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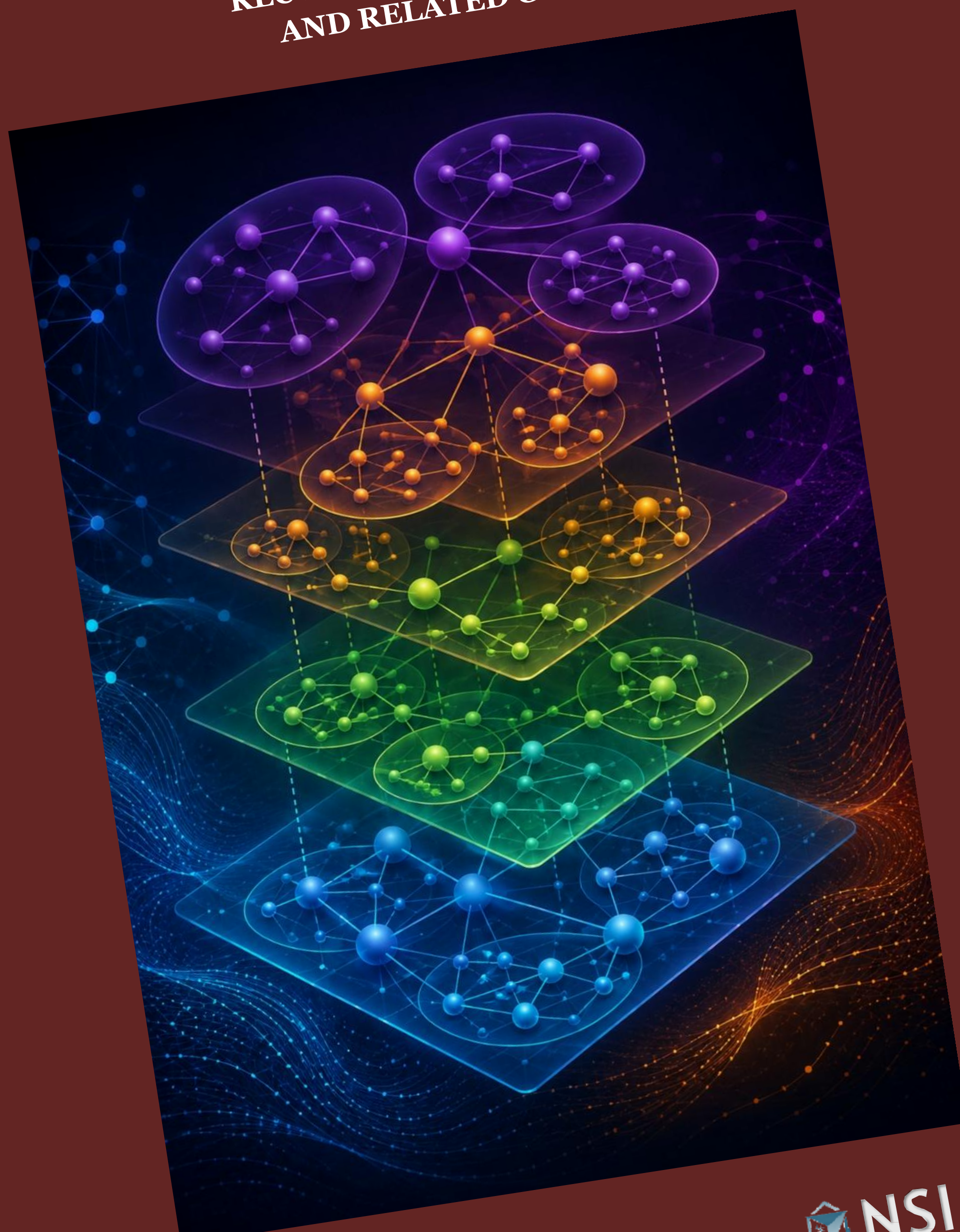
HYPERGRAPH AND SUPERHYPERGRAPH THEORY WITH APPLICATIONS VIII

HIERARCHICAL SUPERHYPERGRAPH, RECURSIVE SUPERHYPERGRAPH, AND RELATED GRAPH THEORY



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HyperGraph and SuperHyperGraph Theory with Applications

VIII

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Chapter 1

Introduction

1.1 Graphs, HyperGraphs, and SuperHyperGraphs

Classical network structures are commonly modeled by *graphs*, where objects are represented by vertices and binary relationships are represented by edges [1]. This framework is highly effective when interactions occur only between pairs of entities. However, in many settings, several entities interact simultaneously, and ordinary graphs are no longer sufficient to describe such relationships adequately. To address this issue, *hypergraphs* extend the classical notion of a graph by allowing each hyperedge to connect an arbitrary nonempty collection of vertices, thereby providing a direct representation of higher-order interactions [2].

Nevertheless, many natural, social, computational, and engineered systems exhibit structures that are not merely higher-order, but also *nested*, *multi-layered*, and *hierarchical*. To represent such rich incidence patterns, F. Smarandache introduced the concept of a *SuperHyperGraph*. Roughly speaking, a SuperHyperGraph is constructed through iterative powerset operations, so that vertices may themselves be set-valued objects, often called *supervertices*, and edges may express connectivity across multiple structural levels [3, 4]. For this reason, SuperHyperGraphs have recently received increasing attention in both theoretical investigations and practical modeling studies [5–10].

Graphs and hypergraphs also serve as intuitive and visually transparent tools for describing complex systems, and they have been applied widely in artificial intelligence, network science, data mining, informatics, chemistry, physics, and related disciplines [11–13]. By explicitly incorporating hierarchical and multi-level incidence, SuperHyperGraphs provide a broader and more adaptable framework for the study of complex structured data (e.g., [14–23]).

Table 1.1 summarizes the main distinctions among graphs, hypergraphs, and superhypergraphs. Throughout this book, unless explicitly stated otherwise, we use

$$\mathbb{N}_0 := \{0, 1, 2, \ldots\}.$$

The structural parameters n, k, r are taken from $\mathbb{N}_0$, except when a positive value is explicitly required.

A more explicit side-by-side comparison of graphs, hypergraphs, and n-superhypergraphs is given in Table 1.2.

Table 1.1: Main differences among graph, hypergraph, and superhypergraph.

Concept	*Notation*	*Edge Family*	*Main Structural Feature*
Graph [1]	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$	Each edge represents a *binary* relation between two vertices.
Hypergraph [24]	$H = (V, \mathcal{E})$	$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	Each hyperedge may connect *any* nonempty subset of vertices, allowing direct modeling of higher-order interactions.
Superhypergraph [3]	$\mathrm{SHG}^{(n)} = (V_0, V, E)$	$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	An n-fold powerset construction allows the representation of *nested*, *hierarchical*, and *multi-level* incidence structures.

Notation. $\mathcal{P}(X) = \{A \mid A \subseteq X\}$, and $\mathcal{P}^0(X) = X,\ \mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

Table 1.2: A concrete comparison of graphs, hypergraphs, and n-superhypergraphs.

Aspect	*Graph* $G = (V, E)$	*Hypergraph* $H = (V, \mathcal{E})$	*n-SuperHyperGraph* $\mathrm{SHG}^{(n)} = (V_0, V, E)$
Vertices	$v \in V$	$v \in V$	$x \in V \subseteq \mathcal{P}^n(V_0)$, so vertices may themselves be nested objects
Edges	$E \subseteq \binom{V}{2}$	$\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$
Incidence	$v \in e$	$v \in e$	$x \in \varepsilon$
Adjacency (typical)	$\{u, v\} \in E$	$\exists e \in \mathcal{E} : \{u, v\} \subseteq e$	$\exists \varepsilon \in E : \{x, y\} \subseteq \varepsilon$
Meaning of one edge	pairwise relation	multi-entity relation	multi-entity relation among supervertices
Distance (typical)	shortest-path distance	Berge or primal distance	super-Berge or primal-type distance
Typical use	binary connectivity	higher-order group structure	nested and hierarchical incidence structure

Notation. $\mathcal{P}^0(X) = X$ and $\mathcal{P}^{k+1}(X) = \mathcal{P}(\mathcal{P}^k(X))$.

1.2 Hierarchical SuperHyperGraphs and Recursive SuperHyperGraphs

A hierarchical SuperHyperGraph admits vertices taken from multiple iterated-powerset levels and allows *mixed-level* edges, while also requiring a coherence condition that preserves consistency between adjacent levels. By contrast, a *Recursive HyperGraph* is a hypergraph in which a hyperedge may contain not only ordinary vertices but also lower-level hyperedges as elements, thereby allowing nested or self-referential incidence up to a prescribed recursion depth [25, 26]. An (n, k)-recursive SuperHyperGraph combines level-n supervertices, generated through iterated powersets, with depth-k recursive superhyperedges that may contain both supervertices and lower-level edges as components [27]. A concise comparison between SuperHyperGraphs and Hierarchical SuperHyperGraphs is presented in Table 1.3, and a concise comparison between SuperHyperGraphs and Recursive SuperHyperGraphs is given in Table 1.4.

Table 1.3: Concise comparison between SuperHyperGraphs and Hierarchical SuperHyperGraphs.

Aspect	*SuperHyperGraph*	*Hierarchical SuperHyperGraph*
Notation	$\mathrm{SHG}^{(n)} = (V, E)$	$\mathbb{H}^{\langle r\rangle} = (V, E)$
Vertex domain	$V \subseteq \mathcal{P}^n(V_0)$ (single fixed level)	$V \subseteq \mathcal{U}_r(V_0)$, where $\mathcal{U}_r(V_0) = \bigcup_{k=0}^{r} \mathcal{P}^{\langle k\rangle}(V_0)$ (multiple nonempty hierarchical levels)
Vertex type	All supervertices lie in the same iterated-powerset layer.	Supervertices may belong to different levels $0, \ldots, r$.
Edge family	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$
Incidence pattern	Incidence is among same-level supervertices.	Edges may connect mixed-level supervertices.
Extra structural condition	No level-coherence condition is imposed in the basic definition.	Downward closure / coherence is required: if $X \in V$ and $\ell(X) \geq 1$, then $X \subseteq V$.
Main emphasis	Nested higher-order objects at one fixed depth.	Multi-level nested structure with explicit hierarchical consistency.
Typical use	Higher-order incidence at a prescribed powerset depth.	Nested, multi-level, and cross-level incidence modeling.

Notation. $\mathcal{P}^{\langle 0\rangle}(V_0) = V_0$, $\mathcal{P}^{\langle k+1\rangle}(V_0) = \mathcal{P}(\mathcal{P}^{\langle k\rangle}(V_0)) \setminus \{\emptyset\}$, $\mathcal{U}_r(V_0) = \bigcup_{k=0}^{r} \mathcal{P}^{\langle k\rangle}(V_0)$, and $\ell(x)$ denotes the level of $x \in \mathcal{U}_r(V_0)$.

