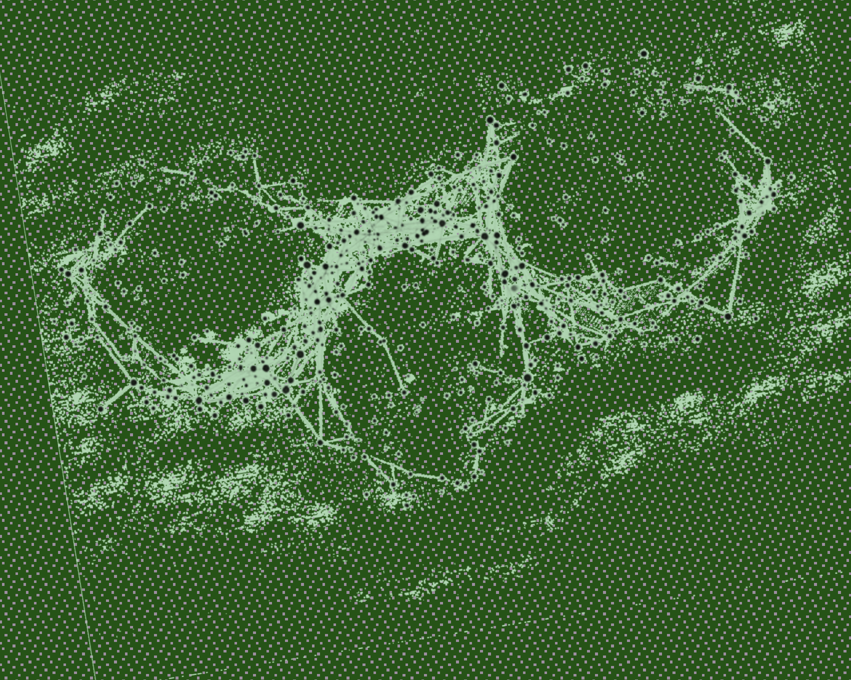
TAKAAKI FUJITA
FLORENTIN SMARANDACHE

HYPERGRAPH AND SUPERHYPERGRAPH THEORY WITH APPLICATIONS IV

UNCERTAIN GRAPH THEORY



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IV

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HyperGraph and SuperHyperGraph Theory with Applications (IV): Uncertain Graph Theory

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Abstract

Graph theory studies networks of vertices and edges and their associated structural and algorithmic properties [1]. To model real-world settings in which relationships are imprecise, a *fuzzy graph* enriches a graph by assigning to each vertex and edge a membership degree in $[0, 1]$. Building on this idea, *neutrosophic* and *quadripartitioned neutrosophic* graphs incorporate multiple components to represent truth, indeterminacy, and falsity (and their refinements), thereby providing greater expressive power than the fuzzy model. *Plithogenic graphs* further broaden this landscape by offering a flexible framework for managing uncertainty through attribute values and degrees of contradiction. Beyond ordinary graphs, a *hypergraph* allows each edge to connect an arbitrary nonempty subset of the vertex set. Iterating the powerset construction yields nested higher-order vertex objects and leads to finite *SuperHyperGraphs*, whose vertices and edges may themselves be set-valued across multiple layers. In this book, we examine the relationships among a wide range of graph, hypergraph, and superhypergraph classes including plithogenic models and we discuss additional related structures within this ecosystem. The present volume is a sequel to [2]. It is also a substantially revised and expanded version of [3]; accordingly, some overlap with [3] should be expected.

Keywords: Neutrosophic graph, Plithogenic graphs, Quadripartitioned Neutrosophic graph, Fuzzy graph, Uncertain Graph

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Chapter 1

Introduction

1.1 Graph Theory

Graph theory is a central branch of mathematics devoted to the study of networks composed of vertices and edges, with particular attention to paths, recurring structural patterns, and fundamental invariants [1]. Over many decades it has developed into a mature discipline and has supported a wide range of applications across numerous fields [4–6]. In particular, in recent years it has played a major role in AI, notably through graph neural networks and related learning paradigms (e.g. [7–11]).

Within graph theory, many families of graphs, structural notions, and algorithmic methodologies have been explored. Representative directions include investigations focused on tree-like structure [12,13], path-based structure [14], and models related to linear layouts [15,16]. These lines of research are typically driven by concrete objectives. A recurring theme is that restricting attention to well-behaved graph classes, rather than arbitrary graphs, often makes it possible to design substantially faster algorithms, highlighting a practical advantage of class-based analysis [17].

1.2 HyperGraph and SuperHyperGraph

Classical graphs can be inadequate for describing complex networks in which three or more entities interact simultaneously. Hypergraphs overcome this limitation by allowing each hyperedge to connect an arbitrary nonempty subset of vertices, thereby capturing higher-order interactions [8]. Despite their expressive power, hypergraphs may still be insufficient for modeling layered, nested, and intrinsically hierarchical relationships that occur in many real-world systems. To address this gap, F. Smarandache introduced the notion of a *SuperHyperGraph* [18,19]. A SuperHyperGraph uses iterative powerset-based constructions to encode nested connectivity patterns and multilevel relations [18,20,21], and it has received substantial attention in recent years [22–27].

Table 1.1 highlights the main distinctions among graphs, hypergraphs, and superhypergraphs. Throughout this book, n denotes a natural number unless stated otherwise (cf. [2]). For further details on SuperHyperGraphs, please refer, as needed, to the literature such as [2].

Table 1.1: Salient differences among graphs, hypergraphs, and superhypergraphs.

Concept	*Notation*	*Edge Type*	*Extension Mechanism*
Graph [1]	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$	Standard edges encode relations between exactly two vertices.
Hypergraph [28]	$H = (V, E)$	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Hyperedges may join any nonempty subset of vertices.
Superhypergraph [18]	$\mathrm{SHG}^{(n)} = (V_0, V, E)$	$V \subseteq \mathcal{P}^n(V_0),\ E \subseteq \mathcal{P}(V)$	An n-fold powerset construction is used to capture nested structure.

Notation. $\mathcal{P}(X) = \{A \subseteq X\}$ and $\mathcal{P}^0(X) = X,\ \mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

1.3 Fuzzy, Neutrosophic, Quadripartitioned Neutrosophic, and Plithogenic Graphs

Many real-world systems involve uncertainty, both in numerical parameters and in the relationships among concepts. Motivated by this, several uncertainty-aware graph formalisms have been introduced and actively studied, including fuzzy graphs, neutrosophic graphs, quadripartitioned neutrosophic graphs, and plithogenic graphs.

A *fuzzy graph* assigns to each vertex and each edge a membership degree in $[0, 1]$, expressing the extent to which that object belongs to the modeled structure [29,30]. Equivalently, a fuzzy graph may be viewed as a graph-theoretic representation of a fuzzy set(cf. [31, 32]). In applications, fuzzy graphs have been used to model imprecise or uncertain relations in settings such as social networks, decision-making, and transportation systems [29, 30]. Their broad applicability has led to sustained research activity.

Within fuzzy graph theory, many refinements and extensions have been proposed, either to enlarge the original framework or to better align the model with application requirements. Typical examples include Intuitionistic Fuzzy Graphs [33], Bipolar Fuzzy Graphs [34], Fuzzy Planar Graphs [35], Irregular Bipolar Fuzzy Graphs [36], General Fuzzy Graphs [37, 38], and Complex Hesitant Fuzzy Graphs [39]. Studying such classes is helpful for identifying shared structural features, developing tailored algorithms, and transferring theoretical results to concrete problem settings.

More generally, a large collection of graph models has been developed to represent uncertainty and subtle conceptual relationships. These include, among others, fuzzy graphs [29, 30], vague graphs [40–42], plithogenic graphs [43–46], probabilistic graphs [47–49], vague hypergraphs [50],, N-graphs [51], N-hypergraphs [52], Markov graphs [53], soft graphs (soft sets) [54,55], hypersoft graphs [56,57], and rough graphs (rough sets) [58,59]. Among these frameworks, the present book focuses primarily on neutrosophic graphs and quadripartitioned neutrosophic graphs [60–62], each of which has been developed with distinct motivations.

In recent years, neutrosophic graphs [60,63] and neutrosophic hypergraphs [64,65] have attracted growing attention within neutrosophic set theory [66, 67]. The term "neutrosophic" refers to an extension of classical and fuzzy logic in which truth, indeterminacy, and falsity are modeled as separate components. As a generalization of fuzzy graphs [29,30], neutrosophic graphs have been studied actively due to their flexibility and their wide range of potential applications, paralleling the appeal of fuzzy graphs. A variety of related neutrosophic graph and hypergraph classes has also been introduced, including Bipolar Neutrosophic Graphs [65,68–71], Neutrosophic Incidence Graphs [72–75], single-valued neutrosophic signed graphs [76], Strong Neutrosophic Graphs [77], m-polar neutrosophic graphs [78–80], Complex Neutrosophic Hypergraphs [64], and Bipolar Neutrosophic Hypergraphs [65].

Plithogenic graphs extend uncertainty-aware graph frameworks by describing each vertex and edge through attribute values together with the corresponding degrees of appurtenance, and by incorporating a *contradiction* (or dissimilarity) function that quantifies incompatibility between distinct attribute values [3, 81–83]. This can be regarded as a graph-theoretic counterpart of the notion of a *Plithogenic Set* [43, 84, 85]. This additional layer supports context-dependent aggregation of heterogeneous and potentially conflicting evaluations on networks, thereby refining classical fuzzy-, intuitionistic fuzzy-, and neutrosophic-graph paradigms (e.g. [3, 86–89]). For convenience, Table 1.2 summarizes, in a unified notation, the canonical information assigned to vertices and edges in several representative graph extensions.

Table 1.2: Representative graph extensions and the canonical information stored on vertices and/or edges.

Graph Type	**Canonical data attached to vertices/edges**
Fuzzy Graph	Vertex membership $\sigma : V \to [0, 1]$ and edge membership $\mu : E \to [0, 1]$ (typically with $\mu(uv) \le \sigma(u) \wedge \sigma(v)$).
Intuitionistic Fuzzy Graph	Vertex degrees $(\mu_A, \nu_A) : V \to [0, 1]^2$ and edge degrees $(\mu_B, \nu_B) : E \to [0, 1]^2$ with $\mu + \nu \le 1$; the residual represents hesitation.
Neutrosophic Graph	Vertex triple $(T_A, I_A, F_A) : V \to [0, 1]^3$ and edge triple $(T_B, I_B, F_B) : E \to [0, 1]^3$ (truth, indeterminacy, falsity).
Quadripartitioned Neutrosophic Graph	Vertex quadruple $(T, C, U, F) : V \to [0, 1]^4$ and edge quadruple $(T, C, U, F) : E \to [0, 1]^4$, typically encoding truth, contradiction, unknown, and falsity.
Pentapartitioned Neutrosophic Graph	Vertex quintuple $(T, C, U, F, S) : V \to [0, 1]^5$ and edge quintuple $(T, C, U, F, S) : E \to [0, 1]^5$, i.e., a five-component refinement of neutrosophic information.
Plithogenic Graph	Vertex structure $PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf})$ and edge structure $PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf})$, where $\mathrm{adf} : M \times M_\ell \to [0, 1]^s$ and $\mathrm{bdf} : N \times N_m \to [0, 1]^s$ encode s-dimensional appurtenance, while aCf and bCf are symmetric contradiction maps in $[0, 1]^t$.

Concepts such as Uncertain Graphs and Functorial Graphs have also been studied in recent years as unified frameworks for treating these notions, including Plithogenic Graphs, in an integrated manner. Given the breadth and rapid growth of the literature in fuzzy mathematics, it is unsurprising that closely related notions may be introduced independently in different venues and at different times. Nevertheless, unifying overlapping concepts is important and, in our view, will materially support further progress in the area. Moreover, both for theoretical work and for applications, it is valuable to compare a wide range of uncertainty-aware graph classes in order to choose the most appropriate framework for a given problem.

1.4 Our Contribution

In view of the above, a systematic study of graph models designed to handle uncertainty is highly relevant. Research on quadripartitioned neutrosophic graphs remains at an early stage and is far less widely known than the corresponding developments for fuzzy graphs, intuitionistic fuzzy graphs, and neutrosophic graphs. In this book, we introduce and study new graph classes aligned with intuitionistic fuzzy graphs and quadripartitioned neutrosophic graphs, namely *General Intuitionistic Fuzzy Graphs*, *General Quadripartitioned Neutrosophic Graphs*, and *Quadripartitioned Neutrosophic Hypergraphs*. We also consider *Pentapartitioned Neutrosophic Graphs*, formulated to handle five uncertainty parameters. Finally, we examine whether most of the graph models mentioned above can be realized as special cases within the broader framework of plithogenic graphs.

Our main conclusion is summarized in the theorem below. In addition, we analyze the inclusion relationships among the relevant graph classes. Moreover, without loss of generality, the same results hold in the settings of hypergraphs and superhypergraphs as well.

Theorem 1.4.1. *Within the graph classes under consideration, the following statements hold.*

- *An empty graph and a null graph can be represented as* 2*-valued graphs and* 3*-valued graphs, respectively.*
- *Every edge-fuzzy graph can be converted into a* 2*-valued graph by thresholding edge membership values.*
- *Every fuzzy graph can be converted into a* 3*-valued graph by mapping vertex and edge membership values to* $\{-1, 0, 1\}$.
- *Every Intuitionistic Fuzzy Graph can be reduced to a Fuzzy Graph by setting the non-membership function* v_A *to* 0 *for all vertices.*
- *Every Neutrosophic Graph can be reduced to an Intuitionistic Fuzzy Graph by setting the indeterminacy component to* 0*.*
- *Every Pentapartitioned Neutrosophic Graph is a generalization of the Quadripartitioned Neutrosophic Graph.*
- *Plithogenic graphs generalize fuzzy graphs, intuitionistic fuzzy graphs, neutrosophic graphs, quadripartitioned neutrosophic graphs, and extended pentapartitioned neutrosophic graphs.*
- *Every general plithogenic graph can be transformed into a General Quadripartitioned Neutrosophic Graph, a General Fuzzy Graph, a General Intuitionistic Fuzzy Graph, a Four-Valued Fuzzy Graph, an Ambiguous Graph, a Picture Fuzzy Graph, a Hesitant Fuzzy Graph, an Intuitionistic Hesitant Fuzzy Graph, a Fuzzy Graph, an Intuitionistic Fuzzy Graph, a Neutrosophic Graph, a Quadripartitioned Neutrosophic Graph, a Pentapartitioned Neutrosophic Graph, a Quadripartitioned Neutrosophic Graph, an Extended Pentapartitioned Neutrosophic Graph, and a Spherical Fuzzy Graph.*

- *Every Uncertain Graph can be transformed into a Plithogenic graph.*

Furthermore, to clarify the relationships among these graph classes, we prove several additional theorems for individual classes as byproducts and also investigate illustrative application examples.

In addition to being a sequel to [2] and a substantially revised and expanded version of [3], this volume differs from [3] mainly in that it incorporates an extended discussion of SuperHyperGraphs and replaces the discussion of Turiyam Neutrosophic Graphs with a discussion of Quadripartitioned Neutrosophic Graphs. We have also improved the proofs and the stated definitions as much as possible.

Chapter 2

Preliminaries

This chapter summarizes the basic definitions, notation, and conventions used throughout the book. In particular, we review core notions for graphs and hypergraphs, together with several uncertainty-aware generalizations, including fuzzy graphs, intuitionistic fuzzy graphs, neutrosophic graphs (and hypergraphs), Quadripartitioned neutrosophic graphs (and hypergraphs), plithogenic graphs, and single-valued neutrosophic graphs. We also record several standard properties associated with these frameworks and, when appropriate, briefly mention representative application contexts. Since our discussion occasionally invokes elementary ideas from set theory in addition to graph theory, the reader may consult standard references on set theory as needed [90].

2.1 Empty Graph and Null graph

We now recall the notions of empty and null graphs.

Definition 2.1.1 (Empty graph and null graph)**.** (cf. [91, 92]) An *empty graph* is a graph $G = (V, E)$ with $E = \emptyset$, i.e., it has vertices but no edges. A *null graph* is the graph $G = (V, E)$ with $V = \emptyset$ and $E = \emptyset$, i.e., it has neither vertices nor edges.

We also record two basic operations that are frequently used in graph algorithms (e.g. [93–95]): vertex addition and edge addition.

Definition 2.1.2 (Graph vertex addition)**.** Let $G = (V, E)$ be a graph. *Vertex addition* produces a graph $G' = (V', E')$ by adjoining a new vertex v_{new} and leaving the edge set unchanged:

$$V' = V \cup \{v_{\text{new}}\}, \qquad E' = E.$$

Definition 2.1.3 (Graph edge addition)**.** Let $G = (V, E)$ be a graph. *Edge addition* produces a graph $G' = (V', E')$ by adjoining a new edge $e_{\text{new}} = (u, v)$ between existing vertices $u, v \in V$:

$$V' = V, \qquad E' = E \cup \{e_{\text{new}}\}.$$

Proposition 2.1.4. *Applying vertex addition to a null graph yields an empty graph.*

Proof. A null graph is $G = (V, E)$ with $V = \emptyset$ and $E = \emptyset$. After adding a single vertex v_{new}, we obtain $V' = \{v_{\text{new}}\}$ while the edge set remains $E' = \emptyset$. Thus the resulting graph has at least one vertex and no edges, which is precisely an empty graph. □

Proposition 2.1.5. *Applying edge addition to an empty graph produces a (nonempty) graph.*

Proof. An empty graph is a graph $G = (V, E)$ with $|V| \geq 1$ and $E = \emptyset$. Choose $u, v \in V$ and add the edge $e_{\text{new}} = (u, v)$. Then the vertex set remains $V' = V$, while the new edge set is $E' = E \cup \{e_{\text{new}}\} = \{e_{\text{new}}\} \neq \emptyset$. Hence the resulting graph has vertices and at least one edge, and therefore is a graph in the usual sense. □

2.2 2-Valued Graph and 3-Valued Graph

In this subsection we outline the notions of *2-valued* and *3-valued* graphs. In classical graph theory, the presence of a vertex or an edge is typically treated as a binary decision: it either exists (coded by 1) or it does not (coded by 0). This viewpoint is closely related to the notion of a *crisp graph* that is often used as the non-fuzzy baseline in fuzzy graph theory [96–99].

Definition 2.2.1 (2-valued representation)**.** Let $G = (V, E)$ be a graph. A *2-valued* description of G may be given by functions

$$f_V : V \to \{0, 1\}, \qquad f_E : E \to \{0, 1\},$$

interpreted as follows:

- for each vertex $v \in V$,

$$f_V(v) = \begin{cases} 1 & \text{if } v \text{ is present,} \\ 0 & \text{if } v \text{ is absent;} \end{cases}$$

- for each edge $e \in E$,

$$f_E(e) = \begin{cases} 1 & \text{if } e \text{ is present,} \\ 0 & \text{if } e \text{ is absent.} \end{cases}$$

In many uncertainty-aware settings, one generalizes this binary viewpoint by assigning richer functions (e.g., membership degrees or multi-parameter values) to vertices and edges; this is a standard mechanism for encoding imprecision or additional semantic states.

Proposition 2.2.2. *Both an empty graph and a null graph admit a 2-valued representation.*

Proof. Empty graph. Let $G = (V, E)$ with $|V| \geq 1$ and $E = \emptyset$. Define $f_V : V \to \{0, 1\}$ by $f_V(v) = 1$ for all $v \in V$ (all listed vertices are present). Since there are no edges, f_E is vacuously represented by $f_E(e) = 0$ for all $e \in E = \emptyset$.

Null graph. In a null graph we have $V = \emptyset$ and $E = \emptyset$. Then there are no vertices or edges to evaluate, and the functions f_V and f_E have empty domains. This still fits the 2-valued scheme (trivially), hence the null graph is also representable in this framework. □

Next we consider a rough 3-valued analogue (cf. [100–102]), where vertices and edges may take one of three states.

Definition 2.2.3 (3-valued representation)**.** Let $G = (V, E)$ be a graph. A *3-valued* description of G may be given by functions

$$f_V : V \to \{-1, 0, 1\}, \qquad f_E : E \to \{-1, 0, 1\},$$

with the interpretation that, for vertices $v \in V$ and edges $e \in E$,

$$f_V(v) = \begin{cases} 1 & \text{if } v \text{ is in a positive state (present),} \\ 0 & \text{if } v \text{ is neutral,} \\ -1 & \text{if } v \text{ is in a negative state (absent or unfavorable),} \end{cases}$$

$$f_E(e) = \begin{cases} 1 & \text{if } e \text{ is in a positive state (present),} \\ 0 & \text{if } e \text{ is neutral,} \\ -1 & \text{if } e \text{ is in a negative state (absent or unfavorable).} \end{cases}$$

Theorem 2.2.4. *A 3-valued graph generalizes a 2-valued graph.*

Proof. A 2-valued graph is obtained by restricting the codomain of the 3-valued functions to the subset $\{0, 1\} \subseteq \{-1, 0, 1\}$ (i.e., by disallowing the value -1). Hence every 2-valued representation is a special case of a 3-valued representation. □

2.3 Hypergraph and SuperHyperGraph Concepts

A *hypergraph* generalizes an ordinary (binary) graph by allowing an edge to connect any finite number of vertices. This additional expressive power is useful for representing genuinely multiway relations that arise in a variety of applications, including computer science and the life sciences [103–106]. Related notions, such as directed hypergraphs [107–111], are also well known. We recall the basic definition and a few standard constructions.

Definition 2.3.1 (Hypergraph [28])**.** A *hypergraph* is a pair $H = (V(H), E(H))$ such that:

- $V(H)$ is a nonempty finite set (the *vertices*), and

- $E(H)$ is a finite family of nonempty subsets of $V(H)$ (the *hyperedges*).

Thus, a hyperedge $e \in E(H)$ may have cardinality $|e| \geq 1$, and in particular $|e|$ need not equal 2.

Example 2.3.2. Let $V(H) = \{A, B, C, D, E\}$ and let

$$E(H) = \{e_1, e_2, e_3\}, \qquad e_1 = \{A, D\}, \quad e_2 = \{D, E\}, \quad e_3 = \{A, B, C\}.$$

Then $H = (V(H), E(H))$ is a hypergraph in which e_1 and e_2 are 2-element hyperedges, while e_3 is a 3-element hyperedge.

Theorem 2.3.3 (Graphs as 2-uniform hypergraphs)**.** *Every finite simple undirected graph $G = (V, E)$ can be represented as a hypergraph $H = (V(H), E(H))$ in which every hyperedge has cardinality* 2*.*

Proof. Let $G = (V, E)$ be a finite simple undirected graph, so $E \subseteq \binom{V}{2}$. Define a hypergraph $H = (V(H), E(H))$ by setting $V(H) := V$ and

$$E(H) := \big\{\{u, v\} \subseteq V : \ \{u, v\} \in E\big\}.$$

Then each $e \in E(H)$ is a nonempty subset of $V(H)$ of size 2, hence H is a hypergraph. Moreover, the correspondence $\{u, v\} \in E \leftrightarrow \{u, v\} \in E(H)$ preserves adjacency, so H encodes exactly the same incidence structure as G. $\square$

Definition 2.3.4 (Induced subhypergraph [28])**.** Let $H = (V(H), E(H))$ be a hypergraph and let $X \subseteq V(H)$. The *induced subhypergraph* of H on X is

$$H[X] \ := \ \big(X, \ \{\, e \cap X : \ e \in E(H), \ e \cap X \neq \emptyset \,\}\big).$$

We also write

$$H \setminus X \ := \ H\big[V(H) \setminus X\big]$$

for the hypergraph obtained by deleting the vertex set X.

Example 2.3.5 (Induced subhypergraph)**.** Let $H = (V(H), E(H))$ be the hypergraph with

$$V(H) = \{A, B, C, D, E\}, \qquad E(H) = \big\{\{A, D\}, \{D, E\}, \{A, B, C\}\big\}.$$

Take the vertex subset $X = \{A, C, D\} \subseteq V(H)$. Then each hyperedge intersects X as follows:

$$\{A, D\} \cap X = \{A, D\}, \qquad \{D, E\} \cap X = \{D\}, \qquad \{A, B, C\} \cap X = \{A, C\}.$$

Hence the induced subhypergraph is

$$H[X] = \big(X, \ \{\{A, D\}, \{D\}, \{A, C\}\}\big),$$

whose vertex set is $\{A, C, D\}$ and whose hyperedges are exactly the nonempty intersections of the original hyperedges with X.

2.4 SuperHyperGraphs

If one iterates the powerset construction further, then vertices themselves may be *nested* set-valued objects. This leads to finite *SuperHyperGraphs*, whose vertex set and edge set can both live at multiple "levels" of set nesting [10, 112]. These hierarchical representations appear naturally in settings where relations are layered or multiscale, for example in molecular design, complex-network analysis, and neural-network modeling, among other applications [11, 24, 113–117]. Related variants have also been studied, including Directed SuperHyperGraphs [21, 118–120] and MetaSuperHyperGraphs [121].

2.4.1 n-SuperHyperGraphs

In this book, unless stated otherwise, the term *SuperHyperGraph* refers to an n*-SuperHyperGraph* for some index n. Throughout, the parameter n appearing in $\mathcal{P}^n(\cdot)$ and in an n-SuperHyperGraph is always a *nonnegative integer*.

Definition 2.4.1 (Base set)**.** A *base set* V_0 is the underlying universe of discourse from which all higher-level objects are built. In particular, every element of the powerset $\mathcal{P}(V_0)$ and of the iterated powersets $\mathcal{P}^n(V_0)$ is ultimately constructed from elements of V_0 by repeated set formation.

Definition 2.4.2 (Iterated powerset and flattening [122])**.** Let V_0 be a finite nonempty set. Define $\mathcal{P}^0(V_0) := V_0$ and, for $k \geq 0$,

$$\mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big).$$

For each $k \geq 0$, the *flattening map*

$$\mathrm{Flat}_k : \mathcal{P}^k(V_0) \setminus \{\emptyset\} \longrightarrow \mathcal{P}(V_0) \setminus \{\emptyset\}$$

is defined recursively by

$$\mathrm{Flat}_0(x) := \{x\} \quad (x \in V_0), \qquad \mathrm{Flat}_{k+1}(X) := \bigcup_{Y \in X} \mathrm{Flat}_k(Y) \quad \big(X \in \mathcal{P}^{k+1}(V_0) \setminus \{\emptyset\}\big).$$

Thus Flat_k sends a nonempty k-level set-object to the (nonempty) set of base elements of V_0 that occur anywhere inside it.

Definition 2.4.3 (n-SuperHyperGraph [18])**.** Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. An n*-SuperHyperGraph on* V_0 is a pair

$$\mathrm{SHG}^{(n)} = (V, E)$$

such that

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

The elements of V are called n*-supervertices*, and the elements of E are called n*-superedges*. Equivalently, each n-superedge $e \in E$ is a nonempty subset of the n-supervertex set V.

Example 2.4.4 (Nested teams and cross-team initiatives as a 2-SuperHyperGraph)**.** Let V_0 be the set of individual employees in a company. For $n = 2$, a 2-supervertex is an element of $\mathcal{P}^2(V_0)$, i.e., a *set of teams*, where each team is a subset of employees.

- A *team* is a subset $T \subseteq V_0$ (e.g., the "Database" team).
- A *department* can be represented as a set of teams, i.e., a 2-supervertex

$$D = \{T_1, T_2, \ldots, T_k\} \in \mathcal{P}^2(V_0),$$

  where each $T_i \subseteq V_0$.

Let V be a collection of departments (each a set of teams), so $V \subseteq \mathcal{P}^2(V_0)$. A 2-superedge $e \in E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ can represent a cross-department initiative (e.g., a company-wide security program) that involves multiple departments simultaneously:

$$e = \{D_{\text{Engineering}}, D_{\text{IT}}, D_{\text{Compliance}}\} \subseteq V.$$

Thus (V, E) is a 2-SuperHyperGraph modeling hierarchical organization (employees $\to$ teams $\to$ departments) together with multiway relations among higher-level units (cross-department projects).

Example 2.4.5 (Multi-level medical cohorts and shared-care pathways as a 1-SuperHyperGraph)**.** Let V_0 be the set of patients in a hospital network. For $n = 1$, a 1-supervertex is a subset of patients, i.e., an element of $\mathcal{P}(V_0)$.

- A 1-supervertex $C \in V \subseteq \mathcal{P}(V_0)$ can represent a *cohort*, such as

  $C_{\text{diabetes}} = \{\text{patients diagnosed with diabetes}\} \subseteq V_0, \qquad C_{\text{cardio}} = \{\text{patients under cardiology care}\} \subseteq V_0.$

- A 1-superedge $e \in E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ can represent a *shared-care pathway* that jointly targets several cohorts, for example an integrated chronic-care program:

$$e = \{C_{\text{diabetes}}, C_{\text{cardio}}, C_{\text{renal}}\} \subseteq V.$$

Hence (V, E) is a 1-SuperHyperGraph in which supervertices are patient cohorts and superedges capture multiway coordination among cohorts in clinical pathways, resource planning, or population-health interventions.

2.4.2 (m, n)-SuperHyperGraph

An (m, n)-SuperHyperGraph is a higher-order network model in which each vertex is an (m, n)-*superhyperfunction* on a fixed base set, and each hyperedge collects several such functions to encode multiway relationships (e.g., shared constraints or contextual couplings) [123]. More generally, an (h, k)-*ary* (m, n)-SuperHyperGraph replaces single functions by (h, k)-ary superhyperfunctions [2]. In this book, we primarily work with n-SuperHyperGraphs.

Notation 2.4.6. *Let S be a nonempty set. Define the iterated powersets recursively by*

$$\mathcal{P}_0(S) := S, \qquad \mathcal{P}_{m+1}(S) := \mathcal{P}\big(\mathcal{P}_m(S)\big) \quad (m \in \mathbb{N}_0),$$

so that $\mathcal{P}_1(S) = \mathcal{P}(S)$, $\mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S))$, and so on. For a set X and $h \in \mathbb{N}$, we also write

$$X^h := \underbrace{X \times \cdots \times X}_{h \text{ copies}}$$

for the h-fold Cartesian power.

Definition 2.4.7 ((m,n)-superhyperfunction)**.** [123, 124] Let $m, n \in \mathbb{N}$ and let $S \neq \emptyset$. An (m,n)-*superhyperfunction on* S is a map

$$f : \ \mathcal{P}_m(S) \longrightarrow \mathcal{P}_n(S).$$

Equivalently, $f \in \mathrm{Hom}\big(\mathcal{P}_m(S),\, \mathcal{P}_n(S)\big)$.

Definition 2.4.8 ((m,n)-SuperHyperGraph)**.** [2] Fix $m, n \in \mathbb{N}$ and a nonempty base set S. Let

$$\mathfrak{F}_{m,n}(S) := \{\, f : \mathcal{P}_m(S) \to \mathcal{P}_n(S) \,\}.$$

An (m,n)-*SuperHyperGraph* is a pair

$$\mathsf{SHG}^{(m,n)} = (V, \mathcal{E}),$$

where $V \subseteq \mathfrak{F}_{m,n}(S)$ is a nonempty vertex set (so each vertex is a concrete (m,n)-superhyperfunction), and

$$\emptyset \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$

is a nonempty family of nonempty *hyperedges*. Thus each $E \in \mathcal{E}$ is a finite nonempty set of vertices, interpreted as a higher-order interaction among the corresponding superhyperfunctions.

Example 2.4.9 (Multi-level access-control policies as a (m,n)-SuperHyperGraph)**.** Let S be the set of all atomic permissions in an organization (e.g., `read-db`, `write-db`, `deploy`, etc.). Consider $m = 1$ and $n = 1$. Then $\mathcal{P}_1(S) = \mathcal{P}(S)$ is the set of all permission bundles.

A vertex $f \in \mathfrak{F}_{1,1}(S)$, i.e. a function

$$f : \mathcal{P}(S) \to \mathcal{P}(S),$$

can model a concrete *policy transformation* that maps a requested bundle of permissions to an approved bundle. For example, f may (i) remove prohibited permissions, (ii) add mandatory monitoring permissions, or (iii) enforce segregation-of-duties constraints.

Let $V \subseteq \mathfrak{F}_{1,1}(S)$ be a collection of such policy functions, e.g.,

$$V = \{f_{\mathrm{HR}}, f_{\mathrm{Eng}}, f_{\mathrm{Finance}}, f_{\mathrm{Security}}, \dots\}.$$

A hyperedge $E \in \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ can represent a *multi-policy coupling* that must be applied jointly, such as a compliance workflow requiring simultaneous satisfaction of security, finance, and legal constraints:

$$E = \{f_{\mathrm{Security}}, f_{\mathrm{Finance}}, f_{\mathrm{Legal}}\}.$$

Thus, the $(1,1)$-SuperHyperGraph $(V, \mathcal{E})$ models higher-order interactions among policy functions, rather than among individuals.

More generally, taking $m = 2$ allows each input to be a *set of bundles* (e.g., alternative requested packages), and taking $n = 2$ allows each output to be a *set of feasible approved packages*, capturing ambiguity and multi-solution policy outcomes.

Example 2.4.10 (Clinical decision rules mapping cohorts to intervention-sets as a (m, n)-SuperHyperGraph)**.** Let S be a set of elementary clinical features (e.g., diagnoses, lab flags, risk factors, symptoms). Fix $m = 2$ and $n = 1$. Then $\mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S))$ may be interpreted as a *collection of patient profiles*, where each profile is a subset of features (an element of $\mathcal{P}(S)$).

A vertex $f \in \mathfrak{F}_{2,1}(S)$, i.e. a function

$$f : \mathcal{P}\big(\mathcal{P}(S)\big) \to \mathcal{P}(S),$$

can represent a concrete *clinical decision rule* that maps a set of profiles (a cohort described by feature-sets) to a recommended set of interventions encoded as features (e.g., `start-med-A`, `order-test-B`, `refer-cardiology`). Different vertices correspond to different guideline modules or specialist rule-sets.

Let $V \subseteq \mathfrak{F}_{2,1}(S)$ be the family of available rule modules, e.g.,

$$V = \{f_{\text{Diabetes}}, f_{\text{Cardio}}, f_{\text{Renal}}, f_{\text{Respiratory}}, \dots\}.$$

A hyperedge $E \in \mathcal{E}$ can represent a *care pathway coupling* in which several rule modules must be applied together, for instance for multi-morbidity management:

$$E = \{f_{\text{Diabetes}}, f_{\text{Cardio}}, f_{\text{Renal}}\}.$$

Hence the $(2, 1)$-SuperHyperGraph $(V, \mathcal{E})$ models higher-order interactions among decision rules (superhyperfunctions), capturing that some clinical pathways arise only from the joint application of multiple guideline components.

2.5 Fuzzy Graph

A Fuzzy Graph captures relationships involving uncertainty by assigning membership degrees to both vertices and edges, enabling flexible and nuanced analysis. It can also be viewed as a graphical representation of a fuzzy set (cf. [125–129]). Due to its significance, Fuzzy Graphs have been the subject of extensive research [130–134].

The Fuzzy framework encompasses a wide range of mathematical and logical structures that have been extensively studied and developed in recent years. The primary concepts include:

1. Fuzzy Set [125, 135–138]

2. Fuzzy Topological Spaces [139, 140]

3. Fuzzy Logics [141, 142]

4. Fuzzy Algebraic Structures [143]

5. Fuzzy Environment [144, 145]

6. Fuzzy Geometry [146]

7. Fuzzy Statistics [143]

8. Fuzzy Physics [147]

9. Fuzzy Control [148–150]

10. Fuzzy system [151, 152]

11. Fuzzy Likert [153, 154]

12. Fuzzy Neural Network [155, 156]

13. Fuzzy Machine Learning [157, 158]

2.5.1 Basic concepts for fuzzy graphs

In this subsection we recall standard preliminaries for fuzzy graphs. Fuzzy graphs are commonly formulated relative to an underlying *crisp* (classical) graph and can be viewed as graphs equipped with fuzzy sets and fuzzy relations [96–99]. We begin with the crisp notion.

Definition 2.5.1 (Crisp (classical) graph)**.** A *crisp graph* is a pair $G^* = (V, E)$, where V is a nonempty set of *vertices* and E is a set of *edges*. In this book we mainly consider finite simple undirected graphs, so $E \subseteq \binom{V}{2}$.

Informally, a fuzzy graph is a crisp graph together with a fuzzy vertex set and a fuzzy relation on the vertex set. We therefore recall fuzzy relations.

Definition 2.5.2 (Fuzzy relation on a fuzzy set)**.** Let S be a nonempty set. A *fuzzy relation* on S is a mapping

$$\mu : S \times S \to [0, 1].$$

Let $\sigma : S \to [0, 1]$ be a fuzzy set on S. We say that μ is a *fuzzy relation on* σ if, for all $x, y \in S$,

$$\mu(x, y) \ \le \ \sigma(x) \wedge \sigma(y), \qquad \text{where} \qquad \sigma(x) \wedge \sigma(y) := \min\{\sigma(x), \sigma(y)\}.$$

Thus $\mu(x, y)$ represents the strength of the relationship between x and y, and the inequality ensures that this strength does not exceed the membership degrees of the endpoints.

We now state the classical definition of a fuzzy graph due to Rosenfeld [29] (see also [159]).

Definition 2.5.3 (Fuzzy graph [29])**.** A *fuzzy graph* is a pair $G = (\sigma, \mu)$ with underlying vertex set V, where:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of V (vertex-membership function), and
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation on σ, i.e.,

$$\mu(x, y) \;\le\; \sigma(x) \wedge \sigma(y) \qquad (\forall x, y \in V).$$

The *underlying crisp graph* associated with G is $G^* = (V^*, E^*)$, where

$$V^* := \{x \in V : \; \sigma(x) > 0\}, \qquad E^* := \big\{\{x, y\} \in \tbinom{V}{2} : \; \mu(x, y) > 0\big\}.$$

Definition 2.5.4 (Fuzzy subgraph [29])**.** Let $G = (\sigma, \mu)$ be a fuzzy graph on V. A *fuzzy subgraph* of G is a fuzzy graph $H = (\sigma', \mu')$ defined on some subset $X \subseteq V$ such that

$$\sigma' = \sigma|_X, \qquad \mu' = \mu|_{X \times X},$$

and hence $\mu'(x, y) \le \sigma'(x) \wedge \sigma'(y)$ for all $x, y \in X$.

Example 2.5.5 (A fuzzy graph)**.** Let $V = \{v_1, v_2, v_3, v_4\}$ and define the vertex-memberships by

$$\sigma(v_1) = 0.1, \quad \sigma(v_2) = 0.3, \quad \sigma(v_3) = 0.2, \quad \sigma(v_4) = 0.4.$$

Define the edge-membership function $\mu : V \times V \to [0, 1]$ by specifying its nonzero values

$$\mu(v_1, v_2) = 0.1, \quad \mu(v_2, v_3) = 0.1, \quad \mu(v_3, v_4) = 0.1, \quad \mu(v_4, v_1) = 0.1, \quad \mu(v_2, v_4) = 0.3,$$

and setting $\mu(x, y) = 0$ for all other pairs (x, y). Then for each listed pair (u, v) we have $\mu(u, v) \le \min\{\sigma(u), \sigma(v)\}$, e.g.,

$$\mu(v_2, v_4) = 0.3 = \min\{\sigma(v_2), \sigma(v_4)\} = \min\{0.3, 0.4\}.$$

Hence $G = (\sigma, \mu)$ is a fuzzy graph in the sense of Definition 2.5.3.

We next recall two frequently used notions.

Definition 2.5.6 (Complete fuzzy graph [160])**.** A fuzzy graph $G = (\sigma, \mu)$ on V is *complete* if, for all distinct $u, v \in V$,

$$\mu(u, v) = \sigma(u) \wedge \sigma(v).$$

Definition 2.5.7 (Strong fuzzy graph [160])**.** A fuzzy graph $G = (\sigma, \mu)$ on V is *strong* if, for all edges $\{u, v\} \in E^*$ of its underlying crisp graph,

$$\mu(u, v) = \sigma(u) \wedge \sigma(v).$$

Theorem 2.5.8. *Every complete fuzzy graph is a strong fuzzy graph, but not every strong fuzzy graph is complete.*

Proof. If G is complete, then $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ holds for *all* distinct $u, v \in V$, hence in particular for all $\{u, v\} \in E^*$; thus G is strong.

Conversely, a strong fuzzy graph only requires the equality on pairs $\{u, v\}$ that appear as edges of the underlying crisp graph G^*. It need not have edges between every pair of vertices (i.e., E^* may be a proper subset of $\binom{V}{2}$), so a strong fuzzy graph need not be complete. □

2.5.2 Other graph classes related to fuzzy graphs

We next record several simple transformations that relate fuzzy graphs to classical (crisp) and discrete-valued graph models. Throughout, let $G = (\sigma, \mu)$ be a fuzzy graph on a finite vertex set V in the sense of Definition 2.5.3.

Proposition 2.5.9 (Degenerate reduction to a null fuzzy graph)**.** *Every fuzzy graph can be reduced to the* null *fuzzy graph by setting all vertex and edge membership values to* 0*.*

Proof. Define $\sigma_0 : V \to [0,1]$ and $\mu_0 : V \times V \to [0,1]$ by $\sigma_0(v) := 0$ for all $v \in V$ and $\mu_0(x,y) := 0$ for all $(x,y) \in V \times V$. Then $\mu_0(x,y) \le \sigma_0(x) \wedge \sigma_0(y) = 0$ for all $x, y \in V$, so $G_0 = (\sigma_0, \mu_0)$ is a fuzzy graph. Its underlying crisp graph has vertex set $\{v : \sigma_0(v) > 0\} = \emptyset$ and edge set $\{\{x,y\} : \mu_0(x,y) > 0\} = \emptyset$, so it is the null (edgeless, vertexless) graph. □

Proposition 2.5.10 (Reduction to an empty crisp graph)**.** *Every fuzzy graph induces an* empty *crisp graph on its underlying vertex support by setting all edge-memberships to* 0*.*

Proof. Let $V^* := \{v \in V : \sigma(v) > 0\}$ be the underlying vertex support. Define $\mu_0 : V \times V \to [0,1]$ by $\mu_0(x,y) := 0$ for all (x,y), and keep the vertex function unchanged. Then (σ, μ_0) is a fuzzy graph because $\mu_0(x,y) = 0 \le \sigma(x) \wedge \sigma(y)$ for all x, y. Its underlying crisp graph has vertex set V^* and no edges, i.e., it is the empty graph on V^*. □

Proposition 2.5.11 (Thresholding to a crisp graph)**.** *Every fuzzy graph $G = (\sigma, \mu)$ induces a crisp graph by taking the positive supports of σ and μ.*

Proof. Define

$$V^* := \{v \in V : \sigma(v) > 0\}, \qquad E^* := \big\{\{u,v\} \in \tbinom{V^*}{2} : \mu(u,v) > 0\big\}.$$

Then $G^* = (V^*, E^*)$ is a (finite) crisp graph. This is exactly the underlying crisp graph associated with G in Definition 2.5.3. □

Proposition 2.5.12 (A 3-valued discretization)**.** *Every fuzzy graph $G = (\sigma, \mu)$ induces a 3-valued vertex/edge labeling by mapping memberships in $[0,1]$ to $\{-1, 0, 1\}$.*

Proof. Fix a threshold $\theta \in (0,1)$ (for concreteness one may take $\theta = \frac{1}{2}$). Define maps $f_V : V \to \{-1,0,1\}$ and $f_E : \binom{V}{2} \to \{-1,0,1\}$ by

$$f_V(v) := \begin{cases} 1, & \sigma(v) > \theta, \\ 0, & \sigma(v) = \theta, \\ -1, & \sigma(v) < \theta, \end{cases} \qquad f_E(\{u,v\}) := \begin{cases} 1, & \mu(u,v) > \theta, \\ 0, & \mu(u,v) = \theta, \\ -1, & \mu(u,v) < \theta. \end{cases}$$

Then (V, f_V, f_E) is a 3-valued representation of G in which labels distinguish "low", "borderline", and "high" membership. This discretization preserves the ordering information relative to the threshold θ while replacing continuous memberships by three states. □

Remark 2.5.13. Proposition 2.5.11 uses the standard support threshold at 0. Proposition 2.5.12 uses an arbitrary threshold θ (often chosen as $\frac{1}{2}$) to obtain a coarse ternary encoding.

Table 2.1: Concise comparison between a (Rosenfeld-type) fuzzy graph and an edge-fuzzy graph.

Aspect	**Fuzzy graph (Rosenfeld-type)**	**Edge-fuzzy graph**
Underlying crisp structure	Typically an underlying crisp graph $G^* = (V, E)$ (explicitly or via edge support)	Explicit crisp graph $G^* = (V, E)$ is part of the data
Data specified	(σ, μ) with $\sigma : V \to [0, 1]$ (vertex membership) and $\mu : E \to [0, 1]$ (edge membership)	(G^*, μ) with $\mu : E \to [0, 1]$ only (no vertex membership function)
Vertex-edge constraint	Imposes compatibility, commonly $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$ for all $uv \in E$ (or equality in strong variants)	No vertex-membership exists, hence no vertex-edge compatibility constraint applies
What is "fuzzy"	Both vertices and edges may carry graded membership degrees	Only edges carry membership degrees; vertices remain crisp
Typical interpretation	Presence/importance of a vertex and strength of a relation are jointly modeled	Vertices are assumed present; uncertainty/intensity is modeled solely on relations
Relation between classes	More structured; can reduce to an edge-fuzzy graph when σ is fixed (e.g., constant) and μ is rescaled	Less structured; can be viewed as a special case of general fuzzy graphs where only edge degrees are retained

2.5.3 Edge-Fuzzy Graph

In many (Rosenfeld-type) fuzzy graph models, fuzziness is assigned to both vertices and edges. In some applications, however, one wishes to keep the vertex set crisp and to represent uncertainty only on edges (or, dually, only on vertices). Such one-sided variants are commonly referred to as *edge-fuzzy graphs* and *vertex-fuzzy graphs* (cf. [161]).

Definition 2.5.14 (Edge-fuzzy graph)**.** An *edge-fuzzy graph* is a pair

$$G = (G^*, \mu), \qquad G^* = (V, E),$$

where G^* is a (finite) crisp graph and $\mu : E \to [0, 1]$ assigns to each edge $e \in E$ a membership degree $\mu(e)$ (interpreted as the strength, reliability, or intensity of the relation represented by e).

For reference, Table 2.1 presents a comparison between a (Rosenfeld-type) fuzzy graph and an edge-fuzzy graph.

Proposition 2.5.15. *Let $G = (\sigma, \mu)$ be a (Rosenfeld-type) fuzzy graph on an underlying crisp graph $G^* = (V, E)$, so that $\sigma : V \to [0, 1]$, $\mu : E \to [0, 1]$, and $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$ for all $uv \in E$. Assume that σ is constant, i.e., $\sigma(v) = c$ for all $v \in V$, for some $c \in [0, 1]$.*

- *If $c = 0$, then $\mu \equiv 0$ on E and the induced edge-fuzzy graph is trivial.*
- *If $c > 0$, then the rescaled mapping*

$$\mu'(uv) := \frac{\mu(uv)}{c} \qquad (uv \in E)$$

defines an edge-fuzzy graph (G^*, μ').

Proof. If $c = 0$, then for every $uv \in E$,

$$\mu(uv) \leq \sigma(u) \wedge \sigma(v) = 0,$$

hence $\mu(uv) = 0$ and the claim follows.

Assume $c > 0$. Since $\mu(uv) \leq c$ for all $uv \in E$, we have $0 \leq \mu(uv)/c \leq 1$. Thus $\mu' : E \to [0, 1]$ is well-defined, and (G^*, μ') is an edge-fuzzy graph by Definition 2.5.14. □

Theorem 2.5.16 (Thresholding to a crisp graph)**.** *Every edge-fuzzy graph (G^*, μ) induces a 2-valued (crisp) graph by thresholding the edge memberships.*

Proof. Let $G^* = (V, E)$ and $\mu : E \to [0, 1]$. Fix a threshold $\tau \in [0, 1]$ and define

$$E_\tau := \{\, e \in E : \ \mu(e) \geq \tau \,\}.$$

Then $G_\tau := (V, E_\tau)$ is a crisp graph. Equivalently, one may encode this as a 2-valued edge map $f_\tau : E \to \{0, 1\}$ given by

$$f_\tau(e) := \begin{cases} 1, & \mu(e) \geq \tau, \\ 0, & \mu(e) < \tau, \end{cases}$$

which records whether e is retained in E_τ. □

More broadly, it is often important to specify *where* fuzziness is introduced (vertices, edges, endpoints, weights, or collections of graphs). For instance, [98] organizes fuzzy graph models into several "types" as follows.

Definition 2.5.17 (Fuzzy graph types [98])**.** A fuzzy graph GF is said to be of the i-th type (or of any combination of types) if it exhibits fuzziness in one of the following ways:

(i) $GF_1 = \{G_1, G_2, \ldots, G_F\}$, where fuzziness occurs *within* each constituent graph G_i.

(ii) $GF_2 = \{V, E_F\}$, where the edge set E_F is fuzzy.

(iii) $GF_3 = \{V, E(t_F, h_F)\}$, where V and E are crisp, but edges have fuzzy tails t_F and fuzzy heads h_F.

(iv) $GF_4 = \{V_F, E\}$, where the vertex set V_F is fuzzy.

(v) $GF_5 = \{V, E(w_F)\}$, where V and E are crisp, but edges carry fuzzy weights w_F.

2.5.4 N-graphs and general fuzzy graphs

This subsection recalls two widely used extensions of fuzzy graphs: N-graphs and (so-called) general fuzzy graphs. In the N-structure approach, the membership (degree) values are *negative* and typically taken in the interval $[-1, 0]$, rather than $[0, 1]$.

Definition 2.5.18 (N-graph [51])**.** Let $G^* = (V, E)$ be a finite simple (undirected) graph. An N*-graph* on G^* is an ordered pair

$$G_N = (\nu, \eta),$$

where $\nu : V \to [-1, 0]$ is an N*-function* on vertices and $\eta : E \to [-1, 0]$ is an N*-relation* on edges, satisfying the compatibility condition

$$\eta(uv) \;\ge\; \max\{\nu(u), \nu(v)\} \qquad (\forall\, uv \in E).$$

Theorem 2.5.19 (Thresholding to a 2-valued N-graph)**.** *Every N-graph $G_N = (\nu, \eta)$ induces a 2-valued N-graph $G_N^{(2)} = (\nu^{(2)}, \eta^{(2)})$ by mapping each negative value to -1 and 0 to 0.*

Proof. Define $\nu^{(2)} : V \to \{-1, 0\}$ and $\eta^{(2)} : E \to \{-1, 0\}$ by

$$\nu^{(2)}(v) := \begin{cases} -1, & \nu(v) < 0, \\ 0, & \nu(v) = 0, \end{cases} \qquad \eta^{(2)}(uv) := \begin{cases} -1, & \eta(uv) < 0, \\ 0, & \eta(uv) = 0. \end{cases}$$

Let $uv \in E$.

If $\eta^{(2)}(uv) = 0$, then $\eta^{(2)}(uv) \ge \max\{\nu^{(2)}(u), \nu^{(2)}(v)\}$ holds trivially because the right-hand side is ≤ 0.

If $\eta^{(2)}(uv) = -1$, then $\eta(uv) < 0$. By Definition 2.5.18,

$$\eta(uv) \;\ge\; \max\{\nu(u), \nu(v)\}.$$

If $\max\{\nu(u), \nu(v)\} = 0$, then the inequality would force $\eta(uv) \ge 0$, contradicting $\eta(uv) < 0$. Hence $\max\{\nu(u), \nu(v)\} < 0$, which implies that at least one of $\nu(u), \nu(v)$ is negative, and therefore

$$\max\{\nu^{(2)}(u), \nu^{(2)}(v)\} = -1.$$

Thus $\eta^{(2)}(uv) = -1 \ge -1 = \max\{\nu^{(2)}(u), \nu^{(2)}(v)\}$.

In both cases, the N-graph constraint holds for $(\nu^{(2)}, \eta^{(2)})$, so $G_N^{(2)}$ is a 2-valued N-graph. □

Definition 2.5.20 (General fuzzy graph [38])**.** Let V be a nonempty finite set. A *general fuzzy graph* on V is a pair

$$G = (\sigma, \mu),$$

where $\sigma : V \to [0, 1]$ is a fuzzy subset of V and $\mu : V \times V \to [0, 1]$ is a fuzzy subset of $V \times V$. In contrast to (Rosenfeld-type) fuzzy graphs, no inequality relating $\mu(u, v)$ to $\sigma(u)$ and $\sigma(v)$ is imposed in the general setting.

2.5.5 Related graph class for fuzzy graph

In this subsection, we explain Related graph classes for fuzzy graph. In the field of Fuzzy Graphs, numerous graph classes have also been proposed. Here, we briefly introduce the related graph classes for fuzzy graphs.

Notation 2.5.21. *In this book, we define the term "Related graph class" as a graph class that either extends or restricts a corresponding graph class in some way.*

Theorem 2.5.22. *The following are examples of related graph classes, including but not limited to:*

- *Bipolar Fuzzy Graphs [34]*
- *Fuzzy Planar Graphs [35, 162, 163]*
- *Irregular Bipolar Fuzzy Graphs [36]*
- *Regular Bipolar Fuzzy Graphs [164]*
- *Picture Fuzzy Tolerance Graphs [165]*
- *Complex Hesitant Fuzzy Graphs [39]*
- *Strong Intuitionistic Fuzzy Graphs [166]*
- *Product Fuzzy Graphs [167]*
- *Partially Total Fuzzy Graphs [168]*
- *Fuzzy Influence Graphs [169]*
- *Picture Fuzzy Directed Hypergraphs [170]*
- *Radio Fuzzy Graphs [171]*
- *Line Regular Fuzzy Semigraphs [172]*
- *Fuzzy Incidence Graphs [173]*

- *Balanced Picture Fuzzy Graphs [174]*
- *Oscillating Polar Fuzzy Graphs [175]*
- *Cayley Fuzzy Graphs [176]*
- *Rough Fuzzy Digraphs [59]*
- *T-Spherical Fuzzy Graphs [177]*
- *Mixed Fuzzy Graphs [178]*
- *Einstein Fuzzy Graphs [179]*
- *Edge-Regular Fuzzy Graphs [180–182]*
- *Robust Fuzzy Graphs [183]*
- *Anti-Product Fuzzy Graphs [184]*
- *Inverse Fuzzy Graphs [185]*
- *Inverse Eccentric Fuzzy Graphs [186]*
- *Cubic Pythagorean Fuzzy Graphs [187]*
- *Complete Fuzzy Graphs [188]*
- *Mixed Picture Fuzzy Graphs [189]*
- *Extended Total Fuzzy Graphs [190]*
- *Pseudo Regular Fuzzy Graphs [191]*
- *Best Fuzzy Graphs [192]*
- *Intuitionistic Felicitous Fuzzy Graphs [193]*
- *Middle Fuzzy Graphs [194, 195]*

- *Bipolar Fuzzy P-Competition Graphs [196]*
- *Fuzzy Intersection Graphs [197–199]*
- *Intuitionistic Fuzzy Soft Expert Graphs [200]*
- *m-Polar Fuzzy Graphs [201, 202]*
- *Balanced Interval-Valued Fuzzy Graphs [203]*
- *Double Layered Fuzzy graph [204]*
- *Triple Layered Fuzzy Graph [205]*
- *Fuzzy Outerplanar Graphs [206, 207]*
- *Inverse fuzzy multigraphs [208]*
- *Bipolar inverse fuzzy graphs [209]*
- *Fuzzy zero divisor graphs [210, 211]*

Considering these fuzzy graph classes enables the identification of shared properties, which can lead to the development of efficient algorithms, deeper analysis, and practical applications across various fields.

Proof. Refer to each reference as needed. □

2.5.6 Fuzzy hypergraph and Fuzzy SuperHyperGraph

A fuzzy hypergraph assigns membership degrees in $[0, 1]$ to vertices and/or hyperedges, modeling uncertain multiway relations among entities. A fuzzy n-SuperHyperGraph is a higher-level network representation in which supervertices and superedges carry membership values for modeling complex interactions (cf. [18, 212]).

Definition 2.5.23 (Fuzzy hypergraph)**.** (cf. [30,213]) Let $H^* = (V, E, \partial)$ be a crisp hypergraph. A *fuzzy hypergraph* on H^* is a sextuple

$$\mathcal{H} \;=\; \big(V, E, \partial;\; \sigma, \mu, \eta\big),$$

with maps

$$\sigma : V \to [0,1], \qquad \mu : E \to [0,1], \qquad \eta : V \times E \to [0,1],$$

such that for all $v \in V$ and $e \in E$,

$$\textit{(support)} \quad \big[v \in \partial(e)\big] \iff \eta(v, e) > 0, \tag{2.1}$$

$$\textit{(incidence bound)} \quad \eta(v, e) \;\le\; \min\{\sigma(v), \mu(e)\}, \tag{2.2}$$

$$\textit{(edge-vertex bound)} \quad \mu(e) \;\le\; \min_{u \in \partial(e)} \sigma(u). \tag{2.3}$$

Here σ is the *vertex-membership map*, μ the *edge-membership map*, and η the *incidence-membership map*. The underlying crisp hypergraph is (V, E, ∂), recoverable via (2.1).

Example 2.5.24 (A fuzzy hypergraph)**.** Let $V = \{a, b, c\}$ and let $E = \{e_1, e_2\}$ with incidence map

$$\partial(e_1) = \{a, b\}, \qquad \partial(e_2) = \{b, c\}.$$

Define vertex- and edge-memberships by

$$\sigma(a) = 0.8,\ \sigma(b) = 0.6,\ \sigma(c) = 0.7,$$

$$\mu(e_1) = 0.6,\ \mu(e_2) = 0.6.$$

Then the edge-vertex bound (2.3) holds since

$$\mu(e_1) = 0.6 \le \min\{\sigma(a), \sigma(b)\} = \min\{0.8, 0.6\} = 0.6,$$

$$\mu(e_2) = 0.6 \le \min\{\sigma(b), \sigma(c)\} = \min\{0.6, 0.7\} = 0.6.$$

Define the incidence-membership map $\eta : V \times E \to [0,1]$ by

$$\eta(a, e_1) = 0.5,\ \eta(b, e_1) = 0.6,\ \eta(b, e_2) = 0.4,\ \eta(c, e_2) = 0.6,$$

and set $\eta(v, e) = 0$ for all other pairs (v, e). Then the support condition (2.1) is satisfied because $\eta(v, e) > 0$ holds exactly for $v \in \partial(e)$, and the incidence bound (2.2) holds since, for example,

$$\eta(a, e_1) = 0.5 \le \min\{\sigma(a), \mu(e_1)\} = \min\{0.8, 0.6\} = 0.6,$$

$$\eta(c, e_2) = 0.6 \le \min\{\sigma(c), \mu(e_2)\} = \min\{0.7, 0.6\} = 0.6,$$

and similarly for the remaining nonzero incidences. Hence $\mathcal{H} = (V, E, \partial; \sigma, \mu, \eta)$ is a fuzzy hypergraph in the sense of Definition 2.5.23.

Definition 2.5.25 (Fuzzy n-SuperHyperGraph)**.** (cf. [18]) Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A *fuzzy n-SuperHyperGraph* is a quadruple

$$(V, E, \sigma, \mu),$$

where $\sigma : V \to [0,1]$ and $\mu : E \to [0,1]$ obey the *admissibility constraint*

$$\mu(e) \;\le\; \min_{v \in e} \sigma(v) \qquad \text{for every } e \in E.$$

Example 2.5.26 (A fuzzy n-SuperHyperGraph)**.** Let $n = 2$ and let $V = \{x, y, z\}$. Consider the 2-SuperHyperGraph $\mathrm{SHG}^{(2)} = (V, E)$ with

$$E = \{e_1, e_2\}, \qquad e_1 = \{x, y\}, \quad e_2 = \{y, z\}.$$

Define $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ by

$$\sigma(x) = 0.9, \quad \sigma(y) = 0.5, \quad \sigma(z) = 0.8,$$

$$\mu(e_1) = 0.5, \quad \mu(e_2) = 0.5.$$

Then the admissibility constraint holds:

$$\mu(e_1) = 0.5 \le \min\{\sigma(x), \sigma(y)\} = \min\{0.9, 0.5\} = 0.5,$$

$$\mu(e_2) = 0.5 \le \min\{\sigma(y), \sigma(z)\} = \min\{0.5, 0.8\} = 0.5.$$

Therefore (V, E, σ, μ) is a fuzzy 2-SuperHyperGraph (hence a fuzzy n-SuperHyperGraph for $n = 2$) according to the stated definition.

2.5.7 Edge-Fuzzy HyperGraph and Edge-Fuzzy SuperHyperGraph

An edge-fuzzy hypergraph assigns membership degrees to hyperedges only, keeping vertices and incidences crisp, modeling uncertain multiway relations. An edge-fuzzy superhypergraph assigns membership degrees to superedges only, with crisp supervertices, modeling fuzzy higher-order relations among supervertices.

Definition 2.5.27 (Edge-fuzzy hypergraph)**.** Let $H^* = (V, E, \partial)$ be a (finite) crisp hypergraph, where V is the vertex set, E is the hyperedge set, and $\partial : E \to \mathcal{P}^*(V)$ assigns to each hyperedge $e \in E$ its (nonempty) incidence set $\partial(e) \subseteq V$. An *edge-fuzzy hypergraph* on H^* is a pair

$$\mathcal{H} = (H^*, \mu) \;=\; (V, E, \partial; \mu),$$

where $\mu : E \to [0, 1]$ assigns to each hyperedge $e \in E$ a membership degree $\mu(e)$. Intuitively, $\mu(e)$ quantifies the strength, reliability, or intensity of the multiway relation represented by e, while the vertices and incidences remain crisp.

Example 2.5.28 (An edge-fuzzy hypergraph)**.** Let $V = \{a, b, c, d\}$ and let $E = \{e_1, e_2, e_3\}$. Define the incidence map $\partial : E \to \mathcal{P}^*(V)$ by

$$\partial(e_1) = \{a, b, c\}, \qquad \partial(e_2) = \{b, d\}, \qquad \partial(e_3) = \{a, c, d\}.$$

Thus $H^* = (V, E, \partial)$ is a crisp hypergraph whose hyperedges represent three multiway relations among the vertices.

Now define an edge-membership map $\mu : E \to [0, 1]$ by

$$\mu(e_1) = 0.8, \qquad \mu(e_2) = 0.3, \qquad \mu(e_3) = 0.6.$$

Then

$$\mathcal{H} = (V, E, \partial; \mu)$$

is an edge-fuzzy hypergraph in the sense of Definition 2.5.27: the vertex set and incidences are crisp, while each hyperedge e_i carries a membership degree $\mu(e_i)$ indicating the strength of the corresponding multiway relation.

Theorem 2.5.29 (Edge-fuzzy hypergraphs generalize edge-fuzzy graphs)**.** *Every edge-fuzzy graph can be viewed canonically as an edge-fuzzy hypergraph. More precisely, for any edge-fuzzy graph $G = (G^*, \mu_G)$ with $G^* = (V, E_G)$, there exists an edge-fuzzy hypergraph $\mathcal{H} = (H^*, \mu_H)$ such that:*

1. *H^* is a 2-uniform hypergraph on the same vertex set V;*

2. *there is a natural bijection between graph-edges and hyperedges preserving membership degrees.*

Proof. Let $G = (G^*, \mu_G)$ be an edge-fuzzy graph with $G^* = (V, E_G)$ and $\mu_G : E_G \to [0, 1]$. Construct a crisp hypergraph

$$H^* = (V, E_H, \partial)$$

as follows. Set $E_H := \{e_{uv} : uv \in E_G\}$ (a renamed copy of the edge set), and define

$$\partial(e_{uv}) := \{u, v\} \qquad (\forall\, uv \in E_G).$$

Then $|\partial(e_{uv})| = 2$ for every $e_{uv} \in E_H$, hence H^* is 2-uniform. Define $\mu_H : E_H \to [0, 1]$ by

$$\mu_H(e_{uv}) := \mu_G(uv) \qquad (\forall\, uv \in E_G).$$

Therefore $\mathcal{H} = (H^*, \mu_H)$ is an edge-fuzzy hypergraph in the sense of Definition 2.5.27.

