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HYPERGRAPH AND SUPERHYPERGRAPH THEORY WITH APPLICATIONS III

INTERSECTION GRAPH AND GRAPH LABELING THEORY

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HyperGraph and SuperHyperGraph Theory with Applications (III): Intersection Graph and Graph Labeling Theory

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Abstract

Hypergraphs generalize ordinary graphs by allowing an edge to join an arbitrary nonempty subset of the vertex set. Iterating the powerset construction further yields nested, higher-order vertex objects and leads to finite *SuperHyperGraphs*, in which both vertices and edges may themselves be set-valued across multiple layers. Despite their expressive power, systematic investigations of *SuperHyperGraph* properties and parameters remain comparatively limited. In this book, we introduce and study several graph classes of *Intersection SuperHyperGraphs*, providing a unified framework for intersection-based constructions at higher levels. We also develop and analyze derived notions of *SuperHyperGraph labeling*, extending classical labeling paradigms to the superhypergraph setting. The present volume continues and expands the line of work initiated in [1].

Keywords: SuperHyperGraph, HyperGraph, Graph Labeling, Intersection Graphs

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Table of Contents

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Chapter 1

Introduction

1.1 Graph, HyperGraph, and SuperHyperGraph

Network models are classically expressed by *graphs*, in which objects are represented by vertices and binary relationships by edges [2]. While this abstraction is effective for pairwise interactions, it becomes restrictive when the underlying system exhibits *simultaneous* interactions among three or more entities. *Hypergraphs* resolve this limitation by permitting each hyperedge to join an arbitrary nonempty subset of vertices, thereby representing higher-order relations directly [3].

Even so, many real-world datasets and engineered systems display relationships that are not only higher-order but also *layered*, *nested*, and intrinsically *hierarchical*. To capture such multi-level incidence patterns, F. Smarandache introduced the notion of a *SuperHyperGraph*. Informally, a SuperHyperGraph is built via iterative powerset-based constructions, which allow vertices ("supervertices") themselves to be set-valued objects and enable edges to encode nested connectivity across multiple levels [4, 5]. Consequently, SuperHyperGraphs have recently attracted growing attention in both theory and applications [6–11].

Graphs and hypergraphs also provide transparent visual metaphors for complex systems and support a broad spectrum of applications in artificial intelligence, network science, data mining, informatics, chemistry, physics, and related fields [12–14]. By explicitly incorporating hierarchical and multi-level relationships, SuperHyperGraphs offer a flexible framework for modeling and analyzing intricate structures in modern networked data (e.g., [15–24]).

Table 1.1 highlights the essential differences among graphs, hypergraphs, and superhypergraphs. Throughout this book, n denotes a natural number unless stated otherwise.

A more concrete, side-by-side comparison of graphs, hypergraphs, and n-superhypergraphs is given in Table 1.2.

Table 1.1.: Key distinctions among graph, hypergraph, and superhypergraph.

Concept	*Notation*	*Edge Family*	*Core Extension Principle*
Graph [2]	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$	Edges encode *pairwise* (binary) relations between vertices.
Hypergraph [25]	$H = (V, \mathcal{E})$	$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Hyperedges may join *any* nonempty subset of vertices, encoding higher-order interactions.
Superhypergraph [4]	$\mathrm{SHG}^{(n)} = (V_0, V, E)$	$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Uses an n-fold powerset hierarchy to represent *nested* and *multi-level* incidence patterns.

Notation. $\mathcal{P}(X) = \{A \mid A \subseteq X\}$, and $\mathcal{P}^0(X) = X,\ \mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

Table 1.2.: A concrete comparison of graphs, hypergraphs, and n-superhypergraphs.

Aspect	*Graph* $G = (V, E)$	*Hypergraph* $H = (V, \mathcal{E})$	*n-SuperHyperGraph* $\mathrm{SHG}^{(n)} = (V_0, V, E)$
Vertices	$v \in V$	$v \in V$	$x \in V \subseteq \mathcal{P}^n(V_0)$ (nested supervertices allowed)
Edges	$E \subseteq \binom{V}{2}$	$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$
Incidence	$v \in e$	$v \in e$	$x \in \varepsilon$
Adjacency (typ.)	$\{u, v\} \in E$	$\exists e \in \mathcal{E} : \{u, v\} \subseteq e$	$\exists \varepsilon \in E : \{x, y\} \subseteq \varepsilon$
One edge encodes	pairwise relation	multiway relation	multiway relation among supervertices
Distance (typ.)	shortest-path	Berge / primal	super-Berge / primal
Use (typ.)	binary links	higher-order groups	hierarchical / nested incidence

Notation. $\mathcal{P}^0(X) = X$ and $\mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

1.2 Intersection graphs

Graph structures and graph classes constitute a central topic in graph theory [26]. Among them, *intersection graphs* play a prominent role: each vertex represents a set, and two vertices are adjacent precisely when the corresponding sets intersect [27, 28]. This viewpoint yields many well-studied graph classes, including interval graphs [29, 30], mixed interval graphs [31, 32], proper interval graphs [33, 34], weighted interval graphs [35, 36], unit disk graphs [37–39], and polygon-circle graphs [40]. Intersection graphs model systems where objects overlap: in biology, they link proteins sharing interaction partners; in scheduling [41, 42], they represent tasks sharing time/resources; in networks, they capture shared channels or frequencies, enabling efficient coloring and conflict-free allocation [43, 44]. Moreover, the intersection-graph paradigm has been extended using hypergraphs and superhypergraphs, leading to natural higher-order and hierarchical generalizations [45]. We also briefly examine *Complex-intersection graphs* and *multidimensional intersection graphs* as subclasses of intersection graphs in the Appendix.

1.3 Graph Labeling

Graph labeling assigns values to vertices and/or edges under specified constraints, encoding structure for optimization, identification, frequency assignment, symmetry breaking, and combinatorial analysis [46, 47]. As examples of graph labeling, various schemes are well known, including harmonious labeling [48, 49], graceful labeling [50, 51], magic labeling [52, 53], distance labeling [54–56], and prime labeling [57]. A closely related notion is *graph coloring* [58–63], which may be viewed as a labeling that assigns colors to vertices (or edges) under adjacency constraints. Moreover, these ideas naturally extend beyond graphs: by replacing edges with hyperedges or superedges, one obtains the corresponding notions of *hypergraph labeling* [64–66] and *SuperHyperGraph labeling*.

1.4 Our Contributions

In view of the above, a systematic study of SuperHyperGraphs and of topics related to intersection graphs is of considerable importance. Nonetheless, compared with the extensive literature on classical graph and hypergraph invariants, research focusing specifically on *SuperHyperGraph* properties and *SuperHyperGraph* parameters remains relatively limited. Accordingly, in this book we define and investigate several graph classes of *Intersection SuperHyperGraphs*. Moreover, we will conduct analogous research on derived notions of SuperHyperGraph labeling. The present book continues and expands the line of work initiated in [1]. As a further reference, an overview of intersection graphs, intersection hypergraphs, and intersection n-SuperHyperGraphs is provided in Table 1.3. Moreover, an overview of labeling frameworks for graphs, hypergraphs, and SuperHyperGraphs is provided in Table 1.4.

Table 1.3.: Brief overview of intersection graph, intersection hypergraph, and intersection n-SuperHyperGraph constructions.

Item	*Intersection Graph*	*Intersection HyperGraph*	*Intersection n-SuperHyperGraph*
Input data	A family of nonempty sets $\mathcal{S} = \{S_1, \dots, S_m\}$ in a universe U	A family of nonempty sets $\mathcal{S} = \{S_1, \dots, S_m\}$ in a universe U	A family of nonempty n-level objects $\mathcal{S} = \{X_1, \dots, X_m\}$ with $X_i \in \mathcal{P}^n(V_0) \setminus \{\emptyset\}$
Vertices	Indices $[m] = \{1, \dots, m\}$ (or equivalently the sets S_i)	Indices $[m] = \{1, \dots, m\}$	The n-supervertices $V = \{X_1, \dots, X_m\} \subseteq \mathcal{P}^n(V_0)$
Edge / (hyper)edge rule	$\{i, j\}$ is an edge iff $S_i \cap S_j \neq \emptyset$	$I \subseteq [m]$ is a hyperedge iff $\|I\| \geq 2$ and $\bigcap_{i \in I} S_i \neq \emptyset$	$\varepsilon \subseteq V$ is a superedge iff $\|\varepsilon\| \geq 2$ and $\bigcap_{X \in \varepsilon} X \neq \emptyset$
What one (hyper)edge encodes	A *pairwise* nonempty intersection	A *common* intersection of a subfamily	A *common* intersection of a subfamily of *nested* objects
Output structure	Simple graph $\mathrm{IG}(\mathcal{S}) = (V, E)$	Hypergraph $\mathrm{IH}(\mathcal{S}) = (V, \mathcal{E})$	n-SuperHyperGraph $\mathrm{ISHG}^{(n)}(\mathcal{S}) = (V, E)$

Notation. $\mathcal{P}(X)$ is the powerset of X, and $\mathcal{P}^0(X) = X$, $\mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

Table 1.4.: Concise overview of labeling frameworks for graphs, hypergraphs, and SuperHyperGraphs.

Structure	**Objects labeled**	**Incidence / adjacency used in constraints**	**Typical labeling schema ingredients**
Graph labeling	Vertices and/or edges of a simple graph $G = (V, E)$	Adjacency $uv \in E$ and graph distance $\mathrm{dist}_G(u, v)$	Predicates built from $uv \in E$, dist_G, (in)equality or arithmetic on labels, and quantification over V and E (e.g. proper coloring, $L(h, k)$, magic, graceful).
Hypergraph labeling	Vertices and/or hyperedges of a hypergraph $H = (V, \mathcal{E})$	Incidence $v \in e$; optionally the primal graph $\mathrm{Pr}(H)$ (2-section) and its distance $\mathrm{dist}_{\mathrm{Pr}(H)}$	Predicates built from $v \in e$, and (optionally) $\mathrm{dist}_{\mathrm{Pr}(H)}$; includes genuine hypergraph constraints (e.g. strong hyperedge coloring, incidence-based restrictions, vertex–hyperedge coupling).
SuperHyperGraph labeling	n-supervertices and/or n-superedges of an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$	Incidence $x \in \varepsilon$ between supervertices $x \in V$ and superedges $\varepsilon \in E$; typically the primal graph $\mathrm{Pr}(\mathrm{SHG}^{(n)})$ and $\mathrm{dist}_{\mathrm{Pr}(\mathrm{SHG}^{(n)})}$	Predicates built from $x \in \varepsilon$, distances in $\mathrm{Pr}(\mathrm{SHG}^{(n)})$, and (optionally) level-n structural features of nested objects (e.g. inclusion relations, sizes/flattenings of constituents, aggregation across superedges).

Chapter 2

Preliminaries

This chapter establishes notation and reviews the fundamental structures used throughout the book.

2.1 SuperHyperGraphs

Classical graph theory models a system of *vertices* linked by *edges*, and studies connectivity, structural invariants, and algorithmic problems motivated by mathematics, computer science, and many applied domains [2]. A *hypergraph* broadens this framework by allowing a single edge to connect an arbitrary nonempty subset of the vertex set; hence it is well suited to represent intrinsically multiway interactions (e.g., relations of arity greater than two) [3, 67]. Such higher-order relations have become especially prominent in contemporary learning and modeling pipelines, including neural architectures that directly leverage hypergraph incidence patterns [3, 68–71].

By iterating the powerset operation, one can also permit *nested* set-valued entities at the vertex level. This leads to finite *SuperHyperGraphs*, in which both vertices and edges may occur at multiple levels of set nesting [72, 73]. Such hierarchical representations arise naturally in layered or multiscale relational settings, for instance in molecular design, complex-network analysis, and neural-network modeling, among other applications [8, 16, 74–78]. Several related generalizations have also been investigated, including Directed SuperHyperGraphs [79, 80] and MetaSuperHyperGraphs [81]. Unless stated otherwise, the index n in $\mathcal{P}^n(\cdot)$ and in the term n-SuperHyperGraph always denotes a nonnegative integer.

Definition 2.1.1 (Base set)**.** A *base set* S is the ambient universe of discourse:

$$S = \{ x \mid x \text{ is an admissible object in the context under consideration} \}.$$

All sets in $\mathcal{P}(S)$ and in the iterated powersets $\mathcal{P}^n(S)$ are ultimately formed from elements of S.

Definition 2.1.2 (Powerset)**.** (see [82]) For a set S, the *powerset* of S is

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

In particular, $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$.

Definition 2.1.3 (Hypergraph)**.** [25, 83] A *hypergraph* is a pair $H = (V, E)$ such that:

- V is a finite set of *vertices*, and
- E is a finite family of nonempty subsets of V, called *hyperedges*.

Thus, a hyperedge may contain more than two vertices, capturing genuinely multiway relations.

Example 2.1.4 (Real-life example of a hypergraph)**.** Consider the problem of organizing university courses and student enrollments. Let V be the set of all students in a department. For each course c, define a hyperedge

$$e_c \;:=\; \{\, s \in V \mid s \text{ is enrolled in course } c \,\}.$$

Then $H = (V, E)$, where $E = \{e_c : c \text{ is a course}\}$, is a hypergraph. Each hyperedge represents a multiway relation: all students jointly participating in the same course.

Definition 2.1.5 (Iterated powerset and flattening)**.** [84] Let V_0 be a finite nonempty set. Define $\mathcal{P}^0(V_0) := V_0$ and

$$\mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \qquad (k \geq 0).$$

For each $k \geq 0$, define the flattening map

$$\mathrm{Flat}_k : \mathcal{P}^k(V_0) \setminus \{\emptyset\} \longrightarrow \mathcal{P}(V_0) \setminus \{\emptyset\}$$

recursively by

$$\mathrm{Flat}_0(x) := \{x\} \quad (x \in V_0), \qquad \mathrm{Flat}_{k+1}(X) := \bigcup_{Y \in X} \mathrm{Flat}_k(Y) \quad (X \in \mathcal{P}^{k+1}(V_0) \setminus \{\emptyset\}).$$

Example 2.1.6 (Iterated powerset and flattening)**.** Let $V_0 := \{a, b, c\}$. Then

$$\mathcal{P}^0(V_0) = \{a, b, c\}, \qquad \mathcal{P}^1(V_0) = \mathcal{P}(V_0), \qquad \mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

Level $k = 0$. For $x = b \in V_0$, by definition

$$\mathrm{Flat}_0(b) = \{b\} \subseteq V_0.$$

Level $k = 1$. Take the 1-level object $Y := \{a, c\} \in \mathcal{P}(V_0) \setminus \{\emptyset\}$. Then

$$\mathrm{Flat}_1(Y) = \bigcup_{y \in Y} \mathrm{Flat}_0(y) = \mathrm{Flat}_0(a) \cup \mathrm{Flat}_0(c) = \{a\} \cup \{c\} = \{a, c\}.$$

Level $k = 2$. Consider the 2-level object

$$X := \big\{\{a\}, \{b, c\}\big\} \in \mathcal{P}^2(V_0) \setminus \{\emptyset\}.$$

Using the recursion,

$$\mathrm{Flat}_2(X) = \bigcup_{Y \in X} \mathrm{Flat}_1(Y) = \mathrm{Flat}_1(\{a\}) \cup \mathrm{Flat}_1(\{b, c\}) = \{a\} \cup \{b, c\} = \{a, b, c\}.$$

Hence X “flattens” to the base-level subset $\{a, b, c\} \subseteq V_0$.

Definition 2.1.7 (n-SuperHyperGraph)**.** (see [4]) Let V_0 be a finite, nonempty base set. Define

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \in \mathbb{N}).$$

For $n \geq 0$, an *n-SuperHyperGraph on V_0* is a pair

$$\mathrm{SHG}^{(n)} = (V, E)$$

such that

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Elements of V are called *n-supervertices*, and elements of E are called *n-superedges* (that is, each n-superedge is a nonempty subset of V).

Example 2.1.8 (Two real-world examples of n-SuperHyperGraphs)**.** We present two concrete scenarios in which nested entities and higher-order relations arise naturally and can be modeled as n-SuperHyperGraphs.

(1) Healthcare delivery: patients, care-teams, and multi-team treatment pathways. Let V_0 be a finite set of *atomic individuals*, e.g.,

$$V_0 = \{\mathsf{Alice}, \mathsf{Bob}, \mathsf{Dr.\ Chen}, \mathsf{Nurse\ D.}, \mathsf{Therapist\ E.}, \mathsf{Pharmacist\ F.}, \dots\}.$$

A *care-team* is a subset of V_0 (doctors, nurses, and staff assigned to a case), so care-teams lie in $\mathcal{P}(V_0) = \mathcal{P}^1(V_0)$. A *multi-disciplinary consortium* (e.g. a tumor board plus rehabilitation unit plus pharmacy panel) is a *set of care-teams*, hence an element of $\mathcal{P}\big(\mathcal{P}(V_0)\big) = \mathcal{P}^2(V_0)$. Thus, taking $n = 2$, we may define a 2-SuperHyperGraph $\mathrm{SHG}^{(2)} = (V, E)$ by letting

$$V \subseteq \mathcal{P}^2(V_0)$$

be a collection of such consortia (each consortium is a set of teams), and letting each superedge $\varepsilon \in E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ represent a *treatment pathway stage* that requires the joint participation of several consortia (for example, a stage combining (i) a surgery consortium, (ii) an oncology consortium, and (iii) a discharge-planning consortium). In this model, a single superedge captures a *multiway coordination* among *nested* objects (consortia of teams of individuals), which is not naturally expressible in an ordinary graph or hypergraph.

(2) Software engineering: services, components, and nested dependency governance. Let V_0 be a finite set of *atomic software artifacts* [85], such as repositories, modules, or packages:

$$V_0 = \{\mathsf{auth\text{-}lib}, \mathsf{payments\text{-}api}, \mathsf{ui\text{-}kit}, \mathsf{db\text{-}migrator}, \dots\}.$$

A *component* is a set of artifacts, so components live in $\mathcal{P}(V_0) = \mathcal{P}^1(V_0)$. A *service* is commonly a set of components (e.g. an API service consisting of an authentication component, a logging component, and a storage component), hence a service is an element of $\mathcal{P}\big(\mathcal{P}(V_0)\big) = \mathcal{P}^2(V_0)$. Taking $n = 2$, define $V \subseteq \mathcal{P}^2(V_0)$ to be a set of services (each service is a set of components), and define superedges $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ by declaring that $\varepsilon = \{S_1, \dots, S_k\} \in E$ whenever the services $S_1, \dots, S_k$ must be governed or updated *together* (e.g. a coordinated release, a shared security patch window, or a joint compliance change). Then $\mathrm{SHG}^{(2)} = (V, E)$ records dependencies and coordination requirements at the level of nested service structures rather than at the level of individual repositories.

In both examples, $\mathcal{P}^n(V_0)$ naturally captures *nested aggregation* (individuals $\to$ teams $\to$ consortia, or artifacts $\to$ components $\to$ services), while superedges capture *higher-order coordination* among these nested objects.

2.2 Intersection SuperHyperGraph

In an intersection graph, vertices stand for sets and adjacency encodes nonempty pairwise intersection [27]. In an intersection hypergraph, hyperedges represent subfamilies with a nonempty *common* intersection [86–88]. An intersection superhypergraph extends this idea to nested set-valued vertices, with superedges determined by nonemptiness of the overall intersection at the chosen nesting level.

Definition 2.2.1 (Intersection graph)**.** [27] Let U be a universe and let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a finite family of nonempty subsets of U. The *intersection graph* of $\mathcal{S}$ is the simple graph

$$\mathrm{IG}(\mathcal{S}) \;:=\; (V, E), \qquad V := \{1, \ldots, m\},$$

whose edges encode pairwise nonempty intersections:

$$\{i, j\} \in E \quad \Longleftrightarrow \quad i \neq j \;\text{ and }\; S_i \cap S_j \neq \emptyset.$$

Equivalently, vertices correspond to members of $\mathcal{S}$ and two vertices are adjacent exactly when the corresponding sets intersect.

Example 2.2.2 (Intersection graph)**.** Let the universe be $U = \{1, 2, 3, 4\}$ and let

$$\mathcal{S} = \{S_1, S_2, S_3\}, \qquad S_1 = \{1, 2\}, \;\; S_2 = \{2, 3\}, \;\; S_3 = \{4\}.$$

Then $S_1 \cap S_2 = \{2\} \neq \emptyset$, while $S_1 \cap S_3 = \emptyset$ and $S_2 \cap S_3 = \emptyset$. Hence the intersection graph $\mathrm{IG}(\mathcal{S}) = (V, E)$ has

$$V = \{1, 2, 3\}, \qquad E = \big\{\{1, 2\}\big\}.$$

That is, the only edge joins the vertices corresponding to S_1 and S_2.

Definition 2.2.3 (Intersection hypergraph)**.** Let U be a universe and let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a finite family of nonempty subsets of U. The *intersection hypergraph* of $\mathcal{S}$ is the hypergraph

$$\mathrm{IH}(\mathcal{S}) \;:=\; (V, \mathcal{E}), \qquad V := \{1, \ldots, m\},$$

with hyperedge family

$$\mathcal{E} \;:=\; \Big\{\, I \subseteq V : \; |I| \geq 2 \;\text{ and }\; \bigcap_{i \in I} S_i \neq \emptyset \,\Big\}.$$

Thus a hyperedge I represents a subfamily $\{S_i\}_{i \in I}$ having a nonempty common intersection.

Example 2.2.4 (Intersection hypergraph)**.** Let $U = \{a, b, c\}$ and

$$\mathcal{S} = \{S_1, S_2, S_3\}, \qquad S_1 = \{a, b\}, \;\; S_2 = \{b, c\}, \;\; S_3 = \{b\}.$$

Then every pair intersects (all pairwise intersections contain b), and moreover the triple intersection is

$$S_1 \cap S_2 \cap S_3 = \{b\} \neq \emptyset.$$

Therefore the intersection hypergraph $\mathrm{IH}(\mathcal{S}) = (V, \mathcal{E})$ has

$$V = \{1, 2, 3\}, \qquad \mathcal{E} = \big\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}.$$

In particular, $\{1, 2, 3\}$ is a hyperedge because S_1, S_2, S_3 share a common element b.

Remark 2.2.5 (Relation to the intersection graph)**.** The 2-uniform truncation of $\mathrm{IH}(\mathcal{S})$ coincides with $\mathrm{IG}(\mathcal{S})$:

$$\{i,j\} \in E\big(\mathrm{IG}(\mathcal{S})\big) \quad \Longleftrightarrow \quad \{i,j\} \in \mathcal{E}\big(\mathrm{IH}(\mathcal{S})\big).$$

Hence, $\mathrm{IG}(\mathcal{S})$ captures only pairwise intersections, whereas $\mathrm{IH}(\mathcal{S})$ captures common intersections of arbitrary size.

Definition 2.2.6 (Intersection n-SuperHyperGraph)**.** Let V_0 be a base universe and fix an integer $n \geq 1$. Let $\mathcal{S} = \{X_1, \ldots, X_m\}$ be a finite family of nonempty n-level objects

$$X_i \in \mathcal{P}^n(V_0) \setminus \{\emptyset\} \qquad (1 \leq i \leq m),$$

so that each X_i is an n-supervertex candidate. The *intersection n-SuperHyperGraph* generated by $\mathcal{S}$ is the n-SuperHyperGraph

$$\mathrm{ISHG}^{(n)}(\mathcal{S}) \;:=\; (V, E), \qquad V := \{X_1, \ldots, X_m\} \subseteq \mathcal{P}^n(V_0),$$

whose n-superedge family is

$$E \;:=\; \Big\{ \varepsilon \subseteq V : \; |\varepsilon| \geq 2 \;\text{ and }\; \bigcap_{X \in \varepsilon} X \neq \emptyset \Big\}.$$

Accordingly, a superedge ε is present precisely when the corresponding supervertices admit a nonempty common intersection (taken in the usual set-theoretic sense at level n).

Example 2.2.7 (Intersection n-SuperHyperGraph)**.** Take $n = 1$ and the base universe $V_0 = \{1, 2, 3\}$. Consider the family of nonempty 1-level objects (ordinary subsets of V_0)

$$\mathcal{S} = \{X_1, X_2, X_3\} \subseteq \mathcal{P}^1(V_0) \setminus \{\emptyset\}, \qquad X_1 = \{1,2\}, \quad X_2 = \{2,3\}, \quad X_3 = \{2\}.$$

Then $X_1 \cap X_2 \cap X_3 = \{2\} \neq \emptyset$, and all pairwise intersections are also nonempty. Thus the intersection 1-SuperHyperGraph $\mathrm{ISHG}^{(1)}(\mathcal{S}) = (V, E)$ has

$$V = \{X_1, X_2, X_3\}, \qquad E = \big\{\{X_1, X_2\}, \{X_1, X_3\}, \{X_2, X_3\}, \{X_1, X_2, X_3\}\big\}.$$

In particular, the superedge $\{X_1, X_2, X_3\}$ exists because the three supervertices share the common element 2 in V_0.

Remark 2.2.8 (Compatibility with the hypergraph case)**.** If $n = 1$ and $X_i = S_i \subseteq V_0$ are ordinary subsets, then $\mathrm{ISHG}^{(1)}(\mathcal{S})$ is an intersection structure whose vertices are the sets themselves. After identifying each set S_i with its index i, Definition 2.2.6 reduces to Definition 2.2.3.

Remark 2.2.9 (A size-reduced variant)**.** Since E may be very large, a common simplification is to retain only the inclusionwise maximal superedges:

$$E_{\max} \;:=\; \{\varepsilon \in E : \;\text{there is no } \varepsilon' \in E \text{ with } \varepsilon \subsetneq \varepsilon'\}.$$

This produces a smaller, albeit coarser, representation of common intersections.

Chapter 3

Results: Some Classes of Intersection SuperHyperGraphs

Many graph notions and classes are known for intersection graphs. The same is true for intersection hypergraphs, and it is therefore natural to expect that analogous extensions can be formulated within the framework of SuperHyperGraphs. Accordingly, in this chapter we define several graph classes of intersection SuperHyperGraphs and provide brief explanations of each.

3.1 Interval SuperHyperGraphs

Interval graphs form a classical family of intersection graphs: each vertex is modeled by an interval on a line, and adjacency encodes nonempty overlap of the corresponding intervals [89–91]. Related concepts are also known, such as proper interval graphs [92, 93]. An interval hypergraph represents vertices by real intervals, and forms each hyperedge from intervals having a nonempty common intersection [94–97]. An interval n-superhypergraph uses nested supervertices represented by interval families, and defines superedges when represented families intersect nonemptily.

Let $(X, \preceq)$ be a finite linearly ordered set. For $a, b \in X$ with $a \preceq b$, define the (closed) order-interval

$$[a, b]_{\preceq} \;:=\; \{x \in X : \; a \preceq x \preceq b\}.$$

A subset $I \subseteq X$ is called an *interval (in* $(X, \preceq)$*)* if $I = [a, b]_{\preceq}$ for some $a \preceq b$. We recall the standard definition below [29, 30, 98].

Definition 3.1.1 (Interval graph)**.** [29,30] An *interval graph* is an undirected graph $G = (V, E)$ for which there exists a family of real intervals $\{I_v\}_{v \in V}$ such that two distinct vertices $u, v \in V$ are adjacent exactly when their intervals overlap:

$$\{u, v\} \in E \quad \Longleftrightarrow \quad I_u \cap I_v \neq \emptyset.$$

Equivalently,

$$E(G) \;=\; \big\{\{u, v\} \subseteq V : \; u \neq v, \; I_u \cap I_v \neq \emptyset\big\}.$$

Example 3.1.2 (Interval graph)**.** Let $V = \{u, v, w, x\}$ and assign to each vertex a real interval

$$I_u = [1, 4], \qquad I_v = [3, 6], \qquad I_w = [5, 7], \qquad I_x = [8, 9].$$

Define $G = (V, E)$ by the overlap rule in Definition (Interval graph). Then

$$I_u \cap I_v = [3, 4] \neq \emptyset, \quad I_v \cap I_w = [5, 6] \neq \emptyset,$$

while $I_u \cap I_w = \emptyset$ and I_x is disjoint from all other intervals. Hence

$$E = \{\{u, v\}, \{v, w\}\},$$

so G is an interval graph consisting of the path $u - v - w$ plus an isolated vertex x.

Definition 3.1.3 (Interval hypergraph)**.** A (finite) hypergraph is a pair $H = (V, \mathcal{E})$, where V is a finite set and $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. We call H an *interval hypergraph* if there exists a linear order $\preceq$ on V such that every hyperedge $e \in \mathcal{E}$ is an interval in $(V, \preceq)$, i.e.,

$$\forall e \in \mathcal{E}\ \exists a_e, b_e \in V \quad (a_e \preceq b_e \ \text{ and } \ e = [a_e, b_e]_{\preceq}).$$

Equivalently, H is interval if there exists an injective map $f : V \to \mathbb{R}$ such that for each $e \in \mathcal{E}$ there is a real interval $I_e \subseteq \mathbb{R}$ with

$$e = \{v \in V : \ f(v) \in I_e\}.$$

Example 3.1.4 (Interval hypergraph)**.** Let $V = \{1, 2, 3, 4, 5\}$ with the natural order $1 \prec 2 \prec 3 \prec 4 \prec 5$. Consider the hyperedge family

$$\mathcal{E} = \{e_1, e_2, e_3\}, \qquad e_1 = \{1, 2, 3\}, \ \ e_2 = \{3, 4\}, \ \ e_3 = \{2, 3, 4, 5\}.$$

Each hyperedge is an order-interval:

$$e_1 = [1, 3]_{\preceq}, \qquad e_2 = [3, 4]_{\preceq}, \qquad e_3 = [2, 5]_{\preceq}.$$

Therefore $H = (V, \mathcal{E})$ is an interval hypergraph.

Definition 3.1.5 (Interval n-SuperHyperGraph)**.** Fix $n \geq 1$ and a finite linear order $(V_0, \preceq)$. Let $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ be a finite set of n-supervertices. Let $\mathcal{J}$ be a finite family of nonempty intervals in $(V_0, \preceq)$.

For each $J \in \mathcal{J}$ define the corresponding n-superedge

$$\varepsilon_J \ := \ \{\, X \in V : \ \mathrm{supp}_n(X) \cap J \neq \emptyset \,\} \ \subseteq V.$$

Set

$$E \ := \ \{\varepsilon_J : \ J \in \mathcal{J}\} \ \subseteq \ \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then $\mathrm{SHG}^{(n)} := (V, E)$ is called an *interval n-SuperHyperGraph* (represented on $(V_0, \preceq)$).

Example 3.1.6 (Interval n-SuperHyperGraph)**.** Take $n = 1$ and let $V_0 = \{1, 2, 3, 4, 5\}$ with the natural order $\preceq$. Let the 1-supervertex set be

$$V = \{X_1, X_2, X_3, X_4\} \subseteq \mathcal{P}(V_0) \setminus \{\emptyset\}, \qquad X_1 = \{1\}, \quad X_2 = \{2, 3\}, \quad X_3 = \{4\}, \quad X_4 = \{5\}.$$

(Here $\mathrm{supp}_1(X) = X$ for $n = 1$.) Let $\mathcal{J} = \{J_1, J_2\}$ be the family of intervals

$$J_1 = [2, 4]_{\preceq} = \{2, 3, 4\}, \qquad J_2 = [4, 5]_{\preceq} = \{4, 5\}.$$

For each $J \in \mathcal{J}$, define the corresponding superedge $\varepsilon_J = \{X \in V : X \cap J \neq \emptyset\}$. Then

$$\varepsilon_{J_1} = \{X_2, X_3\} \quad \text{since} \quad X_2 \cap J_1 = \{2, 3\} \neq \emptyset, \quad X_3 \cap J_1 = \{4\} \neq \emptyset,$$

and

$$\varepsilon_{J_2} = \{X_3, X_4\} \quad \text{since} \quad X_3 \cap J_2 = \{4\} \neq \emptyset, \quad X_4 \cap J_2 = \{5\} \neq \emptyset.$$

Hence the interval 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ has

$$E = \{\varepsilon_{J_1}, \varepsilon_{J_2}\} = \{\{X_2, X_3\}, \{X_3, X_4\}\},$$

which is an interval 1-SuperHyperGraph represented on $(V_0, \preceq)$.

Definition 3.1.7 (Primal graph of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. Its *primal graph* (or 2-*section*) is the simple graph

$$\mathrm{Pr}(\mathrm{SHG}^{(n)}) := (V, F), \qquad \{X, Y\} \in F \iff X \neq Y \text{ and } \exists \varepsilon \in E \text{ with } \{X, Y\} \subseteq \varepsilon.$$

Theorem 3.1.8 (Interval n-SuperHyperGraphs generalize interval hypergraphs and interval graphs)**.** *Fix $n \geq 1$.*

1. ***(Interval hypergraphs embed.)*** *Every interval hypergraph is isomorphic to an interval n-SuperHyperGraph.*

2. ***(Interval graphs arise as primal graphs.)*** *For every interval graph G there exists an interval n-SuperHyperGraph $\mathrm{SHG}^{(n)}$ such that $\mathrm{Pr}(\mathrm{SHG}^{(n)}) \cong G$.*

Proof. **(1)** Let $H = (V_0, \mathcal{E})$ be an interval hypergraph. Thus there exists a linear order $\preceq$ on V_0 such that each $e \in \mathcal{E}$ is an interval in $(V_0, \preceq)$. Define the n-supervertex set

$$V := \{\iota_n(v) : v \in V_0\} \subseteq \mathcal{P}^n(V_0),$$

and take $\mathcal{J} := \mathcal{E}$. For each $e \in \mathcal{E}$, by $\mathrm{supp}_n(\iota_n(v)) = \{v\}$ we obtain

$$\varepsilon_e = \{\, \iota_n(v) \in V : \mathrm{supp}_n(\iota_n(v)) \cap e \neq \emptyset \,\} = \{\, \iota_n(v) : v \in e \,\}.$$

Hence the bijection $\varphi : V \to V_0$ given by $\varphi(\iota_n(v)) = v$ maps each ε_e to e. Therefore φ is a hypergraph isomorphism from the interval n-SuperHyperGraph $(V, \{\varepsilon_e : e \in \mathcal{E}\})$ onto H.

(2) Let $G = (U, F)$ be an interval graph with an interval model $\{I_u \subseteq \mathbb{R} : u \in U\}$. Let A be the set of all endpoints of the I_u, listed increasingly as $A = \{a_1 < \cdots < a_m\}$. For each $k = 1, \ldots, m-1$ choose a point $p_k \in (a_k, a_{k+1})$, and set

$$V_0 := A \cup \{p_1, \ldots, p_{m-1}\},$$

ordered by the usual order on $\mathbb{R}$ (denote it again by $\preceq$). For each $u \in U$ define the discretized trace

$$S_u \ := \ I_u \cap V_0 \ \subseteq \ V_0.$$

By construction, for distinct $u, v \in U$ one has

$$I_u \cap I_v \neq \emptyset \quad \Longleftrightarrow \quad S_u \cap S_v \neq \emptyset,$$

because any nonempty intersection contains either a common endpoint in A or a nontrivial overlap inside some (a_k, a_{k+1}) and hence contains the chosen p_k.

Now define an interval n-SuperHyperGraph as follows. For each $u \in U$ set

$$X_u \ := \ \{ \ \iota_{n-1}(t) \ : \ t \in S_u \ \} \ \in \ \mathcal{P}^n(V_0) \setminus \{\emptyset\},$$

and let $V := \{X_u : u \in U\}$. A direct induction on n gives $\operatorname{supp}_n(X_u) = S_u$. Take the family of base intervals

$$\mathcal{J} \ := \ \{\{t\} : t \in V_0\},$$

and form $E = \{\varepsilon_{\{t\}} : t \in V_0\}$ as in Definition 3.1.5. Then for distinct $u, v \in U$,

$$\{X_u, X_v\} \in E\big(\Pr(\mathrm{SHG}^{(n)})\big) \iff \exists\, t \in V_0 : \ \{X_u, X_v\} \subseteq \varepsilon_{\{t\}}$$

$$\iff \operatorname{supp}_n(X_u) \cap \operatorname{supp}_n(X_v) \neq \emptyset \iff S_u \cap S_v \neq \emptyset.$$

By the equivalence above, this holds if and only if $I_u \cap I_v \neq \emptyset$, i.e., if and only if $\{u, v\} \in F$. Therefore the bijection $u \mapsto X_u$ is a graph isomorphism $G \cong \Pr(\mathrm{SHG}^{(n)})$. □

3.2 Permutation graphs

Permutation graphs admit several equivalent characterizations. One may describe them via inversions in a permutation, or geometrically as intersection graphs of straight-line segments joining two parallel lines [99–102]. A permutation hypergraph represents vertices by permutation segments, and forms hyperedges from subfamilies whose segments share a common intersection [103, 104]. A permutation n-superhypergraph uses nested supervertices represented by permutation-segment families, and adds superedges for common intersections.

Definition 3.2.1 (Permutation graph)**.** [99,101,102] Let $G = (V, E)$ be a graph with $|V| = n$. We call G a *permutation graph* if there exists a permutation π of $\{1, 2, \ldots, n\}$ and a labeling $V = \{1, 2, \ldots, n\}$ such that, for any distinct $i, j \in V$ with $i < j$,

$$\{i, j\} \in E \quad \Longleftrightarrow \quad \pi(i) > \pi(j).$$

That is, two vertices are adjacent precisely when they form an inversion in π.

Equivalently, G is the intersection graph of a collection of line segments drawn between two parallel lines, where each vertex corresponds to one segment and two vertices are adjacent if and only if the corresponding segments intersect.

Example 3.2.2 (Permutation graph)**.** Let $V = \{1, 2, 3, 4\}$ and take the permutation

$$\pi = (3, 1, 4, 2), \quad \text{i.e.,} \quad \pi(1) = 3,\ \pi(2) = 1,\ \pi(3) = 4,\ \pi(4) = 2.$$

Define $G = (V, E)$ by the inversion rule in Definition (Permutation graph): for $i < j$, $\{i, j\} \in E$ iff $\pi(i) > \pi(j)$. The inversions are

$$(1, 2)\ (3 > 1), \qquad (1, 4)\ (3 > 2), \qquad (3, 4)\ (4 > 2),$$

and there are no other inversions. Hence

$$E = \big\{\{1, 2\}, \{1, 4\}, \{3, 4\}\big\}.$$

Therefore G is a permutation graph realized by π.

Definition 3.2.3 (Permutation hypergraph associated with (G, π))**.** Let $G = (V, E)$ be a finite hypergraph with $V = \{v_1, \dots, v_n\}$ and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Let π be a permutation of $[n]$.

Take two vertex-disjoint isomorphic copies $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G via bijections $\varphi_i : V \to V_i$ $(i = 1, 2)$, and set

$$E_3 \ := \ \big\{\{\varphi_1(v_i),\ \varphi_2(v_{\pi(i)})\} \ : \ i \in [n]\big\}.$$

The *permutation hypergraph* of G defined by π is

$$G_\pi \ := \ \big(V_1 \cup V_2,\ E_1 \cup E_2 \cup E_3\big).$$

(Thus E_3 is a perfect matching between the two copies, realized as 2-element hyperedges.)

Example 3.2.4 (Permutation hypergraph associated with (G, π))**.** Let $G = (V, E)$ be the hypergraph with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{e_1, e_2\}, \qquad e_1 = \{v_1, v_2\}, \quad e_2 = \{v_2, v_3\}.$$

Let π be the transposition $\pi = (2, 1, 3)$ on $[3]$. Form two disjoint copies $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ using bijections $\varphi_1(v_i) = a_i$ and $\varphi_2(v_i) = b_i$. Then

$$V_1 = \{a_1, a_2, a_3\}, \quad V_2 = \{b_1, b_2, b_3\},$$
$$E_1 = \big\{\{a_1, a_2\}, \{a_2, a_3\}\big\}, \quad E_2 = \big\{\{b_1, b_2\}, \{b_2, b_3\}\big\}.$$

Moreover,

$$E_3 = \big\{\{a_i, b_{\pi(i)}\} : i \in [3]\big\} = \big\{\{a_1, b_2\}, \{a_2, b_1\}, \{a_3, b_3\}\big\}.$$

Thus the permutation hypergraph G_π is

$$G_\pi = \big(V_1 \cup V_2,\ E_1 \cup E_2 \cup E_3\big),$$

i.e., it consists of the two copied hypergraphs together with the perfect matching E_3 between them.

Definition 3.2.5 (Permutation n-SuperHyperGraph associated with $(\mathrm{SHG}^{(n)}, \pi)$)**.** Let $n \geq 0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph, i.e., V is the set of n-supervertices and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is the family of n-superedges. Fix an ordering $V = \{x_1, \dots, x_m\}$ and let π be a permutation of $[m]$.

Take two vertex-disjoint isomorphic copies $\mathrm{SHG}^{(n)}_1 = (V_1, E_1)$ and $\mathrm{SHG}^{(n)}_2 = (V_2, E_2)$ via bijections $\psi_i : V \to V_i$ $(i = 1, 2)$, and set

$$E_3 \ := \ \big\{\{\psi_1(x_i),\ \psi_2(x_{\pi(i)})\} \ : \ i \in [m]\big\}.$$

The *permutation n-SuperHyperGraph* of $\mathrm{SHG}^{(n)}$ defined by π is

$$\big(\mathrm{SHG}^{(n)}\big)_\pi \ := \ \big(V_1 \cup V_2,\ E_1 \cup E_2 \cup E_3\big).$$

Example 3.2.6 (Permutation n-SuperHyperGraph associated with $(\mathrm{SHG}^{(n)}, \pi)$)**.** Take $n = 1$ and let $\mathrm{SHG}^{(1)} = (V, E)$ be the 1-SuperHyperGraph with

$$V = \{x_1, x_2, x_3\}, \qquad E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{x_1, x_2\}, \quad \varepsilon_2 = \{x_2, x_3\}.$$

Let $\pi = (2, 3, 1)$ be a permutation of $[3]$. Construct disjoint copies $\mathrm{SHG}^{(1)}_1 = (V_1, E_1)$ and $\mathrm{SHG}^{(1)}_2 = (V_2, E_2)$ via

$$\psi_1(x_i) = p_i, \qquad \psi_2(x_i) = q_i \qquad (i = 1, 2, 3).$$

Then

$$V_1 = \{p_1, p_2, p_3\}, \quad V_2 = \{q_1, q_2, q_3\},$$
$$E_1 = \{\{p_1, p_2\}, \{p_2, p_3\}\}, \quad E_2 = \{\{q_1, q_2\}, \{q_2, q_3\}\}.$$

The matching superedge family is

$$E_3 = \{\{p_i, q_{\pi(i)}\} : i \in [3]\} = \{\{p_1, q_2\}, \{p_2, q_3\}, \{p_3, q_1\}\}.$$

Hence the permutation 1-SuperHyperGraph is

$$\left(\mathrm{SHG}^{(1)}\right)_\pi = \left(V_1 \cup V_2,\ E_1 \cup E_2 \cup E_3\right),$$

namely, two copies of $\mathrm{SHG}^{(1)}$ together with the cross-copy perfect matching determined by π.

Theorem 3.2.7 (Permutation n-SuperHyperGraphs generalize the graph and hypergraph cases)**.** *Let $n = 0$.*

1. *If $G = (V, E)$ is a (crisp) hypergraph and π is a permutation of V, then Definition 3.2.5 produces exactly the permutation hypergraph G_π of Definition 3.2.3.*

2. *If $G = (V, E)$ is a (simple) graph (so $E \subseteq \binom{V}{2}$) and π is a permutation of V, then Definition 3.2.5 produces the usual Chartrand–Harary permutation graph construction: two disjoint copies of G plus a perfect matching specified by π.*

Hence, permutation n-SuperHyperGraphs strictly extend both permutation graphs (in this constructional sense) and permutation hypergraphs.

Proof. For $n = 0$, a 0-SuperHyperGraph is precisely a hypergraph: its vertex set V consists of atomic vertices and its edge set satisfies $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$.

(1) Apply Definition 3.2.5 with $n = 0$. The construction takes two disjoint copies of (V, E) and adds the matching family E_3 consisting of 2-element edges $\{v_i^{(1)}, v_{\pi(i)}^{(2)}\}$. This is exactly Definition 3.2.3 (up to renaming the copy maps), so the resulting hypergraphs are isomorphic.

(2) If G is a graph, then every hyperedge in $E_1 \cup E_2$ has size 2, and every matching edge in E_3 also has size 2. Thus the output of Definition 3.2.5 is again a (simple) graph on $2|V|$ vertices obtained from two disjoint copies of G by adding a perfect matching prescribed by π, which is precisely the Chartrand–Harary permutation graph construction.

Therefore, permutation 0-SuperHyperGraphs include permutation hypergraphs, and restricting further to 2-uniform edges recovers the permutation graph construction. □

3.3 Grounded graphs

A *grounded* intersection graph is obtained from geometric objects that are required to touch (be anchored on) a fixed line, while adjacency is still determined by geometric intersection of the objects [105]. A representative and well-studied example is the class of *grounded L-graphs* [106–108]. In the same spirit, one may define grounded *L-hypergraphs* by recording common intersections of multiple curves, and grounded *L-n-SuperHyperGraphs* by allowing n-supervertices while using the same intersection-driven rule to generate superedges.

Fix the *grounding line*

$$\ell \;:=\; \{(x,0)\in\mathbb{R}^2:\; x\in\mathbb{R}\}$$

(the x-axis).

Definition 3.3.1 (Grounded L-curve)**.** A *grounded L-curve* is a rectilinear set $L\subseteq\mathbb{R}^2$ of the form

$$L \;=\; V(b,h)\;\cup\; H(b,h,\lambda),$$

where $b\in\mathbb{R}$, $h\in\mathbb{R}_{>0}$, $\lambda\in\mathbb{R}_{>0}$, and

$$V(b,h)\;:=\;\{(b,y):0\le y\le h\},\qquad H(b,h,\lambda)\;:=\;\{(x,h):b\le x\le b+\lambda\}.$$

The point $(b,0)\in\ell$ is called the *base point* (or *grounding point*) of L.

Example 3.3.2 (A grounded L-curve)**.** Take $b=1$, $h=3$, and $\lambda=2$. Then

$$V(1,3)=\{(1,y):0\le y\le 3\},\qquad H(1,3,2)=\{(x,3):1\le x\le 3\},$$

and the grounded L-curve is

$$L \;=\; V(1,3)\cup H(1,3,2).$$

Its base point is $(1,0)\in\ell$.

For a family $\{L_x\}_{x\in X}$ of subsets of $\mathbb{R}^2$ and $\varepsilon\subseteq X$, write

$$L(\varepsilon)\;:=\;\bigcap_{x\in\varepsilon} L_x.$$

Grounded intersection models.

Definition 3.3.3 (Grounded L-graph)**.** Let $L=\{L_1,\dots,L_m\}$ be a family of grounded L-curves. The *grounded L-graph* associated with L is the simple graph $G(L)=(V,E)$ defined by

$$V=\{v_1,\dots,v_m\},\qquad \{v_i,v_j\}\in E \iff i\neq j \;\text{ and }\; L_i\cap L_j\neq\emptyset.$$

Example 3.3.4 (A grounded L-graph)**.** Let $L = \{L_1, L_2, L_3\}$ be grounded L-curves with parameters

$$L_1 = V(0,2) \cup H(0,2,3), \qquad L_2 = V(1,3) \cup H(1,3,2), \qquad L_3 = V(4,1) \cup H(4,1,1).$$

Then $L_1 \cap L_2 \neq \emptyset$ because the vertical arm of L_2 meets the horizontal arm of L_1 at $(1,2)$, while L_3 is disjoint from both L_1 and L_2. Hence the grounded L-graph $G(L)$ has vertex set $\{v_1, v_2, v_3\}$ and edge set

$$E\big(G(L)\big) = \big\{\{v_1, v_2\}\big\},$$

i.e., it is a single edge plus an isolated vertex.

Definition 3.3.5 (Grounded L-hypergraph (canonical intersection form))**.** Let X be a finite set. A hypergraph $H = (X, \mathcal{E})$ is called a *grounded L-hypergraph* if there exists a family of grounded L-curves $\{L_x\}_{x \in X}$ such that

$$\mathcal{E} \;=\; \Big\{\varepsilon \subseteq X : \; |\varepsilon| \geq 2 \;\text{ and }\; L(\varepsilon) \neq \emptyset\Big\}.$$

Thus each hyperedge is exactly a subfamily of curves having a nonempty common intersection.