Table 1.4: Concise comparison between SuperHyperGraphs and Recursive SuperHyperGraphs.

Aspect	*SuperHyperGraph*	*Recursive SuperHyperGraph*
Notation	$\mathrm{SHG}^{(n)} = (V, E)$	$\mathrm{RSHG}^{(n,k)} = (V, E)$
Vertex set	$V \subseteq \mathcal{P}^n(V_0)$	$V \subseteq \mathcal{P}^n(V_0)$
Edge family	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	$E \subseteq 2_{V,k} \setminus \{\emptyset\}$, with $e \subseteq V \cup E$ for every $e \in E$
Edge contents	Each edge is a nonempty subset of V.	An edge may contain supervertices and lower-level recursive edges.
Extra structure	No recursive depth parameter.	Recursion depth is controlled by k, and recursive edge-membership is acyclic.
Interpretation	Higher-order incidence among n-supervertices.	Nested/self-referential incidence among n-supervertices and recursive edges.
Special case	Basic model.	For $k = 0$, no edge can contain another edge; hence the model reduces to the ordinary n-SuperHyperGraph case.

Notation. $\mathcal{P}^0(V_0) = V_0$, $\mathcal{P}^{m+1}(V_0) = \mathcal{P}\big(\mathcal{P}^m(V_0)\big)$, and $2_{V,k}$ denotes the depth-k powerset universe over V.

1.3 Our Contributions

The main contributions of this book are as follows.

First, we formulate (n, k)-recursive SuperHyperGraphs by combining n-level supervertices with recursive superhyperedges of bounded depth. In this formulation, recursive superhyperedges may contain both supervertices and lower-level recursive superhyperedges, while typed incidence closure and well-foundedness prevent ill-defined circular recursion.

Second, we introduce several extensions of recursive SuperHyperGraphs, including recursive fuzzy, single-valued neutrosophic, uncertain, soft, rough, and weighted SuperHyperGraphs. These variants show how recursive incidence structures can be combined with membership degrees, neutrosophic information, general uncertainty models, parameterized families, approximation structures, and numerical weights.

Third, we develop the framework of Hierarchical SuperHyperGraphs. In this setting, vertices may belong to different iterated-powerset levels, and superhyperedges may connect mixed-level supervertices. A coherence condition is imposed to ensure that higher-level selected objects remain compatible with their immediate constituents.

Fourth, we examine several hierarchical variants, including fuzzy, directed, bidirected, uncertain, soft, rough, and weighted hierarchical SuperHyperGraphs. We also discuss related structural notions such as distance, labeling, domination, and tree-like structures.

Finally, we compare Hierarchical and Recursive SuperHyperGraphs with related frameworks such as Meta-Graphs, Iterated Meta-Graphs, Filtrated Graphs, Iterated MultiGraphs, and Neural Graphs. These comparisons clarify the relationship between SuperHyperGraph-based models and other higher-order or multi-level graph formalisms.

This book is part of a broader series on HyperGraph and SuperHyperGraph theory. For related background and context, see, as needed, works such as [28–32]. Since the present work is mainly theoretical, future studies may further develop algorithms, computational experiments, and concrete applications for the structures introduced here.

Chapter 2

Preliminaries

This chapter fixes the notation and reviews the main structures used throughout the book.

2.1 SuperHyperGraphs

Classical graph theory provides a standard language for describing systems built from *vertices* and *edges*, together with notions such as connectivity, invariants, and structural properties arising in mathematics, computer science, and many application domains [1]. When interactions are not merely pairwise, the classical graph model becomes too restrictive. A *hypergraph* resolves this limitation by allowing a single edge to connect any nonempty collection of vertices, thereby encoding genuinely multiway relations in a direct manner [2, 33]. Such higher-order structures now play an increasingly important role in modern modeling and learning frameworks, including neural architectures that explicitly use hypergraph-based incidence information [2, 34–37].

A further generalization is obtained by iterating the powerset construction. In this setting, vertices themselves may be set-valued and nested, which leads to the notion of a finite *SuperHyperGraph*. This framework is well suited to the representation of hierarchical, multilevel, or nested relational systems [38,39]. Such models naturally arise in areas such as multiscale network analysis, molecular representation, and neural-network design [7,15,40–44]. Related variants have also been studied, including Directed SuperHyperGraphs [45, 46] and MetaSuperHyperGraphs [47]. Unless explicitly stated otherwise, we use

$$\mathbb{N}_0 := \{0, 1, 2, \ldots\}.$$

The parameters n, k, r are assumed to belong to $\mathbb{N}_0$, except when a positive value is explicitly required.

Definition 2.1.1 (Base set)**.** A *base set* S is the underlying universe of admissible objects in the context under consideration:

$$S = \{\, x \mid x \text{ is admissible in the given setting} \,\}.$$

Consequently, every element of $\mathcal{P}(S)$, and more generally of any iterated powerset $\mathcal{P}^n(S)$, is ultimately generated from elements of S.

Definition 2.1.2 (Powerset)**.** (see [48]) For a set S, its *powerset* is the family of all subsets of S:

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

In particular, $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$.

Definition 2.1.3 (Hypergraph)**.** [24,49] A *hypergraph* is an ordered pair $H = (V, E)$ such that:

- V is a finite set of *vertices*, and
- E is a finite family of nonempty subsets of V, called *hyperedges*.

Thus, unlike an ordinary graph, a hyperedge may involve more than two vertices, allowing direct modeling of multiway interactions.

Definition 2.1.4 (Iterated powerset and flattening)**.** [50] Let V_0 be a finite nonempty set. Set

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0).$$

For each $k \geq 0$, define the flattening map

$$\mathrm{Flat}_k : \mathcal{P}^k(V_0) \setminus \{\emptyset\} \longrightarrow \mathcal{P}(V_0) \setminus \{\emptyset\}$$

recursively by

$$\mathrm{Flat}_0(x) := \{x\} \quad (x \in V_0),$$

and

$$\mathrm{Flat}_{k+1}(X) := \bigcup_{Y \in X} \mathrm{Flat}_k(Y) \quad \big(X \in \mathcal{P}^{k+1}(V_0) \setminus \{\emptyset\}\big).$$

Definition 2.1.5 (Nonempty hierarchical powerset levels)**.** Let V_0 be a finite nonempty base set. Define the nonempty hierarchical powerset levels recursively by

$$\mathcal{P}^{\langle 0 \rangle}(V_0) := V_0,$$

and, for $k \geq 0$,

$$\mathcal{P}^{\langle k+1 \rangle}(V_0) := \mathcal{P}\big(\mathcal{P}^{\langle k \rangle}(V_0)\big) \setminus \{\emptyset\}.$$

For $r \in \mathbb{N}_0$, the *hierarchical universe of height* r over V_0 is defined by

$$\mathcal{U}_r(V_0) := \bigcup_{k=0}^{r} \mathcal{P}^{\langle k \rangle}(V_0).$$

An element of $\mathcal{U}_r(V_0)$ is called a *hierarchical supervertex*.

Definition 2.1.6 (Level of a hierarchical supervertex)**.** Let $x \in \mathcal{U}_r(V_0)$. The *level* of x is defined by

$$\ell(x) := \min\{ k \in \{0, 1, \ldots, r\} : x \in \mathcal{P}^{\langle k \rangle}(V_0) \}.$$

When $x \in \mathcal{P}^{\langle k \rangle}(V_0)$ with $k \geq 1$, the elements of x are called the *immediate constituents* of x.

Remark 2.1.7 (Set-theoretic convention)**.** Throughout this book, base elements of V_0 are treated as atomic objects with respect to the iterated powerset construction. Under this convention, the level map ℓ is well-defined. If one works in a setting where base objects may themselves be sets, the same construction can be made fully formal by replacing the union above with a tagged disjoint union of the levels.

Definition 2.1.8 (n-SuperHyperGraph)**.** (see [3]) Let V_0 be a finite nonempty base set. Define

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \in \mathbb{N}).$$

For $n \geq 0$, an *n-SuperHyperGraph on V_0* is a pair

$$\mathrm{SHG}^{(n)} = (V, E)$$

satisfying

$$V \subseteq \mathcal{P}^n(V_0) \qquad \text{and} \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

The elements of V are called *n-supervertices*, while the elements of E are called *n-superedges*. Equivalently, each n-superedge is a nonempty subset of V.

Example 2.1.9 (A concrete 2-SuperHyperGraph)**.** Let

$$V_0 = \{a, b\}.$$

Then

$$\mathcal{P}(V_0) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\},$$

and hence

$$\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

Define three 2-supervertices by

$$x_1 := \big\{\{a\}, \{b\}\big\}, \qquad x_2 := \big\{\{a, b\}\big\}, \qquad x_3 := \big\{\{a\}, \{a, b\}\big\}.$$

Each of x_1, x_2, x_3 is an element of $\mathcal{P}^2(V_0)$. Now set

$$V := \{x_1, x_2, x_3\}.$$

Next, define the superedge family by

$$E := \Big\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}\Big\}.$$

Then every element of E is a nonempty subset of V, so

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Therefore,

$$\mathrm{SHG}^{(2)} = (V, E)$$

is a 2-SuperHyperGraph on the base set V_0.

In this example, the elements x_1, x_2, x_3 are 2-supervertices, and the sets

$$\{x_1, x_2\}, \quad \{x_2, x_3\}, \quad \{x_1, x_2, x_3\}$$

are 2-superedges.

2.2 Fuzzy, Neutrosophic, and Uncertain Graph

A fuzzy set assigns to each element a membership value in the interval $[0, 1]$, thereby modeling graded belonging and partial uncertainty [51, 52]. A fuzzy graph extends this idea to network structures by attaching membership degrees to both vertices and edges, allowing uncertain or gradual connectivity to be represented within a graph-theoretic setting [53, 54].

Definition 2.2.1 (Fuzzy set)**.** [51] Let Y be a nonempty universe. A *fuzzy set* τ on Y is a function

$$\tau : Y \longrightarrow [0, 1],$$

where $\tau(y)$ is the membership degree of $y \in Y$.

A *fuzzy relation* on Y is a fuzzy subset δ of $Y \times Y$. Given a fuzzy set τ on Y, the relation δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\}, \qquad \forall\, y, z \in Y.$$

Definition 2.2.2 (Fuzzy graph)**.** [54] A *fuzzy graph* on a vertex set V is a pair $G = (\sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ is the *vertex membership function*, and
- $\mu : V \times V \to [0, 1]$ is the *edge membership function*, viewed as a fuzzy relation on σ, such that

  $$\mu(x, y) \leq \sigma(x) \wedge \sigma(y), \qquad \forall\, x, y \in V,$$

  where $\wedge$ denotes the minimum operation.

