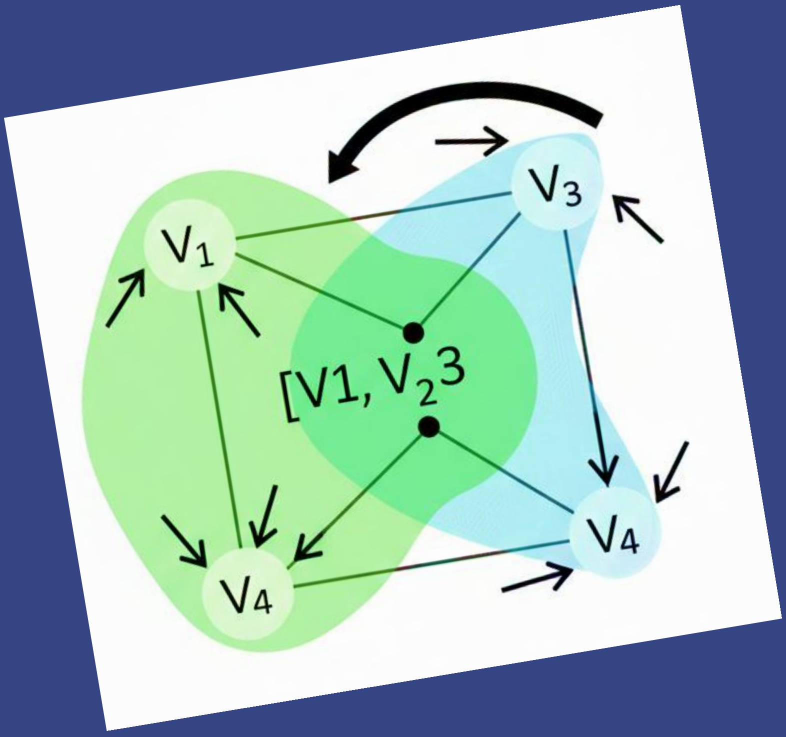
TAKAAKI FUJITA
FLORENTIN SMARANDACHE

HYPERGRAPH AND SUPERHYPERGRAPH THEORY WITH APPLICATIONS VII

ABOUT DIRECTED SUPERHYPERGRAPH



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HyperGraph and SuperHyperGraph Theory with Applications

VII

About Directed SuperHyperGraph

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Chapter 1

Introduction

1.1 Graph, HyperGraph, and SuperHyperGraph

Network models are classically expressed by *graphs*, in which objects are represented by vertices and binary relationships by edges [5]. While this abstraction is effective for pairwise interactions, it becomes restrictive when the underlying system exhibits *simultaneous* interactions among three or more entities. *Hypergraphs* resolve this limitation by permitting each hyperedge to join an arbitrary nonempty subset of vertices, thereby representing higher-order relations directly [6–9].

Even so, many real-world datasets and engineered systems display relationships that are not only higher-order but also *layered*, *nested*, and intrinsically *hierarchical*. To capture such multi-level incidence patterns, F. Smarandache introduced the notion of a *SuperHyperGraph*. Informally, a SuperHyperGraph is built via iterative powerset-based constructions, which allow vertices ("supervertices") themselves to be set-valued objects and enable edges to encode nested connectivity across multiple levels [10,11]. Consequently, SuperHyperGraphs have recently attracted growing attention in both theory and applications [12–18].

Graphs and hypergraphs also provide transparent visual metaphors for complex systems and support a broad spectrum of applications in artificial intelligence, network science, data mining, informatics, chemistry, physics, and related fields [19–21]. By explicitly incorporating hierarchical and multi-level relationships, SuperHyperGraphs offer a flexible framework for modeling and analyzing intricate structures in modern networked data (e.g., [16,22–27]). Table 1.1 summarizes the essential distinctions among graphs, hypergraphs, and n-SuperHyperGraphs.

1.2 Directed SuperHyperGraphs

Graphs, hypergraphs, and superhypergraphs can each be extended by incorporating directionality, yielding directed graphs [29], directed hypergraphs [30, 31], and directed superhypergraphs [11,32], respectively. These are graph-based models enriched with orientation information, and, as with ordinary graphs, they have been widely studied in many theoretical and applied contexts. For reference, an overview of directed graphs, directed hypergraphs, and directed n-SuperHyperGraphs is given in Table 1.2.

Table 1.1: Key distinctions among graphs, hypergraphs, and n-superhypergraphs.

Concept	*Notation*	*Edge family*	*Core extension principle*
Graph [5]	$G = (V, E)$	$E \subseteq \binom{V}{2}$	Edges encode *pairwise* (binary) relations between vertices.
Hypergraph [28]	$H = (V, \mathcal{E})$	$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Hyperedges may join *any* nonempty subset of vertices, encoding higher-order interactions.
n-SuperHyperGraph [10]	$\mathrm{SHG}^{(n)} = (V, E)$ (on a base set V_0)	$V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$, $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Vertices are allowed to be *set-valued objects* living in an n-fold powerset hierarchy, while edges remain subsets of the supervertex set; this supports *nested* and *multi-level* incidence patterns.

Notation. $\mathcal{P}(X) = \{A \mid A \subseteq X\}$, $\binom{V}{2} = \{\{u, v\} \subseteq V \mid u \neq v\}$, and $\mathcal{P}^0(X) = X$, $\mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

Table 1.2: Concise overview of directed graphs, directed hypergraphs, and directed n-SuperHyperGraphs.

Structure	*Standard notation*	*Arcs / hyperarcs / superhyperarcs (direction rule)*
Directed graph	$G = (V, A)$	An arc is an ordered pair $(u, v) \in V \times V$ (often with $u \neq v$); it directs from u to v.
Directed hypergraph	$H = (V, E)$	A hyperarc is an ordered pair $e = (T(e), H(e))$ with nonempty $T(e), H(e) \subseteq V$; it directs from the tail-set to the head-set.
Directed n-SuperHyperGraph	$\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$	Supervertices satisfy $V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}$. Each directed superhyperedge identifier $e \in E$ has $\mathrm{Tail}(e) = \partial^-(e) \in \mathcal{P}(V) \setminus \{\emptyset\}$ and $\mathrm{Head}(e) = \partial^+(e) \in \mathcal{P}(V) \setminus \{\emptyset\}$, directing from $\mathrm{Tail}(e)$ to $\mathrm{Head}(e)$.

1.3 Our Contributions

In view of the above, systematic research on SuperHyperGraphs—and in particular on their *directed* variants—is important. In this book, we provide a unified framework and a structured catalogue of graph-theoretic notions for directed hypergraphs and directed superhypergraphs. More concretely, our contributions are as follows:

(i) **Unified formalization.** We fix a common incidence-based representation of a directed n-SuperHyperGraph via tail/head incidence maps ∂^-, ∂^+, which consistently accommodates parallel superhyperedges and serves as the standard model throughout subsequent constructions.

(ii) **Class taxonomy for directed superhypergraphs.** We introduce and organize a broad family of directed superhypergraph classes—including bidirected, multi, acyclic, weighted, signed, rooted, planar, complete, and tournament-type structures—by specifying, for each class, the defining constraints on ∂^-, ∂^+ and/or additional labels.

(iii) **Directed flow model on superhypergraphs.** We formulate a directed flow notion on directed hypergraphs/superhypergraphs by assigning tail- and head-incidence flows $f^-(x,e)$ and $f^+(e,y)$ and by imposing edge feasibility and vertex conservation constraints, thereby extending classical network-flow primitives beyond pairwise edges.

(iv) **Embedding and specialization results.** We explicitly prove that directed graphs and directed hypergraphs arise as special cases of directed n-SuperHyperGraphs, and we establish corresponding embedding/specialization theorems for several directed classes (e.g., tournaments and their superhypergraph analogues).

(v) **Uncertainty-aware extensions.** We present directed fuzzy, neutrosophic, soft, and rough variants of the above directed structures, and we explain how these models can be treated in a unified manner by varying the underlying uncertainty domain and its order structure.

Also the novelty of this book lies not in presenting a large number of new theorems, but in providing a unified reformulation via a common incidence framework, together with a systematic taxonomy and a reusable template for cross-cutting extensions. We expect that this unified viewpoint and the proposed catalogue of directed superhypergraph notions will stimulate further work on directed graph theory, directed hypergraph theory, and directed SuperHyperGraph theory, including structural theory, algorithms, and applications.

HyperGraph and SuperHyperGraph Theory with Applications (VII): About Directed SuperHyperGraph

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Abstract

Hypergraphs generalize ordinary graphs by allowing an edge to connect an arbitrary nonempty subset of the vertex set. By iterating the powerset construction, one obtains nested, higher-order vertex objects and arrives at finite *S uperHyperGraphs*, i n w hich v ertices a re set-valued across multiple layers and edges encode relations among such set-valued vertices. Graphs, hypergraphs, and superhypergraphs are further extended by introducing directionality, leading to directed graphs, directed hypergraphs, and directed superhypergraphs, which have been actively investigated in the literature. In this book, we examine and survey fundamental graph-theoretic concepts in the settings of directed hypergraphs and directed superhypergraphs.

Keywords: SuperHyperGraph, HyperGraph, Directed HyperGraph, Directed SuperHyperGraph

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Chapter 2

Preliminaries

This chapter establishes notation and reviews the fundamental structures used throughout the book.

2.1 SuperHyperGraphs

Classical graph theory models a system of *vertices* linked by *edges*, and studies connectivity, structural invariants, and algorithmic problems motivated by mathematics, computer science, and many applied domains [5]. A *hypergraph* broadens this framework by allowing a single edge to connect an arbitrary nonempty subset of the vertex set; hence it is well suited to represent intrinsically multiway interactions (e.g., relations of arity greater than two) [6, 8, 28, 33]. Such higher-order relations have become especially prominent in contemporary learning and modeling pipelines, including neural architectures that directly leverage hypergraph incidence patterns [6, 34–37].

By iterating the powerset operation, one can also permit *nested* set-valued entities at the vertex level. This leads to finite *SuperHyperGraphs*, in which both vertices and edges may occur at multiple levels of set nesting [38, 39]. Such hierarchical representations arise naturally in layered or multiscale relational settings, for instance in molecular design, complex-network analysis, and neural-network modeling, among other applications [23, 40–42]. Several related generalizations have also been investigated, including Directed SuperHyperGraphs [43, 44] and MetaSuperHyperGraphs [45].

Definition 2.1.1 (Base set)**.** A *base set* S is the ambient universe of discourse:

$$S = \{ x \mid x \text{ is an admissible object in the context under consideration} \}.$$

All sets in $\mathcal{P}(S)$ and in the iterated powersets $\mathcal{P}^n(S)$ are ultimately formed from elements of S.

Definition 2.1.2 (Powerset)**.** (see [46]) For a set S, the *powerset* of S is

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

In particular, $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$.

Definition 2.1.3 (Hypergraph)**.** [28,47] A *hypergraph* is a pair $H = (V, E)$ such that:

- V is a finite set of *vertices*, and
- E is a finite family of nonempty subsets of V, called *hyperedges*.

Thus, a hyperedge may contain more than two vertices, capturing genuinely multiway relations.

Example 2.1.4 (A small hypergraph with mixed edge sizes)**.** Let

$$V = \{1, 2, 3, 4, 5\}.$$

Define three hyperedges

$$e_1 = \{1, 2, 3\}, \qquad e_2 = \{3, 4\}, \qquad e_3 = \{2, 4, 5\},$$

and set

$$E = \{e_1, e_2, e_3\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then $H = (V, E)$ is a hypergraph in the sense of Definition 2.1.3. Here e_1 and e_3 are 3-uniform hyperedges (triadic interactions), while e_2 is an ordinary 2-edge, illustrating that a hypergraph may contain hyperedges of varying cardinalities. Moreover, the vertex 3 lies in both e_1 and e_2, and the vertex 2 lies in both e_1 and e_3, so the hyperedges overlap nontrivially.

Definition 2.1.5 (Iterated powerset and flattening)**.** [48] Let V_0 be a finite nonempty set. Define $\mathcal{P}^0(V_0) := V_0$ and

$$\mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \qquad (k \geq 0).$$

For each $k \geq 0$, define the flattening map

$$\mathrm{Flat}_k : \mathcal{P}^k(V_0) \setminus \{\emptyset\} \longrightarrow \mathcal{P}(V_0) \setminus \{\emptyset\}$$

recursively by

$$\mathrm{Flat}_0(x) := \{x\} \quad (x \in V_0), \qquad \mathrm{Flat}_{k+1}(X) := \bigcup_{Y \in X} \mathrm{Flat}_k(Y) \quad (X \in \mathcal{P}^{k+1}(V_0) \setminus \{\emptyset\}).$$

Definition 2.1.6 (n-SuperHyperGraph)**.** (see [10]) Let V_0 be a finite, nonempty base set. Define

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{\,k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^{\,k}(V_0)\big) \quad (k \in \mathbb{N}).$$

For $n \geq 0$, an *n-SuperHyperGraph on V_0* is a pair

$$\mathrm{SHG}^{(n)} = (V, E)$$

such that

$$V \subseteq \mathcal{P}^{\,n}(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Elements of V are called *n-supervertices*, and elements of E are called *n-superedges* (that is, each n-superedge is a nonempty subset of V).

Example 2.1.7 (A concrete 2-SuperHyperGraph)**.** Let the base set be

$$V_0 = \{a, b, c\}.$$

Since

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0), \qquad \mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big),$$

a level-2 supervertex is a *set of subsets* of V_0.

Define four 2-supervertices (elements of $\mathcal{P}^2(V_0)$) by

$$v_1 := \{\{a\}, \{b\}\}, \qquad v_2 := \{\{b\}, \{c\}\}, \qquad v_3 := \{\{a, c\}\}, \qquad v_4 := \{\{a, b, c\}\}.$$

Set

$$V := \{v_1, v_2, v_3, v_4\} \subseteq \mathcal{P}^2(V_0).$$

Now define a family of 2-superedges (nonempty subsets of V) by

$$e_1 := \{v_1, v_2\}, \qquad e_2 := \{v_2, v_3, v_4\}, \qquad e_3 := \{v_1, v_4\},$$

and let

$$E := \{e_1, e_2, e_3\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then

$$\mathrm{SHG}^{(2)} = (V, E)$$

is a 2-SuperHyperGraph on the base set V_0 in the sense of Definition 2.1.6.

2.2 Directed SuperHyperGraph

As discussed above, graphs are widely applied across numerous domains. However, when modeling concepts that inherently involve directional information, the use of *directed graphs* becomes essential [49, 50]. These structures have been further extended to *directed hypergraphs* [51–53] and *directed superhypergraphs* [43, 54], which have attracted growing research interest.

Definition 2.2.1 (Directed Hypergraph)**.** (cf. [55, 56]) A *directed hypergraph* is a pair

$$H = (V,\, E),$$

where

- V is a finite set of *vertices.*
- E is a finite set of *hyperarcs*, each hyperarc $e \in E$ being an ordered pair

  $$e = \big(T(e),\, H(e)\big) \in \mathcal{P}(V) \times \mathcal{P}(V),$$

  with

  $$T(e) \subseteq V,\ T(e) \neq \emptyset, \quad H(e) \subseteq V,\ H(e) \neq \emptyset.$$

Intuitively, each $e = (T(e), H(e))$ carries "flow" from all vertices in $T(e)$ (the *tail*) to all vertices in $H(e)$ (the *head*).

Example 2.2.2 (A small directed hypergraph: parallel influence vs. joint requirement)**.** Let

$$V = \{a, b, c, d, e\}.$$

Define a directed hypergraph $H = (V, E)$ with three hyperarcs:

$$e_1 = \big(T(e_1), H(e_1)\big) = \big(\{a, b\}, \{c\}\big), \qquad e_2 = \big(\{c\}, \{d, e\}\big), \qquad e_3 = \big(\{b\}, \{d\}\big).$$

Interpretation. The hyperarc e_1 models a *joint* prerequisite: both a and b together trigger (or supply flow to) c. The hyperarc e_2 models *broadcast*: once c is active, it simultaneously influences d and e. Finally, e_3 represents a direct one-to-one influence from b to d, providing an alternative route to d.

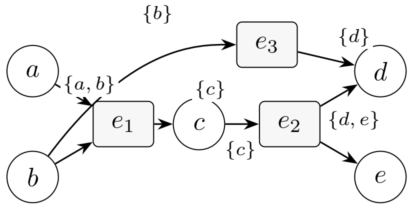

Figure 2.1: A directed hypergraph $H = (V, E)$ with hyperarcs $e_1 : \{a, b\} \to \{c\}$, $e_2 : \{c\} \to \{d, e\}$, and $e_3 : \{b\} \to \{d\}$. Hyperarcs are drawn as small boxes; arrows go from tail-vertices to the box and from the box to head-vertices.

Definition 2.2.3 (Directed n-SuperHyperGraph)**.** Fix an integer $n \geq 0$ and a nonempty base set S. Define iterated powersets by

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^{k+1}(S) = \mathcal{P}(\mathcal{P}^k(S)) \quad (k \geq 0).$$

A *directed n-SuperHyperGraph* is a quadruple

$$\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+),$$

where

$$V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}, \qquad E \text{ is a finite set of directed superedge identifiers,}$$

and

$$\partial^-, \partial^+ : E \longrightarrow \mathcal{P}(V) \setminus \{\emptyset\}$$

are the *tail* and *head* incidence maps. For $e \in E$, write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. Thus each directed n-superhyperedge e carries "flow" from the nonempty tail set $\mathrm{Tail}(e) \subseteq V$ to the nonempty head set $\mathrm{Head}(e) \subseteq V$. *Parallel superhyperedges are permitted*: it may happen that $e \neq e'$ but $\mathrm{Tail}(e) = \mathrm{Tail}(e')$ and $\mathrm{Head}(e) = \mathrm{Head}(e')$.

Remark 2.2.4 (Standard representation used in this book)**.** Throughout this book, the *incidence form* $\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$ is regarded as the standard model for directed n-SuperHyperGraphs, because it (i) treats directed superhyperedges as *identifiers* and hence naturally allows parallel edges, and (ii) cleanly supports additional structures (weights, signs, capacities, flows, etc.) by attaching data to identifiers in E.

Remark 2.2.5 (Ordered-pair shorthand and the simple case)**.** Sometimes we write a directed superhyperedge as an ordered pair

$$e \equiv \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big),$$

as a notational shorthand when this causes no ambiguity. Formally, this identifies an edge $e \in E$ with its *incidence type* $\big(\partial^-(e), \partial^+(e)\big)$. This identification is faithful precisely in the *simple* case:

$$(\partial^-, \partial^+) : E \to \big(\mathcal{P}(V) \setminus \{\emptyset\}\big) \times \big(\mathcal{P}(V) \setminus \{\emptyset\}\big) \quad \text{is injective.}$$

When parallel superhyperedges are present (i.e., the map above is not injective), we keep the identifier $e \in E$ and use the incidence maps ∂^-, ∂^+ explicitly.

Remark 2.2.6 (Multi vs. simple terminology)**.** We say that $\mathrm{DSHG}^{(n)}$ is *multi* if there exist distinct $e, e' \in E$ with $\mathrm{Tail}(e) = \mathrm{Tail}(e')$ and $\mathrm{Head}(e) = \mathrm{Head}(e')$; equivalently, if (∂^-, ∂^+) is not injective. We say it is *simple* if (∂^-, ∂^+) is injective, in which case edges can be identified with their ordered pairs $(\mathrm{Tail}, \mathrm{Head})$ without loss of information.

Example 2.2.7 (A directed 2-SuperHyperGraph with parallel superedges)**.** Let the base set be

$$S = \{a, b, c\}, \qquad n = 2.$$

Define four level-2 supervertices (elements of $\mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S))$) by

$$v_1 := \{\{a\}, \{b\}\}, \quad v_2 := \{\{b\}, \{c\}\}, \quad v_3 := \{\{a, c\}\}, \quad v_4 := \{\{a, b, c\}\},$$

and set

$$V = \{v_1, v_2, v_3, v_4\} \subseteq \mathcal{P}^2(S) \setminus \{\emptyset\}.$$

Let the directed superedge-identifier set be $E = \{e_1, e_2, e_3\}$ and define the incidence maps $\partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}$ by

$$\partial^-(e_1) = \{v_1, v_2\}, \qquad \partial^+(e_1) = \{v_3\},$$
$$\partial^-(e_2) = \{v_1, v_2\}, \qquad \partial^+(e_2) = \{v_4\},$$
$$\partial^-(e_3) = \{v_3\}, \qquad \partial^+(e_3) = \{v_4\}.$$

Then $\mathrm{DSHG}^{(2)} = (V, E, \partial^-, \partial^+)$ is a directed 2-SuperHyperGraph. Note that e_1 and e_2 share the same tail $\{v_1, v_2\}$ but have different heads, illustrating how the identifier-based definition naturally supports multiple distinct directed superedges leaving the same tail-set.

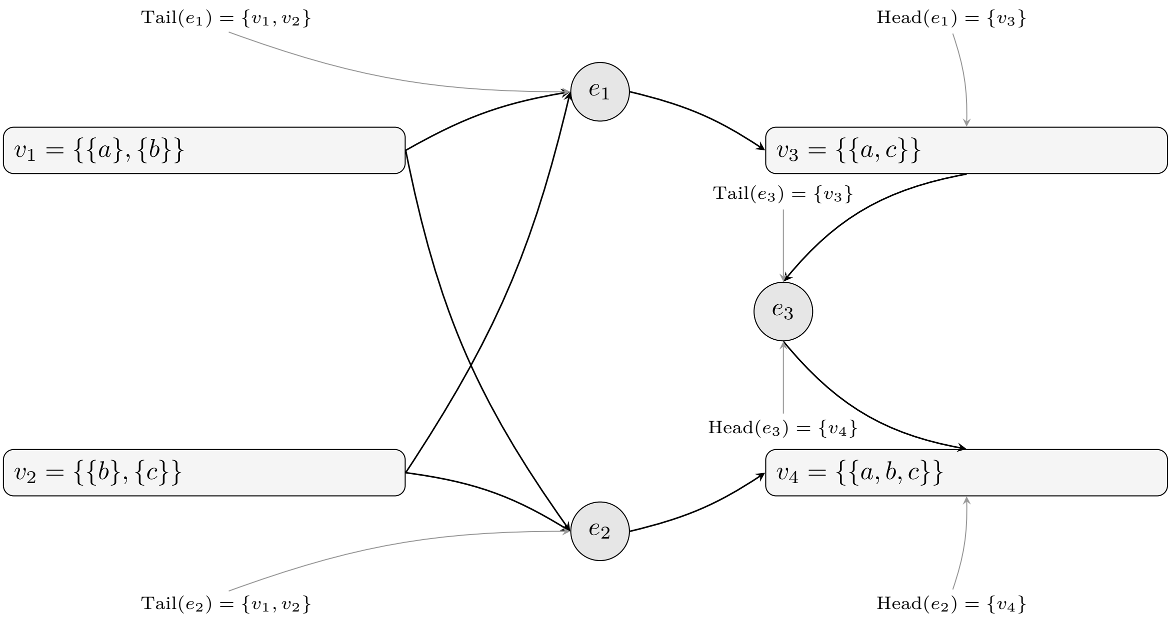

Figure 2.2: A directed 2-SuperHyperGraph $\mathrm{DSHG}^{(2)} = (V, E, \partial^-, \partial^+)$. Supervertices are shown as rounded boxes. Each directed superedge identifier $e_j \in E$ is shown as a circle. Arrows go from tail supervertices to e_j and from e_j to head supervertices; tail/head sets are displayed as external callouts to avoid overlap.

2.2.1 (n, k)-recursive SuperHyperGraph

A *Recursive HyperGraph* is a hypergraph whose hyperedges may contain ordinary vertices and also lower-level hyperedges as elements, enabling nested, self-referential incidence up to a bounded recursion depth (cf. [57, 58]).

Definition 2.2.8 (Depth-k powerset universe)**.** [57, 58] Let S be a (nonempty) set and let $k \in \mathbb{N} \cup \{0\}$. Define a hierarchy of sets $(S_i)_{i \geq 0}$ by

$$S_0 := S, \qquad S_i := \mathcal{P}\Big(\bigcup_{j=0}^{i-1} S_j\Big) \quad (i \geq 1).$$

The *depth-k powerset universe* over S is

$$2_{S,k} \; := \; \mathcal{P}\Big(\bigcup_{i=0}^{k} S_i\Big).$$

Definition 2.2.9 (k-recursive hypergraph)**.** [57, 58] Let V be a (finite) vertex set and let $k \in \mathbb{N} \cup \{0\}$. A *$k$-recursive hypergraph* is a pair

$$H = (V, E)$$

such that

$$E \subseteq 2_{V,k} \setminus \{\emptyset\},$$

where $2_{V,k}$ is the depth-k powerset universe from Definition 2.2.8 applied to $S = V$.

In particular, for $k = 0$ one has $2_{V,0} = \mathcal{P}(V)$ and thus $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, i.e., H reduces to an ordinary hypergraph.

An (n, k)-recursive SuperHyperGraph has level-n supervertices (iterated powersets) and depth-k recursive edges that may include supervertices and lower-level edges as elements. We restrict to well-founded recursive superhyperedges (no membership cycles).

Definition 2.2.10 ((n, k)-recursive SuperHyperGraph)**.** Fix a base (ground) set V_0 and let $n, k \in \mathbb{N} \cup \{0\}$.

(Iterated powersets). Define the iterated powersets by

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{n+1}(V_0) = \mathcal{P}\big(\mathcal{P}^n(V_0)\big) \quad (n \geq 0).$$

A *(n, k)-recursive SuperHyperGraph* is a pair

$$\mathrm{RSHG}^{(n,k)} = (V, E)$$

satisfying:

(i) *(Hierarchical supervertex set).* $V \subseteq \mathcal{P}^n(V_0)$.

(ii) *(Recursive superhyperedge family).* $E \subseteq 2_{V,k} \setminus \{\emptyset\}$, where $2_{V,k}$ is the depth-k powerset universe constructed from $S = V$ as in Definition 2.2.8.

Consistency checks. If $k = 0$, then $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ and one recovers the usual (non-recursive) n-SuperHyperGraph. If $n = 0$, then $V \subseteq V_0$ and one recovers a k-recursive hypergraph on the (ground) vertex set.

Example 2.2.11 (A $(2,1)$-recursive SuperHyperGraph: supervertices at level 2 and one-step edge recursion)**.** Let the ground set be

$$V_0 = \{a, b, c\}, \qquad n = 2, \qquad k = 1.$$

Then $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$, so a 2-supervertex is a *set of subsets* of V_0. Define

$$v_1 := \{\{a\}, \{b\}\}, \qquad v_2 := \{\{b\}, \{c\}\}, \qquad v_3 := \{\{a, c\}\},$$

and set

$$V := \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0).$$

To specify the recursive edge family for depth $k = 1$, recall that the depth-1 universe is

$$U_0 := V, \qquad U_1 := U_0 \sqcup \mathcal{P}(U_0) = V \sqcup \mathcal{P}(V),$$

so a depth-1 superhyperedge may contain *either* supervertices from V *or* (non-recursive) edges from $\mathcal{P}(V)$, treated as distinct objects by disjoint union.

Choose the following two ordinary (non-recursive) superedges (elements of $\mathcal{P}(V) \setminus \{\emptyset\}$):

$$e_0 := \{v_1, v_2\}, \qquad e_0' := \{v_2, v_3\}.$$

Now define two depth-1 recursive superhyperedges by allowing an edge to reference a previous edge-object:

$$E_1 := \{\, v_1,\ e_0 \,\} \ \in\ \mathcal{P}(U_1) \setminus \{\emptyset\}, \qquad E_2 := \{\, v_3,\ e_0' \,\} \ \in\ \mathcal{P}(U_1) \setminus \{\emptyset\}.$$

Let

$$E := \{E_1, E_2\} \ \subseteq\ 2_{V,1} \setminus \{\emptyset\},$$

where $2_{V,1}$ denotes the depth-1 powerset universe over V (Definition 2.2.8).

Then $\mathrm{RSHG}^{(2,1)} = (V, E)$ is a $(2,1)$-recursive SuperHyperGraph: its vertices lie in $\mathcal{P}^2(V_0)$, and its recursive superedges may include both supervertices and (one-step) edge-objects.

Example 2.2.12 (A $(0,2)$-recursive SuperHyperGraph: a 2-recursive hypergraph on a ground vertex set)**.** Let the ground set be

$$V_0 = \{1, 2, 3, 4\}, \qquad n = 0, \qquad k = 2.$$

Then $\mathcal{P}^0(V_0) = V_0$, so we may take

$$V := \{1, 2, 3, 4\} \subseteq \mathcal{P}^0(V_0).$$

Construct the depth-2 powerset universe over V by

$$U_0 := V, \qquad U_1 := U_0 \sqcup \mathcal{P}(U_0) = V \sqcup \mathcal{P}(V), \qquad U_2 := U_1 \sqcup \mathcal{P}(U_1).$$

Thus a depth-2 recursive edge may be a subset of U_2 and can reference objects from V, $\mathcal{P}(V)$, and $\mathcal{P}(U_1)$, all distinguished by disjoint unions.