Finally, the correspondence $\varphi : E_G \to E_H$ given by $\varphi(uv) = e_{uv}$ is a bijection, and by construction $\mu_H(\varphi(uv)) = \mu_G(uv)$ for all $uv \in E_G$. Hence every edge-fuzzy graph is realized as a special case (namely, a 2-uniform case) of an edge-fuzzy hypergraph. □

Remark 2.5.30. Conversely, any edge-fuzzy hypergraph $\mathcal{H} = (V, E, \partial; \mu)$ whose hyperedges all have size 2 (i.e., $|\partial(e)| = 2$ for all $e \in E$) canonically induces an edge-fuzzy graph by identifying each hyperedge with its incident unordered pair.

Definition 2.5.31 (Edge-fuzzy n-SuperHyperGraph)**.** Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph (in the sense of your base definition), so that V is the set of n-supervertices and $E \subseteq \mathcal{P}^*(V)$ is the set of n-superedges. An *edge-fuzzy n-SuperHyperGraph* (or *edge-fuzzy SuperHyperGraph*) on $\mathrm{SHG}^{(n)}$ is a triple

$$\mathfrak{H} \;=\; (V, E, \mu),$$

where $\mu : E \to [0, 1]$ assigns to each superedge $e \in E$ a membership degree $\mu(e)$. No vertex-membership map is assumed; supervertices are treated as crisp objects, while fuzziness is carried only by superedges.

Example 2.5.32 (An edge-fuzzy 2-SuperHyperGraph)**.** Let the base set be

$$V_0 = \{1, 2, 3\}.$$

For $n = 2$, a 2-supervertex is an element of $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$, i.e., a set of subsets of V_0. Define the 2-supervertex set

$$V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0)$$

by

$$v_1 = \{\{1\}, \{1, 2\}\}, \qquad v_2 = \{\{2\}, \{2, 3\}\}, \qquad v_3 = \{\{1, 3\}\}.$$

Define the (crisp) 2-superedge family

$$E = \{e_1, e_2\} \subseteq \mathcal{P}^*(V)$$

by

$$e_1 = \{v_1, v_2\}, \qquad e_2 = \{v_1, v_2, v_3\}.$$

Thus $\mathrm{SHG}^{(2)} = (V, E)$ is a 2-SuperHyperGraph.

Now assign edge-memberships $\mu : E \to [0, 1]$ by

$$\mu(e_1) = 0.4, \qquad \mu(e_2) = 0.8.$$

Then

$$\mathfrak{H} = (V, E, \mu)$$

is an edge-fuzzy 2-SuperHyperGraph (hence an edge-fuzzy n-SuperHyperGraph for $n = 2$) in the sense of Definition 2.5.31: the supervertices v_1, v_2, v_3 are crisp objects in $\mathcal{P}^2(V_0)$, and only the superedges e_1, e_2 carry fuzziness via the membership map μ.

Theorem 2.5.33 (Edge-fuzzy superhypergraphs generalize edge-fuzzy hypergraphs)**.** *Every edge-fuzzy hypergraph induces an edge-fuzzy* 1*-SuperHyperGraph (hence an edge-fuzzy SuperHyperGraph). In particular, edge-fuzzy SuperHyperGraphs generalize edge-fuzzy hypergraphs.*

Proof. Let $\mathcal{H} = (V, E, \partial; \mu)$ be an edge-fuzzy hypergraph in the sense of Definition 2.5.27. Define a base set $V_0 := V$ and form the set of 1-supervertices

$$V^\sharp := \{\{v\} : v \in V_0\} \subseteq \mathcal{P}(V_0).$$

For each hyperedge $e \in E$, define the corresponding superedge

$$\widehat{e} := \{\{v\} \in V^\sharp : v \in \partial(e)\} \subseteq V^\sharp.$$

Let

$$E^\sharp := \{\widehat{e} : e \in E\} \subseteq \mathcal{P}^*(V^\sharp),$$

and define $\mu^\sharp : E^\sharp \to [0, 1]$ by

$$\mu^\sharp(\widehat{e}) := \mu(e) \qquad (\forall\, e \in E).$$

Then $\mathrm{SHG}^{(1)} := (V^\sharp, E^\sharp)$ is a 1-SuperHyperGraph, and

$$\mathfrak{H} := (V^\sharp, E^\sharp, \mu^\sharp)$$

is an edge-fuzzy 1-SuperHyperGraph by Definition 2.5.31.

Moreover, the singleton embedding $\iota : V \to V^\sharp$ given by $\iota(v) = \{v\}$ induces a bijection between hyperedges and superedges via $e \mapsto \widehat{e}$, and the membership degrees are preserved by construction: $\mu^\sharp(\widehat{e}) = \mu(e)$. Hence $\mathcal{H}$ is recovered (up to the canonical identification of v with $\{v\}$) as a special case of an edge-fuzzy SuperHyperGraph. □

2.5.8 N-HyperGraphs and N-SuperHyperGraphs

In the N-structure framework of [51], the degree values are *negative* and lie in $[-1, 0]$. Accordingly, the natural compatibility inequality is written with "$\geq$" (since larger values are closer to 0 and indicate stronger membership).

Definition 2.5.34 (N-HyperGraph [51]). Let $H^* = (V, E)$ be a finite crisp hypergraph, where V is a finite nonempty set and $E \subseteq \mathcal{P}^*(V)$ is a finite family of nonempty hyperedges. An N-*hypergraph* on H^* is a pair

$$H_N = (\nu, \eta),$$

where $\nu : V \to [-1, 0]$ is an N-*function* on vertices and $\eta : E \to [-1, 0]$ is an N-*relation* on hyperedges, such that

$$\eta(e) \ \geq \ \max_{v \in e} \nu(v) \qquad (\forall\, e \in E).$$

Example 2.5.35 (N-HyperGraph). Let $V = \{a, b, c, d\}$ and

$$E = \{e_1, e_2\} \subseteq \mathcal{P}^*(V), \qquad e_1 = \{a, b, c\}, \quad e_2 = \{b, d\}.$$

Define an N-function $\nu : V \to [-1, 0]$ by

$$\nu(a) = -0.9, \qquad \nu(b) = -0.2, \qquad \nu(c) = -0.6, \qquad \nu(d) = -0.4,$$

and define an N-relation $\eta : E \to [-1, 0]$ by

$$\eta(e_1) = -0.2, \qquad \eta(e_2) = -0.2.$$

Then the N-constraint holds:

$$\max_{v \in e_1} \nu(v) = \max\{-0.9, -0.2, -0.6\} = -0.2 \ \leq \ \eta(e_1) = -0.2,$$

$$\max_{v \in e_2} \nu(v) = \max\{-0.2, -0.4\} = -0.2 \ \leq \ \eta(e_2) = -0.2.$$

Hence $H_N = (\nu, \eta)$ is an N-hypergraph in the sense of Definition 2.5.34.

Definition 2.5.36 (N-n-SuperHyperGraph [51]). Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. Let $\mathrm{SHG}^{(n)} = (V, E)$ be a finite n-SuperHyperGraph on V_0, i.e.,

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

An N-n-*SuperHyperGraph* on $\mathrm{SHG}^{(n)}$ is a pair

$$\mathrm{SHG}^{(n)}_N = (\nu, \eta),$$

where $\nu : V \to [-1, 0]$ is an N-function on n-supervertices and $\eta : E \to [-1, 0]$ is an N-relation on n-superedges satisfying

$$\eta(e) \ \geq \ \max_{v \in e} \nu(v) \qquad (\forall\, e \in E).$$

Example 2.5.37 (N-2-SuperHyperGraph)**.** Let $V_0 = \{1, 2, 3\}$ and $n = 2$, so $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$. Define

$$v_1 = \{\{1\}, \{1, 2\}\}, \qquad v_2 = \{\{2\}, \{2, 3\}\}, \qquad v_3 = \{\{1, 3\}\}, \qquad V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0),$$

and

$$E = \{e_1, e_2\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}, \qquad e_1 = \{v_1, v_2\}, \quad e_2 = \{v_1, v_2, v_3\}.$$

Define $\nu : V \to [-1, 0]$ and $\eta : E \to [-1, 0]$ by

$$\nu(v_1) = -0.7, \qquad \nu(v_2) = -0.1, \qquad \nu(v_3) = -0.6,$$

$$\eta(e_1) = -0.1, \qquad \eta(e_2) = -0.1.$$

Then

$$\max_{v \in e_1} \nu(v) = \max\{-0.7, -0.1\} = -0.1 \;\le\; \eta(e_1) = -0.1,$$

$$\max_{v \in e_2} \nu(v) = \max\{-0.7, -0.1, -0.6\} = -0.1 \;\le\; \eta(e_2) = -0.1,$$

so (ν, η) defines an N-2-SuperHyperGraph in the sense of Definition 2.5.36.

Theorem 2.5.38 (N-HyperGraphs generalize N-graphs)**.** *Every N-graph can be realized as an N-hypergraph whose hyperedges all have size* 2.

Proof. Let $G^* = (V, E_G)$ be a finite simple undirected graph and let $G_N = (\nu, \eta_G)$ be an N-graph, where

$$\nu : V \to [-1, 0], \qquad \eta_G : E_G \to [-1, 0], \qquad \eta_G(uv) \ge \max\{\nu(u), \nu(v)\} \;\; (\forall\, uv \in E_G).$$

Form the 2-uniform hypergraph $H^* = (V, E)$ by

$$E := \big\{\{u, v\} \subseteq V : \; uv \in E_G\big\} \subseteq \mathcal{P}^*(V).$$

Define $\eta : E \to [-1, 0]$ by transporting η_G along $uv \leftrightarrow \{u, v\}$:

$$\eta(\{u, v\}) := \eta_G(uv).$$

Then for each hyperedge $e = \{u, v\} \in E$,

$$\eta(e) = \eta_G(uv) \;\ge\; \max\{\nu(u), \nu(v)\} \;=\; \max_{x \in e} \nu(x),$$

so (ν, η) satisfies Definition 2.5.34. Hence G_N is encoded as an N-hypergraph. ☐

Theorem 2.5.39 (N-n-SuperHyperGraphs generalize N-graphs)**.** *Every N-graph can be realized as an N-n-SuperHyperGraph, namely as the case $n = 0$.*

Proof. Let $G_N = (\nu, \eta_G)$ be an N-graph on $G^* = (V, E_G)$ as above. Set $V_0 := V$ and take $n = 0$, so $\mathcal{P}^0(V_0) = V_0$ and we identify the 0-supervertex set with V. Define the 0-SuperHyperGraph by

$$\mathrm{SHG}^{(0)} = (V, E), \qquad E := \big\{\{u, v\} \subseteq V : \; uv \in E_G\big\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Define $\eta : E \to [-1, 0]$ by $\eta(\{u, v\}) := \eta_G(uv)$ and keep the same $\nu : V \to [-1, 0]$. Then for each $e = \{u, v\} \in E$,

$$\eta(e) = \eta_G(uv) \;\ge\; \max\{\nu(u), \nu(v)\} \;=\; \max_{x \in e} \nu(x),$$

so (ν, η) satisfies Definition 2.5.36. Thus G_N is realized as an N-0-SuperHyperGraph. ☐

2.5.9 Applications of fuzzy graphs

This subsection briefly outlines representative application areas of fuzzy graphs. Because fuzzy graphs attach graded membership values to vertices and/or edges, they provide a natural modeling language for networks in which relationships are uncertain, imprecise, or context-dependent. We list several illustrative domains below.

- **Neural networks.** Neural networks are computational models inspired by biological nervous systems, consisting of interconnected processing units (neurons) that learn patterns from data (cf. [214,215]). A number of studies have investigated links between fuzzy graph formalisms and neural-network modeling, learning, and representation [216–219].

- **Decision-making.** Graph-based decision-making represents alternatives, criteria, and dependencies as nodes and edges, enabling structured reasoning about feasible actions and outcomes (cf. [31,220,221]). Since fuzzy graphs can encode partial preference, uncertainty, or ambiguous relationships, they are frequently used in decision-support settings, and many contributions explore this interaction [222–225].

- **COVID-19 and epidemiological networks.** COVID-19 is an infectious disease caused by SARS-CoV-2, transmitted primarily via respiratory droplets and associated with a wide spectrum of symptoms, from mild illness to severe respiratory distress (cf. [226,227]). Several papers have applied fuzzy graph techniques to COVID-19 related modeling and analysis tasks [178,228–230].

- **Communication networks.** Communication networks are interconnected systems that enable data transfer among devices through wired or wireless links, including Internet, telephone, and satellite infrastructures (cf. [231]). Fuzzy graphs have been used to represent uncertain link quality, reliability, and dynamic connectivity in such networks [232–235].

- **Social networking services (SNS).** Social networking services are online platforms in which users maintain profiles, share content, and form connections with other users (cf. [236–238]). Fuzzy graph models are useful for capturing graded social ties, trust, and influence, and multiple studies investigate these perspectives [161,239,240].

- **Fuzzy chemistry:** Fuzzy graphs are also used in chemistry, for example in concepts such as molecular fuzzy graphs [241,242]. Moreover, in chemistry, topological indices have been studied extensively, and research on topological indices in fuzzy graphs is continuously being conducted [243–246].

- **Fuzzy graph signal processing:** Fuzzy graphs are also used in engineering, for instance in signal processing and image processing applications [247].

2.6 Intuitionistic fuzzy graphs

An Intuitionistic Fuzzy Graph (IFG) extends fuzzy graphs by incorporating membership, non-membership, and hesitancy degrees for vertices and edges. An Intuitionistic Fuzzy Graphs have been the subject of extensive research [33, 248]. To put it simply, it is a graph that represents the concept of Intuitionistic Fuzzy Sets (cf. [249, 250]) in graph form.

The Intuitionistic Fuzzy framework encompasses a wide range of mathematical and logical structures that have been extensively studied and developed in recent years. The primary concepts include:

1. Intuitionistic Fuzzy Set [251]

2. Intuitionistic Fuzzy Topological Spaces [252, 253]

3. Intuitionistic Fuzzy Logics [254]

4. Intuitionistic Fuzzy Algebra [255, 256]

5. Intuitionistic Fuzzy Vector [257, 258]

6. Intuitionistic Fuzzy Entropy [259, 260]

7. Intuitionistic Fuzzy Matroid [261, 262]

8. Intuitionistic Fuzzy Control [263, 264]

2.6.1 Definition of Intuitionistic Fuzzy Graph

The definition is provided below.

Definition 2.6.1 (Intuitionistic Fuzzy Graph (IFG))**.** [33] Let $G = (V, E)$ be a classical graph where V denotes the set of vertices and E denotes the set of edges. An *Intuitionistic Fuzzy Graph* (IFG) on G, denoted $G_{IF} = (A, B)$, is defined as follows:

1. (μ_A, v_A) is an *Intuitionistic Fuzzy Set (IFS)* on the vertex set V. For each vertex $x \in V$, the degree of membership $\mu_A(x) \in [0, 1]$ and the degree of non-membership $v_A(x) \in [0, 1]$ satisfy:

$$\mu_A(x) + v_A(x) \leq 1$$

   The value $1 - \mu_A(x) - v_A(x)$ represents the hesitancy or uncertainty regarding the membership of x in the set.

2. (μ_B, v_B) is an *Intuitionistic Fuzzy Relation (IFR)* on the edge set E. For each edge $(x, y) \in E$, the degree of membership $\mu_B(x, y) \in [0, 1]$ and the degree of non-membership $v_B(x, y) \in [0, 1]$ satisfy:
$$\mu_B(x, y) + v_B(x, y) \leq 1$$
Additionally, the following constraints must hold for all $x, y \in V$:
$$\mu_B(x, y) \leq \mu_A(x) \wedge \mu_A(y)$$
$$v_B(x, y) \leq v_A(x) \vee v_A(y)$$

In this definition:

- $\mu_A(x)$ and $v_A(x)$ represent the degree of membership and non-membership of the vertex x, respectively.

- $\mu_B(x, y)$ and $v_B(x, y)$ represent the degree of membership and non-membership of the edge (x, y), respectively.

- If $v_A(x) = 0$ and $v_B(x, y) = 0$ for all $x \in V$ and $(x, y) \in E$, then the Intuitionistic Fuzzy Graph reduces to a Fuzzy Graph.

Example 2.6.2 (Intuitionistic Fuzzy Graph)**.** Consider the Intuitionistic Fuzzy Graph
$$G = (V, E, \mu_A, v_A, \mu_B, v_B)$$
, where
$$V = \{v_1, v_2, v_3, v_4\}$$
and the edges
$$E = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_3, v_4), (v_2, v_4)\}$$
. The membership and non-membership degrees for the vertices are given as:

$$\mu_A(v_1) = 0.1, \quad v_A(v_1) = 0.4$$
$$\mu_A(v_2) = 0.3, \quad v_A(v_2) = 0.3$$
$$\mu_A(v_3) = 0.2, \quad v_A(v_3) = 0.4$$
$$\mu_A(v_4) = 0.4, \quad v_A(v_4) = 0.6$$

For the edges, the membership and non-membership degrees are given as follows:

- $\mu_B(v_1, v_2) = 0.1, \quad v_B(v_1, v_2) = 0.4$

- $\mu_B(v_1, v_4) = 0.1, \quad v_B(v_1, v_4) = 0.6$

- $\mu_B(v_2, v_3) = 0.1, \quad v_B(v_2, v_3) = 0.4$

- $\mu_B(v_3, v_4) = 0.1, \quad v_B(v_3, v_4) = 0.6$
- $\mu_B(v_2, v_4) = 0.3, \quad v_B(v_2, v_4) = 0.6$

This provides an example of how an Intuitionistic Fuzzy Graph can be structured, representing uncertainty in both vertex and edge membership using membership and non-membership degrees.

The following types of graphs are known for Intuitionistic Fuzzy Graphs.

Definition 2.6.3 (Strong Intuitionistic Fuzzy Graph)**.** [166] An Intuitionistic Fuzzy Graph $G = (A, B)$ is called a *strong intuitionistic fuzzy graph* if for all $xy \in E$:

$$\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) = \max(\nu_A(x), \nu_A(y)).$$

Definition 2.6.4 (Complete Intuitionistic Fuzzy Graph)**.** (cf. [265]) An Intuitionistic Fuzzy Graph $G = (A, B)$ is called *complete* if for all $xy \in E$:

$$\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) = \min(\nu_A(x), \nu_A(y)).$$

Proposition 2.6.5. *Every Intuitionistic Fuzzy Graph can be transformed into a Fuzzy Graph by restricting the non-membership function v_A to 0 for all vertices.*

Proof. To transform G_{IF} into a fuzzy graph, we set the non-membership functions $v_A(x) = 0$ for all vertices $x \in V$, and $v_B(x, y) = 0$ for all edges $(x, y) \in E$.

With this restriction, the conditions simplify as follows:

$$\mu_A(x) + v_A(x) = \mu_A(x) \leq 1 \quad \text{for all } x \in V,$$

$$\mu_B(x, y) + v_B(x, y) = \mu_B(x, y) \leq 1 \quad \text{for all } (x, y) \in E.$$

Thus, G_{IF} reduces to a fuzzy graph $G_F = (\sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ is the membership function for vertices, corresponding to μ_A,
- $\mu : E \to [0, 1]$ is the membership function for edges, corresponding to μ_B,

and the constraints:

$$\mu(x, y) \leq \sigma(x) \wedge \sigma(y) \quad \text{for all } (x, y) \in E,$$

are satisfied as in the definition of a fuzzy graph. Therefore, by restricting the non-membership function v_A to 0, we transform the Intuitionistic Fuzzy Graph into a Fuzzy Graph. □

2.6.2 Graph class for Intuitionistic Fuzzy Graph

We consider about Graph class for Intuitionistic Fuzzy Graph.

Theorem 2.6.6. *The following are examples of related graph classes, including but not limited to:*

- *Strong Intuitionistic Fuzzy Graphs [166]*
- *Perfect intuitionistic fuzzy graphs [266]*
- *Intuitionistic fuzzy competition graphs [267]*
- *Intuitionistic fuzzy threshold graphs [268]*
- *Balanced Intuitionistic Fuzzy Graphs [269]*
- *Bipolar Intuitionistic Fuzzy Competition Graphs [270]*
- *Intuitionistic Felicitous Fuzzy Graphs [193]*
- *Intuitionistic Fuzzy Soft Expert Graphs [200]*
- *Intuitionistic fuzzy tolerance graphs [271]*
- *Intuitionistic fuzzy planar graphs [272]*
- *Edge regular intuitionistic fuzzy graph [273]*
- *m-Neighbourly Irregular Instuitionistic Fuzzy Graphs [274]*
- *R-edge regular intuitionistic fuzzy graphs [275]*
- *Perfectly Edge-Regular Intuitionistic Fuzzy Graphs [276]*
- *Edge Regular Intuitionistic Fuzzy M-Polargraphs [277]*
- *Intuitionistic fuzzy labeling graphs [278]*
- *Regular Interval-Valued Intuitionistic Fuzzy Graphs [279]*

- *Anti intuitionistic fuzzy graph [280]*
- *Interval-valued intuitionistic fuzzy graphs [281, 282]*
- *Intuitionistic fuzzy directed hypergraphs [283]*
- *Complex intuitionistic fuzzy graphs [284]*
- *Complex t-Intuitionistic Fuzzy Graph [285]*
- *Intuitionistic fuzzy incidence graphs [286]*
- *Intuitionistic fuzzy k-partite hypergraphs [287, 288]*
- *Irregular Intuitionistic Fuzzy Graphs [289]*
- *intuitionistic L-fuzzy graph [290, 291]*
- *Bipolar intuitionistic anti fuzzy graphs [292]*
- *Intuitionistic anti-fuzzy graphs [293]*
- *intuitionistic product fuzzy graphs [294]*
- *intuitionistic k-partitioned fuzzy graph [295]*
- *intutionistic fuzzy multigraphs [296, 297]*

Proof. Refer to each reference as needed. □

2.6.3 Intuitionistic fuzzy HyperGraph and n-SuperHyperGraph

An *intuitionistic fuzzy hypergraph* generalizes hypergraphs by equipping each vertex and each hyperedge with membership and nonmembership degrees under the same Atanassov framework [298–300]. An intuitionistic superhypergraph is a hierarchical hypergraph in which every n-supervertex and superedge is equipped with both membership and non-membership degrees that satisfy Atanassov's sum and appurtenance constraints. We now give the formal definition of this structure.

Definition 2.6.7 (Intuitionistic Fuzzy Hypergraph)**.** (cf. [298–300]) Let V be a nonempty finite set of *vertices*. An *intuitionistic fuzzy hyperedge* on V is an ordered pair

$$E = (\mu_E, \nu_E),$$

where

$$\mu_E, \nu_E \colon V \longrightarrow [0,1]$$

such that

$$0 \ \leq \ \mu_E(v) + \nu_E(v) \ \leq \ 1, \quad \forall\, v \in V.$$

Its *support* is defined by

$$\operatorname{supp}(E) \ = \ \bigl\{\, v \in V \ \big|\ \mu_E(v) > 0 \text{ or } \nu_E(v) < 1 \bigr\}.$$

An *intuitionistic fuzzy hypergraph* is a pair

$$H = \bigl(V, \mathcal{E}\bigr),$$

where $\mathcal{E} = \{E_1, \ldots, E_m\}$ is a finite family of intuitionistic fuzzy hyperedges on V satisfying the covering condition

$$\bigcup_{j=1}^{m} \operatorname{supp}(E_j) \ = \ V.$$

The elements of V are called *vertices*, and each $E_j \in \mathcal{E}$ is called an *intuitionistic fuzzy hyperedge*. The *order* of H is $|V|$, and the number of hyperedges is $|\mathcal{E}|$.

Definition 2.6.8 (Intuitionistic fuzzy n-SuperHyperGraph)**.** Let

$$\mathrm{SHG}^{(n)} = (V_0, V, E)$$

be an n-SuperHyperGraph on the finite base set V_0, so that $V \subseteq \mathrm{POWS}^n(V_0)$ and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. An *intuitionistic fuzzy n-SuperHyperGraph on* $\mathrm{SHG}^{(n)}$ is a sextuple

$$\mathcal{H} = \bigl(V,\ E,\ \sigma,\ \sigma^c,\ \mu,\ \nu\bigr),$$

where the functions

$$\sigma : V \longrightarrow [0,1], \qquad \sigma^c : V \longrightarrow [0,1],$$
$$\mu : E \longrightarrow [0,1], \qquad \nu : E \longrightarrow [0,1],$$

satisfy, for all $v \in V$ and $e \in E$,

$$0 \ \leq \ \sigma(v) + \sigma^c(v) \ \leq \ 1, \quad 0 \ \leq \ \mu(e) + \nu(e) \ \leq \ 1, \tag{2.4}$$

$$\mu(e) \ \leq \ \min_{v \in e} \sigma(v), \quad \nu(e) \ \leq \ \min_{v \in e} \sigma^c(v). \tag{2.5}$$

Here $\sigma(v)$ and $\sigma^c(v)$ are the membership and non-membership degrees of the n-supervertex v, while $\mu(e)$ and $\nu(e)$ are the membership and non-membership degrees of the n-superedge e. Equation (2.4) is the Atanassov constraint, and (2.5) enforces consistency of edge-values with those of their incident supervertices.

2.6.4 Applications of intuitionistic fuzzy graphs

This subsection briefly highlights representative application areas of intuitionistic fuzzy graphs. Because intuitionistic fuzzy graphs can encode both membership and non-membership information (with an implicit hesitation margin), they are well-suited for networked systems in which relations and interactions are uncertain, incomplete, or conflicting. We list several illustrative domains below.

- **Intuitionistic fuzzy graphs neural network [301, 302]:** It is an extension of fuzzy graph neural networks, and a variety of studies have been conducted.

- **Water supply systems.** Water supply systems form large-scale networks for collecting, treating, and distributing water from sources (e.g., rivers or reservoirs) to residential and industrial consumers (cf. [303, 304]). Recent studies have investigated how intuitionistic fuzzy graphs can support modeling, analysis, and decision-making in such infrastructures under uncertainty [305, 306].

- **Cellular networks.** A cellular network is a wireless communication architecture in which coverage is partitioned into cells served by base stations, enabling mobile connectivity and handover between cells (cf. [307]). Intuitionistic fuzzy graph frameworks have been applied t o capture uncertain link qualities and dynamic interactions in cellular communication settings [308].

- **COVID-19 and epidemiological networks.** Several works have applied intuitionistic fuzzy graphs to represent uncertain contacts, transmission risks, and heterogeneous relationships in COVID-19 related data and decision problems [81, 309].

- **Social networks:** Research has been conducted in this area in a manner similar to fuzzy graphs [212, 310, 311].

2.7 Neutrosophic Graph

We present the concept of a neutrosophic graph [60, 62, 63, 312–314], which builds upon and expands the theory of fuzzy graphs [315]. Neutrosophic graphs offer a powerful tool for addressing uncertainty, making them particularly useful in diverse domains such as social networks and industrial systems. While fuzzy graph theory has already proven its relevance in modern scientific and technological applications, including operations research, neural networks [219, 316], artificial intelligence [219, 316], and decision-making [315], neutrosophic graphs provide an enhanced framework for more complex scenarios.

The Neutrosophic framework encompasses a wide range of mathematical and logical structures that have been extensively studied and developed in recent years. The primary concepts include:

1. Neutrosophic Set [317–320]

2. Neutrosophic Offset [321, 322]

3. Neutrosophic Topological Spaces [323, 324]

4. Neutrosophic Ring [325, 326]

5. Neutrosophic linear equations [327, 328]

6. Neutrosophic Probability [329–331]

7. Neutrosophic Logics [320, 332, 333]

8. Neutrosophic Algebraic Structures [334, 335]

9. Neutrosophic Vector [336–338]

10. Neutrosophic Matroid [339, 340]

11. NeutroGeometry [341–343]

12. Neutrosophic psychology [344–348]

13. Neutrosophic Likert [349–351]

14. Neutrosophic Statistics [321, 352]

15. Neutrosophic Physics [353]

16. Neutrosophic Sociology [354]

17. Neutrosophic Automata [355–357]

A *single-valued neutrosophic graph* (SVNG) enriches a crisp graph by assigning to every vertex and every edge three independent degrees-*truth*, *indeterminacy*, and *falsity*-each taking values in $[0, 1]$. This allows one to model, simultaneously, positive support, uncertainty, and opposition (cf. [358–360]).

Definition 2.7.1 (Single-Valued Neutrosophic Graph (SVNG))**.** [61] Let $G^* = (V, E)$ be a finite crisp graph (directed or undirected). A *single-valued neutrosophic graph* on G^* is a pair $G = (A, B)$ consisting of:

- a *single-valued neutrosophic vertex set*

$$A = (T_A, I_A, F_A), \qquad T_A, I_A, F_A : V \to [0,1],$$

 and

- a *single-valued neutrosophic edge relation*

$$B = (T_B, I_B, F_B), \qquad T_B, I_B, F_B : E \to [0,1].$$

For $v \in V$, the values $T_A(v)$, $I_A(v)$, and $F_A(v)$ represent the truth-, indeterminacy-, and falsity-membership degrees of v. For an edge $e \in E$, the values $T_B(e)$, $I_B(e)$, and $F_B(e)$ represent the corresponding degrees of e.

These data satisfy the endpoint-compatibility inequalities: for every edge $xy \in E$,

$$T_B(xy) \le \min\{T_A(x), T_A(y)\}, \qquad I_B(xy) \le \min\{I_A(x), I_A(y)\}, \qquad F_B(xy) \le \max\{F_A(x), F_A(y)\}.$$

If G^* is undirected, one typically assumes that B is symmetric, i.e., $T_B(xy) = T_B(yx)$, $I_B(xy) = I_B(yx)$, and $F_B(xy) = F_B(yx)$ whenever both arcs are present; otherwise the SVNG is directed.

Example 2.7.2 (An SVNG on a 4-cycle)**.** Let $V = \{v_1, v_2, v_3, v_4\}$ and let $E = \{v_1v_2,\ v_2v_3,\ v_3v_4,\ v_4v_1\}$. Define the vertex labels by

$$A(v_1) = (0.5, 0.1, 0.4), \qquad A(v_2) = (0.6, 0.3, 0.2),$$
$$A(v_3) = (0.2, 0.3, 0.4), \qquad A(v_4) = (0.4, 0.2, 0.5),$$

and the edge labels by

$$B(v_1v_2) = (0.2, 0.1, 0.4), \qquad B(v_2v_3) = (0.2, 0.3, 0.4),$$
$$B(v_3v_4) = (0.2, 0.2, 0.5), \qquad B(v_4v_1) = (0.1, 0.1, 0.5).$$

A direct componentwise check shows that the inequalities in Definition 2.7.1 hold for each edge in E; hence $G = (A, B)$ is a single-valued neutrosophic graph.

Proposition 2.7.3 (A canonical reduction from SVNGs to fuzzy graphs)**.** *Every single-valued neutrosophic graph induces a (Rosenfeld-type) fuzzy graph.*

Proof. Let $G = (A, B)$ be an SVNG on $G^* = (V, E)$. Define $\sigma : V \to [0,1]$ and $\mu : E \to [0,1]$ by

$$\sigma(x) := T_A(x), \qquad \mu(xy) := \max\{0,\ T_B(xy) - I_B(xy) - F_B(xy)\} \qquad (xy \in E).$$

Then $\mu(xy) \in [0,1]$ because $\mu(xy) \le T_B(xy) \le 1$ and $\mu(xy) \ge 0$ by definition. Moreover, since $\mu(xy) \le T_B(xy)$ and

$$T_B(xy) \le \min\{T_A(x), T_A(y)\} = \min\{\sigma(x), \sigma(y)\},$$

we obtain the fuzzy-graph compatibility condition $\mu(xy) \le \sigma(x) \wedge \sigma(y)$ for all $xy \in E$. Hence (V, E, σ, μ) is a fuzzy graph. □

Proposition 2.7.4 (Projection from SVNGs to intuitionistic fuzzy graphs)**.** *Every single-valued neutrosophic graph induces an intuitionistic fuzzy graph by discarding indeterminacy and projecting falsity so that the Atanassov constraint holds.*

Proof. Let $G = (A, B)$ be an SVNG on $G^* = (V, E)$. Define intuitionistic fuzzy vertex functions (μ_A, ν_A) and edge functions (μ_B, ν_B) by

$$\mu_A(x) := T_A(x), \qquad \nu_A(x) := \min\{F_A(x),\, 1 - T_A(x)\} \qquad (x \in V),$$

$$\mu_B(xy) := T_B(xy), \qquad \nu_B(xy) := \min\{F_B(xy),\, 1 - T_B(xy)\} \qquad (xy \in E).$$

Then $\mu_A(x)+\nu_A(x) \le 1$ for all $x \in V$ and $\mu_B(xy)+\nu_B(xy) \le 1$ for all $xy \in E$ by construction.

Furthermore, the SVNG endpoint constraint gives

$$\mu_B(xy) = T_B(xy) \le \min\{T_A(x), T_A(y)\} = \min\{\mu_A(x), \mu_A(y)\}.$$

Also, $\nu_B(xy) \le F_B(xy) \le \max\{F_A(x), F_A(y)\}$ and $\nu_A(\cdot) \le F_A(\cdot)$ pointwise, hence

$$\nu_B(xy) \le \max\{\nu_A(x), \nu_A(y)\}.$$

Therefore $(V, E, \mu_A, \nu_A, \mu_B, \nu_B)$ satisfies the usual intuitionistic-fuzzy graph inequalities. Equivalently, this construction may be interpreted as setting indeterminacy to 0 and then enforcing $\mu + \nu \le 1$ via the above projection of the falsity component. □

2.7.1 Graph class related to neutrosophic graph

We describe the graph classes related to neutrosophic graphs. Due to the applicability and significance of neutrosophic graphs, many related graph classes have been extensively studied. The definitions are provided below.

Theorem 2.7.5. *The following are examples of related Neutrosophic graph classes, including but not limited to:*

- *Bipolar Neutrosophic Graphs [361]*
- *Interval-Valued Neutrosophic Graphs [362]*
- *Single-Valued Neutrosophic Graphs [61, 363]*
- *Interval Complex Neutrosophic Graphs [364, 365]*
- *Neutrosophic Hypergraphs [64, 363, 366, 367]*
- *Neutrosophic Vague Line Graphs [364]*

- *Neutrosophic Vague Graphs [364, 368, 369]*
- *Single-Valued Neutrosophic Signed Graphs [76]*
- *Neutrosophic Soft Rough Graphs [370]*
- *Neutrosophic Incidence Graphs [72, 371]*
- *Fermatean Neutrosophic Graphs [372, 373]*
- *Heptapartitioned Neutrosophic graphs [374, 375]*
- *Interval-valued pentapartitioned neutrosophic graphs [365]*
- *Single-valued pentapartitioned neutrosophic graphs [376]*
- *pentapartitioned neutrosophic graphs citequek2022new*
- *t-Neutrosophic Fuzzy graph [377]*
- *Complex t-Neutrosophic Graph [378]*
- *Regular neutrosophic graphs [63, 379]*
- *Q-neutrosophic soft graphs [54]*
- *Balanced Neutrosophic Graphs [37, 380]*
- *Neutrosophic minimum spanning tree graph [381]*
- *HyperNeutrosophic Graph [20, 382]*
- *SuperHyperNeutrosophic Graph [20]*

Proof. Refer to each reference as needed. □

2.7.2 Single-Valued Neutrosophic Hypergraph

A *single-valued neutrosophic hypergraph* (SVNH) extends a crisp hypergraph by allowing each hyperedge to carry three membership profiles (truth, indeterminacy, falsity) over the vertex set, thereby representing multiway relations together with graded support, uncertainty, and opposition (cf. [363, 363, 366, 367, 383]).

Definition 2.7.6 (Single-Valued Neutrosophic Hypergraph (SVNH))**.** [363] Let $V = \{v_1, \ldots, v_n\}$ be a finite set of vertices. A *single-valued neutrosophic hyperedge* on V is a triple

$$E = (T_E, I_E, F_E), \qquad T_E, I_E, F_E : V \to [0, 1].$$

For $v \in V$, the values $T_E(v)$, $I_E(v)$, and $F_E(v)$ are interpreted as the truth-, indeterminacy-, and falsity-membership degrees of v in the hyperedge E, respectively.

The *support* of E is the set

$$\mathrm{supp}(E) \ := \ \{\, v \in V : \ T_E(v) + I_E(v) + F_E(v) > 0 \,\}.$$

A *single-valued neutrosophic hypergraph* is a pair

$$H = (V, \mathcal{E}),$$

where $\mathcal{E} = \{E_1, \ldots, E_m\}$ is a finite family of single-valued neutrosophic hyperedges on V (cf. [363]).

The following notions are standard for SVNHs.

- **Vertex adjacency.** Two vertices $u, v \in V$ are *adjacent* if there exists $E \in \mathcal{E}$ such that $u, v \in \mathrm{supp}(E)$.
- **Connectivity.** The SVNH H is *connected* if for every pair $u, v \in V$ there exists a sequence $u = x_0, x_1, \ldots, x_k = v$ such that x_{i-1} and x_i are adjacent for each i.
- **Hyperedge adjacency.** Two hyperedges $E_i, E_j \in \mathcal{E}$ are *adjacent* if $\mathrm{supp}(E_i) \cap \mathrm{supp}(E_j) \neq \emptyset$.
- **Order and size.** The *order* of H is $|V|$, and the *size* of H is $|\mathcal{E}|$.
- **Neutrosophic cardinality (degree) of a hyperedge.** For $E \in \mathcal{E}$, define

  $$d_H(E) \ := \ \sum_{v \in V} \big(T_E(v) + I_E(v) + F_E(v)\big) \ = \ \sum_{v \in \mathrm{supp}(E)} \big(T_E(v) + I_E(v) + F_E(v)\big).$$

  (Vertices outside the support contribute 0.)
- **Rank and anti-rank.** The *rank* and *anti-rank* of H are

  $$r(H) := \max_{E \in \mathcal{E}} d_H(E), \qquad \underline{r}(H) := \min_{E \in \mathcal{E}} d_H(E).$$

- **Linearity.** The SVNH H is *linear* if any two distinct hyperedges meet in at most one vertex, i.e.,
$$\left|\operatorname{supp}(E_i) \cap \operatorname{supp}(E_j)\right| \leq 1 \qquad (\forall\, E_i \neq E_j \in \mathcal{E}).$$

Remark. Since $T_E(v), I_E(v), F_E(v) \in [0,1]$, one automatically has $0 \leq T_E(v) + I_E(v) + F_E(v) \leq 3$ for all $v \in V$; thus the usual "membership-sum bound" does not impose an additional restriction in the single-valued setting.

Example 2.7.7 (An explicit SVNH)**.** Let
$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}, \qquad \mathcal{E} = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}.$$
Each hyperedge $E_k = (T_{E_k}, I_{E_k}, F_{E_k})$ is specified by listing its nonzero vertex triples, and setting $(T_{E_k}(v), I_{E_k}(v), F_{E_k}(v)) = (0,0,0)$ for all other $v \in V$.

$$\begin{aligned}
E_1 &: \ (v_1, 0.3, 0.4, 0.6),\ (v_3, 0.7, 0.4, 0.4),\\
E_2 &: \ (v_1, 0.3, 0.4, 0.6),\ (v_2, 0.5, 0.7, 0.6),\\
E_3 &: \ (v_2, 0.5, 0.7, 0.6),\ (v_4, 0.6, 0.4, 0.8),\\
E_4 &: \ (v_3, 0.7, 0.4, 0.4),\ (v_6, 0.4, 0.2, 0.7),\\
E_5 &: \ (v_3, 0.7, 0.4, 0.4),\ (v_5, 0.6, 0.7, 0.5),\\
E_6 &: \ (v_5, 0.6, 0.7, 0.5),\ (v_6, 0.4, 0.2, 0.7),\\
E_7 &: \ (v_4, 0.6, 0.4, 0.8),\ (v_6, 0.4, 0.2, 0.7).
\end{aligned}$$

For instance, $\operatorname{supp}(E_1) = \{v_1, v_3\}$ and $\operatorname{supp}(E_5) = \{v_3, v_5\}$, hence v_1 and v_3 are adjacent, and the hyperedges E_1 and E_5 are adjacent because they share the vertex v_3 in their supports. Moreover,
$$d_H(E_1) = (0.3 + 0.4 + 0.6) + (0.7 + 0.4 + 0.4) = 2.8,$$
and analogous computations give $d_H(E_k)$ for the other hyperedges. Finally, the structure is linear whenever every pair of distinct supports intersects in at most one vertex (e.g., $\operatorname{supp}(E_5) \cap \operatorname{supp}(E_6) = \{v_5\}$ and $\operatorname{supp}(E_4) \cap \operatorname{supp}(E_7) = \{v_6\}$).

2.7.3 Single-valued Neutrosophic n-Superhypergraph

A *Single-valued Neutrosophic n-Superhypergraph* [384] is a concept that generalizes both the Single-valued Neutrosophic graph [385–387] and the Single-valued Neutrosophic hypergraph [388, 389]. It also extends the notion of a Fuzzy n-Superhypergraph. The formal definition and a representative example are given below(cf. [390]).

Definition 2.7.8 (Neutrosophic n-Superhypergraph)**.** (cf. [384, 390]) Let V_0 be a finite *base set* of vertices, and for each integer $k \geq 0$ define
$$\mathcal{P}^0(V_0) = V_0,$$
$$\mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big),$$

where $\mathcal{P}(\cdot)$ denotes the usual powerset. An *n-Superhypergraph* is a pair

$$\mathrm{SHG}^{(n)} = (V,\, E), \quad V \;\subseteq\; \mathcal{P}^n(V_0), \;\; E \;\subseteq\; \mathcal{P}^n(V_0).$$

A *Neutrosophic n-Superhypergraph* is then the tuple

$$\bigl(V,\, E,\, T_V,\, I_V,\, F_V,\, T_E,\, I_E,\, F_E\bigr),$$

where

- $T_V,\, I_V,\, F_V : V \to [0,1]$ assign to each n-supervertex $v \in V$ its truth-, indeterminacy-, and falsity-membership degrees, respectively, subject to

$$0 \;\le\; T_V(v) + I_V(v) + F_V(v) \;\le\; 3,$$
$$\forall v \in V.$$

- $T_E,\, I_E,\, F_E : E \times V \to [0,1]$ assign to each n-superedge $e \in E$ and vertex $v \in e$ its truth-, indeterminacy-, and falsity-membership degrees, respectively, subject to

$$0 \;\le\; T_E(e,v) + I_E(e,v) + F_E(e,v) \;\le\; 3,$$
$$\forall e \in E,\; \forall v \in e.$$

These functions satisfy the *edge-appurtenance constraints*:

$$T_E(e,v) \;\le\; T_V(v),$$
$$I_E(e,v) \;\le\; I_V(v),$$
$$F_E(e,v) \;\le\; F_V(v),$$
$$\forall\, e \in E,\; \forall\, v \in e.$$

Example 2.7.9 (A neutrosophic 2-Superhypergraph)**.** Let the base set be

$$V_0 = \{1,2,3\}, \qquad n = 2,$$

so $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$. Define the 2-supervertex set

$$V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0)$$

by

$$v_1 = \{\{1\},\{1,2\}\}, \qquad v_2 = \{\{2\},\{2,3\}\}, \qquad v_3 = \{\{1,3\}\}.$$

Define the 2-superedge family

$$E = \{e_1, e_2\}, \qquad e_1 = \{v_1, v_2\}, \quad e_2 = \{v_1, v_2, v_3\}.$$

Thus $\mathrm{SHG}^{(2)} = (V,E)$ is a 2-Superhypergraph in the sense of Definition 2.7.8.

Vertex neutrosophic degrees. Define $T_V, I_V, F_V : V \to [0,1]$ by

$$(T_V, I_V, F_V)(v_1) = (0.6, 0.3, 0.4),$$
$$(T_V, I_V, F_V)(v_2) = (0.8, 0.2, 0.5),$$

$$(T_V, I_V, F_V)(v_3) = (0.4, 0.4, 0.6).$$

Each triple satisfies $0 \leq T_V(v) + I_V(v) + F_V(v) \leq 3$; for example, $0.6 + 0.3 + 0.4 = 1.3 \leq 3$.

Edge–vertex neutrosophic degrees. Define $T_E, I_E, F_E : E \times V \to [0, 1]$ by specifying values only when $v \in e$ (and setting the value to 0 otherwise):

$$(T_E, I_E, F_E)(e_1, v_1) = (0.5, 0.2, 0.3), \qquad (T_E, I_E, F_E)(e_1, v_2) = (0.7, 0.1, 0.4),$$
$$(T_E, I_E, F_E)(e_2, v_1) = (0.4, 0.2, 0.3),$$
$$(T_E, I_E, F_E)(e_2, v_2) = (0.6, 0.2, 0.5),$$
$$(T_E, I_E, F_E)(e_2, v_3) = (0.3, 0.3, 0.5).$$

Each assigned triple satisfies $0 \leq T_E(e, v)+I_E(e, v)+F_E(e, v) \leq 3$; for instance, $0.7+0.1+0.4 = 1.2 \leq 3$.

Verification of the edge-appurtenance constraints. For every incidence (e, v) with $v \in e$, we have componentwise

$$T_E(e, v) \leq T_V(v), \qquad I_E(e, v) \leq I_V(v), \qquad F_E(e, v) \leq F_V(v).$$

For example, for (e_1, v_2),

$$0.7 \leq 0.8, \qquad 0.1 \leq 0.2, \qquad 0.4 \leq 0.5,$$

and similarly for the remaining incidences. Therefore

$$\big(V, E, T_V, I_V, F_V, T_E, I_E, F_E\big)$$

is a neutrosophic 2-Superhypergraph in the sense of Definition 2.7.8.

2.7.4 Application of neutrosophic graph

We introduce the applications of Neutrosophic Graphs. Neutrosophic Graphs, which can generalize fuzzy graphs, hold potential for a wide range of applications. Below, we present an example of such applications.

- Decision-making: Several papers have explored the use of Neutrosophic Graphs in Decision-making [391–394].
- COVID-19: Several papers have explored the use of Neutrosophic Graphs in COVID-19 countermeasures [395]
- Hospital Infrastructure Design: A paper has explored the use of Neutrosophic raphs in Hospital Infrastructure Design [396].
- Internet Streaming Services: Internet streaming services deliver digital content like video, music, or live broadcasts to users via the internet in real-time (cf. [397]). The study is conducted in [398].

- Natural disaster: A natural disaster is a catastrophic event caused by natural processes, such as earthquakes, tsunamis, hurricanes, or floods, leading to widespread damage and loss of life (cf. [399, 400]). In the study [401], the potential application of Neutrosophic Graphs in Japan's earthquake response is being explored. Additionally, [402] e xamines response methods for tsunami management in Japan.

- Networks: Several papers have explored the use of Neutrosophic Graphs in mobile network [403] and wireless networks [404, 405].

- **Neural networks:** A wide range of studies has been conducted in this area, similar to those on fuzzy graphs [117, 406–410].

Chapter 3

Reviews and Results of Graph Classes

The results in this book are presented as follows.

3.1 Classes of general intuitionistic fuzzy graphs

In this section, we investigate graph classes associated with general intuitionistic fuzzy graphs.

3.1.1 General intuitionistic fuzzy graphs

Motivated by the notion of *general fuzzy graphs* and related "general" uncertainty-graph formalisms [37,38], we introduce a corresponding generalization for intuitionistic fuzzy graphs. The key idea is to keep the Atanassov constraints (membership + non-membership ≤ 1) on vertices and edges, while *not* enforcing any fixed endpoint inequality linking edge-values to vertex-values.

Definition 3.1.1 (General intuitionistic fuzzy graph)**.** Let V be a nonempty finite set and let $E \subseteq V \times V$ be a (crisp) set of edges. A *general intuitionistic fuzzy graph* on (V, E) is a quadruple

$$G_{GIF} = (\mu_A, \nu_A, \mu_B, \nu_B),$$

where

$$\mu_A, \nu_A : V \to [0, 1], \qquad \mu_B, \nu_B : E \to [0, 1],$$

satisfy the Atanassov bounds

$$0 \leq \mu_A(v) + \nu_A(v) \leq 1 \quad (\forall v \in V), \qquad 0 \leq \mu_B(e) + \nu_B(e) \leq 1 \quad (\forall e \in E).$$

Here μ_A and ν_A are the membership and non-membership functions on vertices, and μ_B and ν_B are the membership and non-membership functions on edges. Unlike the classical intuitionistic fuzzy graph, no endpoint-compatibility constraints are imposed in the general setting.

Definition 3.1.2 (Weak general intuitionistic fuzzy graph)**.** A *weak general intuitionistic fuzzy graph* on (V, E) is a general intuitionistic fuzzy graph $G_{WGIF} = (\mu_A, \nu_A, \mu_B, \nu_B)$ satisfying the additional condition

$$\mu_B(uv) \; < \; \mu_A(u) \wedge \mu_A(v) \qquad (\forall\, uv \in E),$$

where $\wedge$ denotes the minimum operation.

Remark 3.1.3. The strict inequality in Definition 3.1.2 ensures that the edge-membership is not determined by (nor equal to) the standard endpoint expression $\mu_A(u) \wedge \mu_A(v)$. Using "$\neq$" instead of "$<$" does *not* yield a meaningful "weak" class, because then classical intuitionistic fuzzy graphs (which satisfy equality) would fail to be weak. The choice "$<$" aligns with the common usage of "weak" in fuzzy graph theory (compare weak fuzzy graphs vs. strong fuzzy graphs).

Theorem 3.1.4 (Classical $\Rightarrow$ general)**.** *Every (classical) intuitionistic fuzzy graph is a general intuitionistic fuzzy graph.*

Proof. Let G_{IF} be a classical intuitionistic fuzzy graph on (V, E). Then its vertex and edge maps (μ_A, ν_A) and (μ_B, ν_B) satisfy the Atanassov bounds $0 \le \mu_A + \nu_A \le 1$ on V and $0 \le \mu_B + \nu_B \le 1$ on E, together with additional endpoint constraints. Since Definition 3.1.1 requires only the Atanassov bounds, G_{IF} is (a fortiori) a general intuitionistic fuzzy graph. □

Theorem 3.1.5 (Weak general $\Rightarrow$ general)**.** *Every weak general intuitionistic fuzzy graph is a general intuitionistic fuzzy graph.*

Proof. This is immediate from the definitions: a weak general intuitionistic fuzzy graph is, by Definition 3.1.2, a general intuitionistic fuzzy graph with an extra inequality. □

Theorem 3.1.6 (Forgetting non-membership: weak general intuitionistic $\Rightarrow$ weak general fuzzy)**.** *Let $G_{WGIF} = (\mu_A, \nu_A, \mu_B, \nu_B)$ be a weak general intuitionistic fuzzy graph on (V, E). Define $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ by*

$$\sigma(v) := \mu_A(v) \quad (v \in V), \qquad \mu(e) := \mu_B(e) \quad (e \in E).$$

Then $G_{WG} = (\sigma, \mu)$ is a weak general fuzzy graph (i.e., $\mu(uv) < \sigma(u) \wedge \sigma(v)$ for all $uv \in E$).

Proof. By Definition 3.1.2, for each $uv \in E$ we have

$$\mu_B(uv) \; < \; \mu_A(u) \wedge \mu_A(v).$$

With $\sigma = \mu_A$ and $\mu = \mu_B$, this becomes

$$\mu(uv) \; < \; \sigma(u) \wedge \sigma(v),$$

which is exactly the defining inequality of a weak general fuzzy graph. □

Corollary 3.1.7 (Forgetting non-membership: general intuitionistic $\Rightarrow$ general fuzzy)**.** *Every general intuitionistic fuzzy graph induces a general fuzzy graph by ignoring the non-membership functions.*

Proof. Let $G_{GIF} = (\mu_A, \nu_A, \mu_B, \nu_B)$ be a general intuitionistic fuzzy graph. Set $\sigma := \mu_A$ on V and $\mu := \mu_B$ on E. Then (σ, μ) is a general fuzzy graph in the sense that it consists of a fuzzy vertex set and a fuzzy edge set with no endpoint constraint. □

3.1.2 General intuitionistic fuzzy hypergraphs

A general intuitionistic fuzzy hypergraph assigns membership and non-membership degrees to vertices and hyperedges, satisfying Atanassov bounds, without endpoint constraints.

Definition 3.1.8 (General intuitionistic fuzzy hypergraph)**.** Let $H^* = (V, E)$ be a finite crisp hypergraph, where V is a finite nonempty set and $E \subseteq \mathcal{P}^*(V)$ is a finite family of nonempty subsets of V. A *general intuitionistic fuzzy hypergraph* on H^* is a quadruple

$$H_{GIF} = \big(V, E;\ \mu_V, \nu_V, \mu_E, \nu_E\big),$$

where

$$\mu_V, \nu_V : V \to [0, 1], \qquad \mu_E, \nu_E : E \to [0, 1],$$

satisfy the Atanassov bounds

$$0 \le \mu_V(v) + \nu_V(v) \le 1 \quad (\forall v \in V), \qquad 0 \le \mu_E(e) + \nu_E(e) \le 1 \quad (\forall e \in E).$$

No additional constraints linking $\mu_E(e)$ or $\nu_E(e)$ to the values on the vertices of e are required at this level of generality.

Theorem 3.1.9 (General intuitionistic fuzzy hypergraphs generalize crisp hypergraphs)**.** *Every crisp hypergraph $H^* = (V, E)$ can be viewed canonically as a general intuitionistic fuzzy hypergraph.*

Proof. Let $H^* = (V, E)$ be a crisp hypergraph. Define constant maps

$$\mu_V(v) := 1, \quad \nu_V(v) := 0 \qquad (\forall v \in V),$$
$$\mu_E(e) := 1, \quad \nu_E(e) := 0 \qquad (\forall e \in E).$$

Then $\mu_V + \nu_V = 1$ on V and $\mu_E + \nu_E = 1$ on E, so the Atanassov bounds hold. Hence $H_{GIF} = (V, E; \mu_V, \nu_V, \mu_E, \nu_E)$ is a general intuitionistic fuzzy hypergraph in the sense of Definition 3.1.8. This construction preserves the underlying incidence structure because the crisp hyperedges remain exactly the elements of E. □

3.1.3 General intuitionistic fuzzy n-SuperHyperGraphs

A general intuitionistic fuzzy n-SuperHyperGraph assigns membership and non-membership degrees to supervertices and superedges, satisfying Atanassov bounds, independently.

Definition 3.1.10 (General intuitionistic fuzzy n-SuperHyperGraph)**.** Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph on V_0, i.e.,

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

A *general intuitionistic fuzzy n-SuperHyperGraph* on $\mathrm{SHG}^{(n)}$ is a quadruple

$$\mathcal{S}^{(n)}_{GIF} = \big(V, E;\ \mu_V, \nu_V, \mu_E, \nu_E\big),$$

where

$$\mu_V, \nu_V : V \to [0, 1], \qquad \mu_E, \nu_E : E \to [0, 1],$$

satisfy the Atanassov bounds

$$0 \le \mu_V(v) + \nu_V(v) \le 1 \quad (\forall v \in V), \qquad 0 \le \mu_E(e) + \nu_E(e) \le 1 \quad (\forall e \in E).$$

Again, no additional constraints between superedge degrees and the degrees of contained supervertices are imposed in the general model.

Example 3.1.11 (Healthcare pathways across nested patient cohorts as a general intuitionistic fuzzy 2-SuperHyperGraph)**.** Let V_0 be a finite set of patients in a hospital network. Take $n = 2$, so $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$ consists of *sets of cohorts*, where each cohort is a subset of patients.

Supervertices (nested cohorts). Each 2-supervertex is a set of cohorts representing a higher-level clinical category. For instance, define

$$v_1 = \{\, C_{\text{diabetes}},\, C_{\text{hypertension}} \,\}, \qquad v_2 = \{\, C_{\text{cardio}},\, C_{\text{stroke-risk}} \,\}, \qquad v_3 = \{\, C_{\text{renal}},\, C_{\text{elderly}} \,\},$$

where each $C_{\bullet} \subseteq V_0$ is a patient cohort (e.g., all patients diagnosed with diabetes). Let

$$V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0).$$

Superedges (multiway care pathways). Let $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ encode multi-department pathways that jointly target several higher-level categories, e.g.,

$$e_1 = \{v_1, v_2\}, \qquad e_2 = \{v_1, v_2, v_3\}, \qquad E = \{e_1, e_2\}.$$

Here e_2 represents an integrated chronic-care program requiring coordination across metabolic, cardiovascular, and renal cohorts.

General intuitionistic fuzzy degrees. Assign membership and non-membership degrees to supervertices and superedges:

$$(\mu_V, \nu_V)(v_1) = (0.7, 0.2), \quad (\mu_V, \nu_V)(v_2) = (0.6, 0.3), \quad (\mu_V, \nu_V)(v_3) = (0.5, 0.4),$$
$$(\mu_E, \nu_E)(e_1) = (0.4, 0.5), \qquad (\mu_E, \nu_E)(e_2) = (0.6, 0.3).$$

These satisfy the Atanassov bounds:

$$0 \le 0.7 + 0.2 \le 1, \quad 0 \le 0.6 + 0.3 \le 1, \quad 0 \le 0.5 + 0.4 \le 1,$$
$$0 \le 0.4 + 0.5 \le 1, \qquad 0 \le 0.6 + 0.3 \le 1.$$

Therefore

$$\mathcal{S}^{(2)}_{GIF} = \big(V, E; \mu_V, \nu_V, \mu_E, \nu_E\big)$$

is a general intuitionistic fuzzy 2-SuperHyperGraph in the sense of Definition 3.1.10. In this interpretation, μ_V and ν_V quantify inclusion and exclusion evidence for high-level cohort categories, while μ_E and ν_E quantify support and opposition for multiway care pathways, without enforcing any fixed coupling between them.

Theorem 3.1.12 (General intuitionistic fuzzy n-SuperHyperGraphs generalize crisp n-SuperHyperGraphs)**.** *Every crisp n-SuperHyperGraph* $\mathrm{SHG}^{(n)} = (V, E)$ *can be viewed canonically as a general intuitionistic fuzzy n-SuperHyperGraph.*

Proof. Let $\mathrm{SHG}^{(n)} = (V, E)$ be a crisp n-SuperHyperGraph. Define constant maps

$$\mu_V(v) := 1, \quad \nu_V(v) := 0 \qquad (\forall v \in V),$$
$$\mu_E(e) := 1, \quad \nu_E(e) := 0 \qquad (\forall e \in E).$$

Then the Atanassov bounds hold on V and on E, and $\mathcal{S}^{(n)}_{GIF} = (V, E; \mu_V, \nu_V, \mu_E, \nu_E)$ is a general intuitionistic fuzzy n-SuperHyperGraph by Definition 3.1.10. The underlying set-based superhypergraph structure (V, E) is unchanged. □

3.2 Graph Classes of Quadripartitioned Neutrosophic Graphs

In this section, we investigate graph classes associated with quadripartitioned neutrosophic graphs.

3.2.1 Single-Valued Quadripartitioned Neutrosophic Graph

A *Single-Valued Quadripartitioned Neutrosophic Graph* (SVQPNG) is a quadripartitioned neutrosophic graph in which all membership degrees take values in $[0,1]$. Equivalently, it is a QPNG $G = (A, B)$ whose vertex/edge maps range in $[0,1]^4$.

Definition 3.2.1 (Single-Valued Quadripartitioned Neutrosophic Graph)**.** Let $G^* = (V, E)$ be a finite simple graph. A *Single-Valued Quadripartitioned Neutrosophic Graph* is a pair $G = (A, B)$ such that:

- $A : V \to [0,1]^4$, $A(v) = (t_A(v), c_A(v), u_A(v), f_A(v))$,
- $B : E \to [0,1]^4$, $B(uv) = (t_B(uv), c_B(uv), u_B(uv), f_B(uv))$,

and the following constraints hold:

$$0 \le t_A(v) + c_A(v) + u_A(v) + f_A(v) \le 4 \quad (\forall v \in V),$$

$$0 \le t_B(uv) + c_B(uv) + u_B(uv) + f_B(uv) \le 4 \quad (\forall uv \in E),$$

$$t_B(uv) \le \min\{t_A(u), t_A(v)\}, \quad c_B(uv) \le \min\{c_A(u), c_A(v)\},$$

$$u_B(uv) \le \max\{u_A(u), u_A(v)\}, \quad f_B(uv) \le \max\{f_A(u), f_A(v)\} \qquad (\forall uv \in E).$$

Example 3.2.2. Let G^* be the triangle on $V = \{V_1, V_2, V_3\}$ with $E = \{V_1V_2, V_2V_3, V_1V_3\}$. Assign the vertex degrees

$$A(V_1) = (0.3, 0.4, 0.2, 0.5), \qquad A(V_2) = (0.2, 0.3, 0.3, 0.4), \qquad A(V_3) = (0.4, 0.4, 0.3, 0.4),$$

so that $t_A + c_A + u_A + f_A \le 4$ holds at each vertex. Define the edge degrees by

$$B(V_1V_2) = (0.2, 0.2, 0.3, 0.4), \qquad B(V_2V_3) = (0.2, 0.3, 0.3, 0.4), \qquad B(V_1V_3) = (0.3, 0.4, 0.2, 0.5).$$

Then, for instance on V_2V_3,

$$t_B(V_2V_3) = 0.2 \le \min\{0.2, 0.4\} = 0.2, \qquad c_B(V_2V_3) = 0.3 \le \min\{0.3, 0.4\} = 0.3,$$

$$u_B(V_2V_3) = 0.3 \le \max\{0.3, 0.3\} = 0.3, \qquad f_B(V_2V_3) = 0.4 \le \max\{0.4, 0.4\} = 0.4.$$

The other edges are checked similarly; hence $G = (A, B)$ is a SVQPNG.

Theorem 3.2.3 (Projection to a single-valued neutrosophic graph)**.** *Let $G = (A, B)$ be a SVQPNG on $G^* = (V, E)$. Define a triple-valued structure on V and E by*

$$T(v) := t_A(v), \qquad I(v) := \max\{c_A(v), u_A(v)\}, \qquad F(v) := f_A(v),$$

$$T(uv) := t_B(uv), \qquad I(uv) := \max\{c_B(uv), u_B(uv)\}, \qquad F(uv) := f_B(uv).$$

Then T, I, F take values in $[0, 1]$ and satisfy $0 \le T + I + F \le 3$ on both vertices and edges. Moreover, for every $uv \in E$,

$$T(uv) \le \min\{T(u), T(v)\}, \qquad F(uv) \le \max\{F(u), F(v)\},$$

and

$$I(uv) \le \max\{I(u), I(v)\}.$$

In particular, this yields a natural (indeterminacy-aggregating) projection of SVQPNGs to a single-valued neutrosophic-type graph.

Proof. Since $t_A, c_A, u_A, f_A \in [0, 1]$, we have $T(v), I(v), F(v) \in [0, 1]$ and hence $T(v) + I(v) + F(v) \le 3$; similarly for edges.

For the edge constraints, the truth and falsity inequalities follow immediately from the SVQPNG axioms: $T(uv) = t_B(uv) \le \min\{t_A(u), t_A(v)\} = \min\{T(u), T(v)\}$ and $F(uv) = f_B(uv) \le \max\{f_A(u), f_A(v)\} = \max\{F(u), F(v)\}$.

