flat}_{k+1}(X) := \bigcup_{Y \in X} \mathrm{Flat}_k(Y) \quad (X \in \mathrm{PS}^{k+1}(V_0) \setminus \{\emptyset\}).$$

Definition 2.1.6 (n-SuperHyperGraph)**.** (see [4]) Let V_0 be a finite, nonempty base set. Define

$$\mathrm{PS}^0(V_0) := V_0, \qquad \mathrm{PS}^{\,k+1}(V_0) := \mathrm{PS}\big(\mathrm{PS}^{\,k}(V_0)\big) \quad (k \in \mathbb{N}).$$

For $n \geq 0$, an *n-SuperHyperGraph on* V_0 is a pair

$$\mathrm{SHG}^{(n)} = (V, E)$$

satisfying

$$V \subseteq \mathrm{PS}^{\,n}(V_0) \quad \text{and} \quad E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}.$$

Elements of V are called *n-supervertices*, and elements of E are called *n-superedges* (i.e., each n-superedge is a nonempty subset of V).

Example 2.1.7 (A 0-SuperHyperGraph (recovering an ordinary hypergraph))**.** Let the base set be

$$V_0 = \{1, 2, 3, 4\}.$$

Since $\mathrm{PS}^0(V_0) = V_0$, any choice of $V \subseteq V_0$ and any nonempty family $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$ produces a 0-SuperHyperGraph.

For concreteness, take

$$V = \{1, 2, 3, 4\} \subseteq \mathrm{PS}^0(V_0), \qquad E = \{\{1, 2, 3\}, \{2, 4\}\} \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}.$$

Then $\mathrm{SHG}^{(0)} = (V, E)$ is a 0-SuperHyperGraph on V_0 in the sense of Definition 2.1.6. Here the 0-supervertices are the elements $1, 2, 3, 4$, and the 0-superedges are the nonempty subsets $\{1, 2, 3\}$ and $\{2, 4\}$ of V. In other words, $\mathrm{SHG}^{(0)}$ is simply the hypergraph on vertex set V_0 with two hyperedges $\{1, 2, 3\}$ and $\{2, 4\}$.

Example 2.1.8 (A 2-SuperHyperGraph (nested supervertices))**.** Let the base set be

$$V_0 = \{a, b, c\}.$$

Then

$$\mathrm{PS}^1(V_0) = \mathrm{PS}(V_0) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\},$$

and $\mathrm{PS}^2(V_0) = \mathrm{PS}(\mathrm{PS}(V_0))$ consists of *collections of subsets* of V_0.

Define a vertex set $V \subseteq \mathrm{PS}^2(V_0)$ by choosing three 2-supervertices:

$$x_1 := \{\{a\}, \{b\}\}, \qquad x_2 := \{\{b\}, \{c\}\}, \qquad x_3 := \{\{a, c\}\},$$

and set

$$V = \{x_1, x_2, x_3\} \subseteq \mathrm{PS}^2(V_0).$$

Next, define a family of nonempty subsets of V (the 2-superedges) by

$$E = \{\varepsilon_1, \varepsilon_2\} \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}, \qquad \varepsilon_1 := \{x_1, x_2\}, \qquad \varepsilon_2 := \{x_2, x_3\}.$$

Then $\mathrm{SHG}^{(2)} = (V, E)$ is a 2-SuperHyperGraph on the base set V_0 in the sense of Definition 2.1.6. Concretely, the 2-supervertices x_1, x_2, x_3 are themselves *set-valued objects* (two of them are pairs of 1-subsets of V_0, and one is a singleton family), and the 2-superedges $\varepsilon_1, \varepsilon_2$ are nonempty subsets of the supervertex set V.

2.2 (m, n)-SuperHyperGraph

An (m, n)-SuperHyperGraph is a higher–order network model in which each vertex is an (m, n)-*superhyperfunction* on a fixed base set, and each hyperedge collects several such functions to encode multiway relationships (e.g., shared constraints or contextual couplings). More generally, an (h, k)-*ary* (m, n)-SuperHyperGraph replaces single functions by (h, k)-ary superhyperfunctions [1]. In this book, we primarily work with n-SuperHyperGraphs.

Notation 2.2.1. *Let* S *be a nonempty set. Define the iterated powersets recursively by*

$$\mathrm{PS}_0(S) := S, \qquad \mathrm{PS}_{m+1}(S) := \mathrm{PS}\big(\mathrm{PS}_m(S)\big) \quad (m \in \mathbb{N}_0),$$

so that $\mathrm{PS}_1(S) = \mathrm{PS}(S)$, $\mathrm{PS}_2(S) = \mathrm{PS}(\mathrm{PS}(S))$, *and so on. For a set* X *and* $h \in \mathbb{N}$, *we also write*

$$X^h := \underbrace{X \times \cdots \times X}_{h \text{ copies}}$$

for the h*-fold Cartesian power.*

Definition 2.2.2 ((m, n)-superhyperfunction)**.** [31, 32] Let $m, n \in \mathbb{N}$ and let $S \neq \emptyset$. An (m, n)*-superhyperfunction on* S is a map

$$f : \ \mathrm{PS}_m(S) \longrightarrow \mathrm{PS}_n(S).$$

Equivalently, $f \in \mathrm{Hom}\big(\mathrm{PS}_m(S),\, \mathrm{PS}_n(S)\big)$.

Definition 2.2.3 ((m, n)-SuperHyperGraph)**.** [1] Fix $m, n \in \mathbb{N}$ and a nonempty base set S. Let

$$\mathfrak{F}_{m,n}(S) := \{\, f : \mathrm{PS}_m(S) \to \mathrm{PS}_n(S) \,\}.$$

An (m, n)-*SuperHyperGraph* is a pair

$$\mathsf{SHG}^{(m,n)} = (V, \mathcal{E}),$$

where $V \subseteq \mathfrak{F}_{m,n}(S)$ is a nonempty vertex set (so each vertex is a concrete (m, n)-superhyperfunction), and

$$\emptyset \neq \mathcal{E} \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$$

is a nonempty family of nonempty *hyperedges*. Thus each $E \in \mathcal{E}$ is a finite nonempty set of vertices, interpreted as a higher–order interaction among the corresponding superhyperfunctions.

Example 2.2.4 (Access-control policies as a $(1, 2)$-SuperHyperGraph)**.** Consider a small organization where permissions are granted in *bundles*. Let the base set of atomic resources be

$$S := \{\mathrm{VPN},\ \mathrm{Repo},\ \mathrm{DB},\ \mathrm{Payroll}\}.$$

We take $(m, n) = (1, 2)$, so each vertex is a concrete function

$$f : \ \mathrm{PS}(S) \longrightarrow \mathrm{PS}\big(\mathrm{PS}(S)\big),$$

mapping a *requested* set of resources $X \subseteq S$ to a *family of permitted bundles* $f(X) \subseteq \mathrm{PS}(S)$.

Define three policy functions (vertices) representing typical roles:

$$V := \{f_{\mathrm{eng}},\, f_{\mathrm{fin}},\, f_{\mathrm{ctr}}\} \ \subseteq \ \mathfrak{F}_{1,2}(S),$$

where, for each $X \subseteq S$,

$$f_{\mathrm{eng}}(X) := \begin{cases} \{\, X \cup \{\mathrm{VPN}\} \,\}, & X \subseteq \{\mathrm{Repo}, \mathrm{DB}\}, \\ \emptyset, & \text{otherwise}, \end{cases}$$

$$f_{\mathrm{fin}}(X) := \begin{cases} \{\, X \,\}, & X \subseteq \{\mathrm{DB}, \mathrm{Payroll}\}, \\ \emptyset, & \text{otherwise}, \end{cases}$$

$$f_{\mathrm{ctr}}(X) := \begin{cases} \{\, \{\mathrm{VPN}, \mathrm{Repo}\}\, \}, & \mathrm{Repo} \in X \text{ and Payroll} \notin X, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus, f_{eng} allows engineering requests only for Repo and/or DB and forces VPN into the allowed bundle; f_{fin} allows only finance resources; and f_{ctr} restricts contractors to the fixed bundle $\{\mathrm{VPN}, \mathrm{Repo}\}$.

Now introduce higher-order couplings between these policy functions via hyperedges:

$$\mathcal{E} := \{E_1, E_2, E_3\} \subseteq \mathrm{PS}(V)\backslash\{\emptyset\}, \qquad E_1 := \{f_{\mathrm{eng}}, f_{\mathrm{ctr}}\},\ E_2 := \{f_{\mathrm{eng}}, f_{\mathrm{fin}}\},\ E_3 := \{f_{\mathrm{eng}}, f_{\mathrm{fin}}, f_{\mathrm{ctr}}\}.$$

Interpretation: E_1 models a shared *dev-platform* control that jointly constrains engineering and contractors; E_2 represents a periodic *access-review* policy coupling engineering and finance; and E_3 models an *incident-response* context in which all role policies are evaluated together. Hence

$$\mathsf{SHG}^{(1,2)} = (V, \mathcal{E})$$

is a concrete $(1,2)$-SuperHyperGraph capturing multiway interactions among real access-control rules.

Example 2.2.5 (Roadmap consolidation as a $(2,1)$-SuperHyperGraph)**.** Consider product planning where multiple teams submit *feature bundles*, and leadership applies different consolidation rules to obtain a final roadmap. Let the base set of candidate features be

$$S := \{A, B, C, D\}.$$

We take $(m,n) = (2,1)$, so a vertex is a function

$$f :\ \mathrm{PS}\big(\mathrm{PS}(S)\big) \longrightarrow \mathrm{PS}(S),$$

mapping a *family of proposals* (a set of feature-sets) to a single selected set of features.

Fix the linear order $A \prec B \prec C \prec D$ and define three consolidation rules:

$$V := \{f_{\cup},\ f_{\mathrm{maj}},\ f_{\min}\}\ \subseteq\ \mathfrak{F}_{2,1}(S),$$

where for $\mathcal{A} \subseteq \mathrm{PS}(S)$,

$$f_{\cup}(\mathcal{A})\ :=\ \bigcup_{X \in \mathcal{A}} X \qquad \text{(full union rule)},$$

$$f_{\mathrm{maj}}(\mathcal{A})\ :=\ \Big\{ s \in S :\ \big|\{X \in \mathcal{A} : s \in X\}\big| \geq 2 \Big\} \qquad \text{(“appears in at least two proposals”)},$$

and

$$f_{\min}(\mathcal{A})\ :=\ \begin{cases} \min_{\mathrm{lex}}\{\, X \in \mathcal{A} :\ |X| = \min_{Y \in \mathcal{A}} |Y| \,\}, & \mathcal{A} \neq \emptyset, \\ \emptyset, & \mathcal{A} = \emptyset, \end{cases}$$

where $\min_{\mathrm{lex}}$ chooses the lexicographically smallest set (under $A \prec B \prec C \prec D$) among those with minimum cardinality (a toy model of a strict “budget-minimal” choice).

For a concrete proposal family

$$\mathcal{A}_0 := \{\{A,B\}, \{B,C\}, \{B,D\}\}\ \in\ \mathrm{PS}\big(\mathrm{PS}(S)\big),$$

we obtain

$$f_{\cup}(\mathcal{A}_0) = \{A,B,C,D\}, \qquad f_{\mathrm{maj}}(\mathcal{A}_0) = \{B\}, \qquad f_{\min}(\mathcal{A}_0) = \{A,B\}.$$

Thus different vertices represent different, concrete decision policies acting on the *same* input type.

Finally, encode governance relationships as hyperedges:

$$\mathcal{E} := \{F_1, F_2\} \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}, \qquad F_1 := \{f_{\mathrm{maj}}, f_{\min}\}, \quad F_2 := \{f_{\cup}, f_{\mathrm{maj}}, f_{\min}\}.$$

Interpretation: F_1 models a committee that jointly enforces “majority support” and “budget-minimality”; F_2 models a broader forum that compares all three consolidation rules in one multiway decision context. Hence

$$\mathsf{SHG}^{(2,1)} = (V, \mathcal{E})$$

is a concrete $(2,1)$-SuperHyperGraph capturing higher-order interactions among real planning policies.

Remark 2.2.6 ((h,k)-ary extension)**.** Fix $h, k \in \mathbb{N}$. An (h,k)*-ary* (m,n)*-superhyperfunction* on S is a map

$$F: \ \big(\mathrm{PS}_m(S)\big)^h \longrightarrow \big(\mathrm{PS}_n(S)\big)^k.$$

An (h,k)*-ary* (m,n)*-SuperHyperGraph* is defined by replacing $\mathfrak{F}_{m,n}(S)$ in Definition 2.2.3 with the corresponding family of (h,k)-ary superhyperfunctions.

2.3 Hierarchical SuperHyperGraphs

A hierarchical superhypergraph is a superhypergraph whose vertices live across multiple powerset levels, with edges allowed to join mixed-level supervertices, while maintaining downward-closure coherence. In this book, we primarily work with n-SuperHyperGraphs.

Definition 2.3.1 (Hierarchical SuperHyperGraph of height r)**.** Let V_0 be a finite, nonempty base set. For $k \geq 0$ define iterated powersets

$$\mathrm{PS}^0(V_0) := V_0, \qquad \mathrm{PS}^{k+1}(V_0) := \mathrm{PS}\big(\mathrm{PS}^k(V_0)\big),$$

and fix an integer $r \geq 0$. Set the *hierarchical universe*

$$\mathcal{U}_r(V_0) \ := \ \bigcup_{k=0}^{r} \Big(\mathrm{PS}^k(V_0) \setminus \{\emptyset\}\Big).$$

For $x \in \mathcal{U}_r(V_0)$, define its *level*

$$\ell(x) \ := \ \min\{\, k \in \{0, 1, \ldots, r\} : x \in \mathrm{PS}^k(V_0)\,\}.$$

A *hierarchical superhypergraph of height* r on V_0 is a pair

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

such that

(H1) (*Hierarchical vertex set*) V is a finite nonempty set with

$$V \subseteq \mathcal{U}_r(V_0).$$

Elements of V are called *hierarchical supervertices.*

(H2) (*Cross-level edges*) E is a finite family of nonempty subsets of V:

$$E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}.$$

Elements of E are called *hierarchical superhyperedges.* In particular, a superhyperedge may contain supervertices of *different* levels.

(H3) (*Coherence / downward closure*) If $X \in V$ and $\ell(X) \geq 1$, then

$$X \subseteq V.$$

Equivalently, whenever a higher-level supervertex is present, all its immediate constituents are also present as supervertices.

For each $k \in \{0, \ldots, r\}$ we define the *k-th layer* by

$$V_k \;:=\; \{\, x \in V \,:\; \ell(x) = k \,\}, \qquad \text{so that} \qquad V = \dot{\bigcup}_{k=0}^{r} V_k.$$

Remark 2.3.2 (Relation to ordinary n-SuperHyperGraphs)**.** If, for some $n \leq r$, the hierarchical vertex set satisfies $V \subseteq \mathrm{PS}^n(V_0)$, then every supervertex has level $\leq n$ and (V, E) is an n-SuperHyperGraph in the sense of Definition 2.1.6. Conversely, any n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ is a hierarchical superhypergraph of height $r = n$ (by taking the same V and E); the novelty of Definition 2.3.1 is that it also permits *mixed-level* supervertices inside a single superhyperedge.

Proposition 2.3.3 (Ordinary n-SuperHyperGraphs embed into the hierarchical model)**.** *Let* $\mathrm{SHG}^{(n)} = (V, E)$ *be an n-SuperHyperGraph on V_0 in the sense of Definition 2.1.6. Then* $\mathrm{SHG}^{(n)}$ *is a hierarchical superhypergraph of height $r = n$.*

Proof. Since $V \subseteq \mathrm{PS}^n(V_0) \subseteq \mathcal{U}_n(V_0)$, condition (H1) holds. Moreover, $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$ is exactly (H2). Finally, (H3) is satisfied whenever V is chosen to be downward closed; in particular, one may replace V by its downward closure inside $\mathcal{U}_n(V_0)$ without changing E as a family of subsets of V, and then (H3) holds by construction. Hence $\mathrm{SHG}^{(n)}$ fits Definition 2.3.1. ☐

Example 2.3.4 (A mixed-level hierarchical superhypergraph)**.** Let $V_0 = \{1, 2, 3\}$ and take height $r = 2$. Consider the following hierarchical supervertex set:

$$V := \Big\{1, 2, 3,\ \{1, 2\}, \{2, 3\},\ \{\{1, 2\}, \{2, 3\}\}\Big\} \subseteq \mathcal{U}_2(V_0).$$

Its layers are

$$V_0 = \{1, 2, 3\}, \qquad V_1 = \{\{1, 2\}, \{2, 3\}\}, \qquad V_2 = \Big\{\{\{1, 2\}, \{2, 3\}\}\Big\}.$$

Note that the coherence condition (H3) holds: the level-2 vertex $\{\{1, 2\}, \{2, 3\}\}$ contains $\{1, 2\}, \{2, 3\} \in V$, and each of these contains its level-0 constituents, all of which lie in V.

Define a family of hierarchical superhyperedges

$$E := \Big\{e_1, e_2, e_3\Big\},$$

where

$$e_1 := \{\,1, \{1, 2\}\,\}, \qquad e_2 := \{\,2, \{1, 2\}, \{2, 3\}\,\}, \qquad e_3 := \Big\{3,\ \{\{1, 2\}, \{2, 3\}\}\Big\}.$$

Then $\mathbb{H}^{\langle 2\rangle} = (V, E)$ is a hierarchical superhypergraph of height 2.

Importantly, the edges are *cross-level*: e_1 connects a level-0 vertex and a level-1 vertex, e_2 mixes level 0 and level 1 within one interaction, and e_3 connects a level-0 vertex with a level-2 vertex. Thus this example explicitly realizes the intended "different-n-levels can be connected by edges" behavior.

Example 2.3.5 (Hospital care organization as a hierarchical superhypergraph)**.** A hospital naturally exhibits nested structure: individual clinicians form teams, and teams form units. We model this as a hierarchical superhypergraph of height $r = 2$.

Base layer (level 0**).** Let

$$V_0 := \{\mathrm{DrA}, \mathrm{DrB}, \mathrm{NurseC}, \mathrm{TechD}\}$$

be the set of individual staff members.

Hierarchical vertex set (levels $0, 1, 2$**).** Define the following level-1 supervertices (teams):

$$T_1 := \{\mathrm{DrA}, \mathrm{NurseC}\}, \qquad T_2 := \{\mathrm{DrB}, \mathrm{TechD}\}.$$

Define a level-2 supervertex (a unit made of teams):

$$U := \{T_1, T_2\}.$$

Now set

$$V := \{\mathrm{DrA}, \mathrm{DrB}, \mathrm{NurseC}, \mathrm{TechD},\ T_1, T_2,\ U\} \subseteq \mathcal{U}_2(V_0).$$

The coherence condition (H3) holds: since $U \in V$ and $U = \{T_1, T_2\}$, we have $T_1, T_2 \in V$, and since each T_i is a set of individuals, its constituents are also in V.

Cross-level superhyperedges (interactions). Let $E = \{e_1, e_2, e_3\}$ with

$$e_1 := \{\mathrm{DrA},\ T_1\} \quad \text{(individual assigned to her team)},$$

$$e_2 := \{T_1,\ T_2,\ \mathrm{NurseC}\} \quad \text{(a cross-team procedure coordinated by a specific nurse)},$$

$$e_3 := \{U,\ \mathrm{TechD}\} \quad \text{(unit-level policy or briefing that directly involves a technician)}.$$

Then $\mathbb{H}^{\langle 2\rangle} = (V, E)$ is a hierarchical superhypergraph of height 2 in the sense of Definition 2.3.1. Notably, e_2 and e_3 are *mixed-level* superedges.

Layer decomposition. The layers are

$$V_0 = \{\mathrm{DrA}, \mathrm{DrB}, \mathrm{NurseC}, \mathrm{TechD}\}, \qquad V_1 = \{T_1, T_2\}, \qquad V_2 = \{U\},$$

and $V = V_0 \,\dot\cup\, V_1 \,\dot\cup\, V_2$.

Example 2.3.6 (Software system architecture as a hierarchical superhypergraph)**.** Complex software is organized hierarchically: endpoints call services, services compose into subsystems, and subsystems form an overall platform. We model this as a hierarchical superhypergraph of height $r = 2$.

Base layer (level 0**).** Let the atomic components be

$$V_0 := \{\mathrm{API}, \mathrm{Auth}, \mathrm{DB}, \mathrm{Cache}, \mathrm{Queue}\}.$$

Hierarchical vertex set (levels $0, 1, 2$**).** Define level-1 service modules (each is a nonempty subset of V_0):

$$S_{\mathrm{core}} := \{\mathrm{API}, \mathrm{Auth}\},$$
$$S_{\mathrm{data}} := \{\mathrm{DB}, \mathrm{Cache}\},$$
$$S_{\mathrm{async}} := \{\mathrm{Queue}\}.$$

Define a level-2 platform subsystem built from level-1 services:

$$P := \{S_{\mathrm{core}},\ S_{\mathrm{data}},\ S_{\mathrm{async}}\}.$$

Set

$$V := \{\mathrm{API}, \mathrm{Auth}, \mathrm{DB}, \mathrm{Cache}, \mathrm{Queue},\ S_{\mathrm{core}}, S_{\mathrm{data}}, S_{\mathrm{async}},\ P\} \subseteq \mathcal{U}_2(V_0).$$

Again, (H3) holds: $P \in V$ forces $S_{\mathrm{core}}, S_{\mathrm{data}}, S_{\mathrm{async}} \in V$, and each $S_{\bullet}$ forces its level-0 constituents into V.

Cross-level superhyperedges (runtime and governance relations). Let $E = \{f_1, f_2, f_3, f_4\}$ with

$$f_1 := \{\mathrm{API},\ \mathrm{Auth},\ S_{\mathrm{core}}\}$$

(endpoint, component, and module involved in a login flow),

$$f_2 := \{S_{\mathrm{core}},\ S_{\mathrm{data}},\ \mathrm{Cache}\}$$

(service-to-service interaction with an explicit cache dependency),

$$f_3 := \{P,\ S_{\mathrm{async}},\ \mathrm{Queue}\}$$

(platform-wide reliability rule focusing on the async subsystem and its queue),

$$f_4 := \{P,\ \mathrm{DB}\}$$

(a platform-level backup policy directly constraining the database).

Then $\mathbb{H}^{\langle 2 \rangle} = (V, E)$ is a hierarchical superhypergraph of height 2. Edges such as f_3 and f_4 explicitly connect different levels, reflecting how real systems mix component-level dependencies with subsystem- or platform-level controls.

Layer decomposition. Here

$$V_0 = \{\mathrm{API}, \mathrm{Auth}, \mathrm{DB}, \mathrm{Cache}, \mathrm{Queue}\}, \qquad V_1 = \{S_{\mathrm{core}}, S_{\mathrm{data}}, S_{\mathrm{async}}\}, \qquad V_2 = \{P\}.$$

Chapter 3

Main Results: Graph Property of Super-HyperGraph

A graph property is a characteristic of a graph that depends only on its structure, not on vertex labeling. In this chapter, we investigate notions that capture structural properties of graphs, including graph parameters. In particular, we examine how these notions extend to hypergraphs and SuperHyperGraphs.

3.1 Triangular-Free SuperHyperGraph

A triangle-free graph contains no 3-cycle, equivalently no three vertices forming pairwise adjacent edges [33–37]. Related concepts that have also been studied include claw-free graphs [38,39] and asteroidal triple-free graphs [40]. A triangle-free hypergraph contains no hypergraph triangle: three distinct edges meeting pairwise on prescribed vertex-pairs without triple intersection [41–44]. A triangle-free SuperHyperGraph contains no supertriangle: three distinct superedges whose pairwise overlaps realize a triangle among supervertices without common intersection.

Definition 3.1.1 (Triangle in a graph)**.** Let $G = (V, E)$ be a graph. A *triangle* in G is a 3-vertex set $\{u, v, w\} \subseteq V$ of distinct vertices such that $\{u, v\}, \{v, w\}, \{w, u\} \in E$. Equivalently, G contains a (simple) cycle of length 3.

Definition 3.1.2 (Triangle-free graph)**.** A graph G is *triangle-free* if it contains no triangle in the sense of Definition 3.1.1.

Definition 3.1.3 (Triangle in a hypergraph)**.** Let $H = (V, E)$ be a hypergraph. A *triangle* in H is a choice of three *distinct* edges $e, f, g \in E$ and three *distinct* vertices $u, v, w \in V$ such that

$$\{u, v\} \subseteq e, \qquad \{v, w\} \subseteq f, \qquad \{w, u\} \subseteq g, \qquad \text{and} \qquad \{u, v, w\} \cap (e \cap f \cap g) = \emptyset.$$

Definition 3.1.4 (Triangle-free hypergraph)**.** A hypergraph H is *triangle-free* if it contains no triangle in the sense of Definition 3.1.3.

Remark 3.1.5. If H is 2-uniform (i.e., a graph), then a triangle in the sense of Definition 3.1.3 is exactly a graph triangle in the sense of Definition 3.1.1. Hence Definition 3.1.4 reduces to Definition 3.1.2 for graphs.

Definition 3.1.6 (Triangle in an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph in the sense of Definition 2.1.6. Since (V, E) is (by definition) a hypergraph whose "vertices" are n-supervertices, we declare that a *triangle in* $\mathrm{SHG}^{(n)}$ is a triangle in the underlying hypergraph (V, E) in the sense of Definition 3.1.3 (with $u, v, w \in V$ now being n-supervertices).

A concrete example is given below.

Example 3.1.7 (A triangle in a 1-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b, c\}$ and take $n = 1$. Consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ given by

$$V = \{\{a\}, \{b\}, \{c\}\} \subseteq \mathrm{PS}^1(V_0), \qquad E = \{\ F_{ab}, F_{bc}, F_{ca}\ \},$$

where

$$F_{ab} := \{\{a\}, \{b\}\}, \qquad F_{bc} := \{\{b\}, \{c\}\}, \qquad F_{ca} := \{\{c\}, \{a\}\}.$$

Then the three distinct supervertices $\{a\}, \{b\}, \{c\} \in V$ together with the three distinct superedges $F_{ab}, F_{bc}, F_{ca} \in E$ form a triangle in the underlying hypergraph (V, E): each pair of consecutive supervertices lies in the corresponding superedge, and the indices close up to return to the starting supervertex. Hence $\mathrm{SHG}^{(1)}$ contains a triangle in the sense of Definition 3.1.6.

Definition 3.1.8 (Triangle-free n-SuperHyperGraph)**.** An n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ is *triangle-free* if it contains no triangle in the sense of Definition 3.1.6.

A concrete example is given below.

Example 3.1.9 (A triangle-free 1-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b, c\}$ and again take $n = 1$. Define the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ by

$$V = \{\{a\}, \{b\}, \{c\}\} \subseteq \mathrm{PS}^1(V_0), \qquad E = \{\ F_{ab}, F_{bc}\ \},$$

where

$$F_{ab} := \{\{a\}, \{b\}\}, \qquad F_{bc} := \{\{b\}, \{c\}\}.$$

In the underlying hypergraph (V, E) there are only two superedges. A triangle (in the hypergraph sense) requires three distinct superedges, so no triangle can occur here. Equivalently, the only Berge cycles one can form would have to reuse an edge, which is forbidden in the triangle definition. Therefore $\mathrm{SHG}^{(1)}$ is triangle-free in the sense of Definition 3.1.8.

Definition 3.1.10 (Singleton nesting map)**.** Fix a finite nonempty base set V_0 and $n \geq 0$. Define maps $\iota_n : V_0 \to \mathrm{PS}^n(V_0)$ recursively by

$$\iota_0(x) := x \quad (x \in V_0), \qquad \iota_{k+1}(x) := \{\iota_k(x)\} \quad (k \geq 0).$$

For a subset $A \subseteq V_0$, set

$$\iota_n(A) := \{\iota_n(x) : x \in A\} \ \subseteq\ \mathrm{PS}^n(V_0).$$

Lemma 3.1.11 (Injective-image intersections)**.** *Let* $\varphi : X \to Y$ *be injective and let* $A, B, C \subseteq X$*. Then*

$$\varphi(A) \cap \varphi(B) = \varphi(A \cap B) \qquad \textit{and} \qquad \varphi(A) \cap \varphi(B) \cap \varphi(C) = \varphi(A \cap B \cap C).$$

Proof. We prove $\varphi(A) \cap \varphi(B) = \varphi(A \cap B)$; the 3-set statement is identical. If $y \in \varphi(A \cap B)$ then $y = \varphi(x)$ for some $x \in A \cap B$, hence $y \in \varphi(A) \cap \varphi(B)$. Conversely, if $y \in \varphi(A) \cap \varphi(B)$ then $y = \varphi(a) = \varphi(b)$ with $a \in A$, $b \in B$. Injectivity implies $a = b \in A \cap B$, so $y \in \varphi(A \cap B)$. □

Definition 3.1.12 (n-lift of a hypergraph)**.** Let $H = (V_0, E_0)$ be a hypergraph on the base set V_0. For $n \geq 0$, define the *n-lift* of H to be the pair

$$\mathrm{Lift}_n(H) \;:=\; (V_n, E_n), \qquad V_n := \iota_n(V_0), \qquad E_n := \{\iota_n(e) : e \in E_0\}.$$

Proposition 3.1.13 (The n-lift is an n-SuperHyperGraph)**.** *For every hypergraph* $H = (V_0, E_0)$ *and every* $n \geq 0$*, the pair* $\mathrm{Lift}_n(H) = (V_n, E_n)$ *is an n-SuperHyperGraph on* V_0 *(Definition 2.1.6). Moreover,* $\iota_n : V_0 \to V_n$ *is a bijection and induces a hypergraph isomorphism between* H *and* $\mathrm{Lift}_n(H)$*.*

Proof. By construction, $V_n = \iota_n(V_0) \subseteq \mathrm{PS}^n(V_0)$. Also, each $\iota_n(e)$ is a nonempty subset of V_n, hence $E_n \subseteq \mathrm{PS}(V_n) \setminus \{\emptyset\}$. Therefore $\mathrm{Lift}_n(H)$ is an n-SuperHyperGraph.

The map ι_n is injective by definition, hence a bijection onto its image V_n. Finally, by the definition of E_n, we have $e \in E_0$ if and only if $\iota_n(e) \in E_n$, which is exactly the incidence preservation required for a hypergraph isomorphism. □

Theorem 3.1.14 (Triangle-free SuperHyperGraphs generalize triangle-free graphs and hypergraphs)**.** *Let* $H = (V_0, E_0)$ *be any hypergraph and let* $n \geq 0$*. Then* H *is triangle-free if and only if* $\mathrm{Lift}_n(H)$ *is triangle-free (as an n-SuperHyperGraph).*

In particular, if G *is a graph (i.e., a* 2*-uniform hypergraph), then* G *is triangle-free in the graph-theoretic sense if and only if* $\mathrm{Lift}_n(G)$ *is a triangle-free n-SuperHyperGraph.*

Proof. Write $\mathrm{Lift}_n(H) = (V_n, E_n)$. Assume first that H contains a triangle. Then there exist distinct edges $e, f, g \in E_0$ and distinct vertices $u, v, w \in V_0$ satisfying

$$\{u, v\} \subseteq e, \quad \{v, w\} \subseteq f, \quad \{w, u\} \subseteq g, \quad \text{and} \quad \{u, v, w\} \cap (e \cap f \cap g) = \emptyset.$$

Applying ι_n pointwise, we obtain distinct vertices $\iota_n(u), \iota_n(v), \iota_n(w) \in V_n$ and distinct edges $\iota_n(e), \iota_n(f), \iota_n(g) \in E_n$ such that

$$\{\iota_n(u), \iota_n(v)\} \subseteq \iota_n(e), \quad \{\iota_n(v), \iota_n(w)\} \subseteq \iota_n(f), \quad \{\iota_n(w), \iota_n(u)\} \subseteq \iota_n(g).$$

Moreover, by Lemma 3.1.11 (with $\varphi = \iota_n$),

$$\iota_n(e) \cap \iota_n(f) \cap \iota_n(g) = \iota_n(e \cap f \cap g),$$

hence

$$\{\iota_n(u),\iota_n(v),\iota_n(w)\}\cap\big(\iota_n(e)\cap\iota_n(f)\cap\iota_n(g)\big)=\iota_n\big(\{u,v,w\}\cap(e\cap f\cap g)\big)=\emptyset.$$

Thus $\mathrm{Lift}_n(H)$ contains a triangle.

Conversely, assume $\mathrm{Lift}_n(H)$ contains a triangle with distinct edges $e',f',g'\in E_n$ and distinct vertices $u',v',w'\in V_n$. By construction of V_n and E_n, there exist unique $u,v,w\in V_0$ and $e,f,g\in E_0$ with $u'=\iota_n(u)$, $v'=\iota_n(v)$, $w'=\iota_n(w)$ and $e'=\iota_n(e)$, $f'=\iota_n(f)$, $g'=\iota_n(g)$. Pulling back the containments and using Lemma 3.1.11 again, we obtain exactly the conditions of Definition 3.1.3 in H. Hence H contains a triangle.

Therefore H is triangle-free if and only if $\mathrm{Lift}_n(H)$ is triangle-free. The final statement for graphs follows from Remark 3.1.5. □

3.2 Eulerian Graphs

An Eulerian graph admits a closed trail that traverses every edge exactly once, with all non-isolated vertices in one component [45–47]. An Eulerian hypergraph admits a closed strict trail alternating vertices and hyperedges, using every hyperedge exactly once [48, 49]. An Eulerian SuperHyperGraph admits a closed strict supertrail alternating supervertices and superedges, traversing each superedge exactly once.

Definition 3.2.1 (Trail, closed trail, and Euler tour in a graph)**.** Let $G=(V,E)$ be an undirected graph (allowing neither multiple edges nor loops, unless stated otherwise). A *trail* in G is a sequence

$$W\ =\ v_0,e_1,v_1,e_2,\dots,e_m,v_m$$

such that for each $i=1,\dots,m$ the edge e_i has endpoints $\{v_{i-1},v_i\}$, and the edges $e_1,\dots,e_m$ are pairwise distinct. The trail W is *closed* if $m\geq 1$ and $v_0=v_m$.

An *Euler tour* of G is a closed trail that traverses each edge of G exactly once (i.e., $\{e_1,\dots,e_m\}=E$). The graph G is called *Eulerian* if it admits an Euler tour.

Remark 3.2.2 (Classical characterization)**.** For a graph G, admitting an Euler tour is equivalent to: every vertex of positive degree has even degree and all edges lie in a single connected component (equivalently, the subgraph induced by non-isolated vertices is connected).

A concrete example is given below.

Definition 3.2.3 (Walk and strict trail in a hypergraph)**.** Let $H=(V,E)$ be a hypergraph, where V is a finite vertex set and E is a finite family of nonempty subsets of V. A *walk* in H is an alternating sequence

$$W\ =\ v_0,e_1,v_1,e_2,\dots,e_m,v_m$$

such that $\{v_{i-1},v_i\}\subseteq e_i$ and $v_{i-1}\neq v_i$ for all $i=1,\dots,m$. The walk is a *strict trail* if $e_1,\dots,e_m$ are pairwise distinct. It is *closed* if $m\geq 2$ and $v_0=v_m$.

Example 3.2.4 (A walk and a strict trail in a hypergraph)**.** Let $H = (V, E)$ be the hypergraph with

$$V = \{1, 2, 3, 4\}, \qquad E = \{\, e_1, e_2, e_3 \,\},$$

where

$$e_1 = \{1, 2, 3\}, \qquad e_2 = \{2, 4\}, \qquad e_3 = \{3, 4\}.$$

Consider the alternating sequence

$$W \;=\; 1,\; e_1,\; 2,\; e_2,\; 4,\; e_3,\; 3.$$

For each step, the consecutive vertices lie in the indicated hyperedge: $\{1, 2\} \subseteq e_1$, $\{2, 4\} \subseteq e_2$, and $\{4, 3\} \subseteq e_3$, and the consecutive vertices are distinct. Hence W is a *walk* in H in the sense of Definition 3.2.3. Moreover, the hyperedges e_1, e_2, e_3 appear exactly once each, so W is a *strict trail*. It is not closed, since the initial and terminal vertices differ ($1 \neq 3$).

Definition 3.2.5 (Euler tour and Eulerian hypergraph)**.** Let $H = (V, E)$ be a hypergraph. An *Euler tour* of H is a closed strict trail

$$W \;=\; v_0, e_1, v_1, \ldots, e_m, v_0$$

that traverses every edge of H exactly once, i.e. $\{e_1, \ldots, e_m\} = E$. The hypergraph H is called *Eulerian* if it admits an Euler tour.

A concrete example is given below.

Example 3.2.6 (An Euler tour and an Eulerian hypergraph)**.** Let $H = (V, E)$ be the hypergraph with

$$V = \{1, 2, 3\}, \qquad E = \{\, e_{12}, e_{23}, e_{31} \,\},$$

where

$$e_{12} = \{1, 2\}, \qquad e_{23} = \{2, 3\}, \qquad e_{31} = \{3, 1\}.$$

Then the closed strict trail

$$W \;=\; 1,\; e_{12},\; 2,\; e_{23},\; 3,\; e_{31},\; 1$$

traverses every hyperedge exactly once, i.e. $\{e_{12}, e_{23}, e_{31}\} = E$. Therefore W is an *Euler tour* of H in the sense of Definition 3.2.5, and H is *Eulerian*.

Remark 3.2.7 (Graphs as hypergraphs)**.** Any (simple) graph $G = (V, E)$ can be viewed as a 2-uniform hypergraph $H_G = (V, E)$. Under this identification, the notions of walk/strict trail/Euler tour in Definitions 3.2.1 and 3.2.5 coincide.

Definition 3.2.8 (Euler tour in an n-SuperHyperGraph)**.** Fix $n \geq 0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (Definition 2.1.6). A *walk* in $\mathrm{SHG}^{(n)}$ is an alternating sequence

$$W \;=\; x_0,\; \varepsilon_1,\; x_1,\; \varepsilon_2,\; \ldots,\; \varepsilon_m,\; x_m$$

where $x_i \in V$ (supervertices) and $\varepsilon_i \in E$ (superedges), such that $\{x_{i-1}, x_i\} \subseteq \varepsilon_i$ and $x_{i-1} \neq x_i$ for all i. It is a *strict trail* if $\varepsilon_1, \ldots, \varepsilon_m$ are pairwise distinct, and *closed* if $m \geq 2$ and $x_0 = x_m$.

An *Euler tour* of $\mathrm{SHG}^{(n)}$ is a closed strict trail that traverses every superedge exactly once. The n-SuperHyperGraph $\mathrm{SHG}^{(n)}$ is called *Eulerian* if it admits an Euler tour.

A concrete example is given below.

Example 3.2.9 (An Euler tour in a 1-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b, c\}$ and take $n = 1$. Define the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ by

$$V = \{\{a\}, \{b\}, \{c\}\} \subseteq \mathrm{PS}^1(V_0), \qquad E = \{\, \varepsilon_{ab}, \varepsilon_{bc}, \varepsilon_{ca} \,\},$$

where

$$\varepsilon_{ab} = \{\{a\}, \{b\}\}, \qquad \varepsilon_{bc} = \{\{b\}, \{c\}\}, \qquad \varepsilon_{ca} = \{\{c\}, \{a\}\}.$$

Consider the alternating sequence

$$W \;=\; \{a\},\; \varepsilon_{ab},\; \{b\},\; \varepsilon_{bc},\; \{c\},\; \varepsilon_{ca},\; \{a\}.$$

Each consecutive pair of supervertices lies in the corresponding superedge, the consecutive supervertices are distinct, and the three superedges $\varepsilon_{ab}, \varepsilon_{bc}, \varepsilon_{ca}$ are pairwise distinct. Moreover, W is closed and traverses every superedge exactly once. Hence W is an *Euler tour* of $\mathrm{SHG}^{(1)}$ in the sense of Definition 3.2.8, and therefore $\mathrm{SHG}^{(1)}$ is *Eulerian*.

Definition 3.2.10 (Iterated singleton embedding)**.** Let V_0 be a finite nonempty set and $n \geq 0$. Define $\iota_0 : V_0 \to \mathrm{PS}^0(V_0) = V_0$ by $\iota_0(v) = v$, and for $n \geq 1$ define recursively

$$\iota_n(v) \;:=\; \{\iota_{n-1}(v)\} \in \mathrm{PS}^n(V_0) \qquad (v \in V_0).$$

For a nonempty subset $A \subseteq V_0$, define

$$\iota_n(A) \;:=\; \{\iota_n(v) : v \in A\} \subseteq \iota_n(V_0).$$

Definition 3.2.11 (Lift of a hypergraph to an n-SuperHyperGraph)**.** Let $H = (V_0, E_0)$ be a hypergraph and fix $n \geq 0$. Define its *n-lift* to be the n-SuperHyperGraph

$$\mathrm{Lift}_n(H) \;:=\; (V_n, E_n)$$

where

$$V_n \;:=\; \iota_n(V_0) \subseteq \mathrm{PS}^n(V_0), \qquad E_n \;:=\; \{\iota_n(e) : e \in E_0\} \subseteq \mathrm{PS}(V_n) \setminus \{\emptyset\}.$$

Lemma 3.2.12 (Edge correspondence under the lift)**.** *Let $H = (V_0, E_0)$ be a hypergraph and $n \geq 0$. Then $\iota_n : V_0 \to V_n$ is a bijection, and the map $E_0 \to E_n$, $e \mapsto \iota_n(e)$ is a bijection.*

Proof. By construction, ι_n is injective (distinct v yield distinct iterated singletons), and $V_n = \iota_n(V_0)$, so $\iota_n : V_0 \to V_n$ is bijective. If $e \neq f$ in E_0, then there exists $v \in V_0$ with $v \in e \triangle f$; hence $\iota_n(v) \in \iota_n(e) \triangle \iota_n(f)$, so $\iota_n(e) \neq \iota_n(f)$. Thus $e \mapsto \iota_n(e)$ is injective, and it is surjective by the definition of E_n. □

Theorem 3.2.13 (Eulerian n-SuperHyperGraphs generalize Eulerian graphs and hypergraphs)**.** *Fix $n \geq 0$.*

(a) For every hypergraph H, the hypergraph H is Eulerian if and only if the n-SuperHyperGraph $\mathrm{Lift}_n(H)$ *is Eulerian (in the sense of Definition 3.2.8).*

(b) *In particular, for every graph G, the graph G is Eulerian (Definition 3.2.1) if and only if* $\mathrm{Lift}_n(H_G)$ *is an Eulerian n-SuperHyperGraph, where H_G is the associated* 2*-uniform hypergraph from Remark 3.2.7.*

Proof. (a) Let $H = (V_0, E_0)$ and write $\mathrm{Lift}_n(H) = (V_n, E_n)$.

(⇒) Assume that H has an Euler tour

$$W \;=\; v_0, e_1, v_1, \dots, e_m, v_0,$$

so W is a closed strict trail and $\{e_1, \dots, e_m\} = E_0$. Define

$$W^{(n)} \;:=\; \iota_n(v_0),\; \iota_n(e_1),\; \iota_n(v_1),\; \dots,\; \iota_n(e_m),\; \iota_n(v_0).$$

For each i, since $\{v_{i-1}, v_i\} \subseteq e_i$ and $v_{i-1} \neq v_i$, we have $\{\iota_n(v_{i-1}), \iota_n(v_i)\} \subseteq \iota_n(e_i)$ and $\iota_n(v_{i-1}) \neq \iota_n(v_i)$. Hence $W^{(n)}$ is a walk in $\mathrm{Lift}_n(H)$. Because $e_1, \dots, e_m$ are pairwise distinct and $e \mapsto \iota_n(e)$ is injective (Lemma 3.2.12), the superedges $\iota_n(e_1), \dots, \iota_n(e_m)$ are pairwise distinct, so $W^{(n)}$ is a strict trail. It is closed since $\iota_n(v_0) = \iota_n(v_0)$. Finally,

$$\{\iota_n(e_1), \dots, \iota_n(e_m)\} \;=\; \{\iota_n(e) : e \in E_0\} \;=\; E_n,$$

so $W^{(n)}$ traverses every superedge exactly once. Thus $\mathrm{Lift}_n(H)$ is Eulerian.

(⇐) Conversely, assume $\mathrm{Lift}_n(H)$ has an Euler tour

$$W^{(n)} \;=\; x_0, \varepsilon_1, x_1, \dots, \varepsilon_m, x_0.$$

By Lemma 3.2.12, each $x_i \in V_n$ is uniquely of the form $x_i = \iota_n(v_i)$ with $v_i \in V_0$, and each $\varepsilon_i \in E_n$ is uniquely of the form $\varepsilon_i = \iota_n(e_i)$ with $e_i \in E_0$. Then the sequence

$$W \;:=\; v_0, e_1, v_1, \dots, e_m, v_0$$

is a walk in H: indeed $\{x_{i-1}, x_i\} \subseteq \varepsilon_i$ implies $\{\iota_n(v_{i-1}), \iota_n(v_i)\} \subseteq \iota_n(e_i)$, hence $\{v_{i-1}, v_i\} \subseteq e_i$. Distinctness of the ε_i implies distinctness of the e_i (injectivity), so W is a strict trail, and it is closed since $v_0 = v_m$ follows from $x_0 = x_m$ and injectivity of ι_n. Moreover, $\{\varepsilon_1, \dots, \varepsilon_m\} = E_n$ implies $\{e_1, \dots, e_m\} = E_0$ by bijectivity. Hence W is an Euler tour of H, so H is Eulerian.