First choose two ordinary hyperedges on V:

$$e_1 := \{1,2\}, \qquad e_2 := \{2,3,4\} \qquad (\in \mathcal{P}(V) \setminus \{\emptyset\}).$$

At depth 1, we may form a recursive edge that mixes vertices and an edge-object:

$$E^{(1)} := \{\, 3,\ e_1 \,\} \ \in \ \mathcal{P}(U_1) \setminus \{\emptyset\}.$$

At depth 2, we may now reference the depth-1 edge-object $E^{(1)}$ as an element of U_2, and form a second-order recursive edge such as

$$E^{(2)} := \{\, 4,\ e_2,\ E^{(1)} \,\} \ \in \ \mathcal{P}(U_2) \setminus \{\emptyset\}.$$

Finally set

$$E := \{\, E^{(1)},\ E^{(2)} \,\} \ \subseteq \ 2_{V,2} \setminus \{\emptyset\}.$$

Then $\mathrm{RSHG}^{(0,2)} = (V, E)$ is a $(0,2)$-recursive SuperHyperGraph. Here $n = 0$ yields an ordinary ground vertex set, while $k = 2$ permits two-step recursion in the edge objects: the edge $E^{(2)}$ explicitly contains the lower-level edge-object $E^{(1)}$.

Chapter 3

Some Graph Classes of Directed Super-HyperGraphs

In this chapter, we discuss several graph classes of directed SuperHyperGraphs and describe their fundamental properties.

3.1 Bidirected SuperHyperGraph

One of the well-known extended notions of a directed graph is the bidirected graph. A bidirected graph assigns to each vertex–edge incidence a sign indicating whether the edge is locally directed toward or away from that vertex [59–62]. A bidirected hypergraph assigns such signs to vertex–hyperedge incidences, requiring that the signed values on each hyperedge sum to zero [63]. A bidirected superhypergraph assigns signs to supervertex–superedge incidences, again imposing that each superedge has total signed sum zero [11, 63].

Definition 3.1.1 (Bidirected Graph). [59] A *bidirected graph* (also called a *bigraph*) is a pair

$$B = (G, \tau),$$

where $G = (V, E)$ is a simple undirected graph (no loops and no parallel edges), and

$$\tau : V \times E \to \{-1, 0, 1\}$$

is a *bidirection function* such that for every vertex–edge pair (v, e):

1. $\tau(v, e) = 1$ means that the edge e is locally directed *towards* v;

2. $\tau(v, e) = -1$ means that the edge e is locally directed *away from* v;

3. $\tau(v, e) = 0$ means that v is not incident to e.

The graph G is called the *underlying graph* of B.

Definition 3.1.2 (Bidirected Hypergraph). [63] A *bidirected hypergraph* is a triple

$$H = (V, E, \tau),$$

where V is a nonempty set of vertices, E is a family of nonempty subsets of V (hyperedges), and

$$\tau : V \times E \to \{-1, 0, 1\}$$

is a bidirection function satisfying:

$$\tau(v, e) = 0 \iff v \notin e,$$

and, additionally, for each hyperedge $e \in E$ we impose the *balancing condition*

$$\sum_{v \in e} \tau(v, e) = 0.$$

Definition 3.1.3 (Bidirected Superhypergraph). [63] A *bidirected superhypergraph* is a quadruple

$$\mathcal{H} = (V, S, E, \tau),$$

where:

1. V is a nonempty set of (base) vertices;
2. S is a set of nonempty subsets of V, called *supervertices*;
3. E is a family of *superedges*, where each $e \in E$ is a nonempty subset of S;
4. $\tau : S \times E \to \{-1, 0, 1\}$ is a bidirection function such that

   $$\tau(s, e) = 0 \iff s \notin e,$$

   and for each superedge $e \in E$ we impose the *balancing condition*

   $$\sum_{s \in e} \tau(s, e) = 0.$$

Example 3.1.4 (A small bidirected superhypergraph). Let the base vertex set be

$$V = \{1, 2, 3, 4\}.$$

Define three supervertices (nonempty subsets of V) by

$$s_1 := \{1, 2\}, \qquad s_2 := \{2, 3\}, \qquad s_3 := \{3, 4\},$$

and set

$$S := \{s_1, s_2, s_3\}.$$

Let the superedge family be

$$E := \{e_1, e_2\}, \qquad e_1 := \{s_1, s_2\}, \quad e_2 := \{s_2, s_3\}.$$

Define the bidirection function $\tau : S \times E \to \{-1, 0, 1\}$ by specifying its nonzero values as follows:

$$\tau(s_1, e_1) = +1, \qquad \tau(s_2, e_1) = -1,$$

$$\tau(s_2, e_2) = +1, \qquad \tau(s_3, e_2) = -1,$$

and set $\tau(s, e) = 0$ for all other pairs (s, e) (equivalently, whenever $s \notin e$).

Then $\tau(s, e) = 0 \iff s \notin e$ holds by construction. Moreover, the balancing condition is satisfied for each superedge:

$$\sum_{s \in e_1} \tau(s, e_1) = \tau(s_1, e_1) + \tau(s_2, e_1) = +1 + (-1) = 0,$$

$$\sum_{s \in e_2} \tau(s, e_2) = \tau(s_2, e_2) + \tau(s_3, e_2) = +1 + (-1) = 0.$$

Hence

$$\mathcal{H} = (V, S, E, \tau)$$

is a bidirected superhypergraph. Intuitively, on each superedge one incident supervertex is locally oriented "towards" the superedge $(+1)$ and the other is locally oriented "away" from it (-1), and the total signed sum is balanced to zero.

3.2 Directed Acyclic Superhypergraphs (DASH)

A directed acyclic graph is a directed graph containing no directed cycles, admitting a topological ordering respecting all edges globally [64–66]. Directed acyclic graphs model causal, temporal, and dependency structures; they enable topological ordering, efficient scheduling and compilation, and underpin Bayesian networks, workflows, and version-control build systems (cf. [67–69]). A directed acyclic hypergraph is a directed hypergraph whose hyperedges create no directed cycles, allowing hierarchical topological ordering constraints overall [70, 71]. A directed acyclic SuperHypergraph is a directed n-SuperHyperGraph without directed superhyperedge cycles, supporting level-wise topological ordering across all nested structures [44]. The relevant definitions and related notions are presented below.

Definition 3.2.1 (Directed hypercycle)**.** Let $H = (V, E)$ be a directed hypergraph, where each hyperarc $e \in E$ is an ordered pair $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with nonempty $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$. A *directed hypercycle* in H is a sequence of distinct vertices

$$v_1, v_2, \ldots, v_k \in V \qquad (k \geq 2),$$

together with hyperarcs

$$e_1, e_2, \ldots, e_k \in E$$

such that, for each $i = 1, \ldots, k$,

$$v_i \in \mathrm{Tail}(e_i) \qquad \text{and} \qquad v_{i+1} \in \mathrm{Head}(e_i),$$

where indices are taken cyclically, i.e. $v_{k+1} := v_1$.

Definition 3.2.2 (Directed Acyclic Hypergraph (DAH))**.** A *directed acyclic hypergraph* (DAH) is a directed hypergraph $H = (V, E)$ that contains no directed hypercycle (Definition 3.2.1).

Definition 3.2.3 (Directed cycle in a directed n-SuperHyperGraph)**.** Let $\mathrm{DSHG}^{(n)} = (V, E)$ be a directed n-SuperHyperGraph in the tail/head form, where each directed n-superhyperedge $e \in E$ is an ordered pair $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with nonempty $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$ and $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$. A *directed cycle* in $\mathrm{DSHG}^{(n)}$ is a sequence of distinct n-supervertices

$$v_1, v_2, \ldots, v_k \in V \qquad (k \geq 2),$$

together with directed n-superhyperedges

$$e_1, e_2, \ldots, e_k \in E$$

such that, for each $i = 1, \ldots, k$,

$$v_i \in \mathrm{Tail}(e_i) \qquad \text{and} \qquad v_{i+1} \in \mathrm{Head}(e_i),$$

with the convention $v_{k+1} := v_1$.

Example 3.2.4 (A real-life directed cycle in a directed n-SuperHyperGraph)**.** *Scenario (circular approval in an organization).* Consider an organization in which decisions are approved at the level of *committees* (not individual people). Let each n-supervertex represent a committee (a set-valued object encoding its members, subcommittees, or roles), and let each directed n-superhyperedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ mean: "every committee in $\mathrm{Head}(e)$ can approve only after at least one committee in $\mathrm{Tail}(e)$ has approved."

Suppose there are three committees $v_1, v_2, v_3 \in V$ and directed n-superhyperedges

$$e_1 : \{v_1\} \to \{v_2\}, \qquad e_2 : \{v_2\} \to \{v_3\}, \qquad e_3 : \{v_3\} \to \{v_1\}.$$

Then v_1, v_2, v_3 together with e_1, e_2, e_3 form a directed cycle in the sense of Definition 3.2.3: the approval of v_1 requires v_3, v_3 requires v_2, and v_2 requires v_1, producing a circular dependency that blocks the decision.

Definition 3.2.5 (Directed Acyclic SuperHypergraph (DASH))**.** A *directed acyclic superhypergraph* (DASH) is a directed n-SuperHyperGraph $\mathrm{DSHG}^{(n)} = (V, E)$ that contains no directed cycle (Definition 3.2.3).

Example 3.2.6 (A real-life DASH (no circular dependency))**.** *Scenario (hierarchical release pipeline).* Consider a software release pipeline where work proceeds through hierarchical units: feature teams, integration groups, and a release board. Model each unit as an n-supervertex, and let a directed n-superhyperedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ mean: "any unit in $\mathrm{Head}(e)$ may proceed only after at least one unit in $\mathrm{Tail}(e)$ has completed."

Assume the dependencies follow the strict order

$$\text{Feature teams} \longrightarrow \text{Integration} \longrightarrow \text{Release board},$$

so every directed superhyperedge points from an earlier stage to a later stage, and no edge ever points backward. Then there is no directed cycle (no stage depends, directly or indirectly, on a later stage), so the resulting directed n-SuperHyperGraph is a directed acyclic superhypergraph (DASH) in the sense of Definition 3.2.5.

3.3 Multidirected SuperHyperGraph

Multidirected graph is a directed graph that permits multiple parallel directed edges between the same ordered pair of vertices, so edge multiplicity represents repeated or layered interactions [72, 73]. Multidirected hypergraph is a directed hypergraph in which each hyperedge has a tail set and a head set (or head vertex), and multiple identical directed hyperedges are allowed [54]. Multidirected SuperHyperGraph is a multidirected hypergraph whose vertices are level-n supervertices (iterated-powerset objects), with directed superhyperedges between supervertex sets and possible multiplicities [54].

Definition 3.3.1 (Multidirected Graph)**.** [72, 73] A *multidirected graph* is a 5-tuple

$$G = (V, E, s, t, m),$$

where V is a finite set of vertices, E is a finite set of directed edges (allowing repetitions), $s : E \to V$ assigns the *source* of each edge, $t : E \to V$ assigns the *target* of each edge, and

$$m : V \times V \to \mathbb{N}_0$$

is a *multiplicity function* such that $m(u, v)$ counts how many edges are directed from u to v.

Definition 3.3.2 (Multidirected Hypergraph)**.** [54] A *multidirected hypergraph* is a triple

$$H = (V, E, m),$$

where V is a finite vertex set, E is a finite set of directed hyperedges, and $m : E \to \mathbb{N}$ assigns a positive integer multiplicity to each hyperedge. Each hyperedge $e \in E$ is an ordered pair

$$e = \big(T(e), h(e)\big),$$

where $T(e) \subseteq V$ is a nonempty *tail* (a set of sources) and $h(e) \in V$ is a *head* (a single target). The value $m(e)$ records how many parallel instances of e occur.

Definition 3.3.3 (Multidirected n-SuperHyperGraph (Multidirected Superhypergraph))**.** [54] Fix an integer $n \geq 1$ and a finite base set V_0. Define iterated powersets by $\mathcal{P}^0(V_0) := V_0$ and $\mathcal{P}^{k+1}(V_0) := \mathcal{P}(\mathcal{P}^k(V_0))$. A *multidirected n-SuperHyperGraph* is a triple

$$SH = (V, E, m),$$

where

$$V \subseteq \mathcal{P}^n(V_0)$$

is a set of *n-supervertices*, and

$$E \subseteq \mathcal{P}(V) \times \mathcal{P}(V)$$

is a set of directed *n-superhyperedges*. Each $e \in E$ is an ordered pair

$$e = \big(T(e), H(e)\big)$$

with nonempty tail $T(e) \subseteq V$ and nonempty head $H(e) \subseteq V$. Finally, $m : E \to \mathbb{N}$ assigns a positive integer multiplicity to each directed n-superhyperedge.

Example 3.3.4 (Real-life interpretation of a multidirected n-SuperHyperGraph)**.** *Scenario (repeated shipments between hierarchical logistics groups).* Let base items be individual warehouses and hubs, collected in a finite base set V_0. A level-n supervertex $v \in V \subseteq \mathcal{P}^n(V_0)$ represents a *hierarchical logistics group*, for example a set of distribution regions, each region itself being a set of hubs, and so on (nested groupings).

A directed n-superhyperedge $e = (T(e), H(e)) \in E \subseteq \mathcal{P}(V) \times \mathcal{P}(V)$ models an operational rule of the form: "shipments can be dispatched from any group in the tail-set $T(e)$ toward any group in the head-set $H(e)$." The multiplicity $m(e) \in \mathbb{N}$ records how many *parallel* shipment contracts or recurring daily routes exist for the same tail/head pattern.

For instance, if $T(e) = \{v_{\text{Tokyo}}, v_{\text{Osaka}}\}$ and $H(e) = \{v_{\text{Sapporo}}\}$, then $m(e) = 5$ may represent five parallel truck departures per day from the Tokyo/Osaka group toward the Sapporo group, all sharing the same hierarchical origin and destination structure.

3.4 Weighted directed SuperHyperGraphs

A weighted directed graph assigns a numeric weight to each directed edge, representing cost, capacity, strength, or preference, between vertices [74–76]. A weighted directed hypergraph assigns a weight to each directed hyperarc from a tail-vertex set to a head-vertex set [77]. A weighted directed superhypergraph assigns weights to directed superhyperedges between tail and head sets of nested supervertices within n-level powerset hierarchies.

Definition 3.4.1 (Weight domain)**.** A *weight domain* is a nonempty set W together with a distinguished element 0_W. In most applications one takes $W = \mathrm{R}_{\geq 0}$ (nonnegative costs) or $W = \mathrm{R}$ (signed weights). If one wishes to aggregate weights along walks/flows, it is standard to assume that $(W, \oplus, 0_W)$ is a commutative monoid (e.g., $(\mathrm{R}_{\geq 0}, +, 0)$).

Definition 3.4.2 (Weighted directed graph)**.** [74, 75] Let W be a weight domain. A *weighted directed graph* (weighted digraph) is a triple

$$G = (V, A, \mathrm{wt}),$$

where V is a (finite) nonempty set of vertices, $A \subseteq V \times V$ is a set of *arcs*, and $\mathrm{wt} : A \to W$ is a *weight function.* For an arc $a = (u, v) \in A$, we write $\mathrm{Tail}(a) := u$ and $\mathrm{Head}(a) := v$.

If one wishes to forbid loops, additionally require $(v, v) \notin A$ for all $v \in V$.

Definition 3.4.3 (Weighted directed hypergraph)**.** [77] Let W be a weight domain and let V be a (finite) nonempty vertex set. Write

$$\mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\}.$$

A *weighted directed hypergraph* is a triple

$$H = (V, E, \mathrm{wt}),$$

where E is a (finite) set of *directed hyperarcs* and each $e \in E$ is an ordered pair

$$e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big) \in \mathcal{P}^*(V) \times \mathcal{P}^*(V),$$

and $\mathrm{wt} : E \to W$ assigns a weight to each directed hyperarc.

Definition 3.4.4 (Weighted directed n-SuperHyperGraph)**.** Let W be a weight domain. Fix an integer $n \geq 0$ and a finite nonempty base set V_0. Define iterated powersets by $\mathcal{P}^0(V_0) := V_0$ and $\mathcal{P}^{k+1}(V_0) := \mathcal{P}(\mathcal{P}^k(V_0))$. Let

$$V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$$

be a set of *n-supervertices*, and write $\mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\}$.

A *weighted directed n-SuperHyperGraph* is a triple

$$\mathrm{SHG}_{\mathrm{w}}^{(n)} = (V, E, \mathrm{wt}),$$

where E is a (finite) set of *directed n-superhyperedges* and each $e \in E$ is an ordered pair

$$e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big) \in \mathcal{P}^*(V) \times \mathcal{P}^*(V),$$

and $\mathrm{wt} : E \to W$ is a weight function on directed n-superhyperedges.

Example 3.4.5 (Real-life interpretation of a weighted directed n-SuperHyperGraph)**.** *Scenario (hierarchical procurement with bundle-level costs).* Let the base set V_0 be a finite catalogue of individual products (SKUs) in a company. A level-n supervertex $v \in V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ represents a *nested bundle* of products, such as "families of approved bundles" (e.g., a set of bundles, each bundle being a set of SKUs).

A directed n-superhyperedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e)) \in E \subseteq \mathcal{P}^*(V) \times \mathcal{P}^*(V)$ models a transition between *groups of bundles*, for example: "if any bundle-family in $\mathrm{Tail}(e)$ is selected, then procurement must also consider the bundle-families in $\mathrm{Head}(e)$." The weight $\mathrm{wt}(e) \in W$ records the associated cost (or penalty), such as expected total price increase, risk surcharge, or lead-time delay.

For instance, $\mathrm{wt}(e) = 3.5$ (days) may represent an additional expected lead-time incurred when moving from a set of standard bundle-families to a set of premium bundle-families required by compliance constraints.

3.5 Signed Directed SuperHyperGraphs

Signed directed graphs assign each directed edge a sign, modeling activating or inhibiting influence along oriented vertex-to-vertex interactions [78]. Signed directed hypergraphs assign each directed hyperarc a sign, capturing positive or negative multiway influence from tail sets to head sets [79]. Signed directed superhypergraphs assign signs to directed superhyperedges between tail and head supervertex sets, generalizing signed digraphs and hypergraphs.

Definition 3.5.1 (Signed directed graph)**.** A *signed directed graph* is a triple

$$\mathcal{G}^{\pm} = (V, A, \mathrm{sgn}),$$

where $G = (V, A)$ is a finite directed graph (arcs $A \subseteq V \times V$) and

$$\mathrm{sgn} : A \to \{+1, -1\}$$

assigns a sign to each directed arc.

Definition 3.5.2 (Signed directed hypergraph)**.** A *signed directed hypergraph* is a triple

$$\mathcal{H}^{\pm} = (V, E, \mathrm{sgn}),$$

where $H = (V, E)$ is a finite directed hypergraph whose hyperarcs are ordered pairs

$$e = (\mathrm{Tail}(e), \mathrm{Head}(e)), \qquad \emptyset \neq \mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V,$$

and

$$\mathrm{sgn} : E \to \{+1, -1\}$$

assigns a sign to each directed hyperarc.

Definition 3.5.3 (Signed directed n-SuperHyperGraph)**.** Fix $n \geq 0$ and a nonempty base set S. Let

$$\mathrm{DSHG}^{(n)} = (V, E, \partial^{-}, \partial^{+})$$

be a directed n-SuperHyperGraph in incidence form, i.e.

$$V \subseteq \mathcal{P}^{n}(S) \setminus \{\emptyset\}, \qquad \partial^{-}, \partial^{+} : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Write $\mathrm{Tail}(e) := \partial^{-}(e)$ and $\mathrm{Head}(e) := \partial^{+}(e)$. A *signed directed n-SuperHyperGraph* is a quadruple

$$\mathcal{S}^{\pm} = \big(V, E, \partial^{-}, \partial^{+}, \mathrm{sgn}\big),$$

where

$$\mathrm{sgn} : E \to \{+1, -1\}$$

assigns a sign to each directed n-superhyperedge identifier.

Example 3.5.4 (Real-life interpretation of a signed directed n-SuperHyperGraph)**.** *Scenario (positive/negative influence among hierarchical teams in a company).* Let the base set S consist of individual employees. A level-n supervertex $v \in V \subseteq \mathcal{P}^{n}(S) \setminus \{\emptyset\}$ represents a *nested* organizational unit, e.g., a portfolio of squads, each squad being a set of employees.

A directed superhyperedge identifier $e \in E$ with $\mathrm{Tail}(e) \subseteq V$ and $\mathrm{Head}(e) \subseteq V$ models an influence relation from a set of source units to a set of target units, such as: "initiatives led by any unit in $\mathrm{Tail}(e)$ affect the performance of units in $\mathrm{Head}(e)$." The sign map $\mathrm{sgn} : E \to \{+1, -1\}$ encodes the *polarity* of that influence:

$$\mathrm{sgn}(e) = +1 \text{ means the influence is supportive (synergy)},$$

$$\mathrm{sgn}(e) = -1 \text{ means the influence is inhibiting (resource competition)}.$$

For example, a positive superedge may represent that the *Developer Experience* group and the *Platform Tools* group jointly accelerate multiple product teams (synergy), whereas a negative superedge may represent that two large programs draw from the same scarce specialists, reducing the throughput of the affected teams (competition).

Theorem 3.5.5 (Signed directed SuperHyperGraphs generalize signed directed graphs and hypergraphs)**.** *Let*

$$\mathcal{S}^{\pm} = \big(V, E, \partial^{-}, \partial^{+}, \mathrm{sgn}\big)$$

be a signed directed n-SuperHyperGraph (Definition 3.5.3).

(i) ***(Hypergraph specialization.)*** *If* $n = 0$ *and we view* $V \subseteq \mathcal{P}^0(S) = S$ *as an ordinary vertex set, then*

$$\mathcal{S}_0^{\pm} := (V, E, \mathrm{sgn})$$

is a signed directed hypergraph in the sense of Definition 3.5.2, with hyperarcs $(\mathrm{Tail}(e), \mathrm{Head}(e))$ *and the same sign map* sgn*.*

(ii) ***(Graph specialization.)*** *Assume* $n = 0$ *and every superhyperedge identifier* $e \in E$ *has singleton tail and singleton head:*

$$\mathrm{Tail}(e) = \{u\}, \qquad \mathrm{Head}(e) = \{v\} \qquad (u, v \in V).$$

Define the arc set

$$A := \{(u, v) \in V \times V : \exists e \in E \text{ with } \mathrm{Tail}(e) = \{u\},\ \mathrm{Head}(e) = \{v\}\}.$$

If the structure is simple *in the sense that there is at most one* $e \in E$ *for each ordered pair* (u, v)*, then defining* $\mathrm{sgn}(u, v) := \mathrm{sgn}(e)$ *yields a signed directed graph* (V, A, sgn) *in the sense of Definition 3.5.1.*

Proof. **(i)** When $n = 0$, supervertices are ordinary vertices, and $\mathrm{Tail}(e), \mathrm{Head}(e)$ are nonempty subsets of V. Hence each e defines a directed hyperarc $(\mathrm{Tail}(e), \mathrm{Head}(e))$ on V. Keeping the same label map $\mathrm{sgn} : E \to \{+1, -1\}$ gives precisely a signed directed hypergraph.

(ii) Under the singleton assumption, each $e \in E$ determines an ordered pair (u, v) with $\mathrm{Tail}(e) = \{u\}$ and $\mathrm{Head}(e) = \{v\}$. If the directed incidence pattern is simple (no parallel e with the same (u, v)), the assignment $\mathrm{sgn}(u, v) := \mathrm{sgn}(e)$ is well-defined and produces a sign map on the induced arc set A. Therefore (V, A, sgn) satisfies Definition 3.5.1. □

3.6 Rooted Directed SuperHyperGraph

A rooted directed graph is a directed graph with one distinguished vertex called the root; in the common "flow graph" sense, every vertex is reachable from the root by a directed path(cf. [80–82]). A rooted directed hypergraph is a directed hypergraph with a distinguished root vertex; typically, every vertex is reachable from the root by a directed hyperpath following tail-to-head incidence(cf. [83]). A rooted directed superhypergraph is a directed n-superhypergraph with a distinguished root supervertex; usually, every supervertex is reachable from it by a directed superhyperpath through the nested superedge structure.

Definition 3.6.1 (Directed walk and reachability in a digraph)**.** Let $G = (V, A)$ be a directed graph with $A \subseteq V \times V$. A *directed walk* from u to v is a finite sequence of vertices

$$u = v_0,\ v_1,\ \dots,\ v_k = v \qquad (k \geq 0)$$

such that $(v_{i-1}, v_i) \in A$ for all $i = 1, \dots, k$. We say that v is *reachable* from u (denoted $u \rightsquigarrow v$) if such a directed walk exists.

Definition 3.6.2 (Directed hyperwalk and reachability)**.** Let $H = (V, E)$ be a directed hypergraph in the sense that each hyperarc $e \in E$ is an ordered pair $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with $\mathrm{Tail}(e), \mathrm{Head}(e) \in \mathcal{P}(V) \setminus \{\emptyset\}$. A *directed hyperwalk* from u to v is a finite alternating sequence

$$u = x_0,\, e_1,\, x_1,\, e_2,\, \ldots,\, e_k,\, x_k = v \qquad (k \geq 0),$$

where $x_i \in V$ and $e_i \in E$, such that for each $i = 1, \ldots, k$,

$$x_{i-1} \in \mathrm{Tail}(e_i) \qquad \text{and} \qquad x_i \in \mathrm{Head}(e_i).$$

We say that v is *reachable* from u (denoted $u \rightsquigarrow v$) if such a directed hyperwalk exists.

Definition 3.6.3 (Directed superhyperwalk and reachability)**.** Fix $n \geq 0$ and a finite nonempty base set V_0. Let $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ be a set of n-supervertices, and let E be a family of directed n-superhyperedges, where each $e \in E$ is an ordered pair

$$e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big) \in \mathcal{P}(V) \setminus \{\emptyset\} \;\times\; \mathcal{P}(V) \setminus \{\emptyset\}.$$

A *directed superhyperwalk* from u to v (where $u, v \in V$) is a finite alternating sequence

$$u = s_0,\, e_1,\, s_1,\, e_2,\, \ldots,\, e_k,\, s_k = v \qquad (k \geq 0),$$

where $s_i \in V$ and $e_i \in E$, such that for each $i = 1, \ldots, k$,

$$s_{i-1} \in \mathrm{Tail}(e_i) \qquad \text{and} \qquad s_i \in \mathrm{Head}(e_i).$$

We say that v is *reachable* from u (denoted $u \rightsquigarrow v$) if such a directed superhyperwalk exists.

Example 3.6.4 (Real-life directed superhyperwalk and reachability)**.** *Scenario (hierarchical incident response escalation).* Let the base set V_0 be a finite set of on-call engineers. A level-n supervertex $s \in V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ represents a *nested team unit* (e.g., an incident squad, a group of squads, or a portfolio-level response group). A directed n-superhyperedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ encodes an escalation rule: if any unit in $\mathrm{Tail}(e)$ is engaged, then the units in $\mathrm{Head}(e)$ must be notified next.

For instance, an escalation may proceed

$$s_0 \xrightarrow{e_1} s_1 \xrightarrow{e_2} s_2,$$

where s_0 is an initial squad, s_1 is an incident-command group, and s_2 is an executive-response group. This corresponds to a directed superhyperwalk

$$u = s_0,\ e_1,\ s_1,\ e_2,\ v = s_2$$

with $s_0 \in \mathrm{Tail}(e_1)$, $s_1 \in \mathrm{Head}(e_1)$, $s_1 \in \mathrm{Tail}(e_2)$, and $s_2 \in \mathrm{Head}(e_2)$. Thus the executive-response group $v = s_2$ is *reachable* from the initial squad $u = s_0$ in the sense of Definition 3.6.3.

Definition 3.6.5 (Rooted directed graph)**.** A *rooted directed graph* is a triple

$$(G, r) \equiv (V, A, r),$$

where $G = (V, A)$ is a directed graph and $r \in V$ is a distinguished vertex called the *root.*

It is called *accessible* (or a *flow graph*) if

$$\forall\, v \in V\ \exists \text{ a directed walk from } r \text{ to } v, \quad \text{i.e.,} \quad r \rightsquigarrow v \ \text{ for all } v \in V,$$

where $\rightsquigarrow$ is as in Definition 3.6.1.

Definition 3.6.6 (Rooted directed hypergraph)**.** A *rooted directed hypergraph* is a triple

$$(H, r) \equiv (V, E, r),$$

where $H = (V, E)$ is a directed hypergraph (each $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with nonempty tail/head subsets of V) and $r \in V$ is the *root*.

It is called *accessible* if

$$\forall\, v \in V, \quad r \rightsquigarrow v$$

in the sense of Definition 3.6.2.

Definition 3.6.7 (Rooted directed n-SuperHyperGraph)**.** Fix $n \geq 0$ and a finite nonempty base set V_0. A *rooted directed n-SuperHyperGraph* is a triple

$$(\mathrm{SHG}^{(n)}, r) \equiv (V, E, r),$$

where

$$V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$$

is a set of n-supervertices, E is a family of directed n-superhyperedges with

$$e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big), \qquad \mathrm{Tail}(e), \mathrm{Head}(e) \in \mathcal{P}(V) \setminus \{\emptyset\},$$

and $r \in V$ is the distinguished *root supervertex*.

It is called *accessible* if

$$\forall\, v \in V, \quad r \rightsquigarrow v$$

in the sense of Definition 3.6.3.

Example 3.6.8 (Real-life rooted directed n-SuperHyperGraph)**.** *Scenario (rooted workflow hierarchy from a central coordinator).* Let the base set V_0 be a finite set of tasks in a large project. A level-n supervertex represents a nested collection of tasks (e.g., tasks grouped into work-packages, work-packages grouped into milestones, and so on). Let the distinguished root supervertex $r \in V$ represent the central coordination unit (e.g., the overall project board or master milestone).