For indeterminacy, using $c_B(uv) \le \min\{c_A(u), c_A(v)\}$ and $u_B(uv) \le \max\{u_A(u), u_A(v)\}$,

$$I(uv) = \max\{c_B(uv), u_B(uv)\} \le \max\Big(\min\{c_A(u), c_A(v)\},\ \max\{u_A(u), u_A(v)\}\Big).$$

Since $\min\{c_A(u), c_A(v)\} \le c_A(u)$ and $\min\{c_A(u), c_A(v)\} \le c_A(v)$, while $\max\{u_A(u), u_A(v)\} \le \max\{I(u), I(v)\}$, it follows that $I(uv) \le \max\{I(u), I(v)\}$. □

Remark 3.2.4. If your manuscript adopts a *stronger* neutrosophic edge-constraint for indeterminacy such as $I(uv) \le \min\{I(u), I(v)\}$, then a guaranteed reduction is obtained by discarding the unknown-part and setting $I(v) := c_A(v)$, $I(uv) := c_B(uv)$. This keeps the min-type constraint because $c_B(uv) \le \min\{c_A(u), c_A(v)\}$ holds in every SVQPNG.

3.2.2 General Quadripartitioned Neutrosophic Graph

Next, we consider the General Quadripartitioned Neutrosophic Graph. Similar to how a General Fuzzy Graph is a generalization of a Fuzzy Graph, a General Quadripartitioned Neutrosophic Graph is a generalization of a Quadripartitioned Neutrosophic Graph. The definition is provided below.

Definition 3.2.5 (General Quadripartitioned Neutrosophic Graph)**.** Let $G^* = (V, E)$ be a finite simple graph. A *General Quadripartitioned Neutrosophic Graph* is an assignment of four membership degrees

$$t(v), c(v), u(v), f(v) \in [0, 1] \quad (\forall v \in V),$$

$$t(e), c(e), u(e), f(e) \in [0, 1] \quad (\forall e \in E),$$

such that

$$0 \le t(v) + c(v) + u(v) + f(v) \le 4 \quad (\forall v \in V), \qquad 0 \le t(e) + c(e) + u(e) + f(e) \le 4 \quad (\forall e \in E),$$

with *no* further constraints relating edge degrees to endpoint degrees.

Definition 3.2.6 (Weak General Quadripartitioned Neutrosophic Graph)**.** A *Weak General Quadripartitioned Neutrosophic Graph* is a general QPNG in which, for every edge $uv \in E$, the following inequalities hold:

$$t(uv) \le \min\{t(u), t(v)\}, \qquad c(uv) \le \min\{c(u), c(v)\},$$

$$u(uv) \le \max\{u(u), u(v)\}, \qquad f(uv) \le \max\{f(u), f(v)\}.$$

Theorem 3.2.7. *Every (single-valued) quadripartitioned neutrosophic graph is a weak general quadripartitioned neutrosophic graph.*

Proof. This is immediate because the defining edge inequalities of a QPNG are exactly the additional conditions required in the weak general case. □

Theorem 3.2.8. *Every weak general quadripartitioned neutrosophic graph is a general quadripartitioned neutrosophic graph.*

Proof. A weak general QPNG satisfies all axioms of a general QPNG, with extra inequalities relating edges to endpoints; hence it is, in particular, a general QPNG. □

Definition 3.2.9 (Union, c-complement, and $|\mu|$-complement (general case))**.** Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two general quadripartitioned neutrosophic graphs with underlying crisp graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Write $\sigma_k(x) = (t_k(x), c_k(x), u_k(x), f_k(x))$ for vertices and $\mu_k(e) = (t_k(e), c_k(e), u_k(e), f_k(e))$ for edges.

Union. Define $G = G_1 \cup G_2 = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ on $V = V_1 \cup V_2$, $E = E_1 \cup E_2$ by:

$$(\sigma_1 \cup \sigma_2)(x) =$$

$$\begin{cases} \sigma_1(x) & x \in V_1 \setminus V_2, \\ \sigma_2(x) & x \in V_2 \setminus V_1, \\ \big(\max\{t_1(x), t_2(x)\}, \max\{c_1(x), c_2(x)\}, \max\{u_1(x), u_2(x)\}, \max\{f_1(x), f_2(x)\}\big) & x \in V_1 \cap V_2, \end{cases}$$

and similarly for edges:

$$(\mu_1 \cup \mu_2)(e) =$$

$$\begin{cases} \mu_1(e) & e \in E_1 \setminus E_2, \\ \mu_2(e) & e \in E_2 \setminus E_1, \\ \big(\max\{t_1(e), t_2(e)\}, \max\{c_1(e), c_2(e)\}, \max\{u_1(e), u_2(e)\}, \max\{f_1(e), f_2(e)\}\big) & e \in E_1 \cap E_2. \end{cases}$$

c-complement. Define $G^c = (\sigma^c, \mu^c)$ by $\sigma^c = \sigma$ and, for each edge $e \in E$,

$$\mu^c(e) = \begin{cases} (0, 0, 0, 0) & \text{if } \mu(e) = (0, 0, 0, 0), \\ (1 - t(e),\ 1 - c(e),\ 1 - u(e),\ 1 - f(e)) & \text{otherwise.} \end{cases}$$

$|\mu|$-complement. For an edge $uv \in E$, set

$$\sigma(u) \otimes \sigma(v) := \big(\min\{t(u), t(v)\},\ \min\{c(u), c(v)\},\ \max\{u(u), u(v)\},\ \max\{f(u), f(v)\}\big).$$

Then define

$$\mu^{|\mu|}(uv) = \begin{cases} (0,0,0,0) & \text{if } \mu(uv) = (0,0,0,0), \\ \big(|(\sigma(u) \otimes \sigma(v))_t - t(uv)|,\ |(\sigma(u) \otimes \sigma(v))_c - c(uv)|, & \\ \qquad |(\sigma(u) \otimes \sigma(v))_u - u(uv)|,\ |(\sigma(u) \otimes \sigma(v))_f - f(uv)|\big) & \text{otherwise,} \end{cases}$$

where the subscripts $(\cdot)_t, (\cdot)_c, (\cdot)_u, (\cdot)_f$ denote the corresponding components.

Theorem 3.2.10. *General quadripartitioned neutrosophic graphs are closed under c-complement and $|\mu|$-complement.*

Proof. For c-complement, each component $1 - x$ remains in $[0, 1]$, hence the edge-sum is $(1 - t) + (1 - c) + (1 - u) + (1 - f) = 4 - (t + c + u + f) \le 4$, and it is nonnegative.

For $|\mu|$-complement, each component is an absolute difference of two numbers in $[0, 1]$, hence lies in $[0, 1]$; therefore the sum of four components is at most 4. Vertex degrees are unchanged in both operations. □

Theorem 3.2.11. *Under the union operation defined above, the union of two weak general quadripartitioned neutrosophic graphs is again a weak general quadripartitioned neutrosophic graph.*

Proof. Let G_1, G_2 be weak general QPNGs and $G = G_1 \cup G_2$. Because vertex degrees in G are defined by componentwise maxima on overlaps, they are componentwise *not smaller* than the corresponding degrees in either G_1 or G_2. For an edge uv, if it belongs to only one input graph, the required inequalities follow immediately. If $uv \in E_1 \cap E_2$, then each edge component in G is the maximum of two quantities, each of which satisfies the weak general bounds in its own graph; taking maxima preserves the inequalities $t(uv) \le \min\{t(u), t(v)\}$, $c(uv) \le \min\{c(u), c(v)\}$, $u(uv) \le \max\{u(u), u(v)\}$, $f(uv) \le \max\{f(u), f(v)\}$. □

3.2.3 Single-Valued Quadripartitioned Neutrosophic Hypergraph

A *Single-Valued Quadripartitioned Neutrosophic Hypergraph* (SVQNH) generalizes classical hypergraphs by assigning to each vertex-within-a-hyperedge four degrees: truth, contradiction, unknown, and falsity, each in $[0, 1]$.

Definition 3.2.12 (Single-Valued Quadripartitioned Neutrosophic Hypergraph)**.** A *Single-Valued Quadripartitioned Neutrosophic Hypergraph* is a pair $H = (V, E)$ where:

- $V = \{v_1, v_2, \ldots, v_n\}$ is a finite set of vertices.

- $E = \{E_1, E_2, \ldots, E_m\}$ is a finite family of *quadripartitioned neutrosophic subsets* of V. Concretely, each hyperedge E_i is described by four membership functions

$$t_{E_i}, c_{E_i}, u_{E_i}, f_{E_i} : V \to [0, 1],$$

often written as

$$E_i = \{(v,\ t_{E_i}(v),\ c_{E_i}(v),\ u_{E_i}(v),\ f_{E_i}(v)) :\ v \in V\},$$

such that for every $v \in V$,

$$0 \le t_{E_i}(v) + c_{E_i}(v) + u_{E_i}(v) + f_{E_i}(v) \le 4.$$

Define the *support* of a hyperedge E_i by

$$\operatorname{supp}(E_i) := \{v \in V :\ t_{E_i}(v) + c_{E_i}(v) + u_{E_i}(v) > 0\}.$$

(Thus, vertices with purely falsity information and no membership evidence are excluded from the support.)

- **Adjacency of vertices:** Distinct vertices $x, y \in V$ are adjacent if there exists $E_i \in E$ with $x, y \in \operatorname{supp}(E_i)$.

- **Connectivity:** H is connected if every pair of vertices is joined by a finite sequence of pairwise adjacent vertices.

- **Adjacency of hyperedges:** E_i and E_j are adjacent if $\operatorname{supp}(E_i) \cap \operatorname{supp}(E_j) \neq \emptyset$.

- **Order and size:** The order is $|V|$, and the size is $|E|$.

- **Rank and anti-rank:** The rank and anti-rank are defined by

$$\operatorname{rank}(H) = \max_{E_i \in E} |\operatorname{supp}(E_i)|, \qquad \operatorname{arank}(H) = \min_{E_i \in E} |\operatorname{supp}(E_i)|.$$

(Optionally, one may also define a *quadripartitioned weight* of E_i by $\sum_{v \in \operatorname{supp}(E_i)} (t_{E_i}(v) + c_{E_i}(v) + u_{E_i}(v) + f_{E_i}(v))$, but this is distinct from the combinatorial rank.)

- **Linearity:** H is *linear* if

$$|\operatorname{supp}(E_i) \cap \operatorname{supp}(E_j)| \le 1 \qquad (\forall i \neq j).$$

Example 3.2.13. Consider $H = (V, E)$ with

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}, \qquad E = \{E_1, E_2, E_3, E_4, E_5, E_6\},$$

where each E_k lists only its supported vertices (others may be taken as $0, 0, 0, 0$):

$$E_1 = \{(v_1, 0.3, 0.4, 0.2, 0.3), (v_3, 0.4, 0.3, 0.2, 0.4)\},$$
$$E_2 = \{(v_1, 0.3, 0.4, 0.2, 0.3), (v_2, 0.5, 0.6, 0.3, 0.2)\},$$
$$E_3 = \{(v_2, 0.5, 0.6, 0.3, 0.2), (v_4, 0.6, 0.4, 0.3, 0.5)\},$$
$$E_4 = \{(v_3, 0.4, 0.3, 0.2, 0.4), (v_6, 0.3, 0.4, 0.2, 0.5)\},$$
$$E_5 = \{(v_3, 0.4, 0.3, 0.2, 0.4), (v_5, 0.6, 0.5, 0.3, 0.2)\},$$
$$E_6 = \{(v_5, 0.6, 0.5, 0.3, 0.2), (v_6, 0.3, 0.4, 0.2, 0.5)\}.$$

Each listed tuple satisfies $0 \le t + c + u + f \le 4$ (e.g., in E_1, for v_1: $0.3 + 0.4 + 0.2 + 0.3 = 1.2$). Hence this is an SVQNH.

Theorem 3.2.14 (Coarsening SVQNH to a single-valued neutrosophic hypergraph)**.** *Every SVQNH can be transformed into a single-valued neutrosophic hypergraph (SVNH) by collapsing the contradiction and unknown components into a single indeterminacy component.*

Proof. Let $H = (V, E)$ be an SVQNH. For each hyperedge E_i and vertex $v \in V$, define

$$t'_{E_i}(v) = t_{E_i}(v), \qquad i'_{E_i}(v) = \min\{1,\ c_{E_i}(v) + u_{E_i}(v)\}, \qquad f'_{E_i}(v) = f_{E_i}(v).$$

Then $t'_{E_i}, i'_{E_i}, f'_{E_i} : V \to [0,1]$ define a single-valued neutrosophic subset of V for each hyperedge, hence yield a single-valued neutrosophic hypergraph structure on (V, E). □

Corollary 3.2.15. *Every linear SVQNH can be transformed into a linear SVNH by the same construction.*

Proof. The construction in Theorem 3.2.14 does not change the supports $\mathrm{supp}(E_i)$ (since it only aggregates components pointwise), so linearity is preserved. □

3.2.4 Single-Valued Quadripartitioned Neutrosophic n-SuperHyperGraphs

A single-valued quadripartitioned neutrosophic hypergraph (SVQNH) equips each hyperedge with four vertexwise degrees (truth, contradiction, unknown, falsity) in $[0, 1]$. To lift this idea to hierarchical settings, we replace vertices by n-supervertices (elements of an iterated powerset) and allow each *superedge* to carry quadripartitioned neutrosophic information over the supervertex set.

Definition 3.2.16 (Single-Valued Quadripartitioned Neutrosophic n-SuperHyperGraph)**.** Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. Let V be a nonempty finite set of n-supervertices with

$$V \subseteq \mathcal{P}^n(V_0).$$

A *single-valued quadripartitioned neutrosophic n-SuperHyperGraph* (SVQN-nSHG) on V_0 is a pair

$$\mathcal{H}^{(n)}_{\mathrm{Q}} = (V, \mathcal{E}),$$

where $\mathcal{E} = \{E_1, \ldots, E_m\}$ is a finite family of *quadripartitioned neutrosophic subsets* of V. Concretely, each superedge $E_i \in \mathcal{E}$ is specified by four membership maps

$$T_{E_i},\ C_{E_i},\ U_{E_i},\ F_{E_i} : V \to [0, 1],$$

often written as

$$E_i = \big\{(v,\ T_{E_i}(v),\ C_{E_i}(v),\ U_{E_i}(v),\ F_{E_i}(v)) :\ v \in V\big\},$$

such that for every $v \in V$,

$$0 \ \le\ T_{E_i}(v) + C_{E_i}(v) + U_{E_i}(v) + F_{E_i}(v) \ \le\ 4.$$

The *support* of a superedge E_i is defined by

$$\mathrm{supp}(E_i) := \{\, v \in V :\ T_{E_i}(v) + C_{E_i}(v) + U_{E_i}(v) > 0 \,\}.$$

(Thus a supervertex with purely falsity information and no positive/unknown evidence is excluded from the support.)

Example 3.2.17 (Supply-chain risk tiers and cross-tier constraints as an SVQN-2SHG)**.** Let V_0 be a finite set of *atomic suppliers* (individual firms) in a supply network. Take $n = 2$, so a 2-supervertex is an element of $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$, i.e., a *set of supplier groups*, where each supplier group is a subset of V_0.

Supervertices (tiers as sets of groups). Define three 2-supervertices representing high-level *risk tiers*:

$$v_1 = \{G_{\text{critical}}, G_{\text{single-source}}\}, \qquad v_2 = \{G_{\text{regulated}}, G_{\text{high-lead-time}}\}, \qquad v_3 = \{G_{\text{commodity}}, G_{\text{multi-source}}\},$$

where each $G_\bullet \subseteq V_0$ is a group of suppliers (e.g., G_{critical} is the set of suppliers providing critical components). Let

$$V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0).$$

Superedges (multiway constraints among tiers). Let $\mathcal{E} = \{E_1, E_2\}$, where each E_i is a quadripartitioned neutrosophic subset of V. Intuitively, a superedge represents a cross-tier *policy or constraint bundle* involving several tiers simultaneously, such as "export-control compliance" or "multi-tier continuity planning".

Quadripartitioned degrees on supervertices within each superedge. Define E_1 by the nonzero assignments

$$(T_{E_1}, C_{E_1}, U_{E_1}, F_{E_1})(v_1) = (0.8, 0.1, 0.1, 0.2),$$
$$(T_{E_1}, C_{E_1}, U_{E_1}, F_{E_1})(v_2) = (0.6, 0.2, 0.2, 0.3),$$

and set $(T_{E_1}, C_{E_1}, U_{E_1}, F_{E_1})(v_3) = (0, 0, 0, 0)$. Define E_2 by

$$(T_{E_2}, C_{E_2}, U_{E_2}, F_{E_2})(v_1) = (0.5, 0.2, 0.1, 0.4),$$
$$(T_{E_2}, C_{E_2}, U_{E_2}, F_{E_2})(v_2) = (0.4, 0.1, 0.3, 0.3),$$
$$(T_{E_2}, C_{E_2}, U_{E_2}, F_{E_2})(v_3) = (0.3, 0.1, 0.2, 0.2).$$

Each tuple lies in $[0, 1]^4$ and satisfies the total bound $0 \le T + C + U + F \le 4$, e.g., $0.8 + 0.1 + 0.1 + 0.2 = 1.2 \le 4$ and $0.3 + 0.1 + 0.2 + 0.2 = 0.8 \le 4$.

Supports. By Definition 3.2.16, the supports are

$$\operatorname{supp}(E_1) = \{v_1, v_2\} \quad (\text{since } T + C + U > 0 \text{ for } v_1, v_2 \text{ and equals } 0 \text{ for } v_3),$$
$$\operatorname{supp}(E_2) = \{v_1, v_2, v_3\}.$$

Therefore

$$\mathcal{H}_{\mathrm{Q}}^{(2)} = (V, \mathcal{E})$$

is a single-valued quadripartitioned neutrosophic 2-SuperHyperGraph in the sense of Definition 3.2.16. In this interpretation, each superedge encodes a cross-tier policy whose relevance (truth), internal inconsistency (contradiction), unresolved information (unknown), and opposition (falsity) are assessed separately for each tier.

Theorem 3.2.18 (SVQN-nSHGs generalize SVQNHs)**.** *Every single-valued quadripartitioned neutrosophic hypergraph (SVQNH) is a special case of a single-valued quadripartitioned neutrosophic n-SuperHyperGraph, namely the case $n = 0$.*

Proof. Let $H = (V, E)$ be an SVQNH in the sense of Definition 3.2.12. Set the base set $V_0 := V$ and take $n = 0$. Then $\mathcal{P}^0(V_0) = V_0$, so we may identify the 0-supervertex set with V. Define

$$V^{(0)} := V \subseteq \mathcal{P}^0(V_0), \qquad \mathcal{E} := E.$$

Each hyperedge $E_i \in E$ in the SVQNH is already given by four maps $t_{E_i}, c_{E_i}, u_{E_i}, f_{E_i} : V \to [0,1]$ satisfying

$$0 \leq t_{E_i}(v) + c_{E_i}(v) + u_{E_i}(v) + f_{E_i}(v) \leq 4 \qquad (\forall v \in V).$$

Define the corresponding SVQN-0SHG superedge data by

$$T_{E_i} := t_{E_i}, \qquad C_{E_i} := c_{E_i}, \qquad U_{E_i} := u_{E_i}, \qquad F_{E_i} := f_{E_i}.$$

Then $\mathcal{H}_{\mathrm{Q}}^{(0)} = (V^{(0)}, \mathcal{E})$ satisfies Definition 3.2.16, and by construction it carries exactly the same vertexwise quadripartitioned degrees as the original SVQNH. Hence every SVQNH is realized as an SVQN-nSHG with $n = 0$. □

Remark 3.2.19. Conversely, any SVQN-0SHG (i.e., with $n = 0$ and $V \subseteq V_0$) is precisely a single-valued quadripartitioned neutrosophic hypergraph on the vertex set V by the same identification. Thus Theorem 3.2.18 captures an equivalence at level $n = 0$.

3.3 Graph Classes of Single-Valued Pentapartitioned Neutrosophic Graphs

In this section, we discuss graph classes associated with single-valued pentapartitioned neutrosophic graphs.

3.3.1 Single-Valued Pentapartitioned Neutrosophic Graph

A *single-valued pentapartitioned neutrosophic graph* models five components of uncertainty: *truth* (T), *contradiction* (C), *ignorance* (R), *unknown* (U), and *falsity* (F), each taking values in $[0, 1]$.

Definition 3.3.1 (Single-Valued Pentapartitioned Neutrosophic Graph)**.** Let $G^* = (V, E)$ be a finite simple undirected graph, where V is a finite vertex set and $E \subseteq \{\{u, v\} : u, v \in V,\ u \neq v\}$ is the edge set.

A *single-valued pentapartitioned neutrosophic graph* (SVPNG) on G^* is a pair

$$G = (P, Q),$$

where:

- P is a pentapartitioned neutrosophic vertex set on V,

$$P = \{(v, T_P(v), C_P(v), R_P(v), U_P(v), F_P(v)) : v \in V\},$$

  with $T_P, C_P, R_P, U_P, F_P : V \to [0, 1]$ satisfying

$$0 \leq T_P(v) + C_P(v) + R_P(v) + U_P(v) + F_P(v) \leq 5 \qquad (\forall v \in V).$$

- Q is a pentapartitioned neutrosophic edge set on E,

$$Q = \{(e, T_Q(e), C_Q(e), R_Q(e), U_Q(e), F_Q(e)) : e \in E\},$$

with $T_Q, C_Q, R_Q, U_Q, F_Q : E \to [0,1]$ such that for every edge $e = \{u, v\} \in E$,

$$\begin{aligned}
&T_Q(e) \le \min\{T_P(u), T_P(v)\},\\
&C_Q(e) \le \min\{C_P(u), C_P(v)\},\\
&R_Q(e) \ge \max\{R_P(u), R_P(v)\},\\
&U_Q(e) \ge \max\{U_P(u), U_P(v)\},\\
&F_Q(e) \ge \max\{F_P(u), F_P(v)\},\\
&0 \le T_Q(e) + C_Q(e) + R_Q(e) + U_Q(e) + F_Q(e) \le 5.
\end{aligned}$$

Theorem 3.3.2 (SVPNG contains a four-component special case)**.** *Let $G = (P, Q)$ be an SVPNG on $G^* = (V, E)$. Assume that*

$$R_P(v) = 0 \quad (\forall v \in V), \qquad R_Q(e) = 0 \quad (\forall e \in E),$$

and in addition

$$T_P(v) + C_P(v) + U_P(v) + F_P(v) \le 4 \quad (\forall v \in V),$$
$$T_Q(e) + C_Q(e) + U_Q(e) + F_Q(e) \le 4 \quad (\forall e \in E).$$

Then the induced four-tuple labeling

$$(v,\ T_P(v),\ C_P(v),\ U_P(v),\ F_P(v)) \quad \textit{and}$$

$$(e,\ T_Q(e),\ C_Q(e),\ U_Q(e),\ F_Q(e))$$

defines a single-valued quadripartitioned neutrosophic graph structure on G^*.

Proof. Under the stated assumptions, all four components lie in $[0, 1]$ and the vertex/edge sums are bounded by 4. The edge constraints in Definition 3.3.1 remain valid after suppressing the R-component (since R is identically 0 here), hence the four-component constraints hold on G^*. □

Theorem 3.3.3 (Edge truth bound)**.** *In an SVPNG $G = (P, Q)$, for any edge $e = \{u, v\} \in E$ one has*

$$T_Q(e) \le \min\{T_P(u), T_P(v)\}.$$

Proof. This is exactly the first inequality in Definition 3.3.1. □

Theorem 3.3.4 (Total uncertainty is bounded by 5)**.** *In an SVPNG $G = (P, Q)$, the total membership sum of any vertex or edge does not exceed 5:*

$$T_P(v) + C_P(v) + R_P(v) + U_P(v) + F_P(v) \le 5 \quad (\forall v \in V),$$

$$T_Q(e) + C_Q(e) + R_Q(e) + U_Q(e) + F_Q(e) \le 5 \quad (\forall e \in E).$$

Proof. This follows directly from the defining constraints in Definition 3.3.1. □

3.3.2 Single-Valued Pentapartitioned Neutrosophic Hypergraphs and n-SuperHyperGraphs

We now extend the single-valued pentapartitioned neutrosophic graph model (SVPNG, Definition 3.3.1) from graphs to hypergraphs and further to n-SuperHyperGraphs. As in the graph case, each object carries five components of uncertainty: *truth* (T), *contradiction* (C), *ignorance* (R), *unknown* (U), and *falsity* (F), all valued in $[0, 1]$.

Definition 3.3.5 (Single-Valued Pentapartitioned Neutrosophic Hypergraph)**.** Let $H^* = (V, E)$ be a finite crisp hypergraph, where V is a finite nonempty vertex set and $E \subseteq \mathcal{P}^*(V)$ is a finite family of nonempty hyperedges.

A *single-valued pentapartitioned neutrosophic hypergraph* (SVPNH) on H^* is a pair

$$H = (P, Q),$$

where:

- P is a pentapartitioned neutrosophic vertex set on V, specified by maps
$$T_P, C_P, R_P, U_P, F_P : V \to [0, 1]$$
satisfying, for every $v \in V$,
$$0 \le T_P(v) + C_P(v) + R_P(v) + U_P(v) + F_P(v) \le 5.$$

- Q is a pentapartitioned neutrosophic hyperedge set on E, specified by maps
$$T_Q, C_Q, R_Q, U_Q, F_Q : E \to [0, 1]$$
satisfying, for every hyperedge $e \in E$,
$$0 \le T_Q(e) + C_Q(e) + R_Q(e) + U_Q(e) + F_Q(e) \le 5,$$
together with the endpoint-consistency constraints
$$T_Q(e) \le \min_{v \in e} T_P(v), \qquad C_Q(e) \le \min_{v \in e} C_P(v),$$
$$R_Q(e) \ge \max_{v \in e} R_P(v), \qquad U_Q(e) \ge \max_{v \in e} U_P(v), \qquad F_Q(e) \ge \max_{v \in e} F_P(v).$$

Theorem 3.3.6 (SVPNH generalizes crisp hypergraphs)**.** *Every crisp hypergraph $H^* = (V, E)$ can be viewed as a single-valued pentapartitioned neutrosophic hypergraph.*

Proof. Let $H^* = (V, E)$ be a crisp hypergraph. Define vertex maps by

$$T_P(v) = 1,\ C_P(v) = 0,\ R_P(v) = 0,\ U_P(v) = 0,\ F_P(v) = 0 \qquad (\forall v \in V),$$

and hyperedge maps by

$$T_Q(e) = 1,\ C_Q(e) = 0,\ R_Q(e) = 0,\ U_Q(e) = 0,\ F_Q(e) = 0 \qquad (\forall e \in E).$$

Then all values lie in $[0, 1]$, the vertex and hyperedge sums are ≤ 5, and the consistency inequalities hold because

$$T_Q(e) = 1 \le \min_{v \in e} 1, \qquad C_Q(e) = 0 \le \min_{v \in e} 0, \qquad R_Q(e) = U_Q(e) = F_Q(e) = 0 \ge \max_{v \in e} 0.$$

Hence (P, Q) defines an SVPNH structure on the underlying crisp hypergraph. $\square$

Definition 3.3.7 (Single-Valued Pentapartitioned Neutrosophic n-SuperHyperGraph)**.** Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. Let $\mathrm{SHG}^{(n)} = (V, E)$ be a crisp n-SuperHyperGraph on V_0, i.e.,

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

A *single-valued pentapartitioned neutrosophic n-SuperHyperGraph* (SVPN-nSHG) on $\mathrm{SHG}^{(n)}$ is a pair

$$\mathcal{H}_{\mathrm{P}}^{(n)} = (P, Q),$$

where:

- P is a pentapartitioned neutrosophic vertex set on the n-supervertex set V, i.e.,

  $$T_P, C_P, R_P, U_P, F_P : V \to [0,1], \qquad 0 \le T_P(v)+C_P(v)+R_P(v)+U_P(v)+F_P(v) \le 5 \;\; (\forall v \in V);$$

- Q is a pentapartitioned neutrosophic superedge set on E, i.e.,

  $$T_Q, C_Q, R_Q, U_Q, F_Q : E \to [0,1], \qquad 0 \le T_Q(e)+C_Q(e)+R_Q(e)+U_Q(e)+F_Q(e) \le 5 \;\; (\forall e \in E),$$

  and for every superedge $e \in E$,

  $$T_Q(e) \le \min_{v \in e} T_P(v), \qquad C_Q(e) \le \min_{v \in e} C_P(v),$$

  $$R_Q(e) \ge \max_{v \in e} R_P(v), \qquad U_Q(e) \ge \max_{v \in e} U_P(v), \qquad F_Q(e) \ge \max_{v \in e} F_P(v).$$

Theorem 3.3.8 (SVPN-nSHGs generalize crisp n-SuperHyperGraphs)**.** *Every crisp n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ can be viewed canonically as an SVPN-nSHG.*

Proof. Let $\mathrm{SHG}^{(n)} = (V, E)$ be a crisp n-SuperHyperGraph. Define the vertex maps and superedge maps exactly as in the proof of Theorem 3.3.6, namely set $T = 1$ and $C = R = U = F = 0$ on all supervertices and superedges. Then the Atanassov-style bounds (sum ≤ 5) and the min/max consistency inequalities hold trivially. Thus the resulting structure is an SVPN-nSHG on the same underlying (V, E). $\square$

Theorem 3.3.9 (SVPN hypergraphs generalize SVPN graphs)**.** *Every single-valued pentapartitioned neutrosophic graph (SVPNG) is a special case of a single-valued pentapartitioned neutrosophic hypergraph (SVPNH).*

Proof. Let $G = (P, Q)$ be an SVPNG on a finite simple undirected graph $G^* = (V, E_G)$ in the sense of Definition 3.3.1, so $E_G \subseteq \binom{V}{2}$. Construct a hypergraph $H^* = (V, E)$ by identifying each graph edge $\{u, v\} \in E_G$ with the 2-element hyperedge $e_{uv} := \{u, v\}$, and set

$$E := \{\, e_{uv} : \; \{u, v\} \in E_G \,\} \subseteq \mathcal{P}^*(V).$$

Define vertex maps on V by reusing the SVPNG vertex maps:

$$(T_P, C_P, R_P, U_P, F_P) \text{ on } V.$$

Define hyperedge maps on E by transporting the SVPNG edge maps along the identification $\{u, v\} \leftrightarrow e_{uv}$:

$$T_Q(e_{uv}) := T_Q(\{u, v\}), \ \ldots, \ F_Q(e_{uv}) := F_Q(\{u, v\}).$$

Then all bounds and inequalities in Definition 3.3.5 hold, because for a 2-element hyperedge $e_{uv} = \{u, v\}$ one has

$$\min_{x \in e_{uv}} T_P(x) = \min\{T_P(u), T_P(v)\},$$

$$\max_{x \in e_{uv}} R_P(x) = \max\{R_P(u), R_P(v)\},$$

and similarly for the other components. Hence the given SVPNG is realized as an SVPNH whose hyperedges all have size 2. □

Remark 3.3.10. Combining Theorem 3.3.9 with $n = 0$ in Definition 3.3.7, one also obtains an embedding of SVPNG into the $n = 0$ case of SVPN-nSHGs (viewing graphs as 2-uniform hypergraphs).

3.4 Refined Graph for uncertainty

In recent years, refined sets have been introduced to handle various types of uncertainty [411–415]. Refined sets adopt the concept of splitting values. By splitting truth values, they generalize fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets. Based on these sets, we define the Refined Graph as follows.

3.4.1 Refined Fuzzy Graph

A refined fuzzy graph assigns each vertex and edge multiple membership subdegrees in $[0, 1]$, aggregating them to represent nuanced uncertainty.

Definition 3.4.1 (Refined Fuzzy Graph)**.** A *Refined Fuzzy Graph (RFG)* is an extension of a classical fuzzy graph where each vertex and edge is assigned a refined fuzzy value, represented by a Refined Fuzzy Set (RFS).

Let $G = (V, E)$ be a simple graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. The RFG assigns refined fuzzy sets to both vertices and edges, defined as follows:

- **RFS for Vertices**:
  For each vertex $v \in V$, a Refined Fuzzy Set (RFS) is assigned:

  $$\mu_v = (T_1(v), T_2(v), \ldots, T_p(v)),$$

  where $p \geq 2$ and $T_j(v) \in [0, 1]$ for all $j = 1, 2, \ldots, p$. Here, $T_j(v)$ represents the sub-membership degree of vertex v, and the sum of these values is allowed to be adjusted based on the number of values p:

  $$\sum_{j=1}^{p} T_j(v) \leq p.$$

  This allows for greater flexibility in modeling the membership degrees of vertices.

- **RFS for Edges**:
  For each edge $e = (u, v) \in E$, a Refined Fuzzy Set (RFS) is assigned:
  $$\mu_e = (T_1(e), T_2(e), \ldots, T_p(e)),$$
  where $p \geq 2$ and $T_j(e) \in [0, 1]$ for all $j = 1, 2, \ldots, p$. Here, $T_j(e)$ represents the sub-membership degree of edge e, and the sum of these values is similarly allowed to be adjusted:
  $$\sum_{j=1}^{p} T_j(e) \leq p.$$

Example 3.4.2 (A refined fuzzy graph)**.** Let $G = (V, E)$ be the 3-vertex path with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{v_1v_2,\ v_2v_3\}.$$

Fix $p = 3$ (three refined sub-membership components).

Vertex refined memberships. Assign to each vertex $v \in V$ a refined fuzzy value

$$\mu_v = \big(T_1(v), T_2(v), T_3(v)\big) \in [0, 1]^3$$

by

$$\mu_{v_1} = (0.7, 0.2, 0.5), \qquad \mu_{v_2} = (0.6, 0.6, 0.4), \qquad \mu_{v_3} = (0.3, 0.5, 0.2).$$

Each component lies in $[0, 1]$, and the required bound $\sum_{j=1}^{3} T_j(v) \leq 3$ holds for every vertex, e.g., $0.7 + 0.2 + 0.5 = 1.4 \leq 3$.

Edge refined memberships. Assign to each edge $e \in E$ a refined fuzzy value

$$\mu_e = \big(T_1(e), T_2(e), T_3(e)\big) \in [0, 1]^3$$

by

$$\mu_{v_1v_2} = (0.5, 0.2, 0.3), \qquad \mu_{v_2v_3} = (0.2, 0.4, 0.1).$$

Again, each component lies in $[0, 1]$, and $\sum_{j=1}^{3} T_j(e) \leq 3$ holds for every edge, e.g., $0.5 + 0.2 + 0.3 = 1.0 \leq 3$.

Therefore, with these vertex- and edge-level refined fuzzy assignments, G becomes a refined fuzzy graph in the sense of the given definition.

Theorem 3.4.3. *If for every vertex $v \in V$ and edge $e \in E$ in a Refined Fuzzy Graph (RFG), all refined membership degrees $T_j(v)$ and $T_j(e)$ are equal and sum to 1, then the RFG reduces to a classical fuzzy graph.*

Proof. In a classical fuzzy graph, each vertex v and edge e has a single membership degree $\mu(v) \in [0, 1]$ and $\mu(e) \in [0, 1]$.

In the RFG, each vertex v has refined membership degrees:

$$\mu_v = (T_1(v), T_2(v), \ldots, T_p(v)),$$

where $T_j(v) \in [0,1]$ and $\sum_{j=1}^{p} T_j(v) = 1$.

Assuming all $T_j(v)$ are equal, we have:

$$T_j(v) = \frac{1}{p}, \quad \forall j = 1, 2, \ldots, p.$$

Thus:

$$\sum_{j=1}^{p} T_j(v) = p \times \frac{1}{p} = 1.$$

Define the classical membership degree as:

$$\mu(v) = \sum_{j=1}^{p} T_j(v) = 1.$$

Similarly for edges:

$$\mu(e) = \sum_{j=1}^{p} T_j(e) = 1.$$

Therefore, the RFG reduces to a classical fuzzy graph where each vertex and edge has a membership degree of 1. □

3.4.2 Refined Intuitionistic Fuzzy Graph

A refined intuitionistic fuzzy graph assigns to vertices and edges multiple membership and non-membership subdegrees, with total membership plus non-membership bounded.

Definition 3.4.4 (Refined Intuitionistic Fuzzy Graph)**.** A *Refined Intuitionistic Fuzzy Graph (RIFG)* extends a classical intuitionistic fuzzy graph by assigning refined membership and non-membership values to both vertices and edges. Each vertex and edge in the graph is associated with a Refined Intuitionistic Fuzzy Set (RIFS), defined as follows:

- **RIFS for Vertices**:
  For each vertex $v \in V$, the Refined Intuitionistic Fuzzy Set (RIFS) is given by:

  $$(\mu_v, \nu_v) = \left((T_1(v), T_2(v), \ldots, T_p(v)); (F_1(v), F_2(v), \ldots, F_s(v))\right),$$

  where $p \geq 1$, $s \geq 1$, and $T_j(v), F_l(v) \in [0,1]$ for all $j = 1, 2, \ldots, p$ and $l = 1, 2, \ldots, s$. The sum of membership values $\sum_{j=1}^{p} T_j(v)$ and non-membership values $\sum_{l=1}^{s} F_l(v)$ is adjusted based on the number of truth and falsity values:

  $$\sum_{j=1}^{p} T_j(v) + \sum_{l=1}^{s} F_l(v) \leq p + s.$$

- **RIFS for Edges**:
  For each edge $e = (u, v) \in E$, a Refined Intuitionistic Fuzzy Set (RIFS) is given by:

  $$(\mu_e, \nu_e) = ((T_1(e), T_2(e), \ldots, T_p(e)); (F_1(e), F_2(e), \ldots, F_s(e))),$$

  where the same conditions apply as for the vertices, and the sum is similarly adjusted:

  $$\sum_{j=1}^{p} T_j(e) + \sum_{l=1}^{s} F_l(e) \leq p + s.$$

Theorem 3.4.5. *If the sum of the refined membership degrees and non-membership degrees for each vertex and edge in a Refined Intuitionistic Fuzzy Graph (RIFG) is 1, and there is only one refined value for each, then the RIFG reduces to a classical intuitionistic fuzzy graph.*

Proof. In RIFG, for each vertex v, we have:

$$\mu_v = (T_1(v)), \quad \nu_v = (F_1(v)),$$

with $T_1(v), F_1(v) \in [0, 1]$ and $T_1(v) + F_1(v) = 1$.

Similarly for edges.

This aligns with the definition of a classical intuitionistic fuzzy graph, where each vertex and edge has a membership degree $\mu(v)$ and a non-membership degree $\nu(v)$ such that $\mu(v)+\nu(v) = 1$. □

3.4.3 Refined Neutrosophic Graph

A refined neutrosophic graph assigns to each vertex and edge multiple truth, indeterminacy, and falsity subdegrees, totaling at most $r + s + t$.

Definition 3.4.6 (Refined Neutrosophic Graph)**.** (cf. [416]) Let $G = (V, E)$ be a simple graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Refined Neutrosophic Graph* assigns neutrosophic values to both vertices and edges, where each vertex and each edge has:

- A membership degree split into r values $\mu_1, \mu_2, \ldots, \mu_r$,
- An indeterminacy degree split into s values $\sigma_1, \sigma_2, \ldots, \sigma_s$,
- A non-membership degree split into t values $\nu_1, \nu_2, \ldots, \nu_t$,

such that for each vertex or edge x, the following condition holds:

$$0 \leq \sum_{i=1}^{r} \mu_i(x) + \sum_{i=1}^{s} \sigma_i(x) + \sum_{i=1}^{t} \nu_i(x) \leq n,$$

where $n = r + s + t$.

For each vertex $v \in V$, a neutrosophic n-valued refined set (μ_v, σ_v, ν_v) is assigned, where

$$\mu_v = (\mu_1(v), \ldots, \mu_r(v)), \quad \sigma_v = (\sigma_1(v), \ldots, \sigma_s(v)), \quad \nu_v = (\nu_1(v), \ldots, \nu_t(v)).$$

Similarly, for each edge $e = (u, v) \in E$, a neutrosophic n-valued refined set (μ_e, σ_e, ν_e) is assigned, where

$$\mu_e = (\mu_1(e), \ldots, \mu_r(e)), \quad \sigma_e = (\sigma_1(e), \ldots, \sigma_s(e)), \quad \nu_e = (\nu_1(e), \ldots, \nu_t(e)).$$

Example 3.4.7 (A refined neutrosophic graph)**.** Let $G = (V, E)$ be the 3-vertex path with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{v_1v_2,\ v_2v_3\}.$$

Fix $(r, s, t) = (2, 1, 2)$, so $n = r + s + t = 5$.

Vertex labels. Assign to each vertex $v \in V$ a refined neutrosophic value

$$\mu_v = (\mu_1(v), \mu_2(v)), \qquad \sigma_v = (\sigma_1(v)), \qquad \nu_v = (\nu_1(v), \nu_2(v))$$

as follows:

$$(v_1) :\ (0.6, 0.2;\ 0.4;\ 0.1, 0.3), \qquad (v_2) :\ (0.5, 0.4;\ 0.3;\ 0.2, 0.1), \qquad (v_3) :\ (0.3, 0.2;\ 0.6;\ 0.4, 0.1).$$

Each component lies in $[0, 1]$, and the total bound holds for every vertex, e.g. for v_1,

$$0.6 + 0.2 + 0.4 + 0.1 + 0.3 = 1.6 \leq 5.$$

Edge labels. Assign to each edge $e \in E$ a refined neutrosophic value

$$\mu_e = (\mu_1(e), \mu_2(e)), \qquad \sigma_e = (\sigma_1(e)), \qquad \nu_e = (\nu_1(e), \nu_2(e))$$

by

$$(v_1v_2) :\ (0.2, 0.3;\ 0.2;\ 0.4, 0.1), \qquad (v_2v_3) :\ (0.1, 0.2;\ 0.3;\ 0.3, 0.2).$$

Again, each component lies in $[0, 1]$, and the total bound holds for each edge, e.g. for v_1v_2,

$$0.2 + 0.3 + 0.2 + 0.4 + 0.1 = 1.2 \leq 5.$$

Therefore, with $(r, s, t) = (2, 1, 2)$ and the above vertex/edge assignments, G becomes a refined neutrosophic graph in the sense of the definition.

Theorem 3.4.8. *In a Refined Neutrosophic Graph, the total of the refined membership, indeterminacy, and non-membership degrees for any vertex or edge does not exceed* $n = r + s + t$*.*

Proof. By definition, for each vertex or edge x:

$$0 \leq \sum_{i=1}^{r} \mu_i(x) + \sum_{i=1}^{s} \sigma_i(x) + \sum_{i=1}^{t} \nu_i(x) \leq n.$$

This ensures that the combined degrees do not exceed the total number of refined components n. □

Theorem 3.4.9. *If $r = s = t = 1$ in a Refined Neutrosophic Graph, it reduces to a classical neutrosophic graph.*

Proof. With $r = s = t = 1$, each vertex and edge has single degrees:

$$\mu(x) = \mu_1(x), \quad \sigma(x) = \sigma_1(x), \quad \nu(x) = \nu_1(x),$$

and the condition becomes:

$$0 \leq \mu(x) + \sigma(x) + \nu(x) \leq 3.$$

After normalizing by dividing each degree by 3, we obtain degrees in $[0, 1]$, matching the classical neutrosophic graph definition. □

3.4.4 Refined quadripartitioned neutrosophic graph

A refined quadripartitioned neutrosophic graph assigns each vertex and edge multiple truth, contradiction, unknown, and falsity subdegrees within $[0, 1]$.

Definition 3.4.10 (Refined quadripartitioned neutrosophic graph)**.** Let $G^* = (V, E)$ be a finite simple undirected graph. Fix integers $r, s, q, t \geq 1$ and set $N := r + s + q + t$.

A *Refined (single-valued) Quadripartitioned Neutrosophic Graph* is a structure

$$G_{\mathrm{Q}}^{(r,s,q,t)} = \big(V, E;\ T, C, U, F;\ T_E, C_E, U_E, F_E\big),$$

where for each vertex $v \in V$ we assign vectors

$$T(v) = (T_1(v), \ldots, T_r(v)), \quad C(v) = (C_1(v), \ldots, C_s(v)),$$

$$U(v) = (U_1(v), \ldots, U_q(v)), \quad F(v) = (F_1(v), \ldots, F_t(v)),$$

and for each edge $\{u, v\} \in E$ we assign vectors

$$T_E(u, v) = (T_{E,1}(u, v), \ldots, T_{E,r}(u, v)),\ \ldots,\ F_E(u, v) = (F_{E,1}(u, v), \ldots, F_{E,t}(u, v)),$$

such that:

(a) (Range and total bound on vertices) for every $v \in V$,

$$T_i(v), C_j(v), U_k(v), F_\ell(v) \in [0,1], \quad 0 \le \sum_{i=1}^{r} T_i(v) + \sum_{j=1}^{s} C_j(v) + \sum_{k=1}^{q} U_k(v) + \sum_{\ell=1}^{t} F_\ell(v) \le N.$$

(b) (Range and total bound on edges) for every $\{u,v\} \in E$,

$$T_{E,i}(u,v), C_{E,j}(u,v), U_{E,k}(u,v), F_{E,\ell}(u,v) \in [0,1],$$

and

$$0 \le \sum_{i=1}^{r} T_{E,i}(u,v) + \sum_{j=1}^{s} C_{E,j}(u,v) + \sum_{k=1}^{q} U_{E,k}(u,v) + \sum_{\ell=1}^{t} F_{E,\ell}(u,v) \le N.$$

(c) (Refined edge-vertex constraints; componentwise) for every $\{u,v\} \in E$,

$$T_{E,i}(u,v) \le \min\{T_i(u), T_i(v)\} \quad (1 \le i \le r),$$

$$C_{E,j}(u,v) \le \min\{C_j(u), C_j(v)\} \quad (1 \le j \le s),$$

$$U_{E,k}(u,v) \le \max\{U_k(u), U_k(v)\} \quad (1 \le k \le q),$$

$$F_{E,\ell}(u,v) \le \max\{F_\ell(u), F_\ell(v)\} \quad (1 \le \ell \le t).$$

Theorem 3.4.11. *In a refined quadripartitioned neutrosophic graph, the total refined degree (T, C, U, F combined) of any vertex or edge does not exceed $N = r + s + q + t$.*

Proof. This is immediate from Definition 3.4.10(a) for vertices and Definition 3.4.10(b) for edges. □

Theorem 3.4.12. *If $r = s = q = t = 1$ in Definition 3.4.10, then $G_{\mathrm{Q}}^{(1,1,1,1)}$ reduces to a (single-valued) quadripartitioned neutrosophic graph in the usual sense.*

Proof. If $r = s = q = t = 1$, then each vertex has four scalar degrees $T_1(v), C_1(v), U_1(v), F_1(v)$ and each edge has four scalar degrees $T_{E,1}(u,v), C_{E,1}(u,v), U_{E,1}(u,v), F_{E,1}(u,v)$. The total bounds become $0 \le T_1 + C_1 + U_1 + F_1 \le 4$, and the componentwise edge-vertex constraints become exactly the standard min/max constraints for QPNG. □

3.4.5 Refined Single-Valued Pentapartitioned Neutrosophic Graph

A refined single-valued quadripartitioned neutrosophic graph assigns each vertex and edge truth, contradiction, unknown, and falsity subdegrees in $[0,1]$ separately. A refined single-valued pentapartitioned neutrosophic graph assigns each vertex and edge five neutrosophic component subdegrees in $[0,1]$ separately consistently throughout.

Definition 3.4.13 (Refined SVPN-graph)**.** Let $G^* = (V, E)$ be a finite simple undirected graph. Fix integers $r, s, p, q, t \geq 1$ and set $N := r + s + p + q + t$.

A *Refined Single-Valued Pentapartitioned Neutrosophic Graph* (refined SVPN-graph) is a structure

$$\widehat{G}_{\mathrm{P}}^{(r,s,p,q,t)} = \big(V, E;\ TP, CP, RP, UP, FP;\ TQ, CQ, RQ, UQ, FQ\big),$$

where, for each vertex $v \in V$, we assign vectors

$$TP(v) = (TP_1(v), \ldots, TP_r(v)), \quad CP(v) = (CP_1(v), \ldots, CP_s(v)),$$

$$RP(v) = (RP_1(v), \ldots, RP_p(v)),$$

$$UP(v) = (UP_1(v), \ldots, UP_q(v)),$$

$$FP(v) = (FP_1(v), \ldots, FP_t(v)),$$

and for each edge $\{u, v\} \in E$ we assign vectors

$$TQ(u,v) = (TQ_1(u,v), \ldots, TQ_r(u,v)),\ \ldots,$$

$$FQ(u,v) = (FQ_1(u,v), \ldots, FQ_t(u,v)),$$

such that:

(a) (Range and total bound on vertices) for every $v \in V$,

$$TP_i(v), CP_j(v), RP_k(v), UP_\ell(v), FP_m(v) \in [0,1],$$

and

$$0 \leq \sum_{i=1}^{r} TP_i(v) + \sum_{j=1}^{s} CP_j(v) + \sum_{k=1}^{p} RP_k(v) + \sum_{\ell=1}^{q} UP_\ell(v) + \sum_{m=1}^{t} FP_m(v) \leq N.$$

(b) (Range and total bound on edges) for every $\{u, v\} \in E$,

$$TQ_i(u,v), CQ_j(u,v), RQ_k(u,v), UQ_\ell(u,v), FQ_m(u,v) \in [0,1],$$

and

$$0 \leq \sum_{i=1}^{r} TQ_i(u,v) + \sum_{j=1}^{s} CQ_j(u,v) + \sum_{k=1}^{p} RQ_k(u,v) + \sum_{\ell=1}^{q} UQ_\ell(u,v) + \sum_{m=1}^{t} FQ_m(u,v) \leq N.$$

(c) (Refined SVPN edge constraints; componentwise) for every $\{u, v\} \in E$,

$$TQ_i(u,v) \le \min\{TP_i(u), TP_i(v)\} \quad (1 \le i \le r),$$

$$CQ_j(u,v) \le \min\{CP_j(u), CP_j(v)\} \quad (1 \le j \le s),$$
$$RQ_k(u,v) \ge \max\{RP_k(u), RP_k(v)\} \quad (1 \le k \le p),$$
$$UQ_\ell(u,v) \ge \max\{UP_\ell(u), UP_\ell(v)\} \quad (1 \le \ell \le q),$$
$$FQ_m(u,v) \ge \max\{FP_m(u), FP_m(v)\} \quad (1 \le m \le t).$$

Theorem 3.4.14. *In a refined SVPN-graph, the total refined degree (T, C, R, U, F combined) of any vertex or edge does not exceed* $N = r + s + p + q + t$.

Proof. This is immediate from Definition 3.4.13(a) and (b). □

Theorem 3.4.15. *If* $r = s = p = q = t = 1$ *in Definition 3.4.13, then* $\widehat{G}_{\mathrm{P}}^{(1,1,1,1,1)}$ *reduces to a classical SVPN-graph (single-valued pentapartitioned neutrosophic graph).*

Proof. If $r = s = p = q = t = 1$, then each vertex has five scalar degrees $TP_1, CP_1, RP_1, UP_1, FP_1$ and each edge has five scalar degrees $TQ_1, CQ_1, RQ_1, UQ_1, FQ_1$. The total bounds become $0 \le TP_1 + CP_1 + RP_1 + UP_1 + FP_1 \le 5$, and the edge constraints become exactly the standard SVPN inequalities. □

Restricted refined model. Hereafter, when needed, we may assume the balanced refinement

$$r = s = p = q = t = c$$

for a fixed constant $c \ge 1$.

3.5 Iterative Refined Neutrosophic Graph

An Iterative Refined Neutrosophic graph assigns each vertex and edge a leaf-indexed vector of refined truth/indeterminacy/falsity subdegrees.

Definition 3.5.1 (Iterated refinement profile)**.** An *iterated refinement profile* is a pair $\mathcal{P} = (\mathcal{T}, \tau)$ consisting of:

- a finite rooted tree $\mathcal{T}$ (edges oriented away from the root), and
- a type map $\tau : \mathrm{Leaf}(\mathcal{T}) \to \{\mathbf{T}, \mathbf{I}, \mathbf{F}\}$ assigning to each leaf one of the three neutrosophic types: truth, indeterminacy, or falsity.

Write

$$L := \mathrm{Leaf}(\mathcal{T}), \qquad L_{\mathbf{T}} := \tau^{-1}(\mathbf{T}), \quad L_{\mathbf{I}} := \tau^{-1}(\mathbf{I}), \quad L_{\mathbf{F}} := \tau^{-1}(\mathbf{F}),$$

so that $L = L_{\mathbf{T}} \sqcup L_{\mathbf{I}} \sqcup L_{\mathbf{F}}$. We call $|L|$ the *(iterated) refinement dimension* of $\mathcal{P}$.

Definition 3.5.2 (Iterative Refined Neutrosophic value)**.** Let $\mathcal{P} = (\mathcal{T}, \tau)$ be an iterated refinement profile with leaf set L. An *Iterative Refined Neutrosophic value of type* $\mathcal{P}$ is a vector

$$v = (v_\lambda)_{\lambda \in L} \ \in \ [0,1]^L.$$

Optionally (and typically in the single-valued numerical setting), one may impose the total bound

$$0 \ \le \ \sum_{\lambda \in L} v_\lambda \ \le \ |L|.$$

Definition 3.5.3 (Iterative Refined Neutrosophic Set (IRNS))**.** Let X be a nonempty universe and let $\mathcal{P} = (\mathcal{T}, \tau)$ be an iterated refinement profile with leaf set L. An *Iterative Refined Neutrosophic set of type* $\mathcal{P}$ on X is a mapping

$$A_{\mathcal{P}} : X \longrightarrow [0,1]^L, \qquad x \longmapsto A_{\mathcal{P}}(x) = (A_\lambda(x))_{\lambda \in L},$$

where each coordinate function $A_\lambda : X \to [0,1]$ represents the refined degree attached to the leaf λ. If the total bound in Definition 3.5.2 is required, then we additionally assume

$$0 \ \le \ \sum_{\lambda \in L} A_\lambda(x) \ \le \ |L| \qquad (\forall x \in X).$$

Definition 3.5.4 (Iterative Refined Neutrosophic Graph (IRNG))**.** Let $G = (V, E)$ be a finite simple undirected graph and let $\mathcal{P} = (\mathcal{T}, \tau)$ be an iterated refinement profile with leaf set L. An *Iterative Refined Neutrosophic graph of type* $\mathcal{P}$ is a triple

$$G_{\mathcal{P}} = \big(V, E;\ A_{\mathcal{P}}, B_{\mathcal{P}}\big),$$

where

$$A_{\mathcal{P}} : V \to [0,1]^L, \qquad B_{\mathcal{P}} : E \to [0,1]^L$$

assign to each vertex and each edge an Iterative Refined Neutrosophic value of type $\mathcal{P}$. If one adopts the single-valued total bound, then we assume

$$0 \ \le \ \sum_{\lambda \in L} A_\lambda(v) \ \le \ |L| \qquad (\forall v \in V), \qquad 0 \ \le \ \sum_{\lambda \in L} B_\lambda(e) \ \le \ |L| \qquad (\forall e \in E),$$

where $A_\lambda(v)$ (resp. $B_\lambda(e)$) denotes the λ-coordinate of $A_{\mathcal{P}}(v)$ (resp. $B_{\mathcal{P}}(e)$).

Example 3.5.5 (Online marketplace trust under multi-source evidence as an IRNG)**.** Consider an online marketplace in which users (buyers and sellers) interact. Let $G = (V, E)$ be a simple undirected graph whose vertices are user accounts and whose edges represent recent transactions:

$$V = \{\text{accounts}\}, \qquad E = \{\{u, v\} :\ u \text{ and } v \text{ completed a transaction}\}.$$

We model the reliability of users and transactions using an *Iterative Refined Neutrosophic graph.*

Refinement profile. Let $\mathcal{P} = (\mathcal{T}, \tau)$ be a depth-2 refinement profile whose leaf set is

$$L = L_{\mathbf{T}} \sqcup L_{\mathbf{I}} \sqcup L_{\mathbf{F}},$$

with

$$L_{\mathbf{T}} = \{\lambda^{(T)}_{\text{rev}}, \lambda^{(T)}_{\text{deliv}}\}, \quad L_{\mathbf{I}} = \{\lambda^{(I)}_{\text{new}}, \lambda^{(I)}_{\text{mixed}}\}, \quad L_{\mathbf{F}} = \{\lambda^{(F)}_{\text{fraud}}\}.$$

Here the type map τ labels the first two leaves as truth-like, the next two as indeterminacy-like, and the last as falsity-like. Intuitively:

- $\lambda^{(T)}_{\mathrm{rev}}$ aggregates positive evidence from reviews,
- $\lambda^{(T)}_{\mathrm{deliv}}$ aggregates positive evidence from delivery confirmations,
- $\lambda^{(I)}_{\mathrm{new}}$ represents uncertainty due to new accounts with little history,
- $\lambda^{(I)}_{\mathrm{mixed}}$ represents uncertainty due to mixed or contradictory signals,
- $\lambda^{(F)}_{\mathrm{fraud}}$ represents negative evidence from fraud reports.

Vertex labels. For each user $u \in V$, assign a leaf-indexed vector $A_{\mathcal{P}}(u) = (A_\lambda(u))_{\lambda\in L} \in [0,1]^L$. For example, for three accounts $v_1, v_2, v_3 \in V$, one may set

$$A_{\mathcal{P}}(v_1) = (0.8, 0.7, 0.1, 0.1, 0.0),$$
$$A_{\mathcal{P}}(v_2) = (0.4, 0.3, 0.3, 0.2, 0.1),$$
$$A_{\mathcal{P}}(v_3) = (0.2, 0.1, 0.5, 0.3, 0.2),$$

where the coordinates are ordered as

$$(\lambda^{(T)}_{\mathrm{rev}},\ \lambda^{(T)}_{\mathrm{deliv}},\ \lambda^{(I)}_{\mathrm{new}},\ \lambda^{(I)}_{\mathrm{mixed}},\ \lambda^{(F)}_{\mathrm{fraud}}).$$

Each coordinate lies in $[0, 1]$, and the optional total bound holds, e.g.,

$$0.8 + 0.7 + 0.1 + 0.1 + 0.0 = 1.7 \le |L| = 5.$$

Edge labels. For each transaction edge $e = \{u, v\} \in E$, assign a leaf-indexed vector $B_{\mathcal{P}}(e) = (B_\lambda(e))_{\lambda\in L} \in [0,1]^L$. For example, if $e_{12} = \{v_1, v_2\}$ and $e_{23} = \{v_2, v_3\}$ are transaction edges, set

$$B_{\mathcal{P}}(e_{12}) = (0.6, 0.7, 0.1, 0.1, 0.0), \qquad B_{\mathcal{P}}(e_{23}) = (0.3, 0.2, 0.3, 0.2, 0.1),$$

again satisfying the optional total bound, e.g., $0.6 + 0.7 + 0.1 + 0.1 + 0.0 = 1.5 \le 5$.

Therefore, $(V, E; A_{\mathcal{P}}, B_{\mathcal{P}})$ is an Iterative Refined Neutrosophic graph of type $\mathcal{P}$ in the sense of Definition 3.5.4. In this application, truth-like leaves represent distinct sources of positive evidence, indeterminacy-like leaves represent different kinds of uncertainty, and falsity-like leaves represent negative evidence, all tracked simultaneously for users and transactions.

Remark 3.5.6 (Optional endpoint constraints)**.** Definition 3.5.4 is intentionally model-agnostic. If desired, one can impose additional componentwise endpoint-consistency conditions by type: for $e = \{u, v\} \in E$, truth-like and indeterminacy-like coordinates may be bounded by $\min\{A_\lambda(u), A_\lambda(v)\}$, while falsity-like coordinates may be bounded by $\max\{A_\lambda(u), A_\lambda(v)\}$. Such constraints are optional and are not required for the generalization theorem below.

Theorem 3.5.7 (IRNG generalizes refined neutrosophic graphs)**.** *Every refined neutrosophic graph (in the sense of r truth-subdegrees, s indeterminacy-subdegrees, and t falsity-subdegrees) can be viewed as an Iterative Refined Neutrosophic graph.*

Proof. Let $G = (V, E)$ be a refined neutrosophic graph with parameters $r, s, t \geq 1$. Thus for each vertex or edge $x \in V \cup E$ one has refined coordinates

$$\mu_1(x), \ldots, \mu_r(x) \in [0,1], \qquad \sigma_1(x), \ldots, \sigma_s(x) \in [0,1], \qquad \nu_1(x), \ldots, \nu_t(x) \in [0,1],$$

satisfying the total bound

$$0 \;\leq\; \sum_{i=1}^{r} \mu_i(x) \;+\; \sum_{j=1}^{s} \sigma_j(x) \;+\; \sum_{k=1}^{t} \nu_k(x) \;\leq\; n, \qquad n := r + s + t.$$

Construct an iterated refinement profile $\mathcal{P} = (\mathcal{T}, \tau)$ as follows: take $\mathcal{T}$ to be a depth-1 rooted tree whose leaf set is

$$L = \{\mathbf{T}_1, \ldots, \mathbf{T}_r\} \;\sqcup\; \{\mathbf{I}_1, \ldots, \mathbf{I}_s\} \;\sqcup\; \{\mathbf{F}_1, \ldots, \mathbf{F}_t\},$$

and define $\tau(\mathbf{T}_i) = \mathbf{T}$, $\tau(\mathbf{I}_j) = \mathbf{I}$, and $\tau(\mathbf{F}_k) = \mathbf{F}$.

Define $A_{\mathcal{P}} : V \to [0,1]^L$ by setting, for each $v \in V$,

$$A_{\mathbf{T}_i}(v) := \mu_i(v)\ (1 \leq i \leq r), \qquad A_{\mathbf{I}_j}(v) := \sigma_j(v)\ (1 \leq j \leq s), \qquad A_{\mathbf{F}_k}(v) := \nu_k(v)\ (1 \leq k \leq t).$$

Similarly define $B_{\mathcal{P}} : E \to [0,1]^L$ by, for each $e \in E$,

$$B_{\mathbf{T}_i}(e) := \mu_i(e), \qquad B_{\mathbf{I}_j}(e) := \sigma_j(e), \qquad B_{\mathbf{F}_k}(e) := \nu_k(e).$$

Then $A_{\mathcal{P}}(v)$ and $B_{\mathcal{P}}(e)$ are vectors in $[0,1]^L$, and the refined total bound implies

$$0 \;\leq\; \sum_{\lambda \in L} A_\lambda(v) \;\leq\; |L|, \qquad 0 \;\leq\; \sum_{\lambda \in L} B_\lambda(e) \;\leq\; |L|,$$

because $|L| = r+s+t = n$ and the sums over L are exactly the sums of the refined components.

Therefore $G_{\mathcal{P}} = (V, E; A_{\mathcal{P}}, B_{\mathcal{P}})$ satisfies Definition 3.5.4, so it is an Iterative Refined Neutrosophic graph of type $\mathcal{P}$. By construction, it reproduces the original refined neutrosophic data, hence every refined neutrosophic graph is realized as a special case of an IRNG. □

3.6 Iterative Refined Neutrosophic Hypergraphs and n-SuperHyperGraphs

We extend the notion of an Iterative Refined Neutrosophic graph to hypergraphs and to n-SuperHyperGraphs by replacing edges with hyperedges and superedges, while keeping the same *leaf-indexed* refinement mechanism.

Definition 3.6.1 (Iterative Refined Neutrosophic Hypergraph (IRNH))**.** Let $H^* = (V, E)$ be a finite crisp hypergraph, where V is a finite nonempty vertex set and $E \subseteq \mathcal{P}^*(V)$ is a finite family of nonempty hyperedges. Let $\mathcal{P} = (\mathcal{T}, \tau)$ be an iterated refinement profile with leaf set $L = \mathrm{Leaf}(\mathcal{T})$ (Definition 3.5.1).

An *Iterative Refined Neutrosophic hypergraph of type* $\mathcal{P}$ is a pair

$$H_{\mathcal{P}} = \big(H^*;\, A_{\mathcal{P}}, B_{\mathcal{P}}\big),$$

where

$$A_{\mathcal{P}} : V \to [0,1]^L, \qquad B_{\mathcal{P}} : E \to [0,1]^L$$

assign to each vertex and each hyperedge an Iterative Refined Neutrosophic value of type $\mathcal{P}$.