The associated *crisp graph* $G^* = (\sigma^*, \mu^*)$ is given by

$$\sigma^* = \{\, x \in V \mid \sigma(x) > 0 \,\}, \qquad \mu^* = \{\, (x, y) \in V \times V \mid \mu(x, y) > 0 \,\}.$$

A *fuzzy subgraph* $H = (\sigma', \mu')$ of G is obtained by selecting a subset $X \subseteq V$ together with maps

$$\sigma' : X \to [0, 1], \qquad \mu' : X \times X \to [0, 1],$$

such that

$$\mu'(x, y) \leq \sigma'(x) \wedge \sigma'(y), \qquad \forall\, x, y \in X.$$

A Single-Valued Neutrosophic Graph enriches the graph model by attaching to each vertex and edge three degrees: truth, indeterminacy, and falsity [55–58].

Definition 2.2.3 (Single-Valued Neutrosophic Graph)**.** [58] Let $G^* = (V, E)$ be a crisp graph, where V is the vertex set and $E \subseteq V \times V$ is the edge set. A *single-valued neutrosophic graph* (SVNG) on G^* is a pair

$$G = (A, B),$$

where

-
$$A = \{\langle v, T_A(v), I_A(v), F_A(v)\rangle : v \in V\}$$
is the *single-valued neutrosophic vertex set*, with
$$T_A, I_A, F_A : V \to [0,1],$$
representing the truth-membership, indeterminacy-membership, and falsity-membership degrees of vertices, respectively, and satisfying
$$0 \le T_A(v) + I_A(v) + F_A(v) \le 3 \qquad \text{for every } v \in V;$$

-
$$B = \{\langle uv, T_B(uv), I_B(uv), F_B(uv)\rangle : uv \in E\}$$
is the *single-valued neutrosophic edge set*, with
$$T_B, I_B, F_B : E \to [0,1],$$
such that for all $u, v \in V$ with $uv \in E$,
$$T_B(uv) \le \min\{T_A(u), T_A(v)\},$$
$$I_B(uv) \le \min\{I_A(u), I_A(v)\},$$
and
$$F_B(uv) \ge \max\{F_A(u), F_A(v)\}.$$

If B is symmetric, then $G = (A, B)$ is called an *undirected SVNG*; otherwise, it is called a *directed SVNG*.

Remark 2.2.4 (Conventions for neutrosophic edge compatibility)**.** Different conventions are used in the literature for compatibility between vertex degrees and edge degrees in neutrosophic graph models. In some single-valued neutrosophic graph formulations, the falsity degree of an edge is bounded below by the falsity degrees of its endpoints. In the incidence-based SuperHyperGraph formulations used later in this book, however, truth, indeterminacy, and falsity values may all be treated as incidence-type appurtenance degrees and may therefore be bounded above by the corresponding degrees of the incident objects. Whenever such a convention is used, it is specified in the relevant definition.

An Uncertain Set assigns to each element a degree from an uncertainty model, unifying fuzzy, intuitionistic, neutrosophic and plithogenic frameworks [59]. An Uncertain Graph is a graph where vertices or edges carry degrees in an uncertainty model, subsuming fuzzy, intuitionistic, neutrosophic. We first recall the notion of an Uncertain Model, which provides the membership–degree domain.

Definition 2.2.5 (Uncertain Model)**.** [59] Let U denote the class of all *uncertain models*. Each $M \in U$ is specified by

- a nonempty set $\mathrm{Dom}(M) \subseteq [0,1]^k$ of *admissible degree tuples* for some fixed integer $k \ge 1$;

- model–specific algebraic or geometric constraints on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \le 1$ in the intuitionistic fuzzy case, or $T + I + F \le 3$ in the neutrosophic case).

Typical examples include:

- Fuzzy model: $\mathrm{Dom}(M) = [0, 1]$;
- Intuitionistic fuzzy model: $\mathrm{Dom}(M) = \{(\mu, \nu) \in [0, 1]^2 \mid \mu + \nu \le 1\}$;
- Neutrosophic model: $\mathrm{Dom}(M) = \{(T, I, F) \in [0, 1]^3 \mid 0 \le T + I + F \le 3\}$;
- Plithogenic model, and many other extensions.

Definition 2.2.6 (Uncertain Set (U-Set))**.** [59] Let X be a nonempty universe, and let M be a fixed uncertain model with degree–domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$. An *Uncertain Set of type M* (or *U-Set* for short) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is called the *uncertainty–degree function* (or membership map) of $\mathcal{U}$.

For $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ encodes the degree(s) to which x belongs to the uncertain set, according to the model M.

Remark 2.2.7. Special cases:

- If M is the fuzzy model and $\mathrm{Dom}(M) = [0, 1]$, then $\mu_M : X \to [0, 1]$ is a usual fuzzy membership function and $\mathcal{U}$ is a fuzzy set.
- If M is neutrosophic, then $\mu_M(x) = (T(x), I(x), F(x))$ gives a neutrosophic set.
- Other choices of M recover intuitionistic fuzzy sets, picture fuzzy sets, plithogenic sets, and so on.

As noted in the remark, various generalizations are possible. For reference, Table 2.1 presents a catalogue of uncertainty-set families (U-Sets) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$ (cf. [60]).

The definitions and related concepts of Uncertain Graphs are presented below.

Table 2.1: A catalogue of uncertainty-set families (U-Sets) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [60].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [51,61]; N-Fuzzy Set [62–64] Shadowed Set [65–67]
2		Intuitionistic Fuzzy Set [68,69]; Vague Set [70,71]; Bipolar Fuzzy Set (two-component description) [72]; Variable Fuzzy Set [73–75]; Paraconsistent Fuzzy Set [76,77]; Bifuzzy Set [78,79]
3		Single-Valued Neutrosophic Set [80,81]; Picture Fuzzy Set [82,83]; Spherical Fuzzy Set [84,85]; Tripolar Fuzzy Set (three-component formalisms) [86–88]; Neutrosophic Vague Set [89,90]
4		Quadripartitioned Neutrosophic Set [91,92]; Double-Valued Neutrosophic Set [93,94]; Dual Hesitant Fuzzy Set [95,96]; Ambiguous Set [97–99]; Turiyam Neutrosophic Set [100–103]
5		Pentapartitioned Neutrosophic Set [104–106]; Triple-Valued Neutrosophic Set [107–109]
6		Hexapartitioned Neutrosophic Set; Quadruple-Valued Neutrosophic Set [108, 110]
7		Heptapartitioned Neutrosophic Set; Quintuple-Valued Neutrosophic Set [108, 111,112]
8		Octapartitioned Neutrosophic Set [113,114]
9		Nonapartitioned Neutrosophic Set [113,114]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [115]; MultiFuzzy Set [116]; n-Refined Fuzzy Set [117,118]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [118]; Multi-Intuitionistic Fuzzy Set [116]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [118]; Multi-Neutrosophic Set [116,119]

Reading guide. In the U-Set scheme [59], each model M is specified by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ and a membership map $\mu_M : X \to \mathrm{Dom}(M)$. The table groups representative families by the ambient dimension k (i.e., how many numerical components are stored per element).

(a) A widely cited viewpoint is that neutrosophic sets provide a unifying umbrella covering several earlier multi-component fuzzy models (and their generalizations); see [120].

(b) Ambiguous sets are commonly presented as subclasses of certain four-component neutrosophic families; see [91,92,99].

(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [121].

Definition 2.2.8 (Uncertain Graph)**.** Let $G = (V, E)$ be a (finite, undirected, loopless) graph and let M be an uncertain model with degree–domain $\mathrm{Dom}(M)$. An *Uncertain Graph of type M* is a triple

$$\mathcal{G}_M = (V, E, \mu_M),$$

where

$$\mu_M : V \cup E \longrightarrow \mathrm{Dom}(M)$$

assigns to each vertex $v \in V$ and each edge $e \in E$ an uncertainty degree $\mu_M(v)$ or $\mu_M(e)$ in $\mathrm{Dom}(M)$.

Optionally, one may impose model–specific consistency conditions between vertex and edge degrees (for instance, $\mu_M(e)$ bounded in terms of $\mu_M(u)$ and $\mu_M(v)$ for $e = \{u, v\}$ in fuzzy or intuitionistic fuzzy graph models), but these constraints are encoded in the choice of M and are not fixed at the level of this general definition.

Remark 2.2.9. Again, particular choices of M recover well–known graph models:

- Fuzzy graph (when M is fuzzy and $\mu_M : V \cup E \to [0,1]$);

- Intuitionistic fuzzy graph, neutrosophic graph, plithogenic graph, etc., for the corresponding models M.

As a reference, Table 2.2 presents a catalogue of uncertainty-graph families (Uncertain Graphs) organised by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

Table 2.2: A catalogue of uncertainty-graph families (Uncertain Graphs) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$.

k	Representative uncertainty-graph type(s) $\mathcal{G}_M = (V, E, \mu_M)$ with $\mu_M : V \cup E \to \mathrm{Dom}(M) \subseteq [0,1]^k$
1	Fuzzy graph; N-graph; shadowed-graph variants
2	Intuitionistic fuzzy graph [122]; vague graph [123]; bipolar fuzzy graph [124]; intuitionistic evidence graph; variable fuzzy graph; paraconsistent fuzzy graph; bifuzzy graph [125, 126]
3	Neutrosophic graph [58][(a)]; hesitant fuzzy graph [127]; tripolar fuzzy graph; three-way fuzzy graph; picture fuzzy graph [128, 129]; spherical fuzzy graph [84]; inconsistent intuitionistic fuzzy graph; ternary fuzzy / neutrosophic-fuzzy graph; neutrosophic vague graph
4	Quadripartitioned neutrosophic graph [130, 131]; double-valued neutrosophic graph [93]; dual hesitant fuzzy graph [132]; ambiguous graph[(b)]; local-neutrosophic graph; support-neutrosophic graph; turiyam neutrosophic graph [133][(c)]
5	Pentapartitioned neutrosophic graph [134]; triple-valued neutrosophic graph
6	Hexapartitioned neutrosophic graph; quadruple-valued neutrosophic graph
7	Heptapartitioned neutrosophic graph [135]; quintuple-valued neutrosophic graph
8	Octapartitioned neutrosophic graph
9	Nonapartitioned neutrosophic graph
n	n-refined fuzzy graph; multi-valued (fuzzy) graphs; multi-fuzzy graphs [136]
$2n$	n-refined intuitionistic fuzzy graph; multi-intuitionistic fuzzy graphs
$3n$	n-refined neutrosophic graph; multi-neutrosophic graphs

[(a)] Neutrosophic graph models are often treated as broad frameworks that can specialize to many degree-based graph formalisms under suitable constraints.
[(b)] Ambiguous-graph models are commonly presented as subclasses of certain quadripartitioned and also double-valued neutrosophic graph models.
[(c)] Turiyam neutrosophic graphs are reported as subclasses of certain quadripartitioned neutrosophic graph models.