Each directed superhyperedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ encodes an allowed dependency propagation from a set of upstream packages to a set of downstream packages. If the project is organized so that every work-package or milestone can be reached from r via a sequence of such dependency steps, then for every $v \in V$ there exists a directed superhyperwalk from r to v. In this case, $(\mathrm{SHG}^{(n)}, r)$ forms an *accessible* rooted directed n-SuperHyperGraph as in Definition 3.6.7.

Theorem 3.6.9 (Generalization property)**.** *Rooted directed n-SuperHyperGraphs (Definition 3.6.7) generalize rooted directed hypergraphs (Definition 3.6.6) and rooted directed graphs (Definition 3.6.5) in the following precise sense.*

(i) ***(Hypergraph case.)*** *Every rooted directed hypergraph* $(H, r) = (V, E, r)$ *induces a rooted directed* 0*-SuperHyperGraph* $(\mathrm{SHG}^{(0)}, r)$ *by taking* $V_0 := V$*, identifying* $\mathcal{P}^0(V_0) = V_0$*, setting the* 0*-supervertex set equal to* V*, and keeping the same hyperarc set* E *(with the same tails/heads). Conversely, every rooted directed* 0*-SuperHyperGraph is (canonically) a rooted directed hypergraph.*

Moreover, under this identification, reachability $r \rightsquigarrow v$ *via directed hyperwalks coincides with reachability via directed superhyperwalks.*

(ii) ***(Graph case.)*** *Every rooted directed graph* $(G, r) = (V, A, r)$ *induces a rooted directed* 0*-SuperHyperGraph* $(\mathrm{SHG}^{(0)}, r)$ *by taking* $V_0 := V$*, setting* $V := V_0$*, and replacing each arc* $(u, v) \in A$ *with the directed* 0*-superhyperedge*

$$e_{(u,v)} := (\{u\}, \{v\}) \in \mathcal{P}(V) \setminus \{\emptyset\} \times \mathcal{P}(V) \setminus \{\emptyset\}.$$

Conversely, any rooted directed 0*-SuperHyperGraph in which every directed superhyperedge has singleton tail and singleton head corresponds to a rooted directed graph.*

Under this identification, reachability by directed walks in G *coincides with reachability by directed superhyperwalks in* $\mathrm{SHG}^{(0)}$*.*

Proof. **(i) Hypergraph case.** Let $(H, r) = (V, E, r)$ be a rooted directed hypergraph. Put $V_0 := V$ and $n := 0$, so $\mathcal{P}^0(V_0) = V_0 = V$. Define $\mathrm{SHG}^{(0)} := (V, E)$ where each $e \in E$ is already an ordered pair $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with nonempty $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$. This satisfies Definition 3.6.7 for $n = 0$.

Conversely, if $(\mathrm{SHG}^{(0)}, r) = (V, E, r)$ is a rooted directed 0-SuperHyperGraph, then $V \subseteq \mathcal{P}^0(V_0) = V_0$ and each $e \in E$ is an ordered pair of nonempty subsets of V, hence (V, E, r) is a rooted directed hypergraph.

Finally, a directed hyperwalk $r = x_0, e_1, x_1, \ldots, e_k, x_k = v$ satisfies $x_{i-1} \in \mathrm{Tail}(e_i)$ and $x_i \in \mathrm{Head}(e_i)$, which is exactly the condition for a directed superhyperwalk when $n = 0$ (Definition 3.6.3). Thus reachability coincides.

(ii) Graph case. Let $(G, r) = (V, A, r)$ be a rooted directed graph. Put $V_0 := V$ and $n := 0$. Define the directed 0-SuperHyperGraph $\mathrm{SHG}^{(0)} = (V, E)$ by

$$E := \{\, e_{(u,v)} \mid (u, v) \in A \,\}, \qquad e_{(u,v)} := (\{u\}, \{v\}).$$

Then each $e_{(u,v)}$ has nonempty tail/head subsets of V, so $(\mathrm{SHG}^{(0)}, r)$ is a rooted directed 0-SuperHyperGraph.

Conversely, if $(\mathrm{SHG}^{(0)}, r) = (V, E, r)$ is such that every $e \in E$ has the form $e = (\{u\}, \{v\})$ for some $u, v \in V$, define

$$A := \{\, (u, v) \in V \times V \mid (\{u\}, \{v\}) \in E \,\}.$$

Then (V, A, r) is a rooted directed graph.

For reachability, a directed walk $r = v_0, v_1, \ldots, v_k = v$ in G with $(v_{i-1}, v_i) \in A$ corresponds to the directed superhyperwalk

$$r = v_0,\ e_{(v_0,v_1)},\ v_1,\ e_{(v_1,v_2)},\ \ldots,\ e_{(v_{k-1},v_k)},\ v_k = v$$

in $\mathrm{SHG}^{(0)}$, because $v_{i-1} \in \{v_{i-1}\} = \mathrm{Tail}(e_{(v_{i-1},v_i)})$ and $v_i \in \{v_i\} = \mathrm{Head}(e_{(v_{i-1},v_i)})$. The reverse implication is identical. Hence reachability coincides. □

3.7 Poly-SuperHyperGraph

A poly-graph is a higher-dimensional directed graph whose k-cells connect composable paths of (k−1)-cells as boundaries [84–86]. A poly-hypergraph extends a directed hypergraph by allowing higher cells whose sources and targets are composable hyperpaths of lower cells. A poly-superhypergraph extends directed n-superhypergraphs by adding higher cells whose boundaries are composable superhyperpaths among nested supervertices.

Notation 3.7.1 (Directed paths)**.** *Let $G = (V, A)$ be a directed graph. A* directed path *of length $m \geq 0$ is a sequence*

$$v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \xrightarrow{a_m} v_m,$$

where each $a_i = (v_{i-1}, v_i) \in A$. Its source *and* target *are $s(\pi) := v_0$ and $t(\pi) := v_m$. For $m = 0$, the empty path at v has $s = t = v$.*

Notation 3.7.2 (Directed hyperpaths)**.** *(cf. [87, 88]) Let $H = (V, E)$ be a directed hypergraph, where each $e \in E$ is an ordered pair $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with nonempty $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$. A* directed hyperpath *of length $m \geq 0$ is an alternating sequence*

$$v_0,\ e_1,\ v_1,\ e_2,\ \ldots,\ e_m,\ v_m$$

such that for each $i = 1, \ldots, m$,

$$v_{i-1} \in \mathrm{Tail}(e_i) \quad \textit{and} \quad v_i \in \mathrm{Head}(e_i).$$

Its source and target are $s(\pi) := v_0$ and $t(\pi) := v_m$. (For $m = 0$ we again allow the empty path at v.)

Notation 3.7.3 (Directed superhyperpaths)**.** *Let $\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$ be a directed n-SuperHyperGraph in the sense of Definition 2.2.3, and write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. A* directed superhyperpath *of length $m \geq 0$ is an alternating sequence*

$$x_0,\ e_1,\ x_1,\ e_2,\ \ldots,\ e_m,\ x_m \qquad (x_i \in V,\ e_i \in E),$$

such that for each $i = 1, \ldots, m$,

$$x_{i-1} \in \mathrm{Tail}(e_i) \quad \textit{and} \quad x_i \in \mathrm{Head}(e_i).$$

Its source and target are $s(\pi) := x_0$ and $t(\pi) := x_m$.

Definition 3.7.4 (m-polygraph)**.** Fix an integer $m \geq 1$. An *m-polygraph* is a family

$$\mathcal{P} = \left(P_0, P_1, \ldots, P_m;\ s_k, t_k\right)_{k=1}^{m}$$

where P_k is a set of k-cells for each k, and:

- The 1-skeleton (P_0, P_1) is a directed graph, i.e. each $a \in P_1$ has $s_1(a), t_1(a) \in P_0$.

- For each $k \geq 2$, the source and target maps

$$s_k, t_k : P_k \to \mathrm{Path}_{k-1}(\mathcal{P})$$

assign to every k-cell x a *(k–1)-path* built from $(k-1)$-cells, with the *parallelism condition* that the overall endpoints agree:

$$s\big(s_k(x)\big) = s\big(t_k(x)\big), \qquad t\big(s_k(x)\big) = t\big(t_k(x)\big).$$

(Here $s(\cdot), t(\cdot)$ denote the endpoint maps of a path, extended inductively.)

Definition 3.7.5 (m-poly-hypergraph)**.** Fix an integer $m \geq 1$. An *m-poly-hypergraph* is a family

$$\mathcal{H} = \big(H_0, H_1, \dots, H_m;\ s_k, t_k\big)_{k=2}^{m},$$

where:

- H_0 is a finite set of vertices.

- H_1 is a finite set of directed hyperarcs; each $e \in H_1$ has nonempty tail/head subsets

$$\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq H_0.$$

Thus (H_0, H_1) is a directed hypergraph.

- For each $k \geq 2$, the maps

$$s_k, t_k : H_k \to \mathrm{Path}_{k-1}(\mathcal{H})$$

assign to each k-cell x two *parallel* $(k-1)$-paths. In dimension 1, the $(k-1)$-paths are directed hyperpaths in the sense of Notation 3.7.2; in higher dimensions, $(k-1)$-paths are ordinary composable strings of $(k-1)$-cells with matching endpoints. Parallelism means equality of endpoints:

$$s\big(s_k(x)\big) = s\big(t_k(x)\big), \qquad t\big(s_k(x)\big) = t\big(t_k(x)\big).$$

Definition 3.7.6 ((m,n)-poly-superhypergraph)**.** Fix integers $m \geq 1$ and $n \geq 0$, and a finite nonempty base set S. An *(m,n)-poly-superhypergraph* is a family

$$\mathcal{S} = \big(S_0, S_1, \dots, S_m;\ \partial^-, \partial^+,\ s_k, t_k\big)_{k=2}^{m}$$

such that:

- $S_0 =: V$ is a finite set of n-supervertices with

$$V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}.$$

- $S_1 =: E$ is a finite set of directed n-superhyperedge identifiers, equipped with incidence maps

$$\partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. Thus the 1-skeleton $(V, E, \partial^-, \partial^+)$ is a directed n-SuperHyperGraph.

- For each $k \geq 2$, the maps
$$s_k, t_k : S_k \to \mathrm{Path}_{k-1}(\mathcal{S})$$
assign to each k-cell x two *parallel* $(k-1)$-paths. In dimension 1, these are directed superhyperpaths as in Notation 3.7.3; in higher dimensions, they are ordinary composable strings of cells. Parallelism means equality of endpoints:
$$s\big(s_k(x)\big) = s\big(t_k(x)\big), \qquad t\big(s_k(x)\big) = t\big(t_k(x)\big).$$

Example 3.7.7 (Real-life $(2,1)$-poly-superhypergraph: two equivalent approval workflows)**.** *Scenario.* Consider an organization where releases must pass through *groups* of tasks/owners, and where some multi-step approval chains can be replaced by an equivalent single approval (e.g., due to a standing waiver).

Let the base set be a set of atomic tasks/owners
$$S = \{\mathsf{Dev}, \mathsf{QA}, \mathsf{Sec}, \mathsf{Ops}\}, \qquad n = 1, \qquad m = 2.$$
Thus 1-supervertices are subsets of S, i.e. elements of $\mathcal{P}^1(S) = \mathcal{P}(S)$. Define three 1-supervertices (teams) by
$$v_A := \{\mathsf{Dev}, \mathsf{QA}\}, \qquad v_B := \{\mathsf{Sec}\}, \qquad v_C := \{\mathsf{Ops}\},$$
and set
$$V := \{v_A, v_B, v_C\} \subseteq \mathcal{P}(S) \setminus \{\emptyset\}.$$

Directed 1*-superhyperedges (the* 1*-skeleton).* Let $E = \{e_1, e_2, e_3\}$ be directed superedge identifiers and define $\partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}$ by
$$\partial^-(e_1) = \{v_A\}, \quad \partial^+(e_1) = \{v_B\}$$
(“Dev/QA approval triggers Security review”),
$$\partial^-(e_2) = \{v_B\}, \quad \partial^+(e_2) = \{v_C\}$$
(“Security review triggers Ops rollout”),
$$\partial^-(e_3) = \{v_A\}, \quad \partial^+(e_3) = \{v_C\}$$
(“Dev/QA can directly authorize Ops rollout under a waiver”).

Then $(V, E, \partial^-, \partial^+)$ is a directed 1-SuperHyperGraph (the 1-skeleton).

A 2*-cell (poly-level rule).* Let $S_2 := \{\alpha\}$ consist of a single 2-cell α expressing that the two-step workflow e_1 then e_2 is an accepted substitute for the direct workflow e_3. Formally, define $s_2, t_2 : S_2 \to \mathrm{Path}_1(\mathcal{S})$ by
$$s_2(\alpha) := \big(v_A,\ e_1,\ v_B,\ e_2,\ v_C\big), \qquad t_2(\alpha) := \big(v_A,\ e_3,\ v_C\big).$$
Both $s_2(\alpha)$ and $t_2(\alpha)$ have the same endpoints v_A and v_C, so they are parallel 1-paths. Hence
$$\mathcal{S} = \big(S_0, S_1, S_2; \partial^-, \partial^+, s_2, t_2\big) \quad \text{with} \quad S_0 = V,\ S_1 = E$$
is a $(2,1)$-poly-superhypergraph. Intuitively, α records a higher-level equivalence (or rewrite) between two composable directed superhyperpaths in a hierarchical approval pipeline.

Theorem 3.7.8 (Poly-superhypergraphs generalize polygraphs and poly-hypergraphs)**.** *Let $\mathcal{S}$ be a (m,n)-poly-superhypergraph as in Definition 3.7.6.*

(i) ***(Underlying n-SuperHyperGraph structure.)*** *Define the undirected superedge family*

$$E_{\text{und}} := \{ \text{Tail}(e) \cup \text{Head}(e) \mid e \in E \} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then $\text{SHG}_{\mathcal{S}}^{(n)} := (V, E_{\text{und}})$ is an n-SuperHyperGraph on S.

(ii) ***(Reduction to a poly-hypergraph.)*** *If $n = 0$ (so $V \subseteq \mathcal{P}^0(S) = S$ consists of ordinary vertices), then the 1-skeleton $(V, E, \partial^-, \partial^+)$ is a directed hypergraph on V, and the higher-dimensional boundary maps (s_k, t_k) make $\mathcal{S}$ into an m-poly-hypergraph in the sense of Definition 3.7.5.*

(iii) ***(Reduction to a polygraph.)*** *Assume $n = 0$ and, moreover, each directed superhyperedge has singleton tail and head:*

$$\text{Tail}(e) = \{u\}, \qquad \text{Head}(e) = \{v\} \qquad (u, v \in V).$$

Define a directed edge (u,v) for each such e. Then the 1-skeleton is an ordinary directed graph, directed superhyperpaths coincide with ordinary directed paths, and $\mathcal{S}$ specializes to an m-polygraph in the sense of Definition 3.7.4.

Proof. **(i)** By Definition 3.7.6, $V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}$. For each $e \in E$, the sets $\text{Tail}(e), \text{Head}(e)$ are nonempty subsets of V, hence $\text{Tail}(e) \cup \text{Head}(e) \in \mathcal{P}(V) \setminus \{\emptyset\}$. Therefore $E_{\text{und}} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, and $\text{SHG}_{\mathcal{S}}^{(n)} = (V, E_{\text{und}})$ satisfies the definition of an n-SuperHyperGraph.

(ii) If $n = 0$, then $\mathcal{P}^0(S) = S$ and $V \subseteq S$ is an ordinary vertex set. Also $\partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}$ makes (V, E) a directed hypergraph. The higher-cell boundary maps (s_k, t_k) are unchanged, and the notion of 1-paths becomes Notation 3.7.2. Hence $\mathcal{S}$ is exactly an m-poly-hypergraph.

(iii) Under the singleton hypothesis, each $e \in E$ determines a unique arc (u, v). A directed superhyperpath $x_0, e_1, x_1, \ldots, e_m, x_m$ satisfies $x_{i-1} \in \{u_i\}$ and $x_i \in \{v_i\}$, hence $x_{i-1} = u_i$ and $x_i = v_i$, so it is precisely a directed path in the induced digraph. Thus the path notions and the endpoint maps agree, and the higher boundary maps (s_k, t_k) satisfy the polygraph axioms with respect to ordinary paths. □

3.8 Random directed Superhypergraphs

Random directed graphs include each possible arc independently with a specified probability, producing a random digraph on fixed vertices [89–92]. Random directed hypergraphs include each possible tail–head hyperarc independently with prescribed probabilities, producing random directed multiway relations (cf. [51, 93, 94]). Random directed superhypergraphs include directed superhyperedges between supervertex sets independently with prescribed probabilities, generalizing random digraphs and hypergraphs.

Definition 3.8.1 (Random directed graph: Erdős–Rényi digraph)**.** Let V be a finite vertex set and let

$$\overrightarrow{\binom{V}{2}} := \{(u,v) \in V \times V : \ u \neq v\}$$

be the set of possible loopless arcs. Fix $p \in [0,1]$. A *random directed graph* (Erdős–Rényi digraph) on V with parameter p is the random digraph

$$D(V,p) = (V,A),$$

where $\left(\mathbf{1}_{(u,v)\in A}\right)_{(u,v)\in\overrightarrow{\binom{V}{2}}}$ are independent Bernoulli(p) variables. Equivalently, for each $(u,v) \in \overrightarrow{\binom{V}{2}}$ we include (u,v) in A independently with probability p.

Definition 3.8.2 (Random directed hypergraph)**.** Let V be a finite vertex set and write

$$\mathcal{P}^{*}(V) := \mathcal{P}(V) \setminus \{\emptyset\}, \qquad \mathcal{A}(V) := \mathcal{P}^{*}(V) \times \mathcal{P}^{*}(V).$$

Fix parameters $\mathbf{p} = (p_{r,s})_{r,s\geq 1}$ with $p_{r,s} \in [0,1]$. A *random directed hypergraph* on V with parameter $\mathbf{p}$ is the random pair

$$H(V,\mathbf{p}) = (V,E),$$

where for each potential hyperarc $e = (T,H) \in \mathcal{A}(V)$ we include e in E independently with probability

$$\mathbb{P}\big((T,H) \in E\big) = p_{|T|,|H|}.$$

Thus every realized hyperarc $e = (T(e),H(e))$ has nonempty tail/head subsets of V.

Remark 3.8.3 (Degree-constrained (uniform) random directed hypergraphs)**.** Another standard "null model" is the *uniform* distribution on a fixed directed hypergraph space with prescribed degree sequence. For instance, Kraakman–Stegehuis define a directed hypergraph as (V,A) where the hyperarc family is a multiset and each hyperarc has a tail/head that are (possibly) multisets of V, i.e. $a = (a^t,a^h)$ with a^t, a^h multisets of V. They also define spaces $\mathrm{Hy}_x^y(d)$ of directed hypergraphs with a fixed degree sequence d, with optional restrictions on self-loops/degenerate/multi-hyperarcs. Choosing H uniformly at random from such a space yields a degree-preserving random directed hypergraph model.

Definition 3.8.4 (Random directed n-SuperHyperGraph)**.** Fix an integer $n \geq 0$ and a finite nonempty base set V_0. Let $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ be a fixed set of n-supervertices and define

$$\mathcal{P}^{*}(V) := \mathcal{P}(V) \setminus \{\emptyset\}, \qquad \mathcal{A}(V) := \mathcal{P}^{*}(V) \times \mathcal{P}^{*}(V).$$

Fix parameters $\mathbf{p} = (p_{r,s})_{r,s\geq 1}$ with $p_{r,s} \in [0,1]$. A *random directed n-SuperHyperGraph* on (V_0,V) with parameter $\mathbf{p}$ is the random quadruple

$$\mathrm{RDSHG}^{(n)}(V_0,V;\mathbf{p}) = (V,E,\partial^-,\partial^+),$$

constructed as follows:

- For each $(T,H) \in \mathcal{A}(V)$, include (T,H) in E independently with probability $p_{|T|,|H|}$.
- Define incidence maps by projections:

$$\partial^-(T,H) := T, \qquad \partial^+(T,H) := H \qquad \big((T,H) \in E\big).$$

Thus every realized superhyperedge $e \in E$ is directed from the nonempty tail set $\mathrm{Tail}(e) = \partial^-(e)$ to the nonempty head set $\mathrm{Head}(e) = \partial^+(e)$.

Remark 3.8.5 (Underlying SuperHyperGraph)**.** For any outcome $(V, E, \partial^-, \partial^+)$ of $\mathrm{RDSHG}^{(n)}$, forgetting direction yields an undirected n-SuperHyperGraph (V, E_{und}) by setting

$$E_{\mathrm{und}} := \{\ \partial^-(e) \cup \partial^+(e) \mid e \in E\ \} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Example 3.8.6 (Real-life interpretation of a random directed n-SuperHyperGraph)**.** *Scenario (stochastic information propagation between nested communities).* Let the base set V_0 be a finite set of individuals in a city (e.g., residents). A level-n supervertex $v \in V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ represents a *nested community* such as "a collection of neighborhoods," where each neighborhood is itself a subset of residents, and higher levels group neighborhoods into districts, etc.

Suppose a public-health message (e.g., vaccination guidance) can be relayed from one *group of communities* to another, but whether a particular relay happens depends on random circumstances (local events, media exposure, volunteer availability). Model each possible directed relay pattern $(T, H) \in \mathcal{P}^*(V) \times \mathcal{P}^*(V)$ as a potential directed superhyperedge, where T is the set of source communities and H is the set of target communities. Choose parameters $p_{r,s} \in [0, 1]$ so that $p_{r,s}$ is the probability that a relay from any r source-communities to any s target-communities occurs.

Then sampling $\mathrm{RDSHG}^{(n)}(V_0, V; \mathbf{p})$ produces a random directed n-SuperHyperGraph in which each realized directed superhyperedge represents a randomly occurring multi-community relay of information, directed from the active source communities to the influenced target communities.

Theorem 3.8.7 (Random directed superhypergraphs generalize random directed graphs and hypergraphs)**.** *Let* $\mathrm{RDSHG}^{(n)}(V_0, V; \mathbf{p})$ *be as in Definition 3.8.4.*

(i) *(**Hypergraph specialization.**) If $n = 0$ and we take $V_0 := V$ (so $V \subseteq \mathcal{P}^0(V_0) = V_0$ is an ordinary vertex set), then* $\mathrm{RDSHG}^{(0)}(V, V; \mathbf{p})$ *is exactly the random directed hypergraph $H(V, \mathbf{p})$ from Definition 3.8.2 (after identifying E with its set of hyperarcs).*

(ii) *(**Graph specialization.**) Assume $n = 0$ and impose the parameter restriction*

$$p_{r,s} = 0 \quad \textit{whenever } (r, s) \neq (1, 1).$$

Then $\mathrm{RDSHG}^{(0)}(V, V; \mathbf{p})$ *reduces to a random directed graph $D(V, p)$ with $p := p_{1,1}$, by identifying each selected superhyperedge $(\{u\}, \{v\})$ with the arc (u, v).*

Proof. **(i)** When $n = 0$ we have $\mathcal{P}^0(V_0) = V_0$, hence V is an ordinary vertex set. Moreover, $\mathcal{P}^*(V) \times \mathcal{P}^*(V)$ is exactly the set of potential directed hyperarcs on V, and the construction of E in Definition 3.8.4 coincides with Definition 3.8.2. The incidence maps ∂^-, ∂^+ are the projections onto tail/head, so the resulting random object is precisely $H(V, \mathbf{p})$.

(ii) Under $p_{r,s} = 0$ for $(r, s) \neq (1, 1)$, the only hyperarcs that can appear are those with singleton tail and singleton head, i.e. pairs $(\{u\}, \{v\})$ with $u \neq v$. Define the arc set

$$A := \{(u, v) \in V \times V : \ u \neq v, \ (\{u\}, \{v\}) \in E\}.$$

Then for each ordered pair (u, v) with $u \neq v$,

$$\mathbb{P}\big((u, v) \in A\big) = \mathbb{P}\big((\{u\}, \{v\}) \in E\big) = p_{1,1} = p,$$

and the independence of different $(\{u\}, \{v\})$ events implies independence of different arc events. Hence (V, A) has the law of the Erdős–Rényi digraph $D(V, p)$ from Definition 3.8.1. □

3.9 Planar Directed SuperHyperGraphs

A planar directed graph is a digraph whose underlying undirected graph admits a planar embedding without edge crossings in the plane [95–98]. A planar directed hypergraph is a directed hypergraph whose structure graph has a planar embedding, representing each hyperarc without crossings [99]. A planar directed superhypergraph is a directed n-SuperHyperGraph whose structure graph is planar, embedding supervertices and directed superedges without crossings.

Definition 3.9.1 (Planar directed graph)**.** [95, 96] A directed graph (digraph) $G = (V, A)$ is *planar* if its underlying undirected graph

$$G^{\mathrm{und}} = (V, \{\{u, v\} \mid (u, v) \in A \text{ or } (v, u) \in A\})$$

admits a planar embedding (equivalently, G^{und} is a planar graph).

Definition 3.9.2 (Structure graph of a directed hypergraph)**.** Let $H = (V, E)$ be a directed hypergraph, where each hyperarc $e \in E$ is an ordered pair $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with $\emptyset \neq \mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$. Define the *structure graph* of H as the digraph

$$\mathsf{S}(H) := (V \ \cup \ (E \times \{-, +\}), \ B),$$

where we write $e^- := (e, -)$ and $e^+ := (e, +)$, and

$$B := \{(v, e^-) \mid e \in E, \ v \in \mathrm{Tail}(e)\} \ \cup \ \{(e^+, v) \mid e \in E, \ v \in \mathrm{Head}(e)\} \ \cup \ \{(e^-, e^+) \mid e \in E\}.$$

Definition 3.9.3 (Planar directed hypergraph)**.** A directed hypergraph H is *planar* if its structure graph $\mathsf{S}(H)$ is a planar digraph, i.e. if $\mathsf{S}(H)^{\mathrm{und}}$ is planar.

Definition 3.9.4 (Structure graph of a directed n-SuperHyperGraph)**.** Fix $n \geq 0$ and a nonempty base set S. Let

$$\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$$

be a directed n-SuperHyperGraph in incidence form, i.e.

$$V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}, \qquad \partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. Define the *structure graph* of $\mathrm{DSHG}^{(n)}$ as the digraph

$$\mathsf{S}\big(\mathrm{DSHG}^{(n)}\big) := \big(V \ \cup \ (E \times \{-, +\}), \ B\big),$$

where $e^- := (e, -)$ and $e^+ := (e, +)$, and

$$B := \{(x, e^-) \mid e \in E, \ x \in \mathrm{Tail}(e)\} \ \cup \ \{(e^+, y) \mid e \in E, \ y \in \mathrm{Head}(e)\} \ \cup \ \{(e^-, e^+) \mid e \in E\}.$$

Definition 3.9.5 (Planar directed n-SuperHyperGraph)**.** A directed n-SuperHyperGraph $\mathrm{DSHG}^{(n)}$ is *planar* if its structure graph $\mathsf{S}(\mathrm{DSHG}^{(n)})$ is a planar digraph (Definition 3.9.1).

Example 3.9.6 (Real-life planar directed n-SuperHyperGraph)**.** *Scenario (multi-level road-flow on a planar map).* Consider a city map drawn in the plane, where intersections are base vertices and roads are directed by typical traffic flow (one-way streets, preferred directions). Let the base set V_0 be the set of intersections. A level-n supervertex $v \in V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ represents a nested geographic aggregation, such as *lanes within a road segment*, *road segments within a district*, or *districts within a region* (depending on n).

A directed n-superhyperedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ models a multi-to-multi routing rule: flow leaving any aggregated unit in $\mathrm{Tail}(e)$ can enter the aggregated units in $\mathrm{Head}(e)$ (e.g., turning constraints from several incoming road-segment groups into several outgoing groups).

Because all components live on a planar street layout, the corresponding structure graph $\mathsf{S}(\mathrm{DSHG}^{(n)})$ can be drawn on the map by placing each supervertex at its geographic region and routing each incidence connection along the underlying roads, without crossings. Hence the directed n-SuperHyperGraph describing these hierarchical traffic dependencies is *planar* in the sense of Definition 3.9.5.

Theorem 3.9.7 (Planar directed SuperHyperGraphs generalize planar digraphs and planar directed hypergraphs)**.**

(i) *(**Hypergraph specialization.**) Let $n = 0$ and identify $V \subseteq \mathcal{P}^0(S) = S$ with an ordinary vertex set. Then a directed 0-SuperHyperGraph $\mathrm{DSHG}^{(0)} = (V, E, \partial^-, \partial^+)$ is planar in the sense of Definition 3.9.5 if and only if the directed hypergraph $H = (V, E)$ is planar in the sense of Definition 3.9.3 (under the identification $\mathrm{Tail}(e) = \partial^-(e)$, $\mathrm{Head}(e) = \partial^+(e)$).*

(ii) *(**Graph specialization.**) Let $n = 0$ and suppose that every $e \in E$ has singleton tail and singleton head:*

$$\mathrm{Tail}(e) = \{u\}, \qquad \mathrm{Head}(e) = \{v\} \qquad (u, v \in V).$$

Assume further that there is at most one $e \in E$ for each ordered pair (u, v), and define

$$A := \{(u, v) \in V \times V : \ \exists\, e \in E \text{ with } \mathrm{Tail}(e) = \{u\},\ \mathrm{Head}(e) = \{v\}\}.$$

Then (V, A) is a digraph, and $\mathrm{DSHG}^{(0)}$ is planar (Definition 3.9.5) if and only if the digraph (V, A) is planar (Definition 3.9.1).