Optionally (single-valued numerical setting), one may impose the total bounds

$$0 \le \sum_{\lambda \in L} A_\lambda(v) \le |L| \qquad (\forall v \in V), \qquad 0 \le \sum_{\lambda \in L} B_\lambda(e) \le |L| \qquad (\forall e \in E),$$

where $A_\lambda(v)$ and $B_\lambda(e)$ denote the λ-coordinates of $A_{\mathcal{P}}(v)$ and $B_{\mathcal{P}}(e)$, respectively.

Remark 3.6.2 (Optional incidence/containment constraints)**.** Definition 3.6.1 is intentionally model-agnostic. If desired, one may impose additional componentwise constraints relating $B_{\mathcal{P}}(e)$ to $\{A_{\mathcal{P}}(v) : v \in e\}$, for example by bounding truth-/indeterminacy-like coordinates by $\min_{v \in e} A_\lambda(v)$ and falsity-like coordinates by $\max_{v \in e} A_\lambda(v)$. Such conditions are optional and not required below.

Example 3.6.3 (Disaster-response task forces as an IRNH)**.** Consider a municipal disaster-response setting. Let V be a finite set of *agencies/units* (e.g., Fire, Police, Hospital, Power Company, Water Utility, Logistics Team). A *hyperedge* represents a multi-agency task force assembled for a specific mission (e.g., evacuation, medical triage, infrastructure repair). Thus a crisp hypergraph

$$H^* = (V, E), \qquad E \subseteq \mathcal{P}^*(V),$$

models multiway collaboration.

To capture uncertainty in assessments, fix an iterated refinement profile $\mathcal{P} = (\mathcal{T}, \tau)$ with leaf set

$$L = L_{\mathbf{T}} \sqcup L_{\mathbf{I}} \sqcup L_{\mathbf{F}},$$

where we interpret truth-like leaves as *positive evidence*, indeterminacy-like leaves as *unknown/unstable information*, and falsity-like leaves as *negative evidence*. For instance, choose

$$L_{\mathbf{T}} = \{\lambda^{(T)}_{\text{cap}}, \lambda^{(T)}_{\text{avail}}\}, \quad L_{\mathbf{I}} = \{\lambda^{(I)}_{\text{comm}}, \lambda^{(I)}_{\text{weather}}\}, \quad L_{\mathbf{F}} = \{\lambda^{(F)}_{\text{risk}}\},$$

where the labels suggest: capability evidence, availability evidence, communication uncertainty, weather uncertainty, and operational risk evidence.

Crisp hypergraph instance. Let $V = \{f, p, h\}$ represent Fire (f), Police (p), and Hospital (h), and let

$$E = \{e_1, e_2\}, \qquad e_1 = \{f, p, h\}, \quad e_2 = \{f, h\}.$$

Here e_1 models a joint evacuation-and-triage task force, while e_2 models a medical-evacuation subteam.

Iterated refined neutrosophic degrees. Assign vertex labels $A_{\mathcal{P}} : V \to [0,1]^L$ and hyperedge labels $B_{\mathcal{P}} : E \to [0,1]^L$ by (coordinate order $\lambda^{(T)}_{\text{cap}}, \lambda^{(T)}_{\text{avail}}, \lambda^{(I)}_{\text{comm}}, \lambda^{(I)}_{\text{weather}}, \lambda^{(F)}_{\text{risk}}$):

$$A_{\mathcal{P}}(f) = (0.8, 0.7, 0.1, 0.1, 0.0), \quad A_{\mathcal{P}}(p) = (0.7, 0.6, 0.2, 0.1, 0.1), \quad A_{\mathcal{P}}(h) = (0.9, 0.5, 0.2, 0.1, 0.1),$$

$$B_{\mathcal{P}}(e_1) = (0.7, 0.5, 0.2, 0.1, 0.1), \qquad B_{\mathcal{P}}(e_2) = (0.6, 0.4, 0.2, 0.2, 0.1).$$

All coordinates lie in $[0,1]$, and the optional total bound holds, e.g., $0.7 + 0.5 + 0.2 + 0.1 + 0.1 = 1.6 \le |L| = 5$.

Therefore $H_{\mathcal{P}} = (H^*; A_{\mathcal{P}}, B_{\mathcal{P}})$ is an iterative refined neutrosophic hypergraph of type $\mathcal{P}$ in the sense of Definition 3.6.1.

Theorem 3.6.4 (IRNH generalizes crisp hypergraphs)**.** *Every finite crisp hypergraph can be viewed canonically as an Iterative Refined Neutrosophic hypergraph.*

Proof. Let $H^* = (V, E)$ be a crisp hypergraph. Choose a depth-1 rooted tree $\mathcal{T}$ with three leaves

$$L = \{\lambda_{\mathbf{T}}, \lambda_{\mathbf{I}}, \lambda_{\mathbf{F}}\},$$

and define $\tau(\lambda_{\mathbf{T}}) = \mathbf{T}$, $\tau(\lambda_{\mathbf{I}}) = \mathbf{I}$, $\tau(\lambda_{\mathbf{F}}) = \mathbf{F}$. Define

$$A_{\mathcal{P}} : V \to [0,1]^L, \qquad B_{\mathcal{P}} : E \to [0,1]^L$$

by setting, for every $v \in V$ and $e \in E$,

$$A_{\lambda_{\mathbf{T}}}(v) = 1, \quad A_{\lambda_{\mathbf{I}}}(v) = 0, \quad A_{\lambda_{\mathbf{F}}}(v) = 0,$$

$$B_{\lambda_{\mathbf{T}}}(e) = 1, \quad B_{\lambda_{\mathbf{I}}}(e) = 0, \quad B_{\lambda_{\mathbf{F}}}(e) = 0.$$

Then $A_{\mathcal{P}}(v), B_{\mathcal{P}}(e) \in [0,1]^L$ and

$$\sum_{\lambda \in L} A_\lambda(v) = 1 \le |L| = 3, \qquad \sum_{\lambda \in L} B_\lambda(e) = 1 \le 3,$$

so the optional total bounds hold. Hence $H_{\mathcal{P}} = (H^*; A_{\mathcal{P}}, B_{\mathcal{P}})$ is an IRNH in the sense of Definition 3.6.1. The underlying incidence structure (V, E) is unchanged, so the crisp hypergraph is recovered by forgetting the labels. □

Definition 3.6.5 (Iterative Refined Neutrosophic n-SuperHyperGraph)**.** Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. Let $\mathrm{SHG}^{(n)} = (V, E)$ be a crisp n-SuperHyperGraph on V_0, i.e.,

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Let $\mathcal{P} = (\mathcal{T}, \tau)$ be an iterated refinement profile with leaf set L.

An *Iterative Refined Neutrosophic n-SuperHyperGraph of type $\mathcal{P}$* is a pair

$$\mathcal{S}_{\mathcal{P}}^{(n)} = \big(\mathrm{SHG}^{(n)};\, A_{\mathcal{P}}, B_{\mathcal{P}}\big),$$

where

$$A_{\mathcal{P}} : V \to [0,1]^L, \qquad B_{\mathcal{P}} : E \to [0,1]^L$$

assign a leaf-indexed Iterative Refined Neutrosophic value to each n-supervertex and each n-superedge. Optionally, one may impose the total bounds

$$0 \;\le\; \sum_{\lambda \in L} A_\lambda(v) \;\le\; |L| \qquad (\forall v \in V), \qquad 0 \;\le\; \sum_{\lambda \in L} B_\lambda(e) \;\le\; |L| \qquad (\forall e \in E).$$

Example 3.6.6 (Nested project portfolios and cross-portfolio governance as an IRN-2SHG)**.** Consider a large organization managing many projects. Let V_0 be the set of *atomic projects*. A *team portfolio* is a subset of projects, hence an element of $\mathcal{P}(V_0)$. A *division portfolio* is a set of team portfolios, hence an element of $\mathcal{P}^2(V_0)$.

Fix $n = 2$ and let $V \subseteq \mathcal{P}^2(V_0)$ be a finite family of division portfolios (the 2-supervertices). A superedge $e \in E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ represents a *cross-division governance constraint* (e.g., shared compliance requirements, shared resource caps, or joint roadmap dependencies).

Concrete instance. Let $V_0 = \{p_1, p_2, p_3, p_4\}$ be four projects and define 2-supervertices

$$v_1 = \{\{p_1, p_2\}, \{p_3\}\}, \qquad v_2 = \{\{p_2, p_4\}\}, \qquad v_3 = \{\{p_1\}, \{p_4\}\},$$

so $V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0)$. Let

$$E = \{e_1\}, \qquad e_1 = \{v_1, v_2, v_3\},$$

encoding a governance rule that simultaneously constrains three division portfolios.

Iterated refinement profile and labels. Use the same leaf set L as in Example 3.6.3 and interpret leaves as: capability evidence (budget/skills), availability evidence (capacity), communication uncertainty (coordination), weather uncertainty (external volatility), and risk evidence (compliance failure). Assign

$$A_{\mathcal{P}}(v_1) = (0.7, 0.6, 0.2, 0.1, 0.1),$$

$$A_{\mathcal{P}}(v_2) = (0.5, 0.5, 0.3, 0.1, 0.2),$$

$$A_{\mathcal{P}}(v_3) = (0.6, 0.4, 0.2, 0.2, 0.1),$$

and

$$B_{\mathcal{P}}(e_1) = (0.5, 0.4, 0.3, 0.1, 0.2).$$

Again all coordinates lie in $[0, 1]$ and satisfy the optional total bound.

Hence

$$\mathcal{S}^{(2)}_{\mathcal{P}} = \left(\mathrm{SHG}^{(2)};\ A_{\mathcal{P}}, B_{\mathcal{P}}\right)$$

is an iterative refined neutrosophic 2-SuperHyperGraph of type $\mathcal{P}$ in the sense of Definition 3.6.5. In this application, the supervertices are *nested* portfolio objects and the superedge captures a multiway governance interaction, while the leaf-indexed refinement records multiple sources of evidence and uncertainty simultaneously.

Theorem 3.6.7 (IRN-nSHGs generalize crisp n-SuperHyperGraphs)**.** *Every crisp n-SuperHyperGraph can be viewed canonically as an Iterative Refined Neutrosophic n-SuperHyperGraph.*

Proof. Let $\mathrm{SHG}^{(n)} = (V, E)$ be a crisp n-SuperHyperGraph. Choose the same depth-1 profile $\mathcal{P}$ with three leaves $L = \{\lambda_{\mathbf{T}}, \lambda_{\mathbf{I}}, \lambda_{\mathbf{F}}\}$ as in the proof of Theorem 3.6.4. Define $A_{\mathcal{P}} : V \to [0, 1]^L$ and $B_{\mathcal{P}} : E \to [0, 1]^L$ by assigning $(1, 0, 0)$ (in the $(\lambda_{\mathbf{T}}, \lambda_{\mathbf{I}}, \lambda_{\mathbf{F}})$ coordinates) to every supervertex and superedge. Then all values lie in $[0, 1]$, and the optional total bounds hold exactly as before. Hence $\mathcal{S}^{(n)}_{\mathcal{P}} = (\mathrm{SHG}^{(n)}; A_{\mathcal{P}}, B_{\mathcal{P}})$ is an IRN-nSHG. Forgetting labels recovers the original crisp n-SuperHyperGraph. □

Theorem 3.6.8 (IRNH generalizes IRNG)**.** *Every Iterative Refined Neutrosophic graph (IRNG) can be realized as an Iterative Refined Neutrosophic hypergraph (IRNH) whose hyperedges all have size* 2*.*

Proof. Let $G_{\mathcal{P}} = (V, E; A_{\mathcal{P}}, B_{\mathcal{P}})$ be an IRNG of type $\mathcal{P}$ (Definition 3.5.4), where $E \subseteq \binom{V}{2}$. Define a crisp 2-uniform hypergraph

$$H^* = (V, E_H), \qquad E_H := \big\{\{u, v\} \subseteq V : \ \{u, v\} \in E\big\}.$$

Define $A^H_{\mathcal{P}} : V \to [0,1]^L$ by $A^H_{\mathcal{P}} := A_{\mathcal{P}}$. Define $B^H_{\mathcal{P}} : E_H \to [0,1]^L$ by

$$B^H_{\mathcal{P}}(\{u, v\}) := B_{\mathcal{P}}(uv) \qquad (\{u, v\} \in E_H),$$

where uv denotes the corresponding edge of the original graph. Since $B_{\mathcal{P}}$ takes values in $[0,1]^L$ and satisfies the optional total bound (if assumed), the same holds for $B^H_{\mathcal{P}}$. Therefore

$$H_{\mathcal{P}} := (H^*; \, A^H_{\mathcal{P}}, B^H_{\mathcal{P}})$$

is an IRNH of type $\mathcal{P}$ (Definition 3.6.1), and by construction it encodes exactly the same leaf-indexed vertex and edge data. Thus $G_{\mathcal{P}}$ is realized as a 2-uniform special case of an IRNH. □

Chapter 4

Result:Property of Plithogenic graphs

In this chapter, we explore the properties of Plithogenic graphs. Specifically, we define Plithogenic graphs and General Plithogenic graphs, and then examine their relationships.

4.1 Plithogenic graphs

Recently, plithogenic graphs have been proposed as graphical counterparts of plithogenic sets and as a unified framework that can specialize to many uncertainty-aware graph models (fuzzy, intuitionistic fuzzy, neutrosophic, quadripartitioned neutrosophic, pentapartitioned neutrosophic, etc.) [3, 81, 417].

Definition 4.1.1 (Plithogenic set)**.** [43, 418] Let S be a universal set and let $P \subseteq S$. A *plithogenic set* is a tuple

$$PS = (P, v, P_v, \mathrm{pdf}, \mathrm{pCF}),$$

where v is an attribute, P_v is the set of possible values of v,

$$\mathrm{pdf} : P \times P_v \to [0,1]^s \quad \text{(degree of appurtenance function; DAF)},$$

$$\mathrm{pCF} : P_v \times P_v \to [0,1]^t \quad \text{(degree of contradiction function; DCF)},$$

and pCF satisfies (for all $a, b \in P_v$):

$$\mathrm{pCF}(a, a) = 0, \qquad \mathrm{pCF}(a, b) = \mathrm{pCF}(b, a).$$

Tables 4.1 present examples of sets that can be generalized by the Plithogenic Set framework (cf. [419]). Various concepts for handling uncertainty are being continuously defined and studied in the scientific community.

Definition 4.1.2 (Plithogenic graph)**.** [3, 81] Let $G^* = (V, E)$ be a crisp (simple) graph with finite vertex set V and edge set $E \subseteq \{\{u, v\} : u, v \in V,\ u \neq v\}$.

A *plithogenic graph* on G^* is a pair

$$PG = (PM, PN),$$

where $PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf})$ is a plithogenic vertex structure and $PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf})$ is a plithogenic edge structure, with:

Table 4.1: A catalogue of Plithogenic Set families by number of components s.

s	t	Representative type(s)
1	0	Fuzzy Set [30,125]; N-Set [420]; Shadowed Set [421–423]
2	0	Intuitionistic Fuzzy Set [424,425]; Vague Set [50,426]; Bipolar Fuzzy Set [427]; Intuitionistic Evidence Set [428–430]; Variable Fuzzy Set [431–433]; Paraconsistent Fuzzy Set [434,435]; Bifuzzy Set [436,437]
3	0	Neutrosophic Set[(a)] [318,320]; Hesitant Fuzzy Set [438,439]; Tripolar Fuzzy Set [440–442]; Three-way Fuzzy Set [443,444]; Picture Fuzzy Set [165,445]; Spherical Fuzzy Set [131,446]; Inconsistent Intuitionistic Fuzzy Set [447,448]; Ternary Fuzzy Set [60,449]; Neutrosophic Fuzzy Set [450,451]; (Kleene three-valued logic [452,453];) Neutrosophic Vague Set [66,454]
4	0	Quadripartitioned Neutrosophic Set [455,456]; Double-Valued Neutrosophic Set [457,458]; Dual hesitant fuzzy sets [459,460]; Ambiguous Set[(b)] [461–463]; Local-Neutrosophic Set [464]; Support Neutrosophic Set [465]; (Belnap four-valued logic [466,467];) Turiyam Neutrosophic Set[(c)] [468–471]
5	0	Pentapartitioned Neutrosophic Set [472–474]; Triple-valued Neutrosophic Set [475–477]
6	0	Hexapartitioned Neutrosophic Set; Quadruple-Valued Neutrosophic Set [476, 478]
7	0	Heptapartitioned Neutrosophic Set; Quintuple-Valued Neutrosophic Set [476, 479,480]
8	0	Octapartitioned Neutrosophic Set [481]
9	0	Nonapartitioned Neutrosophic Set [481]
n	0	n-Refined Fuzzy Set [411,482]; (n-valued (Łukasiewicz) logic [483];) Multi-valued (Fuzzy) Sets [484]; MultiFuzzy Set [485]
$2n$	0	n-Refined Intuitionistic Fuzzy Set [411]; Multi-Intuitionistic Fuzzy Set [485]
$3n$	0	n-Refined Neutrosophic Set [411]; Multi-Neutrosophic Set [485,486]
1	1	Plithogenic Fuzzy Set [418,487,488]
2	1	Plithogenic Intuitionistic Fuzzy Set [87]
3	1	Plithogenic Neutrosophic Set [489–491]

(a) It is widely recognized that the neutrosophic set generalizes the intuitionistic fuzzy set, the inconsistent intuitionistic fuzzy set (including picture fuzzy and ternary fuzzy sets), the Pythagorean fuzzy set, the spherical fuzzy set, and the q-rung orthopair fuzzy set; similarly, neutrosophication generalizes regret theory, grey system theory, and three-way decision theory [492].
(b) Ambiguous Sets are known to form a subclass of Quadripartitioned Neutrosophic Sets [455,456] as well as of Double-Valued Neutrosophic Sets [463].
(c) Turiyam Neutrosophic Sets are known to constitute a subclass of the existing Quadripartitioned Neutrosophic Sets [493].

- $M \subseteq V$, $N \subseteq E$;
- M_ℓ (resp. N_m) is the domain of attribute values for vertices (resp. edges);
- $\mathrm{adf} : M \times M_\ell \to [0,1]^s$ and $\mathrm{aCf} : M_\ell \times M_\ell \to [0,1]^t$;
- $\mathrm{bdf} : N \times N_m \to [0,1]^s$ and $\mathrm{bCf} : N_m \times N_m \to [0,1]^t$;

and the following constraints hold:

1. **Edge appurtenance constraint:** for all $xy = \{x, y\} \in N$ and for all $(a, b) \in M_\ell \times M_\ell$ (compatible with N_m),
$$\mathrm{bdf}(xy, (a, b)) \;\le\; \min\{\mathrm{adf}(x, a), \mathrm{adf}(y, b)\},$$
where the inequality is interpreted componentwise in $[0,1]^s$.

2. **Reflexivity and symmetry of contradiction:** $\mathrm{aCf}(a,a) = 0$, $\mathrm{aCf}(a,b) = \mathrm{aCf}(b,a)$ for all $a, b \in M_\ell$, and similarly for bCf on N_m.

Tables 4.2 and **??** present representative examples of *graph* families that can be unified and generalized within the Plithogenic Graph framework (cf. [419]). More broadly, the scientific community continues to develop and refine a wide spectrum of uncertainty-aware graph models.

Table 4.2: A catalogue of Plithogenic *graph* families by number of components s.

s	t	Representative type(s)
1	0	Fuzzy graph; N-graph; shadowed-graph variants
2	0	Intuitionistic fuzzy graph [494]; vague graph [495]; bipolar fuzzy graph [34]; intuitionistic evidence graph; variable fuzzy graph; paraconsistent fuzzy graph; bifuzzy graph [496, 497]
3	0	Neutrosophic graph [61][(a)]; hesitant fuzzy graph [498]; tripolar fuzzy graph; three-way fuzzy graph; picture fuzzy graph [174, 499]; spherical fuzzy graph [131]; inconsistent intuitionistic fuzzy graph; ternary fuzzy / neutrosophic-fuzzy graph; neutrosophic vague graph
4	0	Quadripartitioned neutrosophic graph [500, 501]; double-valued neutrosophic graph [457]; dual hesitant fuzzy graph [502]; ambiguous graph[(b)]; local-neutrosophic graph; support-neutrosophic graph; turiyam neutrosophic graph [503][(c)]
5	0	Pentapartitioned neutrosophic graph [365]; triple-valued neutrosophic graph
6	0	Hexapartitioned neutrosophic graph; quadruple-valued neutrosophic graph
7	0	Heptapartitioned neutrosophic graph [504]; quintuple-valued neutrosophic graph
8	0	Octapartitioned Neutrosophic Graph
9	0	Nonapartitioned Neutrosophic Graph
n	0	n-refined fuzzy graph; multi-valued (fuzzy) graphs; multi-fuzzy graphs [505]
$2n$	0	n-refined intuitionistic fuzzy graph; multi-intuitionistic fuzzy graphs
$3n$	0	n-refined neutrosophic graph; multi-neutrosophic graphs
1	1	Plithogenic Fuzzy Graph [418, 487, 488]
2	1	Plithogenic Intuitionistic Fuzzy Graph [87]
3	1	Plithogenic Neutrosophic Graph [489–491]

(a) It is widely recognized that neutrosophic graphs generalize several earlier uncertainty-aware graph models, including intuitionistic fuzzy graphs and various inconsistent intuitionistic variants; moreover, neutrosophication provides a broad unifying methodology across multiple uncertainty theories (cf. [492]).
(b) Ambiguous graphs are commonly treated as a subclass of quadripartitioned neutrosophic graphs and also of double-valued neutrosophic graphs (see, e.g., [455, 456, 463]).
(c) Turiyam neutrosophic graphs are known to constitute a subclass of quadripartitioned neutrosophic graphs (cf. [493]).

Theorem 4.1.3 (From a plithogenic graph to a QPNG)**.** *Assume $PG = (PM, PN)$ is a plithogenic graph on $G^* = (V, E)$ with parameters $s = 4$ and $t = 1$. Assume further that the vertex-attribute domain and edge-attribute domain are singletons, $M_\ell = \{\ell_0\}$ and $N_m = \{m_0\}$, and that the contradiction functions are identically 0 (i.e., no attribute-value contradiction is modeled).*

Define a quadripartitioned neutrosophic vertex-label A on V and edge-label B on E by:

$$(t_A(v), c_A(v), u_A(v), f_A(v)) := \mathrm{adf}(v, \ell_0) \in [0,1]^4 \qquad (\forall v \in V),$$

$$(t_B(e), c_B(e), u_B(e), f_B(e)) := \mathrm{bdf}(e, m_0) \in [0,1]^4 \qquad (\forall e \in E).$$

Then $G = (A, B, V, E)$ is a quadripartitioned neutrosophic graph (QPNG).

Proof. Each component of $\mathrm{adf}(v,\ell_0)$ and $\mathrm{bdf}(e,m_0)$ lies in $[0,1]$ by definition, hence

$$0 \le t_A(v)+c_A(v)+u_A(v)+f_A(v) \le 4, \qquad 0 \le t_B(e)+c_B(e)+u_B(e)+f_B(e) \le 4.$$

Let $e=\{x,y\}\in E$. By the edge appurtenance constraint (componentwise),

$$\mathrm{bdf}(e,m_0) \le \min\{\mathrm{adf}(x,\ell_0),\mathrm{adf}(y,\ell_0)\}.$$

Reading components gives:

$$t_B(e) \le \min\{t_A(x),t_A(y)\}, \quad c_B(e) \le \min\{c_A(x),c_A(y)\},$$

and also

$$u_B(e) \le \min\{u_A(x),u_A(y)\} \le \max\{u_A(x),u_A(y)\}, \quad f_B(e) \le \min\{f_A(x),f_A(y)\} \le \max\{f_A(x),f_A(y)\}.$$

Therefore the defining inequalities of a QPNG are satisfied. □

Theorem 4.1.4 (From a plithogenic graph to a PPNG)**.** *Assume $PG=(PM,PN)$ is a plithogenic graph on $G^*=(V,E)$ with parameters $s=5$ and $t=1$, and assume $M_\ell=\{\ell_0\}$ and $N_m=\{m_0\}$ are singletons.*

Write

$$\mathrm{adf}(v,\ell_0)=(\mu_1(v),\mu_2(v),\mu_3(v),\mu_4(v),\mu_5(v)) \in [0,1]^5,$$
$$\mathrm{bdf}(e,m_0)=(\mu_1(e),\mu_2(e),\mu_3(e),\mu_4(e),\mu_5(e)) \in [0,1]^5.$$

Define a pentapartitioned neutrosophic labeling by

$$t_A(v)=\mu_1(v), \quad c_A(v)=1-\mu_2(v), \quad g_A(v)=1-\mu_3(v), \quad u_A(v)=1-\mu_4(v), \quad f_A(v)=1-\mu_5(v),$$

and similarly on edges $e\in E$:

$$t_B(e)=\mu_1(e), \quad c_B(e)=1-\mu_2(e), \quad g_B(e)=1-\mu_3(e), \quad u_B(e)=1-\mu_4(e), \quad f_B(e)=1-\mu_5(e).$$

Then $G=(A,B,V,E)$ satisfies the standard pentapartitioned neutrosophic graph inequalities:

$$t_B(e) \le \min\{t_A(x),t_A(y)\}, \qquad c_B(e) \ge \max\{c_A(x),c_A(y)\}, \qquad g_B(e) \ge \max\{g_A(x),g_A(y)\},$$
$$u_B(e) \ge \max\{u_A(x),u_A(y)\}, \qquad f_B(e) \ge \max\{f_A(x),f_A(y)\},$$

for every edge $e=\{x,y\}$, and hence yields a pentapartitioned neutrosophic graph (PPNG).

Proof. All vertex and edge components lie in $[0,1]$ since each $\mu_i(\cdot)\in[0,1]$. Hence the component-sum condition $0 \le t+c+g+u+f \le 5$ holds automatically.

Let $e=\{x,y\}\in E$. By the plithogenic edge appurtenance constraint (componentwise),

$$\mu_i(e) \le \min\{\mu_i(x),\mu_i(y)\} \qquad (i=1,2,3,4,5).$$

For $i=1$, this is exactly

$$t_B(e)=\mu_1(e) \le \min\{\mu_1(x),\mu_1(y)\} = \min\{t_A(x),t_A(y)\}.$$

For $i\in\{2,3,4,5\}$, taking complements reverses the inequality:

$$1-\mu_i(e) \;\ge\; 1-\min\{\mu_i(x),\mu_i(y)\} \;=\; \max\{1-\mu_i(x),1-\mu_i(y)\}.$$

Thus

$$c_B(e) \ge \max\{c_A(x),c_A(y)\}, \quad g_B(e) \ge \max\{g_A(x),g_A(y)\},$$
$$u_B(e) \ge \max\{u_A(x),u_A(y)\}, \quad f_B(e) \ge \max\{f_A(x),f_A(y)\}.$$

Therefore the pentapartitioned neutrosophic edge inequalities are satisfied. □

4.2 Other Graph class related to Plithogenic graphs

These Plithogenic graphs are highly manageable, and the following generalizations are also possible. In this subsection, we will explore their relationships with those graph classes.

4.2.1 General Plithogenic Graph

First, we consider the General Plithogenic Graph, which is a relaxed version of the Plithogenic Graph. Simply put, this definition is designed to allow for modifications to each graph by incorporating conditions that are suitable for each specific graph, enabling their application in a more flexible manner.

Definition 4.2.1 (General Plithogenic Graph)**.** [45, 469, 506] A **General Plithogenic Graph** G^{GP} is an extension of the classical Plithogenic Graph that allows for a more flexible and independent treatment of vertices and edges. It is defined as follows:

Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges.

A General Plithogenic Graph $G^{GP} = (PM, PN)$ consists of:

1. **General Plithogenic Vertex Set** PM:

$$PM = (M, l, Ml, adf, aCf)$$

   where:

   - $M \subseteq V$: Set of vertices.
   - l: Attribute associated with the vertices.
   - Ml: Range of possible attribute values.
   - $adf : M \times Ml \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for vertices.
   - $aCf : Ml \times Ml \to [0, 1]^t$: Degree of Contradiction Function (DCF) for vertices.

2. **General Plithogenic Edge Set** PN:

$$PN = (N, m, Nm, bdf, bCf)$$

   where:

   - $N \subseteq E$: Set of edges.

- m: Attribute associated with the edges.
- Nm: Range of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$: Degree of Appurtenance Function (DAF) for edges.
- $bCf : Nm \times Nm \to [0,1]^t$: Degree of Contradiction Function (DCF) for edges.

The General Plithogenic Graph G^{GP} only needs to satisfy the following **Reflexivity and Symmetry** properties of the Contradiction Functions:

- Reflexivity and Symmetry of Contradiction Functions:

$$\begin{aligned} aCf(a,a) &= 0, & \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), & \forall a,b \in Ml \\ bCf(a,a) &= 0, & \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), & \forall a,b \in Nm \end{aligned}$$

Theorem 4.2.2. *Plithogenic Graph is General Plithogenic Graph.*

Proof. Obviously holds. □

We consider transforming this graph into various other graphs. As mentioned earlier, this definition is intended to facilitate modifications to each graph by incorporating conditions tailored to each specific graph, allowing for more flexible application.

4.2.2 Relations to spherical fuzzy graphs

As a representative example, we briefly mention *spherical fuzzy graphs* (SFGs) [131]. Within fuzzy graph theory, spherical fuzzy graphs have attracted considerable attention, and many variants and properties have been studied [177, 507–510].

Theorem 4.2.3 (Related graph classes for spherical fuzzy graphs)**.** *Examples of graph classes related to spherical fuzzy graphs include (but are not limited to) the following:*

- *T-spherical fuzzy graphs [177],*
- *spherical fuzzy labelling graphs [511],*
- *spherical fuzzy digraphs [512],*

- *T-spherical fuzzy Hamacher graphs [513],*
- *regular spherical fuzzy graphs [514],*
- *spherical fuzzy cycle graphs [515],*
- *spherical fuzzy tree graphs [515],*
- *pseudo-regular spherical fuzzy graphs [516],*
- *spherical neutrosophic graphs [517].*

Proof. This theorem is a literature catalogue; see the cited references for the corresponding definitions and results. □

We next recall the standard definition of an SFG [131].

Definition 4.2.4 (Spherical fuzzy graph [131])**.** Let V be a nonempty finite set. A *spherical fuzzy graph* (SFG) on V is a pair $G = (M, N)$, where:

- M is a spherical fuzzy set on V, specified by three maps

  $$\alpha_M,\ \gamma_M,\ \beta_M : V \to [0,1],$$

  called the degrees of *truthness*, *abstinence*, and *falseness*, respectively; and

- N is a spherical fuzzy relation on $V \times V$, specified by three maps

  $$\alpha_N,\ \gamma_N,\ \beta_N : V \times V \to [0,1],$$

  interpreted as the truthness, abstinence, and falseness degrees of (ordered) pairs.

These data satisfy, for all $a, b \in V$,

$$\begin{aligned}\alpha_N(a,b) &\le \min\{\alpha_M(a), \alpha_M(b)\},\\ \gamma_N(a,b) &\le \min\{\gamma_M(a), \gamma_M(b)\},\\ \beta_N(a,b) &\le \max\{\beta_M(a), \beta_M(b)\},\end{aligned}$$

together with the spherical constraint

$$0 \ \le\ \alpha_N(a,b)^2 + \gamma_N(a,b)^2 + \beta_N(a,b)^2 \ \le\ 1.$$

Here M is called the *spherical fuzzy vertex set* and N the *spherical fuzzy edge set* of G.

Example 4.2.5 (A spherical fuzzy graph)**.** Let $V = \{v_1, v_2, v_3\}$ and consider the crisp edge set

$$E = \{v_1v_2,\ v_2v_3\}.$$

Define the spherical fuzzy vertex degrees $(\alpha_M, \gamma_M, \beta_M) : V \to [0,1]^3$ by

$$(\alpha_M, \gamma_M, \beta_M)(v_1) = (0.8, 0.3, 0.2),$$

$$(\alpha_M, \gamma_M, \beta_M)(v_2) = (0.6, 0.4, 0.3),$$

$$(\alpha_M, \gamma_M, \beta_M)(v_3) = (0.5, 0.2, 0.4).$$

Define the spherical fuzzy edge degrees $(\alpha_N, \gamma_N, \beta_N) : V \times V \to [0,1]^3$ by specifying nonzero values only on the edges (and setting all other pairs to $(0,0,0)$):

$$(\alpha_N, \gamma_N, \beta_N)(v_1, v_2) = (0.6, 0.3, 0.3), \qquad (\alpha_N, \gamma_N, \beta_N)(v_2, v_3) = (0.4, 0.2, 0.4).$$

Verification of the vertex–edge bounds. For v_1v_2 we have

$$\alpha_N(v_1, v_2) = 0.6 \le \min\{0.8, 0.6\} = 0.6, \qquad \gamma_N(v_1, v_2) = 0.3 \le \min\{0.3, 0.4\} = 0.3,$$

$$\beta_N(v_1, v_2) = 0.3 \le \max\{0.2, 0.3\} = 0.3.$$

For v_2v_3 we have

$$\alpha_N(v_2, v_3) = 0.4 \le \min\{0.6, 0.5\} = 0.5, \qquad \gamma_N(v_2, v_3) = 0.2 \le \min\{0.4, 0.2\} = 0.2,$$

$$\beta_N(v_2, v_3) = 0.4 \le \max\{0.3, 0.4\} = 0.4.$$

Verification of the spherical constraint. For v_1v_2,

$$\alpha_N(v_1, v_2)^2 + \gamma_N(v_1, v_2)^2 + \beta_N(v_1, v_2)^2 = 0.6^2 + 0.3^2 + 0.3^2 = 0.54 \le 1,$$

and for v_2v_3,

$$0.4^2 + 0.2^2 + 0.4^2 = 0.36 \le 1.$$

Therefore $G = (M, N)$ with the above (α, γ, β) data is a spherical fuzzy graph in the sense of Definition 4.2.4 (with underlying crisp support E).

Theorem 4.2.6 (From general plithogenic graphs to spherical fuzzy graphs)**.** *Let $G^{GP} = (PM, PN)$ be a general plithogenic graph whose DAFs are 3-dimensional, i.e. $s = 3$ (and whose DCF dimension is $t = 1$). Assume that the vertex-DAF and edge-DAF values satisfy the spherical bounds: for every vertex $v \in M$ and every edge $e \in N$,*

$$0 \le a_1(v)^2 + a_2(v)^2 + a_3(v)^2 \le 1, \qquad 0 \le b_1(e)^2 + b_2(e)^2 + b_3(e)^2 \le 1,$$

where $\mathrm{adf}(v, \cdot) = (a_1(v), a_2(v), a_3(v))$ and $\mathrm{bdf}(e, \cdot) = (b_1(e), b_2(e), b_3(e))$. If, moreover, the plithogenic edge appurtenance constraint holds componentwise in these three coordinates, then G^{GP} induces a spherical fuzzy graph.

Proof. Fix compatible attribute values so that each vertex and edge is assigned a single 3-tuple by the DAFs. Define the spherical fuzzy vertex degrees on V by

$$\alpha_M(v) := a_1(v), \qquad \gamma_M(v) := a_2(v), \qquad \beta_M(v) := a_3(v) \qquad (v \in V),$$

setting these values to 0 for $v \notin M$ if $M \subsetneq V$. Define the spherical fuzzy edge degrees on E by

$$\alpha_N(e) := b_1(e), \qquad \gamma_N(e) := b_2(e),$$
$$\beta_N(e) := b_3(e) \qquad (e \in E),$$

setting them to 0 for $e \notin N$ if $N \subsetneq E$. The assumed spherical bounds imply

$$0 \le \alpha_M(v)^2 + \gamma_M(v)^2 + \beta_M(v)^2 \le 1,$$
$$0 \le \alpha_N(e)^2 + \gamma_N(e)^2 + \beta_N(e)^2 \le 1,$$

so the spherical constraint in Definition 4.2.4 is satisfied (in particular for edges).

Finally, the plithogenic edge appurtenance constraint (specialized to the chosen attribute values) yields, componentwise,

$$\alpha_N(uv) \le \min\{\alpha_M(u), \alpha_M(v)\},$$
$$\gamma_N(uv) \le \min\{\gamma_M(u), \gamma_M(v)\},$$
$$\beta_N(uv) \le \max\{\beta_M(u), \beta_M(v)\},$$

for each edge $uv \in N$ (and trivially for $uv \notin N$ under the zero-extension). Therefore $G = (M, N)$ with the above (α, γ, β) data is a spherical fuzzy graph. □

Remark 4.2.7. The condition $t = 1$ (a single-valued contradiction function) is not used in the construction of the spherical fuzzy degrees; rather, the key requirement is that the DAF values are 3-dimensional and satisfy the spherical bounds. If one wishes to obtain a spherical fuzzy graph from an arbitrary general plithogenic graph, one may first apply a normalization mapping $[0, 1]^3 \to [0, 1]^3$ that enforces $x_1^2 + x_2^2 + x_3^2 \le 1$.

4.2.3 Relation for General graphs

General (Intuitionistic) fuzzy graphs, weak General (Intuitionistic) fuzzy graphs, General quadripartitioned Neutrosophic graphs, and weak General Quadripartitioned Neutrosophic graphs can all be generalized using Plithogenic graphs. The following properties hold.

Theorem 4.2.8. *A Plithogenic graph with $s = 1$ and $t = 1$ satisfies all the conditions required to transform it into general fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.2.9. *A General Plithogenic Graph with $s = 1$ and $t = 1$ satisfies all the conditions required to transform it into weak general fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.2.10. *A General Plithogenic Graph with $s = 2$ and $t = 1$ satisfies all the conditions required to transform it into general Intuitionistic fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.2.11. *A General Plithogenic Graph with $s = 2$ and $t = 1$ satisfies all the conditions required to transform it into weak general Intuitionistic fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.2.12. *A General Plithogenic Graph with $s = 4$ and $t = 1$ satisfies all the conditions required to transform it into general Quadripartitioned Neutrosophic graph.*

Proof. Obviously holds. □

Theorem 4.2.13. *A General Plithogenic Graph with $s = 4$ and $t = 1$ satisfies all the conditions required to transform it into weak general quadripartitioned Neutrosophic graph.*

Proof. Obviously holds. □

4.2.4 Relation for Pythagorean fuzzy graphs

Next, we consider the Pythagorean fuzzy graph. Pythagorean fuzzy graphs have also been extensively studied in the field of fuzzy theory [518–522].

Theorem 4.2.14. *The following are examples of related graph classes for Pythagorean fuzzy graphs, including but not limited to:*

- *Cubic Pythagorean fuzzy graphs [187]*
- *Complex Pythagorean Dombi fuzzy graphs [523]*
- *Interval-valued Pythagorean fuzzy graphs [524]*
- *Interval-valued complex Pythagorean fuzzy graph [525]*

- *Pythagorean Dombi fuzzy graphs [526]*
- *Pythagorean fuzzy soft graphs [521]*
- *Pythagorean neutrosophic fuzzy graphs [527]*
- *Complex pythagorean fuzzy planar graphs [528]*
- *Pythagorean neutrosophic Dombi fuzzy graphs [529]*
- *Complex Pythagorean fuzzy threshold graphs [530]*
- *Pythagorean Dombi fuzzy soft graphs [531]*
- *Pythagorean fuzzy incidence graphs [532, 533]*
- *Pythagorean neutrosophic graphs [470, 534, 535]*
- *Fermatean fuzzy graphs [536–541]*
- *Fermatean neutrosophic graphs [372, 542]*
- *q-rung orthopair fuzzy graphs [543–545]*

Proof. Refer to each reference as needed. □

Definition 4.2.15 (Pythagorean fuzzy graph)**.** [518] A *Pythagorean fuzzy graph (PFG)* on a nonempty set V is a pair $G = (P, Q)$, where:

- P is a Pythagorean fuzzy set (PFS) on V,
- Q is a Pythagorean fuzzy relation (PFR) on $V \times V$,

satisfying the following conditions for all $u, v \in V$:

$$Q(uv) \leq P(u) \wedge P(v),$$
$$Q(uv) \geq P(u) \vee P(v),$$

where $Q : V \times V \to [0, 1]$ and $\overline{Q} : V \times V \to [0, 1]$ represent the membership and non-membership functions of Q, respectively. These functions satisfy the Pythagorean fuzzy condition:

$$0 \leq Q^2(uv) + \overline{Q}^2(uv) \leq 1 \quad \forall uv \in E.$$

The following theorem holds.

Theorem 4.2.16. *A General Plithogenic Graph with $s = 2$ and $t = 1$, representing membership and non-membership degrees, can be transformed into a Pythagorean fuzzy graph (PFG).*

Proof. To prove that a General Plithogenic Graph with $s = 2$ and $t = 1$ can be transformed into a Pythagorean fuzzy graph (PFG), we start by recalling the definition of a PFG. A PFG on a nonempty set V is a pair $G = (P, Q)$, where:

- P is a Pythagorean fuzzy set (PFS) on V,
- Q is a Pythagorean fuzzy relation (PFR) on $V \times V$.

In a General Plithogenic Graph with $s = 2$, the two uncertainty components μ_1 and μ_2 represent membership and non-membership degrees, respectively. We can map these directly to the PFG structure by defining:

$$Q(uv) = \mu_1(uv) \quad \text{and} \quad \overline{Q}(uv) = \mu_2(uv),$$

for each edge $uv \in E$.

The Pythagorean condition requires that:

$$0 \leq Q^2(uv) + \overline{Q}^2(uv) \leq 1.$$

Since μ_1 and μ_2 in the Plithogenic Graph satisfy this condition by definition (i.e., $\mu_1^2 + \mu_2^2 \leq 1$), they conform to the Pythagorean fuzzy requirements.

Furthermore, the Degree of Appurtenance Function (DAF) in the Plithogenic Graph ensures that:

$$\mu_1(uv) \leq \min(\mu_1(u), \mu_1(v)),$$

which aligns with the requirement that $Q(uv) \leq \min(P(u), P(v))$ in a PFG. □

4.2.5 Relations to hesitancy fuzzy graphs

We next consider *hesitancy fuzzy graphs* (often studied under closely related "hesitant/hesitancy" terminology). As with many uncertainty-aware graph models, hesitancy fuzzy graphs have been investigated extensively in the literature [498,546–549]. This can be regarded as a graph-theoretic counterpart of the notion of a *Hesitant Fuzzy Set* [438, 439]. The defining feature is that each vertex (and edge) carries, in addition to membership and non-membership degrees, an explicit *hesitancy* (or indeterminacy) component.

Theorem 4.2.17 (Related classes for hesitancy fuzzy graphs)**.** *Examples of graph classes related to hesitancy fuzzy graphs include (but are not limited to) the following:*

- *complex hesitant fuzzy graphs [39],*
- *constant hesitancy fuzzy graphs [550],*
- *bipolar hesitancy fuzzy graphs [551, 552],*
- *hesitancy fuzzy magic labeling graphs [553],*
- *dual hesitant fuzzy graphs [554],*
- *regular hesitancy fuzzy soft graphs [555],*
- *hesitant fuzzy hypergraphs [556].*

Proof. This theorem is a literature catalogue; see the cited references for the corresponding definitions and results. □

We now recall a standard definition [546].

Definition 4.2.18 (Hesitancy fuzzy graph [546])**.** A *hesitancy fuzzy graph* is a structure

$$G = (V, E;\ \mu_1, \gamma_1, \beta_1;\ \mu_2, \gamma_2, \beta_2),$$

where $V = \{v_1, \ldots, v_n\}$ is a finite vertex set, $E \subseteq V \times V$ is an edge set, and:

- $\mu_1, \gamma_1, \beta_1 : V \to [0, 1]$ are, respectively, the *membership*, *non-membership*, and *hesitancy* degrees on vertices, satisfying for every $v \in V$,

$$\mu_1(v) + \gamma_1(v) + \beta_1(v) = 1, \qquad \text{equivalently} \qquad \beta_1(v) = 1 - \big(\mu_1(v) + \gamma_1(v)\big).$$

- $\mu_2, \gamma_2, \beta_2 : V \times V \to [0, 1]$ are, respectively, the membership, non-membership, and hesitancy degrees on ordered pairs, and for every $(v_i, v_j) \in E$ they satisfy

$$\mu_2(v_i, v_j) \le \min\{\mu_1(v_i), \mu_1(v_j)\},$$
$$\gamma_2(v_i, v_j) \le \max\{\gamma_1(v_i), \gamma_1(v_j)\},$$
$$\beta_2(v_i, v_j) \le \min\{\beta_1(v_i), \beta_1(v_j)\},$$

together with the bound

$$0 \le \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) + \beta_2(v_i, v_j) \le 1.$$

Example 4.2.19 (A hesitancy fuzzy graph)**.** Let $V = \{v_1, v_2, v_3\}$ and let the edge set be

$$E = \{(v_1, v_2), (v_2, v_3)\} \subseteq V \times V.$$

Vertex degrees. Define $\mu_1, \gamma_1, \beta_1 : V \to [0, 1]$ by

$$(\mu_1, \gamma_1, \beta_1)(v_1) = (0.7, 0.2, 0.1), \qquad (\mu_1, \gamma_1, \beta_1)(v_2) = (0.5, 0.3, 0.2), \qquad (\mu_1, \gamma_1, \beta_1)(v_3) = (0.4, 0.1, 0.5).$$

Each vertex satisfies $\mu_1(v) + \gamma_1(v) + \beta_1(v) = 1$.

Edge degrees. Define $\mu_2, \gamma_2, \beta_2 : V \times V \to [0, 1]$ by specifying values on E and setting 0 elsewhere:

$$(\mu_2, \gamma_2, \beta_2)(v_1, v_2) = (0.5, 0.2, 0.1), \qquad (\mu_2, \gamma_2, \beta_2)(v_2, v_3) = (0.4, 0.1, 0.2),$$

and $(\mu_2, \gamma_2, \beta_2)(x, y) = (0, 0, 0)$ for all $(x, y) \notin E$.

Verification. For $(v_1, v_2) \in E$,

$$\mu_2(v_1, v_2) = 0.5 \le \min\{0.7, 0.5\} = 0.5, \qquad \gamma_2(v_1, v_2) = 0.2 \le \max\{0.2, 0.3\} = 0.3,$$

$$\beta_2(v_1, v_2) = 0.1 \le \min\{0.1, 0.2\} = 0.1, \qquad 0.5 + 0.2 + 0.1 = 0.8 \le 1.$$

For $(v_2, v_3) \in E$,

$$\mu_2(v_2, v_3) = 0.4 \le \min\{0.5, 0.4\} = 0.4, \qquad \gamma_2(v_2, v_3) = 0.1 \le \max\{0.3, 0.1\} = 0.3,$$

$$\beta_2(v_2, v_3) = 0.2 \le \min\{0.2, 0.5\} = 0.2, \qquad 0.4 + 0.1 + 0.2 = 0.7 \le 1.$$

Hence $G = (V, E; \mu_1, \gamma_1, \beta_1; \mu_2, \gamma_2, \beta_2)$ is a hesitancy fuzzy graph in the sense of Definition 4.2.18.

Theorem 4.2.20 (From general plithogenic graphs to hesitancy fuzzy graphs)**.** *Let $G^{GP} = (PM, PN)$ be a general plithogenic graph whose DAFs are 3-dimensional ($s = 3$) and whose DCF dimension is $t = 1$. Fix compatible attribute values so that each vertex $v \in M$ is assigned a single triple*

$$\mathrm{adf}(v, \cdot) = (a_1(v), a_2(v), a_3(v)) \in [0, 1]^3$$

and each edge $e \in N$ is assigned a single triple

$$\mathrm{bdf}(e, \cdot) = (b_1(e), b_2(e), b_3(e)) \in [0, 1]^3.$$

Assume in addition that these triples satisfy the normalization

$$a_1(v) + a_2(v) + a_3(v) = 1 \quad (\forall v \in M), \qquad 0 \le b_1(e) + b_2(e) + b_3(e) \le 1 \quad (\forall e \in N),$$

and that the plithogenic edge appurtenance constraint holds componentwise in these three coordinates. Then G^{GP} induces a hesitancy fuzzy graph in the sense of Definition 4.2.18.

Proof. Define vertex degrees on V by

$$\mu_1(v) := a_1(v), \qquad \gamma_1(v) := a_2(v), \qquad \beta_1(v) := a_3(v) \qquad (v \in V),$$

setting $(\mu_1, \gamma_1, \beta_1) = (0, 0, 0)$ for vertices not in M if $M \subsetneq V$. Then the assumed normalization yields $\mu_1(v) + \gamma_1(v) + \beta_1(v) = 1$ for all $v \in M$.

Define edge degrees on $V \times V$ by

$$\mu_2(e) := b_1(e), \qquad \gamma_2(e) := b_2(e), \qquad \beta_2(e) := b_3(e) \qquad (e \in E),$$

again extending by zeros outside N if $N \subsetneq E$. The assumed bound $0 \le b_1(e)+b_2(e)+b_3(e) \le 1$ implies

$$0 \le \mu_2(e) + \gamma_2(e) + \beta_2(e) \le 1 \qquad (\forall e \in E).$$

Finally, the plithogenic edge appurtenance constraint specialized to the chosen attribute values yields, componentwise, that for each edge $e = \{u, v\} \in N$,

$$b_1(e) \le \min\{a_1(u), a_1(v)\}, \qquad b_2(e) \le \min\{a_2(u), a_2(v)\}, \qquad b_3(e) \le \min\{a_3(u), a_3(v)\}.$$

Thus

$$\mu_2(e) \le \min\{\mu_1(u), \mu_1(v)\}, \qquad \beta_2(e) \le \min\{\beta_1(u), \beta_1(v)\}.$$

Moreover, since $\gamma_2(e) = b_2(e) \le \min\{a_2(u), a_2(v)\} \le \max\{a_2(u), a_2(v)\} = \max\{\gamma_1(u), \gamma_1(v)\}$, we also obtain

$$\gamma_2(e) \le \max\{\gamma_1(u), \gamma_1(v)\}.$$

Therefore all inequalities in Definition 4.2.18 hold for edges in N (and trivially for edges outside N by the zero-extension). Hence the constructed data form a hesitancy fuzzy graph. □

Remark 4.2.21. Theorem 4.2.20 uses an explicit normalization on vertex DAF triples to match the defining constraint $\mu_1 + \gamma_1 + \beta_1 = 1$ in Definition 4.2.18. If one starts from arbitrary DAF triples in $[0, 1]^3$, a normalization map $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3)/(x_1 + x_2 + x_3)$ (when the denominator is nonzero) yields the required condition.

4.2.6 Double-Valued Neutrosophic Set and Graph

A DVNS assigns truth, falsity, and two indeterminacy degrees (truth-leaning, falsity-leaning) in $[0, 1]$ to each element to model nuanced uncertainty [458, 557–560]. A DVNG labels vertices and edges with DVNS quadruples, enforcing endpoint bounds on truth/indeterminacy and maximal falsity consistency throughout graph.

Definition 4.2.22 (Double-Valued Neutrosophic Set (DVNS))**.** Let X be a nonempty universe. A *double-valued neutrosophic set* (DVNS) A on X is specified by four functions

$$T_A,\ I_A^T,\ I_A^F,\ F_A : X \to [0, 1],$$

and is written as

$$A = \{(x,\, T_A(x),\, I_A^T(x),\, I_A^F(x),\, F_A(x)) :\ x \in X\}.$$

Here $T_A(x)$ is the truth-membership degree, $F_A(x)$ is the falsity-membership degree, and $I_A^T(x)$ (resp. $I_A^F(x)$) is the indeterminacy degree leaning toward truth (resp. falsity). For each $x \in X$ the *single-valued* constraint holds:

$$0 \le T_A(x) + I_A^T(x) + I_A^F(x) + F_A(x) \le 4.$$

Definition 4.2.23 (Double-Valued Neutrosophic Graph (DVNG))**.** Let $G^* = (V, E)$ be a finite simple undirected graph, where $E \subseteq \binom{V}{2}$. A *double-valued neutrosophic graph* (DVNG) on G^* is a pair

$$G = (A, B),$$

where:

- A is a DVNS on V, i.e.,

$$T_A,\ I_A^T,\ I_A^F,\ F_A : V \to [0,1], \qquad 0 \le T_A(v) + I_A^T(v) + I_A^F(v) + F_A(v) \le 4 \ \ (\forall v \in V).$$

- B is a DVNS-type relation on E, i.e.,

$$T_B,\ I_B^T,\ I_B^F,\ F_B : E \to [0,1], \qquad 0 \le T_B(e) + I_B^T(e) + I_B^F(e) + F_B(e) \le 4 \ \ (\forall e \in E).$$

Moreover, A and B are required to satisfy the following endpoint-consistency conditions for every edge $uv \in E$:

$$T_B(uv) \le \min\{T_A(u), T_A(v)\}, \qquad I_B^T(uv) \le \min\{I_A^T(u), I_A^T(v)\},$$
$$I_B^F(uv) \le \min\{I_A^F(u), I_A^F(v)\}, \qquad F_B(uv) \ge \max\{F_A(u), F_A(v)\}.$$

Example 4.2.24 (A double-valued neutrosophic graph)**.** Let $G^* = (V, E)$ be the 3-vertex path with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{\{v_1, v_2\}, \{v_2, v_3\}\}.$$

Vertex DVNS data. Define $T_A, I_A^T, I_A^F, F_A : V \to [0,1]$ by

$$(T_A, I_A^T, I_A^F, F_A)(v_1) = (0.7, 0.2, 0.1, 0.3), \quad (v_2) = (0.5, 0.3, 0.2, 0.4), \quad (v_3) = (0.6, 0.1, 0.2, 0.2).$$

Each vertex satisfies the bound $0 \le T_A + I_A^T + I_A^F + F_A \le 4$, e.g., $0.7+0.2+0.1+0.3 = 1.3 \le 4$.

Edge DVNS-type data. Define $T_B, I_B^T, I_B^F, F_B : E \to [0,1]$ by

$$(T_B, I_B^T, I_B^F, F_B)(v_1v_2) = (0.5, 0.2, 0.1, 0.5), \qquad (T_B, I_B^T, I_B^F, F_B)(v_2v_3) = (0.4, 0.1, 0.2, 0.4),$$

where v_iv_j denotes the edge $\{v_i, v_j\}$. Each edge satisfies $0 \le T_B + I_B^T + I_B^F + F_B \le 4$, e.g., $0.5 + 0.2 + 0.1 + 0.5 = 1.3 \le 4$.

Verification of endpoint-consistency. For $v_1v_2 \in E$,

$$T_B(v_1v_2) = 0.5 \le \min\{0.7, 0.5\} = 0.5, \qquad I_B^T(v_1v_2) = 0.2 \le \min\{0.2, 0.3\} = 0.2,$$
$$I_B^F(v_1v_2) = 0.1 \le \min\{0.1, 0.2\} = 0.1, \qquad F_B(v_1v_2) = 0.5 \ge \max\{0.3, 0.4\} = 0.4.$$

For $v_2v_3 \in E$,

$$T_B(v_2v_3) = 0.4 \le \min\{0.5, 0.6\} = 0.5, \qquad I_B^T(v_2v_3) = 0.1 \le \min\{0.3, 0.1\} = 0.1,$$
$$I_B^F(v_2v_3) = 0.2 \le \min\{0.2, 0.2\} = 0.2, \qquad F_B(v_2v_3) = 0.4 \ge \max\{0.4, 0.2\} = 0.4.$$

Hence $G = (A, B)$ with the above data is a double-valued neutrosophic graph in the sense of the definition.

Theorem 4.2.25 (General Plithogenic graphs subsume double-valued neutrosophic graphs)**.** *Every double-valued neutrosophic graph (DVNG) can be represented as a special case of a general plithogenic graph. More precisely, given a DVNG $G^{DV} = (A, B)$ on a crisp graph $G^* = (V, E)$, one can construct a general plithogenic graph $G^{GP} = (PM, PN)$ on the same underlying graph such that the DVNG vertex- and edge-membership quadruples are recovered from the corresponding plithogenic degrees of appurtenance (DAFs).*

Proof. Let $G^{DV} = (A, B)$ be a DVNG on $G^* = (V, E)$. Thus A assigns to each vertex $v \in V$ a quadruple

$$\big(T_A(v),\, I_A^T(v),\, I_A^F(v),\, F_A(v)\big) \in [0,1]^4,$$

and B assigns to each edge $e \in E$ a quadruple

$$\big(T_B(e),\, I_B^T(e),\, I_B^F(e),\, F_B(e)\big) \in [0,1]^4,$$

together with the DVNG endpoint-consistency constraints.

We now build a general plithogenic graph $G^{GP} = (PM, PN)$ realizing these data.

Step 1: vertex-side plithogenic structure. Set $M := V$. Choose a single vertex-attribute symbol l (any label suffices), and let the attribute-value domain be

$$Ml := \{\mathbf{T}, \mathbf{I}^T, \mathbf{I}^F, \mathbf{F}\}.$$

Fix the number of components to be $s := 4$ (so that DAF values lie in $[0,1]^4$) and take $t := 1$ (any $t \geq 1$ works). Define the vertex DAF

$$adf : M \times Ml \to [0,1]^4$$

by assigning, for each $v \in V$ and each $a \in Ml$,

$$adf(v,a) \;:=\; \begin{cases} \big(T_A(v),\, I_A^T(v),\, I_A^F(v),\, F_A(v)\big), & a = \mathbf{T}, \\ \big(T_A(v),\, I_A^T(v),\, I_A^F(v),\, F_A(v)\big), & a = \mathbf{I}^T, \\ \big(T_A(v),\, I_A^T(v),\, I_A^F(v),\, F_A(v)\big), & a = \mathbf{I}^F, \\ \big(T_A(v),\, I_A^T(v),\, I_A^F(v),\, F_A(v)\big), & a = \mathbf{F}. \end{cases}$$

(That is, we store the entire DVNS quadruple in the DAF value; the attribute-value tag a merely indexes the same quadruple. This is permitted in the *general* plithogenic setting, which does not impose coupling constraints between vertices and edges.)

Define the vertex DCF

$$aCf : Ml \times Ml \to [0,1]^t$$

by

$$aCf(x,y) := 0 \qquad (\forall x, y \in Ml).$$

Then aCf is reflexive and symmetric: $aCf(a,a) = 0$ and $aCf(a,b) = aCf(b,a)$ for all $a, b \in Ml$.

Step 2: edge-side plithogenic structure. Set $N := E$. Choose a single edge-attribute symbol m, and let

$$Nm := \{\mathbf{T}, \mathbf{I}^T, \mathbf{I}^F, \mathbf{F}\}.$$

Define the edge DAF

$$bdf : N \times Nm \to [0,1]^4$$

by assigning, for each $e \in E$ and each $b \in Nm$,

$$bdf(e,b) \; := \; \big(T_B(e),\, I_B^T(e),\, I_B^F(e),\, F_B(e)\big).$$

Define the edge DCF

$$bCf : Nm \times Nm \to [0,1]^t$$

again by $bCf(x,y) := 0$ for all $x, y \in Nm$, which is reflexive and symmetric.

Step 3: assemble G^{GP}. With the above choices, define

$$PM := (M, l, Ml, adf, aCf), \qquad PN := (N, m, Nm, bdf, bCf),$$

and set $G^{GP} := (PM, PN)$. By construction, G^{GP} satisfies the only required axioms in the stated definition of a general plithogenic graph, namely reflexivity and symmetry of the contradiction functions.

Step 4: recovery of the DVNG data. For each vertex $v \in V$, the DVNS quadruple $\big(T_A(v), I_A^T(v), I_A^F(v), F_A(v)\big)$ is read off from $adf(v, \mathbf{T})$ (equivalently, from $adf(v,a)$ for any $a \in Ml$). Similarly, for each edge $e \in E$, the DVNS-type edge quadruple $\big(T_B(e), I_B^T(e), I_B^F(e), F_B(e)\big)$ is read off from $bdf(e, \mathbf{T})$ (or any $b \in Nm$).

Finally, the DVNG endpoint-consistency constraints are properties of the original DVNG labels (A, B) and remain true for the recovered labels, since we have preserved them exactly in the DAF encoding. Therefore the DVNG is realized as a special case of a general plithogenic graph. □

4.2.7 Triple-Valued Neutrosophic Set and Graph

A TVNS assigns truth, falsity, and three indeterminacy degrees (truth-leaning, neutral, falsity-leaning) in $[0,1]$ per element [475–477]. Related concepts include the quadruple-valued neutrosophic set [476] and the quintuple-valued neutrosophic set [476, 480, 561], among others. A TVNG labels vertices and edges with TVNS quintuple degrees, ensuring endpoint-consistency bounds on truth/indeterminacy and maximal falsity.

Definition 4.2.26 (Triple-Valued Neutrosophic Set (TVNS))**.** [475,476] Let X be a nonempty universe. A *triple-valued neutrosophic set* (TVNS) A on X is specified by five functions

$$T_A,\; I_A^T,\; I_A^N,\; I_A^F,\; F_A : X \to [0,1],$$

and is written as

$$A = \big\{(x,\, T_A(x),\, I_A^T(x),\, I_A^N(x),\, I_A^F(x),\, F_A(x)) : \; x \in X\big\}.$$

Here $I_A^T(x)$ is indeterminacy leaning toward truth, $I_A^F(x)$ is indeterminacy leaning toward falsity, and $I_A^N(x)$ is *neutral* indeterminacy (neither leaning toward truth nor falsity). For each $x \in X$ the *single-valued* constraint holds:

$$0 \le T_A(x) + I_A^T(x) + I_A^N(x) + I_A^F(x) + F_A(x) \le 5.$$

Definition 4.2.27 (Triple-Valued Neutrosophic Graph (TVNG))**.** Let $G^* = (V, E)$ be a finite simple undirected graph, where $E \subseteq \binom{V}{2}$. A *triple-valued neutrosophic graph* (TVNG) on G^* is a pair

$$G = (A, B),$$

where:

- A is a TVNS on V, i.e.,

$$T_A,\ I_A^T,\ I_A^N,\ I_A^F,\ F_A : V \to [0,1], \quad 0 \le T_A(v)+I_A^T(v)+I_A^N(v)+I_A^F(v)+F_A(v) \le 5\ (\forall v \in V).$$

- B is a TVNS-type relation on E, i.e.,

$$T_B,\ I_B^T,\ I_B^N,\ I_B^F,\ F_B : E \to [0,1], \quad 0 \le T_B(e)+I_B^T(e)+I_B^N(e)+I_B^F(e)+F_B(e) \le 5\ (\forall e \in E).$$

Moreover, A and B are required to satisfy the following endpoint-consistency conditions for every edge $uv \in E$:

$$T_B(uv) \le \min\{T_A(u), T_A(v)\}, \qquad I_B^T(uv) \le \min\{I_A^T(u), I_A^T(v)\},$$
$$I_B^N(uv) \le \min\{I_A^N(u), I_A^N(v)\}, \qquad I_B^F(uv) \le \min\{I_A^F(u), I_A^F(v)\},$$
$$F_B(uv) \ge \max\{F_A(u), F_A(v)\}.$$

Example 4.2.28 (A triple-valued neutrosophic graph)**.** Let $G^* = (V, E)$ be the 3-vertex path with

$$V = \{v_1, v_2, v_3\}, \qquad E = \{\{v_1, v_2\}, \{v_2, v_3\}\}.$$

Vertex TVNS data. Define $T_A, I_A^T, I_A^N, I_A^F, F_A : V \to [0,1]$ by

$$(T_A, I_A^T, I_A^N, I_A^F, F_A)(v_1) = (0.7, 0.2, 0.1, 0.1, 0.3),$$
$$(T_A, I_A^T, I_A^N, I_A^F, F_A)(v_2) = (0.5, 0.3, 0.2, 0.1, 0.4),$$
$$(T_A, I_A^T, I_A^N, I_A^F, F_A)(v_3) = (0.6, 0.1, 0.2, 0.2, 0.2).$$

Each vertex satisfies the bound $0 \le T_A + I_A^T + I_A^N + I_A^F + F_A \le 5$; for instance, $0.7 + 0.2 + 0.1 + 0.1 + 0.3 = 1.4 \le 5$.