Example 3.3.6 (A grounded L-hypergraph)**.** Let $X = \{a, b, c\}$ and represent its elements by grounded L-curves

$$L_a = V(0,2) \cup H(0,2,4), \qquad L_b = V(1,3) \cup H(1,3,2), \qquad L_c = V(2,2) \cup H(2,2,1).$$

Then $L_a \cap L_b \neq \emptyset$ at $(1,2)$, $L_a \cap L_c \neq \emptyset$ at $(2,2)$, and $L_b \cap L_c = \emptyset$ since L_b runs along height 3 and its vertical arm is at $x = 1$, whereas L_c lies at height 2 with vertical arm at $x = 2$. Moreover,

$$L_a \cap L_b \cap L_c = \emptyset,$$

because any point common to L_b must have $y = 3$ or $x = 1$, while any point common to L_c must have $y = 2$ or $x = 2$, which cannot happen simultaneously. Therefore the canonical grounded L-hypergraph $H = (X, \mathcal{E})$ has

$$\mathcal{E} = \big\{\{a, b\}, \{a, c\}\big\},$$

i.e., exactly the two 2-element hyperedges corresponding to the nonempty pairwise intersections.

Definition 3.3.7 (Grounded L-n-SuperHyperGraph)**.** Fix $n \in \mathbb{N}_0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph, i.e.,

$$\emptyset \neq V \subseteq \mathcal{P}^n(V_0), \qquad \emptyset \neq E \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

for some finite base set V_0. We call $\mathrm{SHG}^{(n)}$ a *grounded L-n-SuperHyperGraph* if there exists an injective map

$$\rho : V \longrightarrow \{\text{grounded } L\text{-curves in } \mathbb{R}^2\}$$

such that the superedges are precisely the intersecting subcollections:

$$E \;=\; \Big\{\varepsilon \subseteq V : \; |\varepsilon| \geq 2 \;\text{ and }\; \bigcap_{x \in \varepsilon} \rho(x) \neq \emptyset\Big\}.$$

In this case, ρ is called a *grounded L-representation* of $\mathrm{SHG}^{(n)}$.

Example 3.3.8 (A grounded L-n-SuperHyperGraph)**.** Let $V_0 = \{1, 2, 3\}$ and take $n = 1$. Consider the 1-supervertex set

$$V := \{\{1\}, \{2\}, \{2, 3\}\} \subseteq \mathcal{P}(V_0).$$

Define an injective grounded L-representation $\rho : V \to \{\text{grounded } L\text{-curves}\}$ by

$$\rho(\{1\}) = V(0, 2) \cup H(0, 2, 3), \qquad \rho(\{2\}) = V(1, 3) \cup H(1, 3, 2), \qquad \rho(\{2, 3\}) = V(2, 2) \cup H(2, 2, 2).$$

Then $\rho(\{1\}) \cap \rho(\{2\}) \neq \emptyset$ at $(1, 2)$, and $\rho(\{1\}) \cap \rho(\{2, 3\}) \neq \emptyset$ at $(2, 2)$, while $\rho(\{2\}) \cap \rho(\{2, 3\}) = \emptyset$. Also, the triple intersection is empty: $\rho(\{1\}) \cap \rho(\{2\}) \cap \rho(\{2, 3\}) = \emptyset$. Hence the grounded L-1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ determined by Definition 3.3.7 has

$$E = \{\{\{1\}, \{2\}\}, \{\{1\}, \{2, 3\}\}\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

i.e., it has exactly two superedges, each recording a nonempty intersection of the corresponding represented curves.

Definition 3.3.9 (Primal graph (2-section))**.** For an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$, its *primal graph* (or 2*-section*) is the simple graph

$$\mathrm{Pr}(\mathrm{SHG}^{(n)}) := (V, F), \qquad \{x, y\} \in F \iff x \neq y \text{ and } \exists\, \varepsilon \in E \text{ with } \{x, y\} \subseteq \varepsilon.$$

Theorem 3.3.10 (Grounded L-n-SuperHyperGraphs subsume the graph and hypergraph cases)**.** *Fix $n \in \mathbb{N}_0$ and let $L = \{L_1, \dots, L_m\}$ be a family of grounded L-curves.*

1. ***(Graphs via primal graphs).*** *There exists a grounded L-n-SuperHyperGraph $\mathrm{SHG}^{(n)}$ such that*
$$\mathrm{Pr}(\mathrm{SHG}^{(n)}) \cong G(L),$$
where $G(L)$ is the grounded L-graph from Definition 3.3.3.

2. ***(Hypergraphs as the $n = 0$ case).*** *The canonical grounded L-hypergraph $H(L) = ([m], \mathcal{E}(L))$ defined by*
$$\mathcal{E}(L) := \Big\{T \subseteq [m] : |T| \geq 2 \text{ and } \bigcap_{i \in T} L_i \neq \emptyset\Big\}$$
is a grounded L-0-SuperHyperGraph in the sense of Definition 3.3.7.

Proof. Let $V_0 := [m] = \{1, \dots, m\}$. Define the iterated singleton embedding by

$$\iota_0(i) := i, \qquad \iota_{k+1}(i) := \{\iota_k(i)\} \quad (k \geq 0).$$

Set

$$V := \{\iota_n(1), \dots, \iota_n(m)\} \subseteq \mathcal{P}^n(V_0), \qquad \rho(\iota_n(i)) := L_i.$$

Define

$$E := \Big\{\varepsilon \subseteq V : |\varepsilon| \geq 2 \text{ and } \bigcap_{x \in \varepsilon} \rho(x) \neq \emptyset\Big\}.$$

Then $\mathrm{SHG}^{(n)} := (V, E)$ is an n-SuperHyperGraph and is grounded L-n by Definition 3.3.7.

(1) For distinct $i, j \in [m]$, the vertices $\iota_n(i)$ and $\iota_n(j)$ are adjacent in $\Pr(\mathrm{SHG}^{(n)})$ iff there exists $\varepsilon \in E$ with $\{\iota_n(i), \iota_n(j)\} \subseteq \varepsilon$, which (by the definition of E) holds iff

$$\rho(\iota_n(i)) \cap \rho(\iota_n(j)) \neq \emptyset \quad \Longleftrightarrow \quad L_i \cap L_j \neq \emptyset.$$

This is exactly the adjacency rule for $G(L)$. Hence the bijection $v_i \mapsto \iota_n(i)$ is a graph isomorphism $G(L) \cong \Pr(\mathrm{SHG}^{(n)})$.

(2) When $n = 0$, we have $V = \{1, \ldots, m\}$, and the same construction gives

$$E = \left\{ T \subseteq [m] : |T| \geq 2 \ \text{ and } \ \bigcap_{i \in T} L_i \neq \emptyset \right\} = \mathcal{E}(L).$$

Thus $([m], \mathcal{E}(L))$ is precisely the grounded L-hypergraph of Definition 3.3.5, viewed as a 0-SuperHyperGraph. □

Remark 3.3.11 (Recovering the graph exactly as a 2-uniform model)**.** If one replaces E by its 2-uniform subfamily

$$E^{(2)} \ := \ \Big\{ \{x, y\} \subseteq V : \ x \neq y \ \text{ and } \ \rho(x) \cap \rho(y) \neq \emptyset \Big\},$$

then $(V, E^{(2)})$ is a (simple) graph isomorphic to the grounded L-graph $G(L)$.

3.4 Ray intersection models

Ray graphs form a geometric intersection class in which vertices are represented by planar rays (half-lines), and adjacency records whether the corresponding rays intersect [109–113]. In the same spirit, one may define a *ray hypergraph* by taking hyperedges to be exactly those subfamilies of rays having a nonempty common intersection, and a *ray n-SuperHyperGraph* by allowing n-supervertices (nested objects) while still generating superedges by nonempty common intersection of the associated rays.

Definition 3.4.1 (Ray in the plane)**.** A *ray* in $\mathbb{R}^2$ is a set of the form

$$R(p, d) \ := \ \{\, p + td : \ t \in [0, \infty) \,\},$$

where $p \in \mathbb{R}^2$ is the *apex* and $d \in \mathbb{R}^2 \setminus \{0\}$ is the *direction vector*. Two rays R, R' *intersect* if $R \cap R' \neq \emptyset$.

Definition 3.4.2 (Ray graph)**.** [109, 113] A graph $G = (V, E)$ is a *ray graph* if there exists a collection $\mathcal{R} = \{R_v\}_{v \in V}$ of rays in $\mathbb{R}^2$ such that for any distinct $u, v \in V$,

$$\{u, v\} \in E \quad \Longleftrightarrow \quad R_u \cap R_v \neq \emptyset.$$

Equivalently, G is the intersection graph of $\mathcal{R}$.

For a family $\{R_x\}_{x \in X}$ of subsets of $\mathbb{R}^2$ and $\varepsilon \subseteq X$, write

$$R(\varepsilon) \ := \ \bigcap_{x \in \varepsilon} R_x.$$

Definition 3.4.3 (Ray hypergraph (canonical intersection form)). Let X be a finite set. A hypergraph $H = (X, \mathcal{E})$ is called a *ray hypergraph* if there exists a family of rays $\{R_x\}_{x\in X}$ in $\mathbb{R}^2$ such that

$$\mathcal{E} \;=\; \Big\{\varepsilon \subseteq X :\ |\varepsilon| \geq 2 \ \text{ and } \ R(\varepsilon) \neq \emptyset\Big\}.$$

Thus the hyperedges are precisely the subfamilies of rays having a nonempty common intersection.

Definition 3.4.4 (Ray n-SuperHyperGraph). Fix $n \in \mathbb{N}_0$ and a finite base set V_0. Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph, i.e.,

$$\emptyset \neq V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

We call $\mathrm{SHG}^{(n)}$ a *ray n-SuperHyperGraph* if there exists an injective map

$$\rho : V \longrightarrow \{\text{rays in } \mathbb{R}^2\}$$

such that the superedge family is exactly the set of intersecting subcollections:

$$E \;=\; \Big\{\varepsilon \subseteq V :\ |\varepsilon| \geq 2 \ \text{ and } \ \bigcap_{x\in\varepsilon} \rho(x) \neq \emptyset\Big\}.$$

In this case, ρ is called a *ray representation* of $\mathrm{SHG}^{(n)}$.

Definition 3.4.5 (Primal graph (2-section)). For an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$, its *primal graph* is the simple graph

$$\mathrm{Pr}(\mathrm{SHG}^{(n)}) \;:=\; (V, F), \qquad \{x, y\} \in F \iff x \neq y \ \text{ and } \ \exists\, \varepsilon \in E \ \text{ with } \ \{x, y\} \subseteq \varepsilon.$$

Theorem 3.4.6 (Ray n-SuperHyperGraphs generalize ray graphs). *Fix $n \in \mathbb{N}_0$. If $G = (V_G, E_G)$ is a ray graph, then there exists a ray n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ such that*

$$\mathrm{Pr}(\mathrm{SHG}^{(n)}) \;\cong\; G.$$

Hence the ray-intersection graph class embeds into ray n-SuperHyperGraphs via the primal-graph reduction.

Proof. Since G is a ray graph, there exists a family of rays $\{R_v\}_{v\in V_G}$ in $\mathbb{R}^2$ such that, for distinct $u, v \in V_G$,

$$\{u, v\} \in E_G \iff R_u \cap R_v \neq \emptyset.$$

Let $V_0 := V_G$ and define the iterated singleton embedding $\iota_0(v) := v$ and $\iota_{k+1}(v) := \{\iota_k(v)\}$. Set

$$V \;:=\; \{\iota_n(v) : v \in V_G\} \subseteq \mathcal{P}^n(V_0), \qquad \rho(\iota_n(v)) \;:=\; R_v.$$

Define

$$E \;:=\; \Big\{\varepsilon \subseteq V :\ |\varepsilon| \geq 2 \ \text{ and } \ \bigcap_{x\in\varepsilon} \rho(x) \neq \emptyset\Big\}.$$

Then $\mathrm{SHG}^{(n)} = (V, E)$ is a ray n-SuperHyperGraph by Definition 3.4.4.

Now take distinct $u, v \in V_G$. In $\mathrm{Pr}(\mathrm{SHG}^{(n)})$, the vertices $\iota_n(u)$ and $\iota_n(v)$ are adjacent iff there exists $\varepsilon \in E$ with $\{\iota_n(u), \iota_n(v)\} \subseteq \varepsilon$, which holds iff

$$\rho(\iota_n(u)) \cap \rho(\iota_n(v)) \neq \emptyset \iff R_u \cap R_v \neq \emptyset \iff \{u, v\} \in E_G.$$

Therefore $u \mapsto \iota_n(u)$ is an isomorphism $G \cong \mathrm{Pr}(\mathrm{SHG}^{(n)})$. □

3.5 String intersection models

String graphs are among the most flexible geometric intersection classes: vertices are represented by simple planar curves ("strings"), and adjacency records whether two curves meet [114–120]. In the same spirit, a *string hypergraph* represents vertices by planar curves and takes hyperedges to be exactly those subfamilies having a nonempty common intersection (cf. [121]). Finally, a *string n-SuperHyperGraph* allows n-supervertices (nested objects) while still generating superedges by common intersection of the associated strings.

Definition 3.5.1 (String (simple planar curve))**.** A *string* in $\mathbb{R}^2$ is a set $C \subseteq \mathbb{R}^2$ that is the image of a continuous injective map $\gamma : [0,1] \to \mathbb{R}^2$; that is,

$$C = \gamma([0,1]).$$

Equivalently, a string is a Jordan arc (it has no self-intersections). Two strings C, C' *intersect* if $C \cap C' \neq \emptyset$.

Definition 3.5.2 (String graph)**.** (cf. [114, 115]) A graph $G = (V, E)$ is a *string graph* if there exists a family of strings $\mathcal{C} = \{C_v\}_{v \in V}$ in $\mathbb{R}^2$ such that for any distinct $u, v \in V$,

$$\{u, v\} \in E \iff C_u \cap C_v \neq \emptyset.$$

Equivalently, G is the intersection graph of $\mathcal{C}$.

Remark 3.5.3 (Optional general-position assumptions)**.** Many works additionally impose a general-position condition, e.g. forbidding triple intersections (no point lies on three strings). A *1-string representation* requires that any two strings intersect at most once; a *1-string graph* is a string graph admitting such a representation. These assumptions are optional and will be invoked only when explicitly stated.

Example 3.5.4 (A string graph)**.** Let G be the path P_4 on four vertices

$$V = \{v_1, v_2, v_3, v_4\}, \qquad E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}.$$

We give an explicit string representation in $\mathbb{R}^2$.

Define four strings $C_{v_i} = \gamma_i([0,1])$ by the injective continuous maps

$$\gamma_1(t) := (t, 0), \qquad \gamma_2(t) := (t, 0.5), \qquad \gamma_3(t) := (t, 1.0), \qquad \gamma_4(t) := (t, 1.5),$$

each restricted to the x-interval $[0,1]$, and then add the following vertical "connectors" as part of the same strings:

C_{v_2} also contains the segment $\{(0.5, s) : s \in [0,1]\}$, $\qquad C_{v_3}$ also contains the segment $\{(0.5, s) : s \in [0.5, 1.5]\}$.

Equivalently, one may describe C_{v_2} as the union of the horizontal segment $\{(x, 0.5) : x \in [0,1]\}$ and the vertical segment $\{(0.5, y) : y \in [0,1]\}$, and similarly for C_{v_3}.

Then $C_{v_1} \cap C_{v_2} = \{(0.5, 0)\} \neq \emptyset$, $C_{v_2} \cap C_{v_3} = \{(0.5, 0.5)\} \neq \emptyset$, and $C_{v_3} \cap C_{v_4} = \{(0.5, 1.5)\} \neq \emptyset$. Moreover, $C_{v_1} \cap C_{v_3} = \emptyset$, $C_{v_1} \cap C_{v_4} = \emptyset$, and $C_{v_2} \cap C_{v_4} = \emptyset$. Hence, by Definition 3.5.2, G is a string graph.

For a family $\{C_x\}_{x\in X}$ of subsets of $\mathbb{R}^2$ and $\varepsilon \subseteq X$, write

$$C(\varepsilon) \ := \ \bigcap_{x\in\varepsilon} C_x.$$

Definition 3.5.5 (String hypergraph (canonical intersection form))**.** Let X be a finite set. A hypergraph $H = (X, \mathcal{E})$ is called a *string hypergraph* if there exists a family of strings $\{C_x\}_{x\in X}$ in $\mathbb{R}^2$ such that

$$\mathcal{E} \ = \ \Big\{\varepsilon \subseteq X : \ |\varepsilon| \geq 2 \ \text{ and } \ C(\varepsilon) \neq \emptyset\Big\}.$$

Thus the hyperedges are precisely the subfamilies of strings having a nonempty common intersection.

Example 3.5.6 (A string hypergraph in canonical intersection form)**.** Let $X = \{a, b, c\}$ and consider the hypergraph $H = (X, \mathcal{E})$ with

$$\mathcal{E} = \big\{\{a, b\}, \{a, c\}, \{a, b, c\}\big\}.$$

We realize H as a string hypergraph as in Definition 3.5.5.

Work in $\mathbb{R}^2$ and define three strings C_a, C_b, C_c as unions of line segments (each is a Jordan arc):

$$C_a := \{(0, t) : t \in [-1, 2]\},$$

$$C_b := \{(t, 0) : t \in [-1, 1]\}, \qquad C_c := \{(t, 1) : t \in [-1, 1]\}.$$

Then $C_a \cap C_b = \{(0, 0)\} \neq \emptyset$ and $C_a \cap C_c = \{(0, 1)\} \neq \emptyset$, while $C_b \cap C_c = \emptyset$ since they lie on distinct horizontal lines.

Now modify C_a by adding (within the same simple arc) a short segment connecting the two intersection points, so that C_a contains the vertical segment between $(0, 0)$ and $(0, 1)$. This does not change the above pairwise intersections but ensures that

$$C_a \cap C_b \cap C_c \ = \ \{(0, 0)\} \cap C_c \ = \ \emptyset$$

would fail; hence, to obtain a *triple* intersection, instead shift C_c so that it passes through $(0, 0)$:

$$C'_c := \{(t, 0) : t \in [-1, 1]\} \cup \{(0, s) : s \in [0, 1]\},$$

which is still a Jordan arc after smoothing at $(0, 0)$.

With C_a and C_b as above and C'_c in place of C_c, we have

$$C_a \cap C_b \cap C'_c \ = \ \{(0, 0)\} \neq \emptyset, \qquad C_b \cap C'_c \neq \emptyset,$$

so we must avoid creating the undesired hyperedge $\{b, c\}$. To do so, replace C_b by a horizontal arc that meets C_a at $(0, 0)$ but stops short of the x-axis portion of C'_c:

$$C'_b := \{(t, 0) : t \in [-1, -0.1]\} \ \cup \ \{(t, 0) : t \in [-0.1, 0]\},$$

so that C'_b meets C_a at $(0, 0)$ and does not meet the right-going part of C'_c.

Then the nonempty common intersections are exactly:

$$C_a \cap C'_b \neq \emptyset, \qquad C_a \cap C'_c \neq \emptyset, \qquad C_a \cap C'_b \cap C'_c \neq \emptyset,$$

and $C'_b \cap C'_c = \emptyset$. Therefore,

$$\mathcal{E} = \Big\{\varepsilon \subseteq X : \ |\varepsilon| \geq 2 \ \text{ and } \ \bigcap_{x\in\varepsilon} C_x \neq \emptyset\Big\},$$

so H is a string hypergraph in the sense of Definition 3.5.5.

Definition 3.5.7 (String n-SuperHyperGraph)**.** Fix $n \in \mathbb{N}_0$ and a finite base set V_0. Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph, i.e.,

$$\emptyset \neq V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

We call $\mathrm{SHG}^{(n)}$ a *string n-SuperHyperGraph* if there exists an injective map

$$\rho : V \longrightarrow \{\text{strings in } \mathbb{R}^2\}$$

such that the superedges are exactly the intersecting subcollections:

$$E \;=\; \Big\{\varepsilon \subseteq V :\; |\varepsilon| \geq 2 \;\text{ and }\; \bigcap_{x \in \varepsilon} \rho(x) \neq \emptyset\Big\}.$$

In this case, ρ is called a *string representation* of $\mathrm{SHG}^{(n)}$.

Example 3.5.8 (A string n-SuperHyperGraph)**.** Fix $n \in \mathbb{N}_0$ and let the base set be

$$V_0 = \{1, 2, 3\}.$$

Let $\iota_n : V_0 \to \mathcal{P}^n(V_0)$ be the iterated singleton embedding $\iota_0(v) = v$ and $\iota_{t+1}(v) = \{\iota_t(v)\}$. Define the n-supervertex set

$$V := \{\iota_n(1),\ \iota_n(2),\ \iota_n(3)\} \subseteq \mathcal{P}^n(V_0),$$

and the superedge family

$$E := \big\{\{\iota_n(1), \iota_n(2)\},\ \{\iota_n(2), \iota_n(3)\},\ \{\iota_n(1), \iota_n(2), \iota_n(3)\}\big\}.$$

Thus $\mathrm{SHG}^{(n)} := (V, E)$ is an n-SuperHyperGraph.

We now give a string representation $\rho : V \to \{\text{strings in } \mathbb{R}^2\}$. Let C_1, C_2, C_3 be the following Jordan arcs (again, unions of segments can be smoothed at junctions):

$$C_1 := \{(t, 0) : t \in [0, 2]\}, \qquad C_2 := \{(1, t) : t \in [-1, 1]\}, \qquad C_3 := \{(t, 1) : t \in [0, 2]\}.$$

Then $C_1 \cap C_2 = \{(1, 0)\} \neq \emptyset$ and $C_2 \cap C_3 = \{(1, 1)\} \neq \emptyset$, while $C_1 \cap C_3 = \emptyset$. To enforce a nonempty triple intersection, modify C_2 so that it passes through $(1, 0)$ and $(1, 1)$ and add a short detour causing C_1 and C_3 also to meet at a single common point $p := (1, 0.5)$; for instance, replace C_1 and C_3 by arcs that each touch p once, without creating additional intersections.

Define $\rho(\iota_n(1)) := C_1$, $\rho(\iota_n(2)) := C_2$, and $\rho(\iota_n(3)) := C_3$ (after the above local modification), so that

$$\rho(\iota_n(1)) \cap \rho(\iota_n(2)) \neq \emptyset, \qquad \rho(\iota_n(2)) \cap \rho(\iota_n(3)) \neq \emptyset, \qquad \rho(\iota_n(1)) \cap \rho(\iota_n(2)) \cap \rho(\iota_n(3)) \neq \emptyset,$$

and no other pair among $\{\rho(\iota_n(1)), \rho(\iota_n(2)), \rho(\iota_n(3))\}$ intersects. Hence, by Definition 3.5.7,

$$E = \Big\{\varepsilon \subseteq V :\; |\varepsilon| \geq 2 \;\text{ and }\; \bigcap_{x \in \varepsilon} \rho(x) \neq \emptyset\Big\},$$

so $\mathrm{SHG}^{(n)}$ is a string n-SuperHyperGraph.

Definition 3.5.9 (Primal graph (2-section))**.** For an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$, its *primal graph* is the simple graph

$$\mathrm{Pr}(\mathrm{SHG}^{(n)}) \;:=\; (V, F), \qquad \{x, y\} \in F \iff x \neq y \;\text{ and }\; \exists\, \varepsilon \in E \;\text{ with }\; \{x, y\} \subseteq \varepsilon.$$

Theorem 3.5.10 (String n-SuperHyperGraphs generalize string graphs)**.** *Fix* $n \in \mathbb{N}_0$*. If* $G = (V_G, E_G)$ *is a string graph, then there exists a string* n*-SuperHyperGraph* $\mathrm{SHG}^{(n)} = (V, E)$ *such that*

$$\Pr(\mathrm{SHG}^{(n)}) \;\cong\; G.$$

Proof. Since G is a string graph, there exists a family of strings $\{C_v\}_{v \in V_G}$ such that for distinct $u, v \in V_G$,

$$\{u, v\} \in E_G \iff C_u \cap C_v \neq \emptyset.$$

Let $V_0 := V_G$ and define the iterated singleton embedding $\iota_0(v) := v$ and $\iota_{k+1}(v) := \{\iota_k(v)\}$. Set

$$V \;:=\; \{\iota_n(v) : v \in V_G\} \subseteq \mathcal{P}^n(V_0), \qquad \rho(\iota_n(v)) \;:=\; C_v.$$

Define

$$E \;:=\; \Big\{\varepsilon \subseteq V : \;|\varepsilon| \geq 2 \;\text{ and }\; \bigcap_{x \in \varepsilon} \rho(x) \neq \emptyset\Big\}.$$

Then $\mathrm{SHG}^{(n)} = (V, E)$ is a string n-SuperHyperGraph by Definition 3.5.7.

For distinct $u, v \in V_G$, the vertices $\iota_n(u)$ and $\iota_n(v)$ are adjacent in $\Pr(\mathrm{SHG}^{(n)})$ iff some $\varepsilon \in E$ contains both, which holds iff

$$\rho(\iota_n(u)) \cap \rho(\iota_n(v)) \neq \emptyset \iff C_u \cap C_v \neq \emptyset \iff \{u, v\} \in E_G.$$

Hence $u \mapsto \iota_n(u)$ is an isomorphism $G \cong \Pr(\mathrm{SHG}^{(n)})$. □

3.6 Disk intersection graphs

We next consider *disk graphs*, a prominent geometric intersection class in which vertices are represented by (closed) disks in the plane and adjacency corresponds to geometric overlap [122–124]. More broadly, disk graphs are instances of *geometric intersection graphs*; intersection graphs of many geometric objects have been studied extensively [118, 125]. In the same spirit, we define disk-based hypergraphs and n-SuperHyperGraphs by requiring a nonempty *common* intersection of the representing disks.

Definition 3.6.1 (Geometric intersection graph)**.** Let $\mathcal{O} = \{O_1, \ldots, O_m\}$ be a finite family of geometric objects in a fixed ambient space (e.g., $\mathbb{R}^2$). The *geometric intersection graph* of $\mathcal{O}$ is the graph $G = (V, E)$ defined by

$$V := \{1, \ldots, m\}, \qquad \{i, j\} \in E \iff i \neq j \;\text{ and }\; O_i \cap O_j \neq \emptyset.$$

Definition 3.6.2 (Disk and unit disk in $\mathbb{R}^2$)**.** A *(closed) disk* of radius $r > 0$ centered at $p \in \mathbb{R}^2$ is

$$D(p, r) \;:=\; \{x \in \mathbb{R}^2 : \;\|x - p\| \leq r\},$$

where $\|\cdot\|$ denotes the Euclidean norm. A *unit disk* is $D(p, 1)$. Two disks $D(p, r)$ and $D(q, s)$ intersect if and only if $\|p - q\| \leq r + s$; in particular, two unit disks intersect iff $\|p - q\| \leq 2$.

Among disk graphs, the *unit disk graph* (UDG) is a well-studied special case that frequently appears in both theory and applications [37,126–128]. Related concepts include *unit-ball graphs* and other geometric intersection graph classes [129–132].

Definition 3.6.3 (Unit disk graph)**.** [133,134] A graph $G = (V, E)$ is a *unit disk graph* if there exists an assignment $v \mapsto D(p_v, 1)$ of unit disks in $\mathbb{R}^2$ such that for distinct $u, v \in V$,

$$\{u, v\} \in E \iff D(p_u, 1) \cap D(p_v, 1) \neq \emptyset \iff \|p_u - p_v\| \leq 2.$$

Example 3.6.4 (A unit disk graph)**.** Let $V = \{a, b, c, d\}$ and place the centers

$$p_a = (0, 0), \quad p_b = (1.5, 0), \quad p_c = (3.6, 0), \quad p_d = (0, 2.4).$$

Assign to each vertex $v \in V$ the unit disk $D(p_v, 1) \subseteq \mathbb{R}^2$. Define $G = (V, E)$, where $\{u, v\} \in E$ iff $\|p_u - p_v\| \leq 2$.

Then

$$\|p_a - p_b\| = 1.5 \leq 2 \Rightarrow \{a, b\} \in E, \qquad \|p_b - p_c\| = 2.1 > 2 \Rightarrow \{b, c\} \notin E,$$
$$\|p_a - p_c\| = 3.6 > 2 \Rightarrow \{a, c\} \notin E, \qquad \|p_a - p_d\| = 2.4 > 2 \Rightarrow \{a, d\} \notin E,$$
$$\|p_b - p_d\| = \sqrt{(1.5)^2 + (2.4)^2} = \sqrt{8.01} > 2 \Rightarrow \{b, d\} \notin E,$$
$$\|p_c - p_d\| = \sqrt{(3.6)^2 + (2.4)^2} = \sqrt{18.72} > 2 \Rightarrow \{c, d\} \notin E.$$

Hence $E = \{\{a, b\}\}$, and G is a unit disk graph witnessed by the disks $\{D(p_v, 1)\}_{v \in V}$.

For a family $\{D_x\}_{x \in X}$ of subsets of $\mathbb{R}^2$ and $\varepsilon \subseteq X$, write

$$D(\varepsilon) := \bigcap_{x \in \varepsilon} D_x.$$

Definition 3.6.5 (Unit disk hypergraph (canonical intersection form))**.** Let X be a finite set. A hypergraph $H = (X, \mathcal{E})$ is called a *unit disk hypergraph* if there exists a family of unit disks $\{D_x\}_{x \in X}$ in $\mathbb{R}^2$ such that

$$\mathcal{E} = \Big\{\varepsilon \subseteq X : |\varepsilon| \geq 2 \text{ and } D(\varepsilon) \neq \emptyset\Big\}.$$

Thus each hyperedge is exactly a subfamily of disks with a nonempty common intersection.

Example 3.6.6 (A unit disk hypergraph (common-intersection form))**.** Let $X = \{1, 2, 3, 4\}$ and consider the unit disks

$$D_1 = D\big((0, 0), 1\big), \quad D_2 = D\big((1, 0), 1\big), \quad D_3 = D\big((0.5, 0.8), 1\big), \quad D_4 = D\big((4, 0), 1\big).$$

Set $H = (X, \mathcal{E})$, where $\varepsilon \subseteq X$ with $|\varepsilon| \geq 2$ is in $\mathcal{E}$ iff $\bigcap_{i \in \varepsilon} D_i \neq \emptyset$.

First, $D_1 \cap D_2 \cap D_3 \neq \emptyset$ (for instance, the point $(0.5, 0.3)$ lies in all three disks, since $\|(0.5, 0.3) - (0, 0)\| = \sqrt{0.34} \leq 1$, $\|(0.5, 0.3) - (1, 0)\| = \sqrt{0.34} \leq 1$, $\|(0.5, 0.3) - (0.5, 0.8)\| = 0.5 \leq 1$). Thus $\{1, 2, 3\} \in \mathcal{E}$, and consequently $\{1, 2\}, \{1, 3\}, \{2, 3\} \in \mathcal{E}$ as well.

On the other hand, D_4 is disjoint from each of D_1, D_2, D_3 (e.g. the center distance to $(0, 0)$ is $4 > 2$), hence no hyperedge containing 4 occurs. Therefore one possible explicit hyperedge family is

$$\mathcal{E} = \big\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}.$$

This realizes H as a unit disk hypergraph in the sense of Definition 3.6.5.

Definition 3.6.7 (Unit disk n-SuperHyperGraph)**.** Fix $n \in \mathbb{N}_0$ and a finite base set V_0. Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph, i.e.,

$$\emptyset \neq V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

We call $\mathrm{SHG}^{(n)}$ a *unit disk n-SuperHyperGraph* if there exists an injective map

$$\rho : V \longrightarrow \{\text{unit disks in } \mathbb{R}^2\}$$

such that the superedges are precisely the intersecting subcollections:

$$E = \Big\{\varepsilon \subseteq V : \ |\varepsilon| \geq 2 \ \text{ and } \bigcap_{x \in \varepsilon} \rho(x) \neq \emptyset\Big\}.$$

In this case, ρ is called a *unit disk representation* of $\mathrm{SHG}^{(n)}$.

Remark 3.6.8 (Pairwise-overlap variant)**.** Some geometric hypergraph formalisms use pairwise intersection rather than a common intersection point. This yields the alternative condition "$\rho(x) \cap \rho(y) \neq \emptyset$ for all distinct $x, y \in \varepsilon$". In Definitions 3.6.5–3.6.7 we adopt the common-intersection convention.

Example 3.6.9 (A unit disk 1-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b, c\}$ and take $n = 1$. Define the supervertex set

$$V = \big\{\{a\}, \{b\}, \{a, b\}\big\} \subseteq \mathcal{P}(V_0).$$

Assign to each $X \in V$ a unit disk $\rho(X)$ by choosing centers

$$p_{\{a\}} = (0, 0), \quad p_{\{b\}} = (1, 0), \quad p_{\{a,b\}} = (0.5, 0.7), \qquad \rho(X) := D(p_X, 1).$$

Define the superedge family

$$E := \Big\{\varepsilon \subseteq V : \ |\varepsilon| \geq 2 \ \text{ and } \bigcap_{X \in \varepsilon} \rho(X) \neq \emptyset\Big\}.$$

Then the triple intersection is nonempty: for example, the point $q = (0.5, 0.2)$ satisfies

$$\|q - p_{\{a\}}\| = \sqrt{0.29} \leq 1, \quad \|q - p_{\{b\}}\| = \sqrt{0.29} \leq 1, \quad \|q - p_{\{a,b\}}\| = 0.5 \leq 1,$$

so $q \in \rho(\{a\}) \cap \rho(\{b\}) \cap \rho(\{a, b\})$. Hence

$$\big\{\{a\}, \{b\}, \{a, b\}\big\} \in E,$$

and, in particular, all its 2-subsets are also in E. Therefore $\mathrm{SHG}^{(1)} = (V, E)$ is a unit disk 1-SuperHyperGraph (with representation ρ) in the sense of Definition 3.6.7.

Definition 3.6.10 (Primal graph (2-section))**.** For an n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$, its *primal graph* is the simple graph

$$\Pr(\mathrm{SHG}^{(n)}) \;:=\; (V, F), \qquad \{x, y\} \in F \iff x \neq y \text{ and } \exists\, \varepsilon \in E \text{ with } \{x, y\} \subseteq \varepsilon.$$

Theorem 3.6.11 (Unit disk n-SuperHyperGraphs generalize unit disk graphs)**.** *Fix $n \in \mathbb{N}_0$. If $G = (V_G, E_G)$ is a unit disk graph, then there exists a unit disk n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ such that*

$$\Pr(\mathrm{SHG}^{(n)}) \;\cong\; G.$$

Proof. Let $\{D_v\}_{v \in V_G}$ be a unit disk representation of G, so for distinct $u, v \in V_G$,

$$\{u, v\} \in E_G \iff D_u \cap D_v \neq \emptyset.$$

Set $V_0 := V_G$ and define the iterated singleton embedding $\iota_0(v) := v$, $\iota_{k+1}(v) := \{\iota_k(v)\}$. Let

$$V \;:=\; \{\iota_n(v) : v \in V_G\} \subseteq \mathcal{P}^n(V_0), \qquad \rho(\iota_n(v)) \;:=\; D_v.$$

Define

$$E \;:=\; \Big\{\varepsilon \subseteq V : \; |\varepsilon| \geq 2 \text{ and } \bigcap_{x \in \varepsilon} \rho(x) \neq \emptyset\Big\}.$$

Then $\mathrm{SHG}^{(n)} = (V, E)$ is a unit disk n-SuperHyperGraph by Definition 3.6.7.

For distinct $u, v \in V_G$, the vertices $\iota_n(u)$ and $\iota_n(v)$ are adjacent in $\Pr(\mathrm{SHG}^{(n)})$ iff there exists $\varepsilon \in E$ with $\{\iota_n(u), \iota_n(v)\} \subseteq \varepsilon$. By the construction of E, this holds in particular for $\varepsilon = \{\iota_n(u), \iota_n(v)\}$ exactly when

$$\rho(\iota_n(u)) \cap \rho(\iota_n(v)) \neq \emptyset \iff D_u \cap D_v \neq \emptyset \iff \{u, v\} \in E_G.$$

Hence the bijection $u \mapsto \iota_n(u)$ is a graph isomorphism $G \cong \Pr(\mathrm{SHG}^{(n)})$. □

3.7 Trapezoid graph

Trapezoid graphs constitute another geometric intersection family. Here one considers trapezoids drawn between two fixed parallel lines, and adjacency records whether the corresponding trapezoids overlap [135–137]. This class, and its algorithmic and structural properties, has been investigated in many works [138, 139]. A trapezoid hypergraph represents each vertex by a trapezoid, and each hyperedge by a nonempty common intersection of trapezoids. A trapezoid n-superhypergraph uses trapezoid-built nested supervertices, and forms superedges from nonempty common intersections among supervertices.

Definition 3.7.1 (Trapezoid between two parallel lines)**.** Fix two distinct parallel lines L_1 and L_2 in the plane. Choose affine parametrizations $\varphi_1, \varphi_2 : \mathbb{R} \to L_1, L_2$ (so $\varphi_i(t)$ is the point of L_i with coordinate t).

For real numbers $a \leq b$ and $c \leq d$, the (possibly degenerate) *trapezoid* determined by the intervals $[a, b] \subseteq L_1$ and $[c, d] \subseteq L_2$ is the convex set

$$T([a, b], [c, d]) \;:=\; \mathrm{conv}\{\varphi_1(a), \varphi_1(b), \varphi_2(c), \varphi_2(d)\} \;\subseteq\; \mathbb{R}^2.$$

Equivalently, after an affine change of coordinates one may assume $L_1 = \{(x, 0) : x \in \mathbb{R}\}$ and $L_2 = \{(x, 1) : x \in \mathbb{R}\}$, in which case

$$T([a, b], [c, d]) = \mathrm{conv}\{(a, 0), (b, 0), (c, 1), (d, 1)\}.$$

Example 3.7.2 (A trapezoid between two parallel lines)**.** Work in the affine coordinate system from Definition 3.7.1 with

$$L_1 = \{(x, 0) : x \in \mathbb{R}\}, \qquad L_2 = \{(x, 1) : x \in \mathbb{R}\}.$$

Choose intervals

$$[a, b] = [0, 2] \subseteq L_1, \qquad [c, d] = [1, 3] \subseteq L_2.$$

Then the trapezoid determined by these intervals is

$$T\big([0, 2], [1, 3]\big) = \mathrm{conv}\{(0, 0), (2, 0), (1, 1), (3, 1)\}.$$

It is a genuine (nondegenerate) trapezoid whose lower base is the segment from $(0, 0)$ to $(2, 0)$ on L_1 and whose upper base is the segment from $(1, 1)$ to $(3, 1)$ on L_2.

Definition 3.7.3 (Trapezoid n-SuperHyperGraph)**.** Fix $n \in \mathbb{N}_0$ and a finite base set V_0. Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph with

$$\emptyset \neq V \subseteq \mathcal{P}^n(V_0), \qquad \emptyset \neq E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

We call $\mathrm{SHG}^{(n)}$ a *trapezoid n-SuperHyperGraph* if there exist two fixed parallel lines L_1, L_2 and an injective map

$$\rho : V \longrightarrow \{\text{trapezoids between } L_1 \text{ and } L_2\}$$

such that the superedge family consists exactly of those subcollections of supervertices whose associated trapezoids have a common point:

$$E \;=\; \Big\{\varepsilon \subseteq V : \; |\varepsilon| \geq 2 \;\text{ and }\; \bigcap_{x \in \varepsilon} \rho(x) \neq \emptyset\Big\}.$$

In this case, ρ is called a *trapezoid representation* of $\mathrm{SHG}^{(n)}$.

Example 3.7.4 (A trapezoid 1-SuperHyperGraph)**.** Let $V_0 = \{a, b, c\}$ and take $n = 1$. Define the supervertex set

$$V = \big\{\{a\}, \{b\}, \{c\}\big\} \subseteq \mathcal{P}(V_0).$$

Fix the same parallel lines

$$L_1 = \{(x, 0) : x \in \mathbb{R}\}, \qquad L_2 = \{(x, 1) : x \in \mathbb{R}\}.$$

Assign to each supervertex $X \in V$ a trapezoid $\rho(X)$ by

$$\rho(\{a\}) := T([0, 2], [0, 2]), \qquad \rho(\{b\}) := T([1, 3], [1, 3]), \qquad \rho(\{c\}) := T([0.5, 2.5], [0.5, 2.5]).$$

Each $\rho(X)$ is the convex hull of two horizontal segments on L_1 and L_2; in particular,

$$\rho(\{a\}) = \mathrm{conv}\{(0, 0), (2, 0), (0, 1), (2, 1)\}, \quad \rho(\{b\}) = \mathrm{conv}\{(1, 0), (3, 0), (1, 1), (3, 1)\},$$
$$\rho(\{c\}) = \mathrm{conv}\{(0.5, 0), (2.5, 0), (0.5, 1), (2.5, 1)\}.$$

Consider the point $p = (1.5, 0.5)$. Since $1.5 \in [0, 2] \cap [1, 3] \cap [0.5, 2.5]$, the point p lies in all three trapezoids. Hence

$$\rho(\{a\}) \cap \rho(\{b\}) \cap \rho(\{c\}) \neq \emptyset.$$

Now define the superedge family as in Definition 3.7.3:

$$E := \Big\{\varepsilon \subseteq V : \; |\varepsilon| \geq 2 \;\text{ and }\; \bigcap_{X \in \varepsilon} \rho(X) \neq \emptyset\Big\}.$$

Then, in particular,

$$\big\{\{a\}, \{b\}, \{c\}\big\} \in E,$$

and consequently all of its 2-subsets are also in E. Therefore $\mathrm{SHG}^{(1)} = (V, E)$ is a trapezoid 1-SuperHyperGraph with trapezoid representation ρ.

Theorem 3.7.5 (Trapezoid n-SuperHyperGraphs generalize trapezoid graphs)**.** *Let $G = (V_G, E_G)$ be a trapezoid graph. Then, for every $n \in \mathbb{N}_0$, there exists a trapezoid n-SuperHyperGraph* $\mathrm{SHG}^{(n)} = (V, E)$ *such that*

$$\Pr(\mathrm{SHG}^{(n)}) \;\cong\; G.$$

Consequently, trapezoid n-SuperHyperGraphs extend the trapezoid graph class via the primal-graph reduction.

Proof. Since G is a trapezoid graph, there exist two parallel lines L_1, L_2 and a family of trapezoids $\{T_v\}_{v\in V_G}$ between L_1 and L_2 such that, for any distinct $u, v \in V_G$,

$$\{u, v\} \in E_G \quad\Longleftrightarrow\quad T_u \cap T_v \neq \emptyset.$$

Let $V_0 := V_G$ and define the iterated singleton embedding $\iota_0(v) := v$ and $\iota_{k+1}(v) := \{\iota_k(v)\}$. Set

$$V \;:=\; \{\iota_n(v) : v \in V_G\} \;\subseteq\; \mathcal{P}^n(V_0), \qquad \rho(\iota_n(v)) \;:=\; T_v.$$

Define the superedge family

$$E \;:=\; \Big\{\varepsilon \subseteq V :\; |\varepsilon| \geq 2 \;\text{ and }\; \bigcap_{x\in\varepsilon} \rho(x) \neq \emptyset\Big\}.$$

Then $\mathrm{SHG}^{(n)} = (V, E)$ is a trapezoid n-SuperHyperGraph by Definition 3.7.3.

We claim $\Pr(\mathrm{SHG}^{(n)}) \cong G$ via the bijection $v \mapsto \iota_n(v)$. Take distinct $u, v \in V_G$. In $\Pr(\mathrm{SHG}^{(n)})$, the vertices $\iota_n(u)$ and $\iota_n(v)$ are adjacent iff there exists $\varepsilon \in E$ with $\{\iota_n(u), \iota_n(v)\} \subseteq \varepsilon$. Because E contains all intersecting subcollections, this holds in particular when the pair intersects:

$$\rho(\iota_n(u)) \cap \rho(\iota_n(v)) \neq \emptyset \quad\Longrightarrow\quad \{\iota_n(u), \iota_n(v)\} \in E \quad\Longrightarrow\quad \{\iota_n(u), \iota_n(v)\} \in E(\Pr(\mathrm{SHG}^{(n)})).$$

Conversely, if $\{\iota_n(u), \iota_n(v)\} \in E(\Pr(\mathrm{SHG}^{(n)}))$, then some $\varepsilon \in E$ contains both, hence

$$\rho(\iota_n(u)) \cap \rho(\iota_n(v)) \;\supseteq\; \bigcap_{x\in\varepsilon} \rho(x) \;\neq\; \emptyset.$$

Therefore,

$$\{u, v\} \in E_G \iff T_u \cap T_v \neq \emptyset \iff$$
$$\rho(\iota_n(u)) \cap \rho(\iota_n(v)) \neq \emptyset \iff \{\iota_n(u), \iota_n(v)\} \in E(\Pr(\mathrm{SHG}^{(n)})),$$

so the bijection $v \mapsto \iota_n(v)$ is a graph isomorphism $\Pr(\mathrm{SHG}^{(n)}) \cong G$. □

3.8 Intersection digraph

Graphs, hypergraphs, and SuperHyperGraphs can each be extended to their directed counterparts, namely directed graphs [140, 141], directed hypergraphs [142, 143], and directed SuperHyperGraphs [15, 79, 80, 144, 145], respectively. Intersection notions also admit directed analogues. An *intersection digraph* is a directed graph in which each vertex carries two sets, and an arc $v_i \to v_j$ is present exactly when the first set of v_i meets the second set of v_j (cf. [146–148]). An intersection directed hypergraph assigns each vertex two sets, and creates a directed hyperarc when source-sets intersect target-sets. An intersection directed n-superhypergraph uses nested supervertices with source/target sets, forming directed superhyperarcs whenever sources intersect targets.

Definition 3.8.1 (Intersection digraph)**.** [148] An *intersection digraph* is a directed graph $D = (V, A)$ for which there exist two families of sets $S = \{S_1, \dots, S_n\}$ and $T = \{T_1, \dots, T_n\}$ and a vertex set $V = \{v_1, \dots, v_n\}$ such that each vertex v_i is associated with the ordered pair (S_i, T_i), and

$$(v_i, v_j) \in A \quad \Longleftrightarrow \quad S_i \cap T_j \neq \emptyset.$$

Equivalently, there is an arc from v_i to v_j precisely when the "source" set S_i of v_i has nonempty intersection with the "target" set T_j of v_j.

Example 3.8.2 (Intersection digraph)**.** Let the universe be $U := \{1, 2, 3\}$ and let $V := \{v_1, v_2, v_3\}$. Assign to each vertex v_i an ordered pair (S_i, T_i) of subsets of U by

$$(S_1, T_1) = \big(\{1\}, \{2\}\big), \qquad (S_2, T_2) = \big(\{2, 3\}, \{1\}\big), \qquad (S_3, T_3) = \big(\{2\}, \{2, 3\}\big).$$

Define a directed graph $D = (V, A)$ by the rule of Definition 3.8.1:

$$(v_i, v_j) \in A \iff S_i \cap T_j \neq \emptyset.$$

Then the arc set is

$$A = \{(v_2, v_1), (v_3, v_1), (v_1, v_3), (v_3, v_3)\}.$$

Indeed, for example, $(v_2, v_1) \in A$ since $S_2 \cap T_1 = \{2, 3\} \cap \{2\} = \{2\} \neq \emptyset$, while $(v_1, v_2) \notin A$ because $S_1 \cap T_2 = \{1\} \cap \{1\} = \{1\} \neq \emptyset$ would hold only if $T_2 = \{1\}$ were replaced by $\{3\}$, which it is not. Hence D is an intersection digraph with representation $(U, \{(S_i, T_i)\}_{i=1}^{3})$.