Proof. **(i)** When $n = 0$, the vertex set V is an ordinary set, and $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$. By Definitions 3.9.2 and 3.9.4, the two structure graphs $\mathsf{S}(H)$ and $\mathsf{S}(\mathrm{DSHG}^{(0)})$ are identical (same vertex set $V \cup (E \times \{-, +\})$ and same arc set B). Hence planarity is equivalent.

(ii) Under the singleton hypothesis, each $e \in E$ corresponds to a unique arc $(u, v) \in A$. In $\mathsf{S}(\mathrm{DSHG}^{(0)})^{\mathrm{und}}$, each such arc is represented by the undirected path

$$u \,-\, e^- \,-\, e^+ \,-\, v,$$

i.e. by subdividing the undirected edge $\{u, v\}$ twice. Therefore, $\mathsf{S}(\mathrm{DSHG}^{(0)})^{\mathrm{und}}$ is a subdivision of the underlying undirected graph $(V, A)^{\mathrm{und}}$. Subdividing edges preserves planarity, and conversely suppressing degree-2 subdivision vertices also preserves planarity. Thus $\mathsf{S}(\mathrm{DSHG}^{(0)})^{\mathrm{und}}$ is planar if and only if $(V, A)^{\mathrm{und}}$ is planar, which is equivalent to planarity of the digraph (V, A) by Definition 3.9.1. □

3.10 Oriented SuperHyperGraph

An oriented graph is a loopless digraph with no opposite arc pairs, obtained by orienting edges of a simple graph [100–102]. An oriented hypergraph assigns each vertex–hyperedge incidence a sign, equivalently partitioning every hyperedge into tail and head parts [103–105]. An oriented superhypergraph assigns incidence signs on n-supervertices and superhyperedges, equivalently partitioning each superhyperedge into tail and head [106].

Definition 3.10.1 (Oriented graph)**.** An *oriented graph* is a directed graph $D = (V, A)$ such that

$$(v, v) \notin A \quad (\forall v \in V), \qquad \text{and for all distinct } u, v \in V, \ \ \neg\big((u, v) \in A \ \wedge \ (v, u) \in A\big).$$

Equivalently, an oriented graph is obtained by assigning a single direction to each edge of a simple undirected graph (no 2-cycles and no loops).

Definition 3.10.2 (Oriented hypergraph)**.** Let V be a finite set and let $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ be a finite family of nonempty subsets of V (hyperedges). An *oriented hypergraph* is a triple

$$\mathcal{H} = (V, E, \sigma),$$

where the *incidence orientation* is a map

$$\sigma : \{(v, e) \mid e \in E, \ v \in e\} \longrightarrow \{+1, -1\}.$$

Equivalently, each hyperedge $e \in E$ is equipped with a disjoint ordered partition

$$e \ = \ T(e) \ \sqcup \ H(e),$$

and one sets

$$\sigma(v, e) = \begin{cases} +1, & v \in T(e), \\ -1, & v \in H(e). \end{cases}$$

Definition 3.10.3 (Oriented n-SuperHyperGraph (oriented superhypergraph))**.** Let V_0 be a finite nonempty base set and fix an integer $n \geq 0$. Define iterated powersets by

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)) \quad (k \geq 0).$$

An *oriented n-SuperHyperGraph* is a triple

$$\mathcal{G}^{(n)} = (V, E, \sigma),$$

where

$$V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}, \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

and the incidence orientation is

$$\sigma : \{(v, e) \mid e \in E, \ v \in e\} \longrightarrow \{+1, -1\}.$$

Equivalently, each n-superhyperedge $e \in E$ admits a disjoint ordered partition

$$e \ = \ T(e) \ \sqcup \ H(e),$$

and $\sigma(v, e) = +1$ on $T(e)$ and $\sigma(v, e) = -1$ on $H(e)$.

Example 3.10.4 (Real-life oriented 2-SuperHyperGraph: stakeholders pushing or blocking a policy bundle)**.** *Scenario.* Consider a city council discussing a policy package that combines multiple sub-policies (budget items, safety rules, and infrastructure projects). Let the base set V_0 be a finite set of individual stakeholders:

$$V_0 = \{\mathsf{Mayor},\ \mathsf{Treasury},\ \mathsf{Police},\ \mathsf{Transit},\ \mathsf{Residents}\}.$$

Take $n = 2$. A 2-supervertex is a *set of subsets* of V_0, representing a nested coalition (e.g., a portfolio of committees, each committee being a subset of stakeholders). Define four 2-supervertices:

$$v_1 := \big\{\{\mathsf{Mayor}, \mathsf{Treasury}\}, \{\mathsf{Transit}\}\big\} \quad \text{(executive + transit committee portfolio)},$$

$$v_2 := \big\{\{\mathsf{Police}\}\big\} \quad \text{(public-safety committee)}, \qquad v_3 := \big\{\{\mathsf{Residents}\}\big\} \quad \text{(citizen group)},$$

$$v_4 := \big\{\{\mathsf{Treasury}\}, \{\mathsf{Residents}\}\big\} \quad \text{(fiscal oversight + citizen review)}.$$

Set

$$V := \{v_1, v_2, v_3, v_4\} \subseteq \mathcal{P}^2(V_0) \setminus \{\emptyset\}.$$

Superhyperedges and incidence orientation. Let

$$e_1 := \{v_1, v_2, v_4\}, \qquad e_2 := \{v_1, v_3, v_4\}, \qquad E := \{e_1, e_2\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Interpretation: each superhyperedge is a negotiation arena involving several coalition-units. An incidence sign $\sigma(v, e) \in \{+1, -1\}$ records whether coalition v is acting as a *supporter* $(+1)$ or an *opposer* (-1) within that arena.

Define ordered partitions (tails/heads) for each superhyperedge by

$$T(e_1) = \{v_1, v_4\}, \quad H(e_1) = \{v_2\}, \qquad T(e_2) = \{v_1, v_3\}, \quad H(e_2) = \{v_4\}.$$

Thus, in e_1 the executive–finance portfolio v_1 and the oversight coalition v_4 push the package, while the safety committee v_2 blocks it unless its demands are met; in e_2 the executive portfolio v_1 and citizen group v_3 push transparency measures, while v_4 blocks until additional fiscal safeguards are added. Equivalently, set

$$\sigma(v, e) = \begin{cases} +1, & v \in T(e), \\ -1, & v \in H(e), \end{cases}$$

for $e \in \{e_1, e_2\}$. Then $\mathcal{G}^{(2)} = (V, E, \sigma)$ is an oriented 2-SuperHyperGraph in the sense of Definition 3.10.3.

Theorem 3.10.5 (Oriented superhypergraphs generalize oriented graphs and oriented hypergraphs)**.** *Let $\mathcal{G}^{(n)} = (V, E, \sigma)$ be an oriented n-SuperHyperGraph.*

(i) *(**Hypergraph specialization.**) If $n = 0$ (so $V \subseteq \mathcal{P}^0(V_0) = V_0$ is an ordinary vertex set), then $\mathcal{G}^{(0)} = (V, E, \sigma)$ is exactly an oriented hypergraph in the sense of Definition 3.10.2.*

(ii) *(**Graph specialization.**) Assume $n = 0$ and every hyperedge $e \in E$ has size 2 and is oriented by singletons, i.e.*

$$|e| = 2, \qquad |T(e)| = |H(e)| = 1 \quad (\forall e \in E).$$

Define an arc set

$$A := \{(u, v) \in V \times V \mid \exists\, e \in E \text{ with } e = \{u, v\},\ T(e) = \{u\},\ H(e) = \{v\}\}.$$

Then (V, A) is an oriented graph in the sense of Definition 3.10.1.

(iii) ***(Underlying*** n***-SuperHyperGraph structure.)*** *If we forget the incidence orientation* σ*, then* (V, E) *is an* n*-SuperHyperGraph.*

Proof. **(i)** When $n = 0$, we have $V \subseteq V_0$ and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Thus (V, E) is a (crisp) hypergraph and σ is an incidence-sign map on $\{(v, e) \mid v \in e\}$, which is precisely Definition 3.10.2.

(ii) For each $e \in E$ with $e = \{u, v\}$, the hypothesis $|T(e)| = |H(e)| = 1$ forces either $T(e) = \{u\}, H(e) = \{v\}$ or $T(e) = \{v\}, H(e) = \{u\}$, hence it induces exactly one directed arc between u and v. Therefore $(u, v) \in A$ implies $(v, u) \notin A$, and no loop is created, so (V, A) is an oriented graph.

(iii) By Definition 3.10.3, $V \subseteq \mathcal{P}^n(V_0)$ and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Forgetting σ leaves exactly the pair (V, E), which satisfies the definition of an n-SuperHyperGraph. □

3.11 Complete Directed Graphs

A complete directed graph has every arc $u \to v$ for all distinct vertices $u \neq v$, in both directions [107–110]. A complete directed hypergraph contains every admissible hyperarc (T, H) with nonempty disjoint tail and head subsets of vertices (cf. [111–113]). A complete directed superhypergraph contains every admissible superhyperedge (T, H) between nonempty disjoint tail and head subfamilies of supervertices.

Definition 3.11.1 (Complete directed graph)**.** [107,108] Let V be a finite set with $|V| = n \geq 1$. The *complete (loopless) directed graph* on V is the digraph

$$K^{\to}(V) = (V, A), \qquad A := \{(u, v) \in V \times V : \ u \neq v\}.$$

Equivalently, $K^{\to}(V)$ is the *complete symmetric digraph* (it contains *all* possible arcs between distinct vertices).

Definition 3.11.2 (Complete directed hypergraph)**.** Let V be a finite vertex set and write $\mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\}$. A *directed hyperarc* is an ordered pair $(T, H) \in \mathcal{P}^*(V) \times \mathcal{P}^*(V)$. In this book we adopt the *loopless (partition) convention* that

$$T \cap H = \emptyset \quad \text{(tail and head are disjoint).}$$

The *complete (loopless) directed hypergraph* on V is

$$\overrightarrow{\mathcal{K}}(V) := \big(V, \mathcal{A}(V)\big), \qquad \mathcal{A}(V) := \{(T, H) \in \mathcal{P}^*(V) \times \mathcal{P}^*(V) : \ T \cap H = \emptyset\}.$$

Thus every admissible nonempty tail-set and nonempty disjoint head-set occurs as a directed hyperarc.

Definition 3.11.3 (Complete directed n-SuperHyperGraph)**.** Fix an integer $n \geq 0$ and a finite nonempty base set V_0. Let

$$V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$$

be a (finite) set of n-supervertices and write $\mathcal{P}^*(V) := \mathcal{P}(V) \setminus \{\emptyset\}$.

The *complete (loopless) directed n-SuperHyperGraph* on the supervertex set V is the incidence structure

$$\overrightarrow{\mathrm{SHG}}^{(n)}_{\mathrm{comp}}(V) := (V, E, \partial^-, \partial^+),$$

defined by

$$E := \{(T, H) \in \mathcal{P}^*(V) \times \mathcal{P}^*(V) : \ T \cap H = \emptyset\},$$

and the projection incidence maps

$$\partial^-(T, H) := T, \qquad \partial^+(T, H) := H \qquad \big((T, H) \in E\big).$$

Hence every admissible ordered pair (nonempty disjoint tail/head subfamilies of V) appears as a directed n-superhyperedge.

Remark 3.11.4 (Underlying undirected complete n-SuperHyperGraph)**.** Forgetting direction in Definition 3.11.3 yields the undirected n-SuperHyperGraph

$$\mathrm{SHG}^{(n)}_{\mathrm{comp}}(V) := (V, E_{\mathrm{und}}), \qquad E_{\mathrm{und}} := \{\ T \cup H \mid (T, H) \in E\ \} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Example 3.11.5 (Real-life interpretation of a complete directed n-SuperHyperGraph)**.** *Scenario (all-to-all routing possibilities among hierarchical cloud-service groups).* Let V_0 be a finite set of concrete microservices in a cloud platform, for example

$$V_0 = \{\mathsf{Auth}, \mathsf{Billing}, \mathsf{Search}, \mathsf{Storage}, \mathsf{Analytics}\}.$$

Fix $n = 2$. A 2-supervertex is a set of subsets of V_0, which can represent a *portfolio of service clusters* (e.g., clusters are subsets of services, portfolios are sets of clusters). Suppose $V \subseteq \mathcal{P}^2(V_0) \setminus \{\emptyset\}$ lists all portfolios under consideration, such as: a "core platform" portfolio, an "observability" portfolio, a "customer-facing" portfolio, etc.

In a stress-test planning phase, engineers may wish to consider *every possible* directed interaction rule between *groups of portfolios*: for any nonempty set $T \subseteq V$ of source portfolios and any nonempty disjoint set $H \subseteq V$ of target portfolios, they allow the possibility that traffic or dependencies can be routed from T to H (for example, to model any conceivable cross-portfolio call pattern under fault injection). Including *all* such admissible ordered pairs (T, H) corresponds exactly to the complete (loopless) directed n-SuperHyperGraph $\overrightarrow{\mathrm{SHG}}^{(n)}_{\mathrm{comp}}(V)$ in Definition 3.11.3. Here each directed n-superhyperedge (T, H) represents a potential many-to-many routing/dependency relation from the source portfolio-set T to the disjoint target portfolio-set H.

Theorem 3.11.6 (Complete directed superhypergraphs generalize complete digraphs and complete directed hypergraphs)**.** *Let $\overrightarrow{\mathrm{SHG}}^{(n)}_{\mathrm{comp}}(V) = (V, E, \partial^-, \partial^+)$ be as in Definition 3.11.3.*

(i) ***(Hypergraph specialization.)*** *If $n = 0$ and we identify $V \subseteq \mathcal{P}^0(V_0) = V_0$ with an ordinary vertex set, then*

$$(V, E) \equiv (V, \mathcal{A}(V))$$

is exactly the complete directed hypergraph $\overrightarrow{\mathcal{K}}(V)$ of Definition 3.11.2.

(ii) ***(Graph specialization.)*** *Assume $n = 0$ and restrict attention to the subfamily of superhyperedges with singleton tail and singleton head:*

$$E_{1,1} := \{(\{u\}, \{v\}) \in E : \ u, v \in V, \ u \neq v\}.$$

Then $(V, E_{1,1})$ canonically identifies with the complete directed graph $K^{\rightarrow}(V)$ of Definition 3.11.1 via

$$(\{u\}, \{v\}) \longleftrightarrow (u, v).$$

Proof. **(i)** When $n = 0$, $\mathcal{P}^0(V_0) = V_0$, so V is an ordinary vertex set. By Definition 3.11.3,

$$E = \{(T, H) \in \mathcal{P}^*(V) \times \mathcal{P}^*(V) : \ T \cap H = \emptyset\},$$

which is exactly the hyperarc set $\mathcal{A}(V)$ in Definition 3.11.2. Thus the structures coincide (with ∂^-, ∂^+ being the projections).

(ii) Still with $n = 0$, consider only edges $(\{u\}, \{v\})$. Because $T \cap H = \emptyset$ in the loopless convention, we necessarily have $u \neq v$. Therefore $E_{1,1}$ is in bijection with

$$A = \{(u, v) \in V \times V : \ u \neq v\},$$

and under the identification $(\{u\}, \{v\}) \leftrightarrow (u, v)$ we obtain precisely the complete directed graph $K^{\rightarrow}(V)$. □

3.12 Regular Directed SuperHyperGraph

A regular directed graph has constant outdegree and indegree at every vertex, optionally equal, so each vertex is balanced [114–118]. A regular directed hypergraph has constant tail-degree and head-degree at every vertex across all hyperarcs, possibly balanced equally [119]. A regular directed superhypergraph has constant tail-incidence and head-incidence counts at every supervertex across directed superhyperedges, balanced optionally.

Notation 3.12.1 (Degrees in a digraph)**.** *Let $G = (V, A)$ be a finite directed graph (digraph), where $A \subseteq V \times V$ is the arc set. For $v \in V$, define the out-degree and in-degree by*

$$d_G^+(v) := \big|\{\, u \in V : \ (v, u) \in A \,\}\big|, \qquad d_G^-(v) := \big|\{\, u \in V : \ (u, v) \in A \,\}\big|.$$

Definition 3.12.2 (Regular directed graph)**.** A digraph $G = (V, A)$ is (r^-, r^+)*-regular* if there exist integers $r^-, r^+ \geq 0$ such that

$$d_G^-(v) = r^- \quad \text{and} \quad d_G^+(v) = r^+ \qquad (\forall v \in V).$$

If $r^- = r^+ = r$, then G is called *(balanced) r-regular.*

Notation 3.12.3 (Tail/head degrees in a directed hypergraph)**.** *Let $H = (V, E)$ be a directed hypergraph, where each hyperarc is an ordered pair*

$$e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big), \qquad \emptyset \neq \mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V.$$

(Compare the in-set/out-set terminology for directed hyperarcs and the induced in-/out-degree conventions in the directed-hypergraph literature. For $v \in V$ define

$$d_H^+(v) := \big|\{\, e \in E : \ v \in \mathrm{Tail}(e) \,\}\big|, \qquad d_H^-(v) := \big|\{\, e \in E : \ v \in \mathrm{Head}(e) \,\}\big|.$$

Definition 3.12.4 (Regular directed hypergraph)**.** A directed hypergraph $H = (V, E)$ is (r^-, r^+)-*regular* if there exist integers $r^-, r^+ \geq 0$ such that

$$d_H^-(v) = r^- \quad \text{and} \quad d_H^+(v) = r^+ \qquad (\forall v \in V),$$

where $d_H^\pm$ are as in Notation 3.12.3. If $r^- = r^+ = r$, then H is called *(balanced) r-regular.*

Notation 3.12.5 (Tail/head degrees in a directed n-SuperHyperGraph)**.** *Fix an integer $n \geq 0$ and a nonempty base set S. Let* $\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$ *be a directed n-SuperHyperGraph in incidence form, i.e.*

$$V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}, \qquad \partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Write $\mathrm{Tail}(e) := \partial^-(e)$ *and* $\mathrm{Head}(e) := \partial^+(e)$*. For $v \in V$ define*

$$d^+_{\mathrm{DSHG}}(v) := \bigl|\{\, e \in E : \ v \in \mathrm{Tail}(e) \,\}\bigr|, \qquad d^-_{\mathrm{DSHG}}(v) := \bigl|\{\, e \in E : \ v \in \mathrm{Head}(e) \,\}\bigr|.$$

Definition 3.12.6 (Regular directed n-SuperHyperGraph)**.** A directed n-SuperHyperGraph $\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$ is (r^-, r^+)-*regular* if there exist integers $r^-, r^+ \geq 0$ such that

$$d^-_{\mathrm{DSHG}}(v) = r^- \quad \text{and} \quad d^+_{\mathrm{DSHG}}(v) = r^+ \qquad (\forall v \in V),$$

where $d^\pm_{\mathrm{DSHG}}$ are as in Notation 3.12.5. If $r^- = r^+ = r$, then $\mathrm{DSHG}^{(n)}$ is called *(balanced) r-regular.*

Example 3.12.7 (Real-life regular directed n-SuperHyperGraph)**.** *Scenario (standardized cross-team request policy in a large organization).* Let the base set S be a finite set of individual employees, and let each n-supervertex $v \in V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}$ represent a nested organizational unit (e.g., a department composed of teams, each team composed of employees).

Assume the company enforces a uniform collaboration policy:

- each unit must *submit* requests to exactly r^+ other units per planning cycle (e.g., integration requests, reviews, or service tickets), and
- each unit must *receive* requests from exactly r^- other units per cycle, because the workload is load-balanced by design.

Model each request pattern as a directed n-superhyperedge identifier $e \in E$ with $\mathrm{Tail}(e) \subseteq V$ the set of requesting units and $\mathrm{Head}(e) \subseteq V$ the set of receiving units. If the policy is implemented so that, for every unit $v \in V$, the number of incident outgoing superhyperedges

$$d^+_{\mathrm{DSHG}}(v) = \bigl|\{e \in E : \ v \in \mathrm{Tail}(e)\}\bigr|$$

equals the same constant r^+, and the number of incident incoming superhyperedges

$$d^-_{\mathrm{DSHG}}(v) = \bigl|\{e \in E : \ v \in \mathrm{Head}(e)\}\bigr|$$

equals the same constant r^-, then the resulting directed n-SuperHyperGraph is (r^-, r^+)-regular in the sense of Definition 3.12.6. In the special case $r^- = r^+$, every unit sends and receives the same number of requests, yielding a balanced regular structure.

Theorem 3.12.8 (Regular directed SuperHyperGraphs generalize regular digraphs and regular directed hypergraphs)**.** *Let* $\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$ *be a directed* n*-SuperHyperGraph.*

(i) ***(Hypergraph specialization.)*** *If* $n = 0$*, then* $V \subseteq \mathcal{P}^0(S) = S$ *is an ordinary vertex set and every* $e \in E$ *determines a directed hyperarc* $(\mathrm{Tail}(e), \mathrm{Head}(e))$ *on* V*. Under this identification,*

$$\mathrm{DSHG}^{(0)} \textit{ is } (r^-, r^+)\textit{-regular} \iff H = (V, \{(\mathrm{Tail}(e), \mathrm{Head}(e)) \mid e \in E\}) \textit{ is } (r^-, r^+)\textit{-regular.}$$

(ii) ***(Graph specialization.)*** *Assume* $n = 0$ *and every* $e \in E$ *has singleton tail and singleton head:*

$$\mathrm{Tail}(e) = \{u\}, \qquad \mathrm{Head}(e) = \{v\} \qquad (u, v \in V),$$

and assume simplicity *(no parallel identifiers for the same ordered pair* (u, v)*). Let*

$$A := \{(u, v) \in V \times V : \ \exists e \in E \textit{ with } \mathrm{Tail}(e) = \{u\}, \ \mathrm{Head}(e) = \{v\}\}.$$

Then (V, A) *is a digraph, and*

$$\mathrm{DSHG}^{(0)} \textit{ is } (r^-, r^+)\textit{-regular} \iff G = (V, A) \textit{ is } (r^-, r^+)\textit{-regular.}$$

Proof. **(i)** When $n = 0$, supervertices are ordinary vertices and $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$ are nonempty. Hence e is exactly a directed hyperarc on V. Moreover, for each $v \in V$,

$$d^+_{\mathrm{DSHG}}(v) = |\{e \in E : v \in \mathrm{Tail}(e)\}|, \qquad d^-_{\mathrm{DSHG}}(v) = |\{e \in E : v \in \mathrm{Head}(e)\}|,$$

which coincide with the corresponding tail/head degree counts in Notation 3.12.3 for the induced directed hypergraph. Therefore the constancy conditions defining (r^-, r^+)-regularity are equivalent.

(ii) Under the singleton hypothesis, each $e \in E$ corresponds to a unique arc (u, v) with $\mathrm{Tail}(e) = \{u\}$ and $\mathrm{Head}(e) = \{v\}$. By simplicity, this correspondence is well-defined. For any $w \in V$,

$$d^+_{\mathrm{DSHG}}(w) = |\{e \in E : w \in \mathrm{Tail}(e)\}| = |\{(w, x) \in A\}| = d^+_G(w),$$

and similarly $d^-_{\mathrm{DSHG}}(w) = d^-_G(w)$. Hence $\mathrm{DSHG}^{(0)}$ is (r^-, r^+)-regular if and only if G is (r^-, r^+)-regular. □

3.13 Bipartite Directed SuperHyperGraph

A bipartite directed graph partitions vertices into two parts, and every arc goes between parts, never within a part [120,121]. A bipartite directed hypergraph partitions vertices into two parts, and each hyperarc's tail and head lie entirely in opposite parts (cf. [122, 123]). A bipartite directed superhypergraph partitions supervertices into two parts, and each directed superedge's tail and head lie in opposite parts.

Definition 3.13.1 (Bipartite directed graph)**.** [120,121] A directed graph (digraph) $D = (V, A)$ is called *bipartite* if there exists a partition

$$V = U \,\dot\cup\, W, \qquad U \cap W = \emptyset,$$

such that every arc has its endpoints in different parts:

$$(u, v) \in A \implies \bigl(u \in U,\ v \in W\bigr) \text{ or } \bigl(u \in W,\ v \in U\bigr).$$

Equivalently, $A \cap (U \times U) = \emptyset$ and $A \cap (W \times W) = \emptyset$.

Definition 3.13.2 (Bipartite directed hypergraph)**.** Let $H = (V, E)$ be a directed hypergraph, where each hyperarc is an ordered pair

$$e = \bigl(\mathrm{Tail}(e), \mathrm{Head}(e)\bigr), \qquad \emptyset \neq \mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V.$$

We call H *bipartite* if there exists a partition $V = U \dot\cup W$ such that for every $e \in E$,

$$\bigl(\mathrm{Tail}(e) \subseteq U \ \wedge\ \mathrm{Head}(e) \subseteq W\bigr) \quad \text{or} \quad \bigl(\mathrm{Tail}(e) \subseteq W \ \wedge\ \mathrm{Head}(e) \subseteq U\bigr).$$

In other words, each directed hyperarc goes from one part entirely into the other part.

Definition 3.13.3 (Bipartite directed n-SuperHyperGraph)**.** Fix an integer $n \geq 0$ and a nonempty base set S. Let

$$\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$$

be a directed n-SuperHyperGraph in incidence form, so that

$$V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}, \qquad \partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. We call $\mathrm{DSHG}^{(n)}$ *bipartite* if there exists a partition

$$V = V_1 \,\dot\cup\, V_2$$

such that for every $e \in E$,

$$\bigl(\mathrm{Tail}(e) \subseteq V_1 \ \wedge\ \mathrm{Head}(e) \subseteq V_2\bigr) \quad \text{or} \quad \bigl(\mathrm{Tail}(e) \subseteq V_2 \ \wedge\ \mathrm{Head}(e) \subseteq V_1\bigr).$$

Example 3.13.4 (Real-life bipartite directed n-SuperHyperGraph)**.** *Scenario (two-sided marketplace with hierarchical groupings).* Consider an online marketplace that connects *sellers* and *buyers*. Let the base set S be a finite set of individual accounts. A level-n supervertex $v \in V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}$ represents a nested group, for example a *seller coalition* (stores grouped into brands, brands grouped into categories) or a *buyer coalition* (customers grouped into segments, segments grouped into regions).

Partition the supervertices into two classes:

$$V_1 = \{\text{seller-side supervertices}\}, \qquad V_2 = \{\text{buyer-side supervertices}\}, \qquad V = V_1 \dot\cup V_2.$$

A directed superhyperedge $e \in E$ with $\mathrm{Tail}(e) \subseteq V_1$ and $\mathrm{Head}(e) \subseteq V_2$ can model *supply influence*, such as "promotions launched by these seller groups reach those buyer segments." Conversely, a superhyperedge with $\mathrm{Tail}(e) \subseteq V_2$ and $\mathrm{Head}(e) \subseteq V_1$ can model *demand feedback*, such as "interest signals from these buyer segments trigger restocking by those seller groups."

Because every directed superhyperedge goes strictly from seller-side groups to buyer-side groups or vice versa, and never stays within the same side, the resulting directed n-SuperHyperGraph is bipartite in the sense of Definition 3.13.3.

Theorem 3.13.5 (Bipartite directed superhypergraphs generalize bipartite digraphs and bipartite directed hypergraphs)**.** *Let* $\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$ *be a bipartite directed n-SuperHyperGraph in the sense of Definition 3.13.3.*

(i) ***(Hypergraph specialization.)*** *If* $n = 0$ *and we identify* $V \subseteq \mathcal{P}^0(S) = S$ *with an ordinary vertex set, then the induced directed hypergraph*

$$H := (V, E_H), \qquad E_H := \{(\mathrm{Tail}(e), \mathrm{Head}(e)) \mid e \in E\},$$

is bipartite in the sense of Definition 3.13.2.

(ii) ***(Graph specialization.)*** *Assume* $n = 0$ *and every* $e \in E$ *has singleton tail and singleton head:*

$$\mathrm{Tail}(e) = \{u\}, \qquad \mathrm{Head}(e) = \{v\} \qquad (u, v \in V),$$

and assume simplicity *(for each ordered pair* (u, v) *there is at most one* $e \in E$ *with* $\mathrm{Tail}(e) = \{u\}$ *and* $\mathrm{Head}(e) = \{v\}$*). Define*

$$A := \{(u, v) \in V \times V \mid \exists\, e \in E \text{ with } \mathrm{Tail}(e) = \{u\},\ \mathrm{Head}(e) = \{v\}\}.$$

Then $D := (V, A)$ *is a bipartite directed graph in the sense of Definition 3.13.1.*

Proof. **(i)** Let $n = 0$. By Definition 3.13.3, there exists a partition $V = V_1 \dot\cup V_2$ such that for every $e \in E$,

$$\mathrm{Tail}(e) \subseteq V_1,\ \mathrm{Head}(e) \subseteq V_2 \quad \text{or} \quad \mathrm{Tail}(e) \subseteq V_2,\ \mathrm{Head}(e) \subseteq V_1.$$

Since each hyperarc in H is exactly $(\mathrm{Tail}(e), \mathrm{Head}(e))$ for some $e \in E$, the same partition witnesses that H is bipartite in the sense of Definition 3.13.2.