Edge TVNS-type data. Define $T_B, I_B^T, I_B^N, I_B^F, F_B : E \to [0,1]$ by

$$(T_B, I_B^T, I_B^N, I_B^F, F_B)(v_1v_2) = (0.5, 0.2, 0.1, 0.1, 0.5),$$
$$(T_B, I_B^T, I_B^N, I_B^F, F_B)(v_2v_3) = (0.4, 0.1, 0.2, 0.1, 0.4),$$

where v_iv_j denotes the edge $\{v_i, v_j\}$. Each edge satisfies $0 \le T_B + I_B^T + I_B^N + I_B^F + F_B \le 5$; for example, $0.5 + 0.2 + 0.1 + 0.1 + 0.5 = 1.4 \le 5$.

Verification of endpoint-consistency. For $v_1v_2 \in E$,

$$T_B(v_1v_2) = 0.5 \le \min\{0.7, 0.5\} = 0.5, \qquad I_B^T(v_1v_2) = 0.2 \le \min\{0.2, 0.3\} = 0.2,$$

$$I_B^N(v_1v_2) = 0.1 \le \min\{0.1, 0.2\} = 0.1, \qquad I_B^F(v_1v_2) = 0.1 \le \min\{0.1, 0.1\} = 0.1,$$
$$F_B(v_1v_2) = 0.5 \ge \max\{0.3, 0.4\} = 0.4.$$

For $v_2v_3 \in E$,

$$T_B(v_2v_3) = 0.4 \le \min\{0.5, 0.6\} = 0.5, \qquad I_B^T(v_2v_3) = 0.1 \le \min\{0.3, 0.1\} = 0.1,$$
$$I_B^N(v_2v_3) = 0.2 \le \min\{0.2, 0.2\} = 0.2, \qquad I_B^F(v_2v_3) = 0.1 \le \min\{0.1, 0.2\} = 0.1,$$
$$F_B(v_2v_3) = 0.4 \ge \max\{0.4, 0.2\} = 0.4.$$

Hence $G = (A, B)$ with the above data is a triple-valued neutrosophic graph in the sense of the definition.

Theorem 4.2.29 (General Plithogenic graphs subsume triple-valued neutrosophic graphs)**.** *Every triple-valued neutrosophic graph (TVNG) can be realized as a special case of a general plithogenic graph. More precisely, given a TVNG $G^{TV} = (A, B)$ on a crisp graph $G^* = (V, E)$, one can construct a general plithogenic graph $G^{GP} = (PM, PN)$ on the same underlying graph such that the vertex- and edge-labellings of the TVNG are recovered from the corresponding plithogenic degrees of appurtenance (DAFs).*

Proof. Let $G^{TV} = (A, B)$ be a TVNG on $G^* = (V, E)$. Thus A assigns to each vertex $v \in V$ a quintuple

$$\big(T_A(v),\, I_A^T(v),\, I_A^N(v),\, I_A^F(v),\, F_A(v)\big) \in [0, 1]^5,$$

and B assigns to each edge $e \in E$ a quintuple

$$\big(T_B(e),\, I_B^T(e),\, I_B^N(e),\, I_B^F(e),\, F_B(e)\big) \in [0, 1]^5,$$

together with the endpoint-consistency constraints of Definition (TVNG).

We now build a general plithogenic graph $G^{GP} = (PM, PN)$ encoding exactly these data.

Step 1: vertex-side plithogenic structure. Set $M := V$. Choose a single vertex-attribute symbol l. Let the attribute-value domain be

$$Ml := \{\mathbf{T}, \mathbf{I}^T, \mathbf{I}^N, \mathbf{I}^F, \mathbf{F}\}.$$

Fix $s := 5$ (so that vertex DAF values lie in $[0, 1]^5$) and take any $t \ge 1$ (for concreteness, $t := 1$). Define the vertex DAF

$$adf : M \times Ml \to [0, 1]^5$$

by storing the entire TVNS quintuple:

$$adf(v, a) \;:=\; \big(T_A(v),\, I_A^T(v),\, I_A^N(v),\, I_A^F(v),\, F_A(v)\big) \qquad (\forall\, v \in V,\ \forall\, a \in Ml).$$

Define the vertex DCF

$$aCf : Ml \times Ml \to [0, 1]^t$$

as the identically zero function:

$$aCf(x, y) := 0 \qquad (\forall x, y \in Ml).$$

Then $aCf(a, a) = 0$ and $aCf(a, b) = aCf(b, a)$ for all $a, b \in Ml$, so the required reflexivity and symmetry hold.

Step 2: edge-side plithogenic structure. Set $N := E$. Choose a single edge-attribute symbol m and let

$$Nm := \{\mathbf{T}, \mathbf{I}^T, \mathbf{I}^N, \mathbf{I}^F, \mathbf{F}\}.$$

Define the edge DAF

$$bdf : N \times Nm \to [0,1]^5$$

by

$$bdf(e,b) \;:=\; \big(T_B(e),\, I_B^T(e),\, I_B^N(e),\, I_B^F(e),\, F_B(e)\big) \qquad (\forall\, e \in E,\ \forall\, b \in Nm).$$

Define the edge DCF

$$bCf : Nm \times Nm \to [0,1]^t$$

by $bCf(x,y) := 0$ for all $x, y \in Nm$, which again is reflexive and symmetric.

Step 3: assemble G^{GP}. Define

$$PM := (M, l, Ml, adf, aCf), \qquad PN := (N, m, Nm, bdf, bCf),$$

and set $G^{GP} := (PM, PN)$. By construction, G^{GP} satisfies the axioms in the definition of a general plithogenic graph, namely reflexivity and symmetry of aCf and bCf.

Step 4: recovery of the TVNG data and preservation of constraints. For each vertex $v \in V$, the TVNS quintuple is recovered by reading

$$\big(T_A(v), I_A^T(v), I_A^N(v), I_A^F(v), F_A(v)\big) = adf(v, \mathbf{T})$$

(and similarly from $adf(v,a)$ for any $a \in Ml$). For each edge $e \in E$, the edge quintuple is recovered from

$$\big(T_B(e), I_B^T(e), I_B^N(e), I_B^F(e), F_B(e)\big) = bdf(e, \mathbf{T})$$

(and similarly from $bdf(e,b)$ for any $b \in Nm$).

Since these recovered values coincide *exactly* with the original TVNG labels, all endpoint-consistency inequalities of the TVNG (for every $uv \in E$),

$$T_B(uv) \le \min\{T_A(u), T_A(v)\}, \quad I_B^T(uv) \le \min\{I_A^T(u), I_A^T(v)\}, \quad I_B^N(uv) \le \min\{I_A^N(u), I_A^N(v)\},$$

$$I_B^F(uv) \le \min\{I_A^F(u), I_A^F(v)\}, \quad F_B(uv) \ge \max\{F_A(u), F_A(v)\},$$

remain valid after embedding. Therefore the given TVNG is represented as a special case of the general plithogenic graph G^{GP}. □

4.2.8 Relation for Intuitionistic Hesitant Fuzzy Graphs

Next, we consider the Intuitionistic Hesitant Fuzzy Graph, a newly proposed class of graphs in recent years. It is known to be a generalized class of Hesitant Fuzzy Graphs. The definition is provided below [562, 563].

Definition 4.2.30 (Intuitionistic Hesitancy Fuzzy Graph)**.** [562] An Intuitionistic Hesitancy Fuzzy Graph is $G = (V, E, \sigma, \mu)$, where:

- V is the vertex set.

- $\lambda_1, \delta_1, \rho_1 : V \to [0,1]$ represent the degree of membership (MS), non-membership (NMS), and hesitancy of $v \in V$, respectively, satisfying:

$$0 \leq \lambda_1(v) + \delta_1(v) + \rho_1(v) \leq 1$$

- $\lambda_2, \delta_2, \rho_2 : V \times V \to [0,1]$ represent the degree of membership (MS), non-membership (NMS), and hesitancy of $x = (u,v) \in V \times V$, respectively, satisfying:

$$\lambda_2(x) \leq \min\{\lambda_1(u), \lambda_1(v)\}$$
$$\delta_2(x) \leq \max\{\delta_1(u), \delta_1(v)\}$$
$$\rho_2(x) \leq \min\{\rho_1(u), \rho_1(v)\}$$

  and

$$0 \leq \lambda_2(x) + \delta_2(x) + \rho_2(x) \leq 1, \quad \forall x \in V \times V$$

Theorem 4.2.31. *A General Plithogenic Graph with $s = 3$ and $t = 1$, representing membership and non-membership degrees, can be transformed into a Intuitionistic Hesitancy Fuzzy Graphs.*

Proof. It can be proven in the same way as the previous theorem. □

4.2.9 Relations to picture fuzzy graphs

We next consider *picture fuzzy graphs* (PFGs), which extend fuzzy graphs by recording three degrees for each vertex and edge, typically interpreted as *positive membership*, *neutral membership*, and *negative membership*. Picture fuzzy graphs have been studied extensively, and a range of applications has also been explored [564, 565]. These can be regarded as graph-theoretic counterparts of the notion of a *Picture Fuzzy Set* [445, 566, 567].

Theorem 4.2.32 (Related classes for picture fuzzy graphs)**.** *Examples of graph classes related to picture fuzzy graphs include (but are not limited to) the following:*

- *picture Dombi fuzzy graphs [568],*
- *picture fuzzy line graphs [569],*
- *picture fuzzy planar graphs [570],*
- *picture fuzzy tolerance graphs [165],*
- *picture fuzzy φ-tolerance competition graphs [571],*

- *interval-valued picture fuzzy graphs [572],*
- *picture fuzzy incidence graphs [573],*
- *picture fuzzy threshold graphs [165],*
- *picture fuzzy soft graphs [223],*
- *Cayley picture fuzzy graphs [574],*
- *m-polar picture fuzzy graphs [575],*
- *interval-valued picture (S, T)-fuzzy graphs [576],*
- *q-rung picture fuzzy graphs [577],*
- *mixed picture fuzzy graphs [189],*
- *picture fuzzy directed hypergraphs [170],*
- *picture fuzzy cubic graphs [578],*
- *picture fuzzy digraphs [512],*
- *balanced picture fuzzy graphs [579].*

Proof. This theorem is a literature catalogue; see the cited references for the corresponding definitions and results. □

We recall a standard definition below [562].

Definition 4.2.33 (Picture fuzzy graph [562])**.** Let $G^* = (V, E)$ be a (finite) graph. A *picture fuzzy graph* on G^* is a pair $G = (A, B)$, where

$$A = (\mu_A, \eta_A, \nu_A) \quad \text{is a picture fuzzy set on } V, \qquad B = (\mu_B, \eta_B, \nu_B) \quad \text{is a picture fuzzy set on } E,$$

with maps $\mu_A, \eta_A, \nu_A : V \to [0, 1]$ and $\mu_B, \eta_B, \nu_B : E \to [0, 1]$ satisfying, for each edge $uv \in E$,

$$\mu_B(uv) \le \min\{\mu_A(u), \mu_A(v)\},$$
$$\eta_B(uv) \le \min\{\eta_A(u), \eta_A(v)\},$$
$$\nu_B(uv) \ge \max\{\nu_A(u), \nu_A(v)\}.$$

Theorem 4.2.34 (From general plithogenic graphs to picture fuzzy graphs)**.** *Let $G^{GP} = (PM, PN)$ be a general plithogenic graph whose DAFs are 2-dimensional ($s = 2$) and whose DCF dimension is $t = 1$. Fix compatible attribute values so that each vertex $v \in M$ is assigned a single pair*

$$\mathrm{adf}(v, \cdot) = (a_1(v), a_2(v)) \in [0, 1]^2$$

and each edge $e \in N$ is assigned a single pair

$$\mathrm{bdf}(e, \cdot) = (b_1(e), b_2(e)) \in [0, 1]^2.$$

Assume the normalization

$$a_1(v) + a_2(v) \leq 1 \quad (\forall v \in M), \qquad b_1(e) + b_2(e) \leq 1 \quad (\forall e \in N),$$

and assume that the plithogenic edge appurtenance constraint holds componentwise in these two coordinates. Then G^{GP} induces a picture fuzzy graph in the sense of Definition 4.2.33.

Proof. Define vertex degrees on V by

$$\mu_A(v) := a_1(v), \qquad \eta_A(v) := a_2(v), \qquad \nu_A(v) := 1 - a_1(v) - a_2(v) \qquad (v \in V),$$

setting $(\mu_A, \eta_A, \nu_A) = (0, 0, 1)$ for $v \notin M$ if $M \subsetneq V$. The assumed bound $a_1(v) + a_2(v) \leq 1$ guarantees $\nu_A(v) \in [0, 1]$.

Similarly, define edge degrees on E by

$$\mu_B(e) := b_1(e), \qquad \eta_B(e) := b_2(e), \qquad \nu_B(e) := 1 - b_1(e) - b_2(e) \qquad (e \in E),$$

setting $(\mu_B, \eta_B, \nu_B) = (0, 0, 1)$ for $e \notin N$ if $N \subsetneq E$. Again $b_1(e) + b_2(e) \leq 1$ ensures $\nu_B(e) \in [0, 1]$.

Let $uv \in N$. By the componentwise plithogenic edge appurtenance constraint,

$$b_1(uv) \leq \min\{a_1(u), a_1(v)\}, \qquad b_2(uv) \leq \min\{a_2(u), a_2(v)\}.$$

Thus $\mu_B(uv) \leq \min\{\mu_A(u), \mu_A(v)\}$ and $\eta_B(uv) \leq \min\{\eta_A(u), \eta_A(v)\}$.

Moreover, the above inequalities imply

$$b_1(uv)+b_2(uv) \ \leq \ \min\{a_1(u), a_1(v)\}+\min\{a_2(u), a_2(v)\} \ \leq \ \min\{a_1(u)+a_2(u),\ a_1(v)+a_2(v)\}.$$

Therefore,

$$\nu_B(uv) = 1 - b_1(uv) - b_2(uv) \ \geq \ 1 - \min\{a_1(u) + a_2(u), a_1(v) + a_2(v)\} = \max\{\nu_A(u), \nu_A(v)\},$$

which is exactly the third inequality in Definition 4.2.33. Hence $G = (A, B)$ is a picture fuzzy graph. □

Remark 4.2.35. The normalization $a_1 + a_2 \leq 1$ (and similarly for $b_1 + b_2 \leq 1$) is needed to ensure that the induced third component $\nu = 1 - \mu - \eta$ lies in $[0, 1]$. If one starts from arbitrary pairs in $[0, 1]^2$, one may enforce the bound by a rescaling $(x_1, x_2) \mapsto (x_1, x_2)/\max\{1, x_1 + x_2\}$.

4.2.10 Relations to vague graphs

Vague graphs provide a graph-theoretic realization of *vague sets* and have been investigated in a variety of directions [40, 41, 50, 580, 581]. From a modeling viewpoint, they are particularly close to intuitionistic fuzzy graphs, since both frameworks encode a pair of degrees that can be interpreted as evidence "for" and "against" membership, together with an induced hesitation interval.

Theorem 4.2.36 (Related classes for vague graphs)**.** *Examples of graph classes related to vague graphs include (but are not limited to) the following:*

- *edge-irregular product vague graphs [582],*
- *vague incidence graphs [583],*
- *complex vague graphs [584],*
- *irregular vague graphs [585],*
- *neutrosophic vague graphs [586],*
- *neutrosophic vague line graphs [587],*
- *cubic vague graphs [588],*
- *vague influence graphs [589],*
- *neutrosophic vague incidence graphs [587],*
- *neutrosophic bipolar vague incidence graphs [590],*
- *neutrosophic vague soft graphs [55].*

Proof. This theorem is a literature catalogue; see the cited references for the corresponding definitions and results. □

We recall the definition of a vague graph for completeness.

Definition 4.2.37 (Vague graph)**.** [50] Let $G^* = (V, E)$ be a finite simple undirected graph. A *vague graph* on G^* is a pair $G = (A, B)$, where

$$A = (t_A, f_A) \text{ is a vague set on } V, \qquad B = (t_B, f_B) \text{ is a vague set on } E,$$

that is,

$$t_A, f_A : V \to [0, 1], \qquad t_B, f_B : E \to [0, 1],$$

such that for all $v \in V$ and $e \in E$,

$$0 \le t_A(v) + f_A(v) \le 1, \qquad 0 \le t_B(e) + f_B(e) \le 1.$$

Moreover, the vertex- and edge-data satisfy the endpoint-consistency conditions: for every edge $uv \in E$,

$$t_B(uv) \ \le \ \min\{t_A(u), t_A(v)\}, \qquad f_B(uv) \ \ge \ \max\{f_A(u), f_A(v)\}.$$

Equivalently, each vertex v and edge uv carries the *vague value intervals*

$$V_A(v) = [\, t_A(v),\, 1 - f_A(v)\,], \qquad V_B(uv) = [\, t_B(uv),\, 1 - f_B(uv)\,],$$

so that an edge cannot have stronger "true evidence" than its endpoints, while its "false evidence" dominates that of at least one endpoint.

Theorem 4.2.38 (From general plithogenic graphs to vague graphs)**.** *Let $G^{GP} = (PM, PN)$ be a general plithogenic graph whose DAFs are 2-dimensional ($s = 2$) and whose DCF dimension is $t = 1$. Fix compatible attribute values so that each vertex $v \in M$ is assigned a single pair*

$$\mathrm{adf}(v, \cdot) = (a_1(v), a_2(v)) \in [0, 1]^2$$

and each edge $e \in N$ is assigned a single pair

$$\mathrm{bdf}(e, \cdot) = (b_1(e), b_2(e)) \in [0, 1]^2.$$

Assume that the plithogenic edge appurtenance constraint holds componentwise in these two coordinates. Then G^{GP} induces a vague graph on the same underlying crisp graph.

Proof. Define the vague vertex functions $t_A, f_A : V \to [0, 1]$ by

$$t_A(v) := a_1(v), \qquad f_A(v) := a_2(v) \qquad (v \in V),$$

setting $(t_A, f_A) = (0, 0)$ for $v \notin M$ if $M \subsetneq V$. Similarly, define the vague edge functions $t_B, f_B : E \to [0, 1]$ by

$$t_B(e) := b_1(e), \qquad f_B(e) := b_2(e) \qquad (e \in E),$$

setting $(t_B, f_B) = (0, 0)$ for $e \notin N$ if $N \subsetneq E$. These maps clearly take values in $[0, 1]$.

Let $uv \in N$. The componentwise plithogenic edge appurtenance constraint yields

$$b_1(uv) \le \min\{a_1(u), a_1(v)\}, \qquad b_2(uv) \le \min\{a_2(u), a_2(v)\}.$$

Hence

$$t_B(uv) \le \min\{t_A(u), t_A(v)\} \quad \text{and} \quad f_B(uv) = b_2(uv) \le \min\{f_A(u), f_A(v)\} \le \max\{f_A(u), f_A(v)\}.$$

Thus the t-inequality in Definition 4.2.37 holds, and the required f-inequality also holds. Finally, since $t_A, f_A, t_B, f_B \in [0, 1]$, the bounds

$$0 \le t_A(v) + f_A(v) \le 2, \qquad 0 \le t_B(e) + f_B(e) \le 2$$

are automatic; if one wishes to enforce the stricter classical vague-set condition $t_A(v) + f_A(v) \le 1$ and $t_B(e) + f_B(e) \le 1$, it suffices to assume (or normalize) the DAF pairs so that $a_1(v) + a_2(v) \le 1$ for vertices and $b_1(e) + b_2(e) \le 1$ for edges. Therefore the constructed pair (A, B) is a vague graph on G^*. □

4.2.11 Relations to multi-fuzzy graphs

We next discuss *multi-fuzzy graphs*, which extend fuzzy graphs by attaching a *vector* of membership degrees to each vertex and edge, in the spirit of multi-fuzzy sets [591–593]. Closely related directions include n-dimensional fuzzy sets [594], multidimensional fuzzy sets [595–597], and probabilistic hesitant fuzzy sets [598–600]. We record a standard definition and then relate it to general plithogenic graphs.

Definition 4.2.39 (Multi-fuzzy graph)**.** Let $G^* = (V, E)$ be a finite simple undirected graph and fix an integer $k \geq 1$. A *k-dimensional multi-fuzzy graph* on G^* is a pair

$$G^{(k)} = (\sigma, \mu),$$

where

$$\sigma : V \to [0, 1]^k, \qquad \mu : E \to [0, 1]^k,$$

are the *vertex* and *edge* multi-membership maps. Writing $\sigma(v) = (\sigma_1(v), \ldots, \sigma_k(v)$ and $\mu(uv) = (\mu_1(uv), \ldots, \mu_k(uv))$, one typically imposes the componentwise endpoint constraints

$$\mu_i(uv) \;\leq\; \min\{\sigma_i(u), \sigma_i(v)\} \qquad (\forall\, uv \in E,\ \forall\, 1 \leq i \leq k),$$

so that each coordinate behaves like a fuzzy graph membership degree.

Remark 4.2.40. Some authors define multi-fuzzy graphs purely by specifying the vector-valued memberships on vertices and edges, without requiring endpoint constraints; the above formulation includes the common "Rosenfeld-type" compatibility condition coordinatewise.

Theorem 4.2.41 (From general plithogenic graphs to multi-fuzzy graphs)**.** *Let $G^{GP} = (PM, PN)$ be a general plithogenic graph whose DAFs are k-dimensional ($s = k$) and whose DCF dimension is $t = 1$. Fix compatible attribute values so that each vertex $v \in M$ is assigned a single vector*

$$\mathrm{adf}(v, \cdot) = \mathbf{a}(v) \in [0, 1]^k$$

and each edge $e \in N$ is assigned a single vector

$$\mathrm{bdf}(e, \cdot) = \mathbf{b}(e) \in [0, 1]^k.$$

Assume that the plithogenic edge appurtenance constraint holds componentwise in these k coordinates. Then G^{GP} induces a k-dimensional multi-fuzzy graph on the same underlying crisp graph.

Proof. Define $\sigma : V \to [0, 1]^k$ and $\mu : E \to [0, 1]^k$ by

$$\sigma(v) := \mathbf{a}(v) \quad (v \in V), \qquad \mu(e) := \mathbf{b}(e) \quad (e \in E),$$

extending by $\sigma(v) = \mathbf{0}$ for $v \notin M$ and $\mu(e) = \mathbf{0}$ for $e \notin N$ if $M \subsetneq V$ or $N \subsetneq E$. Clearly σ and μ take values in $[0, 1]^k$.

Let $e = uv \in N$. The plithogenic edge appurtenance constraint specialized to the chosen attribute values yields

$$\mathbf{b}(uv) \;\leq\; \min\{\mathbf{a}(u), \mathbf{a}(v)\},$$

where the minimum and inequality are interpreted componentwise in $[0,1]^k$. Hence, for each coordinate $1 \leq i \leq k$,

$$\mu_i(uv) \ \leq \ \min\{\sigma_i(u), \sigma_i(v)\}.$$

Therefore (σ, μ) satisfies Definition 4.2.39, so it is a k-dimensional multi-fuzzy graph.

Finally, the assumption $t = 1$ (single-valued contradiction functions) plays no role in the vector-valued membership construction; it merely indicates that contradiction is scalar-valued and does not obstruct the mapping. □

Remark 4.2.42. If one adopts the constraint-free variant of multi-fuzzy graphs, then Theorem 4.2.41 holds without assuming the plithogenic edge appurtenance inequality. Conversely, if one wishes to ensure endpoint constraints for all edges in E (not only those in N), one may take $N = E$ or extend $\mathbf{b}$ by zero on $E \setminus N$, as done above.

4.2.12 Relations to intuitionistic multi-fuzzy graphs

We now consider *intuitionistic multi-fuzzy graphs*, which extend intuitionistic fuzzy graphs by attaching, to each vertex and edge, a *family* of intuitionistic degrees indexed by a fixed dimension k (cf. intuitionistic multi-fuzzy sets) [601–603]. Intuitively, each index $i \in \{1, \ldots, k\}$ provides a separate membership/non-membership assessment, together with an induced hesitation margin.

Definition 4.2.43 (Intuitionistic multi-fuzzy graph)**.** Let $G^* = (V, E)$ be a finite simple (undirected) graph and fix an integer $k \geq 1$. An *intuitionistic k-multi-fuzzy graph* on G^* is a pair

$$G^{(k)}_{IMF} = (A, B),$$

where:

- $A = (\mu_A, \nu_A)$ is an intuitionistic k-multi-fuzzy vertex assignment, i.e.,

  $$\mu_A, \nu_A : V \to [0,1]^k, \qquad \mu_A(v) = (\mu_{A,1}(v), \ldots, \mu_{A,k}(v)), \quad \nu_A(v) = (\nu_{A,1}(v), \ldots, \nu_{A,k}(v)),$$

  such that for every $v \in V$ and every $1 \leq i \leq k$,

  $$0 \leq \mu_{A,i}(v) + \nu_{A,i}(v) \leq 1.$$

- $B = (\mu_B, \nu_B)$ is an intuitionistic k-multi-fuzzy edge assignment, i.e.,

  $$\mu_B, \nu_B : E \to [0,1]^k, \qquad \mu_B(e) = (\mu_{B,1}(e), \ldots, \mu_{B,k}(e)), \quad \nu_B(e) = (\nu_{B,1}(e), \ldots, \nu_{B,k}(e)),$$

  such that for every $e \in E$ and every $1 \leq i \leq k$,

  $$0 \leq \mu_{B,i}(e) + \nu_{B,i}(e) \leq 1.$$

For each level i, the induced *hesitation* (indeterminacy) degrees are

$$\pi_{A,i}(v) := 1 - \mu_{A,i}(v) - \nu_{A,i}(v) \qquad (v \in V), \qquad \pi_{B,i}(e) := 1 - \mu_{B,i}(e) - \nu_{B,i}(e) \qquad (e \in E).$$

Optionally (in the Rosenfeld/Atanassov-style setting), one may impose the componentwise endpoint constraints: for every edge $uv \in E$ and every $1 \leq i \leq k$,

$$\mu_{B,i}(uv) \ \leq \ \min\{\mu_{A,i}(u), \mu_{A,i}(v)\}, \qquad \nu_{B,i}(uv) \ \leq \ \max\{\nu_{A,i}(u), \nu_{A,i}(v)\}.$$

Remark 4.2.44. Some authors work with the unconstrained version (only the per-level Atanassov bounds $\mu + \nu \leq 1$), while others additionally require endpoint constraints; Definition 4.2.43 records both by treating the latter as optional.

Theorem 4.2.45 (From general plithogenic graphs to intuitionistic multi-fuzzy graphs)**.** *Let $G^{GP} = (PM, PN)$ be a general plithogenic graph whose DAFs are $2k$-dimensional ($s = 2k$) and whose DCF dimension is $t = 1$. Fix compatible attribute values so that each vertex $v \in M$ is assigned a single vector*

$$\mathrm{adf}(v, \cdot) = \big(a_1(v), \dots, a_{2k}(v)\big) \in [0,1]^{2k}$$

and each edge $e \in N$ is assigned a single vector

$$\mathrm{bdf}(e, \cdot) = \big(b_1(e), \dots, b_{2k}(e)\big) \in [0,1]^{2k}.$$

Assume the per-level Atanassov bounds

$$a_{2i-1}(v) + a_{2i}(v) \leq 1 \quad (\forall v \in M,\ 1 \leq i \leq k), \qquad b_{2i-1}(e) + b_{2i}(e) \leq 1 \quad (\forall e \in N,\ 1 \leq i \leq k),$$

and assume that the plithogenic edge appurtenance constraint holds componentwise in these $2k$ coordinates. Then G^{GP} induces an intuitionistic k-multi-fuzzy graph on the same underlying crisp graph.

Proof. Define vertex maps $\mu_A, \nu_A : V \to [0,1]^k$ by, for $1 \leq i \leq k$,

$$\mu_{A,i}(v) := a_{2i-1}(v), \qquad \nu_{A,i}(v) := a_{2i}(v) \qquad (v \in V),$$

extending by $(\mu_A, \nu_A) = (\mathbf{0}, \mathbf{0})$ on $V \setminus M$ if $M \subsetneq V$. The assumed bounds $a_{2i-1}(v) + a_{2i}(v) \leq 1$ imply $0 \leq \mu_{A,i}(v) + \nu_{A,i}(v) \leq 1$ for all v and i.

Similarly, define edge maps $\mu_B, \nu_B : E \to [0,1]^k$ by

$$\mu_{B,i}(e) := b_{2i-1}(e), \qquad \nu_{B,i}(e) := b_{2i}(e) \qquad (e \in E),$$

extending by zeros on $E \setminus N$ if $N \subsetneq E$. Then $0 \leq \mu_{B,i}(e) + \nu_{B,i}(e) \leq 1$ holds for all e and i by the assumed bounds.

This defines an intuitionistic k-multi-fuzzy graph $G^{(k)}_{IMF} = (A, B)$ in the sense of Definition 4.2.43. If one additionally adopts the optional endpoint constraints, they follow componentwise from the plithogenic edge appurtenance constraint exactly as in the multi-fuzzy case (applied separately to the μ- and ν-coordinates at each level i). □

Remark 4.2.46. The scalar DCF condition $t = 1$ is not essential for the construction; the key point is that the DAF dimension $s = 2k$ provides k pairs of coordinates that can be interpreted as (μ_i, ν_i).

4.3 Plithogenic SuperHyperGraph

Plithogenic hypergraph generalizes plithogenic graphs, assigning plithogenic memberships to hyperedges connecting arbitrary vertex subsets under contradictory attributes and uncertainties simultaneously [604]. Plithogenic SuperHyperGraph equips multi-level supervertices and superedges with plithogenic attribute degrees, capturing hierarchical contradictions across nested interaction structures faithfully everywhere [18, 115, 605, 606].

Definition 4.3.1 (Plithogenic Hypergraph)**.** (cf. [604]) Let V be a finite set of vertices and $E \subseteq \mathcal{P}(V)$ a family of hyperedges. A *plithogenic vertex system* is a tuple

$$PM = (V, \ell, M_\ell, \text{adf}, \text{ aCf}),$$

where

- ℓ is a vertex-attribute,
- M_ℓ is the finite set of possible attribute values,
- $\text{adf} : V \times M_\ell \to [0,1]^s$ is the degree-of-appurtenance function,
- $\text{aCf} : M_\ell \times M_\ell \to [0,1]^t$ is the degree-of-contradiction function,

satisfying $\text{aCf}(a,a) = 0$ and symmetry. Similarly, a *plithogenic hyperedge system* is

$$PN = (E, m, N_m, \text{bdf}, \text{ bCf}),$$

with

- m an edge-attribute,
- N_m its value set,
- $\text{bdf} : E \times N_m \to [0,1]^s$ the edge-appurtenance function,
- $\text{bCf} : N_m \times N_m \to [0,1]^t$ the edge-contradiction function,

satisfying analogous reflexivity and symmetry. The tuple

$$H = (PM,\ PN)$$

is called a *plithogenic hypergraph* if for every $e = \{x, y, \dots\} \in E$ and every attribute-value combination $(a, b, \dots) \in M_\ell \times N_m$ the following hold:

$$\text{bdf}(e, (a, b, \dots)) \le \min\{\text{adf}(x, a),\ \text{adf}(y, b), \dots\},$$
$$\text{bCf}(\alpha, \beta) \le \min\{\text{aCf}(\alpha),\ \text{aCf}(\beta)\}.$$

Example 4.3.2 (A plithogenic hypergraph)**.** Let $V = \{a, b, c\}$ and let the hyperedge family be

$$E = \{e_1, e_2\} \subseteq \mathcal{P}(V), \qquad e_1 = \{a, b, c\}, \quad e_2 = \{b, c\}.$$

Fix $s = 1$ and $t = 1$. Let the vertex-attribute be $\ell =$“risk” with value set

$$M_\ell = \{\mathrm{L}, \mathrm{H}\} \quad \text{(Low/High)}.$$

Define the vertex appurtenance map $\mathrm{adf} : V \times M_\ell \to [0, 1]$ by

$$\mathrm{adf}(a, \mathrm{H}) = 0.8,\ \mathrm{adf}(a, \mathrm{L}) = 0.2, \qquad \mathrm{adf}(b, \mathrm{H}) = 0.6,\ \mathrm{adf}(b, \mathrm{L}) = 0.4, \qquad \mathrm{adf}(c, \mathrm{H}) = 0.3,\ \mathrm{adf}(c, \mathrm{L}) = 0.7.$$

Let the vertex contradiction map $\mathrm{aCf} : M_\ell \times M_\ell \to [0, 1]$ be

$$\mathrm{aCf}(\mathrm{L}, \mathrm{L}) = \mathrm{aCf}(\mathrm{H}, \mathrm{H}) = 0, \qquad \mathrm{aCf}(\mathrm{L}, \mathrm{H}) = \mathrm{aCf}(\mathrm{H}, \mathrm{L}) = 0.9,$$

which is reflexive and symmetric.

For hyperedges, let the edge-attribute be $m =$“severity” with value set

$$N_m = \{\mathrm{S}_1, \mathrm{S}_2\}.$$

Define $\mathrm{bdf} : E \times N_m \to [0, 1]$ by

$$\mathrm{bdf}(e_1, \mathrm{S}_1) = 0.2, \quad \mathrm{bdf}(e_1, \mathrm{S}_2) = 0.5, \qquad \mathrm{bdf}(e_2, \mathrm{S}_1) = 0.3, \quad \mathrm{bdf}(e_2, \mathrm{S}_2) = 0.4.$$

Let $\mathrm{bCf} : N_m \times N_m \to [0, 1]$ be given by

$$\mathrm{bCf}(\mathrm{S}_1, \mathrm{S}_1) = \mathrm{bCf}(\mathrm{S}_2, \mathrm{S}_2) = 0, \qquad \mathrm{bCf}(\mathrm{S}_1, \mathrm{S}_2) = \mathrm{bCf}(\mathrm{S}_2, \mathrm{S}_1) = 0.6,$$

again reflexive and symmetric.

Finally, choose a single attribute-value combination for each hyperedge, e.g. $(\mathrm{H}, \mathrm{H}, \mathrm{L})$ for e_1 and (H, L) for e_2, and check the plithogenic appurtenance constraint:

$$\mathrm{bdf}(e_1, \mathrm{S}_1) = 0.2 \le \min\{\mathrm{adf}(a, \mathrm{H}), \mathrm{adf}(b, \mathrm{H}), \mathrm{adf}(c, \mathrm{L})\} = \min\{0.8, 0.6, 0.7\} = 0.6,$$

$$\mathrm{bdf}(e_2, \mathrm{S}_1) = 0.3 \le \min\{\mathrm{adf}(b, \mathrm{H}), \mathrm{adf}(c, \mathrm{L})\} = \min\{0.6, 0.7\} = 0.6.$$

Thus $H = (PM, PN)$ with the above data is a plithogenic hypergraph in the sense of the definition.

Definition 4.3.3 (Plithogenic n-SuperHyperGraph)**.** [18] Let V_0 be a finite base set and let $n \in \mathbb{N}_0$. Consider an n-SuperHyperGraph over V_0 in the sense of Definition **??**, that is,

$$\mathrm{SHG}^{(n)} \ = \ (V, E, \partial),$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of n-supervertices;
- E is a finite set of (super)edge identifiers;
- $\partial : E \to \mathcal{P}^*(V)$ is the incidence map, so that for each $e \in E$, the set $\partial(e) \subseteq V$ is the nonempty incidence set of e.

Fix the same dimensions $s, t \in \mathbb{N}$ as above.

A *plithogenic vertex system* on V is a tuple

$$PM^{(n)} = \bigl(V, \ell, M_\ell, \mathrm{adf}^{(n)}, \mathrm{aCF}\bigr),$$

where

- ℓ is a vertex attribute;
- M_ℓ is a nonempty finite set of possible attribute values for vertices;
- $\mathrm{adf}^{(n)} : V \times M_\ell \to [0,1]^s$ is the (vertex) degree–of–appurtenance function; for $v \in V$ and $a \in M_\ell$, $\mathrm{adf}^{(n)}(v,a)$ encodes the (possibly vector–valued) membership degree of v having attribute value a;
- $\mathrm{aCF} : M_\ell \times M_\ell \to [0,1]^t$ is the (vertex) degree–of–contradiction function, satisfying

$$\mathrm{aCF}(a,a) = 0, \qquad \mathrm{aCF}(a,b) = \mathrm{aCF}(b,a) \quad \text{for all } a,b \in M_\ell.$$

A *plithogenic superedge system* on E is a tuple

$$PN^{(n)} = \bigl(E, m, N_m, \mathrm{bdf}^{(n)}, \mathrm{bCF}\bigr),$$

where

- m is a superedge attribute;
- N_m is a nonempty finite set of possible attribute values for superedges;
- $\mathrm{bdf}^{(n)} : E \times N_m \to [0,1]^s$ is the (superedge) degree–of–appurtenance function; for $e \in E$ and $u \in N_m$, $\mathrm{bdf}^{(n)}(e,u)$ encodes the membership degree of e having attribute value u;
- $\mathrm{bCF} : N_m \times N_m \to [0,1]^t$ is the (superedge) degree–of–contradiction function, satisfying

$$\mathrm{bCF}(u,u) = 0, \qquad \mathrm{bCF}(u,v) = \mathrm{bCF}(v,u) \quad \text{for all } u,v \in N_m.$$

For each n-superedge $e \in E$ we assume a prescribed *plithogenic aggregation rule*

$$\beta_e : M_\ell^{\partial(e)} \longrightarrow N_m,$$

which assigns to every family of vertex–attribute values

$$\alpha = (\alpha_v)_{v \in \partial(e)} \in M_\ell^{\partial(e)}$$

a superedge–attribute value $\beta_e(\alpha) \in N_m$.

The triple

$$\mathsf{Plith\text{-}SHG}^{(n)} := \big(\mathrm{SHG}^{(n)},\, PM^{(n)},\, PN^{(n)}\big)$$

is called a *Plithogenic n-SuperHyperGraph* if, for every $e \in E$ and every family $\alpha = (\alpha_v)_{v\in\partial(e)} \in M_\ell^{\partial(e)}$, the following *plithogenic appurtenance–compatibility* condition holds, componentwise in $[0,1]^s$:

$$\mathrm{bdf}^{(n)}\big(e, \beta_e(\alpha)\big) \;\le\; \min_{v\in\partial(e)} \mathrm{adf}^{(n)}\big(v, \alpha_v\big).$$

Here the minimum is taken pointwise in $\mathbb{R}^s$, that is, for each coordinate $j \in \{1, \ldots, s\}$,

$$\mathrm{bdf}^{(n)}\big(e, \beta_e(\alpha)\big)_j \;\le\; \min_{v\in\partial(e)} \mathrm{adf}^{(n)}\big(v, \alpha_v\big)_j.$$

We call $\mathsf{Plith\text{-}SHG}^{(n)}$ a *Plithogenic n-SuperHyperGraph.* For $n = 1$ and $V \subseteq V_0$, E corresponding to nonempty subsets of V_0, this construction reduces to a Plithogenic hypergraph, while for $n = 0$ and $|E| = 1$ it recovers a single plithogenic set on the vertex universe.

Example 4.3.4 (A plithogenic 2-SuperHyperGraph)**.** Let the base set be

$$V_0 = \{1, 2, 3\}, \qquad n = 2,$$

so $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$. Define three 2-supervertices by

$$v_1 = \{\{1\}, \{1, 2\}\}, \qquad v_2 = \{\{2\}, \{2, 3\}\}, \qquad v_3 = \{\{1, 3\}\},$$

and set $V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0)$.

Let $E = \{e_1, e_2\}$ be a set of superedge identifiers with incidence map

$$\partial(e_1) = \{v_1, v_2\}, \qquad \partial(e_2) = \{v_1, v_2, v_3\}.$$

Fix $s = 1$ and $t = 1$.

Vertex system. Let the vertex attribute be $\ell =$"priority" with value set $M_\ell = \{A, B\}$. Define $\mathrm{adf}^{(2)} : V \times M_\ell \to [0, 1]$ by

$$\mathrm{adf}^{(2)}(v_1, A) = 0.7, \quad \mathrm{adf}^{(2)}(v_2, A) = 0.5, \quad \mathrm{adf}^{(2)}(v_3, A) = 0.4,$$

$$\mathrm{adf}^{(2)}(v_1, B) = 0.3, \quad \mathrm{adf}^{(2)}(v_2, B) = 0.6, \quad \mathrm{adf}^{(2)}(v_3, B) = 0.8.$$

Let $\mathrm{aCF} : M_\ell \times M_\ell \to [0, 1]$ be

$$\mathrm{aCF}(A, A) = \mathrm{aCF}(B, B) = 0, \qquad \mathrm{aCF}(A, B) = \mathrm{aCF}(B, A) = 0.4.$$

Superedge system. Let the superedge attribute be $m =$"mode" with value set $N_m = \{X, Y\}$. Define $\mathrm{bdf}^{(2)} : E \times N_m \to [0, 1]$ by

$$\mathrm{bdf}^{(2)}(e_1, X) = 0.5, \quad \mathrm{bdf}^{(2)}(e_1, Y) = 0.2,$$

$$\mathrm{bdf}^{(2)}(e_2, X) = 0.3, \quad \mathrm{bdf}^{(2)}(e_2, Y) = 0.4,$$

and let $\mathrm{bCF}(X, X) = \mathrm{bCF}(Y, Y) = 0$ and $\mathrm{bCF}(X, Y) = \mathrm{bCF}(Y, X) = 0.6$.

Aggregation rules and compatibility check. Define aggregation rules β_{e_1} and β_{e_2} by

$$\beta_{e_1}(A, A) = X, \qquad \beta_{e_2}(B, A, A) = Y.$$

Then the plithogenic appurtenance–compatibility holds: for e_1 with $\alpha_{v_1} = \alpha_{v_2} = A$,

$$\mathrm{bdf}^{(2)}\big(e_1, \beta_{e_1}(A, A)\big) = \mathrm{bdf}^{(2)}(e_1, X) = 0.5$$

$$\leq \min\{\mathrm{adf}^{(2)}(v_1, A), \mathrm{adf}^{(2)}(v_2, A)\} = \min\{0.7, 0.5\} = 0.5,$$

and for e_2 with $(\alpha_{v_1}, \alpha_{v_2}, \alpha_{v_3}) = (B, A, A)$,

$$\mathrm{bdf}^{(2)}\big(e_2, \beta_{e_2}(B, A, A)\big) = \mathrm{bdf}^{(2)}(e_2, Y) = 0.4$$

$$\leq \min\{\mathrm{adf}^{(2)}(v_1, B), \mathrm{adf}^{(2)}(v_2, A), \mathrm{adf}^{(2)}(v_3, A)\} = \min\{0.3, 0.5, 0.4\} = 0.3.$$

To enforce the inequality for this second case, choose $\beta_{e_2}(B, A, A) = X$ instead (so that $\mathrm{bdf}^{(2)}(e_2, X) = 0.3$), yielding

$$0.3 \leq \min\{0.3, 0.5, 0.4\} = 0.3.$$

Hence, with this choice, $\mathsf{Plith\text{-}SHG}^{(2)}$ satisfies the compatibility condition and is a plithogenic 2-SuperHyperGraph.

4.4 Restricted Refined Plithogenic graphs

As noted earlier, refined versions of uncertainty models for sets and graphs have been developed actively in recent years (cf. [414, 415, 607, 608]). In this section we introduce a *restricted refined* variant of plithogenic graphs, in which each uncertainty component is split into r subcomponents in a uniform way.

Definition 4.4.1 (Restricted refined plithogenic graph (RRPG))**.** Let $G^* = (V, E)$ be a finite (crisp) graph, where V is the vertex set and $E \subseteq V \times V$ is the edge set. Fix integers $s \geq 1$ (number of uncertainty components), $t \geq 1$ (dimension of contradiction information), and $r \geq 1$ (refinement depth).

A *restricted refined plithogenic graph* is a pair

$$RRPG = (RPM, RPN),$$

where:

1. the *restricted refined plithogenic vertex structure* is

$$RPM = (M, \ell, M_\ell, \mathrm{adf}_r, \mathrm{aCf}_r),$$

with $M \subseteq V$, an attribute symbol ℓ, an attribute-value domain M_ℓ, and maps

$$\mathrm{adf}_r : M \times M_\ell \to [0,1]^{s\times r}, \qquad \mathrm{aCf}_r : M_\ell \times M_\ell \to [0,1]^{t\times r};$$

2. the *restricted refined plithogenic edge structure* is

$$RPN = (N, m, N_m, \mathrm{bdf}_r, \mathrm{bCf}_r),$$

with $N \subseteq E$, an attribute symbol m, an attribute-value domain N_m, and maps

$$\mathrm{bdf}_r : N \times N_m \to [0,1]^{s\times r}, \qquad \mathrm{bCf}_r : N_m \times N_m \to [0,1]^{t\times r}.$$

For clarity, we write

$$\mathrm{adf}_r(v,a) = \big(a_{i,j}(v,a)\big)_{\substack{1\le i\le s\\ 1\le j\le r}} \in [0,1]^{s\times r}, \qquad \mathrm{bdf}_r(e,b) = \big(b_{i,j}(e,b)\big)_{\substack{1\le i\le s\\ 1\le j\le r}} \in [0,1]^{s\times r},$$

and interpret i as the component index and j as the refinement (split) index.

The following axioms are required:

(A1) **Edge appurtenance constraint (refined, componentwise).** For every edge $xy \in N$ and every compatible choice of attribute values $(a,b) \in M_\ell \times M_\ell$ (as in the base plithogenic model),

$$\mathrm{bdf}_r(xy,(a,b)) \;\le\; \min\{\mathrm{adf}_r(x,a), \mathrm{adf}_r(y,b)\},$$

where the minimum and inequality are taken componentwise in $[0,1]^{s\times r}$; equivalently, for all $1 \le i \le s$ and $1 \le j \le r$,

$$b_{i,j}(xy,(a,b)) \;\le\; \min\{a_{i,j}(x,a), a_{i,j}(y,b)\}.$$

(A2) **Reflexivity and symmetry of refined contradiction.** For all $a, b \in M_\ell$,

$$\mathrm{aCf}_r(a,a) = \mathbf{0}, \qquad \mathrm{aCf}_r(a,b) = \mathrm{aCf}_r(b,a),$$

and for all $c, d \in N_m$,

$$\mathrm{bCf}_r(c,c) = \mathbf{0}, \qquad \mathrm{bCf}_r(c,d) = \mathrm{bCf}_r(d,c),$$

where $\mathbf{0}$ denotes the zero element of $[0,1]^{t\times r}$.

Theorem 4.4.2 (Reduction to classical plithogenic graphs when $r = 1$)**.** *If $r = 1$, then a restricted refined plithogenic graph reduces to a classical plithogenic graph.*

Proof. Assume $r = 1$. Then $[0,1]^{s\times r} = [0,1]^{s\times 1} \cong [0,1]^s$ and $[0,1]^{t\times r} = [0,1]^{t\times 1} \cong [0,1]^t$. Hence the maps in Definition 4.4.1 become

$$\mathrm{adf}_1 : M \times M_\ell \to [0,1]^s, \qquad \mathrm{aCf}_1 : M_\ell \times M_\ell \to [0,1]^t,$$

$$\mathrm{bdf}_1 : N \times N_m \to [0,1]^s, \qquad \mathrm{bCf}_1 : N_m \times N_m \to [0,1]^t,$$

which are exactly the degrees of appurtenance and contradiction of the classical plithogenic graph model. The refined axioms (A1)–(A2) reduce to their classical counterparts, because "componentwise in $s \times 1$" is the same as "componentwise in s". Therefore the RRPG structure coincides with a classical plithogenic graph. □

Theorem 4.4.3 (Reductions to restricted refined uncertainty-graph families)**.** *With appropriate parameter choices, the restricted refined plithogenic graph model specializes to the corresponding restricted refined fuzzy-, intuitionistic fuzzy-, neutrosophic-, quadripartitioned neutrosophic-, or pentapartitioned neutrosophic-graph families. In particular, taking $t = 1$ and choosing $s \in \{1,2,3,4,5\}$ yields:*

- $s = 1$*: restricted refined fuzzy graphs;*

- $s = 2$: *restricted refined intuitionistic fuzzy graphs;*
- $s = 3$: *restricted refined neutrosophic graphs;*
- $s = 4$: *restricted refined quadripartitioned neutrosophic graphs;*
- $s = 5$: *restricted refined pentapartitioned neutrosophic graphs,*

each with refinement depth $r \geq 1$.

Proof. Fix $t = 1$ and interpret the s components of adf_r and bdf_r as the corresponding uncertainty coordinates (truth-like, indeterminacy-like, falsity-like, etc.), each split into r refined subcomponents. Under this identification, the RRPG edge appurtenance constraint (A1) becomes exactly the componentwise endpoint constraint that defines the corresponding restricted refined graph family, while (A2) is compatible with the usual (single-valued) symmetry requirement for contradiction/dissimilarity information. Therefore the RRPG model contains these restricted refined uncertainty-graph classes as special cases under the stated parameter choices. □

4.5 Restricted Refined Plithogenic Hypergraphs and n-SuperHyperGraphs

This section lifts the restricted refined plithogenic graph (RRPG) model (Definition 4.4.1) from graphs to hypergraphs and then to n-SuperHyperGraphs. The guiding principle is unchanged: degrees of appurtenance and contradiction are *refined* by splitting each of the s uncertainty components into r subcomponents.

Definition 4.5.1 (Restricted refined plithogenic hypergraph (RRPH))**.** Let $H^* = (V, E)$ be a finite crisp hypergraph, where V is a finite nonempty vertex set and $E \subseteq \mathcal{P}^*(V)$ is a finite family of nonempty hyperedges. Fix integers $s \geq 1$, $t \geq 1$, and $r \geq 1$.

A *restricted refined plithogenic hypergraph* is a pair

$$RRPH = (RPM_H, RPN_H),$$

where:

1. the *restricted refined plithogenic vertex structure* is

$$RPM_H = (V, \ell, V_\ell, \mathrm{adf}_r, \mathrm{aCf}_r),$$

with an attribute symbol ℓ, an attribute-value domain V_ℓ, and maps

$$\mathrm{adf}_r : V \times V_\ell \to [0,1]^{s\times r}, \qquad \mathrm{aCf}_r : V_\ell \times V_\ell \to [0,1]^{t\times r};$$

2. the *restricted refined plithogenic hyperedge structure* is

$$RPN_H = (E, m, E_m, \mathrm{bdf}_r, \mathrm{bCf}_r),$$

with an attribute symbol m, an attribute-value domain E_m, and maps

$$\mathrm{bdf}_r : E \times E_m \to [0,1]^{s\times r}, \qquad \mathrm{bCf}_r : E_m \times E_m \to [0,1]^{t\times r}.$$

Write $\mathrm{adf}_r(v,a) = \big(a_{i,j}(v,a)\big)_{1\le i\le s,\ 1\le j\le r}$ and $\mathrm{bdf}_r(e,b) = \big(b_{i,j}(e,b)\big)_{1\le i\le s,\ 1\le j\le r}$. The required axioms are:

(H1) **Hyperedge appurtenance constraint (refined, componentwise).** For every hyperedge $e \in E$ and every admissible attribute choice $(a_v)_{v\in e}$ (compatible with E_m),

$$\mathrm{bdf}_r\big(e, (a_v)_{v\in e}\big) \ \le\ \min_{v\in e} \mathrm{adf}_r(v, a_v),$$

where the minimum and inequality are taken componentwise in $[0,1]^{s\times r}$; equivalently, for all $1 \le i \le s$ and $1 \le j \le r$,

$$b_{i,j}\big(e, (a_v)_{v\in e}\big) \ \le\ \min_{v\in e} a_{i,j}(v, a_v).$$

(When $|e| = 2$ and $e = \{x, y\}$, this reduces to the graph constraint (A1) in Definition 4.4.1.)

(H2) **Reflexivity and symmetry of refined contradiction.** For all $a, b \in V_\ell$,

$$\mathrm{aCf}_r(a,a) = \mathbf{0}, \qquad \mathrm{aCf}_r(a,b) = \mathrm{aCf}_r(b,a),$$

and for all $c, d \in E_m$,

$$\mathrm{bCf}_r(c,c) = \mathbf{0}, \qquad \mathrm{bCf}_r(c,d) = \mathrm{bCf}_r(d,c),$$

where $\mathbf{0}$ denotes the zero element of $[0,1]^{t\times r}$.

Example 4.5.2 (A restricted refined plithogenic hypergraph (RRPH))**.** Let $V = \{a, b, c\}$ and let

$$E = \{e_1, e_2\} \subseteq \mathcal{P}^{*}(V), \qquad e_1 = \{a, b, c\}, \quad e_2 = \{b, c\}.$$

Fix $s = 2$, $t = 1$, and $r = 2$.

Vertex structure. Let the vertex attribute be $\ell =$"status" with value set

$$V_\ell = \{\mathrm{L}, \mathrm{H}\}.$$

Define $\mathrm{adf}_r : V \times V_\ell \to [0,1]^{2\times 2}$ by

$$\mathrm{adf}_r(a,\mathrm{H}) = \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.2 \end{pmatrix}, \quad \mathrm{adf}_r(b,\mathrm{H}) = \begin{pmatrix} 0.5 & 0.4 \\ 0.2 & 0.1 \end{pmatrix}, \quad \mathrm{adf}_r(c,\mathrm{L}) = \begin{pmatrix} 0.4 & 0.3 \\ 0.6 & 0.5 \end{pmatrix},$$

and assign arbitrary values to the remaining pairs $(v, a) \in V \times V_\ell$ (they will not be used below). Let the refined contradiction map $\mathrm{aCf}_r : V_\ell \times V_\ell \to [0,1]^{1\times 2}$ be

$$\mathrm{aCf}_r(\mathrm{L},\mathrm{L}) = \mathrm{aCf}_r(\mathrm{H},\mathrm{H}) = (0,0), \qquad \mathrm{aCf}_r(\mathrm{L},\mathrm{H}) = \mathrm{aCf}_r(\mathrm{H},\mathrm{L}) = (0.6, 0.6).$$

This is reflexive and symmetric.

Hyperedge structure. Let the hyperedge attribute be $m =$"mode" with value set

$$E_m = \{\mathrm{X}, \mathrm{Y}\}.$$

Define $\mathrm{bdf}_r : E \times E_m \to [0,1]^{2\times 2}$ by

$$\mathrm{bdf}_r(e_1, \mathrm{X}) = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \end{pmatrix}, \qquad \mathrm{bdf}_r(e_2, \mathrm{X}) = \begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.1 \end{pmatrix},$$

and define $\mathrm{bCf}_r : E_m \times E_m \to [0,1]^{1\times 2}$ by

$$\mathrm{bCf}_r(\mathrm{X},\mathrm{X}) = \mathrm{bCf}_r(\mathrm{Y},\mathrm{Y}) = (0,0), \qquad \mathrm{bCf}_r(\mathrm{X},\mathrm{Y}) = \mathrm{bCf}_r(\mathrm{Y},\mathrm{X}) = (0.4, 0.4),$$

again reflexive and symmetric.

Verification of the refined hyperedge appurtenance constraint. Choose the admissible attribute assignment $(a_v)_{v\in e_1}$ given by

$$(a_a, a_b, a_c) = (\mathrm{H}, \mathrm{H}, \mathrm{L}),$$

and take the corresponding hyperedge attribute to be X. Then

$$\min_{v\in e_1} \mathrm{adf}_r(v, a_v) = \min \left\{ \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.2 \end{pmatrix}, \begin{pmatrix} 0.5 & 0.4 \\ 0.2 & 0.1 \end{pmatrix}, \begin{pmatrix} 0.4 & 0.3 \\ 0.6 & 0.5 \end{pmatrix} \right\} = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \end{pmatrix},$$

where the minimum is taken componentwise. Hence

$$\mathrm{bdf}_r(e_1, \mathrm{X}) = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \end{pmatrix} \leq \min_{v\in e_1} \mathrm{adf}_r(v, a_v),$$

so (H1) holds for this admissible choice. A similar (and simpler) check holds for e_2 using $(a_b, a_c) = (\mathrm{H}, \mathrm{L})$ and the same hyperedge attribute X, since

$$\min \left\{ \begin{pmatrix} 0.5 & 0.4 \\ 0.2 & 0.1 \end{pmatrix}, \begin{pmatrix} 0.4 & 0.3 \\ 0.6 & 0.5 \end{pmatrix} \right\} = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.1 \end{pmatrix} \geq \begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.1 \end{pmatrix} = \mathrm{bdf}_r(e_2, \mathrm{X}).$$

Therefore (RPM_H, RPN_H) defines an RRPH on $H^* = (V, E)$ in the sense of Definition 4.5.1.

Theorem 4.5.3 (RRPH generalizes crisp hypergraphs)**.** *Every crisp hypergraph $H^* = (V, E)$ can be viewed canonically as a restricted refined plithogenic hypergraph.*

Proof. Let $H^* = (V, E)$ be a crisp hypergraph and fix $s, t, r \geq 1$. Choose any attribute-value domains V_ℓ and E_m and define constant refined functions by

$$\mathrm{adf}_r(v, a) := \mathbf{1}_{s\times r} \quad (\forall v \in V, \forall a \in V_\ell), \qquad \mathrm{bdf}_r(e, b) := \mathbf{1}_{s\times r} \quad (\forall e \in E, \forall b \in E_m),$$

where $\mathbf{1}_{s\times r}$ denotes the all-ones element of $[0,1]^{s\times r}$. Define aCf_r and bCf_r to be identically $\mathbf{0}$ on $V_\ell \times V_\ell$ and $E_m \times E_m$, respectively. Then (H2) holds trivially, and (H1) holds because for any hyperedge e,

$$\mathbf{1}_{s\times r} \;=\; \mathrm{bdf}_r(e, \cdot) \;\leq\; \min_{v\in e} \mathbf{1}_{s\times r} \;=\; \min_{v\in e} \mathrm{adf}_r(v, \cdot).$$

Thus (RPM_H, RPN_H) defines an RRPH on the same underlying hypergraph (V, E). □

Theorem 4.5.4 (RRPH generalizes RRPG)**.** *Every restricted refined plithogenic graph RRPG can be realized as a special case of a restricted refined plithogenic hypergraph in which all hyperedges have size* 2*.*

Proof. Let $RRPG = (RPM, RPN)$ be a restricted refined plithogenic graph on a crisp graph $G^* = (V, E_G)$ as in Definition 4.4.1, so $E_G \subseteq \binom{V}{2}$ (in the undirected simple case). Form a 2-uniform hypergraph $H^* = (V, E)$ by taking $E = \{\{u, v\} : \{u, v\} \in E_G\}$. Reuse the same refined vertex structure on V. For each graph edge $\{u, v\}$, view it as the hyperedge $e = \{u, v\}$; then the hyperedge constraint (H1) reduces to (A1). Hence the RRPG data define an RRPH with 2-element hyperedges. □

Definition 4.5.5 (Restricted refined plithogenic n-SuperHyperGraph (RRP-nSHG))**.** Let V_0 be a finite nonempty base set and let $n \in \mathbb{N}_0$. Let $\mathrm{SHG}^{(n)} = (V, E)$ be a crisp n-SuperHyperGraph on V_0, i.e.,

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Fix integers $s, t, r \geq 1$.

A *restricted refined plithogenic n-SuperHyperGraph* is a pair

$$\mathit{RRP\text{-}SHG}^{(n)} = (RPM^{(n)}, RPN^{(n)}),$$

where:

1. $RPM^{(n)} = (V, \ell, V_\ell, \mathrm{adf}_r, \mathrm{aCf}_r)$ is a restricted refined plithogenic structure on the n-supervertex set V, with
$$\mathrm{adf}_r : V \times V_\ell \to [0,1]^{s \times r}, \qquad \mathrm{aCf}_r : V_\ell \times V_\ell \to [0,1]^{t \times r};$$

2. $RPN^{(n)} = (E, m, E_m, \mathrm{bdf}_r, \mathrm{bCf}_r)$ is a restricted refined plithogenic structure on the n-superedge set E, with
$$\mathrm{bdf}_r : E \times E_m \to [0,1]^{s \times r}, \qquad \mathrm{bCf}_r : E_m \times E_m \to [0,1]^{t \times r}.$$

The axioms are the direct superhypergraph analogues of Definition 4.5.1:

(S1) **Superedge appurtenance constraint (refined, componentwise).** For every superedge $e \in E$ and every admissible attribute assignment $(a_v)_{v \in e}$,

$$\mathrm{bdf}_r\big(e, (a_v)_{v \in e}\big) \;\leq\; \min_{v \in e} \mathrm{adf}_r(v, a_v)$$

(componentwise in $[0,1]^{s \times r}$).

(S2) **Reflexivity and symmetry of refined contradiction.** aCf_r and bCf_r are reflexive and symmetric (componentwise) exactly as in (H2).

Example 4.5.6 (A restricted refined plithogenic 2-SuperHyperGraph (RRP-2SHG))**.** Let the base set be

$$V_0 = \{1, 2, 3\}, \qquad n = 2,$$

so $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$. Define three 2-supervertices

$$v_1 = \{\{1\}, \{1, 2\}\}, \qquad v_2 = \{\{2\}, \{2, 3\}\}, \qquad v_3 = \{\{1, 3\}\}, \qquad V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0).$$

Let the superedge family be

$$E = \{e_1, e_2\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}, \qquad e_1 = \{v_1, v_2\}, \quad e_2 = \{v_1, v_2, v_3\}.$$

Fix $s = 1$, $t = 1$, and $r = 2$.

Supervertex structure. Let the attribute be ℓ =“priority” with $V_\ell = \{A, B\}$ and define $\mathrm{adf}_r : V \times V_\ell \to [0,1]^{1\times 2}$ by

$$\mathrm{adf}_r(v_1, A) = (0.7, 0.6), \quad \mathrm{adf}_r(v_2, A) = (0.5, 0.4), \quad \mathrm{adf}_r(v_3, A) = (0.4, 0.3),$$

$$\mathrm{adf}_r(v_1, B) = (0.3, 0.2), \quad \mathrm{adf}_r(v_2, B) = (0.6, 0.5), \quad \mathrm{adf}_r(v_3, B) = (0.8, 0.7).$$

Let $\mathrm{aCf}_r(A, A) = \mathrm{aCf}_r(B, B) = (0, 0)$ and $\mathrm{aCf}_r(A, B) = \mathrm{aCf}_r(B, A) = (0.4, 0.4)$.

Superedge structure. Let m =“mode” with $E_m = \{X, Y\}$ and define $\mathrm{bdf}_r : E \times E_m \to [0,1]^{1\times 2}$ by

$$\mathrm{bdf}_r(e_1, X) = (0.5, 0.4), \qquad \mathrm{bdf}_r(e_2, X) = (0.3, 0.3),$$

and let $\mathrm{bCf}_r(X, X) = \mathrm{bCf}_r(Y, Y) = (0, 0)$ and $\mathrm{bCf}_r(X, Y) = \mathrm{bCf}_r(Y, X) = (0.6, 0.6)$.

Verification of the refined superedge constraint. For $e_1 = \{v_1, v_2\}$ choose $(a_{v_1}, a_{v_2}) = (A, A)$. Then

$$\min_{v\in e_1} \mathrm{adf}_r(v, a_v) = \min\{(0.7, 0.6), (0.5, 0.4)\} = (0.5, 0.4) = \mathrm{bdf}_r(e_1, X).$$

For $e_2 = \{v_1, v_2, v_3\}$ choose $(a_{v_1}, a_{v_2}, a_{v_3}) = (B, A, A)$. Then

$$\min_{v\in e_2} \mathrm{adf}_r(v, a_v) = \min\{(0.3, 0.2), (0.5, 0.4), (0.4, 0.3)\} = (0.3, 0.2).$$

To satisfy (S1) with equality, set $\mathrm{bdf}_r(e_2, X) = (0.3, 0.2)$ (instead of $(0.3, 0.3)$). With this adjustment,

$$\mathrm{bdf}_r(e_2, X) = (0.3, 0.2) \le (0.3, 0.2) = \min_{v\in e_2} \mathrm{adf}_r(v, a_v).$$

Therefore, after this consistent choice, $(RPM^{(2)}, RPN^{(2)})$ defines an RRP-2SHG in the sense of Definition 4.5.5.