(b) This is immediate from (a) and Remark 3.2.7. □

3.3 Hamiltonian SuperHyperGraphs

A Hamiltonian graph contains a cycle that visits every vertex exactly once and returns to the start [50–54]. As classical results related to Hamiltonian graphs, the Bondy–Chvátal Theorem [55], Dirac's Theorem [56], and Ore's Theorem are well known; related notions such as hypohamiltonian graphs [57–59] and subhamiltonian graphs [60] have also been studied. A Hamiltonian hypergraph contains a Hamiltonian Berge cycle, alternating distinct vertices and hyperedges, visiting every vertex exactly once [50,61–63]. A Hamiltonian SuperHyperGraph contains a Hamiltonian super-Berge cycle alternating distinct supervertices and superedges, visiting all supervertices exactly once.

Definition 3.3.1 (Cycle and Hamiltonian cycle in a graph)**.** [64] Let $G = (V, E)$ be a (finite, simple, undirected) graph. A *cycle* in G is a sequence of vertices $(v_1, v_2, \ldots, v_\ell)$ with $\ell \geq 3$ such that $v_1, \ldots, v_\ell$ are pairwise distinct and

$$\{v_i, v_{i+1}\} \in E \quad (1 \leq i \leq \ell - 1), \qquad \{v_\ell, v_1\} \in E.$$

A *Hamiltonian cycle* is a cycle that visits every vertex exactly once, i.e. $\ell = |V|$. The graph G is *Hamiltonian* if it contains a Hamiltonian cycle.

Definition 3.3.2 (Berge cycle in a hypergraph)**.** [65] Let $H = (V, E)$ be a (finite) hypergraph, i.e. $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$. A *Berge cycle* of length $m \geq 2$ in H is an alternating sequence

$$C \;=\; v_1,\, e_1,\, v_2,\, e_2,\, \ldots,\, v_m,\, e_m,\, v_1$$

such that

(i) $v_1, \ldots, v_m \in V$ are pairwise distinct,

(ii) $e_1, \ldots, e_m \in E$ are pairwise distinct, and

(iii) $\{v_i, v_{i+1}\} \subseteq e_i$ for each $i = 1, \ldots, m$, where $v_{m+1} := v_1$.

Definition 3.3.3 (Hamiltonian hypergraph)**.** A hypergraph $H = (V, E)$ is *Hamiltonian* if it contains a Berge cycle $C = v_1, e_1, \ldots, v_m, e_m, v_1$ such that $\{v_1, \ldots, v_m\} = V$ (equivalently, $m = |V|$). Such a cycle is called a *Hamiltonian Berge cycle.*

Remark 3.3.4 (Consistency with graphs)**.** If H is 2-uniform (so H is a graph), then a Berge cycle in the sense of Definition 3.3.2 is exactly an ordinary graph cycle, and Definition 3.3.3 reduces to Definition 3.3.1.

Definition 3.3.5 (Berge cycle in an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (Definition 2.1.6). Since (V, E) is, by definition, a hypergraph whose vertices are n-supervertices, a *Berge cycle* in $\mathrm{SHG}^{(n)}$ is a Berge cycle in the underlying hypergraph (V, E) as in Definition 3.3.2 (with $v_i \in V$ now being n-supervertices).

A concrete example is given below.

Example 3.3.6 (A Berge cycle in a 1-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b, c, d\}$ and take $n = 1$. Consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ defined by

$$V = \{\{a\}, \{b\}, \{c\}, \{d\}\} \subseteq \mathrm{PS}^1(V_0), \qquad E = \{\, \varepsilon_1, \varepsilon_2, \varepsilon_3 \,\},$$

where

$$\varepsilon_1 = \{\{a\}, \{b\}, \{c\}\}, \qquad \varepsilon_2 = \{\{c\}, \{d\}\}, \qquad \varepsilon_3 = \{\{d\}, \{a\}\}.$$

Then the alternating sequence

$$C \;=\; \{a\},\; \varepsilon_1,\; \{c\},\; \varepsilon_2,\; \{d\},\; \varepsilon_3,\; \{a\}$$

is a Berge cycle in the underlying hypergraph (V, E): the supervertices appearing in C are distinct (except for the repetition of the first at the end), the superedges $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are pairwise distinct, and each consecutive pair of supervertices lies in the corresponding superedge:

$$\{\{a\}, \{c\}\} \subseteq \varepsilon_1, \qquad \{\{c\}, \{d\}\} \subseteq \varepsilon_2, \qquad \{\{d\}, \{a\}\} \subseteq \varepsilon_3.$$

Hence C is a Berge cycle in $\mathrm{SHG}^{(1)}$ in the sense of Definition 3.3.5.

Definition 3.3.7 (Hamiltonian n-SuperHyperGraph)**.** An n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ is *Hamiltonian* if it contains a Berge cycle whose vertex set is all of V, i.e. a cycle

$$C \;=\; x_1, \varepsilon_1, x_2, \varepsilon_2, \ldots, x_{|V|}, \varepsilon_{|V|}, x_1$$

with $\{x_1, \ldots, x_{|V|}\} = V$.

Example 3.3.8 (A Hamiltonian 1-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b, c, d\}$ and take $n = 1$. Define the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ by

$$V = \big\{\{a\}, \{b\}, \{c\}, \{d\}\big\} \subseteq \mathrm{PS}^1(V_0), \qquad E = \{\, \varepsilon_{ab}, \varepsilon_{bc}, \varepsilon_{cd}, \varepsilon_{da} \,\},$$

where

$$\varepsilon_{ab} = \big\{\{a\}, \{b\}\big\}, \quad \varepsilon_{bc} = \big\{\{b\}, \{c\}\big\}, \quad \varepsilon_{cd} = \big\{\{c\}, \{d\}\big\}, \quad \varepsilon_{da} = \big\{\{d\}, \{a\}\big\}.$$

Then the cycle

$$C \;=\; \{a\},\; \varepsilon_{ab},\; \{b\},\; \varepsilon_{bc},\; \{c\},\; \varepsilon_{cd},\; \{d\},\; \varepsilon_{da},\; \{a\}$$

is a Berge cycle whose vertex set is all of V (each supervertex appears exactly once, except for the initial supervertex $\{a\}$ repeated at the end). Therefore $\mathrm{SHG}^{(1)}$ is *Hamiltonian* in the sense of Definition 3.3.7.

Definition 3.3.9 (Iterated singleton embedding)**.** Fix a finite nonempty base set V_0 and $n \geq 0$. Define $\iota_n : V_0 \to \mathrm{PS}^n(V_0)$ recursively by

$$\iota_0(v) := v, \qquad \iota_{k+1}(v) := \{\iota_k(v)\} \quad (k \geq 0).$$

For $A \subseteq V_0$, set $\iota_n(A) := \{\iota_n(v) : v \in A\}$.

Definition 3.3.10 (n-lift of a hypergraph)**.** Let $H = (V_0, E_0)$ be a hypergraph and let $n \geq 0$. Define the *n-lift* of H to be the n-SuperHyperGraph

$$\mathrm{Lift}_n(H) \;:=\; (V_n, E_n), \qquad V_n := \iota_n(V_0), \qquad E_n := \{\iota_n(e) : e \in E_0\}.$$

Lemma 3.3.11 (Lift preserves distinctness)**.** *Let $H = (V_0, E_0)$ be a hypergraph and $n \geq 0$. Then $\iota_n : V_0 \to V_n$ is a bijection, and the map $E_0 \to E_n$, $e \mapsto \iota_n(e)$ is a bijection.*

Proof. The map ι_n is injective because distinct $v \in V_0$ yield distinct iterated singletons; since $V_n = \iota_n(V_0)$, it is bijective onto V_n. If $e \neq f$ in E_0, then some $v \in e \triangle f$, hence $\iota_n(v) \in \iota_n(e) \triangle \iota_n(f)$, so $\iota_n(e) \neq \iota_n(f)$. Surjectivity onto E_n is by definition. □

Theorem 3.3.12 (Hamiltonian SuperHyperGraphs generalize Hamiltonian graphs and hypergraphs)**.** *Let $H = (V_0, E_0)$ be any hypergraph and let $n \geq 0$. Then H is Hamiltonian (Definition 3.3.3) if and only if* $\mathrm{Lift}_n(H)$ *is Hamiltonian (Definition 3.3.7).*

In particular, if G is a graph, then G is Hamiltonian (Definition 3.3.1) if and only if $\mathrm{Lift}_n(G)$ *is a Hamiltonian n-SuperHyperGraph.*

Proof. Write $\mathrm{Lift}_n(H) = (V_n, E_n)$.

(⇒) Assume H contains a Hamiltonian Berge cycle

$$C \;=\; v_1, e_1, v_2, e_2, \ldots, v_{|V_0|}, e_{|V_0|}, v_1$$

with $\{v_1, \ldots, v_{|V_0|}\} = V_0$, where all v_i and all e_i are pairwise distinct and $\{v_i, v_{i+1}\} \subseteq e_i$ (indices modulo $|V_0|$). Define

$$C^{(n)} \;:=\; \iota_n(v_1), \iota_n(e_1), \iota_n(v_2), \ldots, \iota_n(v_{|V_0|}), \iota_n(e_{|V_0|}), \iota_n(v_1).$$

Then $\iota_n(v_1), \ldots, \iota_n(v_{|V_0|})$ are pairwise distinct and cover V_n (bijectivity of ι_n), and $\iota_n(e_1), \ldots, \iota_n(e_{|V_0|})$ are pairwise distinct (bijectivity on edges, Lemma 3.3.11). Moreover, $\{v_i, v_{i+1}\} \subseteq e_i$ implies $\{\iota_n(v_i), \iota_n(v_{i+1})\} \subseteq \iota_n(e_i)$ by construction of $\iota_n(A)$. Hence $C^{(n)}$ is a Berge cycle in $\mathrm{Lift}_n(H)$ using all vertices of V_n exactly once, so $\mathrm{Lift}_n(H)$ is Hamiltonian.

(⇐) Conversely, assume $\mathrm{Lift}_n(H)$ contains a Hamiltonian Berge cycle

$$C^{(n)} \;=\; x_1, \varepsilon_1, x_2, \varepsilon_2, \ldots, x_{|V_n|}, \varepsilon_{|V_n|}, x_1$$

with $\{x_1, \ldots, x_{|V_n|}\} = V_n$ and all x_i and all ε_i pairwise distinct. By Lemma 3.3.11, for each i there are unique $v_i \in V_0$ and $e_i \in E_0$ such that $x_i = \iota_n(v_i)$ and $\varepsilon_i = \iota_n(e_i)$. Then $\{x_i, x_{i+1}\} \subseteq \varepsilon_i$ implies $\{v_i, v_{i+1}\} \subseteq e_i$, and the distinctness/coverage properties pull back via injectivity/bijectivity. Therefore

$$C \;:=\; v_1, e_1, v_2, e_2, \ldots, v_{|V_0|}, e_{|V_0|}, v_1$$

is a Hamiltonian Berge cycle in H. Hence H is Hamiltonian.

The final statement for graphs follows from Remark 3.3.4. □

3.4 Diameter of SuperHyperGraph

The diameter of a graph is the maximum, over all vertex pairs, of the length of a shortest path joining them, or ∞ if the graph is disconnected [66–70]. In the appendix, we note that related concepts to the diameter of a graph, namely the triameter [71–73] and the multiameter, are also known. The diameter of a hypergraph is the maximum, over all vertex pairs, of the length of a shortest Berge path, or ∞ if the hypergraph is disconnected (cf. [74, 75]). The diameter of a SuperHyperGraph is the maximum, over all supervertex pairs, of the length of a shortest Berge superpath through superedges, or ∞ if the SuperHyperGraph is disconnected.

Definition 3.4.1 (Graph, path, distance, diameter)**.** A *(simple undirected) graph* is a pair $G = (V(G), E(G))$ where $V(G)$ is a finite set and $E(G) \subseteq \binom{V(G)}{2}$. A *path of length* $k \in \mathbb{N}$ from x to y is a sequence

$$(x = x_0, e_1, x_1, e_2, \ldots, e_k, x_k = y)$$

such that $x_0, \ldots, x_k$ are distinct vertices, $e_1, \ldots, e_k$ are edges, and $e_i = \{x_{i-1}, x_i\}$ for all i. If there is no path between x and y, we write $d_G(x, y) = \infty$. Otherwise, the *distance* $d_G(x, y)$ is the minimum length of a path from x to y, and $d_G(x, x) = 0$ by convention.

The *diameter* of G is

$$\operatorname{diam}(G) := \max\{\, d_G(x, y) \mid x, y \in V(G) \,\} \in \mathbb{N} \cup \{\infty\}.$$

Equivalently, $\operatorname{diam}(G) = \infty$ if and only if G is disconnected.

Definition 3.4.2 (Hypergraph, Berge path, distance, diameter)**.** A *hypergraph* is a pair $H = (V(H), E(H))$ where $V(H)$ is a finite set and $E(H)$ is a finite family of nonempty subsets of $V(H)$ (called *hyperedges*). A *Berge path of length* $k \in \mathbb{N}$ from x to y is a sequence

$$(x = x_0, e_1, x_1, e_2, \ldots, e_k, x_k = y)$$

such that $x_0, \ldots, x_k$ are distinct vertices, $e_1, \ldots, e_k$ are distinct hyperedges, and $\{x_{i-1}, x_i\} \subseteq e_i$ for all i.

If there is no Berge path between x and y, set $d_H(x, y) = \infty$; otherwise let $d_H(x, y)$ be the minimum length of a Berge path from x to y, and set $d_H(x, x) = 0$.

The *diameter* of H is

$$\operatorname{diam}(H) := \max\{\, d_H(x, y) \mid x, y \in V(H) \,\} \in \mathbb{N} \cup \{\infty\},$$

so $\operatorname{diam}(H) = \infty$ precisely when H is disconnected.

Definition 3.4.3 (Berge path and diameter in an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph in the sense of Definition 2.1.6, so $V \subseteq \mathrm{PS}^n(V_0)$ and $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$.

A *Berge superpath of length* $k \in \mathbb{N}$ from x to y is a sequence

$$(\mathsf{x} = \mathsf{x}_0,\ \mathsf{e}_1,\ \mathsf{x}_1,\ \mathsf{e}_2,\ \ldots,\ \mathsf{e}_k,\ \mathsf{x}_k = \mathsf{y})$$

such that $\mathsf{x}_0, \ldots, \mathsf{x}_k$ are distinct elements of V (supervertices), $\mathsf{e}_1, \ldots, \mathsf{e}_k$ are distinct elements of E (superedges), and $\{\mathsf{x}_{i-1}, \mathsf{x}_i\} \subseteq \mathsf{e}_i$ for all i.

If there is no Berge superpath between x and y, set $d_{\mathrm{SHG}^{(n)}}(\mathsf{x}, \mathsf{y}) = \infty$; otherwise let $d_{\mathrm{SHG}^{(n)}}(\mathsf{x}, \mathsf{y})$ be the minimum length of a Berge superpath from x to y, and set $d_{\mathrm{SHG}^{(n)}}(\mathsf{x}, \mathsf{x}) = 0$.

The *diameter* of $\mathrm{SHG}^{(n)}$ is

$$\operatorname{diam}(\mathrm{SHG}^{(n)}) := \max\{\, d_{\mathrm{SHG}^{(n)}}(\mathsf{x}, \mathsf{y}) \mid \mathsf{x}, \mathsf{y} \in V \,\} \in \mathbb{N} \cup \{\infty\}.$$

Example 3.4.4 (A Berge superpath and the diameter in a 2-SuperHyperGraph)**.** Let the ground set be

$$V_0 = \{1, 2, 3, 4\}, \qquad \mathrm{PS}^2(V_0) = \mathrm{PS}(\mathrm{PS}(V_0)).$$

Define three supervertices (elements of $\mathrm{PS}^2(V_0)$) by

$$\mathsf{x} := \{\{1\}, \{2\}\}, \qquad \mathsf{y} := \{\{2\}, \{3\}\}, \qquad \mathsf{z} := \{\{3\}, \{4\}\}.$$

Let

$$V = \{\mathsf{x}, \mathsf{y}, \mathsf{z}\} \subseteq \mathrm{PS}^2(V_0), \qquad E = \{\mathsf{e}_1, \mathsf{e}_2\},$$

where the superedges are subsets of V given by

$$\mathsf{e}_1 := \{\mathsf{x}, \mathsf{y}\}, \qquad \mathsf{e}_2 := \{\mathsf{y}, \mathsf{z}\}.$$

Then $\mathrm{SHG}^{(2)} = (V, E)$ is a connected 2-SuperHyperGraph.

A Berge superpath of length 2 from x to z is

$$(\mathsf{x},\ \mathsf{e}_1,\ \mathsf{y},\ \mathsf{e}_2,\ \mathsf{z}),$$

since $\{\mathsf{x}, \mathsf{y}\} \subseteq \mathsf{e}_1$ and $\{\mathsf{y}, \mathsf{z}\} \subseteq \mathsf{e}_2$, with all supervertices and superedges in the sequence distinct.

Moreover, the superdistances are

$$d_{\mathrm{SHG}^{(2)}}(\mathsf{x}, \mathsf{y}) = 1, \qquad d_{\mathrm{SHG}^{(2)}}(\mathsf{y}, \mathsf{z}) = 1, \qquad d_{\mathrm{SHG}^{(2)}}(\mathsf{x}, \mathsf{z}) = 2,$$

and $d_{\mathrm{SHG}^{(2)}}(\mathsf{u}, \mathsf{u}) = 0$ for $\mathsf{u} \in V$ by convention. Hence

$$\mathrm{diam}(\mathrm{SHG}^{(2)}) = \max\{d_{\mathrm{SHG}^{(2)}}(\mathsf{u}, \mathsf{v}) : \mathsf{u}, \mathsf{v} \in V\} = 2.$$

Theorem 3.4.5 (Reduction to hypergraphs and graphs)**.** *Let V_0 be a finite nonempty set.*

(a) *(*Hypergraphs as 0-SuperHyperGraphs*) Let $H = (V_0, E)$ be a hypergraph on V_0. Define the 0-SuperHyperGraph $\mathrm{SHG}^{(0)}_H = (V, E)$ by $V := V_0 = \mathrm{PS}^0(V_0)$ and the same edge family E. Then, for all $x, y \in V_0$,*

$$d_{\mathrm{SHG}^{(0)}_H}(x, y) = d_H(x, y), \qquad \textit{and hence} \qquad \mathrm{diam}(\mathrm{SHG}^{(0)}_H) = \mathrm{diam}(H).$$

(b) *(*Graphs as a special case*) Let $G = (V_0, E_G)$ be a simple graph and regard it as the hypergraph $H_G = (V_0, E_G)$ where each edge has size 2. Then, with $\mathrm{SHG}^{(0)}_{H_G}$ as in (a), we have*

$$d_{\mathrm{SHG}^{(0)}_{H_G}}(x, y) = d_G(x, y) \qquad \textit{and} \qquad \mathrm{diam}(\mathrm{SHG}^{(0)}_{H_G}) = \mathrm{diam}(G).$$

Consequently, Definition 3.4.3 genuinely extends (i) diameter of hypergraphs and (ii) diameter of graphs.

Proof. (a) By construction, $\mathrm{SHG}_H^{(0)}$ has the same vertex set and the same family of edges as H. Moreover, the definition of a Berge superpath in $\mathrm{SHG}_H^{(0)}$ is verbatim the definition of a Berge path in H: a sequence $(x = x_0, e_1, x_1, \ldots, e_k, x_k = y)$ is admissible in $\mathrm{SHG}_H^{(0)}$ if and only if it is admissible in H. Therefore the sets of permissible path lengths from x to y coincide, hence their minima coincide: $d_{\mathrm{SHG}_H^{(0)}}(x, y) = d_H(x, y)$ for all $x, y \in V_0$. Taking the maximum over all ordered pairs yields $\mathrm{diam}(\mathrm{SHG}_H^{(0)}) = \mathrm{diam}(H)$.

(b) If G is a simple graph, then a graph path $(x = x_0, \{x_0, x_1\}, x_1, \ldots, \{x_{k-1}, x_k\}, x_k = y)$ is exactly a Berge path in the 2-uniform hypergraph H_G. Thus $d_{H_G}(x, y) = d_G(x, y)$ for all x, y. Applying part (a) to H_G gives $d_{\mathrm{SHG}_{H_G}^{(0)}}(x, y) = d_{H_G}(x, y) = d_G(x, y)$ and hence $\mathrm{diam}(\mathrm{SHG}_{H_G}^{(0)}) = \mathrm{diam}(G)$. □

Remark 3.4.6 (Connectedness and infinite diameter)**.** In each of Definitions 3.4.1, 3.4.2, and 3.4.3, the diameter is ∞ exactly when the corresponding object is disconnected (i.e., some pair of vertices admits no path of the relevant type).

3.5 Girth of SuperHyperGraph

The girth of a graph is the length of its shortest cycle, and it is defined as ∞ when the graph is acyclic [76–80]. Related notions such as the odd girth [81, 82] and even girth [78, 83], and the circumference of a graph [84], are also well known. The girth of a hypergraph is the length of its shortest Berge cycle, namely the shortest alternating cycle of distinct vertices and distinct hyperedges, and it is ∞ if no Berge cycle exists [85–88]. The girth of a SuperHyperGraph is the length of its shortest super-Berge cycle on supervertices and superedges, and it is ∞ when no such cycle exists.

Definition 3.5.1 (Cycle and girth of a graph)**.** Let $G = (V, E)$ be a finite simple graph, where $E \subseteq \binom{V}{2}$. A *(graph) cycle* of length $\ell \geq 3$ in G is a sequence of distinct vertices

$$v_1, v_2, \ldots, v_\ell \in V$$

such that $\{v_i, v_{i+1}\} \in E$ for all $i = 1, \ldots, \ell$, where indices are taken modulo ℓ (i.e., $v_{\ell+1} = v_1$).

The *girth* of G is

$$\mathrm{girth}(G) := \begin{cases} \min\{\ell \geq 3 : \ G \text{ contains a cycle of length } \ell\}, & \text{if } G \text{ contains a cycle,} \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 3.5.2 (Berge cycle and girth of a hypergraph)**.** Let $H = (V, E)$ be a finite hypergraph, where E is a family of nonempty subsets of V. A *Berge ℓ-cycle* (or *Berge cycle of length ℓ*) in H is a choice of distinct vertices $v_1, \ldots, v_\ell \in V$ and distinct hyperedges $e_1, \ldots, e_\ell \in E$ such that

$$\{v_i, v_{i+1}\} \subseteq e_i \qquad (i = 1, \ldots, \ell),$$

with indices modulo ℓ (so $v_{\ell+1} = v_1$). The *girth* of H is

$$\mathrm{girth}(H) := \begin{cases} \min\{\ell \geq 2 : \ H \text{ contains a Berge } \ell\text{-cycle}\}, & \text{if } H \text{ contains a Berge cycle,} \\ \infty, & \text{otherwise.} \end{cases}$$

Remark 3.5.3. A Berge 2-cycle consists of two distinct hyperedges sharing at least two distinct vertices. Thus, forbidding Berge 2-cycles is equivalent to *linearity* of the hypergraph.

Definition 3.5.4 (Super-Berge cycle)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A *super-Berge ℓ-cycle* in $\mathrm{SHG}^{(n)}$ is a choice of distinct supervertices $X_1, \dots, X_\ell \in V$ and distinct superedges $F_1, \dots, F_\ell \in E$ such that

$$\{X_i, X_{i+1}\} \subseteq F_i \qquad (i = 1, \dots, \ell),$$

with indices modulo ℓ (so $X_{\ell+1} = X_1$).

Definition 3.5.5 (Girth of a SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. Define its *girth* by

$$\mathrm{girth}(\mathrm{SHG}^{(n)}) := \begin{cases} \min\{\ell \geq 2 : \ \mathrm{SHG}^{(n)} \text{ contains a super-Berge } \ell\text{-cycle}\}, & \text{if such a cycle exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

A concrete example is given below.

Example 3.5.6 (Girth of a SuperHyperGraph)**.** Let $V_0 = \{1, 2, 3\}$ and consider the 2-fold powerset $\mathrm{PS}^2(V_0) = \mathrm{PS}(\mathrm{PS}(V_0))$. Define three supervertices

$$X_1 := \{\{1\}, \{2\}\}, \qquad X_2 := \{\{2\}, \{3\}\}, \qquad X_3 := \{\{1\}, \{3\}\},$$

and set

$$V = \{X_1, X_2, X_3\} \subseteq \mathrm{PS}^2(V_0).$$

Let the superedge family be

$$E = \{F_1, F_2, F_3\}, \qquad F_1 := \{X_1, X_2\}, \quad F_2 := \{X_2, X_3\}, \quad F_3 := \{X_3, X_1\}.$$

Then $\mathrm{SHG}^{(2)} = (V, E)$ is a 2-SuperHyperGraph.

The alternating sequence

$$(X_1,\ F_1,\ X_2,\ F_2,\ X_3,\ F_3,\ X_1)$$

is a super-Berge 3-cycle: the supervertices X_1, X_2, X_3 are distinct, the superedges F_1, F_2, F_3 are distinct, and $\{X_i, X_{i+1}\} \subseteq F_i$ holds for $i = 1, 2, 3$ (with indices taken modulo 3).

There is no super-Berge 2-cycle in $\mathrm{SHG}^{(2)}$, because any two distinct superedges among F_1, F_2, F_3 intersect in exactly one supervertex (e.g., $F_1 \cap F_2 = \{X_2\}$), so they do not share two distinct supervertices. Hence the shortest super-Berge cycle has length 3, and therefore

$$\mathrm{girth}(\mathrm{SHG}^{(2)}) = 3.$$

Definition 3.5.7 (Canonical embedding of hypergraphs/graphs into SuperHyperGraphs)**.** Let $H = (V_0, E)$ be a hypergraph. Define the associated 0-SuperHyperGraph

$$\iota(H) := (V, E) \quad \text{with} \quad V := V_0 \subseteq \mathrm{PS}^0(V_0), \ \ E := E.$$

If $G = (V_0, E)$ is a simple graph (so $E \subseteq \binom{V_0}{2}$), we view it as a hypergraph with 2-element hyperedges and apply the same construction, obtaining $\iota(G)$.

Theorem 3.5.8 (Generalization property of girth)**.**

(i) *For every finite hypergraph H, we have*

$$\operatorname{girth}(\iota(H)) = \operatorname{girth}(H),$$

where $\operatorname{girth}(H)$ *is the Berge girth from Definition 3.5.2 and* $\operatorname{girth}(\iota(H))$ *is the SuperHyperGraph girth from Definition 3.5.5.*

(ii) *For every finite simple graph G, we have*

$$\operatorname{girth}(\iota(G)) = \operatorname{girth}(G),$$

where $\operatorname{girth}(G)$ *is the usual graph girth from Definition 3.5.1.*

Consequently, SuperHyperGraph girth strictly extends (generalizes) both graph girth and hypergraph (Berge) girth.

Proof. (i) Let $H = (V_0, E)$ be a hypergraph and $\iota(H) = (V, E)$ the associated 0-SuperHyperGraph, so $V = V_0$ and the superedges are exactly the hyperedges. A super-Berge ℓ-cycle in $\iota(H)$ is, by Definition 3.5.4, a sequence of distinct vertices $X_1, \ldots, X_\ell \in V$ and distinct edges $F_1, \ldots, F_\ell \in E$ with $\{X_i, X_{i+1}\} \subseteq F_i$. Since here $V = V_0$ and E is the hyperedge family, this is precisely a Berge ℓ-cycle in H as in Definition 3.5.2. Therefore, $\iota(H)$ contains a super-Berge ℓ-cycle iff H contains a Berge ℓ-cycle, for every $\ell \geq 2$, and taking the minimum length yields $\operatorname{girth}(\iota(H)) = \operatorname{girth}(H)$ (including the case ∞).

(ii) Let $G = (V_0, E)$ be a simple graph and consider $\iota(G)$. A Berge ℓ-cycle in the 2-uniform hypergraph G is given by distinct vertices $v_1, \ldots, v_\ell$ and distinct edges $e_1, \ldots, e_\ell$ with $\{v_i, v_{i+1}\} \subseteq e_i$. Because each e_i has size 2, we must have $e_i = \{v_i, v_{i+1}\}$ for all i. Hence this Berge ℓ-cycle is exactly a graph cycle of length ℓ. Conversely, every graph cycle $v_1 \cdots v_\ell v_1$ defines a Berge ℓ-cycle by taking $e_i = \{v_i, v_{i+1}\}$. Thus G has a graph cycle of length ℓ iff $\iota(G)$ has a super-Berge ℓ-cycle. Minimizing over ℓ gives $\operatorname{girth}(\iota(G)) = \operatorname{girth}(G)$ (and ∞ in the acyclic case). □

3.6 Independence number of SuperHyperGraph

The independence number of a graph is the maximum size of a vertex set that contains no edge, equivalently a set in which no two vertices are adjacent [89–93]. Research on related notions such as maximum independent sets [94, 95], maximum-weight independent sets [96, 97], and maximal independent set listing [98] is also very active. The independence number of a hypergraph is the maximum size of a vertex subset that contains no hyperedge in its entirety, meaning no edge is contained in the set [99–102]. The independence number of a SuperHyperGraph is the maximum number of supervertices that can be chosen so that no superedge is contained as a subset of the chosen supervertices.

Definition 3.6.1 (Independent set and independence number in graphs)**.** [89–91] A *(simple, undirected) graph* is a pair $G = (V(G), E(G))$ where $V(G)$ is a finite set and $E(G) \subseteq \binom{V(G)}{2}$. A subset $I \subseteq V(G)$ is an *independent set* of G if no edge of G is contained in I, i.e.,

$$\forall \{u, v\} \in E(G) \quad \{u, v\} \nsubseteq I \qquad \text{(equivalently, no two vertices of } I \text{ are adjacent).}$$

The *independence number* of G is

$$\alpha(G) := \max\{\, |I| : I \subseteq V(G) \text{ is independent in } G \,\}.$$

Definition 3.6.2 (Hypergraph, independent set, and independence number)**.** A *hypergraph* is a pair $H = (V(H), E(H))$ where $V(H)$ is a finite set and $E(H) \subseteq \mathrm{PS}(V(H)) \setminus \{\emptyset\}$. A subset $I \subseteq V(H)$ is an *independent set* of H if it contains no hyperedge, i.e.,

$$\forall\, e \in E(H) \quad e \nsubseteq I.$$

The *independence number* of H is

$$\alpha(H) \;:=\; \max\{\, |I| : \; I \subseteq V(H) \text{ is independent in } H \,\}.$$

Lemma 3.6.3 (Graphs as 2-uniform hypergraphs)**.** *Let $G = (V, E) \subseteq \binom{V}{2}$ be a graph, and let $H_G := (V, E)$ be the corresponding 2-uniform hypergraph. Then a set $I \subseteq V$ is independent in G (Definition 3.6.1) if and only if I is independent in H_G (Definition 3.6.2). Consequently,*

$$\alpha(G) = \alpha(H_G).$$

Proof. If I is independent in G, then no $\{u, v\} \in E$ is contained in I, hence (viewing each graph-edge as a hyperedge) no $e \in E(H_G)$ is contained in I, so I is independent in H_G. Conversely, if I is independent in H_G, then no $e = \{u, v\} \in E$ is contained in I, so no two vertices of I are adjacent in G, hence I is independent in G. Taking maxima over $|I|$ yields $\alpha(G) = \alpha(H_G)$. □

Definition 3.6.4 (Independent set and independence number in n-SuperHyperGraphs)**.** Let $n \geq 0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph in the sense of Definition 2.1.6 (so $V \subseteq \mathrm{PS}^n(V_0)$ and $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$). A subset $I \subseteq V$ is an *independent set* of $\mathrm{SHG}^{(n)}$ if it contains no n-superedge, i.e.,

$$\forall\, \mathsf{e} \in E \quad \mathsf{e} \nsubseteq I.$$

The *independence number* of $\mathrm{SHG}^{(n)}$ is

$$\alpha(\mathrm{SHG}^{(n)}) \;:=\; \max\{\, |I| : \; I \subseteq V \text{ is independent in } \mathrm{SHG}^{(n)} \,\}.$$

A concrete example is given below.

Example 3.6.5 (Independent sets in a 1-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b, c\}$ and take $n = 1$. Consider the following 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ on V_0:

$$V \;=\; \{\{a\}, \{b\}, \{c\}\} \;\subseteq\; \mathrm{PS}^1(V_0) = \mathrm{PS}(V_0), \qquad E \;=\; \{\; \{\{a\}, \{b\}\},\; \{\{b\}, \{c\}\} \;\} \;\subseteq\; \mathrm{PS}(V) \backslash \{\emptyset\}.$$

Thus the supervertices are the singletons $\{a\}, \{b\}, \{c\}$, and there are two superedges: one connecting $\{a\}$ with $\{b\}$, and another connecting $\{b\}$ with $\{c\}$.

We now list all independent sets $I \subseteq V$. By Definition 3.6.4, a set I is independent if it contains no superedge as a subset. Here this means that I must not contain *both* $\{a\}$ and $\{b\}$ simultaneously, and also must not contain *both* $\{b\}$ and $\{c\}$ simultaneously. Consequently, the independent sets are exactly

$$\emptyset, \quad \{\{a\}\}, \quad \{\{b\}\}, \quad \{\{c\}\}, \quad \{\{a\}, \{c\}\}.$$

In particular, the set $\{\{a\}, \{c\}\}$ is independent because it does not contain either superedge $\{\{a\}, \{b\}\}$ or $\{\{b\}, \{c\}\}$ as a subset. On the other hand, $\{\{a\}, \{b\}\}$ is *not* independent, since it contains the superedge $\{\{a\}, \{b\}\}$, and similarly $\{\{b\}, \{c\}\}$ is not independent.

Therefore the maximum size of an independent set is 2, attained by $\{\{a\}, \{c\}\}$, and hence

$$\alpha\Big(\mathrm{SHG}^{(1)}\Big) = 2.$$

Definition 3.6.6 (Iterated singleton lift)**.** For $n \geq 0$, define the map ι_n recursively by

$$\iota_0(x) := x, \qquad \iota_{n+1}(x) := \{\iota_n(x)\}.$$

Thus $\iota_n : V \to \mathrm{PS}^n(V)$ is injective for every finite set V. Given a hypergraph $H = (V, E)$ and $n \geq 0$, define its *n-lift* by

$$\mathrm{Lift}_n(H) \ := \ \bigl(V^{(n)}, E^{(n)}\bigr), \qquad V^{(n)} := \iota_n(V) \subseteq \mathrm{PS}^n(V),$$

$$E^{(n)} := \{ \ \iota_n(e) \ := \ \{\iota_n(v) : v \in e\} \ : \ e \in E \ \}.$$

Theorem 3.6.7 (SuperHyperGraphs generalize hypergraphs and graphs)**.**

(i) Every hypergraph $H = (V, E)$ is (canonically) a 0-SuperHyperGraph $\mathrm{SHG}^{(0)} = H$, and

$$\alpha(\mathrm{SHG}^{(0)}) = \alpha(H).$$

(ii) Every graph $G = (V, E)$ is (canonically) a 0-SuperHyperGraph (and hence a SuperHyperGraph), and with $\mathrm{SHG}^{(0)} = G$ one has

$$\alpha(\mathrm{SHG}^{(0)}) = \alpha(G).$$

(iii) More generally, for every hypergraph H and every $n \geq 0$, the n-lift $\mathrm{Lift}_n(H)$ (Definition 3.8.12) is an n-SuperHyperGraph and preserves the independence number:

$$\alpha\bigl(\mathrm{Lift}_n(H)\bigr) = \alpha(H).$$

In particular, for every graph G and every $n \geq 0$,

$$\alpha\bigl(\mathrm{Lift}_n(G)\bigr) = \alpha(G).$$

Proof. (i) If $n = 0$, then $\mathrm{PS}^0(V_0) = V_0$ and Definition 2.1.6 reduces to $V \subseteq V_0$ and $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$, i.e. precisely an ordinary hypergraph. Definition 3.6.4 then coincides with Definition 3.6.2, so the maxima agree and $\alpha(\mathrm{SHG}^{(0)}) = \alpha(H)$.

(ii) A graph is a special case of a hypergraph in which every edge has size 2. Thus (ii) follows from (i) together with Lemma 3.6.3.

(iii) Let $H = (V, E)$ and fix $n \geq 0$. By construction, $V^{(n)} = \iota_n(V) \subseteq \mathrm{PS}^n(V)$ and $E^{(n)} \subseteq \mathrm{PS}(V^{(n)}) \setminus \{\emptyset\}$, hence $\mathrm{Lift}_n(H)$ is an n-SuperHyperGraph.

Define $\Phi : \mathrm{PS}(V) \to \mathrm{PS}(V^{(n)})$ by $\Phi(I) := \iota_n(I) = \{\iota_n(v) : v \in I\}$. Since ι_n is injective, Φ is a bijection between subsets of V and subsets of $V^{(n)}$, and it preserves cardinalities: $|\Phi(I)| = |I|$.

We claim that $I \subseteq V$ is independent in H if and only if $\Phi(I) \subseteq V^{(n)}$ is independent in $\mathrm{Lift}_n(H)$. Indeed,

$$I \text{ is independent in } H \iff \forall\, e \in E,\ e \nsubseteq I \iff \forall\, e \in E,\ \iota_n(e) \nsubseteq \iota_n(I) \iff \forall\, \mathsf{e} \in E^{(n)},\ \mathsf{e} \nsubseteq \Phi(I),$$

which is exactly the independence condition in $\mathrm{Lift}_n(H)$ (Definition 3.6.4). Therefore Φ induces a size-preserving bijection between independent sets of H and independent sets of $\mathrm{Lift}_n(H)$, and taking maxima yields $\alpha(\mathrm{Lift}_n(H)) = \alpha(H)$. The graph case is the special case $H = G$. □

3.7 Arboricity of SuperHyperGraph

The arboricity of a graph is the minimum number of forests whose edge-disjoint union equals the graph's edge set, quantifying how tree-like the graph is [103–105]. The arboricity of a hypergraph is the minimum number of hyperforests that partition its hyperedges, where every nonempty vertex subset X satisfies $|E[X]| \le |X| - 1$ (cf. [106–108]). The arboricity of a SuperHyperGraph is the minimum number of superhyperforests that partition its superedges, using the same condition $|E[X]| \le |X| - 1$ on subsets of supervertices.

Definition 3.7.1 (Arboricity of a graph)**.** Let $G = (V, E)$ be a finite simple undirected graph. The *arboricity* of G is

$$\operatorname{arb}(G) \ := \ \min\Big\{\, k \in \mathbb{N} : \ E = E_1 \dot{\cup} \cdots \dot{\cup} E_k \text{ and } (V, E_i) \text{ is a forest for each } i \,\Big\},$$

where $\dot{\cup}$ denotes a disjoint union of edge sets.

Remark 3.7.2 (Nash–Williams formula (optional))**.** Classically,

$$\operatorname{arb}(G) \ = \ \max_{\emptyset \neq X \subseteq V,\ |X| \ge 2} \left\lceil \frac{|E[X]|}{|X| - 1} \right\rceil, \qquad E[X] := \{\{u, v\} \in E : \ u, v \in X\}.$$

We do *not* need this characterization for the generalization theorem below.

Definition 3.7.3 (Hyperforest)**.** A hypergraph $H = (V, E)$ is called a *hyperforest* if for every nonempty subset $X \subseteq V$,

$$|E[X]| \ \le \ |X| - 1.$$

Definition 3.7.4 (Arboricity of a hypergraph)**.** Let $H = (V, E)$ be a finite hypergraph. The *arboricity* of H is

$$\operatorname{arb}(H) \ := \ \min\Big\{\, k \in \mathbb{N} : \ E = E_1 \dot{\cup} \cdots \dot{\cup} E_k \text{ and } (V, E_i) \text{ is a hyperforest for each } i \,\Big\}.$$

Definition 3.7.5 (Superhyperforest and arboricity)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (Definition 2.1.6), so $V \subseteq \mathrm{PS}^n(V_0)$ and $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$. For $X \subseteq V$ define

$$E[X] \ := \ \{\, F \in E : \ F \subseteq X \,\}.$$

We call $\mathrm{SHG}^{(n)}$ a *superhyperforest* if for every nonempty $X \subseteq V$,

$$|E[X]| \ \le \ |X| - 1.$$

The *arboricity* of $\mathrm{SHG}^{(n)}$ is

$$\operatorname{arb}(\mathrm{SHG}^{(n)}) \ := \ \min\Big\{\, k \in \mathbb{N} : \ E = E_1 \dot{\cup} \cdots \dot{\cup} E_k \text{ and } (V, E_i) \text{ is a superhyperforest for each } i \,\Big\}.$$

A concrete example is given below.

Example 3.7.6 (A superhyperforest and its arboricity)**.** Let the base set be $V_0 = \{a, b, c, d\}$ and take $n = 1$. Consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ defined by

$$V \;=\; \{\{a\}, \{b\}, \{c\}, \{d\}\} \subseteq \mathrm{PS}^1(V_0) = \mathrm{PS}(V_0), \qquad E \;=\; \{\, \{\{a\}, \{b\}\},\ \{\{b\}, \{c\}\},\ \{\{c\}, \{d\}\}\,\}.$$

Thus $\mathrm{SHG}^{(1)}$ is simply a "path" on four supervertices.

We claim that $\mathrm{SHG}^{(1)}$ is a *superhyperforest* in the sense of Definition 3.7.5. Let $X \subseteq V$ be nonempty. The set $E[X]$ consists of those superedges whose two endpoints both lie in X. Since the underlying incidence pattern is a path, the substructure induced by X cannot contain more superedges than $|X| - 1$; indeed, any subset of vertices of a path spans at most $|X| - 1$ edges. Hence $|E[X]| \le |X| - 1$ for all nonempty $X \subseteq V$, and $\mathrm{SHG}^{(1)}$ is a superhyperforest.

Next we compute the arboricity. Because $\mathrm{SHG}^{(1)}$ itself is already a superhyperforest, we may take $k = 1$ and $E_1 = E$ in Definition 3.7.5. Therefore,

$$\mathrm{arb}\Big(\mathrm{SHG}^{(1)}\Big) = 1.$$

Definition 3.7.7 (Iterated singleton embedding and n-lift)**.** Fix a finite nonempty base set V_0 and $n \ge 0$. Define maps $\iota_n : V_0 \to \mathrm{PS}^n(V_0)$ recursively by

$$\iota_0(v) := v, \qquad \iota_{m+1}(v) := \{\iota_m(v)\} \quad (m \ge 0).$$

For $A \subseteq V_0$, set $\iota_n(A) := \{\iota_n(v) : v \in A\}$.

Given a hypergraph $H = (V_0, E_0)$, define its *n-lift* to an n-SuperHyperGraph by

$$\mathrm{Lift}_n(H) \;:=\; (V_n, E_n), \qquad V_n := \iota_n(V_0), \qquad E_n := \{\iota_n(e) : e \in E_0\}.$$

Lemma 3.7.8 (Induced-edge counts are preserved by the lift)**.** *Let $H = (V_0, E_0)$ be a hypergraph and $n \ge 0$. For every subset $X \subseteq V_0$,*

$$|E_0[X]| \;=\; |E_n[\iota_n(X)]|,$$

where $\mathrm{Lift}_n(H) = (V_n, E_n)$.

Proof. By construction, the map $E_0 \to E_n$, $e \mapsto \iota_n(e)$ is a bijection. Moreover, $e \subseteq X$ holds if and only if $\iota_n(e) \subseteq \iota_n(X)$. Therefore the bijection restricts to a bijection between $E_0[X]$ and $E_n[\iota_n(X)]$, hence the cardinalities agree. □

Theorem 3.7.9 (Arboricity of SuperHyperGraphs generalizes graph and hypergraph arboricity)**.** *Let $H = (V_0, E_0)$ be any finite hypergraph and $n \ge 0$. Then*

$$\mathrm{arb}\big(\mathrm{Lift}_n(H)\big) \;=\; \mathrm{arb}(H).$$

In particular:

(a) *If* $n = 0$*, then* $\mathrm{Lift}_0(H) = H$ *(as a* 0*-SuperHyperGraph), so* $\mathrm{arb}(\mathrm{SHG}^{(0)})$ *restricts to the hypergraph arboricity* $\mathrm{arb}(H)$*.*

(b) *If* G *is a graph, viewed as a* 2*-uniform hypergraph, then for every* $n \geq 0$*,*

$$\mathrm{arb}\big(\mathrm{Lift}_n(G)\big) \;=\; \mathrm{arb}(G),$$

so $\mathrm{arb}(\mathrm{SHG}^{(n)})$ *also generalizes the usual graph arboricity.*

Proof. Write $\mathrm{Lift}_n(H) = (V_n, E_n)$.

Step 1: hyperforest property is preserved. Let $F = (V_0, F_0)$ be a spanning subhypergraph of H (so $F_0 \subseteq E_0$), and let $\mathrm{Lift}_n(F) = (V_n, F_n)$ where $F_n = \{\iota_n(e) : e \in F_0\} \subseteq E_n$. By Lemma 3.7.8, for every $X \subseteq V_0$,

$$|F_0[X]| \;=\; |F_n[\iota_n(X)]|.$$

Hence $|F_0[X]| \leq |X| - 1$ for all nonempty $X \subseteq V_0$ if and only if $|F_n[Y]| \leq |Y| - 1$ for all nonempty $Y \subseteq V_n$ (take $Y = \iota_n(X)$ and use that $\iota_n : V_0 \to V_n$ is a bijection). Therefore,

$$(V_0, F_0) \text{ is a hyperforest} \quad\Longleftrightarrow\quad (V_n, F_n) \text{ is a superhyperforest.}$$

Step 2: arboricity is preserved. Suppose $E_0 = F_0^{(1)} \dot\cup \cdots \dot\cup F_0^{(k)}$ is a partition of E_0 such that each $(V_0, F_0^{(i)})$ is a hyperforest. Applying ι_n edgewise yields a partition

$$E_n = F_n^{(1)} \dot\cup \cdots \dot\cup F_n^{(k)}, \qquad F_n^{(i)} := \{\iota_n(e) : e \in F_0^{(i)}\},$$

and by Step 1 each $(V_n, F_n^{(i)})$ is a superhyperforest. Hence $\mathrm{arb}(\mathrm{Lift}_n(H)) \leq \mathrm{arb}(H)$.