(ii) Under the singleton hypothesis, each $e \in E$ determines an ordered pair (u, v) via $\mathrm{Tail}(e) = \{u\}$ and $\mathrm{Head}(e) = \{v\}$, and the simplicity assumption ensures that A is well-defined as a set of arcs. Let $V = V_1 \dot\cup V_2$ witness bipartiteness of $\mathrm{DSHG}^{(0)}$. If $(u, v) \in A$, choose $e \in E$ with $\mathrm{Tail}(e) = \{u\}$ and $\mathrm{Head}(e) = \{v\}$. Then $\{u\} = \mathrm{Tail}(e)$ and $\{v\} = \mathrm{Head}(e)$ lie in opposite parts, hence u and v lie in different parts. Therefore no arc in A has both endpoints in V_1 or both in V_2, and $D = (V, A)$ is bipartite (Definition 3.13.1). □

3.14 Line Directed SuperHyperGraph

A line directed graph has arcs as vertices; it connects two arc-vertices when the first arc's head equals the second arc's tail [124–129]. A line directed hypergraph has hyperarcs as vertices; it links $e_1 \to e_2$ when e_1's head lies in e_2's tail. A line directed superhypergraph has superhyperedges as vertices; it links $e_1 \to e_2$ when $\mathrm{Head}(e_1) \subseteq \mathrm{Tail}(e_2)$.

Definition 3.14.1 (Line directed graph)**.** [124–126] Let $D = (V, A)$ be a finite directed graph (digraph), where $A \subseteq V \times V$ is the arc set. The *line directed graph* of D, denoted $L(D)$, is the digraph

$$L(D) = (V_L, A_L),$$

whose vertex set is the arc set $V_L := A$, and whose arc set is

$$A_L := \{\, ((u, v), (v, w)) \in A \times A \;\mid\; (u, v) \in A,\ (v, w) \in A \,\}.$$

Equivalently, there is an arc $e_1 \to e_2$ in $L(D)$ if and only if the head of e_1 equals the tail of e_2.

Definition 3.14.2 (Line directed hypergraph)**.** Let $H = (V, E)$ be a directed hypergraph in the *single-head convention*, i.e. each hyperarc has the form

$$e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big), \qquad \emptyset \neq \mathrm{Tail}(e) \subseteq V, \quad \mathrm{Head}(e) \in V.$$

The *line directed hypergraph* of H, denoted $\mathrm{LDHG}(H)$, is the directed hypergraph

$$\mathrm{LDHG}(H) := \big(E,\, E_{\mathrm{line}}\big),$$

whose vertex set is the hyperarc set E, and whose hyperarc set is

$$E_{\mathrm{line}} := \big\{\, (\{e_1\}, e_2) \;\big|\; e_1, e_2 \in E,\ \mathrm{Head}(e_1) \in \mathrm{Tail}(e_2) \,\big\}.$$

Thus $\mathrm{LDHG}(H)$ records composability of hyperarcs: $e_1 \to e_2$ iff the output $\mathrm{Head}(e_1)$ is an input of e_2.

Definition 3.14.3 (Line directed n-SuperHyperGraph)**.** Fix $n \geq 0$ and let $\mathrm{DSH}_n = (V, E)$ be a directed n-SuperHyperGraph in the tail–head object form, so each directed n-superhyperedge is an ordered pair

$$e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big) \in \mathcal{P}^n(V) \times \mathcal{P}^n(V).$$

The *line directed superhypergraph* of DSH_n, denoted $\mathrm{LDSH}(\mathrm{DSH}_n)$, is the directed 1-SuperHyperGraph

$$\mathrm{LDSH}(\mathrm{DSH}_n) := \big(E,\, E_{\mathrm{line}}\big),$$

with base set E (the superhyperedges of DSH_n) and

$$E_{\mathrm{line}} := \big\{\, (\{e_1\}, \{e_2\}) \in \mathcal{P}(E) \times \mathcal{P}(E) \;\big|\; e_1 = (T_1, H_1),\ e_2 = (T_2, H_2) \in E,\ H_1 \subseteq T_2 \,\big\}.$$

Hence there is a line superhyperedge $e_1 \to e_2$ precisely when the head object of e_1 is contained in the tail object of e_2.

Example 3.14.4 (Real-life line directed n-SuperHyperGraph)**.** *Scenario (workflow chaining between hierarchical actions in a CI/CD pipeline).* Consider a CI/CD system in which *actions* operate on hierarchical artifacts (files, modules, services, and service groups). Model each hierarchical artifact as an n-level object, and let a directed n-superhyperedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ represent an action that consumes a set of prerequisite artifacts $\mathrm{Tail}(e)$ and produces or enables a set of artifacts $\mathrm{Head}(e)$.

For example, let

$$e_1 :\ \mathrm{Tail}(e_1) = \{\text{compiled core modules}\} \;\longrightarrow\; \mathrm{Head}(e_1) = \{\text{container image}\},$$

$$e_2 :\ \mathrm{Tail}(e_2) = \{\text{container image, deployment manifest}\} \;\longrightarrow\; \mathrm{Head}(e_2) = \{\text{staging deployment}\},$$

$$e_3 :\ \mathrm{Tail}(e_3) = \{\text{staging deployment}\} \;\longrightarrow\; \mathrm{Head}(e_3) = \{\text{production rollout}\}.$$

Then $\mathrm{Head}(e_1) \subseteq \mathrm{Tail}(e_2)$ (the container image produced by e_1 is required by e_2), and $\mathrm{Head}(e_2) \subseteq \mathrm{Tail}(e_3)$ (the staging deployment enabled by e_2 is required by e_3). In the line directed superhypergraph $\mathrm{LDSH}(\mathrm{DSH}_n)$, the original actions become vertices, and there are directed line edges

$$e_1 \to e_2, \qquad e_2 \to e_3,$$

capturing the fact that action e_2 can follow e_1, and action e_3 can follow e_2. Thus $\mathrm{LDSH}(\mathrm{DSH}_n)$ models the *action-composability graph* of the pipeline at a hierarchical level.

Theorem 3.14.5 (Line directed superhypergraphs generalize line digraphs and line directed hypergraphs)**.**

(i) ***(Line digraph as a special case.)*** *Let $D = (V, A)$ be a digraph and embed it as a directed 1-SuperHyperGraph* $\mathrm{DSH}_1(D) = (V, E)$ *by*

$$E := \{ e_{(u,v)} := (\{u\}, \{v\}) \mid (u, v) \in A \} \subseteq \mathcal{P}(V) \times \mathcal{P}(V).$$

Then $\mathrm{LDSH}(\mathrm{DSH}_1(D))$ *is canonically isomorphic to the line directed graph* $L(D)$*.*

(ii) ***(Line directed hypergraph as a special case.)*** *Let $H = (V, E)$ be a directed hypergraph in the single-head convention and embed it as a directed 1-SuperHyperGraph* $\mathrm{DSH}_1(H) = (V, \widehat{E})$ *by*

$$\widehat{E} := \{ \widehat{e} = (T, \{h\}) \mid e = (T, h) \in E \} \subseteq \mathcal{P}(V) \times \mathcal{P}(V).$$

Then $\mathrm{LDSH}(\mathrm{DSH}_1(H))$ *agrees with* $\mathrm{LDHG}(H)$ *after identifying each vertex $\widehat{e}$ with the original hyperarc e.*

Proof. **(i)** Define $\varphi : A \to E$ by $\varphi((u, v)) := e_{(u,v)} = (\{u\}, \{v\})$. This is a bijection by construction. By Definition 3.14.3, there is a line edge

$$\{e_{(u,v)}\} \longrightarrow \{e_{(x,y)}\} \quad \text{in } \mathrm{LDSH}(\mathrm{DSH}_1(D))$$

if and only if $\{v\} \subseteq \{x\}$, i.e. $v = x$. Therefore, under φ^{-1}, such a line edge exists if and only if

$$(u, v) \to (v, y) \quad \text{in } L(D),$$

which is exactly the adjacency rule of Definition 3.14.1. Hence φ yields a canonical isomorphism.

(ii) Define $\psi : E \to \widehat{E}$ by $\psi((T, h)) := \widehat{e} = (T, \{h\})$; this is a bijection. A line edge $\{\widehat{e}_1\} \to \{\widehat{e}_2\}$ exists in $\mathrm{LDSH}(\mathrm{DSH}_1(H))$ iff

$$\mathrm{Head}(\widehat{e}_1) = \{h_1\} \subseteq \mathrm{Tail}(\widehat{e}_2) = T_2 \iff h_1 \in T_2,$$

which is precisely the defining condition for $(\{e_1\}, e_2) \in E_{\text{line}}$ in Definition 3.14.2. Therefore the edge sets coincide under ψ, giving the claimed agreement. □

3.15 Arborescence in directed graph

Arborescence in a directed graph is a rooted digraph where every vertex has exactly one directed walk from the root, forming a directed rooted tree [130–134]. Arborescence in a directed hypergraph is a rooted directed hypergraph where every vertex has exactly one directed hyperwalk from the root, under tail-to-head reachability (cf. [135–137]). Arborescence in a directed superhypergraph is a rooted directed n-SuperHyperGraph where every supervertex has exactly one directed superhyperwalk from the root through superhyperedges.

Definition 3.15.1 (Arborescence in a directed graph)**.** [130, 131] Let $D = (V, A)$ be a finite directed graph and let $r \in V$. We call (D, r) an *(out-)arborescence rooted at* r if for every $v \in V$ there exists *exactly one* directed walk from r to v.

Definition 3.15.2 (Arborescence in a directed hypergraph)**.** Let $H = (V, E)$ be a finite directed hypergraph where each hyperarc $e \in E$ is an ordered pair $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with $\emptyset \neq \mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$. Fix a root $r \in V$.

A *directed hyperwalk* from r to v is an alternating sequence

$$r = x_0,\ e_1,\ x_1,\ e_2,\ \ldots,\ e_k,\ x_k = v \qquad (k \geq 0),$$

where $x_i \in V$, $e_i \in E$, and $x_{i-1} \in \mathrm{Tail}(e_i)$, $x_i \in \mathrm{Head}(e_i)$ for all i.

We call (H, r) a *(hyper)arborescence rooted at* r if for every $v \in V$ there exists *exactly one* directed hyperwalk from r to v.

Definition 3.15.3 (Arborescence in a directed n-SuperHyperGraph)**.** Fix $n \geq 0$ and a finite nonempty base set V_0. Let $\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$ be a directed n-SuperHyperGraph in incidence form:

$$V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}, \qquad \partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. Fix a root supervertex $r \in V$.

A *directed superhyperwalk* from r to v (where $v \in V$) is an alternating sequence

$$r = s_0,\ e_1,\ s_1,\ e_2,\ \ldots,\ e_k,\ s_k = v \qquad (k \geq 0),$$

where $s_i \in V$, $e_i \in E$, and $s_{i-1} \in \mathrm{Tail}(e_i)$, $s_i \in \mathrm{Head}(e_i)$ for all i.

We call $(\mathrm{DSHG}^{(n)}, r)$ an *(out-)arborescence rooted at* r if for every $v \in V$ there exists *exactly one* directed superhyperwalk from r to v.

Example 3.15.4 (A small arborescence in a directed 2-SuperHyperGraph)**.** Let the finite base set be

$$V_0 = \{a, b, c\}, \qquad n = 2.$$

Define four 2-supervertices (elements of $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$) by

$$r := \{\{a\}, \{b\}\}, \qquad v_2 := \{\{b\}, \{c\}\}, \qquad v_3 := \{\{a, c\}\}, \qquad v_4 := \{\{a, b, c\}\},$$

and set

$$V := \{r, v_2, v_3, v_4\} \subseteq \mathcal{P}^2(V_0) \setminus \{\emptyset\}.$$

Let the directed superedge identifier set be $E = \{e_1, e_2, e_3\}$, and define incidence maps $\partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}$ by

$$\partial^-(e_1) = \{r\}, \qquad \partial^+(e_1) = \{v_2\},$$
$$\partial^-(e_2) = \{r\}, \qquad \partial^+(e_2) = \{v_3\},$$
$$\partial^-(e_3) = \{v_2\}, \qquad \partial^+(e_3) = \{v_4\}.$$

Write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. Then $\mathrm{DSHG}^{(2)} = (V, E, \partial^-, \partial^+)$ is a directed 2-SuperHyperGraph with root r.

Uniqueness of directed superhyperwalks from the root.

- To reach r, there is exactly one directed superhyperwalk of length 0: (r).

- To reach v_2, the only possible walk is
$$r,\ e_1,\ v_2,$$
because $v_2 \in \mathrm{Head}(e_1)$ and no other superedge has v_2 in its head.

- To reach v_3, the only possible walk is
$$r,\ e_2,\ v_3,$$
since $v_3 \in \mathrm{Head}(e_2)$ and there is no alternative incoming superedge to v_3.

- To reach v_4, the only possible walk is
$$r,\ e_1,\ v_2,\ e_3,\ v_4,$$
because the only incoming superedge to v_4 is e_3, and the only way to reach its tail-vertex v_2 from r is via e_1.

Hence every $v \in V$ is reachable from r by exactly one directed superhyperwalk, so
$$(\mathrm{DSHG}^{(2)}, r)$$
is an (out-)arborescence rooted at r in the sense of Definition 3.15.3.

Theorem 3.15.5 (Generalization property)**.** *Arborescences in directed n-SuperHyperGraphs generalize arborescences in directed graphs and directed hypergraphs.*

(i) ***(Hypergraph specialization.)*** *If $n = 0$, then a directed 0-SuperHyperGraph arborescence is exactly a directed-hypergraph arborescence under the identification $e \leftrightarrow (\mathrm{Tail}(e), \mathrm{Head}(e))$.*

(ii) ***(Graph specialization.)*** *If $n = 0$ and every $e \in E$ has singleton tail and singleton head, i.e. $\mathrm{Tail}(e) = \{u\}$ and $\mathrm{Head}(e) = \{v\}$, and if the incidence is simple (no parallel identifiers for the same (u, v)), then the induced digraph $D = (V, A)$ with $A = \{(u, v) : (\{u\}, \{v\}) \in E\}$ is an arborescence rooted at r in the usual sense.*

Proof. **(i)** When $n = 0$, supervertices are ordinary vertices and superhyperwalks are exactly hyperwalks, since the conditions $s_{i-1} \in \mathrm{Tail}(e_i)$ and $s_i \in \mathrm{Head}(e_i)$ are identical in both settings. Hence existence and uniqueness of walks from r to any v coincide.

(ii) Under the singleton-tail/head assumption, each superhyperedge corresponds to a unique arc (u, v). Deleting edge identifiers and singleton braces gives a bijection between superhyperwalks in $\mathrm{DSHG}^{(0)}$ and directed walks in $D = (V, A)$. Therefore, existence and uniqueness of walks from r to each vertex are preserved, so D is an arborescence rooted at r. □

3.16 Recursive directed hypergraph

A (n, k)-recursive directed SuperHyperGraph has level-n supervertices and depth-k directed superhyperedges whose tails/heads may include supervertices and lower-level directed edges.

Definition 3.16.1 (Depth-k recursive directed hypergraph)**.** Let V be a finite set and let $k \in \mathbb{N}$. Set $U_0 := V$ and define recursively, for $i = 1, \ldots, k$,

$$\mathcal{A}_i \subseteq \mathcal{P}^*(U_{i-1}) \times \mathcal{P}^*(U_{i-1}), \qquad U_i := U_{i-1} \sqcup \mathcal{A}_i,$$

where $\sqcup$ denotes disjoint union (so that vertices and lower-level hyperarcs are treated as distinct objects). The tuple

$$\mathrm{RDH}^{(k)} = \big(V; \mathcal{A}_1, \ldots, \mathcal{A}_k\big)$$

is called a *depth-k recursive directed hypergraph.* The overall set of hyperarcs is $\mathcal{A} := \bigcup_{i=1}^{k} \mathcal{A}_i$.

Interpretation. A level-i hyperarc $(T, H) \in \mathcal{A}_i$ may point from/to *any nonempty collections of objects already available at earlier levels*, i.e., elements of U_{i-1}, which include both vertices and lower-level hyperarcs. Thus recursion is well-founded and bounded by k.

Definition 3.16.2 ((n, k)-recursive directed SuperHyperGraph)**.** Fix a finite nonempty ground set V_0 and let $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. Define the iterated powersets

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{n+1}(V_0) = \mathcal{P}\big(\mathcal{P}^n(V_0)\big).$$

Choose a (nonempty) supervertex set

$$\emptyset \neq V \subseteq \mathcal{P}^n(V_0).$$

A *(n, k)-recursive directed SuperHyperGraph* is a depth-k recursive directed hypergraph

$$\mathrm{RDSHG}^{(n,k)} = \big(V; \mathcal{A}_1, \ldots, \mathcal{A}_k\big)$$

in the sense of Definition 3.16.1, with $U_0 := V$. Equivalently, setting $U_0 := V$ and, for $i = 1, \ldots, k$,

$$\mathcal{A}_i \subseteq \mathcal{P}^*(U_{i-1}) \times \mathcal{P}^*(U_{i-1}), \qquad U_i := U_{i-1} \sqcup \mathcal{A}_i,$$

each level-i directed superhyperedge may reference supervertices and lower-level directed superhyperedges, thereby combining vertex hierarchy (level n) with bounded edge recursion (depth k).

Example 3.16.3 (A $(1, 2)$-recursive directed SuperHyperGraph)**.** Let the finite ground set be

$$V_0 = \{a, b, c\}, \qquad n = 1, \qquad k = 2.$$

Then $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$. Choose the following nonempty 1-supervertices:

$$v_1 := \{a, b\}, \qquad v_2 := \{b, c\}, \qquad v_3 := \{a, c\},$$

and set

$$V := \{v_1, v_2, v_3\} \subseteq \mathcal{P}^1(V_0).$$

Set $U_0 := V$.

Level 1 directed superhyperedges. Let $\mathcal{A}_1 \subseteq \mathcal{P}^*(U_0) \times \mathcal{P}^*(U_0)$ consist of the two hyperarcs

$$\alpha_1 := (\{v_1\}, \{v_2\}), \qquad \alpha_2 := (\{v_2\}, \{v_3\}).$$

These are well-defined because $\{v_1\}, \{v_2\}, \{v_3\} \in \mathcal{P}^*(U_0)$. Form

$$U_1 := U_0 \sqcup \mathcal{A}_1,$$

so that vertices in V and hyperarcs in $\mathcal{A}_1$ are treated as distinct objects.

Level 2 directed superhyperedges (edge recursion). At level 2 we may reference objects already available in U_1, including the level-1 hyperarcs α_1, α_2. Define one level-2 hyperarc

$$\beta := (\{\, v_1, \alpha_1 \,\}, \{\, v_3, \alpha_2 \,\}) \;\in\; \mathcal{P}^*(U_1) \times \mathcal{P}^*(U_1).$$

Intuitively, β records a higher-order directed relation whose tail contains a supervertex v_1 *and* a lower-level directed superhyperedge α_1, while its head contains v_3 and α_2. Set

$$\mathcal{A}_2 := \{\beta\}, \qquad U_2 := U_1 \sqcup \mathcal{A}_2.$$

Then

$$\mathrm{RDSHG}^{(1,2)} = (V; \mathcal{A}_1, \mathcal{A}_2)$$

is a $(1, 2)$-recursive directed SuperHyperGraph in the sense of Definition 3.16.2: level-1 hyperarcs connect supervertices, and the level-2 hyperarc β demonstrates bounded recursion by explicitly referencing level-1 hyperarcs as objects.

A comparison between a directed n-SuperHyperGraph and an (n, k)-recursive directed Super-HyperGraph is given in Table 3.1.

3.17 Symmetric directed graphs

A symmetric directed graph contains both arcs $u \to v$ and $v \to u$ whenever either direction is present (cf. [109, 138–141]). A symmetric directed hypergraph contains each hyperarc (T, H) together with its reverse (H, T), for all nonempty tail/head sets. A symmetric directed super-hypergraph contains each directed superhyperedge with a reverse superhyperedge swapping tail and head supervertex sets, always.

Definition 3.17.1 (Symmetric directed graph)**.** A directed graph (digraph) $D = (V, A)$ is *symmetric* if for every ordered pair of distinct vertices $u, v \in V$,

$$(u, v) \in A \implies (v, u) \in A.$$

Equivalently, A is closed under reversal: $A = A^{-1}$ where $A^{-1} := \{(v, u) \mid (u, v) \in A\}$.

Definition 3.17.2 (Symmetric directed hypergraph)**.** Let $H = (V, E)$ be a directed hypergraph in the tail–head form, i.e. each hyperarc is an ordered pair

$$e = (T(e), H(e)), \qquad \emptyset \neq T(e), H(e) \subseteq V.$$

The directed hypergraph H is *symmetric* if it is closed under reversal: for every $e = (T, H) \in E$, the reversed hyperarc

$$e^{\mathrm{rev}} := (H, T)$$

also belongs to E. Equivalently, $E = E^{\mathrm{rev}}$ where $E^{\mathrm{rev}} := \{(H, T) \mid (T, H) \in E\}$.

Table 3.1: Comparison between a directed n-SuperHyperGraph and an (n,k)-recursive directed SuperHyperGraph.

Aspect	*Directed n-SuperHyperGraph*	*(n,k)-recursive directed SuperHyperGraph*
Base/vertex universe	Fix $V_0 \neq \emptyset$, choose $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ of n-supervertices.	Same n-level vertex hierarchy: $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$, then set $U_0 := V$.
Edge objects	Edges are directed superhyperedges between *subsets of* V: $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with $\mathrm{Tail}(e), \mathrm{Head}(e) \in \mathcal{P}^*(V)$.	Edges are *stratified by recursion depth*: $\mathcal{A}_i \subseteq \mathcal{P}^*(U_{i-1}) \times \mathcal{P}^*(U_{i-1})$ and new edge-objects are adjoined: $U_i := U_{i-1} \sqcup \mathcal{A}_i$.
What tails/heads may contain	Only n-supervertices: $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$.	Any previously available objects: tails/heads may include n-supervertices and lower-level edges (elements of U_{i-1}).
Recursion in edges	None (edges do not reference edges).	Bounded recursion of depth k: level-i edges may reference level-$< i$ edges.
Parameter controlling hierarchy	Vertex nesting depth n.	Vertex nesting depth n *and* edge-recursion depth k.
Special-case relation	Baseline model.	If one restricts each $\mathcal{A}_i$ to use only vertices (no lower-level edges in tails/heads), then the model collapses to an ordinary directed n-SuperHyperGraph (up to bundling $\bigcup_i \mathcal{A}_i$ as the edge set).

Definition 3.17.3 (Symmetric directed n-SuperHyperGraph)**.** Fix an integer $n \geq 0$ and a nonempty base set S. Let

$$\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$$

be a directed n-SuperHyperGraph in incidence form, where

$$V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}, \qquad \partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. We call $\mathrm{DSHG}^{(n)}$ *symmetric* if there exists an involution

$$\rho : E \to E, \qquad \rho(\rho(e)) = e \quad (\forall e \in E),$$

such that for every $e \in E$,

$$\mathrm{Tail}(\rho(e)) = \mathrm{Head}(e) \qquad \text{and} \qquad \mathrm{Head}(\rho(e)) = \mathrm{Tail}(e).$$

In this case, $\rho(e)$ is called the *reverse* of e.

Example 3.17.4 (Real-life symmetric directed n-SuperHyperGraph)**.** *Scenario (reciprocal data-sharing agreements among hierarchical institutions).* Let the base set S be a finite set of individual departments across several organizations. A level-n supervertex $v \in V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}$ represents a nested institutional unit, such as "a consortium of labs," where each lab is itself a set of departments and higher levels group labs into institutes.

Suppose institutions sign *reciprocal* data-sharing agreements: whenever a group of units $T \subseteq V$ is allowed to send data to another group $H \subseteq V$, the reverse transfer $H \to T$ is also contractually allowed. Model each agreement as a directed n-superhyperedge identifier $e \in E$ with $\mathrm{Tail}(e) = T$

(senders) and $\mathrm{Head}(e) = H$ (receivers). Reciprocity means that for each $e \in E$ there is a corresponding reverse agreement $\rho(e) \in E$ satisfying

$$\mathrm{Tail}(\rho(e)) = \mathrm{Head}(e), \qquad \mathrm{Head}(\rho(e)) = \mathrm{Tail}(e),$$

and reversing twice returns the original agreement, $\rho(\rho(e)) = e$. Thus the resulting directed n-SuperHyperGraph is *symmetric* in the sense of Definition 3.17.3.

Theorem 3.17.5 (Symmetric directed superhypergraphs generalize symmetric digraphs and symmetric directed hypergraphs)**.** *Let* $\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$ *be a symmetric directed* n*-SuperHyperGraph (Definition 3.17.3).*

(i) ***(Hypergraph specialization.)*** *If* $n = 0$ *and we identify* $V \subseteq \mathcal{P}^0(S) = S$ *with an ordinary vertex set, then the induced directed hypergraph*

$$H = (V, E_H), \qquad E_H := \{(\mathrm{Tail}(e), \mathrm{Head}(e)) \mid e \in E\}$$

is symmetric in the sense of Definition 3.17.2.

(ii) ***(Graph specialization.)*** *Assume* $n = 0$ *and every* $e \in E$ *has singleton tail and singleton head:*

$$\mathrm{Tail}(e) = \{u\}, \qquad \mathrm{Head}(e) = \{v\} \qquad (u, v \in V),$$

and assume simplicity *of incidence: for each ordered pair* (u, v) *there is at most one* $e \in E$ *with* $\mathrm{Tail}(e) = \{u\}$ *and* $\mathrm{Head}(e) = \{v\}$*. Define the arc set*

$$A := \{(u, v) \in V \times V : \exists e \in E \text{ with } \mathrm{Tail}(e) = \{u\},\ \mathrm{Head}(e) = \{v\}\}.$$

Then $D = (V, A)$ *is a symmetric directed graph in the sense of Definition 3.17.1.*

Proof. **(i)** Let $n = 0$. For any $e \in E$, write $(T, H) := (\mathrm{Tail}(e), \mathrm{Head}(e))$. By symmetry of $\mathrm{DSHG}^{(0)}$, there exists $\rho(e) \in E$ with

$$(\mathrm{Tail}(\rho(e)), \mathrm{Head}(\rho(e))) = (H, T).$$

Hence $(H, T) \in E_H$ whenever $(T, H) \in E_H$, so $H = (V, E_H)$ is symmetric as a directed hypergraph.

(ii) Under the singleton hypothesis, each $e \in E$ determines a unique ordered pair (u, v) by $\mathrm{Tail}(e) = \{u\}$ and $\mathrm{Head}(e) = \{v\}$, and by the simplicity assumption this correspondence is well-defined on arcs. If $(u, v) \in A$, choose $e \in E$ with $\mathrm{Tail}(e) = \{u\}$ and $\mathrm{Head}(e) = \{v\}$. By symmetry there is $\rho(e) \in E$ with

$$\mathrm{Tail}(\rho(e)) = \mathrm{Head}(e) = \{v\}, \qquad \mathrm{Head}(\rho(e)) = \mathrm{Tail}(e) = \{u\},$$

so $(v, u) \in A$. Therefore A is closed under reversal, i.e. $D = (V, A)$ is a symmetric digraph. □

Chapter 4

Some Concepts for Directed SuperHyper-Graphs

In this chapter, we examine several fundamental concepts for directed SuperHyperGraphs.

4.1 Feedback arc set in a directed SuperHyperGraph

A feedback arc set in a directed graph is a set of arcs whose removal intersects every directed cycle, making it acyclic [142–145]. A feedback arc set in a directed hypergraph is a set of hyperarcs whose removal hits every directed hypercycle, eliminating all cyclic reachability. A feedback arc set in a directed superhypergraph is a set of directed superhyperedges whose removal intersects every directed supercycle, yielding an acyclic structure.

Definition 4.1.1 (Feedback arc set in a directed graph)**.** [142, 143] Let $G = (V, A)$ be a directed graph. A subset $F \subseteq A$ is called a *feedback arc set* (FAS) of G if every directed cycle in G contains at least one arc from F, i.e.,

$$\forall \text{ directed cycles } C \subseteq G, \qquad A(C) \cap F \neq \emptyset.$$

Equivalently, the digraph $G - F := (V,\ A \setminus F)$ is acyclic.

Definition 4.1.2 (Directed cycle in a directed hypergraph)**.** Let $H = (V, E)$ be a directed hypergraph in which each hyperarc $e \in E$ is an ordered pair $e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big)$ with $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$ nonempty. A *directed (hyper)cycle* in H is a sequence

$$v_1, v_2, \dots, v_k \in V \qquad (k \geq 2)$$

of distinct vertices together with hyperarcs

$$e_1, e_2, \dots, e_k \in E$$

such that, for each $i = 1, \dots, k$,

$$v_i \in \mathrm{Tail}(e_i) \qquad \text{and} \qquad v_{i+1} \in \mathrm{Head}(e_i),$$

where indices are taken cyclically, i.e. $v_{k+1} := v_1$.

Definition 4.1.3 (Feedback arc set in a directed hypergraph)**.** Let $H = (V, E)$ be a directed hypergraph and let directed hypercycles be as in Definition 4.1.2. A subset $F \subseteq E$ is called a *feedback hyperarc set* of H if every directed hypercycle contains at least one hyperarc from F, i.e.,

$$\forall \text{ directed hypercycles } C, \qquad E(C) \cap F \neq \emptyset.$$

Equivalently, the directed hypergraph $H - F := (V,\ E \setminus F)$ has no directed hypercycle.