Theorem 4.5.7 (RRP-nSHGs generalize crisp n-SuperHyperGraphs)**.** *Every crisp n-SuperHyperGraph $\mathrm{SHG}^{(n)} = (V, E)$ can be viewed canonically as a restricted refined plithogenic n-SuperHyperGraph.*

Proof. The proof is identical to Theorem 4.5.3, replacing the vertex set V by the n-supervertex set $V \subseteq \mathcal{P}^n(V_0)$ and hyperedges by superedges $e \in E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Assign constant $\mathbf{1}_{s\times r}$ to all refined appurtenance maps and $\mathbf{0}$ to all refined contradiction maps. Then the refined superedge constraint (S1) and symmetry/reflexivity (S2) hold trivially. □

Theorem 4.5.8 (RRP-nSHGs generalize RRPHs)**.** *Every restricted refined plithogenic hypergraph is a special case of a restricted refined plithogenic n-SuperHyperGraph, namely the case $n = 0$.*

Proof. Let $RRPH$ be a restricted refined plithogenic hypergraph on a crisp hypergraph (V, E). Set $V_0 := V$ and $n = 0$, so $\mathcal{P}^0(V_0) = V_0$. Identify the 0-supervertex set with V and the superedge set with the hyperedge family E. Under this identification, the axioms in Definition 4.5.5 coincide with those in Definition 4.5.1. □

4.6 Plithogenic Graph Types

Following the philosophy of "fuzzy graph types" introduced in [98], we propose an analogous *type classification* for plithogenic graphs. From a practical standpoint, it is often essential to decide *where* uncertainty should be placed (vertices, edges, endpoints, weights, or collections of graphs) and *what* should remain crisp, since different applications and datasets naturally support different modelling choices.

Definition 4.6.1 (Plithogenic graph type)**.** A *plithogenic graph* PG is said to be of the *i-th plithogenic type* (written "PG is of type PG_i") if it exhibits plithogenic information in the i-th manner below, or in any combination of these manners:

(i) **Type PG_1 (family-level plithogenicity).**

$$PG_1 = \{G_1, G_2, \ldots, G_P\},$$

where each constituent graph G_j carries plithogenic vertex/edge structures (possibly with different attribute sets and contradiction maps).

(ii) **Type PG_2 (plithogenic edge set).**

$$PG_2 = \{V, E_P\},$$

where V is crisp and the edge set is plithogenic in the sense that edges are equipped with attributes together with degrees of appurtenance and contradiction information (i.e., the plithogenic structure is concentrated on edges).

(iii) **Type PG_3 (plithogenic endpoints).**

$$PG_3 = \{V,\ E(t_P, h_P)\},$$

where V and E are crisp, but each edge $e \in E$ has a *plithogenic tail* $t_P(e)$ and a *plithogenic head* $h_P(e)$ with respect to chosen attributes (i.e., fuzziness/uncertainty is attached to the endpoints or directions of edges rather than to vertices/edges globally).

(iv) **Type PG_4 (plithogenic vertex set).**

$$PG_4 = \{V_P, E\},$$

where E is crisp and the vertex set is plithogenic, meaning each vertex carries attributes with degrees of appurtenance and contradiction information.

(v) **Type PG_5 (plithogenic edge weights).**

$$PG_5 = \{V,\ E(w_P)\},$$

where V and E are crisp, but each edge $e \in E$ is endowed with a plithogenic weight $w_P(e)$ that encodes attribute-based degrees of appurtenance and contradiction.

Remark 4.6.2. Definition 4.6.1 is intended as a modelling taxonomy. Depending on the data and the application domain, one may select the most appropriate type (or combination of types). By specializing the plithogenic degrees to particular uncertainty models, one obtains corresponding "types" of fuzzy graphs, intuitionistic fuzzy graphs, neutrosophic graphs, quadripartitioned neutrosophic graphs, and pentapartitioned neutrosophic graphs, in direct analogy with [98].

Example 4.6.3 (Type interpretation via the parameters (s, t))**.** In a plithogenic graph, the parameters s and t control the dimension of the degree of appurtenance and the contradiction information, respectively. When $t = 1$ and s is chosen as below, one obtains the following commonly used subclasses (each of which may appear in any of the types PG_1–PG_5):

- $s = 1$: **Plithogenic fuzzy graphs** (PFG), in which the appurtenance degree is scalar-valued and behaves as a fuzzy-type weight.

- $s = 2$: **Plithogenic intuitionistic fuzzy graphs** (PIFG), in which two coordinates may be interpreted as membership and non-membership degrees.

- $s = 3$: **Plithogenic neutrosophic graphs** (PNG), in which three coordinates represent truth, indeterminacy, and falsity degrees.

- $s = 4$: **Plithogenic quadripartitioned neutrosophic graphs**, in which four coordinates encode a quadripartitioned refinement (e.g., truth, contradiction, unknown, falsity).

- $s = 5$: **Plithogenic pentapartitioned neutrosophic graphs**, in which five coordinates encode a pentapartitioned refinement.

Chapter 5

Uncertain Graph and Functorial Graph

5.1 Uncertain Graphs, Hypergraphs, and SuperHyperGraphs

An *uncertain set* associates with each element a degree taken from a chosen uncertainty model, thereby providing a unifying umbrella for fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related frameworks [609, 610]. An *uncertain graph* is a graph in which vertices and/or edges are equipped with degrees from such a model, encompassing, as special cases, fuzzy, intuitionistic fuzzy, and neutrosophic graph formalisms. Likewise, an *uncertain hypergraph* assigns uncertainty-model degrees to vertices and hyperedges, capturing higher-order interactions when the available information is incomplete [2]. Finally, an *uncertain n-SuperHyperGraph* equips each supervertex and superedge of an n-SuperHyperGraph with uncertainty-model degrees, enabling a systematic and rigorous treatment of hierarchical uncertainty [2]. We begin by recalling the notion of an *uncertain model*, which specifies the domain of admissible degree-tuples.

Definition 5.1.1 (Uncertain model)**.** [609] Let U denote the class of all *uncertain models*. Each $M \in U$ is determined by:

- a nonempty set $\mathrm{Dom}(M) \subseteq [0, 1]^k$ of *admissible degree tuples* for some fixed integer $k \geq 1$; and
- model-specific algebraic or geometric constraints imposed on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \leq 1$ in the intuitionistic fuzzy setting, or $0 \leq T + I + F \leq 3$ in the neutrosophic setting).

Typical instances include:

- **Fuzzy model:** $\mathrm{Dom}(M) = [0, 1]$;
- **Intuitionistic fuzzy model:** $\mathrm{Dom}(M) = \{(\mu, \nu) \in [0, 1]^2 : \ \mu + \nu \leq 1\}$;

- **Neutrosophic model:** $\mathrm{Dom}(M) = \{(T, I, F) \in [0,1]^3 :\ 0 \le T + I + F \le 3\}$;

- **Plithogenic model,** and many further extensions.

Definition 5.1.2 (Uncertain set (U-set))**.** [609] Let X be a nonempty universe, and fix an uncertain model M with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *uncertain set of type* M (briefly, a *U-set*) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is the *uncertainty-degree function* (membership map) of $\mathcal{U}$. For $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ encodes the degree(s) to which x belongs to $\mathcal{U}$, as prescribed by the model M.

We now state the corresponding graph-theoretic notions.

Definition 5.1.3 (Uncertain graph)**.** Let $G = (V, E)$ be a finite, undirected, loopless graph, and let M be an uncertain model with degree-domain $\mathrm{Dom}(M)$. An *uncertain graph of type* M is a triple

$$\mathcal{G}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns an uncertainty degree in $\mathrm{Dom}(M)$ to each vertex $v \in V$ and each edge $e \in E$. Optionally, one may impose model-dependent consistency relations between vertex- and edge-degrees (e.g., bounding $\mu_M(e)$ in terms of $\mu_M(u)$ and $\mu_M(v)$ for $e = \{u, v\}$ in fuzzy or intuitionistic fuzzy settings), but such constraints are dictated by the chosen model M and are not fixed at the level of this general definition.

For convenience, Table 5.1 lists representative uncertainty-graph families, organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

Definition 5.1.4 (Uncertain hypergraph)**.** [2] Let $H = (V, E)$ be a hypergraph and let M be an uncertain model with degree-domain $\mathrm{Dom}(M)$. An *uncertain hypergraph of type* M is a triple

$$\mathcal{H}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns an uncertainty degree to each vertex $v \in V$ and each hyperedge $e \in E$. As in the graph case, any relations between vertex-degrees and hyperedge-degrees (e.g., bounds of $\mu_M(e)$ in terms of $\mu_M(v)$ for $v \in e$) are governed by the selected model M and its constraints.

Table 5.1: A catalogue of uncertainty-graph families (uncertain graphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-graph type(s) $\mathcal{G}_M = (V, E, \mu_M)$ with $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$
1	Fuzzy graph; N-graph; shadowed-graph variants
2	Intuitionistic fuzzy graph [494]; vague graph [495]; bipolar fuzzy graph [34]; intuitionistic evidence graph; variable fuzzy graph; paraconsistent fuzzy graph; bifuzzy graph [496, 497]
3	Neutrosophic graph [61][(a)]; hesitant fuzzy graph [498]; tripolar fuzzy graph; three-way fuzzy graph; picture fuzzy graph [174, 499]; spherical fuzzy graph [131]; inconsistent intuitionistic fuzzy graph; ternary fuzzy / neutrosophic-fuzzy graph; neutrosophic vague graph
4	Quadripartitioned neutrosophic graph [500, 501]; double-valued neutrosophic graph [457]; dual hesitant fuzzy graph [502]; ambiguous graph[(b)]; local-neutrosophic graph; support-neutrosophic graph; turiyam neutrosophic graph [503][(c)]
5	Pentapartitioned neutrosophic graph [365]; triple-valued neutrosophic graph
6	Hexapartitioned neutrosophic graph; quadruple-valued neutrosophic graph
7	Heptapartitioned neutrosophic graph [504]; quintuple-valued neutrosophic graph
8	Octapartitioned neutrosophic graph
9	Nonapartitioned neutrosophic graph
n	n-refined fuzzy graph; multi-valued (fuzzy) graphs; multi-fuzzy graphs [505]
$2n$	n-refined intuitionistic fuzzy graph; multi-intuitionistic fuzzy graphs
$3n$	n-refined neutrosophic graph; multi-neutrosophic graphs

(a) Neutrosophic graph models are often treated as broad frameworks that can specialize to many degree-based graph formalisms under suitable constraints.
(b) Ambiguous-graph models are commonly presented as subclasses of certain quadripartitioned and also double-valued neutrosophic graph models.
(c) Turiyam neutrosophic graphs are reported as subclasses of certain quadripartitioned neutrosophic graph models.

Definition 5.1.5 (Uncertain n-SuperHyperGraph)**.** [2] Let V_0 be a finite base set and let $n \in \mathbb{N}_0$. Assume that an n-SuperHyperGraph on V_0 is given by

$$\mathrm{SHG}^{(n)} = (V_n, E),$$

where

$$\emptyset \neq V_n \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad \emptyset \neq E \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\},$$

so that each n-superedge $e \in E$ is a nonempty subset of the n-supervertex set V_n.

Fix an uncertain model M with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *uncertain n-SuperHyperGraph of type M* is a triple

$$\mathcal{S}_M^{(n)} = (V_n, E, \mu_M),$$

where

$$\mu_M : V_n \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns an uncertainty degree in $\mathrm{Dom}(M)$ to each n-supervertex $v \in V_n$ and each n-superedge $e \in E$.

Any additional relations between superedge-degrees and the degrees of the supervertices they contain (for example, model-specific bounds or aggregation rules) are determined by the chosen uncertain model M and are not fixed at the level of this general definition.

For $n = 0$ and $V_0 = V_n$, this notion reduces to an uncertain hypergraph of type M.

Theorem 5.1.6 (Uncertain graphs generalize plithogenic graphs)**.** *Every plithogenic graph can be encoded as an uncertain graph of a suitable uncertain-model type. More precisely, let $PG = (PM, PN)$ be a plithogenic graph on a crisp graph $G^* = (V, E)$ in the sense of Definition 4.1.2. Then there exist an uncertain model M with degree-domain* $\mathrm{Dom}(M)$ *and an uncertain graph $\mathcal{G}_M = (V, E, \mu_M)$ such that the plithogenic vertex and edge data (in particular, the degrees of appurtenance) are recovered from μ_M by a canonical decoding map. Hence the class of uncertain graphs subsumes the class of plithogenic graphs.*

Proof. Let $PG = (PM, PN)$ be a plithogenic graph on $G^* = (V, E)$, where

$$PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf}), \qquad PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf}),$$

with $M \subseteq V$, $N \subseteq E$, and where $\mathrm{adf} : M \times M_\ell \to [0,1]^s$ and $\mathrm{bdf} : N \times N_m \to [0,1]^s$ are the vertex and edge degrees of appurtenance, respectively.

Step 1: define the uncertain model. Set

$$\mathrm{Dom}(M) \;:=\; \{0\} \;\cup\; \big(\{1\} \times [0,1]^s\big) \;\subseteq\; [0,1]^{s+1}.$$

We view an element of $\mathrm{Dom}(M)$ either as the distinguished symbol 0 (representing "no data / not selected"), or as a pair $(1, \mathbf{d})$ with $\mathbf{d} \in [0,1]^s$ (representing a recorded s-tuple of degrees). No further algebraic constraints are imposed; thus M is a valid uncertain model in the sense of Definition 5.1.3 (via its specified degree-domain).

Step 2: encode vertices. Define μ_M on vertices by

$$\mu_M(v) \;:=\; \begin{cases} (1, \mathbf{d}_v), & v \in M, \\ 0, & v \in V \setminus M, \end{cases}$$

where $\mathbf{d}_v \in [0,1]^s$ is the concatenation of the appurtenance degrees of v across all attribute-values:

$$\mathbf{d}_v \;:=\; \big(\mathrm{adf}(v, a)\big)_{a \in M_\ell} \;\in\; [0,1]^{s \cdot |M_\ell|}.$$

If one prefers to keep the ambient dimension fixed to s, choose and fix a single attribute-value $a_0 \in M_\ell$ and set $\mathbf{d}_v := \mathrm{adf}(v, a_0) \in [0,1]^s$. Either choice yields a well-defined encoding; below we adopt the single-value version for notational simplicity:

$$\mathbf{d}_v := \mathrm{adf}(v, a_0) \in [0,1]^s.$$

Step 3: encode edges. Similarly, define μ_M on edges by

$$\mu_M(e) \;:=\; \begin{cases} (1, \mathbf{d}_e), & e \in N, \\ 0, & e \in E \setminus N, \end{cases}$$

where $\mathbf{d}_e \in [0,1]^s$ is obtained from the edge appurtenance data. Fix a compatible attribute-pair (a_0, b_0) in the sense of Definition 4.1.2 and set

$$\mathbf{d}_e := \mathrm{bdf}(e, (a_0, b_0)) \in [0,1]^s.$$

Thus $\mu_M : V \cup E \to \mathrm{Dom}(M)$ is well-defined.

Step 4: define the uncertain graph and decode the plithogenic data. Let

$$\mathcal{G}_M \ := \ (V, E, \mu_M).$$

Then $\mathcal{G}_M$ is an uncertain graph of type M by Definition 5.1.3. To recover the plithogenic degrees, define a decoding map

$$\pi : [0,1]^{s+1} \longrightarrow [0,1]^s, \qquad \pi(0) := \mathbf{0}, \quad \pi(1, \mathbf{d}) := \mathbf{d}.$$

For $v \in M$, we obtain $\pi(\mu_M(v)) = \mathrm{adf}(v, a_0)$, and for $e \in N$, we obtain $\pi(\mu_M(e)) = \mathrm{bdf}(e, (a_0, b_0))$. Hence the (chosen) plithogenic appurtenance data are recovered exactly.

Step 5: interpretation of plithogenic constraints. The uncertain-graph definition does not force a fixed relation between vertex and edge degrees; such relations may be imposed as additional model-dependent constraints. In particular, the plithogenic edge appurtenance constraint

$$\mathrm{bdf}(xy, (a_0, b_0)) \le \min\{\mathrm{adf}(x, a_0), \mathrm{adf}(y, b_0)\}$$

(componentwise in $[0,1]^s$) is preserved under the embedding, since it is a property of the encoded tuples:

$$\pi(\mu_M(xy)) \le \min\{\pi(\mu_M(x)), \pi(\mu_M(y))\}.$$

Likewise, reflexivity and symmetry of aCf and bCf remain available as auxiliary structure, even though they are not required by the uncertain-graph axioms.

Therefore, every plithogenic graph gives rise (canonically, after fixing representative attribute-values) to an uncertain graph whose degree map encodes the plithogenic degrees. This proves that uncertain graphs generalize plithogenic graphs. □

5.1.1 Refined Uncertain Graph

A refined uncertain graph assigns each vertex and edge refined subcomponent degrees whose aggregation reconstructs an admissible uncertainty-model tuple, enabling finer-grained uncertainty representation. Throughout, let M be an *uncertain model* with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (for some $k \ge 1$). An *uncertain graph of type M* on a crisp graph $G^* = (V, E)$ is a triple $\mathcal{G}_M = (V, E, \mu_M)$ with $\mu_M : V \cup E \to \mathrm{Dom}(M)$, as in Definition 5.1.3.

Definition 5.1.7 (Refinement signature)**.** Fix $k \ge 1$. A *refinement signature* for k components is a family

$$\mathcal{R} = (L_1, \dots, L_k)$$

of finite nonempty index sets. Put

$$L := \bigsqcup_{j=1}^{k} L_j$$

(disjoint union). Elements of L_j are interpreted as refined subcomponents of the j-th base component.

Definition 5.1.8 (Canonical aggregation associated with a refinement signature)**.** Given $\mathcal{R} = (L_1, \ldots, L_k)$ as in Definition 5.1.7, define the aggregation map

$$\mathrm{Agg}_{\mathcal{R}} : [0,1]^L \longrightarrow [0,1]^k$$

by

$$\mathrm{Agg}_{\mathcal{R}}(x)_j \;:=\; \frac{1}{|L_j|} \sum_{\lambda \in L_j} x_\lambda \qquad (1 \le j \le k),$$

i.e., the j-th base component is the arithmetic mean of its refined subcomponents.

Definition 5.1.9 (Refined uncertain model)**.** Let M be an uncertain model with $\mathrm{Dom}(M) \subseteq [0,1]^k$ and let $\mathcal{R}$ be a refinement signature for k components. The *refined uncertain model* of type $(M, \mathcal{R})$ has refined degree-domain

$$\mathrm{Dom}(M, \mathcal{R}) \;:=\; \Big\{ x \in [0,1]^L : \; \mathrm{Agg}_{\mathcal{R}}(x) \in \mathrm{Dom}(M) \Big\}.$$

(Optionally, one may also impose the standard bound $0 \le \sum_{\lambda \in L} x_\lambda \le |L|$; this is automatic for $x \in [0,1]^L$.)

Definition 5.1.10 (Refined uncertain graph (RUG))**.** Let $G^* = (V, E)$ be a finite simple undirected graph. A *refined uncertain graph of type* $(M, \mathcal{R})$ is a triple

$$\mathcal{G}_{M,\mathcal{R}} = \big(V, E, \mu_{M,\mathcal{R}}\big),$$

where

$$\mu_{M,\mathcal{R}} : V \cup E \longrightarrow \mathrm{Dom}(M, \mathcal{R})$$

assigns to each vertex and each edge a refined degree vector indexed by $L = \bigsqcup_{j=1}^{k} L_j$. Model-dependent endpoint constraints (analogous to those in fuzzy/intuitionistic/neutrosophic graph theories) may be added if desired, but are not built into this general definition.

Theorem 5.1.11 (Refined uncertain graphs generalize uncertain graphs)**.** *Let $\mathcal{G}_M = (V, E, \mu_M)$ be an uncertain graph of type M. Then $\mathcal{G}_M$ can be viewed canonically as a refined uncertain graph of type $(M, \mathcal{R}_{\mathrm{triv}})$, where $\mathcal{R}_{\mathrm{triv}}$ is the trivial refinement signature $L_j = \{\lambda_j\}$ for $1 \le j \le k$.*

Proof. Let $\mathcal{R}_{\mathrm{triv}} = (\{\lambda_1\}, \ldots, \{\lambda_k\})$. Then $L = \bigsqcup_{j=1}^{k} \{\lambda_j\}$ has size k, and $\mathrm{Agg}_{\mathcal{R}_{\mathrm{triv}}}$ is the identity map on $[0,1]^k$. Hence

$$\mathrm{Dom}(M, \mathcal{R}_{\mathrm{triv}}) = \{x \in [0,1]^k : \mathrm{Agg}_{\mathcal{R}_{\mathrm{triv}}}(x) \in \mathrm{Dom}(M)\} = \mathrm{Dom}(M).$$

Define $\mu_{M,\mathcal{R}_{\mathrm{triv}}} : V \cup E \to \mathrm{Dom}(M, \mathcal{R}_{\mathrm{triv}})$ by $\mu_{M,\mathcal{R}_{\mathrm{triv}}} := \mu_M$. Then $(V, E, \mu_{M,\mathcal{R}_{\mathrm{triv}}})$ is a refined uncertain graph and coincides with the original uncertain graph. □

Theorem 5.1.12 (Refined uncertain graphs generalize refined neutrosophic graphs)**.** *Every refined neutrosophic graph (with r truth-subdegrees, s indeterminacy-subdegrees, and t falsity-subdegrees) can be represented as a refined uncertain graph of a suitable type $(M, \mathcal{R})$.*

Proof. Let $G = (V, E)$ be a refined neutrosophic graph with functions

$$\mu_1, \ldots, \mu_r,\ \sigma_1, \ldots, \sigma_s,\ \nu_1, \ldots, \nu_t : V \cup E \to [0,1]$$

as in the given definition.

Define an uncertain model M_{N} by

$$\mathrm{Dom}(M_{\mathrm{N}}) \ := \ \{(T, I, F) \in [0,1]^3 :\ 0 \le T + I + F \le 3\}.$$

(Equivalently, one may take $\mathrm{Dom}(M_{\mathrm{N}}) = [0,1]^3$; the inequality $T + I + F \le 3$ is automatic.)

Define the refinement signature $\mathcal{R} = (L_{\mathbf{T}}, L_{\mathbf{I}}, L_{\mathbf{F}})$ by

$$L_{\mathbf{T}} = \{\mathbf{T}_1, \ldots, \mathbf{T}_r\}, \qquad L_{\mathbf{I}} = \{\mathbf{I}_1, \ldots, \mathbf{I}_s\}, \qquad L_{\mathbf{F}} = \{\mathbf{F}_1, \ldots, \mathbf{F}_t\},$$

so that $L = L_{\mathbf{T}} \sqcup L_{\mathbf{I}} \sqcup L_{\mathbf{F}}$.

Define $\mu_{M_{\mathrm{N}},\mathcal{R}} : V \cup E \to [0,1]^L$ by, for each $x \in V \cup E$,

$$\big(\mu_{M_{\mathrm{N}},\mathcal{R}}(x)\big)_{\mathbf{T}_i} := \mu_i(x), \quad \big(\mu_{M_{\mathrm{N}},\mathcal{R}}(x)\big)_{\mathbf{I}_j} := \sigma_j(x), \quad \big(\mu_{M_{\mathrm{N}},\mathcal{R}}(x)\big)_{\mathbf{F}_\ell} := \nu_\ell(x).$$

Since each $\mu_i(x), \sigma_j(x), \nu_\ell(x) \in [0,1]$, we have $\mu_{M_{\mathrm{N}},\mathcal{R}}(x) \in [0,1]^L$. Moreover, the aggregated triple

$$\mathrm{Agg}_{\mathcal{R}}\big(\mu_{M_{\mathrm{N}},\mathcal{R}}(x)\big) = \left(\frac{1}{r}\sum_{i=1}^{r} \mu_i(x),\ \frac{1}{s}\sum_{j=1}^{s} \sigma_j(x),\ \frac{1}{t}\sum_{\ell=1}^{t} \nu_\ell(x)\right)$$

lies in $[0,1]^3$, hence in $\mathrm{Dom}(M_{\mathrm{N}})$. Therefore $\mu_{M_{\mathrm{N}},\mathcal{R}}(x) \in \mathrm{Dom}(M_{\mathrm{N}}, \mathcal{R})$ for all $x \in V \cup E$.

Consequently, $\mathcal{G}_{M_{\mathrm{N}},\mathcal{R}} = (V, E, \mu_{M_{\mathrm{N}},\mathcal{R}})$ is a refined uncertain graph (Definition 5.1.10) whose refined coordinates reproduce exactly the refined neutrosophic data. □

5.1.2 Iterative Refined Uncertain Graph

An iterative refined uncertain graph assigns each vertex and edge a leaf-indexed refined degree vector whose aggregated components lie in an uncertain model's domain.

Definition 5.1.13 (Iterative refinement profile)**.** Let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *iterative refinement profile* for M is a pair $\mathcal{P} = (\mathcal{T}, \pi)$ consisting of:

- a finite rooted tree $\mathcal{T}$ (edges oriented away from the root), and
- a labeling map $\pi : \mathrm{Leaf}(\mathcal{T}) \to \{1, 2, \ldots, k\}$ assigning to each leaf the index of the base component it refines.

Let $L := \mathrm{Leaf}(\mathcal{T})$ and $L_j := \pi^{-1}(j)$ for $1 \le j \le k$; then $(L_1, \ldots, L_k)$ is a refinement signature.

Definition 5.1.14 (Iterative refined uncertain model)**.** Let M and $\mathcal{P} = (\mathcal{T}, \pi)$ be as in Definition 5.1.13, and let $\mathcal{R}_{\mathcal{P}} = (L_1, \dots, L_k)$ be the induced refinement signature. The *iterative refined uncertain model* of type $(M, \mathcal{P})$ has degree-domain

$$\mathrm{Dom}(M, \mathcal{P}) \ := \ \mathrm{Dom}\big(M, \mathcal{R}_{\mathcal{P}}\big) \ = \ \Big\{x \in [0,1]^L : \ \mathrm{Agg}_{\mathcal{R}_{\mathcal{P}}}(x) \in \mathrm{Dom}(M)\Big\}.$$

Definition 5.1.15 (Iterative refined uncertain graph (IRUG))**.** Let $G^* = (V, E)$ be a finite simple undirected graph and let $\mathcal{P} = (\mathcal{T}, \pi)$ be an iterative refinement profile for M with leaf set L. An *iterative refined uncertain graph of type* $(M, \mathcal{P})$ is a triple

$$\mathcal{G}_{M,\mathcal{P}} = \big(V, E, \mu_{M,\mathcal{P}}\big),$$

where

$$\mu_{M,\mathcal{P}} : V \cup E \longrightarrow \mathrm{Dom}(M, \mathcal{P})$$

assigns to each vertex and each edge a leaf-indexed refined degree vector.

Example 5.1.16 (An iterative refined uncertain graph)**.** Let the base uncertain model be the intuitionistic fuzzy model

$$\mathrm{Dom}(M) \ = \ \{(\mu, \nu) \in [0,1]^2 : \ \mu + \nu \le 1\},$$

so $k = 2$.

Step 1: choose an iterative refinement profile. Let $\mathcal{T}$ be a rooted tree of depth 1 with three leaves

$$L = \{\lambda_1, \lambda_2, \lambda_3\}.$$

Define $\pi : L \to \{1, 2\}$ by

$$\pi(\lambda_1) = 1, \qquad \pi(\lambda_2) = 1, \qquad \pi(\lambda_3) = 2.$$

Thus $L_1 = \{\lambda_1, \lambda_2\}$ refines the first base component (membership) and $L_2 = \{\lambda_3\}$ refines the second base component (non-membership). With the standard aggregation (arithmetic mean),

$$\mathrm{Agg}_{\mathcal{R}_{\mathcal{P}}}(x) = \big(\tfrac{x_{\lambda_1} + x_{\lambda_2}}{2}, \ x_{\lambda_3}\big) \quad (x \in [0,1]^L),$$

the refined degree-domain becomes

$$\mathrm{Dom}(M, \mathcal{P}) = \Big\{x \in [0,1]^L : \ \tfrac{x_{\lambda_1} + x_{\lambda_2}}{2} + x_{\lambda_3} \le 1\Big\}.$$

Step 2: choose a crisp graph. Let $G^* = (V, E)$ be the path on three vertices:

$$V = \{v_1, v_2, v_3\}, \qquad E = \{v_1v_2, \ v_2v_3\}.$$

Step 3: assign leaf-indexed refined degrees. Define $\mu_{M,\mathcal{P}} : V \cup E \to \mathrm{Dom}(M, \mathcal{P})$ by the following leaf-vectors:

Vertices:

$$\mu_{M,\mathcal{P}}(v_1) = (0.8, 0.6, 0.1), \quad \mu_{M,\mathcal{P}}(v_2) = (0.4, 0.2, 0.3), \quad \mu_{M,\mathcal{P}}(v_3) = (0.5, 0.3, 0.2),$$

where the coordinates correspond to $(\lambda_1, \lambda_2, \lambda_3)$.

Edges:

$$\mu_{M,\mathcal{P}}(v_1v_2) = (0.3, 0.1, 0.4), \qquad \mu_{M,\mathcal{P}}(v_2v_3) = (0.2, 0.2, 0.2).$$

Verification. For each $x \in V \cup E$, the aggregated pair

$$\mathrm{Agg}_{\mathcal{R}_{\mathcal{P}}}\big(\mu_{M,\mathcal{P}}(x)\big) = \Big(\tfrac{x_{\lambda_1}+x_{\lambda_2}}{2},\ x_{\lambda_3}\Big)$$

lies in $\mathrm{Dom}(M)$ because the inequality $\frac{x_{\lambda_1}+x_{\lambda_2}}{2} + x_{\lambda_3} \le 1$ holds. For instance,

$$\mathrm{Agg}\big(\mu_{M,\mathcal{P}}(v_1)\big) = \Big(\tfrac{0.8+0.6}{2}, 0.1\Big) = (0.7, 0.1), \quad 0.7 + 0.1 = 0.8 \le 1,$$

and

$$\mathrm{Agg}\big(\mu_{M,\mathcal{P}}(v_1v_2)\big) = \Big(\tfrac{0.3+0.1}{2}, 0.4\Big) = (0.2, 0.4), \quad 0.2 + 0.4 = 0.6 \le 1.$$

Hence $\mu_{M,\mathcal{P}}(x) \in \mathrm{Dom}(M, \mathcal{P})$ for all $x \in V \cup E$. Therefore

$$\mathcal{G}_{M,\mathcal{P}} = (V, E, \mu_{M,\mathcal{P}})$$

is an iterative refined uncertain graph of type $(M, \mathcal{P})$ in the sense of Definition 5.1.15.

Theorem 5.1.17 (Iterative refined uncertain graphs generalize iterative refined neutrosophic graphs)**.** *Every iterative (iterated) refined neutrosophic graph with a leaf-indexed refinement profile can be represented as an iterative refined uncertain graph of a suitable type* $(M, \mathcal{P})$*.*

Proof. Let $G = (V, E)$ be an Iterative Refined Neutrosophic graph given by a refinement profile $\mathcal{P}_{\mathrm{N}} = (\mathcal{T}, \tau)$, where $\tau : \mathrm{Leaf}(\mathcal{T}) \to \{\mathbf{T}, \mathbf{I}, \mathbf{F}\}$. Write $L = \mathrm{Leaf}(\mathcal{T})$ and let the vertex/edge labels be

$$A_{\mathcal{P}_{\mathrm{N}}} : V \to [0,1]^L, \qquad B_{\mathcal{P}_{\mathrm{N}}} : E \to [0,1]^L.$$

Define the uncertain model M_{N} as in Theorem 5.1.12, with $\mathrm{Dom}(M_{\mathrm{N}}) \subseteq [0,1]^3$. Convert τ into a leaf-labeling $\pi : L \to \{1, 2, 3\}$ by the identification

$$\pi(\lambda) = \begin{cases} 1, & \tau(\lambda) = \mathbf{T}, \\ 2, & \tau(\lambda) = \mathbf{I}, \\ 3, & \tau(\lambda) = \mathbf{F}. \end{cases}$$

Let $\mathcal{P} = (\mathcal{T}, \pi)$ be the resulting iterative refinement profile for M_{N} (Definition 5.1.13).

Now define $\mu_{M_{\mathrm{N}},\mathcal{P}} : V \cup E \to [0,1]^L$ by

$$\mu_{M_{\mathrm{N}},\mathcal{P}}(v) := A_{\mathcal{P}_{\mathrm{N}}}(v) \quad (v \in V), \qquad \mu_{M_{\mathrm{N}},\mathcal{P}}(e) := B_{\mathcal{P}_{\mathrm{N}}}(e) \quad (e \in E).$$

For any $x \in V \cup E$, the aggregated triple $\mathrm{Agg}_{\mathcal{R}_{\mathcal{P}}}\big(\mu_{M_{\mathrm{N}},\mathcal{P}}(x)\big)$ is obtained by averaging the leaf values over the truth-, indeterminacy-, and falsity-labeled leaves. Since all leaf coordinates lie in $[0,1]$, each average lies in $[0,1]$, and the sum of the three averages is at most 3. Hence the aggregated triple lies in $\mathrm{Dom}(M_{\mathrm{N}})$, so $\mu_{M_{\mathrm{N}},\mathcal{P}}(x) \in \mathrm{Dom}(M_{\mathrm{N}}, \mathcal{P})$.

Therefore $\mathcal{G}_{M_{\mathrm{N}},\mathcal{P}} = (V, E, \mu_{M_{\mathrm{N}},\mathcal{P}})$ is an iterative refined uncertain graph (Definition 5.1.15) that reproduces the given iterative refined neutrosophic labels. □

Theorem 5.1.18 (Iterative refined uncertain graphs also generalize uncertain graphs)**.** *Every uncertain graph of type M can be viewed as an iterative refined uncertain graph of type $(M, \mathcal{P}_{\mathrm{triv}})$, where $\mathcal{P}_{\mathrm{triv}}$ is the depth-1 tree whose leaf set has exactly one leaf per base component.*

Proof. Let $\mathcal{T}$ be a rooted tree of depth 1 with exactly k leaves $\lambda_1, \ldots, \lambda_k$. Define $\pi(\lambda_j) = j$ for $1 \le j \le k$. Then $\mathcal{P}_{\mathrm{triv}} = (\mathcal{T}, \pi)$ induces the trivial refinement signature $L_j = \{\lambda_j\}$, so $\mathrm{Agg}_{\mathcal{R}_{\mathcal{P}_{\mathrm{triv}}}}$ is the identity on $[0,1]^k$ and $\mathrm{Dom}(M, \mathcal{P}_{\mathrm{triv}}) = \mathrm{Dom}(M)$. Given an uncertain graph $\mathcal{G}_M = (V, E, \mu_M)$, set $\mu_{M,\mathcal{P}_{\mathrm{triv}}} := \mu_M$. Then $(V, E, \mu_{M,\mathcal{P}_{\mathrm{triv}}})$ is an iterative refined uncertain graph and coincides with $\mathcal{G}_M$. □

5.1.3 Iterative Refined Uncertain Hypergraphs and n-SuperHyperGraphs

This subsection extends iterative refined uncertain graphs (Definition 5.1.15) to hypergraphs and to n-SuperHyperGraphs. The guiding idea is unchanged: vertices and edge-like objects carry *leaf-indexed* refined degree vectors whose componentwise aggregations lie in the base uncertain model.

Definition 5.1.19 (Iterative refined uncertain hypergraph (IRUH))**.** Let $H^* = (V, E)$ be a finite crisp hypergraph, where V is a finite nonempty set and $E \subseteq \mathcal{P}^*(V)$ is a finite family of nonempty hyperedges. Let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$, and let $\mathcal{P} = (\mathcal{T}, \pi)$ be an iterative refinement profile for M with leaf set $L = \mathrm{Leaf}(\mathcal{T})$ (Definition 5.1.13). Recall that the induced iterative refined uncertain model has degree-domain $\mathrm{Dom}(M, \mathcal{P})$ (Definition 5.1.14).

An *iterative refined uncertain hypergraph of type* $(M, \mathcal{P})$ is a triple

$$\mathcal{H}_{M,\mathcal{P}} = \big(V, E, \mu_{M,\mathcal{P}}\big),$$

where

$$\mu_{M,\mathcal{P}} : V \cup E \longrightarrow \mathrm{Dom}(M, \mathcal{P})$$

assigns to each vertex $v \in V$ and each hyperedge $e \in E$ a leaf-indexed refined degree vector.

Example 5.1.20 (An iterative refined uncertain hypergraph (IRUH))**.** Let the base uncertain model be the intuitionistic fuzzy model

$$\mathrm{Dom}(M) \;=\; \{(\mu, \nu) \in [0,1]^2 : \; \mu + \nu \le 1\},$$

so $k = 2$.

Step 1: choose an iterative refinement profile. Let $\mathcal{T}$ be a rooted tree of depth 1 with four leaves

$$L = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}.$$

Define $\pi : L \to \{1, 2\}$ by

$$\pi(\lambda_1) = 1, \qquad \pi(\lambda_2) = 1, \qquad \pi(\lambda_3) = 1, \qquad \pi(\lambda_4) = 2.$$

Thus $L_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ refines the first base component (membership) and $L_2 = \{\lambda_4\}$ refines the second base component (non-membership). With the standard aggregation (arithmetic mean),

$$\mathrm{Agg}_{\mathcal{R}_\mathcal{P}}(x) = \Big(\tfrac{x_{\lambda_1}+x_{\lambda_2}+x_{\lambda_3}}{3},\ x_{\lambda_4}\Big) \quad (x \in [0,1]^L),$$

the refined degree-domain is

$$\mathrm{Dom}(M,\mathcal{P}) = \Big\{x \in [0,1]^L :\ \tfrac{x_{\lambda_1}+x_{\lambda_2}+x_{\lambda_3}}{3} + x_{\lambda_4} \le 1\Big\}.$$

Step 2: choose a crisp hypergraph. Let $V = \{a, b, c, d\}$ and let

$$E = \{e_1, e_2\} \subseteq \mathcal{P}^*(V), \qquad e_1 = \{a, b, c\}, \quad e_2 = \{b, d\}.$$

Thus $H^* = (V, E)$ is a finite crisp hypergraph.

Step 3: assign leaf-indexed refined degrees. Define $\mu_{M,\mathcal{P}} : V \cup E \to \mathrm{Dom}(M,\mathcal{P})$ by the following leaf-vectors (in the coordinate order $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$):

Vertices:

$$\mu_{M,\mathcal{P}}(a) = (0.9, 0.6, 0.6, 0.1), \quad \mu_{M,\mathcal{P}}(b) = (0.5, 0.4, 0.3, 0.2),$$
$$\mu_{M,\mathcal{P}}(c) = (0.4, 0.2, 0.4, 0.3), \quad \mu_{M,\mathcal{P}}(d) = (0.6, 0.3, 0.3, 0.2).$$

Hyperedges:

$$\mu_{M,\mathcal{P}}(e_1) = (0.3, 0.3, 0.2, 0.4), \qquad \mu_{M,\mathcal{P}}(e_2) = (0.2, 0.1, 0.2, 0.3).$$

Verification. For each $x \in V \cup E$, compute

$$\mathrm{Agg}_{\mathcal{R}_\mathcal{P}}\big(\mu_{M,\mathcal{P}}(x)\big) = \Big(\tfrac{x_{\lambda_1}+x_{\lambda_2}+x_{\lambda_3}}{3},\ x_{\lambda_4}\Big) \in [0,1]^2.$$

The defining inequality holds, hence the aggregate lies in $\mathrm{Dom}(M)$. For example,

$$\mathrm{Agg}\big(\mu_{M,\mathcal{P}}(a)\big) = \Big(\tfrac{0.9+0.6+0.6}{3}, 0.1\Big) = (0.7, 0.1), \quad 0.7 + 0.1 = 0.8 \le 1,$$

and

$$\mathrm{Agg}\big(\mu_{M,\mathcal{P}}(e_1)\big) = \Big(\tfrac{0.3+0.3+0.2}{3}, 0.4\Big) = \Big(\tfrac{0.8}{3}, 0.4\Big) \approx (0.267, 0.4), \quad 0.267 + 0.4 \le 1.$$

Therefore $\mu_{M,\mathcal{P}}(x) \in \mathrm{Dom}(M,\mathcal{P})$ for all $x \in V \cup E$, and

$$\mathcal{H}_{M,\mathcal{P}} = (V, E, \mu_{M,\mathcal{P}})$$

is an iterative refined uncertain hypergraph of type $(M,\mathcal{P})$ in the sense of Definition 5.1.19.

Theorem 5.1.21 (IRUH generalizes crisp hypergraphs)**.** *Every finite crisp hypergraph can be viewed canonically as an iterative refined uncertain hypergraph of a suitable type* $(M,\mathcal{P})$.

Proof. Let $H^* = (V, E)$ be a crisp hypergraph. Choose any uncertain model M with nonempty degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ and fix a point $d_0 \in \mathrm{Dom}(M)$. Let $\mathcal{P}_{\mathrm{triv}} = (\mathcal{T}, \pi)$ be the depth-1 profile with leaves $\lambda_1, \ldots, \lambda_k$ and $\pi(\lambda_j) = j$. Then $\mathrm{Dom}(M, \mathcal{P}_{\mathrm{triv}}) = \mathrm{Dom}(M)$. Define $\mu_{M,\mathcal{P}_{\mathrm{triv}}} : V \cup E \to \mathrm{Dom}(M)$ by $\mu_{M,\mathcal{P}_{\mathrm{triv}}}(x) := d_0$ for all $x \in V \cup E$. Hence $\mathcal{H}_{M,\mathcal{P}_{\mathrm{triv}}} = (V, E, \mu_{M,\mathcal{P}_{\mathrm{triv}}})$ is an IRUH, and forgetting the degrees recovers the original crisp hypergraph. □

Definition 5.1.22 (Iterative refined uncertain n-SuperHyperGraph (IRU-nSHG))**.** Let V_0 be a finite nonempty base set, let $n \in \mathbb{N}_0$, and let $\mathrm{SHG}^{(n)} = (V, E)$ be a crisp n-SuperHyperGraph on V_0, i.e.,

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Let M be an uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ and let $\mathcal{P} = (\mathcal{T}, \pi)$ be an iterative refinement profile for M with leaf set L.

An *iterative refined uncertain n-SuperHyperGraph of type* $(M, \mathcal{P})$ is a triple

$$\mathcal{S}^{(n)}_{M,\mathcal{P}} = \big(V, E, \mu_{M,\mathcal{P}}\big),$$

where

$$\mu_{M,\mathcal{P}} : V \cup E \longrightarrow \mathrm{Dom}(M, \mathcal{P})$$

assigns to each n-supervertex $v \in V$ and each n-superedge $e \in E$ a leaf-indexed refined degree vector.

Theorem 5.1.23 (IRU-nSHGs generalize crisp n-SuperHyperGraphs)**.** *Every crisp n-SuperHyperGraph can be viewed canonically as an iterative refined uncertain n-SuperHyperGraph of a suitable type $(M, \mathcal{P})$.*

Proof. Let $\mathrm{SHG}^{(n)} = (V, E)$ be a crisp n-SuperHyperGraph. Choose any uncertain model M with nonempty $\mathrm{Dom}(M)$ and fix $d_0 \in \mathrm{Dom}(M)$. Take the trivial profile $\mathcal{P}_{\mathrm{triv}}$ as in Theorem 5.1.21, so that $\mathrm{Dom}(M, \mathcal{P}_{\mathrm{triv}}) = \mathrm{Dom}(M)$. Define $\mu_{M,\mathcal{P}_{\mathrm{triv}}} : V \cup E \to \mathrm{Dom}(M)$ by $\mu_{M,\mathcal{P}_{\mathrm{triv}}}(x) := d_0$ for all $x \in V \cup E$. Then $(V, E, \mu_{M,\mathcal{P}_{\mathrm{triv}}})$ is an IRU-nSHG, and forgetting the degrees recovers the original crisp superhypergraph. □

Theorem 5.1.24 (IRUH generalizes iterative refined uncertain graphs)**.** *Every iterative refined uncertain graph of type $(M, \mathcal{P})$ is a special case of an iterative refined uncertain hypergraph of the same type, namely the case where all hyperedges have size* 2.

Proof. Let $\mathcal{G}_{M,\mathcal{P}} = (V, E_G, \mu_{M,\mathcal{P}})$ be an iterative refined uncertain graph on a finite simple graph, so $E_G \subseteq \binom{V}{2}$. Form the 2-uniform hypergraph $H^* = (V, E)$ with

$$E := \big\{\{u, v\} : \{u, v\} \in E_G\big\} \subseteq \mathcal{P}^*(V).$$

Define $\mathcal{H}_{M,\mathcal{P}} = (V, E, \mu'_{M,\mathcal{P}})$ by setting

$$\mu'_{M,\mathcal{P}}(v) := \mu_{M,\mathcal{P}}(v) \quad (v \in V), \qquad \mu'_{M,\mathcal{P}}(\{u, v\}) := \mu_{M,\mathcal{P}}(\{u, v\}) \quad (\{u, v\} \in E).$$

Then $\mu'_{M,\mathcal{P}} : V \cup E \to \mathrm{Dom}(M, \mathcal{P})$ is well-defined and preserves all degrees. Hence $\mathcal{G}_{M,\mathcal{P}}$ is realized as the 2-uniform special case of the hypergraph $\mathcal{H}_{M,\mathcal{P}}$. □

5.1.4 Uncertain Graph Types

In analogy with the "graph type" viewpoint and with the plithogenic-type taxonomy introduced earlier, we now define *uncertain graph types*. The guiding principle is the same: in applications, it is often crucial to decide *where* uncertainty is introduced (vertices, edges, endpoints, weights, or even collections of graphs), while keeping the remaining structure crisp.

Definition 5.1.25 (Uncertain graph type)**.** Fix an uncertain model M with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *uncertain graph of type* M is said to be of the *i-th uncertain graph type* (denoted $UG_i(M)$) if it exhibits uncertainty degrees in the i-th manner below, or in any combination of these manners:

(i) **Type $UG_1(M)$ (family-level uncertainty).**

$$UG_1(M) = \{\, \mathcal{G}_M^{(1)}, \mathcal{G}_M^{(2)}, \dots, \mathcal{G}_M^{(P)} \,\},$$

a finite family of uncertain graphs of type M, where each $\mathcal{G}_M^{(j)} = (V_j, E_j, \mu_M^{(j)})$ is an uncertain graph of type M.

(ii) **Type $UG_2(M)$ (uncertain edge set).**

$$UG_2(M) = \big(V,\ E,\ \mu_E\big),$$

where $G^* = (V, E)$ is a crisp graph and $\mu_E : E \to \mathrm{Dom}(M)$ assigns uncertainty degrees to edges only (vertices are kept crisp).

(iii) **Type $UG_3(M)$ (uncertain endpoints / incidence-based uncertainty).**

$$UG_3(M) = \big(V,\ E,\ \mu_M^{\mathrm{end}}\big),$$

where $G^* = (V, E)$ is crisp and

$$\mu_M^{\mathrm{end}} : V \times E \to \mathrm{Dom}(M)$$

assigns degrees to incidences (endpoints), i.e., to pairs (v, e) with v incident to e. This type captures models in which uncertainty is attached to edge endpoints (e.g. heads/tails, signed/bidirected endpoints, or role-dependent attachments).

(iv) **Type $UG_4(M)$ (uncertain vertex set).**

$$UG_4(M) = \big(V,\ E,\ \mu_V\big),$$

where $G^* = (V, E)$ is crisp and $\mu_V : V \to \mathrm{Dom}(M)$ assigns uncertainty degrees to vertices only (edges are kept crisp).

(v) **Type $UG_5(M)$ (uncertain edge weights).**

$$UG_5(M) = \big(V,\ E,\ w_M\big),$$

where $G^* = (V, E)$ is crisp and $w_M : E \to \mathrm{Dom}(M)$ assigns uncertainty degrees interpreted specifically as edge-weights (cost, reliability, intensity, etc.).

When more than one of (i)–(v) is present simultaneously (e.g. degrees on both vertices and edges), we say the uncertain graph is of the corresponding *combined* uncertain type.

Remark 5.1.26. Type $UG_3(M)$ is formulated incidence-wise to transparently accommodate directed, bidirected, and multi-role edge models by encoding uncertainty at endpoints. If one prefers to remain within ordinary graphs without explicit incidences, one may specialize $UG_3(M)$ to directed graphs by restricting to ordered pairs $(u, v) \in E \subseteq V \times V$ and reading μ_M^{end} as a degree assignment to ordered endpoints.

Theorem 5.1.27 (Uncertain graph types generalize plithogenic graph types)**.** *Every plithogenic graph type PG_i (Definition 4.6.1) can be realized as an uncertain graph type $UG_i(M)$ for a suitable uncertain model M and an appropriate choice of degree map(s). In particular, uncertain graph types strictly subsume plithogenic graph types.*

Proof. Let PG be a plithogenic graph on a crisp graph $G^* = (V, E)$, written

$$PG = (PM, PN), \qquad PM = (M, \ell, M_\ell, \mathrm{adf}, \mathrm{aCf}), \qquad PN = (N, m, N_m, \mathrm{bdf}, \mathrm{bCf}),$$

as in Definition 4.1.2. We construct an uncertain model $M^\dagger$ and encode the plithogenic degrees into uncertainty degrees.

Step 1: choose the uncertain model. Define the degree-domain

$$\mathrm{Dom}(M^\dagger) \ := \ [0,1]^s,$$

with no additional constraints. This is a valid uncertain model in the sense of Definition (Uncertain model).

Step 2: define uncertain degrees from plithogenic appurtenance. Fix compatible attribute values so that each vertex $v \in M$ is assigned a single s-tuple $\mathrm{adf}(v, \cdot) \in [0,1]^s$ and each edge $e \in N$ is assigned a single s-tuple $\mathrm{bdf}(e, \cdot) \in [0,1]^s$ (as in the standard specialization arguments used earlier).

- (*Vertex uncertainty*) Define $\mu_V : V \to \mathrm{Dom}(M^\dagger)$ by

$$\mu_V(v) := \begin{cases} \mathrm{adf}(v, \cdot), & v \in M, \\ \mathbf{0}, & v \in V \setminus M, \end{cases}$$

  where $\mathbf{0}$ is the zero vector in $[0,1]^s$.

- (*Edge uncertainty*) Define $\mu_E : E \to \mathrm{Dom}(M^\dagger)$ by

$$\mu_E(e) := \begin{cases} \mathrm{bdf}(e, \cdot), & e \in N, \\ \mathbf{0}, & e \in E \setminus N. \end{cases}$$

If one wishes to include endpoint uncertainty (type UG_3), define $\mu_M^{\mathrm{end}}(v, e)$ by assigning to an incidence (v, e) the relevant endpoint tuple derived from the plithogenic data (e.g. by restricting the edge-DAF to the coordinate(s) associated with v).

Step 3: match types. We now show that each plithogenic type PG_i embeds into the corresponding uncertain type $UG_i(M^\dagger)$.

(i) If PG is of type $PG_1 = \{G_1, \ldots, G_P\}$ (a family of plithogenic graphs), apply the above encoding to each G_j, obtaining uncertain graphs $\mathcal{G}_{M^\dagger}^{(j)}$. The family $\{\mathcal{G}_{M^\dagger}^{(1)}, \ldots, \mathcal{G}_{M^\dagger}^{(P)}\}$ is precisely type $UG_1(M^\dagger)$.

(ii) If PG is of type $PG_2 = \{V, E_P\}$ (plithogenic edges on a crisp vertex set), take the uncertain degree map to be $\mu_E : E \to \mathrm{Dom}(M^\dagger)$ and keep vertices crisp. This is exactly type $UG_2(M^\dagger)$.

(iii) If PG is of type $PG_3 = \{V, E(t_P, h_P)\}$ (plithogenic tails/heads on a crisp graph), encode the tail/head information as endpoint degrees on incidences. This yields type $UG_3(M^\dagger)$.

(iv) If PG is of type $PG_4 = \{V_P, E\}$ (plithogenic vertices on a crisp edge set), use $\mu_V : V \to \mathrm{Dom}(M^\dagger)$ and keep edges crisp, yielding type $UG_4(M^\dagger)$.

(v) If PG is of type $PG_5 = \{V, E(w_P)\}$ (plithogenic weights on crisp edges), interpret μ_E as the weight map $w_M : E \to \mathrm{Dom}(M^\dagger)$, yielding type $UG_5(M^\dagger)$.

Thus, for each $i \in \{1, \ldots, 5\}$, every instance of PG_i can be represented as an instance of $UG_i(M^\dagger)$ for a suitable uncertain model. Therefore uncertain graph types generalize plithogenic graph types. □

Remark 5.1.28. The embedding above uses only the plithogenic degrees of appurtenance (DAF) as uncertainty degrees. The contradiction functions aCf and bCf can also be encoded in an uncertain framework by enlarging the degree-domain (e.g. $\mathrm{Dom}(M) \subseteq [0, 1]^{s+t}$) or by treating them as auxiliary data attached to vertices/edges.

5.2 Functorial Graphs, Hypergraphs, and SuperHyperGraphs

A *functorial set* is a functor that assigns to each object of a category a set, and transports elements along morphisms in a structure-preserving manner [609]. A *functorial graph* assigns to every object a graph and to every morphism a graph homomorphism, with identities and compositions preserved. Similarly, a *functorial hypergraph* associates to each object a hypergraph and sends morphisms to hypergraph homomorphisms, again respecting the categorical laws. A *functorial superhypergraph* assigns to each object a superhypergraph and maps morphisms to homomorphisms that preserve supervertices, superedges, and the underlying hierarchical structure.

Definition 5.2.1 (Functorial set)**.** [609] Let $\mathcal{C}$ be a category and let

$$F : \mathcal{C} \longrightarrow \mathbf{Set}$$

be a covariant functor. We call F a *functorial set* on $\mathcal{C}$.

For each object $X \in \mathrm{Ob}(\mathcal{C})$, the set $F(X)$ is interpreted as the collection of "F-sets over X", and every element $s \in F(X)$ is an individual F-set based at X.

Each morphism $f : X \to Y$ in $\mathcal{C}$ induces a *pushforward*

$$F(f) : F(X) \longrightarrow F(Y), \qquad s \longmapsto F(f)(s),$$

and the functoriality axioms

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \qquad F(g \circ f) = F(g) \circ F(f)$$

hold for all composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Example 5.2.2 (Functorial set: the powerset functor)**.** Let $\mathcal{C} = \mathbf{Set}$ be the category of (small) sets and functions. Define the covariant functor

$$\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$$

by

$$\mathcal{P}(X) := \{ A \mid A \subseteq X \}, \qquad \mathcal{P}(f)(A) := f[A] = \{f(a) \mid a \in A\} \quad (A \subseteq X),$$

for every function $f : X \to Y$. Then $\mathcal{P}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{P}(X)}$ and $\mathcal{P}(g \circ f) = \mathcal{P}(g) \circ \mathcal{P}(f)$, so $\mathcal{P}$ is a functorial set on **Set**.

Definition 5.2.3 (Functorial graph)**.** Let **Graph** denote the category whose objects are finite, simple, undirected graphs $G = (V, E)$ with

$$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\},$$

and whose morphisms $\varphi : G \to G'$ are *graph homomorphisms*, i.e. vertex maps $\varphi : V \to V'$ satisfying

$$\{u, v\} \in E \implies \{\varphi(u), \varphi(v)\} \in E'.$$

Let $\mathcal{C}$ be a category. A *functorial graph* on $\mathcal{C}$ is a covariant functor

$$\mathsf{G} : \mathcal{C} \longrightarrow \mathbf{Graph}.$$

Equivalently, to each object $X \in \mathrm{Ob}(\mathcal{C})$ it assigns a graph

$$\mathsf{G}(X) = (V_X, E_X),$$

and to each morphism $f : X \to Y$ it assigns a graph homomorphism

$$\mathsf{G}(f) : \mathsf{G}(X) \longrightarrow \mathsf{G}(Y),$$

such that

$$\mathsf{G}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{G}(X)}, \qquad \mathsf{G}(g \circ f) = \mathsf{G}(g) \circ \mathsf{G}(f)$$

for all composable f, g in $\mathcal{C}$.

Example 5.2.4 (Functorial graph: the complete-graph functor)**.** Let $\mathcal{C} = \mathbf{Set}$. Define a covariant functor

$$\mathsf{K} : \mathbf{Set} \longrightarrow \mathbf{Graph}$$

by assigning to each set X the complete graph on X,

$$\mathsf{K}(X) := \big(X,\ \{\{x, x'\} \subseteq X :\ x \neq x'\}\big),$$

and to each function $f : X \to Y$ the vertex map

$$\mathsf{K}(f) := f : X \to Y.$$

If $\{x, x'\}$ is an edge of $\mathsf{K}(X)$, then $x \neq x'$, hence either $f(x) \neq f(x')$ (so $\{f(x), f(x')\}$ is an edge in $\mathsf{K}(Y)$) or $f(x) = f(x')$ (in which case the image is a loop-free degenerate pair that can be ignored in **Graph**); in particular, when restricting to injective morphisms in $\mathcal{C}$, $\mathsf{K}(f)$ is a genuine graph homomorphism. Moreover, $\mathsf{K}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{K}(X)}$ and $\mathsf{K}(g \circ f) = \mathsf{K}(g) \circ \mathsf{K}(f)$. Thus K provides a concrete example of a functorial graph on $\mathcal{C}$.

Definition 5.2.5 (Functorial hypergraph)**.** Let **HGraph** denote the category whose objects are finite hypergraphs $H = (V, E)$ with

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

and whose morphisms $\psi : H \to H'$ are *hypergraph homomorphisms*, i.e. vertex maps $\psi : V \to V'$ such that

$$\forall\, e \in E :\ \psi[e] \in E',$$

where $\psi[e] := \{\psi(v) \mid v \in e\}$ is the image of the hyperedge e. Let $\mathcal{C}$ be a category. A *functorial hypergraph* on $\mathcal{C}$ is a covariant functor

$$\mathsf{H} : \mathcal{C} \longrightarrow \mathbf{HGraph}.$$

Equivalently, for each object $X \in \mathrm{Ob}(\mathcal{C})$ it assigns a hypergraph

$$\mathsf{H}(X) = (V_X, E_X),$$

and for each morphism $f : X \to Y$ it assigns a hypergraph homomorphism

$$\mathsf{H}(f) : \mathsf{H}(X) \longrightarrow \mathsf{H}(Y),$$

such that

$$\mathsf{H}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{H}(X)}, \qquad \mathsf{H}(g \circ f) = \mathsf{H}(g) \circ \mathsf{H}(f)$$

for all composable f, g in $\mathcal{C}$.

Example 5.2.6 (Functorial hypergraph: image of a family under set maps)**.** Let $\mathcal{C} = \mathbf{Set}$. Fix a finite nonempty index set Λ and define a covariant functor

$$\mathsf{H}_\Lambda : \mathbf{Set} \longrightarrow \mathbf{HGraph}$$

as follows. For each set X, let

$$\mathsf{H}_\Lambda(X) := \big(X,\ E_X\big), \qquad E_X := \{\, A \subseteq X :\ A \neq \emptyset,\ |A| \leq |\Lambda| \,\},$$

i.e., the hyperedges are all nonempty subsets of X of size at most $|\Lambda|$. For a function $f : X \to Y$, define $\mathsf{H}_\Lambda(f) : X \to Y$ to be the same underlying map. If $e \in E_X$, then $f[e] \subseteq Y$ is nonempty and has size at most $|e| \leq |\Lambda|$, hence $f[e] \in E_Y$. Therefore $\mathsf{H}_\Lambda(f)$ is a hypergraph homomorphism. Functoriality is immediate because H_Λ acts by the identity on arrows: $\mathsf{H}_\Lambda(\mathrm{id}_X) = \mathrm{id}_{\mathsf{H}_\Lambda(X)}$ and $\mathsf{H}_\Lambda(g \circ f) = \mathsf{H}_\Lambda(g) \circ \mathsf{H}_\Lambda(f)$.

Definition 5.2.7 (Functorial SuperHyperGraph)**.** Fix an integer $n \geq 1$. Let $\mathbf{SHGraph}_n$ denote the category whose objects are finite level-n SuperHyperGraphs. Concretely, an object is a triple

$$\mathsf{SH} = (V_0, V, E),$$

where:

- V_0 is a finite base set;
- $V \subseteq \mathcal{P}^n(V_0)$ is a nonempty set of n-supervertices; and
- $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is a nonempty family of n-superedges, each superedge being a nonempty subset of V.

Thus the supervertices lie in the n-th iterated powerset of V_0, whereas superedges are ordinary (nonempty) subsets of the supervertex set V.

A morphism

$$\Phi : (V_0, V, E) \longrightarrow (V_0', V', E')$$

in $\mathbf{SHGraph}_n$ is a *superhypergraph homomorphism*, namely a base map $\varphi_0 : V_0 \to V_0'$ such that the induced map on the n-th iterated powerset,

$$\varphi_n := \mathcal{P}^n(\varphi_0) : \mathcal{P}^n(V_0) \longrightarrow \mathcal{P}^n(V_0'),$$

satisfies

$$\varphi_n(V) \subseteq V' \quad \text{and} \quad \varphi_n[e] := \{\varphi_n(v) \mid v \in e\} \in E' \quad \text{for all } e \in E.$$

Let $\mathcal{C}$ be a category. A *functorial SuperHyperGraph of level* n on $\mathcal{C}$ is a covariant functor

$$\mathsf{SH} : \mathcal{C} \longrightarrow \mathbf{SHGraph}_n.$$

For each object $X \in \mathrm{Ob}(\mathcal{C})$, the value

$$\mathsf{SH}(X) = (V_0^X, V_X, E_X)$$

is a level-n SuperHyperGraph, and for each morphism $f : X \to Y$ the arrow

$$\mathsf{SH}(f) : \mathsf{SH}(X) \longrightarrow \mathsf{SH}(Y)$$

is a superhypergraph homomorphism as above, satisfying the functoriality identities

$$\mathsf{SH}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{SH}(X)}, \qquad \mathsf{SH}(g \circ f) = \mathsf{SH}(g) \circ \mathsf{SH}(f)$$

for all composable f, g in $\mathcal{C}$.

In particular, when $n = 0$ and $V = V_0$, a functorial SuperHyperGraph reduces to a functorial hypergraph.

Example 5.2.8 (Functorial SuperHyperGraph: a singleton-lift to level n)**.** Fix $n \geq 1$ and let $\mathcal{C} = \mathbf{Set}_{\text{fin}}$ be the category of finite sets and functions. Define a covariant functor

$$\mathsf{SH}^{(n)} : \mathbf{Set}_{\text{fin}} \longrightarrow \mathbf{SHGraph}_n$$

by sending a finite set X to the level-n SuperHyperGraph

$$\mathsf{SH}^{(n)}(X) := (X,\ V_X,\ E_X), \qquad V_X := \big\{\mathcal{P}^{n-1}(\{x\}) : x \in X\big\} \ \subseteq\ \mathcal{P}^n(X),$$

and

$$E_X := \mathcal{P}(V_X) \setminus \{\emptyset\}.$$

Thus the supervertices are the canonical "n-level singletons" inside $\mathcal{P}^n(X)$, and every nonempty subset of V_X is taken as a superedge.