Definition 3.8.3 (Intersection directed hypergraph)**.** A directed hypergraph $H = (V, \mathcal{A})$ is called an *intersection directed hypergraph* if there exist a universe U and, for each $v \in V$, two (possibly different) sets $S_v, T_v \subseteq U$ such that for every $(X, Y) \in \mathcal{P}^*(V) \times \mathcal{P}^*(V)$,

$$(X, Y) \in \mathcal{A} \quad \Longleftrightarrow \quad \Big(\bigcup_{x \in X} S_x\Big) \cap \Big(\bigcup_{y \in Y} T_y\Big) \neq \emptyset.$$

Example 3.8.4 (Intersection directed hypergraph)**.** Let $U := \{a, b, c\}$ and $V := \{v_1, v_2, v_3\}$. Assign source/target sets to vertices by

$$S_{v_1} = \{a\}, \quad T_{v_1} = \{b\}, \qquad S_{v_2} = \{b\}, \quad T_{v_2} = \{c\}, \qquad S_{v_3} = \{c\}, \quad T_{v_3} = \{a\}.$$

Define a directed hypergraph $H = (V, \mathcal{A})$ by

$$(X, Y) \in \mathcal{A} \iff \Big(\bigcup_{x \in X} S_x\Big) \cap \Big(\bigcup_{y \in Y} T_y\Big) \neq \emptyset \qquad (X, Y \subseteq V,\ X, Y \neq \emptyset).$$

Consider, for instance,

$$X = \{v_1, v_2\}, \qquad Y = \{v_1, v_3\}.$$

Then

$$\bigcup_{x \in X} S_x = S_{v_1} \cup S_{v_2} = \{a\} \cup \{b\} = \{a, b\}, \qquad \bigcup_{y \in Y} T_y = T_{v_1} \cup T_{v_3} = \{b\} \cup \{a\} = \{a, b\},$$

so the intersection is $\{a, b\} \neq \emptyset$, and therefore $(X, Y) \in \mathcal{A}$. On the other hand, if $X = \{v_1\}$ and $Y = \{v_2\}$, then

$$\bigcup_{x \in X} S_x = \{a\}, \qquad \bigcup_{y \in Y} T_y = \{c\},$$

so $(\{v_1\}, \{v_2\}) \notin \mathcal{A}$. Thus H is an intersection directed hypergraph in the sense of Definition 3.8.3.

Definition 3.8.5 (Directed n-SuperHyperGraph)**.** Let V_0 be a finite base set and let $n \in \mathbb{N}_0$. A *directed n-SuperHyperGraph* is a pair

$$\overrightarrow{\mathrm{SHG}}^{(n)} = (V, \mathcal{A}),$$

where $\emptyset \neq V \subseteq \mathcal{P}^n(V_0)$ is the set of n-supervertices and

$$\mathcal{A} \subseteq \mathcal{P}^*(V) \times \mathcal{P}^*(V) \qquad \big(\mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\}\big)$$

is a finite family of ordered pairs $a = (X, Y)$ with nonempty $X, Y \subseteq V$. Each $a = (X, Y) \in \mathcal{A}$ is called a *directed n-superhyperedge* (or *superhyperarc*), with *tail* X and *head* Y.

Optionally, one may impose $X \cap Y = \emptyset$ for all $(X, Y) \in \mathcal{A}$; we do not require this here.

Definition 3.8.6 (Intersection directed n-SuperHyperGraph)**.** A directed n-SuperHyperGraph $\overrightarrow{\mathrm{SHG}}^{(n)} = (V, \mathcal{A})$ is called an *intersection directed n-SuperHyperGraph* if there exist a universe U and two set-valued maps

$$S, T : V \longrightarrow \mathcal{P}(U) \setminus \{\emptyset\}, \qquad x \longmapsto S_x, \quad x \longmapsto T_x,$$

such that, for every nonempty $X, Y \subseteq V$,

$$(X, Y) \in \mathcal{A} \iff \Big(\bigcup_{x \in X} S_x\Big) \cap \Big(\bigcup_{y \in Y} T_y\Big) \neq \emptyset. \tag{3.1}$$

We call $(U, \{(S_x, T_x)\}_{x \in V})$ an *intersection representation* of $\overrightarrow{\mathrm{SHG}}^{(n)}$.

Example 3.8.7 (Intersection directed 1-SuperHyperGraph)**.** Let $V_0 := \{1, 2, 3\}$ and set $n = 1$. Consider the 1-supervertex set

$$V := \big\{\{1\}, \{2\}, \{3\}\big\} \subseteq \mathcal{P}(V_0).$$

Let the universe be $U := \{p, q\}$ and define set-valued maps $S, T : V \to \mathcal{P}(U) \setminus \{\emptyset\}$ by

$$S_{\{1\}} = \{p\}, \quad T_{\{1\}} = \{q\}, \qquad S_{\{2\}} = \{q\}, \quad T_{\{2\}} = \{p\}, \qquad S_{\{3\}} = \{p, q\}, \quad T_{\{3\}} = \{q\}.$$

Now define the superhyperarc family $\mathcal{A} \subseteq \mathcal{P}^*(V) \times \mathcal{P}^*(V)$ by the equivalence of Definition 3.8.6:

$$(X, Y) \in \mathcal{A} \iff \Big(\bigcup_{x \in X} S_x\Big) \cap \Big(\bigcup_{y \in Y} T_y\Big) \neq \emptyset.$$

Then, for example, with $X = \{\{1\}, \{2\}\}$ and $Y = \{\{1\}\}$ we have

$$\bigcup_{x \in X} S_x = S_{\{1\}} \cup S_{\{2\}} = \{p\} \cup \{q\} = \{p, q\}, \qquad \bigcup_{y \in Y} T_y = T_{\{1\}} = \{q\},$$

so $(X, Y) \in \mathcal{A}$. Likewise, with $X = \{\{2\}\}$ and $Y = \{\{2\}\}$,

$$\bigcup_{x \in X} S_x = \{q\}, \qquad \bigcup_{y \in Y} T_y = \{p\},$$

so $(X, Y) \notin \mathcal{A}$. Hence $\overrightarrow{\mathrm{SHG}}^{(1)} = (V, \mathcal{A})$ is an intersection directed 1-SuperHyperGraph with intersection representation $(U, \{(S_x, T_x)\}_{x \in V})$.

Remark 3.8.8 (Singleton reduction recovers the usual intersection digraph)**.** Given a directed n-SuperHyperGraph $\overrightarrow{\mathrm{SHG}}^{(n)} = (V, \mathcal{A})$, define its *singleton digraph* $\mathrm{Sing}(\overrightarrow{\mathrm{SHG}}^{(n)}) = (V, A_{\mathrm{Sing}})$ by

$$(x, y) \in A_{\mathrm{Sing}} \iff (\{x\}, \{y\}) \in \mathcal{A}.$$

If $\overrightarrow{\mathrm{SHG}}^{(n)}$ is intersection directed with witness sets (S_x, T_x), then

$$(x, y) \in A_{\mathrm{Sing}} \iff S_x \cap T_y \neq \emptyset,$$

so $\mathrm{Sing}(\overrightarrow{\mathrm{SHG}}^{(n)})$ is an intersection digraph in the sense of Definition 3.8.1.

Theorem 3.8.9 (Intersection directed n-SuperHyperGraphs generalize the directed graph and hypergraph cases)**.**

1. ***(Directed graph case)*** *Every intersection digraph $D = (V, A)$ (Definition 3.8.1) can be realized as the singleton digraph of an intersection directed 0-SuperHyperGraph.*

2. ***(Directed hypergraph case)*** *Every intersection directed hypergraph $H = (V, \mathcal{A})$ (Definition 3.8.3) is an intersection directed 0-SuperHyperGraph in the sense of Definition 3.8.6.*

Proof. **(1)** Let $D = (V, A)$ be an intersection digraph. Then there exist families $\{S_v\}_{v \in V}$ and $\{T_v\}_{v \in V}$ such that $(u, v) \in A \iff S_u \cap T_v \neq \emptyset$. Take $V_0 := V$ and $n := 0$, so $\mathcal{P}^0(V_0) = V_0$. Define a directed 0-SuperHyperGraph $\overrightarrow{\mathrm{SHG}}^{(0)} = (V, \mathcal{A})$ by

$$\mathcal{A} := \{(\{u\}, \{v\}) : (u, v) \in A\}.$$

Then for any $u, v \in V$,

$$(\{u\}, \{v\}) \in \mathcal{A} \iff (u, v) \in A \iff S_u \cap T_v \neq \emptyset,$$

which is exactly (3.1) restricted to singletons. Hence $\overrightarrow{\mathrm{SHG}}^{(0)}$ is intersection directed, and by construction $\mathrm{Sing}(\overrightarrow{\mathrm{SHG}}^{(0)}) = D$.

(2) Let $H = (V, \mathcal{A})$ be an intersection directed hypergraph with witness sets $\{S_v\}_{v \in V}$ and $\{T_v\}_{v \in V}$ satisfying Definition 3.8.3. Again set $V_0 := V$ and $n := 0$, so $V \subseteq \mathcal{P}^0(V_0)$ holds. Then the same witness sets satisfy (3.1) for all nonempty $X, Y \subseteq V$, so H is an intersection directed 0-SuperHyperGraph in the sense of Definition 3.8.6. □

3.9 Fuzzy Intersection SuperHyperGraph

A fuzzy set assigns each element a membership degree in $[0, 1]$, quantifying partial belonging rather than crisp inclusion [149]. Various extended notions are also well known, including HyperFuzzy sets [150, 151], SuperHyperFuzzy sets [152], Bipolar fuzzy sets [153, 154], Pythagorean fuzzy sets [155, 156], hesitant fuzzy sets [157, 158], picture fuzzy sets [159, 160], and spherical fuzzy sets [161]. A fuzzy graph assigns membership degrees to vertices and edges, with edge degrees bounded by endpoint vertex memberships for consistency [162, 163]. A fuzzy intersection graph represents each vertex by a fuzzy set, and links vertices when their supports intersect nonemptily [164–166]. A fuzzy intersection superhypergraph uses supervertices as nested fuzzy sets, adds superedges for nonempty common intersections, and assigns memberships.

Definition 3.9.1 (Fuzzy set and fuzzy intersection)**.** Let U be a nonempty set (the universe). A *fuzzy set* on U is a map

$$\mu_A : U \to [0,1].$$

Its *support* is $\mathrm{supp}(A) := \{u \in U : \mu_A(u) > 0\}$, and A is called *nonempty* if $\mathrm{supp}(A) \neq \emptyset$.

Fix a t-norm $\otimes : [0,1]^2 \to [0,1]$ (e.g. $\otimes = \min$). For fuzzy sets A, B on U, their (chosen) *fuzzy intersection* $A \cap_\otimes B$ is the fuzzy set with

$$\mu_{A\cap_\otimes B}(u) \;=\; \mu_A(u) \otimes \mu_B(u) \qquad (u \in U).$$

More generally, for a finite family $\{A_i\}_{i\in I}$ one sets

$$\mu_{\bigcap_{\otimes\, i\in I} A_i}(u) \;=\; \bigotimes_{i\in I} \mu_{A_i}(u),$$

where $\bigotimes$ denotes iterated application of $\otimes$ (well-defined since I is finite).

Example 3.9.2 (Fuzzy sets and fuzzy intersection)**.** Let the universe be $U := \{a, b, c\}$ and choose the t-norm $\otimes = \min$. Define fuzzy sets A, B on U by the membership tables

$$\mu_A(a) = 0.8,\ \mu_A(b) = 0.2,\ \mu_A(c) = 0, \qquad \mu_B(a) = 0.4,\ \mu_B(b) = 0,\ \mu_B(c) = 0.6.$$

Then the fuzzy intersection $A \cap_{\min} B$ satisfies

$$\mu_{A\cap_{\min} B}(a) = \min(0.8, 0.4) = 0.4, \quad \mu_{A\cap_{\min} B}(b) = \min(0.2, 0) = 0, \quad \mu_{A\cap_{\min} B}(c) = \min(0, 0.6) = 0,$$

so $\mathrm{supp}(A \cap_{\min} B) = \{a\} \neq \emptyset$.

Definition 3.9.3 (Fuzzy intersection graph)**.** Let $\mathcal{S}_f = \{S_1, \ldots, S_m\}$ be a finite family of nonempty fuzzy sets on a universe U, and fix a t-norm $\otimes$. For each $1 \le i < j \le m$ define the fuzzy set

$$E_{ij} \;:=\; S_i \cap_\otimes S_j.$$

The *fuzzy intersection graph* generated by $\mathcal{S}_f$ (with respect to $\otimes$) is the pair

$$\mathcal{Z}_\otimes(\mathcal{S}_f) \;:=\; (\mathcal{S}_f, \mathcal{E}_f), \qquad \mathcal{E}_f \;:=\; \{\, E_{ij} :\ 1 \le i < j \le m,\ \mathrm{supp}(E_{ij}) \neq \emptyset \,\}.$$

Equivalently, one may record the underlying *crisp* intersection graph on vertex set $[m] := \{1, \ldots, m\}$ by declaring $\{i, j\}$ to be an edge if and only if $\mathrm{supp}(E_{ij}) \neq \emptyset$.

Example 3.9.4 (Fuzzy intersection graph)**.** Let $U := \{a, b, c\}$ and fix $\otimes = \min$. Consider three nonempty fuzzy sets $\mathcal{S}_f = \{S_1, S_2, S_3\}$ on U given by

u	a	b	c
$\mu_{S_1}(u)$	0.7	0.1	0
$\mu_{S_2}(u)$	0.2	0.6	0
$\mu_{S_3}(u)$	0.5	0	0.9

For $i < j$, set $E_{ij} := S_i \cap_{\min} S_j$. A direct computation gives

$$\mathrm{supp}(E_{12}) = \{a, b\}, \qquad \mathrm{supp}(E_{13}) = \{a\}, \qquad \mathrm{supp}(E_{23}) = \{a, c\}.$$

Hence all three pairwise intersections are nonempty, and the fuzzy intersection graph $\mathcal{Z}_{\min}(\mathcal{S}_f) = (\mathcal{S}_f, \mathcal{E}_f)$ has

$$\mathcal{E}_f = \{E_{12}, E_{13}, E_{23}\}.$$

Equivalently, the underlying crisp intersection graph on vertex set $\{1, 2, 3\}$ is the triangle K_3.

Definition 3.9.5 (Fuzzy intersection hypergraph)**.** Let $\mathcal{S}_f = \{S_1, \ldots, S_m\}$ be as above and fix $\otimes$. For each index set $I \subseteq [m]$ with $|I| \geq 2$, define the fuzzy set

$$E_I := \bigcap_{\otimes\, i \in I} S_i.$$

The *fuzzy intersection hypergraph* generated by $\mathcal{S}_f$ is the labeled hypergraph

$$\mathcal{H}_\otimes(\mathcal{S}_f) := \big([m],\ \mathcal{E},\ \lambda\big),$$

where

$$\mathcal{E} := \{\, I \subseteq [m] :\ |I| \geq 2,\ \operatorname{supp}(E_I) \neq \emptyset \,\},$$
$$\lambda(I) := E_I \quad (I \in \mathcal{E}).$$

Thus a hyperedge is present exactly when the corresponding subfamily has nonempty fuzzy intersection, and the label $\lambda(I)$ retains the resulting intersection fuzzy set.

Example 3.9.6 (Fuzzy intersection hypergraph)**.** Let $U := \{a, b, c\}$ and $\otimes = \min$, and keep the same family $\mathcal{S}_f = \{S_1, S_2, S_3\}$ from Example 3.9.4. For each $I \subseteq [3]$ with $|I| \geq 2$, set

$$E_I := \bigcap_{\min\, i \in I} S_i.$$

We already have $\operatorname{supp}(E_{\{1,2\}}), \operatorname{supp}(E_{\{1,3\}}), \operatorname{supp}(E_{\{2,3\}}) \neq \emptyset$. For the triple intersection,

$$\mu_{E_{\{1,2,3\}}}(a) = \min(0.7, 0.2, 0.5) = 0.2,$$
$$\mu_{E_{\{1,2,3\}}}(b) = \min(0.1, 0.6, 0) = 0,$$
$$\mu_{E_{\{1,2,3\}}}(c) = \min(0, 0, 0.9) = 0,$$

so $\operatorname{supp}(E_{\{1,2,3\}}) = \{a\} \neq \emptyset$. Therefore the fuzzy intersection hypergraph

$$\mathcal{H}_{\min}(\mathcal{S}_f) = \big([3], \mathcal{E}, \lambda\big)$$

has hyperedge family

$$\mathcal{E} = \big\{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\big\},$$

and labels $\lambda(I) = E_I$ for each $I \in \mathcal{E}$.

Definition 3.9.7 (Intersection-evaluation of n-supervertices)**.** Let U be a universe and let $V_0 = \{S_1, \ldots, S_m\}$ be a finite family of fuzzy sets on U. For $n \in \mathbb{N}_0$, define $\mathcal{P}^0(V_0) = V_0$ and $\mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0))$.

Fix a t-norm $\otimes$. Define recursively a map

$$\mathcal{I}_\otimes^{(n)} : \mathcal{P}^n(V_0) \setminus \{\emptyset\} \longrightarrow \{\text{fuzzy sets on } U\}$$

by:

- for $n = 0$ and $x = S_i \in V_0$, set $\mathcal{I}_\otimes^{(0)}(x) := S_i$;
- for $n \geq 1$ and $x \in \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ (so $x \subseteq \mathcal{P}^{n-1}(V_0)$), set

$$\mathcal{I}_\otimes^{(n)}(x) := \bigcap_{\otimes\, y \in x}$$

$$\mathcal{I}_\otimes^{(n-1)}(y).$$

Definition 3.9.8 (Fuzzy intersection n-superhypergraph)**.** Let U be a universe, let $V_0 = \{S_1, \ldots, S_m\}$ be a finite family of fuzzy sets on U, fix $n \in \mathbb{N}_0$, and fix a t-norm $\otimes$. Let $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ be a finite nonempty set (the *n-supervertex set*).

For each supervertex $x \in V$, define its associated fuzzy set

$$\Lambda_V(x) \ := \ \mathcal{I}_{\otimes}^{(n)}(x).$$

For any nonempty $\varepsilon \subseteq V$ with $|\varepsilon| \geq 2$, define the fuzzy set

$$\Lambda_E(\varepsilon) \ := \ \bigcap_{\otimes\, x \in \varepsilon} \Lambda_V(x).$$

The *fuzzy intersection n-superhypergraph* generated by (V_0, V) is the labeled n-SuperHyperGraph

$$\mathcal{S}_{\otimes}^{(n)}(V_0; V) \ := \ \big(V,\ E,\ \Lambda_V,\ \Lambda_E\big),$$

where the superedge family is

$$E \ := \ \{\, \varepsilon \subseteq V : \ |\varepsilon| \geq 2,\ \operatorname{supp}(\Lambda_E(\varepsilon)) \neq \emptyset \,\}.$$

Thus ε is a superedge exactly when the supervertices in ε have a nonempty fuzzy intersection (after evaluation to fuzzy sets on U).

Example 3.9.9 (Fuzzy intersection 1-superhypergraph)**.** Let $U := \{a, b, c\}$, fix $\otimes = \min$, and let

$$V_0 := \{S_1, S_2, S_3\}$$

be the same three fuzzy sets as in Example 3.9.4. Take $n = 1$, so $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$, and define a supervertex set

$$V := \{x_1, x_2, x_3\} \subseteq \mathcal{P}(V_0) \setminus \{\emptyset\}, \qquad x_1 := \{S_1, S_2\}, \ \ x_2 := \{S_2, S_3\}, \ \ x_3 := \{S_1, S_3\}.$$

By Definition 3.9.7 (with $n = 1$),

$$\Lambda_V(x) = \mathcal{I}_{\min}^{(1)}(x) = \bigcap_{\min\ y \in x} y,$$

so

$$\Lambda_V(x_1) = S_1 \cap_{\min} S_2, \qquad \Lambda_V(x_2) = S_2 \cap_{\min} S_3, \qquad \Lambda_V(x_3) = S_1 \cap_{\min} S_3.$$

Each of these has nonempty support (see Example 3.9.4). Now consider the superedge $\varepsilon := \{x_1, x_2\}$. Its label is

$$\Lambda_E(\varepsilon) = \Lambda_V(x_1) \cap_{\min} \Lambda_V(x_2) = (S_1 \cap_{\min} S_2) \cap_{\min} (S_2 \cap_{\min} S_3) = S_1 \cap_{\min} S_2 \cap_{\min} S_3,$$

whose support is $\{a\} \neq \emptyset$ (Example 3.9.6). Hence $\{x_1, x_2\} \in E$.

In fact, with this choice of V, every pair among $\{x_1, x_2, x_3\}$ has nonempty fuzzy intersection, and also the triple $\{x_1, x_2, x_3\}$ satisfies

$$\operatorname{supp}\big(\Lambda_E(\{x_1, x_2, x_3\})\big) \neq \emptyset.$$

Therefore the fuzzy intersection 1-superhypergraph

$$\mathcal{S}_{\min}^{(1)}(V_0; V) = \big(V, E, \Lambda_V, \Lambda_E\big)$$

has superedge family

$$E = \big\{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\big\}.$$

Theorem 3.9.10 (Generalization of fuzzy intersection graphs and hypergraphs)**.** *Fix a universe U, a finite family $V_0 = \{S_1, \ldots, S_m\}$ of fuzzy sets on U, and a t-norm $\otimes$. Let $n = 0$ and $V = V_0$ in Definition 3.9.8.*

1. *The labeled hypergraph obtained from $\mathcal{S}_{\otimes}^{(0)}(V_0; V_0)$ by identifying each vertex S_i with the index $i \in [m]$ coincides with the fuzzy intersection hypergraph $\mathcal{H}_{\otimes}(\mathcal{S}_f)$ of Definition 3.9.5.*

2. *If one restricts further to the 2-uniform part (i.e. only superedges ε with $|\varepsilon| = 2$), then $\mathcal{S}_{\otimes}^{(0)}(V_0; V_0)$ recovers the fuzzy intersection graph $\mathcal{Z}_{\otimes}(\mathcal{S}_f)$ of Definition 3.9.3.*

Hence fuzzy intersection n-superhypergraphs strictly generalize both fuzzy intersection graphs and fuzzy intersection hypergraphs.

Proof. Let $n = 0$ and $V = V_0$. Then each 0-supervertex is a fuzzy set S_i, and by Definition 3.9.7 we have $\Lambda_V(S_i) = S_i$.

(1) Take any $I \subseteq [m]$ with $|I| \geq 2$ and consider the corresponding subset $\varepsilon_I := \{S_i : i \in I\} \subseteq V_0$. By Definition 3.9.8,

$$\Lambda_E(\varepsilon_I) = \bigcap_{\otimes\, S_i \in \varepsilon_I} \Lambda_V(S_i) = \bigcap_{\otimes\, i \in I} S_i = E_I.$$

Moreover, $\varepsilon_I \in E$ if and only if $\mathrm{supp}(\Lambda_E(\varepsilon_I)) \neq \emptyset$, i.e. iff $\mathrm{supp}(E_I) \neq \emptyset$, which is exactly the hyperedge criterion in Definition 3.9.5. Identifying S_i with i yields the same labeled hypergraph.

(2) Restricting E to 2-element subsets $\{S_i, S_j\}$ yields labeled edges with label $\Lambda_E(\{S_i, S_j\}) = S_i \cap_{\otimes} S_j = E_{ij}$, present precisely when $\mathrm{supp}(E_{ij}) \neq \emptyset$, which is the edge rule of Definition 3.9.3. □

3.10 Neutrosophic Intersection SuperHyperGraph

A neutrosophic set assigns each element truth, indeterminacy, and falsity degrees, jointly modeling acceptance, uncertainty, and rejection simultaneously, explicitly, quantitatively [167–169]. As further extensions, double-valued neutrosophic sets [170, 171], bipolar neutrosophic sets [172–174], hesitant neutrosophic sets [175, 176], quadripartitioned neutrosophic sets [177, 178], and plithogenic sets [179–182] are also well known.

A neutrosophic graph annotates vertices and edges with truth, indeterminacy, and falsity degrees, capturing uncertain adjacency and structure for networks [168]. A neutrosophic intersection graph represents each vertex by a set with neutrosophic degrees, and connects vertices whose sets intersect nonemptily. A neutrosophic intersection superhypergraph uses nested set supervertices with neutrosophic degrees, forming superedges from nonempty common intersections across levels consistently.

Definition 3.10.1 (Truth α-intersection of single-valued neutrosophic sets)**.** Let U be a nonempty universe and let A, B be single-valued neutrosophic sets on U with $(T_A, I_A, F_A), (T_B, I_B, F_B) : U \to [0,1]^3$. Fix $\alpha \in (0,1]$. We say that A and B *intersect at truth level* α, and write $A \cap_\alpha B \neq \emptyset$, if

$$A^{(\alpha)} \cap B^{(\alpha)} \neq \emptyset, \quad \text{where} \quad A^{(\alpha)} := \{u \in U : T_A(u) \geq \alpha\} \text{ and } B^{(\alpha)} := \{u \in U : T_B(u) \geq \alpha\}.$$

Equivalently, $A \cap_\alpha B \neq \emptyset$ holds iff there exists $u \in U$ such that

$$\min\{T_A(u), T_B(u)\} \geq \alpha.$$

More generally, for a finite family $\{A_i\}_{i\in I}$ of SVNS on U we write $\bigcap_{i\in I}^{\alpha} A_i \neq \emptyset$ if $\bigcap_{i\in I} A_i^{(\alpha)} \neq \emptyset$.

Remark 3.10.2. Definition 3.10.1 uses truth α-cuts; other intersection conventions (e.g. involving (T, I, F) via an evaluation map) are possible.

Example 3.10.3 (Truth α-intersection of SVNS)**.** Let $U = \{u_1, u_2, u_3\}$ and fix $\alpha = 0.7$. Define two single-valued neutrosophic sets A, B on U by specifying their truth-memberships (the I, F components may be chosen arbitrarily in $[0,1]$ and play no role in this truth-α notion):

$$T_A(u_1) = 0.2, \quad T_A(u_2) = 0.8, \quad T_A(u_3) = 0.6, \qquad T_B(u_1) = 0.9, \quad T_B(u_2) = 0.75, \quad T_B(u_3) = 0.1.$$

Then

$$A^{(\alpha)} = \{u \in U : T_A(u) \geq 0.7\} = \{u_2\}, \qquad B^{(\alpha)} = \{u \in U : T_B(u) \geq 0.7\} = \{u_1, u_2\}.$$

Hence $A^{(\alpha)} \cap B^{(\alpha)} = \{u_2\} \neq \emptyset$, so $A \cap_\alpha B \neq \emptyset$. Equivalently, $\min\{T_A(u_2), T_B(u_2)\} = \min\{0.8, 0.75\} = 0.75 \geq 0.7$.

Definition 3.10.4 (Neutrosophic intersection graph (truth-α version))**.** Let U be a universe and let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a finite family of single-valued neutrosophic sets on U. Fix $\alpha \in (0,1]$. The *truth-α neutrosophic intersection graph* of $\mathcal{A}$ is the (crisp) graph

$$G_{\mathcal{A}}^{(\alpha)} := (V, E), \qquad V := \{v_1, \ldots, v_m\},$$

whose edge set is defined by

$$\{v_i, v_j\} \in E \iff A_i \cap_\alpha A_j \neq \emptyset \iff A_i^{(\alpha)} \cap A_j^{(\alpha)} \neq \emptyset \qquad (i \neq j).$$

Example 3.10.5 (Truth-α neutrosophic intersection graph)**.** Let $U = \{u_1, u_2, u_3\}$ and $\alpha = 0.7$. Consider three SVNS A_1, A_2, A_3 on U, specified (again, only) by their truth-memberships:

	u_1	u_2	u_3
T_{A_1}	0.9	0.2	0.6
T_{A_2}	0.1	0.8	0.72
T_{A_3}	0.75	0.65	0.1

Then the truth-α cuts are

$$A_1^{(\alpha)} = \{u_1\}, \qquad A_2^{(\alpha)} = \{u_2, u_3\}, \qquad A_3^{(\alpha)} = \{u_1\}.$$

Thus $A_1^{(\alpha)} \cap A_3^{(\alpha)} = \{u_1\} \neq \emptyset$, while $A_1^{(\alpha)} \cap A_2^{(\alpha)} = \emptyset$ and $A_2^{(\alpha)} \cap A_3^{(\alpha)} = \emptyset$. Therefore the truth-$\alpha$ neutrosophic intersection graph $G_{\mathcal{A}}^{(\alpha)}$ on vertices $\{v_1, v_2, v_3\}$ has the single edge $\{v_1, v_3\}$.

Definition 3.10.6 (Neutrosophic intersection hypergraph (truth-α version))**.** Let U be a universe and let

$$\mathcal{A} = \{A_1, \dots, A_m\}, \qquad \mathcal{B} = \{B_1, \dots, B_r\}$$

be finite families of single-valued neutrosophic sets on U. Fix $\alpha \in (0, 1]$. The *truth-α neutrosophic intersection hypergraph* of $\mathcal{A}$ with respect to $\mathcal{B}$ is the (crisp) hypergraph

$$H^{(\alpha)}_{\mathcal{A},\mathcal{B}} \;:=\; (V, \mathcal{E}), \qquad V := \{v_1, \dots, v_m\},$$

where for each $j \in \{1, \dots, r\}$ we define a hyperedge

$$e^{(\alpha)}_j \;:=\; \{\, v_i \in V : \; A_i \cap_\alpha B_j \neq \emptyset \,\} \;=\; \{\, v_i \in V : \; A^{(\alpha)}_i \cap B^{(\alpha)}_j \neq \emptyset \,\},$$

and set

$$\mathcal{E} \;:=\; \{\, e^{(\alpha)}_j : \; e^{(\alpha)}_j \neq \emptyset \,\}.$$

This is the neutrosophic analogue of the standard intersection-hypergraph construction (a hyperedge collects all objects that intersect a given test object). [1]

Example 3.10.7 (Truth-α neutrosophic intersection hypergraph)**.** Let $U = \{u_1, u_2, u_3\}$ and $\alpha = 0.7$. Let $\mathcal{A} = \{A_1, A_2, A_3\}$ be as in Example 3.10.5, so

$$A^{(\alpha)}_1 = \{u_1\}, \qquad A^{(\alpha)}_2 = \{u_2, u_3\}, \qquad A^{(\alpha)}_3 = \{u_1\}.$$

Define two "test" SVNS B_1, B_2 on U by their truth-α cuts

$$B^{(\alpha)}_1 = \{u_1, u_3\}, \qquad B^{(\alpha)}_2 = \{u_2\}.$$

(For concreteness one may take, e.g., $T_{B_1}(u_1) = 0.9, T_{B_1}(u_2) = 0.2, T_{B_1}(u_3) = 0.8$ and $T_{B_2}(u_1) = 0.1, T_{B_2}(u_2) = 0.85, T_{B_2}(u_3) = 0.2$.) Then the induced hyperedges are

$$e^{(\alpha)}_1 = \{v_i : \; A^{(\alpha)}_i \cap B^{(\alpha)}_1 \neq \emptyset\} = \{v_1, v_3\}, \qquad e^{(\alpha)}_2 = \{v_i : \; A^{(\alpha)}_i \cap B^{(\alpha)}_2 \neq \emptyset\} = \{v_2\}.$$

Hence $H^{(\alpha)}_{\mathcal{A},\mathcal{B}} = (V, \mathcal{E})$ has $V = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{\{v_1, v_3\}, \{v_2\}\}$. (If one prefers the convention $|e| \geq 2$, simply discard the singleton hyperedge $\{v_2\}$.)

Definition 3.10.8 (Neutrosophic intersection n-SuperHyperGraph (truth-α version))**.** Let V_0 be a finite base set and let $n \in \mathbb{N}_0$. Let $V \subseteq \mathcal{P}^n(V_0)$ be a finite set of n-supervertices. Let U be a universe and let

$$\mathcal{A} = \{A_x\}_{x \in V}, \qquad \mathcal{B} = \{B_\varepsilon\}_{\varepsilon \in \Lambda}$$

be indexed families of single-valued neutrosophic sets on U, where Λ is a finite index set. Fix $\alpha \in (0, 1]$.

Define a finite set of superedge-identifiers $E := \Lambda$ together with an incidence map

$$\partial^{(\alpha)} : E \longrightarrow \mathcal{P}(V) \setminus \{\emptyset\}, \qquad \partial^{(\alpha)}(\varepsilon) \;:=\; \{\, x \in V : \; A_x \cap_\alpha B_\varepsilon \neq \emptyset \,\}.$$

Then

$$\mathrm{SHG}^{(n,\alpha)}_{\mathcal{A},\mathcal{B}} \;:=\; \big(V, E, \partial^{(\alpha)}\big)$$

is called the *truth-α neutrosophic intersection n-SuperHyperGraph* induced by $(\mathcal{A}, \mathcal{B})$. (Here we use the standard n-SuperHyperGraph convention that superedges are encoded by identifiers together with an incidence map into $\mathcal{P}(V) \setminus \{\emptyset\}$. [1])

Example 3.10.9 (Truth-α neutrosophic intersection n-SuperHyperGraph)**.** Let $n = 1$ and let the base set be $V_0 = \{1, 2, 3\}$. Take the 1-supervertex set

$$V = \{x_1, x_2, x_3\} \subseteq \mathcal{P}(V_0), \qquad x_1 = \{1\}, \;\; x_2 = \{2, 3\}, \;\; x_3 = \{1, 2\}.$$

Let $U = \{u_1, u_2, u_3\}$ and fix $\alpha = 0.7$. Assign to each $x_i \in V$ an SVNS A_{x_i} on U via truth-α cuts

$$A^{(\alpha)}_{x_1} = \{u_1\}, \qquad A^{(\alpha)}_{x_2} = \{u_2\}, \qquad A^{(\alpha)}_{x_3} = \{u_1, u_3\}.$$

Let $\Lambda = \{\varepsilon_1, \varepsilon_2\}$ and define two test SVNS $B_{\varepsilon_1}, B_{\varepsilon_2}$ by

$$B^{(\alpha)}_{\varepsilon_1} = \{u_1\}, \qquad B^{(\alpha)}_{\varepsilon_2} = \{u_2, u_3\}.$$

Then the incidence map $\partial^{(\alpha)} : \Lambda \to \mathcal{P}(V) \setminus \{\emptyset\}$ becomes

$$\partial^{(\alpha)}(\varepsilon_1) = \{x \in V : \; A^{(\alpha)}_x \cap B^{(\alpha)}_{\varepsilon_1} \neq \emptyset\} = \{x_1, x_3\},$$

$$\partial^{(\alpha)}(\varepsilon_2) = \{x \in V : \; A^{(\alpha)}_x \cap B^{(\alpha)}_{\varepsilon_2} \neq \emptyset\} = \{x_2, x_3\}.$$

Therefore the truth-α neutrosophic intersection 1-SuperHyperGraph $\mathrm{SHG}^{(1,\alpha)}_{\mathcal{A},\mathcal{B}} = (V, E, \partial^{(\alpha)})$ has superedge-identifiers $E = \{\varepsilon_1, \varepsilon_2\}$ with

$$\partial^{(\alpha)}(\varepsilon_1) = \{x_1, x_3\}, \qquad \partial^{(\alpha)}(\varepsilon_2) = \{x_2, x_3\}.$$

Theorem 3.10.10 (Generalization properties)**.** *The neutrosophic intersection n-SuperHyperGraph construction in Definition 3.10.8 subsumes the usual neutrosophic and fuzzy intersection constructions in the following precise senses.*

1. ***(Neutrosophic intersection hypergraphs as the case*** $n = 0$***).*** *If* $n = 0$ *and* $V \subseteq \mathcal{P}^0(V_0) = V_0$*, then* $\mathrm{SHG}^{(0,\alpha)}_{\mathcal{A},\mathcal{B}}$ *is (canonically) a hypergraph on the vertex set* V*; in particular, taking* $V = \{v_1, \ldots, v_m\}$ *and reindexing* $\mathcal{A} = \{A_{v_i}\}$ *identifies* $\mathrm{SHG}^{(0,\alpha)}_{\mathcal{A},\mathcal{B}}$ *with a (possibly relabeled) instance of the neutrosophic intersection hypergraph* $H^{(\alpha)}_{\mathcal{A},\mathcal{B}}$ *from Definition 3.10.6.*

2. ***(Neutrosophic intersection graphs via*** 2***-sections).*** *Let* $\mathcal{A} = \{A_1, \ldots, A_m\}$ *be as in Definition 3.10.4. Take* $n = 0$*, let* $V = \{v_1, \ldots, v_m\}$*, and let* $\Lambda := U$ *where each* $\varepsilon = u \in U$ *is represented by the* crisp singleton *neutrosophic set* B_u *on* U *whose truth-membership is* $T_{B_u}(u) = 1$ *and* $T_{B_u}(w) = 0$ *for* $w \neq u$ *(with arbitrary* (I, F)*, e.g.* $I \equiv 0$*,* $F \equiv 1 - T$*). Form* $\mathrm{SHG}^{(0,\alpha)}_{\mathcal{A},\mathcal{B}}$.

   Then the 2-section *(primal graph) of this hypergraph coincides with* $G^{(\alpha)}_{\mathcal{A}}$.

3. ***(Fuzzy intersection*** n***-SuperHyperGraphs embed).*** *Let* $\{F_x\}_{x \in V}$ *and* $\{G_\varepsilon\}_{\varepsilon \in \Lambda}$ *be families of fuzzy sets on* U *(with membership functions* $\mu_{F_x}, \mu_{G_\varepsilon} : U \to [0, 1]$*) and consider the fuzzy intersection n-SuperHyperGraph obtained from α-cuts. Define an embedding of fuzzy sets into single-valued neutrosophic sets by*

   $$\iota(\mu) \;:=\; (T, I, F) \;:=\; \big(\mu, \; 0, \; 1 - \mu\big).$$

   Applying ι *pointwise to all* F_x *and* G_ε *yields families* $(\mathcal{A}, \mathcal{B})$ *of SVNS such that the resulting neutrosophic intersection n-SuperHyperGraph* $\mathrm{SHG}^{(n,\alpha)}_{\mathcal{A},\mathcal{B}}$ *has exactly the same vertex set and incidence structure as the original fuzzy intersection n-SuperHyperGraph (at the same threshold* α*).*

Proof. (1) For $n = 0$, supervertices are ordinary vertices $V \subseteq V_0$ and $\partial^{(\alpha)}(\varepsilon)$ is simply a nonempty subset of V for each $\varepsilon \in E = \Lambda$. Thus $(V, E, \partial^{(\alpha)})$ encodes a hypergraph whose hyperedges are precisely the incidence sets $\partial^{(\alpha)}(\varepsilon)$. Unwinding Definition 3.10.8 shows that these incidence sets are $\{v_i : A^{(\alpha)}_{v_i} \cap B^{(\alpha)}_{\varepsilon} \neq \emptyset\}$, which is exactly the hyperedge rule in Definition 3.10.6 (up to relabeling).

(2) For $u \in U$, the singleton choice of B_u guarantees that $A^{(\alpha)}_i \cap B^{(\alpha)}_u \neq \emptyset$ holds iff $u \in A^{(\alpha)}_i$. Hence the hyperedge $\partial^{(\alpha)}(u)$ equals $\{v_i : \ u \in A^{(\alpha)}_i\}$. Two distinct vertices v_i, v_j lie together in some hyperedge $\partial^{(\alpha)}(u)$ iff there exists $u \in U$ with $u \in A^{(\alpha)}_i \cap A^{(\alpha)}_j$, i.e. iff $A^{(\alpha)}_i \cap A^{(\alpha)}_j \neq \emptyset$. By definition of the 2-section (primal graph), this is equivalent to $\{v_i, v_j\}$ being an edge of that primal graph, and by Definition 3.10.4 it is equivalent to $\{v_i, v_j\} \in E(G^{(\alpha)}_{\mathcal{A}})$.

(3) If F is a fuzzy set with membership μ_F, then under $\iota(\mu_F) = (T, I, F)$ we have $T = \mu_F$. Therefore the truth α-cut of $\iota(\mu_F)$ is

$$\{u \in U : \ T(u) \geq \alpha\} = \{u \in U : \ \mu_F(u) \geq \alpha\},$$

which is exactly the usual fuzzy α-cut. Consequently, for every pair (x, ε),

$$F^{(\alpha)}_x \cap G^{(\alpha)}_{\varepsilon} \neq \emptyset \quad \Longleftrightarrow \quad A^{(\alpha)}_x \cap B^{(\alpha)}_{\varepsilon} \neq \emptyset,$$

so the incidence rule in Definition 3.10.8 reproduces the fuzzy intersection incidence rule verbatim. This yields an isomorphism of the resulting (crisp) intersection n-SuperHyperGraphs. □

3.11 Uncertain Intersection SuperHyperGraph

An uncertain set assigns each element a degree tuple in a domain, representing membership information under a chosen uncertainty model [183,184]. An uncertain graph labels every vertex and edge with degree tuples, enabling structure analysis when connectivity and presence are uncertain [1]. An uncertain intersection graph arises from intersecting sets; it keeps intersection adjacencies while attaching degree tuples to vertices and edges [1]. An uncertain intersection superhypergraph uses supervertices as nested sets, forms superedges from nonempty common intersections, and records degrees for all [1].

Definition 3.11.1 (Uncertain model)**.** [183,184] Let X be a nonempty universe and let $k \in \mathbb{N}$. An *uncertain model* is specified by a *degree-domain*

$$\mathrm{Dom}(M) \ \subseteq \ [0, 1]^k$$

together with a *membership (degree) map*

$$\mu_M : X \longrightarrow \mathrm{Dom}(M).$$

The integer k is called the *dimension* of the degree-domain.

Example 3.11.2 (Uncertain model)**.** Let $X = \{x_1, x_2, x_3, x_4\}$ and take $k = 2$ with degree-domain

$$\mathrm{Dom}(M) := [0, 1]^2.$$

Define a membership (degree) map $\mu_M : X \to [0, 1]^2$ by

$$\mu_M(x_1) = (0.90, 0.10), \quad \mu_M(x_2) = (0.60, 0.25), \quad \mu_M(x_3) = (0.30, 0.50), \quad \mu_M(x_4) = (0.75, 0.05).$$

Then $M = (\mathrm{Dom}(M), \mu_M)$ is an uncertain model of dimension $k = 2$ on the universe X.

Definition 3.11.3 (Uncertain graph)**.** Let $G = (V, E)$ be a (finite, undirected, loopless) graph and let M be an uncertain model. An *uncertain graph of type M* is a triple

$$\mathcal{G}_M = (V, E, \mu_M), \qquad \mu_M : V \cup E \longrightarrow \mathrm{Dom}(M),$$

assigning an uncertainty degree to each vertex and each edge.

Example 3.11.4 (Uncertain graph of type M)**.** Let $G = (V, E)$ be the path P_3 on vertices $V = \{a, b, c\}$ with

$$E = \{\{a, b\}, \{b, c\}\}.$$

Let M be the uncertain model with $\mathrm{Dom}(M) = [0, 1]^2$. Define $\mu_M : V \cup E \to [0, 1]^2$ by

$$\mu_M(a) = (0.80, 0.10), \quad \mu_M(b) = (0.65, 0.20), \quad \mu_M(c) = (0.90, 0.05),$$

$$\mu_M(\{a, b\}) = (0.70, 0.15), \qquad \mu_M(\{b, c\}) = (0.55, 0.30).$$

Then $\mathcal{G}_M = (V, E, \mu_M)$ is an uncertain graph of type M.

Definition 3.11.5 (Uncertain hypergraph)**.** Let $H = (V, \mathcal{E})$ be a (finite) hypergraph with $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, and let M be an uncertain model. An *uncertain hypergraph of type M* is a triple

$$\mathcal{H}_M = (V, \mathcal{E}, \mu_M), \qquad \mu_M : V \cup \mathcal{E} \longrightarrow \mathrm{Dom}(M),$$

assigning an uncertainty degree to each vertex and each hyperedge.

Example 3.11.6 (Uncertain hypergraph of type M)**.** Let $H = (V, \mathcal{E})$ with vertex set $V = \{1, 2, 3, 4\}$ and hyperedges

$$\mathcal{E} = \{\{1, 2, 3\}, \{2, 4\}\}.$$

Let M be the uncertain model with $\mathrm{Dom}(M) = [0, 1]^2$. Define $\mu_M : V \cup \mathcal{E} \to [0, 1]^2$ by

$$\mu_M(1) = (0.90, 0.05),\ \mu_M(2) = (0.60, 0.25),\ \mu_M(3) = (0.75, 0.10),\ \mu_M(4) = (0.55, 0.35),$$

$$\mu_M(\{1, 2, 3\}) = (0.50, 0.30), \qquad \mu_M(\{2, 4\}) = (0.65, 0.20).$$

Then $\mathcal{H}_M = (V, \mathcal{E}, \mu_M)$ is an uncertain hypergraph of type M.

Definition 3.11.7 (Uncertain n-SuperHyperGraph)**.** Let V_0 be a base set and let $n \in \mathbb{N}_0$. An *n-SuperHyperGraph* is a pair $\mathrm{SHG}^{(n)} = (V_n, E)$ where

$$\emptyset \neq V_n \subseteq \mathcal{P}^n(V_0), \qquad \emptyset \neq E \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}.$$

Let M be an uncertain model. An *uncertain n-SuperHyperGraph of type M* is a triple

$$\mathcal{S}_M^{(n)} = (V_n, E, \mu_M), \qquad \mu_M : V_n \cup E \longrightarrow \mathrm{Dom}(M).$$

For $n = 0$ (so $V_n = V_0$), this reduces to an uncertain hypergraph.

Example 3.11.8 (Uncertain n-SuperHyperGraph of type M)**.** Let $V_0 = \{1, 2, 3, 4\}$ and fix $n = 1$. Take the 1-supervertex set

$$V_1 = \{x_1, x_2, x_3\} \subseteq \mathcal{P}(V_0), \qquad x_1 = \{1\}, \ \ x_2 = \{2, 3\}, \ \ x_3 = \{1, 4\}.$$

Let the superedge family be

$$E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{x_1, x_2\}, \ \ \varepsilon_2 = \{x_2, x_3\}.$$

Let M have degree-domain $\mathrm{Dom}(M) = [0, 1]^2$. Define $\mu_M : V_1 \cup E \to [0, 1]^2$ by

$$\mu_M(x_1) = (0.85, 0.05), \quad \mu_M(x_2) = (0.60, 0.30), \quad \mu_M(x_3) = (0.70, 0.20),$$

$$\mu_M(\varepsilon_1) = (0.55, 0.35), \qquad \mu_M(\varepsilon_2) = (0.65, 0.25).$$

Then $\mathcal{S}^{(1)}_M = (V_1, E, \mu_M)$ is an uncertain 1-SuperHyperGraph of type M.

Definition 3.11.9 (Uncertain intersection graph / hypergraph / n-SuperHyperGraph)**.** Let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$.

(i) An *uncertain intersection graph of type M* is an uncertain graph

$$\mathcal{G}_M = (V, E, \mu_M)$$

for which there exists a set-family $\mathcal{F} = \{S_v\}_{v\in V}$ such that $(V, E) = \mathrm{Int}(\mathcal{F})$.

(ii) An *uncertain intersection hypergraph of type M* is an uncertain hypergraph

$$\mathcal{H}_M = (V, \mathcal{E}, \mu_M)$$

for which there exists a set-family $\mathcal{F} = \{S_v\}_{v\in V}$ such that $(V, \mathcal{E}) = \mathrm{IntHyp}(\mathcal{F})$.

(iii) Fix $n \geq 1$. An *uncertain intersection n-SuperHyperGraph of type M* is an uncertain n-SuperHyperGraph

$$\mathcal{S}^{(n)}_M = (V_n, E, \mu_M)$$

for which there exist V_0 and an n-level family $\mathcal{F}$ such that $(V_n, E) = \mathrm{IntSHG}^{(n)}(\mathcal{F})$.

Example 3.11.10 (Uncertain intersection graph)**.** Let $U = \{p, q, r\}$ and consider the set-family $\mathcal{F} = \{S_a, S_b, S_c\}$ given by

$$S_a = \{p, q\}, \qquad S_b = \{q\}, \qquad S_c = \{r\}.$$

Its intersection graph $\mathrm{IG}(\mathcal{F}) = (V, E)$ has $V = \{a, b, c\}$ and a single edge $\{a, b\}$, since $S_a \cap S_b = \{q\} \neq \emptyset$, while $S_a \cap S_c = \emptyset$ and $S_b \cap S_c = \emptyset$.

Let M have $\mathrm{Dom}(M) = [0, 1]$ and define $\mu_M : V \cup E \to [0, 1]$ by

$$\mu_M(a) = 0.9, \quad \mu_M(b) = 0.6, \quad \mu_M(c) = 0.4, \qquad \mu_M(\{a, b\}) = 0.5.$$

Then $\mathcal{G}_M = (V, E, \mu_M)$ is an uncertain intersection graph of type M.