Definition 4.1.4 (Feedback arc set in a directed n-SuperHyperGraph)**.** Fix $n \geq 0$ and a finite nonempty base set V_0. Let $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ be a set of n-supervertices, and let E be a family of directed n-superhyperedges, where each $e \in E$ is an ordered pair $e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big)$ with $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$ nonempty.

A subset $F \subseteq E$ is called a *feedback superarc set* (feedback n-superhyperedge set) if every directed cycle contains at least one directed n-superhyperedge from F. Equivalently, removing F makes the directed n-SuperHyperGraph acyclic (no directed cycle remains).

Example 4.1.5 (Real-life feedback superarc set in a directed n-SuperHyperGraph)**.** *Scenario (breaking circular dependencies in a multi-team software project).* Consider a large software project where work is organized hierarchically: individual tasks belong to modules, modules belong to subsystems, and subsystems belong to products. Let the base set V_0 be the set of atomic tasks. A level-n supervertex $v \in V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ represents a nested work package, such as a subsystem-level bundle of modules (each module is itself a set of tasks).

A directed n-superhyperedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ represents a dependency rule: "any package in $\mathrm{Head}(e)$ cannot be completed until at least one package in $\mathrm{Tail}(e)$ has been completed." Suppose three subsystem packages $v_1, v_2, v_3 \in V$ satisfy

$$e_{12} : \ \{v_1\} \to \{v_2\}, \qquad e_{23} : \ \{v_2\} \to \{v_3\}, \qquad e_{31} : \ \{v_3\} \to \{v_1\},$$

forming a directed cycle (a circular dependency) among the packages.

A practical intervention is to *cut* at least one dependency by redesigning an interface or introducing a stub. For example, choose

$$F := \{e_{31}\} \subseteq E,$$

meaning "remove (or override) the dependency $v_3 \to v_1$ by decoupling v_1 from v_3." Then every directed cycle in this dependency superhypergraph is intersected by F, and removing F breaks all cycles, yielding an acyclic dependency structure. Hence F is a feedback superarc set in the sense of Definition 4.1.4.

Theorem 4.1.6 (SuperHyperGraph feedback arc sets generalize graph/hypergraph feedback arc sets)**.** *Let V be a finite set.*

(i) ***(Hypergraph** $\Rightarrow$ 0**-SuperHyperGraph.)*** *Let $H = (V, E)$ be a directed hypergraph. Form the directed* 0*-SuperHyperGraph* $\mathrm{SHG}^{(0)} = (V, E)$ *by taking $V_0 := V$ and viewing each hyperarc $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ as a directed* 0*-superhyperedge with* $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$*. Then, for any $F \subseteq E$,*

$$F \textit{ is a feedback hyperarc set of } H \iff F \textit{ is a feedback superarc set of } \mathrm{SHG}^{(0)}.$$

(ii) ***(Graph* $\Rightarrow$ 0*-SuperHyperGraph.)*** *Let* $G = (V, A)$ *be a directed graph. Define a directed* 0*-SuperHyperGraph* $\mathrm{SHG}^{(0)} = (V, E)$ *by taking* $V_0 := V$ *and*

$$E := \{ e_{(u,v)} \mid (u, v) \in A \}, \qquad e_{(u,v)} := \big(\{u\}, \{v\}\big).$$

Then, for any $F \subseteq A$*, letting*

$$\iota(F) := \{ e_{(u,v)} \in E \mid (u, v) \in F \} \subseteq E,$$

one has

$$F \text{ is a feedback arc set of } G \iff \iota(F) \text{ is a feedback superarc set of } \mathrm{SHG}^{(0)}.$$

Hence, Definition 4.1.4 strictly extends the standard feedback arc set notions for directed graphs and directed hypergraphs.

Proof. **(i)** In the case $n = 0$, supervertices are ordinary vertices, and a directed 0-superhyperedge is exactly a directed hyperarc $(\mathrm{Tail}(e), \mathrm{Head}(e))$ with nonempty tail/head subsets of V. Moreover, the directed-cycle condition used for $\mathrm{SHG}^{(0)}$ specializes to the alternating vertex–edge criterion of Definition 4.1.2. Therefore, the collections of directed cycles in H and in $\mathrm{SHG}^{(0)}$ coincide, so a subset $F \subseteq E$ hits all directed cycles in H if and only if it hits all directed cycles in $\mathrm{SHG}^{(0)}$.

(ii) A directed cycle in G is a sequence of distinct vertices $v_1, \ldots, v_k$ with arcs $(v_i, v_{i+1}) \in A$ (cyclic indices). Under the embedding $(u, v) \mapsto e_{(u,v)} = (\{u\}, \{v\})$, this becomes a directed cycle in $\mathrm{SHG}^{(0)}$ witnessed by the same vertices and the corresponding directed superhyperedges $e_{(v_i,v_{i+1})}$, because

$$v_i \in \{v_i\} = \mathrm{Tail}\big(e_{(v_i,v_{i+1})}\big), \qquad v_{i+1} \in \{v_{i+1}\} = \mathrm{Head}\big(e_{(v_i,v_{i+1})}\big).$$

Conversely, any directed cycle in $\mathrm{SHG}^{(0)}$ that uses only singleton-to-singleton edges $e_{(u,v)}$ projects to a directed cycle in G by forgetting braces. Thus cycles correspond bijectively under ι, and a set $F \subseteq A$ intersects every directed cycle of G if and only if $\iota(F)$ intersects every directed cycle of $\mathrm{SHG}^{(0)}$. □

4.2 Flow SuperHyperNetwork

A flow network is a capacitated directed graph with source and sink; feasible flows satisfy capacity bounds and conservation constraints [146–149]. A flow hypernetwork is a capacitated directed hypergraph; hyperflows traverse tail-to-head hyperarcs and satisfy throughput and vertex-conservation constraints. A flow superhypernetwork is a capacitated directed n-superhypergraph; superhyperflows respect hierarchical supervertices, superedge capacities, and conservation constraints.

Definition 4.2.1 (Flow network and feasible s–t flow)**.** [146–149] A *flow network* is a quadruple

$$\mathcal{N} = (G, c, s, t),$$

where $G = (V, A)$ is a finite directed graph, $c : A \to \mathrm{R}_{\geq 0}$ is a *capacity function*, and $s, t \in V$ are distinguished vertices (source and sink), $s \neq t$.

A *feasible (s–t) flow* on $\mathcal{N}$ is a function $f : A \to \mathrm{R}_{\geq 0}$ such that

$$0 \leq f(a) \leq c(a) \qquad (\forall a \in A) \qquad \text{(capacity)}$$

$$\sum_{(u,v)\in A} f(u, v) \;=\; \sum_{(v,w)\in A} f(v, w) \qquad (\forall v \in V \setminus \{s, t\}) \qquad \text{(conservation)}$$

(where $f(u, v)$ denotes $f((u, v))$). The *value* of f is

$$|f| \;:=\; \sum_{(u,t)\in A} f(u, t) \;-\; \sum_{(t,w)\in A} f(t, w) \;=\; \sum_{(s,w)\in A} f(s, w) \;-\; \sum_{(u,s)\in A} f(u, s).$$

Definition 4.2.2 (Flow hypernetwork and feasible hyperflow)**.** Let $H = (V, E)$ be a directed hypergraph in the sense of Definition 2.2.1, so each hyperarc $e \in E$ is $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with $\mathrm{Tail}(e), \mathrm{Head}(e) \in \mathcal{P}(V) \setminus \{\emptyset\}$.

A *flow hypernetwork* is a quadruple

$$\mathcal{H} = (H, c, s, t),$$

where $c : E \to \mathrm{R}_{\geq 0}$ is a capacity function and $s, t \in V$ are distinct vertices.

A *feasible (s–t) hyperflow* on $\mathcal{H}$ is specified by two nonnegative incidence maps

$$f^- : \{(v, e) \mid e \in E,\ v \in \mathrm{Tail}(e)\} \to \mathrm{R}_{\geq 0}, \qquad f^+ : \{(e, v) \mid e \in E,\ v \in \mathrm{Head}(e)\} \to \mathrm{R}_{\geq 0},$$

such that

$$\sum_{v\in\mathrm{Tail}(e)} f^-(v, e) \;=\; \sum_{v\in\mathrm{Head}(e)} f^+(e, v) \;\leq\; c(e) \qquad (\forall e \in E) \qquad \text{(hyperarc capacity/throughput)}$$

$$\sum_{e:\ v\in\mathrm{Head}(e)} f^+(e, v) \;=\; \sum_{e:\ v\in\mathrm{Tail}(e)} f^-(v, e) \qquad (\forall v \in V \setminus \{s, t\}) \quad \text{(vertex conservation)}$$

The *value* of (f^-, f^+) is the net inflow into t:

$$|f| \;:=\; \sum_{e:\ t\in\mathrm{Head}(e)} f^+(e, t) \;-\; \sum_{e:\ t\in\mathrm{Tail}(e)} f^-(t, e),$$

(which equals the net outflow from s by conservation at all other vertices).

Definition 4.2.3 (Flow n-superhypernetwork and feasible superhyperflow)**.** Fix an integer $n \geq 0$ and a finite nonempty base set V_0. Let $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ be a set of n-supervertices, and let E be a family of directed n-superhyperedges, each written

$$e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big), \qquad \mathrm{Tail}(e), \mathrm{Head}(e) \in \mathcal{P}(V) \setminus \{\emptyset\}.$$

(That is, the underlying directed n-SuperHyperGraph is given in the same incidence form used throughout this book.)

A *flow n-superhypernetwork* is a quadruple

$$\mathcal{S} = (\mathrm{SHG}^{(n)}_{\mathrm{dir}}, c, s, t) \equiv ((V, E), c, s, t),$$

where $c : E \to \mathrm{R}_{\geq 0}$ is a capacity function and $s, t \in V$ are distinct n-supervertices.

A *feasible (s–t) superhyperflow* on $\mathcal{S}$ is specified by nonnegative incidence maps

$$f^- : \{(x, e) \mid e \in E,\ x \in \mathrm{Tail}(e)\} \to \mathrm{R}_{\geq 0}, \qquad f^+ : \{(e, y) \mid e \in E,\ y \in \mathrm{Head}(e)\} \to \mathrm{R}_{\geq 0},$$

such that

$$\sum_{x \in \mathrm{Tail}(e)} f^-(x, e) \ = \sum_{y \in \mathrm{Head}(e)} f^+(e, y) \ \leq \ c(e) \qquad (\forall e \in E)$$

(superedge capacity/throughput)

$$\sum_{e:\ v \in \mathrm{Head}(e)} f^+(e, v) \ = \sum_{e:\ v \in \mathrm{Tail}(e)} f^-(v, e) \qquad (\forall v \in V \setminus \{s, t\})$$

(supervertex conservation)

The *value* is

$$|f| \ := \sum_{e:\ t \in \mathrm{Head}(e)} f^+(e, t) \ - \sum_{e:\ t \in \mathrm{Tail}(e)} f^-(t, e).$$

Example 4.2.4 (Real-life flow n-superhypernetwork)**.** *Scenario (hierarchical content delivery in a streaming platform).* Consider a video streaming service with a hierarchical delivery architecture. Let the base set V_0 be the set of concrete servers and caches (edge POPs, regional caches, origin servers). A level-n supervertex $v \in V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ represents a *nested delivery unit*, for example: servers grouped into racks, racks grouped into data centers, and data centers grouped into regions.

A directed n-superhyperedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ models a permissible transfer pattern between *sets of delivery units*, such as "traffic can be shifted from these regional cache groups to those edge cache groups." The capacity $c(e)$ represents the maximum throughput (e.g., in Gbps) that can be routed along that pattern. Choose a source supervertex s representing the origin-region group and a sink supervertex t representing a specific edge-region group serving end users.

A feasible superhyperflow (f^-, f^+) assigns nonnegative throughput contributions from each supervertex in $\mathrm{Tail}(e)$ into e and from e into each supervertex in $\mathrm{Head}(e)$, satisfying: (i) for every transfer pattern e, the total routed throughput through e is conserved and does not exceed $c(e)$, and (ii) for every intermediate delivery group $v \notin \{s, t\}$, the total incoming throughput equals the total outgoing throughput (no net creation or loss of traffic at that level of aggregation). The flow value $|f|$ is then the net delivered streaming throughput reaching the sink group t, quantifying end-to-end delivery performance under hierarchical routing constraints.

Theorem 4.2.5 (SuperHyperGraph structure and generalization of flow models)**.**

(i) ***(Underlying SuperHyperGraph structure.)*** *If $\mathcal{S}$ is a flow n-superhypernetwork as in Definition 4.2.3, then forgetting (c, s, t) yields a directed n-SuperHyperGraph $\mathrm{SHG}^{(n)}_{\mathrm{dir}} = (V, E)$ (on the base set V_0), i.e. $\mathcal{S}$ is a directed n-SuperHyperGraph equipped with capacities and terminals.*

(ii) ***(Hypernetwork as the*** $n = 0$ ***special case.)*** *Let* $\mathcal{H} = (H, c, s, t)$ *be a flow hypernetwork (Definition 4.2.2). Set* $V_0 := V$ *and* $n := 0$*. Then* $\mathcal{H}$ *is (canonically) a flow* 0*-superhypernetwork: its vertex set is* $V \subseteq \mathcal{P}^0(V_0) = V_0$*, its directed superhyperedges are exactly the hyperarcs of* H*, and feasible hyperflows coincide with feasible superhyperflows.*

(iii) ***(Flow network as the singleton-tail/head subcase.)*** *Let* $\mathcal{N} = (G, c, s, t)$ *be a flow network (Definition 4.2.1) with* $G = (V, A)$*. Define a directed* 0*-SuperHyperGraph on* $V_0 := V$ *by replacing each arc* $(u, v) \in A$ *with the directed superhyperedge* $e_{(u,v)} := (\{u\}, \{v\})$*, and set* $c(e_{(u,v)}) := c(u, v)$*. Then* $\mathcal{N}$ *embeds into a flow* 0*-superhypernetwork* $\mathcal{S}_0$*. Moreover, feasible flows* $f : A \to \mathrm{R}_{\geq 0}$ *on* $\mathcal{N}$ *correspond bijectively to feasible superhyperflows* (f^-, f^+) *on* $\mathcal{S}_0$ *via*

$$f^-(u, e_{(u,v)}) = f(u, v), \qquad f^+(e_{(u,v)}, v) = f(u, v),$$

and all other incidences are 0*. Under this correspondence, the flow values are equal.*

Proof. **(i)** This is immediate from Definition 4.2.3: the data (V, E) already satisfy $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ and each $e \in E$ is an ordered pair $(\mathrm{Tail}(e), \mathrm{Head}(e))$ with $\mathrm{Tail}(e), \mathrm{Head}(e) \in \mathcal{P}(V) \setminus \{\emptyset\}$, which is precisely the directed n-SuperHyperGraph incidence structure used in the book. The remaining data (c, s, t) are additional annotations.

(ii) Put $n = 0$ and $V_0 := V$. Then $\mathcal{P}^0(V_0) = V_0 = V$, so the supervertex universe is exactly the vertex set. A directed hyperarc $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ in H is already an ordered pair of nonempty subsets of V, hence it is a directed 0-superhyperedge in the sense of Definition 4.2.3. The capacity map c and terminals s, t are unchanged. Finally, the feasibility constraints in Definitions 4.2.2 and 4.2.3 are identical after this identification; therefore feasible hyperflows and feasible superhyperflows coincide.

(iii) Build $\mathcal{S}_0$ as stated. Given a feasible graph flow $f : A \to \mathrm{R}_{\geq 0}$, define (f^-, f^+) by $f^-(u, e_{(u,v)}) = f(u, v)$ and $f^+(e_{(u,v)}, v) = f(u, v)$, with all other incidences 0. For each $e_{(u,v)}$ we have

$$\sum_{x \in \mathrm{Tail}(e_{(u,v)})} f^-(x, e_{(u,v)}) = f^-(u, e_{(u,v)}) = f(u, v) = f^+(e_{(u,v)}, v) = \sum_{y \in \mathrm{Head}(e_{(u,v)})} f^+(e_{(u,v)}, y),$$

and $f(u, v) \leq c(u, v) = c(e_{(u,v)})$, so the superedge capacity/throughput constraints hold. For any $w \in V \setminus \{s, t\}$, the conservation constraint becomes

$$\sum_{(u,w) \in A} f(u, w) \;=\; \sum_{(w,v) \in A} f(w, v),$$

which is exactly graph-flow conservation. The equality of values follows by expanding the definition of $|f|$ in both models.

Conversely, if (f^-, f^+) is a feasible superhyperflow on $\mathcal{S}_0$, then each superedge has singleton tail/head, so define $f(u, v) := f^-(u, e_{(u,v)}) = f^+(e_{(u,v)}, v)$; the superedge constraints force these two quantities to be equal. Vertex conservation in $\mathcal{S}_0$ becomes graph-flow conservation, and capacity bounds carry over. Hence we obtain a feasible graph flow, and the constructions are inverse to each other. □

4.3 SuperHyperTournaments

A tournament is a complete directed graph: for each distinct vertex pair, exactly one directed edge is chosen [150–154]. A k-hypertournament assigns exactly one ordering to every k-vertex subset, orienting each k-uniform hyperedge uniquely [155–158]. A superhypertournament orients every k-subset of n-supervertices uniquely, forming an oriented complete k-uniform n-SuperHyperGraph.

Notation 4.3.1. *For a set X and an integer $k \geq 2$, write*

$$\binom{X}{k} := \{\, S \subseteq X \mid |S| = k \,\} \qquad \textit{and} \qquad X^{\underline{k}} := \{(x_1, \dots, x_k) \in X^k \mid x_i \neq x_j \ (i \neq j)\}.$$

For $S = \{x_1, \dots, x_k\} \in \binom{X}{k}$, let $\mathrm{Lin}(S)$ denote the set of all linear orderings of S, equivalently all $k!$ permutations of its elements viewed as k-tuples in $X^{\underline{k}}$.

Definition 4.3.2 (Tournament). [150, 151, 159] A *tournament* is a directed graph $T = (V, A)$ (with no loops) such that for every two distinct vertices $u, v \in V$ exactly one of (u, v) or (v, u) belongs to A. Equivalently,

$$\forall \{u, v\} \in \binom{V}{2}, \quad \big|\{(u, v), (v, u)\} \cap A\big| = 1.$$

Definition 4.3.3 (k-Hypertournament). Fix an integer $k \geq 2$. A *k-hypertournament* is a pair $H = (V, \mathcal{A})$ where V is a finite set and $\mathcal{A} \subseteq V^{\underline{k}}$ is a set of *arcs* (ordered k-tuples) such that for every k-subset $S \in \binom{V}{k}$,

$$|\mathcal{A} \cap \mathrm{Lin}(S)| = 1.$$

In words: each k-subset of vertices appears with exactly one of its $k!$ possible orientations.

Remark 4.3.4 (Special case $k = 2$). When $k = 2$, a 2-hypertournament is exactly a tournament: for each $\{u, v\} \in \binom{V}{2}$, exactly one of (u, v) or (v, u) is chosen.

Definition 4.3.5 ((n, k)-SuperHypertournament). Fix integers $n \geq 0$ and $k \geq 2$, and a finite nonempty base set V_0. Let $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ be a set of n-supervertices.

A (n, k)-*superhypertournament* is a pair

$$\mathcal{T}^{(n,k)} = (V, \mathcal{A}),$$

where $\mathcal{A} \subseteq V^{\underline{k}}$ is a set of *superarcs* such that for every k-subset $S \in \binom{V}{k}$,

$$|\mathcal{A} \cap \mathrm{Lin}(S)| = 1.$$

We call $\mathcal{T}^{(n,k)}$ simply a *superhypertournament* when n, k are understood.

Example 4.3.6 (Real-life (n,k)-superhypertournament: ranking hierarchical teams from k-way comparisons)**.** *Scenario (product-group ranking from* 3*-way evaluations).* Suppose a company evaluates several *hierarchical product groups*, where each group consists of multiple teams, and each team consists of individual contributors. Let the base set V_0 be the set of individuals, and let $n \geq 1$ be fixed. An n-supervertex $v \in V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ represents a nested product group (e.g., a set of teams, each team itself a set of individuals).

Assume the company uses *three-way* (i.e., $k = 3$) comparative reviews: for every triple of distinct product groups $\{x, y, z\} \subseteq V$, a committee produces a strict ranking of the three groups (best $\succ$ middle $\succ$ worst) based on KPIs and qualitative judgment. Equivalently, the outcome for $\{x, y, z\}$ is recorded as exactly one ordered triple

$$(x, y, z) \quad \text{or} \quad (x, z, y) \quad \text{or} \quad (y, x, z) \quad \text{or} \quad \cdots \quad \in V^{\underline{3}},$$

representing the unique linear order chosen on that 3-set.

Collecting, for every 3-subset $S \in \binom{V}{3}$, exactly one ordering in $\mathrm{Lin}(S)$, we obtain a set of superarcs $\mathcal{A} \subseteq V^{\underline{3}}$ satisfying

$$|\mathcal{A} \cap \mathrm{Lin}(S)| = 1 \qquad (\forall S \in \binom{V}{3}),$$

so $\mathcal{T}^{(n,3)} = (V, \mathcal{A})$ is an $(n,3)$-superhypertournament in the sense of Definition 4.3.5.

Theorem 4.3.7 (Superhypertournaments generalize tournaments and hypertournaments)**.** *Let $\mathcal{T}^{(n,k)} = (V, \mathcal{A})$ be a (n,k)-superhypertournament.*

(i) *If $n = 0$ and $k = 2$, then $\mathcal{T}^{(0,2)}$ is precisely a tournament on V.*

(ii) *If $n = 0$ and $k \geq 2$, then $\mathcal{T}^{(0,k)}$ is precisely a k-hypertournament on V.*

Proof. If $n = 0$, then $\mathcal{P}^0(V_0) = V_0$ and $V \subseteq V_0$, so the elements of V are ordinary vertices.

(i) Assume $k = 2$. For each $\{u, v\} \in \binom{V}{2}$, the defining condition $|\mathcal{A} \cap \mathrm{Lin}(\{u, v\})| = 1$ says exactly one of (u, v) or (v, u) belongs to $\mathcal{A}$. Thus $(V, \mathcal{A})$ satisfies Definition 4.3.2.

(ii) For general $k \geq 2$, the condition $|\mathcal{A} \cap \mathrm{Lin}(S)| = 1$ for every $S \in \binom{V}{k}$ is exactly Definition 4.3.3. □

Theorem 4.3.8 (Underlying n-SuperHyperGraph structure)**.** *Let $\mathcal{T}^{(n,k)} = (V, \mathcal{A})$ be a (n,k)-superhypertournament on a base set V_0. Define*

$$E := \binom{V}{k} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then $\mathrm{SHG}^{(n)}_{\mathcal{T}} := (V, E)$ is an n-SuperHyperGraph on V_0 in the sense of Definition 2.1.6. Moreover, $\mathcal{A}$ is an orientation *of this complete k-uniform n-SuperHyperGraph, assigning to each $e \in E$ exactly one linear ordering of its k incident supervertices.*

Proof. By Definition 4.3.5, we have $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$. By construction, $E = \binom{V}{k}$ is a family of nonempty subsets of V, hence $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Therefore (V, E) satisfies Definition 2.1.6, so it is an n-SuperHyperGraph on V_0.

Finally, for each $e \in E$ (a k-subset of V), the defining axiom of a (n, k)-superhypertournament asserts $|\mathcal{A} \cap \mathrm{Lin}(e)| = 1$, i.e. exactly one ordering of e is chosen. This is precisely an orientation of the complete k-uniform superhypergraph (V, E). □

Chapter 5

Uncertain Directed SuperHyperGraphs

In this chapter, we investigate uncertain directed SuperHyperGraphs.

5.1 Fuzzy Directed n-Superhypergraphs

A fuzzy set assigns to each element a membership degree in $[0, 1]$, thereby modeling partial belonging rather than crisp inclusion or exclusion [160–162]. As concepts related to fuzzy sets, several well-known extensions have been studied, including intuitionistic fuzzy sets [163, 164], hesitant fuzzy sets [165, 166], picture fuzzy sets [167, 168], and spherical fuzzy sets [169, 170].

Fuzzy directed graphs extend directed graphs by assigning membership degrees to vertices and arcs in order to capture uncertain directional relations [171–173]. Fuzzy directed hypergraphs further generalize this idea by allowing directed hyperarcs from a tail-set of vertices to a head-set of vertices, together with membership degrees describing uncertain multiway influence [52, 174, 175]. In this book, we define fuzzy directed n-Superhypergraphs by combining fuzzy membership assignments with the hierarchical vertex universe of n-Superhypergraphs.

Definition 5.1.1 (Fuzzy directed hypergraph)**.** (cf. [52, 174, 176, 177]) Let $V \neq \emptyset$ be a vertex set. A *fuzzy directed hypergraph* is a quadruple

$$H = \big(V, E, \sigma, \mu\big),$$

where:

- E is a finite set of directed hyperarcs. Each $e \in E$ is an ordered pair

  $$e = \big(T(e), H(e)\big), \qquad \emptyset \neq T(e) \subseteq V,\ \emptyset \neq H(e) \subseteq V,$$

  called the *tail* and *head* of e, respectively.

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex.

- $\mu : E \to [0, 1]$ assigns a membership degree to each hyperarc.

These functions satisfy the *edge-appurtenance (consistency) constraint*

$$\mu(e) \ \le \min_{x \in T(e) \cup H(e)} \sigma(x), \qquad \forall e \in E.$$

Remark 5.1.2 (On the disjointness convention)**.** Some authors impose additional conventions such as $H(e) \subseteq V \setminus T(e)$ (disjoint tail and head). This is optional and model-dependent; the present definition allows overlap unless explicitly required.

Definition 5.1.3 (Fuzzy directed n-Superhypergraph)**.** Let $S \neq \emptyset$ be a base set and let $n \geq 0$. Define iterated powersets by

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^{k+1}(S) = \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \geq 0).$$

Let $\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$ be a directed n-SuperHyperGraph in incidence form, i.e.

$$V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}, \qquad \partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. A *fuzzy directed n-Superhypergraph* is a quadruple

$$\big(V, E, \sigma, \mu\big),$$

where $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ satisfy, for every $e \in E$,

$$\mu(e) \ \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x).$$

Example 5.1.4 (Real-life fuzzy directed n-Superhypergraph)**.** *Scenario (uncertain escalation in a hierarchical incident-response organization).* Let the base set S be a finite set of on-call engineers. A level-n supervertex $v \in V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}$ represents a *nested* response unit (e.g., engineers grouped into squads, squads grouped into incident domains, and domains grouped into a company-wide response group).

The "availability" or "readiness" of each unit is uncertain (shift overlaps, fatigue, and competing incidents). Encode this by a membership map

$$\sigma : V \to [0, 1],$$

where $\sigma(v)$ is the estimated probability (or confidence level) that unit v can respond promptly.

A directed n-superhyperedge identifier $e \in E$ models an escalation rule: "if any unit in $\mathrm{Tail}(e)$ is engaged, then units in $\mathrm{Head}(e)$ are next candidates to be paged." The reliability of executing this escalation is also uncertain (paging may fail, or handoff may be delayed), so define

$$\mu : E \to [0, 1],$$

where $\mu(e)$ quantifies how reliably escalation e can be carried out.

The constraint

$$\mu(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x)$$

captures the natural requirement that an escalation cannot be more reliable than the least-available unit participating in it. Hence $\big(V, E, \sigma, \mu\big)$ forms a fuzzy directed n-Superhypergraph in the sense of Definition 5.1.3.

5.2 Single-valued Neutrosophic Directed n-Superhypergraph

A single-valued neutrosophic set assigns to each element degrees of truth, indeterminacy, and falsity in $[0,1]$, typically satisfying $T + I + F \leq 3$ for uncertainty modeling [178–180]. Related concepts include interval-valued neutrosophic sets [181, 182], hesitant neutrosophic sets [183], quadripartitioned neutrosophic sets [184–186], double-valued neutrosophic sets [187–189], and plithogenic sets [190, 191].

A single-valued neutrosophic directed graph orients edges and assigns each vertex and arc triple degrees of truth, indeterminacy, falsity levels [192, 193]. A single-valued neutrosophic directed hypergraph uses oriented hyperarcs between vertex sets with neutrosophic truth, indeterminacy, falsity degrees modeling uncertainty precisely [193]. We define the *Single-valued Neutrosophic Directed n-Superhypergraph* as an extension of the classical *Single-valued Neutrosophic Directed Hypergraph* by incorporating the hierarchical structure of n-Superhypergraphs [54].