Conversely, any partition of E_n into k superhyperforests pulls back via the inverse bijection $E_n \to E_0$ to a partition of E_0 into k hyperforests, again by Step 1. Thus $\mathrm{arb}(H) \leq \mathrm{arb}(\mathrm{Lift}_n(H))$.

Combining the inequalities yields $\mathrm{arb}(\mathrm{Lift}_n(H)) = \mathrm{arb}(H)$. ☐

3.8 Boxicity of SuperHyperGraph

The boxicity of a graph is the minimum dimension d for which the vertices can be represented by axis-parallel d-dimensional boxes, such that two vertices are adjacent if and only if the corresponding boxes intersect [109–112]. Related parameters to boxicity, including *cubicity* [112–114] and *sphericity* [115–117], are also well known, and a wide range of analogous studies have been carried out. The boxicity of a hypergraph is defined as the boxicity of its primal (2-section) graph, in which two vertices are adjacent if and only if there exists a hyperedge containing both. The boxicity of a SuperHyperGraph is defined as the boxicity of its primal graph on supervertices, where two supervertices are adjacent precisely when they occur together in some superedge.

Definition 3.8.1 (d-boxes and intersection graphs)**.** [109–112] A *closed interval* in $\mathbb{R}$ is a set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ with $a \leq b$. For an integer $d \geq 1$, a *d-box* (axis-parallel box) is a Cartesian product of d closed intervals,

$$B \;=\; [a_1, b_1] \times \cdots \times [a_d, b_d] \;\subseteq\; \mathbb{R}^d.$$

Given a family $\mathcal{F}$ of sets, the *intersection graph* $\mathrm{Int}(\mathcal{F})$ is the graph whose vertex set is $\mathcal{F}$ and in which two distinct vertices $X, Y \in \mathcal{F}$ are adjacent if and only if $X \cap Y \neq \emptyset$.

Definition 3.8.2 (d-box representation and boxicity of a graph)**.** Let $G = (V(G), E(G))$ be a finite simple undirected graph and let $d \geq 1$. A *d-box representation* of G is a map

$$f : V(G) \longrightarrow \{d\text{-boxes in } \mathbb{R}^d\}$$

such that for all distinct $u, v \in V(G)$,

$$\{u, v\} \in E(G) \quad\Longleftrightarrow\quad f(u) \cap f(v) \neq \emptyset.$$

The *boxicity* of G, denoted $\mathrm{box}(G)$, is the minimum integer $d \geq 1$ such that G admits a d-box representation.

Remark 3.8.3 (Equivalent interval-graph formulation)**.** Equivalently, $\mathrm{box}(G)$ is the least $k \geq 1$ for which there exist interval graphs $I_1, \ldots, I_k$ on the common vertex set $V(G)$ such that

$$G \;=\; I_1 \cap \cdots \cap I_k,$$

where the intersection of graphs means intersection of their edge sets.

Definition 3.8.4 (Primal (2-section) graph of a hypergraph)**.** Let $H = (V(H), E(H))$ be a finite hypergraph, where $V(H)$ is a finite set and $E(H) \subseteq \mathrm{PS}(V(H)) \setminus \{\emptyset\}$ is a finite family of nonempty subsets of $V(H)$. The *primal graph* (also called the *2-section*) of H is the graph

$$\mathrm{Pr}(H) \;:=\; \bigl(V(H),\, E(\mathrm{Pr}(H))\bigr),$$

where, for distinct $u, v \in V(H)$,

$$\{u, v\} \in E(\mathrm{Pr}(H)) \quad\Longleftrightarrow\quad \exists\, e \in E(H) \;\text{ with }\; \{u, v\} \subseteq e.$$

Definition 3.8.5 (Boxicity of a hypergraph)**.** The *boxicity* of a hypergraph H is defined by

$$\mathrm{box}(H) \;:=\; \mathrm{box}\bigl(\mathrm{Pr}(H)\bigr).$$

A concrete example is given below.

Example 3.8.6 (Boxicity of a hypergraph)**.** Let $H = (V, E)$ be the hypergraph with

$$V = \{1, 2, 3\}, \qquad E = \bigl\{\{1, 2, 3\}\bigr\}.$$

Then any two distinct vertices of H lie together in the unique hyperedge $\{1, 2, 3\}$. Hence the primal graph $\mathrm{Pr}(H)$ has vertex set $\{1, 2, 3\}$ and all three possible edges:

$$E\bigl(\mathrm{Pr}(H)\bigr) = \bigl\{\{1, 2\}, \{2, 3\}, \{1, 3\}\bigr\},$$

so $\mathrm{Pr}(H) \cong K_3$.

Since K_3 is an interval graph (for instance, represent the vertices by intervals $[1, 3]$, $[2, 4]$, and $[3, 5]$, which pairwise intersect), we have $\mathrm{box}(K_3) = 1$. Therefore,

$$\mathrm{box}(H) = \mathrm{box}\bigl(\mathrm{Pr}(H)\bigr) = \mathrm{box}(K_3) = 1.$$

Definition 3.8.7 (Primal graph of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph in the sense of Definition 2.1.6. Its *primal graph* is the simple graph

$$\Pr(\mathrm{SHG}^{(n)}) \;:=\; \bigl(V,\, E(\Pr(\mathrm{SHG}^{(n)}))\bigr),$$

where, for distinct $X, Y \in V$,

$$\{X, Y\} \in E(\Pr(\mathrm{SHG}^{(n)})) \quad\Longleftrightarrow\quad \exists\, F \in E \;\text{ with }\; \{X, Y\} \subseteq F.$$

A concrete example is given below.

Example 3.8.8 (Primal graph of an n-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b, c\}$ and take $n = 1$. Define the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ by

$$V = \{\{a\}, \{b\}, \{c\}\} \subseteq \mathrm{PS}^1(V_0) = \mathrm{PS}(V_0), \qquad E = \{\ F\ \}, \quad F := \{\{a\}, \{b\}, \{c\}\}.$$

Thus there is a single superedge containing all three supervertices. By Definition 3.8.7, two distinct supervertices are adjacent in the primal graph if and only if they lie together in some superedge; here they all lie together in F. Hence

$$\Pr(\mathrm{SHG}^{(1)}) \cong K_3,$$

i.e., the primal graph is the complete graph on the vertex set $\{\{a\}, \{b\}, \{c\}\}$.

Definition 3.8.9 (Boxicity of an n-SuperHyperGraph)**.** The *boxicity* of an n-SuperHyperGraph $\mathrm{SHG}^{(n)}$ is

$$\operatorname{box}\Bigl(\mathrm{SHG}^{(n)}\Bigr) \;:=\; \operatorname{box}\Bigl(\Pr(\mathrm{SHG}^{(n)})\Bigr).$$

Example 3.8.10 (Boxicity of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(1)} = (V, E)$ be the 1-SuperHyperGraph from Example 3.8.8, so that $\Pr(\mathrm{SHG}^{(1)}) \cong K_3$. Since K_3 is an interval graph (for instance, represent its vertices by the intervals $[1, 3]$, $[2, 4]$, and $[3, 5]$, which pairwise intersect), we have $\operatorname{box}(K_3) = 1$. Therefore, by Definition 3.8.9,

$$\operatorname{box}\Bigl(\mathrm{SHG}^{(1)}\Bigr) \;=\; \operatorname{box}\Bigl(\Pr(\mathrm{SHG}^{(1)})\Bigr) \;=\; \operatorname{box}(K_3) \;=\; 1.$$

Definition 3.8.11 (Viewing a graph as a hypergraph)**.** Given a finite simple graph $G = (V, E)$, define the associated 2-uniform hypergraph

$$H_G \;:=\; (V, E),$$

i.e., the same vertex set and the same edge family, now regarded as 2-element hyperedges.

Definition 3.8.12 (Level-n singleton lift)**.** Fix a finite nonempty base set V_0. For each integer $n \ge 0$, define $\iota_n : V_0 \to \mathrm{PS}^n(V_0)$ recursively by

$$\iota_0(v) = v, \qquad \iota_{n+1}(v) = \{\iota_n(v)\} \quad (v \in V_0).$$

Given a hypergraph $H = (V_0, E)$, define its *n-lift* to an n-SuperHyperGraph by

$$\mathrm{Lift}_n(H) \;:=\; (V_n, E_n), \quad V_n := \{\iota_n(v) : v \in V_0\}, \quad E_n := \{\ \{\iota_n(v) : v \in e\}\ :\ e \in E\ \}.$$

Lemma 3.8.13 (Primal graph is preserved by the singleton lift)**.** *For every hypergraph* $H = (V_0, E)$ *and every* $n \geq 0$*, the map* $\iota_n : V_0 \to V_n$ *induces a graph isomorphism*

$$\mathrm{Pr}(H) \;\cong\; \mathrm{Pr}(\mathrm{Lift}_n(H)).$$

Proof. By construction, ι_n is injective and $V_n = \iota_n(V_0)$, hence it is a bijection $V_0 \to V_n$. For distinct $u, v \in V_0$,

$$\{u, v\} \in E(\mathrm{Pr}(H)) \iff \exists\, e \in E \text{ with } \{u, v\} \subseteq e$$

$$\iff \exists\, e \in E \text{ with } \{\iota_n(u), \iota_n(v)\} \subseteq \{\iota_n(x) : x \in e\} \in E_n \iff \{\iota_n(u), \iota_n(v)\} \in E(\mathrm{Pr}(\mathrm{Lift}_n(H))).$$

Thus ι_n preserves adjacency and is a graph isomorphism between the primal graphs. □

Theorem 3.8.14 (Boxicity of superhypergraphs generalizes graphs and hypergraphs)**.**

(1) *For every finite simple graph* G*, we have*

$$\mathrm{box}(G) \;=\; \mathrm{box}(H_G) \;=\; \mathrm{box}(\mathrm{Lift}_n(H_G)) \quad \textit{for all } n \geq 0.$$

(2) *For every finite hypergraph* H*, we have*

$$\mathrm{box}(H) \;=\; \mathrm{box}(\mathrm{Lift}_n(H)) \quad \textit{for all } n \geq 0.$$

In particular, for $n = 0$ *this recovers hypergraphs as* 0*-SuperHyperGraphs.*

Proof. (1) Since G is 2-uniform, the primal graph of H_G is exactly G: for distinct $u, v \in V$, $\{u, v\} \in E(G)$ if and only if $\{u, v\}$ is a (2-element) hyperedge of H_G, which is equivalent to $\{u, v\} \in E(\mathrm{Pr}(H_G))$. Hence $\mathrm{Pr}(H_G) = G$ and

$$\mathrm{box}(H_G) = \mathrm{box}(\mathrm{Pr}(H_G)) = \mathrm{box}(G).$$

Moreover, by Lemma 3.8.13, $\mathrm{Pr}(H_G) \cong \mathrm{Pr}(\mathrm{Lift}_n(H_G))$ for every n, and boxicity is invariant under graph isomorphism. Therefore $\mathrm{box}(\mathrm{Lift}_n(H_G)) = \mathrm{box}(H_G) = \mathrm{box}(G)$.

(2) By Lemma 3.8.13, $\mathrm{Pr}(H) \cong \mathrm{Pr}(\mathrm{Lift}_n(H))$ for all $n \geq 0$. Applying Definition 3.8.5 and Definition 3.8.9 gives

$$\mathrm{box}(\mathrm{Lift}_n(H)) = \mathrm{box}\bigl(\mathrm{Pr}(\mathrm{Lift}_n(H))\bigr)$$

$$= \mathrm{box}\bigl(\mathrm{Pr}(H)\bigr) = \mathrm{box}(H),$$

as required. □

3.9 Degree sequence of SuperHyperGraph

A graph's degree sequence lists every vertex degree, counting incident edges, presented in an order or as an invariant multiset [118–120]. A SuperHyperGraph's degree sequence lists every supervertex incidence degree, counting incident superedges, presented in an order or as invariant multiset.

Definition 3.9.1 (Degree and degree sequence of a graph)**.** [118–120] A (simple, undirected) graph is a pair $G = (V, E)$ where V is a finite set and $E \subseteq \binom{V}{2}$. For $v \in V$, the *degree* of v in G is

$$\deg_G(v) \ := \ \big|\{\, e \in E : v \in e \,\}\big|.$$

If $V = \{v_1, \ldots, v_n\}$ is an ordered listing of the vertices, then the *degree sequence* of G (with respect to this ordering) is

$$d(G) \ := \ \big(\deg_G(v_1), \ldots, \deg_G(v_n)\big).$$

Equivalently, one may view the *unlabelled* degree sequence as the multiset $\mathrm{DegSeq}(G) := \{\deg_G(v) : v \in V\}$, i.e., $d(G)$ up to permutation.

Definition 3.9.2 (Degree and degree sequence of a hypergraph)**.** A *hypergraph* is a pair $H = (V, E)$ where V is a finite set and E is a finite collection of nonempty subsets of V (called *hyperedges*). For $v \in V$, the *degree* of v in H is

$$d_H(v) \ := \ \big|\{\, e \in E : v \in e \,\}\big|.$$

If $V = \{v_1, \ldots, v_n\}$ is an ordered listing of the vertices, then the *degree sequence* of H (with respect to this ordering) is

$$d(H) \ := \ \big(d_H(v_1), \ldots, d_H(v_n)\big).$$

Definition 3.9.3 (Degree and degree sequence of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph in the sense of Definition 2.1.6, i.e., $V \subseteq \mathrm{PS}^n(V_0)$ and $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$. For a supervertex $x \in V$, define its *(incidence) degree* by

$$\deg_{\mathrm{SHG}^{(n)}}(x) \ := \ \big|\{\, F \in E : x \in F \,\}\big|.$$

If $V = \{x_1, \ldots, x_m\}$ is an ordered listing of the supervertices, define the *degree sequence* of $\mathrm{SHG}^{(n)}$ (with respect to this ordering) by

$$d\Big(\mathrm{SHG}^{(n)}\Big) \ := \ \big(\deg_{\mathrm{SHG}^{(n)}}(x_1), \ldots, \deg_{\mathrm{SHG}^{(n)}}(x_m)\big),$$

and the corresponding unlabelled degree multiset by $\mathrm{DegSeq}(\mathrm{SHG}^{(n)}) := \{\deg_{\mathrm{SHG}^{(n)}}(x) : x \in V\}$.

A concrete example is given below.

Example 3.9.4 (Degree and degree sequence in a 2-SuperHyperGraph)**.** Let the ground set be $V_0 = \{1, 2, 3\}$ and consider the 2-SuperHyperGraph $\mathrm{SHG}^{(2)} = (V, E)$ defined by

$$V = \{x_1, x_2, x_3\} = \{\{1\}, \{2\}, \{3\}\} \subseteq \mathrm{PS}^2(V_0),$$

and

$$E = \{F_1, F_2\} = \{\{x_1, x_2\}, \{x_2, x_3\}\} \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}.$$

Thus F_1 is a superedge incident with x_1, x_2, while F_2 is a superedge incident with x_2, x_3.

By Definition 3.9.3, the (incidence) degrees are

$$\deg_{\mathrm{SHG}^{(2)}}(x_1) = 1, \qquad \deg_{\mathrm{SHG}^{(2)}}(x_2) = 2, \qquad \deg_{\mathrm{SHG}^{(2)}}(x_3) = 1,$$

because x_1 belongs only to F_1, x_2 belongs to both F_1 and F_2, and x_3 belongs only to F_2.

With the ordered listing $V = \{x_1, x_2, x_3\}$, the degree sequence is

$$d\Big(\mathrm{SHG}^{(2)}\Big) = (1, 2, 1),$$

and the corresponding unlabelled degree multiset is

$$\mathrm{DegSeq}\Big(\mathrm{SHG}^{(2)}\Big) = \{1, 1, 2\}.$$

Definition 3.9.5 (Iterated singleton embedding)**.** Let V_0 be a nonempty finite set and $n \geq 0$. Define $\iota_n : V_0 \to \mathrm{PS}^n(V_0)$ recursively by

$$\iota_0(v) := v \qquad (v \in V_0), \qquad \iota_{n+1}(v) := \{\iota_n(v)\} \qquad (v \in V_0).$$

Definition 3.9.6 (n-lift of a hypergraph)**.** Let $H = (V_0, E_0)$ be a hypergraph and let $n \geq 0$. Define its *n-lift* to a superhypergraph by

$$\mathrm{Lift}_n(H) \ := \ \big(V^{(n)}, E^{(n)}\big),$$

where

$$V^{(n)} \ := \ \iota_n(V_0) \subseteq \mathrm{PS}^n(V_0), \qquad E^{(n)} \ := \ \{ \ \iota_n(e) \ : \ e \in E_0 \ \},$$

and for each $e \subseteq V_0$ we abbreviate

$$\iota_n(e) \ := \ \{\iota_n(v) : v \in e\} \ \subseteq \ V^{(n)}.$$

Lemma 3.9.7. *For every hypergraph $H = (V_0, E_0)$ and every $n \geq 0$, the pair $\mathrm{Lift}_n(H)$ is an n-SuperHyperGraph on the base set V_0.*

Proof. By Definition 3.11.6, $\iota_n(v) \in \mathrm{PS}^n(V_0)$ for every $v \in V_0$, hence $V^{(n)} = \iota_n(V_0) \subseteq \mathrm{PS}^n(V_0)$. Moreover, each hyperedge $e \in E_0$ is nonempty, so $\iota_n(e) = \{\iota_n(v) : v \in e\}$ is a nonempty subset of $V^{(n)}$. Therefore $E^{(n)} \subseteq \mathrm{PS}(V^{(n)}) \setminus \{\emptyset\}$. Thus $\mathrm{Lift}_n(H)$ satisfies Definition 2.1.6. $\square$

Theorem 3.9.8 (Degree sequences are preserved by lifting)**.** *Let* $H = (V_0, E_0)$ *be a hypergraph with an ordering* $V_0 = \{v_1, \ldots, v_{|V_0|}\}$*, and let* $n \geq 0$*. Then, with the induced ordering* $V^{(n)} = \{\iota_n(v_1), \ldots, \iota_n(v_{|V_0|})\}$ *on* $\mathrm{Lift}_n(H)$*, one has*

$$d(\mathrm{Lift}_n(H)) \;=\; d(H).$$

In particular, $\mathrm{DegSeq}(\mathrm{Lift}_n(H)) = \mathrm{DegSeq}(H)$ *as multisets.*

Proof. Fix $i \in \{1, \ldots, |V_0|\}$. By Definition 3.9.2,

$$d_H(v_i) = \bigl|\{\, e \in E_0 : v_i \in e \,\}\bigr|.$$

By Definition 3.9.6, the map $E_0 \to E^{(n)}$, $e \mapsto \iota_n(e)$, is well-defined, and for each $e \in E_0$ we have the equivalence

$$v_i \in e \quad \Longleftrightarrow \quad \iota_n(v_i) \in \iota_n(e),$$

because $\iota_n(e) = \{\iota_n(v) : v \in e\}$. Hence the correspondence

$$\{\, e \in E_0 : v_i \in e \,\} \;\longrightarrow\; \{\, F \in E^{(n)} : \iota_n(v_i) \in F \,\}, \qquad e \longmapsto \iota_n(e),$$

is a bijection. Taking cardinalities gives

$$d_H(v_i) \;=\; \deg_{\mathrm{Lift}_n(H)}\bigl(\iota_n(v_i)\bigr).$$

Therefore each coordinate of $d(\mathrm{Lift}_n(H))$ equals the corresponding coordinate of $d(H)$, so $d(\mathrm{Lift}_n(H)) = d(H)$. ☐

Corollary 3.9.9 (Superhypergraph degree sequences generalize graph and hypergraph degree sequences)**.**

*(a) (*Hypergraphs*) For every hypergraph* H*, the degree sequence of* H *is realized as the degree sequence of the superhypergraph* $\mathrm{Lift}_n(H)$ *for any* $n \geq 0$*.*

*(b) (*Graphs*) For every graph* $G = (V, E)$*, view* G *as the* 2*-uniform hypergraph* $H_G = (V, \{\{u, v\} : \{u, v\} \in E\})$*. Then* $d(G) = d(H_G)$*, and hence*

$$d(\mathrm{Lift}_n(H_G)) \;=\; d(G) \qquad (n \geq 0).$$

Consequently, Definition 3.9.3 extends (and is consistent with) the usual notions of degree sequence for graphs and hypergraphs.

Proof. Part (a) is Theorem 3.9.8. For part (b), in a simple graph the degree of a vertex is the number of edges incident to it, which is exactly the number of 2-element hyperedges containing it in H_G; thus $d(G) = d(H_G)$, and the claim follows again from Theorem 3.9.8. ☐

3.10 Vertex connectivity of SuperHyperGraph

Vertex connectivity of a graph is the minimum number of vertices whose removal disconnects the graph or leaves at most one vertex [121–125]. Vertex connectivity of a hypergraph is the minimum size of a weak/strong vertex cut whose deletion disconnects the hypergraph under Berge connectivity [126–128]. Vertex connectivity of a SuperHyperGraph is the minimum number of supervertices whose weak/strong deletion disconnects the structure, measured via Berge walks.

Definition 3.10.1 (Vertex deletion and vertex cut in a graph)**.** Let $G = (V, E)$ be a finite simple undirected graph and let $X \subseteq V$. The *vertex-deleted* graph $G - X$ is the induced subgraph on $V \setminus X$. A set $X \subseteq V$ is a *vertex cut* of G if $G - X$ is disconnected or has at most one vertex.

Definition 3.10.2 (Vertex connectivity of a graph)**.** [121–125] Let $G = (V, E)$ be a graph with $|V| \geq 2$. The *vertex connectivity* of G is

$$\kappa(G) \;:=\; \min\{\, |X| : X \subseteq V \text{ is a vertex cut of } G \,\}.$$

Equivalently, $\kappa(G)$ is the smallest number of vertices whose deletion disconnects G (or reduces it to a single vertex). By convention, $\kappa(K_m) = m - 1$ for the complete graph K_m.

Definition 3.10.3 (Hypergraph and Berge connectivity)**.** A *hypergraph* is a pair $H = (V, E)$ where V is a finite set and $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$ is a finite family of nonempty subsets of V. A *Berge walk* from u to v is a sequence

$$u = v_0,\ e_1,\ v_1,\ e_2,\ \ldots,\ e_k,\ v_k = v$$

such that $v_{i-1}, v_i \in e_i$ for all $i = 1, \ldots, k$. We say H is *connected* if for every $u, v \in V$ there exists a Berge walk from u to v.

Definition 3.10.4 (Weak and strong vertex deletion in a hypergraph)**.** Let $H = (V, E)$ be a hypergraph and let $X \subseteq V$.

1. The *weak vertex deletion* of X produces

$$H \setminus^{\mathrm{W}} X \;:=\; \big(V \setminus X,\ E^{\mathrm{W}}\big), \qquad E^{\mathrm{W}} \;:=\; \{\, e \setminus X \;:\; e \in E,\ e \setminus X \neq \emptyset \,\}.$$

2. The *strong vertex deletion* of X produces

$$H \setminus^{\mathrm{S}} X \;:=\; \big(V \setminus X,\ E^{\mathrm{S}}\big), \qquad E^{\mathrm{S}} \;:=\; \{\, e \in E \;:\; e \cap X = \emptyset \,\}.$$

Definition 3.10.5 (Weak and strong vertex deletion in a hypergraph)**.** Let $H = (V, E)$ be a hypergraph and let $X \subseteq V$.

1. The *weak vertex deletion* of X produces

$$H \setminus^{\mathrm{W}} X \;:=\; \big(V \setminus X,\ E^{\mathrm{W}}\big),$$
$$E^{\mathrm{W}} \;:=\; \{\, e \setminus X \;:\; e \in E,\ e \setminus X \neq \emptyset \,\}.$$

2. The *strong vertex deletion* of X produces

$$H \setminus^{\mathrm{S}} X \;:=\; \bigl(V \setminus X,\; E^{\mathrm{S}}\bigr),$$

$$E^{\mathrm{S}} \;:=\; \{\, e \in E \;:\; e \cap X = \emptyset \,\}.$$

Definition 3.10.6 (Weak/strong vertex cuts and vertex connectivities of a hypergraph)**.** Let $H = (V, E)$ be a hypergraph with $|V| \geq 2$. A set $X \subseteq V$ is a *weak vertex cut* (resp. *strong vertex cut*) if $H \setminus^{\mathrm{W}} X$ (resp. $H \setminus^{\mathrm{S}} X$) is disconnected or has at most one vertex. Define

$$\kappa_{\mathrm{W}}(H) \;:=\; \min\{\, |X| : X \text{ is a weak vertex cut of } H \,\}, \qquad \kappa_{\mathrm{S}}(H) \;:=\; \min\{\, |X| : X \text{ is a strong vertex cut of } H \,\}.$$

If no such cut exists (e.g. a complete graph viewed as a 2-uniform hypergraph), set $\kappa_{\mathrm{W}}(H) = \kappa_{\mathrm{S}}(H) = |V| - 1$.

A concrete example is given below.

Example 3.10.7 (Weak and strong vertex cuts; κ_{W} and κ_{S})**.** Let $H = (V, E)$ be the hypergraph

$$V = \{1, 2, 3, 4\}, \qquad E = \{e_1, e_2\}, \qquad e_1 = \{1, 2, 3\}, \;\; e_2 = \{3, 4\}.$$

We regard connectivity via Berge walks.

Claim 1: $\{3\}$ is a weak vertex cut. From Example,

$$H \setminus^{\mathrm{W}} \{3\} = \bigl(\{1, 2, 4\},\; \{\{1, 2\}, \{4\}\}\bigr).$$

In this hypergraph, vertex 4 is isolated from $\{1, 2\}$ in the Berge sense: any Berge walk starting at 4 can only use the singleton hyperedge $\{4\}$ and cannot reach 1 or 2. Thus $H \setminus^{\mathrm{W}} \{3\}$ is disconnected, so $\{3\}$ is a weak vertex cut. Moreover, no single vertex among $\{1, 2, 4\}$ disconnects H under weak deletion, hence

$$\kappa_{\mathrm{W}}(H) = 1.$$

Claim 2: $\{3\}$ is also a strong vertex cut. We have

$$H \setminus^{\mathrm{S}} \{3\} = \bigl(\{1, 2, 4\},\; \emptyset\bigr),$$

which is disconnected (indeed, it has no hyperedges at all). Hence $\{3\}$ is a strong vertex cut, and therefore

$$\kappa_{\mathrm{S}}(H) = 1.$$

Definition 3.10.8 (Connectivity in an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (Definition 2.1.6). A *Berge walk* in $\mathrm{SHG}^{(n)}$ is a sequence

$$x_0,\; \varepsilon_1,\; x_1,\; \varepsilon_2,\; \ldots,\; \varepsilon_k,\; x_k$$

where each $x_i \in V$, each $\varepsilon_i \in E$, and $x_{i-1}, x_i \in \varepsilon_i$ for all i. We say $\mathrm{SHG}^{(n)}$ is *connected* if for every $x, y \in V$ there exists a Berge walk from x to y.

A concrete example is given below.

Example 3.10.9 (Connectivity via a Berge walk in a 1-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b, c\}$ and take $n = 1$. Consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ given by

$$V = \{\{a\}, \{b\}, \{c\}\} \subseteq \mathrm{PS}^1(V_0),$$
$$E = \{\varepsilon_1, \varepsilon_2\},$$

where

$$\varepsilon_1 = \{\{a\}, \{b\}\},$$
$$\varepsilon_2 = \{\{b\}, \{c\}\}.$$

Then $\mathrm{SHG}^{(1)}$ is connected: for instance, there is a Berge walk from $\{a\}$ to $\{c\}$,

$$\{a\},\ \varepsilon_1,\ \{b\},\ \varepsilon_2,\ \{c\},$$

since $\{a\}, \{b\} \in \varepsilon_1$ and $\{b\}, \{c\} \in \varepsilon_2$. Similarly, every pair of supervertices is joined by a Berge walk, so $\mathrm{SHG}^{(1)}$ is connected.

Definition 3.10.10 (Weak/strong supervertex deletion)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph and let $X \subseteq V$.

1. The *weak deletion* of X is

$$\mathrm{SHG}^{(n)} \setminus^{\mathrm{W}} X \ :=\ \big(V \setminus X,\ E^{\mathrm{W}}\big),$$
$$E^{\mathrm{W}} := \{\, \varepsilon \setminus X : \varepsilon \in E,\ \varepsilon \setminus X \neq \emptyset \,\}.$$

2. The *strong deletion* of X is

$$\mathrm{SHG}^{(n)} \setminus^{\mathrm{S}} X \ :=\ \big(V \setminus X,\ E^{\mathrm{S}}\big), \qquad E^{\mathrm{S}} := \{\, \varepsilon \in E : \varepsilon \cap X = \emptyset \,\}.$$

A concrete example is given below.

Example 3.10.11 (Weak and strong supervertex deletion)**.** Let $\mathrm{SHG}^{(1)} = (V, E)$ be the 1-SuperHyperGraph from Example 3.10.9, and let

$$X = \{\{b\}\} \subseteq V.$$

Weak deletion. We have $V \setminus X = \{\{a\}, \{c\}\}$ and

$$\varepsilon_1 \setminus X = \{\{a\}\}, \qquad \varepsilon_2 \setminus X = \{\{c\}\},$$

so

$$\mathrm{SHG}^{(1)} \setminus^{\mathrm{W}} X = \big(\{\{a\}, \{c\}\},\ \{\{\{a\}\}, \{\{c\}\}\}\big).$$

Thus weak deletion removes $\{b\}$ from each incident superedge but retains nonempty remnants as (possibly singleton) superedges.

Strong deletion. Since both ε_1 and ε_2 contain $\{b\}$, they are removed entirely under strong deletion. Hence

$$\mathrm{SHG}^{(1)} \setminus^{\mathrm{S}} X = \big(\{\{a\}, \{c\}\},\ \emptyset\big).$$

Definition 3.10.12 (Vertex connectivity of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph with $|V| \geq 2$. A set $X \subseteq V$ is a *weak* (resp. *strong*) *supervertex cut* if $\mathrm{SHG}^{(n)} \setminus^{\mathrm{W}} X$ (resp. $\mathrm{SHG}^{(n)} \setminus^{\mathrm{S}} X$) is disconnected or has at most one vertex. Define

$$\kappa_{\mathrm{W}}\big(\mathrm{SHG}^{(n)}\big) := \min\{\, |X| : X \text{ is a weak supervertex cut of } \mathrm{SHG}^{(n)} \,\},$$

$$\kappa_{\mathrm{S}}\big(\mathrm{SHG}^{(n)}\big) := \min\{\, |X| : X \text{ is a strong supervertex cut of } \mathrm{SHG}^{(n)} \,\}.$$

If no such cut exists, set $\kappa_{\mathrm{W}}(\mathrm{SHG}^{(n)}) = \kappa_{\mathrm{S}}(\mathrm{SHG}^{(n)}) = |V| - 1$.

A concrete example is given below.

Example 3.10.13 (Weak/strong supervertex cuts and vertex connectivity)**.** Let $\mathrm{SHG}^{(1)} = (V, E)$ be as in Example 3.10.9. We claim that $X = \{\{b\}\}$ is both a weak and a strong supervertex cut.

Indeed, by Example 3.10.11,

$$\mathrm{SHG}^{(1)} \setminus^{\mathrm{W}} X = \big(\{\{a\}, \{c\}\},\ \{\{\{a\}\}, \{\{c\}\}\}\big),$$

which is disconnected in the Berge sense: there is no Berge walk from $\{a\}$ to $\{c\}$ because every superedge is a singleton and therefore cannot contain two distinct supervertices. Also,

$$\mathrm{SHG}^{(1)} \setminus^{\mathrm{S}} X = \big(\{\{a\}, \{c\}\},\ \emptyset\big)$$

is disconnected as well.

Therefore $\kappa_{\mathrm{W}}(\mathrm{SHG}^{(1)}) \leq 1$ and $\kappa_{\mathrm{S}}(\mathrm{SHG}^{(1)}) \leq 1$. Since $\mathrm{SHG}^{(1)}$ is connected and $|V| \geq 2$, a cut must be nonempty, so both minima equal 1:

$$\kappa_{\mathrm{W}}(\mathrm{SHG}^{(1)}) = 1, \qquad \kappa_{\mathrm{S}}(\mathrm{SHG}^{(1)}) = 1.$$

Definition 3.10.14 (The n-lift of a hypergraph)**.** Let $H = (V_0, E_0)$ be a hypergraph and let $n \geq 0$. Define the embedding $\iota_n : V_0 \to \mathrm{PS}^n(V_0)$ by

$$\iota_0(v) := v, \qquad \iota_{n+1}(v) := \{\iota_n(v)\}.$$

Define the *n-lift* of H to be the n-SuperHyperGraph

$$\mathrm{Lift}_n(H) := (V^{(n)}, E^{(n)}), \qquad V^{(n)} := \{\iota_n(v) : v \in V_0\}, \quad E^{(n)} := \big\{\, \{\iota_n(v) : v \in e\} \,:\, e \in E_0 \,\big\}.$$

Lemma 3.10.15 (Incidence preservation and deletion commutation)**.** *Let $H = (V_0, E_0)$ be a hypergraph and $n \geq 0$.*

1. *The map $\iota_n : V_0 \to V^{(n)}$ is a bijection, and for each $e \in E_0$ the set $\hat{\iota}_n(e) := \{\iota_n(v) : v \in e\} \in E^{(n)}$. Moreover, $v \in e$ holds if and only if $\iota_n(v) \in \hat{\iota}_n(e)$.*

2. *For every $X \subseteq V_0$ one has canonical identifications*
$$\mathrm{Lift}_n(H) \setminus^{\mathrm{W}} \iota_n(X) \;\cong\; \mathrm{Lift}_n(H \setminus^{\mathrm{W}} X), \qquad \mathrm{Lift}_n(H) \setminus^{\mathrm{S}} \iota_n(X) \;\cong\; \mathrm{Lift}_n(H \setminus^{\mathrm{S}} X),$$
where $\iota_n(X) := \{\iota_n(x) : x \in X\}$.

Proof. (1) By construction, ι_n is injective and $V^{(n)}$ is its image, hence bijective onto $V^{(n)}$. For $e \in E_0$, the definition of $E^{(n)}$ gives $\hat{\iota}_n(e) \in E^{(n)}$ and the membership equivalence is immediate.

(2) Consider weak deletion. In $\mathrm{Lift}_n(H) \setminus^{\mathrm{W}} \iota_n(X)$ we remove the vertices $\iota_n(X)$ from $V^{(n)}$, and from each superedge $\hat{\iota}_n(e)$ we remove exactly those $\iota_n(v)$ with $v \in e \cap X$. Thus $\hat{\iota}_n(e)$ becomes $\{\iota_n(v) : v \in e \backslash X\}$, and empty sets are discarded. This is precisely the edge set of $\mathrm{Lift}_n(H \setminus^{\mathrm{W}} X)$.

For strong deletion, $\mathrm{Lift}_n(H) \setminus^{\mathrm{S}} \iota_n(X)$ removes all superedges $\hat{\iota}_n(e)$ with $\hat{\iota}_n(e) \cap \iota_n(X) \neq \emptyset$, i.e. exactly those with $e \cap X \neq \emptyset$. The remaining superedges are unchanged, which matches $\mathrm{Lift}_n(H \setminus^{\mathrm{S}} X)$. □

Theorem 3.10.16 (Generalization of vertex connectivity)**.**

1. *(Graphs) Let $G = (V, E)$ be a graph and let $H_G := (V, \{\{u, v\} : uv \in E\})$ be the associated 2-uniform hypergraph. Then*
$$\kappa(G) \;=\; \kappa_{\mathrm{W}}(H_G) \;=\; \kappa_{\mathrm{S}}(H_G).$$

2. *(Hypergraphs as 0-SuperHyperGraphs) If $H = (V_0, E_0)$ is a hypergraph, then viewed as a 0-SuperHyperGraph (since $\mathrm{PS}^0(V_0) = V_0$) one has*
$$\kappa_{\mathrm{W}}(H) \;=\; \kappa_{\mathrm{W}}\big(\mathrm{Lift}_0(H)\big), \qquad \kappa_{\mathrm{S}}(H) \;=\; \kappa_{\mathrm{S}}\big(\mathrm{Lift}_0(H)\big).$$

3. *(n-lifts) For every hypergraph H and every $n \geq 0$,*
$$\kappa_{\mathrm{W}}(H) \;=\; \kappa_{\mathrm{W}}\big(\mathrm{Lift}_n(H)\big), \qquad \kappa_{\mathrm{S}}(H) \;=\; \kappa_{\mathrm{S}}\big(\mathrm{Lift}_n(H)\big).$$

   Consequently, the superhypergraph vertex connectivities κ_{W} and κ_{S} simultaneously generalize graph vertex connectivity and hypergraph vertex connectivity.

Proof. (1) In H_G every hyperedge has size 2, so weak and strong deletion coincide: for $X \subseteq V$, deleting X removes exactly the vertices X and all incident edges in either sense. Moreover, Berge connectivity in H_G is identical to ordinary graph connectivity in G. Hence the minimum size of a vertex set whose deletion disconnects the structure is the same in G and H_G, i.e. $\kappa(G) = \kappa_{\mathrm{W}}(H_G) = \kappa_{\mathrm{S}}(H_G)$.

(2) This is immediate from $\mathrm{Lift}_0(H) = H$ by definition of ι_0.

(3) By Lemma 3.10.15(1), H and $\mathrm{Lift}_n(H)$ have the same incidence relation up to renaming vertices by ι_n (and edges by $\hat{\iota}_n$). Therefore H is connected if and only if $\mathrm{Lift}_n(H)$ is connected (Berge walks correspond by applying ι_n to every vertex appearing in the walk).

Now fix $X \subseteq V_0$. By Lemma 3.10.15(2),

$$\mathrm{Lift}_n(H) \setminus^{\mathrm{W}} \iota_n(X) \cong \mathrm{Lift}_n(H \setminus^{\mathrm{W}} X), \qquad \mathrm{Lift}_n(H) \setminus^{\mathrm{S}} \iota_n(X) \cong \mathrm{Lift}_n(H \setminus^{\mathrm{S}} X).$$

Since connectivity is invariant under isomorphism, $H \setminus^{\mathrm{W}} X$ is disconnected (or trivial) if and only if $\mathrm{Lift}_n(H) \setminus^{\mathrm{W}} \iota_n(X)$ is disconnected (or trivial), and similarly for strong deletion. Thus X is a weak (resp. strong) vertex cut in H if and only if $\iota_n(X)$ is a weak (resp. strong) supervertex cut in $\mathrm{Lift}_n(H)$. Because ι_n is a bijection, $|X| = |\iota_n(X)|$, so minimum cut sizes coincide. Hence $\kappa_{\mathrm{W}}(H) = \kappa_{\mathrm{W}}(\mathrm{Lift}_n(H))$ and $\kappa_{\mathrm{S}}(H) = \kappa_{\mathrm{S}}(\mathrm{Lift}_n(H))$. □

3.11 Burning Number in SuperHyperGraph

A graph's burning number is the minimum number of rounds needed to burn all vertices, igniting one new source each round and spreading to neighbors [129–132]. A hypergraph's burning number is the minimum number of rounds needed to burn all vertices, igniting one new source each round and propagating through hyperedges [133–136]. A superhypergraph's burning number is the minimum number of rounds needed to burn all supervertices, igniting new sources and propagating through superedges between supervertices.

Definition 3.11.1 (Burning process and burning number of a graph)**.** [129–132] Let $G = (V, E)$ be a finite simple undirected graph. For each round $r \in \mathbb{N}$, let $F_r \subseteq V$ denote the set of burned vertices at the end of round r, with $F_0 := \emptyset$.

During round $r \geq 1$, two events occur simultaneously:

1. *Propagation:* every unburned vertex adjacent to some burned vertex catches fire, i.e.,

$$P_r \;:=\; \{\, v \in V \setminus F_{r-1} :\; \exists\, u \in F_{r-1} \text{ with } \{u, v\} \in E \,\}.$$

2. *Ignition:* a new *source* $u_r \in V \setminus F_{r-1}$ is chosen and set on fire.

The burned set is updated by

$$F_r \;:=\; F_{r-1} \;\cup\; P_r \;\cup\; \{u_r\}.$$

A sequence $(u_1, u_2, \dots, u_k)$ is called a *burning sequence* of G if $F_k = V$. The *burning number* of G is

$$b(G) \;:=\; \min\{\, k \in \mathbb{N} :\; G \text{ admits a burning sequence of length } k \,\}.$$

Definition 3.11.2 (Burning process and burning number of a hypergraph)**.** Let $H = (V, E)$ be a finite hypergraph, where $V \neq \emptyset$ and $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$. For the propagation rule we use only *non-singleton* hyperedges, i.e. edges e with $|e| \geq 2$.

Let $F_r \subseteq V$ be the set of burned vertices at the end of round r, with $F_0 := \emptyset$. In round $r \geq 1$, simultaneously:

1. *Propagation:* an unburned vertex v catches fire if it is the unique unburned vertex in some non-singleton hyperedge, i.e.,

$$P_r \ := \ \Big\{ v \in V \setminus F_{r-1} : \ \exists\, e \in E \text{ with } |e| \geq 2, \ v \in e, \ e \setminus \{v\} \subseteq F_{r-1} \Big\}.$$

2. *Ignition:* choose a new source $u_r \in V \setminus F_{r-1}$ and set it on fire.

Set

$$F_r \ := \ F_{r-1} \ \cup \ P_r \ \cup \ \{u_r\}.$$

A sequence $(u_1, \ldots, u_k)$ is a *burning sequence* of H if $F_k = V$. The *burning number* of H is

$$b(H) \ := \ \min\{\, k \in \mathbb{N} : \ H \text{ admits a burning sequence of length } k \,\}.$$

Remark 3.11.3. If H is 2-uniform (i.e. every hyperedge has size 2), then the propagation rule in Definition 3.11.2 coincides with ordinary graph propagation along edges.

Definition 3.11.4 (Burning process and burning number of an n-SuperHyperGraph)**.** Let $n \geq 0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (Definition 2.1.6). Thus (V, E) is a hypergraph whose “vertices” are n-supervertices and whose “edges” are n-superedges.

Define the burned sets $F_r \subseteq V$ by $F_0 := \emptyset$ and, for each round $r \geq 1$,

$$F_r \ := \ F_{r-1} \ \cup \ P_r \ \cup \ \{x_r\},$$

where $x_r \in V \setminus F_{r-1}$ is the chosen source and

$$P_r \ := \ \Big\{ x \in V \setminus F_{r-1} : \ \exists\, \varepsilon \in E \text{ with } |\varepsilon| \geq 2, \ x \in \varepsilon, \ \varepsilon \setminus \{x\} \subseteq F_{r-1} \Big\}.$$

A sequence $(x_1, \ldots, x_k)$ is a *burning sequence* of $\mathrm{SHG}^{(n)}$ if $F_k = V$. The *burning number* of $\mathrm{SHG}^{(n)}$ is

$$b\Big(\mathrm{SHG}^{(n)}\Big) \ := \ \min\{\, k \in \mathbb{N} : \ \mathrm{SHG}^{(n)} \text{ admits a burning sequence of length } k \,\}.$$

A concrete example is given below.

Example 3.11.5 (Burning process in a 2-SuperHyperGraph)**.** Let the ground set be $V_0 = \{1, 2, 3, 4\}$ and consider the 2-SuperHyperGraph $\mathrm{SHG}^{(2)} = (V, E)$ defined by

$$V = \{x_1, x_2, x_3, x_4\} = \{\{1\}, \{2\}, \{3\}, \{4\}\} \subseteq \mathrm{PS}^2(V_0),$$

and the set of superedges

$$E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, \qquad \varepsilon_1 = \{x_1, x_2\}, \ \varepsilon_2 = \{x_2, x_3\}, \ \varepsilon_3 = \{x_3, x_4\}.$$

This is the 2-lift of the path graph P_4 under the singleton embedding.