2.3 Soft Graph

Soft graph is a parameterized graph structure assigning to each parameter a subgraph, enabling flexible modeling of systems whose relations vary across contexts and scenarios [92, 137, 138].

Definition 2.3.1 (Soft Graph)**.** Let $G^* = (V, E)$ be a simple graph, and let A be a nonempty set of parameters. A *soft graph* over G^* is a quadruple

$$G = (G^*, F, K, A),$$

where

$$F : A \to \mathcal{P}(V), \qquad K : A \to \mathcal{P}(E),$$

such that, for every $a \in A$, the pair

$$H(a) = \big(F(a), K(a)\big)$$

is a subgraph of G^*.

2.4 Rough Graph

A rough graph represents a graph through lower and upper approximation graphs under an equivalence relation, thereby modeling indiscernibility, vagueness, and boundary uncertainty structurally formally [139, 140].

Definition 2.4.1 (Rough Graph)**.** Let $U = (V, E)$ be a universe graph, and let R be an equivalence relation on E, inducing edge equivalence classes $[e]_R$ for $e \in E$. For a graph $T = (W, X)$ with $W \subseteq V$ and $X \subseteq E$, define

$$\underline{R}(X) = \{\, e \in E : [e]_R \subseteq X \,\}, \qquad \overline{R}(X) = \{\, e \in E : [e]_R \cap X \neq \varnothing \,\}.$$

Then the pair

$$\big(\underline{R}(T), \overline{R}(T)\big) = \big((W, \underline{R}(X)), (W, \overline{R}(X))\big)$$

is called the *rough graph* associated with T. If X is not a union of R-equivalence classes, then T is said to be an *R-rough graph*.

Chapter 3

Recursive SuperHyperGraph and Related Concepts

In this chapter, we present several recursive SuperHyperGraph-related concepts.

3.1 Recursive HyperGraph

A *Recursive HyperGraph* is a hypergraph in which hyperedges are allowed to contain not only ordinary vertices but also lower-level hyperedges as elements, thereby supporting nested (self-referential) incidence up to a bounded recursion depth [25, 26]. An (n, k)-recursive SuperHyperGraph combines level-n supervertices (via iterated powersets) with depth-k recursive superhyperedges that may contain supervertices and lower-level edges as elements [27].

Definition 3.1.1 (Depth-k powerset universe)**.** [25, 26] Let S be a nonempty set and let $k \in \mathbb{N} \cup \{0\}$. Define a hierarchy of sets $(S_i)_{i \geq 0}$ by

$$S_0 := S, \qquad S_i := \mathcal{P}\Big(\bigcup_{j=0}^{i-1} S_j\Big) \quad (i \geq 1).$$

The *depth-k powerset universe* over S is

$$2_{S,k} \;:=\; \mathcal{P}\Big(\bigcup_{i=0}^{k} S_i\Big).$$

Definition 3.1.2 (k-recursive hypergraph)**.** [25, 26] Let V be a finite vertex set and let $k \in \mathbb{N} \cup \{0\}$. A *$k$-recursive hypergraph* is a pair

$$H = (V, E)$$

such that

$$E \subseteq 2_{V,k} \setminus \{\emptyset\},$$

where $2_{V,k}$ is the depth-k powerset universe from Definition 3.1.1 applied to $S = V$.

In particular, when $k = 0$ one has $2_{V,0} = \mathcal{P}(V)$ and therefore $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, so H reduces to an ordinary hypergraph.

Definition 3.1.3 ((n,k)-recursive SuperHyperGraph)**.** Let V_0 be a finite nonempty base set, and let $n, k \in \mathbb{N}_0$.

Define the iterated powersets by

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{m+1}(V_0) = \mathcal{P}\big(\mathcal{P}^m(V_0)\big) \quad (m \geq 0).$$

An (n,k)*-recursive SuperHyperGraph* on V_0 is a pair

$$\mathrm{RSHG}^{(n,k)} = (V, E)$$

satisfying the following axioms:

(R1) *Supervertex set.* V is a finite nonempty set such that

$$V \subseteq \mathcal{P}^n(V_0).$$

The elements of V are called n*-supervertices.*

(R2) *Recursive superhyperedge family.* E is a finite family of nonempty recursive superhyperedges such that

$$E \subseteq 2_{V,k} \setminus \{\emptyset\},$$

where $2_{V,k}$ is the depth-k powerset universe from Definition 3.1.1 applied to $S = V$.

(R3) *Typed incidence closure.* Every element of every recursive superhyperedge is either an n-supervertex or a recursive superhyperedge:

$$\forall e \in E, \qquad e \subseteq V \cup E.$$

(R4) *Well-foundedness and depth bound.* Define a relation $\prec$ on E by

$$f \prec e \quad \Longleftrightarrow \quad f \in e \qquad (f, e \in E).$$

The relation $\prec$ is required to be acyclic, and every $\prec$-chain

$$e_0 \prec e_1 \prec \cdots \prec e_t$$

has length at most k.

Equivalently, there exists a rank map

$$h : E \to \{0, 1, \ldots, k\}$$

such that

$$f \in e \cap E \quad \Longrightarrow \quad h(f) < h(e).$$

Remark 3.1.4. Axiom (R3) ensures that every object occurring inside a recursive superhyperedge has an admissible type: it is either a supervertex or another recursive superhyperedge. Axiom (R4) excludes circular definitions such as $e \in e$ or $e_1 \in e_2 \in e_1$, and it also ensures that recursion is bounded by the prescribed depth k.

Example 3.1.5 (A concrete $(1,1)$-recursive SuperHyperGraph)**.** Let

$$V_0 = \{a, b, c\}.$$

Then

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0).$$

Define three 1-supervertices by

$$x := \{a\}, \qquad y := \{b\}, \qquad z := \{a, c\}.$$

Set

$$V := \{x, y, z\}.$$

Clearly,

$$V \subseteq \mathcal{P}(V_0) = \mathcal{P}^1(V_0).$$

Now define two recursive superhyperedges by

$$e_1 := \{x, y\}, \qquad e_2 := \{e_1, z\}.$$

Since $e_1 \subseteq V$, one has

$$e_1 \in \mathcal{P}(V).$$

For $k = 1$, the depth-1 powerset universe over V is

$$2_{V,1} = \mathcal{P}\big(V \cup \mathcal{P}(V)\big).$$

Because

$$e_1 \in \mathcal{P}(V) \subseteq V \cup \mathcal{P}(V) \qquad \text{and} \qquad z \in V \subseteq V \cup \mathcal{P}(V),$$

it follows that

$$e_2 = \{e_1, z\} \subseteq V \cup \mathcal{P}(V),$$

and hence

$$e_2 \in 2_{V,1}.$$

Also,

$$e_1 \in \mathcal{P}(V) \subseteq 2_{V,1}.$$

Therefore, if we set

$$E := \{e_1, e_2\},$$

then

$$E \subseteq 2_{V,1} \setminus \{\emptyset\}.$$

Moreover, the typed incidence closure condition holds. Indeed,

$$e_1 = \{x, y\} \subseteq V,$$

and

$$e_2 = \{e_1, z\} \subseteq V \cup E,$$

because $e_1 \in E$ and $z \in V$.

The recursive edge-membership relation has the only nontrivial relation

$$e_1 \prec e_2,$$

since $e_1 \in e_2$. Hence the relation $\prec$ is acyclic, and the longest chain has length 1. Equivalently, the rank map

$$h : E \to \{0, 1\}, \qquad h(e_1) = 0, \quad h(e_2) = 1,$$

satisfies

$$f \in e \cap E \Longrightarrow h(f) < h(e).$$

Hence

$$\mathrm{RSHG}^{(1,1)} = (V, E)$$

is a $(1, 1)$-recursive SuperHyperGraph.

In this example, e_1 is an ordinary superhyperedge on the supervertex set V, while e_2 is genuinely recursive because it contains the lower-level edge e_1 as one of its elements.

3.2 Recursive Fuzzy SuperHyperGraphs

A fuzzy n-SuperHyperGraph is a higher-level network representation in which supervertices and superedges carry membership values for modeling complex interactions (cf. [3, 141]).

Definition 3.2.1 (Fuzzy n-SuperHyperGraph)**.** (cf. [3]) Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A *fuzzy n-SuperHyperGraph* is a quadruple

$$(V, E, \sigma, \mu),$$

where $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ obey the *admissibility constraint*

$$\mu(e) \ \leq \ \min_{v \in e} \sigma(v) \qquad \text{for every } e \in E.$$

Definition 3.2.2 ((n, k)-recursive Fuzzy SuperHyperGraph)**.** Let V_0 be a finite nonempty base set, and let $n, k \in \mathbb{N}_0$. Let

$$\mathrm{RSHG}^{(n,k)} = (V, E)$$

be an (n, k)-recursive SuperHyperGraph in the sense of Definition 3.1.3.

A (n, k)*-recursive fuzzy SuperHyperGraph* is a quadruple

$$\mathrm{RFSHG}^{(n,k)} = (V, E, \sigma, \mu),$$

where

$$\sigma : V \to [0, 1], \qquad \mu : E \to [0, 1],$$

are membership maps satisfying the recursive fuzzy admissibility condition

$$\mu(e) \leq \min_{x \in e} \widehat{\sigma}(x) \qquad (e \in E),$$

where

$$\widehat{\sigma} : V \cup E \to [0, 1]$$

is defined by

$$\widehat{\sigma}(x) := \begin{cases} \sigma(x), & x \in V, \\ \mu(x), & x \in E. \end{cases}$$

Remark 3.2.3. Condition (C3) is the natural recursive extension of the usual fuzzy-hypergraph constraint "edge membership cannot exceed the membership of any incident endpoint": if e contains lower-level edges as *elements*, then those elements also constrain $\mu(e)$. Condition (C2) guarantees that this recursive interpretation is well-founded.

Remark 3.2.4. The typed incidence closure and the well-foundedness of recursive edge membership are inherited from the underlying (n, k)-recursive SuperHyperGraph. Thus the only additional datum in the fuzzy version is the membership assignment together with the recursive admissibility constraint.

Notation 3.2.5 (Typed object universe)**.** *For a recursive SuperHyperGraph* (V, E)*, we write*

$$V \sqcup E := (V \times \{0\}) \cup (E \times \{1\})$$

for the typed disjoint union of supervertices and recursive superhyperedges. This notation distinguishes a supervertex from a recursive superhyperedge even if they are represented by the same underlying set.