For a function $f : X \to Y$, define $\mathsf{SH}^{(n)}(f)$ to be the superhypergraph homomorphism induced by the base map $\varphi_0 := f$, i.e.,

$$\mathsf{SH}^{(n)}(f) \ = \ \Phi_f \quad \text{with} \quad \varphi_n = \mathcal{P}^n(f) : \mathcal{P}^n(X) \to \mathcal{P}^n(Y).$$

Then $\varphi_n(V_X) \subseteq V_Y$ because

$$\varphi_n\big(\mathcal{P}^{n-1}(\{x\})\big) = \mathcal{P}^{n-1}(\{f(x)\}),$$

and for any superedge $e \in E_X$ we have $\varphi_n[e] \in E_Y$ since E_Y contains *all* nonempty subsets of V_Y. Functoriality follows from the functoriality of $\mathcal{P}^n$:

$$\mathsf{SH}^{(n)}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{SH}^{(n)}(X)}, \qquad \mathsf{SH}^{(n)}(g \circ f) = \mathsf{SH}^{(n)}(g) \circ \mathsf{SH}^{(n)}(f).$$

Chapter 6

Additional Results: A unified framework for (multi)directed graphs, hypergraphs, and n-SuperHyperGraphs

This chapter introduces a single incidence-based object that can model, within one definition, *graphs, hypergraphs, n-SuperHyperGraphs, directed variants, bidirected variants, and multidirected (multi-edge) variants.* The unification mechanism is to represent every edge-like object by its *incidences* (vertex-edge attachments), each incidence carrying an explicit *role* (direction/type), and each edge carrying an optional *multiplicity.*

6.1 Role-incidence (n, R)-SuperHyperMultigraph

A role-incidence (n, R)-SuperHyperMultigraph models hierarchical n-level supervertices and (super)edges via typed incidences, where each incidence carries a role in R and edges may have multiplicity.

Definition 6.1.1 (Role-incidence (n, R)-SuperHyperMultigraph)**.** Fix an integer $n \in \mathbb{N}_0$ and a finite nonempty set R of *incidence roles.* Let V_0 be a finite base set, and define iterated powersets by

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0).$$

A *role-incidence* (n, R)*-SuperHyperMultigraph* is an octuple

$$\mathfrak{G} = (V_0,\ V,\ E,\ I,\ s,\ t,\ \rho,\ \kappa),$$

where:

- V is a finite set of *n-supervertices* with $V \subseteq \mathcal{P}^n(V_0)$;
- E is a finite set of *(super)edges*;
- I is a finite set of *incidences*;

- $s : I \to E$ is the incidence-to-edge map and $t : I \to V$ is the incidence-to-vertex map;
- $\rho : I \to R$ is the *role map* assigning a role to each incidence;
- $\kappa : E \to \mathbb{N}$ is a *multiplicity* map (with $\kappa(e) = 1$ meaning “single” and $\kappa(e) > 1$ meaning “multi”).

For each edge $e \in E$, define its incidence fiber and its role-wise supports by

$$I_e := s^{-1}(e) \subseteq I, \qquad \mathrm{supp}_r(e) := \{\, t(i) : i \in I_e,\ \rho(i) = r \,\} \subseteq V \quad (r \in R).$$

We require that every edge has at least one incidence:

$$I_e \neq \emptyset \qquad (\forall\, e \in E).$$

The (undirected) support of e is

$$\mathrm{supp}(e) := \bigcup_{r \in R} \mathrm{supp}_r(e) = \{t(i) : i \in I_e\} \subseteq V.$$

Example 6.1.2 (A role-incidence $(0, \{\text{tail}, \text{head}\})$-SuperHyperMultigraph)**.** Let $n = 0$ and let the base set be

$$V_0 = \{a, b, c\}.$$

Then $\mathcal{P}^0(V_0) = V_0$. Set $V := V_0$ and choose the role set

$$R = \{\text{tail}, \text{head}\}.$$

Edges and multiplicities. Let $E = \{e_1, e_2, e_3\}$ and define multiplicities

$$\kappa(e_1) = 1, \qquad \kappa(e_2) = 2, \qquad \kappa(e_3) = 1.$$

Intuitively, e_2 represents two parallel directed connections of the same “type” (captured by $\kappa(e_2) = 2$).

Incidences. Let the incidence set be

$$I = \{i_1, i_2, i_3, i_4, i_5, i_6\}.$$

Define the incidence-to-edge map $s : I \to E$ by

$$s(i_1) = s(i_2) = e_1, \quad s(i_3) = s(i_4) = e_2, \quad s(i_5) = s(i_6) = e_3,$$

so that each edge has exactly two incidences. Define the incidence-to-vertex map $t : I \to V$ by

$$t(i_1) = a,\ t(i_2) = b; \qquad t(i_3) = b,\ t(i_4) = c; \qquad t(i_5) = a,\ t(i_6) = c.$$

Finally define the role map $\rho : I \to R$ by

$$\rho(i_1) = \text{tail},\ \rho(i_2) = \text{head};$$
$$\rho(i_3) = \text{tail},\ \rho(i_4) = \text{head};$$
$$\rho(i_5) = \text{tail},\ \rho(i_6) = \text{head}.$$

Supports. For each edge $e \in E$, the incidence fiber $I_e = s^{-1}(e)$ is nonempty, and the role-wise supports are:

$$\mathrm{supp}_{\mathrm{tail}}(e_1) = \{a\},\ \mathrm{supp}_{\mathrm{head}}(e_1) = \{b\},$$
$$\mathrm{supp}_{\mathrm{tail}}(e_2) = \{b\},\ \mathrm{supp}_{\mathrm{head}}(e_2) = \{c\},$$
$$\mathrm{supp}_{\mathrm{tail}}(e_3) = \{a\},\ \mathrm{supp}_{\mathrm{head}}(e_3) = \{c\}.$$

Thus $\mathfrak{G} = (V_0, V, E, I, s, t, \rho, \kappa)$ is a role-incidence $(0, R)$-SuperHyperMultigraph in the sense of Definition 6.1.1; it is, concretely, a directed multigraph on $\{a, b, c\}$ with multiplicity on the $b \to c$ connection.

Remark 6.1.3 (How standard graph notions appear as special cases)**.** By choosing n, the role set R, and simple constraints on the incidence fibers I_e, Definition 6.1.1 recovers many familiar structures.

1. **(Undirected) graph.** Take $n = 0$, $V = V_0$, and $R = \{\bullet\}$. Impose: (i) $\kappa \equiv 1$; (ii) for each $e \in E$, the support has size $|\,\mathrm{supp}(e)| = 2$; and (iii) $t|_{I_e}$ is injective (no repeated endpoints). Then each edge joins exactly two distinct vertices.

2. **Hypergraph.** Take $n = 0$, $V = V_0$, and $R = \{\bullet\}$, with $\kappa \equiv 1$. Allow $|\,\mathrm{supp}(e)| \geq 1$ (and still often take $t|_{I_e}$ injective to model set-based hyperedges). Then each edge is a hyperedge incident with a nonempty set of vertices.

3. **n-SuperHyperGraph.** Take $n \geq 1$, $V \subseteq \mathcal{P}^n(V_0)$, $R = \{\bullet\}$, and $\kappa \equiv 1$. If one further requires $t|_{I_e}$ injective for all e, then each e corresponds to the subset $\mathrm{supp}(e) \subseteq V$, recovering the usual set-based view "$E \subseteq \mathcal{P}^*(V)$".

4. **Directed graph.** Take $n = 0$, $V = V_0$, and $R = \{\mathrm{tail}, \mathrm{head}\}$. Impose $\kappa \equiv 1$ and
$$|\,\mathrm{supp}_{\mathrm{tail}}(e)| = |\,\mathrm{supp}_{\mathrm{head}}(e)| = 1 \quad (\forall e \in E),$$
so each edge has a unique tail and a unique head.

5. **Directed hypergraph / directed superhypergraph.** Take $R = \{\mathrm{tail}, \mathrm{head}\}$ and allow $|\,\mathrm{supp}_{\mathrm{tail}}(e)| \geq 1$ and $|\,\mathrm{supp}_{\mathrm{head}}(e)| \geq 1$. With $n = 0$ this is a directed hypergraph; with $n \geq 1$ it is a directed n-SuperHyperGraph.

6. **Bidirected (Super)HyperGraph.** Take $R = \{+, -\}$ (local endpoint orientations). Impose $|\,\mathrm{supp}(e)| = 2$ for each e and allow the two incidences to carry roles independently (e.g., both $+$, both $-$, or one of each). With $n = 0$ this yields bidirected graphs; with $n \geq 1$ it yields bidirected SuperHyperGraphs.

7. **Multidirected (Super)HyperGraph.** Parallel edges are captured either by allowing distinct edges with the same incidence pattern, or by using $\kappa(e) > 1$. For example, taking $R = \{\mathrm{tail}, \mathrm{head}\}$ and requiring one tail and one head per edge gives a multidirected graph (or, for $n \geq 1$, a multidirected SuperHyperGraph) in which $\kappa(e)$ records multiplicity.

Theorem 6.1.4 (Universality of the role-incidence formalism)**.** *Every finite structure listed below can be represented (canonically) as a special case of a role-incidence* (n, R)*-SuperHyperMultigraph in the sense of Definition 6.1.1:*

graph, hypergraph, n-SuperHyperGraph, directed variants, bidirected variants, multidirected variants.

Proof. We sketch the canonical encodings; in each case one specifies n, R, and the incidence set I.

(1) Hypergraph (including graphs). Given a hypergraph $H = (V, E_H)$ with $E_H \subseteq \mathcal{P}^*(V)$, set $n = 0$, $V_0 = V$, $R = \{\bullet\}$, and $\kappa \equiv 1$. Let E be a renamed copy of E_H, and define

$$I := \{(v, e) : e \in E,\ v \in e\}, \qquad s(v, e) := e, \qquad t(v, e) := v, \qquad \rho(v, e) := \bullet.$$

Then $\mathrm{supp}(e) = e$ for each e, so the incidence object recovers the hypergraph. If additionally $|e| = 2$ for all e, it recovers an undirected graph.

(2) Directed graph / directed hypergraph. Take $R = \{\mathrm{tail}, \mathrm{head}\}$. For a directed graph with arcs (u, v), define each edge e to have exactly two incidences, one with role tail attached to u and one with role head attached to v. For a directed hypergraph with hyperarcs (T, H) (tail/head sets), let incidences connect every $u \in T$ with role tail and every $v \in H$ with role head.

(3) n-SuperHyperGraph and directed/bidirected/multidirected variants. Choose $n \geq 1$ and view supervertices as elements of $\mathcal{P}^n(V_0)$. Then apply the same incidence constructions as in (1) and (2), but with $t(i) \in V \subseteq \mathcal{P}^n(V_0)$. Bidirectedness is obtained by choosing $R = \{+, -\}$ and allowing independent endpoint roles. Multidirectedness is encoded by either multiple edges with the same incidence pattern or by $\kappa(e) > 1$.

In all cases, the resulting data satisfy the axioms of Definition 6.1.1 by construction, and the original structure is recovered from the supports $\mathrm{supp}_r(e)$ together with the chosen constraints (e.g. arity-2, one tail/one head, etc.). □

6.2 A unified framework for uncertain and (multi)directed uncertain SuperHyperGraphs

We now incorporate uncertainty degrees in a model-agnostic way, so that *uncertain graphs, uncertain hypergraphs, uncertain n-SuperHyperGraphs*, and their directed/bidirected/multidirected counterparts become instances of a single definition.

Definition 6.2.1 (Uncertain model (degree domain))**.** An *uncertain model* M consists of a nonempty degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$ (for some $k \geq 1$), together with model-specific admissibility constraints on $\mathrm{Dom}(M)$ (e.g. $\mu + \nu \leq 1$ in the intuitionistic fuzzy case).

Definition 6.2.2 (Uncertain role-incidence (n, R)-SuperHyperMultigraph of type M)**.** Let M be an uncertain model with degree-domain $\mathrm{Dom}(M)$. An *uncertain role-incidence* (n, R)*-SuperHyperMultigraph of type* M is a pair

$$\mathfrak{G}_M = (\mathfrak{G}, \mu_M),$$

where $\mathfrak{G} = (V_0, V, E, I, s, t, \rho, \kappa)$ is a role-incidence (n, R)-SuperHyperMultigraph (Definition 6.1.1) and

$$\mu_M : V \cup E \cup I \longrightarrow \mathrm{Dom}(M)$$

is a degree assignment. (If one wishes to model uncertainty only on vertices and edges, one may fix μ_M on I to a constant value in $\mathrm{Dom}(M)$.)

Any additional consistency relations between vertex-, edge-, and incidence-degrees (e.g. fuzzy/intuitionistic/neutrosophic bounds of edge degrees in terms of endpoint degrees) are treated as *model-dependent axioms* and are not imposed at the level of this general definition.

Remark 6.2.3 (Recovery of standard uncertain families)**.** By combining Remark 6.1.3 with Definition 6.2.2, one obtains:

- **Uncertain graph:** take $n = 0$, $R = \{\bullet\}$, impose arity-2 constraints on edges, and use μ_M on $V \cup E$.
- **Uncertain hypergraph:** take $n = 0$, $R = \{\bullet\}$ and allow larger supports.
- **Uncertain n-SuperHyperGraph:** take $n \geq 1$, $R = \{\bullet\}$.
- **Directed / bidirected / multidirected uncertain variants:** choose R (e.g. $\{\mathrm{tail}, \mathrm{head}\}$ or $\{+, -\}$), impose the corresponding structural constraints, and allow κ to encode multiplicity when needed.

In particular, a *multidirected uncertain n-SuperHyperGraph* is obtained by taking $R = \{\mathrm{tail}, \mathrm{head}\}$, allowing $\kappa(e) > 1$, and assigning degrees μ_M to supervertices and superedges (and optionally incidences).

Example 6.2.4 (A role-incidence $(0, \{\mathrm{tail}, \mathrm{head}\})$-SuperHyperMultigraph)**.** Let $n = 0$ and let the base set be

$$V_0 = \{a, b, c\}.$$

Then $\mathcal{P}^0(V_0) = V_0$. Set $V := V_0$ and choose the role set

$$R = \{\mathrm{tail}, \mathrm{head}\}.$$

Edges and multiplicities. Let $E = \{e_1, e_2, e_3\}$ and define multiplicities

$$\kappa(e_1) = 1, \qquad \kappa(e_2) = 2, \qquad \kappa(e_3) = 1.$$

Intuitively, e_2 represents two parallel directed connections of the same "type" (captured by $\kappa(e_2) = 2$).

Incidences. Let the incidence set be

$$I = \{i_1, i_2, i_3, i_4, i_5, i_6\}.$$

Define the incidence-to-edge map $s : I \to E$ by

$$s(i_1) = s(i_2) = e_1, \quad s(i_3) = s(i_4) = e_2, \quad s(i_5) = s(i_6) = e_3,$$

so that each edge has exactly two incidences. Define the incidence-to-vertex map $t : I \to V$ by

$$t(i_1) = a,\ t(i_2) = b; \qquad t(i_3) = b,\ t(i_4) = c; \qquad t(i_5) = a,\ t(i_6) = c.$$

Finally define the role map $\rho : I \to R$ by

$$\rho(i_1) = \text{tail},\ \rho(i_2) = \text{head};$$

$$\rho(i_3) = \text{tail},\ \rho(i_4) = \text{head};$$

$$\rho(i_5) = \text{tail},\ \rho(i_6) = \text{head}.$$

Supports. For each edge $e \in E$, the incidence fiber $I_e = s^{-1}(e)$ is nonempty, and the role-wise supports are:

$$\mathrm{supp}_{\text{tail}}(e_1) = \{a\},\ \mathrm{supp}_{\text{head}}(e_1) = \{b\},$$
$$\mathrm{supp}_{\text{tail}}(e_2) = \{b\},\ \mathrm{supp}_{\text{head}}(e_2) = \{c\},$$
$$\mathrm{supp}_{\text{tail}}(e_3) = \{a\},\ \mathrm{supp}_{\text{head}}(e_3) = \{c\}.$$

Thus $\mathfrak{G} = (V_0, V, E, I, s, t, \rho, \kappa)$ is a role-incidence $(0, R)$-SuperHyperMultigraph in the sense of Definition 6.1.1; it is, concretely, a directed multigraph on $\{a, b, c\}$ with multiplicity on the $b \to c$ connection.

Theorem 6.2.5 (Universality for uncertain directed/bidirected/multidirected superhyperstructures)**.** *Every finite uncertain graph, uncertain hypergraph, uncertain n-SuperHyperGraph, and their directed/bidirected/multidirected variants can be represented as an uncertain role-incidence (n, R)-SuperHyperMultigraph of a suitable type M (Definition 6.2.2).*

Proof. Take the corresponding crisp structure and encode it as $\mathfrak{G}$ using Theorem 6.1.4. If the original uncertain object assigns degrees in $\mathrm{Dom}(M)$ to its vertices and edges (and possibly to incidences), define μ_M on $V \cup E \cup I$ by the same assignments (and, if incidences are not present in the original model, set μ_M constant on I). Then $(\mathfrak{G}, \mu_M)$ satisfies Definition 6.2.2. All additional axioms specific to the chosen uncertainty theory (fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, etc.) can be imposed as extra constraints on μ_M without changing the underlying unified representation. □

Chapter 7

Conclusions

In this book, we introduced and investigated new graph classes related to intuitionistic fuzzy graphs and quadripartitioned neutrosophic graphs, namely *General Intuitionistic Fuzzy Graphs*, *General Quadripartitioned Neutrosophic Graphs*, and *Quadripartitioned Neutrosophic Hypergraphs*. We also examined the concept of *Pentapartitioned Neutrosophic Graphs*. In addition, we explored how these graph types may be accommodated and further generalized within the broader framework of *Plithogenic Graphs*.

Looking ahead, we expect further progress on algorithmic aspects of these graph models, deeper investigations of their mathematical structures, and continued development of applications, including studies related to graph neural networks and other learning paradigms.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

Appendix (List of Tables)

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Bibliography

[1] Reinhard Diestel. *Graph theory.* Springer (print edition); Reinhard Diestel (eBooks), 2024.

[2] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications.* Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[3] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume).* Biblio Publishing, 2024.

[4] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications.* Chapman and Hall/CRC, 2018.

[5] Georgios A Pavlopoulos, Maria Secrier, Charalampos N Moschopoulos, Theodoros G Soldatos, Sophia Kossida, Jan Aerts, Reinhard Schneider, and Pantelis G Bagos. Using graph theory to analyze biological networks. *BioData mining*, 4:1–27, 2011.

[6] Max Franz, Christian T Lopes, Gerardo Huck, Yue Dong, Onur Sumer, and Gary D Bader. Cytoscape. js: a graph theory library for visualisation and analysis. *Bioinformatics*, 32(2):309–311, 2016.

[7] Eli Chien, Chao Pan, Jianhao Peng, and Olgica Milenkovic. You are allset: A multiset function framework for hypergraph neural networks. *ArXiv*, abs/2106.13264, 2021.

[8] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, pages 3558–3565, 2019.

[9] Yue Gao, Yifan Feng, Shuyi Ji, and Rongrong Ji. Hgnn+: General hypergraph neural networks. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 45(3):3181–3199, 2022.

[10] Takaaki Fujita. Multi-superhypergraph neural networks: A generalization of multi-hypergraph neural networks. *Neutrosophic Computing and Machine Learning*, 39:328–347, 2025.

[11] Takaaki Fujita and Arif Mehmood. Superhypergraph attention networks. *Neutrosophic Computing and Machine Learning*, 40(1):10–27, 2025.

[12] Richard Cummins. Hamilton circuits in tree graphs. *IEEE Transactions on Circuit Theory*, 13(1):82–90, 1966.

[13] Michael J Duff. Quantum tree graphs and the schwarzschild solution. *Physical Review D*, 7(8):2317, 1973.

[14] Haitze J Broersma and Cornelis Hoede. Path graphs. *Journal of graph theory*, 13(4):427–444, 1989.

[15] Vida Dujmović and David R Wood. On linear layouts of graphs. *Discrete Mathematics & Theoretical Computer Science*, 6, 2004.

[16] Philipp de Col, Fabian Klute, and Martin Nöllenburg. Mixed linear layouts: Complexity, heuristics, and experiments. In *International Symposium on Graph Drawing and Network Visualization*, pages 460–467. Springer, 2019.

[17] Andreas Brandstädt, Van Bang Le, and Jeremy P Spinrad. *Graph classes: a survey.* SIAM, 1999.

[18] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra.* Infinite Study, 2020.

[19] Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today.* Infinite Study, 2022.

[20] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond.* Biblio Publishing, 2025.

[21] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[22] Takaaki Fujita and Florentin Smarandache. Fundamental computational problems and algorithms for superhypergraphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 240, 2024.

[23] Takaaki Fujita. Telecommunications hypernetwork and telecommunications superhypernetwork. *Intelligence Modeling in Electromechanical Systems*, 2(1):16–31, 2025.

[24] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.

[25] Shouxian Zhu. Neutrosophic n-superhypernetwork: A new approach for evaluating short video communication effectiveness in media convergence. *Neutrosophic Sets and Systems*, 85:1004–1017, 2025.

[26] Takaaki Fujita, Arif Mehmood, and Volkan Duran. Generalizing multidimensional networks: A framework of multidimensional hypernetworks and superhypernetworks. *Neutrosophic Knowledge*, 8:62–79, 2025.

[27] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[28] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[29] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[30] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[31] Hongxing Li and Vincent C Yen. *Fuzzy sets and fuzzy decision-making*. CRC press, 1995.

[32] Didier Dubois and Henri Prade. *Fundamentals of fuzzy sets*, volume 7. Springer Science & Business Media, 2012.

[33] R Parvathi and MG Karunambigai. Intuitionistic fuzzy graphs. In *Computational Intelligence, Theory and Applications: International Conference 9th Fuzzy Days in Dortmund, Germany, Sept. 18–20, 2006 Proceedings*, pages 139–150. Springer, 2006.

[34] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[35] Sovan Samanta and Madhumangal Pal. Fuzzy planar graphs. *IEEE Transactions on Fuzzy Systems*, 23(6):1936–1942, 2015.

[36] Sovan Samanta and Madhumangal Pal. Irregular bipolar fuzzy graphs. *arXiv preprint arXiv:1209.1682*, 2012.

[37] Takaaki Fujita and Florentin Smarandache. General, general weak, anti, balanced, and semi-neutrosophic graph. *Neutrosophic Sets and Systems*, 85(1):23, 2025.

[38] TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. *Ratio Mathematica*, 47, 2023.

[39] Eman A AbuHijleh. Complex hesitant fuzzy graph. *Fuzzy Information and Engineering*, 15(2):149–161, 2023.

[40] Sovan Samanta, Madhumangal Pal, Hossein Rashmanlou, and Rajab Ali Borzooei. Vague graphs and strengths. *Journal of Intelligent & Fuzzy Systems*, 30(6):3675–3680, 2016.

[41] Hossein Rashmanlou and Rajab Ali Borzooei. Vague graphs with application. *Journal of Intelligent & Fuzzy Systems*, 30(6):3291–3299, 2016.

[42] Rajab Ali Borzooei and Hossein Rashmanlou. New concepts of vague graphs. *International Journal of Machine Learning and Cybernetics*, 8:1081–1092, 2017.

[43] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[44] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[45] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[46] Prem Kumar Singh. *Intuitionistic Plithogenic Graph*. Infinite Study, 2022.

[47] George Kollios, Michalis Potamias, and Evimaria Terzi. Clustering large probabilistic graphs. *IEEE Transactions on Knowledge and Data Engineering*, 25(2):325–336, 2011.

[48] Harary Frank. Shortest paths in probabilistic graphs. *operations research*, 17(4):583–599, 1969.

[49] Silviu Maniu, Reynold Cheng, and Pierre Senellart. An indexing framework for queries on probabilistic graphs. *ACM Transactions on Database Systems (TODS)*, 42(2):1–34, 2017.

[50] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[51] M. Akram and K. H. Dar. On n-graphs. *Southeast Asian Bulletin of Mathematics*, 38:35–49, 2014.

[52] Muhammad Akram, Wenjuan Chen, and Bijan Davvaz. On n-hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(6):2937–2944, 2014.

[53] Sergiy Kozerenko. On markov graphs. *Algebra and discrete mathematics*, 16(1), 2018.

[54] Vakkas Uluçay. Q-neutrosophic soft graphs in operations management and communication network. *Soft Computing*, 25(13):8441–8459, 2021.

[55] S Satham Hussain, R Jahir Hussain, Ghulam Muhiuddin, and P Anitha. Operations on neutrosophic vague soft graphs. *Neutrosophic Sets and Systems, vol. 51/2022: An International Journal in Information Science and Engineering*, page 254, 2022.

[56] Muhammad Saeed, Atiqe Ur Rahman, Muhammad Arshad, and Alok Dhital. A novel approach to neutrosophic hypersoft graphs with properties. *Neutrosophic Sets and Systems*, 46:336–355, 2021.

[57] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[58] R Aruna Devi and K Anitha. Construction of rough graph to handle uncertain pattern from an information system. *arXiv preprint arXiv:2205.10127*, 2022.

[59] Uzma Ahmad and Tahira Batool. Domination in rough fuzzy digraphs with application. *Soft Computing*, 27(5):2425–2442, 2023.

[60] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group (revisited). *Neutrosophic sets and Systems*, 26(1):2, 2019.

[61] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, 10:86–101, 2016.

[62] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[63] Liangsong Huang, Yu Hu, Yuxia Li, PK Kishore Kumar, Dipak Koley, and Arindam Dey. A study of regular and irregular neutrosophic graphs with real life applications. *Mathematics*, 7(6):551, 2019.

[64] Anam Luqman, Muhammad Akram, and Florentin Smarandache. Complex neutrosophic hypergraphs: new social network models. *Algorithms*, 12(11):234, 2019.

[65] Muhammad Akram and KP Shum. *Bipolar neutrosophic planar graphs*. Infinite Study, 2017.

[66] Shawkat Alkhazaleh. Neutrosophic vague set theory. *Critical Review*, 10:29–39, 2015.

[67] Metawee Songsaeng and Aiyared Iampan. *Neutrosophic set theory applied to UP-algebras*. Infinite Study, 2019.

[68] Takaaki Fujita. Revisiting bipolar neutrosophic graph and interval-valued neutrosophic graph. *Neutrosophic Systems with Applications*, 25, 2025.

[69] Takaaki Fujita. Survey of trees, forests, and paths in fuzzy and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 477, 2015.

[70] Siti Nurul Fitriah Mohamad, Roslan Hasni, and Binyamin Yusoff. On dominating energy in bipolar single-valued neutrosophic graph. *Neutrosophic Sets and Systems*, 56(1):10, 2023.

[71] Elvir Akhmetshin, Ilyos Abdullayev, Aleksey Ilyin, Denis Shakhov, and Tatyana Khorolskaya. Towards sustainable economy: Boosting financial credit risk forecasting using bipolar single-valued neutrosophic graph sets approach. *International Journal of Neutrosophic Science (IJNS)*, 26(2), 2025.

[72] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[73] Muhammad Akram, Nabeela Ishfaq, Florentin Smarandache, and Said Broumi. Application of bipolar neutrosophic sets to incidence graphs. *Infinite Study*, 2019.

[74] Siti Nurul Fitriah Mohamad, Roslan Hasni, Florentin Smarandache, and Binyamin Yusoff. Novel concept of energy in bipolar single-valued neutrosophic graphs with applications. *Axioms*, 10(3):172, 2021.

[75] Siti Nurul Fitriah Mohamad, Roslan Hasni, and Florentin Smarandache. Novel concepts on domination in neutrosophic incidence graphs with some applications. *Journal of Advanced Computational Intelligence and Intelligent Informatics*, 27(5):837–847, 2023.

[76] Seema Mehra and Manjeet Singh. *Single valued neutrosophic signed graphs*. Infinite Study, 2017.

[77] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Strong neutrosophic graphs and subgraph topological subspaces*. Infinite Study, 2016.

[78] Basavaraj V Hiremath, Durga Nagarajan, Satham Hussain, Hossein Rashmanlou, and Farshid Mofidnakhaei. m-polar quadripartitioned neutrosophic graphs with applications in decision-making for mobile network selection. *Neutrosophic Sets and Systems*, 82:458–477, 2025.

[79] S Satham Hussain, Durga Nagarajan, Hossein Rashmanlou, and Farshid Mofidnakhaei. Novel supply chain decision making model under m-polar quadripartitioned neutrosophic environment. *Journal of Applied Mathematics and Computing*, 71(1):1051–1076, 2025.

[80] Kartick Mohanta, Arindam Dey, Anita Pal, Hoang Viet Long, and Le Hoang Son. A study of m-polar neutrosophic graph with applications. *Journal of Intelligent & Fuzzy Systems*, 38(4):4809–4828, 2020.

[81] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.

[82] Prem Kumar Singh et al. Dark data analysis using intuitionistic plithogenic graphs. *International Journal of Neutrosophic Sciences*, 16(2):80–100, 2021.

[83] Takaaki Fujita. Claw-free graph and at-free graph in fuzzy, neutrosophic, and plithogenic graphs. *Information Sciences with Applications*, 5:40–55, 2025.

[84] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics.* Infinite Study, 2017.

[85] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making.* Infinite Study, 2020.

[86] Nivetha Martin, R Priya, and Florentin Smarandache. Generalized plithogenic sets in multi-attribute decision making. In *Neutrosophic and Plithogenic Inventory Models for Applied Mathematics*, pages 519–546. IGI Global Scientific Publishing, 2025.

[87] T Bharathi and Arul Priya Sampath. Plithogenic product intuitionistic fuzzy graph. *Afrika Matematika*, 36(4):1–18, 2025.

[88] N Angel, P Pandiammal, and Nivetha Martin. Plithogenic hypersoft based plithogenic cognitive maps in sustainable industrial development. *Neutrosophic Sets and Systems*, 73(1):7, 2024.

[89] Takaaki Fujita and Florentin Smarandache. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, 74:457–479, 2024.

[90] Thomas Jech. *Set theory: The third millennium edition, revised and expanded.* Springer, 2003.

[91] AT Shahida and MS Sunitha. On the metric dimension of join of a graph with empty graph (op). *Electronic Notes in Discrete Mathematics*, 63:435–445, 2017.

[92] Frank Harary and Ronald C Read. Is the null-graph a pointless concept? In *Graphs and Combinatorics: Proceedings of the Capital Conference on Graph Theory and Combinatorics at the George Washington University June 18–22, 1973*, pages 37–44. Springer, 2006.

[93] Sadik Delen and Ismail Naci Cangul. Effect of edge and vertex addition on albertson and bell indices. *AIMS Mathematics*, 6(1):925–938, 2021.

[94] Hovhannes A Harutyunyan. An efficient vertex addition method for broadcast networks. *Internet Mathematics*, 5(3):211–225, 2008.

[95] John M Boyer and Wendy J Myrvold. Simplified o (n) planarity by edge addition. *Graph Algorithms Appl*, 5:241, 2006.

[96] Ann Mary Philip, Sunny Joseph Kalayathankal, and Joseph Varghese Kureethara. The interval valued fuzzy graph associated with a crisp graph. *Italian Journal of Pure and Applied Mathematics*, page 205.

[97] Linh Anh Nguyen. Computing crisp simulations and crisp directed simulations for fuzzy graph-based structures. *arXiv preprint arXiv:2012.01845*, 2020.

[98] M Blue, Brian Bush, and J Puckett. Unified approach to fuzzy graph problems. *Fuzzy sets and Systems*, 125(3):355–368, 2002.

[99] Muhammad Umar Mirza, Rukhshanda Anjum, Maged Z Youssef, and Turki Alsuraiheed. A comprehensive study on fuzzy and crisp graph indices: generalized formulae, proximity and accuracy analysis. *AIMS Mathematics*, 8(12):30922–30939, 2023.

[100] Johannes H Hattingh, Michael A Henning, and Peter J Slater. Three-valued k-neighborhood domination in graphs. *Australas. J Comb.*, 9:233–242, 1994.

[101] MA Malkov. Finite closed sets of functions in multi-valued logic. *Pure and Appl. Math J*, 6(1):14–24, 2017.

[102] Hailong Liu and Liang Sun. On the minus domination number of graphs. *Czechoslovak Mathematical Journal*, 54:883–887, 2004.

[103] Song Feng, Emily Heath, Brett Jefferson, Cliff Joslyn, Henry Kvinge, Hugh D Mitchell, Brenda Praggastis, Amie J Eisfeld, Amy C Sims, Larissa B Thackray, et al. Hypergraph models of biological networks to identify genes critical to pathogenic viral response. *BMC bioinformatics*, 22(1):287, 2021.

[104] Nicole Pearcy, Jonathan J Crofts, and Nadia Chuzhanova. Hypergraph models of metabolism. *International Journal of Biological, Veterinary, Agricultural and Food Engineering*, 8(8):752–756, 2014.

[105] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: A survey. In *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázne, Czech Republic, August 27–31, 2001 Proceedings 26*, pages 37–57. Springer, 2001.

[106] Sathyanarayanan Gopalakrishnan, Supriya Sridharan, Soumya Ranjan Nayak, Janmenjoy Nayak, and Swaminathan Venkataraman. Central hubs prediction for bio networks by directed hypergraph-ga with validation to covid-19 ppi. *Pattern Recognition Letters*, 153:246–253, 2022.

[107] Spencer Krieger and John Kececioglu. Shortest hyperpaths in directed hypergraphs for reaction pathway inference. *Journal of Computational Biology*, 30(11):1198–1225, 2023.

[108] Spencer Krieger and John D. Kececioglu. Shortest hyperpaths in directed hypergraphs for reaction pathway inference. *Journal of computational biology : a journal of computational molecular cell biology*, 2023.

[109] Jamie Burke, Ashley Akbari, Rowena Bailey, Kevin Fasusi, Ronan A. Lyons, Jonathan Pearson, James Rafferty, and Daniel Schofield. Representing multimorbid disease progressions using directed hypergraphs. In *medRxiv*, 2023.

[110] Giorgio Gallo, Claudio Gentile, Daniele Pretolani, and Gabriella Rago. Max horn sat and the minimum cut problem in directed hypergraphs. *Mathematical Programming*, 80:213–237, 1998.

[111] Heechan Moon, Hyunju Kim, Sunwoo Kim, and Kijung Shin. Four-set hypergraphlets for characterization of directed hypergraphs. *ArXiv*, abs/2311.14289, 2023.

[112] Julio Cesar Méndez Bravo, Claudia Jeaneth Bolanos Piedrahita, Manuel Alberto Méndez Bravo, and Luis Manuel Pilacuan-Bonete. Integrating smed and industry 4.0 to optimize processes with plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84:328–340, 2025.

[113] Berrocal Villegas Salomón Marcos, Montalvo Fritas Willner, Berrocal Villegas Carmen Rosa, Flores Fuentes Rivera María Yissel, Espejo Rivera Roberto, Laura Daysi Bautista Puma, and Dante Manuel Macazana Fernández. Using plithogenic n-superhypergraphs to assess the degree of relationship between information skills and digital competencies. *Neutrosophic Sets and Systems*, 84:513–524, 2025.

[114] Nelly Hodelín Amable, Elizabeth Esther Vergel De Salazar, Martha Gloria Martínez Isaac, Olivia Catalina Olavarría Sánchez, and Johanna Mariuxi Solís Palma. Representation of motivational dynamics in school environments through plithogenic n-superhypergraphs with family participation. *Neutrosophic Sets and Systems*, 92:570–583, 2025.

[115] Salomón Marcos Berrocal Villegas, Willner Montalvo Fritas, Carmen Rosa Berrocal Villegas, María Yissel Flores Fuentes Rivera, Roberto Espejo Rivera, Laura Daysi Bautista Puma, and Dante Manuel Macazana Fernández. Using plithogenic n-superhypergraphs to assess the degree of relationship between information skills and digital competencies. *Neutrosophic Sets and Systems*, 84(1):41, 2025.

[116] Ehab Roshdy, Marwa Khashaba, and Mariam Emad Ahmed Ali. Neutrosophic super-hypergraph fusion for proactive cyberattack countermeasures: A soft computing framework. *Neutrosophic Sets and Systems*, 94:232–252, 2025.

[117] Takaaki Fujita and Florentin Smarandache. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 5:577, 2025.

[118] Takaaki Fujita and Florentin Smarandache. Soft directed n-superhypergraphs with some real-world applications. *European Journal of Pure and Applied Mathematics*, 18(4):6643–6643, 2025.

[119] Takaaki Fujita. Review of some superhypergraph classes: Directed, bidirected, soft, and rough. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[120] Takaaki Fujita. Directed acyclic superhypergraphs (dash): A general framework for hierarchical dependency modeling. *Neutrosophic Knowledge*, 6:72–86, 2025.

[121] Takaaki Fujita. Metahypergraphs, metasuperhypergraphs, and iterated metagraphs: Modeling graphs of graphs, hypergraphs of hypergraphs, superhypergraphs of superhypergraphs, and beyond. *Current Research in Interdisciplinary Studies*, 4(5):1–23, 2025.

[122] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[123] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[124] Maissam Jdid, Florentin Smarandache, and Takaaki Fujita. A linear mathematical model of the vocational training problem in a company using neutrosophic logic, hyperfunctions, and superhyperfunction. *Neutrosophic Sets and Systems*, 87:1–11, 2025.

[125] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[126] YY Yao. A comparative study of fuzzy sets and rough sets. *Information sciences*, 109(1-4):227–242, 1998.

[127] Didier Dubois and Henri Prade. Fuzzy sets and systems: theory and applications. In *Mathematics in Science and Engineering*, 2011.

[128] Witold Pedrycz. *Fuzzy sets engineering*. CRC press, 1995.

[129] Robert Lin. Note on fuzzy sets. *Yugoslav Journal of Operations Research*, 24:299–303, 2014.

[130] Talal Al-Hawary and Laith AlMomani. -balanced fuzzy graphs. *arXiv preprint arXiv:1804.08677*, 2018.

[131] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[132] TALAL ALI AL-HAWARY and MAREF YM ALZOUBI. β-product of product fuzzy graphs. *Journal of applied mathematics & informatics*, 42(2):283–290, 2024.

[133] Talal Ali Al-Hawary. Certain classes of fuzzy graphs. *European Journal of Pure and Applied Mathematics*, 10(3):552–560, 2017.

[134] T Al-Hawary, S Al-Shalaldeh, and MUHAMMAD Akram. Certain matrices and energies of fuzzy graphs. *TWMS Journal of Applied and Engineering Mathematics*, 2:2146–1147, 2021.

[135] George J Klir. *Fuzzy set theory*. Wiley-IEEE Press, 2006.

[136] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

[137] Charles C Ragin. *Fuzzy-set social science*. University of Chicago Press, 2000.

[138] J Maiers and Yosef S Sherif. Applications of fuzzy set theory. *IEEE Transactions on Systems, Man, and Cybernetics*, (1):175–189, 1985.

[139] R Lowen. Fuzzy topological spaces and fuzzy compactness. *Journal of Mathematical analysis and applications*, 56(3):621–633, 1976.

[140] Alexander P Šostak. On a fuzzy topological structure. In *Proceedings of the 13th Winter School on Abstract Analysis*, pages 89–103. Circolo Matematico di Palermo, 1985.

[141] Richard E Bellman and Lotfi Asker Zadeh. *Local and fuzzy logics.* Springer, 1977.

[142] Petr Hájek. *Metamathematics of fuzzy logic*, volume 4. Springer Science & Business Media, 2013.

[143] Abraham Kandel and William J Byatt. Fuzzy sets, fuzzy algebra, and fuzzy statistics. *Proceedings of the IEEE*, 66(12):1619–1639, 1978.

[144] Richard E Bellman and Lotfi Asker Zadeh. Decision-making in a fuzzy environment. *Management science*, 17(4):B–141, 1970.

[145] Ram Narasimhan. Goal programming in a fuzzy environment. *Decision sciences*, 11(2):325–336, 1980.

[146] Azriel Rosenfeld. The fuzzy geometry of image subsets. In *Readings in Fuzzy Sets for Intelligent Systems*, pages 633–639. Elsevier, 1993.

[147] S Baez, AP Balachandran, S Vaidya, and B Ydri. Monopoles and solitons in fuzzy physics. *Communications in Mathematical Physics*, 208:787–798, 2000.

[148] Kevin M Passino, Stephen Yurkovich, and Michael Reinfrank. *Fuzzy control*, volume 42. Addison-wesley Reading, MA, 1998.

[149] Dimiter Driankov, Hans Hellendoorn, and Michael Reinfrank. *An introduction to fuzzy control.* Springer Science & Business Media, 2013.

[150] Witold Pedrycz. *Fuzzy control and fuzzy systems.* Research Studies Press Ltd., 1993.

[151] Guanrong Chen and Trung Tat Pham. *Introduction to fuzzy systems.* CRC Press, 2005.

[152] Earl Cox, Michael O'Hagan, Rodman Taber, and Michael O'Hagen. *The fuzzy systems handkbook with cdrom.* Academic Press, Inc., 1998.

[153] Konul Memmedova and Banu Ertuna. Development of a fuzzy likert scales to measure variables in social sciences. *Information Sciences*, 654:119792, 2024.

[154] Qing Li. A novel likert scale based on fuzzy sets theory. *Expert systems with applications*, 40(5):1609–1618, 2013.

[155] Ayad Hendalianpour and Jafar Razmi. Customer satisfaction measurement using fuzzy neural network. *Decis Sci Lett*, 6(2):193–206, 2017.

[156] MM Gupta and DH Rao. On the principles of fuzzy neural networks. *Fuzzy sets and systems*, 61(1):1–18, 1994.

[157] Meysam Gheisarnejad, Ardashir Mohammadzadeh, Hamed Farsizadeh, and Mohammad Hassan Khooban. Stabilization of 5g telecom converter-based deep type-3 fuzzy machine learning control for telecom applications. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 69:544–548, 2022.

[158] Li Liu and Fanzhang Li. A survey on dynamic fuzzy machine learning. *ACM Computing Surveys*, 55:1 – 42, 2022.

[159] John N Mordeson and Sunil Mathew. *Advanced topics in fuzzy graph theory*, volume 375. Springer, 2019.

[160] A Nagoor Gani and K Radha. On regular fuzzy graphs. *Journal of Physical Sciences*, 12:33–40, 2008.

[161] Georgios Drakopoulos, Eleanna Kafeza, Phivos Mylonas, Spyros Sioutas, G Cong, and M Ramanath. A graph neural network for fuzzy twitter graphs. In *CIKM Workshops*, 2021.

[162] Takaaki Fujita and Florentin Smarandache. *Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs.* Infinite Study, 2025.

[163] Muhammad Akram, Ayesha Bashir, and Sovan Samanta. Complex pythagorean fuzzy planar graphs. *International Journal of Applied and Computational Mathematics*, 6:1–27, 2020.

[164] Muhammad Akram and Wieslaw A. Dudek. Regular bipolar fuzzy graphs. *Neural Computing and Applications*, 21:197–205, 2012.

[165] Sankar Das, Ganesh Ghorai, and Madhumangal Pal. Picture fuzzy tolerance graphs with application. *Complex & Intelligent Systems*, 8(1):541–554, 2022.

[166] Muhammad Akram and Bijan Davvaz. Strong intuitionistic fuzzy graphs. *Filomat*, 26(1):177–196, 2012.

[167] Talal Al-Hawary and Bayan Horani. On product fuzzy graphs. *Annals of fuzzy mathematics and Informatics*, 12(2):279–294, 2016.

[168] Fekadu Tesgera Agama, VN SrinivasaRao Repalle, and Laxmi Rathour. New results of partially total fuzzy graph. *BMC Research Notes*, 16(1):168, 2023.

[169] Sunil Mathew and John N Mordeson. Fuzzy influence graphs. *New Mathematics and Natural Computation*, 13(03):311–325, 2017.

[170] Waheed Ahmad Khan, Waqar Arif, Quoc Hung NGUYEN, Thanh Trung Le, and Hai Van Pham. Picture fuzzy directed hypergraphs with applications towards decision-making and managing hazardous chemicals. *IEEE Access*, 2024.

[171] Rupkumar Mahapatra, Sovan Samanta, Tofigh Allahviranloo, and Madhumangal Pal. Radio fuzzy graphs and assignment of frequency in radio stations. *Computational and Applied Mathematics*, 38:1–20, 2019.

[172] S Archana and PREETHI KUTTIPULACKAL. Line regular fuzzy semigraphs. *Baghdad Science Journal*, 20(1 (SI)):0288–0288, 2023.

[173] T Dinesh. Fuzzy incidence graph structures. *Adv. Fuzzy Math.(AFM)*, 15(1):21–30, 2020.

[174] Sk Amanathulla, Biswajit Bera, and Madhumangal Pal. Balanced picture fuzzy graph with application. *Artificial Intelligence Review*, 54(7):5255–5281, 2021.

[175] A Anthoni Amali and J Jesintha Rosline. Oscillating polar fuzzy graph (opfg) and its applications. In *AIP Conference Proceedings*. AIP Publishing, 2023.

[176] AA Talebi. Cayley fuzzy graphs on the fuzzy groups. *Computational and Applied Mathematics*, 37:4611–4632, 2018.

[177] Abhishek Guleria and Rakesh Kumar Bajaj. T-spherical fuzzy graphs: operations and applications in various selection processes. *Arabian Journal for Science and Engineering*, 45(3):2177–2193, 2020.

[178] Sankar Sahoo. Colouring of mixed fuzzy graph and its application in covid19. *Journal of Multiple-Valued Logic & Soft Computing*, 35, 2020.

[179] Jebin Jacob, Thampy Abraham, and KP Jose. Einstein fuzzy graphs. In *AIP Conference Proceedings*. AIP Publishing, 2023.

[180] Michael Cary. Perfectly regular and perfectly edge-regular fuzzy graphs. *Annals of Pure and Applied Mathematics*, 16(2):461–469, 2018.

[181] Toto Nusantara, Desi Rahmadani, Mochammad Hafiizh, Elita Dia Cahyanti, and Abdullah Bin Gani. On vertice and edge regular anti fuzzy graphs. In *Journal of Physics: Conference Series*. IOP Publishing, 2021.

[182] Talal Ali Al-Hawary. Density results for perfectly regular and perfectly edge-regular fuzzy graphs. *Journal of Discrete Mathematical Sciences and Cryptography*, pages 1–8, 2022.

[183] A Sudha and P Sundararajan. Robust fuzzy graph. *Ratio Mathematica*, 46, 2023.

[184] T Ali Al-Hawary. Anti product fuzzy graphs. *International Journal of Applied Mathematics*, 36(6):747, 2023.

[185] Rajab Ali Borzooei, R Almallah, Young Bae Jun, and H Ghaznavi. Inverse fuzzy graphs with applications. *New Mathematics and Natural Computation*, 16(02):397–418, 2020.

[186] N Meenal and J Jeromi Jovita. A first look at inverse eccentric fuzzy graph. In *AIP Conference Proceedings*, volume 2829. AIP Publishing, 2023.

[187] G Muhiuddin, Saira Hameed, Ayman Rasheed, and Uzma Ahmad. Cubic planar graph and its application to road network. *Mathematical Problems in Engineering*, 2022(1):5251627, 2022.

[188] Talal Al-Hawary. Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, 4:26, 2011.

[189] K. K. Myithili and C. Nandhini. Mixed picture fuzzy graph. *Ratio Mathematica*, 42:225, 2022.

[190] Fekadu Tesgera Agama and VN SrinivasaRao Repalle. A study on an extended total fuzzy graph. *Appl. Math*, 16(4):511–518, 2022.

[191] NR Santhi Maheswari and C Sekar. On pseudo regular fuzzy graphs. *Annals of pure and Applied Mathematics*, 11(1):105–113, 2016.

[192] R Sumathi, R Sujatha, and R Sundareswaran. Best fuzzy graph with application in fuzzy social network. In *AIP Conference Proceedings*. AIP Publishing, 2022.

[193] T Bharathi, S Felixia, and S Leo. Intuitionistic felicitous fuzzy graphs. *Eur. Chem. Bull.*, 12(5), 2023.

[194] S Yahya Mohamed and S Suganthi. Fair fuzzy matching in middle fuzzy graph. *Ratio Mathematica*, 43:176–182, 2022.

[195] S Yahya Mohamed and S Suganthi. Fair fuzzy matching in middle fuzzy graph. *Ratio Mathematica*, 43:176–182, 2022.

[196] Deva Nithyanandham and Felix Augustin. A bipolar fuzzy p-competition graph based aras technique for prioritizing covid-19 vaccines. *Applied Soft Computing*, 146:110632, 2023.

[197] MK Sen, G Chowdhury, and DS Malik. Fuzzy intersection graphs of fuzzy semigroups. *New Mathematics and Natural Computation*, 2(01):1–10, 2006.

[198] MLN McAllister. Fuzzy intersection graphs. *Computers & Mathematics with Applications*, 15(10):871–886, 1988.

[199] Sreenanda Raut and Madhumangal Pal. Fuzzy intersection graph: a geometrical approach. *Journal of Ambient Intelligence and Humanized Computing*, pages 1–25, 2022.

[200] Vakkas Ulucay and Memet Sahin. Intuitionistic fuzzy soft expert graphs with application. *Uncertainty discourse and applications*, 1(1):1–10, 2024.

[201] Ganesh Ghorai and Madhumangal Pal. Some isomorphic properties of m-polar fuzzy graphs with applications. *SpringerPlus*, 5, 2016.

[202] Muhammad Akram, Muhammad Akram, and Di Cecco. *m-polar fuzzy graphs*. Springer, 2019.

[203] Hossein Rashmanlou and Madhumangal Pal. Balanced interval-valued fuzzy graphs. *Journal of Physical Sciences*, 17:43–57, 2013.

[204] A. Kirupa and J. Jesintha Rosline. Domination in double layered fuzzy graphs. *THE 11TH NATIONAL CONFERENCE ON MATHEMATICAL TECHNIQUES AND APPLICATIONS*, 2019.

[205] J Jesintha Rosline and T Pathinathan. Triple layered fuzzy graph. *Int. J. Fuzzy Mathematical Arhive*, 8(1):36–42, 2015.

[206] Deivanai Jaisankar, Sujatha Ramalingam, Deivanayagampillai Nagarajan, and Tadesse Walelign. Fuzzy outerplanar graphs and its applications. *Int. J. Comput. Intell. Syst.*, 17:231, 2024.

[207] Takaaki Fujita. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. *ResearchGate*, July 2024.

[208] Rajab Ali Borzooei and R. Almallah. Inverse fuzzy multigraphs and planarity with application in decision-making. *Soft Computing*, 26:1531 – 1539, 2021.

[209] Majid Khalili, Rajabali Borzooei, and D. Ebrahimibagha. Bipolar inverse fuzzy graphs. *Fuzzy Systems and its Applications*, 6(1):269–285, 2023.

[210] A Kuppan and J Ravi Sankar. Fuzzy zero divisor graph in a commutative ring. *TWMS Journal of Applied and Engineering Mathematics*, 11(Jaem Vol 11-Special Issue, 2021):42, 2021.

[211] A Kuppan and J Ravi Sankar. Decomposition of fuzzy zero divisor graph. In *Journal of Physics: Conference Series*. IOP Publishing, 2021.

[212] Mohammed Alqahtani. Intuitionistic fuzzy quasi-supergraph integration for social network decision making. *International Journal of Analysis and Applications*, 23:137–137, 2025.

[213] Muhammad Akram and Noura Omair Alshehri. Tempered interval-valued fuzzy hypergraphs. *University politehnica of bucharest scientific bulletin-series a-applied mathematics and physics*, 77(1):39–48, 2015.

[214] Kevin Gurney. *An introduction to neural networks*. CRC press, 2018.

[215] Oludare Isaac Abiodun, Aman Jantan, Abiodun Esther Omolara, Kemi Victoria Dada, Nachaat AbdElatif Mohamed, and Humaira Arshad. State-of-the-art in artificial neural network applications: A survey. *Heliyon*, 4(11), 2018.

[216] Tong Wei, Junlin Hou, and Rui Feng. Fuzzy graph neural network for few-shot learning. In *2020 International joint conference on neural networks (IJCNN)*, pages 1–8. IEEE, 2020.

[217] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[218] Dalibor Krleža and Krešimir Fertalj. Graph matching using hierarchical fuzzy graph neural networks. *Ieee transactions on fuzzy systems*, 25(4):892–904, 2016.

[219] Puyin Liu and Hong-Xing Li. *Fuzzy neural network theory and application*, volume 59. World Scientific, 2004.

[220] João MC Sousa. *Fuzzy decision making in modeling and control*, volume 27. World Scientific, 2002.

[221] Walter JM Kickert. Fuzzy theories on decision making: A critical review. 1979.

[222] Sundas Shahzadi, Musavarah Sarwar, and Muhammad Akram. Decision-making approach with fuzzy type-2 soft graphs. *Journal of Mathematics*, 2020(1):8872446, 2020.

[223] P Chellamani, D Ajay, Said Broumi, and T Antony Alphonse Ligori. An approach to decision-making via picture fuzzy soft graphs. *Granular computing*, pages 1–22, 2021.

[224] Uzma Ahmad and Muhammad Sabir. Multicriteria decision-making based on the degree and distance-based indices of fuzzy graphs. *Granular Computing*, 8(4):793–807, 2023.

[225] Ali NA Koam, Muhammad Akram, and Peide Liu. Decision-making analysis based on fuzzy graph structures. *Mathematical Problems in Engineering*, 2020(1):6846257, 2020.

[226] Marco Ciotti, Massimo Ciccozzi, Alessandro Terrinoni, Wen-Can Jiang, Cheng-Bin Wang, and Sergio Bernardini. The covid-19 pandemic. *Critical reviews in clinical laboratory sciences*, 57(6):365–388, 2020.

[227] Sumitra Pokhrel and Roshan Chhetri. A literature review on impact of covid-19 pandemic on teaching and learning. *Higher education for the future*, 8(1):133–141, 2021.

[228] Ali Asghar Talebi, Janusz Kacprzyk, Hossein Rashmanlou, Seyed Hossein Sadati, and Farshid Mofidnakhaei. A novel application of domination in fuzzy graphs in medical covid-19 related immunization. *Journal of Multiple-Valued Logic & Soft Computing*, 37, 2021.

[229] Nurfarhana Hassan, Tahir Ahmad, Azmirul Ashaari, Siti Rahmah Awang, Siti Salwana Mamat, Wan Munirah Wan Mohamad, and Amirul Aizad Ahmad Fuad. A fuzzy graph approach analysis for covid-19 outbreak. *Results in Physics*, 25:104267, 2021.

[230] S Ramkumar and R Sridevi. On improper m- polar soft fuzzy graph and its applications for medical diagnosis in the current covid-19 scenario. *Indian Journal of Science and Technology*, 16(34):2681–2692, 2023.

[231] Peter R Monge and Noshir S Contractor. *Theories of communication networks*. Oxford University Press, USA, 2003.

[232] Siddhartha Sankar Biswas, Bashir Alam, and Mohammad Najmud Doja. Real time multigraphs for communication networks: An intuitionistic fuzzy mathematical model. *Journal of Computer Science*, 9(7):847, 2013.

[233] Junye Ma, Qingguo Li, and Xiangnan Zhou. Fuzzy edge connectivity and fuzzy local edge connectivity with applications to communication networks. *Fuzzy Sets and Systems*, 410:109–125, 2021.

[234] Naeem Jan, Tahir Mahmood, Lemnaouar Zedam, Kifayat Ullah, José Carlos Rodríguez Alcantud, and Bijan Davvaz. Analysis of social networks, communication networks and shortest path problems in the environment of interval-valued q-rung ortho pair fuzzy graphs. *International Journal of Fuzzy Systems*, 21:1687–1708, 2019.

[235] Junye Ma, Lin Li, and Jing Li. Fuzzy average edge connectivity with its application to communication networks. *Soft Computing*, 27(3):1367–1378, 2023.

[236] Alexander Richter and Michael Koch. Functions of social networking services. *From CSCW to Web 2.0: European Developments in Collaborative Design Selected Papers from COOP08*, 2008.

[237] Yong-Yeol Ahn, Seungyeop Han, Haewoon Kwak, Sue Moon, and Hawoong Jeong. Analysis of topological characteristics of huge online social networking services. In *Proceedings of the 16th international conference on World Wide Web*, pages 835–844, 2007.

[238] Philippa Collin, Kitty Rahilly, Ingrid Richardson, and Amanda Third. The benefits of social networking services. 2011.

[239] Amita Jain, Sunny Rai, Ankita Manaktala, and Lokender Sarna. Analysing asymmetrical associations using fuzzy graph and discovering hidden connections in facebook. *Global Journal of Enterprise Information System*, 9(1):1–12, 2017.

[240] Zi Lu, Ruiling Han, Weilu Du, and Dianshuang Wu. A fuzzy social network analysis method and a case study on tianya. *Advances in Intelligent Systems and Computing*, 2014.

[241] Takaaki Fujita. Molecular fuzzy graphs, hypergraphs, and superhypergraphs. *Journal of Intelligent Decision and Computational Modelling*, 1(3):158–171, 2025.

[242] Sk Rabiul Islam and Madhumangal Pal. An investigation of edge f-index on fuzzy graphs and application in molecular chemistry. *Complex & Intelligent Systems*, 9(2):2043–2063, 2023.

[243] SR Islam and M Pal. Second zagreb index for fuzzy graphs and its application in mathematical chemistry. *Iranian Journal of Fuzzy Systems*, 20(1):119–136, 2023.

[244] Zeeshan Saleem Mufti, Ali Tabraiz, Muhammad Farhan Hanif, et al. Molecular insights into tetracene through fuzzy topological indices in chemical graph theory. *Chemical Papers*, 79(5):2937–2953, 2025.

[245] Ali Tabraiz, Zeeshan Saleem Mufti, Muhammad Nauman Aslam, Naeem Saleem, and Hasan Hosseinzadeh. Fuzzy computational analysis of flower graph via fuzzy topological indices. *Journal of Mathematics*, 2023(1):8077729, 2023.

[246] Mehri Hasani and Masoud Ghods. Investigating fuzzy topological indices of linear and multi-cyclic naphtalene. *Journal of Interdisciplinary Mathematics*, 28(5):1779–1793, 2025.

[247] Saima Anis, Madad Khan, Muhammad Mazhar, Muhammad Naveed, and Kostaq Hila. Complex fuzzy dynamical graphs and their applications in signals processing. *International Journal of Analysis and Applications*, 23:9–9, 2025.

[248] Bijan Davvaz, Naeem Jan, Tahir Mahmood, and Kifayat Ullah. Intuitionistic fuzzy graphs of n th type with applications. *Journal of Intelligent & Fuzzy Systems*, 36(4):3923–3932, 2019.

[249] Krassimir T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20:87–96, 1986.

[250] P. A. Ejegwa, S. O. Akowe, P. M. Otene, and J. M. Ikyule. An overview on intuitionistic fuzzy sets. *International Journal of Scientific & Technology Research*, 3:142–145, 2014.

[251] Gaisi Takeuti and Satoko Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *The journal of symbolic logic*, 49(3):851–866, 1984.

[252] Reza Saadati and Jin Han Park. On the intuitionistic fuzzy topological spaces. *Chaos, Solitons & Fractals*, 27(2):331–344, 2006.

[253] Doğan Çoker. An introduction to intuitionistic fuzzy topological spaces. *Fuzzy sets and systems*, 88(1):81–89, 1997.

[254] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[255] Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.

[256] Chiranjibe Jana, Tapan Senapati, Monoranjan Bhowmik, and Madhumangal Pal. On intuitionistic fuzzy g-subalgebras of g-algebras. *Fuzzy Information and Engineering*, 7(2):195–209, 2015.

[257] Moumita Chiney and SK Samanta. Intuitionistic fuzzy basis of an intuitionistic fuzzy vector space. *Notes on Intuitionistic Fuzzy Sets*, 23(4):62–74, 2017.

[258] Moumita Chiney and SK Samanta. Intuitionistic fuzzy dimension of an intuitionistic fuzzy vector space. *Notes on Intuitionistic Fuzzy Sets*, 24(1):21–29, 2018.

[259] Jun Ye. Two effective measures of intuitionistic fuzzy entropy. *Computing*, 87(1):55–62, 2010.

[260] Kiran Pandey, Arunodaya Mishra, Pratibha Rani, Jabir Ali, and Ripon Chakrabortty. Selecting features by utilizing intuitionistic fuzzy entropy method. *Decision Making: Applications in Management and Engineering*, 6(1):111–133, 2023.

[261] Xiaonan Li and Huangjian Yi. Intuitionistic fuzzy matroids. *J. Intell. Fuzzy Syst.*, 33:3653–3663, 2017.

[262] Yonghong Li, Li Li, Jiang Li, Dong Qiu, and Huiming Duan. Bases of g-v intuitionistic fuzzy matroids. *Mathematics*, 8(9):1392, 2020.

[263] Oscar Castillo, Fatih Kutlu, and Özkan Atan. Intuitionistic fuzzy control of twin rotor multiple input multiple output systems. *Journal of Intelligent & Fuzzy Systems*, 38(1):821–833, 2020.

[264] Mohamed Hamdy, Mohamed Magdy, and Salah Helmy. Control and synchronization for two chua systems based on intuitionistic fuzzy control scheme: A comparative study. *Transactions of the Institute of Measurement and Control*, 43(7):1650–1667, 2021.

[265] VV Vijesh and R Muthuraj. Some characteristics on join of intuitionistic fuzzy graphs. *IOSR Journal of Mathematics (IOSR-JM), e-ISSN*, pages 2278–5728, 2023.

[266] A. Nagoor Gani and S. Shajitha Begum. Perfect intuitionistic fuzzy graphs. *Bulletin of Pure & Applied Sciences- Mathematics and Statistics*, pages 145–152, 2011.

[267] Sankar Sahoo and Madhumangal Pal. Intuitionistic fuzzy competition graphs. *Journal of Applied Mathematics and Computing*, 52:37–57, 2016.

[268] Lanzhen Yang and Hua Mao. Intuitionistic fuzzy threshold graphs. *J. Intell. Fuzzy Syst.*, 36:6641–6651, 2019.

[269] M. G. Karunambigai, Muhammad Akram, S. Sivasankar, and K. Palanivel. Balanced intuitionistic fuzzy graphs. *Applied mathematical sciences*, 7:2501–2514, 2013.

[270] N Deva and A. Felix. Bipolar intuitionistic fuzzy competition graphs. *Journal of Physics: Conference Series*, 2267, 2022.

[271] Sankar Sahoo and Madhumangal Pal. Intuitionistic fuzzy tolerance graphs with application. *Journal of Applied Mathematics and Computing*, 55:495–511, 2017.

[272] Noura Alshehri and andMuhammad Akram. Intuitionistic fuzzy planar graphs. *Discrete Dynamics in Nature and Society*, 2014:1–9, 2014.

[273] MG Karunambigai, K Palanivel, and S Sivasankar. Edge regular intuitionistic fuzzy graph. *Advances in Fuzzy Sets and Systems*, 20(1):25–46, 2015.

[274] NR Maheswari and C Sekar. On m-neighbourly irregular instuitionistic fuzzy graphs. *International Journal of Mathematical Combinatorics*, 3, 2016.

[275] A Al-Jarrah, MO Massa'deh, AEK Baareh, et al. A study on r-edge regular intuitionistic fuzzy graphs. *J Math Comput Sci*, 23(4):279–288, 2021.

[276] Michael Cary. Perfectly regular and perfectly edge-regular fuzzy graphs. *Annals of Pure and Applied Mathematics*, 16(2):461–469, 2018.

[277] S. P. Geetha and S. Praveena. Distinct concept of edge regular intuitionistic fuzzy m-polargraphs. *ScieXplore: International Journal of Research in Science*, 2018.

[278] Sankar Sahoo and Madhumangal Pal. Intuitionistic fuzzy labeling graphs. *TWMS Journal of Applied and Engineering Mathematics*, 8(2):466–476, 2018.

[279] SN Mishra and A Pal. Regular interval-valued intuitionistic fuzzy graphs. *Journal of Informatics & Mathematical Sciences*, 9(3), 2017.

[280] M. Mohammed Jabarulla and K. Asmiya Banu. Anti intuitionistic fuzzy graph. *Advances and Applications in Mathematical Sciences*, 20(4):597–604, 2021.