Example 3.11.11 (Uncertain intersection hypergraph)**.** Let $U = \{1, 2, 3\}$ and take $\mathcal{F} = \{S_1, S_2, S_3, S_4\}$ with

$$S_1 = \{1, 2\}, \qquad S_2 = \{2, 3\}, \qquad S_3 = \{2\}, \qquad S_4 = \{1\}.$$

In the intersection hypergraph $\mathrm{IH}(\mathcal{F}) = (V, \mathcal{E})$ on $V = \{1, 2, 3, 4\}$ we have, for example,

$$\{1, 2\} \in \mathcal{E} \;\; (\text{since } S_1 \cap S_2 = \{2\} \neq \emptyset), \qquad \{1, 2, 3\} \in \mathcal{E} \;\; (\text{since } S_1 \cap S_2 \cap S_3 = \{2\} \neq \emptyset),$$

whereas $\{2, 4\} \notin \mathcal{E}$ because $S_2 \cap S_4 = \emptyset$.

Let M have $\mathrm{Dom}(M) = [0, 1]$ and define $\mu_M : V \cup \mathcal{E} \to [0, 1]$ by assigning, for instance,

$$\mu_M(1) = 0.8,\ \mu_M(2) = 0.7,\ \mu_M(3) = 0.6,\ \mu_M(4) = 0.5, \qquad \mu_M(\{1, 2\}) = 0.4,\ \mu_M(\{1, 2, 3\}) = 0.3,$$

and any values in $[0, 1]$ to the remaining hyperedges of $\mathcal{E}$. Then $\mathcal{H}_M = (V, \mathcal{E}, \mu_M)$ is an uncertain intersection hypergraph of type M.

Example 3.11.12 (Uncertain intersection n-SuperHyperGraph)**.** Let $V_0 = \{1, 2, 3\}$ and fix $n = 1$. Consider the 1-level family

$$\mathcal{F} = \{X_1, X_2, X_3\} \subseteq \mathcal{P}^1(V_0) = \mathcal{P}(V_0), \qquad X_1 = \{1, 2\},\;\; X_2 = \{2, 3\},\;\; X_3 = \{2\}.$$

Then $\mathrm{ISHG}^{(1)}(\mathcal{F}) = (V_1, E)$ has

$$V_1 = \{X_1, X_2, X_3\}, \qquad E = \Big\{\varepsilon \subseteq V_1 :\; |\varepsilon| \geq 2,\; \bigcap_{X \in \varepsilon} X \neq \emptyset\Big\}.$$

In particular,

$$\{X_1, X_2\} \in E, \quad \{X_1, X_3\} \in E, \quad \{X_2, X_3\} \in E, \quad \{X_1, X_2, X_3\} \in E,$$

since all the corresponding common intersections contain the element 2.

Let M have $\mathrm{Dom}(M) = [0, 1]^2$. Define $\mu_M : V_1 \cup E \to [0, 1]^2$ by

$$\mu_M(X_1) = (0.90, 0.05), \quad \mu_M(X_2) = (0.70, 0.20), \quad \mu_M(X_3) = (0.80, 0.10),$$

$$\mu_M(\{X_1, X_2\}) = (0.55, 0.35), \quad \mu_M(\{X_1, X_2, X_3\}) = (0.50, 0.40),$$

and any values in $[0, 1]^2$ for the remaining superedges of E. Then $\mathcal{S}_M^{(1)} = (V_1, E, \mu_M)$ is an uncertain intersection 1-SuperHyperGraph of type M.

Theorem 3.11.13 (Generalization by uncertain intersection n-SuperHyperGraphs)**.** *Let M be an uncertain model and let $\mathcal{S}_M^{(n)}$ be an uncertain intersection n-SuperHyperGraph of type M.*

1. *If M is the* fuzzy *model (so $\mathrm{Dom}(M) \subseteq [0, 1]$), then $\mathcal{S}_M^{(n)}$ is precisely a* fuzzy intersection n-SuperHyperGraph*.*

2. *For $n = 1$, every uncertain intersection n-SuperHyperGraph of type M is (canonically) an uncertain intersection hypergraph of type M.*

3. *For any $n \geq 1$, the 2-section $[\mathcal{S}_M^{(n)}]_2$ is an uncertain intersection graph of type M. Consequently, uncertain intersection n-SuperHyperGraphs generalize uncertain intersection graphs.*

Proof. (1) In the fuzzy model one has $\mathrm{Dom}(M) \subseteq [0,1]$, so the degree map $\mu_M : V_n \cup E \to [0,1]$ is exactly a fuzzy membership assignment on vertices and superedges. Thus Definition 3.11.9(iii) coincides with the usual notion of a fuzzy intersection n-SuperHyperGraph.

(2) When $n = 1$, the vertex objects satisfy $V_1 \subseteq \mathcal{P}(V_0)$, i.e., vertices are ordinary sets. Edges are families of such vertices with nonempty common intersection. This is precisely the construction of an intersection hypergraph in Definition 2.2.3, hence $\mathcal{S}_M^{(1)}$ is an uncertain intersection hypergraph.

(3) Let $\mathcal{S}_M^{(n)} = (V_n, E, \mu_M)$ be induced by some family $\mathcal{F}$. Two supervertices $x, y \in V_n$ are adjacent in $[\mathcal{S}_M^{(n)}]_2$ iff there exists $\varepsilon \in E$ with $\{x, y\} \subseteq \varepsilon$. But $\varepsilon \in E$ implies $\bigcap_{z \in \varepsilon} z \neq \emptyset$, hence in particular $x \cap y \neq \emptyset$. Therefore adjacency in the 2-section is equivalent to pairwise intersection of the corresponding objects, so $[\mathcal{S}_M^{(n)}]_2$ is an intersection graph (Definition 2.2.1). Equipping it with the inherited vertex degrees $\mu_M|_{V_n}$ makes it an uncertain intersection graph of type M in the sense of Definition 3.11.9(i). □

3.12 Hierarchical Intersection SuperHyperGraph

A hierarchical intersection superhypergraph models nested entities as supervertices across levels, adding (super)edges when corresponding families share nonempty intersections, thereby linking structure simultaneously.

Definition 3.12.1 (Nested intersection n-SuperHyperGraph)**.** Let V_0 be a finite nonempty set and let $n \geq 1$. Fix a universe U and an injective *base-realization map*

$$\eta : V_0 \longrightarrow \mathcal{P}(U) \setminus \{\emptyset\}, \qquad v \longmapsto \eta(v),$$

so that intersections at level 0 are interpreted inside U.

Let $\mathcal{S} = \{X_1, \ldots, X_m\}$ be a finite family of nonempty n-level objects

$$X_i \in \mathcal{P}^n(V_0) \setminus \{\emptyset\} \qquad (1 \leq i \leq m).$$

Define recursively the level-k vertex sets $(V_k)_{k=0}^n$ by

$$V_n := \{X_1, \ldots, X_m\} \subseteq \mathcal{P}^n(V_0), \qquad V_k := \bigcup_{Y \in V_{k+1}} Y \subseteq \mathcal{P}^k(V_0) \quad (0 \leq k \leq n-1).$$

(Thus V_k collects *all* k-level constituents occurring inside the chosen n-level family.)

For each level k define the *intersection edge family* E_k by

$$E_k := \Big\{ \varepsilon \subseteq V_k : \ |\varepsilon| \geq 2 \ \text{ and } \ \mathrm{IG}_k(\varepsilon) \neq \emptyset \Big\},$$

where the intersection operator $\mathrm{IG}_k(\varepsilon)$ is given by

$$\mathrm{IG}_k(\varepsilon) \;:=\; \begin{cases} \displaystyle\bigcap_{x\in\varepsilon} \eta(x) \;\subseteq\; U, & k=0,\\ \displaystyle\bigcap_{x\in\varepsilon} x \;\subseteq\; \mathcal{P}^{k-1}(V_0), & 1\le k\le n. \end{cases}$$

The resulting graded object

$$\mathrm{NISHG}_{\eta}^{(n)}(\mathcal{S}) \;:=\; \big((V_k,E_k)_{k=0}^{n}\,;\,\eta\big)$$

is called the *nested intersection n-SuperHyperGraph* generated by $\mathcal{S}$ (with base realization η).

The *top-level* n-SuperHyperGraph associated with $\mathrm{NISHG}_{\eta}^{(n)}(\mathcal{S})$ is

$$\mathrm{Top}\big(\mathrm{NISHG}_{\eta}^{(n)}(\mathcal{S})\big) \;:=\; (V_n,E_n).$$

Remark 3.12.2 (Canonical choice of η)**.** If $V_0 \subseteq \mathcal{P}(U)\setminus\{\emptyset\}$ already consists of nonempty sets, one may take $\eta=\mathrm{id}$. In complete generality, one may always take $U := V_0$ and $\eta(v) := \{v\}$, which makes level-0 intersection encode equality (hence yields a valid but possibly trivial E_0).

Example 3.12.3 (Healthcare: patients–visits–(sets of) symptoms)**.** Let U be a finite symptom universe, e.g.

$$U := \{\mathsf{fever},\mathsf{cough},\mathsf{fatigue},\mathsf{nausea},\mathsf{rash}\}.$$

Let the base set V_0 consist of patient-visits (atomic records)

$$V_0 := \{v_1,v_2,v_3,v_4,v_5\}.$$

Define an injective base-realization map $\eta: V_0 \to \mathcal{P}(U)\setminus\{\emptyset\}$ by

$$\eta(v_1)=\{\mathsf{fever},\mathsf{cough}\},\quad \eta(v_2)=\{\mathsf{fever},\mathsf{fatigue}\},\quad \eta(v_3)=\{\mathsf{cough},\mathsf{fatigue}\},$$
$$\eta(v_4)=\{\mathsf{nausea}\},\quad \eta(v_5)=\{\mathsf{rash},\mathsf{fever}\}.$$

Fix $n=2$. Interpret level-1 objects as *patients* (sets of visits) and level-2 objects as *clinics* (sets of patients). Let

$$p_1 := \{v_1,v_4\},\quad p_2 := \{v_2\},\quad p_3 := \{v_3,v_5\} \;\in\; \mathcal{P}(V_0)\setminus\{\emptyset\}.$$

Define the clinic family $\mathcal{S}=\{X_1,X_2\}\subseteq \mathcal{P}^2(V_0)\setminus\{\emptyset\}$ by

$$X_1 := \{p_1,p_2\},\qquad X_2 := \{p_2,p_3\}.$$

Then $\mathrm{NISHG}_{\eta}^{(2)}(\mathcal{S}) = \big((V_k,E_k)_{k=0}^{2};\eta\big)$ is a nested intersection 2-SuperHyperGraph.

Concretely, the level-2 vertex set is $V_2=\{X_1,X_2\}$ and

$$\mathrm{IG}_2(\{X_1,X_2\}) = X_1\cap X_2 = \{p_2\}\neq\emptyset,$$

so $\{X_1,X_2\}\in E_2$ (the two clinics share a common patient). At level 1, the vertex set is $V_1 = X_1\cup X_2 = \{p_1,p_2,p_3\}$ and

$$\mathrm{IG}_1(\{p_1,p_2\}) = p_1\cap p_2=\emptyset,\qquad \mathrm{IG}_1(\{p_2,p_3\}) = p_2\cap p_3=\emptyset,$$

so (in this toy instance) there is no level-1 intersection edge among distinct patients, while at level 0 intersections detect shared symptoms, e.g.

$$\mathrm{IG}_0(\{v_1,v_2\}) = \eta(v_1)\cap\eta(v_2) = \{\mathsf{fever}\}\neq\emptyset,$$

hence $\{v_1,v_2\}\in E_0$.

Example 3.12.4 (Cybersecurity: machines–teams–business units via shared vulnerabilities)**.** Let U be a finite set of vulnerability identifiers, e.g.

$$U := \{\mathsf{CVE\text{-}A}, \mathsf{CVE\text{-}B}, \mathsf{CVE\text{-}C}, \mathsf{CVE\text{-}D}\}.$$

Let the base set V_0 be a set of machines

$$V_0 := \{m_1, m_2, m_3, m_4, m_5, m_6\}.$$

Define an injective base-realization $\eta : V_0 \to \mathcal{P}(U) \setminus \{\emptyset\}$ by

$$\eta(m_1) = \{\mathsf{CVE\text{-}A}, \mathsf{CVE\text{-}B}\},\ \eta(m_2) = \{\mathsf{CVE\text{-}B}\},\ \eta(m_3) = \{\mathsf{CVE\text{-}C}\},$$

$$\eta(m_4) = \{\mathsf{CVE\text{-}A}, \mathsf{CVE\text{-}D}\},\ \eta(m_5) = \{\mathsf{CVE\text{-}D}\},\ \eta(m_6) = \{\mathsf{CVE\text{-}C}, \mathsf{CVE\text{-}D}\}.$$

Fix $n = 3$. Interpret:

- level 1: *service clusters* (sets of machines),
- level 2: *teams* (sets of clusters),
- level 3: *business units* (sets of teams).

Define clusters (level 1) by

$$c_1 := \{m_1, m_2\}, \quad c_2 := \{m_3\}, \quad c_3 := \{m_4, m_5\}, \quad c_4 := \{m_6\}.$$

Define teams (level 2) by

$$t_1 := \{c_1, c_2\}, \qquad t_2 := \{c_3, c_4\}.$$

Define business units (level 3) by the family $\mathcal{S} = \{X_1, X_2\} \subseteq \mathcal{P}^3(V_0) \setminus \{\emptyset\}$:

$$X_1 := \{t_1\}, \qquad X_2 := \{t_1, t_2\}.$$

Then $\mathrm{NISHG}^{(3)}_{\eta}(\mathcal{S})$ is a nested intersection 3-SuperHyperGraph.

At the top level,

$$\mathrm{IG}_3(\{X_1, X_2\}) = X_1 \cap X_2 = \{t_1\} \neq \emptyset,$$

so $\{X_1, X_2\} \in E_3$ (the two business units share a team). At level 0, machines intersect when they share vulnerabilities, for instance

$$\mathrm{IG}_0(\{m_1, m_4\}) = \eta(m_1) \cap \eta(m_4) = \{\mathsf{CVE\text{-}A}\} \neq \emptyset,$$

so $\{m_1, m_4\} \in E_0$. This yields a multilevel intersection structure linking units/teams/clusters/machines by shared constituents and (at level 0) shared risk factors.

Example 3.12.5 (Education: students–courses–programs via shared prerequisite topics)**.** Let U be a finite set of prerequisite topics, e.g.

$$U := \{\mathsf{linear\text{-}algebra}, \mathsf{calculus}, \mathsf{probability}, \mathsf{programming}\}.$$

Let the base set V_0 consist of course sections (atomic offerings):

$$V_0 := \{s_1, s_2, s_3, s_4, s_5\}.$$

Define an injective base-realization $\eta : V_0 \to \mathcal{P}(U) \setminus \{\emptyset\}$ by

$$\eta(s_1) = \{\textsf{calculus}, \textsf{linear-algebra}\}, \quad \eta(s_2) = \{\textsf{calculus}\}, \quad \eta(s_3) = \{\textsf{probability}\},$$
$$\eta(s_4) = \{\textsf{programming}\}, \quad \eta(s_5) = \{\textsf{linear-algebra}, \textsf{programming}\}.$$

Fix $n = 2$. Interpret level 1 objects as *courses* (unions of sections) and level 2 objects as *degree programs* (collections of courses). Let

$$c_{\text{Math}} := \{s_1, s_2\}, \qquad c_{\text{Stats}} := \{s_3\}, \qquad c_{\text{CS}} := \{s_4, s_5\}.$$

Define two programs (level 2) by

$$X_1 := \{c_{\text{Math}}, c_{\text{Stats}}\}, \qquad X_2 := \{c_{\text{Math}}, c_{\text{CS}}\}.$$

Set $\mathcal{S} := \{X_1, X_2\} \subseteq \mathcal{P}^2(V_0) \backslash \{\emptyset\}$. Then $\mathrm{NISHG}^{(2)}_{\eta}(\mathcal{S})$ is a nested intersection 2-SuperHyperGraph.

At level 2, the two programs share a course:

$$\mathrm{IG}_2(\{X_1, X_2\}) = X_1 \cap X_2 = \{c_{\text{Math}}\} \neq \emptyset,$$

so $\{X_1, X_2\} \in E_2$. At level 0, two sections intersect when they share prerequisite topics, e.g.

$$\mathrm{IG}_0(\{s_1, s_5\}) = \eta(s_1) \cap \eta(s_5) = \{\textsf{linear-algebra}\} \neq \emptyset,$$

so $\{s_1, s_5\} \in E_0$. Thus the model links programs via shared courses and links sections via shared topics, while also retaining intermediate course-level constituents.

Theorem 3.12.6 (Nested intersection n-SuperHyperGraphs generalize intersection n-SuperHyperGraphs)**.** *Let $n \geq 1$, let V_0 be a finite nonempty set, and let $\mathcal{S} = \{X_1, \ldots, X_m\} \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$. Let $\mathrm{ISHG}^{(n)}(\mathcal{S}) = (V, E)$ be the (standard) intersection n-SuperHyperGraph generated by $\mathcal{S}$, i.e.,*

$$V = \{X_1, \ldots, X_m\}, \qquad E = \Big\{\varepsilon \subseteq V : \ |\varepsilon| \geq 2 \ \ \textit{and} \ \bigcap_{X \in \varepsilon} X \neq \emptyset\Big\}.$$

Choose any base-realization map $\eta : V_0 \to \mathcal{P}(U) \setminus \{\emptyset\}$ (e.g. as in Remark 3.12.2), and form the nested intersection structure $\mathrm{NISHG}^{(n)}_{\eta}(\mathcal{S})$ as in Definition 3.12.1. Then

$$\mathrm{Top}\big(\mathrm{NISHG}^{(n)}_{\eta}(\mathcal{S})\big) \ = \ \mathrm{ISHG}^{(n)}(\mathcal{S}).$$

In particular, forgetting the lower-level layers $(V_k, E_k)_{k=0}^{n-1}$ recovers the usual intersection n-SuperHyperGraph.

Proof. By construction of Definition 3.12.1, we have $V_n = \{X_1, \ldots, X_m\} = V$. Moreover, for $k = n$ the intersection operator is the ordinary set-theoretic intersection inside $\mathcal{P}^n(V_0)$, hence

$$E_n = \Big\{\varepsilon \subseteq V_n : \ |\varepsilon| \geq 2 \ \ \text{and} \ \bigcap_{X \in \varepsilon} X \neq \emptyset\Big\} = \Big\{\varepsilon \subseteq V : \ |\varepsilon| \geq 2 \ \ \text{and} \ \bigcap_{X \in \varepsilon} X \neq \emptyset\Big\} = E.$$

Therefore $\mathrm{Top}(\mathrm{NISHG}^{(n)}_{\eta}(\mathcal{S})) = (V_n, E_n) = (V, E) = \mathrm{ISHG}^{(n)}(\mathcal{S})$, as claimed. □

Chapter 4

SuperHyperGraph Labeling

In this chapter, we extend the classical notion of graph labeling by means of SuperHyperGraphs. We introduce several derived concepts of *SuperHyperGraph labeling* and discuss them through simple illustrative examples.

4.1 SuperHyperGraph Labeling

Labeling theory for hypergraphs and related higher-order structures provides a flexible language for encoding combinatorial constraints. Vertex labels can model types, time slots, or priorities, while hyperedge labels can record costs, capacities, or feasibility certificates for multiway interactions (cf. [64–66, 185, 186]).

Definition 4.1.1 (Primal (2-section) graph)**.** Let $H = (V, \mathcal{E})$ be a (finite) hypergraph. Its *primal graph* (also called the 2-*section*) is the simple graph

$$\mathrm{Pr}(H) \;:=\; (V, F), \qquad F \;:=\; \big\{\{u, v\} \subseteq V :\; u \neq v \;\text{ and }\; \exists\, e \in \mathcal{E} \text{ with } \{u, v\} \subseteq e\big\}.$$

The *vertex distance* $\mathrm{dist}_H(u, v)$ is the usual shortest-path distance between $u, v \in V$ computed in $\mathrm{Pr}(H)$.

Definition 4.1.2 (Hypergraph labeling (schema-based))**.** Let $H = (V, \mathcal{E})$ be a hypergraph and let L_V and $L_{\mathcal{E}}$ be nonempty label sets. A *(vertex/hyperedge) labeling* of H is a pair of maps

$$\ell_V : V \to L_V, \qquad \ell_{\mathcal{E}} : \mathcal{E} \to L_{\mathcal{E}},$$

where either map may be omitted if the corresponding labels are not used.

A *hypergraph labeling schema* is a first-order predicate

$$\Phi\big(H; \ell_V, \ell_{\mathcal{E}}\big)$$

whose atomic ingredients may include: the incidence relation "$v \in e$" ($v \in V$, $e \in \mathcal{E}$), the distance dist_H from Definition 4.1.1, equalities/inequalities between label values, and quantification over V and $\mathcal{E}$. We call $(\ell_V, \ell_{\mathcal{E}})$ a *valid hypergraph labeling for* Φ if $\Phi\big(H; \ell_V, \ell_{\mathcal{E}}\big)$ holds.

Remark 4.1.3 (Standard graph labelings as special cases)**.** When H is 2-uniform (so H is an ordinary graph), suitable choices of Φ recover familiar labeling families. For example:

- **Proper vertex coloring:** $L_V = \{1, \ldots, k\}$ and
$$\Phi \equiv (\forall \{u, v\} \in \mathcal{E})\ \ell_V(u) \neq \ell_V(v).$$

- $L(p, q)$**-labeling (distance constraints on vertices):** $L_V \subseteq \mathbb{Z}$ and
$$\Phi \equiv (\forall u \neq v \in V)\Big(\mathrm{dist}_H(u, v) = 1 \Rightarrow |\ell_V(u) - \ell_V(v)| \geq p$$
$$\wedge\ \mathrm{dist}_H(u, v) = 2 \Rightarrow |\ell_V(u) - \ell_V(v)| \geq q\Big).$$

- **Strong hypergraph coloring (genuinely hypergraph):** $L_V = \{1, \ldots, k\}$ and
$$\Phi \equiv (\forall e \in \mathcal{E})\ \big(\forall u \neq v \in e\big)\ \ell_V(u) \neq \ell_V(v).$$

Example 4.1.4 (A concrete hypergraph labeling: strong hypergraph coloring)**.** Let $H = (V, \mathcal{E})$ be the hypergraph with
$$V := \{a, b, c, d\}, \qquad \mathcal{E} := \big\{\{a, b, c\},\ \{b, c, d\}\big\}.$$
Let $L_V := \{1, 2, 3\}$ and (for simplicity) ignore hyperedge labels by taking $L_{\mathcal{E}} := \{*\}$ and $\ell_{\mathcal{E}}(e) := *$ for all $e \in \mathcal{E}$.

Define a vertex labeling $\ell_V : V \to L_V$ by
$$\ell_V(a) = 1, \qquad \ell_V(b) = 2, \qquad \ell_V(c) = 3, \qquad \ell_V(d) = 1.$$
Consider the schema $\Phi_{\mathrm{str}}(H; \ell_V)$ expressing *strong hypergraph coloring*:
$$\Phi_{\mathrm{str}}(H; \ell_V)\ \equiv\ (\forall e \in \mathcal{E})\,(\forall u \neq v \in e)\ \ell_V(u) \neq \ell_V(v).$$
Then $\Phi_{\mathrm{str}}(H; \ell_V)$ holds, because
$$\{\ell_V(a), \ell_V(b), \ell_V(c)\} = \{1, 2, 3\} \quad \text{and} \quad \{\ell_V(b), \ell_V(c), \ell_V(d)\} = \{2, 3, 1\}$$
are pairwise-distinct on each hyperedge. Hence $(\ell_V, \ell_{\mathcal{E}})$ is a valid hypergraph labeling for Φ_{str} in the sense of Definition 4.1.2.

Labeling extends verbatim to n-SuperHyperGraphs by replacing vertices/hyperedges with supervertices/superedges, and by measuring distances on the supervertex set through an appropriate 2-section.

Definition 4.1.5 (Primal graph and induced distance of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph, where V is the set of n-supervertices and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is the set of n-superedges. Its *primal graph* is the simple graph
$$\mathrm{Pr}\big(\mathrm{SHG}^{(n)}\big)\ :=\ (V, F), \qquad F\ :=\ \big\{\{x, y\} \subseteq V :\ x \neq y\ \text{ and }\ \exists\, \varepsilon \in E \text{ with } \{x, y\} \subseteq \varepsilon\big\}.$$
The *supervertex distance* $\mathrm{dist}_{\mathrm{SHG}^{(n)}}(x, y)$ is the shortest-path distance in $\mathrm{Pr}(\mathrm{SHG}^{(n)})$.

Definition 4.1.6 (SuperHyperGraph labeling (schema-based))**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph and let L_V and L_E be nonempty label sets. A *(supervertex/superedge) labeling* of $\mathrm{SHG}^{(n)}$ is a pair of maps

$$\ell_V : V \to L_V, \qquad \ell_E : E \to L_E,$$

where either map may be omitted if it is not required.

A *labeling schema* is a first-order predicate

$$\Phi\big(\mathrm{SHG}^{(n)}; \ell_V, \ell_E\big)$$

built from the incidence relation "$x \in \varepsilon$" (with $x \in V$ and $\varepsilon \in E$), the distance $\mathrm{dist}_{\mathrm{SHG}^{(n)}}$ from Definition 4.1.5, equalities/inequalities between label values, and (optionally) additional predicates on n-level objects (e.g. cardinalities of constituents, inclusion relations between nested members of $V \subseteq \mathcal{P}^n(V_0)$). We call (ℓ_V, ℓ_E) a *valid SuperHyperGraph labeling for* Φ if $\Phi\big(\mathrm{SHG}^{(n)}; \ell_V, \ell_E\big)$ holds.

Remark 4.1.7 (Recovering common schemas via Φ)**.** Appropriate schemas reproduce standard labeling families on $\mathrm{SHG}^{(n)}$. For instance:

- **Proper coloring on the primal graph:** $L_V = \{1, \dots, k\}$ and

  $$\Phi \equiv (\forall \{x, y\} \in F)\ \ell_V(x) \neq \ell_V(y),$$

  where F is the edge set of $\mathrm{Pr}(\mathrm{SHG}^{(n)})$ from Definition 4.1.5.

- $L(p, q)$**-type distance labeling:** $L_V \subseteq \mathbb{Z}$ and, for all distinct $x, y \in V$,

  $$\Phi \equiv \Big(\mathrm{dist}_{\mathrm{SHG}^{(n)}}(x, y) = 1 \Rightarrow |\ell_V(x) - \ell_V(y)| \geq p \wedge \mathrm{dist}_{\mathrm{SHG}^{(n)}}(x, y) = 2 \Rightarrow |\ell_V(x) - \ell_V(y)| \geq q\Big).$$

- **Strong hypercoloring on superedges:** $L_V = \{1, \dots, k\}$ and

  $$\Phi \equiv (\forall \varepsilon \in E)\ \big(\forall x \neq y \in \varepsilon\big)\ \ell_V(x) \neq \ell_V(y).$$

Example 4.1.8 (A concrete n-SuperHyperGraph labeling: a distance–$L(2, 1)$ labeling on supervertices)**.** Fix $n := 1$ and a base set $V_0 := \{1, 2, 3, 4\}$. Define three 1-supervertices

$$X_1 := \{1, 2\}, \qquad X_2 := \{2, 3\}, \qquad X_3 := \{3, 4\},$$

and set

$$V := \{X_1, X_2, X_3\} \ \subseteq\ \mathcal{P}^1(V_0) = \mathcal{P}(V_0).$$

Let the superedge family be

$$E := \big\{\{X_1, X_2\},\ \{X_2, X_3\}\big\} \ \subseteq\ \mathcal{P}(V) \setminus \{\emptyset\},$$

so $\mathrm{SHG}^{(1)} := (V, E)$ is a 1-SuperHyperGraph.

By Definition 4.1.5, the primal graph $\mathrm{Pr}(\mathrm{SHG}^{(1)})$ has vertex set V and edges $\{X_1, X_2\}$ and $\{X_2, X_3\}$; hence

$$\mathrm{dist}_{\mathrm{SHG}^{(1)}}(X_1, X_2) = 1, \quad \mathrm{dist}_{\mathrm{SHG}^{(1)}}(X_2, X_3) = 1, \quad \mathrm{dist}_{\mathrm{SHG}^{(1)}}(X_1, X_3) = 2.$$

Let $L_V := \mathbb{Z}_{\geq 0}$ and ignore superedge labels by taking $L_E := \{*\}$ with $\ell_E(\varepsilon) = *$. Define a supervertex labeling $\ell_V : V \to L_V$ by

$$\ell_V(X_1) = 0, \qquad \ell_V(X_2) = 2, \qquad \ell_V(X_3) = 1.$$

Consider the schema $\Phi_{2,1}(\mathrm{SHG}^{(1)}; \ell_V)$ stating an $L(2,1)$-type distance constraint on supervertices:

$$\Phi_{2,1}(\mathrm{SHG}^{(1)}; \ell_V) \;\equiv\; (\forall x \neq y \in V)$$

$$\Big(\mathrm{dist}_{\mathrm{SHG}^{(1)}}(x, y) = 1 \Rightarrow |\ell_V(x) - \ell_V(y)| \geq 2 \;\wedge\; \mathrm{dist}_{\mathrm{SHG}^{(1)}}(x, y) = 2 \Rightarrow |\ell_V(x) - \ell_V(y)| \geq 1\Big).$$

This schema holds for the above ℓ_V, since

$$|\ell_V(X_1) - \ell_V(X_2)| = |0 - 2| = 2 \geq 2,$$

$$|\ell_V(X_2) - \ell_V(X_3)| = |2 - 1| = 1 \not\geq 2,$$

so we adjust slightly by instead setting $\ell_V(X_3) = 4$. Then

$$|\ell_V(X_1) - \ell_V(X_2)| = 2 \geq 2,$$

$$|\ell_V(X_2) - \ell_V(X_3)| = |2 - 4| = 2 \geq 2,$$

$$|\ell_V(X_1) - \ell_V(X_3)| = |0 - 4| = 4 \geq 1,$$

and therefore $\Phi_{2,1}(\mathrm{SHG}^{(1)}; \ell_V)$ holds. Hence (ℓ_V, ℓ_E) is a valid 1-SuperHyperGraph labeling for $\Phi_{2,1}$ in the sense of Definition 4.1.6.

4.2 SuperHyperGraph MultiLabeling

Multi-labeling enriches a combinatorial structure by assigning several labels simultaneously to each vertex-like object and each edge-like object. The resulting *label vector* can encode multiple constraints at once (e.g. scheduling, capacities, priorities), and it naturally supports optimization problems in which different coordinates interact [187].

Definition 4.2.1 (Graph MultiLabeling)**.** [187] Fix integers $p, q \geq 0$. Let $G = (V, E)$ be a finite (simple) graph. Choose nonempty *vertex alphabets* $L_V^{(1)}, \ldots, L_V^{(p)}$ and nonempty *edge alphabets* $L_E^{(1)}, \ldots, L_E^{(q)}$. A *Graph MultiLabeling* on G is a pair of label-tuples

$$\boldsymbol{\ell}_V = \big(\ell_V^{(1)}, \ldots, \ell_V^{(p)}\big), \qquad \ell_V^{(a)} : V \to L_V^{(a)} \;\; (1 \leq a \leq p),$$

$$\boldsymbol{\ell}_E = \big(\ell_E^{(1)}, \ldots, \ell_E^{(q)}\big), \qquad \ell_E^{(b)} : E \to L_E^{(b)} \;\; (1 \leq b \leq q).$$

Equivalently, one may package the coordinates into product-valued maps

$$\ell_V : V \to \prod_{a=1}^{p} L_V^{(a)}, \qquad \ell_E : E \to \prod_{b=1}^{q} L_E^{(b)},$$

where $\ell_V(v) = \big(\ell_V^{(1)}(v), \ldots, \ell_V^{(p)}(v)\big)$ and similarly for edges.

A *MultiLabeling schema* is a first-order predicate

$$\Phi\big(G; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E\big)$$

constructed from the adjacency relation in G, graph distances (when used), and the label components $\ell_V^{(a)}(v)$, $\ell_E^{(b)}(e)$ together with fixed relations/operations on the alphabets (e.g. equality, order, arithmetic, or application-specific constraints). We call $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ a *valid Graph MultiLabeling for* Φ if $\Phi\big(G; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E\big)$ holds.

Remark 4.2.2 (Typical coordinatewise and coupled constraints)**.** Many standard labelings appear as special cases on a single coordinate, for example:

- **Proper coloring on coordinate** a**:** $L_V^{(a)} = \{1, \dots, k\}$ and $(\forall \{u, v\} \in E)\ \ell_V^{(a)}(u) \neq \ell_V^{(a)}(v)$.
- $L(h, k)$**-type spacing on coordinate** a**:** $L_V^{(a)} \subseteq \mathbb{Z}$ and

$$\mathrm{dist}_G(u, v) = 1 \Rightarrow |\ell_V^{(a)}(u) - \ell_V^{(a)}(v)| \geq h, \qquad \mathrm{dist}_G(u, v) = 2 \Rightarrow |\ell_V^{(a)}(u) - \ell_V^{(a)}(v)| \geq k.$$

- **Edge capacities on coordinate** b**:** $L_E^{(b)} = \{1, \dots, C\}$ together with cross-constraints such as $\ell_E^{(b)}(\{u, v\}) \geq f\big(\ell_V^{(a)}(u), \ell_V^{(a)}(v)\big)$ for a fixed function f.

More generally, schemas may *couple* coordinates, requiring that multiple label components jointly avoid conflicts or satisfy feasibility constraints.

Example 4.2.3 (Graph MultiLabeling on a path with a coupled constraint)**.** Let $G = (V, E)$ be the path P_4 with

$$V := \{v_1, v_2, v_3, v_4\}, \qquad E := \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\big\}.$$

Fix $p = 2$ and $q = 1$. Choose alphabets

$$L_V^{(1)} := \{1, 2\} \quad \text{(a “color” coordinate)}, \qquad L_V^{(2)} := \mathbb{Z}_{\geq 0} \quad \text{(a “time” coordinate)},$$

$$L_E^{(1)} := \mathbb{Z}_{\geq 0} \quad \text{(an “edge capacity” coordinate)}.$$

Define the vertex-label coordinates by

$$\ell_V^{(1)}(v_1) = 1,\ \ell_V^{(1)}(v_2) = 2,\ \ell_V^{(1)}(v_3) = 1,\ \ell_V^{(1)}(v_4) = 2,$$

$$\ell_V^{(2)}(v_1) = 0,\ \ell_V^{(2)}(v_2) = 2,\ \ell_V^{(2)}(v_3) = 4,\ \ell_V^{(2)}(v_4) = 6,$$

and define the edge-label coordinate by

$$\ell_E^{(1)}(\{u, v\}) \ := \ \big|\ell_V^{(2)}(u) - \ell_V^{(2)}(v)\big| \qquad (\{u, v\} \in E).$$

Consider the schema $\Phi_G(G; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ asserting:

- (proper coloring on coordinate 1) for every $\{u, v\} \in E$, $\ell_V^{(1)}(u) \neq \ell_V^{(1)}(v)$;

- ($L(2,1)$-type spacing on coordinate 2) for all $u \neq v$,

$$\mathrm{dist}_G(u,v) = 1 \Rightarrow |\ell_V^{(2)}(u) - \ell_V^{(2)}(v)| \geq 2, \qquad \mathrm{dist}_G(u,v) = 2 \Rightarrow |\ell_V^{(2)}(u) - \ell_V^{(2)}(v)| \geq 1;$$

- (coupling) for each $\{u,v\} \in E$, the edge label satisfies $\ell_E^{(1)}(\{u,v\}) = |\ell_V^{(2)}(u) - \ell_V^{(2)}(v)|$.

Then Φ_G holds for the above maps (adjacent time-differences equal 2, and distance-2 differences equal 4), so $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ is a valid Graph MultiLabeling in the sense of Definition 4.2.1.

Hypergraph multi-labeling extends the same idea by replacing pairwise adjacency with vertex–hyperedge incidence and (optionally) an induced distance on the vertex set [187].

Definition 4.2.4 (HyperGraph MultiLabeling)**.** [187] Fix integers $p, q \geq 0$. Let $H = (V, \mathcal{E})$ be a finite hypergraph. Choose nonempty *vertex alphabets* $L_V^{(1)}, \ldots, L_V^{(p)}$ and nonempty *hyperedge alphabets* $L_{\mathcal{E}}^{(1)}, \ldots, L_{\mathcal{E}}^{(q)}$. A *HyperGraph MultiLabeling* on H is given by

$$\boldsymbol{\ell}_V = \big(\ell_V^{(1)}, \ldots, \ell_V^{(p)}\big), \qquad \ell_V^{(a)} : V \to L_V^{(a)} \quad (1 \leq a \leq p),$$

$$\boldsymbol{\ell}_{\mathcal{E}} = \big(\ell_{\mathcal{E}}^{(1)}, \ldots, \ell_{\mathcal{E}}^{(q)}\big), \qquad \ell_{\mathcal{E}}^{(b)} : \mathcal{E} \to L_{\mathcal{E}}^{(b)} \quad (1 \leq b \leq q).$$

Equivalently, one may use product-valued maps $\ell_V : V \to \prod_{a=1}^{p} L_V^{(a)}$ and $\ell_{\mathcal{E}} : \mathcal{E} \to \prod_{b=1}^{q} L_{\mathcal{E}}^{(b)}$.

A *MultiLabeling schema* is a first-order predicate

$$\Phi\big(H; \boldsymbol{\ell}_V, \boldsymbol{\ell}_{\mathcal{E}}\big)$$

built from the incidence relation $v \in e$, optionally a chosen vertex-distance on V (e.g. via the primal graph/2-section), and the label components together with fixed relations/operations on the alphabets. We call $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_{\mathcal{E}})$ a *valid HyperGraph MultiLabeling for* Φ if $\Phi\big(H; \boldsymbol{\ell}_V, \boldsymbol{\ell}_{\mathcal{E}}\big)$ holds.

Remark 4.2.5 (Examples of schemas in the hypergraph setting)**.** Depending on the application, Φ may encode:

- **Strong hyperedge coloring** on a vertex coordinate a: for each $e \in \mathcal{E}$, the labels $\{\ell_V^{(a)}(v) : v \in e\}$ are pairwise distinct.

- $L(h,k)$**-type separation** on a vertex coordinate a using a chosen distance dist_H: $\mathrm{dist}_H(u,v) = 1 \Rightarrow |\ell_V^{(a)}(u) - \ell_V^{(a)}(v)| \geq h$ and $\mathrm{dist}_H(u,v) = 2 \Rightarrow |\ell_V^{(a)}(u) - \ell_V^{(a)}(v)| \geq k$.

- **Vertex–hyperedge coupling**: for each $e \in \mathcal{E}$, the hyperedge label $\ell_{\mathcal{E}}^{(b)}(e)$ is constrained by an aggregate of $\{\ell_V^{(a)}(v) : v \in e\}$ (sum, maximum, count, etc.).

Example 4.2.6 (HyperGraph MultiLabeling with strong hyperedge coloring and vertex–hyperedge coupling)**.** Let $H = (V, \mathcal{E})$ be the hypergraph with

$$V := \{a, b, c, d\}, \qquad \mathcal{E} := \{e_1, e_2\}, \qquad e_1 := \{a, b, c\}, \;\; e_2 := \{b, c, d\}.$$

Fix $p = 2$ and $q = 1$. Choose alphabets

$$L_V^{(1)} := \{1, 2, 3\} \quad \text{(a "role"/color coordinate)},$$
$$L_V^{(2)} := \{10, 20, 30, 40\} \quad \text{(a "priority" coordinate)},$$
$$L_{\mathcal{E}}^{(1)} := \{0, 1\} \quad \text{(a binary hyperedge flag)}.$$

Define vertex coordinates by

$$\ell_V^{(1)}(a) = 1, \; \ell_V^{(1)}(b) = 2, \; \ell_V^{(1)}(c) = 3, \; \ell_V^{(1)}(d) = 1,$$
$$\ell_V^{(2)}(a) = 10, \; \ell_V^{(2)}(b) = 20, \; \ell_V^{(2)}(c) = 30, \; \ell_V^{(2)}(d) = 40,$$

and define the hyperedge coordinate by the coupling rule

$$\ell_{\mathcal{E}}^{(1)}(e) \;:=\; \begin{cases} 1, & \text{if } \max\{\ell_V^{(2)}(v) : v \in e\} \geq 30, \\ 0, & \text{otherwise}, \end{cases} \qquad (e \in \mathcal{E}).$$

Let $\Phi_H(H; \boldsymbol{\ell}_V, \boldsymbol{\ell}_{\mathcal{E}})$ be the schema asserting:

- (strong hyperedge coloring on coordinate 1) $(\forall e \in \mathcal{E})\, (\forall u \neq v \in e)\; \ell_V^{(1)}(u) \neq \ell_V^{(1)}(v)$;
- (coupling) $(\forall e \in \mathcal{E})$ the value $\ell_{\mathcal{E}}^{(1)}(e)$ equals the threshold test defined above.

Then Φ_H holds: indeed, the role labels on $e_1 = \{a, b, c\}$ are $\{1, 2, 3\}$ (all distinct) and on $e_2 = \{b, c, d\}$ are $\{2, 3, 1\}$ (all distinct), and the hyperedge flag is consistent with the maximum-priority rule. Hence we obtain a valid HyperGraph MultiLabeling in the sense of Definition 4.2.4.

We now formulate the analogous notion for n-SuperHyperGraphs. Since superedges are set-systems on the supervertex set, it is convenient to measure "distance" between supervertices via the primal graph.

Definition 4.2.7 (Primal graph and induced distance for an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph, i.e., $\emptyset \neq V \subseteq \mathcal{P}^n(V_0)$ and $\emptyset \neq E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Its *primal graph* (or 2*-section*) is the simple graph

$$\mathrm{Pr}\big(\mathrm{SHG}^{(n)}\big) := (V, F), \qquad \{x, y\} \in F \iff x \neq y \;\text{ and }\; \exists\, \varepsilon \in E \;\text{ with }\; \{x, y\} \subseteq \varepsilon.$$

The *(supervertex) distance* $\mathrm{dist}_{\mathrm{SHG}^{(n)}}(x, y)$ is the usual shortest-path distance between $x, y \in V$ computed in $\mathrm{Pr}(\mathrm{SHG}^{(n)})$.

Definition 4.2.8 (Flattening (base support) of an n-level object)**.** Let V_0 be a base set and $n \in \mathbb{N}_0$. Define recursively a map $\mathrm{Flat}_n : \mathcal{P}^n(V_0) \to \mathcal{P}(V_0)$ by

$$\mathrm{Flat}_0(v) := \{v\} \quad (v \in V_0), \qquad \mathrm{Flat}_{n+1}(X) := \bigcup_{Y \in X} \mathrm{Flat}_n(Y) \quad (X \in \mathcal{P}^{n+1}(V_0)).$$

Thus $\mathrm{Flat}_n(X)$ collects the base elements of V_0 that appear anywhere inside the nested object X.

Definition 4.2.9 (SuperHyperGraph MultiLabeling)**.** Fix integers $p, q \geq 0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. Choose nonempty *supervertex alphabets* $L_V^{(1)}, \ldots, L_V^{(p)}$ and nonempty *superedge alphabets* $L_E^{(1)}, \ldots, L_E^{(q)}$. A *SuperHyperGraph MultiLabeling* on $\mathrm{SHG}^{(n)}$ is given by

$$\boldsymbol{\ell}_V = \big(\ell_V^{(1)}, \ldots, \ell_V^{(p)}\big), \qquad \ell_V^{(a)} : V \to L_V^{(a)} \ \ (1 \leq a \leq p),$$

$$\boldsymbol{\ell}_E = \big(\ell_E^{(1)}, \ldots, \ell_E^{(q)}\big), \qquad \ell_E^{(b)} : E \to L_E^{(b)} \ \ (1 \leq b \leq q).$$

Equivalently, one may use product-valued maps

$$\ell_V : V \to \prod_{a=1}^{p} L_V^{(a)}, \qquad \ell_E : E \to \prod_{b=1}^{q} L_E^{(b)}.$$

A *MultiLabeling schema* is a first-order predicate

$$\Phi\big(\mathrm{SHG}^{(n)}; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E\big)$$

constructed from the incidence relation $x \in \varepsilon$ (with $x \in V$ and $\varepsilon \in E$), the distance $\mathrm{dist}_{\mathrm{SHG}^{(n)}}$ from Definition 4.2.7, the flattening operator Flat_n from Definition 4.2.8, basic set-theoretic or cardinality operations on $\mathcal{P}(V_0)$, and fixed relations/operations on the alphabets. We call $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ a *valid SuperHyperGraph MultiLabeling for* Φ if $\Phi\big(\mathrm{SHG}^{(n)}; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E\big)$ holds.

Remark 4.2.10 (Common patterns of constraints in Φ)**.** Schemas for SuperHyperGraph MultiLabeling frequently impose:

- **Distance-sensitive separation on a coordinate** a**:** $\mathrm{dist}_{\mathrm{SHG}^{(n)}}(x, y) = 1 \Rightarrow |\ell_V^{(a)}(x) - \ell_V^{(a)}(y)| \geq \lambda_1$ and $\mathrm{dist}_{\mathrm{SHG}^{(n)}}(x, y) = 2 \Rightarrow |\ell_V^{(a)}(x) - \ell_V^{(a)}(y)| \geq \lambda_2$.

- **Support-aware vertex constraints:** for instance $\ell_V^{(a)}(X) = |\,\mathrm{Flat}_n(X)|$ for all $X \in V$.

- **Superedge aggregation constraints:** $\ell_E^{(b)}(\varepsilon) = g\big(\{\mathrm{Flat}_n(X) : X \in \varepsilon\}\big)$ for a fixed aggregator g (e.g. $|\bigcup_{X \in \varepsilon} \mathrm{Flat}_n(X)|$, $|\bigcap_{X \in \varepsilon} \mathrm{Flat}_n(X)|$, or a statistic of vertex-label coordinates).

Example 4.2.11 (n-SuperHyperGraph MultiLabeling with a support-aware coordinate)**.** Fix $n := 1$ and a base set $V_0 := \{1, 2, 3, 4\}$. Consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ with supervertices

$$X_1 := \{1, 2\}, \qquad X_2 := \{2, 3\}, \qquad X_3 := \{3, 4\}, \qquad V := \{X_1, X_2, X_3\} \subseteq \mathcal{P}(V_0),$$

and superedges

$$E := \big\{\{X_1, X_2\}, \{X_2, X_3\}\big\}.$$

(Thus the primal graph on V is the path X_1–X_2–X_3.)

Fix $p = 2$ and $q = 1$. Choose alphabets

$$L_V^{(1)} := \mathbb{Z}_{\geq 0} \quad \text{(a “support-size” coordinate)}, \qquad L_V^{(2)} := \mathbb{Z}_{\geq 0} \quad \text{(a “distance label” coordinate)},$$

$$L_E^{(1)} := \mathbb{Z}_{\geq 0} \quad \text{(a "union-size" superedge coordinate).}$$

For $n = 1$ the flattening/base-support satisfies $\mathrm{Flat}_1(X) = X \subseteq V_0$. Define the supervertex coordinates by

$$\ell_V^{(1)}(X) \;:=\; |\,\mathrm{Flat}_1(X)| = |X|, \qquad \ell_V^{(2)}(X_1) = 0,\; \ell_V^{(2)}(X_2) = 2,\; \ell_V^{(2)}(X_3) = 4,$$

and define the superedge coordinate by the support-union aggregator

$$\ell_E^{(1)}(\varepsilon) \;:=\; \Big|\bigcup_{X\in\varepsilon} \mathrm{Flat}_1(X)\Big| \qquad (\varepsilon \in E).$$

Let $\Phi_{\mathrm{SHG}}(\mathrm{SHG}^{(1)}; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ be the schema asserting:

- (support-aware coordinate) for all $X \in V$, $\ell_V^{(1)}(X) = |\,\mathrm{Flat}_1(X)|$;
- (distance separation on coordinate 2) for all $X \neq Y \in V$,
$$\mathrm{dist}_{\mathrm{SHG}^{(1)}}(X, Y) = 1 \Rightarrow |\ell_V^{(2)}(X) - \ell_V^{(2)}(Y)| \geq 2;$$
- (superedge aggregation) for all $\varepsilon \in E$, $\ell_E^{(1)}(\varepsilon) = \big|\bigcup_{X\in\varepsilon} \mathrm{Flat}_1(X)\big|$.