Definition 5.2.1 (Single-valued Neutrosophic Directed Hypergraph)**.** (cf. [192]) A *single-valued neutrosophic directed hypergraph* on a nonempty set X is an ordered pair

$$G' = \big(G,\ \{F_j\}_{j=1}^n\big),$$

where

$$G = \{\,G_j\}_{j=1}^n, \quad G_j = \big(T(G_j),\, H(G_j)\big)$$

is a family of nontrivial single-valued neutrosophic subsets of X, with

$$T(G_j) = \{\,\big(v, \alpha_G(v), \beta_G(v), \gamma_G(v)\big) \mid v \in X\},$$

$$H(G_j) = \{\,\big(v', \alpha_G(v'), \beta_G(v'), \gamma_G(v')\big) \mid v' \in X\},$$

and each neutrosophic *hyperarc* is

$$F_j\big(T(G_j), H(G_j)\big) = (\alpha_{F_j},\, \beta_{F_j},\, \gamma_{F_j})$$

satisfying, for all j,

$$\alpha_{F_j} \leq \bigwedge_{v\in T(G_j),\, v'\in H(G_j)} \big(\alpha_G(v) \wedge \alpha_G(v')\big),$$
$$\beta_{F_j} \leq \bigwedge_{v\in T(G_j),\, v'\in H(G_j)} \big(\beta_G(v) \wedge \beta_G(v')\big),$$
$$\gamma_{F_j} \leq \bigvee_{v\in T(G_j),\, v'\in H(G_j)} \big(\gamma_G(v) \wedge \gamma_G(v')\big),$$

and

$$X = \bigcup_{j=1}^n \operatorname{supp}(G_j).$$

Definition 5.2.2 (Neutrosophic Directed Superhypergraph)**.** Let S be a nonempty *base set* and $n \geq 0$ an integer. Define

$$\mathcal{P}^0(S) = S,$$
$$\mathcal{P}^{k+1}(S) = \mathcal{P}(\mathcal{P}^k(S)) \quad (k \geq 0).$$

A *directed n-Superhypergraph* is a pair $\mathrm{DSHG}^{(n)} = (V, E)$ with

$$V \subseteq \mathcal{P}^n(S),$$

$$E \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S),$$

where each $e \in E$ is $\big(\mathrm{Tail}(e), \mathrm{Head}(e)\big)$. A *single-valued neutrosophic directed n-Superhypergraph* is the septuple

$$\big(V,\, E,\, T_V,\, I_V,\, F_V,\, T_E,\, I_E,\, F_E\big),$$

where

$$\begin{aligned}
&T_V,\, I_V,\, F_V : V \to [0,1],\\
&T_V(v) + I_V(v) + F_V(v) \le 3,\ \forall v \in V,\\
&T_E,\, I_E,\, F_E : E \to [0,1],\\
&\quad T_E(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} T_V(x),\\
&\quad I_E(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} I_V(x),\\
&\quad F_E(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} F_V(x), \quad \forall e \in E.
\end{aligned}$$

Example 5.2.3 (Real-life single-valued neutrosophic directed n-Superhypergraph)**.** *Scenario (credibility of hierarchical intelligence reports and their directed influence).* Let the base set S be a finite collection of raw information sources (sensor feeds, eyewitness accounts, open-source reports). A level-n supervertex $v \in V \subseteq \mathcal{P}^n(S)$ represents a *nested analytic unit*, for example: sources grouped into dossiers, dossiers grouped into cases, and cases grouped into a strategic assessment portfolio (depending on n).

For each unit $v \in V$, assign a single-valued neutrosophic credibility profile

$$\big(T_V(v),\, I_V(v),\, F_V(v)\big) \in [0,1]^3,$$

where $T_V(v)$ measures how strongly the unit is believed reliable, $F_V(v)$ measures how strongly it is believed unreliable, and $I_V(v)$ measures the degree of indeterminacy (insufficient or conflicting evidence).

A directed n-superedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e)) \in E$ models an inference or propagation step: "information aggregated in the tail-units is used to update or justify assessments in the head-units." Assign to each such step a neutrosophic quality triple

$$\big(T_E(e),\, I_E(e),\, F_E(e)\big) \in [0,1]^3,$$

where $T_E(e)$ reflects confidence that the inference is valid, $F_E(e)$ reflects confidence that it is invalid, and $I_E(e)$ reflects indeterminacy due to missing context or conflicting interpretations.

The constraints

$$T_E(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} T_V(x), \quad I_E(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} I_V(x), \quad F_E(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} F_V(x)$$

express that the strength (or indeterminacy, or falsity) assigned to an inference cannot exceed the least corresponding degree among the participating units. For instance, if a head-unit is highly indeterminate, then the inference step inherits that indeterminacy bound. Thus $\big(V, E, T_V, I_V, F_V, T_E, I_E, F_E\big)$ forms a single-valued neutrosophic directed n-Superhypergraph in the sense of Definition 5.2.2.

5.3 Uncertain Directed SuperHyperGraph

Uncertain sets assign each element a model-specific degree tuple in $\mathrm{Dom}(M) \subseteq [0,1]^k$, encoding membership uncertainty under constraints [3, 194]. Uncertain directed graphs assign degree tuples to vertices and arcs, typically constraining each arc's degree by its endpoint degrees. Uncertain directed hypergraphs assign degree tuples to vertices and hyperarcs, typically bounding each hyperarc's degree by incident vertices' degrees. Uncertain directed superhypergraphs assign degree tuples to supervertices and directed superhyperedges, bounding each superedge's degree by its tail-head supervertices' degrees.

Definition 5.3.1 (Uncertain model)**.** [194] Let U denote the class of all *uncertain models.* Each $M \in U$ is determined by:

- a nonempty set $\mathrm{Dom}(M) \subseteq [0,1]^k$ of *admissible degree tuples* for some fixed integer $k \geq 1$; and
- model-specific algebraic or geometric constraints imposed on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \leq 1$ in the intuitionistic fuzzy setting, or $0 \leq T + I + F \leq 3$ in the neutrosophic setting).

Typical instances include:

- **Fuzzy model:** $\mathrm{Dom}(M) = [0,1]$;
- **Intuitionistic fuzzy model:** $\mathrm{Dom}(M) = \{(\mu,\nu) \in [0,1]^2 : \ \mu + \nu \leq 1\}$;
- **Neutrosophic model:** $\mathrm{Dom}(M) = \{(T,I,F) \in [0,1]^3 : \ 0 \leq T + I + F \leq 3\}$;
- **Plithogenic model,** and many further extensions.

Definition 5.3.2 (Uncertain set (U-set))**.** [194] Let X be a nonempty universe, and fix an uncertain model M with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *uncertain set of type M* (briefly, a *U-set*) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is the *uncertainty-degree function* (membership map) of $\mathcal{U}$. For $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ encodes the degree(s) to which x belongs to $\mathcal{U}$, as prescribed by the model M.

Definition 5.3.3 (Uncertain graph)**.** Let $G = (V, E)$ be a finite, undirected, loopless graph, and let M be an uncertain model with degree-domain $\mathrm{Dom}(M)$. An *uncertain graph of type* M is a triple

$$\mathcal{G}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns an uncertainty degree in $\mathrm{Dom}(M)$ to each vertex $v \in V$ and each edge $e \in E$. Optionally, one may impose model-dependent consistency relations between vertex- and edge-degrees (e.g., bounding $\mu_M(e)$ in terms of $\mu_M(u)$ and $\mu_M(v)$ for $e = \{u, v\}$ in fuzzy or intuitionistic fuzzy settings), but such constraints are dictated by the chosen model M and are not fixed at the level of this general definition.

For convenience, Table 5.1 lists representative uncertainty-graph families, organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$.

Table 5.1: A catalogue of uncertainty-graph families (uncertain graphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$.

k	Representative uncertainty-graph type(s) $\mathcal{G}_M = (V, E, \mu_M)$ with $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0, 1]^k$
1	Fuzzy graph [161]; N-graph; shadowed-graph variants
2	Intuitionistic fuzzy graph [195]; vague graph [196]; bipolar fuzzy graph [197]; intuitionistic evidence graph; variable fuzzy graph; paraconsistent fuzzy graph; bifuzzy graph [198, 199]
3	Neutrosophic graph [180][(a)]; hesitant fuzzy graph [200]; tripolar fuzzy graph; three-way fuzzy graph; picture fuzzy graph [201, 202]; spherical fuzzy graph [169]; inconsistent intuitionistic fuzzy graph; ternary fuzzy / neutrosophic-fuzzy graph; neutrosophic vague graph
4	Quadripartitioned neutrosophic graph [203, 204]; double-valued neutrosophic graph [205]; dual hesitant fuzzy graph [206]; ambiguous graph[(b)]; local-neutrosophic graph; support-neutrosophic graph; turiyam neutrosophic graph [207][(c)]
5	Pentapartitioned neutrosophic graph [208]; triple-valued neutrosophic graph
6	Hexapartitioned neutrosophic graph; quadruple-valued neutrosophic graph
7	Heptapartitioned neutrosophic graph [209]; quintuple-valued neutrosophic graph
8	Octapartitioned neutrosophic graph
9	Nonapartitioned neutrosophic graph
n	n-refined fuzzy graph; multi-valued (fuzzy) graphs; multi-fuzzy graphs [210]
$2n$	n-refined intuitionistic fuzzy graph; multi-intuitionistic fuzzy graphs
$3n$	n-refined neutrosophic graph; multi-neutrosophic graphs

(a) Neutrosophic graph models are often treated as broad frameworks that can specialize to many degree-based graph formalisms under suitable constraints.
(b) Ambiguous-graph models are commonly presented as subclasses of certain quadripartitioned and also double-valued neutrosophic graph models.
(c) Turiyam neutrosophic graphs are reported as subclasses of certain quadripartitioned neutrosophic graph models.

Notation 5.3.4 (Product order)**.** *Fix an uncertain model M with* $\mathrm{Dom}(M) \subseteq [0, 1]^k$. *For* $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k) \in [0, 1]^k$ *write*

$$a \preceq b \quad :\Longleftrightarrow \quad a_i \le b_i \ \text{ for all } i = 1, \ldots, k$$

(the product/coordinatewise order). For any finite nonempty family $\{a^{(j)}\}_{j \in J} \subseteq [0, 1]^k$ *define its* meet *(greatest lower bound)*

$$\bigwedge_{j \in J} a^{(j)} \ := \ \big(\min_{j \in J} a_1^{(j)}, \ \ldots, \ \min_{j \in J} a_k^{(j)}\big) \in [0, 1]^k.$$

Remark 5.3.5 (On closure)**.** The meet is taken in the ambient lattice $([0,1]^k, \preceq)$, hence it always exists. In most standard uncertain models (fuzzy, intuitionistic fuzzy, neutrosophic, etc.), $\mathrm{Dom}(M)$ is closed under coordinatewise minimum, so $\bigwedge a^{(j)} \in \mathrm{Dom}(M)$ whenever all $a^{(j)} \in \mathrm{Dom}(M)$. This closure is *not* required for the definitions below (only $\mathrm{Dom}(M) \subseteq [0,1]^k$ is used).

Definition 5.3.6 (Uncertain directed graph of type M)**.** Let M be an uncertain model with $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *uncertain directed graph of type M* is a quadruple

$$G_M = (V, A, \sigma, \mu),$$

where (V, A) is a directed graph (arcs $A \subseteq V \times V$),

$$\sigma : V \to \mathrm{Dom}(M), \qquad \mu : A \to \mathrm{Dom}(M),$$

and the following *incidence admissibility* condition holds for every arc $a = (u, v) \in A$:

$$\mu(a) \;\preceq\; \sigma(u) \wedge \sigma(v) \quad \text{in } [0,1]^k.$$

For convenience, Table 5.2 lists representative *uncertainty–digraph* families (uncertain directed graphs), organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

Table 5.2: A catalogue of uncertainty–digraph families by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty–digraph type(s) $\mathcal{D}_M = (V, A, \sigma, \mu)$ with $\sigma : V \to \mathrm{Dom}(M)$ and $\mu : A \to \mathrm{Dom}(M)$, where A is a set of arcs
1	Fuzzy directed graph[(a)] [172, 211–213]; shadowed directed graph variants
2	Intuitionistic fuzzy directed graph [214–216]; vague directed graph [217, 218]; bipolar fuzzy directed graph [219, 220]; bifuzzy directed graph; paraconsistent fuzzy directed graph; interval-valued intuitionistic fuzzy directed graph [163, 221, 222](as a 2-domain subclass)
3	Neutrosophic directed graph[(b)] [192, 223–225]; hesitant fuzzy directed graph; tripolar fuzzy directed graph; picture fuzzy directed graph [226–228]; spherical fuzzy directed graph [229]; inconsistent intuitionistic fuzzy directed graph; ternary fuzzy / neutrosophic-fuzzy directed graph
4	Quadripartitioned neutrosophic directed graph [230]; double-valued neutrosophic directed graph; dual hesitant fuzzy directed graph; ambiguous directed graph[(c)]; local-neutrosophic / support-neutrosophic directed graphs; turiyam neutrosophic directed graph [231][(d)]
5	Pentapartitioned neutrosophic directed graph [231]; triple-valued neutrosophic directed graph
6	Hexapartitioned neutrosophic directed graph; quadruple-valued neutrosophic directed graph
7	Heptapartitioned neutrosophic directed graph; quintuple-valued neutrosophic directed graph
8	Octapartitioned neutrosophic directed graph
9	Nonapartitioned neutrosophic directed graph
n	n-refined fuzzy directed graph; multi-valued (fuzzy) directed graphs; multi-fuzzy directed graphs
$2n$	n-refined intuitionistic fuzzy directed graph; multi-intuitionistic fuzzy directed graphs
$3n$	n-refined neutrosophic directed graph; multi-neutrosophic directed graphs

(a) When $k = 1$, an uncertainty–digraph reduces to a fuzzy/valued digraph with a single membership degree on vertices and arcs.
(b) Neutrosophic directed graphs are often treated as umbrella frameworks that specialize to many degree-based digraph formalisms under suitable constraints.
(c) Ambiguous directed graphs are commonly presented as subclasses of certain quadripartitioned and also double-valued neutrosophic directed graph models.
(d) Turiyam neutrosophic directed graphs are reported as subclasses of certain quadripartitioned neutrosophic directed graph models.

Definition 5.3.7 (Uncertain directed hypergraph of type M)**.** Let M be an uncertain model with $\mathrm{Dom}(M) \subseteq [0,1]^k$. Let $H = (V, E)$ be a directed hypergraph in the sense of Definition 2.2.1, so each $e \in E$ is $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ with nonempty $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$. An *uncertain directed hypergraph of type M* is a quadruple

$$H_M = (V, E, \sigma, \mu),$$

where

$$\sigma : V \to \mathrm{Dom}(M), \qquad \mu : E \to \mathrm{Dom}(M),$$

and for every hyperarc $e \in E$,

$$\mu(e) \preceq \bigwedge_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x) \quad \text{in } [0,1]^k.$$

Definition 5.3.8 (Uncertain directed n-SuperHyperGraph of type M)**.** Let M be an uncertain model with $\mathrm{Dom}(M) \subseteq [0,1]^k$. Fix $n \geq 0$ and a finite nonempty base set V_0. Let $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ be a set of n-supervertices. Let E be a set of directed n-superhyperedges, each written

$$e = (\mathrm{Tail}(e), \mathrm{Head}(e)), \qquad \mathrm{Tail}(e), \mathrm{Head}(e) \in \mathcal{P}(V) \setminus \{\emptyset\}.$$

(Thus the underlying directed structure is a directed n-SuperHyperGraph in incidence form.)

An *uncertain directed n-SuperHyperGraph of type M* is a quadruple

$$\mathrm{SHG}_M^{(n)} = (V, E, \sigma, \mu),$$

where

$$\sigma : V \to \mathrm{Dom}(M), \qquad \mu : E \to \mathrm{Dom}(M),$$

and for every directed n-superhyperedge $e \in E$,

$$\mu(e) \preceq \bigwedge_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x) \quad \text{in } [0,1]^k.$$

Remark 5.3.9 (Underlying SuperHyperGraph)**.** Forgetting (σ, μ) from $\mathrm{SHG}_M^{(n)} = (V, E, \sigma, \mu)$ yields the directed n-SuperHyperGraph (V, E); hence every uncertain directed n-SuperHyperGraph carries a canonical SuperHyperGraph structure.

Theorem 5.3.10 (Uncertain directed SuperHyperGraphs unify graph/hypergraph/fuzzy/neutrosophic cases)**.** *Let M be an uncertain model with* $\mathrm{Dom}(M) \subseteq [0,1]^k$ *and let*

$$\mathrm{SHG}_M^{(n)} = (V, E, \sigma, \mu)$$

be an uncertain directed n-SuperHyperGraph of type M (Definition 5.3.8).

(i) ***(Hypergraph specialization.)*** *If* $n = 0$ *and* $V_0 := V$, *then* $\mathrm{SHG}_M^{(0)}$ *is exactly an uncertain directed hypergraph of type M (Definition 5.3.7), with the same vertex set, hyperarc set, and degree maps.*

(ii) ***(Graph specialization.)*** *Assume $n = 0$ and every directed superhyperedge $e \in E$ has singleton tail and singleton head:*

$$e = (\{u\}, \{v\}) \quad \text{for some } u, v \in V.$$

Define the arc set $A := \{(u, v) \in V \times V \mid (\{u\}, \{v\}) \in E\}$ and set $\mu((u, v)) := \mu((\{u\}, \{v\}))$. Then (V, A, σ, μ) is an uncertain directed graph of type M (Definition 5.3.6), and the incidence admissibility constraints coincide.

(iii) ***(Fuzzy directed n-SuperHyperGraph as a special uncertain model.)*** *Let M_{fz} be the fuzzy model with* $\mathrm{Dom}(M_{\mathrm{fz}}) = [0, 1]$ *(thus $k = 1$). Then an uncertain directed n-SuperHyperGraph of type M_{fz} is exactly a fuzzy directed n-SuperHyperGraph in the sense of Definition 5.1.3.*

(iv) ***(Neutrosophic directed n-SuperHyperGraph as a special uncertain model.)*** *Let M_{neu} be the neutrosophic model with*

$$\mathrm{Dom}(M_{\mathrm{neu}}) = \{(T, I, F) \in [0, 1]^3 : \ 0 \le T + I + F \le 3\}.$$

Write $\sigma(v) = (T_V(v), I_V(v), F_V(v))$ and $\mu(e) = (T_E(e), I_E(e), F_E(e))$. Then the single uncertain constraint

$$\mu(e) \preceq \bigwedge_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x)$$

is equivalent (coordinatewise) to the three neutrosophic edge constraints in Definition 5.2.2. Hence uncertain directed n-SuperHyperGraphs of type M_{neu} coincide with neutrosophic directed n-SuperHyperGraphs.

Consequently, Definition 5.3.8 simultaneously generalizes uncertain directed graphs, uncertain directed hypergraphs, fuzzy directed n-SuperHyperGraphs, and neutrosophic directed n-SuperHyperGraphs, while retaining an underlying directed n-SuperHyperGraph structure.

Proof. **(i)** If $n = 0$, then $\mathcal{P}^0(V_0) = V_0$ and $V \subseteq V_0$ consists of ordinary vertices. A directed 0-superhyperedge $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ has $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$ nonempty, i.e. it is precisely a directed hyperarc on V. The admissibility inequality in Definition 5.3.8 matches that of Definition 5.3.7 verbatim.

(ii) Under the singleton condition, each $e \in E$ is uniquely of the form $e = (\{u\}, \{v\})$. Then

$$\bigwedge_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x) = \sigma(u) \wedge \sigma(v),$$

so the superhyperedge admissibility inequality becomes $\mu((u, v)) \preceq \sigma(u) \wedge \sigma(v)$, which is exactly Definition 5.3.6.

(iii) For the fuzzy model, $k = 1$ and $\preceq$ is the usual order on $[0, 1]$. Thus

$$\mu(e) \preceq \bigwedge_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x) \iff \mu(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} \sigma(x),$$

which is the edge–appurtenance constraint in Definition 5.1.3. No other change occurs, so the structures coincide.

(iv) For the neutrosophic model, $k = 3$ and $\preceq$ is componentwise. Hence

$$(T_E(e), I_E(e), F_E(e)) \preceq \Big(\min_x T_V(x),\ \min_x I_V(x),\ \min_x F_V(x)\Big), \quad x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e),$$

which is equivalent to the three coordinate inequalities

$$T_E(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} T_V(x), \quad I_E(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} I_V(x), \quad F_E(e) \le \min_{x \in \mathrm{Tail}(e) \cup \mathrm{Head}(e)} F_V(x),$$

i.e. exactly the neutrosophic edge constraints already imposed in Definition 5.2.2. The vertex-domain constraint $0 \le T_V + I_V + F_V \le 3$ (and similarly for edges, if imposed) is ensured by $\sigma(v), \mu(e) \in \mathrm{Dom}(M_{\mathrm{neu}})$. □

5.4 Soft Directed SuperHyperGraph

Soft sets model parameterized uncertainty: each parameter selects a subset of the universe, without requiring numeric membership degrees [232, 233]. Soft directed graphs model parameter-dependent subdigraphs: each parameter selects vertex and arc subsets forming an induced directed subgraph [81, 234–237]. Soft directed hypergraphs model parameter-dependent subhypergraphs: each parameter selects vertices and directed hyperarcs with tails and heads inside them [238–240]. Soft directed superhypergraphs model parameter-dependent directed n-SuperHyperGraphs: each parameter selects supervertices and superedges preserving tail-head incidence constraints [43].

Definition 5.4.1 (Soft directed graph)**.** [81, 234] Let $G = (V, A)$ be a (simple) directed graph, where V is a finite vertex set and $A \subseteq V \times V$ is the arc set. Let $C \neq \emptyset$ be a parameter set. A *soft directed graph* over G with parameters C is a quadruple

$$(G,\ C,\ \mathcal{A},\ \mathcal{B}),$$

where

$$\mathcal{A} : C \to \mathcal{P}(V), \qquad \mathcal{B} : C \to \mathcal{P}(A),$$

such that for every $c \in C$,

$$\mathcal{A}(c) \subseteq V, \qquad \mathcal{B}(c) \subseteq \{(u, v) \in A :\ u \in \mathcal{A}(c),\ v \in \mathcal{A}(c)\}.$$

For each $c \in C$, the pair $\big(\mathcal{A}(c), \mathcal{B}(c)\big)$ is called the *soft induced subdigraph* at parameter c.

Definition 5.4.2 (Soft directed hypergraph)**.** Let $H^* = (\Gamma, \Xi)$ be a (simple) directed hypergraph, where Γ is the vertex set and

$$\Xi \subseteq \big(\mathcal{P}(\Gamma) \setminus \{\emptyset\}\big) \times \big(\mathcal{P}(\Gamma) \setminus \{\emptyset\}\big)$$

is the set of directed hyperarcs. Let $C \neq \emptyset$ be a parameter set. A *soft directed hypergraph* over H^* with parameters C is a quadruple

$$(H^*,\ C,\ \mathcal{A},\ \mathcal{B}),$$

where

$$\mathcal{A} : C \to \mathcal{P}(\Gamma), \qquad \mathcal{B} : C \to \mathcal{P}(\Xi),$$

and for every $c \in C$,

$$\mathcal{A}(c) \subseteq \Gamma, \qquad \mathcal{B}(c) \subseteq \{(T, H) \in \Xi :\ T \subseteq \mathcal{A}(c),\ H \subseteq \mathcal{A}(c)\}.$$

Each pair $\big(\mathcal{A}(c), \mathcal{B}(c)\big)$ is called the *soft induced subhypergraph* at parameter c.

Definition 5.4.3 (Soft directed n-SuperHyperGraph)**.** Fix a nonempty base set S and an integer $n \geq 0$. Let

$$\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$$

be a directed n-SuperHyperGraph in incidence form, where

$$V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}, \qquad \partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$.

Let $C \neq \emptyset$ be a parameter set. A *soft directed n-SuperHyperGraph* over $\mathrm{DSHG}^{(n)}$ (with parameters C) is a quadruple

$$(\mathrm{DSHG}^{(n)},\ C,\ \mathcal{A},\ \mathcal{B}),$$

where

$$\mathcal{A} : C \to \mathcal{P}(V), \qquad \mathcal{B} : C \to \mathcal{P}(E),$$

such that for every $c \in C$,

$$\mathcal{A}(c) \subseteq V, \qquad \mathcal{B}(c) \subseteq \{\, e \in E : \ \mathrm{Tail}(e) \subseteq \mathcal{A}(c),\ \mathrm{Head}(e) \subseteq \mathcal{A}(c) \,\}.$$

For each $c \in C$, the pair $\big(\mathcal{A}(c), \mathcal{B}(c)\big)$ is called the *soft induced directed sub-n-SuperHyperGraph* at parameter c.

Example 5.4.4 (Real-life soft directed n-SuperHyperGraph)**.** *Scenario (parameterized access-control policies in a hierarchical IT organization).* Let the base set S be a finite set of user accounts and resources (e.g., employees, service accounts, and critical systems). A level-n supervertex $v \in V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}$ represents a nested organizational/security unit, such as "users grouped into teams," "teams grouped into departments," and "departments grouped into business divisions".

Let a directed n-superhyperedge $e \in E$ encode an allowed permission propagation rule from a set of units $\mathrm{Tail}(e)$ to a set of units $\mathrm{Head}(e)$, for example: "if any unit in $\mathrm{Tail}(e)$ is granted access, then the units in $\mathrm{Head}(e)$ inherit access (or must be audited)."

In practice, the active policy depends on a *parameter* such as the security posture:

$$C = \{\mathsf{Normal},\ \mathsf{HighAlert},\ \mathsf{IncidentMode}\}.$$

For each posture $c \in C$, define $\mathcal{A}(c) \subseteq V$ as the set of supervertices that are considered in-scope (e.g., only production divisions under HighAlert), and $\mathcal{B}(c) \subseteq E$ as the set of directed superhyperedges whose tails and heads stay entirely within $\mathcal{A}(c)$ (e.g., stricter propagation rules and additional audit routes under IncidentMode).

Then $(\mathrm{DSHG}^{(n)}, C, \mathcal{A}, \mathcal{B})$ is a soft directed n-SuperHyperGraph: each parameter c selects a concrete directed sub-n-SuperHyperGraph $\big(\mathcal{A}(c), \mathcal{B}(c)\big)$ representing the access-control policy in that posture.

5.5 Rough Directed SuperHyperGraph

Rough sets represent a subset via lower and upper approximations induced by an equivalence relation, capturing indiscernibility-based uncertainty [241,242]. Rough directed graphs represent uncertain arc selection using lower and upper approximations of a distinguished arc subset under an arc-equivalence relation [220, 243–246]. Rough directed hypergraphs represent uncertain hyperarc selection using lower and upper approximations of a distinguished hyperarc subset under a hyperarc-equivalence relation. Rough directed superhypergraphs represent uncertain directed superedge selection using lower and upper approximations of a distinguished superedge subset under a superedge-equivalence relation.

Definition 5.5.1 (Lower and upper approximations in an approximation space)**.** Let $U \neq \emptyset$ and let $\sim$ be an equivalence relation on U (an approximation space $(U, \sim)$). For $x \in U$, write $[x]_\sim := \{y \in U : y \sim x\}$. For any subset $X \subseteq U$, define

$$\underline{\sim}(X) \; := \; \{x \in U : \; [x]_\sim \subseteq X\}, \qquad \overline{\sim}(X) \; := \; \{x \in U : \; [x]_\sim \cap X \neq \emptyset\}.$$

We call $\underline{\sim}(X)$ and $\overline{\sim}(X)$ the *lower* and *upper* approximations of X, respectively. The set X is *rough* (w.r.t. $\sim$) if $\underline{\sim}(X) \neq \overline{\sim}(X)$.

Definition 5.5.2 (Rough directed graph)**.** Let $D = (V, A)$ be a finite directed graph (digraph), and let $\sim_A$ be an equivalence relation on the arc set A. Fix a distinguished arc family $A_0 \subseteq A$. Define

$$A\_ := \underline{\sim_A}(A_0), \qquad A^* := \overline{\sim_A}(A_0).$$

The pair of digraphs

$$D\_ := (V, A\_), \qquad D^* := (V, A^*)$$

is called the *rough directed graph* induced by $(D, \sim_A, A_0)$. (Equivalently, a rough directed graph may be presented as the triple $(V, A\_, A^*)$ with $A\_ \subseteq A^* \subseteq V \times V$.)

Definition 5.5.3 (Rough directed hypergraph)**.** Let $H = (V, E)$ be a finite directed hypergraph, where each hyperarc $e \in E$ is an ordered pair $e = (T(e), H(e))$ with nonempty $T(e), H(e) \subseteq V$. Let $\sim_E$ be an equivalence relation on the hyperarc set E, and fix a distinguished hyperarc family $E_0 \subseteq E$. Define

$$E\_ := \underline{\sim_E}(E_0), \qquad E^* := \overline{\sim_E}(E_0).$$

Then

$$H\_ := (V, E\_), \qquad H^* := (V, E^*)$$

(with the inherited tail/head data from H) are directed hypergraphs. The pair $(H\_, H^*)$ is called the *rough directed hypergraph* induced by $(H, \sim_E, E_0)$.