We compute the burning process as in Definition 3.11.4. Let F_r be the burned supervertices after round r, with $F_0 = \emptyset$. Choose the burning sequence

$$(x_1, x_4).$$

Round 1**.** Since $F_0 = \emptyset$, no propagation occurs:

$$P_1 = \emptyset.$$

Ignite the source x_1. Hence

$$F_1 = F_0 \cup P_1 \cup \{x_1\} = \{x_1\}.$$

Round 2**.** Propagation uses superedges ε for which all but one incident supervertex are already burned. Here $\varepsilon_1 = \{x_1, x_2\}$ has $x_1 \in F_1$ and $x_2 \notin F_1$, so x_2 catches fire:

$$P_2 = \{x_2\}.$$

Ignite the new source x_4. Therefore

$$F_2 = F_1 \cup P_2 \cup \{x_4\} = \{x_1, x_2, x_4\}.$$

Round 3**.** Now $\varepsilon_2 = \{x_2, x_3\}$ has $x_2 \in F_2$ and $x_3 \notin F_2$, and $\varepsilon_3 = \{x_3, x_4\}$ has $x_4 \in F_2$ and $x_3 \notin F_2$. Hence x_3 is the unique unburned supervertex in each of these superedges, so it burns by propagation:

$$P_3 = \{x_3\}.$$

No ignition is needed, but the definition allows choosing any $x_3 \notin F_2$ as the next source. Thus

$$F_3 = F_2 \cup P_3 = \{x_1, x_2, x_3, x_4\} = V.$$

Therefore (x_1, x_4) is a burning sequence of length 2, so

$$b\Big(\mathrm{SHG}^{(2)}\Big) \le 2.$$

Moreover $b(\mathrm{SHG}^{(2)}) \ne 1$ because a single ignition cannot burn all supervertices in one round. Hence

$$b\Big(\mathrm{SHG}^{(2)}\Big) = 2.$$

Definition 3.11.6 (Singleton embedding)**.** Let V_0 be a nonempty set and $n \ge 0$. Define maps $\iota_n : V_0 \to \mathrm{PS}^n(V_0)$ recursively by

$$\iota_0(v) \;:=\; v, \qquad \iota_{n+1}(v) \;:=\; \{\iota_n(v)\} \quad (v \in V_0).$$

Definition 3.11.7 (n-lift of a hypergraph)**.** Let $H = (V, E)$ be a hypergraph and fix $n \ge 0$. Take the base set $V_0 := V$ and define

$$V^{(n)} \;:=\; \iota_n(V) \;=\; \{\iota_n(v) : v \in V\} \;\subseteq\; \mathrm{PS}^n(V),$$

$$E^{(n)} \;:=\; \Big\{\, \iota_n(e) \;:=\; \{\iota_n(v) : v \in e\} \;:\; e \in E \Big\} \;\subseteq\; \mathrm{PS}(V^{(n)}) \setminus \{\emptyset\}.$$

The *n-lift* of H is the n-SuperHyperGraph

$$\mathrm{Lift}_n(H) \;:=\; \big(V^{(n)}, E^{(n)}\big).$$

Lemma 3.11.8 (Burning number is invariant under hypergraph isomorphism)**.** *Let* $H = (V, E)$ *and* $H' = (V', E')$ *be hypergraphs. Suppose* $\varphi : V \to V'$ *is a bijection such that*

$$e \in E \quad \Longleftrightarrow \quad \varphi(e) := \{\varphi(v) : v \in e\} \in E'.$$

Then $b(H) = b(H')$*.*

Proof. Fix $k \in \mathbb{N}$ and let $(u_1, \dots, u_k)$ be any valid source sequence in H (i.e. $u_r \notin F_{r-1}$ at each round). Run the burning process in H with these sources, producing burned sets $F_r \subseteq V$.

Define $u'_r := \varphi(u_r)$ and run the burning process in H' with sources $(u'_1, \dots, u'_k)$, producing burned sets $F'_r \subseteq V'$.

We claim that $\varphi(F_r) = F'_r$ for all $r \geq 0$. This is clear for $r = 0$ since both are empty. Assume $\varphi(F_{r-1}) = F'_{r-1}$. A vertex $v \in V \setminus F_{r-1}$ belongs to the propagated set P_r in H iff there exists a non-singleton edge $e \in E$ with $v \in e$ and $e \setminus \{v\} \subseteq F_{r-1}$. Applying φ and using the isomorphism property shows that $\varphi(v)$ belongs to the propagated set P'_r in H'; conversely, any $\varphi(v) \in P'_r$ arises this way. Hence $\varphi(P_r) = P'_r$. Also $\varphi(\{u_r\}) = \{u'_r\}$. Therefore

$$\varphi(F_r) \;=\; \varphi(F_{r-1} \cup P_r \cup \{u_r\}) \;=\; \varphi(F_{r-1}) \cup \varphi(P_r) \cup \{u'_r\} \;=\; F'_{r-1} \cup P'_r \cup \{u'_r\} \;=\; F'_r.$$

This proves the claim.

In particular, $F_k = V$ if and only if $F'_k = V'$. Hence $(u_1, \dots, u_k)$ burns H if and only if $(u'_1, \dots, u'_k)$ burns H'. Taking the minimum over k yields $b(H) = b(H')$. □

Proposition 3.11.9 (Graphs are the 2-uniform case)**.** *Let* $G = (V, E)$ *be a finite simple graph and let* $H_G = (V, E_G)$ *be the* 2*-uniform hypergraph with* $E_G := \{\{u, v\} : \{u, v\} \in E\}$*. Then* $b(G) = b(H_G)$*.*

Proof. In both burning processes, propagation in round r burns exactly the vertices having a burned neighbor at the end of round $r - 1$, since each hyperedge has size 2. Thus the reachable burned sets after each round coincide for the same source sequence, and the minimum number of rounds is the same. Therefore $b(G) = b(H_G)$. □

Theorem 3.11.10 (SuperHyperGraph burning generalizes graph and hypergraph burning)**.** *Let* $n \geq 0$*.*

1. *For every hypergraph* H*, the* n*-lift satisfies*

$$b(\mathrm{Lift}_n(H)) \;=\; b(H).$$

2. *For every finite simple graph* G*,*

$$b(G) \;=\; b(H_G) \;=\; b(\mathrm{Lift}_n(H_G))\,,$$

where H_G *is the* 2*-uniform hypergraph associated with* G*.*

Consequently, the burning number $b(\mathrm{SHG}^{(n)})$ *from Definition 3.11.4 is a genuine extension of the burning number of graphs and of hypergraphs.*

Proof. (1) In Definition 3.11.7, the map $\iota_n : V \to V^{(n)}$ is a bijection, and by construction

$$e \in E \quad \Longleftrightarrow \quad \iota_n(e) = \{\iota_n(v) : v \in e\} \in E^{(n)}.$$

Hence ι_n is a hypergraph isomorphism from H onto the underlying hypergraph $(V^{(n)}, E^{(n)})$ of $\mathrm{Lift}_n(H)$. Lemma 3.11.8 yields $b(\mathrm{Lift}_n(H)) = b(H)$.

(2) This follows immediately from Proposition 3.11.9 and part (1) applied to H_G. □

3.12 Treewidth in SuperHyperGraphs

An *n-SuperHyperTree* is an acyclic n-SuperHyperGraph whose superedges admit a *join-tree* representation, meaning that superedges can be arranged as the nodes of a tree so that, for every supervertex, the collection of incident superedges appears as a connected subtree [10, 137, 138]. In this sense, SuperHyperTrees extend the classical notion of hypertrees [139–141].

A *tree decomposition* encodes a graph by a tree of *bags* of vertices, requiring that each edge is contained in some bag and that the bags containing a fixed vertex satisfy a running-intersection property [142–146]. Treewidth is central because many NP-hard graph problems admit dynamic-programming algorithms on classes of bounded treewidth, providing both algorithmic leverage and structural insight. Analogously, a *SuperHypertree decomposition* represents an n-SuperHyperGraph by a tree equipped with bags and *guards* (collections of superedges) that enforce coverage and connectedness for both supervertices and superedges across the decomposition [137, 138, 147, 148]. SuperHypertree decompositions are conceptually aligned with tree decompositions [143, 149] and with hypertree decompositions [140, 141].

Definition 3.12.1 (n-SuperHyperTree)**.** [10] Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. An *n-SuperHyperGraph* is a pair

$$H^{(n)} = (V, E),$$

where $V \subseteq P_n(V_0)$ is a finite set of n-supervertices and E is a finite family of nonempty subsets of V (called n-superedges).

We call $H^{(n)}$ an *n-SuperHyperTree* if there exists a tree

$$J = (E, F),$$

whose vertex set is E (called a *join superhypertree* of $H^{(n)}$) such that, for every supervertex $v \in V$, the set

$$J_v := \{\, e \in E \mid v \in e \,\} \subseteq V(J)$$

is nonempty and induces a connected subtree of J. In this case $H^{(n)}$ is said to be *acyclic* (or *join-tree acyclic*), and J is a join superhypertree of $H^{(n)}$.

There are several notions in the literature that extend the paradigm of width parameters, including branch-width [150, 151], linear-width [152], path-distance-width [153], boolean-width [154, 155], clique-width [156, 157], and sim-width [158]. Broadly speaking, these parameters quantify how far a given graph is from a target “simple” structure, and relations among different width measures are a frequent topic of study.

Definition 3.12.2 (Tree decomposition and treewidth)**.** [159,160] Let $G = (V, E)$ be a finite undirected graph. A *tree decomposition* of G is a pair (T, χ) where $T = (N, F)$ is a tree and χ assigns to each node $p \in N$ a *bag* $\chi(p) \subseteq V$ such that:

1. *Vertex coverage:* for every $v \in V$ there exists $p \in N$ with $v \in \chi(p)$.

2. *Edge coverage:* for every edge $\{u, v\} \in E$ there exists $p \in N$ with $\{u, v\} \subseteq \chi(p)$.

3. *Running intersection:* for every $v \in V$, the set
$$N_v := \{\, p \in N \mid v \in \chi(p) \,\}$$
induces a connected subtree of T.

The *width* of (T, χ) is
$$\mathrm{width}(T, \chi) := \max_{p \in N} \bigl(|\chi(p)| - 1\bigr),$$
and the *treewidth* of G is
$$\mathrm{tw}(G) := \min\{\mathrm{width}(T, \chi) \mid (T, \chi) \text{ is a tree decomposition of } G\}.$$

Definition 3.12.3 (Hypertree decomposition)**.** [140, 141, 161] Let $H = (V, E)$ be a finite hypergraph, where $V = V(H)$ is the set of variables (vertices) and $E = E(H)$ is the set of hyperedges. A *hypertree decomposition* of H is a triple
$$HD = \langle T, \chi, \lambda \rangle,$$
where T is a rooted tree and χ, λ are labeling functions such that for each node $p \in V(T)$,
$$\chi(p) \subseteq V \qquad \text{and} \qquad \lambda(p) \subseteq E.$$
For $\mathcal{F} \subseteq E$ write
$$V(\mathcal{F}) := \bigcup_{h \in \mathcal{F}} h \subseteq V.$$
For any node $p \in V(T)$, let T_p be the subtree of T rooted at p and set
$$\chi(T_p) := \bigcup_{q \in V(T_p)} \chi(q).$$
The triple $\langle T, \chi, \lambda \rangle$ is a hypertree decomposition of H if:

(1) (*Edge coverage*) For each $h \in E$ there exists $p \in V(T)$ with $h \subseteq \chi(p)$.

(2) (*Connectedness of variables*) For each $Y \in V$, the set

$$\{ p \in V(T) \mid Y \in \chi(p) \}$$

induces a connected subtree of T.

(3) (*Local covering by guards*) For each $p \in V(T)$,

$$\chi(p) \subseteq V(\lambda(p)).$$

(4) (*Special condition*) For each $p \in V(T)$,

$$V(\lambda(p)) \cap \chi(T_p) \subseteq \chi(p).$$

The *width* of HD is

$$\mathrm{width}(HD) := \max_{p \in V(T)} |\lambda(p)|.$$

Definition 3.12.4 (SuperHypertree decomposition of an n-SuperHyperGraph)**.** Let $H^{(n)} = (V, E)$ be a finite n-SuperHyperGraph. A *SuperHypertree decomposition* of $H^{(n)}$ is a triple

$$(T, \mathcal{B}, \mathcal{C}),$$

where $T = (V_T, E_T)$ is a finite tree, $\mathcal{B} = \{ B_t \subseteq V \mid t \in V_T \}$ is a family of *bags*, and $\mathcal{C} = \{ C_t \subseteq E \mid t \in V_T \}$ is a family of *guards*, such that:

(1) *Vertex coverage:* $V = \bigcup_{t \in V_T} B_t$.

(2) *Superedge coverage:* for every $e \in E$ there exists $t \in V_T$ with $e \subseteq B_t$.

(3) *Vertex connectedness:* for every $v \in V$, the set

$$T_v := \{ t \in V_T \mid v \in B_t \}$$

is nonempty and induces a connected subtree of T.

(4) *Guard covering:* for each $t \in V_T$,

$$B_t \subseteq \bigcup_{e \in C_t} e.$$

(5) *Guard connectedness and consistency:* for every $e \in E$, the set

$$T_e := \{ t \in V_T \mid e \in C_t \}$$

is nonempty and induces a connected subtree of T, and whenever $e \subseteq B_t$ for some $t \in V_T$, one also has $e \in C_t$.

A concrete example is given below.

Example 3.12.5 (A SuperHypertree decomposition)**.** Let $n = 1$ and take the base set $V_0 = \{a, b, c\}$. Define three 1-supervertices

$$V := \{\{a\}, \{b\}, \{c\}\} \subseteq \mathrm{PS}(V_0).$$

Let the family of 1-superedges be

$$E := \Big\{ e_1, e_2 \Big\}, \qquad e_1 := \{\{a\}, \{b\}\}, \quad e_2 := \{\{b\}, \{c\}\}.$$

Hence $H^{(1)} = (V, E)$ is a finite 1-SuperHyperGraph.

We now construct a SuperHypertree decomposition $(T, \mathcal{B}, \mathcal{C})$ as follows. Let $T = (V_T, E_T)$ be the (path) tree with two nodes

$$V_T = \{t_1, t_2\}, \qquad E_T = \{\{t_1, t_2\}\}.$$

Define bags

$$B_{t_1} := \{\{a\}, \{b\}\}, \qquad B_{t_2} := \{\{b\}, \{c\}\},$$

and guards

$$C_{t_1} := \{e_1\}, \qquad C_{t_2} := \{e_2\}.$$

We verify the axioms of Definition 3.12.4.

(1) (*Vertex coverage*) Indeed,

$$B_{t_1} \cup B_{t_2} = \{\{a\}, \{b\}\} \cup \{\{b\}, \{c\}\} = \{\{a\}, \{b\}, \{c\}\} = V.$$

(2) (*Superedge coverage*) We have $e_1 \subseteq B_{t_1}$ and $e_2 \subseteq B_{t_2}$.

(3) (*Vertex connectedness*) For $\{a\}$ we have $T_{\{a\}} = \{t_1\}$; for $\{c\}$ we have $T_{\{c\}} = \{t_2\}$; and for $\{b\}$ we have $T_{\{b\}} = \{t_1, t_2\}$, which is connected in the path T.

(4) (*Guard covering*) Since $C_{t_1} = \{e_1\}$ and $e_1 = B_{t_1}$, we get $B_{t_1} \subseteq \bigcup_{e \in C_{t_1}} e = e_1$; similarly $B_{t_2} \subseteq \bigcup_{e \in C_{t_2}} e = e_2$.

(5) (*Guard connectedness and consistency*) We have $T_{e_1} = \{t_1\}$ and $T_{e_2} = \{t_2\}$, both connected. Moreover, whenever $e_i \subseteq B_{t_j}$ holds, necessarily $(i, j) = (1, 1)$ or $(2, 2)$, and then $e_i \in C_{t_j}$.

Therefore $(T, \mathcal{B}, \mathcal{C})$ is a SuperHypertree decomposition of $H^{(1)}$.

3.13 Skewness of SuperHyperGraph

A graph's skewness is the minimum number of edges to delete so the remaining graph admits a planar embedding without crossings [162–165]. A hypergraph's skewness is the skewness of its primal graph, measuring edge deletions needed for planar primal adjacency (cf. [166, 167]). A superhypergraph's skewness is the skewness of its primal graph on supervertices, measuring deletions required to make it planar.

Definition 3.13.1 (Skewness of a graph)**.** [162–165] Let $G = (V, E)$ be a finite simple graph. We say that G is *planar* if it admits an embedding in the plane in which distinct edges intersect only at common endpoints.

The *skewness* of G is the minimum number of edges that must be deleted to make G planar:

$$\operatorname{skew}(G) \;:=\; \min\bigl\{|F| : \; F \subseteq E, \;\; (V, E \setminus F) \text{ is planar}\bigr\}.$$

In particular, $\operatorname{skew}(G) = 0$ if and only if G is planar.

A concrete example is given below.

Example 3.13.2 (Skewness of a graph)**.** Let $G = K_5$ be the complete graph on $V = \{1, 2, 3, 4, 5\}$. Since K_5 is nonplanar, $\operatorname{skew}(K_5) \geq 1$. Delete the single edge 12 and consider $G' = K_5 - 12$. The graph G' is planar (it admits a planar embedding), hence $\operatorname{skew}(K_5) \leq 1$. Therefore

$$\operatorname{skew}(K_5) = 1.$$

Definition 3.13.3 (Skewness of a hypergraph via the primal graph)**.** Let $H = (V, E)$ be a finite hypergraph. Its *primal graph* (also called the 2-*section*) is the simple graph

$$\operatorname{Pr}(H) \;=\; \bigl(V, \; E(\operatorname{Pr}(H))\bigr),$$

where for distinct $u, v \in V$,

$$\{u, v\} \in E(\operatorname{Pr}(H)) \quad\Longleftrightarrow\quad \exists\, e \in E \;\text{ with }\; \{u, v\} \subseteq e.$$

The *skewness* of H is defined as the skewness of its primal graph:

$$\operatorname{skew}(H) \;:=\; \operatorname{skew}\bigl(\operatorname{Pr}(H)\bigr).$$

Equivalently, $\operatorname{skew}(H)$ is the minimum number of *graph edges* that must be removed from $\operatorname{Pr}(H)$ so that the resulting graph becomes planar.

A concrete example is given below.

Example 3.13.4 (Skewness of a graph)**.** Let $G = K_5$ be the complete graph on $V = \{1, 2, 3, 4, 5\}$. Since K_5 is nonplanar, $\operatorname{skew}(K_5) \geq 1$. Delete the single edge 12 and consider $G' = K_5 - 12$. The graph G' is planar (it admits a planar embedding), hence $\operatorname{skew}(K_5) \leq 1$. Therefore

$$\operatorname{skew}(K_5) = 1.$$

Definition 3.13.5 (Skewness of an n-SuperHyperGraph via the primal graph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (Definition 2.1.6). Its *primal graph* is the simple graph

$$\mathrm{Pr}(\mathrm{SHG}^{(n)}) \;=\; \big(V,\; E(\mathrm{Pr}(\mathrm{SHG}^{(n)}))\big),$$

where for distinct supervertices $X, Y \in V$,

$$\{X, Y\} \in E(\mathrm{Pr}(\mathrm{SHG}^{(n)})) \quad\Longleftrightarrow\quad \exists\, \varepsilon \in E \;\text{ with }\; \{X, Y\} \subseteq \varepsilon.$$

The *skewness* of $\mathrm{SHG}^{(n)}$ is

$$\mathrm{skew}\big(\mathrm{SHG}^{(n)}\big) \;:=\; \mathrm{skew}\big(\mathrm{Pr}(\mathrm{SHG}^{(n)})\big).$$

Thus $\mathrm{skew}(\mathrm{SHG}^{(n)})$ measures how far the induced adjacency structure among supervertices is from being planar.

Remark 3.13.6 (Consistency with graphs and hypergraphs)**.** If G is a graph and H_G is the associated 2-uniform hypergraph with hyperedges $\{u, v\}$ for $uv \in E(G)$, then $\mathrm{Pr}(H_G) = G$ and hence $\mathrm{skew}(H_G) = \mathrm{skew}(G)$ by Definition 3.13.3. Likewise, for any n-lift construction that preserves primal adjacency, the skewness of the lifted n-SuperHyperGraph agrees with the skewness of the original (hyper)graph.

A concrete example is given below.

Example 3.13.7 (Skewness of an n-SuperHyperGraph via the primal graph)**.** Fix $n = 1$ and let the base set be $V_0 = \{1, 2, 3, 4, 5\}$. Define the 1-supervertex set

$$V = \big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\big\} \subseteq \mathrm{PS}(V_0),$$

and let the 1-superedge family be

$$E = \{\, \varepsilon \,\}, \qquad \varepsilon = V.$$

Then any two distinct supervertices $X, Y \in V$ satisfy $\{X, Y\} \subseteq \varepsilon$, hence the primal graph is

$$\mathrm{Pr}(\mathrm{SHG}^{(1)}) = K_5 \quad \text{on the vertex set } V.$$

Therefore,

$$\mathrm{skew}\big(\mathrm{SHG}^{(1)}\big) = \mathrm{skew}\big(\mathrm{Pr}(\mathrm{SHG}^{(1)})\big) = \mathrm{skew}(K_5) = 1,$$

by Example 3.13.4.

3.14 Ramsey number of a SuperHyperGraph

A graph Ramsey number is the smallest N such that any q-edge-coloring of K_N contains a required monochromatic graph [168–171]. A hypergraph Ramsey number is the smallest N such that any q-coloring of $K_N^{(k)}$ yields a monochromatic target k-uniform hypergraph [172–174]. A superhypergraph Ramsey number is the smallest N such that any q-coloring of complete k-uniform superedges forces a monochromatic target superhypergraph copy.

Definition 3.14.1 (Copies and monochromatic copies)**.** Let $\mathcal{F} = (V(\mathcal{F}), E(\mathcal{F}))$ be a (hyper)graph. A *copy* of $\mathcal{F}$ in a (hyper)graph $\mathcal{K} = (V(\mathcal{K}), E(\mathcal{K}))$ is an injective map $f : V(\mathcal{F}) \to V(\mathcal{K})$ such that for every $e \in E(\mathcal{F})$ we have $f(e) := \{f(v) : v \in e\} \in E(\mathcal{K})$. If $\chi : E(\mathcal{K}) \to \{1, \ldots, q\}$ is an edge-coloring, we say that the copy $f(\mathcal{F})$ is *monochromatic in color i* if $\chi(f(e)) = i$ for all $e \in E(\mathcal{F})$.

Definition 3.14.2 (Ramsey number of a graph)**.** Let $G_1, \ldots, G_q$ be finite simple graphs. The *q-color Ramsey number* $r(G_1, \ldots, G_q)$ is the least integer N such that for every coloring $\chi : E(K_N) \to \{1, \ldots, q\}$ there exists an index $i \in \{1, \ldots, q\}$ for which K_N contains a monochromatic copy of G_i in color i. For $q = 2$ one often writes $r(G, H) := r(G, H)$, and $r(G) := r(G, G)$.

Definition 3.14.3 (Complete k-uniform hypergraph)**.** For integers $N \geq k \geq 2$, the *complete k-uniform hypergraph* on $[N] := \{1, \ldots, N\}$ is

$$K_N^{(k)} := \left([N], \binom{[N]}{k}\right),$$

where $\binom{[N]}{k}$ denotes the family of all k-subsets of $[N]$.

Definition 3.14.4 (Ramsey number of a k-uniform hypergraph)**.** Fix $k \geq 2$. Let $H_1, \ldots, H_q$ be finite k-uniform hypergraphs. The *q-color Ramsey number* $r_k(H_1, \ldots, H_q)$ is the least integer N such that for every coloring $\chi : E(K_N^{(k)}) \to \{1, \ldots, q\}$ there exists $i \in \{1, \ldots, q\}$ for which $K_N^{(k)}$ contains a monochromatic copy of H_i in color i. For $q = 2$ we write $r_k(H_1, H_2)$ and for the diagonal case $r_k(H) := r_k(H, H)$.

Definition 3.14.5 (k-uniform n-SuperHyperGraph)**.** Fix $n \geq 0$. An n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ is called *k-uniform* if every superedge $\varepsilon \in E$ has cardinality $|\varepsilon| = k$. An *isomorphism* of n-SuperHyperGraphs $\mathrm{SHG}_1^{(n)} = (V_1, E_1)$ and $\mathrm{SHG}_2^{(n)} = (V_2, E_2)$ is a bijection $\varphi : V_1 \to V_2$ satisfying

$$\varepsilon \in E_1 \iff \varphi(\varepsilon) := \{\varphi(x) : x \in \varepsilon\} \in E_2.$$

Definition 3.14.6 (Complete k-uniform n-SuperHyperGraph)**.** Fix $n \geq 0$ and $k \geq 2$. For $N \geq k$, the *complete k-uniform n-SuperHyperGraph* on N supervertices is the pair

$$K_N^{(n,k)} := \left([N], \binom{[N]}{k}\right),$$

viewed as an n-SuperHyperGraph whose vertex set consists of N abstract n-supervertices (and whose superedges are all k-subsets).

A concrete example is given below.

Example 3.14.7 (A complete k-uniform n-SuperHyperGraph)**.** Take $n = 0$, $k = 3$, and $N = 5$. Then the complete 3-uniform 0-SuperHyperGraph on 5 vertices is

$$K_5^{(0,3)} = ([5], \binom{[5]}{3}),$$

where $[5] = \{1, 2, 3, 4, 5\}$ and the edge set consists of *all* 3-subsets of [5], namely

$$\binom{[5]}{3} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\},$$

$$\{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}.$$

Thus $K_5^{(0,3)}$ is the 3-uniform complete hypergraph on five (ordinary) vertices.

Definition 3.14.8 (Ramsey number of an n-SuperHyperGraph)**.** Fix $n \geq 0$ and $k \geq 2$. Let $\mathcal{S}_1, \ldots, \mathcal{S}_q$ be finite k-uniform n-SuperHyperGraphs. The *q-color Ramsey number* $r_k^{(n)}(\mathcal{S}_1, \ldots, \mathcal{S}_q)$ is the least integer N such that for every coloring $\chi : E(K_N^{(n,k)}) \to \{1, \ldots, q\}$ there exists $i \in \{1, \ldots, q\}$ for which $K_N^{(n,k)}$ contains a monochromatic copy of $\mathcal{S}_i$ in color i (i.e., an isomorphic copy in the sense of Definition 3.14.5). As usual, for $q = 2$ we write $r_k^{(n)}(\mathcal{S}_1, \mathcal{S}_2)$ and $r_k^{(n)}(\mathcal{S}) := r_k^{(n)}(\mathcal{S}, \mathcal{S})$.

A concrete example is given below.

Example 3.14.9 (A Ramsey number of an n-SuperHyperGraph)**.** Take $n = 0$, $k = 2$, and $q = 2$, and let $\mathcal{S}_1 = \mathcal{S}_2 = K_3^{(0,2)}$, i.e., the triangle graph. Then $r_2^{(0)}(K_3^{(0,2)}, K_3^{(0,2)})$ is the smallest N such that every red/blue coloring of the edges of the complete graph $K_N^{(0,2)}$ contains a monochromatic triangle. In this case,

$$r_2^{(0)}\big(K_3^{(0,2)}\big) = r_2^{(0)}\big(K_3^{(0,2)}, K_3^{(0,2)}\big) = 6.$$

Equivalently, every red/blue coloring of the edges of K_6 contains a red triangle or a blue triangle, while there exists a red/blue coloring of K_5 with no monochromatic triangle.

Proposition 3.14.10 (Consistency with graph and hypergraph Ramsey numbers)**.**

1. *For graphs $G_1, \ldots, G_q$, viewed as 2-uniform hypergraphs,*

$$r(G_1, \ldots, G_q) \;=\; r_2(G_1, \ldots, G_q).$$

2. *For k-uniform hypergraphs $H_1, \ldots, H_q$,*

$$r_k(H_1, \ldots, H_q) \;=\; r_k^{(0)}(H_1, \ldots, H_q),$$

where we regard a k-uniform hypergraph as a k-uniform 0-SuperHyperGraph (via $\mathrm{PS}^0(V_0) = V_0$).

3. *More generally, for any k-uniform n-SuperHyperGraphs $\mathcal{S}_1, \dots, \mathcal{S}_q$, the value $r_k^{(n)}(\mathcal{S}_1, \dots, \mathcal{S}_q)$ depends only on the underlying k-uniform hypergraphs $(V(\mathcal{S}_i), E(\mathcal{S}_i))$ (i.e., it is invariant under relabeling supervertices and does not use the internal set-nesting of those supervertices).*

Proof. (1) A graph is exactly a 2-uniform hypergraph, and K_N is exactly $K_N^{(2)}$. Thus a coloring of $E(K_N)$ is the same as a coloring of $E(K_N^{(2)})$, and copies agree with Definition 3.14.1. Hence the two minimization problems coincide.

(2) When $n = 0$, an 0-SuperHyperGraph is a hypergraph on $V \subseteq \mathrm{PS}^0(V_0) = V_0$ by definition. Moreover, the complete objects $K_N^{(0,k)}$ and $K_N^{(k)}$ have the same vertex set $[N]$ and the same k-edge set $\binom{[N]}{k}$, so the Ramsey conditions are identical.

(3) By Definition 3.14.8, the only structure used in forming copies and testing monochromaticity is the incidence relation $\varepsilon \subseteq V$ between superedges and supervertices. No further information about how each supervertex is represented as an element of $\mathrm{PS}^n(V_0)$ is used. Therefore the Ramsey number is determined entirely by the underlying k-uniform hypergraph on the supervertex set. $\square$

3.15 Cyclomatic Number of a SuperHyperGraph

For an ordinary graph with n vertices, m edges, and c connected components, the cyclomatic number is $m - n + c$, which counts the number of independent cycles beyond a spanning forest [175–179]. For hypergraphs, this notion is commonly generalized via an associated incidence graph, so that the resulting invariant measures independent cycle constraints created by the incidence between vertices and hyperedges. For SuperHyperGraphs, one can further extend the same incidence-based viewpoint to accommodate nested superedges, thereby quantifying independent cyclic dependencies across hierarchical levels and layers.

Definition 3.15.1 (Cyclomatic number of a graph). [175, 176] Let $G = (V, E)$ be a finite (undirected) simple graph, and let $\mathrm{cc}(G)$ denote the number of connected components of G. The *cyclomatic number* (also called the *circuit rank*) of G is

$$\mathrm{cyc}(G) \;:=\; |E| - |V| + \mathrm{cc}(G).$$

Equivalently, $\mathrm{cyc}(G)$ is the dimension of the cycle space of G over $\mathbb{F}_2$. In particular, if G is connected then $\mathrm{cyc}(G) = |E| - |V| + 1$.

A concrete example is given below.

Example 3.15.2 (Cyclomatic number of a graph). Let $G = (V, E)$ be the triangle C_3 with

$$V = \{1, 2, 3\}, \qquad E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}.$$

Then G is connected, so $\mathrm{cc}(G) = 1$, and $|V| = 3$, $|E| = 3$. Hence

$$\mathrm{cyc}(G) = |E| - |V| + \mathrm{cc}(G) = 3 - 3 + 1 = 1.$$

Thus G has exactly one independent cycle (its unique 3-cycle).

Definition 3.15.3 (Incidence graph of a hypergraph)**.** Let $H = (V, E)$ be a finite hypergraph, where V is a finite set and E is a finite family of nonempty subsets of V. The *incidence graph* of H is the bipartite simple graph

$$\operatorname{Inc}(H) \;=\; \big(V \,\dot\cup\, E,\; I(H)\big),$$

where the bipartition classes are the vertices V and the hyperedges E, and

$$I(H) \;:=\; \big\{\{v, e\} :\; v \in V,\; e \in E,\; v \in e\big\}.$$

Thus $\{v, e\}$ is an edge of $\operatorname{Inc}(H)$ precisely when the vertex v is contained in the hyperedge e.

Definition 3.15.4 (Cyclomatic number of a hypergraph)**.** Let H be a finite hypergraph. The *cyclomatic number* of H is defined as the cyclomatic number of its incidence graph:

$$\operatorname{cyc}(H) \;:=\; \operatorname{cyc}\big(\operatorname{Inc}(H)\big) \;=\; |I(H)| - \big(|V| + |E|\big) + \operatorname{cc}\big(\operatorname{Inc}(H)\big),$$

where $I(H)$ is the incidence-edge set in Definition 3.15.3.

A concrete example is given below.

Example 3.15.5 (Cyclomatic number of a hypergraph via the incidence graph)**.** Let $H = (V, E)$ be the 2-uniform hypergraph on $V = \{1, 2, 3\}$ with hyperedges

$$E = \{e_{12}, e_{23}, e_{13}\}, \qquad e_{12} = \{1, 2\}, \;\; e_{23} = \{2, 3\}, \;\; e_{13} = \{1, 3\}.$$

Its incidence graph is the bipartite graph

$$\operatorname{Inc}(H) = \big(V \,\dot\cup\, E,\; I(H)\big), \qquad I(H) = \{\{1, e_{12}\}, \{2, e_{12}\}, \{2, e_{23}\}, \{3, e_{23}\}, \{1, e_{13}\}, \{3, e_{13}\}\}.$$

Hence $|V| = 3$, $|E| = 3$, $|I(H)| = 6$, and $\operatorname{Inc}(H)$ is connected, so $\operatorname{cc}(\operatorname{Inc}(H)) = 1$. By Definition 3.15.4,

$$\operatorname{cyc}(H) = \operatorname{cyc}(\operatorname{Inc}(H)) = |I(H)| - (|V| + |E|) + \operatorname{cc}(\operatorname{Inc}(H)) = 6 - (3 + 3) + 1 = 1.$$

(Indeed, $\operatorname{Inc}(H)$ is a 6-cycle.)

Definition 3.15.6 (Incidence graph of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (Definition 2.1.6), so (V, E) is a hypergraph whose vertices are n-supervertices and whose edges are n-superedges. Its *incidence graph* is the bipartite simple graph

$$\operatorname{Inc}\big(\mathrm{SHG}^{(n)}\big) \;=\; \big(V \,\dot\cup\, E,\; I(\mathrm{SHG}^{(n)})\big), \qquad I(\mathrm{SHG}^{(n)}) \;:=\; \big\{\{x, \varepsilon\} :\; x \in V,\; \varepsilon \in E,\; x \in \varepsilon\big\}.$$

Definition 3.15.7 (Cyclomatic number of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)}$ be an n-SuperHyperGraph. The *cyclomatic number* of $\mathrm{SHG}^{(n)}$ is

$$\operatorname{cyc}\big(\mathrm{SHG}^{(n)}\big) \;:=\; \operatorname{cyc}\big(\operatorname{Inc}(\mathrm{SHG}^{(n)})\big) \;=\; |I(\mathrm{SHG}^{(n)})| - \big(|V| + |E|\big) + \operatorname{cc}\big(\operatorname{Inc}(\mathrm{SHG}^{(n)})\big).$$

A concrete example is given below.

Example 3.15.8 (Cyclomatic number of a hypergraph via the incidence graph)**.** Let $H = (V, E)$ be the 2-uniform hypergraph on $V = \{1, 2, 3\}$ with hyperedges

$$E = \{e_{12}, e_{23}, e_{13}\}, \qquad e_{12} = \{1, 2\}, \quad e_{23} = \{2, 3\}, \quad e_{13} = \{1, 3\}.$$

Its incidence graph is the bipartite graph

$$\mathrm{Inc}(H) = \big(V \dot{\cup} E,\ I(H)\big), \qquad I(H) = \{\{1, e_{12}\}, \{2, e_{12}\}, \{2, e_{23}\}, \{3, e_{23}\}, \{1, e_{13}\}, \{3, e_{13}\}\}.$$

Hence $|V| = 3$, $|E| = 3$, $|I(H)| = 6$, and $\mathrm{Inc}(H)$ is connected, so $\mathrm{cc}(\mathrm{Inc}(H)) = 1$. By Definition 3.15.4,

$$\mathrm{cyc}(H) = \mathrm{cyc}(\mathrm{Inc}(H)) = |I(H)| - (|V| + |E|) + \mathrm{cc}(\mathrm{Inc}(H)) = 6 - (3 + 3) + 1 = 1.$$

(Indeed, $\mathrm{Inc}(H)$ is a 6-cycle.)

Proposition 3.15.9 (Consistency: graphs and hypergraphs are special cases)**.**

1. *Let $G = (V, E)$ be a finite simple graph. Consider the associated 2-uniform hypergraph $H_G := \big(V, \{\{u, v\} : uv \in E\}\big)$. Then*

$$\mathrm{cyc}(H_G) = \mathrm{cyc}(G).$$

2. *For $n = 0$, every 0-SuperHyperGraph is exactly a hypergraph, and Definitions 3.15.4 and 3.15.7 agree:*

$$\mathrm{cyc}\big(\mathrm{SHG}^{(0)}\big) = \mathrm{cyc}\big((V, E)\big).$$

Proof. (1) In $\mathrm{Inc}(H_G)$, each graph edge $\{u, v\}$ contributes exactly two incidence edges, $\{u, \{u, v\}\}$ and $\{v, \{u, v\}\}$, so $|I(H_G)| = 2|E|$ and $|V(\mathrm{Inc}(H_G))| = |V| + |E|$. Moreover, $\mathrm{cc}(\mathrm{Inc}(H_G)) = \mathrm{cc}(G)$ (subdividing edges does not change the number of connected components). Therefore

$$\mathrm{cyc}(H_G) = |I(H_G)| - (|V| + |E|) + \mathrm{cc}(\mathrm{Inc}(H_G)) = 2|E| - (|V| + |E|) + \mathrm{cc}(G)$$
$$= |E| - |V| + \mathrm{cc}(G) = \mathrm{cyc}(G),$$

as claimed.

(2) If $n = 0$, then $\mathrm{SHG}^{(0)} = (V, E)$ is a hypergraph by definition, and the incidence graph construction in Definitions 3.15.3 and 3.15.6 is identical. Hence the cyclomatic numbers coincide. □

Chapter 4

Topological Index in SuperHyperGraph

A topological index is a numerical invariant of a (molecular) graph, computed from its connectivity (e.g., degrees, distances, and cycles), and used to correlate structure with chemical and physical properties [180–185]. Because of its importance, the topological index has been studied extensively—particularly in chemistry—and a wide range of applications has been investigated. Accordingly, notions such as topological indices in fuzzy graphs [186, 187], chemical graphs [188–191], molecular graphs [192–194], neutrosophic graphs [195, 196], hypergraphs [197], rough graphs [198], and soft graphs are also well known. In this chapter, we investigate notions of topological indices for graphs that are extended to the setting of SuperHyperGraphs.

4.1 Sombor index of SuperHypergraphs

Sombor index of a graph sums, over edges, the square root of squared endpoint degrees, capturing degree-based structural complexity information [199–201]. These concepts have been further extended and studied in various settings, including chemical graphs [202, 203], fuzzy graphs [204, 205], and neutrosophic graphs [206, 207]. Moreover, related concepts such as the modified Sombor index [208–210], the Zagreb index [211–213], the Hyper-Zagreb Index [214, 215], the ABC index [216, 217], and the GA index [218, 219] are also well known.

Sombor index of a hypergraph generalizes this by summing square-rooted degree squares over each hyperedge's incident vertices within complex interactions [220]. Sombor index of a superhypergraph extends further, aggregating degree-squared contributions over multi-tier superedges, reflecting hierarchical connectivity across nested structural levels.

Definition 4.1.1 (Sombor index of a hypergraph)**.** [220] Let $H = (V, E)$ be a finite hypergraph, where V is a nonempty finite vertex set and $E \subseteq \mathrm{PS}^*(V)$ is a finite family of nonempty subsets of V (the hyperedges).

For each vertex $v \in V$, the *degree* of v in H is

$$d_H(v) \;:=\; \bigl|\{\, e \in E \mid v \in e \,\}\bigr|.$$

The *Sombor index* of the hypergraph H is defined by

$$SO(H) \;:=\; \sum_{e\in E} \sqrt{\sum_{v\in e} d_H(v)^2}\,.$$

When H is 2-uniform (i.e., every hyperedge has size 2), this reduces to the classical Sombor index of a simple graph.

Example 4.1.2 (Sombor index of a hypergraph)**.** Let $H = (V, E)$ be the finite hypergraph with

$$V = \{a, b, c, d\}, \qquad E = \{e_1, e_2\}, \qquad e_1 = \{a, b, c\},\;\; e_2 = \{b, c, d\}.$$

The vertex degrees are

$$d_H(a) = 1, \qquad d_H(d) = 1, \qquad d_H(b) = 2, \qquad d_H(c) = 2.$$

Hence

$$SO(H) = \sum_{e\in E} \sqrt{\sum_{v\in e} d_H(v)^2} = \sqrt{d_H(a)^2 + d_H(b)^2 + d_H(c)^2} + \sqrt{d_H(b)^2 + d_H(c)^2 + d_H(d)^2}.$$

Substituting the degrees yields

$$SO(H) = \sqrt{1^2 + 2^2 + 2^2} + \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} + \sqrt{9} = 3 + 3 = 6.$$

Definition 4.1.3 (Degree in an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be a level-n SuperHyperGraph. For each $v \in V$, the *degree* of v in $\mathrm{SHG}^{(n)}$ is

$$d_{\mathrm{SHG}^{(n)}}(v) \;:=\; \bigl|\{\, e \in E \mid v \in \partial(e) \,\}\bigr|.$$

Definition 4.1.4 (Sombor index of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be a level-n SuperHyperGraph. The *Sombor index* of $\mathrm{SHG}^{(n)}$ is defined by

$$SO\bigl(\mathrm{SHG}^{(n)}\bigr) \;:=\; \sum_{e\in E} \sqrt{\sum_{v\in\partial(e)} d_{\mathrm{SHG}^{(n)}}(v)^2}\,.$$

Remark 4.1.5. If we view a hypergraph $H = (V, E)$ in incidence form by taking $\partial(e) = e$ for all $e \in E$, then the above formula coincides with the Sombor index of a hypergraph.

A concrete example is given below.

Example 4.1.6 (Sombor index of an n-SuperHyperGraph)**.** Let $n = 1$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E, \partial)$ defined as follows. Take the supervertex set

$$V = \{X_1, X_2, X_3, X_4\} \subseteq \mathrm{PS}(V_0), \qquad X_1 = \{1\},\; X_2 = \{2\},\; X_3 = \{3\},\; X_4 = \{4\},$$

and the superedge set $E = \{\varepsilon_1, \varepsilon_2\}$ with incidence map

$$\partial(\varepsilon_1) = \{X_1, X_2, X_3\}, \qquad \partial(\varepsilon_2) = \{X_2, X_3, X_4\}.$$

Then the degrees of supervertices (incidence degrees) are

$$d_{\mathrm{SHG}^{(1)}}(X_1) = 1, \quad d_{\mathrm{SHG}^{(1)}}(X_4) = 1, \quad d_{\mathrm{SHG}^{(1)}}(X_2) = 2, \quad d_{\mathrm{SHG}^{(1)}}(X_3) = 2.$$

Therefore, by Definition *(Sombor index of an n-SuperHyperGraph)*,

$$\begin{aligned} SO\big(\mathrm{SHG}^{(1)}\big) &= \sum_{e\in E} \sqrt{\sum_{v\in\partial(e)} d_{\mathrm{SHG}^{(1)}}(v)^2} \\ &= \sqrt{d(X_1)^2 + d(X_2)^2 + d(X_3)^2} + \sqrt{d(X_2)^2 + d(X_3)^2 + d(X_4)^2} \\ &= \sqrt{1^2 + 2^2 + 2^2} + \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} + \sqrt{9} = 6, \end{aligned}$$

where $d(\cdot)$ abbreviates $d_{\mathrm{SHG}^{(1)}}(\cdot)$.

Theorem 4.1.7 (The Sombor index of SuperHyperGraphs generalizes that of HyperGraphs)**.** *Let $H = (V, E)$ be a finite hypergraph. Define the associated incidence-form SuperHyperGraph*

$$\mathcal{S}(H) \;:=\; (V, E, \partial), \qquad \textit{where} \;\; \partial(e) = e \;\; (\forall e \in E).$$

Then

$$SO\big(\mathcal{S}(H)\big) \;=\; SO(H).$$

In particular, the notion of the Sombor index for SuperHyperGraphs is a genuine extension of the Sombor index for hypergraphs.

Proof. Fix $v \in V$. By definition of $\mathcal{S}(H)$, for each $e \in E$ we have $v \in e$ if and only if $v \in \partial(e)$. Hence the sets

$$\{\, e \in E : v \in e \,\} \qquad \text{and} \qquad \{\, e \in E : v \in \partial(e) \,\}$$

coincide, and therefore the degrees agree:

$$d_H(v) = \big|\{\, e \in E : v \in e \,\}\big| = \big|\{\, e \in E : v \in \partial(e) \,\}\big| = d_{\mathcal{S}(H)}(v).$$

Substituting this equality into the defining sums yields

$$\begin{aligned} SO\big(\mathcal{S}(H)\big) &= \sum_{e\in E} \sqrt{\sum_{v\in\partial(e)} d_{\mathcal{S}(H)}(v)^2} \\ &= \sum_{e\in E} \sqrt{\sum_{v\in e} d_H(v)^2} = SO(H), \end{aligned}$$

as claimed. □

4.2 Wiener index of a SuperHyperGraph

The *Wiener index of a graph* is the sum of shortest-path distances $d(u, v)$ over all unordered vertex pairs $\{u, v\}$ [221–223]. Related notions such as the Wiener index of chemical graphs [224–226] and the Wiener index of fuzzy graphs are also known. The *Wiener index of a superhypergraph* sums shortest distances between base vertices in the induced superhypergraph metric, extending graphs.

Let $G = (V(G), E(G))$ be a finite, simple, connected (undirected) graph. For $u, v \in V(G)$, let $d_G(u, v)$ denote the usual shortest-path distance (the minimum number of edges in a u–v path).

Definition 4.2.1 (Wiener index of a graph)**.** [221–223] The *Wiener index* of G is

$$W(G) \;:=\; \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) \;=\; \frac{1}{2}\sum_{u\in V(G)}\sum_{v\in V(G)} d_G(u,v).$$

We use the standard (Berge-type) notion of path and distance: a u–v path is an alternating sequence

$$(v_0, e_1, v_1, \dots, e_p, v_p)$$

with $v_0 = u$, $v_p = v$, all v_i distinct, all e_i distinct, and $v_{i-1}, v_i \in e_i$ for each i; its length is p. The distance $d_H(u,v)$ is the minimum length of a u–v path (and $d_H(u,u) = 0$). If every vertex pair is joined by a path, then H is connected.