Then Φ_{SHG} holds: adjacent supervertices differ by 2 on coordinate 2, and

$$\ell_E^{(1)}(\{X_1, X_2\}) = |\{1,2\} \cup \{2,3\}| = 3, \qquad \ell_E^{(1)}(\{X_2, X_3\}) = |\{2,3\} \cup \{3,4\}| = 3.$$

Therefore $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ is a valid 1-SuperHyperGraph MultiLabeling in the sense of Definition 4.2.9.

4.3 SuperHyperGraph L(h,k)-Labeling

A graph $L(h, k)$-labeling assigns integers to vertices so adjacent vertices differ by at least h and distance-two vertices by k [188–191]. A hypergraph $L(h, k)$-labeling labels vertices so co-hyperedge vertices differ by at least h, and incidence-distance-two vertices differ by k likewise. A SuperHyperGraph $L(h, k)$-labeling labels supervertices so superedge-adjacent supervertices differ by at least h, and primal-distance-two supervertices differ by k too.

Definition 4.3.1 (Graph $L(h, k)$-labeling)**.** [188, 192] Let $G = (V, E)$ be a finite simple graph and let $h, k \in \mathbb{R}_{\geq 0}$. An $L(h, k)$-*labeling* of G is a map

$$\ell : V \longrightarrow \mathbb{Z}_{\geq 0}$$

such that for all distinct $u, v \in V$,

$$\mathrm{dist}_G(u, v) = 1 \implies |\ell(u) - \ell(v)| \geq h, \qquad \mathrm{dist}_G(u, v) = 2 \implies |\ell(u) - \ell(v)| \geq k.$$

Equivalently, the second condition can be stated as:

$$\big(\exists\, w \in V :\; uw \in E \text{ and } wv \in E\big) \implies |\ell(u) - \ell(v)| \geq k,$$

which is the usual "common-neighbor'' formulation.

The *span* of ℓ is

$$\mathrm{span}(\ell) := \max_{v\in V} \ell(v) - \min_{v\in V} \ell(v),$$

and (without loss of generality) one may assume $\min \ell = 0$. We write $\lambda_{h,k}(G)$ for the minimum span among all $L(h, k)$-labelings of G.

Example 4.3.2 (A graph $L(h,k)$-labeling)**.** Fix $(h,k) = (3,5)$ and let G be the path P_4 with

$$V(G) = \{v_1, v_2, v_3, v_4\}, \qquad E(G) = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\big\}.$$

Define $\ell : V(G) \to \mathbb{Z}_{\geq 0}$ by

$$\ell(v_1) = 0, \qquad \ell(v_2) = 3, \qquad \ell(v_3) = 8, \qquad \ell(v_4) = 11.$$

Then adjacent vertices satisfy $|\ell(v_i) - \ell(v_{i+1})| \geq 3$ for $i = 1, 2, 3$, and the distance-2 pairs (v_1, v_3) and (v_2, v_4) satisfy

$$|\ell(v_1) - \ell(v_3)| = 8 \geq 5, \qquad |\ell(v_2) - \ell(v_4)| = 8 \geq 5.$$

Hence ℓ is an $L(3,5)$-labeling of G.

Definition 4.3.3 (Hypergraph $L(h,k)$-labeling via the primal graph)**.** Let $H = (V, \mathcal{E})$ be a finite hypergraph and let $\mathrm{Pr}(H)$ denote its *primal graph* (or 2-section), i.e. the simple graph on vertex set V where $\{u, v\}$ is an edge iff $u \neq v$ and there exists $e \in \mathcal{E}$ with $\{u, v\} \subseteq e$.

An $L(h,k)$*-labeling* of H is an $L(h,k)$-labeling of the graph $\mathrm{Pr}(H)$; namely, it is a map

$$\ell : V \longrightarrow \mathbb{Z}_{\geq 0}$$

such that for all distinct $u, v \in V$,

$$\mathrm{dist}_{\mathrm{Pr}(H)}(u,v) = 1 \implies |\ell(u) - \ell(v)| \geq h, \qquad \mathrm{dist}_{\mathrm{Pr}(H)}(u,v) = 2 \implies |\ell(u) - \ell(v)| \geq k.$$

We set $\lambda_{h,k}(H) := \lambda_{h,k}(\mathrm{Pr}(H))$.

Remark 4.3.4 (Berge distance agrees with primal-graph distance)**.** If one defines the distance between vertices of a hypergraph using Berge paths (vertex–hyperedge alternating paths), then the resulting Berge distance equals $\mathrm{dist}_{\mathrm{Pr}(H)}$. Indeed, each Berge step $u \in e \ni v$ corresponds to an edge uv in $\mathrm{Pr}(H)$, and conversely each edge of $\mathrm{Pr}(H)$ is witnessed by some hyperedge of H. Thus Definition 4.3.3 is equivalent to defining $L(h,k)$-constraints using Berge distance.

Example 4.3.5 (A hypergraph $L(h,k)$-labeling via the primal graph)**.** Fix $(h,k) = (3,5)$ and let $H = (V, \mathcal{E})$ be the hypergraph

$$V = \{a, b, c, d\}, \qquad \mathcal{E} = \{e_1, e_2\}, \qquad e_1 = \{a, b, c\}, \quad e_2 = \{c, d\}.$$

Its primal graph $\mathrm{Pr}(H)$ has edge set

$$E\big(\mathrm{Pr}(H)\big) = \big\{\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\big\}.$$

Define $\ell : V \to \mathbb{Z}_{\geq 0}$ by

$$\ell(a) = 0, \qquad \ell(b) = 3, \qquad \ell(c) = 8, \qquad \ell(d) = 11.$$

Along each edge of $\mathrm{Pr}(H)$ we have $|\ell(u) - \ell(v)| \geq 3$, and the distance-2 pairs in $\mathrm{Pr}(H)$ are (a, d) and (b, d) (via c), for which

$$|\ell(a) - \ell(d)| = 11 \geq 5, \qquad |\ell(b) - \ell(d)| = 8 \geq 5.$$

Therefore ℓ is an $L(3,5)$-labeling of H in the sense of Definition 4.3.3.

Definition 4.3.6 (n-SuperHyperGraph $L(h,k)$-labeling)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be a finite n-SuperHyperGraph. Let $\Pr(\mathrm{SHG}^{(n)})$ be its primal graph (the 2-section): vertices are V, and distinct $x, y \in V$ are adjacent iff there exists $\varepsilon \in E$ with $\{x, y\} \subseteq \varepsilon$.

An $L(h,k)$*-labeling* of $\mathrm{SHG}^{(n)}$ is an $L(h,k)$-labeling of $\Pr(\mathrm{SHG}^{(n)})$; that is, it is a map

$$\ell : V \longrightarrow \mathbb{Z}_{\geq 0}$$

such that for all distinct $x, y \in V$,

$$\mathrm{dist}_{\Pr(\mathrm{SHG}^{(n)})}(x,y) = 1 \implies |\ell(x)-\ell(y)| \geq h, \qquad \mathrm{dist}_{\Pr(\mathrm{SHG}^{(n)})}(x,y) = 2 \implies |\ell(x)-\ell(y)| \geq k.$$

We write $\lambda_{h,k}(\mathrm{SHG}^{(n)}) := \lambda_{h,k}\big(\Pr(\mathrm{SHG}^{(n)})\big)$.

Example 4.3.7 (A hypergraph $L(h,k)$-labeling via the primal graph)**.** Fix $(h,k) = (3,5)$ and let $H = (V, \mathcal{E})$ be the hypergraph

$$V = \{a, b, c, d\}, \qquad \mathcal{E} = \{e_1, e_2\}, \qquad e_1 = \{a, b, c\}, \quad e_2 = \{c, d\}.$$

Its primal graph $\Pr(H)$ has edge set

$$E\big(\Pr(H)\big) = \big\{\{a,b\}, \{a,c\}, \{b,c\}, \{c,d\}\big\}.$$

Define $\ell : V \to \mathbb{Z}_{\geq 0}$ by

$$\ell(a) = 0, \qquad \ell(b) = 3, \qquad \ell(c) = 8, \qquad \ell(d) = 11.$$

Along each edge of $\Pr(H)$ we have $|\ell(u) - \ell(v)| \geq 3$, and the distance-2 pairs in $\Pr(H)$ are (a,d) and (b,d) (via c), for which

$$|\ell(a) - \ell(d)| = 11 \geq 5, \qquad |\ell(b) - \ell(d)| = 8 \geq 5.$$

Therefore ℓ is an $L(3,5)$-labeling of H in the sense of Definition 4.3.3.

Theorem 4.3.8 (n-SuperHyperGraph $L(h,k)$-labeling generalizes the graph and hypergraph cases)**.** *Fix $n \in \mathbb{N}_0$ and $h, k \in \mathbb{R}_{\geq 0}$.*

1. ***(Graphs embed.)*** *For every finite simple graph $G = (V_G, E_G)$ there exists an n-SuperHyperGraph $\mathrm{SHG}^{(n)}_G$ such that*

$$\Pr(\mathrm{SHG}^{(n)}_G) \;\cong\; G.$$

Consequently, $L(h,k)$-labelings of G are in bijection with $L(h,k)$-labelings of $\mathrm{SHG}^{(n)}_G$ (via transport along the isomorphism), and

$$\lambda_{h,k}(\mathrm{SHG}^{(n)}_G) = \lambda_{h,k}(G).$$

2. ***(Hypergraphs embed.)*** *For every finite hypergraph $H = (V_H, \mathcal{E}_H)$ there exists an n-SuperHyperGraph $\mathrm{SHG}^{(n)}_H$ such that*

$$\Pr(\mathrm{SHG}^{(n)}_H) \;\cong\; \Pr(H).$$

Hence $L(h,k)$-labelings of H (Definition 4.3.3) are in bijection with $L(h,k)$-labelings of $\mathrm{SHG}^{(n)}_H$, and

$$\lambda_{h,k}(\mathrm{SHG}^{(n)}_H) = \lambda_{h,k}(H).$$

3. ***(Recovery at level*** $n = 0$***.)*** *When* $n = 0$*, an* n*-SuperHyperGraph is (canonically) a hypergraph, so Definition 4.3.6 reduces to Definition 4.3.3.*

Proof. Define the iterated singleton embedding $\iota_0(x) := x$ and $\iota_{t+1}(x) := \{\iota_t(x)\}$.

(1) Graphs. Let $G = (V_G, E_G)$ and set $V_0 := V_G$ and

$$V := \{\iota_n(v) : v \in V_G\} \subseteq \mathcal{P}^n(V_0).$$

Define the superedge family

$$E := \{\{\iota_n(u), \iota_n(v)\} : \{u, v\} \in E_G\}.$$

Then $\mathrm{SHG}^{(n)}_G := (V, E)$ is an n-SuperHyperGraph. By construction, distinct $\iota_n(u), \iota_n(v) \in V$ are adjacent in $\Pr(\mathrm{SHG}^{(n)}_G)$ iff $\{u, v\} \in E_G$. Thus the bijection $\varphi : V_G \to V$ given by $\varphi(v) = \iota_n(v)$ is a graph isomorphism $G \cong \Pr(\mathrm{SHG}^{(n)}_G)$. In particular, $\mathrm{dist}_G(u, v) = \mathrm{dist}_{\Pr(\mathrm{SHG}^{(n)}_G)}(\iota_n(u), \iota_n(v))$ for all $u, v \in V_G$, so an $L(h, k)$-labeling of G pulls back/pushes forward to an $L(h, k)$-labeling of $\mathrm{SHG}^{(n)}_G$ with the same span, proving $\lambda_{h,k}(\mathrm{SHG}^{(n)}_G) = \lambda_{h,k}(G)$.

(2) Hypergraphs. Let $H = (V_H, \mathcal{E}_H)$. Again set $V_0 := V_H$ and

$$V := \{\iota_n(v) : v \in V_H\} \subseteq \mathcal{P}^n(V_0), \qquad E := \{\{\iota_n(v) : v \in e\} : e \in \mathcal{E}_H\}.$$

Then $\mathrm{SHG}^{(n)}_H := (V, E)$ is an n-SuperHyperGraph. Two distinct vertices $u, v \in V_H$ are adjacent in $\Pr(H)$ iff there exists $e \in \mathcal{E}_H$ with $\{u, v\} \subseteq e$, which holds iff there exists the corresponding superedge $\varepsilon_e = \{\iota_n(x) : x \in e\} \in E$ containing $\{\iota_n(u), \iota_n(v)\}$. Hence the bijection $\varphi(v) = \iota_n(v)$ yields an isomorphism $\Pr(H) \cong \Pr(\mathrm{SHG}^{(n)}_H)$. Distances are preserved, so $L(h, k)$-labelings (and their optimal spans) correspond: $\lambda_{h,k}(\mathrm{SHG}^{(n)}_H) = \lambda_{h,k}(\Pr(H)) = \lambda_{h,k}(H)$.

(3) The case $n = 0$**.** For $n = 0$ one has $V \subseteq \mathcal{P}^0(V_0) = V_0$ and superedges are subsets of V, i.e. a hypergraph structure; the standard identification of level-0 SuperHyperGraphs with hypergraphs is canonical. Therefore Definition 4.3.6 specializes to Definition 4.3.3. □

4.4 SuperHyperGraph Graceful labeling

A *graceful labeling of a graph* injectively labels vertices by $\{0, \ldots, |E|\}$ so absolute edge differences produce all labels $1, \ldots, |E|$ exactly once [50, 51, 193–195]. A *graceful labeling of a hypergraph* injectively labels vertices by $\{0, \ldots, |\mathcal{E}|\}$ so each hyperedge's minimum pairwise difference yields a bijection onto $[|\mathcal{E}|]$. A *graceful labeling of a superhypergraph* injectively labels supervertices by $\{0, \ldots, |E|\}$ so every superedge's minimum internal difference forms a bijection onto $[|E|]$.

Definition 4.4.1 (Graceful labeling of a graph)**.** Let $G = (V, E)$ be a finite simple graph and put $m := |E|$. A *graceful labeling* of G is an injective map

$$f : V \longrightarrow \{0, 1, \ldots, m\}$$

such that the induced edge-labeling

$$f_E : E \longrightarrow \{1, 2, \ldots, m\}, \qquad f_E(\{u, v\}) := |f(u) - f(v)|$$

is a bijection. A graph admitting a graceful labeling is called *graceful.*

Example 4.4.2 (Graceful labeling of a graph)**.** Let $G = P_3$ be the path on three vertices,

$$V = \{v_1, v_2, v_3\}, \qquad E = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\},$$

so $m = |E| = 2$. Define $f : V \to \{0, 1, 2\}$ by

$$f(v_1) = 0, \qquad f(v_2) = 2, \qquad f(v_3) = 1.$$

Then the induced edge-labels are

$$f_E(\{v_1, v_2\}) = |0 - 2| = 2, \qquad f_E(\{v_2, v_3\}) = |2 - 1| = 1,$$

so f_E is a bijection $E \to \{1, 2\}$. Hence f is a graceful labeling of G in the sense of Definition 4.4.1.

Remark 4.4.3 (A hypergraph convention)**.** There are several non-equivalent extensions of graceful labeling from graphs to hypergraphs. In this section we fix the following *min-difference* convention; it agrees with Definition 4.4.1 on 2-uniform hypergraphs (i.e., graphs).

Definition 4.4.4 (Graceful labeling of a hypergraph (min-difference version))**.** Let $H = (V, \mathcal{E})$ be a finite hypergraph and put $m := |\mathcal{E}|$. A *graceful labeling* of H (in the sense of Remark 4.4.3) is an injective map

$$f : V \longrightarrow \{0, 1, \ldots, m\}$$

such that the induced hyperedge-labeling

$$f_{\mathcal{E}} : \mathcal{E} \longrightarrow \{1, 2, \ldots, m\}, \qquad f_{\mathcal{E}}(e) \ := \ \min\big\{|f(u) - f(v)| : \ u, v \in e, \ u \neq v\big\}$$

is a bijection. A hypergraph admitting such a labeling is called *graceful*.

Example 4.4.5 (Graceful labeling of a hypergraph (min-difference version))**.** Let $H = (V, \mathcal{E})$ be the hypergraph

$$V = \{a, b, c\}, \qquad \mathcal{E} = \{e_1, e_2\}, \qquad e_1 = \{a, b, c\}, \ \ e_2 = \{b, c\},$$

so $m = |\mathcal{E}| = 2$. Define $f : V \to \{0, 1, 2\}$ by

$$f(a) = 0, \qquad f(b) = 2, \qquad f(c) = 1.$$

Compute the induced hyperedge-labels (Definition 4.4.4):

$$f_{\mathcal{E}}(e_1) = \min\{|f(u) - f(v)| : \ u \neq v, \ u, v \in \{a, b, c\}\} = \min\{2, 1, 1\} = 1,$$

$$f_{\mathcal{E}}(e_2) = \min\{|f(u) - f(v)| : \ u \neq v, \ u, v \in \{b, c\}\} = |2 - 1| = 1.$$

Thus $f_{\mathcal{E}}$ is *not* injective. To obtain a graceful labeling, keep the same vertex set but use the two hyperedges

$$\mathcal{E}' = \{e'_1, e'_2\}, \qquad e'_1 = \{a, c\}, \ \ e'_2 = \{a, b, c\},$$

and define $f'(a) = 0$, $f'(b) = 2$, $f'(c) = 1$ as above. Then

$$f'_{\mathcal{E}'}(e'_1) = \min\{|f'(a) - f'(c)|\} = 1, \qquad f'_{\mathcal{E}'}(e'_2) = \min\{2, 1, 1\} = 1,$$

still fails, so instead set

$$f''(a) = 0, \qquad f''(b) = 1, \qquad f''(c) = 2.$$

Now

$$f''_{\mathcal{E}'}(e'_1) = |0 - 2| = 2, \qquad f''_{\mathcal{E}'}(e'_2) = \min\{1, 2, 1\} = 1,$$

so $f''_{\mathcal{E}'}$ is a bijection $\mathcal{E}' \to \{1, 2\}$. Hence f'' is a graceful labeling of the hypergraph $H' = (V, \mathcal{E}')$ in the min-difference sense of Definition 4.4.4.

Definition 4.4.6 (Graceful labeling of an n-SuperHyperGraph (min-difference version))**.** Fix $n \in \mathbb{N}_0$. Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (so V is the set of n-supervertices and each superedge is a nonempty subset of V). Put $m := |E|$. A *graceful labeling* of $\mathrm{SHG}^{(n)}$ (in the sense of Remark 4.4.3) is an injective map

$$f : V \longrightarrow \{0, 1, \dots, m\}$$

such that the induced superedge-labeling

$$f_E : E \longrightarrow \{1, 2, \dots, m\}, \qquad f_E(\varepsilon) \;:=\; \min\bigl\{|f(x) - f(y)| :\ x, y \in \varepsilon,\ x \neq y\bigr\}$$

is a bijection. An n-SuperHyperGraph admitting such a labeling is called *graceful*.

Example 4.4.7 (Graceful labeling of an n-SuperHyperGraph (min-difference version))**.** Fix $n = 1$ and let $\mathrm{SHG}^{(1)} = (V, E)$ have supervertex set

$$V = \{X_1, X_2, X_3\}, \qquad X_1 = \{1\},\ \ X_2 = \{2\},\ \ X_3 = \{3\} \subseteq \mathcal{P}^1(V_0),$$

and superedge family

$$E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{X_1, X_3\},\ \ \varepsilon_2 = \{X_1, X_2, X_3\}.$$

Then $m = |E| = 2$ and define $f : V \to \{0, 1, 2\}$ by

$$f(X_1) = 0, \qquad f(X_2) = 1, \qquad f(X_3) = 2.$$

The induced superedge-labels are

$$f_E(\varepsilon_1) = \min\{|f(X_1) - f(X_3)|\} = |0 - 2| = 2, \qquad f_E(\varepsilon_2) = \min\{1, 2, 1\} = 1.$$

Hence f_E is a bijection $E \to \{1, 2\}$, and f is a graceful labeling of $\mathrm{SHG}^{(1)}$ in the sense of Definition 4.4.6.

Definition 4.4.8 (Iterated singleton embedding)**.** Let V_0 be a finite set and fix $n \in \mathbb{N}_0$. Define $\iota_0 : V_0 \to \mathcal{P}^0(V_0) = V_0$ by $\iota_0(v) = v$, and for $k \geq 0$ define $\iota_{k+1}(v) := \{\iota_k(v)\}$. Then $\iota_n : V_0 \to \mathcal{P}^n(V_0)$ is injective.

Definition 4.4.9 (n-lift of a hypergraph)**.** Let $H = (V_0, \mathcal{E})$ be a hypergraph and fix $n \in \mathbb{N}_0$. Define the n-supervertex set

$$V^{(n)} \;:=\; \{\iota_n(v) : v \in V_0\} \;\subseteq\; \mathcal{P}^n(V_0),$$

and the n-superedge set

$$E^{(n)} \;:=\; \{\; \iota_n(e) \;:=\; \{\iota_n(v) : v \in e\} \;:\; e \in \mathcal{E} \;\}.$$

Then $\mathrm{Lift}_n(H) := (V^{(n)}, E^{(n)})$ is an n-SuperHyperGraph, called the *n-lift* of H. (If H is a graph, this definition applies with $\mathcal{E} = E$.)

Theorem 4.4.10 (n-SuperHyperGraph graceful labeling generalizes graph/hypergraph graceful labeling)**.** *Fix $n \in \mathbb{N}_0$.*

1. *If a graph G is graceful (Definition 4.4.1), then its n-lift* $\mathrm{Lift}_n(G)$ *is graceful as an n-SuperHyperGraph (Definition 4.4.6).*

2. *If a hypergraph H is graceful (Definition 4.4.4), then its n-lift* $\mathrm{Lift}_n(H)$ *is graceful as an n-SuperHyperGraph (Definition 4.4.6).*

3. *For $n = 0$, Definition 4.4.6 coincides with Definition 4.4.4 (hence also with Definition 4.4.1 in the 2-uniform case).*

Proof. Let $H = (V_0, \mathcal{E})$ be a hypergraph (a graph is the 2-uniform special case), and fix $n \in \mathbb{N}_0$. Write $\mathrm{Lift}_n(H) = (V^{(n)}, E^{(n)})$ as in Definition 4.4.9. Set $m := |\mathcal{E}| = |E^{(n)}|$.

Assume H has a graceful labeling $f : V_0 \to \{0, 1, \dots, m\}$ in the sense of Definition 4.4.4 (or Definition 4.4.1 if H is a graph). Define a labeling on $V^{(n)}$ by transport along ι_n:

$$\widetilde{f} : V^{(n)} \longrightarrow \{0, 1, \dots, m\},$$

$$\widetilde{f}(\iota_n(v)) := f(v).$$

Since ι_n and f are injective, so is $\widetilde{f}$.

Now take a hyperedge $e \in \mathcal{E}$ and its lifted superedge $\iota_n(e) \in E^{(n)}$. By definition of the induced label in Definition 4.4.6,

$$\begin{aligned}\widetilde{f}_E(\iota_n(e)) &= \min\{|\widetilde{f}(\iota_n(u)) - \widetilde{f}(\iota_n(v))| : \ u, v \in e, \ u \neq v\} \\ &= \min\{|f(u) - f(v)| : \ u, v \in e, \ u \neq v\} = f_{\mathcal{E}}(e).\end{aligned}$$

Hence the induced superedge-labeling $\widetilde{f}_E : E^{(n)} \to \{1, \dots, m\}$ is exactly $f_{\mathcal{E}}$ transported along the bijection $e \mapsto \iota_n(e)$ between $\mathcal{E}$ and $E^{(n)}$. Therefore, $\widetilde{f}_E$ is a bijection if and only if $f_{\mathcal{E}}$ is a bijection. This proves (2). In the graph case, every edge has size 2, so the "min" is taken over a singleton set and

$$\widetilde{f}_E(\{\iota_n(u), \iota_n(v)\}) = |f(u) - f(v)|,$$

which yields (1).

Finally, for $n = 0$ we have $V^{(0)} = V_0$ and $E^{(0)} = \mathcal{E}$, so the notion of graceful labeling in Definition 4.4.6 is literally Definition 4.4.4. This proves (3). □

4.5 SuperHyperGraph Harmonious Labeling

A *graph harmonious labeling* injectively maps vertices into $\mathbb{Z}_{|E|}$ so each edge-sum $f(u) + f(v)$ yields distinct residues [47, 48, 48, 49, 49, 196, 197]. A *hypergraph harmonious labeling* injectively maps vertices into $\mathbb{Z}_{|\mathcal{E}|}$ so each hyperedge-sum $\sum_{v \in e} f(v)$ is bijective. A *superhypergraph harmonious labeling* injectively maps supervertices into $\mathbb{Z}_{|E|}$ so each superedge-sum $\sum_{x \in \varepsilon} f(x)$ is bijective. For an integer $m \geq 1$, write $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ for the cyclic group of order m, and compute all sums modulo m.

Definition 4.5.1 (Graph harmonious labeling)**.** Let $G = (V, E)$ be a finite (simple, undirected) graph with $q := |E| \geq 1$. A map $f : V \to \mathbb{Z}_q$ is called a *harmonious labeling* of G if

1. f is injective, and

2. the induced edge-labeling $f^* : E \to \mathbb{Z}_q$ defined by

$$f^*(\{u, v\}) \;:=\; f(u) + f(v) \qquad (\{u, v\} \in E)$$

   is bijective (equivalently, all edge labels are pairwise distinct in $\mathbb{Z}_q$).

If such an f exists, then G is called a *harmonious graph*.

Example 4.5.2 (Harmonious labeling of a graph)**.** Let $G = P_4$ be the path on four vertices,

$$V = \{v_1, v_2, v_3, v_4\}, \qquad E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\big\},$$

so $q := |E| = 3$ and $\mathbb{Z}_q = \mathbb{Z}_3$. Define $f : V \to \mathbb{Z}_3$ by

$$f(v_1) = 0, \qquad f(v_2) = 1, \qquad f(v_3) = 2, \qquad f(v_4) = 0.$$

Then f is injective on V? (No: $f(v_1) = f(v_4)$.) So we instead take a graph with $|V| \leq q$ to allow injectivity.

Let $G = P_3$ be the path on three vertices,

$$V = \{v_1, v_2, v_3\}, \qquad E = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\},$$

so $q = 2$ and $\mathbb{Z}_2 = \{0, 1\}$. Define

$$f(v_1) = 0, \qquad f(v_2) = 1, \qquad f(v_3) = 0.$$

Again f is not injective. Hence, for the injective (standard) notion, take the triangle $G = K_3$, where $q = 3$ and $|V| = 3$.

Let $G = K_3$ with $V = \{v_1, v_2, v_3\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}$, so $q = 3$. Define $f : V \to \mathbb{Z}_3$ by

$$f(v_1) = 0, \qquad f(v_2) = 1, \qquad f(v_3) = 2.$$

Then f is injective, and the induced edge-labels (modulo 3) are

$$f^*(\{v_1, v_2\}) = 0 + 1 = 1, \quad f^*(\{v_2, v_3\}) = 1 + 2 = 0, \quad f^*(\{v_1, v_3\}) = 0 + 2 = 2,$$

which are all distinct and hence form a bijection $E \to \mathbb{Z}_3$. Therefore f is a harmonious labeling of K_3.

Definition 4.5.3 (Hypergraph harmonious labeling)**.** Let $H = (V, \mathcal{E})$ be a finite hypergraph with $m := |\mathcal{E}| \geq 1$. A map $f : V \to \mathbb{Z}_m$ is called a *harmonious labeling* of H if

1. f is injective, and

2. the induced hyperedge-labeling $f^* : \mathcal{E} \to \mathbb{Z}_m$ defined by

$$f^*(e) \;:=\; \sum_{v \in e} f(v) \qquad (e \in \mathcal{E})$$

   is bijective.

If such an f exists, then H is called a *harmonious hypergraph*.

Example 4.5.4 (Harmonious labeling of a hypergraph)**.** Let $H = (V, \mathcal{E})$ be the hypergraph

$$V = \{a, b, c\}, \qquad \mathcal{E} = \{e_1, e_2\}, \qquad e_1 = \{a, b\}, \;\; e_2 = \{a, c\}.$$

Here $m := |\mathcal{E}| = 2$ and we work in $\mathbb{Z}_2$. Define $f : V \to \mathbb{Z}_2$ by

$$f(a) = 0, \qquad f(b) = 1, \qquad f(c) = 0.$$

Then f is not injective, so we choose a hypergraph with $|V| \le m$.

Let $V = \{a, b\}$ and $\mathcal{E} = \{e_1, e_2\}$ with

$$e_1 = \{a\}, \qquad e_2 = \{b\}.$$

However, hyperedges are usually assumed nonempty and may be singletons, but then sums are trivial. To keep $|V| \le m$ and have $|e| \ge 2$, take $m = 3$ instead.

Let $H = (V, \mathcal{E})$ with

$$V = \{a, b, c\}, \qquad \mathcal{E} = \{e_1, e_2, e_3\}, \quad e_1 = \{a, b\}, \; e_2 = \{a, c\}, \; e_3 = \{b, c\}.$$

Then $m = 3$ and we use $\mathbb{Z}_3$. Define $f : V \to \mathbb{Z}_3$ by

$$f(a) = 0, \qquad f(b) = 1, \qquad f(c) = 2.$$

This f is injective, and the induced hyperedge-labels are

$$f^*(e_1) = f(a)+f(b) = 1, \quad f^*(e_2) = f(a)+f(c) = 2, \quad f^*(e_3) = f(b)+f(c) = 1+2 = 0 \pmod 3,$$

which are all distinct and hence give a bijection $\mathcal{E} \to \mathbb{Z}_3$. Therefore f is a harmonious labeling of H.

Definition 4.5.5 (n-SuperHyperGraph harmonious labeling)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be a finite n-SuperHyperGraph with $m := |E| \ge 1$. A map $f : V \to \mathbb{Z}_m$ is called a *harmonious labeling* of $\mathrm{SHG}^{(n)}$ if

1. f is injective, and

2. the induced superedge-labeling $f^* : E \to \mathbb{Z}_m$ defined by

$$f^*(\varepsilon) \;:=\; \sum_{x \in \varepsilon} f(x) \qquad (\varepsilon \in E)$$

   is bijective.

If such an f exists, then $\mathrm{SHG}^{(n)}$ is called *harmonious.*

Example 4.5.6 (Harmonious labeling of an n-SuperHyperGraph)**.** Fix $n = 1$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ with

$$V = \{X_1, X_2, X_3\}, \qquad E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\},$$

where

$$\varepsilon_1 = \{X_1, X_2\}, \qquad \varepsilon_2 = \{X_1, X_3\}, \qquad \varepsilon_3 = \{X_2, X_3\}.$$

Thus $m := |E| = 3$ and we work in $\mathbb{Z}_3$. Define $f : V \to \mathbb{Z}_3$ by

$$f(X_1) = 0, \qquad f(X_2) = 1, \qquad f(X_3) = 2.$$

Then f is injective, and the induced superedge-labels are

$$f^*(\varepsilon_1) = 0 + 1 = 1, \quad f^*(\varepsilon_2) = 0 + 2 = 2, \quad f^*(\varepsilon_3) = 1 + 2 = 0 \pmod 3,$$

which are pairwise distinct and hence bijective $E \to \mathbb{Z}_3$. Therefore f is a harmonious labeling of $\mathrm{SHG}^{(1)}$ in the sense of Definition 4.5.5.

Theorem 4.5.7 (n-SuperHyperGraph harmonious labeling generalizes the graph/hypergraph notions)**.**

1. ***(Reduction to*** $n = 0$***.)*** *When* $n = 0$*, Definition 4.5.5 coincides with Definition 4.5.3. If, in addition, every hyperedge has size* 2*, it coincides with Definition 4.5.1.*

2. ***(Lifting graphs to level*** n***.)*** *Let* $G = (V_G, E_G)$ *be a harmonious graph and fix* $n \in \mathbb{N}_0$*. Then there exists a harmonious* n*-SuperHyperGraph* $\mathrm{SHG}^{(n)}$ *whose supervertices are iterated singletons of* V_G *and whose superedges correspond bijectively to* E_G*.*

3. ***(Lifting hypergraphs to level*** n***.)*** *Let* $H = (V_H, \mathcal{E}_H)$ *be a harmonious hypergraph and fix* $n \in \mathbb{N}_0$*. Then there exists a harmonious* n*-SuperHyperGraph* $\mathrm{SHG}^{(n)}$ *whose supervertices are iterated singletons of* V_H *and whose superedges correspond bijectively to* $\mathcal{E}_H$*.*

Proof. (1) If $n = 0$, then $V \subseteq \mathcal{P}^0(V_0) = V_0$, and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, so (V, E) is an ordinary hypergraph. The induced labeling rule in Definition 4.5.5 is exactly the hyperedge-sum rule of Definition 4.5.3, hence the notions coincide. If moreover each $\varepsilon \in E$ has $|\varepsilon| = 2$, then (V, E) is a simple graph and the rule reduces to the edge-sum rule of Definition 4.5.1.

(2) Let $G = (V_G, E_G)$ be harmonious and set $q := |E_G|$. Choose a harmonious labeling $f : V_G \to \mathbb{Z}_q$. Let $V_0 := V_G$ and define the iterated singleton embedding by

$$\iota_0(v) := v, \qquad \iota_{k+1}(v) := \{\iota_k(v)\} \qquad (v \in V_0,\ k \geq 0).$$

Define

$$V \ := \ \{\iota_n(v) : v \in V_G\} \ \subseteq \ \mathcal{P}^n(V_0), \qquad E \ := \ \big\{\{\iota_n(u), \iota_n(v)\} : \{u, v\} \in E_G\big\}.$$

Then $\mathrm{SHG}^{(n)} := (V, E)$ is an n-SuperHyperGraph and $|E| = |E_G| = q$. Define $f_n : V \to \mathbb{Z}_q$ by $f_n(\iota_n(v)) := f(v)$. Since ι_n is injective and f is injective, so is f_n.

For any edge $\{u, v\} \in E_G$, the corresponding superedge is $\varepsilon_{uv} := \{\iota_n(u), \iota_n(v)\} \in E$, and

$$f_n^*(\varepsilon_{uv}) \;=\; f_n(\iota_n(u)) + f_n(\iota_n(v)) \;=\; f(u) + f(v) \;=\; f^*(\{u, v\}) \quad \text{in } \mathbb{Z}_q.$$

Thus $f_n^* : E \to \mathbb{Z}_q$ is bijective because $f^* : E_G \to \mathbb{Z}_q$ is bijective. Hence $\mathrm{SHG}^{(n)}$ is harmonious.

(3) Let $H = (V_H, \mathcal{E}_H)$ be harmonious and set $m := |\mathcal{E}_H|$. Choose a harmonious labeling $f : V_H \to \mathbb{Z}_m$. As above, set $V_0 := V_H$ and define ι_n by iterated singletons. Let

$$V \;:=\; \{\iota_n(v) : v \in V_H\} \subseteq \mathcal{P}^n(V_0), \qquad E \;:=\; \Big\{ \,\widehat{e} \;:=\; \{\iota_n(v) : v \in e\} \;:\; e \in \mathcal{E}_H \,\Big\}.$$

Then $|E| = |\mathcal{E}_H| = m$ and $\widehat{(\cdot)} : \mathcal{E}_H \to E$ is a bijection. Define $f_n(\iota_n(v)) := f(v)$. For any $e \in \mathcal{E}_H$,

$$f_n^*(\widehat{e}) \;=\; \sum_{x \in \widehat{e}} f_n(x) \;=\; \sum_{v \in e} f(v) \;=\; f^*(e) \quad \text{in } \mathbb{Z}_m.$$

Therefore $f_n^* : E \to \mathbb{Z}_m$ is bijective because $f^* : \mathcal{E}_H \to \mathbb{Z}_m$ is bijective. Hence $\mathrm{SHG}^{(n)} = (V, E)$ is harmonious. □

4.6 SuperHyperGraph Lucky Labeling

A graph lucky labeling assigns positive integers to vertices so adjacent vertices have different neighbor-sum values; the minimum range is lucky number [198–203]. A hypergraph lucky labeling labels vertices so any two vertices sharing a hyperedge have distinct sums of labels in their hyper-neighborhoods. A SuperHyperGraph lucky labeling labels supervertices so any pair co-contained in a superedge have different neighbor-sums computed in the primal graph.

Definition 4.6.1 (Lucky labeling of a graph)**.** Let $G = (V, E)$ be a finite simple graph and let $f : V \to \mathbb{N}$ be a vertex labeling. For each $v \in V$, define the *neighborhood sum*

$$S_f(v) \;:=\; \sum_{u \in N_G(v)} f(u),$$

with the convention $S_f(v) := 0$ if v is isolated (i.e. $N_G(v) = \emptyset$). We call f a *lucky labeling* of G if

$$S_f(u) \;\neq\; S_f(v) \qquad \text{for every edge } uv \in E.$$

The *lucky number* of G, denoted $\eta(G)$, is the least positive integer k such that G admits a lucky labeling $f : V \to \{1, 2, \ldots, k\}$.

Example 4.6.2 (A lucky labeling of a graph)**.** Let $G = P_3$ be the path on three vertices

$$V = \{v_1, v_2, v_3\}, \qquad E = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\}.$$

Define $f : V \to \mathbb{N}$ by

$$f(v_1) = 1, \qquad f(v_2) = 1, \qquad f(v_3) = 2.$$

Then the neighborhood sums are

$$S_f(v_1) = f(v_2) = 1, \qquad S_f(v_2) = f(v_1) + f(v_3) = 3, \qquad S_f(v_3) = f(v_2) = 1.$$

Hence, for each edge of G,

$$S_f(v_1) \neq S_f(v_2) \quad \text{and} \quad S_f(v_2) \neq S_f(v_3),$$

so f is a lucky labeling of G. In particular, f uses only labels in $\{1, 2\}$, so $\eta(G) \leq 2$; moreover $\eta(G) \neq 1$, thus $\eta(G) = 2$.

Definition 4.6.3 (Lucky labeling of a hypergraph)**.** Let $H = (V, \mathcal{E})$ be a finite hypergraph, and let $\mathrm{Pr}(H)$ denote its *primal graph* (the simple graph on vertex set V in which $u \neq v$ are adjacent iff $\{u, v\} \subseteq e$ for some hyperedge $e \in \mathcal{E}$). A labeling $f : V \to \mathbb{N}$ is called a *lucky labeling* of H if it is a lucky labeling of the primal graph $\mathrm{Pr}(H)$, i.e. if, with

$$S_f^H(v) \;:=\; \sum_{u \in N_{\mathrm{Pr}(H)}(v)} f(u) \qquad (v \in V),$$

(and $S_f^H(v) := 0$ when $N_{\mathrm{Pr}(H)}(v) = \emptyset$), we have

$$S_f^H(u) \;\neq\; S_f^H(v) \qquad \text{for every edge } uv \in E\big(\mathrm{Pr}(H)\big).$$

The *lucky number* of H is defined by $\eta(H) := \eta\big(\mathrm{Pr}(H)\big)$.

Example 4.6.4 (A lucky labeling of a hypergraph (via the primal graph))**.** Let $H = (V, \mathcal{E})$ be the hypergraph with

$$V = \{a, b, c, d\}, \qquad \mathcal{E} = \{e_1, e_2\}, \qquad e_1 = \{a, b, c\}, \quad e_2 = \{c, d\}.$$

Its primal graph $\mathrm{Pr}(H)$ has edges

$$\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\},$$

i.e., $\mathrm{Pr}(H)$ is a triangle on $\{a, b, c\}$ with a pendant vertex d attached to c.

Define $f : V \to \mathbb{N}$ by

$$f(a) = 1, \qquad f(b) = 2, \qquad f(c) = 1, \qquad f(d) = 1.$$

Compute neighborhood sums in $\mathrm{Pr}(H)$:

$$S_f^H(a) = f(b) + f(c) = 3, \qquad S_f^H(b) = f(a) + f(c) = 2,$$

$$S_f^H(c) = f(a) + f(b) + f(d) = 4, \qquad S_f^H(d) = f(c) = 1.$$

Along every primal edge we have distinct sums:

$$S_f^H(a) \neq S_f^H(b), \; S_f^H(a) \neq S_f^H(c),$$

$$S_f^H(b) \neq S_f^H(c), \; S_f^H(c) \neq S_f^H(d).$$

Therefore f is a lucky labeling of H (Definition 4.6.3). In particular, f uses only labels in $\{1, 2\}$, so $\eta(H) \leq 2$, and clearly $\eta(H) \neq 1$; hence $\eta(H) = 2$.

Definition 4.6.5 (Lucky labeling of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be a finite n-SuperHyperGraph, and let $\Pr(\mathrm{SHG}^{(n)})$ be its primal graph (on vertex set V, where $x \neq y$ are adjacent iff $\{x, y\} \subseteq \varepsilon$ for some superedge $\varepsilon \in E$). A labeling $f : V \to \mathbb{N}$ is a *lucky labeling* of $\mathrm{SHG}^{(n)}$ if it is a lucky labeling of $\Pr(\mathrm{SHG}^{(n)})$, i.e. if the neighborhood sums

$$S_f^{(n)}(x) \ := \sum_{y \in N_{\Pr(\mathrm{SHG}^{(n)})}(x)} f(y) \qquad (x \in V),$$

(with $S_f^{(n)}(x) := 0$ when $N_{\Pr(\mathrm{SHG}^{(n)})}(x) = \emptyset$) satisfy

$$S_f^{(n)}(x) \ \neq \ S_f^{(n)}(y) \qquad \text{for every edge } xy \in E\big(\Pr(\mathrm{SHG}^{(n)})\big).$$

The *lucky number* of $\mathrm{SHG}^{(n)}$, denoted $\eta(\mathrm{SHG}^{(n)})$, is the least $k \in \mathbb{N}$ such that there exists a lucky labeling $f : V \to \{1, 2, \ldots, k\}$.

Example 4.6.6 (A lucky labeling of an n-SuperHyperGraph (primal-graph version))**.** Fix $n = 1$ and let $V_0 = \{a, b, c\}$. Consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ with

$$V = \{X_1, X_2, X_3\} \subseteq \mathcal{P}(V_0), \qquad X_1 = \{a\},\ X_2 = \{b\},\ X_3 = \{c\},$$

and with a single superedge

$$E = \{\varepsilon\}, \qquad \varepsilon = \{X_1, X_2, X_3\}.$$

Then the primal graph $\Pr(\mathrm{SHG}^{(1)})$ is the complete graph K_3 on $\{X_1, X_2, X_3\}$.

Define $f : V \to \mathbb{N}$ by

$$f(X_1) = 1, \qquad f(X_2) = 2, \qquad f(X_3) = 3.$$

Since $\Pr(\mathrm{SHG}^{(1)}) \cong K_3$, the neighborhood sums are

$$S_f^{(1)}(X_1) = f(X_2) + f(X_3) = 5,$$

$$S_f^{(1)}(X_2) = f(X_1) + f(X_3) = 4,$$

$$S_f^{(1)}(X_3) = f(X_1) + f(X_2) = 3.$$

Thus, for every edge X_iX_j in $\Pr(\mathrm{SHG}^{(1)})$, one has $S_f^{(1)}(X_i) \neq S_f^{(1)}(X_j)$. Hence f is a lucky labeling of $\mathrm{SHG}^{(1)}$ in the sense of Definition 4.6.5. Moreover, $\eta(\mathrm{SHG}^{(1)}) = 3$ because $\Pr(\mathrm{SHG}^{(1)}) \cong K_3$ admits no lucky labeling using only $\{1, 2\}$.

Theorem 4.6.7 (n-SuperHyperGraph lucky labeling generalizes the graph and hypergraph notions)**.**

(i) (Hypergraphs as the case $n = 0$.) For every hypergraph $H = (V, \mathcal{E})$, define the level-0 SuperHyperGraph

$$\mathrm{SHG}_H^{(0)} \ := \ (V, \mathcal{E}).$$

Then a labeling $f : V \to \mathbb{N}$ is a lucky labeling of H (Definition 4.6.3) if and only if f is a lucky labeling of $\mathrm{SHG}_H^{(0)}$ (Definition 4.6.5 with $n = 0$). Consequently,

$$\eta(H) \ = \ \eta\big(\mathrm{SHG}_H^{(0)}\big).$$

(ii) (Graphs embedded at any level n.) Let $G = (V_G, E_G)$ be a finite simple graph and fix $n \in \mathbb{N}_0$. Define the iterated singleton embedding $\iota_0(v) := v$ and $\iota_{k+1}(v) := \{\iota_k(v)\}$, and set

$$V^{(n)} \ := \ \{\iota_n(v) : v \in V_G\}, \qquad E^{(n)} \ := \ \big\{\{\iota_n(u), \iota_n(v)\} : uv \in E_G\big\}.$$

Then $\mathrm{SHG}_G^{(n)} := (V^{(n)}, E^{(n)})$ is an n-SuperHyperGraph and

$$\mathrm{Pr}\big(\mathrm{SHG}_G^{(n)}\big) \ \cong \ G.$$

In particular, G admits a lucky labeling with label set $\{1, \ldots, k\}$ if and only if $\mathrm{SHG}_G^{(n)}$ admits a lucky labeling with the same label set, and hence

$$\eta(G) \ = \ \eta\big(\mathrm{SHG}_G^{(n)}\big).$$

Proof. (i) By definition, a labeling is lucky for a hypergraph H precisely when it is lucky on the primal graph $\mathrm{Pr}(H)$. When $n = 0$ and $\mathrm{SHG}_H^{(0)} = (V, \mathcal{E})$, the primal graph $\mathrm{Pr}(\mathrm{SHG}_H^{(0)})$ coincides with $\mathrm{Pr}(H)$ (both have vertex set V and edges given by pairwise co-membership in some $e \in \mathcal{E}$). Therefore the neighborhood-sum conditions in Definitions 4.6.3 and 4.6.5 (with $n = 0$) are identical, proving the equivalence and the equality of lucky numbers.

(ii) The pair $(V^{(n)}, E^{(n)})$ satisfies $V^{(n)} \subseteq \mathcal{P}^n(V_G)$ and $E^{(n)} \subseteq \mathcal{P}(V^{(n)}) \setminus \{\emptyset\}$, so $\mathrm{SHG}_G^{(n)}$ is an n-SuperHyperGraph. Moreover, two distinct vertices $\iota_n(u), \iota_n(v) \in V^{(n)}$ are adjacent in $\mathrm{Pr}(\mathrm{SHG}_G^{(n)})$ if and only if $\{\iota_n(u), \iota_n(v)\} \in E^{(n)}$, which holds if and only if $uv \in E_G$. Hence the bijection $V_G \to V^{(n)}$, $v \mapsto \iota_n(v)$, is a graph isomorphism $G \cong \mathrm{Pr}(\mathrm{SHG}_G^{(n)})$.

Finally, by Definition 4.6.5, a labeling of $\mathrm{SHG}_G^{(n)}$ is lucky exactly when the corresponding labeling of its primal graph is lucky; transporting labels along the above isomorphism gives a lucky labeling of G and conversely. Therefore $\eta(G) = \eta(\mathrm{SHG}_G^{(n)})$. □

4.7 SuperHyperGraph Magic labeling

A *magic labeling of a graph* is a bijection on vertices and edges such that every edge has the same constant total of its endpoint labels and its edge label [52,53,204,205]. Related notions are also known, such as *antimagic graphs* [206–208]. A *magic labeling of a hypergraph* is a bijection on vertices and hyperedges such that each hyperedge has identical weight, namely its label plus the sum of labels of its incident vertices. A *magic labeling of a superhypergraph* is a bijection on supervertices and superedges such that every superedge attains one common weight, computed as its label plus the sum of labels of its member supervertices.

Definition 4.7.1 (Magic labeling of a graph)**.** Let $G = (V, E)$ be a finite simple graph. A *(edge-)magic total labeling* of G is a bijection

$$f : \ V \sqcup E \ \longrightarrow \ [\,|V| + |E|\,] := \{1, 2, \ldots, |V| + |E|\}$$

for which there exists an integer c (called a *magic constant*) such that for every edge $uv \in E$,

$$f(u) \ + \ f(uv) \ + \ f(v) \ = \ c.$$

If such an f exists, we say that G is *(edge-)magic*.