Example 3.2.6 (A concrete $(1, 1)$-recursive fuzzy SuperHyperGraph)**.** Let

$$V_0 = \{a, b, c\}.$$

Then

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0).$$

Define three 1-supervertices by

$$x := \{a\}, \qquad y := \{b\}, \qquad z := \{a, c\},$$

and set

$$V := \{x, y, z\}.$$

Clearly,

$$V \subseteq \mathcal{P}(V_0) = \mathcal{P}^1(V_0).$$

Next, define two recursive superhyperedges by

$$e_1 := \{x, y\}, \qquad e_2 := \{e_1, z\}.$$

Let

$$E := \{e_1, e_2\}.$$

For $k = 1$, we have

$$2_{V,1} = \mathcal{P}\big(V \cup \mathcal{P}(V)\big).$$

Since $e_1 \subseteq V$, one has $e_1 \in \mathcal{P}(V) \subseteq 2_{V,1}$. Also, because $e_1 \in \mathcal{P}(V)$ and $z \in V$, it follows that

$$e_2 = \{e_1, z\} \in 2_{V,1}.$$

Hence

$$E \subseteq 2_{V,1} \setminus \{\emptyset\},$$

so (V, E) is a $(1, 1)$-recursive SuperHyperGraph.

Now define the fuzzy vertex-membership map $\sigma : V \to [0, 1]$ by

$$\sigma(x) = 0.9, \qquad \sigma(y) = 0.7, \qquad \sigma(z) = 0.8,$$

and the fuzzy edge-membership map $\mu : E \to [0, 1]$ by

$$\mu(e_1) = 0.6, \qquad \mu(e_2) = 0.5.$$

We verify the required conditions.

(C1) Typed incidence. For $e_1 = \{x, y\}$, both x and y belong to V. For $e_2 = \{e_1, z\}$, one element e_1 belongs to E and the other element z belongs to V. Thus every element of every recursive superhyperedge lies in $V \cup E$.

(C2) Well-foundedness and depth bound. The only recursive edge-membership relation among edges is

$$e_1 \prec e_2,$$

because $e_1 \in e_2$, while $e_2 \notin e_1$ and $e_1 \notin e_1$, $e_2 \notin e_2$. Hence the relation $\prec$ is acyclic. Moreover, the longest chain has length 1, so the recursion depth is at most 1. Equivalently, the rank map

$$h : E \to \{0, 1\}, \qquad h(e_1) = 0, \quad h(e_2) = 1,$$

satisfies

$$f \in e \cap E \Longrightarrow h(f) < h(e).$$

(C3) Recursive min-constraint. Define $\widehat{\sigma} : V \cup E \to [0, 1]$ by

$$\widehat{\sigma}(u) = \begin{cases} \sigma(u), & u \in V, \\ \mu(u), & u \in E. \end{cases}$$

For the edge $e_1 = \{x, y\}$,

$$\mu(e_1) = 0.6 \le \min\{\widehat{\sigma}(x), \widehat{\sigma}(y)\} = \min\{\sigma(x), \sigma(y)\} = \min\{0.9, 0.7\} = 0.7.$$

For the edge $e_2 = \{e_1, z\}$,

$$\mu(e_2) = 0.5 \le \min\{\widehat{\sigma}(e_1), \widehat{\sigma}(z)\} = \min\{\mu(e_1), \sigma(z)\} = \min\{0.6, 0.8\} = 0.6.$$

Therefore,

$$\mathrm{RFSHG}^{(1,1)} = (V, E, \sigma, \mu)$$

is a concrete $(1, 1)$-recursive fuzzy SuperHyperGraph.

In this example, e_1 is an ordinary fuzzy superhyperedge on the supervertex set V, while e_2 is genuinely recursive because it contains the lower-level edge e_1 as one of its elements.

3.3 Recursive Neutrosophic SuperHyperGraphs

A neutrosophic n-superhypergraph assigns truth, indeterminacy, and falsity degrees to n-supervertices and superhyperedge incidences. The definition of a Neutrosophic n-Superhypergraph is given as follows [3, 142].

Definition 3.3.1 (Single-Valued Neutrosophic Hypergraph)**.** (cf. [143]) Let $V = \{v_1, \ldots, v_n\}$ be a finite vertex set and let $E = \{E_i\}_{i=1}^m$ be a family of nontrivial single-valued neutrosophic subsets of V such that

$$V \;=\; \bigcup_{i=1}^{m} \operatorname{supp}(E_i).$$

Each neutrosophic hyperedge E_i is given by

$$E_i \;=\; \bigl\{(v, T_{E_i}(v), I_{E_i}(v), F_{E_i}(v)) : v \in V\bigr\},$$

where

$$T_{E_i},\, I_{E_i},\, F_{E_i} : V \longrightarrow [0,1] \quad \text{satisfy} \quad 0 \;\le\; T_{E_i}(v) + I_{E_i}(v) + F_{E_i}(v) \;\le\; 3 \;\; \forall\, v \in V.$$

Then $H = (V, E)$ is called a *single-valued neutrosophic hypergraph.*

Definition 3.3.2 (Neutrosophic n-SuperHyperGraph)**.** (cf. [3]) Let V_0 be a finite base set, and define iteratively

$$\mathcal{P}^0(V_0) = V_0,$$
$$\mathcal{P}^{k+1}(V_0) = \mathcal{P}\bigl(\mathcal{P}^k(V_0)\bigr).$$

An *n-SuperHyperGraph* is a pair $\mathrm{SHG}^{(n)} = (V, E)$ such that

$$V \subseteq \mathcal{P}^n(V_0),$$

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Elements of V are called *n-supervertices*, and elements of E are called *n-superhyperedges* (each $e \in E$ is a nonempty subset of V).

A *Neutrosophic n-SuperHyperGraph* is the tuple

$$\mathcal{NSHG}^{(n)} = \bigl(V,\, E,\, T_V,\, I_V,\, F_V,\, T_E,\, I_E,\, F_E\bigr),$$

where

- $T_V, I_V, F_V : V \to [0,1]$ assign to each n-supervertex v its truth-membership $T_V(v)$, indeterminacy $I_V(v)$, and falsity $F_V(v)$, with $0 \le T_V(v) + I_V(v) + F_V(v) \le 3$.

- $T_E, I_E, F_E : E \times V \to [0,1]$ assign to each pair (e, v) an *incidence* neutrosophic triple

  $$T_E(e, v), I_E(e, v), F_E(e, v)$$

  , with

  $$0 \le T_E(e, v) + I_E(e, v) + F_E(e, v) \le 3$$

  .

These functions satisfy the following *incidence consistency* conditions:

$$\begin{aligned}&\text{(support)}\quad v\notin e \implies T_E(e,v)=I_E(e,v)=F_E(e,v)=0,\\&\text{(edge-appurtenance)}\quad v\in e \implies T_E(e,v)\le T_V(v),\ I_E(e,v)\le I_V(v),\ F_E(e,v)\le F_V(v),\\&\qquad \forall\, e\in E,\ \forall\, v\in V.\end{aligned}$$

An (n,k)-recursive neutrosophic superhypergraph models n-level nested supervertices and k-depth recursive superhyperedges with truth/indeterminacy/falsity memberships.

Definition 3.3.3 (Single-valued neutrosophic set on a universe)**.** Let X be a nonempty set. A *single-valued neutrosophic set* (SVNS) on X is a triple

$$A=(T_A,I_A,F_A),\qquad T_A,\ I_A,\ F_A: X\longrightarrow[0,1],$$

such that for every $x\in X$,

$$0\le T_A(x)+I_A(x)+F_A(x)\le 3.$$

Its (neutrosophic) support is

$$\operatorname{supp}(A):=\{\,x\in X\mid T_A(x)+I_A(x)+F_A(x)>0\,\}.$$

Definition 3.3.4 (Recursive universe over a supervertex set)**.** Let V be a nonempty finite set and let $k\in\mathbb{N}\cup\{0\}$. Let $(S_i)_{i\ge 0}$ be the hierarchy from Definition 3.1.1:

$$S_0:=V,\qquad S_i:=\mathcal{P}\Big(\bigcup_{j=0}^{i-1}S_j\Big)\quad (i\ge 1).$$

Define the *depth-k recursive universe* over V by

$$\mathcal{U}_{V,k}:=\bigcup_{i=0}^{k}S_i.$$

Note that $2_{V,k}=\mathcal{P}(\mathcal{U}_{V,k})$ (Definition 3.1.1).

Definition 3.3.5 ((n,k)-recursive single-valued Neutrosophic SuperHyperGraph)**.** Fix a finite nonempty base set V_0 and integers $n,k\in\mathbb{N}\cup\{0\}$. An *$(n,k)$-recursive single-valued Neutrosophic SuperHyperGraph* (abbrev. *(n,k)-recursive NSHG*) on V_0 is a tuple

$$\mathrm{RNSHG}^{(n,k)}=\big(V,\ E,\ T_{\mathcal{U}},I_{\mathcal{U}},F_{\mathcal{U}},\ (T_e,I_e,F_e)_{e\in E}\big)$$

satisfying the following axioms.

(i) *(Hierarchical supervertices).* V is a finite nonempty set with

$$V\subseteq\mathcal{P}^n(V_0).$$

(ii) *(Recursive universe).* Let $\mathcal{U}_{V,k}$ be the depth-k recursive universe over V (Definition 3.3.4).

(iii) *(Neutrosophic availability on the universe).*

$$T_{\mathcal{U}},\ I_{\mathcal{U}},\ F_{\mathcal{U}} : \mathcal{U}_{V,k} \longrightarrow [0,1] \quad \text{satisfy} \quad 0 \le T_{\mathcal{U}}(x) + I_{\mathcal{U}}(x) + F_{\mathcal{U}}(x) \le 3 \ \ \forall x \in \mathcal{U}_{V,k}.$$

(Interpretation: $T_{\mathcal{U}}, I_{\mathcal{U}}, F_{\mathcal{U}}$ bound how strongly an object $x \in \mathcal{U}_{V,k}$ can participate in higher relations.)