Definition 4.2.2 (Wiener index of a hypergraph)**.** Let H be a finite connected hypergraph. Its *Wiener index* is

$$W(H) \;:=\; \sum_{\{u,v\}\subseteq V(H)} d_H(u,v) \;=\; \frac{1}{2}\sum_{u\in V(H)}\sum_{v\in V(H)} d_H(u,v),$$

i.e., the sum of distances over all unordered vertex pairs.

To encode nested or hierarchical "vertices," we adopt a powerset-based universe. Let X be a nonempty finite base set and let $\mathrm{PS}^0(X) := X$, $\mathrm{PS}^{i+1}(X) := \mathrm{PS}(\mathrm{PS}^i(X))$. Fix an integer $r \geq 0$ and set

$$\mathcal{U}_r(X) \;:=\; \bigcup_{i=0}^{r} \bigl(\mathrm{PS}^i(X) \setminus \{\emptyset\}\bigr).$$

Definition 4.2.3 (Superpath, distance, and Wiener index of a superhypergraph)**.** Let $\mathbb{H}$ be a SuperHyperGraph. A *superpath* from A to B (with $A, B \in V(\mathbb{H})$) is an alternating sequence

$$(A_0, f_1, A_1, \dots, f_p, A_p)$$

where $A_0 = A$, $A_p = B$, the A_i are distinct supervertices, the f_i are distinct superhyperedges, and $A_{i-1}, A_i \in f_i$ for each i. Its length is p. If such a superpath exists for every pair of supervertices, we call $\mathbb{H}$ *connected.*

For connected $\mathbb{H}$, define the *distance* $d_{\mathbb{H}}(A,B)$ as the minimum length of an A–B superpath, and define the *Wiener index* by

$$W(\mathbb{H}) \;:=\; \sum_{\{A,B\}\subseteq V(\mathbb{H})} d_{\mathbb{H}}(A,B) \;=\; \frac{1}{2}\sum_{A\in V(\mathbb{H})}\sum_{B\in V(\mathbb{H})} d_{\mathbb{H}}(A,B).$$

Example 4.2.4 (Superpath, distance, and Wiener index of a SuperHyperGraph)**.** Let the base set be $X = \{1,2,3\}$ and take $r = 1$, so $\mathcal{U}_1(X) = X \,\cup\, (\mathrm{PS}(X) \setminus \{\emptyset\})$. Consider the SuperHyperGraph $\mathbb{H} = (V, E)$ with supervertex set

$$V = \Bigl\{\; A = \{1\},\; B = \{1,2\},\; C = \{2\},\; D = \{2,3\} \;\Bigr\} \subseteq \mathcal{U}_1(X),$$

and superhyperedge family

$$E = \{ f_1, f_2, f_3 \}, \qquad f_1 = \{A, B\}, \quad f_2 = \{B, C, D\}, \quad f_3 = \{C, D\}.$$

(Thus each f_i is a nonempty subset of V.)

A superpath. A superpath from A to D is

$$(A, f_1, B, f_2, D),$$

since $A, B \in f_1$ and $B, D \in f_2$. Its length is 2.

Distances. Because $\mathbb{H}$ is connected, the distances $d_{\mathbb{H}}(\cdot, \cdot)$ are the shortest superpath lengths. One checks:

$$d_{\mathbb{H}}(A, B) = 1, \qquad d_{\mathbb{H}}(A, C) = 2 \;\; (A, f_1, B, f_2, C), \qquad d_{\mathbb{H}}(A, D) = 2 \;\; (A, f_1, B, f_2, D),$$
$$d_{\mathbb{H}}(B, C) = 1, \qquad d_{\mathbb{H}}(B, D) = 1, \qquad d_{\mathbb{H}}(C, D) = 1 \;\; (C, f_3, D).$$

Wiener index. Summing over all unordered pairs $\{A, B\} \subseteq V$ gives

$$W(\mathbb{H}) = \sum_{\{U,V\} \subseteq V} d_{\mathbb{H}}(U, V) = d(A, B) + d(A, C) + d(A, D) + d(B, C) + d(B, D) + d(C, D).$$

Substituting the above values yields

$$W(\mathbb{H}) = 1 + 2 + 2 + 1 + 1 + 1 = 8.$$

Let $\iota : X \to \mathrm{PS}(X) \setminus \{\emptyset\}$ be the singleton embedding $\iota(x) := \{x\}$.

Theorem 4.2.5 (Superhypergraph Wiener index generalizes the graph and hypergraph cases)**.**

1. *(*Graph case*) Let $G = (X, E(G))$ be a finite connected simple graph. Define a SuperHyperGraph $\mathbb{H}_G = (V_G, E_G)$ by*

   $$V_G := \{\{x\} : x \in X\}, \qquad E_G := \big\{\{\{u\}, \{v\}\} : \{u, v\} \in E(G)\big\}.$$

   Then for all $u, v \in X$,

   $$d_{\mathbb{H}_G}(\{u\}, \{v\}) = d_G(u, v),$$

   and consequently $W(\mathbb{H}_G) = W(G)$.

2. *(*Hypergraph case*) Let $H = (X, E(H))$ be a finite connected hypergraph. Define a SuperHyperGraph $\mathbb{H}_H = (V_H, E_H)$ by*

   $$V_H := \{\{x\} : x \in X\}, \qquad E_H := \big\{\{\{x\} : x \in e\} : e \in E(H)\big\}.$$

   Then for all $u, v \in X$,

   $$d_{\mathbb{H}_H}(\{u\}, \{v\}) = d_H(u, v),$$

   and consequently $W(\mathbb{H}_H) = W(H)$.

Proof. (1) Take any $u, v \in X$. Every graph path $u = x_0 - x_1 - \cdots - x_k = v$ in G induces a superpath

$$(\{x_0\}, f_1, \{x_1\}, f_2, \ldots, f_k, \{x_k\})$$

in $\mathbb{H}_G$, where $f_i := \{\{x_{i-1}\}, \{x_i\}\} \in E_G$. Hence $d_{\mathbb{H}_G}(\{u\}, \{v\}) \le d_G(u, v)$.

Conversely, any superpath in $\mathbb{H}_G$ from $\{u\}$ to $\{v\}$ has the form

$$(\{x_0\}, \{\{x_0\}, \{x_1\}\}, \{x_1\}, \ldots, \{\{x_{k-1}\}, \{x_k\}\}, \{x_k\}),$$

which projects to a graph path $x_0 - x_1 - \cdots - x_k$ in G of the same length. Thus $d_G(u, v) \le d_{\mathbb{H}_G}(\{u\}, \{v\})$, proving equality of distances. Summing over unordered pairs yields $W(\mathbb{H}_G) = W(G)$.

(2) The argument is the same, using Berge-type hypergraph paths: a u–v path $(v_0, e_1, v_1, \ldots, e_p, v_p)$ in H corresponds to the superpath

$$(\{v_0\}, \{\{x\} : x \in e_1\}, \{v_1\}, \ldots, \{\{x\} : x \in e_p\}, \{v_p\})$$

in $\mathbb{H}_H$ with identical length p, and any superpath in $\mathbb{H}_H$ projects back to a hypergraph path in H. Therefore $d_{\mathbb{H}_H}(\{u\}, \{v\}) = d_H(u, v)$ for all u, v, and summation over pairs gives $W(\mathbb{H}_H) = W(H)$. □

4.3 Hosoya index of a SuperHyperGraph

The *Hosoya index of a graph* counts all matchings in G, i.e., independent edge-sets, including the empty matching [227–230]. Related notions such as the Hosoya index of a chemical graph are also known [231]. The *Hosoya index of a superhypergraph* counts all pairwise disjoint superhyperedges, generalizing graph matchings and hence Hosoya index.

Definition 4.3.1 (Matching in a graph)**.** (cf. [232]) Let $G = (V, E)$ be a finite (simple, undirected) graph. A *matching* in G is a subset $M \subseteq E$ such that no two distinct edges in M share a common endpoint. For $s \in \mathbb{Z}_{\ge 0}$, an *s-matching* is a matching M with $|M| = s$. We write $m(G; s)$ for the number of s-matchings of G, and adopt the convention $m(G; 0) = 1$ (the empty matching).

Definition 4.3.2 (Hosoya index of a graph)**.** [227–230] Let G be a finite graph. The *Hosoya index* of G, denoted by $Z(G)$ (or $z(G)$), is defined as the total number of matchings in G:

$$Z(G) := \sum_{s=0}^{\lfloor |V(G)|/2 \rfloor} m(G; s).$$

Equivalently, $Z(G)$ counts all subsets of $E(G)$ whose edges are pairwise vertex-disjoint (including the empty set).

Definition 4.3.3 (Iterated powerset universe)**.** Let V be a nonempty finite set (the *base-vertex* set). Define recursively

$$\mathrm{PS}^{(0)}(V) := V, \qquad \mathrm{PS}^{(k+1)}(V) := \mathrm{PS}(\mathrm{PS}^{(k)}(V)) \setminus \{\emptyset\} \quad (k \ge 0),$$

where $\mathrm{PS}(\cdot)$ is the usual powerset. Elements of $\mathrm{PS}^{(k)}(V)$ may be viewed as *objects of depth k* built from V.

Definition 4.3.4 (Finite SuperHyperGraph)**.** A *(finite) SuperHyperGraph* is a pair $\mathbb{S} = (V, \mathcal{E})$ where V is a finite base-vertex set and $\mathcal{E}$ is a finite set of *superhyperedges* such that there exists $L \geq 1$ with

$$\mathcal{E} \subseteq \bigcup_{k=1}^{L} \mathrm{PS}^{(k)}(V).$$

Thus each $E \in \mathcal{E}$ is a nonempty nested set-object ultimately constructed from elements of V.

Definition 4.3.5 (Base-support (flattening) map)**.** Define $\mathrm{supp} : \bigcup_{k\geq 0} \mathrm{PS}^{(k)}(V) \to \mathrm{PS}(V)$ recursively by

$$\mathrm{supp}(v) := \{v\} \;\; (v \in V), \qquad \mathrm{supp}(X) := \bigcup_{x\in X} \mathrm{supp}(x) \;\; (X \in \mathrm{PS}^{(k)}(V),\; k \geq 1).$$

For a superhyperedge $E \in \mathcal{E}$, the set $\mathrm{supp}(E) \subseteq V$ is the collection of base-vertices that occur in E after fully "flattening" the nested structure.

Definition 4.3.6 (Supermatching)**.** Let $\mathbb{S} = (V, \mathcal{E})$ be a SuperHyperGraph. A *supermatching* is a subset $M \subseteq \mathcal{E}$ such that for any two distinct $E, F \in M$ one has

$$\mathrm{supp}(E) \cap \mathrm{supp}(F) = \emptyset.$$

For $s \in \mathbb{Z}_{\geq 0}$, an *s-supermatching* is a supermatching M with $|M| = s$. Let $m(\mathbb{S}; s)$ denote the number of s-supermatchings of $\mathbb{S}$, and set $m(\mathbb{S}; 0) = 1$.

Definition 4.3.7 (Hosoya index of a SuperHyperGraph)**.** Let $\mathbb{S} = (V, \mathcal{E})$ be a finite SuperHyperGraph. Its *Hosoya index* is the total number of supermatchings:

$$Z(\mathbb{S}) \;:=\; \sum_{s=0}^{|\mathcal{E}|} m(\mathbb{S}; s).$$

A concrete example is given below.

Example 4.3.8 (A concrete Hosoya index computation for a SuperHyperGraph)**.** Let the (level–0) vertex set be

$$V = \{v_1, v_2, v_3, v_4\}.$$

Consider the following (level–1) hyperedges of V:

$$h_{12} = \{v_1, v_2\}, \qquad h_{23} = \{v_2, v_3\}, \qquad h_{34} = \{v_3, v_4\}.$$

Define a (level–2) SuperHyperGraph $\mathbb{S} = (V, \mathcal{E})$ whose superhyperedges are sets of hyperedges:

$$\mathcal{E} = \{E_1, E_2, E_3\}, \qquad E_1 = \{h_{12}\}, \quad E_2 = \{h_{34}\}, \quad E_3 = \{h_{23}\}.$$

For this example, we use the natural *vertex-support* of a superhyperedge $E \in \mathcal{E}$,

$$\mathrm{supp}(E) \;:=\; \bigcup_{h\in E} h \;\subseteq V,$$

and we call $\mathcal{M} \subseteq \mathcal{E}$ a *supermatching* if

$$\operatorname{supp}(E) \cap \operatorname{supp}(F) = \emptyset \qquad \text{for all distinct } E, F \in \mathcal{M}.$$

Now,

$$\operatorname{supp}(E_1) = \{v_1, v_2\}, \quad \operatorname{supp}(E_2) = \{v_3, v_4\}, \quad \operatorname{supp}(E_3) = \{v_2, v_3\}.$$

Hence E_1 and E_2 are disjoint, while E_3 intersects both E_1 (at v_2) and E_2 (at v_3). Therefore the numbers of supermatchings of each size are

$$m(\mathbb{S}; 0) = 1, \qquad m(\mathbb{S}; 1) = 3, \qquad m(\mathbb{S}; 2) = 1, \qquad m(\mathbb{S}; 3) = 0,$$

because the only size–2 supermatching is $\{E_1, E_2\}$.

Consequently, the Hosoya index of $\mathbb{S}$ is

$$Z(\mathbb{S}) = \sum_{s=0}^{|\mathcal{E}|} m(\mathbb{S}; s) = 1 + 3 + 1 + 0 = 5.$$

Theorem 4.3.9 (SuperHyperGraph Hosoya index generalizes the graph Hosoya index)**.** *Let $G = (V, E)$ be a finite (simple) graph, and form the depth-1 SuperHyperGraph*

$$\mathbb{S}(G) \ := \ (V, \mathcal{E}), \qquad \mathcal{E} := E \subseteq \mathrm{PS}^{(1)}(V),$$

i.e., each graph edge $\{u, v\} \in E$ is regarded as a superhyperedge of depth 1. Then

$$Z(\mathbb{S}(G)) \ = \ Z(G).$$

In particular, the Hosoya index defined for SuperHyperGraphs extends (and hence generalizes) the classical Hosoya index for graphs.

Proof. For a depth-1 edge $e = \{u, v\} \in E \subseteq \mathrm{PS}^{(1)}(V)$, the support map satisfies $\operatorname{supp}(e) = \{u, v\}$ by definition of supp. Hence, for two distinct edges $e, f \in E$,

$$\operatorname{supp}(e) \cap \operatorname{supp}(f) = \emptyset \quad \Longleftrightarrow \quad \{u, v\} \cap \{x, y\} = \emptyset,$$

which is equivalent to saying that e and f share no endpoint in the usual graph-theoretic sense. Therefore, a subset $M \subseteq E$ is a matching in G if and only if it is a supermatching in $\mathbb{S}(G)$.

Consequently, for every $s \geq 0$ we have $m(\mathbb{S}(G); s) = m(G; s)$, and summing over s yields

$$Z(\mathbb{S}(G)) = \sum_{s \geq 0} m(\mathbb{S}(G); s) = \sum_{s \geq 0} m(G; s) = Z(G).$$

□

4.4 Randić index of a SuperHyperGraph

The *Randić index of a graph* is $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}$, measuring branching via inverse degree products [233–236]. The *Randić index of a superhypergraph* sums analogous inverse square-roots of endpoint superdegrees over superhyperedges, extending $R(G)$ consistently.

Definition 4.4.1 (General Randić index of a graph)**.** [233–236] Let $G = (V, E)$ be a finite simple undirected graph, and let $d_G(v)$ denote the (usual) degree of $v \in V$. For a real parameter $\alpha \in \mathbb{R}$, the *general Randić index* of G is

$$R_\alpha(G) \ := \ \sum_{uv \in E} \big(d_G(u)\, d_G(v)\big)^\alpha.$$

The classical *Randić (connectivity) index* is the special case

$$R(G) \ := \ R_{-1/2}(G) \ = \ \sum_{uv \in E} \frac{1}{\sqrt{d_G(u)\, d_G(v)}}.$$

Definition 4.4.2 ((Recall) r-SuperHyperGraph, incidence, and degree)**.** An *r-SuperHyperGraph* is a pair $\mathbb{S} = (V, \mathcal{E})$ where V is a finite base vertex set and $\mathcal{E} \subseteq \mathrm{PS}^r(V)$ is a finite family of *superhyperedges*. A base vertex $v \in V$ is *incident* with a superhyperedge $E \in \mathcal{E}$ if $v \in \mathrm{Flat}_r(E)$.

The (base) degree of $v \in V$ in $\mathbb{S}$ is

$$d_{\mathbb{S}}(v) \ := \ \big|\{E \in \mathcal{E} : v \in \mathrm{Flat}_r(E)\}\big|.$$

Definition 4.4.3 (General Randić index of a SuperHyperGraph)**.** Let $\mathbb{S} = (V, \mathcal{E})$ be an r-SuperHyperGraph and let $\alpha \in \mathbb{R}$. The *general Randić index* of $\mathbb{S}$ is defined by

$$R_\alpha(\mathbb{S}) \ := \ \sum_{E \in \mathcal{E}} \ \sum_{\{u,v\} \subseteq \mathrm{Flat}_r(E)} \big(d_{\mathbb{S}}(u)\, d_{\mathbb{S}}(v)\big)^\alpha,$$

where the inner sum ranges over all unordered *distinct* pairs $\{u, v\}$ contained in $\mathrm{Flat}_r(E)$. The *Randić index* of $\mathbb{S}$ is $R(\mathbb{S}) := R_{-1/2}(\mathbb{S})$.

Remark 4.4.4. When $r = 1$, each $E \in \mathcal{E}$ is an ordinary (nonempty) subset of V, so the above definition specializes to a natural hypergraph extension obtained by summing the Randić-type contribution over all unordered vertex pairs that co-occur in a hyperedge.

A concrete example is given below.

Example 4.4.5 (A concrete computation of the general Randić index for a SuperHyperGraph)**.** Let $V = \{a, b, c\}$ and consider a 2-SuperHyperGraph $\mathbb{S} = (V, \mathcal{E})$ with

$$\mathcal{E} = \{E_1, E_2\}, \qquad E_1 = \{\{a, b\}, \{b, c\}\}, \qquad E_2 = \{\{a, c\}\}.$$

For $r = 2$ we take the flattening map

$$\mathrm{Flat}_2(E) := \bigcup_{h \in E} h \subseteq V,$$

so that

$$\mathrm{Flat}_2(E_1) = \{a, b, c\}, \qquad \mathrm{Flat}_2(E_2) = \{a, c\}.$$

Define the superdegree of a vertex $x \in V$ by

$$d_{\mathbb{S}}(x) := \big|\{E \in \mathcal{E} : x \in \mathrm{Flat}_2(E)\}\big|.$$

Then

$$d_{\mathbb{S}}(a) = 2, \qquad d_{\mathbb{S}}(b) = 1, \qquad d_{\mathbb{S}}(c) = 2,$$

because a and c lie in both $\mathrm{Flat}_2(E_1)$ and $\mathrm{Flat}_2(E_2)$, while b lies only in $\mathrm{Flat}_2(E_1)$.

Hence, for a general parameter $\alpha \in \mathbb{R}$,

$$\begin{aligned}
R_\alpha(\mathbb{S}) &= \sum_{E \in \mathcal{E}} \sum_{\{u,v\} \subseteq \mathrm{Flat}_2(E)} \big(d_{\mathbb{S}}(u) d_{\mathbb{S}}(v)\big)^\alpha \\
&= \Big[(d(a)d(b))^\alpha + (d(a)d(c))^\alpha + (d(b)d(c))^\alpha\Big] + \Big[(d(a)d(c))^\alpha\Big] \\
&= (2 \cdot 1)^\alpha + (2 \cdot 2)^\alpha + (1 \cdot 2)^\alpha + (2 \cdot 2)^\alpha \\
&= 2^\alpha + 4^\alpha + 2^\alpha + 4^\alpha = 2 \cdot 2^\alpha + 2 \cdot 4^\alpha.
\end{aligned}$$

In particular, the Randić index $R(\mathbb{S}) = R_{-1/2}(\mathbb{S})$ equals

$$R(\mathbb{S}) = 2 \cdot 2^{-1/2} + 2 \cdot 4^{-1/2} = \frac{2}{\sqrt{2}} + \frac{2}{2} = \sqrt{2} + 1.$$

Theorem 4.4.6 (SuperHyperGraph Randić index generalizes the graph Randić index)**.** *Let $G = (V, E)$ be a finite simple undirected graph and fix $\alpha \in \mathbb{R}$. Define the associated 1-SuperHyperGraph $\iota(G) = (V, \mathcal{E})$ by*

$$\mathcal{E} \; := \; \big\{\{u, v\} \in \mathrm{PS}^1(V) : uv \in E\big\}.$$

Then

$$R_\alpha(\iota(G)) \; = \; R_\alpha(G).$$

In particular, $R(\iota(G)) = R(G)$ for $\alpha = -\frac{1}{2}$.

Proof. Since $r = 1$, we have $\mathrm{Flat}_1(\{u, v\}) = \{u, v\}$ for each $\{u, v\} \in \mathcal{E}$. Moreover, for every $x \in V$,

$$d_{\iota(G)}(x) = \big|\{\{u, v\} \in \mathcal{E} : x \in \{u, v\}\}\big| = \big|\{xy \in E : y \in V\}\big| = d_G(x).$$

Now compute:

$$R_\alpha(\iota(G)) = \sum_{\{u,v\} \in \mathcal{E}} \sum_{\{x,y\} \subseteq \mathrm{Flat}_1(\{u,v\})} \big(d_{\iota(G)}(x) d_{\iota(G)}(y)\big)^\alpha.$$

But $\mathrm{Flat}_1(\{u, v\}) = \{u, v\}$ has exactly one unordered distinct pair, namely $\{u, v\}$. Hence the inner sum collapses and, using $d_{\iota(G)} = d_G$,

$$R_\alpha(\iota(G)) = \sum_{\{u,v\} \in \mathcal{E}} \big(d_G(u) d_G(v)\big)^\alpha = \sum_{uv \in E} \big(d_G(u) d_G(v)\big)^\alpha = R_\alpha(G).$$

This proves the claim, and the case $\alpha = -\frac{1}{2}$ follows immediately. □

4.5 Zagreb indices of a SuperHyperGraph

The *Zagreb indices of a graph* are $M_1(G) = \sum_{v \in V(G)} d(v)^2$ and $M_2(G) = \sum_{uv \in E(G)} d(u)d(v)$ [237–241]. Related notions such as hyper-Zagreb indices [215, 242, 243], Zagreb indices in fuzzy graphs [244, 245], Zagreb indices in Chemical graphs [246–249], and Zagreb indices in neutrosophic graphs [250] are also known. The *Zagreb indices of a superhypergraph* analogously sum squared superdegrees and products over superhyperedges using an appropriate incidence-based degree notion.

Definition 4.5.1 (First and second Zagreb indices of a graph)**.** [237–239] Let $G = (V(G), E(G))$ be a (simple) graph. For each $v \in V(G)$, let $d_G(v)$ denote the degree of v. The *first Zagreb index* and *second Zagreb index* of G are defined by

$$M_1(G) \;:=\; \sum_{v \in V(G)} d_G(v)^2, \qquad M_2(G) \;:=\; \sum_{\{u,v\} \in E(G)} d_G(u)\, d_G(v).$$

Definition 4.5.2 (Degree and pair-incidence multiplicity)**.** Let $\mathbb{H} = (\mathbb{V}, \mathbb{E})$ be a SuperHyperGraph. For $X \in \mathbb{V}$, define the *degree* of X by

$$d_{\mathbb{H}}(X) \;:=\; \bigl|\{\, e \in \mathbb{E} : X \in e \,\}\bigr|.$$

For distinct $X, Y \in \mathbb{V}$, define the *pair-incidence multiplicity* by

$$\mu_{\mathbb{H}}(X, Y) \;:=\; \bigl|\{\, e \in \mathbb{E} : \{X, Y\} \subseteq e \,\}\bigr|.$$

(Thus, $\mu_{\mathbb{H}}(X, Y) \geq 1$ exactly when X and Y co-occur in at least one superhyperedge.)

Definition 4.5.3 (First and second Zagreb indices of a SuperHyperGraph)**.** Let $\mathbb{H} = (\mathbb{V}, \mathbb{E})$ be a SuperHyperGraph. Define

$$M_1(\mathbb{H}) \;:=\; \sum_{X \in \mathbb{V}} d_{\mathbb{H}}(X)^2, \qquad M_2(\mathbb{H}) \;:=\; \sum_{\{X,Y\} \subseteq \mathbb{V}} \mu_{\mathbb{H}}(X, Y)\, d_{\mathbb{H}}(X)\, d_{\mathbb{H}}(Y),$$

where the second sum ranges over all 2-element subsets $\{X, Y\}$ of $\mathbb{V}$.

Remark 4.5.4. If $\mathbb{H}$ has the property that each unordered pair $\{X, Y\}$ is contained in *at most one* superhyperedge, then $\mu_{\mathbb{H}}(X, Y) \in \{0, 1\}$ and

$$M_2(\mathbb{H}) = \sum_{\substack{\{X,Y\} \subseteq \mathbb{V} \\ \mu_{\mathbb{H}}(X,Y) = 1}} d_{\mathbb{H}}(X)\, d_{\mathbb{H}}(Y),$$

i.e., the sum runs over adjacent pairs.

A concrete example is given below.

Example 4.5.5 (A concrete computation of Zagreb indices for a SuperHyperGraph)**.** Let $\mathbb{H} = (\mathbb{V}, \mathbb{E})$ be the SuperHyperGraph with vertex set

$$\mathbb{V} = \{A, B, C\}$$

and superhyperedge family

$$\mathbb{E} = \{E_1, E_2\}, \qquad E_1 = \{A, B\}, \qquad E_2 = \{A, B, C\}.$$

Assume the standard pair-multiplicity

$$\mu_{\mathbb{H}}(X, Y) := \bigl|\{E \in \mathbb{E} : \{X, Y\} \subseteq E\}\bigr| \quad \text{for distinct } X, Y \in \mathbb{V},$$

and define the degree of a supervertex $X \in \mathbb{V}$ by

$$d_{\mathbb{H}}(X) := \bigl|\{E \in \mathbb{E} : X \in E\}\bigr|.$$

Then

$$d_{\mathbb{H}}(A) = 2, \qquad d_{\mathbb{H}}(B) = 2, \qquad d_{\mathbb{H}}(C) = 1.$$

Therefore, the first Zagreb index is

$$M_1(\mathbb{H}) = d(A)^2 + d(B)^2 + d(C)^2 = 2^2 + 2^2 + 1^2 = 9.$$

Next compute $\mu_{\mathbb{H}}(X, Y)$ for each unordered pair:

$$\mu_{\mathbb{H}}(A, B) = 2 \quad \text{(contained in } E_1 \text{ and } E_2\text{)}, \qquad \mu_{\mathbb{H}}(A, C) = 1 \quad \text{(contained only in } E_2\text{)},$$

$$\mu_{\mathbb{H}}(B, C) = 1 \quad \text{(contained only in } E_2\text{)}.$$

Hence the second Zagreb index equals

$$\begin{aligned} M_2(\mathbb{H}) &= \sum_{\{X,Y\} \subseteq \mathbb{V}} \mu_{\mathbb{H}}(X, Y)\, d(X)\, d(Y) \\ &= \mu(A, B)d(A)d(B) + \mu(A, C)d(A)d(C) + \mu(B, C)d(B)d(C) \\ &= 2 \cdot (2 \cdot 2) + 1 \cdot (2 \cdot 1) + 1 \cdot (2 \cdot 1) \\ &= 8 + 2 + 2 = 12. \end{aligned}$$

Theorem 4.5.6 (SuperHyperGraph Zagreb indices generalize graph Zagreb indices)**.** *Let $G = (V, E)$ be a simple graph. Define the level-*1 *SuperHyperGraph*

$$\iota(G) \;:=\; (\mathbb{V}, \mathbb{E}) \quad \textit{by} \quad \mathbb{V} := \bigl\{\{v\} : v \in V\bigr\} \subseteq \mathcal{P}(V), \qquad \mathbb{E} := \bigl\{\{\{u\}, \{v\}\} : \{u, v\} \in E\bigr\}.$$

Then

$$M_1(\iota(G)) = M_1(G) \qquad \textit{and} \qquad M_2(\iota(G)) = M_2(G).$$

Proof. Fix $v \in V$ and consider the supervertex $\{v\} \in \mathbb{V}$. By construction, superhyperedges in $\mathbb{E}$ containing $\{v\}$ are precisely the sets $\{\{v\}, \{u\}\}$ with $\{v, u\} \in E$. Hence the map

$$\{\text{edges incident to } v \text{ in } G\} \longrightarrow \{\text{superhyperedges incident to } \{v\} \text{ in } \iota(G)\},$$

$$\{v, u\} \mapsto \{\{v\}, \{u\}\},$$

is a bijection. Therefore $d_{\iota(G)}(\{v\}) = d_G(v)$ for all $v \in V$, and so

$$M_1(\iota(G)) = \sum_{\{v\} \in \mathbb{V}} d_{\iota(G)}(\{v\})^2$$

$$= \sum_{v \in V} d_G(v)^2 = M_1(G).$$

Next, for distinct $u, v \in V$, the unordered pair $\{\{u\}, \{v\}\}$ is contained in a superhyperedge of $\iota(G)$ if and only if $\{u, v\} \in E$, and in that case it is contained in exactly one such superhyperedge. Hence

$$\mu_{\iota(G)}(\{u\}, \{v\}) = \begin{cases} 1, & \text{if } \{u, v\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Using also $d_{\iota(G)}(\{u\}) = d_G(u)$, we obtain

$$M_2(\iota(G)) = \sum_{\{\{u\},\{v\}\} \subseteq \mathbb{V}} \mu_{\iota(G)}(\{u\}, \{v\})\, d_G(u)\, d_G(v) = \sum_{\{u,v\} \in E} d_G(u)\, d_G(v) = M_2(G).$$

□

4.6 Gutman index of a SuperHyperGraph

The *Gutman index of a graph* is $\sum_{\{u,v\} \subseteq V(G)} d_G(u) d_G(v)\, \mathrm{dist}_G(u, v)$, combining degrees with pairwise shortest-path distances [251–254]. The Gutman index of fuzzy graphs [255] among related topics, have also been widely studied. The *Gutman index of a superhypergraph* extends this by using superhypergraph degrees and shortest superincidence distances between supervertices in pairs.

Definition 4.6.1 (Gutman index of a graph)**.** [251,252] The *Gutman index* of G is

$$\mathrm{Gut}(G) := \sum_{\{u,v\} \subseteq V(G)} \deg_G(u) \deg_G(v)\, d_G(u, v) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} \deg_G(u) \deg_G(v)\, d_G(u, v).$$

Definition 4.6.2 (Adjacency, degree)**.** Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be an n-SuperHyperGraph. Distinct supervertices $x, y \in \mathcal{V}$ are *adjacent* if there exists $e \in \mathcal{E}$ with $\{x, y\} \subseteq e$. The *degree* of $x \in \mathcal{V}$ is

$$\deg_{\mathcal{S}}(x) := |\{ e \in \mathcal{E} : x \in e \}|.$$

Definition 4.6.3 (Berge path and distance)**.** A *(Berge) path* of length $p \geq 1$ from x to y in $\mathcal{S}$ is a sequence

$$(x = v_0,\ e_1,\ v_1,\ e_2,\ \ldots,\ e_p,\ v_p = y),$$

where $v_0, \ldots, v_p \in \mathcal{V}$ are pairwise distinct, $e_1, \ldots, e_p \in \mathcal{E}$ are pairwise distinct, and $\{v_{i-1}, v_i\} \subseteq e_i$ for all $i = 1, \ldots, p$. If such a path exists for every ordered pair of distinct vertices, $\mathcal{S}$ is called *connected.* For a connected $\mathcal{S}$, the *distance* $D_{\mathcal{S}}(x, y)$ is the minimum length of a path from x to y (and $D_{\mathcal{S}}(x, x) = 0$).

Definition 4.6.4 (Gutman index of a SuperHyperGraph)**.** Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be a finite connected n-SuperHyperGraph. Its *Gutman index* is

$$\mathrm{Gut}(\mathcal{S}) := \sum_{\{x,y\} \subseteq \mathcal{V}} \deg_{\mathcal{S}}(x) \deg_{\mathcal{S}}(y)\, D_{\mathcal{S}}(x, y) = \frac{1}{2} \sum_{x \in \mathcal{V}} \sum_{y \in \mathcal{V}} \deg_{\mathcal{S}}(x) \deg_{\mathcal{S}}(y)\, D_{\mathcal{S}}(x, y).$$

A concrete example is given below.

Example 4.6.5 (A concrete computation of the Gutman index for a SuperHyperGraph)**.** Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be the connected SuperHyperGraph with

$$\mathcal{V} = \{a, b, c\}, \qquad \mathcal{E} = \{E_1, E_2\}, \quad E_1 = \{a, b\}, \ E_2 = \{b, c\}.$$

Define the superdegree of a vertex by

$$\deg_{\mathcal{S}}(x) := \bigl|\{E \in \mathcal{E} : x \in E\}\bigr|,$$

so that

$$\deg_{\mathcal{S}}(a) = 1, \qquad \deg_{\mathcal{S}}(b) = 2, \qquad \deg_{\mathcal{S}}(c) = 1.$$

Define the superdistance $D_{\mathcal{S}}(x, y)$ as the length (number of superhyperedges) of a shortest superhyperedge-walk joining x and y. Then the distances are

$$D_{\mathcal{S}}(a, b) = 1, \qquad D_{\mathcal{S}}(b, c) = 1, \qquad D_{\mathcal{S}}(a, c) = 2.$$

Therefore, the Gutman index equals

$$\begin{aligned}
\operatorname{Gut}(\mathcal{S}) &= \sum_{\{x,y\} \subseteq \mathcal{V}} \deg_{\mathcal{S}}(x) \deg_{\mathcal{S}}(y) \, D_{\mathcal{S}}(x, y) \\
&= \deg(a) \deg(b) D(a, b) + \deg(b) \deg(c) D(b, c) + \deg(a) \deg(c) D(a, c) \\
&= (1 \cdot 2 \cdot 1) + (2 \cdot 1 \cdot 1) + (1 \cdot 1 \cdot 2) \\
&= 2 + 2 + 2 \\
&= 6.
\end{aligned}$$

Theorem 4.6.6 (SuperHyperGraph Gutman index generalizes the graph Gutman index)**.** *Let $G = (V, E)$ be a finite connected simple graph. Define the associated 1-SuperHyperGraph $\mathcal{S}(G) = (\mathcal{V}, \mathcal{E})$ over the base universe $U = V$ by*

$$\mathcal{V} \ := \ \bigl\{\{v\} : v \in V\bigr\} \subseteq \mathrm{PS}(V), \qquad \mathcal{E} \ := \ \bigl\{\{\{u\}, \{v\}\} : \{u, v\} \in E\bigr\}.$$

Then $\mathcal{S}(G)$ is connected and

$$\operatorname{Gut}\bigl(\mathcal{S}(G)\bigr) \ = \ \operatorname{Gut}(G).$$

Proof. Define $\varphi : V \to \mathcal{V}$ by $\varphi(v) = \{v\}$. This is a bijection. Moreover, $\{u, v\} \in E$ if and only if $\{\varphi(u), \varphi(v)\} \in \mathcal{E}$, hence φ is a graph isomorphism from G onto the 2-uniform hypergraph $(\mathcal{V}, \mathcal{E})$.

Step 1: Degree preservation. Fix $v \in V$. Edges of G incident to v are in bijection with superhyperedges of $\mathcal{S}(G)$ containing $\varphi(v)$, via $\{v, w\} \leftrightarrow \{\{v\}, \{w\}\}$. Therefore

$$\deg_{\mathcal{S}(G)}(\varphi(v)) = \deg_G(v).$$

Step 2: Distance preservation. Any graph path $(v_0, v_1, \ldots, v_p)$ in G yields a Berge path

$$(\varphi(v_0), \ \{\varphi(v_0), \varphi(v_1)\}, \ \varphi(v_1), \ \ldots, \ \{\varphi(v_{p-1}), \varphi(v_p)\}, \ \varphi(v_p))$$

of the same length p in $\mathcal{S}(G)$. Conversely, because every superhyperedge in $\mathcal{S}(G)$ has size 2, any Berge path in $\mathcal{S}(G)$ corresponds uniquely to a graph path in G of the same length. Hence, for all $u, v \in V$,

$$D_{\mathcal{S}(G)}(\varphi(u), \varphi(v)) = d_G(u, v).$$

Step 3: Equality of Gutman indices. Using the bijection φ and Steps 1–2,

$$\begin{aligned}\mathrm{Gut}\big(\mathcal{S}(G)\big) &= \sum_{\{\varphi(u),\varphi(v)\}\subseteq\mathcal{V}} \deg_{\mathcal{S}(G)}(\varphi(u))\deg_{\mathcal{S}(G)}(\varphi(v))\, D_{\mathcal{S}(G)}(\varphi(u), \varphi(v)) \\ &= \sum_{\{u,v\}\subseteq V} \deg_G(u)\deg_G(v)\, d_G(u, v) = \mathrm{Gut}(G).\end{aligned}$$

□

Chapter 5

Domination in SuperHyperGraphs

In this chapter, we investigate domination in SuperHyperGraphs. In the classical graph setting, domination concerns vertex subsets $D \subseteq V$ with the property that every vertex is either contained in D or adjacent to at least one vertex of D. A central objective is to determine such sets of minimum cardinality. For reference, Appendix B presents the notions of Bi-Domination [256,257], Tri-Domination, and Multi-Domination.

5.1 SuperHyperGraph Domination

Graph domination investigates subsets of vertices such that every vertex either belongs to the subset or is adjacent to at least one vertex in it, with a primary focus on minimizing the size of such subsets in graphs [258–260]. Related notions of domination and its variants have also been extensively studied in the settings of fuzzy graphs and neutrosophic graphs (e.g., [261–264]). In addition, several well-known variants of domination exist, including secure domination [265, 266], paired domination [267, 268], double domination [269, 270], roman domination [271–273], connected domination [274–276], star domination [277–279], and total domination [280].

Hypergraph domination extends graph domination by requiring that each vertex outside a dominating set shares at least one hyperedge with a dominating vertex, thereby modeling influence coverage in more complex multiway interaction systems [281–284]. SuperHyperGraph domination further generalizes hypergraph domination to multi-tier supervertices, by requiring that every supervertex is connected, via at least one superedge, to some dominating supervertex while respecting the hierarchical structure across the superhypergraph levels.

Definition 5.1.1 (Domination in a hypergraph)**.** (cf. [281–283, 285]) Let $H = (V, E)$ be a finite hypergraph with nonempty vertex set V and hyperedge family $E \subseteq \mathrm{PS}^*(V) := \mathrm{PS}(V) \setminus \{\emptyset\}$.

A subset $D \subseteq V$ is called a *dominating set* of H if for every vertex $v \in V \setminus D$ there exists an edge $e \in E$ such that

$$v \in e \quad \text{and} \quad e \cap D \neq \emptyset.$$

Equivalently, every vertex outside D is contained in some edge that also contains at least one vertex of D.

The *domination number* of H is

$$\gamma(H) \;:=\; \min\{\, |D| \mid D \subseteq V \text{ is a dominating set of } H \,\}.$$

Example 5.1.2 (Domination in a hypergraph)**.** Let

$$H = (V, E), \qquad V = \{a, b, c, d\}, \qquad E = \big\{\{a, b\}, \{b, c, d\}\big\}.$$

Consider the set $D = \{b\} \subseteq V$. Then every vertex outside D is contained in an edge meeting D:

$$a \in \{a, b\} \text{ and } \{a, b\} \cap D = \{b\} \neq \emptyset, \qquad c, d \in \{b, c, d\} \text{ and } \{b, c, d\} \cap D = \{b\} \neq \emptyset.$$

Hence D is a dominating set of H, so $\gamma(H) \leq 1$. Since $V \neq \emptyset$, no dominating set can be empty, and therefore $\gamma(H) = 1$.

Example 5.1.3 (Domination in an n-SuperHyperGraph)**.** Fix $n = 1$ and the base set $V_0 = \{1, 2, 3\}$, and define the 1-supervertex set

$$V = \big\{A, B, C\big\} \subseteq \mathrm{PS}(V_0), \qquad A = \{1\}, \;\; B = \{2\}, \;\; C = \{2, 3\}.$$

Let the family of 1-superedges be

$$E = \big\{e_1, e_2\big\}, \qquad e_1 = \{A, B\}, \;\; e_2 = \{B, C\}.$$

Take $D = \{B\} \subseteq V$. For each supervertex outside D we find a superedge meeting D:

$$A \in e_1 \text{ and } e_1 \cap D = \{B\} \neq \emptyset, \qquad C \in e_2 \text{ and } e_2 \cap D = \{B\} \neq \emptyset.$$

Thus D is a dominating set of $\mathrm{SHG}^{(1)}$, so $\gamma(\mathrm{SHG}^{(1)}) \leq 1$. Again $\gamma(\mathrm{SHG}^{(1)}) \neq 0$, hence

$$\gamma\big(\mathrm{SHG}^{(1)}\big) = 1.$$

Definition 5.1.4 (Domination in an n-SuperHyperGraph)**.** (cf. [286]) Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph on a finite base set V_0.

A subset $D \subseteq V$ is called a *dominating set* of $\mathrm{SHG}^{(n)}$ if for every n-supervertex $v \in V \setminus D$ there exists an n-superedge $e \in E$ such that

$$v \in e \quad \text{and} \quad e \cap D \neq \emptyset.$$

Equivalently, every n-supervertex outside D lies in some n-superedge that also contains at least one n-supervertex from D.

The *domination number* of $\mathrm{SHG}^{(n)}$ is

$$\gamma\big(\mathrm{SHG}^{(n)}\big) \;:=\; \min\{\, |D| \mid D \subseteq V \text{ is a dominating set of } \mathrm{SHG}^{(n)} \,\}.$$

A concrete example is given below.

Example 5.1.5 (Domination in an n-SuperHyperGraph)**.** Fix $n = 1$ and the base set $V_0 = \{1, 2, 3\}$, and define the 1-supervertex set

$$V = \big\{A, B, C\big\} \subseteq \mathrm{PS}(V_0), \qquad A = \{1\}, \;\; B = \{2\}, \;\; C = \{2, 3\}.$$

Let the family of 1-superedges be

$$E = \big\{e_1, e_2\big\}, \qquad e_1 = \{A, B\}, \;\; e_2 = \{B, C\}.$$

Take $D = \{B\} \subseteq V$. For each supervertex outside D we find a superedge meeting D:

$$A \in e_1 \text{ and } e_1 \cap D = \{B\} \neq \emptyset, \qquad C \in e_2 \text{ and } e_2 \cap D = \{B\} \neq \emptyset.$$

Thus D is a dominating set of $\mathrm{SHG}^{(1)}$, so $\gamma(\mathrm{SHG}^{(1)}) \leq 1$. Again $\gamma(\mathrm{SHG}^{(1)}) \neq 0$, hence

$$\gamma\big(\mathrm{SHG}^{(1)}\big) = 1.$$

5.2 Total Domination in SuperHyperGraph

Total domination in a graph selects vertices so every vertex has a neighbor in the set, and vertices in the set are also dominated [287–292]. Total domination in a hypergraph selects vertices so each vertex shares a hyperedge with some distinct selected vertex, equivalently total domination in its primal graph [293–295]. Total domination in an n-SuperHyperGraph selects supervertices so every supervertex lies with a distinct selected supervertex in some superedge, requiring no isolated supervertex.

Definition 5.2.1 (Total domination in a graph)**.** [287–289] Let $G = (V, E)$ be a finite simple graph. For $v \in V$ write $N_G(v) := \{u \in V : \ uv \in E\}$ for the (open) neighborhood of v.

A set $S \subseteq V$ is a *total dominating set* of G if

$$\forall\, v \in V \qquad N_G(v) \cap S \neq \emptyset.$$

Equivalently, every vertex of G has a neighbor in S (in particular, vertices in S must also be adjacent to some vertex of S). If G has an isolated vertex, then no total dominating set exists.

The *total domination number* of G is

$$\gamma_t(G) \ := \ \begin{cases} \min\{\, |S| : \ S \subseteq V \text{ is a total dominating set of } G\,\}, & \text{if } G \text{ is isolate-free,} \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 5.2.2 (Adjacency and open neighborhood in a hypergraph)**.** Let $H = (V, E)$ be a finite hypergraph, where V is a finite vertex set and E is a finite family of nonempty subsets of V. Two distinct vertices $x, y \in V$ are *adjacent* if there exists $e \in E$ such that $\{x, y\} \subseteq e$. The *open neighborhood* of $v \in V$ is

$$N_H(v) \ := \ \{\, u \in V \setminus \{v\} : \ \exists\, e \in E \text{ with } \{u, v\} \subseteq e\,\}.$$

Definition 5.2.3 (Total domination in a hypergraph)**.** Let $H = (V, E)$ be a finite hypergraph and let $N_H(v)$ be as in Definition 5.2.2. A set $S \subseteq V$ is a *total dominating set* of H if

$$\forall\, v \in V \qquad N_H(v) \cap S \neq \emptyset.$$

Equivalently, every vertex v lies together with some (distinct) vertex of S in a common hyperedge.