Definition 5.5.4 (Rough directed n-SuperHyperGraph)**.** Fix $n \geq 0$ and a nonempty base set S. Let

$$\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$$

be a finite directed n-SuperHyperGraph in incidence form, where

$$V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}, \qquad \partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}.$$

Write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. Let $\sim_E$ be an equivalence relation on the directed superedge identifier set E, and fix $E_0 \subseteq E$. Define

$$E\_ := \underline{\sim_E}(E_0), \qquad E^* := \overline{\sim_E}(E_0).$$

To keep incidence well-typed, further set

$$\widehat{E}\_ := \{\, e \in E\_ : \ \mathrm{Tail}(e) \cup \mathrm{Head}(e) \subseteq V \,\}, \qquad \widehat{E}^* := \{\, e \in E^* : \ \mathrm{Tail}(e) \cup \mathrm{Head}(e) \subseteq V \,\},$$

(which holds automatically here since $\mathrm{Tail}(e), \mathrm{Head}(e) \subseteq V$ by definition of $\partial^{\pm}$). Then

$$\mathrm{DSHG}^{(n)}_{\_} := (V, \widehat{E}\_, \partial^- |_{\widehat{E}\_}, \partial^+ |_{\widehat{E}\_}), \qquad \mathrm{DSHG}^{(n)*} := (V, \widehat{E}^*, \partial^- |_{\widehat{E}^*}, \partial^+ |_{\widehat{E}^*})$$

are directed n-SuperHyperGraphs. The pair $(\mathrm{DSHG}^{(n)}_{\_}, \mathrm{DSHG}^{(n)*})$ is called the *rough directed n-SuperHyperGraph* induced by $(\mathrm{DSHG}^{(n)}, \sim_E, E_0)$.

Example 5.5.5 (Real-life rough directed n-SuperHyperGraph). *Scenario (partially observed dependency rules in a hierarchical supply chain).* Let the base set S be a finite set of individual facilities (factories, warehouses, ports). A level-n supervertex $v \in V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}$ represents a nested logistics unit, for example: facilities grouped into routes, routes grouped into regions, and regions grouped into a global network.

A directed n-superhyperedge identifier $e \in E$ with tail $\mathrm{Tail}(e) \subseteq V$ and head $\mathrm{Head}(e) \subseteq V$ represents an operational dependency pattern: "if any unit in $\mathrm{Tail}(e)$ is disrupted, then the units in $\mathrm{Head}(e)$ are likely impacted next."

In practice, only a subset $E_0 \subseteq E$ of dependency patterns is directly observed from incident reports, and many patterns are indistinguishable at the resolution of the data (e.g., two routes are logged under the same coarse category). Model this by an equivalence relation $\sim_E$ on E where $e_1 \sim_E e_2$ means "the monitoring system cannot reliably distinguish e_1 from e_2."

Then the lower approximation $E\_ = \underline{\sim_E}(E_0)$ contains the superedges whose entire equivalence class is observed, i.e., the dependencies that are *certainly present* given the data. The upper approximation $E^* = \overline{\sim_E}(E_0)$ contains the superedges whose class intersects the observed set, i.e., the dependencies that are *possibly present*. Thus $\mathrm{DSHG}^{(n)}_{\_}$ models the confirmed disruption-propagation rules, while $\mathrm{DSHG}^{(n)*}$ models all plausible rules consistent with observations, forming a rough directed n-SuperHyperGraph pair as in Definition 5.5.4.

Theorem 5.5.6 (Specializations).

1. *If $n = 0$, then a rough directed n-SuperHyperGraph in Definition 5.5.4 reduces to a rough directed hypergraph in Definition 5.5.3 (by identifying $V \subseteq \mathcal{P}^0(S) = S$ with an ordinary vertex set).*

2. *If $n = 0$ and every directed superedge has singleton tail and singleton head, i.e. for each $e \in E$ there exist $u, v \in V$ with $\mathrm{Tail}(e) = \{u\}$ and $\mathrm{Head}(e) = \{v\}$, then the rough directed hypergraph reduces to a rough directed graph in Definition 5.5.2.*

Proof. (1) When $n = 0$, supervertices are ordinary vertices and directed superedges are hyperarcs with tail/head nonempty subsets of V; restricting to lower/upper approximations of E_0 yields exactly the lower/upper directed hypergraphs.

(2) Under the singleton condition, each hyperarc is an ordinary arc $u \to v$. Thus lower/upper approximations of the edge-identifier set correspond to lower/upper approximations of the induced arc set, giving the rough directed graph pair $(D\_, D^*)$. □

Chapter 6

Directed (m, n)-SuperHyperGraphs and Hierarchical Directed SuperHyperGraphs

In this chapter, we further investigate directed (m, n)-SuperHyperGraphs and hierarchical directed SuperHyperGraphs.

6.1 Undirected (m, n)-SuperHyperGraph

A (m, n)-SuperHyperGraph is a mathematical structure in which each vertex corresponds to an (m, n)-superhyperfunction defined on a base set, while the hyperedges group such functions together to represent higher-order relationships and contextual connections. An (h, k)-ary (m, n)-SuperHyperGraph further generalizes this idea by taking vertices as (h, k)-ary (m, n)-superhyperfunctions [1].

Notation 6.1.1. *For a nonempty base set S define*

$$\mathcal{P}_0(S) := S, \qquad \mathcal{P}_{m+1}(S) := \mathcal{P}\big(\mathcal{P}_m(S)\big) \quad (m \in \mathbb{N}_0),$$

so $\mathcal{P}_1(S) = \mathcal{P}(S)$, $\mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S))$, etc. We also use the Cartesian power $X^h := \underbrace{X \times \cdots \times X}_{h \text{ copies}}$ for $h \in \mathbb{N}$.

Definition 6.1.2 (*(m, n)-superhyperfunction*)**.** [247,248] Let $m, n \in \mathbb{N}$ and $S \neq \varnothing$. An (m, n)-*superhyperfunction on S* is a map

$$f : \mathcal{P}_m(S) \longrightarrow \mathcal{P}_n(S).$$

Equivalently, $f \in \mathrm{Hom}\big(\mathcal{P}_m(S), \mathcal{P}_n(S)\big)$ as functions of sets.

Definition 6.1.3 (*(m, n)-SuperHyperGraph*)**.** [1] Fix $m, n \in \mathbb{N}$ and a nonempty base set S. Let

$$\mathfrak{F}_{m,n}(S) := \Big\{ f : \mathcal{P}_m(S) \to \mathcal{P}_n(S) \Big\}.$$

An (m, n)-*SuperHyperGraph* is a pair

$$\mathsf{SHG}^{(m,n)} := (V, \mathcal{E}),$$

where $V \subseteq \mathfrak{F}_{m,n}(S)$ is a nonempty set of vertices (each vertex is a concrete (m, n)-superhyperfunction) and

$$\varnothing \neq \mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$$

is a nonempty family of nonempty *hyperedges*. Each hyperedge $E \in \mathcal{E}$ groups a finite, nonempty set of superhyperfunctions to encode higher-order relations/constraints among them.

Example 6.1.4 ((1,1)-SuperHyperGraph for purchase-approval policies)**.** *Real-life setting.* Consider an enterprise purchase workflow. Let the base set

$$S = \{\mathsf{Budget},\ \mathsf{Legal},\ \mathsf{Security},\ \mathsf{Procurement}\}$$

encode four approval/constraint tags. Take $(m, n) = (1, 1)$, so

$$\mathcal{P}_1(S) = \mathcal{P}(S), \qquad \mathfrak{F}_{1,1}(S) = \{\, f : \mathcal{P}(S) \to \mathcal{P}(S) \,\}.$$

Interpret a subset $X \subseteq S$ as the *set of detected constraints* for a purchase request, and interpret $f(X) \subseteq S$ as the *set of approvals required* by a policy f.

Define three concrete policies (superhyperfunctions):

$$f_{\text{std}}(X) := \{\mathsf{Budget}, \mathsf{Procurement}\} \cup \big(X \cap \{\mathsf{Legal}, \mathsf{Security}\}\big),$$

$$f_{\text{fast}}(X) := \{\mathsf{Budget}\} \cup \big(X \cap \{\mathsf{Security}\}\big), \qquad f_{\text{strict}}(X) := S \quad (\forall X \subseteq S).$$

Here f_{std} is the normal policy, f_{fast} is an expedited policy, and f_{strict} forces every approval.

Set

$$V := \{f_{\text{std}}, f_{\text{fast}}, f_{\text{strict}}\} \subseteq \mathfrak{F}_{1,1}(S).$$

Let

$$\mathcal{E} := \{\ \{f_{\text{std}}, f_{\text{fast}}\},\ \{f_{\text{std}}, f_{\text{strict}}\}\ \} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then $\mathsf{SHG}^{(1,1)} = (V, \mathcal{E})$ is a $(1, 1)$-SuperHyperGraph. Each hyperedge groups policies that are considered together in governance, e.g. "approved operational policies" $\{f_{\text{std}}, f_{\text{fast}}\}$ versus "compliance-sensitive policies" $\{f_{\text{std}}, f_{\text{strict}}\}$.

Example 6.1.5 ((1,2)-SuperHyperGraph for clinical decision bundles)**.** *Real-life setting.* Consider a simple clinical triage guideline that maps observed symptoms to *alternative care bundles*. Let

$$S = \{\mathsf{Fever},\ \mathsf{Cough},\ \mathsf{PCR},\ \mathsf{Isolation},\ \mathsf{Antipyretic}\}.$$

Take $(m, n) = (1, 2)$. Then

$$\mathcal{P}_1(S) = \mathcal{P}(S), \qquad \mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S)), \qquad \mathfrak{F}_{1,2}(S) = \{\, f : \mathcal{P}(S) \to \mathcal{P}(\mathcal{P}(S)) \,\}.$$

Interpret $X \subseteq S$ as the *observed symptom/flag set*, and interpret $f(X) \in \mathcal{P}(\mathcal{P}(S))$ as a *set of feasible care bundles* (each bundle is a subset of S).

Define two superhyperfunctions:

$$g_{\text{resp}}(X) := \begin{cases} \{\{\mathsf{PCR}, \mathsf{Isolation}\},\ \{\mathsf{Antipyretic}\}\}, & \{\mathsf{Fever}, \mathsf{Cough}\} \subseteq X, \\ \{\{\mathsf{Antipyretic}\}\}, & \mathsf{Fever} \in X \text{ and } \mathsf{Cough} \notin X, \\ \{\emptyset\}, & \text{otherwise,} \end{cases}$$

$$g_{\text{conservative}}(X) := \begin{cases} \{\{\mathsf{PCR}, \mathsf{Isolation}, \mathsf{Antipyretic}\}\}, & \mathsf{Fever} \in X, \\ \{\emptyset\}, & \text{otherwise.} \end{cases}$$

Set

$$V := \{g_{\text{resp}}, g_{\text{conservative}}\} \subseteq \mathfrak{F}_{1,2}(S), \qquad \mathcal{E} := \{\{g_{\text{resp}}, g_{\text{conservative}}\}\}.$$

Then $\mathsf{SHG}^{(1,2)} = (V, \mathcal{E})$ is a $(1, 2)$-SuperHyperGraph. The (single) hyperedge groups the two clinical policies because they are jointly compared during guideline review, auditing, and update cycles.

6.2 Hierarchical Undirected SuperHyperGraph

A hierarchical superhypergraph is a superhypergraph whose vertices live across multiple powerset levels, with edges allowed to join mixed-level supervertices, while maintaining downward-closure coherence. In this book, we primarily work with n-SuperHyperGraphs.

Definition 6.2.1 (Hierarchical SuperHyperGraph of height r)**.** Let V_0 be a finite, nonempty base set. For $k \geq 0$ define iterated powersets

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}(\mathcal{P}^k(V_0)),$$

and fix an integer $r \geq 0$. Set the *hierarchical universe*

$$\mathcal{U}_r(V_0) \ := \ \bigcup_{k=0}^{r} \Big(\mathcal{P}^k(V_0) \setminus \{\emptyset\}\Big).$$

For $x \in \mathcal{U}_r(V_0)$, define its *level*

$$\ell(x) \ := \ \min\{\, k \in \{0, 1, \ldots, r\} : x \in \mathcal{P}^k(V_0) \,\}.$$

A *hierarchical superhypergraph of height* r on V_0 is a pair

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

such that

(H1) (*Hierarchical vertex set*) V is a finite nonempty set with

$$V \subseteq \mathcal{U}_r(V_0).$$

Elements of V are called *hierarchical supervertices.*

(H2) (*Cross-level edges*) E is a finite family of nonempty subsets of V:

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Elements of E are called *hierarchical superhyperedges.* In particular, a superhyperedge may contain supervertices of *different* levels.

(H3) (*Coherence / downward closure*) If $X \in V$ and $\ell(X) \geq 1$, then

$$X \subseteq V.$$

Equivalently, whenever a higher-level supervertex is present, all its immediate constituents are also present as supervertices.

For each $k \in \{0, \ldots, r\}$ we define the *k-th layer* by

$$V_k := \{ x \in V : \ell(x) = k \}, \qquad \text{so that} \qquad V = \dot{\bigcup}_{k=0}^{r} V_k.$$

Example 6.2.2 (Real-life hierarchical SuperHyperGraph of height 2)**.** *Scenario (university course structure with students, study groups, and programs).* Let the base set V_0 be a finite set of students enrolled in a course:

$$V_0 = \{\mathsf{Alice}, \mathsf{Bob}, \mathsf{Chen}, \mathsf{Dana}\}.$$

Take height $r = 2$. Level 1 objects represent *study groups* (subsets of students), and level 2 objects represent *program cohorts* (sets of study groups).

Define hierarchical supervertices

$$g_1 := \{\mathsf{Alice}, \mathsf{Bob}\}, \qquad g_2 := \{\mathsf{Chen}, \mathsf{Dana}\} \qquad \text{(study groups, level 1)},$$

$$p := \{g_1, g_2\} \qquad \text{(a cohort of groups, level 2)},$$

and include the level 0 students as well. Set

$$V := \{\mathsf{Alice}, \mathsf{Bob}, \mathsf{Chen}, \mathsf{Dana},\ g_1, g_2,\ p\} \subseteq \mathcal{U}_2(V_0).$$

The *downward-closure* requirement holds because $p \in V$ implies $g_1, g_2 \in V$, and each $g_i \in V$ implies its member students lie in V.

Now define cross-level superhyperedges that capture interactions across these layers:

$$e_1 := \{\mathsf{Alice}, g_1\}, \qquad e_2 := \{g_1, g_2, p\}, \qquad e_3 := \{\mathsf{Dana}, g_2, p\},$$

and let

$$E := \{e_1, e_2, e_3\} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Interpretation: e_1 links a student to her study group, e_2 links both groups to the cohort (a mixed-level edge), and e_3 links a student, her group, and the cohort, reflecting a program-level activity involving all three levels. Hence $\mathbb{H}^{\langle 2 \rangle} = (V, E)$ is a hierarchical SuperHyperGraph of height 2 in the sense of Definition 6.2.1.

6.3 Directed (m, n)-SuperHyperGraph

A directed (m, n)-SuperHyperGraph has vertices as (m, n)-superhyperfunctions $\mathcal{P}_m(S) \to \mathcal{P}_n(S)$, with directed hyperedges between vertex-subsets encoding flow.

Definition 6.3.1 (Directed (m, n)-SuperHyperGraph)**.** Fix $m, n \in \mathbb{N}_0$ and a nonempty base set S. Let

$$\mathfrak{F}_{m,n}(S) \;:=\; \{\, f : \mathcal{P}_m(S) \to \mathcal{P}_n(S) \,\}.$$

A *directed* (m, n)*-SuperHyperGraph* on S is a quadruple

$$\mathsf{DSHG}^{(m,n)} = (V, E, \partial^-, \partial^+),$$

where

$$\emptyset \neq V \subseteq \mathfrak{F}_{m,n}(S), \qquad E \neq \emptyset \text{ is a finite set of directed superedge identifiers,}$$

and

$$\partial^-, \partial^+ : E \longrightarrow \mathcal{P}(V) \setminus \{\emptyset\}$$

are the *tail* and *head* incidence maps. For $e \in E$ write $\mathrm{Tail}(e) := \partial^-(e)$ and $\mathrm{Head}(e) := \partial^+(e)$. Thus each directed superedge e carries "flow" from the nonempty tail set $\mathrm{Tail}(e) \subseteq V$ to the nonempty head set $\mathrm{Head}(e) \subseteq V$.

Remark 6.3.2 (Underlying undirected structure)**.** Given $\mathsf{DSHG}^{(m,n)} = (V, E, \partial^-, \partial^+)$, define

$$E_{\mathrm{und}} \;:=\; \{\ \mathrm{Tail}(e) \cup \mathrm{Head}(e) \mid e \in E\ \} \;\subseteq\; \mathcal{P}(V) \setminus \{\emptyset\}.$$

Then (V, E_{und}) is an undirected hypergraph on V (hence an undirected (m, n)-SuperHyperGraph in the sense of Definition 6.1.3).

Example 6.3.3 (A directed $(1, 1)$-SuperHyperGraph of set-valued transformations)**.** Let the base set be

$$S = \{a, b\}, \qquad m = n = 1.$$

Then

$$\mathcal{P}_1(S) = \mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Consider the following three $(1, 1)$-superhyperfunctions (set maps) in $\mathfrak{F}_{1,1}(S) = \{\, f : \mathcal{P}(S) \to \mathcal{P}(S) \,\}$:

$$f_{\mathrm{id}}(X) := X \quad \text{(identity)}, \qquad f_{\mathrm{c}}(X) := S \backslash X \quad \text{(complement)}, \qquad f_{\emptyset}(X) := \emptyset \quad \text{(constant-empty)}.$$

Set the vertex set

$$V := \{f_{\mathrm{id}},\, f_{\mathrm{c}},\, f_{\emptyset}\} \subseteq \mathfrak{F}_{1,1}(S).$$

Let the directed superedge-identifier set be

$$E := \{e_1, e_2\},$$

and define incidence maps $\partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}$ by

$$\partial^-(e_1) = \{f_{\mathrm{id}}, f_{\mathrm{c}}\}, \qquad \partial^+(e_1) = \{f_{\emptyset}\},$$

$$\partial^-(e_2) = \{f_{\emptyset}\}, \qquad \partial^+(e_2) = \{f_{\mathrm{id}}\}.$$

Then

$$\mathsf{DSHG}^{(1,1)} = (V, E, \partial^-, \partial^+)$$

is a directed $(1, 1)$-SuperHyperGraph on S. Here e_1 represents a directed superedge from the tail-set $\{f_{\mathrm{id}}, f_{\mathrm{c}}\}$ to the head-set $\{f_{\emptyset}\}$, and e_2 represents a directed superedge from $\{f_{\emptyset}\}$ to $\{f_{\mathrm{id}}\}$.

Theorem 6.3.4 (Directed (m, n)-SuperHyperGraphs generalize directed n-SuperHyperGraphs)**.** *Fix $n \geq 0$ and a base set $S \neq \emptyset$. Let*

$$\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$$

be a directed n-SuperHyperGraph in the sense of Definition 2.2.3 (so $V \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}$ and $\partial^\pm : E \to \mathcal{P}(V) \setminus \{\emptyset\}$).

Fix any $m \in \mathbb{N}_0$ and define an embedding

$$\iota : \mathcal{P}^n(S) \setminus \{\emptyset\} \longrightarrow \mathfrak{F}_{m,n}(S), \qquad \iota(x)(A) := x \quad (A \in \mathcal{P}_m(S)).$$

Let $V^\iota := \iota[V] \subseteq \mathfrak{F}_{m,n}(S)$ and define

$$\partial_\iota^-(e) := \iota\big[\partial^-(e)\big], \qquad \partial_\iota^+(e) := \iota\big[\partial^+(e)\big] \qquad (e \in E).$$

Then

$$\mathsf{DSHG}_\iota^{(m,n)} := \big(V^\iota, E, \partial_\iota^-, \partial_\iota^+\big)$$

is a directed (m, n)-SuperHyperGraph, and ι induces an isomorphism of directed incidence structures between $\mathrm{DSHG}^{(n)}$ and $\mathsf{DSHG}_\iota^{(m,n)}$.

Proof. For each $x \in \mathcal{P}^n(S) \setminus \{\emptyset\}$, the map $\iota(x)$ is a well-defined function $\mathcal{P}_m(S) \to \mathcal{P}_n(S) = \mathcal{P}^n(S)$, hence $\iota(x) \in \mathfrak{F}_{m,n}(S)$. Thus $V^\iota \subseteq \mathfrak{F}_{m,n}(S)$ and is nonempty.

Since $\partial^-(e), \partial^+(e) \in \mathcal{P}(V)\setminus\{\emptyset\}$, their images under ι are nonempty subsets of V^ι, hence $\partial_\iota^-, \partial_\iota^+ : E \to \mathcal{P}(V^\iota) \setminus \{\emptyset\}$ are valid incidence maps. Therefore $\mathsf{DSHG}_\iota^{(m,n)}$ satisfies Definition 6.3.1.

Finally, $\iota : V \to V^\iota$ is bijective by construction, and for every $e \in E$ and $v \in V$,

$$v \in \partial^-(e) \iff \iota(v) \in \iota[\partial^-(e)] = \partial_\iota^-(e),$$

and similarly for ∂^+. Hence ι preserves and reflects incidence, i.e. it is an isomorphism of the directed hyperedge incidence structures. □

6.4 Hierarchical Directed SuperHyperGraph

A hierarchical directed SuperHyperGraph of height r uses supervertices from powerset levels r, enforces downward closure, and allows directed superedges between mixed-level supervertices.

Definition 6.4.1 (Hierarchical directed SuperHyperGraph of height r)**.** Let V_0 be a finite, nonempty base set. For $k \geq 0$ define iterated powersets

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big),$$

and fix an integer $r \geq 0$. Set the hierarchical universe

$$\mathcal{U}_r(V_0) \;:=\; \bigcup_{k=0}^{r} \Big(\mathcal{P}^k(V_0) \setminus \{\emptyset\}\Big).$$

For $x \in \mathcal{U}_r(V_0)$ define its level

$$\ell(x) \;:=\; \min\{\, k \in \{0, 1, \ldots, r\} : x \in \mathcal{P}^k(V_0)\,\}.$$

A *hierarchical directed SuperHyperGraph of height* r on V_0 is a quadruple

$$\mathbb{H}_{\to}^{\langle r\rangle} = (V, E, \partial^-, \partial^+)$$

such that:

(HD1) (Hierarchical vertex set) V is a finite nonempty set with

$$V \subseteq \mathcal{U}_r(V_0).$$

(HD2) (Directed cross-level superedges) E is a finite nonempty set of directed superedge identifiers and

$$\partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}$$

are tail/head incidence maps. A directed superedge e may join supervertices of *different* levels.

(HD3) (Coherence / downward closure) If $X \in V$ and $\ell(X) \geq 1$, then

$$X \subseteq V.$$

Remark 6.4.2 (Underlying directed hypergraph)**.** Forgetting the level function and viewing vertices simply as elements of a set, any $\mathbb{H}_{\to}^{\langle r \rangle} = (V, E, \partial^-, \partial^+)$ is a directed hypergraph in incidence form (tails/heads are nonempty subsets of V).

Example 6.4.3 (Hierarchical directed SuperHyperGraph (height 2): incident-response escalation across levels)**.** *Scenario.* Let the base set V_0 be a finite set of on-call engineers:

$$V_0 = \{\mathsf{Aki}, \mathsf{Bo}, \mathsf{Chen}, \mathsf{Dana}\}.$$

Interpret level 1 supervertices as *squads* and level 2 supervertices as *groups of squads* (e.g., incident domains). Define

$$s_1 := \{\mathsf{Aki}, \mathsf{Bo}\}, \qquad s_2 := \{\mathsf{Chen}, \mathsf{Dana}\},$$

$$g := \{s_1, s_2\}.$$

Let the hierarchical vertex set include all constituents (downward closure):

$$V := \{\mathsf{Aki}, \mathsf{Bo}, \mathsf{Chen}, \mathsf{Dana},\ s_1, s_2,\ g\} \subseteq \mathcal{U}_2(V_0).$$

Let $E = \{e_1, e_2\}$ be directed superedge identifiers and define tail/head incidence maps by

$$\partial^-(e_1) = \{s_1\}, \qquad \partial^+(e_1) = \{g\},$$

$$\partial^-(e_2) = \{\mathsf{Dana}\}, \qquad \partial^+(e_2) = \{s_2\}.$$

Interpretation: e_2 models an escalation from an individual engineer to her squad, while e_1 models an escalation from a squad to the higher-level incident domain g. These superedges join mixed levels ($\ell(\mathsf{Dana}) = 0$, $\ell(s_2) = 1$, $\ell(g) = 2$), so $\mathbb{H}_{\to}^{\langle 2 \rangle} = (V, E, \partial^-, \partial^+)$ is a hierarchical directed SuperHyperGraph of height 2 in the sense of Definition 6.4.1.

Example 6.4.4 (Hierarchical directed SuperHyperGraph (height 3): supply-chain disruption propagation)**.** *Scenario.* Let the base set V_0 be a finite set of facilities:

$$V_0 = \{\mathsf{Plant}, \mathsf{Port}, \mathsf{Warehouse}, \mathsf{Retail}\}.$$

Interpret level 1 supervertices as *local clusters* of facilities, level 2 as *regions* (sets of clusters), and level 3 as a *global network unit* (a set of regions). Define

$$c_1 := \{\mathsf{Plant}, \mathsf{Port}\}, \qquad c_2 := \{\mathsf{Warehouse}, \mathsf{Retail}\},$$
$$R_1 := \{c_1\}, \qquad R_2 := \{c_2\}, \qquad G := \{R_1, R_2\}.$$

Let

$$V := \{\mathsf{Plant}, \mathsf{Port}, \mathsf{Warehouse}, \mathsf{Retail},\ c_1, c_2,\ R_1, R_2,\ G\} \subseteq \mathcal{U}_3(V_0),$$

which is downward closed by construction.

Let $E = \{e_a, e_b\}$ and set

$$\partial^-(e_a) = \{\mathsf{Port}\}, \qquad \partial^+(e_a) = \{c_1\},$$
$$\partial^-(e_b) = \{R_1\}, \qquad \partial^+(e_b) = \{G\}.$$

Interpretation: e_a models that a disruption at the port triggers alerts for the local cluster c_1, and e_b models that a regional disruption triggers a global coordination response at G. Again the superedges may connect different levels, so this structure is a hierarchical directed SuperHyperGraph of height 3 in the sense of Definition 6.4.1.

Notation 6.4.5 (Downward closure)**.** *Let $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$ for some $n \geq 0$. Define the* downward closure $\mathrm{Down}(V) \subseteq \mathcal{U}_n(V_0)$ *recursively by*

$$W_n := V, \qquad W_{k-1} := \bigcup_{X \in W_k} X \ \cup\ W_{k-1} \quad (k = n, n-1, \dots, 1), \qquad \mathrm{Down}(V) := \bigcup_{k=0}^{n} W_k.$$

Equivalently, $\mathrm{Down}(V)$ *is the smallest subset of $\mathcal{U}_n(V_0)$ that contains V and is downward closed: if $X \in \mathrm{Down}(V)$ and $\ell(X) \geq 1$, then $X \subseteq \mathrm{Down}(V)$.*

Theorem 6.4.6 (Hierarchical directed SuperHyperGraphs generalize directed n-SuperHyperGraphs)**.** *Let*

$$\mathrm{DSHG}^{(n)} = (V, E, \partial^-, \partial^+)$$

be a directed n-SuperHyperGraph on a finite base set V_0 (Definition 2.2.3), so $V \subseteq \mathcal{P}^n(V_0) \setminus \{\emptyset\}$.

Set $V^\star := \mathrm{Down}(V)$ (Notation 6.4.5) and keep the same directed superedge identifiers E. View ∂^-, ∂^+ as maps into $\mathcal{P}(V^\star) \setminus \{\emptyset\}$ via the inclusions $\partial^\pm(e) \subseteq V \subseteq V^\star$. Then

$$\mathbb{H}^{\langle n \rangle}_{\to} := (V^\star, E, \partial^-, \partial^+)$$

is a hierarchical directed SuperHyperGraph of height n (Definition 6.4.1). Moreover, the original directed n-SuperHyperGraph $\mathrm{DSHG}^{(n)}$ *is recovered as the induced substructure on the top layer V (i.e. on the vertices of level n).*

Proof. By construction, $V^\star = \mathrm{Down}(V) \subseteq \mathcal{U}_n(V_0)$ and is finite and nonempty, so (HD1) holds. Because $\partial^\pm(e) \subseteq V \subseteq V^\star$ for all $e \in E$, the same maps $\partial^-, \partial^+ : E \to \mathcal{P}(V^\star) \setminus \{\emptyset\}$ define directed cross-level superedges, so (HD2) holds. The set $V^\star$ is downward closed by definition of $\mathrm{Down}(V)$, hence (HD3) holds. Therefore $\mathbb{H}^{\langle n \rangle}_{\to}$ is a hierarchical directed SuperHyperGraph of height n.

Finally, all tails and heads lie inside the original top-level vertex set V; hence restricting $\mathbb{H}^{\langle n \rangle}_{\to}$ to the induced substructure on V reproduces exactly the same directed incidence data $(V, E, \partial^-, \partial^+)$, i.e. the original $\mathrm{DSHG}^{(n)}$. □

Chapter 7

Conclusion

In this book, we examined and surveyed fundamental graph-theoretic concepts in the settings of directed hypergraphs and directed superhypergraphs. We hope that future research will further advance these topics, including studies supported by computational experiments based on the present framework.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

Appendix (List of Tables)

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Hypergraphs generalize ordinary graphs by allowing edges to connect arbitrary, non-empty subsets of a vertex set. By iteratively applying the powerset construction, one can generate nested, higher-order vertex objects. This leads to the framework of finite SuperHyperGraphs, where vertices are set-valued across multiple layers and edges encode relations among these complex structures. Furthermore, the introduction of directionality extends graphs, hypergraphs, and superhypergraphs into their directed counterparts—fields of study that have seen significant growth in recent literature. This book examines and surveys fundamental graph-theoretic concepts specifically within the evolving contexts of directed hypergraphs and directed superhypergraphs.

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