Example 4.7.2 (Magic labeling of a graph)**.** Let $G = (V, E)$ be the graph with two vertices and one edge,

$$V = \{u, v\}, \qquad E = \{\{u, v\}\}.$$

Define a bijection $f : V \sqcup E \to [3] = \{1, 2, 3\}$ by

$$f(u) = 1, \qquad f(v) = 2, \qquad f(uv) = 3,$$

where we write uv for the edge $\{u, v\}$. Then for the unique edge $uv \in E$,

$$f(u) + f(uv) + f(v) = 1 + 3 + 2 = 6.$$

Hence f is an (edge-)magic total labeling of G with magic constant $c = 6$.

Definition 4.7.3 (Magic labeling of a hypergraph)**.** Let $H = (V, \mathcal{E})$ be a finite hypergraph. A *magic total labeling* of H is a bijection

$$f : \; V \sqcup \mathcal{E} \;\longrightarrow\; [\,|V| + |\mathcal{E}|\,]$$

such that there exists an integer c with the property that every hyperedge $e \in \mathcal{E}$ has the same *hyperedge weight*

$$w_f(e) \;:=\; f(e) \;+\; \sum_{v \in e} f(v) \;=\; c.$$

If such an f exists, we say that H is *magic*.

Example 4.7.4 (Magic labeling of a hypergraph)**.** Let $H = (V, \mathcal{E})$ be the hypergraph

$$V = \{a, b, c\}, \qquad \mathcal{E} = \{e_1, e_2\}, \qquad e_1 = \{a, b\}, \;\; e_2 = \{b, c\}.$$

Define a bijection $f : V \sqcup \mathcal{E} \to [5] = \{1, 2, 3, 4, 5\}$ by

$$f(a) = 2, \qquad f(b) = 1, \qquad f(c) = 3, \qquad f(e_1) = 5, \qquad f(e_2) = 4.$$

Compute the hyperedge weights:

$$w_f(e_1) = f(e_1)+f(a)+f(b) = 5+2+1 = 8, \qquad w_f(e_2) = f(e_2)+f(b)+f(c) = 4+1+3 = 8.$$

Thus every hyperedge has the same weight $c = 8$, so f is a magic total labeling of H.

Definition 4.7.5 (Magic labeling of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be a finite n-SuperHyperGraph. A *magic total labeling* of $\mathrm{SHG}^{(n)}$ is a bijection

$$f : \; V \sqcup E \;\longrightarrow\; [\,|V| + |E|\,]$$

such that there exists an integer c satisfying, for every superedge $\varepsilon \in E$,

$$w_f(\varepsilon) \;:=\; f(\varepsilon) \;+\; \sum_{x \in \varepsilon} f(x) \;=\; c.$$

If such an f exists, we say that $\mathrm{SHG}^{(n)}$ is *magic*.

Example 4.7.6 (Magic labeling of an n-SuperHyperGraph)**.** Take $n = 1$ and a base set $V_0 = \{a, b, c\}$. Consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ with

$$V = \{X_1, X_2, X_3\} \subseteq \mathcal{P}(V_0), \qquad X_1 = \{a\}, \;\; X_2 = \{b\}, \;\; X_3 = \{c\},$$

and superedge family

$$E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{X_1, X_2\}, \;\; \varepsilon_2 = \{X_2, X_3\}.$$

Define a bijection $f : V \sqcup E \to [5] = \{1, 2, 3, 4, 5\}$ by

$$f(X_1) = 2, \qquad f(X_2) = 1, \qquad f(X_3) = 3, \qquad f(\varepsilon_1) = 5, \qquad f(\varepsilon_2) = 4.$$

Then the superedge weights satisfy

$$w_f(\varepsilon_1) = f(\varepsilon_1) + f(X_1) + f(X_2) = 5 + 2 + 1 = 8,$$
$$w_f(\varepsilon_2) = f(\varepsilon_2) + f(X_2) + f(X_3) = 4 + 1 + 3 = 8.$$

Hence f is a magic total labeling of $\mathrm{SHG}^{(1)}$ with magic constant $c = 8$.

Theorem 4.7.7 (n-SuperHyperGraph magic labeling generalizes the graph and hypergraph notions)**.**

*(i) (*Hypergraphs as the case $n = 0$.*) Let $H = (V, \mathcal{E})$ be a hypergraph, viewed as the* 0*-SuperHyperGraph* $\mathrm{SHG}^{(0)} := (V, \mathcal{E})$*. Then a map $f : V \sqcup \mathcal{E} \to [|V| + |\mathcal{E}|]$ is a magic labeling of H (Definition 4.7.3) if and only if it is a magic labeling of* $\mathrm{SHG}^{(0)}$ *(Definition 4.7.5 with $n = 0$). In particular, H is magic if and only if* $\mathrm{SHG}^{(0)}$ *is magic.*

*(ii) (*Graphs embedded at any level n.*) Let $G = (V_G, E_G)$ be a finite simple graph and fix $n \in \mathbb{N}_0$. Define $\iota_0(v) := v$ and $\iota_{k+1}(v) := \{\iota_k(v)\}$, and set*

$$V^{(n)} \; := \; \{\iota_n(v) : v \in V_G\}, \qquad E^{(n)} \; := \; \big\{\{\iota_n(u), \iota_n(v)\} : \; uv \in E_G\big\}.$$

Then $\mathrm{SHG}_G^{(n)} := (V^{(n)}, E^{(n)})$ *is an n-SuperHyperGraph, and G is magic (in the sense of Definition 4.7.1) if and only if* $\mathrm{SHG}_G^{(n)}$ *is magic (in the sense of Definition 4.7.5). Moreover, magic constants are preserved under this correspondence.*

Proof. (i) When $n = 0$, the "supervertices" are the vertices V and the "superedges" are exactly the hyperedges $\mathcal{E}$. The weight condition

$$f(e) + \sum_{v \in e} f(v) = c$$

is literally identical in Definitions 4.7.3 and 4.7.5. Hence the two notions coincide.

(ii) By construction, $V^{(n)} \subseteq \mathcal{P}^n(V_G)$ and $E^{(n)}$ is a family of nonempty subsets of $V^{(n)}$, so $\mathrm{SHG}_G^{(n)}$ is an n-SuperHyperGraph. Each edge $uv \in E_G$ corresponds to the unique superedge $\{\iota_n(u), \iota_n(v)\} \in E^{(n)}$.

Suppose first that $f : V_G \sqcup E_G \to [|V_G| + |E_G|]$ is a magic labeling of G with magic constant c. Define $\tilde{f} : V^{(n)} \sqcup E^{(n)} \to [|V_G| + |E_G|]$ by

$$\tilde{f}(\iota_n(v)) := f(v) \quad (v \in V_G), \qquad \tilde{f}(\{\iota_n(u), \iota_n(v)\}) := f(uv) \quad (uv \in E_G).$$

This is a bijection because $v \mapsto \iota_n(v)$ and $uv \mapsto \{\iota_n(u), \iota_n(v)\}$ are bijections between the corresponding parts. For any $uv \in E_G$ we then have

$$\tilde{f}(\iota_n(u)) + \tilde{f}(\{\iota_n(u), \iota_n(v)\}) + \tilde{f}(\iota_n(v)) \;=\; f(u) + f(uv) + f(v) \;=\; c,$$

so $\tilde{f}$ is a magic labeling of $\mathrm{SHG}_G^{(n)}$ with the same constant c.

Conversely, given a magic labeling $\tilde{f} : V^{(n)} \sqcup E^{(n)} \to [|V^{(n)}| + |E^{(n)}|]$ with constant c, define $f : V_G \sqcup E_G \to [|V_G| + |E_G|]$ by $f(v) := \tilde{f}(\iota_n(v))$ and $f(uv) := \tilde{f}(\{\iota_n(u), \iota_n(v)\})$. The same computation shows $f(u) + f(uv) + f(v) = c$ for every $uv \in E_G$, hence f is a magic labeling of G.

Therefore G is magic if and only if $\mathrm{SHG}_G^{(n)}$ is magic, and the magic constant is preserved. □

4.8 Fuzzy SuperHyperGraph Labeling

Fuzzy Graph Labeling assigns distinct membership degrees in $[0, 1]$ to vertices and edges, requiring each edge degree to be bounded by its two endpoint degrees [209–213]. Fuzzy HyperGraph Labeling assigns distinct degrees in $[0, 1]$ to vertices and hyperedges, requiring every hyperedge degree to be bounded by all incident vertex degrees. Fuzzy SuperHyperGraph Labeling assigns distinct degrees in $[0, 1]$ to supervertices and superedges, requiring each superedge degree to be bounded by all supervertices contained in it.

Definition 4.8.1 (Fuzzy labeling of a graph)**.** Let $G^* = (V, E^*)$ be a finite simple graph. A *fuzzy labeling* of G^* is an injective map

$$\omega : V \cup E^* \longrightarrow [0, 1]$$

such that, writing

$$\sigma_\omega(v) := \omega(v) \qquad (v \in V), \qquad \mu_\omega(\{u, v\}) := \omega(\{u, v\}) \qquad (\{u, v\} \in E^*),$$

we have for every edge $\{u, v\} \in E^*$ the admissibility inequality

$$\mu_\omega(\{u, v\}) \;<\; \sigma_\omega(u) \wedge \sigma_\omega(v),$$

where $\wedge$ denotes the minimum on $[0, 1]$. The resulting fuzzy graph is denoted $G_\omega := (\sigma_\omega, \mu_\omega)$, and we say that G^* is a *fuzzy labeling graph* if it admits such an ω.

Example 4.8.2 (Fuzzy labeling of a graph)**.** Let $G^* = (V, E^*)$ be the path P_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E^* = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\}.$$

Define an injective map $\omega : V \cup E^* \to [0, 1]$ by

$$\omega(v_1) = 0.70, \qquad \omega(v_2) = 0.90, \qquad \omega(v_3) = 0.80,$$

$$\omega(\{v_1, v_2\}) = 0.60, \qquad \omega(\{v_2, v_3\}) = 0.75.$$

Then $\sigma_\omega(v) = \omega(v)$ for $v \in V$ and $\mu_\omega(e) = \omega(e)$ for $e \in E^*$. For each edge we check the admissibility inequality:

$$\mu_\omega(\{v_1, v_2\}) = 0.60 < \min\{0.70, 0.90\} = 0.70,$$

$$\mu_\omega(\{v_2, v_3\}) = 0.75 < \min\{0.90, 0.80\} = 0.80.$$

Hence ω is a fuzzy labeling of G^* in the sense of Definition 4.8.1.

Definition 4.8.3 (Fuzzy labeling of a hypergraph)**.** Let $H^* = (V, \mathcal{E}^*)$ be a finite (undirected) hypergraph. A *fuzzy labeling* of H^* is an injective map

$$\omega : V \cup \mathcal{E}^* \longrightarrow [0,1]$$

such that, with

$$\sigma_\omega(v) := \omega(v) \qquad (v \in V), \qquad \mu_\omega(e) := \omega(e) \qquad (e \in \mathcal{E}^*),$$

one has for every $e \in \mathcal{E}^*$ the hyperedge admissibility inequality

$$\mu_\omega(e) \; < \; \bigwedge_{v \in e} \sigma_\omega(v).$$

We call $H_\omega := (V, \mathcal{E}^*, \sigma_\omega, \mu_\omega)$ the induced fuzzy-labeled hypergraph.

Example 4.8.4 (Fuzzy labeling of a graph)**.** Let $G^* = (V, E^*)$ be the path P_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E^* = \{\{v_1, v_2\}, \{v_2, v_3\}\}.$$

Define an injective map $\omega : V \cup E^* \to [0,1]$ by

$$\omega(v_1) = 0.70, \qquad \omega(v_2) = 0.90, \qquad \omega(v_3) = 0.80,$$

$$\omega(\{v_1, v_2\}) = 0.60, \qquad \omega(\{v_2, v_3\}) = 0.75.$$

Then $\sigma_\omega(v) = \omega(v)$ for $v \in V$ and $\mu_\omega(e) = \omega(e)$ for $e \in E^*$. For each edge we check the admissibility inequality:

$$\mu_\omega(\{v_1, v_2\}) = 0.60 < \min\{0.70, 0.90\} = 0.70,$$

$$\mu_\omega(\{v_2, v_3\}) = 0.75 < \min\{0.90, 0.80\} = 0.80.$$

Hence ω is a fuzzy labeling of G^* in the sense of Definition 4.8.1.

Definition 4.8.5 (Fuzzy labeling of an n-SuperHyperGraph)**.** Let $n \in \mathbb{N}_0$ and let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (as in Definition 2.1.7). A *fuzzy labeling* of $\mathrm{SHG}^{(n)}$ is an injective map

$$\omega : V \cup E \longrightarrow [0,1]$$

such that, writing

$$\sigma_\omega(x) := \omega(x) \qquad (x \in V), \qquad \mu_\omega(\varepsilon) := \omega(\varepsilon) \qquad (\varepsilon \in E),$$

we have for every superedge $\varepsilon \in E$ the admissibility inequality

$$\mu_\omega(\varepsilon) \; < \; \bigwedge_{x \in \varepsilon} \sigma_\omega(x).$$

In this case, we call $(\mathrm{SHG}^{(n)}, \omega)$ a *fuzzy-labeled n-SuperHyperGraph.*

Remark 4.8.6 (On strict vs. non-strict admissibility)**.** In Definitions 4.8.1–4.8.5 one may replace "$<$" by "$\leq$" without affecting the embedding/generalization results below. We keep the strict form to match common presentations of fuzzy labelings.

Example 4.8.7 (Fuzzy labeling of an n-SuperHyperGraph)**.** Take $n = 1$ and a base set $V_0 = \{a, b, c, d\}$. Consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E)$ with

$$V = \{X_1, X_2, X_3, X_4\} \subseteq \mathcal{P}(V_0), \qquad X_1 = \{a\},\ X_2 = \{b\},\ X_3 = \{c\},\ X_4 = \{d\},$$

and with two superedges

$$E = \{\varepsilon_1, \varepsilon_2\}, \qquad \varepsilon_1 = \{X_1, X_2, X_3\}, \quad \varepsilon_2 = \{X_2, X_4\}.$$

Define an injective map $\omega : V \cup E \to [0, 1]$ by

$$\omega(X_1) = 0.80, \quad \omega(X_2) = 0.90, \quad \omega(X_3) = 0.70, \quad \omega(X_4) = 0.60,$$

$$\omega(\varepsilon_1) = 0.65, \qquad \omega(\varepsilon_2) = 0.55.$$

(These values are all distinct, so ω is injective.) With $\sigma_\omega(x) = \omega(x)$ for $x \in V$ and $\mu_\omega(\varepsilon) = \omega(\varepsilon)$ for $\varepsilon \in E$, we verify admissibility for each superedge:

$$\mu_\omega(\varepsilon_1) = 0.65 < \min\{\sigma_\omega(X_1), \sigma_\omega(X_2), \sigma_\omega(X_3)\} = \min\{0.80, 0.90, 0.70\} = 0.70,$$

$$\mu_\omega(\varepsilon_2) = 0.55 < \min\{\sigma_\omega(X_2), \sigma_\omega(X_4)\} = \min\{0.90, 0.60\} = 0.60.$$

Hence ω is a fuzzy labeling of $\mathrm{SHG}^{(1)}$ in the sense of Definition 4.8.5.

Theorem 4.8.8 (n-SuperHyperGraph fuzzy labelings generalize graph and hypergraph fuzzy labelings)**.** *Fix $n \in \mathbb{N}_0$.*

1. ***(Hypergraphs embed.)*** *Let (H^*, ω) be a fuzzy-labeled hypergraph in the sense of Definition 4.8.3, where $H^* = (V, \mathcal{E}^*)$. Then there exists a fuzzy-labeled n-SuperHyperGraph $(\mathrm{SHG}^{(n)}, \omega^{(n)})$ whose underlying incidence structure is canonically isomorphic to H^*.*

2. ***(Graphs embed; primal graph recovers the graph.)*** *Let (G^*, ω) be a fuzzy-labeled graph in the sense of Definition 4.8.1, where $G^* = (V, E^*)$. Then there exists a fuzzy-labeled n-SuperHyperGraph $(\mathrm{SHG}^{(n)}, \omega^{(n)})$ such that the primal graph (the 2-section) of $\mathrm{SHG}^{(n)}$ is isomorphic to G^*, and the labeling $\omega^{(n)}$ transports ω under this isomorphism.*

Proof. Define the iterated singleton embedding $\iota_0 : V \to V$ by $\iota_0(v) = v$ and

$$\iota_{k+1}(v) := \{\iota_k(v)\} \qquad (k \geq 0,\ v \in V).$$

Then $\iota_n(v) \in \mathcal{P}^n(V)$ for every $v \in V$.

(1) Hypergraph case. Let $H^* = (V, \mathcal{E}^*)$ and let ω be as in Definition 4.8.3. Set the n-supervertex set

$$V^{(n)} \ :=\ \{\iota_n(v) : v \in V\} \ \subseteq\ \mathcal{P}^n(V),$$

and for each hyperedge $e \in \mathcal{E}^*$ define the corresponding n-superedge

$$\widehat{e} \ :=\ \{\iota_n(v) : v \in e\} \ \subseteq\ V^{(n)}.$$

Let

$$E^{(n)} \ :=\ \{\widehat{e} : e \in \mathcal{E}^*\}.$$

Then $\mathrm{SHG}^{(n)} := (V^{(n)}, E^{(n)})$ is an n-SuperHyperGraph.

Define $\omega^{(n)} : V^{(n)} \cup E^{(n)} \to [0,1]$ by

$$\omega^{(n)}(\iota_n(v)) := \omega(v) \quad (v \in V), \qquad \omega^{(n)}(\widehat{e}) := \omega(e) \quad (e \in \mathcal{E}^*).$$

Injectivity of $\omega^{(n)}$ follows from injectivity of ω and the bijective correspondences $v \leftrightarrow \iota_n(v)$ and $e \leftrightarrow \widehat{e}$. Moreover, for each $\widehat{e} \in E^{(n)}$ we have

$$\mu_{\omega^{(n)}}(\widehat{e}) = \omega^{(n)}(\widehat{e}) = \omega(e) < \bigwedge_{v \in e} \omega(v) = \bigwedge_{x \in \widehat{e}} \omega^{(n)}(x) = \bigwedge_{x \in \widehat{e}} \sigma_{\omega^{(n)}}(x),$$

so $(\mathrm{SHG}^{(n)}, \omega^{(n)})$ is a fuzzy-labeled n-SuperHyperGraph. Finally, the map $v \mapsto \iota_n(v)$ gives an incidence-preserving isomorphism $H^* \cong (V^{(n)}, E^{(n)})$.

(2) Graph case. Apply the same construction to $G^* = (V, E^*)$, viewing E^* as a 2-uniform hyperedge family. Then every superedge $\widehat{e} \in E^{(n)}$ has size 2. Hence two distinct vertices $\iota_n(u), \iota_n(v) \in V^{(n)}$ lie together in some superedge of $E^{(n)}$ if and only if $\{u, v\} \in E^*$. Equivalently, the 2-section (primal graph) of $\mathrm{SHG}^{(n)}$ is isomorphic to G^* via $u \mapsto \iota_n(u)$, and the same computation as above shows that $\omega^{(n)}$ satisfies the edge-admissibility inequalities. □

4.9 Neutrosophic SuperHyperGraph Labeling

Neutrosophic Graph Labeling assigns distinct truth–indeterminacy–falsity triples to vertices and edges, with each edge triple constrained by its endpoints [214–220]. Neutrosophic HyperGraph Labeling assigns distinct (T, I, F) triples to vertices and hyperedges, with each hyperedge triple bounded by the incident vertex triples. Neutrosophic SuperHyperGraph Labeling assigns distinct (T, I, F) triples to supervertices and superedges, with each superedge triple constrained by all contained supervertices.

Definition 4.9.1 (Neutrosophic labeling of a graph)**.** Let $G = (V, E)$ be a finite (simple, undirected) graph. A *neutrosophic graph labeling* is a pair of maps

$$\sigma : V \to [0,1]^3, \qquad \mu : E \to [0,1]^3,$$

written componentwise as

$$\sigma(v) = (T_V(v), I_V(v), F_V(v)), \qquad \mu(e) = (T_E(e), I_E(e), F_E(e)).$$

They must satisfy, for every edge $e = \{u, v\} \in E$,

$$T_E(e) \le \min\{T_V(u), T_V(v)\}, \qquad I_E(e) \le \min\{I_V(u), I_V(v)\}, \qquad F_E(e) \le \max\{F_V(u), F_V(v)\},$$

and the neutrosophic feasibility constraints

$$0 \le T_V(x) + I_V(x) + F_V(x) \le 3 \quad (x \in V), \qquad 0 \le T_E(e) + I_E(e) + F_E(e) \le 3 \quad (e \in E).$$

To emphasize the *labeling* character, we additionally require that each of the six component maps T_V, I_V, F_V on V and T_E, I_E, F_E on E is injective (equivalently, all corresponding component values are pairwise distinct on the appropriate domain).

Example 4.9.2 (Neutrosophic labeling of a graph)**.** Let $G = (V, E)$ be the path P_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{\{v_1, v_2\}, \{v_2, v_3\}\}.$$

Define $\sigma : V \to [0, 1]^3$ by

$$\sigma(v_1) = (0.70,\ 0.20,\ 0.30), \qquad \sigma(v_2) = (0.90,\ 0.10,\ 0.50), \qquad \sigma(v_3) = (0.80,\ 0.30,\ 0.40),$$

and define $\mu : E \to [0, 1]^3$ by

$$\mu(\{v_1, v_2\}) = (0.60,\ 0.05,\ 0.45), \qquad \mu(\{v_2, v_3\}) = (0.75,\ 0.08,\ 0.55).$$

Each vertex triple and edge triple satisfies the feasibility bounds $0 \le T + I + F \le 3$. Moreover, for $e_{12} = \{v_1, v_2\}$ we have

$$T_E(e_{12}) = 0.60 \le \min\{0.70, 0.90\} = 0.70, \qquad I_E(e_{12}) = 0.05 \le \min\{0.20, 0.10\} = 0.10,$$

$$F_E(e_{12}) = 0.45 \le \max\{0.30, 0.50\} = 0.50,$$

and for $e_{23} = \{v_2, v_3\}$ we have

$$T_E(e_{23}) = 0.75 \le \min\{0.90, 0.80\} = 0.80, \qquad I_E(e_{23}) = 0.08 \le \min\{0.10, 0.30\} = 0.10,$$

$$F_E(e_{23}) = 0.55 \le \max\{0.50, 0.40\} = 0.50 \quad \text{fails.}$$

To enforce the required constraint $F_E(e) \le \max\{F_V(u), F_V(v)\}$, replace the last component by 0.49:

$$\mu(\{v_2, v_3\}) = (0.75,\ 0.08,\ 0.49).$$

With this adjustment, all three edge inequalities hold for both edges.

Finally, each component map is injective on its domain:

$$T_V : \{0.70, 0.90, 0.80\},\ I_V : \{0.20, 0.10, 0.30\},\ F_V : \{0.30, 0.50, 0.40\}$$

are pairwise distinct on V, and

$$T_E : \{0.60, 0.75\},\ I_E : \{0.05, 0.08\},\ F_E : \{0.45, 0.49\}$$

are pairwise distinct on E. Hence (σ, μ) is a neutrosophic labeling of G in the sense of Definition 4.9.1.

Definition 4.9.3 (Neutrosophic labeling of a hypergraph)**.** Let $H = (V, \mathcal{E})$ be a finite hypergraph. A *neutrosophic hypergraph labeling* is a pair of maps

$$\sigma : V \to [0, 1]^3, \qquad \mu : \mathcal{E} \to [0, 1]^3,$$

with $\sigma(v) = (T_V(v), I_V(v), F_V(v))$ and $\mu(e) = (T_E(e), I_E(e), F_E(e))$, such that for every hyperedge $e \in \mathcal{E}$,

$$T_E(e) \le \min_{v \in e} T_V(v), \qquad I_E(e) \le \min_{v \in e} I_V(v), \qquad F_E(e) \le \max_{v \in e} F_V(v),$$

and

$$0 \le T_V(v) + I_V(v) + F_V(v) \le 3 \quad (v \in V), \qquad 0 \le T_E(e) + I_E(e) + F_E(e) \le 3 \quad (e \in \mathcal{E}).$$

As in Definition 4.9.1, we further require injectivity of the component maps T_V, I_V, F_V on V and T_E, I_E, F_E on $\mathcal{E}$.

Example 4.9.4 (Neutrosophic labeling of a graph)**.** Let $G = (V, E)$ be the path P_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{\{v_1, v_2\}, \{v_2, v_3\}\}.$$

Define $\sigma : V \to [0,1]^3$ by

$$\sigma(v_1) = (0.70,\ 0.20,\ 0.30), \qquad \sigma(v_2) = (0.90,\ 0.10,\ 0.50), \qquad \sigma(v_3) = (0.80,\ 0.30,\ 0.40),$$

and define $\mu : E \to [0,1]^3$ by

$$\mu(\{v_1, v_2\}) = (0.60,\ 0.05,\ 0.45), \qquad \mu(\{v_2, v_3\}) = (0.75,\ 0.08,\ 0.55).$$

Each vertex triple and edge triple satisfies the feasibility bounds $0 \le T + I + F \le 3$. Moreover, for $e_{12} = \{v_1, v_2\}$ we have

$$T_E(e_{12}) = 0.60 \le \min\{0.70, 0.90\} = 0.70, \qquad I_E(e_{12}) = 0.05 \le \min\{0.20, 0.10\} = 0.10,$$

$$F_E(e_{12}) = 0.45 \le \max\{0.30, 0.50\} = 0.50,$$

and for $e_{23} = \{v_2, v_3\}$ we have

$$T_E(e_{23}) = 0.75 \le \min\{0.90, 0.80\} = 0.80, \qquad I_E(e_{23}) = 0.08 \le \min\{0.10, 0.30\} = 0.10,$$

$$F_E(e_{23}) = 0.55 \le \max\{0.50, 0.40\} = 0.50 \quad \text{fails.}$$

To enforce the required constraint $F_E(e) \le \max\{F_V(u), F_V(v)\}$, replace the last component by 0.49:

$$\mu(\{v_2, v_3\}) = (0.75,\ 0.08,\ 0.49).$$

With this adjustment, all three edge inequalities hold for both edges.

Finally, each component map is injective on its domain:

$$T_V : \{0.70, 0.90, 0.80\},\ I_V : \{0.20, 0.10, 0.30\},\ F_V : \{0.30, 0.50, 0.40\}$$

are pairwise distinct on V, and

$$T_E : \{0.60, 0.75\},\ I_E : \{0.05, 0.08\},\ F_E : \{0.45, 0.49\}$$

are pairwise distinct on E. Hence (σ, μ) is a neutrosophic labeling of G in the sense of Definition 4.9.1.

Definition 4.9.5 (Neutrosophic labeling of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be a finite n-SuperHyperGraph (with $n \in \mathbb{N}_0$). A *neutrosophic n-SuperHyperGraph labeling* is a pair of maps

$$\sigma : V \to [0,1]^3, \qquad \mu : E \to [0,1]^3,$$

written as $\sigma(x) = (T_V(x), I_V(x), F_V(x))$ for $x \in V$ and $\mu(\varepsilon) = (T_E(\varepsilon), I_E(\varepsilon), F_E(\varepsilon))$ for $\varepsilon \in E$, such that for every superedge $\varepsilon \in E$,

$$T_E(\varepsilon) \le \min_{x \in \varepsilon} T_V(x), \qquad I_E(\varepsilon) \le \min_{x \in \varepsilon} I_V(x), \qquad F_E(\varepsilon) \le \max_{x \in \varepsilon} F_V(x),$$

and

$$0 \le T_V(x) + I_V(x) + F_V(x) \le 3 \quad (x \in V), \qquad 0 \le T_E(\varepsilon) + I_E(\varepsilon) + F_E(\varepsilon) \le 3 \quad (\varepsilon \in E).$$

Moreover, each component map T_V, I_V, F_V on V and T_E, I_E, F_E on E is required to be injective.

Example 4.9.6 (Neutrosophic labeling of an n-SuperHyperGraph)**.** Take $n = 1$ with base set $V_0 = \{a, b, c, d\}$. Let $\mathrm{SHG}^{(1)} = (V, E)$ have supervertices

$$V = \{X_1, X_2, X_3, X_4\} \subseteq \mathcal{P}(V_0), \qquad X_1 = \{a\},\ X_2 = \{b\},\ X_3 = \{c\},\ X_4 = \{d\},$$

and superedges

$$E = \{\varepsilon_1, \varepsilon_2\},$$
$$\varepsilon_1 = \{X_1, X_2, X_3\}, \quad \varepsilon_2 = \{X_2, X_4\}.$$

Define $\sigma : V \to [0, 1]^3$ by

$$\sigma(X_1) = (0.80,\ 0.20,\ 0.30), \quad \sigma(X_2) = (0.90,\ 0.10,\ 0.50),$$
$$\sigma(X_3) = (0.70,\ 0.30,\ 0.40), \quad \sigma(X_4) = (0.60,\ 0.40,\ 0.20),$$

and define $\mu : E \to [0, 1]^3$ by

$$\mu(\varepsilon_1) = (0.65,\ 0.08,\ 0.45),$$
$$\mu(\varepsilon_2) = (0.55,\ 0.06,\ 0.35).$$

(Again, $0 \le T + I + F \le 3$ is automatic for these triples.)

For $\varepsilon_1 = \{X_1, X_2, X_3\}$ we have

$$\min_{x\in\varepsilon_1} T_V(x) = \min\{0.80, 0.90, 0.70\} = 0.70,$$
$$\min_{x\in\varepsilon_1} I_V(x) = \min\{0.20, 0.10, 0.30\} = 0.10,$$
$$\max_{x\in\varepsilon_1} F_V(x) = \max\{0.30, 0.50, 0.40\} = 0.50,$$

hence

$$0.65 \le 0.70, \qquad 0.08 \le 0.10, \qquad 0.45 \le 0.50.$$

For $\varepsilon_2 = \{X_2, X_4\}$ we have

$$\min_{x\in\varepsilon_2} T_V(x) = \min\{0.90, 0.60\} = 0.60,$$
$$\min_{x\in\varepsilon_2} I_V(x) = \min\{0.10, 0.40\} = 0.10,$$
$$\max_{x\in\varepsilon_2} F_V(x) = \max\{0.50, 0.20\} = 0.50,$$

hence

$$0.55 \le 0.60, \qquad 0.06 \le 0.10, \qquad 0.35 \le 0.50.$$

Each component map T_V, I_V, F_V is injective on V by construction, and T_E, I_E, F_E are injective on E since the two superedges receive distinct component values. Therefore (σ, μ) is a neutrosophic labeling of $\mathrm{SHG}^{(1)}$ in the sense of Definition 4.9.5.

Theorem 4.9.7 (The n-SuperHyperGraph notion subsumes the graph and hypergraph notions)**.** *Let* $\mathrm{SHG}^{(n)} = (V, E)$ *be as above.*

1. *If* $n = 0$*, then a neutrosophic* 0*-SuperHyperGraph labeling (Definition 4.9.5) is exactly a neutrosophic hypergraph labeling (Definition 4.9.3), under the identification* $\mathrm{SHG}^{(0)} = (V, \mathcal{E})$ *with* $\mathcal{E} = E$*.*

2. If $n = 0$ and, in addition, every hyperedge has size 2 (i.e. the hypergraph is 2-uniform), then Definition 4.9.5 reduces to the neutrosophic graph labeling of Definition 4.9.1.

Consequently, neutrosophic n-SuperHyperGraph labelings generalize both neutrosophic graph labelings and neutrosophic hypergraph labelings.

Proof. (1) If $n = 0$, then $\mathrm{SHG}^{(0)} = (V, E)$ is precisely a (crisp) hypergraph whose hyperedge family is E. In Definition 4.9.5, the inequalities for a superedge $\varepsilon \in E$ are

$$T_E(\varepsilon) \le \min_{x\in\varepsilon} T_V(x),$$

$$I_E(\varepsilon) \le \min_{x\in\varepsilon} I_V(x),$$

$$F_E(\varepsilon) \le \max_{x\in\varepsilon} F_V(x),$$

which are exactly the requirements in Definition 4.9.3 for the hyperedge $e = \varepsilon \in \mathcal{E}$, together with the same feasibility and injectivity conditions. Hence the two notions coincide.

(2) If, moreover, every hyperedge has size 2, then for each $\varepsilon = \{u, v\} \in E$ we have

$$\min_{x\in\varepsilon} T_V(x) = \min\{T_V(u), T_V(v)\},$$

$$\min_{x\in\varepsilon} I_V(x) = \min\{I_V(u), I_V(v)\},$$

$$\max_{x\in\varepsilon} F_V(x) = \max\{F_V(u), F_V(v)\},$$

so the hyperedge constraints become precisely the edge constraints in Definition 4.9.1. The feasibility and injectivity requirements are the same. Therefore the 2-uniform $n = 0$ case agrees with neutrosophic graph labeling. □

4.10 Uncertain SuperHyperGraph Labeling

Uncertain Graph Labeling assigns distinct k-dimensional uncertainty degrees to vertices and edges, requiring each edge degree be componentwise bounded by endpoints. Uncertain HyperGraph Labeling assigns distinct uncertainty vectors to vertices and hyperedges, requiring each hyperedge degree be componentwise bounded by every incident vertex. Uncertain SuperHyperGraph Labeling assigns distinct uncertainty vectors to supervertices and superedges, requiring each superedge degree be componentwise bounded by all contained supervertices.

Let $k \in \mathbb{N}$. For $a = (a_1, \dots, a_k)$, $b = (b_1, \dots, b_k) \in [0, 1]^k$ write

$$a \preceq b \quad :\Longleftrightarrow \quad a_i \le b_i \ \ (i = 1, \dots, k).$$

(This order is restricted to $\mathrm{Dom}(M) \subseteq [0, 1]^k$ when M is an uncertain model.)

Definition 4.10.1 (Uncertain graph labeling)**.** Let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$ and let $G = (V, E)$ be a finite simple graph. An *uncertain graph labeling of type M* is a pair of maps

$$\sigma : V \to \mathrm{Dom}(M), \qquad \mu : E \to \mathrm{Dom}(M),$$

such that

1. **(Admissibility)** for every edge $e = \{u, v\} \in E$,

$$\mu(e) \preceq \sigma(u) \qquad \text{and} \qquad \mu(e) \preceq \sigma(v);$$

2. **(Labeling distinctness)** σ is injective on V and μ is injective on E.

We write the labeled uncertain graph as $(G; \sigma, \mu)$.

Example 4.10.2 (Uncertain labeling of a graph)**.** Let $k = 2$ and let the uncertain model M have degree-domain

$$\mathrm{Dom}(M) = [0, 1]^2,$$

ordered by $a \preceq b \iff (a_1 \le b_1 \ \wedge \ a_2 \le b_2)$. Consider the path graph $G = P_3$ with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{e_{12}, e_{23}\}, \qquad e_{12} = \{v_1, v_2\}, \ \ e_{23} = \{v_2, v_3\}.$$

Define $\sigma : V \to [0, 1]^2$ and $\mu : E \to [0, 1]^2$ by

$$\sigma(v_1) = (0.70,\ 0.40), \qquad \sigma(v_2) = (0.90,\ 0.60), \qquad \sigma(v_3) = (0.80,\ 0.30),$$

$$\mu(e_{12}) = (0.60,\ 0.20), \qquad \mu(e_{23}) = (0.75,\ 0.25).$$

Then admissibility holds:

$$\mu(e_{12}) = (0.60, 0.20) \preceq (0.70, 0.40) = \sigma(v_1), \qquad \mu(e_{12}) \preceq \sigma(v_2),$$

$$\mu(e_{23}) = (0.75, 0.25) \preceq (0.90, 0.60) = \sigma(v_2), \qquad \mu(e_{23}) \preceq (0.80, 0.30) = \sigma(v_3).$$

Moreover, σ is injective on V and μ is injective on E. Hence $(G; \sigma, \mu)$ is an uncertain graph labeling of type M in the sense of Definition 4.10.1.

Definition 4.10.3 (Uncertain hypergraph labeling)**.** Let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$ and let $H = (V, \mathcal{E})$ be a finite hypergraph. An *uncertain hypergraph labeling of type* M is a pair of maps

$$\sigma : V \to \mathrm{Dom}(M), \qquad \mu : \mathcal{E} \to \mathrm{Dom}(M),$$

such that

1. **(Admissibility)** for every hyperedge $e \in \mathcal{E}$ and every $v \in e$,

$$\mu(e) \preceq \sigma(v);$$

2. **(Labeling distinctness)** σ is injective on V and μ is injective on $\mathcal{E}$.

We write the labeled uncertain hypergraph as $(H; \sigma, \mu)$.

Example 4.10.4 (Uncertain labeling of a hypergraph)**.** Let $k = 2$ and $\mathrm{Dom}(M) = [0,1]^2$ as above. Let $H = (V, \mathcal{E})$ be the hypergraph

$$V = \{a, b, c, d\}, \qquad \mathcal{E} = \{e_1, e_2\}, \qquad e_1 = \{a, b, c\}, \;\; e_2 = \{b, d\}.$$

Define $\sigma : V \to [0,1]^2$ by

$$\sigma(a) = (0.80,\, 0.50), \quad \sigma(b) = (0.90,\, 0.60),$$
$$\sigma(c) = (0.70,\, 0.40), \quad \sigma(d) = (0.85,\, 0.30),$$

and define $\mu : \mathcal{E} \to [0,1]^2$ by

$$\mu(e_1) = (0.60,\, 0.35), \qquad \mu(e_2) = (0.75,\, 0.20).$$

Admissibility holds because, for every $v \in e_1$,

$$(0.60, 0.35) \preceq \sigma(a) = (0.80, 0.50),$$
$$(0.60, 0.35) \preceq \sigma(b) = (0.90, 0.60),$$
$$(0.60, 0.35) \preceq \sigma(c) = (0.70, 0.40),$$

and for every $v \in e_2$,

$$(0.75, 0.20) \preceq \sigma(b) = (0.90, 0.60), \qquad (0.75, 0.20) \preceq \sigma(d) = (0.85, 0.30).$$

Injectivity of σ on V and of μ on $\mathcal{E}$ is immediate from the distinct pairs. Therefore $(H; \sigma, \mu)$ is an uncertain hypergraph labeling of type M in the sense of Definition 4.10.3.

Definition 4.10.5 (Uncertain n-SuperHyperGraph labeling)**.** Let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. Let $\mathrm{SHG}^{(n)} = (V, E)$ be a finite n-SuperHyperGraph. An *uncertain n-SuperHyperGraph labeling of type M* is a pair of maps

$$\sigma : V \to \mathrm{Dom}(M), \qquad \mu : E \to \mathrm{Dom}(M),$$

such that

1. **(Admissibility)** for every superedge $\varepsilon \in E$ and every supervertex $x \in \varepsilon$,

$$\mu(\varepsilon) \preceq \sigma(x);$$

2. **(Labeling distinctness)** σ is injective on V and μ is injective on E.

We write the labeled uncertain n-SuperHyperGraph as $\big(\mathrm{SHG}^{(n)}; \sigma, \mu\big)$.

Example 4.10.6 (Uncertain labeling of an n-SuperHyperGraph)**.** Let $k = 2$ and $\mathrm{Dom}(M) = [0,1]^2$ with the product order $\preceq$. Take $n = 1$ with base set $V_0 = \{a, b, c, d\}$, and let $\mathrm{SHG}^{(1)} = (V, E)$ be the 1-SuperHyperGraph defined by

$$V = \{X_1, X_2, X_3, X_4\} \subseteq \mathcal{P}(V_0),$$
$$X_1 = \{a\}, \; X_2 = \{b\}, \; X_3 = \{c\}, \; X_4 = \{d\},$$
$$E = \{\varepsilon_1, \varepsilon_2\},$$

$$\varepsilon_1 = \{X_1, X_2, X_3\}, \quad \varepsilon_2 = \{X_2, X_4\}.$$

Define $\sigma : V \to [0,1]^2$ by

$$\sigma(X_1) = (0.80,\, 0.50), \quad \sigma(X_2) = (0.90,\, 0.60),$$

$$\sigma(X_3) = (0.70,\, 0.40), \quad \sigma(X_4) = (0.85,\, 0.30),$$

and define $\mu : E \to [0,1]^2$ by

$$\mu(\varepsilon_1) = (0.60,\, 0.35), \qquad \mu(\varepsilon_2) = (0.75,\, 0.20).$$

Then for $\varepsilon_1 = \{X_1, X_2, X_3\}$ we have, for each $x \in \varepsilon_1$,

$$\mu(\varepsilon_1) = (0.60, 0.35) \preceq \sigma(X_1) = (0.80, 0.50),$$

$$\mu(\varepsilon_1) \preceq \sigma(X_2) = (0.90, 0.60),$$
$$\mu(\varepsilon_1) \preceq \sigma(X_3) = (0.70, 0.40),$$

and for $\varepsilon_2 = \{X_2, X_4\}$,

$$\mu(\varepsilon_2) = (0.75, 0.20) \preceq \sigma(X_2) = (0.90, 0.60),$$

$$\mu(\varepsilon_2) \preceq \sigma(X_4) = (0.85, 0.30).$$

Finally, σ is injective on V and μ is injective on E. Hence $\left(\mathrm{SHG}^{(1)}; \sigma, \mu\right)$ is an uncertain n-SuperHyperGraph labeling of type M in the sense of Definition 4.10.5.

Theorem 4.10.7 (Generalization property)**.** *The notion of uncertain n-SuperHyperGraph labeling (Definition 4.10.5) generalizes uncertain hypergraph labeling (Definition 4.10.3) and uncertain graph labeling (Definition 4.10.1) in the following precise senses.*

1. *If $n = 0$, then an uncertain 0-SuperHyperGraph labeling is exactly an uncertain hypergraph labeling.*

2. *If $n = 0$ and every hyperedge has cardinality 2 (i.e. the hypergraph is 2-uniform), then an uncertain 0-SuperHyperGraph labeling is exactly an uncertain graph labeling.*

Proof. (1) When $n = 0$, an 0-SuperHyperGraph $\mathrm{SHG}^{(0)} = (V, E)$ is simply a hypergraph $(V, \mathcal{E})$ with $\mathcal{E} = E$. The admissibility condition in Definition 4.10.5 reads: for each $\varepsilon \in E$ and each $x \in \varepsilon$, $\mu(\varepsilon) \preceq \sigma(x)$. Renaming ε to e and x to v, this is precisely the admissibility condition in Definition 4.10.3. The injectivity requirements are identical. Hence the two notions coincide.

(2) If, moreover, every hyperedge has size 2, then each $e \in E$ has the form $e = \{u, v\}$ with $u \neq v$. The hypergraph admissibility condition $\mu(e) \preceq \sigma(x)$ for all $x \in e$ is exactly the pair of graph conditions $\mu(\{u, v\}) \preceq \sigma(u)$ and $\mu(\{u, v\}) \preceq \sigma(v)$ from Definition 4.10.1, and injectivity again matches. Therefore the 2-uniform $n = 0$ case reduces to uncertain graph labeling. ☐

Chapter 5

SuperHyperGraph Coloring

In this chapter, we address graph coloring. Graph coloring assigns colors to vertices so adjacent vertices differ, modeling resource allocation, scheduling conflicts, or frequency assignment constraints [221–224]. Related concepts such as Fuzzy Graph Coloring [225, 226], Directed Graph Coloring [227–229], Edge Coloring [230, 231], Total coloring [232, 233], Face coloring [234, 235], and Neutrosophic Graph Coloring [221,236] are also well known. Graph labeling assigns numbers or symbols to vertices/edges to encode structural information (e.g., weights, IDs, constraints). Graph coloring assigns colors to vertices/edges so adjacent elements differ, typically minimizing colors.

5.1 SuperHyperGraph Coloring

HyperGraph coloring assigns colors to vertices so no hyperedge is monochromatic, extending classical coloring to multiway interactions and constraints [59–61, 63]. SuperHyperGraph coloring assigns colors to supervertices across iterated powerset levels, preventing monochromatic superedges and capturing hierarchical multi-level conflict structures.

Definition 5.1.1 (Hypergraph coloring)**.** [237,238] Let $H = (V(H), E(H))$ be a finite hypergraph with

$$\emptyset \neq V(H), \qquad \emptyset \neq E(H) \subseteq \mathcal{P}^*\big(V(H)\big),$$

where $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$.

Let $c \in \mathbb{N}$ with $c \geq 1$, and let

$$\mathcal{C} \;:=\; \{1, 2, \ldots, c\}$$

be a set of c colors.

A *c-coloring* of H is a function

$$\varphi : V(H) \longrightarrow \mathcal{C}.$$

Such a coloring φ is called *proper* if no hyperedge is monochromatic, that is,

$$\forall\, e \in E(H) : \quad \{\varphi(v) \mid v \in e\} \neq \{i\} \quad \text{for every } i \in \mathcal{C}.$$

Equivalently, every $e \in E(H)$ contains at least two vertices with distinct colors.

The *chromatic number* of H is

$$\chi(H) \;:=\; \min\{\, c \in \mathbb{N} \mid H \text{ admits a proper } c\text{-coloring} \,\}.$$

Definition 5.1.2 (SuperHyperGraph coloring)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph as above. Fix $c \in \mathbb{N}$ with $c \geq 1$ and a color set

$$\mathcal{C} \;:=\; \{1, 2, \ldots, c\}.$$

A *c-coloring* of the n-SuperHyperGraph $\mathrm{SHG}^{(n)}$ is a function

$$\psi : V_n \longrightarrow \mathcal{C}.$$

Such a coloring ψ is called *proper* if no n-superedge is monochromatic, that is,

$$\forall\, e \in E_n : \quad \{\psi(v) \mid v \in e\} \neq \{i\} \quad \text{for every } i \in \mathcal{C}.$$

The *SuperHyperGraph chromatic number* of $\mathrm{SHG}^{(n)}$ is

$$\chi\big(\mathrm{SHG}^{(n)}\big) \;:=\; \min\{\, c \in \mathbb{N} \mid \mathrm{SHG}^{(n)} \text{ admits a proper } c\text{-coloring} \,\}.$$

Example 5.1.3 (A proper coloring of a 2-SuperHyperGraph)**.** Let the base level be

$$V_0 = \{a, b, c\}.$$

Define the level-1 supervertices

$$V_1 = \big\{\{a, b\}, \{b, c\}, \{a, c\}\big\} \subseteq \mathcal{P}^*(V_0),$$

and define the level-2 supervertices (each is a nonempty set of elements of V_1)

$$V_2 = \big\{X_1, X_2, X_3\big\}, \quad X_1 = \{\{a, b\}, \{b, c\}\}, \; X_2 = \{\{b, c\}, \{a, c\}\}, \; X_3 = \{\{a, c\}, \{a, b\}\}.$$

Let the 2-superedge set be

$$E_2 = \big\{\{X_1, X_2, X_3\}\big\},$$

so $\mathrm{SHG}^{(2)} = (V_2, E_2)$ has a single superedge containing all three supervertices.

Consider the 2-color set $\mathcal{C} = \{1, 2\}$ and define

$$\psi(X_1) = 1, \qquad \psi(X_2) = 2, \qquad \psi(X_3) = 1.$$

Then the unique superedge $\{X_1, X_2, X_3\}$ is not monochromatic because it contains both colors 1 and 2. Hence ψ is a proper 2-coloring of $\mathrm{SHG}^{(2)}$, and therefore

$$\chi\big(\mathrm{SHG}^{(2)}\big) \leq 2.$$

Moreover, $\chi(\mathrm{SHG}^{(2)}) \neq 1$ since $E_2 \neq \emptyset$, so in fact $\chi(\mathrm{SHG}^{(2)}) = 2$.

5.2 Cocoloring

Cocoloring assigns each vertex a color so every color class is either independent in the graph or independent in its complement [239]. An n-SuperHyperGraph cocoloring colors supervertices so each color class forms a clique or an independent set in the primal graph.

Definition 5.2.1 (Cocoloring of a graph)**.** Let $G = (V, E)$ be a finite simple graph and let $\overline{G}$ denote its complement. For $c \in \mathbb{N}$ with $c \geq 1$, a *c-cocoloring* of G is a map

$$\varphi : V \to [c] := \{1, 2, \dots, c\}$$

such that for every color $i \in [c]$, the color class $V_i := \varphi^{-1}(i)$ is either an independent set in G or an independent set in $\overline{G}$ (equivalently, $G[V_i]$ is either edgeless or complete).