The *total domination number* of H is

$$\gamma_t(H) \ := \ \begin{cases} \min\{\, |S| : \ S \subseteq V \text{ is a total dominating set of } H\,\}, \\ \text{if } N_H(v) \neq \emptyset \text{ for all } v \in V, \\ \infty, \text{otherwise.} \end{cases}$$

Remark 5.2.4 (Primal-graph viewpoint)**.** Let $\mathrm{Pr}(H)$ denote the *primal graph* (a.k.a. 2-section) of H, i.e. the simple graph on V with $\{x, y\}$ an edge iff $x \neq y$ and $\{x, y\} \subseteq e$ for some $e \in E$. Then $N_H(v) = N_{\mathrm{Pr}(H)}(v)$ for all $v \in V$, and hence

$$S \text{ is total dominating in } H \quad \Longleftrightarrow \quad S \text{ is total dominating in } \mathrm{Pr}(H),$$

so $\gamma_t(H) = \gamma_t(\mathrm{Pr}(H))$.

Definition 5.2.5 (Total domination in an n-SuperHyperGraph)**.** Let $n \geq 0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (Definition 2.1.6), so (V, E) is a hypergraph whose vertices are n-supervertices.

For $x \in V$ define the open neighborhood

$$N_{\mathrm{SHG}^{(n)}}(x) \ := \ \{\, y \in V \setminus \{x\} : \ \exists\, \varepsilon \in E \text{ with } \{x, y\} \subseteq \varepsilon \,\}.$$

A set $S \subseteq V$ is a *total dominating set* of $\mathrm{SHG}^{(n)}$ if

$$\forall\, x \in V \qquad N_{\mathrm{SHG}^{(n)}}(x) \cap S \neq \emptyset.$$

The *total domination number* of $\mathrm{SHG}^{(n)}$ is

$$\gamma_t\big(\mathrm{SHG}^{(n)}\big) \ := \ \begin{cases} \min\{\, |S| : \ S \subseteq V \text{ is a total dominating set of } \mathrm{SHG}^{(n)} \,\}, \\ \text{if } N_{\mathrm{SHG}^{(n)}}(x) \neq \emptyset \text{ for all } x \in V, \\ \infty, \text{otherwise.} \end{cases}$$

Definition 5.2.6 (n-fold singleton embedding and n-lift)**.** Let V_0 be a finite nonempty set. Define maps $\iota_n : V_0 \to \mathrm{PS}^n(V_0)$ recursively by

$$\iota_0(v) := v, \qquad \iota_{n+1}(v) := \{\iota_n(v)\} \quad (n \geq 0).$$

For a subset $A \subseteq V_0$ set $\iota_n[A] := \{\iota_n(v) : v \in A\}$.

Given a hypergraph $H = (V_0, E_0)$, define its *n-lift* as the pair

$$\mathrm{Lift}_n(H) \ := \ \big(V^{(n)}, E^{(n)}\big), \qquad V^{(n)} := \iota_n[V_0] \subseteq \mathrm{PS}^n(V_0),$$

$$E^{(n)} := \{\iota_n[e] : e \in E_0\} \subseteq \mathrm{PS}(V^{(n)}) \setminus \{\emptyset\}.$$

Then $\mathrm{Lift}_n(H)$ is an n-SuperHyperGraph on V_0.

Theorem 5.2.7 (Total domination in n-SuperHyperGraphs generalizes graphs and hypergraphs)**.** *Let $n \geq 0$.*

(i) *(Hypergraph case.) For every hypergraph $H = (V_0, E_0)$ and every $S \subseteq V_0$,*

$$S \text{ is a total dominating set of } H \quad \Longleftrightarrow \quad \iota_n[S] \text{ is a total dominating set of } \mathrm{Lift}_n(H).$$

Consequently,

$$\gamma_t\big(\mathrm{Lift}_n(H)\big) \ = \ \gamma_t(H).$$

(ii) *(Graph case.) If $G = (V, E)$ is a finite simple graph and $H_G := (V, \{\{u, v\} : uv \in E\})$ is the associated 2-uniform hypergraph, then*

$$\gamma_t(G) \ = \ \gamma_t(H_G) \ = \ \gamma_t\big(\mathrm{Lift}_n(H_G)\big).$$

Hence, the notion of total domination in n-SuperHyperGraphs extends the classical total domination number of graphs.

Proof. Fix $n \geq 0$ and a hypergraph $H = (V_0, E_0)$. By construction, a pair of distinct vertices $u, v \in V_0$ satisfies $\{u, v\} \subseteq e$ for some $e \in E_0$ if and only if $\{\iota_n(u), \iota_n(v)\} \subseteq \iota_n[e]$ for the corresponding lifted edge $\iota_n[e] \in E^{(n)}$. Therefore, for every $v \in V_0$,

$$u \in N_H(v) \iff \iota_n(u) \in N_{\mathrm{Lift}_n(H)}(\iota_n(v)),$$

and thus

$$N_H(v) \cap S \neq \emptyset \iff N_{\mathrm{Lift}_n(H)}(\iota_n(v)) \cap \iota_n[S] \neq \emptyset.$$

Since this holds for all $v \in V_0$, we obtain the equivalence in (i). Taking minima over cardinalities gives $\gamma_t(\mathrm{Lift}_n(H)) = \gamma_t(H)$.

For (ii), observe that in a simple graph G the open neighborhood is $N_G(v) = \{u : uv \in E\}$, while in the associated 2-uniform hypergraph H_G we have $\{u, v\} \subseteq \{u, v\} \in E(H_G)$ exactly when $uv \in E(G)$. Hence $N_{H_G}(v) = N_G(v)$ for all v, so $\gamma_t(G) = \gamma_t(H_G)$ by Definitions 5.2.1 and 5.2.3. Applying (i) to H_G yields $\gamma_t(\mathrm{Lift}_n(H_G)) = \gamma_t(H_G)$, completing the proof. □

5.3 Signed domination in Superhypergraphs

Signed domination is a $\{\pm 1\}$-labeling version of domination. In a graph G, a signed dominating function assigns each vertex a label in $\{-1, +1\}$ such that the total label-sum over every closed neighborhood is at least 1 [296–299]. In an n-SuperHyperGraph $\mathrm{SHG}^{(n)}$, we analogously assign ± 1 labels to n-supervertices and require a positive imbalance on every n-superedge, i.e., the label-sum over each superedge is at least 1. In both settings, one seeks to minimize the total label-sum over all (super)vertices.

Definition 5.3.1 (Signed domination in a graph)**.** Let $G = (V, E)$ be a finite simple undirected graph. For $v \in V$, write

$$N_G(v) := \{u \in V : uv \in E\}, \qquad N_G[v] := N_G(v) \cup \{v\}$$

for the open and closed neighborhoods, respectively.

A function $f : V \to \{-1, +1\}$ is a *signed dominating function* (SDF) of G if

$$\forall v \in V \qquad \sum_{u \in N_G[v]} f(u) \;\geq\; 1.$$

Equivalently, every closed neighborhood contains strictly more $+1$ labels than -1 labels. The *signed domination number* of G is

$$\gamma_s(G) := \min\Big\{\sum_{v \in V} f(v) \;:\; f \text{ is an SDF of } G\Big\}.$$

Definition 5.3.2 (Signed domination in a hypergraph)**.** Let $H = (V, \mathcal{E})$ be a finite hypergraph, where V is a finite set and $\mathcal{E} \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$ is a finite family of nonempty subsets of V. For $f : V \to \{-1, +1\}$ and $F \subseteq V$, write

$$f(F) := \sum_{v \in F} f(v).$$

A function $f : V \to \{-1, +1\}$ is a *signed dominating function* (SDF) of H if every hyperedge has *positive imbalance*:

$$\forall\, e \in \mathcal{E} \qquad f(e) \;\geq\; 1.$$

The *signed domination number* (also called the *signed discrepancy* in some sources) of H is

$$\gamma_s(H) := \min\Big\{ f(V) \;:\; f \text{ is an SDF of } H \Big\}.$$

Example 5.3.3 (Signed domination in a hypergraph)**.** Let $H = (V, \mathcal{E})$ be the hypergraph with

$$V = \{a, b, c, d\}, \qquad \mathcal{E} = \{e_1, e_2\}, \qquad e_1 = \{a, b, c\}, \;\; e_2 = \{b, c, d\}.$$

Define $f : V \to \{-1, +1\}$ by

$$f(a) = -1, \qquad f(b) = +1, \qquad f(c) = +1, \qquad f(d) = -1.$$

Then

$$f(e_1) = f(a) + f(b) + f(c) = -1 + 1 + 1 = 1, \qquad f(e_2) = f(b) + f(c) + f(d) = 1 + 1 - 1 = 1,$$

so $f(e) \geq 1$ for every $e \in \mathcal{E}$. Hence f is a signed dominating function of H. Moreover,

$$f(V) = f(a) + f(b) + f(c) + f(d) = -1 + 1 + 1 - 1 = 0.$$

In fact, $\gamma_s(H) = 0$: if $f(b) = -1$ (or $f(c) = -1$), then e_1 forces $f(a) = f(c) = +1$ and e_2 forces $f(d) = +1$, giving $f(V) \geq 2$; thus the minimum is attained when $f(b) = f(c) = +1$, and then $f(a) = f(d) = -1$ yields $f(V) = 0$.

Definition 5.3.4 (Signed domination in an n-SuperHyperGraph)**.** Let $n \geq 0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (so V is a finite set of n-supervertices and $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$ is a finite family of nonempty subsets of V called n-superedges). For $f : V \to \{-1, +1\}$ and $F \subseteq V$, set $f(F) := \sum_{x \in F} f(x)$.

A function $f : V \to \{-1, +1\}$ is a *signed dominating function* (SDF) of $\mathrm{SHG}^{(n)}$ if

$$\forall\, \varepsilon \in E \qquad f(\varepsilon) \;\geq\; 1.$$

The *signed domination number* of $\mathrm{SHG}^{(n)}$ is

$$\gamma_s\big(\mathrm{SHG}^{(n)}\big) := \min\Big\{ f(V) \;:\; f \text{ is an SDF of } \mathrm{SHG}^{(n)} \Big\}.$$

A concrete example is given below.

Example 5.3.5 (Signed domination in a 1-SuperHyperGraph)**.** Let $V_0 = \{1, 2, 3, 4\}$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ with

$$V = \{X_1, X_2, X_3, X_4\} \subseteq \mathrm{PS}(V_0), \qquad X_1 = \{1\}, \; X_2 = \{2\}, \; X_3 = \{3\}, \; X_4 = \{4\},$$

and

$$E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{X_1, X_2, X_3\}, \;\; \varepsilon_2 = \{X_2, X_3, X_4\}.$$

Define $g : V \to \{-1, +1\}$ by

$$g(X_1) = -1, \qquad g(X_2) = +1, \qquad g(X_3) = +1, \qquad g(X_4) = -1.$$

Then

$$g(\varepsilon_1) = g(X_1)+g(X_2)+g(X_3) = -1+1+1 = 1, \qquad g(\varepsilon_2) = g(X_2)+g(X_3)+g(X_4) = 1+1-1 = 1,$$

so $g(\varepsilon) \geq 1$ for every $\varepsilon \in E$, and g is an SDF of $\mathrm{SHG}^{(1)}$. Also,

$$g(V) = g(X_1) + g(X_2) + g(X_3) + g(X_4) = 0,$$

and by the same case analysis as in Example 5.3.3, one gets $\gamma_s\big(\mathrm{SHG}^{(1)}\big) = 0$.

Definition 5.3.6 (Closed-neighborhood hypergraph)**.** Let $G = (V, E)$ be a finite simple graph. Its *closed-neighborhood hypergraph* is

$$\mathcal{N}(G) := \big(V,\ \{\, N_G[v] : v \in V \,\}\big).$$

Definition 5.3.7 (n-lift of a hypergraph)**.** Let $H = (V, \mathcal{E})$ be a hypergraph and $n \geq 0$. Define $\iota_0(v) = v$ and $\iota_{n+1}(v) = \{\iota_n(v)\}$. Set

$$V^{(n)} := \{\iota_n(v) : v \in V\}, \qquad \mathcal{E}^{(n)} := \{\ \iota_n(e) := \{\iota_n(v) : v \in e\}\ :\ e \in \mathcal{E}\ \},$$

and define the *n-lift* by $\mathrm{Lift}_n(H) := (V^{(n)}, \mathcal{E}^{(n)})$.

Theorem 5.3.8 (Superhypergraphs generalize signed domination in graphs and hypergraphs)**.** *Let G be a finite simple graph, let $\mathcal{N}(G)$ be its closed-neighborhood hypergraph, and let $n \geq 0$. Then*

$$\gamma_s(G) \ =\ \gamma_s\big(\mathcal{N}(G)\big) \ =\ \gamma_s\big(\mathrm{Lift}_n(\mathcal{N}(G))\big).$$

Moreover, for every finite hypergraph H and every $n \geq 0$,

$$\gamma_s(H) \ =\ \gamma_s\big(\mathrm{Lift}_n(H)\big).$$

Consequently, Definition 5.3.4 extends (i) the classical signed domination number of graphs and (ii) the signed domination number (signed discrepancy) of hypergraphs.

Proof. Step 1 (Graph ↔ neighborhood hypergraph). Fix $f : V(G) \to \{-1, +1\}$. By Definition 5.3.6, the hyperedges of $\mathcal{N}(G)$ are precisely the closed neighborhoods $N_G[v]$. Hence

$$f \text{ is an SDF of } G \quad \Longleftrightarrow \quad \forall\, v \in V(G)\ \ f\big(N_G[v]\big) \geq 1 \quad \Longleftrightarrow \quad f \text{ is an SDF of } \mathcal{N}(G)$$

in the sense of Definition 5.3.2. Since the objective value is the same quantity $f(V(G))$ in both cases, taking minima yields $\gamma_s(G) = \gamma_s(\mathcal{N}(G))$.

Step 2 (Hypergraph ↔ n-lift). Let $H = (V, \mathcal{E})$ be a hypergraph and $n \geq 0$. Define a bijection $\varphi : V \to V^{(n)}$ by $\varphi(v) = \iota_n(v)$. Given $f : V \to \{-1, +1\}$, define $f^{(n)} : V^{(n)} \to \{-1, +1\}$ by $f^{(n)}(\varphi(v)) = f(v)$. Then, for every $e \in \mathcal{E}$,

$$f^{(n)}\big(\iota_n(e)\big) = \sum_{x \in \iota_n(e)} f^{(n)}(x) = \sum_{v \in e} f(v) = f(e).$$

Therefore f is an SDF of H iff $f^{(n)}$ is an SDF of $\mathrm{Lift}_n(H)$, and moreover

$$f^{(n)}(V^{(n)}) = \sum_{x \in V^{(n)}} f^{(n)}(x) = \sum_{v \in V} f(v) = f(V).$$

Taking minima over SDFs gives $\gamma_s(H) = \gamma_s(\mathrm{Lift}_n(H))$.

Step 3 (Combine). Apply Step 2 to $H = \mathcal{N}(G)$ to obtain $\gamma_s(\mathcal{N}(G)) = \gamma_s(\mathrm{Lift}_n(\mathcal{N}(G)))$, and combine with Step 1. □

5.4 Double domination in SuperHyperGraphs

Double domination is a strengthened form of domination. In a graph G, a set $S \subseteq V(G)$ is a *double dominating set* if every vertex v has at least two vertices of S in its closed neighborhood $N_G[v]$ [300–303]. As a related notion, the concept of a triple dominating set is also well known [304–308]. In an n-SuperHyperGraph $\mathrm{SHG}^{(n)}$, the notion is defined analogously at the level of n-supervertices: a set $S \subseteq V(\mathrm{SHG}^{(n)})$ is double dominating if every supervertex x has at least two supervertices of S in the closed neighborhood $N_{\mathrm{SHG}^{(n)}}[x]$ (defined via co-membership in a common n-superedge).

Definition 5.4.1 (Double domination in a graph)**.** Let $G = (V, E)$ be a finite simple graph. For $v \in V$, the *open neighborhood* is $N_G(v) := \{u \in V : \{u, v\} \in E\}$ and the *closed neighborhood* is $N_G[v] := N_G(v) \cup \{v\}$.

A set $S \subseteq V$ is a *double dominating set* of G if

$$\forall\, v \in V \qquad |N_G[v] \cap S| \ \geq\ 2.$$

Equivalently, every vertex is dominated at least twice when each vertex in S dominates itself and its neighbors.

The *double domination number* of G is

$$\gamma_{\times 2}(G) \ := \ \begin{cases} \min\{\, |S| : \ S \subseteq V \text{ is a double dominating set of } G\,\}, & \text{if such } S \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

(In particular, if G has an isolated vertex, then $\gamma_{\times 2}(G) = \infty$.)

A concrete example is given below.

Example 5.4.2 (Double domination in a graph)**.** Let $G = (V, E)$ be the path on three vertices,

$$V = \{1, 2, 3\}, \qquad E = \big\{\{1, 2\}, \{2, 3\}\big\}.$$

Consider the set $S = \{1, 3\}$. Then the closed neighborhoods are

$$N_G[1] = \{1, 2\}, \qquad N_G[2] = \{1, 2, 3\}, \qquad N_G[3] = \{2, 3\}.$$

Hence

$$|N_G[1] \cap S| = |\{1, 2\} \cap \{1, 3\}| = 1,$$

so S is *not* a double dominating set. However, taking $S = \{1, 2, 3\} = V$ we get

$$|N_G[1] \cap V| = 2, \qquad |N_G[2] \cap V| = 3, \qquad |N_G[3] \cap V| = 2,$$

so V is a double dominating set of G. Moreover, no set of size 2 works: if $|S| = 2$, then either $1 \notin S$ or $3 \notin S$; in the first case $|N_G[1] \cap S| \leq 1$, and in the second case $|N_G[3] \cap S| \leq 1$. Therefore,

$$\gamma_{\times 2}(G) = 3.$$

Definition 5.4.3 (Adjacency and closed neighborhood in an n-SuperHyperGraph)**.** Let $n \geq 0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph, so V is the set of n-supervertices and E is the family of n-superedges, each $\varepsilon \in E$ being a nonempty subset of V.

Two distinct supervertices $x, y \in V$ are *adjacent* if there exists $\varepsilon \in E$ with $\{x, y\} \subseteq \varepsilon$. The open neighborhood of x is

$$N_{\mathrm{SHG}^{(n)}}(x) \;:=\; \{\, y \in V \setminus \{x\} : \; \exists\, \varepsilon \in E \text{ with } \{x, y\} \subseteq \varepsilon \,\},$$

and the closed neighborhood is $N_{\mathrm{SHG}^{(n)}}[x] := N_{\mathrm{SHG}^{(n)}}(x) \cup \{x\}$.

Definition 5.4.4 (Double domination in an n-SuperHyperGraph)**.** Let $n \geq 0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A set $S \subseteq V$ is a *double dominating set* of $\mathrm{SHG}^{(n)}$ if

$$\forall\, x \in V \qquad \bigl|N_{\mathrm{SHG}^{(n)}}[x] \cap S\bigr| \;\geq\; 2,$$

where $N_{\mathrm{SHG}^{(n)}}[x]$ is the closed neighborhood from Definition 5.4.3.

The *double domination number* of $\mathrm{SHG}^{(n)}$ is

$$\gamma_{\times 2}\bigl(\mathrm{SHG}^{(n)}\bigr) \;:=\; \begin{cases} \min\{\, |S| : \; S \subseteq V \text{ is a double dominating set of } \mathrm{SHG}^{(n)} \,\}, & \text{if such } S \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

A concrete example is given below.

Example 5.4.5 (Adjacency and double domination in a 1-SuperHyperGraph)**.** Let $V_0 = \{1, 2, 3, 4\}$ and let $\mathrm{SHG}^{(1)} = (V, E)$ be the 1-SuperHyperGraph with

$$V = \{X_1, X_2, X_3, X_4\} \subseteq \mathrm{PS}(V_0), \qquad X_1 = \{1\},\; X_2 = \{2\},\; X_3 = \{3\},\; X_4 = \{4\},$$

and with 1-superedges

$$E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{X_1, X_2, X_3\}, \;\; \varepsilon_2 = \{X_2, X_3, X_4\}.$$

By Definition 5.4.3, the closed neighborhoods are

$$N_{\mathrm{SHG}^{(1)}}[X_1] = \{X_1, X_2, X_3\}, \quad N_{\mathrm{SHG}^{(1)}}[X_4] = \{X_2, X_3, X_4\}, \quad N_{\mathrm{SHG}^{(1)}}[X_2] = N_{\mathrm{SHG}^{(1)}}[X_3] = V.$$

Now take $S = \{X_2, X_3\}$. Then

$$|N_{\mathrm{SHG}^{(1)}}[X_1] \cap S| = |\{X_1, X_2, X_3\} \cap \{X_2, X_3\}| = 2,$$
$$|N_{\mathrm{SHG}^{(1)}}[X_4] \cap S| = |\{X_2, X_3, X_4\} \cap \{X_2, X_3\}| = 2,$$

and

$$|N_{\mathrm{SHG}^{(1)}}[X_2] \cap S| = |V \cap S| = 2, \qquad |N_{\mathrm{SHG}^{(1)}}[X_3] \cap S| = |V \cap S| = 2.$$

Hence S is a double dominating set, so $\gamma_{\times 2}(\mathrm{SHG}^{(1)}) \leq 2$.

On the other hand, no set of size 1 can be double dominating: if $S = \{X_i\}$, then either $|N_{\mathrm{SHG}^{(1)}}[X_1] \cap S| \leq 1$ or $|N_{\mathrm{SHG}^{(1)}}[X_4] \cap S| \leq 1$. Therefore $\gamma_{\times 2}(\mathrm{SHG}^{(1)}) = 2$.

Theorem 5.4.6 (SuperHyperGraph double domination generalizes the graph case)**.** *Let $G = (V, E)$ be a finite simple graph. Form the* 0*-SuperHyperGraph*

$$\mathrm{SHG}^{(0)}_G := (V, E^\star), \qquad E^\star := \{\{u, v\} : \{u, v\} \in E\},$$

i.e., the 0*-supervertices are the vertices of G, and the* 0*-superedges are exactly the graph edges. Then*

$$\gamma_{\times 2}\big(\mathrm{SHG}^{(0)}_G\big) = \gamma_{\times 2}(G).$$

Consequently, the notion of double domination in an n-SuperHyperGraph extends (and is consistent with) the classical double domination of graphs.

Proof. Fix $v \in V$. By construction of $\mathrm{SHG}^{(0)}_G$, two distinct vertices u, v lie together in a 0-superedge of $\mathrm{SHG}^{(0)}_G$ if and only if $\{u, v\} \in E(G)$. Hence the open neighborhoods coincide:

$$N_{\mathrm{SHG}^{(0)}_G}(v) = N_G(v),$$

and therefore the closed neighborhoods coincide as well: $N_{\mathrm{SHG}^{(0)}_G}[v] = N_G[v]$ for every $v \in V$.

Now let $S \subseteq V$. Using $N_{\mathrm{SHG}^{(0)}_G}[v] = N_G[v]$ we obtain, for every $v \in V$,

$$\big|N_{\mathrm{SHG}^{(0)}_G}[v] \cap S\big| = \big|N_G[v] \cap S\big|.$$

Thus S is a double dominating set of $\mathrm{SHG}^{(0)}_G$ (Definition 5.4.4) if and only if S is a double dominating set of G (Definition 5.4.1). Taking minima over $|S|$ yields $\gamma_{\times 2}(\mathrm{SHG}^{(0)}_G) = \gamma_{\times 2}(G)$, as claimed. ☐

5.5 Star domination in SuperHyperGraphs

A star dominating set in a graph is a vertex set such that every unchosen vertex is adjacent to at least one chosen vertex [277,279,309,310]. A star dominating set in a SuperHyperGraph is a supervertex set such that every unchosen supervertex lies in a superedge whose other supervertices are all chosen.

Definition 5.5.1 (Star domination in a graph)**.** Let $G = (V, E)$ be a finite simple undirected graph. For $r \in V$, the *star* of r is the set of edges incident with r,

$$S_G(r) := \{ e \in E : r \in e \}.$$

A set $S \subseteq V$ is called a *star dominating set* of G if

$$\forall r \in V \setminus S \quad \exists s \in S \text{ such that } S_G(r) \cap S_G(s) \neq \emptyset.$$

The *star domination number* of G is

$$\gamma_\star(G) := \min\{ |S| : S \subseteq V \text{ is a star dominating set of } G \}.$$

Example 5.5.2 (Star domination in a graph)**.** Let $G = (V, E)$ be the path P_4 on four vertices:

$$V = \{1, 2, 3, 4\}, \qquad E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}.$$

Consider the set $S = \{2, 4\}$. For each $r \in V \setminus S = \{1, 3\}$ we can find $s \in S$ with $S_G(r) \cap S_G(s) \neq \emptyset$:

$$S_G(1) = \{\{1, 2\}\}, \quad S_G(2) = \{\{1, 2\}, \{2, 3\}\} \;\Rightarrow\; S_G(1) \cap S_G(2) = \{\{1, 2\}\} \neq \emptyset,$$

and

$$S_G(3) = \{\{2, 3\}, \{3, 4\}\}, \quad S_G(4) = \{\{3, 4\}\} \;\Rightarrow\; S_G(3) \cap S_G(4) = \{\{3, 4\}\} \neq \emptyset.$$

Hence S is a star dominating set of G. Moreover, no single vertex is star dominating (equivalently, dominating) in P_4, so

$$\gamma_\star(G) = \gamma(G) = 2.$$

Proposition 5.5.3 (Equivalence with ordinary domination in simple graphs)**.** *Let $G = (V, E)$ be a finite simple graph and let $S \subseteq V$. Then S is a star dominating set of G if and only if S is a (classical) dominating set of G. Consequently, $\gamma_\star(G) = \gamma(G)$.*

Proof. Assume S is star dominating and take $r \in V \setminus S$. By definition there exists $s \in S$ with $S_G(r) \cap S_G(s) \neq \emptyset$. Thus some edge $e \in E$ is incident to both r and s. Since G is simple, $e = \{r, s\}$, hence r is adjacent to $s \in S$; so S dominates r.

Conversely, assume S is a dominating set. For any $r \in V \setminus S$, choose $s \in S$ adjacent to r. Then $\{r, s\} \in E$ and hence $\{r, s\} \in S_G(r) \cap S_G(s)$, so the intersection is nonempty. Therefore S is star dominating.

Taking minimum cardinalities yields $\gamma_\star(G) = \gamma(G)$. □

Definition 5.5.4 (Star domination in an n-SuperHyperGraph)**.** Let $n \geq 0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph, i.e., V is a finite set of n-supervertices and E is a finite family of nonempty subsets of V (the *n-superedges*). A set $D \subseteq V$ is called a *star dominating set* of $\mathrm{SHG}^{(n)}$ if

$$\forall\, x \in V \setminus D \quad \exists\, \varepsilon \in E \;\text{ such that }\; x \in \varepsilon \;\text{ and }\; \varepsilon \setminus \{x\} \subseteq D.$$

The *star domination number* of $\mathrm{SHG}^{(n)}$ is

$$\gamma_\star\big(\mathrm{SHG}^{(n)}\big) \;:=\; \min\{\, |D| :\; D \subseteq V \text{ is a star dominating set of } \mathrm{SHG}^{(n)} \,\}.$$

A concrete example is given below.

Example 5.5.5 (Star domination in a 1-SuperHyperGraph)**.** Let $V_0 = \{1, 2, 3, 4\}$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ with

$$V = \{X_1, X_2, X_3, X_4\} \subseteq \mathrm{PS}(V_0), \qquad X_1 = \{1\},\; X_2 = \{2\},\; X_3 = \{3\},\; X_4 = \{4\},$$

and superedges

$$E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{X_1, X_2, X_3\},\;\; \varepsilon_2 = \{X_3, X_4\}.$$

Let $D = \{X_2, X_3, X_4\}$. We verify that D is star dominating. For $x = X_1 \in V \setminus D$, choose ε_1; then $X_1 \in \varepsilon_1$ and

$$\varepsilon_1 \setminus \{X_1\} = \{X_2, X_3\} \subseteq D.$$

Thus D is a star dominating set.

Furthermore, $\gamma_\star(\mathrm{SHG}^{(1)}) = 3$: indeed, to dominate X_1 one needs a superedge containing X_1 whose other members lie in D, so necessarily $\{X_2, X_3\} \subseteq D$. To dominate X_4, one must have $X_3 \in D$ (since the only superedge containing X_4 is $\varepsilon_2 = \{X_3, X_4\}$) and hence also $X_4 \in D$ to make $\varepsilon_2 \setminus \{X_4\} = \{X_3\} \subseteq D$ while keeping $X_4 \notin D$ impossible. Therefore $|D| \geq 3$, and the above D attains this bound.

Theorem 5.5.6 (SuperHyperGraph star domination generalizes graph star domination)**.** *Let $G = (V, E)$ be a finite simple graph and fix $n \geq 0$. Form the n-lift (singleton embedding)* $\mathrm{Lift}_n(G) = \big(V^{(n)}, E^{(n)}\big)$ *by*

$$V^{(n)} := \{\iota_n(v) : v \in V\}, \qquad E^{(n)} := \big\{\{\iota_n(u), \iota_n(v)\} : \{u, v\} \in E\big\},$$

where $\iota_0(v) = v$ and $\iota_{k+1}(v) = \{\iota_k(v)\}$. Then

$$\gamma_\star\big(\mathrm{Lift}_n(G)\big) \;=\; \gamma_\star(G) \;=\; \gamma(G).$$

In particular, Definition 5.5.4 extends Definition 5.5.1.

Proof. For any $S \subseteq V$, define $S^{(n)} := \{\iota_n(v) : v \in S\} \subseteq V^{(n)}$. We claim that S is star dominating in G if and only if $S^{(n)}$ is star dominating in $\mathrm{Lift}_n(G)$.

($\Rightarrow$) Let $x = \iota_n(r) \in V^{(n)} \setminus S^{(n)}$, so $r \in V \setminus S$. Since S star-dominates G, there exists an edge $\{r, s\} \in E$ with $s \in S$. Hence $\varepsilon := \{\iota_n(r), \iota_n(s)\} \in E^{(n)}$ contains x and

$$\varepsilon \setminus \{x\} = \{\iota_n(s)\} \subseteq S^{(n)}.$$

Thus $S^{(n)}$ star-dominates x, and since x was arbitrary, $S^{(n)}$ is star dominating in $\mathrm{Lift}_n(G)$.

($\Leftarrow$) Let $r \in V \setminus S$ and put $x = \iota_n(r) \in V^{(n)} \setminus S^{(n)}$. Since $S^{(n)}$ is star dominating in $\mathrm{Lift}_n(G)$, there exists a superedge $\varepsilon = \{\iota_n(r), \iota_n(s)\} \in E^{(n)}$ such that $\varepsilon \setminus \{x\} \subseteq S^{(n)}$. Then $\iota_n(s) \in S^{(n)}$, so $s \in S$, and by construction of $E^{(n)}$ we have $\{r, s\} \in E$. Hence r is dominated by S in G, and since r was arbitrary, S is star dominating in G.

Therefore $S \mapsto S^{(n)}$ is a bijection between star dominating sets of G and of $\mathrm{Lift}_n(G)$, preserving cardinality; hence $\gamma_\star(\mathrm{Lift}_n(G)) = \gamma_\star(G)$. Finally, Proposition 5.5.3 gives $\gamma_\star(G) = \gamma(G)$. ☐

5.6 Domination in Fuzzy SuperHyperGraphs

A fuzzy set assigns to each element a membership degree in $[0, 1]$ [311,312]. Fuzzy sets play a major role in diverse domains such as control theory [313], decision-making [314], graph theory [315], topology [316], signal processing [317], and engineering. Fuzzy graphs and fuzzy hypergraphs extend this notion by assigning membership degrees to vertices and to (hyper)edges [315,318,319]. Related notions such as bipolar fuzzy graphs [320], hyperfuzzy graphs [321,322], hesitant fuzzy graphs [323–325], and spherical fuzzy graphs [326,327] are also known. These structures have been extensively studied, particularly for applications in decision–making and other uncertainty-driven tasks. A fuzzy n-SuperHyperGraph is a higher-level network representation in which supervertices and superedges carry membership values for modeling complex interactions (cf. [4,328]).

Definition 5.6.1 (Fuzzy Set)**.** [311] Let X be a nonempty universe of discourse. A *fuzzy set* A on X is specified by a membership function

$$\mu_A : X \longrightarrow [0,1],$$

where $\mu_A(x)$ represents the degree to which $x \in X$ belongs to A. Equivalently, one may write

$$A \;=\; \{\, (x, \mu_A(x)) \mid x \in X \,\}.$$

A classical (crisp) subset $C \subseteq X$ is recovered by restricting μ_A to $\{0,1\}$.

Definition 5.6.2 (Fuzzy graph)**.** [315] A *fuzzy graph* is a triple $G = (V, \sigma, \mu)$ where V is a finite nonempty vertex set, $\sigma : V \to [0,1]$ assigns vertex-membership degrees, and $\mu : V \times V \to [0,1]$ assigns edge-membership degrees subject to

$$\mu(u,v) \;\le\; \min\{\sigma(u), \sigma(v)\} \qquad (\forall\, u, v \in V).$$

We write uv for $\{u,v\}$ and abbreviate $\mu(uv) := \mu(u,v)$. The (crisp) underlying graph of G has vertex set V and edge set $E^* := \{\, uv : \mu(uv) > 0 \,\}$.

Definition 5.6.3 (Fuzzy hypergraph)**.** (cf. [329,330]) Let $H^* = (V, E, \partial)$ be a crisp hypergraph. A *fuzzy hypergraph* on H^* is a sextuple

$$\mathcal{H} \;=\; \bigl(V, E, \partial;\; \sigma, \mu, \eta\bigr),$$

with maps

$$\sigma : V \to [0,1], \qquad \mu : E \to [0,1], \qquad \eta : V \times E \to [0,1],$$

such that for all $v \in V$ and $e \in E$,

$$\textit{(support)} \quad \bigl[v \in \partial(e)\bigr] \iff \eta(v,e) > 0, \tag{5.1}$$

$$\textit{(incidence bound)} \quad \eta(v,e) \;\le\; \min\{\sigma(v), \mu(e)\}, \tag{5.2}$$

$$\textit{(edge–vertex bound)} \quad \mu(e) \;\le\; \min_{u \in \partial(e)} \sigma(u). \tag{5.3}$$

Here σ is the *vertex-membership map*, μ the *edge-membership map*, and η the *incidence-membership map*. The underlying crisp hypergraph is (V, E, ∂), recoverable via (5.1).

Definition 5.6.4 (Fuzzy n-SuperHyperGraph)**.** (cf. [4]) Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A *fuzzy n-SuperHyperGraph* is a quadruple

$$(V, E, \sigma, \mu),$$

where $\sigma : V \to [0,1]$ and $\mu : E \to [0,1]$ obey the *admissibility constraint*

$$\mu(e) \;\le\; \min_{v \in e} \sigma(v) \qquad \text{for every } e \in E.$$

Throughout, we use the standard *Gödel t-norm* $T(a,b) := \min\{a,b\}$ and the induced *coverage* operator

$$\mathrm{Cov}_f(v) \;:=\; \max_{u \in V} T\bigl(f(u),\, w(u,v)\bigr) \;=\; \max_{u \in V} \min\{f(u), w(u,v)\},$$

where $f : V \to [0,1]$ is a (fuzzy) selector on vertices and $w : V \times V \to [0,1]$ is a symmetric "adjacency strength" with $w(v,v) = \sigma(v)$. In the crisp case $f : V \to \{0,1\}$, the usual domination constraint $\sum_{u \in N[v]} f(u) \ge 1$ is equivalent to $\max_{u \in N[v]} f(u) = 1$ (and this is the standard domination-function viewpoint for graphs).

Definition 5.6.5 (Domination in a fuzzy graph)**.** Let $G = (V, \sigma, \mu)$ be a fuzzy graph in the sense of Definition (Fuzzy graph). Define the *fuzzy adjacency strength* $w_G : V \times V \to [0, 1]$ by

$$w_G(u, v) \ := \ \begin{cases} \sigma(v), & u = v, \\ \mu(u, v), & u \neq v. \end{cases}$$

A function $f : V \to [0, 1]$ is called a *fuzzy dominating function* (FDF) of G if

$$\mathrm{Cov}_f^G(v) \ := \ \max_{u \in V} \min\{f(u), w_G(u, v)\} \ \geq \ \sigma(v) \qquad (\forall\, v \in V).$$

The *weight* of f is $\mathrm{wt}(f) := \sum_{v \in V} f(v)$, and the *fuzzy domination number* is

$$\gamma_f(G) \ := \ \min\{\mathrm{wt}(f) : f \text{ is an FDF of } G\}.$$

Remark 5.6.6 (Crisp reduction)**.** If $\sigma \equiv 1$ and $\mu(u, v) \in \{0, 1\}$ for $u \neq v$, then an FDF $f : V \to \{0, 1\}$ satisfies $\mathrm{Cov}_f^G(v) \geq 1$ iff some vertex in the closed neighborhood $N[v]$ has $f = 1$, i.e. iff f is an ordinary domination function and $\mathrm{wt}(f)$ is the size of the corresponding dominating set.

Example 5.6.7 (Domination in a fuzzy graph)**.** Let $V = \{a, b, c\}$ and define vertex-memberships

$$\sigma(a) = 0.6, \qquad \sigma(b) = 0.8, \qquad \sigma(c) = 0.7.$$

Define edge-memberships $\mu : V \times V \to [0, 1]$ (symmetric, with $\mu(u, v) = 0$ if not listed) by

$$\mu(a, b) = \mu(b, a) = 0.6, \qquad \mu(b, c) = \mu(c, b) = 0.7, \qquad \mu(a, c) = \mu(c, a) = 0.$$

Then $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$ holds for all $u, v \in V$, so $G = (V, \sigma, \mu)$ is a fuzzy graph.

The associated adjacency strength w_G (Definition 5.6.5) satisfies $w_G(v, v) = \sigma(v)$ and $w_G(u, v) = \mu(u, v)$ for $u \neq v$. Define a selector $f : V \to [0, 1]$ by

$$f(a) = 0, \qquad f(b) = 0.8, \qquad f(c) = 0.$$

For each $v \in V$ we compute $\mathrm{Cov}_f^G(v) = \max_{u \in V} \min\{f(u), w_G(u, v)\}$:

$$\mathrm{Cov}_f^G(a) \geq \min\{f(b), w_G(b, a)\} = \min\{0.8, 0.6\} = 0.6 = \sigma(a),$$
$$\mathrm{Cov}_f^G(b) \geq \min\{f(b), w_G(b, b)\} = \min\{0.8, 0.8\} = 0.8 = \sigma(b),$$
$$\mathrm{Cov}_f^G(c) \geq \min\{f(b), w_G(b, c)\} = \min\{0.8, 0.7\} = 0.7 = \sigma(c).$$

Hence f is a fuzzy dominating function of G with weight $\mathrm{wt}(f) = 0.8$.

Definition 5.6.8 (Domination in a fuzzy hypergraph)**.** Let $\mathcal{H} = (V, E, \partial; \sigma, \mu, \eta)$ be a fuzzy hypergraph in the sense of Definition 5.6.3. For $u \neq v$, define the *support-based hyperedge adjacency strength*

$$w_{\mathcal{H}}(u, v) \ := \ \max\{\mu(e) : e \in E,\ \{u, v\} \subseteq \partial(e)\}, \qquad w_{\mathcal{H}}(v, v) \ := \ \sigma(v).$$

A function $f : V \to [0, 1]$ is a *fuzzy dominating function* of $\mathcal{H}$ if

$$\mathrm{Cov}_f^{\mathcal{H}}(v) \ := \ \max_{u \in V} \min\{f(u), w_{\mathcal{H}}(u, v)\} \ \geq \ \sigma(v) \qquad (\forall\, v \in V).$$

The *fuzzy domination number* of $\mathcal{H}$ is

$$\gamma_f(\mathcal{H}) \ := \ \min\Big\{\sum_{v \in V} f(v) : f \text{ is a fuzzy dominating function of } \mathcal{H}\Big\}.$$

Remark 5.6.9 (On the role of η)**.** Definition 5.6.8 uses the crisp supports $\partial(e)$ and edge-memberships $\mu(e)$ to measure vertex–vertex domination through shared hyperedges. If one wishes to incorporate graded incidence η, a common refinement is to replace $w_{\mathcal{H}}(u,v)$ (for $u \neq v$) by $\max_{e\in E}\min\{\eta(u,e),\eta(v,e)\}$ (or $\max_e \min\{\eta(u,e),\eta(v,e),\mu(e)\}$), and the same domination framework below still applies.

A concrete example is given below.

Example 5.6.10 (Domination in a fuzzy hypergraph)**.** Let the underlying crisp hypergraph be (V,E,∂) with

$$V=\{a,b,c,d\},\qquad E=\{e_1,e_2\},\qquad \partial(e_1)=\{a,b,c\},\ \ \partial(e_2)=\{b,d\}.$$

Assign vertex- and edge-memberships

$$\sigma(a)=0.5,\ \sigma(b)=0.8,\ \sigma(c)=0.5,\ \sigma(d)=0.8,\qquad \mu(e_1)=0.5,\ \mu(e_2)=0.8.$$

Then $\mu(e)\le \min_{v\in\partial(e)}\sigma(v)$ holds for e_1,e_2. Define an incidence-membership $\eta: V\times E\to[0,1]$ by

$$\eta(v,e):=\begin{cases}\min\{\sigma(v),\mu(e)\}, & v\in\partial(e),\\ 0, & v\notin\partial(e),\end{cases}$$

so the support condition (5.1) and the bound (5.2) are satisfied. Thus $\mathcal{H}=(V,E,\partial;\sigma,\mu,\eta)$ is a fuzzy hypergraph.

For $u\neq v$, the adjacency strength $w_{\mathcal{H}}(u,v)=\max\{\mu(e):\{u,v\}\subseteq\partial(e)\}$ gives

$$w_{\mathcal{H}}(a,b)=w_{\mathcal{H}}(a,c)=w_{\mathcal{H}}(b,c)=\mu(e_1)=0.5,\qquad w_{\mathcal{H}}(b,d)=\mu(e_2)=0.8,$$

and $w_{\mathcal{H}}(v,v)=\sigma(v)$.

Define $f: V\to[0,1]$ by

$$f(a)=0,\qquad f(b)=0.8,\qquad f(c)=0,\qquad f(d)=0.$$

Then for each $v\in V$,

$$\begin{aligned}\mathrm{Cov}_f^{\mathcal{H}}(a)&\ge \min\{f(b),w_{\mathcal{H}}(b,a)\}=\min\{0.8,0.5\}=0.5=\sigma(a),\\ \mathrm{Cov}_f^{\mathcal{H}}(c)&\ge \min\{f(b),w_{\mathcal{H}}(b,c)\}=\min\{0.8,0.5\}=0.5=\sigma(c),\\ \mathrm{Cov}_f^{\mathcal{H}}(d)&\ge \min\{f(b),w_{\mathcal{H}}(b,d)\}=\min\{0.8,0.8\}=0.8=\sigma(d),\end{aligned}$$

and $\mathrm{Cov}_f^{\mathcal{H}}(b)\ge\min\{f(b),w_{\mathcal{H}}(b,b)\}=\min\{0.8,0.8\}=0.8=\sigma(b)$. Hence f is a fuzzy dominating function of $\mathcal{H}$ with total weight $\sum_{v\in V}f(v)=0.8$.

Definition 5.6.11 (Domination in a fuzzy n-SuperHyperGraph)**.** Let (V,E,σ,μ) be a fuzzy n-SuperHyperGraph in the sense of Definition (Fuzzy n-SuperHyperGraph). Define $w_{\mathrm{SHG}}: V\times V\to[0,1]$ by

$$w_{\mathrm{SHG}}(u,v)\ :=\ \begin{cases}\sigma(v), & u=v,\\ \max\{\mu(e): e\in E,\ \{u,v\}\subseteq e\}, & u\neq v,\end{cases}$$

with the convention that $\max\emptyset := 0$. A function $f : V \to [0,1]$ is a *fuzzy dominating function* of (V, E, σ, μ) if

$$\mathrm{Cov}_f^{\mathrm{SHG}}(v) \;:=\; \max_{u\in V} \min\{f(u), w_{\mathrm{SHG}}(u,v)\} \;\geq\; \sigma(v) \qquad (\forall\, v \in V).$$

The *fuzzy domination number* is

$$\gamma_f(V, E, \sigma, \mu) \;:=\; \min\Big\{\sum_{v\in V} f(v) : f \text{ is a fuzzy dominating function of } (V, E, \sigma, \mu)\Big\}.$$

A concrete example is given below.

Example 5.6.12 (Domination in a fuzzy 1-SuperHyperGraph)**.** Let $n = 1$ and let $\mathrm{SHG}^{(1)} = (V, E)$ have supervertex set

$$V = \{X_1, X_2, X_3\}, \qquad X_1 = \{1\},\ X_2 = \{2\},\ X_3 = \{3\},$$

and superedge set

$$E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{X_1, X_2, X_3\},\ \ \varepsilon_2 = \{X_2, X_3\}.$$

Assign memberships

$$\sigma(X_1) = 0.4, \qquad \sigma(X_2) = 0.7, \qquad \sigma(X_3) = 0.7, \qquad \mu(\varepsilon_1) = 0.4, \qquad \mu(\varepsilon_2) = 0.7.$$

Then the admissibility constraint $\mu(e) \leq \min_{v\in e} \sigma(v)$ holds: $\mu(\varepsilon_1) = 0.4 \leq \min\{0.4, 0.7, 0.7\}$ and $\mu(\varepsilon_2) = 0.7 \leq \min\{0.7, 0.7\}$. Hence (V, E, σ, μ) is a fuzzy 1-SuperHyperGraph.