(iv) *(Recursive neutrosophic superhyperedges).* E is a finite nonempty family of SVNSs on $\mathcal{U}_{V,k}$: for each $e \in E$,

$$e \equiv (T_e, I_e, F_e) \quad \text{with} \quad T_e, I_e, F_e : \mathcal{U}_{V,k} \to [0,1],\ \ 0 \le T_e(x)+I_e(x)+F_e(x) \le 3 \ \forall x \in \mathcal{U}_{V,k}.$$

Moreover, each edge has nonempty support:

$$\operatorname{supp}(e) \neq \emptyset.$$

(v) *(Appurtenance constraints).* For all $e \in E$ and all $x \in \mathcal{U}_{V,k}$,

$$T_e(x) \le T_{\mathcal{U}}(x), \qquad I_e(x) \le I_{\mathcal{U}}(x), \qquad F_e(x) \le F_{\mathcal{U}}(x).$$

(vi) *(Vertex coverage; optional but often assumed).*

$$V \subseteq \bigcup_{e \in E} \operatorname{supp}(e).$$

Remark 3.3.6 (Crisp underlying edge and recursion depth)**.** Given $\mathrm{RNSHG}^{(n,k)}$ as in Definition 3.3.5, each $e \in E$ induces a *crisp* recursive superhyperedge

$$\underline{e} := \operatorname{supp}(e) \subseteq \mathcal{U}_{V,k}.$$

Since $\underline{e} \neq \emptyset$, one has $\underline{e} \in \mathcal{P}(\mathcal{U}_{V,k}) \setminus \{\emptyset\} = 2_{V,k} \setminus \{\emptyset\}$, i.e., $\underline{e}$ is admissible as a depth-k recursive edge.

Example 3.3.7 (A concrete $(1,1)$-recursive single-valued Neutrosophic SuperHyperGraph)**.** Let

$$V_0 = \{a, b\}.$$

Define two 1-supervertices by

$$x := \{a\}, \qquad y := \{b\},$$

and set

$$V := \{x, y\}.$$

Then

$$V \subseteq \mathcal{P}(V_0) = \mathcal{P}^1(V_0).$$

For $k = 1$, the depth-1 recursive universe over V is

$$\mathcal{U}_{V,1} = V \cup \mathcal{P}(V) = \{\, x,\ y,\ \emptyset,\ \{x\},\ \{y\},\ \{x, y\} \,\}.$$

Next, define the neutrosophic availability functions

$$T_{\mathcal{U}},\ I_{\mathcal{U}},\ F_{\mathcal{U}} : \mathcal{U}_{V,1} \to [0,1]$$

by

$$T_{\mathcal{U}}(u) = 0.9, \qquad I_{\mathcal{U}}(u) = 0.3, \qquad F_{\mathcal{U}}(u) = 0.2 \qquad \text{for all } u \in \mathcal{U}_{V,1}.$$

Hence, for every $u \in \mathcal{U}_{V,1}$,

$$0 \le T_{\mathcal{U}}(u) + I_{\mathcal{U}}(u) + F_{\mathcal{U}}(u) = 0.9 + 0.3 + 0.2 = 1.4 \le 3.$$

Now define two recursive single-valued neutrosophic superhyperedges

$$e_1 \equiv (T_{e_1}, I_{e_1}, F_{e_1}) \qquad \text{and} \qquad e_2 \equiv (T_{e_2}, I_{e_2}, F_{e_2})$$

on $\mathcal{U}_{V,1}$ as follows.

For e_1, set

$$(T_{e_1}(u), I_{e_1}(u), F_{e_1}(u)) = \begin{cases} (0.7, 0.2, 0.1), & u = x, \\ (0.6, 0.2, 0.2), & u = y, \\ (0,0,0), & \text{otherwise.} \end{cases}$$

For e_2, set

$$(T_{e_2}(u), I_{e_2}(u), F_{e_2}(u)) = \begin{cases} (0.5, 0.2, 0.1), & u = y, \\ (0.6, 0.1, 0.2), & u = \{x, y\}, \\ (0,0,0), & \text{otherwise.} \end{cases}$$

Let

$$E := \{e_1, e_2\}.$$

We verify the axioms.

(i) Hierarchical supervertices. Already,

$$V = \{x, y\} \subseteq \mathcal{P}^1(V_0).$$

(ii) Recursive universe. The universe $\mathcal{U}_{V,1}$ has been specified above.

(iii) Neutrosophic availability on the universe. For every $u \in \mathcal{U}_{V,1}$,

$$0 \le T_{\mathcal{U}}(u) + I_{\mathcal{U}}(u) + F_{\mathcal{U}}(u) = 1.4 \le 3.$$

(iv) Recursive neutrosophic superhyperedges. Each of e_1 and e_2 is an SVNS on $\mathcal{U}_{V,1}$, and for every $u \in \mathcal{U}_{V,1}$,

$$0 \le T_{e_i}(u) + I_{e_i}(u) + F_{e_i}(u) \le 3 \qquad (i = 1, 2).$$

Moreover, both supports are nonempty:

$$\mathrm{supp}(e_1) = \{x, y\} \neq \emptyset, \qquad \mathrm{supp}(e_2) = \{y, \{x, y\}\} \neq \emptyset.$$

(v) Appurtenance constraints. For every $u \in \mathcal{U}_{V,1}$,

$$T_{e_i}(u) \leq T_{\mathcal{U}}(u) = 0.9, \qquad I_{e_i}(u) \leq I_{\mathcal{U}}(u) = 0.3, \qquad F_{e_i}(u) \leq F_{\mathcal{U}}(u) = 0.2 \qquad (i = 1, 2).$$

Hence the required appurtenance conditions hold.

(vi) Vertex coverage. Since

$$x \in \mathrm{supp}(e_1), \qquad y \in \mathrm{supp}(e_1) \cup \mathrm{supp}(e_2),$$

we obtain

$$V \subseteq \mathrm{supp}(e_1) \cup \mathrm{supp}(e_2) = \bigcup_{e \in E} \mathrm{supp}(e).$$

Therefore,

$$\mathrm{RNSHG}^{(1,1)} = \big(V,\ E,\ T_{\mathcal{U}}, I_{\mathcal{U}}, F_{\mathcal{U}},\ (T_e, I_e, F_e)_{e \in E}\big)$$

is a concrete $(1, 1)$-recursive single-valued Neutrosophic SuperHyperGraph.

In this example, the edge e_2 is genuinely recursive, because it assigns positive neutrosophic membership to the object $\{x, y\} \in \mathcal{P}(V) \subseteq \mathcal{U}_{V,1}$.

3.4 Recursive Uncertain SuperHyperGraph

We combine the notions of an (n, k)-recursive SuperHyperGraph and an Uncertain Set / Uncertain Graph of type M into a single framework.

Definition 3.4.1 (Typed object universe of a recursive SuperHyperGraph)**.** Let

$$\mathrm{RSHG}^{(n,k)} = (V, E)$$

be an (n, k)-recursive SuperHyperGraph.

Define the *typed object universe* of $\mathrm{RSHG}^{(n,k)}$ by

$$V \sqcup E \ :=\ (V \times \{0\}) \cup (E \times \{1\}).$$

Its elements are called *typed recursive objects*. Thus $(v, 0)$ represents a supervertex $v \in V$, while $(e, 1)$ represents a recursive superhyperedge $e \in E$.

This tagged disjoint union distinguishes vertices from edges even in the case where some vertex and some edge may coincide as pure sets.

Definition 3.4.2 (Recursive Uncertain SuperHyperGraph)**.** Fix a finite nonempty base set V_0, integers $n, k \in \mathbb{N} \cup \{0\}$, and an uncertain model M with degree-domain $\mathrm{Dom}(M)$.

A *Recursive Uncertain SuperHyperGraph of type* M is a triple

$$\mathrm{RUSHG}_M^{(n,k)} \;=\; (V, E, \mu_M)$$

satisfying the following conditions:

(RU1) *(Underlying recursive SuperHyperGraph)*

$$(V, E) = \mathrm{RSHG}^{(n,k)}$$

is an (n, k)-recursive SuperHyperGraph; that is,

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq 2_{V,k} \setminus \{\emptyset\}.$$

(RU2) *(Typed incidence closure)* every element of every recursive superhyperedge is either a supervertex or a recursive superhyperedge:

$$\forall e \in E,\ \forall x \in e, \qquad x \in V \text{ or } x \in E.$$

Equivalently,

$$e \subseteq V \cup E \qquad \text{for every } e \in E.$$

(RU3) *(Well-founded recursion and depth bound)* define a binary relation $\prec$ on E by

$$f \prec e \quad \Longleftrightarrow \quad f \in e \qquad (f, e \in E).$$

We require that $\prec$ is acyclic and that every $\prec$-chain

$$e_0 \prec e_1 \prec \cdots \prec e_t$$

has length at most k. Equivalently, there exists a rank map

$$h : E \longrightarrow \{0, 1, \ldots, k\}$$

such that

$$f \in e \cap E \quad \Longrightarrow \quad h(f) < h(e) \qquad \text{for all } e, f \in E.$$

(RU4) *(Uncertainty-degree assignment)*

$$\mu_M : V \sqcup E \longrightarrow \mathrm{Dom}(M)$$

is a function assigning to each typed recursive object a degree tuple admissible in the model M.

We write

$$\mu_M^V(v) := \mu_M(v, 0) \qquad (v \in V),$$

and

$$\mu_M^E(e) := \mu_M(e, 1) \qquad (e \in E),$$

for the induced vertex-side and edge-side uncertainty maps.

Remark 3.4.3. Definition 3.4.2 is the natural generalization of:

- the (n, k)-recursive SuperHyperGraph structure on (V, E),
- the recursive typed-incidence and well-foundedness conditions used for recursive fuzzy SuperHyperGraphs, and
- the general uncertain-set viewpoint, in which one assigns values in $\mathrm{Dom}(M)$ through a map into the chosen uncertainty model.

Unlike the fuzzy case, no universal order-based admissibility inequality is imposed here, because a general uncertain model M need not carry a canonical minimum operation or partial order suitable for such a constraint. If the chosen model M has additional algebraic structure, one may impose further model-specific compatibility axioms.

Example 3.4.4 (A concrete $(1, 1)$-recursive Uncertain SuperHyperGraph)**.** Let M_{fuz} be the fuzzy uncertain model, with degree-domain

$$\mathrm{Dom}(M_{\mathrm{fuz}}) = [0, 1].$$

Take the base set

$$V_0 = \{a, b, c\}.$$

Define three 1-supervertices by

$$x := \{a\}, \qquad y := \{b\}, \qquad z := \{a, c\},$$

and set

$$V := \{x, y, z\}.$$

Then

$$V \subseteq \mathcal{P}(V_0) = \mathcal{P}^1(V_0).$$

Next, define two recursive superhyperedges by

$$e_1 := \{x, y\}, \qquad e_2 := \{e_1, z\},$$

and let

$$E := \{e_1, e_2\}.$$

For $k = 1$, the depth-1 powerset universe over V is

$$2_{V,1} = \mathcal{P}\big(V \cup \mathcal{P}(V)\big).$$

Since $e_1 \subseteq V$, one has

$$e_1 \in \mathcal{P}(V) \subseteq 2_{V,1}.$$

Also, because $e_1 \in \mathcal{P}(V)$ and $z \in V$, it follows that

$$e_2 = \{e_1, z\} \subseteq V \cup \mathcal{P}(V),$$

hence

$$e_2 \in 2_{V,1}.$$

Therefore,

$$E \subseteq 2_{V,1} \setminus \{\emptyset\},$$

so (V, E) is a $(1, 1)$-recursive SuperHyperGraph.