[281] Hao Guan, Waheed Ahmad Khan, Shazia Saleem, Waqar Arif, Jana Shafi, and Aysha Khan. Some connectivity parameters of interval-valued intuitionistic fuzzy graphs with applications. *Axioms*, 12(12):1120, 2023.

[282] P. K. Kishore Kumar, S. Lavanya, Hossein Rashmanlou, and Mostafa Nouri Jouybari. Magic labeling on interval-valued intuitionistic fuzzy graphs. *J. Intell. Fuzzy Syst.*, 33:3999–4006, 2017.

[283] K. K. Myithili, Rangasamy Parvathi, and Muhammad Akram. Certain types of intuitionistic fuzzy directed hypergraphs. *International Journal of Machine Learning and Cybernetics*, 7:287–295, 2016.

[284] Abida Anwar and Faryal Chaudhry. On certain products of complex intuitionistic fuzzy graphs. *Journal of Function Spaces*, 2021.

[285] Murugan Kaviyarasu, Mohammed Alqahtani, Murugesan Rajeshwari, and Gopikumar Sudalaimuthu. Complex t-intuitionistic fuzzy graph with applications of rubber industrial water wastes. *Mathematics*, 2024.

[286] Irfan Nazeer and Tabasam Rashid. Connectivity concepts in intuitionistic fuzzy incidence graphs with application. *International Journal of Applied and Computational Mathematics*, 8, 2022.

[287] K. K. Myithili and R. Keerthika. Regularity and duality of intuitionistic fuzzy k-partite hypergraphs. *Notes on Intuitionistic Fuzzy Sets*, 2023.

[288] KK Myithili and R Keerthika. Types of intuitionistic fuzzy k-partite hypergraphs. In *AIP Conference Proceedings*. AIP Publishing LLC, 2020.

[289] S Yahya Mohamed and R JahirHussain. Irregularity and density of irregular intuitionistic fuzzy graphs. *IOSR Journal of Mathematics*, 2014.

[290] VS Sreedevi and Bloomy Joseph. Degree of intuitionistic l-fuzzy graph. *J. Math. Comput. Sci.*, 11(1):1011–1017, 2021.

[291] Tintumol Sunny, Pramada Ramachandran, and Sr Magie Jose. On level subsets of intuitionistic l-fuzzy graphs. *Malaya Journal of Matematik*, 8(04):1942–1946, 2020.

[292] N Deva and A Felix. Bipolar intuitionistic anti fuzzy graphs. In *AIP Conference Proceedings*. AIP Publishing, 2022.

[293] R Muthuraj, VV Vijesh, and S Sujith. Some peculiar dominations in intuitionistic anti-fuzzy graphs. *Malaya Journal of Matematik*, 9(01):178–184, 2021.

[294] AL-Hawary Talal and Bayan Hourani. On intuitionistic product fuzzy graphs. *Italian Journal of Pure and Applied Mathematics*, page 113, 2017.

[295] T PATHINATHAN and A KIRUPA. Intuitionistic k-partitioned fuzzy graph. *ADVANCES IN MATHEMATICS*, 10(1):181–192, 2020.

[296] Siddhartha Sankar Biswas, Bashir Alam, and MN Doja. Intuitionistic fuzzy real time multigraphs for communication networks: a theoretical model. *AASRI Procedia*, 5:114–119, 2013.

[297] Krassimir T Atanassov, Basil K Papadopoulos, and Apostolos Syropoulos. An application of the theory of intutionistic fuzzy multigraphs. *Mathware & soft computing. 2004 Vol. 11 Núm. 1*, 2004.

[298] Muhammad Akram and Wieslaw A. Dudek. Intuitionistic fuzzy hypergraphs with applications. *Inf. Sci.*, 218:182–193, 2013.

[299] Rangasamy Parvathi, S. Thilagavathi, and M. G. Karunambigai. Operations on intuitionistic fuzzy hypergraphs. *International Journal of Computer Applications*, 51:46–54, 2012.

[300] P. M. Dhanya, A. Sreekumar, M. Jathavedan, and P. B. Ramkumar. Algebra of morphological dilation on intuitionistic fuzzy hypergraphs. *International journal of scientific research in science, engineering and technology*, 4:300–308, 2018.

[301] Krassimir Atanassov, Sotir Sotirov, and Tania Pencheva. Intuitionistic fuzzy deep neural network. *Mathematics*, 11(3):716, 2023.

[302] RJ Kuo and WC Cheng. An intuitionistic fuzzy neural network with gaussian membership function. *Journal of Intelligent & Fuzzy Systems*, 36(6):6731–6741, 2019.

[303] Bernardete Coelho and António Andrade-Campos. Efficiency achievement in water supply systems—a review. *Renewable and Sustainable Energy Reviews*, 30:59–84, 2014.

[304] AO Lambert, Timothy G Brown, M Takizawa, and D Weimer. A review of performance indicators for real losses from water supply systems. *Journal of Water Supply: Research and Technology—AQUA*, 48(6):227–237, 1999.

[305] Zehui Shao, Saeed Kosari, Hossein Rashmanlou, and Muhammad Shoaib. New concepts in intuitionistic fuzzy graph with application in water supplier systems. *Mathematics*, 8(8):1241, 2020.

[306] Murugan Kaviyarasu, Mohammed Alqahtani, Murugesan Rajeshwari, and Gopikumar Sudalaimuthu. Complex t-intuitionistic fuzzy graph with applications of rubber industrial water wastes. *Mathematics*, 12(13):1950, 2024.

[307] Jingyuan Zhang and Ivan Stojmenovic. Cellular networks. *University of Alabama, University of Ottawa, Canada*, 2005.

[308] Naveed Yaqoob, Muhammad Gulistan, Seifedine Kadry, and Hafiz Abdul Wahab. Complex intuitionistic fuzzy graphs with application in cellular network provider companies. *Mathematics*, 7(1):35, 2019.

[309] Hritika Sharma, Saket Shanker, Akhilesh Barve, Kamalakanta Muduli, Anil Kumar, and Sunil Luthra. Interval-valued intuitionistic fuzzy digraph-matrix approach with perman algorithm for measuring covid-19 impact on perishable food supply chain. *Environment, Development and Sustainability*, pages 1–40, 2022.

[310] Waheed Ahmad Khan, Yusra Arooj, and Hai Van Pham. Modeling academic social networks using covering and matching in intuitionistic fuzzy influence graphs. *Symmetry*, 17(5):785, 2025.

[311] Mohammed Alqahtani, R Keerthana, S Venkatesh, and M Kaviyarasu. Hesitant bipolar-valued intuitionistic fuzzy graphs for identifying the dominant person in social media groups. *Symmetry*, 16(10):1293, 2024.

[312] Muhammad Akram and Gulfam Shahzadi. *Operations on single-valued neutrosophic graphs*. Infinite Study, 2017.

[313] Rıdvan Şahin. An approach to neutrosophic graph theory with applications. *Soft Computing*, 23(2):569–581, 2019.

[314] Muhammad Gulistan, Naveed Yaqoob, Zunaira Rashid, Florentin Smarandache, and Hafiz Abdul Wahab. A study on neutrosophic cubic graphs with real life applications in industries. *Symmetry*, 10(6):203, 2018.

[315] Cengiz Kahraman. *Fuzzy multi-criteria decision making: theory and applications with recent developments*, volume 16. Springer Science & Business Media, 2008.

[316] Shigeo Abe. *Neural networks and fuzzy systems: theory and applications*. Springer Science & Business Media, 2012.

[317] Nancy El-Hefenawy, Mohamed A Metwally, Zenat M Ahmed, and Ibrahim M El-Henawy. A review on the applications of neutrosophic sets. *Journal of Computational and Theoretical Nanoscience*, 13(1):936–944, 2016.

[318] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[319] Said Broumi, Assia Bakali, and Ayoub Bahnasse. Neutrosophic sets: An overview. *Infinite Study*, 2018.

[320] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[321] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[322] Florentin Smarandache. Interval-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 117, 2022.

[323] AA Salama and SA Alblowi. Neutrosophic set and neutrosophic topological spaces. *IOSR Journal of Mathematics*, 2012.

[324] I Arokiarani, R Dhavaseelan, S Jafari, and M Parimala. On some new notions and functions in neutrosophic topological spaces. *Neutrosophic sets and systems*, 16:16–19, 2017.

[325] Shawqi Al-lkami and A Al_odhari. Characteristics neutrosophic homomorphism for neutrosophic rings: On review. *Prospects for Applied Mathematics and data Analysis (PAMDA)*, 3(1):63–71, 2024.

[326] Hasan Sankari and Mohammad Abobala. *AH-homomorphisms in neutrosophic rings and refined neutrosophic rings*, volume 38. Infinite Study, 2020.

[327] SA Edalatpanah. Systems of neutrosophic linear equations. *Neutrosophic sets and systems*, 33(1):92–104, 2020.

[328] AA Elsayed and AK Kolankeh. Solving single-valued trapezoidal neutrosophic linear equations: An embedding approach for uncertain systems. *J. Oper. Strateg Anal*, 3(1):1–13, 2025.

[329] S Sudha, B Felcia Merlin, B Shoba, and A Rajkumar. Quadripartitioned neutrosophic probability distributions. *International Journal of Neutrosophic Science (IJNS)*, 25(2), 2025.

[330] Rafif Alhabib, Moustafa Mzher Ranna, Haitham Farah, and AA Salama. *Some neutrosophic probability distributions*. Infinite Study, 2018.

[331] Florentin Smarandache and Yanhui Guo. *New Development of Neutrosophic Probability, Neutrosophic Statistics, Neutrosophic Algebraic Structures, and Neutrosophic Plithogenic Optimizations*. Infinite Study, 2022.

[332] Umberto Rivieccio. Neutrosophic logics: Prospects and problems. *Fuzzy sets and systems*, 159(14):1860–1868, 2008.

[333] Charles Ashbacher. *Introduction to Neutrosophic logic*. Infinite Study, 2014.

[334] WB Vasantha Kandasamy and Florentin Smarandache. *Some neutrosophic algebraic structures and neutrosophic N-algebraic structures*. Infinite Study, 2006.

[335] Riad K Al-Hamido. A new neutrosophic algebraic structures. *Journal of Computational and cognitive Engineering*, 2(2):150–154, 2023.

[336] AAA Agboola and SA Akinleye. Neutrosophic vector spaces. *Neutrosophic Sets and Systems*, 4:9–18, 2014.

[337] Muritala A Ibrahim, BS Badmus, SA Akinleye, et al. On refined neutrosophic vector spaces i. *International Journal of Neutrosophic Science*, 7:97–109, 2020.

[338] Mohammad Abobala. *Ah-subspaces in neutrosophic vector spaces*. Infinite Study, 2020.

[339] Takaaki Fujita. Neutrosophic matroids, antimatroids, and greedoids: A unified framework for combinatorial structures under uncertainty. *Abhath Journal of Basic and Applied Sciences*, 4(1):73–93, 2025.

[340] Takaaki Fujita. Short note of neutrosophic closure matroids. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond: Third volume*, 146, 2024.

[341] Florentin Smarandache. *Real Examples of NeutroGeometry & AntiGeometry*. Infinite Study, 2023.

[342] Erick Gonzalez-Caballero. *Applications of NeutroGeometry and AntiGeometry in Real World*. Infinite Study, 2023.

[343] Florentin Smarandache. *NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries (revisited)*, volume 5. Infinite Study, 2021.

[344] Carmen Verónica Valenzuela Chicaiza, Olga Germania Arciniegas Paspuel, Paola Yesenia, Carrera Cuesta, and Sary Del Rocío Álvarez Hernández. Neutrosophic psychology for emotional intelligence analysis in students of the autonomous university of los andes, ecuador. *Neutrosophic Sets and Systems*, 34:1–8, 2020.

[345] Edward David Ticona Ponce, Adán Humberto Estela Estela, Manuel Cesar Vildoso Villega, Pierre Sean Rojas Padilla, Virgilio Vildoso Gonzales, Hortencio Flores Flores, and Carlos Fretel Martínez. Neutrosophic psychology of teachers: Addressing perceptions of uncertainty and motivation in the implementation of thinking strategies. *Neutrosophic Sets and Systems*, 84:623–635, 2025.

[346] Giselle Rodríguez Caballero, Martha Silvia Mancebo Calzado, Elsy Rodríguez Revelo, Lissett Aide Perea Tenorio, and Cecilia Denisse Anangonó Corozo. Neutrosophic psychology in playful spelling learning: Understanding cognitive and emotional dynamics in spelling competence. *Neutrosophic Sets and Systems*, 92:559–569, 2025.

[347] Takaaki Fujita and Florentin Smarandache. *A Review and Introduction to Neutrosophic Applications across Various Scientific Fields*. Infinite Study, 2025.

[348] Florentin Smarandache. *Neutropsychic Personality: A mathematical approach to psychology*. Infinite Study, 2018.

[349] Seher Bodur, Selçuk Topal, Hacı Gürkan, and Seyyed Ahmad Edalatpanah. A novel neutrosophic likert scale analysis of perceptions of organizational distributive justice via a score function: a complete statistical study and symmetry evidence using real-life survey data. *Symmetry*, 16(5):598, 2024.

[350] Iyo A Cruz Piza, Manuel Ricardo Velásquez, Javier D Bosquez Remache, and Marina Abdurashidova. Neutrosophic likert scale for evaluating environmental sanctions. *International Journal of Neutrosophic Science (IJNS)*, 24(1), 2024.

[351] Carlos Jacinto La Rosa Longobardi, Livia Cristina Piñas Rivera, Lida Violeta Asencios Trujillo, Djamila Gallegos Espinoza, Juan Víctor Soras Valdivia, Felipe Aguirre Chávez, and Elizabeth Gladys Garro Palomino. Factors affecting educational quality: A study using neutrosophic likert scales and fuzzy set qualitative comparative analysis. *Neutrosophic Sets and Systems*, 71(1):4, 2024.

[352] Adrián Alejandro Alvaracín Jarrín, David Santiago Proaño Tamayo, Salomón Alejandro Montecé Giler, Juan Carlos Arandia Zambrano, and Dante Manuel Macazana. Neutrosophic statistics applied in social science. *Neutrosophic Sets and Systems*, 44(1):1, 2021.

[353] Florentin Smarandache. *Neutrosophic Physics: More Problems, More Solutions*. North-European Scientific Publishers, Hanko, Finland, 2010.

[354] Florentin Smarandache. *Introduction to Neutrosophic Sociology (Neutrosociology)*. Infinite Study, 2019.

[355] Anil Kumar Ram, Anupam K Singh, and Bikky Kumar. Neutrosophic automata and reverse neutrosophic automata. *Neutrosophic Sets and Systems*, 63:219–238, 2024.

[356] Anil Kr Ram and Anupam K Singh. A neutrosophic topology for neutrosophic automata. *SuperHyper-Topologies and SuperHyperStructures with their Applications*, page 253, 2025.

[357] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[358] Ruxiang Liu. Study on single-valued neutrosophic graph with application in shortest path problem. *CAAI Transactions on Intelligence Technology*, 5(4):308–313, 2020.

[359] Faruk Karaaslan and Bijan Davvaz. Properties of single-valued neutrosophic graphs. *Journal of Intelligent & Fuzzy Systems*, 34(1):57–79, 2018.

[360] Muhammad Akram and Muzzamal Sitara. Novel applications of single-valued neutrosophic graph structures in decision-making. *Journal of Applied Mathematics and Computing*, 56:501–532, 2018.

[361] Muhammad Akram, Musavarah Sarwar, Wieslaw A Dudek, Muhammad Akram, Musavarah Sarwar, and Wieslaw A Dudek. Bipolar neutrosophic graph structures. *Graphs for the Analysis of Bipolar Fuzzy Information*, pages 393–446, 2021.

[362] Prem Kumar Singh. Interval-valued neutrosophic graph representation of concept lattice and its (α, β, γ)-decomposition. *Arabian Journal for Science and Engineering*, 43(2):723–740, 2018.

[363] Muhammad Akram, Sundas Shahzadi, and AB Saeid. Single-valued neutrosophic hypergraphs. *TWMS Journal of Applied and Engineering Mathematics*, 8(1):122–135, 2018.

[364] S Satham Hussain, R Jahir Hussain, and M Vignesh Babu. Neutrosophic vague incidence graph. *International Journal of Neutrosophic Science*, 12(1):29–38, 2020.

[365] Said Broumi, D Ajay, P Chellamani, Lathamaheswari Malayalan, Mohamed Talea, Assia Bakali, Philippe Schweizer, and Saeid Jafari. Interval valued pentapartitioned neutrosophic graphs with an application to mcdm. *Operational Research in Engineering Sciences: Theory and Applications*, 5(3):68–91, 2022.

[366] Muhammad Aslam Malik, Ali Hassan, Said Broumi, Assia Bakali, Mohamed Talea, and Florentin Smarandache. Isomorphism of bipolar single valued neutrosophic hypergraphs. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 72, 2022.

[367] Muhammad Akram and Anam Luqman. Intuitionistic single-valued neutrosophic hypergraphs. *Opsearch*, 54:799–815, 2017.

[368] P Chitra Devi and P Anitha. *Neutrosophic Bipolar Vague Line Graph*. Infinite Study, 2021.

[369] S Satham Hussain, R Jahir Hussain, and Florentin Smarandache. *On neutrosophic vague graphs*. Infinite Study, 2019.

[370] Muhammad Akram, Hafsa M Malik, Sundas Shahzadi, and Florentin Smarandache. Neutrosophic soft rough graphs with application. *Axioms*, 7(1):14, 2018.

[371] Muhammad Akram, Sidra Sayed, and Florentin Smarandache. Neutrosophic incidence graphs with application. *Axioms*, 7(3):47, 2018.

[372] Prasanta Kumar Raut, Surapati Pramanik, and Bhabani S Mohanty. An overview of fermatean neutrosophic graphs. *Neutrosophic Sets and Systems*, 82(1):40, 2025.

[373] D Sasikala and B Divya. A newfangled interpretation on fermatean neutrosophic dombi fuzzy graphs. *Neutrosophic Systems With Applications*, 7:36–53, 2023.

[374] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.

[375] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, and Haitham Rizk Fadlallah. *MADM Strategy Application of Bipolar Single Valued Heptapartitioned Neutrosophic Set*. Infinite Study, 2023.

[376] Suman Das, Rakhal Das, and Surapati Pramanik. Single valued pentapartitioned neutrosophic graphs. *Neutrosophic Sets and Systems*, 50(1):225–238, 2022.

[377] M Kaviyarasu, Hossein Rashmanlou, Saeed Kosari, Said Broumi, R Venitha, M Rajeshwa, and Farshid Mofidnakhaei. Circular economy strategies to promote sustainable development using t-neutrosophic fuzzy graph. *Neutrosophic Sets and Systems*, 72:186–210, 2024.

[378] Murugan Kaviyarasu, Luminiţa-Ioana Cotîrlă, Daniel Breaz, Murugesan Rajeshwari, and Eleonora Rapeanu. A study on complex t-neutrosophic graph with intention to preserve biodiversity. *Symmetry*, 16(8):1033, 2024.

[379] Muhammad Akram. Graphs under neutrosophic environment. In *Single-Valued Neutrosophic Graphs*, pages 1–81. Springer, 2018.

[380] S Sivasankar and Said Broumi. Balanced neutrosophic graphs. *Neutrosophic Sets and Systems*, 50:309–319, 2022.

[381] Xiao Qun Liao, Tong Su, and Li Ma. Application of neutrosophic minimum spanning tree in electrical power distribution network. *CAAI Transactions on Intelligence Technology*, 5(2):99–105, 2020.

[382] Takaaki Fujita. Hyperfuzzy graph neural networks and hyperplithogenic graph neural networks: Theoretical foundations. In *SuperHyperTopologies and SuperHyperStructures with their Applications*. Biblio Publishing, 2025.

[383] Muhammad Akram and Anam Luqman. Certain networks models using single-valued neutrosophic directed hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 33(1):575–588, 2017.

[384] Takaaki Fujita and Florentin Smarandache. *Neutrosophic soft n-super-hypergraphs with real-world applications*. Infinite Study, 2025.

[385] Juanjuan Ding, Wenhui Bai, and Chao Zhang. A new multi-attribute decision making method with single-valued neutrosophic graphs. *International Journal of Neutrosophic Science*, 2021.

[386] M Hamidi and A Borumand Saeid. Accessible single-valued neutrosophic graphs. *Journal of Applied Mathematics and Computing*, 57:121–146, 2018.

[387] S Satham Hussain, N Durga, Rahmonlou Hossein, and Ghorai Ganesh. New concepts on quadripartitioned single-valued neutrosophic graph with real-life application. *International Journal of Fuzzy Systems*, 24(3):1515–1529, 2022.

[388] Muhammad Akram and Hafiza Saba Nawaz. Implementation of single-valued neutrosophic soft hypergraphs on human nervous system. *Artificial Intelligence Review*, 56(2):1387–1425, 2023.

[389] Muhammad Akram and Hafiza Saba Nawaz. Algorithms for the computation of regular single-valued neutrosophic soft hypergraphs applied to supranational asian bodies. *Journal of Applied Mathematics and Computing*, 68(6):4479–4506, 2022.

[390] Shouxian Zhu. Neutrosophic n-superhypernetwork: A new approach for evaluating short video communication effectiveness in media convergence. *Neutrosophic Sets and Systems*, 85:1004–1017, 2025.

[391] Vakkas Uluçay, Mehmet Sahin, Said Broumi, Assia Bakali, Mohamed Talea, and Florentin Smarandache. Decision-making method based on neutrosophic soft expert graphs. *viXra*, pages 33–76, 2016.

[392] Muhammad Akram and Musavarah Sarwar. Novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. *viXra*, 2017.

[393] Jianming Zhan, Muhammad Akram, and Muzzamal Sitara. Novel decision-making method based on bipolar neutrosophic information. *Soft Computing*, 23:9955 – 9977, 2018.

[394] Said Broumi, Swaminathan Mohanaselvi, Tomasz Witczak, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Complex fermatean neutrosophic graph and application to decision making. *Decision Making: Applications in Management and Engineering*, 2023.

[395] Shio Gai Quek, Ganeshsree Selvachandran, D Ajay, P Chellamani, David Taniar, Hamido Fujita, Phet Duong, Le Hoang Son, and Nguyen Long Giang. New concepts of pentapartitioned neutrosophic graphs and applications for determining safest paths and towns in response to covid-19. *Computational and Applied Mathematics*, 41(4):151, 2022.

[396] Mohammed Alqahtani, M Kaviyarasu, Anas Al-Masarwah, and M Rajeshwari. Application of complex neutrosophic graphs in hospital infrastructure design. *Mathematics*, 12(5):719, 2024.

[397] Dapeng Oliver Wu, Yiwei Thomas Hou, Wenwu Zhu, Ya-Qin Zhang, and Jon M. Peha. Streaming video over the internet: approaches and directions. *IEEE Trans. Circuits Syst. Video Technol.*, 11:282–300, 2001.

[398] Murugan Kaviyarasu. Unveiling internet streaming services: A comparison using neutrosophic graphs. *Fuzzy Information and Engineering*, 2024.

[399] David Alexander. *Natural disasters*. Routledge, 2018.

[400] Okada Norio, Tao Ye, Yoshio Kajitani, Peijun Shi, and Hirokazu Tatano. The 2011 eastern japan great earthquake disaster: Overview and comments. *International Journal of Disaster Risk Science*, 2:34–42, 2011.

[401] Wadei Faris AL-Omeri and M Kaviyarasu. Study on neutrosophic graph with application on earthquake response center in japan. *Symmetry*, 16(6):743, 2024.

[402] Xiaolong Shi, Saeed Kosari, Saira Hameed, Abdul Ghafar Shah, and Samee Ullah. Application of connectivity index of cubic fuzzy graphs for identification of danger zones of tsunami threat. *Plos one*, 19(1):e0297197, 2024.

[403] Said Broumi, Kifayat Ullah, Assia Bakali, Mohamed Talea, Prem Kumar Singh, Tahir Mahmood, Florentin Smarandache, Ayoub Bahnasse, Santanu Kumar Patro, and Angelo de Oliveira. Novel system and method for telephone network planing based on neutrosophic graph. *Infinite Study*, 10(4):403–434, 2018.

[404] Mohammad Hamidi and Arsham Borumand Saeid. Achievable single-valued neutrosophic graphs in wireless sensor networks. *New Mathematics and Natural Computation*, 14(02):157–185, 2018.

[405] Said Broumi, Prem Kumar Singh, Mohamed Talea, Assia Bakali, Florentin Smarandache, and V. Venkateswara Rao. Single-valued neutrosophic techniques for analysis of wifi connection. *Advances in Intelligent Systems and Computing*, 2018.

[406] A Meenakshi, J Shivangi Mishra, Jeong Gon Lee, Antonios Kalampakas, and Sovan Samanta. Advanced risk prediction in healthcare: Neutrosophic graph neural networks for disease transmission. *Complex & Intelligent Systems*, 11(9):413, 2025.

[407] B Kalaiselvi, K Sivakumar, K Selvakumari, VR Pathmavathi, P Kalarani, and A Kesavan. Deep neural network-enhanced trapezoidal neutrosophic shortest-path optimization in uncertain graphs. In *2025 3rd International Conference on Sustainable Computing and Data Communication Systems (ICSCDS)*, pages 1988–1993. IEEE, 2025.

[408] Yasmine M Ibrahim, Reem Essameldin, and Saad M Darwish. An adaptive hate speech detection approach using neutrosophic neural networks for social media forensics. *Computers, Materials & Continua*, 79(1), 2024.

[409] OM Khaled, AA Salama, Mostafa Herajy, MM El-Kirany, Huda E Khalid, Ahmed K Essa, and Ramiz Sabbagh. A novel approach for cyber-attack detection in iot networks with neutrosophic neural networks. *Neutrosophic Sets and Systems*, 86(1):48, 2025.

[410] Shabbir Hassan. Machine learning vs. neutrosophic machine learning. In *Soft Computing and Machine Learning*, pages 102–135. CRC Press, 2025.

[411] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[412] Muhammad Arshad, Atiqe Ur Rahman, and Muhammad Saeed. An abstract approach to convex and concave sets under refined neutrosophic set environment. *Neutrosophic sets and systems*, 53(1):17, 2023.

[413] Hasan Sankari and Mohammad Abobala. *n-Refined neutrosophic modules*, volume 36. Infinite Study, 2020.

[414] Ilanthenral Kandasamy, WB Vasantha, Jagan M Obbineni, and Florentin Smarandache. Sentiment analysis of tweets using refined neutrosophic sets. *Computers in Industry*, 115:103180, 2020.

[415] Irfan Deli. Refined neutrosophic sets and refined neutrosophic soft sets: theory and applications. In *Handbook of research on generalized and hybrid set structures and applications for soft computing*, pages 321–343. IGI Global, 2016.

[416] Florentin Smarandache. *Nidus Idearum. Scilogs, I: de neutrosophia*, volume 1. Infinite Study, 2016.

[417] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Plithogenic Graphs*. Infinite Study, 2020.

[418] Muhammad Azeem, Humera Rashid, Muhammad Kamran Jamil, Selma Gütmen, and Erfan Babaee Tirkolaee. Plithogenic fuzzy graph: A study of fundamental properties and potential applications. *Journal of Dynamics and Games*, pages 0–0, 2024.

[419] Takaaki Fujita. Short survey on the hierarchical uncertainty of fuzzy, neutrosophic, and plithogenic sets. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 285, 2025.

[420] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. N-hyper sets. *Mathematics*, 6(6):87, 2018.

[421] Witold Pedrycz. Shadowed sets: representing and processing fuzzy sets. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 28(1):103–109, 1998.

[422] Mingjie Cai, Qingguo Li, and Guangming Lang. Shadowed sets of dynamic fuzzy sets. *Granular Computing*, 2:85 – 94, 2016.

[423] Gianpiero Cattaneo and Davide Ciucci. Shadowed sets and related algebraic structures. *Fundamenta Informaticae*, 55(3-4):255–284, 2003.

[424] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[425] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[426] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[427] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pages 305–309, 1994.

[428] Yangxue Li and Yong Deng. Intuitionistic evidence sets. *IEEE Access*, 7:106417–106426, 2019.

[429] Yige Xue and Yong Deng. On the conjunction of possibility measures under intuitionistic evidence sets. *Journal of Ambient Intelligence and Humanized Computing*, 12(7):7827–7836, 2021.

[430] Jiawei Zou. Decision-making under extremely uncertain environments based on intuitionistic evidence set. In *2022 7th International Conference on Cloud Computing and Big Data Analytics (ICCCBDA)*, pages 73–80. IEEE, 2022.

[431] Chen Shouyu and Guo Yu. Variable fuzzy sets and its application in comprehensive risk evaluation for flood-control engineering system. *Fuzzy Optimization and Decision Making*, 5(2):153–162, 2006.

[432] Wen-chuan Wang, Dong-mei Xu, Kwok-wing Chau, and Guan-jun Lei. Assessment of river water quality based on theory of variable fuzzy sets and fuzzy binary comparison method. *Water resources management*, 28(12):4183–4200, 2014.

[433] Yunhai Fang, Xilai Zheng, Hui Peng, Huan Wang, and Jia Xin. A new method of the relative membership degree calculation in variable fuzzy sets for water quality assessment. *Ecological indicators*, 98:515–522, 2019.

[434] Vasile Patrascu. A new penta-valued logic based knowledge representation. *arXiv preprint arXiv:1502.05562*, 2015.

[435] Vasile Patrascu. Similarity, cardinality and entropy for bipolar fuzzy set in the framework of penta-valued representation. *arXiv preprint arXiv:1506.02060*, 2015.

[436] Yong-Uk Cho, Young-Bae Jun, and Seok-Zun Song. Bifuzzy ideals of pseudo mv-algebras. *JOURNAL OF APPLIED MATHEMATICS AND COMPUTING*, 22(1/2):475, 2006.

[437] Tadeusz Gerstenkorn and Jacek Mańko. Bifuzzy probabilistic sets. *Fuzzy sets and systems*, 71(2):207–214, 1995.

[438] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[439] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[440] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a γ-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[441] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, 2019.

[442] M Murali Krishna Rao. Tripolar fuzzy interior ideals of γ-semigroup. *Ann. Fuzzy Math. Inform*, 15(2):199–206, 2018.

[443] Xue Rong Zhao and Yiyu Yao. Three-way fuzzy partitions defined by shadowed sets. *Information Sciences*, 497:23–37, 2019.

[444] Qingqing Hu and Xiaohong Zhang. Three-way fuzzy sets and their applications (iii). *Axioms*, 12(1):57, 2023.

[445] Bui Cong Cuong. Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, 30:409, 2015.

[446] Fatma Kutlu Gündoğdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *Journal of intelligent & fuzzy systems*, 36(1):337–352, 2019.

[447] Chris J Hinde, Robert Steven Patching, and Stephen A McCoy. Semantic transfer and contradictory evidence in intuitionistic fuzzy sets. In *2008 IEEE International Conference on Fuzzy Systems (IEEE World Congress on Computational Intelligence)*, pages 2095–2102. IEEE, 2008.

[448] CJ Hinde, RS Patching, and SA McCoy. Inconsistent intuitionistic fuzzy sets and mass assignment. *Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Foundations*, pages 133–153, 2007.

[449] Chao Wang, Minghu Ha, and Xiaowei Liu. A mathematical model of ternary fuzzy set for voting. *Journal of Intelligent & Fuzzy Systems*, 29(6):2381–2386, 2015.

[450] M Kavitha and R Irene Hepzibah. New operations on pythagorean neutrosophic fuzzy sets. *TWMS Journal of Applied and Engineering Mathematics*, 2025.

[451] Sujit Das, Bikash Koli Roy, Mohuya B Kar, Samarjit Kar, and Dragan Pamučar. Neutrosophic fuzzy set and its application in decision making. *Journal of Ambient Intelligence and Humanized Computing*, 11(11):5017–5029, 2020.

[452] Melvin Fitting. Kleene's three valued logics and their children. *Fundamenta informaticae*, 20(1-3):113–131, 1994.

[453] John T Kearns. The strong completeness of a system for kleene's three-valued logic. *Mathematical Logic Quarterly*, 25(3-6):61–68, 1979.

[454] Ashraf Al-Quran, Hazwani Hashim, and Lazim Abdullah. A hybrid approach of interval neutrosophic vague sets and dematel with new linguistic variable. *Symmetry*, 12(2):275, 2020.

[455] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.

[456] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[457] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[458] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[459] Bin Zhu and Zeshui Xu. Some results for dual hesitant fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 26(4):1657–1668, 2014.

[460] José Carlos R Alcantud, Gustavo Santos-García, Xindong Peng, and Jianming Zhan. Dual extended hesitant fuzzy sets. *Symmetry*, 11(5):714, 2019.

[461] Florentin Smarandache. Ambiguous set is a subclass of the double refined indeterminacy neutrosophic set, and of the refined neutrosophic set in general. *Neutrosophic Sets & Systems*, 58, 2023.

[462] Pritpal Singh et al. Ambiguous set theory: A new approach to deal with unconsciousness and ambiguousness of human perception. *Full Length Article*, 5(1):52–2, 2023.

[463] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16, 2023.

[464] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[465] Nguyen Xuan Thao, Florentin Smarandache, and Nguyen Van Dinh. *Support-neutrosophic set: a new concept in soft computing*. Infinite Study, 2017.

[466] Yuri Kaluzhny and Alexei Yu Muravitsky. A knowledge representation based on the belnap's four-valued logic. *Journal of Applied Non-Classical Logics*, 3(2):189–203, 1993.

[467] Marcelo E Coniglio and Abilio Rodrigues. From belnap-dunn four-valued logic to six-valued logics of evidence and truth. *Studia Logica*, 112(3):561–606, 2024.

[468] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[469] Takaaki Fujita. Review of rough turiyam neutrosophic directed graphs and rough pentapartitioned neutrosophic directed graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:48–79, 2025.

[470] Takaaki Fujita and Florentin Smarandache. Pythagorean, fermatean, and complex turiyam neutrosophic graphs. *SciNexuses*, 2:39–63, 2025.

[471] Prem Kumar Singh, Naveen Surathu, Ghattamaneni Surya Prakash, et al. Turiyam based four way unknown profile characterization on social networks. *Full Length Article*, 10(2):27–7, 2024.

[472] Rama Mallick and Surapati Pramanik. Pentapartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 35:49, 2020.

[473] Radha R. and Stanis Arul Mary. Pentapartitioned neutrosophic pythagorean set. *International Research Journal on Advanced Science Hub*, 2021.

[474] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.

[475] Jiarong Jia. Triple-valued neutrosophic off for public digital cultural service level evaluation in craft and art museums. *Neutrosophic Sets and Systems*, 88(1):44, 2025.

[476] Takaaki Fujita. Triple-valued neutrosophic set, quadruple-valued neutrosophic set, quintuple-valued neutrosophic set, and double-valued indetermsoft set. *Neutrosophic Systems with Applications*, 25(5):3, 2025.

[477] Hongxin Wang. Professional identity formation in traditional chinese medicine students: An educational perspective using triple-valued neutrosophic set. *Neutrosophic Sets and Systems*, 88:83–92, 2025.

[478] Fang Sun, Mifeng Ren, Yujing Shi, and Xuanbai Feng. Quadruple-valued neutrosophic offset for maintenance platforms design quality evaluation of digital monitoring and remote operation in new energy power generation systems. *Neutrosophic Sets and Systems*, 88:940–953, 2025.

[479] Wenwen Meng. Quintuple-valued neutrosophic offset for quality evaluation of cross-border e-commerce talent training based on artificial intelligence. *Neutrosophic Sets and Systems*, 88(1):54, 2025.

[480] Yuan Sun. Assessing the performance of basic education informatization in the digital age using quintuple-valued neutrosophic set. *Neutrosophic Sets and Systems*, 88(1):18, 2025.

[481] Takaaki Fujita. Advanced partitioned neutrosophic sets: Formalization of hexa-, hepta-, octa-, nona-, and deca-partitioned structures. *Abhath Journal of Basic and Applied Sciences*, 4(2):40–60, 2025.

[482] Muhammad Arshad, Muhammad Saeed, and Atiqe Ur Rahman. Convexity cum concavity on refined fuzzy set with some properties. *Uncertainty Discourse and Applications*, 1(1):140–150, 2024.

[483] Ryszard Wójcicki. A theorem of the finiteness of the degree of maximality of the n-valued lukasiewicz logic. *Bulletin of the Section of Logic*, 4(1):19–23, 1975.

[484] Giovanni Panti. Multi-valued logics. In *Quantified representation of uncertainty and imprecision*, pages 25–74. Springer, 1998.

[485] Fujita Takaaki and Arif Mehmood. Iterative multifuzzy set, iterative multineutrosophic set, iterative multisoft set, and multiplithogenic sets. *Neutrosophic Computing and Machine Learning*, 41:1–30, 2025.

[486] Florentin Smarandache. *Introduction to the multineutrosophic set*. Infinite Study, 2023.

[487] Raúl González Salas, Jhael Mikaela Gómez Ashqui, and Marco Paúl Medina Valencia. Assessing viral susceptibility in tilapia species using plithogenic fuzzy soft set: a multi-expert approach for tilv. *Neutrosophic Sets and Systems*, 89:185–196, 2025.

[488] Manuel Michael Beraún-Espíritu, Ketty Marilú Moscoso-Paucarchuco, Humberto Dax Bonilla-Mancilla, Edson Hilmer Julca-Marcelo, Anani Basaldua Galarza, Jean Pierre Espeza-Gavilán, Lizve Vilcapoma Ureta, and Jacqueline Denisse Llacza-Molina. Environmental sustainability and organizational commitment in rainbow trout production in junín, peru: An approach from plithogenic fuzzy soft sets. *Neutrosophic Sets and Systems*, 84(1):33, 2025.

[489] SP Priyadharshini and F Nirmala Irudayam. *A New Approach of Multi-Dimensional Single Valued Plithogenic Neutrosophic Set in Multi Criteria Decision Making*. Infinite Study, 2021.

[490] Ruixuan Chen, Qun Liu, Yucang Zhang, and Wuyin Weng. A probabilistic plithogenic neutrosophic rough set for uncertainty-aware food safety analysis. *Neutrosophic Sets & Systems*, 90, 2025.

[491] V. S. Subha and R. Selvakumar. A novel approach to plithogenic neutrosophic hypersoft rough set and its application to decision making problem with reference to bakery industry. *Communications in Mathematics and Applications*, 2024.

[492] Florentin Smarandache. Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set), pythagorean fuzzy set, spherical fuzzy set, and q-rung orthopair fuzzy set, while neutrosophication is a generalization of regret theory, grey system theory, and three-ways decision (revisited). *Journal of New Theory*, 29:1–31, 2019.

[493] Florentin Smarandache. *Nidus Idearum. Scilogs, XIII: Structure/NeutroStructure/AntiStructure*. Infinite Study, 2024.

[494] Hossein Rashmanlou, Sovan Samanta, Madhumangal Pal, and Rajab Ali Borzooei. Intuitionistic fuzzy graphs with categorical properties. *Fuzzy information and Engineering*, 7(3):317–334, 2015.

[495] Zehui Shao, Saeed Kosari, Muhammad Shoaib, and Hossein Rashmanlou. Certain concepts of vague graphs with applications to medical diagnosis. *Frontiers in physics*, 8:357, 2020.

[496] Sundas Shahzadi and Muhammad Akram. Coloring of bifuzzy graphs. *Ital. J. Pure Appl. Math*, 36:429–444, 2016.

[497] Muhammad Akram and Karamat H Dar. Interval-valued bifuzzy graphs. In *Proceedings Of The International Conference On Algebra 2010: Advances in Algebraic Structures*, pages 1–10. World Scientific, 2012.

[498] Xinyu Guo, Bingjie Tian, and Xuedong Tian. Hfgnn-proto: Hesitant fuzzy graph neural network-based prototypical network for few-shot text classification. *Electronics*, 11(15):2423, 2022.

[499] Waqar Arif, Waheed Ahmad Khan, Asghar Khan, and Hossein Rashmanlou. Some indices of picture fuzzy graphs and their applications. *Computational and Applied Mathematics*, 42:1–23, 2023.

[500] M Ramya, S Murali, R Radha, and R Princy. A note on different types of product of fermatean quadripartitioned neutrosophic graphs. *Neutrosophic Sets and Systems*, 97:237–264, 2026.

[501] V Divya and J Jesintha Rosline. Fermatean quadripartitioned neutrosophic fuzzy graph. *Reliability: Theory & Applications*, 20(3 (86)):816–828, 2025.

[502] Sumera Naz, Muhammad Akram, Samirah Alsulami, and Faiza Ziaa. Decision-making analysis under interval-valued q-rung orthopair dual hesitant fuzzy environment. *Int. J. Comput. Intell. Syst.*, 14(1):332–357, 2021.

[503] Takaaki Fujita and Florentin Smarandache. *A compact exploration of turiyam neutrosophic competition graphs*. Infinite Study, 2024.

[504] M Myvizhi, Ahmed Abdelhafeez, Mohamed Refaat Abdellah, Mohamed Eassa, Mohamed S Sawah, Hussam Elbehiery, et al. Single-valued neutrosophic graph with heptapartitioend structure. *Neutrosophic Sets and Systems*, 80:728–748, 2025.

[505] R Muthuraj, K Krithika, and S Revathi. Various product on multi fuzzy graphs. *Ratio Mathematica*, 44:231, 2022.

[506] Takaaki Fujita. General plithogenic soft rough graphs and some related graph classes. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 437, 2024.

[507] Parimala Mani, Ibtesam Alshammari, and Halimah Alshehri. Shortest path algorithm of a network via spherical fuzzy digraphs. *Journal of Engineering Research*, 9, 2021.

[508] B Mohamed Harif and A Nazeera Begam. On regular spherical fuzzy graphs. *Journal of Hyperstructures*, 12(2):244–256, 2023.

[509] Faruk Karaaslan and Fatih Karamaz. Some new product operations of t-spherical fuzzy graphs and an application of t-spherical fuzzy graphs in mcgdm. *Artificial Intelligence Review*, 56(11):13663–13689, 2023.

[510] P Chellamani and D Ajay. Spherical fuzzy labelling graphs. In *Journal of Physics: Conference Series*. IOP Publishing, 2021.

[511] P. Chellamani and D. Ajay. Spherical fuzzy labelling graphs. *Journal of Physics: Conference Series*, 1964, 2021.

[512] Parimala Mani, Ibtesam Alshammari, and Halimah Alshehri. Shortest path algorithm of a network via spherical fuzzy digraphs. *Journal of Engineering Research*, 2021.

[513] Khushbakhat Asif, Muhammad Kamran Jamil, Hanen Karamti, Muhammad Azeem, and Kifayat Ullah. Randić energies for t-spherical fuzzy hamacher graphs and their applications in decision making for business plans. *Computational and Applied Mathematics*, 42:1–32, 2023.

[514] B. Mohamed Harif and Nazeera Begam. On regular spherical fuzzy graph. *Journal of Hyperstructures*, 12(2):244–256, 2023.

[515] T. Lavanya, M. Rowthri, and D. Amsaveni. Spherical fuzzy cycle and spherical fuzzy tree. *Journal of Computational Analysis and Applications (JoCAAA)*, 33(05):610–615, 2024.

[516] B. Mohamed Harif and A. Nazeera Begam. On pseudo regular spherical fuzzy graphs. *Ratio Mathematica*, 50, 2023.

[517] K. Akalyadevi, C. Antony Crispin Sweety, and A. R. Sudamani Ramaswamy. Spherical neutrosophic graph coloring. *RECENT TRENDS IN SCIENCE AND ENGINEERING*, 2022.

[518] Muhammad Akram and Sumera Naz. Energy of pythagorean fuzzy graphs with applications. *Mathematics*, 6(8):136, 2018.

[519] Muhammad Akram, Farwa Ilyas, and Arsham Borumand Saeid. Certain notions of pythagorean fuzzy graphs. *Journal of Intelligent & Fuzzy Systems*, 36(6):5857–5874, 2019.

[520] Rajkumar Verma, José M Merigó, and Manoj Sahni. Pythagorean fuzzy graphs: some results. *arXiv preprint arXiv:1806.06721*, 2018.

[521] Gulfam Shahzadi, Muhammad Akram, and Bijan Davvaz. Pythagorean fuzzy soft graphs with applications. *Journal of Intelligent & Fuzzy Systems*, 38(4):4977–4991, 2020.

[522] Muhammad Akram and Ayesha Khan. Complex pythagorean dombi fuzzy graphs for decision making. *Granular Computing*, 6:645–669, 2021.

[523] Muhammad Akram and Ayesha Khan. Complex pythagorean dombi fuzzy graphs for decision making. *Granular Computing*, 6:645–669, 2021.

[524] Muhammad Akram, Sumera Naz, and Bijan Davvaz. Simplified interval-valued pythagorean fuzzy graphs with application. *Complex & intelligent systems*, 5:229–253, 2019.

[525] Suber Bathusha SN and S Angelin Kavitha Raj. Climatic analysis based on interval-valued complex pythagorean fuzzy graph structure. *Annals of Mathematics and Computer Science*, 19:10–33, 2023.

[526] Muhammad Akram, Jawaria Mohsan Dar, and Sumera Naz. Pythagorean dombi fuzzy graphs. *Complex & Intelligent Systems*, 6:29–54, 2020.

[527] D Ajay, P Chellamani, et al. Pythagorean neutrosophic fuzzy graphs. *International journal of neutrosophic science*, 11(2):108–114, 2020.

[528] Muhammad Akram, Ayesha Bashir, and Sovan Samanta. Complex pythagorean fuzzy planar graphs. *International Journal of Applied and Computational Mathematics*, 6, 2020.

[529] P Chellamani and D Ajay. Pythagorean neutrosophic dombi fuzzy graphs with an application to mcdm. *Neutrosophic Sets and Systems*, 47:411–431, 2021.

[530] Muhammad Akram, Uzma Ahmad, Rukhsar, and Faruk Karaaslan. Complex pythagorean fuzzy threshold graphs with application in petroleum replenishment. *Journal of Applied Mathematics and Computing*, 68:2125–2150, 2021.

[531] Muhammad Akram and Gulfam Shahzadi. Decision-making approach based on pythagorean dombi fuzzy soft graphs. *Granular Computing*, 6:671 – 689, 2020.

[532] Ayesha Shareef, Uzma Ahmad, Saba Siddique, and Mohammed M. Ali Al-Shamiri. Pythagorean fuzzy incidence graphs with application in illegal wildlife trade. *AIMS Mathematics*, 2023.

[533] Muhammad Akram, Ayesha Shareef, and Ahmad N. Al-kenani. Pythagorean fuzzy incidence graphs with application in one-way toll road network. *Granular Computing*, 2024.

[534] D Ajay, P Chellamani, G Rajchakit, N Boonsatit, and P Hammachukiattikul. Regularity of pythagorean neutrosophic graphs with an illustration in mcdm. *AIMS mathematics*, 7(5):9424–9442, 2022.

[535] Said Broumi, R Sundareswaran, M Shanmugapriya, P Chellamani, Assia Bakali, and Mohamed Talea. Determination of various factors to evaluate a successful curriculum design using interval-valued pythagorean neutrosophic graphs. *Soft Computing*, pages 1–20, 2023.

[536] Prabuddha Giri, Sk Amanathulla, and Kalyani Maity Das. An analysis of fermatean fuzzy graph and its application in a car company. *Journal of Applied Mathematics and Computing*, 70(4):3575–3602, 2024.

[537] Broumi Said, S Mohanaselvi, Assia Bakali, and Mohamed Talea. Complex fermatean fuzzy graph and its application in networking problems. In *Sustainable Technologies for Transforming Businesses and Societies: The Keynote Lectures of the 6th International Conference on Entrepreneurship, Innovation and Leadership (ICEIL 2024)*, pages 111–130. Springer, 2025.

[538] Francis Remigius Perpetua Mary, Swaminathan Mohanaselvi, and Said Broumi. A solution approach to minimum spanning tree problem under fermatean fuzzy environment. *Bulletin of Electrical Engineering and Informatics*, 12(3):1738–1746, 2023.

[539] Sivaranjani Krishnaraj, OV Sundaram, S Bhavani, Prasantha Bharathi Dhandapani, and Seenith Sivasundaram. A decision-making approach using seidel laplacian energy within the framework of fermatean fuzzy graphs. *Mathematics in Engineering, Science & Aerospace (MESA)*, 16(3), 2025.

[540] Said Broumi, R Sundareswaran, M Shanmugapriya, Assia Bakali, and Mohamed Talea. Theory and applications of fermatean neutrosophic graphs. *Neutrosophic sets and systems*, 50:248–286, 2022.

[541] G Thamizhendhi, C Kiruthica, and S Suresh. Fermatean fuzzy hypergraph. *Journal of Hunan University Natural Sciences* , 48, 2021.

[542] Wadei Fares AL-omeri, M Kaviyarasu, and M Rajeshwari. Fermatean neutrosophic fuzzy graphs: A study on the winner index with enhancing election analysis. *Neutrosophic Sets and Systems*, 80:128–148, 2025.

[543] Anam Luqman, Muhammad Akram, and Ahmad N. Al-Kenani. q-rung orthopair fuzzy hypergraphs with applications. *Mathematics*, 7(3):260, 2019.

[544] Yongsheng Rao, Saeed Kosari, Saira Hameed, and Zulqarnain Yousaf. Multi-attribute decision-making using q-rung orthopair fuzzy zagreb index. *Artificial Intelligence Review*, 58(5):1–31, 2025.

[545] Waheed Ahmad Khan, Sagheer Abbas, Akhlaq Ahmed, Madhumangal Pal, Muhammad Asif, and Muhammad Saeed Khan. Some novel concepts of interval-valued q-rung orthopair fuzzy graphs and computational framework of fuzzy air conditioning system. *PloS one*, 20(4):e0317676, 2025.

[546] T Pathinathan, J Jon Arockiaraj, and J Jesintha Rosline. Hesitancy fuzzy graphs. *Indian Journal of Science and Technology*, 8(35):1–5, 2015.

[547] R Jahir Hussain, S Satham Hussain, Sankar Sahoo, and Madhumangal Pal. Domination in hesitancy fuzzy graphs. *International Journal of Advanced Intelligence Paradigms*, 25(1-2):11–23, 2023.

[548] N Rajagopal Reddy and S Sharief Basha. A cosine similarity measures between hesitancy fuzzy graphs and its application to decision making. *AIMS Mathematics*, 8(5):11799–11821, 2023.

[549] A Prasanna, MA Rifayathali, and S Ismail Mohideen. Hesitancy fuzzy graph coloring. *International Journal of Fuzzy Mathematical Archive*, 14(2):179–189, 2017.

[550] T Pathinathan, J Jon Arockiaraj, and E Mike Dison. Constant hesitancy fuzzy graphs. *Global Journal of Pure and Applied Mathematics (GJPAM)*, 12(2):2016, 2016.

[551] R Jahir Hussain, TC Mujeeburahman, and D Dhamodharan. Some edge domination parameters in bipolar hesitancy fuzzy graph. *Ratio Mathematica*, 42:157, 2022.

[552] Sakshi Dev Pandey, AS Ranadive, and Sovan Samanta. Bipolar-valued hesitant fuzzy graph and its application. *Social Network Analysis and Mining*, 12(1):14, 2022.

[553] MA Rifayathali, A Prasanna, and S Ismail Mohideen. Hesitancy fuzzy magic labeling graph. *The Journal of Analysis*, 27:39–53, 2019.

[554] Wenhui Bai, Juanjuan Ding, and Chao Zhang. Dual hesitant fuzzy graphs with applications to multi-attribute decision making. *International Journal of Cognitive Computing in Engineering*, 1:18–26, 2020.

[555] L Subha Lakshmi and NR Santhi Maheswari. A study on constant and regular hesitancy fuzzy soft graphs. *International J. Math. Combin. Vol*, 4:33–43, 2023.

[556] Zengtai Gong and Junhu Wang. Hesitant fuzzy graphs, hesitant fuzzy hypergraphs and fuzzy graph decisions. *Journal of Intelligent & Fuzzy Systems*, 40(1):865–875, 2021.

[557] Lin Wei. An integrated decision-making framework for blended teaching quality evaluation in college english courses based on the double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:3259–3266, 2023.

[558] Kanghua Du and Yuming Du. Research on performance evaluation of intangible assets operation and management in sports events with double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:2813–2822, 2023.

[559] Lingling Lin. Double-valued neutrosophic offset for enhancing humanistic competence: An effectiveness study of humanities instruction in vocational college students. *Neutrosophic Sets and Systems*, 88:680–689, 2025.

[560] Heng He. A novel approach to assessing art education teaching quality in vocational colleges based on double-valued neutrosophic numbers and multi-attribute decision-making with tree soft sets. *Neutrosophic Sets and Systems*, 78:206–218, 2025.

[561] Wenwen Meng. Quintuple-valued neutrosophic offset for quality evaluation of cross-border e-commerce talent training based on artificial intelligence. *Neutrosophic Sets and Systems*, 88:834–844, 2025.

[562] Sunil MP and J Suresh Kumar. On intuitionistic hesitancy fuzzy graphs. *Mathematics and Statistics*, 12:48–54, 2024.

[563] Sunil MP and J Suresh Kumar. On beta product of hesitancy fuzzy graphs and intuitionistic hesitancy fuzzy graphs. *Korean Journal of Mathematics*, 31(4):485–494, 2023.

[564] László T. Kóczy, Naeem Jan, Tahir Mahmood, and Kifayat Ullah. Analysis of social networks and wi-fi networks by using the concept of picture fuzzy graphs. *Soft Computing*, 24:16551 – 16563, 2020.

[565] Sk Amanathulla, G Muhiuddin, D Al-Kadi, and M Pal. Multiple attribute decision-making problem using picture fuzzy graph. *Mathematical Problems in Engineering*, 2021(1):9937828, 2021.

[566] R. Ramesh, S. Krishnaprakash, Nikola Ivkovic, Mario Konecki, and Chiranjibe Jana. Spherical picture fuzzy sets: Enhancing decision-making with geometric bonferroni methods. *International Journal of Analysis and Applications*, 2025.

[567] Ahmed Mostafa Khalil, Shenggang Li, Harish Garg, Hong xia Li, and Sheng quan Ma. New operations on interval-valued picture fuzzy set, interval-valued picture fuzzy soft set and their applications. *IEEE Access*, 7:51236–51253, 2019.

[568] Kartick Mohanta, Arindam Dey, and Anita Pal. A study on picture dombi fuzzy graph. *Decision Making: Applications in Management and Engineering*, 3(2):119–130, 2020.

[569] Zhihua Chen, Waheed Ahmad Khan, and Aysha Khan. Concepts of picture fuzzy line graphs and their applications in data analysis. *Symmetry*, 15(5):1018, 2023.

[570] Biswajit Bera, Sk Amanathulla, and Sanat Kumar Mahato. A comprehensive study of picture fuzzy planar graphs with real-world applications. *Journal of uncertain Systems*, 16(04):2350009, 2023.

[571] Sankar Das, Soumitra Poulik, and Ganesh Ghorai. Picture fuzzy φ-tolerance competition graphs with its application. *Journal of Ambient Intelligence and Humanized Computing*, 15(1):547–559, 2024.

[572] Yongsheng Rao, Ruxiang Chen, Waheed Ahmad Khan, and Alishba Zahid. Some new concepts of interval-valued picture fuzzy graphs and their application toward the selection criteria. *International Journal of Fuzzy Systems*, 2024.

[573] Irfan Nazeer and Tabasam Rashid. Picture fuzzy incidence graphs with application. *Punjab University Journal of Mathematics*, 2021.

[574] Waheed Ahmad Khan, Khurram Faiz, and Abdelghani Taouti. Cayley picture fuzzy graphs and interconnected networks. *Intelligent Automation & Soft Computing*, 2022.

[575] Jasminara Khatun, Sk Amanathulla, and Madhumangal Pal. A comprehensive study on m-polar picture fuzzy graphs and its application. *Journal of Uncertain Systems*, page 2450016, 2024.

[576] Waqar Arif, Waheed Ahmad Khan, Hossein Rashmanlou, Asghar Khan, and Adnan Muhammad. Multi attribute decision-making and interval-valued picture (s, t)-fuzzy graphs. *Journal of Applied Mathematics and Computing*, 69:2831–2856, 2023.

[577] Muhammad Akram and Amna Habib. q-rung picture fuzzy graphs: a creative view on regularity with applications. *Journal of Applied Mathematics and Computing*, 61:235 – 280, 2019.

[578] Jasminara Khatun, Sk Amanathulla, and Madhumangal Pal. Picture fuzzy cubic graphs and their applications. *Journal of Intelligent & Fuzzy Systems*, pages 1–18, 2024.

[579] Sk Amanathulla, Biswajit Bera, and Madhumangal Pal. Balanced picture fuzzy graph with application. *Artificial Intelligence Review*, 54:5255 – 5281, 2021.

[580] Rajab Ali Borzooei, Hossein Rashmanlou, Sovan Samanta, and Madhumangal Pal. Regularity of vague graphs. *Journal of Intelligent & Fuzzy Systems*, 30(6):3681–3689, 2016.

[581] Rajab Ali Borzooei and Hossein Rashmanlou. Domination in vague graphs and its applications. *Journal of Intelligent & Fuzzy Systems*, 29(5):1933–1940, 2015.

[582] Abolfazl Lakdashti, Hossein Rashmanlou, PK Kishore Kumar, Ganesh Ghorai, and Madhumangal Pal. Some results on edge irregular product vague graphs. *International Journal of Advanced Intelligence Paradigms*, 27(1):18–28, 2024.

[583] Nastaran Farhang and Yahya Talebi. Novel concepts in vague incidence graphs with application. *TWMS Journal Of Applied And Engineering Mathematics*, 2024.

[584] Shouzhen Zeng, Muhammad Shoaib, Shahbaz Ali, Qamar Abbas, and Muhammad Shahzad Nadeem. Complex vague graphs and their application in decision-making problems. *IEEE access*, 8:174094–174104, 2020.

[585] S Banitalebi. Irregular vague graphs. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2(2):73–90, 2021.

[586] S Satham Hussain, Saeid Jafari, Said Broumi, and N Durga. Operations on neutrosophic vague graphs. *Neutrosophic Sets and Systems*, 35(1):368–386, 2020.

[587] S Satham Hussain, R Jahir Hussain, and Ghulam Muhiuddin. *Neutrosophic vague line graphs*, volume 36. Infinite Study, 2020.

[588] Morteza Taheri and Yahya Talebi. New concepts of cubic vague graphs with novel application. *Journal of Multiple-Valued Logic & Soft Computing*, 36(6), 2021.

[589] Xiaolong Shi, Ruiqi Cai, Ali Asghar Talebi, Masomeh Mojahedfar, and Chanjuan Liu. A novel domination in vague influence graphs with an application. *Axioms*, 13(3):150, 2024.

[590] P Anitha and P Chitra Devi. Neutrosophic bipolar vague incidence graph. *Gorteria UGC-CARE (Group-II) Journal*, 34(8):13, 2021.

[591] Yousef Al-Qudah and Nasruddin Hassan. Operations on complex multi-fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 33(3):1527–1540, 2017.

[592] Sabu Sebastian and Robert John. Multi-fuzzy sets and their correspondence to other sets. *Ann. Fuzzy Math. Inform*, 11(2):341–348, 2016.

[593] Sabu Sebastian, TV Ramakrishnan, KK Gireesan, and SJ Sangeeth. Multi-fuzzy sets and neural networks: a collaborative tool for artificial intelligence. *International Journal of Information Technology*, pages 1–5, 2024.

[594] You-guang Shang, Xue-hai Yuan, and E Stanley Lee. The n-dimensional fuzzy sets and zadeh fuzzy sets based on the finite valued fuzzy sets. *Computers & Mathematics with Applications*, 60(3):442–463, 2010.

[595] Annaxsuel Lima, Eduardo S Palmeira, Benjamín Bedregal, and Humberto Bustince. Multidimensional fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 29(8):2195–2208, 2020.

[596] Landerson Santiago and Benjamin Bedregal. Multidimensional fuzzy sets: Negations and an algorithm for multi-attribute group decision making. *International Journal of Approximate Reasoning*, 169:109171, 2024.

[597] Mübariz Eminli and Nevin Guler. An improved takagi-sugeno fuzzy model with multidimensional fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 21(5):277–287, 2010.

[598] Bahram Farhadinia, Uwe Aickelin, and Hadi Akbarzadeh Khorshidi. Uncertainty measures for probabilistic hesitant fuzzy sets in multiple criteria decision making. *International Journal of Intelligent Systems*, 35(11):1646–1679, 2020.

[599] Chenxia Jin, Jusheng Mi, Fachao Li, and Meishe Liang. A novel probabilistic hesitant fuzzy rough set based multi-criteria decision-making method. *Information Sciences*, 608:489–516, 2022.

[600] Muhammad Ali Khan, Shahzaib Ashraf, Saleem Abdullah, and Fazal Ghani. Applications of probabilistic hesitant fuzzy rough set in decision support system. *Soft computing*, 24:16759–16774, 2020.

[601] Sujit Das, Mohuya B Kar, and Samarjit Kar. Group multi-criteria decision making using intuitionistic multi-fuzzy sets. *Journal of Uncertainty Analysis and Applications*, 1:1–16, 2013.

[602] Ajoy Kanti Das and Carlos Granados. Fp-intuitionistic multi fuzzy n-soft set and its induced fp-hesitant n soft set in decision-making. *Decision making: applications in management and engineering*, 5(1):67–89, 2022.

[603] Sujit Das and Samarjit Kar. Intuitionistic multi fuzzy soft set and its application in decision making. In *Pattern Recognition and Machine Intelligence: 5th International Conference, PReMI 2013, Kolkata, India, December 10-14, 2013. Proceedings 5*, pages 587–592. Springer, 2013.

[604] Nivetha Martin and Florentin Smarandache. Concentric plithogenic hypergraph based on plithogenic hypersoft sets ? a novel outlook. *Neutrosophic Sets and Systems*, 33:5, 2020.

[605] Min Huang, Fenghua Li, et al. Optimizing ai-driven digital resources in vocational english learning using plithogenic n-superhypergraph structures for adaptive content recommendation. *Neutrosophic Sets and Systems*, 88:283–295, 2025.

[606] Eduardo Martín Campoverde Valencia, Jessica Paola Chuisaca Vásquez, and Francisco Ángel Becerra Lois. Multineutrosophic analysis of the relationship between survival and business growth in the manufacturing sector of azuay province, 2020–2023, using plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84(1):28, 2025.

[607] Vakkas Uluçay. Some concepts on interval-valued refined neutrosophic sets and their applications. *Journal of Ambient Intelligence and Humanized Computing*, 12(7):7857–7872, 2021.

[608] Irfan Deli, Said Broumi, and Florentin Smarandache. On neutrosophic refined sets and their applications in medical diagnosis. *Journal of new theory*, 6:88–98, 2015.

[609] Takaaki Fujita and Florentin Smarandache. A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.

[610] Takaaki Fujita and Florentin Smarandache. *A Dynamic Survey of Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Plithogenic, and Extensional Sets.* Neutrosophic Science International Association (NSIA), 2025.

Graph theory studies networks of vertices and edges and their associated structural and algorithmic properties. To model real-world settings in which relationships are imprecise, a fuzzy graph enriches a graph by assigning to each vertex and edge a membership degree in [0, 1]. Build-ing on this idea, neutrosophic and quadripartitioned neutrosophic graphs incorporate multiple components to represent truth, indeterminacy, and falsity (and their refinements), thereby pro-viding greater expressive power than the fuzzy model. Plithogenic graphs further broaden this landscape by offering a flexible framework for managing uncertainty through attribute values and degrees of contradiction. Beyond ordinary graphs, a hypergraph allows each edge to connect an arbitrary nonempty subset of the vertex set. Iterating the powerset construction yields nested higher-order vertex objects and leads to finite SuperHyperGraphs, whose vertices and edges may themselves be set-valued across multiple layers. In this book, we examine the relationships among a wide range of graph, hypergraph, and superhypergraph classes including plithogenic models and we discuss additional related structures within this ecosystem.

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