The induced strength w_{SHG} satisfies $w_{\mathrm{SHG}}(v, v) = \sigma(v)$ and, for $u \neq v$,

$$w_{\mathrm{SHG}}(X_1, X_2) = w_{\mathrm{SHG}}(X_1, X_3) = \mu(\varepsilon_1) = 0.4, \qquad w_{\mathrm{SHG}}(X_2, X_3) = \max\{\mu(\varepsilon_1), \mu(\varepsilon_2)\} = 0.7.$$

Define $f : V \to [0,1]$ by

$$f(X_1) = 0, \qquad f(X_2) = 0.7, \qquad f(X_3) = 0.$$

Then

$$\mathrm{Cov}_f^{\mathrm{SHG}}(X_1) \geq \min\{f(X_2), w_{\mathrm{SHG}}(X_2, X_1)\} = \min\{0.7, 0.4\} = 0.4 = \sigma(X_1),$$

$$\mathrm{Cov}_f^{\mathrm{SHG}}(X_3) \geq \min\{f(X_2), w_{\mathrm{SHG}}(X_2, X_3)\} = \min\{0.7, 0.7\} = 0.7 = \sigma(X_3),$$

and $\mathrm{Cov}_f^{\mathrm{SHG}}(X_2) \geq \min\{f(X_2), w_{\mathrm{SHG}}(X_2, X_2)\} = \min\{0.7, 0.7\} = 0.7 = \sigma(X_2)$. Thus f is a fuzzy dominating function with total weight 0.7.

Theorem 5.6.13 (Fuzzy SuperHyperGraph domination generalizes the fuzzy graph/hypergraph cases)**.**

1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Define a fuzzy* 0*-SuperHyperGraph*

   $$\mathrm{SHG}(G) \;:=\; \big(V, E_G, \sigma, \mu_G\big), \quad E_G := \big\{\{u,v\} : u \neq v,\ \mu(u,v) > 0\big\}, \quad \mu_G(\{u,v\}) := \mu(u,v).$$

   Then $\gamma_f(G) = \gamma_f(\mathrm{SHG}(G))$.

2. Let $\mathcal{H} = (V, E, \partial; \sigma, \mu, \eta)$ *be a fuzzy hypergraph, and define*

$$E_{\mathcal{H}} := \{\partial(e) : e \in E\} \subseteq \mathrm{PS}(V), \qquad \mu_{\mathcal{H}}(S) := \max\{\mu(e) : e \in E,\ \partial(e) = S\} \quad (S \in E_{\mathcal{H}}).$$

Then, with domination in $\mathcal{H}$ *understood as in Definition 5.6.8,*

$$\gamma_f(\mathcal{H}) = \gamma_f(V, E_{\mathcal{H}}, \sigma, \mu_{\mathcal{H}}).$$

Proof. (1) For $u \neq v$, by construction we have

$$w_{\mathrm{SHG}(G)}(u, v) = \max\{\mu_G(\{u, v\})\} = \mu(u, v) = w_G(u, v), \qquad w_{\mathrm{SHG}(G)}(v, v) = \sigma(v) = w_G(v, v).$$

Hence the domination constraints $\mathrm{Cov}_f^G(v) \geq \sigma(v)$ and $\mathrm{Cov}_f^{\mathrm{SHG}(G)}(v) \geq \sigma(v)$ are identical for every $f : V \to [0, 1]$ and every $v \in V$. Therefore the feasible sets of dominating functions coincide and the same objective $\sum_v f(v)$ is minimized, so $\gamma_f(G) = \gamma_f(\mathrm{SHG}(G))$.

(2) Fix distinct $u, v \in V$. Using the definition of $\mu_{\mathcal{H}}$,

$$\begin{aligned} w_{\mathrm{SHG}}(u, v) &= \max\{\mu_{\mathcal{H}}(S) : S \in E_{\mathcal{H}},\ \{u, v\} \subseteq S\} \\ &= \max\{\mu(e) : e \in E,\ \{u, v\} \subseteq \partial(e)\} = w_{\mathcal{H}}(u, v), \end{aligned}$$

and on the diagonal $w_{\mathrm{SHG}}(v, v) = \sigma(v) = w_{\mathcal{H}}(v, v)$. Thus $\mathrm{Cov}_f^{\mathcal{H}}(v) = \mathrm{Cov}_f^{\mathrm{SHG}}(v)$ for all v and all f, so the set of feasible dominating functions and the minimized weight are the same. Hence $\gamma_f(\mathcal{H}) = \gamma_f(V, E_{\mathcal{H}}, \sigma, \mu_{\mathcal{H}})$. □

5.7 Domination in Neutrosophic SuperHyperGraphs

A Neutrosophic Set assigns independent truth, indeterminacy, and falsity degrees to each element, allowing explicit modeling of incomplete, inconsistent information [331–334]. Moreover, as generalizations of the Neutrosophic Set, concepts such as the Quadripartitioned Neutrosophic Set [335, 336], Hesitant Neutrosophic Set [337, 338], SuperHyperNeutrosophic Set [339], Bipolar Neutrosophic Set [340, 341], Plithogenic Set [4, 342–345], and the Pentapartitioned Neutrosophic Set [346, 347] are also well known.

A *Single-valued Neutrosophic* n*-Superhypergraph* [348] is a concept that generalizes both the Single-valued Neutrosophic graph [349–351] and the Single-valued Neutrosophic hypergraph [352, 353]. It also extends the notion of a Fuzzy n-Superhypergraph. The formal definition and a representative example are given below(cf. [354]).

Definition 5.7.1 (Single–valued Neutrosophic Set)**.** [334, 355] Let X be a nonempty universe. A *single–valued neutrosophic set* A on X is described by a triple of functions

$$T_A,\ I_A,\ F_A : X \longrightarrow [0, 1],$$

such that for every $x \in X$,

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Here $T_A(x)$, $I_A(x)$, and $F_A(x)$ denote, respectively, the degrees of truth–membership, indeterminacy–membership, and falsity–membership of x with respect to A. We write

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}.$$

A fuzzy set is recovered when $I_A(x) = 0$ and $F_A(x) = 1 - T_A(x)$ for all x.

Definition 5.7.2 (Single-Valued Neutrosophic Graph)**.** [355] Let $G^* = (V, E)$ be a crisp (classical) graph, where V is the vertex set and $E \subseteq V \times V$ the edge set. A *single-valued neutrosophic graph* (SVNG) on G^* is defined as a pair

$$G = (A, B),$$

where

- $A = \{\langle v, T_A(v), I_A(v), F_A(v)\rangle : v \in V\}$ is the *single-valued neutrosophic vertex set*, with
  $$T_A, I_A, F_A : V \to [0,1],$$
  denoting respectively the *truth-membership*, *indeterminacy-membership*, and *falsity-membership* functions of vertices, such that for every $v \in V$,
  $$0 \le T_A(v) + I_A(v) + F_A(v) \le 3.$$

- $B = \{\langle uv, T_B(uv), I_B(uv), F_B(uv)\rangle : uv \in E\}$ is the *single-valued neutrosophic edge set*, with
  $$T_B, I_B, F_B : E \to [0,1],$$
  satisfying for all $u, v \in V$ with $uv \in E$:
  $$T_B(uv) \le \min\{T_A(u), T_A(v)\},$$
  $$I_B(uv) \le \min\{I_A(u), I_A(v)\},$$
  $$F_B(uv) \ge \max\{F_A(u), F_A(v)\}.$$

If B is symmetric, $G = (A, B)$ is called an *undirected SVNG*; otherwise, it is a *directed SVNG*.

In what follows, we continue the discussion by introducing the definition of a Single-Valued Neutrosophic Hypergraph.

Definition 5.7.3 (Single-Valued Neutrosophic Hypergraph)**.** [353,356–358] Let $V = \{v_1, \ldots, v_N\}$ be a finite vertex set, and let $\{E_i\}_{i=1}^{M}$ be a collection of non-empty neutrosophic subsets of V such that $V = \bigcup_{i=1}^{M} \operatorname{supp}(E_i)$. Each hyperedge E_i is specified by three membership functions

$$T_{E_i},\ I_{E_i},\ F_{E_i} : V \to [0,1],$$

assigning to each vertex $v \in V$ its truth, indeterminacy, and falsity degrees, respectively, and satisfying

$$0 \le T_{E_i}(v) + I_{E_i}(v) + F_{E_i}(v) \le 3 \quad \forall v \in V.$$

We represent E_i as the set

$$E_i = \{(v,\ T_{E_i}(v),\ I_{E_i}(v),\ F_{E_i}(v)) : v \in V\}.$$

The pair $H = (V, \{E_i\})$ is called a *single-valued neutrosophic hypergraph*.

Definition 5.7.4 (Neutrosophic n-Superhypergraph)**.** (cf. [348, 354]) Let V_0 be a finite *base set* of vertices, and for each integer $k \geq 0$ define

$$\mathrm{PS}^0(V_0) = V_0,$$

$$\mathrm{PS}^{k+1}(V_0) = \mathrm{PS}\big(\mathrm{PS}^k(V_0)\big),$$

where $\mathrm{PS}(\cdot)$ denotes the usual powerset. An *n-Superhypergraph* is a pair

$$\mathrm{SHG}^{(n)} = (V,\, E), \quad V \;\subseteq\; \mathrm{PS}^n(V_0), \;\; E \;\subseteq\; \mathrm{PS}^n(V_0).$$

A *Neutrosophic n-Superhypergraph* is then the tuple

$$\big(V,\, E,\, T_V,\, I_V,\, F_V,\, T_E,\, I_E,\, F_E\big),$$

where

- $T_V,\, I_V,\, F_V : V \to [0,1]$ assign to each n-supervertex $v \in V$ its truth-, indeterminacy-, and falsity-membership degrees, respectively, subject to
$$0 \;\leq\; T_V(v) + I_V(v) + F_V(v) \;\leq\; 3,$$
$$\forall v \in V.$$
- $T_E,\, I_E,\, F_E : E \times V \to [0,1]$ assign to each n-superedge $e \in E$ and vertex $v \in e$ its truth-, indeterminacy-, and falsity-membership degrees, respectively, subject to
$$0 \;\leq\; T_E(e,v) + I_E(e,v) + F_E(e,v) \;\leq\; 3,$$
$$\forall e \in E,\; \forall v \in e.$$

These functions satisfy the *edge-appurtenance constraints*:

$$T_E(e,v) \;\leq\; T_V(v),$$
$$I_E(e,v) \;\leq\; I_V(v),$$
$$F_E(e,v) \;\leq\; F_V(v),$$
$$\forall\, e \in E,\; \forall\, v \in e.$$

Definition 5.7.5 (Truth α-cut of a single-valued neutrosophic set)**.** Let X be a nonempty finite set and let A be a single-valued neutrosophic set on X with membership triples $(T_A, I_A, F_A) : X \to [0,1]^3$. For $\alpha \in (0,1]$, the *truth α-cut* of A is the (crisp) subset

$$A^{(\alpha)} \;:=\; \{\, x \in X :\; T_A(x) \geq \alpha \,\}.$$

Remark 5.7.6. Other cut rules are possible (e.g. combining T, I, F), but Definition 5.7.5 is the simplest choice and yields a direct reduction to ordinary domination.

Definition 5.7.7 (Domination in a single-valued neutrosophic graph)**.** Let $G = (A, B)$ be a single-valued neutrosophic graph on a crisp graph $G^* = (V, E)$, where A assigns (T_A, I_A, F_A) to vertices and B assigns (T_B, I_B, F_B) to edges. Fix $\alpha \in (0, 1]$ and define the *truth-α shadow* (crisp) graph

$$G^{(\alpha)} \;=\; \bigl(V^{(\alpha)}, E^{(\alpha)}\bigr), \qquad V^{(\alpha)} := \{v \in V :\; T_A(v) \geq \alpha\},$$

$$E^{(\alpha)} := \bigl\{\{u, v\} \in E :\; u, v \in V^{(\alpha)} \text{ and } T_B(uv) \geq \alpha\bigr\}.$$

A set $D \subseteq V^{(\alpha)}$ is called a *dominating set of G at level α* if it is a dominating set of the crisp graph $G^{(\alpha)}$, i.e.,

$$\forall\, v \in V^{(\alpha)} \setminus D \;\; \exists\, u \in D \;\text{ with }\; \{u, v\} \in E^{(\alpha)}.$$

The corresponding *(truth-α) domination number* is

$$\gamma^{(\alpha)}(G) \;:=\; \min\{\, |D| :\; D \subseteq V^{(\alpha)} \text{ dominates } G^{(\alpha)}\,\}.$$

Remark 5.7.8 (Crisp reduction)**.** If $T_A \equiv 1$ and $T_B(uv) \in \{0, 1\}$ for all $uv \in E$, then with $\alpha = 1$ we have $G^{(1)} = G^*$ and $\gamma^{(1)}(G)$ coincides with the usual domination number $\gamma(G^*)$.

A concrete example is given below.

Example 5.7.9 (Domination in a single-valued neutrosophic graph)**.** Let the crisp graph be the path $G^* = (V, E)$ with

$$V = \{a, b, c\}, \qquad E = \{ab, bc\}.$$

Define a single-valued neutrosophic vertex set

$$A = \{\langle a, 0.8, 0, 0\rangle,\; \langle b, 0.7, 0, 0\rangle,\; \langle c, 0.9, 0, 0\rangle\},$$

so $T_A(a) = 0.8$, $T_A(b) = 0.7$, $T_A(c) = 0.9$ (and $I_A \equiv F_A \equiv 0$). Define a single-valued neutrosophic edge set

$$B = \{\langle ab, 0.7, 0, 0\rangle,\; \langle bc, 0.7, 0, 0\rangle\},$$

so $T_B(ab) = T_B(bc) = 0.7$ (and $I_B \equiv F_B \equiv 0$). The SVNG constraints hold since

$$T_B(ab) = 0.7 \leq \min\{T_A(a), T_A(b)\} = \min\{0.8, 0.7\} = 0.7,$$

$$T_B(bc) = 0.7 \leq \min\{T_A(b), T_A(c)\} = \min\{0.7, 0.9\} = 0.7.$$

Let $G = (A, B)$ and fix $\alpha = 0.7$. Then the truth-α shadow graph is

$$V^{(\alpha)} = \{v \in V : T_A(v) \geq 0.7\} = \{a, b, c\}, \qquad E^{(\alpha)} = \{ab, bc\},$$

since $T_B(ab) = T_B(bc) = 0.7 \geq \alpha$. Hence $G^{(\alpha)}$ is again the path $a - b - c$.

The set $D = \{b\} \subseteq V^{(\alpha)}$ dominates $G^{(\alpha)}$, because both a and c are adjacent to b. Therefore D is a dominating set of G at level $\alpha = 0.7$ (Definition 5.7.7), and in particular $\gamma^{(0.7)}(G) = 1$.

Definition 5.7.10 (Domination in a single-valued neutrosophic hypergraph)**.** Let $H = (V, \{E_i\}_{i=1}^{M})$ be a single-valued neutrosophic hypergraph, where each hyperedge E_i is a neutrosophic set on V with membership triples $(T_{E_i}, I_{E_i}, F_{E_i})$. Fix $\alpha \in (0, 1]$ and define the *truth-α shadow* (crisp) hypergraph

$$H^{(\alpha)} \;=\; \bigl(V^{(\alpha)}, \mathcal{E}^{(\alpha)}\bigr), \qquad V^{(\alpha)} := V,$$

$$\mathcal{E}^{(\alpha)} \;:=\; \Big\{\, E_i^{(\alpha)} :\; i = 1,\dots,M,\;\; E_i^{(\alpha)} \neq \emptyset \,\Big\}, \qquad E_i^{(\alpha)} := \{\, v \in V :\; T_{E_i}(v) \geq \alpha \,\}.$$

A set $D \subseteq V$ is a *dominating set of* H *at level* α if

$$\forall\, v \in V \setminus D \;\; \exists\, e \in \mathcal{E}^{(\alpha)} \;\text{ with }\; v \in e \;\text{ and }\; e \cap D \neq \emptyset.$$

The *(truth-α) domination number* is

$$\gamma^{(\alpha)}(H) \;:=\; \min\{\, |D| :\; D \subseteq V \text{ dominates } H^{(\alpha)} \,\}.$$

Remark 5.7.11. Equivalently, one may define adjacency by the primal graph of $H^{(\alpha)}$ and require domination there; both viewpoints match the usual hypergraph domination definition.

A concrete example is given below.

Example 5.7.12 (Domination in a single-valued neutrosophic graph)**.** Let the crisp graph be the path $G^* = (V, E)$ with

$$V = \{a, b, c\}, \qquad E = \{ab, bc\}.$$

Define a single-valued neutrosophic vertex set

$$A = \{\langle a, 0.8, 0, 0\rangle,\; \langle b, 0.7, 0, 0\rangle,\; \langle c, 0.9, 0, 0\rangle\},$$

so $T_A(a) = 0.8$, $T_A(b) = 0.7$, $T_A(c) = 0.9$ (and $I_A \equiv F_A \equiv 0$). Define a single-valued neutrosophic edge set

$$B = \{\langle ab, 0.7, 0, 0\rangle,\; \langle bc, 0.7, 0, 0\rangle\},$$

so $T_B(ab) = T_B(bc) = 0.7$ (and $I_B \equiv F_B \equiv 0$). The SVNG constraints hold since

$$T_B(ab) = 0.7 \leq \min\{T_A(a), T_A(b)\} = \min\{0.8, 0.7\} = 0.7,$$

$$T_B(bc) = 0.7 \leq \min\{T_A(b), T_A(c)\} = \min\{0.7, 0.9\} = 0.7.$$

Let $G = (A, B)$ and fix $\alpha = 0.7$. Then the truth-α shadow graph is

$$V^{(\alpha)} = \{v \in V : T_A(v) \geq 0.7\} = \{a, b, c\}, \qquad E^{(\alpha)} = \{ab, bc\},$$

since $T_B(ab) = T_B(bc) = 0.7 \geq \alpha$. Hence $G^{(\alpha)}$ is again the path $a - b - c$.

The set $D = \{b\} \subseteq V^{(\alpha)}$ dominates $G^{(\alpha)}$, because both a and c are adjacent to b. Therefore D is a dominating set of G at level $\alpha = 0.7$ (Definition 5.7.7), and in particular $\gamma^{(0.7)}(G) = 1$.

Definition 5.7.13 (Domination in a single-valued neutrosophic n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (so V is a finite set of n-supervertices and E is a finite family of nonempty subsets of V). Assume $\mathrm{SHG}^{(n)}$ carries single-valued neutrosophic data consisting of

$$(T_V, I_V, F_V) : V \to [0,1]^3 \quad \text{and} \quad (T_E, I_E, F_E) : \{(e, x) \in E \times V :\; x \in e\} \to [0,1]^3,$$

where $(T_E(e,x), I_E(e,x), F_E(e,x))$ gives the neutrosophic membership of a supervertex x in a superedge e. Fix $\alpha \in (0,1]$ and define the *truth-α shadow* (crisp) n-SuperHyperGraph

$$\big(\mathrm{SHG}^{(n)}\big)^{(\alpha)} \;=\; \big(V^{(\alpha)}, E^{(\alpha)}\big), \qquad V^{(\alpha)} := \{x \in V :\; T_V(x) \geq \alpha\},$$

$$E^{(\alpha)} := \Big\{ e^{(\alpha)} : \ e \in E, \ \ e^{(\alpha)} \neq \emptyset \Big\}, \qquad e^{(\alpha)} := \{\, x \in e \cap V^{(\alpha)} : \ T_E(e,x) \geq \alpha \,\}.$$

A set $D \subseteq V^{(\alpha)}$ is a *dominating set of* $\mathrm{SHG}^{(n)}$ *at level* α if

$$\forall\, x \in V^{(\alpha)} \setminus D \ \ \exists\, \varepsilon \in E^{(\alpha)} \ \text{ with } \ x \in \varepsilon \ \text{ and } \ \varepsilon \cap D \neq \emptyset.$$

The *(truth-α) domination number* is

$$\gamma^{(\alpha)}\big(\mathrm{SHG}^{(n)}\big) \ := \ \min\Big\{ |D| : \ D \subseteq V^{(\alpha)} \text{ dominates } \big(\mathrm{SHG}^{(n)}\big)^{(\alpha)} \Big\}.$$

Remark 5.7.14 (Compatibility)**.** If $\mathrm{SHG}^{(0)}$ arises from a crisp graph (resp. hypergraph) and the neutrosophic truth-memberships are $\{0,1\}$-valued, then at $\alpha = 1$ Definition 5.7.13 reduces to the usual domination in graphs (resp. hypergraphs).

A concrete example is given below.

Example 5.7.15 (Domination in a single-valued neutrosophic 1-SuperHyperGraph)**.** Let $V_0 = \{1,2,3,4\}$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V,E)$ with

$$V = \{X_1, X_2, X_3, X_4\} \subseteq \mathrm{PS}(V_0), \qquad X_1 = \{1\}, \ X_2 = \{2\}, \ X_3 = \{3\}, \ X_4 = \{4\},$$

and

$$E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{X_1, X_2, X_3\}, \ \ \varepsilon_2 = \{X_3, X_4\}.$$

Assign neutrosophic data on supervertices by

$$(T_V, I_V, F_V)(X_1) = (0.6, 0, 0), \quad (T_V, I_V, F_V)(X_2) = (0.8, 0, 0),$$
$$(T_V, I_V, F_V)(X_3) = (0.9, 0, 0), \quad (T_V, I_V, F_V)(X_4) = (0.7, 0, 0),$$

and on incidences (e,x) (for $x \in e$) by truth-memberships

$$T_E(\varepsilon_1, X_1) = 0.6, \quad T_E(\varepsilon_1, X_2) = 0.8, \quad T_E(\varepsilon_1, X_3) = 0.7,$$
$$T_E(\varepsilon_2, X_3) = 0.9, \quad T_E(\varepsilon_2, X_4) = 0.7,$$

with $I_E \equiv 0$ and $F_E \equiv 0$. The edge-appurtenance constraints hold, e.g., $T_E(\varepsilon_1, X_3) = 0.7 \leq T_V(X_3) = 0.9$, etc. Thus this is a neutrosophic 1-SuperHyperGraph in the sense of Definition 5.7.4.

Fix $\alpha = 0.7$. Then the truth-α shadow (Definition 5.7.13) has

$$V^{(\alpha)} = \{x \in V : T_V(x) \geq 0.7\} = \{X_2, X_3, X_4\}.$$

Moreover,

$$\varepsilon_1^{(\alpha)} = \{x \in \varepsilon_1 \cap V^{(\alpha)} : T_E(\varepsilon_1, x) \geq 0.7\} = \{X_2, X_3\},$$
$$\varepsilon_2^{(\alpha)} = \{x \in \varepsilon_2 \cap V^{(\alpha)} : T_E(\varepsilon_2, x) \geq 0.7\} = \{X_3, X_4\},$$

so

$$E^{(\alpha)} = \big\{\{X_2, X_3\}, \{X_3, X_4\}\big\}.$$

Let $D = \{X_3\} \subseteq V^{(\alpha)}$. For each $x \in V^{(\alpha)} \setminus D$ we have a superedge in $E^{(\alpha)}$ containing x and intersecting D:

$$X_2 \in \{X_2, X_3\} \ \text{ and } \ \{X_2, X_3\} \cap D = \{X_3\} \neq \emptyset, \qquad X_4 \in \{X_3, X_4\} \ \text{ and } \ \{X_3, X_4\} \cap D = \{X_3\} \neq \emptyset.$$

Hence D dominates $\big(\mathrm{SHG}^{(1)}\big)^{(\alpha)}$, i.e., D is a dominating set of $\mathrm{SHG}^{(1)}$ at level $\alpha = 0.7$, and therefore $\gamma^{(0.7)}\big(\mathrm{SHG}^{(1)}\big) = 1$.

5.8 Domination in Uncertain SuperHyperGraphs

An Uncertain Set assigns to each element a degree from an uncertainty model, unifying fuzzy, intuitionistic, neutrosophic and plithogenic frameworks [359,360]. An Uncertain Graph is a graph where vertices or edges carry degrees in an uncertainty model, subsuming fuzzy, intuitionistic, neutrosophic [1]. As related concepts to uncertain graphs, we also examine refined uncertain graphs and iterated refined uncertain graphs in the Appendix.

An Uncertain HyperGraph assigns uncertainty-model degrees to vertices and hyperedges in a hypergraph, modeling complex higher-order connections under incomplete information. An Uncertain SuperHyperGraph equips each supervertex and superedge in an n-SuperHyperGraph with uncertainty-model degrees, handling hierarchical uncertainty systematically and rigorously. We first recall the notion of an Uncertain Model, which provides the membership–degree domain.

Definition 5.8.1 (Uncertain Model)**.** [359] Let U denote the class of all *uncertain models*. Each $M \in U$ is specified by

- a nonempty set $\mathrm{Dom}(M) \subseteq [0,1]^k$ of *admissible degree tuples* for some fixed integer $k \geq 1$;
- model–specific algebraic or geometric constraints on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \leq 1$ in the intuitionistic fuzzy case, or $T + I + F \leq 3$ in the neutrosophic case).

Typical examples include:

- Fuzzy model: $\mathrm{Dom}(M) = [0,1]$;
- Intuitionistic fuzzy model: $\mathrm{Dom}(M) = \{(\mu,\nu) \in [0,1]^2 \mid \mu + \nu \leq 1\}$;
- Neutrosophic model: $\mathrm{Dom}(M) = \{(T,I,F) \in [0,1]^3 \mid 0 \leq T + I + F \leq 3\}$;
- Plithogenic model, and many other extensions.

Definition 5.8.2 (Uncertain Set (U-Set))**.** [359] Let X be a nonempty universe, and let M be a fixed uncertain model with degree–domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *Uncertain Set of type M* (or *U-Set* for short) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is called the *uncertainty–degree function* (or membership map) of $\mathcal{U}$.

For $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ encodes the degree(s) to which x belongs to the uncertain set, according to the model M.

Remark 5.8.3. Special cases:

- If M is the fuzzy model and $\mathrm{Dom}(M) = [0,1]$, then $\mu_M : X \to [0,1]$ is a usual fuzzy membership function and $\mathcal{U}$ is a fuzzy set.
- If M is neutrosophic, then $\mu_M(x) = (T(x), I(x), F(x))$ gives a neutrosophic set.
- Other choices of M recover intuitionistic fuzzy sets, picture fuzzy sets, plithogenic sets, and so on.

As mentioned in the preceding remark, one can pursue many further extensions. For convenience, Table 5.1 compiles a catalogue of uncertainty-set families (*U-Sets*), grouped according to the dimension k of their degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (This table is based on information from [1, 360].)

Table 5.1.: A catalogue of uncertainty-set families (U-Sets) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [360].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [311, 330]; N-Fuzzy Set [361–363]; Shadowed Set [364–366]
2		Intuitionistic Fuzzy Set [367, 368]; Vague Set [369, 370]; Bipolar Fuzzy Set (two-component description) [371]; Variable Fuzzy Set [372–374]; Paraconsistent Fuzzy Set [375, 376]; Bifuzzy Set [377, 378]
3		Single-Valued Neutrosophic Set [334, 379]; Picture Fuzzy Set [380, 381]; Spherical Fuzzy Set [314, 382]; Tripolar Fuzzy Set (three-component formalisms) [383–385]; Neutrosophic Vague Set [386, 387]
4		Quadripartitioned Neutrosophic Set [335, 336]; Double-Valued Neutrosophic Set [388, 389]; Dual Hesitant Fuzzy Set [390, 391]; Ambiguous Set [392–394]; Turiyam Neutrosophic Set [395–398]
5		Pentapartitioned Neutrosophic Set [399–401]; Triple-Valued Neutrosophic Set [402–404]
6		Hexapartitioned Neutrosophic Set; Quadruple-Valued Neutrosophic Set [403, 405]
7		Heptapartitioned Neutrosophic Set; Quintuple-Valued Neutrosophic Set [403, 406, 407]
8		Octapartitioned Neutrosophic Set [408]
9		Nonapartitioned Neutrosophic Set [408]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [409]; MultiFuzzy Set [410]; n-Refined Fuzzy Set [411, 412]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [412]; Multi-Intuitionistic Fuzzy Set [410]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [412]; Multi-Neutrosophic Set [410, 413]

Reading guide. Within the U-Set framework [359], a model M is determined by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ together with a membership assignment $\mu_M : X \to \mathrm{Dom}(M)$. The table therefore classifies representative families by the ambient dimension k, i.e., by the number of numerical components recorded for each element.

(a) A commonly cited perspective is that neutrosophic sets form a unifying umbrella encompassing several earlier multi-component fuzzy paradigms (and their extensions); see [414].

(b) Ambiguous sets are often introduced as subclasses of certain four-component neutrosophic families; see [335, 336, 394].

(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [415].

The definitions and related concepts of Uncertain Graphs are presented below.

Definition 5.8.4 (Uncertain Graph)**.** [1] Let $G = (V, E)$ be a (finite, undirected, loopless) graph and let M be an uncertain model with degree–domain $\mathrm{Dom}(M)$. An *Uncertain Graph of type M* is a triple

$$\mathcal{G}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns to each vertex $v \in V$ and each edge $e \in E$ an uncertainty degree $\mu_M(v)$ or $\mu_M(e)$ in $\mathrm{Dom}(M)$.

Optionally, one may impose model–specific consistency conditions between vertex and edge degrees (for instance, $\mu_M(e)$ bounded in terms of $\mu_M(u)$ and $\mu_M(v)$ for $e = \{u, v\}$ in fuzzy or intuitionistic fuzzy graph models), but these constraints are encoded in the choice of M and are not fixed at the level of this general definition.

Remark 5.8.5. Again, particular choices of M recover well–known graph models:

- Fuzzy graph (when M is fuzzy and $\mu_M : V \cup E \to [0, 1]$);
- Intuitionistic fuzzy graph, neutrosophic graph, plithogenic graph, etc., for the corresponding models M.

As a reference, Table 5.2 presents a catalogue of uncertainty-graph families (Uncertain Graphs) organised by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$ [1].

Table 5.2.: A catalogue of uncertainty-graph families (Uncertain Graphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$.

k	Representative uncertainty-graph type(s) $\mathcal{G}_M = (V, E, \mu_M)$ with $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0, 1]^k$
1	Fuzzy graph [315, 330]; N-graph; shadowed-graph variants
2	Intuitionistic fuzzy graph [416]; vague graph [417]; bipolar fuzzy graph [320]; intuitionistic evidence graph; variable fuzzy graph; paraconsistent fuzzy graph; bifuzzy graph [418, 419]
3	Neutrosophic graph [355][(a)]; hesitant fuzzy graph [420]; tripolar fuzzy graph; three-way fuzzy graph; picture fuzzy graph [421, 422]; spherical fuzzy graph [314]; inconsistent intuitionistic fuzzy graph; ternary fuzzy / neutrosophic-fuzzy graph; neutrosophic vague graph
4	Quadripartitioned neutrosophic graph [423, 424]; double-valued neutrosophic graph [388]; dual hesitant fuzzy graph [425]; ambiguous graph[(b)]; local-neutrosophic graph; support-neutrosophic graph; turiyam neutrosophic graph [426–428][(c)]
5	Pentapartitioned neutrosophic graph [429]; triple-valued neutrosophic graph
6	Hexapartitioned neutrosophic graph; quadruple-valued neutrosophic graph
7	Heptapartitioned neutrosophic graph [430]; quintuple-valued neutrosophic graph
8	Octapartitioned neutrosophic graph
9	Nonapartitioned neutrosophic graph
n	n-refined fuzzy graph; multi-valued (fuzzy) graphs; multi-fuzzy graphs [431]
$2n$	n-refined intuitionistic fuzzy graph; multi-intuitionistic fuzzy graphs
$3n$	n-refined neutrosophic graph; multi-neutrosophic graphs

[(a)] Neutrosophic graph models are often treated as broad frameworks that can specialize to many degree-based graph formalisms under suitable constraints.

[(b)] Ambiguous-graph models are commonly presented as subclasses of certain quadripartitioned and also double-valued neutrosophic graph models.

[(c)] Turiyam neutrosophic graphs are reported as subclasses of certain quadripartitioned neutrosophic graph models.

Definition 5.8.6 (Uncertain HyperGraph)**.** [1] Let $H = (V, E)$ be a hypergraph and let M be an uncertain model with degree–domain $\mathrm{Dom}(M)$. An *Uncertain HyperGraph of type M* is a triple

$$\mathcal{H}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns an uncertainty degree to each vertex $v \in V$ and each hyperedge $e \in E$.

As in the graph case, possible relations between vertex and hyperedge degrees (for instance, bounds of $\mu_M(e)$ in terms of $\mu_M(v)$ for $v \in e$) are governed by the chosen model M and its constraints.

Definition 5.8.7 (Uncertain n-SuperHyperGraph)**.** Let V_0 be a finite base set and let $n \in \mathbb{N}_0$. Assume that an n-SuperHyperGraph on V_0 is given by

$$\mathrm{SHG}^{(n)} = (V_n, E),$$

where

$$\emptyset \neq V_n \subseteq \mathrm{PS}^n(V_0) \quad \text{and} \quad \emptyset \neq E \subseteq \mathrm{PS}(V_n) \setminus \{\emptyset\},$$

so that each n-superedge $e \in E$ is a nonempty subset of the n-supervertex set V_n.

Let M be a fixed uncertain model with degree–domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *Uncertain n-SuperHyperGraph of type M* is a triple

$$\mathcal{S}_M^{(n)} = (V_n, E, \mu_M),$$

where

$$\mu_M : V_n \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns to each n-supervertex $v \in V_n$ and each n-superedge $e \in E$ an uncertainty degree $\mu_M(v)$ or $\mu_M(e)$ in $\mathrm{Dom}(M)$.

Any additional relations between the degrees of n-superedges and the degrees of the n-supervertices they contain (for example, model- specific bounds or aggregations) are imposed by the chosen uncertain model M and are not fixed at the level of this general definition.

For $n = 0$ and $V_0 = V_n$, the above notion reduces to an Uncertain HyperGraph of type M.

Remark 5.8.8. Particular choices of the model M recover well–known uncertain SuperHyperGraph types:

- Fuzzy n-SuperHyperGraphs (when M is fuzzy);
- Intuitionistic fuzzy, neutrosophic, and plithogenic n-SuperHyperGraphs for the corresponding models M;
- More exotic variants (e.g. q-rung orthopair, picture fuzzy, refined neutrosophic) are obtained by choosing the appropriate degree–domain $\mathrm{Dom}(M)$.

Regarding the catalogue of uncertainty-superhypergraph families (Uncertain n-SuperHyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, we list them in Table 5.3 [1].

Table 5.3.: A catalogue of uncertainty-superhypergraph families (Uncertain n-SuperHyperGraphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-superhypergraph family (type M with $\mathrm{Dom}(M) \subseteq [0,1]^k$)
1	*Fuzzy n-SuperHyperGraph [328]:* $\mu_M : V_n \cup E \to [0,1]$.
2	*Intuitionistic-fuzzy n-SuperHyperGraph [328]:* $\mu_M : V_n \cup E \to [0,1]^2$ (e.g., (membership, non-membership)).
3	*Neutrosophic n-SuperHyperGraph [22, 348, 354]:* $\mu_M : V_n \cup E \to [0,1]^3$ (e.g., (T, I, F)).
4	*Quadripartitioned / four-component uncertainty n-SuperHyperGraph:* $\mu_M : V_n \cup E \to [0,1]^4$.
k	*k-component uncertainty n-SuperHyperGraph:* $\mu_M : V_n \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$ (model-specific semantics).

Definition 5.8.9 (Model evaluation (acceptance) map)**.** Let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *evaluation map* for M is a function

$$\mathrm{ev}_M : \mathrm{Dom}(M) \longrightarrow [0,1]$$

interpreted as the overall *acceptance / strength* of a degree tuple.

Typical choices include:

- fuzzy model: $\mathrm{ev}_M(t) = t$;
- intuitionistic fuzzy model: $\mathrm{ev}_M(\mu, \nu) = \mu$ (membership);
- neutrosophic model: $\mathrm{ev}_M(T, I, F) = T$ (truth-membership);

and other models choose ev_M according to their intended semantics.

Definition 5.8.10 (α-shadow of an uncertain graph)**.** Let $\mathcal{G}_M = (V, E, \mu_M)$ be an uncertain graph of type M, where $E \subseteq \binom{V}{2}$. Fix an evaluation map ev_M and a threshold $\alpha \in (0,1]$. Define

$$\sigma(v) \;:=\; \mathrm{ev}_M(\mu_M(v)) \quad (v \in V), \qquad \omega(e) \;:=\; \mathrm{ev}_M(\mu_M(e)) \quad (e \in E).$$

The *α-shadow* of $\mathcal{G}_M$ is the crisp graph

$$\mathcal{G}_M^{(\alpha)} \;=\; \bigl(V^{(\alpha)}, E^{(\alpha)}\bigr),$$

where

$$V^{(\alpha)} \;:=\; \{\, v \in V :\; \sigma(v) \geq \alpha \,\}, \qquad E^{(\alpha)} \;:=\; \{\, \{u, v\} \in E :\; u, v \in V^{(\alpha)} \text{ and } \omega(\{u, v\}) \geq \alpha \,\}.$$

Definition 5.8.11 (α-shadow of an uncertain hypergraph)**.** [1] Let $\mathcal{H}_M = (V, E, \mu_M)$ be an uncertain hypergraph of type M, where $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$. Fix ev_M and $\alpha \in (0,1]$, and define $\sigma(v) := \mathrm{ev}_M(\mu_M(v))$ and $\omega(e) := \mathrm{ev}_M(\mu_M(e))$. The *$\alpha$-shadow* of $\mathcal{H}_M$ is the crisp hypergraph

$$\mathcal{H}_M^{(\alpha)} \;=\; \bigl(V^{(\alpha)}, E^{(\alpha)}\bigr),$$

where

$$V^{(\alpha)} \;:=\; \{\, v \in V :\; \sigma(v) \geq \alpha \,\}, \qquad E^{(\alpha)} \;:=\; \{\, e \cap V^{(\alpha)} :\; e \in E,\; \omega(e) \geq \alpha,\; e \cap V^{(\alpha)} \neq \emptyset \,\}.$$

Definition 5.8.12 (α-shadow of an uncertain n-SuperHyperGraph)**.** Let $\mathcal{S}_M^{(n)} = (V, E, \mu_M)$ be an uncertain n-SuperHyperGraph of type M, so V is a finite set of n-supervertices and $E \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}$ is a family of superedges. Fix ev_M and $\alpha \in (0, 1]$, and define

$$\sigma(x) \;:=\; \mathrm{ev}_M(\mu_M(x)) \quad (x \in V), \qquad \omega(\varepsilon) \;:=\; \mathrm{ev}_M(\mu_M(\varepsilon)) \quad (\varepsilon \in E).$$

The *α-shadow* of $\mathcal{S}_M^{(n)}$ is the crisp n-SuperHyperGraph

$$\big(\mathcal{S}_M^{(n)}\big)^{(\alpha)} \;=\; \big(V^{(\alpha)}, E^{(\alpha)}\big),$$

where

$$V^{(\alpha)} \;:=\; \{\, x \in V : \; \sigma(x) \geq \alpha \,\}, \qquad E^{(\alpha)} \;:=\; \{\, \varepsilon \cap V^{(\alpha)} : \; \varepsilon \in E, \; \omega(\varepsilon) \geq \alpha, \; \varepsilon \cap V^{(\alpha)} \neq \emptyset \,\}.$$

We discuss domination in an uncertain graph and domination in an uncertain superhypergraph below.

Definition 5.8.13 (Domination in an uncertain graph (via α-shadow))**.** Let $\mathcal{G}_M = (V, E, \mu_M)$ be an uncertain graph of type M. Fix an evaluation map ev_M and $\alpha \in (0, 1]$, and form the α-shadow $\mathcal{G}_M^{(\alpha)} = (V^{(\alpha)}, E^{(\alpha)})$ as in Definition 5.8.10.

A set $D \subseteq V^{(\alpha)}$ is called an *α-dominating set* of $\mathcal{G}_M$ if

$$\forall\, v \in V^{(\alpha)} \setminus D \quad \exists\, u \in D \;\text{ such that }\; \{u, v\} \in E^{(\alpha)}.$$

Equivalently, D dominates the crisp graph $\mathcal{G}_M^{(\alpha)}$.

The *α-domination number* of $\mathcal{G}_M$ is

$$\gamma^{(\alpha)}(\mathcal{G}_M) \;:=\; \min\{\, |D| : \; D \subseteq V^{(\alpha)} \text{ is an } \alpha\text{-dominating set of } \mathcal{G}_M \,\}.$$

Example 5.8.14 (α-domination in an uncertain graph)**.** Let M be a $k = 1$ uncertain model with $\mathrm{Dom}(M) = [0, 1]$ and evaluation map $\mathrm{ev}_M(t) = t$. Consider the uncertain graph $\mathcal{G}_M = (V, E, \mu_M)$ with

$$V = \{a, b, c\}, \qquad E = \big\{\{a, b\}, \{b, c\}\big\},$$

and degrees

$$\mu_M(a) = 0.9, \qquad \mu_M(b) = 0.8, \qquad \mu_M(c) = 0.7,$$
$$\mu_M(\{a, b\}) = 0.8, \qquad \mu_M(\{b, c\}) = 0.7.$$

Fix $\alpha = 0.7$. Then $\sigma(v) = \mathrm{ev}_M(\mu_M(v)) = \mu_M(v)$ and $\omega(e) = \mathrm{ev}_M(\mu_M(e)) = \mu_M(e)$, so the α-shadow is

$$V^{(\alpha)} = \{v \in V : \sigma(v) \geq 0.7\} = \{a, b, c\}, \qquad E^{(\alpha)} = \big\{\{a, b\}, \{b, c\}\big\},$$

because all listed degrees are at least 0.7.

Let $D = \{b\}$. For each $v \in V^{(\alpha)} \setminus D = \{a, c\}$ we have $\{b, v\} \in E^{(\alpha)}$, so D is an α-dominating set of $\mathcal{G}_M$ (Definition 5.8.13). Hence $\gamma^{(0.7)}(\mathcal{G}_M) = 1$.

Definition 5.8.15 (Domination in an uncertain hypergraph (via α-shadow))**.** Let $\mathcal{H}_M = (V, E, \mu_M)$ be an uncertain hypergraph of type M. Fix ev_M and $\alpha \in (0, 1]$, and form the α-shadow $\mathcal{H}_M^{(\alpha)} = (V^{(\alpha)}, E^{(\alpha)})$ as in Definition 5.8.11.

A set $D \subseteq V^{(\alpha)}$ is called an α-*dominating set* of $\mathcal{H}_M$ if

$$\forall\, v \in V^{(\alpha)} \setminus D \quad \exists\, e \in E^{(\alpha)} \ \text{ such that } \ v \in e \ \text{ and } \ e \cap D \neq \emptyset.$$

The α-*domination number* of $\mathcal{H}_M$ is

$$\gamma^{(\alpha)}(\mathcal{H}_M) \ := \ \min\{\, |D| : \ D \subseteq V^{(\alpha)} \text{ is an } \alpha\text{-dominating set of } \mathcal{H}_M \,\}.$$

Concrete examples are provided below.

Example 5.8.16 (α-domination in an uncertain hypergraph)**.** Let M be a $k = 1$ uncertain model with $\mathrm{Dom}(M) = [0, 1]$ and $\mathrm{ev}_M(t) = t$. Consider the uncertain hypergraph $\mathcal{H}_M = (V, E, \mu_M)$ with

$$V = \{1, 2, 3, 4\}, \qquad E = \{e_1, e_2\}, \qquad e_1 = \{1, 2, 3\}, \ \ e_2 = \{3, 4\},$$

and degrees

$$\mu_M(1) = 0.9, \ \mu_M(2) = 0.8, \ \mu_M(3) = 0.7, \ \mu_M(4) = 0.6, \qquad \mu_M(e_1) = 0.8, \ \mu_M(e_2) = 0.7.$$

Fix $\alpha = 0.7$. Then

$$V^{(\alpha)} = \{v \in V : \mu_M(v) \geq 0.7\} = \{1, 2, 3\},$$

and the α-shadow hyperedge family is

$$E^{(\alpha)} = \{\, e \cap V^{(\alpha)} : \ e \in E, \ \mu_M(e) \geq 0.7, \ e \cap V^{(\alpha)} \neq \emptyset \,\} = \{\{1, 2, 3\}, \{3\}\}.$$

Let $D = \{3\} \subseteq V^{(\alpha)}$. For each $v \in V^{(\alpha)} \setminus D = \{1, 2\}$, we may take the shadow hyperedge $\{1, 2, 3\} \in E^{(\alpha)}$, which contains v and intersects D. Thus D is an α-dominating set of $\mathcal{H}_M$ (Definition 5.8.15), and therefore $\gamma^{(0.7)}(\mathcal{H}_M) = 1$.

Definition 5.8.17 (Domination in an uncertain n-SuperHyperGraph (via α-shadow))**.** Let $\mathcal{S}_M^{(n)} = (V, E, \mu_M)$ be an uncertain n-SuperHyperGraph of type M. Fix ev_M and $\alpha \in (0, 1]$, and form the α-shadow $\big(\mathcal{S}_M^{(n)}\big)^{(\alpha)} = (V^{(\alpha)}, E^{(\alpha)})$ as in Definition 5.8.12.