Now consider the typed object universe

$$V \sqcup E = (V \times \{0\}) \cup (E \times \{1\}) = \{(x, 0), (y, 0), (z, 0), (e_1, 1), (e_2, 1)\}.$$

Define the uncertainty-degree assignment

$$\mu_{M_{\text{fuz}}} : V \sqcup E \longrightarrow [0, 1]$$

by

$$\mu_{M_{\text{fuz}}}(x, 0) = 0.9, \qquad \mu_{M_{\text{fuz}}}(y, 0) = 0.8, \qquad \mu_{M_{\text{fuz}}}(z, 0) = 0.7,$$

and

$$\mu_{M_{\text{fuz}}}(e_1, 1) = 0.6, \qquad \mu_{M_{\text{fuz}}}(e_2, 1) = 0.5.$$

Equivalently, the induced vertex-side and edge-side maps are

$$\mu^V_{M_{\text{fuz}}}(x) = 0.9, \qquad \mu^V_{M_{\text{fuz}}}(y) = 0.8, \qquad \mu^V_{M_{\text{fuz}}}(z) = 0.7,$$

and

$$\mu^E_{M_{\text{fuz}}}(e_1) = 0.6, \qquad \mu^E_{M_{\text{fuz}}}(e_2) = 0.5.$$

We verify the axioms.

(RU1) Underlying recursive SuperHyperGraph. As shown above,

$$V \subseteq \mathcal{P}^1(V_0), \qquad E \subseteq 2_{V,1} \setminus \{\emptyset\}.$$

Hence $(V, E) = \mathrm{RSHG}^{(1,1)}$.

(RU2) Typed incidence closure. For the edge $e_1 = \{x, y\}$, both elements x and y belong to V. For the edge $e_2 = \{e_1, z\}$, one element e_1 belongs to E and the other element z belongs to V. Thus every element of every recursive superhyperedge lies in $V \cup E$.

(RU3) Well-founded recursion and depth bound. The only recursive edge-membership relation among edges is

$$e_1 \prec e_2,$$

because $e_1 \in e_2$, while $e_2 \notin e_1$, $e_1 \notin e_1$, and $e_2 \notin e_2$. Hence the relation $\prec$ is acyclic. Moreover, the longest $\prec$-chain has length 1, so the recursion depth is at most 1. Equivalently, the rank map

$$h : E \to \{0, 1\}, \qquad h(e_1) = 0, \quad h(e_2) = 1,$$

satisfies

$$f \in e \cap E \Longrightarrow h(f) < h(e).$$

(RU4) Uncertainty-degree assignment. The map

$$\mu_{M_{\mathrm{fuz}}} : V \sqcup E \to [0,1] = \mathrm{Dom}(M_{\mathrm{fuz}})$$

is well-defined, and each assigned value lies in the admissible degree-domain of the fuzzy uncertain model.

Therefore,

$$\mathrm{RUSHG}^{(1,1)}_{M_{\mathrm{fuz}}} = \big(V, E, \mu_{M_{\mathrm{fuz}}}\big)$$

is a concrete $(1,1)$-recursive Uncertain SuperHyperGraph of type M_{fuz}.

In this example, e_1 is an ordinary recursive superhyperedge on V, whereas e_2 is genuinely recursive because it contains the lower-level edge e_1 as one of its elements.

Theorem 3.4.5 (Well-definedness of Recursive Uncertain SuperHyperGraphs)**.** *Let*

$$\mathrm{RUSHG}^{(n,k)}_{M} = (V, E, \mu_M)$$

be a Recursive Uncertain SuperHyperGraph of type M in the sense of Definition 3.4.2. Then the following statements hold:

(i) *the pair (V, E) is a well-defined (n, k)-recursive SuperHyperGraph;*

(ii) *the typed object universe*

$$V \sqcup E = (V \times \{0\}) \cup (E \times \{1\})$$

is a well-defined finite set;

(iii) *the pair*

$$\big(V \sqcup E, \mu_M\big)$$

is a well-defined Uncertain Set of type M;

(iv) *therefore* $\mathrm{RUSHG}^{(n,k)}_{M}$ *is a well-defined mathematical structure.*

Proof. We verify the claims one by one.

(i) By (RU1), the pair (V, E) satisfies

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq 2_{V,k} \setminus \{\emptyset\},$$

which is precisely the defining condition of an (n, k)-recursive SuperHyperGraph. Hence (V, E) is well-defined at the structural level.

Moreover, condition (RU2) ensures that the elements appearing inside recursive superhyperedges are of the permitted two types only, namely supervertices or recursive superhyperedges. Condition (RU3) guarantees that recursive edge-membership is well-founded and bounded by depth k, so no ill-founded cyclic recursion occurs. Thus the recursive interpretation of edges is mathematically sound.

(ii) Since $V \subseteq \mathcal{P}^n(V_0)$ and V_0 is finite, the set V is finite. Because $2_{V,k}$ is constructed from finitely many iterated powersets over the finite set V, it is also finite; hence $E \subseteq 2_{V,k} \setminus \{\emptyset\}$ is finite.

Therefore $V \times \{0\}$ and $E \times \{1\}$ are finite sets. They are also disjoint, because their second coordinates differ. Consequently,

$$V \sqcup E = (V \times \{0\}) \cup (E \times \{1\})$$

is a well-defined finite set.

(iii) By (RU4), the map

$$\mu_M : V \sqcup E \to \mathrm{Dom}(M)$$

is a function. Since $\mathrm{Dom}(M)$ is the degree-domain of the uncertain model M, every value $\mu_M(x)$ belongs to the admissible set $\mathrm{Dom}(M)$. Hence μ_M is an uncertainty-degree function on the universe $V \sqcup E$.

By the general definition of an Uncertain Set of type M, any pair

$$(X, \nu) \quad \text{with} \quad \nu : X \to \mathrm{Dom}(M)$$

is a U-Set of type M. Applying this with $X = V \sqcup E$ and $\nu = \mu_M$, we conclude that

$$\big(V \sqcup E, \mu_M\big)$$

is a well-defined Uncertain Set of type M.

(iv) Combining (i)–(iii), the triple

$$(V, E, \mu_M)$$

consists of a well-defined recursive SuperHyperGraph together with a well-defined uncertainty assignment on its typed object universe. Therefore

$$\mathrm{RUSHG}_M^{(n,k)} = (V, E, \mu_M)$$

is a well-defined Recursive Uncertain SuperHyperGraph of type M. □

Corollary 3.4.6 (Underlying crisp recursive structure)**.** *Every Recursive Uncertain SuperHyperGraph*

$$\mathrm{RUSHG}_M^{(n,k)} = (V, E, \mu_M)$$

has a canonical underlying crisp recursive SuperHyperGraph obtained by forgetting the uncertainty-degree map:

$$\mathsf{Und}\Big(\mathrm{RUSHG}_M^{(n,k)}\Big) \ := \ (V, E).$$

Proof. Immediate from Theorem 3.4.5(i). □

Remark 3.4.7. Special choices of the uncertain model M recover corresponding recursive fuzzy, recursive intuitionistic fuzzy, recursive neutrosophic, or other uncertainty-aware recursive SuperHyperGraphs. In particular, when $\mathrm{Dom}(M) = [0, 1]$, one obtains the scalar-valued fuzzy-type setting; additional fuzzy admissibility inequalities may then be imposed as extra axioms if desired.

3.5 Recursive Soft SuperHyperGraph

We next introduce a soft-set version of recursive SuperHyperGraphs. The basic idea is that each parameter selects a recursive substructure of a fixed (n, k)-recursive SuperHyperGraph.

Definition 3.5.1 (Recursive Soft SuperHyperGraph)**.** Fix a finite nonempty base set V_0, integers $n, k \in \mathbb{N} \cup \{0\}$, and a nonempty parameter set A.

Let

$$\mathrm{RSHG}^{(n,k)} = (V, E)$$

be an (n, k)-recursive SuperHyperGraph, i.e.

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq 2_{V,k} \setminus \{\emptyset\}.$$

A *Recursive Soft SuperHyperGraph* over $\mathrm{RSHG}^{(n,k)}$ is a quadruple

$$\mathcal{RS}\mathrm{SHG}^{(n,k)} = \big(\mathrm{RSHG}^{(n,k)}, F, K, A\big),$$

where

$$F : A \longrightarrow \mathcal{P}(V) \setminus \{\emptyset\}, \qquad K : A \longrightarrow \mathcal{P}(E),$$

such that for every parameter $a \in A$, the following conditions hold:

(RS1) *(Parameter-vertex set)*

$$F(a) \subseteq V.$$

(RS2) *(Parameter-edge family)*

$$K(a) \subseteq E.$$

(RS3) *(Recursive closure over the selected vertices)* every selected edge is admissible as a depth-k recursive edge over $F(a)$:

$$K(a) \subseteq 2_{F(a),k} \setminus \{\emptyset\}.$$

Equivalently, each $e \in K(a)$ is a nonempty recursive object built from the selected supervertices $F(a)$ with recursion depth at most k.

For each $a \in A$, the pair

$$\mathrm{RSHG}_a^{(n,k)} := \big(F(a), K(a)\big)$$

is called the *parameter section* of $\mathcal{RS}\mathrm{SHG}^{(n,k)}$ at a.

Remark 3.5.2. Condition (RS3) is the soft analogue of the typed-incidence and well-foundedness requirements used in recursive fuzzy SuperHyperGraphs. Indeed, it guarantees that every parameter section is itself a legitimate depth-k recursive SuperHyperGraph over the selected vertex set.

Example 3.5.3 (A concrete $(1,1)$-recursive Soft SuperHyperGraph)**.** Let

$$V_0 = \{a, b, c\}, \qquad A = \{\alpha, \beta\}.$$

Since

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0),$$

define three 1-supervertices by

$$x := \{a\}, \qquad y := \{b\}, \qquad z := \{a, c\},$$

and set

$$V := \{x, y, z\}.$$

Then

$$V \subseteq \mathcal{P}(V_0) = \mathcal{P}^1(V_0).$$

Next, define two recursive superhyperedges by

$$e_1 := \{x, y\}, \qquad e_2 := \{e_1, z\},$$

and let

$$E := \{e_1, e_2\}.$$

For $k = 1$, we have

$$2_{V,1} = \mathcal{P}\big(V \cup \mathcal{P}(V)\big).$$

Since $e_1 \subseteq V$, it follows that

$$e_1 \in \mathcal{P}(V) \subseteq 2_{V,1}.$$

Also, because $e_1 \in \mathcal{P}(V)$ and $z \in V$, one has

$$e_2 = \{e_1, z\} \subseteq V \cup \mathcal{P}(V),$$

hence

$$e_2 \in 2_{V,1}.$$

Therefore,

$$E \subseteq 2_{V,1} \setminus \{\emptyset\},$$

so

$$\mathrm{RSHG}^{(1,1)} = (V, E)$$

is a $(1,1)$-recursive SuperHyperGraph.