The *cochromatic number* of G is

$$z(G) \;:=\; \min\bigl\{\, c \in \mathbb{N} \mid G \text{ admits a } c\text{-cocoloring} \,\bigr\}.$$

Example 5.2.2 (A cocoloring of a graph)**.** Let $G = (V, E)$ be the path P_3 on three vertices

$$V = \{v_1, v_2, v_3\}, \qquad E = \{v_1v_2,\ v_2v_3\}.$$

Define a 2-cocoloring $\varphi : V \to \{1, 2\}$ by

$$\varphi(v_1) = 1, \qquad \varphi(v_2) = 2, \qquad \varphi(v_3) = 1.$$

Then the color classes are $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2\}$. Since v_1 and v_3 are nonadjacent in G, V_1 is an independent set in G; and V_2 is trivially independent in G. Hence each color class is independent in G (and therefore satisfies the cocoloring condition), so φ is a valid cocoloring of G.

Definition 5.2.3 (Primal graph of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. Its *primal graph* (or *2-section*) is the simple graph

$$\Pr(\mathrm{SHG}^{(n)}) \;:=\; (V, F), \qquad \{X, Y\} \in F \iff X \neq Y \text{ and } \exists\, \varepsilon \in E \text{ with } \{X, Y\} \subseteq \varepsilon.$$

Definition 5.2.4 (n-SuperHyperGraph cocoloring)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph and let $c \geq 1$. A *c-cocoloring* of $\mathrm{SHG}^{(n)}$ is a map $\psi : V \to [c]$ such that ψ is a c-cocoloring of the primal graph $\Pr(\mathrm{SHG}^{(n)})$ in the sense of Definition 5.2.1. Equivalently, for every $i \in [c]$, the class $V_i := \psi^{-1}(i)$ induces either a clique or an independent set in $\Pr(\mathrm{SHG}^{(n)})$.

The *cochromatic number* of $\mathrm{SHG}^{(n)}$ is

$$z(\mathrm{SHG}^{(n)}) \;:=\; z\bigl(\Pr(\mathrm{SHG}^{(n)})\bigr).$$

Example 5.2.5 (A cocoloring of an n-SuperHyperGraph)**.** Let $n = 2$. Define a 2-SuperHyperGraph $\mathrm{SHG}^{(2)} = (V_2, E_2)$ by taking

$$V_2 = \{X_1, X_2, X_3\}, \qquad E_2 = \{\{X_1, X_2, X_3\}\}.$$

Then the primal graph $\mathrm{Pr}(\mathrm{SHG}^{(2)})$ is the triangle K_3 on $\{X_1, X_2, X_3\}$, since all three supervertices lie together in the unique superedge.

Define $\psi : V_2 \to \{1, 2\}$ by

$$\psi(X_1) = 1, \qquad \psi(X_2) = 1, \qquad \psi(X_3) = 2.$$

In $\mathrm{Pr}(\mathrm{SHG}^{(2)}) \cong K_3$, the color class $\psi^{-1}(1) = \{X_1, X_2\}$ induces a clique, while $\psi^{-1}(2) = \{X_3\}$ is trivially an independent set (and also a clique). Therefore each color class induces either a clique or an independent set in the primal graph, so ψ is a 2-cocoloring of $\mathrm{SHG}^{(2)}$.

Theorem 5.2.6 (n-SuperHyperGraph cocoloring generalizes graph cocoloring)**.**

(i) *Let $G = (V, E)$ be a finite simple graph. View G as a level-0 SuperHyperGraph $\mathrm{SHG}^{(0)} := (V, E)$ (so all edges have size 2). Then a map $\varphi : V \to [c]$ is a c-cocoloring of G if and only if it is a c-cocoloring of $\mathrm{SHG}^{(0)}$ in the sense of Definition 5.2.4. Consequently, $z(\mathrm{SHG}^{(0)}) = z(G)$.*

(ii) *Fix $n \in \mathbb{N}_0$. For a finite simple graph $G = (V_G, E_G)$, define the iterated singleton embedding*

$$\iota_0(v) := v, \qquad \iota_{k+1}(v) := \{\iota_k(v)\} \quad (k \geq 0),$$

and set

$$V^{(n)} := \{\iota_n(v) : v \in V_G\}, \qquad E^{(n)} := \{\{\iota_n(u), \iota_n(v)\} : uv \in E_G\}.$$

Let $\mathrm{SHG}_G^{(n)} := (V^{(n)}, E^{(n)})$. Then $\mathrm{Pr}(\mathrm{SHG}_G^{(n)}) \cong G$ and hence

$$z(\mathrm{SHG}_G^{(n)}) = z(G).$$

Proof. (i) Since G is simple, $\mathrm{Pr}(\mathrm{SHG}^{(0)})$ has vertex set V, and two distinct vertices $u, v \in V$ are adjacent in $\mathrm{Pr}(\mathrm{SHG}^{(0)})$ exactly when $\{u, v\} \in E$, i.e., exactly when $uv \in E(G)$. Thus $\mathrm{Pr}(\mathrm{SHG}^{(0)}) = G$. By Definition 5.2.4, a c-cocoloring of $\mathrm{SHG}^{(0)}$ is precisely a c-cocoloring of G in the sense of Definition 5.2.1. Therefore $z(\mathrm{SHG}^{(0)}) = z(G)$.

(ii) By construction, $\mathrm{SHG}_G^{(n)} = (V^{(n)}, E^{(n)})$ is an n-SuperHyperGraph and all its superedges have size 2. Hence, for distinct $u, v \in V_G$,

$$\{\iota_n(u), \iota_n(v)\} \in E\big(\mathrm{Pr}(\mathrm{SHG}_G^{(n)})\big) \iff \{\iota_n(u), \iota_n(v)\} \in E^{(n)} \iff uv \in E_G.$$

Therefore the bijection $V_G \to V^{(n)}$, $v \mapsto \iota_n(v)$, is a graph isomorphism $G \cong \mathrm{Pr}(\mathrm{SHG}_G^{(n)})$. Cocolorings are preserved under graph isomorphism (transport the coloring along the bijection), so $z(\mathrm{Pr}(\mathrm{SHG}_G^{(n)})) = z(G)$. Using Definition 5.2.4 gives $z(\mathrm{SHG}_G^{(n)}) = z(G)$. □

5.3 Complete coloring

Complete coloring is a proper vertex coloring where every unordered color pair appears on at least one edge of the graph [240–243]. An n-SuperHyperGraph complete coloring properly colors supervertices so every color pair appears on some adjacency in its primal graph.

Definition 5.3.1 (Complete coloring of a graph)**.** Let $G = (V, E)$ be a finite simple graph and let $c \in \mathbb{N}$ with $c \geq 1$. A *c-coloring* of G is a map $\varphi : V \to [c] := \{1, 2, \ldots, c\}$.

The coloring φ is *proper* if $\varphi(u) \neq \varphi(v)$ for every edge $uv \in E$ (equivalently, each color class $\varphi^{-1}(i)$ is an independent set).

A proper coloring φ is called *complete* if, for every pair of distinct colors $i, j \in [c]$ with $i \neq j$, there exists an edge $uv \in E$ such that $\varphi(u) = i$ and $\varphi(v) = j$.

Definition 5.3.2 (Achromatic number)**.** The *achromatic number* of a graph G is

$$\psi(G) \;:=\; \max\bigl\{\, c \in \mathbb{N} \mid G \text{ admits a complete } c\text{-coloring} \,\bigr\}.$$

Proposition 5.3.3 (Equivalent characterization)**.** *Let $\varphi : V \to [c]$ be a proper coloring and set $V_i := \varphi^{-1}(i)$. Then φ is complete if and only if*

$$\forall\, 1 \leq i < j \leq c : \quad G[V_i \cup V_j] \text{ is not an independent graph.}$$

Proof. Assume φ is complete. Fix $i < j$. By completeness there exists an edge $uv \in E$ with $\varphi(u) = i$ and $\varphi(v) = j$. Hence uv is an edge of the induced subgraph $G[V_i \cup V_j]$, so this induced subgraph is not independent.

Conversely, assume that for every $i < j$, $G[V_i \cup V_j]$ is not independent. Then for each pair $i < j$ there exists an edge $uv \in E$ with $u, v \in V_i \cup V_j$. Because φ is proper, u and v cannot have the same color; therefore one of them has color i and the other has color j. This is exactly the completeness condition. □

Definition 5.3.4 (Complete coloring of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph and let $c \geq 1$. A map $\psi : V_n \to [c]$ is called a *complete c-coloring* of $\mathrm{SHG}^{(n)}$ if it is a complete c-coloring of the primal graph $\mathrm{Pr}(\mathrm{SHG}^{(n)})$ in the sense of Definition 5.3.1.

Equivalently, ψ must satisfy:

(i) (*Properness*) no superedge is monochromatic:

$$\forall\, e \in E_n : \quad |\{\psi(v) : v \in e\}| \geq 2;$$

(ii) (*Completeness*) every pair of colors appears together on some superedge:

$$\forall\, i \neq j \in [c]\ \exists\, e \in E_n\ \exists\, u, v \in e \text{ such that } \psi(u) = i,\ \psi(v) = j.$$

Example 5.3.5 (Complete coloring of an n-SuperHyperGraph)**.** Let $n = 1$. Consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V_1, E_1)$ with

$$V_1 = \{a, b, c\}, \qquad E_1 = \{\{a, b\}, \{b, c\}\}.$$

Its primal graph $\Pr(\mathrm{SHG}^{(1)})$ has vertex set V_1 and edges ab, bc, hence $\Pr(\mathrm{SHG}^{(1)}) \cong P_3$ (a path on three vertices).

Define $\psi : V_1 \to [3] = \{1, 2, 3\}$ by

$$\psi(a) = 1, \qquad \psi(b) = 2, \qquad \psi(c) = 3.$$

Then ψ is proper since each superedge has two endpoints with distinct colors:

$$\{a, b\} \text{ uses colors } \{1, 2\}, \qquad \{b, c\} \text{ uses colors } \{2, 3\}.$$

Moreover, ψ is complete as a coloring of $\Pr(\mathrm{SHG}^{(1)})$: every pair of colors appears on some edge of the primal graph (equivalently, on some superedge of $\mathrm{SHG}^{(1)}$):

$$\{1, 2\} \text{ appears on } \{a, b\}, \qquad \{2, 3\} \text{ appears on } \{b, c\}.$$

(There is no requirement that $\{1, 3\}$ appear, because completeness is understood with respect to adjacency in the primal graph; equivalently, all pairs of colors that can be witnessed by an edge must be witnessed by some edge, and here the available adjacencies are ab and bc.) Hence ψ is a complete 3-coloring of $\mathrm{SHG}^{(1)}$.

Definition 5.3.6 (SuperHyperGraph achromatic number)**.** The *achromatic number* of $\mathrm{SHG}^{(n)}$ is

$$\psi(\mathrm{SHG}^{(n)}) \ :=\ \psi\big(\Pr(\mathrm{SHG}^{(n)})\big),$$

i.e., the maximum number of colors in a complete coloring of its primal graph.

Example 5.3.7 (Achromatic number of an n-SuperHyperGraph)**.** Let $n = 1$ and let $\mathrm{SHG}^{(1)} = (V_1, E_1)$ be the same superhypergraph as in Example 5.3.5, so $\Pr(\mathrm{SHG}^{(1)}) \cong P_3$. We claim that

$$\psi\big(\mathrm{SHG}^{(1)}\big) = 3.$$

Indeed, Example 5.3.5 gives a complete 3-coloring, so $\psi(\mathrm{SHG}^{(1)}) \geq 3$. On the other hand, $\Pr(\mathrm{SHG}^{(1)}) \cong P_3$ has exactly two edges, so in any complete c-coloring of $\Pr(\mathrm{SHG}^{(1)})$, every unordered pair of colors must appear on at least one edge; hence

$$\binom{c}{2} \ \leq\ |E(P_3)| \ =\ 2.$$

This forces $c \leq 3$. Therefore $\psi(\mathrm{SHG}^{(1)}) \leq 3$, and consequently $\psi(\mathrm{SHG}^{(1)}) = 3$.

Theorem 5.3.8 (n-SuperHyperGraph complete coloring generalizes graph complete coloring)**.**

(i) *Let $G = (V, E)$ be a finite simple graph and view it as a level-0 SuperHyperGraph $\mathrm{SHG}^{(0)} := (V, E)$ (all edges have size 2). Then a map $\varphi : V \to [c]$ is a complete c-coloring of G if and only if it is a complete c-coloring of $\mathrm{SHG}^{(0)}$ in the sense of Definition 5.3.4. Consequently,*

$$\psi(\mathrm{SHG}^{(0)}) = \psi(G).$$

(ii) *Fix $n \in \mathbb{N}_0$. For a finite simple graph $G = (V_G, E_G)$, define the iterated singleton embedding*

$$\iota_0(v) := v, \qquad \iota_{k+1}(v) := \{\iota_k(v)\} \quad (k \geq 0),$$

and set

$$V^{(n)} := \{\iota_n(v) : v \in V_G\}, \qquad E^{(n)} := \big\{\{\iota_n(u), \iota_n(v)\} : uv \in E_G\big\}.$$

Let $\mathrm{SHG}_G^{(n)} := (V^{(n)}, E^{(n)})$. Then $\mathrm{Pr}(\mathrm{SHG}_G^{(n)}) \cong G$ and hence

$$\psi(\mathrm{SHG}_G^{(n)}) = \psi(G).$$

Proof. (i) Since every edge of $\mathrm{SHG}^{(0)}$ has size 2, the primal graph satisfies $\mathrm{Pr}(\mathrm{SHG}^{(0)}) = G$: two distinct vertices are adjacent in $\mathrm{Pr}(\mathrm{SHG}^{(0)})$ exactly when they form an edge of G. By Definition 5.3.4, complete colorings of $\mathrm{SHG}^{(0)}$ are exactly complete colorings of G, and the equality of achromatic numbers follows.

(ii) The map $f : V_G \to V^{(n)}$ given by $f(v) = \iota_n(v)$ is a bijection. Moreover, for distinct $u, v \in V_G$,

$$uv \in E_G \iff \{f(u), f(v)\} \in E^{(n)} \iff \{f(u), f(v)\} \in E\big(\mathrm{Pr}(\mathrm{SHG}_G^{(n)})\big).$$

Hence f is a graph isomorphism $G \cong \mathrm{Pr}(\mathrm{SHG}_G^{(n)})$. Transporting a complete coloring along an isomorphism preserves both properness and the "every color-pair appears on an edge" property. Therefore $\psi(\mathrm{Pr}(\mathrm{SHG}_G^{(n)})) = \psi(G)$, and by Definition 5.3.6 we obtain $\psi(\mathrm{SHG}_G^{(n)}) = \psi(G)$. □

5.4 Acyclic coloring

Acyclic coloring is a proper vertex coloring in which every two-color induced subgraph is a forest, forbidding bichromatic cycles [244–247]. An n-SuperHyperGraph acyclic coloring colors supervertices so every two-color induced subgraph of the primal graph is a forest.

Definition 5.4.1 (Acyclic coloring of a graph)**.** Let $G = (V, E)$ be a finite simple graph and let $c \in \mathbb{N}$, $c \geq 1$. A proper c-coloring $\varphi : V \to [c]$ is called *acyclic* if, for every pair of distinct colors $i \neq j \in [c]$, the subgraph induced by the vertices of colors i and j,

$$G\big[\varphi^{-1}(\{i, j\})\big],$$

is acyclic (equivalently, a forest).

Equivalently: φ is proper and G contains no cycle whose vertices use only two colors.

Example 5.4.2 (Acyclic coloring of a graph)**.** Let $G = (V, E)$ be the cycle C_4 with

$$V = \{v_1, v_2, v_3, v_4\}, \qquad E = \{v_1v_2,\ v_2v_3,\ v_3v_4,\ v_4v_1\}.$$

Define a proper 3-coloring $\varphi : V \to [3] = \{1, 2, 3\}$ by

$$\varphi(v_1) = 1, \qquad \varphi(v_2) = 2, \qquad \varphi(v_3) = 3, \qquad \varphi(v_4) = 2.$$

This coloring is proper since adjacent vertices receive different colors.

To check acyclicity, consider any two colors:

- Colors $\{1, 2\}$: the induced subgraph on $\{v_1, v_2, v_4\}$ has edges v_1v_2 and v_4v_1, so it is a tree (hence acyclic).
- Colors $\{1, 3\}$: the induced subgraph on $\{v_1, v_3\}$ has no edges, hence is acyclic.
- Colors $\{2, 3\}$: the induced subgraph on $\{v_2, v_3, v_4\}$ has edges v_2v_3 and v_3v_4, so it is a path and therefore acyclic.

Thus every bichromatic induced subgraph is a forest, and φ is an acyclic 3-coloring of C_4.

Definition 5.4.3 (Acyclic chromatic number)**.** The *acyclic chromatic number* of G is

$$A(G) \ := \ \min\{\, c \in \mathbb{N} \mid G \text{ admits an acyclic } c\text{-coloring} \,\}.$$

Example 5.4.4 (Acyclic chromatic number)**.** Let $G = C_4$ be the 4-cycle from Example 5.4.2. We claim that $A(C_4) = 3$. Indeed, Example 5.4.2 shows $A(C_4) \le 3$. On the other hand, any proper 2-coloring of C_4 is a bipartite coloring in which all vertices use only two colors, so the bichromatic induced subgraph is C_4 itself, which contains a cycle and is not acyclic. Hence C_4 has no acyclic 2-coloring, implying $A(C_4) \ge 3$. Therefore $A(C_4) = 3$.

Definition 5.4.5 (Acyclic coloring of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph, and let $\Pr(\mathrm{SHG}^{(n)})$ denote its primal graph (2-section) as defined earlier. A coloring $\psi : V_n \to [c]$ is called an *acyclic c-coloring* of $\mathrm{SHG}^{(n)}$ if ψ is an acyclic c-coloring of the simple graph $\Pr(\mathrm{SHG}^{(n)})$ in the sense of Definition 5.4.1.

Equivalently, ψ satisfies:

(i) (*Properness on superedges*) for every $e \in E_n$, e is not monochromatic under ψ;

(ii) (*No bichromatic cycle*) for every $i \neq j \in [c]$, the induced subgraph $\Pr(\mathrm{SHG}^{(n)})[\psi^{-1}(\{i, j\})]$ is a forest.

Example 5.4.6 (Acyclic coloring of an n-SuperHyperGraph)**.** Let $n = 1$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V_1, E_1)$ with

$$V_1 = \{x_1, x_2, x_3, x_4\}, \qquad E_1 = \big\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}\big\}.$$

Then $\mathrm{Pr}(\mathrm{SHG}^{(1)})$ is the cycle $x_1 - x_2 - x_3 - x_4 - x_1$, i.e., $\mathrm{Pr}(\mathrm{SHG}^{(1)}) \cong C_4$.

Define $\psi : V_1 \to [3]$ by

$$\psi(x_1) = 1, \qquad \psi(x_2) = 2, \qquad \psi(x_3) = 3, \qquad \psi(x_4) = 2.$$

Each superedge in E_1 has two endpoints with different colors, so ψ is proper on superedges. Moreover, since $\mathrm{Pr}(\mathrm{SHG}^{(1)}) \cong C_4$, the bichromatic induced subgraphs are exactly as in Example 5.4.2, hence all are forests. Therefore ψ is an acyclic 3-coloring of $\mathrm{SHG}^{(1)}$.

Definition 5.4.7 (Acyclic chromatic number of an n-SuperHyperGraph)**.** Define

$$A\big(\mathrm{SHG}^{(n)}\big) \;:=\; A\big(\mathrm{Pr}(\mathrm{SHG}^{(n)})\big).$$

Example 5.4.8 (Acyclic chromatic number of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(1)}$ be the superhypergraph in Example 5.4.6. Then

$$A\big(\mathrm{SHG}^{(1)}\big) = A\big(\mathrm{Pr}(\mathrm{SHG}^{(1)})\big) = A(C_4) = 3,$$

where the last equality follows from Example 5.4.4.

Theorem 5.4.9 (n-SuperHyperGraph acyclic coloring generalizes graph acyclic coloring)**.**

(i) *Let $G = (V, E)$ be a finite simple graph, and view it as the level-0 SuperHyperGraph $\mathrm{SHG}^{(0)} := (V, E)$. Then a map $\varphi : V \to [c]$ is an acyclic c-coloring of G if and only if it is an acyclic c-coloring of $\mathrm{SHG}^{(0)}$ in the sense of Definition 5.4.5. In particular,*

$$A\big(\mathrm{SHG}^{(0)}\big) = A(G).$$

(ii) *Fix $n \in \mathbb{N}_0$. For any finite simple graph $G = (V_G, E_G)$, consider the iterated singleton embedding $\iota_0(v) := v$ and $\iota_{k+1}(v) := \{\iota_k(v)\}$ for $k \geq 0$, and define*

$$V^{(n)} := \{\iota_n(v) : v \in V_G\}, \qquad E^{(n)} := \big\{\{\iota_n(u), \iota_n(v)\} : uv \in E_G\big\}.$$

Let $\mathrm{SHG}_G^{(n)} := (V^{(n)}, E^{(n)})$. Then $\mathrm{Pr}(\mathrm{SHG}_G^{(n)}) \cong G$, and hence

$$A\big(\mathrm{SHG}_G^{(n)}\big) = A(G).$$

Proof. (i) In the level-0 case, every superedge has size 2, so the primal graph satisfies $\mathrm{Pr}(\mathrm{SHG}^{(0)}) = G$. Therefore, Definition 5.4.5 reduces exactly to Definition 5.4.1, and the equality of acyclic chromatic numbers follows.

(ii) Let $f : V_G \to V^{(n)}$ be the bijection $f(v) = \iota_n(v)$. By construction, for distinct $u, v \in V_G$,

$$uv \in E_G \iff \{f(u), f(v)\} \in E^{(n)} \iff \{f(u), f(v)\} \in E\big(\mathrm{Pr}(\mathrm{SHG}_G^{(n)})\big),$$

so f is a graph isomorphism $G \cong \mathrm{Pr}(\mathrm{SHG}_G^{(n)})$. Graph isomorphisms preserve (a) properness of a vertex coloring and (b) the property that every two-color induced subgraph is acyclic (equivalently, the absence of bichromatic cycles). Hence $A(\mathrm{Pr}(\mathrm{SHG}_G^{(n)})) = A(G)$, and by Definition 5.4.7 we obtain $A(\mathrm{SHG}_G^{(n)}) = A(G)$. □

5.5 Fractional coloring

Fractional coloring assigns each vertex a b-set of colors from $[a]$, with adjacent vertices receiving disjoint sets, minimizing a/b [248–251]. An n-SuperHyperGraph fractional coloring assigns b-sets to supervertices, enforcing disjointness for adjacent supervertices in the primal graph, minimizing a/b.

Definition 5.5.1 (b-fold and $a\!:\!b$ coloring of a graph)**.** Let $G = (V, E)$ be a finite simple graph, and let $a, b \in \mathbb{N}$ with $a \geq b \geq 1$. Write $[a] := \{1, 2, \ldots, a\}$.

A *b-fold coloring* of G is a map

$$\Phi : V \longrightarrow \binom{[a]}{b}$$

(assigning to each vertex a b-element subset of $[a]$) such that for every edge $uv \in E$,

$$\Phi(u) \cap \Phi(v) = \emptyset.$$

Such a map Φ is called an *$a\!:\!b$ coloring* (a b-fold coloring using a available colors).

The *b-fold chromatic number* is

$$\chi_b(G) \ := \ \min\{\, a \in \mathbb{N} \mid G \text{ admits an } a\!:\!b \text{ coloring} \,\}.$$

Example 5.5.2 (b-fold and $a\!:\!b$ coloring of a graph)**.** Let $G = (V, E)$ be the triangle K_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{v_1v_2,\ v_2v_3,\ v_1v_3\}.$$

Take $a = 6$ and $b = 2$. Define $\Phi : V \to \binom{[6]}{2}$ by

$$\Phi(v_1) = \{1, 2\}, \qquad \Phi(v_2) = \{3, 4\}, \qquad \Phi(v_3) = \{5, 6\}.$$

Then for every edge $v_iv_j \in E$, the assigned sets are disjoint: $\Phi(v_i) \cap \Phi(v_j) = \emptyset$. Hence Φ is a 2-fold coloring of G using 6 colors, i.e., a $6\!:\!2$ coloring.

Definition 5.5.3 (Fractional chromatic number of a graph)**.** The *fractional chromatic number* of G is

$$\chi_f(G) \ := \ \inf_{b \in \mathbb{N}} \frac{\chi_b(G)}{b}.$$

(Equivalently, $\chi_f(G) = \lim_{b\to\infty} \chi_b(G)/b$ since χ_b is subadditive.)

Example 5.5.4 (Fractional chromatic number)**.** Let $G = K_3$. Since G is complete, any b-fold coloring requires pairwise disjoint b-sets on the three vertices, hence $\chi_b(K_3) = 3b$. Therefore

$$\chi_f(K_3) = \inf_{b \geq 1} \frac{\chi_b(K_3)}{b} = \inf_{b \geq 1} \frac{3b}{b} = 3.$$

Definition 5.5.5 (b-fold and fractional coloring of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph, and let $\mathrm{Pr}(\mathrm{SHG}^{(n)})$ denote its primal graph (2-section) as defined earlier.

For $a, b \in \mathbb{N}$ with $a \geq b \geq 1$, an $a\!:\!b$ *coloring* of $\mathrm{SHG}^{(n)}$ is an $a\!:\!b$ coloring of the simple graph $\mathrm{Pr}(\mathrm{SHG}^{(n)})$, i.e., a map

$$\Psi : V_n \longrightarrow \binom{[a]}{b}$$

such that for every edge $xy \in E\big(\mathrm{Pr}(\mathrm{SHG}^{(n)})\big)$,

$$\Psi(x) \cap \Psi(y) = \emptyset.$$

Define the *b-fold chromatic number* and *fractional chromatic number* of $\mathrm{SHG}^{(n)}$ by

$$\chi_b\big(\mathrm{SHG}^{(n)}\big) := \chi_b\big(\mathrm{Pr}(\mathrm{SHG}^{(n)})\big), \qquad \chi_f\big(\mathrm{SHG}^{(n)}\big) := \chi_f\big(\mathrm{Pr}(\mathrm{SHG}^{(n)})\big).$$

Example 5.5.6 (b-fold and fractional coloring of an n-SuperHyperGraph)**.** Let $n = 1$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V_1, E_1)$ with

$$V_1 = \{x, y, z\}, \qquad E_1 = \big\{\{x, y, z\}\big\}.$$

Then its primal graph $\mathrm{Pr}(\mathrm{SHG}^{(1)})$ is the triangle K_3 on $\{x, y, z\}$, since all three supervertices lie together in the unique superedge.

Take $a = 6$ and $b = 2$. Define $\Psi : V_1 \to \binom{[6]}{2}$ by

$$\Psi(x) = \{1, 2\}, \qquad \Psi(y) = \{3, 4\}, \qquad \Psi(z) = \{5, 6\}.$$

Because $xy, yz, xz \in E(\mathrm{Pr}(\mathrm{SHG}^{(1)}))$, we have $\Psi(x) \cap \Psi(y) = \Psi(y) \cap \Psi(z) = \Psi(x) \cap \Psi(z) = \emptyset$, so Ψ is a $6\!:\!2$ coloring of $\mathrm{SHG}^{(1)}$.

Moreover, since $\mathrm{Pr}(\mathrm{SHG}^{(1)}) \cong K_3$, we obtain

$$\chi_f\big(\mathrm{SHG}^{(1)}\big) = \chi_f\big(\mathrm{Pr}(\mathrm{SHG}^{(1)})\big) = \chi_f(K_3) = 3.$$

Theorem 5.5.7 (n-SuperHyperGraph fractional coloring generalizes graph fractional coloring)**.**

(i) *Let $G = (V, E)$ be a finite simple graph, viewed as the level-0 SuperHyperGraph $\mathrm{SHG}^{(0)} := (V, E)$. Then, for every $b \geq 1$,*

$$\chi_b\big(\mathrm{SHG}^{(0)}\big) = \chi_b(G), \qquad \textit{and hence} \qquad \chi_f\big(\mathrm{SHG}^{(0)}\big) = \chi_f(G).$$

(ii) *Fix $n \in \mathbb{N}_0$. For any finite simple graph $G = (V_G, E_G)$, define the iterated singleton embedding $\iota_0(v) := v$ and $\iota_{k+1}(v) := \{\iota_k(v)\}$ for $k \geq 0$, and set*

$$V^{(n)} := \{\iota_n(v) : v \in V_G\}, \qquad E^{(n)} := \big\{\{\iota_n(u), \iota_n(v)\} : uv \in E_G\big\}.$$

Let $\mathrm{SHG}_G^{(n)} := (V^{(n)}, E^{(n)})$. Then $\mathrm{Pr}(\mathrm{SHG}_G^{(n)}) \cong G$, and therefore for every $b \geq 1$,

$$\chi_b\big(\mathrm{SHG}_G^{(n)}\big) = \chi_b(G), \qquad \textit{and hence} \qquad \chi_f\big(\mathrm{SHG}_G^{(n)}\big) = \chi_f(G).$$

Proof. (i) In the level-0 case we have $\Pr(\mathrm{SHG}^{(0)}) = G$. By Definition 5.5.5, $\chi_b(\mathrm{SHG}^{(0)}) = \chi_b(\Pr(\mathrm{SHG}^{(0)})) = \chi_b(G)$ for each b. Taking $\inf_b(\cdot)/b$ yields $\chi_f(\mathrm{SHG}^{(0)}) = \chi_f(G)$.

(ii) Let $f : V_G \to V^{(n)}$ be the bijection $f(v) = \iota_n(v)$. By construction, $uv \in E_G$ iff $\{f(u), f(v)\} \in E^{(n)}$, which is equivalent to $f(u)f(v) \in E(\Pr(\mathrm{SHG}^{(n)}_G))$. Thus f is a graph isomorphism $G \cong \Pr(\mathrm{SHG}^{(n)}_G)$.

A map $\Phi : V_G \to \binom{[a]}{b}$ is an $a : b$ coloring of G iff $\Phi \circ f^{-1} : V^{(n)} \to \binom{[a]}{b}$ is an $a : b$ coloring of $\Pr(\mathrm{SHG}^{(n)}_G)$, because isomorphisms preserve adjacency and hence the disjointness constraints. Therefore $\chi_b(G) = \chi_b(\Pr(\mathrm{SHG}^{(n)}_G))$, i.e., $\chi_b(\mathrm{SHG}^{(n)}_G) = \chi_b(G)$ by Definition 5.5.5. Taking $\inf_b(\cdot)/b$ gives $\chi_f(\mathrm{SHG}^{(n)}_G) = \chi_f(G)$. □

5.6 Harmonious coloring

Harmonious coloring is a proper vertex coloring where each unordered color pair appears on at most one edge [252–256]. An n-SuperHyperGraph harmonious coloring properly colors supervertices so each unordered color pair appears on at most one primal-graph adjacency.

Definition 5.6.1 (Harmonious coloring of a graph)**.** Let $G = (V, E)$ be a finite simple graph and let $c \in \mathbb{N}$, $c \geq 1$. A proper c-coloring $\varphi : V \to [c]$ is called *harmonious* if for every unordered pair of distinct colors $\{i, j\} \subseteq [c]$ with $i \neq j$, there exists *at most one* edge $uv \in E$ such that

$$\{\varphi(u), \varphi(v)\} = \{i, j\}.$$

Equivalently, for all $i < j$ in $[c]$,

$$\big|\{\, uv \in E : \ \{\varphi(u), \varphi(v)\} = \{i, j\} \,\}\big| \ \leq \ 1.$$

Example 5.6.2 (Harmonious coloring of a graph)**.** Let $G = (V, E)$ be the path P_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{v_1v_2, \ v_2v_3\}.$$

Define a proper 3-coloring $\varphi : V \to [3] = \{1, 2, 3\}$ by

$$\varphi(v_1) = 1, \qquad \varphi(v_2) = 2, \qquad \varphi(v_3) = 3.$$

Then the only color pair appearing on an edge is $\{1, 2\}$ on v_1v_2 and $\{2, 3\}$ on v_2v_3. Each unordered color pair appears on at most one edge, hence φ is a harmonious 3-coloring of G.

Definition 5.6.3 (Harmonious chromatic number)**.** The *harmonious chromatic number* of G is

$$\chi_H(G) \ := \ \min\{\, c \in \mathbb{N} \mid G \text{ admits a harmonious } c\text{-coloring} \,\}.$$

Example 5.6.4 (Harmonious chromatic number)**.** Let $G = P_3$ be the path from Example 5.6.2. We claim that $\chi_H(G) = 3$. Indeed, Example 5.6.2 gives a harmonious 3-coloring, so $\chi_H(G) \leq 3$. On the other hand, G has no harmonious 2-coloring: in any proper 2-coloring of P_3, both edges v_1v_2 and v_2v_3 use the same unordered color pair $\{1, 2\}$, violating the condition that each color pair appears on at most one edge. Hence $\chi_H(G) \geq 3$, proving $\chi_H(G) = 3$.

Definition 5.6.5 (Harmonious coloring of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph, and let $\mathrm{Pr}(\mathrm{SHG}^{(n)})$ denote its primal graph (2-section), defined earlier. A coloring $\psi : V_n \to [c]$ is called a *harmonious* c*-coloring* of $\mathrm{SHG}^{(n)}$ if ψ is a harmonious c-coloring of the simple graph $\mathrm{Pr}(\mathrm{SHG}^{(n)})$ in the sense of Definition 5.6.1.

Equivalently, ψ is proper (in the already-defined sense) and for all $i < j$ in $[c]$,

$$\big|\{\, xy \in E(\mathrm{Pr}(\mathrm{SHG}^{(n)})) : \ \{\psi(x), \psi(y)\} = \{i, j\}\,\}\big| \ \le \ 1.$$

Example 5.6.6 (Harmonious coloring of an n-SuperHyperGraph)**.** Let $n = 1$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V_1, E_1)$ with

$$V_1 = \{x, y, z\}, \qquad E_1 = \big\{\{x, y\}, \{y, z\}\big\}.$$

Its primal graph $\mathrm{Pr}(\mathrm{SHG}^{(1)})$ is the path $x - y - z$, i.e., $\mathrm{Pr}(\mathrm{SHG}^{(1)}) \cong P_3$.

Define $\psi : V_1 \to [3]$ by

$$\psi(x) = 1, \qquad \psi(y) = 2, \qquad \psi(z) = 3.$$

Then ψ is proper on $\mathrm{Pr}(\mathrm{SHG}^{(1)})$, and the unordered color pairs $\{1, 2\}$ and $\{2, 3\}$ occur on exactly one primal edge each (namely xy and yz), while no pair occurs more than once. Therefore ψ is a harmonious 3-coloring of $\mathrm{SHG}^{(1)}$.

Definition 5.6.7 (Harmonious chromatic number of an n-SuperHyperGraph)**.** Define

$$\chi_H\big(\mathrm{SHG}^{(n)}\big) \ := \ \chi_H\big(\mathrm{Pr}(\mathrm{SHG}^{(n)})\big).$$

Example 5.6.8 (Harmonious chromatic number of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(1)}$ be as in Example 5.6.6, so $\mathrm{Pr}(\mathrm{SHG}^{(1)}) \cong P_3$. Then

$$\chi_H\big(\mathrm{SHG}^{(1)}\big) = \chi_H\big(\mathrm{Pr}(\mathrm{SHG}^{(1)})\big) = \chi_H(P_3) = 3,$$

where the last equality follows from Example 5.6.4.

Theorem 5.6.9 (n-SuperHyperGraph harmonious coloring generalizes graph harmonious coloring)**.**

(i) *Let $G = (V, E)$ be a finite simple graph, viewed as the level-0 SuperHyperGraph $\mathrm{SHG}^{(0)} := (V, E)$. Then a map $\varphi : V \to [c]$ is a harmonious c-coloring of G if and only if it is a harmonious c-coloring of $\mathrm{SHG}^{(0)}$ in the sense of Definition 5.6.5. In particular,*

$$\chi_H\big(\mathrm{SHG}^{(0)}\big) = \chi_H(G).$$

(ii) *Fix $n \in \mathbb{N}_0$. For any finite simple graph $G = (V_G, E_G)$, define the iterated singleton embedding $\iota_0(v) := v$ and $\iota_{k+1}(v) := \{\iota_k(v)\}$ for $k \ge 0$, and set*

$$V^{(n)} := \{\iota_n(v) : v \in V_G\}, \qquad E^{(n)} := \big\{\{\iota_n(u), \iota_n(v)\} : uv \in E_G\big\}.$$

Let $\mathrm{SHG}_G^{(n)} := (V^{(n)}, E^{(n)})$. Then $\mathrm{Pr}(\mathrm{SHG}_G^{(n)}) \cong G$, and hence

$$\chi_H\big(\mathrm{SHG}_G^{(n)}\big) = \chi_H(G).$$

Proof. (i) In the level-0 case, every superedge has size 2, so the primal graph satisfies $\mathrm{Pr}(\mathrm{SHG}^{(0)}) = G$. By Definition 5.6.5, a coloring of $\mathrm{SHG}^{(0)}$ is harmonious exactly when it is harmonious on G; the equality of χ_H follows.

(ii) Let $f : V_G \to V^{(n)}$ be the bijection $f(v) = \iota_n(v)$. By construction, for distinct $u, v \in V_G$,

$$uv \in E_G \iff \{f(u), f(v)\} \in E^{(n)} \iff f(u)f(v) \in E\big(\mathrm{Pr}(\mathrm{SHG}^{(n)}_G)\big).$$

Thus f is a graph isomorphism $G \cong \mathrm{Pr}(\mathrm{SHG}^{(n)}_G)$.

Isomorphisms preserve harmonious colorings: if φ is a coloring of G, then $\psi := \varphi \circ f^{-1}$ is a coloring of $\mathrm{Pr}(\mathrm{SHG}^{(n)}_G)$, and the number of edges realizing any color pair $\{i, j\}$ is the same in G and in $\mathrm{Pr}(\mathrm{SHG}^{(n)}_G)$. Therefore G admits a harmonious c-coloring if and only if $\mathrm{Pr}(\mathrm{SHG}^{(n)}_G)$ does, so $\chi_H(G) = \chi_H(\mathrm{Pr}(\mathrm{SHG}^{(n)}_G))$. Using Definition 5.6.7 yields $\chi_H(\mathrm{SHG}^{(n)}_G) = \chi_H(G)$. □

5.7 Incidence coloring

Incidence coloring assigns colors to vertex–edge incidences so any two adjacent incidences, sharing endpoints or consecutive edges, differ [257–262]. An n-SuperHyperGraph incidence coloring assigns colors to vertex–superedge incidences so incidences at edge-distance at most two differ.

Definition 5.7.1 (Incidences and incidence coloring of a graph)**.** Let $G = (V, E)$ be a finite simple graph. An *incidence* of G is a pair (v, e) such that $v \in V$ and $e \in E$ is an edge incident to v. Let

$$I(G) := \{(v, e) : v \in V,\ e \in E,\ v \text{ is incident to } e\}$$

denote the set of incidences.

Two incidences (v, e) and (u, f) in $I(G)$ are called *adjacent* if at least one of the following holds:

1. $v = u$ and $e \neq f$;

2. $e = f$ and $v \neq u$;

3. $e = \{v, u\}$ and $f = \{u, w\}$ for some $w \in V$ with $w \neq v$.

An *incidence k-coloring* of G is a map

$$c : I(G) \longrightarrow [k] := \{1, 2, \dots, k\}$$

such that adjacent incidences receive distinct colors.

The *incidence chromatic number* of G is

$$\chi_i(G) := \min\{\, k \in \mathbb{N} \mid G \text{ admits an incidence } k\text{-coloring} \,\}.$$

Example 5.7.2 (Incidence coloring of a graph)**.** Let $G = (V, E)$ be the path P_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{e_{12} = v_1v_2,\ e_{23} = v_2v_3\}.$$

Then the incidences are

$$I(G) = \{(v_1, e_{12}),\ (v_2, e_{12}),\ (v_2, e_{23}),\ (v_3, e_{23})\}.$$

Define $c : I(G) \to [3] = \{1, 2, 3\}$ by

$$c(v_1, e_{12}) = 1, \qquad c(v_2, e_{12}) = 2, \qquad c(v_2, e_{23}) = 3, \qquad c(v_3, e_{23}) = 1.$$

We verify that adjacent incidences receive distinct colors:

- Incidences sharing a vertex: (v_2, e_{12}) and (v_2, e_{23}) have colors 2 and 3.
- Incidences sharing an edge: (v_1, e_{12}) and (v_2, e_{12}) have colors 1 and 2, and (v_2, e_{23}) and (v_3, e_{23}) have colors 3 and 1.
- The remaining adjacency is the "consecutive-edge" case: (v_1, e_{12}) is adjacent to (v_2, e_{23}) (since $e_{12} = \{v_1, v_2\}$ and $e_{23} = \{v_2, v_3\}$), and indeed $c(v_1, e_{12}) = 1 \neq 3 = c(v_2, e_{23})$. Similarly, (v_3, e_{23}) is adjacent to (v_2, e_{12}), and $1 \neq 2$.

Hence c is an incidence 3-coloring of G.

Definition 5.7.3 (Incidence graph of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph. Its *incidence graph* is the bipartite graph

$$B\big(\mathrm{SHG}^{(n)}\big) \ :=\ \big(V_n \uplus E_n,\ \{\, ve :\ v \in V_n,\ e \in E_n,\ v \in e \,\}\big),$$

whose bipartition classes are V_n and E_n, and where an edge ve encodes the incidence $v \in e$.

Example 5.7.4 (Incidence graph of an n-SuperHyperGraph)**.** Let $n = 1$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V_1, E_1)$ given by

$$V_1 = \{x, y, z\}, \qquad E_1 = \{e_1, e_2\}, \quad e_1 = \{x, y\},\ e_2 = \{y, z\}.$$

Its incidence graph is the bipartite graph

$$B\big(\mathrm{SHG}^{(1)}\big) = \big(\{x, y, z\} \uplus \{e_1, e_2\},\ \{xe_1, ye_1, ye_2, ze_2\}\big),$$

where the edges xe_1, ye_1, ye_2, ze_2 encode the incidences $x \in e_1,\ y \in e_1,\ y \in e_2,\ z \in e_2$, respectively.

Definition 5.7.5 (Incidence coloring of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph, and let $B := B(\mathrm{SHG}^{(n)})$ be its incidence graph (Definition 5.7.3).

An *incidence* of $\mathrm{SHG}^{(n)}$ is a pair (v, e) with $v \in V_n$, $e \in E_n$, and $v \in e$. We identify (v, e) with the corresponding edge $ve \in E(B)$.

Two incidences (v, e) and (u, f) are called *adjacent* if the corresponding edges ve and uf in B have *edge-distance at most* 2 in B; equivalently, if either

1. they share an endpoint in B (i.e., $v = u$ with $e \neq f$, or $e = f$ with $v \neq u$); or

2. there exists a third incidence (x, g) such that ve is adjacent to xg in B and xg is adjacent to uf in B.

An *incidence k-coloring* of $\mathrm{SHG}^{(n)}$ is a map

$$c: I\big(\mathrm{SHG}^{(n)}\big) \longrightarrow [k]$$

assigning colors to incidences so that adjacent incidences receive distinct colors.

The *incidence chromatic number* of $\mathrm{SHG}^{(n)}$ is

$$\chi_i\big(\mathrm{SHG}^{(n)}\big) := \min\big\{ k \in \mathbb{N} \mid \mathrm{SHG}^{(n)} \text{ admits an incidence } k\text{-coloring} \big\}.$$

Example 5.7.6 (Incidence coloring of an n-SuperHyperGraph)**.** Continue with $\mathrm{SHG}^{(1)} = (V_1, E_1)$ from Example 5.7.4. The incidence set is

$$I\big(\mathrm{SHG}^{(1)}\big) = \{(x, e_1), (y, e_1), (y, e_2), (z, e_2)\}.$$

In the incidence graph $B = B(\mathrm{SHG}^{(1)})$, these incidences correspond to the edges xe_1, ye_1, ye_2, ze_2.

Define $c : I(\mathrm{SHG}^{(1)}) \to [3]$ by

$$c(x, e_1) = 1, \qquad c(y, e_1) = 2, \qquad c(y, e_2) = 3, \qquad c(z, e_2) = 1.$$

We check the adjacency rule "edge-distance at most 2" in B:

- xe_1 and ye_1 share the endpoint e_1 (distance 1), and $1 \neq 2$.
- ye_1 and ye_2 share the endpoint y (distance 1), and $2 \neq 3$.
- ye_2 and ze_2 share the endpoint e_2 (distance 1), and $3 \neq 1$.
- xe_1 and ye_2 have edge-distance 2 via the intermediate edge ye_1 (since xe_1 meets ye_1 at e_1, and ye_1 meets ye_2 at y); indeed $1 \neq 3$. Similarly, ze_2 and ye_1 have edge-distance 2 via ye_2, and $1 \neq 2$.

Thus adjacent incidences receive distinct colors, so c is an incidence 3-coloring of $\mathrm{SHG}^{(1)}$.

Theorem 5.7.7 (Incidence coloring of n-SuperHyperGraphs generalizes graph incidence coloring)**.** *Let $G = (V, E)$ be a finite simple graph, viewed as the level-0 SuperHyperGraph $\mathrm{SHG}^{(0)} := (V, E)$. Then a map $c : I(G) \to [k]$ is an incidence k-coloring of G in the sense of Definition 5.7.1 if and only if it is an incidence k-coloring of $\mathrm{SHG}^{(0)}$ in the sense of Definition 5.7.5. In particular,*

$$\chi_i\big(\mathrm{SHG}^{(0)}\big) = \chi_i(G).$$

More generally, if G is embedded into an n-SuperHyperGraph via the iterated singleton embedding introduced earlier, then the corresponding incidence graphs are isomorphic and hence the incidence chromatic numbers coincide.

Proof. In the level-0 case, $\mathrm{SHG}^{(0)} = (V, E)$ has the same vertex set and edge family as G, and its incidence graph $B(\mathrm{SHG}^{(0)})$ is exactly the usual vertex–edge incidence bipartite graph $B(G)$.

Under the identification of incidences (v, e) with edges ve of $B(G)$, the adjacency rules in Definition 5.7.1 match precisely the condition that the corresponding edges of $B(G)$ have edge-distance at most 2:

- If $v = u$ and $e \neq f$, or if $e = f$ and $v \neq u$, then ve and uf share an endpoint in $B(G)$, so their edge-distance is 1.

- In the third case, $e = \{v, u\}$ and $f = \{u, w\}$ with $w \neq v$. Then in $B(G)$ the edges ve and uf are not adjacent, but they are linked by the intermediate incidence-edge ue: indeed ve is adjacent to ue (sharing e), and ue is adjacent to uf (sharing u), so the edge-distance between ve and uf is 2. Conversely, an edge-distance 2 witness in $B(G)$ forces exactly such a length-2 incidence chain, which in a simple graph corresponds to consecutive edges meeting at a vertex, i.e., the third rule.

Therefore, the two notions of incidence k-coloring coincide, and hence so do the minima χ_i.

For the stated embedding into level n, the iterated singleton map induces an isomorphism between the vertex–edge incidence bipartite graphs (hence preserves edge-distances), so the existence of incidence k-colorings and the value of χ_i are preserved. □

5.8 SuperHyperGraph List coloring

List coloring chooses each vertex's color from its prescribed list, ensuring adjacent vertices receive different colors under all list assignments [263–267]. An n-SuperHyperGraph list coloring chooses each supervertex's color from its list, ensuring primal-adjacent supervertices differ across all list assignments. Note that we discuss MultiList Coloring, an extension of list coloring of a graph, in the Appendix.

Definition 5.8.1 (List assignment and list coloring of a graph)**.** Let $G = (V, E)$ be a finite simple graph. A *list assignment* on G is a map

$$L : V \longrightarrow \mathcal{P}(\mathcal{C}) \setminus \{\emptyset\},$$

where $\mathcal{C}$ is a (finite or infinite) set of colors and $L(v)$ is the set of colors allowed at v.

An *L-coloring* of G is a vertex-coloring $\varphi : V \to \mathcal{C}$ such that

$$\varphi(v) \in L(v) \qquad (\forall v \in V),$$

and φ is *proper* (in the already-defined sense).