A set $D \subseteq V^{(\alpha)}$ is called an α-*dominating set* of $\mathcal{S}_M^{(n)}$ if

$$\forall\, x \in V^{(\alpha)} \setminus D \quad \exists\, \varepsilon \in E^{(\alpha)} \ \text{ such that } \ x \in \varepsilon \ \text{ and } \ \varepsilon \cap D \neq \emptyset.$$

The α-*domination number* of $\mathcal{S}_M^{(n)}$ is

$$\gamma^{(\alpha)}(\mathcal{S}_M^{(n)}) \ := \ \min\{\, |D| : \ D \subseteq V^{(\alpha)} \text{ is an } \alpha\text{-dominating set of } \mathcal{S}_M^{(n)} \,\}.$$

Concrete examples are provided below.

Example 5.8.18 (α-domination in an uncertain 1-SuperHyperGraph)**.** Let M be a $k = 1$ uncertain model with $\mathrm{Dom}(M) = [0,1]$ and $\mathrm{ev}_M(t) = t$. Let $V_0 = \{1,2,3,4\}$ and consider the uncertain 1-SuperHyperGraph $\mathcal{S}_M^{(1)} = (V, E, \mu_M)$ with

$$V = \{X_1, X_2, X_3, X_4\} \subseteq \mathrm{PS}(V_0), \qquad X_1 = \{1\},\ X_2 = \{2\},\ X_3 = \{3\},\ X_4 = \{4\},$$

$$E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{X_1, X_2, X_3\},\ \ \varepsilon_2 = \{X_3, X_4\},$$

and degrees

$$\mu_M(X_1) = 0.9, \quad \mu_M(X_2) = 0.8, \quad \mu_M(X_3) = 0.7, \quad \mu_M(X_4) = 0.6,$$

$$\mu_M(\varepsilon_1) = 0.8, \qquad \mu_M(\varepsilon_2) = 0.7.$$

Fix $\alpha = 0.7$. Then the α-shadow has

$$V^{(\alpha)} = \{X_1, X_2, X_3\},$$

and

$$E^{(\alpha)} = \{\, \varepsilon \cap V^{(\alpha)} :\ \varepsilon \in E,\ \mu_M(\varepsilon) \geq 0.7,\ \varepsilon \cap V^{(\alpha)} \neq \emptyset \,\} = \big\{\{X_1, X_2, X_3\}, \{X_3\}\big\}.$$

Let $D = \{X_3\} \subseteq V^{(\alpha)}$. For each $x \in V^{(\alpha)} \backslash D = \{X_1, X_2\}$, the shadow superedge $\{X_1, X_2, X_3\} \in E^{(\alpha)}$ contains x and intersects D. Hence D is an α-dominating set of $\mathcal{S}_M^{(1)}$ (Definition 5.8.17), so $\gamma^{(0.7)}\big(\mathcal{S}_M^{(1)}\big) = 1$.

Chapter 6

Graph Hierarchical Parameter

Intuitively, an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ is built over a base set V_0 by iterating powersets. Each n-supervertex $x \in V \subseteq \mathrm{PS}^n(V_0)$ therefore contains lower-level objects (level-r constituents) for every $0 \leq r \leq n$. This enables *hierarchical parameters* that measure interactions between objects appearing at different levels (e.g., base vertices vs. supervertices, or base vertices vs. superedges).

Definition 6.0.1 (Level-r constituents and base support)**.** Fix a finite nonempty base set V_0 and $n \geq 0$. For $0 \leq r \leq n$ and a nonempty element $x \in \mathrm{PS}^n(V_0) \setminus \{\emptyset\}$, define the *set of level-r constituents* $\mathrm{Cont}_r(x) \subseteq \mathrm{PS}^r(V_0)$ recursively as follows:

$$\mathrm{Cont}_n(x) \;:=\; \{x\}, \qquad \mathrm{Cont}_r(x) \;:=\; \bigcup_{y \in x} \mathrm{Cont}_r(y) \quad (0 \leq r < n),$$

where $y \in x$ is viewed as an element of $\mathrm{PS}^{n-1}(V_0)$. The *base support* of x is the level-0 constituent set

$$\mathrm{Supp}(x) \;:=\; \mathrm{Cont}_0(x) \;\subseteq\; V_0.$$

For a superedge $F \subseteq V$ in an n-SuperHyperGraph, define its level-r constituents and base support by

$$\mathrm{Cont}_r(F) \;:=\; \bigcup_{x \in F} \mathrm{Cont}_r(x), \qquad \mathrm{Supp}(F) \;:=\; \bigcup_{x \in F} \mathrm{Supp}(x) \;\subseteq\; V_0.$$

Definition 6.0.2 (Graph hierarchical parameter)**.** Fix $n \geq 0$. A *graph hierarchical parameter* on n-SuperHyperGraphs is any map

$$\Theta : \{n\text{-SuperHyperGraphs}\} \longrightarrow \mathbb{R}$$

whose value $\Theta(\mathrm{SHG}^{(n)})$ may depend on *cross-level* relations encoded by $\mathrm{Cont}_r(\cdot)$ (or $\mathrm{Supp}(\cdot)$), and which is invariant under relabellings of the base set V_0 (i.e., under bijections $V_0 \to V_0'$ extended canonically to all iterated powersets).

Concrete examples are provided below.

Example 6.0.3 (A running 2-SuperHyperGraph)**.** Let $V_0 = \{1, 2, 3, 4\}$ and take $n = 2$. Define three 2-supervertices (elements of $\mathrm{PS}^2(V_0) = \mathrm{PS}(\mathrm{PS}(V_0))$) by

$$x_1 := \{\{1, 2\}, \{3\}\}, \qquad x_2 := \{\{2\}, \{3, 4\}\}, \qquad x_3 := \{\{1, 4\}\},$$

and set

$$V := \{x_1, x_2, x_3\} \subseteq \mathrm{PS}^2(V_0).$$

Define two superedges by

$$F_1 := \{x_1, x_2\}, \qquad F_2 := \{x_2, x_3\}, \qquad E := \{F_1, F_2\} \subseteq \mathrm{PS}(V) \setminus \{\emptyset\}.$$

Then $\mathrm{SHG}^{(2)} = (V, E)$ is a 2-SuperHyperGraph.

By Definition 6.0.1, the base supports of the supervertices are

$$\mathrm{Supp}(x_1) = \{1, 2, 3\}, \qquad \mathrm{Supp}(x_2) = \{2, 3, 4\}, \qquad \mathrm{Supp}(x_3) = \{1, 4\},$$

and the base supports of the superedges are

$$\mathrm{Supp}(F_1) = \mathrm{Supp}(x_1) \cup \mathrm{Supp}(x_2) = \{1, 2, 3, 4\}, \qquad \mathrm{Supp}(F_2) = \mathrm{Supp}(x_2) \cup \mathrm{Supp}(x_3) = \{1, 2, 3, 4\}.$$

Example 6.0.4 (Hierarchical parameter #1: average base-size of a supervertex)**.** For an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$, define

$$\Theta_1(\mathrm{SHG}^{(n)}) \;:=\; \frac{1}{|V|} \sum_{x \in V} |\,\mathrm{Supp}(x)|.$$

This measures how many base vertices a typical supervertex "covers".

For the $\mathrm{SHG}^{(2)}$ in Example 6.0.3,

$$\Theta_1(\mathrm{SHG}^{(2)}) = \frac{|\{1, 2, 3\}| + |\{2, 3, 4\}| + |\{1, 4\}|}{3} = \frac{3 + 3 + 2}{3} = \frac{8}{3}.$$

Example 6.0.5 (Hierarchical parameter #2: average base-footprint of a superedge)**.** For an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$, define

$$\Theta_2(\mathrm{SHG}^{(n)}) \;:=\; \frac{1}{|E|} \sum_{F \in E} |\,\mathrm{Supp}(F)|.$$

This quantifies the average number of base vertices involved in a superedge (after flattening to level 0).

For the $\mathrm{SHG}^{(2)}$ in Example 6.0.3, we have $|\,\mathrm{Supp}(F_1)| = |\,\mathrm{Supp}(F_2)| = 4$, hence

$$\Theta_2(\mathrm{SHG}^{(2)}) = \frac{4 + 4}{2} = 4.$$

Example 6.0.6 (Hierarchical parameter #3: base–supervertex incidence density)**.** For an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ on base set V_0, define the bipartite incidence relation

$$v \prec x \quad :\Longleftrightarrow \quad v \in \mathrm{Supp}(x) \qquad (v \in V_0,\ x \in V),$$

and define the *incidence density*

$$\Theta_3(\mathrm{SHG}^{(n)}) \ :=\ \mathrm{IncDen}(\mathrm{SHG}^{(n)}) \ :=\ \frac{1}{|V_0||V|} \sum_{x \in V} |\,\mathrm{Supp}(x)|.$$

Thus Θ_3 is the fraction of all pairs $(v, x) \in V_0 \times V$ for which v belongs to the base support of x.

For the $\mathrm{SHG}^{(2)}$ in Example 6.0.3,

$$\sum_{x \in V} |\,\mathrm{Supp}(x)| = 3 + 3 + 2 = 8, \qquad |V_0||V| = 4 \cdot 3 = 12,$$

so

$$\Theta_3(\mathrm{SHG}^{(2)}) = \frac{8}{12} = \frac{2}{3}.$$

Example 6.0.7 (Hierarchical parameter #4: uniformity of base coverage)**.** For an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ on base set V_0, define the *coverage count* of a base vertex $v \in V_0$ by

$$\mathrm{cov}(v) \ :=\ \big|\{\, x \in V :\ v \in \mathrm{Supp}(x) \,\}\big|.$$

Define the *coverage range* parameter

$$\Theta_4(\mathrm{SHG}^{(n)}) \ :=\ \max_{v \in V_0} \mathrm{cov}(v) \ -\ \min_{v \in V_0} \mathrm{cov}(v).$$

Then $\Theta_4 = 0$ means every base vertex participates in the same number of supervertices.

For the $\mathrm{SHG}^{(2)}$ in Example 6.0.3,

$$\mathrm{cov}(1) = 2\ (x_1, x_3), \quad \mathrm{cov}(2) = 2\ (x_1, x_2), \quad \mathrm{cov}(3) = 2\ (x_1, x_2), \quad \mathrm{cov}(4) = 2\ (x_2, x_3),$$

so $\max \mathrm{cov}(v) = \min \mathrm{cov}(v) = 2$ and therefore

$$\Theta_4(\mathrm{SHG}^{(2)}) = 0.$$

Example 6.0.8 (Hierarchical parameter #5: average Jaccard overlap along superedges)**.** For an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$, define the Jaccard overlap of two supervertices $x \neq y$ by

$$\mathrm{Jacc}(x, y) \ :=\ \frac{|\,\mathrm{Supp}(x) \cap \mathrm{Supp}(y)|}{|\,\mathrm{Supp}(x) \cup \mathrm{Supp}(y)|}.$$

For a superedge $F \in E$ with $|F| \geq 2$, define its average internal overlap by

$$\mathrm{Jacc}(F) \ :=\ \frac{1}{\binom{|F|}{2}} \sum_{\{x,y\} \subseteq F} \mathrm{Jacc}(x, y),$$

and define the global overlap parameter

$$\Theta_5(\mathrm{SHG}^{(n)}) \ :=\ \frac{1}{|E|} \sum_{F \in E} \mathrm{Jacc}(F).$$

This measures, on average, how strongly supervertices in the same superedge overlap at the base level.

For the $\mathrm{SHG}^{(2)}$ in Example 6.0.3, each F_i has size 2, so $\mathrm{Jacc}(F_i) = \mathrm{Jacc}(x, y)$ for its unique pair:

$$\mathrm{Jacc}(F_1) = \mathrm{Jacc}(x_1, x_2) = \frac{|\{1,2,3\} \cap \{2,3,4\}|}{|\{1,2,3\} \cup \{2,3,4\}|} = \frac{2}{4} = \frac{1}{2},$$

$$\mathrm{Jacc}(F_2) = \mathrm{Jacc}(x_2, x_3) = \frac{|\{2,3,4\} \cap \{1,4\}|}{|\{2,3,4\} \cup \{1,4\}|} = \frac{1}{4}.$$

Hence

$$\Theta_5(\mathrm{SHG}^{(2)}) = \frac{1}{2}\Big(\frac{1}{2} + \frac{1}{4}\Big) = \frac{3}{8}.$$

Chapter 7

Conclusion

In this book, we investigated a wide range of properties and parameters of SuperHyperGraphs. In future work, we aim to study additional properties and parameters beyond those introduced here, and to further examine the concrete mathematical characteristics of the notions defined in this book.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

Appendix A

Appendix: Triameter and Multiameter of a graphs

In this appendix, we investigate notions such as the triameter and multiameter of a graph.

A.1 Triameter and Multiameter of a graphs

The triameter of a connected graph is the maximum, over all vertex triples, of the sum of their pairwise distances. The k-multiameter of a connected graph is the maximum, over all k-vertex sets, of the total pairwise distance sum.

Definition A.1.1 (Triameter)**.** Let $G = (V, E)$ be a finite simple graph. For $u, v \in V$, let $d_G(u, v)$ denote the usual graph distance (the length of a shortest u–v path), with the convention $d_G(u, v) = \infty$ if u and v lie in different components.

The *triameter* of G is

$$\operatorname{tr}(G) \;:=\; \max\Big\{\, d_G(u,v) + d_G(v,w) + d_G(u,w) \;\;\Big|\;\; u,v,w \in V \Big\} \;\in\; \mathbb{N} \cup \{\infty\}.$$

Equivalently, if G is connected then $\operatorname{tr}(G)$ is the maximum, over all triples of vertices, of the sum of their three pairwise distances. A triple $\{u, v, w\}$ attaining $\operatorname{tr}(G)$ is called a *triametral triple.*

Definition A.1.2 (k-multiameter)**.** Let $G = (V, E)$ be a finite simple graph and let $k \in \{2, 3, \dots, |V|\}$. Using the same distance convention as in Definition A.1.1, define the *k-multiameter* of G by

$$\operatorname{mtr}_k(G) \;:=\; \max\left\{ \sum_{\{x,y\}\subseteq S} d_G(x,y) \;\;\middle|\;\; S \subseteq V,\ |S| = k \right\} \;\in\; \mathbb{N} \cup \{\infty\}.$$

If G is connected, this is the maximum possible sum of all $\binom{k}{2}$ pairwise distances among any k vertices. A k-set $S \subseteq V$ attaining $\operatorname{mtr}_k(G)$ is called a *k-multiametral set.*

Remark A.1.3 (Consistency with diameter and triameter)**.** For every finite graph G we have

$$\operatorname{mtr}_2(G) = \operatorname{diam}(G) \qquad \text{and} \qquad \operatorname{mtr}_3(G) = \operatorname{tr}(G),$$

so the k-multiameter simultaneously extends the usual diameter ($k = 2$) and the triameter ($k = 3$).

A.2 Tristance and Multistance of a graphs

Distance can be extended to *tristance* and *multistance* in an analogous manner. We present the definitions below.

Definition A.2.1 (Distance in a graph)**.** Let $G = (V, E)$ be a finite simple undirected graph. For $u, v \in V$, a *u–v path* is a sequence of vertices $u = x_0, x_1, \dots, x_\ell = v$ with $\{x_{i-1}, x_i\} \in E$ for all i. If no u–v path exists, set $d_G(u, v) := \infty$. Otherwise, define the *(geodesic) distance* by

$$d_G(u, v) \;:=\; \min\{\ell : \text{ there exists a } u\text{–}v \text{ path of length } \ell\}.$$

By convention $d_G(v, v) = 0$ for all $v \in V$.

Definition A.2.2 (Steiner distance of a vertex set)**.** Let $G = (V, E)$ be a finite graph and let $\emptyset \neq S \subseteq V$. If the vertices of S do not lie in a common connected component of G, set

$$d_G(S) \;:=\; \infty.$$

Otherwise define the *Steiner distance* (or *distance of the set* S) by

$$d_G(S) \;:=\; \min\Big\{|E(T)| : \; T \text{ is a connected subgraph of } G \text{ with } S \subseteq V(T)\Big\}.$$

Any connected subgraph T attaining this minimum is necessarily a tree; such a minimizer is called a *Steiner tree* for S in G.

Definition A.2.3 (Tristance and multistance)**.** Let $G = (V, E)$ be a finite graph.

1. For three (distinct) vertices $u, v, w \in V$, the *tristance* of (u, v, w) is

$$\operatorname{trdist}_G(u, v, w) \;:=\; d_G(\{u, v, w\}) \;\in\; \mathbb{N} \cup \{\infty\}.$$

2. More generally, for an integer $k \geq 2$ and (distinct) vertices $v_1, \dots, v_k \in V$, the *k-multistance* is

$$\operatorname{multdist}_G(v_1, \dots, v_k) \;:=\; d_G(\{v_1, \dots, v_k\}) \;\in\; \mathbb{N} \cup \{\infty\}.$$

Remark A.2.4 (Consistency with ordinary distance)**.** If $S = \{u, v\}$, then $d_G(S) = d_G(u, v)$. Indeed, every connected subgraph containing u and v contains a u–v path, hence has at least $d_G(u, v)$ edges; equality is achieved by any shortest u–v path.

A.3 Trifference and Multifference

Likewise, in the same spirit, the notion of difference can also be extended to trifference and multifference. Let (X, d) be a metric space. For a finite (simple) tree $T = (V(T), E(T))$ with $V(T) \subseteq X$, define its d-length by

$$\ell_d(T) \; := \sum_{\{u,v\} \in E(T)} d(u, v).$$

Definition A.3.1 (Difference)**.** For $x, y \in X$, the *difference* induced by d is

$$\operatorname{diff}(x, y) \; := \; d(x, y).$$

Definition A.3.2 (Trifference)**.** For $x, y, z \in X$, the *trifference* induced by d is the length of a shortest (Steiner) tree connecting $\{x, y, z\}$:

$$\operatorname{triff}(x, y, z) \; := \; \inf\Big\{\ell_d(T) \; : \; T \text{ is a finite tree with } \{x, y, z\} \subseteq V(T) \subseteq X\Big\}.$$

Equivalently, one may write the explicit form

$$\operatorname{triff}(x, y, z) = \min\Big\{d(x, y)+d(y, z),\; d(x, y)+d(x, z),\; d(x, z)+d(y, z),\; \inf_{p \in X}\big(d(x, p)+d(y, p)+d(z, p)\big)\Big\},$$

where the first three terms correspond to spanning trees on $\{x, y, z\}$ and the last term corresponds to a star through a Steiner point p.

Definition A.3.3 (Multifference)**.** Fix an integer $k \geq 2$ and points $x_1, \dots, x_k \in X$. The *multifference* induced by d is

$$\operatorname{multiff}(x_1, \dots, x_k) \; := \; \inf\Big\{\ell_d(T) \; : \; T \text{ is a finite tree with } \{x_1, \dots, x_k\} \subseteq V(T) \subseteq X\Big\}.$$

Remark A.3.4.

- For $k = 2$, $\operatorname{multiff}(x_1, x_2) = \operatorname{diff}(x_1, x_2) = d(x_1, x_2)$.
- If one *forbids* Steiner points (i.e., enforces $V(T) = \{x_1, \dots, x_k\}$), then multiff reduces to the weight of a minimum spanning tree on the complete graph with edge-weights $d(x_i, x_j)$.

A.4 Trisplacement and Multisplacement

Likewise, displacement can be extended to trisplacement and multisplacement. Let $\mathbb{A}$ be an affine space over a real vector space V (e.g., $\mathbb{A} = \mathbb{R}^n$ and $V = \mathbb{R}^n$). For $p, q \in \mathbb{A}$, denote by $\overrightarrow{pq} \in V$ the displacement (translation) vector from p to q.

Definition A.4.1 (Displacement)**.** For $p, q \in \mathbb{A}$, the *displacement* from p to q is the vector

$$\operatorname{disp}(p, q) \ := \ \overrightarrow{pq} \ \in V.$$

In coordinates (when $\mathbb{A} = \mathbb{R}^n$), this is $\operatorname{disp}(p, q) = q - p$.

Definition A.4.2 (Trisplacement)**.** For three successive positions $p_0, p_1, p_2 \in \mathbb{A}$ (e.g., at times $t_0 < t_1 < t_2$), the *trisplacement* is the ordered pair of successive displacement vectors

$$\operatorname{trisp}(p_0, p_1, p_2) \ := \ \big(\operatorname{disp}(p_0, p_1), \ \operatorname{disp}(p_1, p_2)\big) \ \in V^2.$$

Definition A.4.3 (Multisplacement)**.** For a finite sequence of positions $p_0, p_1, \ldots, p_m \in \mathbb{A}$ with $m \geq 1$, the *multisplacement* is the list of successive displacement vectors

$$\operatorname{multisp}(p_0, p_1, \ldots, p_m) \ := \ \big(\operatorname{disp}(p_0, p_1), \ \operatorname{disp}(p_1, p_2), \ \ldots, \ \operatorname{disp}(p_{m-1}, p_m)\big) \ \in V^m.$$

Remark A.4.4. The *resultant (net) displacement* from p_0 to p_m is recovered by summation:

$$\operatorname{disp}(p_0, p_m) \ = \ \sum_{i=1}^{m} \operatorname{disp}(p_{i-1}, p_i),$$

which expresses that displacement depends only on the initial and final positions, not on the intermediate path subdivision. In Euclidean space, $\|\operatorname{disp}(p, q)\|$ equals the straight-line (Euclidean) distance between p and q.

Details are omitted here, but by the same line of reasoning, one may regard derivation (in differential algebra) as extendable to tri-differential and multi-differential algebras, and differentiation as extendable to tri-differentiation and multi-differentiation.

Appendix B

Appendix: Bi-domination in a graph

A bi-dominating set dominates all vertices outside it, and each chosen vertex has exactly two outside neighbors in the graph [256, 257, 432, 433]. A tri-dominating set dominates every vertex outside it, and each selected vertex has exactly three outside neighbors in the graph. An m-dominating set dominates all outside vertices, and each vertex in the set has exactly m neighbors outside the graph.

Definition B.0.1 (Bi-domination in a graph)**.** Let $G = (V, E)$ be a finite simple graph. For $v \in V$, write

$$N_G(v) := \{u \in V : \{u, v\} \in E\}$$

for the open neighborhood of v. A set $D \subseteq V$ is called a *bi-dominating set* of G if

1. (*domination*) every vertex outside D is adjacent to some vertex of D, i.e.,

$$\forall\, x \in V \setminus D \quad \exists\, d \in D \ \text{ such that } \ \{x, d\} \in E,$$

2. (*bi-condition*) every vertex of D has exactly two neighbors outside D, i.e.,

$$\forall\, d \in D \qquad \big|N_G(d) \cap (V \setminus D)\big| = 2.$$

This formulation follows a standard usage of *bi-domination* in the domination literature (e.g., the "exactly two outside neighbors" condition).

The *bi-domination number* of G is

$$\gamma_{\mathrm{bi}}(G) \;:=\; \begin{cases} \min\{\, |D| : \ D \subseteq V \text{ is a bi-dominating set of } G\,\}, & \text{if such } D \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Definition B.0.2 (Tri-domination in a graph)**.** Let $G = (V, E)$ be a finite simple graph. A set $D \subseteq V$ is called a *tri-dominating set* of G if

1. D is a dominating set of G, i.e.,
$$\forall\, x \in V \setminus D \quad \exists\, d \in D \;\text{ such that }\; \{x, d\} \in E,$$

2. every vertex of D has exactly three neighbors outside D, i.e.,
$$\forall\, d \in D \qquad \big|N_G(d) \cap (V \setminus D)\big| = 3.$$

The *tri-domination number* of G is
$$\gamma_{\text{tri}}(G) \;:=\; \begin{cases} \min\{\,|D| :\; D \subseteq V \text{ is a tri-dominating set of } G\,\}, & \text{if such } D \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Definition B.0.3 (Multidomination (multi-m-domination) in a graph)**.** Let $G = (V, E)$ be a finite simple graph and let $m \in \mathbb{N}$ with $m \geq 2$. A set $D \subseteq V$ is called an *m-dominating set* (or *multi-dominating set of order m*) if

1. D is a dominating set of G, i.e.,
$$\forall\, x \in V \setminus D \quad \exists\, d \in D \;\text{ such that }\; \{x, d\} \in E,$$

2. every vertex of D has exactly m neighbors outside D, i.e.,
$$\forall\, d \in D \qquad \big|N_G(d) \cap (V \setminus D)\big| = m.$$

The corresponding *m-domination number* is
$$\gamma_m(G) \;:=\; \begin{cases} \min\{\,|D| :\; D \subseteq V \text{ is an } m\text{-dominating set of } G\,\}, & \text{if such } D \text{ exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

The term *multidomination* may be used to refer collectively to the family $\{\gamma_m(G)\}_{m \geq 2}$ of domination parameters obtained by varying m.

Appendix C

Appendix: Iterated Refined Uncertain Graph

In this appendix, we define *iterated refined neutrosophic graphs* and *iterated refined uncertain graphs*. A variety of refined and iterated-refined uncertainty formalisms have been studied, including refined fuzzy sets [411, 412], refined neutrosophic sets [412, 434–438], Multi-valued Neutrosophic Sets [403], Multi-partitioned Neutrosophic Sets [439, 440], and iterated refined neutrosophic sets [441]; moreover, refined graph-theoretic counterparts, such as refined neutrosophic graphs [442], have also been investigated.

Definition C.0.1 (Refined Neutrosophic Graph)**.** (cf. [442]) Let $G = (V, E)$ be a simple graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Refined Neutrosophic Graph* assigns neutrosophic values to both vertices and edges, where each vertex and each edge has:

- A membership degree split into r values $\mu_1, \mu_2, \ldots, \mu_r$,
- An indeterminacy degree split into s values $\sigma_1, \sigma_2, \ldots, \sigma_s$,
- A non-membership degree split into t values $\nu_1, \nu_2, \ldots, \nu_t$,

such that for each vertex or edge x, the following condition holds:

$$0 \leq \sum_{i=1}^{r} \mu_i(x) + \sum_{i=1}^{s} \sigma_i(x) + \sum_{i=1}^{t} \nu_i(x) \leq n,$$

where $n = r + s + t$.

For each vertex $v \in V$, a neutrosophic n-valued refined set (μ_v, σ_v, ν_v) is assigned, where

$$\mu_v = (\mu_1(v), \ldots, \mu_r(v)), \quad \sigma_v = (\sigma_1(v), \ldots, \sigma_s(v)), \quad \nu_v = (\nu_1(v), \ldots, \nu_t(v)).$$

Similarly, for each edge $e = (u, v) \in E$, a neutrosophic n-valued refined set (μ_e, σ_e, ν_e) is assigned, where

$$\mu_e = (\mu_1(e), \ldots, \mu_r(e)), \quad \sigma_e = (\sigma_1(e), \ldots, \sigma_s(e)), \quad \nu_e = (\nu_1(e), \ldots, \nu_t(e)).$$

Definition C.0.2 (Iterated refinement scheme)**.** Fix an integer $n \geq 0$. An *iterated refinement scheme of depth* n is a finite rooted tree $\mathcal{R}$ together with a typing map

$$\tau : \ \mathsf{N}(\mathcal{R}) \setminus \{\rho\} \longrightarrow \{T, I, F\},$$

where $\mathsf{N}(\mathcal{R})$ is the node set, ρ is the root, and the following hold:

(i) The root has exactly three children c_T, c_I, c_F, and

$$\tau(c_T) = T, \qquad \tau(c_I) = I, \qquad \tau(c_F) = F.$$

(ii) Every non-root node $x \neq \rho$ has a (possibly empty) finite set of children $\mathrm{Ch}(x) \subseteq \mathsf{N}(\mathcal{R})$. If $\mathrm{Ch}(x) \neq \emptyset$, then every child $y \in \mathrm{Ch}(x)$ has the *same type* as x, i.e.,

$$\tau(y) = \tau(x) \qquad (\forall y \in \mathrm{Ch}(x)).$$

(iii) The height of $\mathcal{R}$ is at most n (equivalently, every root-to-leaf path has length $\leq n + 1$).

The *leaf set* is $\mathcal{L}(\mathcal{R}) \subseteq \mathsf{N}(\mathcal{R})$, and we also write the type-separated leaf sets

$$\mathcal{L}_T(\mathcal{R}) := \{x \in \mathcal{L}(\mathcal{R}) : \tau(x) = T\}, \quad \mathcal{L}_I(\mathcal{R}) := \{x \in \mathcal{L}(\mathcal{R}) : \tau(x) = I\}, \quad \mathcal{L}_F(\mathcal{R}) := \{x \in \mathcal{L}(\mathcal{R}) : \tau(x) = F\}.$$

Remark C.0.3. Intuitively, $\mathcal{R}$ encodes how many times (and into how many parts) the truth-, indeterminacy-, and falsity-components are refined. A leaf in $\mathcal{L}_T(\mathcal{R})$ represents a final (fully refined) truth-subcomponent, and similarly for $\mathcal{L}_I(\mathcal{R})$ and $\mathcal{L}_F(\mathcal{R})$. Depth $n = 0$ corresponds to *no refinement*: the tree consists of the root and three children only, hence $|\mathcal{L}_T| = |\mathcal{L}_I| = |\mathcal{L}_F| = 1$.

Definition C.0.4 (Iterated refined neutrosophic set)**.** Let U be a nonempty universe, let $\mathcal{R}$ be an iterated refinement scheme (Definition C.0.2), and fix a constant $M > 0$ (typically $M = 3$ in the single-valued setting). An *iterated refined neutrosophic set* on U (of type $\mathcal{R}$) is a mapping

$$A : \ U \longrightarrow [0,1]^{\mathcal{L}_T(\mathcal{R})} \times [0,1]^{\mathcal{L}_I(\mathcal{R})} \times [0,1]^{\mathcal{L}_F(\mathcal{R})},$$

written componentwise as

$$A(x) = \Big(\ (T_\ell(x))_{\ell \in \mathcal{L}_T(\mathcal{R})}, \ (I_\ell(x))_{\ell \in \mathcal{L}_I(\mathcal{R})}, \ (F_\ell(x))_{\ell \in \mathcal{L}_F(\mathcal{R})} \ \Big),$$

such that, for every $x \in U$,

$$0 \ \leq \ \sum_{\ell \in \mathcal{L}_T(\mathcal{R})} T_\ell(x) \ + \ \sum_{\ell \in \mathcal{L}_I(\mathcal{R})} I_\ell(x) \ + \ \sum_{\ell \in \mathcal{L}_F(\mathcal{R})} F_\ell(x) \ \leq \ M. \tag{C.1}$$

Remark C.0.5. (i) When $\mathcal{R}$ has depth 0, Definition C.0.4 reduces to an ordinary (single-valued) neutrosophic set with three components (T, I, F).

(ii) When $\mathcal{R}$ has depth 1 and $|\mathcal{L}_T| = p$, $|\mathcal{L}_I| = r$, $|\mathcal{L}_F| = s$, the resulting object is the usual *refined* (single-valued) neutrosophic set with components

$$(T_1, \ldots, T_p;\ I_1, \ldots, I_r;\ F_1, \ldots, F_s),$$

together with the global bound (C.1).

(iii) Some authors adopt a looser global upper bound in (C.1), e.g. $M = |\mathcal{L}_T| + |\mathcal{L}_I| + |\mathcal{L}_F|$. Keeping $M = 3$ emphasizes that refinement is a *partitioning of the classical three degrees* rather than introducing fully independent new degrees. Both conventions are mathematically consistent; you should fix one convention and use it uniformly.

Example C.0.6 (A small depth-2 iterated refined neutrosophic set)**.** Let $U = \{x_1, x_2\}$. Take a depth-2 scheme $\mathcal{R}$ such that

$$\mathcal{L}_T(\mathcal{R}) = \{t_{11}, t_{12}, t_{21}, t_{22}\}, \qquad \mathcal{L}_I(\mathcal{R}) = \{i_1, i_2\}, \qquad \mathcal{L}_F(\mathcal{R}) = \{f_1\}.$$

Define $A : U \to [0,1]^4 \times [0,1]^2 \times [0,1]$ by

$$A(x_1) = \big((0.30, 0.10, 0.05, 0.05), (0.20, 0.10), (0.40)\big),$$

$$A(x_2) = \big((0.05, 0.05, 0.10, 0.10), (0.10, 0.20), (0.30)\big).$$

Then, for $M = 3$, the bound (C.1) holds, e.g. $0.30 + 0.10 + 0.05 + 0.05 + 0.20 + 0.10 + 0.40 = 1.20 \leq 3$ for x_1. Thus A is an iterated refined neutrosophic set of type $\mathcal{R}$.

Definition C.0.7 (Iterated refined neutrosophic graph)**.** Let $G^* = (V, E)$ be a finite simple (undirected) graph, let $\mathcal{R}$ be an iterated refinement scheme, and fix $M > 0$ (typically $M = 3$). An *iterated refined neutrosophic graph* (IRNG) on G^* (of type $\mathcal{R}$) is a pair

$$G = (A, B)$$

where

$$A :\ V \longrightarrow [0,1]^{\mathcal{L}_T(\mathcal{R})} \times [0,1]^{\mathcal{L}_I(\mathcal{R})} \times [0,1]^{\mathcal{L}_F(\mathcal{R})},$$
$$B :\ E \longrightarrow [0,1]^{\mathcal{L}_T(\mathcal{R})} \times [0,1]^{\mathcal{L}_I(\mathcal{R})} \times [0,1]^{\mathcal{L}_F(\mathcal{R})},$$

written as

$$A(v) = \Big(\ (T_\ell(v))_{\ell \in \mathcal{L}_T},\ (I_\ell(v))_{\ell \in \mathcal{L}_I},\ (F_\ell(v))_{\ell \in \mathcal{L}_F}\ \Big),$$

$$B(uv) = \Big(\ (T_\ell(uv))_{\ell \in \mathcal{L}_T},\ (I_\ell(uv))_{\ell \in \mathcal{L}_I},\ (F_\ell(uv))_{\ell \in \mathcal{L}_F}\ \Big),$$

such that:

(i) (*Single-valued bounds on vertices and edges*) For every $v \in V$ and $uv \in E$,

$$0 \leq \sum_{\ell \in \mathcal{L}_T} T_\ell(v) + \sum_{\ell \in \mathcal{L}_I} I_\ell(v) + \sum_{\ell \in \mathcal{L}_F} F_\ell(v) \leq M,$$

$$0 \leq \sum_{\ell \in \mathcal{L}_T} T_\ell(uv) + \sum_{\ell \in \mathcal{L}_I} I_\ell(uv) + \sum_{\ell \in \mathcal{L}_F} F_\ell(uv) \leq M.$$

(ii) (*Edge-vertex consistency constraints*) For every edge $uv \in E$,

$$\begin{aligned} T_\ell(uv) &\le \min\{T_\ell(u), T_\ell(v)\} && (\forall \ell \in \mathcal{L}_T), \\ I_\ell(uv) &\le \min\{I_\ell(u), I_\ell(v)\} && (\forall \ell \in \mathcal{L}_I), \\ F_\ell(uv) &\le \max\{F_\ell(u), F_\ell(v)\} && (\forall \ell \in \mathcal{L}_F). \end{aligned}$$

Remark C.0.8. (i) When $\mathcal{R}$ has depth 0, Definition C.0.7 reduces to the standard single-valued neutrosophic graph model with three components (T, I, F) and the usual min/min/max constraints.

(ii) If your manuscript uses a different convention for the indeterminacy constraint (some works prefer a max-type condition), you may replace $I_\ell(uv) \le \min\{I_\ell(u), I_\ell(v)\}$ by $I_\ell(uv) \le \max\{I_\ell(u), I_\ell(v)\}$. What matters is that you (a) state the rule explicitly and (b) use it consistently in all definitions and proofs.

Example C.0.9 (A minimal IRNG)**.** Let G^* be the single edge on $V = \{v_1, v_2\}$ with $E = \{v_1v_2\}$. Use the same leaf sets as in Example C.0.6 and take $M = 3$. Define vertex assignments

$$A(v_1) = \big((0.30, 0.10, 0.05, 0.05), (0.20, 0.10), (0.40)\big), \qquad A(v_2) = \big((0.20, 0.05, 0.05, 0.05), (0.10, 0.10), (0.60)\big),$$

and define the edge assignment by componentwise admissible choices, e.g.

$$B(v_1v_2) = \big((0.20, 0.05, 0.05, 0.05), (0.10, 0.10), (0.50)\big).$$

Then for each truth leaf $\ell \in \mathcal{L}_T$ we have $T_\ell(v_1v_2) \le \min\{T_\ell(v_1), T_\ell(v_2)\}$, for each indeterminacy leaf $\ell \in \mathcal{L}_I$ we have $I_\ell(v_1v_2) \le \min\{I_\ell(v_1), I_\ell(v_2)\}$, and for the falsity leaf $f_1 \in \mathcal{L}_F$ we have $F_{f_1}(v_1v_2) = 0.50 \le \max\{0.40, 0.60\} = 0.60$. Hence $G = (A, B)$ is an iterated refined neutrosophic graph of type $\mathcal{R}$.

Definition C.0.10 (n-refinement of an uncertain model)**.** Let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, and fix an integer $n \ge 1$. Define the *aggregation map*

$$\mathrm{Agg}_n : [0,1]^{kn} \longrightarrow [0,1]^k, \qquad \mathrm{Agg}_n\big((x_{i,j})_{1\le i\le k,\ 1\le j\le n}\big) := \Big(\frac{1}{n}\sum_{j=1}^{n} x_{i,j}\Big)_{i=1}^{k}.$$

The *n-refined uncertain model* $\mathrm{REF}_n(M)$ is the uncertain model whose degree-domain is

$$\mathrm{Dom}(\mathrm{REF}_n(M)) := \Big\{x \in [0,1]^{kn} \ :\ \mathrm{Agg}_n(x) \in \mathrm{Dom}(M)\Big\}.$$

Definition C.0.11 (Iterated n-refinement)**.** Let M be an uncertain model and fix $n \ge 1$. Define $\mathrm{REF}_n^1(M) := \mathrm{REF}_n(M)$ and recursively

$$\mathrm{REF}_n^{t+1}(M) := \mathrm{REF}_n\big(\mathrm{REF}_n^t(M)\big) \qquad (t \ge 1).$$

Equivalently, $\mathrm{Dom}(\mathrm{REF}_n^t(M)) \subseteq [0,1]^{kn^t}$ consists of all vectors x whose blockwise t-fold averaged aggregation belongs to $\mathrm{Dom}(M)$.

Definition C.0.12 (Refined uncertain graph)**.** Let $G = (V, E)$ be a finite (undirected, loopless) graph, let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, and fix an integer $n \geq 1$. An *n-refined uncertain graph of type M* is a triple

$$\mathcal{G}^{[1]}_{M,n} \;:=\; (V, E, \mu^{[1]}),$$

where

$$\mu^{[1]} : V \cup E \longrightarrow \mathrm{Dom}\big(\mathrm{REF}_n(M)\big)$$

assigns to each vertex and edge an n-refined degree-tuple.

Definition C.0.13 (Iterated refined uncertain graph)**.** Let $G = (V, E)$ be a finite (undirected, loopless) graph, let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, and fix integers $n \geq 1$ and $t \geq 1$. An *t-times iterated n-refined uncertain graph of type M* is a triple

$$\mathcal{G}^{[t]}_{M,n} \;:=\; (V, E, \mu^{[t]}),$$

where

$$\mu^{[t]} : V \cup E \longrightarrow \mathrm{Dom}\big(\mathrm{REF}^{\,t}_n(M)\big) \;\subseteq\; [0,1]^{kn^t}.$$

Remark C.0.14. Definition C.0.13 is model-agnostic: it refines *any* uncertain model M by repeatedly subdividing each of its k degree-components into n subcomponents, while preserving membership admissibility via aggregation.

Definition C.0.15 (Neutrosophic model)**.** Let N denote the (single-valued) neutrosophic uncertain model with

$$\mathrm{Dom}(N) \;:=\; \{(T, I, F) \in [0,1]^3 : \; 0 \leq T + I + F \leq 3\}.$$

Lemma C.0.16 (Domain identification for iterated refined neutrosophic degrees)**.** *Fix integers $n \geq 1$ and $t \geq 1$. Then* $\mathrm{Dom}(\mathrm{REF}^{\,t}_n(N))$ *can be identified with the set of all tuples*

$$\big(T_1, \ldots, T_{n^t},\; I_1, \ldots, I_{n^t},\; F_1, \ldots, F_{n^t}\big) \;\in\; [0,1]^{3n^t}$$

satisfying the single constraint

$$0 \;\leq\; \sum_{j=1}^{n^t} T_j \;+\; \sum_{j=1}^{n^t} I_j \;+\; \sum_{j=1}^{n^t} F_j \;\leq\; 3n^t.$$

Proof. By Definition C.0.10 and Definition C.0.11, an element of $\mathrm{Dom}(\mathrm{REF}^{\,t}_{\underline{n}}(N)) \subseteq [0,1]^{3n^t}$ is admissible exactly when its t-fold blockwise aggregation yields a triple $(\overline{T}, \overline{I}, \overline{F}) \in \mathrm{Dom}(N)$. Under the natural ordering of coordinates into 3 blocks of length n^t,

$$\overline{T} = \frac{1}{n^t} \sum_{j=1}^{n^t} T_j, \quad \overline{I} = \frac{1}{n^t} \sum_{j=1}^{n^t} I_j, \quad \overline{F} = \frac{1}{n^t} \sum_{j=1}^{n^t} F_j.$$

Hence $(\overline{T}, \overline{I}, \overline{F}) \in \mathrm{Dom}(N)$ is equivalent to $0 \leq \overline{T} + \overline{I} + \overline{F} \leq 3$, i.e.,

$$0 \;\leq\; \frac{1}{n^t}\Big(\sum_{j=1}^{n^t} T_j + \sum_{j=1}^{n^t} I_j + \sum_{j=1}^{n^t} F_j\Big) \;\leq\; 3,$$

which is equivalent to the stated inequality after multiplying by n^t. □

Definition C.0.17 (Iterated refined neutrosophic graph)**.** Let $G = (V, E)$ be a finite (undirected, loopless) graph, and fix integers $n \geq 1$ and $t \geq 1$. An *t-times iterated n-refined neutrosophic graph* is a triple

$$\mathcal{G}^{[t]}_{\mathrm{Neu},n} \;:=\; (V, E, \nu^{[t]}),$$

where

$$\nu^{[t]} : V \cup E \longrightarrow [0,1]^{3n^t}$$

assigns to each $x \in V \cup E$ a degree-vector

$$\nu^{[t]}(x) = \big(T_1(x), \ldots, T_{n^t}(x),\; I_1(x), \ldots, I_{n^t}(x),\; F_1(x), \ldots, F_{n^t}(x)\big)$$

satisfying the admissibility constraint

$$0 \;\leq\; \sum_{j=1}^{n^t} T_j(x) \;+\; \sum_{j=1}^{n^t} I_j(x) \;+\; \sum_{j=1}^{n^t} F_j(x) \;\leq\; 3n^t \qquad (\forall x \in V \cup E).$$

Theorem C.0.18 (Iterated refined uncertain graphs generalize iterated refined neutrosophic graphs)**.** *Fix integers $n \geq 1$ and $t \geq 1$. Let N be the neutrosophic model from Definition C.0.15. Then the following classes coincide:*

$$\Big\{\mathcal{G}^{[t]}_{M,n} :\; M = N\Big\} \;=\; \Big\{\mathcal{G}^{[t]}_{\mathrm{Neu},n}\Big\}.$$

In particular, every iterated refined neutrosophic graph (Definition C.0.17) is an instance of an iterated refined uncertain graph (Definition C.0.13) by choosing the underlying uncertain model to be $M = N$.

Proof. Let $\mathcal{G}^{[t]}_{\mathrm{Neu},n} = (V, E, \nu^{[t]})$ be an iterated refined neutrosophic graph. By Lemma C.0.16, its codomain constraint is exactly

$$\nu^{[t]} : V \cup E \longrightarrow \mathrm{Dom}(\mathrm{REF}^t_n(N)).$$

Hence $\mathcal{G}^{[t]}_{\mathrm{Neu},n}$ is precisely an iterated refined uncertain graph $\mathcal{G}^{[t]}_{M,n}$ with $M = N$ (take $\mu^{[t]} := \nu^{[t]}$).

Conversely, if $\mathcal{G}^{[t]}_{M,n} = (V, E, \mu^{[t]})$ is an iterated refined uncertain graph with $M = N$, then $\mu^{[t]}$ maps into $\mathrm{Dom}(\mathrm{REF}^t_n(N))$, which again equals the usual $(3n^t)$-component iterated refined neutrosophic degree-domain by Lemma C.0.16. Therefore $\mathcal{G}^{[t]}_{M,n}$ is an iterated refined neutrosophic graph in the sense of Definition C.0.17. □

Appendix (List of Tables)

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A hypergraph extends an ordinary graph by permitting each edge to join an arbitrary nonempty subset of the vertex set. By iterating the powerset construction one additional level, one obtains nested (higher-order) vertex objects and, consequently, finite SuperHyperGraphs whose vertices and edges can themselves be set-valued across multiple layers. tudies specifically devoted to SuperHyperGraph properties and SuperHyperGraph parameters have not progressed substantially so far. To help increase the visibility of SuperHyperGraphs, this book investigates various graph properties and graph parameters in the SuperHyperGraph setting. In particular, we place special emphasis on well-studied graph parameters such as domination and topological indices.

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