For $k \in \mathbb{N}$, the graph G is *k-choosable* if for every list assignment L with $|L(v)| \geq k$ for all $v \in V$, there exists an L-coloring of G. The *list chromatic number* (choosability) of G is

$$\mathrm{ch}(G) \;:=\; \min\{\, k \in \mathbb{N} : \; G \text{ is } k\text{-choosable} \,\}.$$

More generally, for a function $f : V \to \mathbb{N}$, G is *f-choosable* if for every list assignment L with $|L(v)| \geq f(v)$ for all v, there exists an L-coloring.

Example 5.8.2 (List coloring of a graph)**.** Let $G = (V, E)$ be the path P_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{v_1v_2,\ v_2v_3\}.$$

Let the color set be $\mathcal{C} = \{\text{red}, \text{blue}, \text{green}\}$ and define a list assignment $L : V \to \mathcal{P}(\mathcal{C}) \setminus \{\emptyset\}$ by

$$L(v_1) = \{\text{red}, \text{blue}\}, \qquad L(v_2) = \{\text{blue}, \text{green}\}, \qquad L(v_3) = \{\text{red}, \text{green}\}.$$

Define $\varphi : V \to \mathcal{C}$ by

$$\varphi(v_1) = \text{red}, \qquad \varphi(v_2) = \text{blue}, \qquad \varphi(v_3) = \text{green}.$$

Then $\varphi(v_i) \in L(v_i)$ for each $i \in \{1, 2, 3\}$, and φ is proper because adjacent vertices receive different colors. Hence φ is an L-coloring of G.

Definition 5.8.3 (List coloring of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph, and let $\mathrm{Pr}(\mathrm{SHG}^{(n)})$ denote its primal graph (2-section), defined earlier.

A *list assignment* on $\mathrm{SHG}^{(n)}$ is a map

$$L : V_n \longrightarrow \mathcal{P}(\mathcal{C}) \setminus \{\emptyset\}.$$

An *L-coloring* of $\mathrm{SHG}^{(n)}$ is a map $\psi : V_n \to \mathcal{C}$ such that

$$\psi(v) \in L(v) \qquad (\forall v \in V_n),$$

and ψ is a *proper* vertex-coloring of $\mathrm{Pr}(\mathrm{SHG}^{(n)})$ (equivalently: ψ is proper on the primal adjacency relation).

For $k \in \mathbb{N}$, $\mathrm{SHG}^{(n)}$ is *k-choosable* if for every list assignment L with $|L(v)| \geq k$ for all $v \in V_n$, there exists an L-coloring of $\mathrm{SHG}^{(n)}$. Define its *list chromatic number* by

$$\mathrm{ch}\big(\mathrm{SHG}^{(n)}\big) \ :=\ \mathrm{ch}\big(\mathrm{Pr}(\mathrm{SHG}^{(n)})\big).$$

More generally, for $f : V_n \to \mathbb{N}$, define f-choosability analogously.

Example 5.8.4 (List coloring of an n-SuperHyperGraph)**.** Let $n = 1$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V_1, E_1)$ with

$$V_1 = \{x, y, z\}, \qquad E_1 = \big\{\{x, y\}, \{y, z\}\big\}.$$

Its primal graph $\mathrm{Pr}(\mathrm{SHG}^{(1)})$ is the path $x - y - z$.

Let $\mathcal{C} = \{\text{red}, \text{blue}, \text{green}\}$ and define a list assignment $L : V_1 \to \mathcal{P}(\mathcal{C}) \setminus \{\emptyset\}$ by

$$L(x) = \{\text{red}, \text{blue}\}, \qquad L(y) = \{\text{blue}, \text{green}\}, \qquad L(z) = \{\text{red}, \text{green}\}.$$

Define $\psi : V_1 \to \mathcal{C}$ by

$$\psi(x) = \text{red}, \qquad \psi(y) = \text{blue}, \qquad \psi(z) = \text{green}.$$

Then $\psi(v) \in L(v)$ for all $v \in V_1$, and ψ is proper on $\mathrm{Pr}(\mathrm{SHG}^{(1)})$ since the adjacent pairs (x, y) and (y, z) receive different colors. Therefore ψ is an L-coloring of $\mathrm{SHG}^{(1)}$.

Theorem 5.8.5 (n-SuperHyperGraph list coloring generalizes graph list coloring)**.**

(i) *Let* $G = (V, E)$ *be a finite simple graph, viewed as the level-*0 *SuperHyperGraph* $\mathrm{SHG}^{(0)} := (V, E)$*. Then for every list assignment* $L : V \to \mathcal{P}(\mathcal{C}) \setminus \{\emptyset\}$*, a map* $\varphi : V \to \mathcal{C}$ *is an* L*-coloring of* G *if and only if it is an* L*-coloring of* $\mathrm{SHG}^{(0)}$ *in the sense of Definition 5.8.3. Consequently,*

$$\mathrm{ch}\big(\mathrm{SHG}^{(0)}\big) = \mathrm{ch}(G).$$

(ii) *Fix* $n \in \mathbb{N}_0$*. For any finite simple graph* $G = (V_G, E_G)$*, let* $\mathrm{SHG}_G^{(n)}$ *denote the embedded* n*-SuperHyperGraph obtained from* G *via the iterated singleton embedding introduced earlier. Then* $\mathrm{Pr}(\mathrm{SHG}_G^{(n)}) \cong G$*, and hence*

$$\mathrm{ch}\big(\mathrm{SHG}_G^{(n)}\big) = \mathrm{ch}(G).$$

Proof. (i) In the level-0 case, every superedge has size 2, hence the primal graph satisfies $\mathrm{Pr}(\mathrm{SHG}^{(0)}) = G$. Therefore, the properness constraint and the list constraint $\varphi(v) \in L(v)$ coincide for G and $\mathrm{SHG}^{(0)}$, giving the equivalence of L-colorings and the equality of list chromatic numbers.

(ii) Let $f : V_G \to V(\mathrm{SHG}_G^{(n)})$ be the bijection induced by the iterated singleton embedding. By construction, f is a graph isomorphism $G \cong \mathrm{Pr}(\mathrm{SHG}_G^{(n)})$. Given any list assignment L on V_G, transport it to a list assignment L' on $V(\mathrm{SHG}_G^{(n)})$ by $L'(f(v)) := L(v)$. Then φ is an L-coloring of G if and only if $\psi := \varphi \circ f^{-1}$ is an L'-coloring of $\mathrm{Pr}(\mathrm{SHG}_G^{(n)})$, hence of $\mathrm{SHG}_G^{(n)}$. Thus k-choosability (and therefore $\mathrm{ch}(\cdot)$) is preserved, yielding $\mathrm{ch}(\mathrm{SHG}_G^{(n)}) = \mathrm{ch}(G)$. □

5.9 Radio coloring

Radio coloring assigns positive integer labels so adjacent vertices differ by at least two and distance-two vertices differ by one [268–272]. An n-SuperHyperGraph radio coloring assigns labels to supervertices so the same distance-based separation holds in the primal graph.

Definition 5.9.1 (Radio coloring of a graph)**.** Let $G = (V, E)$ be a finite simple undirected graph, and let $d_G(\cdot, \cdot)$ denote the (already-defined) graph distance. A *radio coloring* of G is a labeling

$$\varphi : V \longrightarrow \mathbb{Z}_{\geq 1}$$

such that for all distinct $u, v \in V$ with $d_G(u, v) \leq 2$,

$$|\varphi(u) - \varphi(v)| \;\geq\; 3 - d_G(u, v).$$

Equivalently:

$$d_G(u, v) = 1 \;\Rightarrow\; |\varphi(u) - \varphi(v)| \geq 2, \qquad d_G(u, v) = 2 \;\Rightarrow\; |\varphi(u) - \varphi(v)| \geq 1.$$

The *span* of φ is

$$\mathrm{span}(\varphi) \;:=\; \max_{v \in V} \varphi(v).$$

The *radio coloring number* of G is

$$\mathrm{rc}(G) \;:=\; \min\big\{\mathrm{span}(\varphi) \;:\; \varphi \text{ is a radio coloring of } G\big\}.$$

Example 5.9.2 (Radio coloring of a graph)**.** Let $G = (V, E)$ be the path P_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{v_1v_2,\ v_2v_3\}.$$

The distances satisfy $d_G(v_1, v_2) = d_G(v_2, v_3) = 1$ and $d_G(v_1, v_3) = 2$. Define a labeling $\varphi : V \to \mathbb{Z}_{\geq 1}$ by

$$\varphi(v_1) = 1, \qquad \varphi(v_2) = 3, \qquad \varphi(v_3) = 5.$$

Then for adjacent vertices the difference is at least 2:

$$|\varphi(v_1) - \varphi(v_2)| = 2, \qquad |\varphi(v_2) - \varphi(v_3)| = 2,$$

and for the distance-2 pair we have

$$|\varphi(v_1) - \varphi(v_3)| = 4 \geq 1.$$

Hence φ is a radio coloring of G, with span $\operatorname{span}(\varphi) = 5$.

Definition 5.9.3 (Radio coloring of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph, and let $\Pr(\mathrm{SHG}^{(n)})$ be its primal graph (2-section). A *radio coloring* of $\mathrm{SHG}^{(n)}$ is a labeling

$$\psi : V_n \longrightarrow \mathbb{Z}_{\geq 1}$$

such that for all distinct $x, y \in V_n$ with $d_{\Pr(\mathrm{SHG}^{(n)})}(x, y) \leq 2$,

$$|\psi(x) - \psi(y)| \ \geq \ 3 - d_{\Pr(\mathrm{SHG}^{(n)})}(x, y).$$

Its *span* is $\operatorname{span}(\psi) := \max_{v \in V_n} \psi(v)$, and the *radio coloring number* of $\mathrm{SHG}^{(n)}$ is

$$\operatorname{rc}\bigl(\mathrm{SHG}^{(n)}\bigr) \ := \ \operatorname{rc}\bigl(\Pr(\mathrm{SHG}^{(n)})\bigr) \ = \ \min\bigl\{\operatorname{span}(\psi) : \ \psi \text{ is a radio coloring of } \mathrm{SHG}^{(n)}\bigr\}.$$

Example 5.9.4 (Radio coloring of an n-SuperHyperGraph)**.** Let $n = 1$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V_1, E_1)$ with

$$V_1 = \{x, y, z\}, \qquad E_1 = \bigl\{\{x, y\}, \{y, z\}\bigr\}.$$

Its primal graph $\Pr(\mathrm{SHG}^{(1)})$ is the path $x - y - z$, hence $d_{\Pr}(x, y) = d_{\Pr}(y, z) = 1$ and $d_{\Pr}(x, z) = 2$.

Define $\psi : V_1 \to \mathbb{Z}_{\geq 1}$ by

$$\psi(x) = 1, \qquad \psi(y) = 3, \qquad \psi(z) = 5.$$

Then the distance-based constraints are satisfied in $\Pr(\mathrm{SHG}^{(1)})$:

$$|\psi(x) - \psi(y)| = 2, \qquad |\psi(y) - \psi(z)| = 2, \qquad |\psi(x) - \psi(z)| = 4 \geq 1.$$

Therefore ψ is a radio coloring of $\mathrm{SHG}^{(1)}$, and $\operatorname{span}(\psi) = 5$.

Theorem 5.9.5 (n-SuperHyperGraph radio coloring generalizes graph radio coloring)**.**

(i) *Let $G = (V, E)$ be a finite simple graph, viewed as the level-0 SuperHyperGraph $\mathrm{SHG}^{(0)} := (V, E)$. Then a labeling $\varphi : V \to \mathbb{Z}_{\geq 1}$ is a radio coloring of G (Definition 5.9.1) if and only if it is a radio coloring of $\mathrm{SHG}^{(0)}$ (Definition 5.9.3). Consequently,*

$$\mathrm{rc}\big(\mathrm{SHG}^{(0)}\big) = \mathrm{rc}(G).$$

(ii) *Fix $n \in \mathbb{N}_0$. For any finite simple graph G, let $\mathrm{SHG}_G^{(n)}$ denote the embedded n-SuperHyperGraph obtained from G via the iterated singleton embedding introduced earlier. Then $\mathrm{Pr}(\mathrm{SHG}_G^{(n)}) \cong G$, and hence*

$$\mathrm{rc}\big(\mathrm{SHG}_G^{(n)}\big) = \mathrm{rc}(G).$$

Proof. (i) For $n = 0$, the primal graph satisfies $\mathrm{Pr}(\mathrm{SHG}^{(0)}) = G$. Therefore the distances $d_{\mathrm{Pr}(\mathrm{SHG}^{(0)})}(\cdot,\cdot)$ and $d_G(\cdot,\cdot)$ coincide, and the constraint $|\cdot| \geq 3 - d(\cdot,\cdot)$ for all pairs at distance at most 2 is identical in both definitions. Thus the sets of feasible labelings coincide, and minimizing span yields $\mathrm{rc}(\mathrm{SHG}^{(0)}) = \mathrm{rc}(G)$.

(ii) Let $f : V(G) \to V(\mathrm{SHG}_G^{(n)})$ be the bijection induced by the iterated singleton embedding. By construction, f is a graph isomorphism $G \cong \mathrm{Pr}(\mathrm{SHG}_G^{(n)})$, hence it preserves distances:

$$d_G(u, v) = d_{\mathrm{Pr}(\mathrm{SHG}_G^{(n)})}\big(f(u), f(v)\big) \qquad (\forall u, v \in V(G)).$$

Given a radio coloring φ of G, define $\psi := \varphi \circ f^{-1}$ on $V(\mathrm{SHG}_G^{(n)})$. The distance-preservation implies that ψ satisfies the same distance-≤ 2 inequalities, so ψ is a radio coloring of $\mathrm{SHG}_G^{(n)}$. Conversely, any radio coloring ψ of $\mathrm{SHG}_G^{(n)}$ pulls back to a radio coloring $\varphi := \psi \circ f$ of G. Moreover $\max \psi = \max \varphi$. Therefore the minimum achievable span is the same on both sides, giving $\mathrm{rc}(\mathrm{SHG}_G^{(n)}) = \mathrm{rc}(G)$. □

5.10 Total coloring

Total coloring assigns colors to every vertex and edge, forbidding equal colors on adjacent or incident elements, using fewest colors [232, 273–275]. n-SuperHyperGraph total coloring assigns colors to all supervertices and superedges, forbidding equal colors on adjacent or incident super-elements, minimizing colors.

Definition 5.10.1 (Total coloring of a graph)**.** Let $G = (V, E)$ be a finite simple undirected graph and let $k \in \mathbb{N}$ with $k \geq 1$. A *(proper) total k-coloring* of G is a map

$$\tau : \; V \cup E \;\longrightarrow\; [k] := \{1, 2, \ldots, k\}$$

such that:

(T1) if $u, v \in V$ and $uv \in E$, then $\tau(u) \neq \tau(v)$;

(T2) if $e, f \in E$ are adjacent edges (i.e., $e \cap f \neq \emptyset$), then $\tau(e) \neq \tau(f)$;

(T3) if $v \in V$ and $e \in E$ are incident (i.e., $v \in e$), then $\tau(v) \neq \tau(e)$.

The *total chromatic number* of G is

$$\chi''(G) \ := \ \min\{\, k \in \mathbb{N} \ : \ G \text{ admits a total } k\text{-coloring} \,\}.$$

Example 5.10.2 (Total coloring of a graph)**.** Let $G = (V, E)$ be the path P_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{e_{12} = v_1v_2,\ e_{23} = v_2v_3\}.$$

Define $\tau : V \cup E \to [3] = \{1, 2, 3\}$ by

$$\tau(v_1) = 1, \qquad \tau(v_2) = 2, \qquad \tau(v_3) = 1,$$

$$\tau(e_{12}) = 3, \qquad \tau(e_{23}) = 3.$$

Then (T1) holds because adjacent vertices receive different colors: $\tau(v_1) \neq \tau(v_2)$ and $\tau(v_2) \neq \tau(v_3)$. Condition (T2) holds because the adjacent edges e_{12} and e_{23} share the endpoint v_2, so they must have distinct colors; indeed, here they both have color 3, so we adjust by setting

$$\tau(e_{12}) = 3, \qquad \tau(e_{23}) = 1.$$

Now $\tau(e_{12}) \neq \tau(e_{23})$, so (T2) holds. Finally, (T3) holds because each edge color differs from the colors of its endpoints:

$$\tau(e_{12}) = 3 \neq 1 = \tau(v_1), \quad \tau(e_{12}) = 3 \neq 2 = \tau(v_2),$$

$$\tau(e_{23}) = 1 \neq 2 = \tau(v_2), \quad \tau(e_{23}) = 1 \neq 1 = \tau(v_3) \text{ fails},$$

so we instead set $\tau(e_{23}) = 3$ and recolor v_3 by $\tau(v_3) = 1$ would still fail; therefore choose

$$\tau(v_1) = 1, \quad \tau(v_2) = 2, \quad \tau(v_3) = 3, \qquad \tau(e_{12}) = 3, \quad \tau(e_{23}) = 1.$$

With this final assignment, (T1)–(T3) all hold, hence τ is a proper total 3-coloring of G.

Definition 5.10.3 (Total coloring of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph and let $k \in \mathbb{N}$ with $k \geq 1$. A *(proper) total k-coloring* of $\mathrm{SHG}^{(n)}$ is a map

$$\Theta : \ V_n \cup E_n \ \longrightarrow \ [k]$$

such that:

(S1) if $x, y \in V_n$ are adjacent (i.e., $\exists\, e \in E_n$ with $\{x, y\} \subseteq e$), then $\Theta(x) \neq \Theta(y)$;

(S2) if $e, f \in E_n$ are adjacent superedges (i.e., $e \cap f \neq \emptyset$), then $\Theta(e) \neq \Theta(f)$;

(S3) if $x \in V_n$ and $e \in E_n$ are incident (i.e., $x \in e$), then $\Theta(x) \neq \Theta(e)$.

The *total chromatic number* of $\mathrm{SHG}^{(n)}$ is

$$\chi''\big(\mathrm{SHG}^{(n)}\big) \ := \ \min\{\, k \in \mathbb{N} \ : \ \mathrm{SHG}^{(n)} \text{ admits a total } k\text{-coloring} \,\}.$$

Example 5.10.4 (Total coloring of an n-SuperHyperGraph)**.** Let $n = 1$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V_1, E_1)$ with

$$V_1 = \{x, y, z\}, \qquad E_1 = \{e_1, e_2\}, \quad e_1 = \{x, y\},\ e_2 = \{y, z\}.$$

(Thus the primal adjacency on V_1 is the path $x - y - z$, and the superedges e_1, e_2 are adjacent since $e_1 \cap e_2 = \{y\} \neq \emptyset$.)

Define $\Theta : V_1 \cup E_1 \to [4] = \{1, 2, 3, 4\}$ by

$$\Theta(x) = 1, \qquad \Theta(y) = 2, \qquad \Theta(z) = 3,$$
$$\Theta(e_1) = 3, \qquad \Theta(e_2) = 1.$$

Then:

- (S1) holds since adjacent supervertices must differ: $\Theta(x) \neq \Theta(y)$ and $\Theta(y) \neq \Theta(z)$.
- (S2) holds since e_1 and e_2 are adjacent superedges and $\Theta(e_1) \neq \Theta(e_2)$.
- (S3) holds because each superedge color differs from the colors of its incident supervertices:
$$\Theta(e_1) = 3 \neq \Theta(x) = 1,\ \Theta(e_1) = 3 \neq \Theta(y) = 2,$$
$$\Theta(e_2) = 1 \neq \Theta(y) = 2,\ \Theta(e_2) = 1 \neq \Theta(z) = 3.$$

Hence Θ is a proper total 4-coloring of $\mathrm{SHG}^{(1)}$.

Theorem 5.10.5 (n-SuperHyperGraph total coloring generalizes graph total coloring)**.** *Let $G = (V, E)$ be a finite simple graph, viewed as the level-0 SuperHyperGraph $\mathrm{SHG}^{(0)} := (V, E)$. Then $\tau : V \cup E \to [k]$ is a total k-coloring of G in Definition 5.10.1 if and only if it is a total k-coloring of $\mathrm{SHG}^{(0)}$ in Definition 5.10.3. Consequently,*

$$\chi''(\mathrm{SHG}^{(0)}) \;=\; \chi''(G).$$

Proof. For $n = 0$ we have $V_0 = V$ and $E_0 = E$, and each edge is a 2-element set.

- Condition (S1) says that adjacent vertices (those contained together in some $e \in E$) must receive different colors, which is exactly (T1).
- In a simple graph, two distinct edges are adjacent precisely when they share an endpoint, i.e., their intersection is nonempty. Thus (S2) is exactly (T2).
- Incidence $x \in e$ is the usual vertex–edge incidence, so (S3) is exactly (T3).

Hence the feasible total k-colorings coincide for G and $\mathrm{SHG}^{(0)}$, and minimizing k yields $\chi''(\mathrm{SHG}^{(0)}) = \chi''(G)$. □

5.11 Exact coloring

Exact coloring is a proper vertex coloring in which every unordered pair of distinct colors occurs on exactly one edge of the graph [276–279]. An n-SuperHyperGraph exact coloring is a proper coloring of supervertices such that every unordered color pair occurs exactly once among adjacent supervertices in its primal graph.

Definition 5.11.1 (Exact k-coloring of a graph)**.** Let $G = (V, E)$ be a finite simple graph and let $k \in \mathbb{N}$, $k \geq 1$. A proper vertex k-coloring $\varphi : V \to [k]$ is called an *exact k-coloring* if for every unordered pair of distinct colors $\{i, j\} \subseteq [k]$ with $i < j$, there exists *exactly one* edge whose endpoints receive colors i and j, i.e.,

$$\left|\{\, uv \in E :\ \{\varphi(u), \varphi(v)\} = \{i, j\}\,\}\right| \;=\; 1 \qquad (\forall\, 1 \leq i < j \leq k).$$

Example 5.11.2 (Exact coloring of a graph)**.** Let $G = (V, E)$ be the 3-vertex path P_3 with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{v_1v_2,\ v_2v_3\}.$$

Define $\varphi : V \to [3] = \{1, 2, 3\}$ by

$$\varphi(v_1) = 1, \qquad \varphi(v_2) = 2, \qquad \varphi(v_3) = 3.$$

Then φ is proper. Moreover, the unordered color pairs $\{1, 2\}$ and $\{2, 3\}$ appear on exactly one edge each:

$$\{1, 2\} \text{ appears only on } v_1v_2, \qquad \{2, 3\} \text{ appears only on } v_2v_3,$$

and no other edge exists. Hence φ is an exact 3-coloring of G in the sense of Definition 5.11.1.

Definition 5.11.3 (Exact k-coloring of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph. Let $\Pr(\mathrm{SHG}^{(n)})$ denote its primal graph (2-section) defined earlier, with vertex set V_n. A map $\psi : V_n \to [k]$ is called an *exact k-coloring* of $\mathrm{SHG}^{(n)}$ if it is a proper vertex coloring of $\Pr(\mathrm{SHG}^{(n)})$ and, for every $1 \leq i < j \leq k$,

$$\left|\{\, xy \in E(\Pr(\mathrm{SHG}^{(n)})) :\ \{\psi(x), \psi(y)\} = \{i, j\}\,\}\right| \;=\; 1.$$

Example 5.11.4 (Exact coloring of an n-SuperHyperGraph)**.** Let $n = 1$ and consider the 1-SuperHyperGraph $\mathrm{SHG}^{(1)} = (V_1, E_1)$ with

$$V_1 = \{X_1, X_2, X_3\}, \qquad E_1 = \{\{X_1, X_2\}, \{X_2, X_3\}\}.$$

Its primal graph $\Pr(\mathrm{SHG}^{(1)})$ is the path $X_1 - X_2 - X_3$, i.e., $\Pr(\mathrm{SHG}^{(1)}) \cong P_3$.

Define $\psi : V_1 \to [3]$ by

$$\psi(X_1) = 1, \qquad \psi(X_2) = 2, \qquad \psi(X_3) = 3.$$

Then ψ is proper on $\Pr(\mathrm{SHG}^{(1)})$, and the unordered color pairs $\{1, 2\}$ and $\{2, 3\}$ occur on exactly one adjacency each in the primal graph:

$$\{1, 2\} \text{ occurs only on } X_1X_2, \qquad \{2, 3\} \text{ occurs only on } X_2X_3.$$

Therefore ψ is an exact 3-coloring of $\mathrm{SHG}^{(1)}$ in the sense of Definition 5.11.3.

Theorem 5.11.5 (Generalization to graphs)**.** *Let $G = (V, E)$ be a finite simple graph.*

(i) *If G is viewed as the level-0 SuperHyperGraph $\mathrm{SHG}^{(0)} := (V, E)$, then a map $\varphi : V \to [k]$ is an exact k-coloring of G (Definition 5.11.1) if and only if it is an exact k-coloring of $\mathrm{SHG}^{(0)}$ (Definition 5.11.3).*

(ii) *More generally, let $\mathrm{SHG}^{(n)}_G$ be any n-SuperHyperGraph embedding of G such that $\mathrm{Pr}(\mathrm{SHG}^{(n)}_G) \cong G$. Then G has an exact k-coloring if and only if $\mathrm{SHG}^{(n)}_G$ has an exact k-coloring.*

Proof. (i) In the level-0 case, every (super)edge has size 2, hence $\mathrm{Pr}(\mathrm{SHG}^{(0)}) = G$. Therefore the set of edges counted in Definition 5.11.1 coincides with the set of edges counted in Definition 5.11.3, for each color pair $\{i, j\}$. Thus the two notions of exact k-coloring are identical.

(ii) Let $f : V(G) \to V(\mathrm{SHG}^{(n)}_G)$ be an isomorphism $G \cong \mathrm{Pr}(\mathrm{SHG}^{(n)}_G)$. Given a coloring $\varphi : V(G) \to [k]$, define $\psi := \varphi \circ f^{-1}$ on $V(\mathrm{SHG}^{(n)}_G)$. Because f preserves adjacency, φ is proper on G if and only if ψ is proper on $\mathrm{Pr}(\mathrm{SHG}^{(n)}_G)$. Moreover, for each $i < j$, the map f induces a bijection between

$$\{\, uv \in E(G) : \{\varphi(u), \varphi(v)\} = \{i, j\} \,\} \quad \text{and} \quad \{\, xy \in E(\mathrm{Pr}(\mathrm{SHG}^{(n)}_G)) : \{\psi(x), \psi(y)\} = \{i, j\} \,\},$$

so the cardinalities are equal. Hence the "equals 1" condition holds for φ if and only if it holds for ψ, proving the claim. □

5.12 Hierarchical coloring

Hierarchical coloring assigns colors across multiple levels of a hierarchical graph so intra-level conflicts and inter-level parent–child constraints are simultaneously satisfied.

Definition 5.12.1 (Layered view and descendant operators)**.** Let $\mathrm{SHG}^{(n)} = (V_n, E_n)$ be an n-SuperHyperGraph built over iterated powerset levels. Assume that the vertex set is stratified into levels

$$V_0,\ V_1,\ \ldots,\ V_n, \qquad \text{with } V_i \subseteq \mathcal{P}^*(V_{i-1}) \ \ (1 \le i \le n),$$

so that each $X \in V_i$ is a nonempty set of $(i-1)$-level objects.

For $1 \le i \le n$, define the *one-step flattening* map

$$\mathrm{Flat}_i : V_i \longrightarrow \mathcal{P}^*(V_{i-1}), \qquad \mathrm{Flat}_i(X) := X,$$

and for $0 \le j < i \le n$ define the $(i{\downarrow}j)$-*descendant set* recursively by

$$\mathrm{Flat}_{i\downarrow i}(X) := \{X\}, \qquad \mathrm{Flat}_{i\downarrow j}(X) := \bigcup_{Y \in \mathrm{Flat}_i(X)} \mathrm{Flat}_{(i-1)\downarrow j}(Y) \quad (j < i).$$

Thus, $\mathrm{Flat}_{i\downarrow j}(X) \subseteq V_j$ is the set of all level-j objects contained in X by iterated membership.

Example 5.12.2 (Layered structure and descendant sets)**.** Let $n = 2$ and start from the base level

$$V_0 = \{a, b, c\}.$$

Define the level-1 supervertices

$$V_1 = \{A, B, C\} \quad \text{with} \quad A = \{a, b\},\ B = \{b, c\},\ C = \{a, c\}.$$

Define the level-2 supervertices

$$V_2 = \{X, Y\} \quad \text{with} \quad X = \{A, B\}, \qquad Y = \{B, C\}.$$

Then $\mathrm{Flat}_2(X) = X = \{A, B\} \subseteq V_1$ and $\mathrm{Flat}_2(Y) = Y = \{B, C\} \subseteq V_1$. Moreover, the descendant sets satisfy

$$\mathrm{Flat}_{2\downarrow 1}(X) = \{A, B\}, \qquad \mathrm{Flat}_{2\downarrow 0}(X) = \mathrm{Flat}_{1\downarrow 0}(A) \cup \mathrm{Flat}_{1\downarrow 0}(B) = \{a, b\} \cup \{b, c\} = \{a, b, c\},$$

and similarly

$$\mathrm{Flat}_{2\downarrow 1}(Y) = \{B, C\}, \qquad \mathrm{Flat}_{2\downarrow 0}(Y) = \{b, c\} \cup \{a, c\} = \{a, b, c\}.$$

Hence $\mathrm{Flat}_{i\downarrow j}$ correctly collects all level-j objects contained in a higher-level supervertex by iterated membership.

Definition 5.12.3 (Hierarchical c-coloring of an n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)}$ be as in Definition 5.12.1, and fix $c \in \mathbb{N}$ with color set $\mathcal{C} = [c] = \{1, 2, \ldots, c\}$.

A *hierarchical c-coloring* of $\mathrm{SHG}^{(n)}$ is a family of maps

$$\Psi = (\psi_0, \psi_1, \ldots, \psi_n), \qquad \psi_i : V_i \longrightarrow \mathcal{C},$$

satisfying the following *intra-level* and *inter-level* constraints:

(H1) Intra-level non-monochromaticity. For each level $i \in \{0, 1, \ldots, n\}$ and every hyperedge $e \in E_i$ (where $E_0 = E(G)$ in the base graph case),

$$\big|\{\psi_i(x) : x \in e\}\big| \ \geq\ 2.$$

(Equivalently, no edge/superedge at level i is monochromatic.)

(H2) Parent–child separation across levels. For each $i \in \{1, 2, \ldots, n\}$, each $X \in V_i$, and each $Y \in \mathrm{Flat}_i(X) \subseteq V_{i-1}$,

$$\psi_i(X) \ \neq\ \psi_{i-1}(Y).$$

(H3) Cross-level projection constraint (optional but often useful). For each $i \in \{1, 2, \ldots, n\}$ and each $e \in E_i$, the projected descendant set one level down,

$$\mathrm{Proj}_{i-1}(e) \ :=\ \bigcup_{X \in e} \mathrm{Flat}_i(X) \ \subseteq\ V_{i-1},$$

is not monochromatic under ψ_{i-1}, i.e.,

$$\big|\{\psi_{i-1}(Y) : Y \in \mathrm{Proj}_{i-1}(e)\}\big| \ \geq\ 2.$$

(When included, this enforces that a non-monochromatic constraint at level i also induces a non-monochromatic constraint on the immediate constituents.)

The least c for which a hierarchical c-coloring exists (with (H1)+(H2), and optionally (H3)) is called the *hierarchical chromatic number* and is denoted by $\chi_{\mathrm{hier}}(\mathrm{SHG}^{(n)})$.

Remark 5.12.4. If $n = 0$, then there is only one level V_0, the inter-level constraint (H2) is void, and (H1) reduces to the usual (graph) proper coloring condition because every edge has size 2. Thus hierarchical coloring extends standard graph coloring by allowing a single coherent coloring framework across all powerset levels.

Example 5.12.5 (A hierarchical coloring)**.** Continue with the layered sets from Example 5.12.2 and define a 2-SuperHyperGraph $\mathrm{SHG}^{(2)}$ whose level-2 superedge set is

$$E_2 = \{\{X, Y\}\}.$$

(Thus, at level 2 we require X and Y not to be monochromatic.)

Let the color set be $\mathcal{C} = [3] = \{1, 2, 3\}$. Define a family of maps $\Psi = (\psi_0, \psi_1, \psi_2)$ by

$$\psi_0(a) = 1, \quad \psi_0(b) = 2, \quad \psi_0(c) = 3,$$

$$\psi_1(A) = 3, \quad \psi_1(B) = 1, \quad \psi_1(C) = 2,$$

$$\psi_2(X) = 2, \quad \psi_2(Y) = 3.$$

We verify the hierarchical constraints (H1)+(H2) (and also (H3) for this example).

- **(H2) Parent–child separation:** For $A = \{a, b\}$ we have $\psi_1(A) = 3 \neq 1 = \psi_0(a)$ and $\psi_1(A) = 3 \neq 2 = \psi_0(b)$; similarly for $B = \{b, c\}$ and $C = \{a, c\}$. For $X = \{A, B\}$ we have $\psi_2(X) = 2 \neq 3 = \psi_1(A)$ and $\psi_2(X) = 2 \neq 1 = \psi_1(B)$; for $Y = \{B, C\}$ we have $\psi_2(Y) = 3 \neq 1 = \psi_1(B)$ and $\psi_2(Y) = 3 \neq 2 = \psi_1(C)$.

- **(H1) Intra-level non-monochromaticity (at level 2):** The unique level-2 superedge $\{X, Y\}$ uses colors $\{2, 3\}$, hence is not monochromatic.

- **(H3) Cross-level projection (optional):** Here $\mathrm{Proj}_1(\{X, Y\}) = \mathrm{Flat}_2(X) \cup \mathrm{Flat}_2(Y) = \{A, B\} \cup \{B, C\} = \{A, B, C\}$, and $\{\psi_1(A), \psi_1(B), \psi_1(C)\} = \{3, 1, 2\}$ has at least two colors.

Therefore Ψ is a hierarchical 3-coloring of $\mathrm{SHG}^{(2)}$ in the sense of Definition 5.12.3.

Chapter 6

Conclusion

In this book, we defined and investigated several graph classes of *Intersection SuperHyperGraphs* and SuperHyperGraph Labeling, SuperHyperGraph Coloring. We hope that future work will further develop their mathematical properties and advance practical studies, including those supported by computational experiments. Moreover, we expect that future work will further develop extensions based on HyperFuzzy sets [150, 151, 280], HyperNeutrosophic sets [281–284], HyperSoft sets [179, 285, 286], rough sets [287, 288], and Plithogenic sets [180, 181, 289].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

Appendix A

Appendix: Complex and n-dimensional intersection graph

A *complex intersection graph* is an intersection graph equipped with a fixed complex-valued evaluation of each represented intersection, encoding magnitude and phase information. An *n-dimensional intersection graph* is an intersection graph realized by nonempty subsets of $\mathbb{R}^n$, where adjacency holds exactly when the subsets intersect.

Definition A.0.1 (Complex intersection graph)**.** Let U be a nonempty set (the universe). Fix a *complex valuation*

$$\psi : U \longrightarrow \mathbb{C}.$$

A *complex intersection graph* (with respect to ψ) is a complex-weighted simple graph

$$G_\psi = (V, E, w), \qquad w : E \to \mathbb{C},$$

for which there exists a family of nonempty sets $\mathcal{F} = \{S_v\}_{v \in V}$ with $S_v \subseteq U$ such that, for all distinct $u, v \in V$,

$$\{u, v\} \in E \iff S_u \cap S_v \neq \emptyset,$$

and the edge-weight is induced from the intersection by

$$w(\{u, v\}) \;:=\; \sum_{x \in S_u \cap S_v} \psi(x) \;\in\; \mathbb{C}.$$

(If U is infinite, one may replace the finite sum by a complex measure/integral; in this book we state the finite form for simplicity.) We call $(U, \psi, \{S_v\}_{v \in V})$ a *complex intersection representation* of G_ψ.

The *underlying (crisp) graph* of G_ψ is the ordinary intersection graph of $\mathcal{F}$ obtained by forgetting w.

Remark A.0.2 (How the imaginary part is "introduced")**.** The imaginary component enters through ψ and hence through $w(\{u, v\})$. For instance, if $\psi = \mathbf{1}$ then $w(\{u, v\}) = |S_u \cap S_v| \in \mathbb{Z}_{\geq 0}$ and one recovers the usual intersection graph together with real intersection-size weights. If ψ takes nonreal values, then w records both magnitude and phase information about intersections.

Example A.0.3 (A complex intersection graph)**.** Let

$$U := \{a, b, c, d\}, \qquad \psi(a) = 1,\ \psi(b) = i,\ \psi(c) = 2,\ \psi(d) = 1 - i.$$

Let $V := \{v_1, v_2, v_3\}$ and define the set-family $\mathcal{F} = \{S_{v_1}, S_{v_2}, S_{v_3}\}$ by

$$S_{v_1} := \{a, b\}, \qquad S_{v_2} := \{b, c\}, \qquad S_{v_3} := \{c, d\}.$$

Then

$$S_{v_1} \cap S_{v_2} = \{b\} \neq \emptyset, \qquad S_{v_2} \cap S_{v_3} = \{c\} \neq \emptyset, \qquad S_{v_1} \cap S_{v_3} = \emptyset.$$

Hence the underlying (crisp) intersection graph is the path $v_1 - v_2 - v_3$, i.e.,

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}\}.$$

By Definition A.0.1, the complex edge-weights are

$$w(\{v_1, v_2\}) = \sum_{x \in S_{v_1} \cap S_{v_2}} \psi(x) = \psi(b) = i, \qquad w(\{v_2, v_3\}) = \sum_{x \in S_{v_2} \cap S_{v_3}} \psi(x) = \psi(c) = 2.$$

Therefore $G_\psi = (V, E, w)$ is a complex intersection graph (with respect to ψ), and $(U, \psi, \{S_v\}_{v \in V})$ is a complex intersection representation.

Definition A.0.4 (n-dimensional intersection graph)**.** Fix an integer $n \geq 1$. A finite simple graph $G = (V, E)$ is called an *n-dimensional intersection graph* if there exists a family of nonempty sets $\mathcal{F} = \{S_v\}_{v \in V}$ with

$$S_v \subseteq \mathbb{R}^n \qquad (v \in V)$$

such that, for all distinct $u, v \in V$,

$$\{u, v\} \in E \iff S_u \cap S_v \neq \emptyset.$$

In this case, $\{S_v\}_{v \in V}$ is called an *n-dimensional intersection representation* of G.

Remark A.0.5 (Optional geometric restrictions)**.** One often studies subclasses by restricting S_v (e.g. convex sets, balls, boxes, segments). Definition A.0.4 imposes no restriction beyond $S_v \subseteq \mathbb{R}^n$ and nonemptiness.

Example A.0.6 (A 2-dimensional intersection graph)**.** Let $n = 2$ and consider the graph $G = (V, E)$ with

$$V := \{v_1, v_2, v_3\}, \qquad E := \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\},$$

i.e., $G \cong K_3$. Define subsets of $\mathbb{R}^2$ by

$$S_{v_1} := \{(0, 0)\}, \qquad S_{v_2} := \{(0, 0), (1, 0)\}, \qquad S_{v_3} := \{(0, 0), (0, 1)\}.$$

Then every pairwise intersection is nonempty:

$$S_{v_1} \cap S_{v_2} = \{(0, 0)\}, \qquad S_{v_1} \cap S_{v_3} = \{(0, 0)\}, \qquad S_{v_2} \cap S_{v_3} = \{(0, 0)\}.$$

Thus, for all distinct $u, v \in V$, we have $S_u \cap S_v \neq \emptyset$, and hence

$$\{u, v\} \in E \iff S_u \cap S_v \neq \emptyset.$$

Therefore $\{S_v\}_{v \in V}$ is a 2-dimensional intersection representation of G in the sense of Definition A.0.4, and G is a 2-dimensional intersection graph.

Moreover, by extending the notion of a Complex Intersection Graph, one can define a Quaternion intersection graph. We present the definition below.

Definition A.0.7 (Quaternion intersection graph)**.** Let U be a nonempty set (the universe). Fix a *quaternion valuation*

$$\psi : U \longrightarrow \mathbb{H},$$

where $\mathbb{H}$ denotes the (real) quaternion algebra. A *quaternion intersection graph* (with respect to ψ) is a quaternion-weighted simple graph

$$G_\psi = (V, E, w), \qquad w : E \to \mathbb{H},$$

for which there exists a family of nonempty sets $\mathcal{F} = \{S_v\}_{v \in V}$ with $S_v \subseteq U$ such that, for all distinct $u, v \in V$,

$$\{u, v\} \in E \quad \Longleftrightarrow \quad S_u \cap S_v \neq \emptyset,$$

and the edge-weight is induced from the intersection by

$$w(\{u, v\}) \;:=\; \sum_{x \in S_u \cap S_v} \psi(x) \;\in\; \mathbb{H}.$$

(If U is infinite, one may replace the finite sum by a quaternion-valued measure/integral; we state the finite form for simplicity.) We call $(U, \psi, \{S_v\}_{v \in V})$ a *quaternion intersection representation* of G_ψ.

The *underlying (crisp) graph* of G_ψ is the ordinary intersection graph of $\mathcal{F}$ obtained by forgetting w.

Remark A.0.8 (How quaternionic information enters)**.** The "non-real" information is introduced through ψ and hence through the weights $w(\{u, v\})$. Note that quaternion *addition* is commutative, so the finite sum over $S_u \cap S_v$ is unambiguous (independent of any ordering of the intersection elements), even though quaternion multiplication is noncommutative.

Example A.0.9 (A quaternion intersection graph)**.** Let

$$U := \{a, b, c\},$$

and fix the valuation $\psi : U \to \mathbb{H}$ by

$$\psi(a) = 1 + \mathbf{i}, \qquad \psi(b) = \mathbf{j}, \qquad \psi(c) = 2\mathbf{k},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Let $V := \{v_1, v_2, v_3\}$ and define the set-family $\mathcal{F} = \{S_{v_1}, S_{v_2}, S_{v_3}\}$ by

$$S_{v_1} := \{a, b\}, \qquad S_{v_2} := \{b, c\}, \qquad S_{v_3} := \{a, c\}.$$

Then all pairwise intersections are nonempty:

$$S_{v_1} \cap S_{v_2} = \{b\}, \qquad S_{v_1} \cap S_{v_3} = \{a\}, \qquad S_{v_2} \cap S_{v_3} = \{c\}.$$

Hence the underlying (crisp) graph is the triangle K_3 on $\{v_1, v_2, v_3\}$, i.e.,

$$E = \big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\big\}.$$

By Definition A.0.7, the quaternion edge-weights are

$$w(\{v_1, v_2\}) = \sum_{x \in \{b\}} \psi(x) = \psi(b) = \mathbf{j}, \qquad w(\{v_1, v_3\}) = \sum_{x \in \{a\}} \psi(x) = \psi(a) = 1 + \mathbf{i},$$

$$w(\{v_2, v_3\}) = \sum_{x \in \{c\}} \psi(x) = \psi(c) = 2\mathbf{k}.$$

Therefore $G_\psi = (V, E, w)$ is a quaternion intersection graph with a concrete quaternion intersection representation $(U, \psi, \{S_v\}_{v \in V})$.

Appendix B

Appendix: Multilist coloring of a graph

Multilist coloring selects, at each vertex, one permissible color-list from several options, then properly colors using chosen lists.

Definition B.0.1 (MultiList assignment and Multilist coloring of a graph)**.** Let $G = (V, E)$ be a finite simple graph, and let $\mathcal{C}$ be a (finite or infinite) set of colors.

A *MultiList assignment* on G is a map

$$\mathcal{L} : V \longrightarrow \mathcal{P}\big(\mathcal{P}(\mathcal{C}) \setminus \{\emptyset\}\big) \setminus \{\emptyset\},$$

such that for each vertex $v \in V$, the value $\mathcal{L}(v)$ is a nonempty family of nonempty color-sets. Each set $S \in \mathcal{L}(v)$ is interpreted as a *feasible list* that may be chosen at v.

A *vertex-choice function* for $\mathcal{L}$ is a map

$$\Theta : V \longrightarrow \mathcal{P}(\mathcal{C}) \setminus \{\emptyset\} \qquad \text{such that} \qquad \Theta(v) \in \mathcal{L}(v) \ \ (\forall v \in V).$$

Given such a choice function Θ, an $(\mathcal{L}, \Theta)$*-coloring* of G is a proper vertex-coloring $\varphi : V \to \mathcal{C}$ satisfying

$$\varphi(v) \in \Theta(v) \qquad (\forall v \in V).$$

A *Multilist coloring* of G with respect to $\mathcal{L}$ is a proper coloring $\varphi : V \to \mathcal{C}$ for which there exists at least one vertex-choice function Θ such that φ is an $(\mathcal{L}, \Theta)$-coloring. Equivalently, φ is a Multilist coloring for $\mathcal{L}$ iff

$$\forall v \in V \ \ \exists S_v \in \mathcal{L}(v) \ \text{ with } \ \varphi(v) \in S_v.$$

For $k \in \mathbb{N}$, we say that G is *k-multi-choosable* if for every MultiList assignment $\mathcal{L}$ satisfying

$$\min\{|S| : \ S \in \mathcal{L}(v)\} \ \geq \ k \qquad (\forall v \in V),$$

there exists a Multilist coloring of G with respect to $\mathcal{L}$. The *multilist chromatic number* of G is

$$\mathrm{ch}_M(G) \;:=\; \min\{\, k \in \mathbb{N} : \; G \text{ is } k\text{-multi-choosable}\,\}.$$

More generally, for a function $f : V \to \mathbb{N}$, the graph G is *f-multi-choosable* if for every MultiList assignment $\mathcal{L}$ with

$$\min\{|S| : \; S \in \mathcal{L}(v)\} \;\geq\; f(v) \qquad (\forall v \in V),$$

there exists a Multilist coloring of G with respect to $\mathcal{L}$.

Theorem B.0.2 (Multilist coloring generalizes list coloring)**.** *Let $G = (V, E)$ be a finite simple graph.*

(i) Every list assignment $L : V \to \mathcal{P}(\mathcal{C}) \setminus \{\emptyset\}$ induces a MultiList assignment $\mathcal{L}_L$ defined by

$$\mathcal{L}_L(v) \;:=\; \{\, L(v)\,\} \qquad (\forall v \in V),$$

and the L-colorings of G are exactly the Multilist colorings with respect to $\mathcal{L}_L$.

(ii) For every $k \in \mathbb{N}$, if G is k-multi-choosable, then G is k-choosable. In particular, $\mathrm{ch}(G) \leq \mathrm{ch}_M(G)$.

Proof. (i) Define $\mathcal{L}_L(v) = \{L(v)\}$. A coloring $\varphi : V \to \mathcal{C}$ is a Multilist coloring for $\mathcal{L}_L$ iff for each v there exists $S_v \in \mathcal{L}_L(v)$ with $\varphi(v) \in S_v$. But $\mathcal{L}_L(v)$ has the unique element $L(v)$, hence this condition is equivalent to $\varphi(v) \in L(v)$ for all v, i.e. φ is an L-coloring. Properness is the same in both notions, so the sets of colorings coincide.

(ii) Let G be k-multi-choosable and let L be any list assignment with $|L(v)| \geq k$ for all v. Consider the induced MultiList assignment $\mathcal{L}_L(v) = \{L(v)\}$. Then

$$\min\{|S| : \; S \in \mathcal{L}_L(v)\} = |L(v)| \geq k \qquad (\forall v \in V).$$

By k-multi-choosability, G has a Multilist coloring for $\mathcal{L}_L$, which is an L-coloring by (i). Therefore G is k-choosable. The inequality $\mathrm{ch}(G) \leq \mathrm{ch}_M(G)$ follows by taking minima over k. $\square$

Appendix (List of Tables)

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Hypergraphs generalize ordinary graphs by allowing an edge to join an arbitrary nonempty subset of the vertex set. Iterating the powerset construction further yields nested, higher-order vertex objects and leads to finite SuperHyperGraphs, in which both vertices and edges may themselves be set-valued across multiple layers. Despite their expressive power, systematic investigations of SuperHyperGraph properties and parameters remain comparatively limited. In this book, we introduce and study several graph classes of Intersection SuperHyperGraphs, providing a unified framework for intersection-based constructions at higher levels. We also develop and analyze derived notions of SuperHyperGraph labeling, extending classical labeling paradigms to the superhypergraph setting.

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