Now define the soft maps

$$F : A \to \mathcal{P}(V) \setminus \{\emptyset\}, \qquad K : A \to \mathcal{P}(E),$$

by

$$F(\alpha) = \{x, y, z\}, \qquad K(\alpha) = \{e_1, e_2\},$$

and

$$F(\beta) = \{x, y\}, \qquad K(\beta) = \{e_1\}.$$

We verify the required conditions.

For the parameter α:

$$F(\alpha) = \{x, y, z\} \subseteq V, \qquad K(\alpha) = \{e_1, e_2\} \subseteq E.$$

Moreover,

$$e_1 \in 2_{F(\alpha),1} \setminus \{\emptyset\}, \qquad e_2 \in 2_{F(\alpha),1} \setminus \{\emptyset\},$$

because $e_1 = \{x, y\}$ is built from vertices in $F(\alpha)$, and

$$e_2 = \{e_1, z\}$$

is a depth-1 recursive edge over $F(\alpha)$. Hence

$$K(\alpha) \subseteq 2_{F(\alpha),1} \setminus \{\emptyset\}.$$

For the parameter β:

$$F(\beta) = \{x, y\} \subseteq V, \qquad K(\beta) = \{e_1\} \subseteq E.$$

Since

$$e_1 = \{x, y\}$$

is a nonempty recursive edge over $F(\beta)$, we have

$$e_1 \in 2_{F(\beta),1} \setminus \{\emptyset\},$$

and therefore

$$K(\beta) \subseteq 2_{F(\beta),1} \setminus \{\emptyset\}.$$

Thus,

$$\mathcal{RS}\mathrm{SHG}^{(1,1)} = \big(\mathrm{RSHG}^{(1,1)}, F, K, A\big)$$

is a Recursive Soft SuperHyperGraph.

Its parameter sections are

$$\mathrm{RSHG}^{(1,1)}_{\alpha} = \big(F(\alpha), K(\alpha)\big) = \big(\{x, y, z\}, \{e_1, e_2\}\big),$$

and

$$\mathrm{RSHG}^{(1,1)}_{\beta} = \big(F(\beta), K(\beta)\big) = \big(\{x, y\}, \{e_1\}\big).$$

In this example, the parameter α selects the full recursive structure, whereas the parameter β selects only the smaller non-recursive part generated by the edge e_1.

Theorem 3.5.4 (Well-definedness of parameter sections)**.** *Let*

$$\mathcal{RS}\mathrm{SHG}^{(n,k)} = \big(\mathrm{RSHG}^{(n,k)}, F, K, A\big)$$

be a Recursive Soft SuperHyperGraph in the sense of Definition 3.5.1. Then, for every $a \in A$, the parameter section

$$\mathrm{RSHG}^{(n,k)}_{a} = \big(F(a), K(a)\big)$$

is a well-defined (n, k)-recursive SuperHyperGraph on the same base set V_0.

Proof. Fix $a \in A$.

First, by (RS1),

$$F(a) \subseteq V.$$

Since (V, E) is an (n, k)-recursive SuperHyperGraph, one has

$$V \subseteq \mathcal{P}^n(V_0).$$

Therefore

$$F(a) \subseteq \mathcal{P}^n(V_0).$$

Hence the selected parameter-vertex set $F(a)$ is admissible as an n-supervertex set over V_0.

Second, by (RS3),

$$K(a) \subseteq 2_{F(a),k} \setminus \{\emptyset\}.$$

Thus every element of $K(a)$ is a nonempty depth-k recursive edge over the selected vertex set $F(a)$.

Combining these two facts, the pair

$$\big(F(a), K(a)\big)$$

satisfies exactly the defining conditions of an (n, k)-recursive SuperHyperGraph:

$$F(a) \subseteq \mathcal{P}^n(V_0), \qquad K(a) \subseteq 2_{F(a),k} \setminus \{\emptyset\}.$$

Hence

$$\mathrm{RSHG}_a^{(n,k)} = \big(F(a), K(a)\big)$$

is well-defined.

Since $a \in A$ was arbitrary, every parameter section is a well-defined (n, k)-recursive SuperHyperGraph. □

Corollary 3.5.5 (Well-definedness of the whole soft structure)**.** *Let*

$$\mathcal{RS}\mathrm{SHG}^{(n,k)} = \big(\mathrm{RSHG}^{(n,k)}, F, K, A\big)$$

be as in Definition 3.5.1. Then $\mathcal{RS}\mathrm{SHG}^{(n,k)}$ is a well-defined soft family of (n, k)-recursive SuperHyperGraphs indexed by the parameter set A.

Proof. By assumption, A is a nonempty set, and F and K are functions with domains A. By Theorem 3.5.4, for each $a \in A$, the image pair

$$\big(F(a), K(a)\big)$$

is a well-defined (n, k)-recursive SuperHyperGraph. Therefore

$$a \longmapsto \big(F(a), K(a)\big)$$

defines a well-defined parameterized family of recursive SuperHyperGraphs. Hence the whole soft structure is well-defined. □

Remark 3.5.6. The above definition is the natural recursive soft analogue of the usual idea of a soft graph or soft hypergraph: each parameter $a \in A$ selects a substructure. Here the selected substructure is not merely a subhypergraph, but a recursive SuperHyperGraph whose recursive depth remains bounded by k.

3.6 Recursive Rough SuperHyperGraph

We next introduce a rough-set version of recursive SuperHyperGraphs. The idea is to keep a fixed selected supervertex set W and to approximate a chosen recursive superhyperedge family X by means of an equivalence relation on the local recursive edge universe over W.

Definition 3.6.1 (Local recursive edge universe over a selected supervertex set)**.** Let

$$\mathrm{RSHG}^{(n,k)} = (V, E)$$

be an (n, k)-recursive SuperHyperGraph on the base set V_0, and let $W \subseteq V$.

Define the *local recursive edge universe over* W by

$$E[W] \;:=\; E \cap \big(2_{W,k} \setminus \{\emptyset\}\big).$$

Thus $E[W]$ consists precisely of those recursive superhyperedges of E that are admissible as depth-k recursive edges over the selected supervertex set W.

Definition 3.6.2 (Recursive Rough SuperHyperGraph)**.** Fix a finite nonempty base set V_0 and integers $n, k \in \mathbb{N} \cup \{0\}$. Let

$$U = \mathrm{RSHG}^{(n,k)} = (V, E)$$

be an (n, k)-recursive SuperHyperGraph on V_0.

Let $W \subseteq V$, and let

$$X \subseteq E[W] = E \cap \big(2_{W,k} \setminus \{\emptyset\}\big).$$

Suppose that R is an equivalence relation on the set $E[W]$. For each $e \in E[W]$, write $[e]_R$ for its R-equivalence class.

Define the *lower* and *upper recursive edge approximations* of X by

$$\underline{R}(X) \;:=\; \{\, e \in E[W] : [e]_R \subseteq X \,\}, \qquad \overline{R}(X) \;:=\; \{\, e \in E[W] : [e]_R \cap X \neq \emptyset \,\}.$$

Then the pair

$$\big(\underline{R}(T), \overline{R}(T)\big) \;:=\; \big((W, \underline{R}(X)),\ (W, \overline{R}(X))\big)$$

is called the *Recursive Rough SuperHyperGraph* associated with

$$T = (W, X)$$

with respect to the equivalence relation R.

If X is not a union of R-equivalence classes in $E[W]$, then $T = (W, X)$ is called an *R-rough recursive SuperHyperGraph*.

Remark 3.6.3. The definition above is the recursive analogue of the usual rough graph construction. The essential difference is that the approximated objects are not ordinary edges but recursive superhyperedges, so the ambient universe must be the local recursive edge universe $E[W]$, not merely the whole edge family E.

Example 3.6.4 (A concrete Recursive Rough SuperHyperGraph)**.** Let

$$V_0 = \{a, b, c\}, \qquad n = 1, \qquad k = 1.$$

Define three 1-supervertices by

$$x := \{a\}, \qquad y := \{b\}, \qquad z := \{a, c\},$$

and set

$$V := \{x, y, z\}.$$

Then

$$V \subseteq \mathcal{P}(V_0) = \mathcal{P}^1(V_0).$$

Next, define two recursive superhyperedges by

$$e_1 := \{x, y\}, \qquad e_2 := \{e_1, z\},$$

and let

$$E := \{e_1, e_2\}.$$

For $k = 1$, one has

$$2_{V,1} = \mathcal{P}\big(V \cup \mathcal{P}(V)\big).$$

Since $e_1 = \{x, y\} \subseteq V$, we obtain

$$e_1 \in \mathcal{P}(V) \subseteq 2_{V,1}.$$

Moreover, because $e_1 \in \mathcal{P}(V)$ and $z \in V$,

$$e_2 = \{e_1, z\} \subseteq V \cup \mathcal{P}(V),$$

and hence

$$e_2 \in 2_{V,1}.$$

Therefore,

$$E \subseteq 2_{V,1} \setminus \{\emptyset\},$$

so

$$U = \mathrm{RSHG}^{(1,1)} = (V, E)$$

is a $(1, 1)$-recursive SuperHyperGraph.

Now choose

$$W := \{x, y, z\} = V.$$

Then

$$E[W] = E \cap \big(2_{W,1} \setminus \{\emptyset\}\big) = E = \{e_1, e_2\}.$$

Let

$$X := \{e_1\} \subseteq E[W].$$

Define an equivalence relation R on $E[W]$ by declaring that e_1 and e_2 are equivalent; that is,

$$[e_1]_R = [e_2]_R = \{e_1, e_2\}.$$

We now compute the lower and upper recursive edge approximations of X.

Since

$$[e_1]_R = \{e_1, e_2\} \not\subseteq X, \qquad [e_2]_R = \{e_1, e_2\} \not\subseteq X,$$

it follows that

$$\underline{R}(X) = \{\, e \in E[W] : [e]_R \subseteq X \,\} = \emptyset.$$

On the other hand,

$$[e_1]_R \cap X = \{e_1\} \neq \emptyset, \qquad [e_2]_R \cap X = \{e_1\} \neq \emptyset,$$

so

$$\overline{R}(X) = \{\, e \in E[W] : [e]_R \cap X \neq \emptyset \,\} = \{e_1, e_2\}.$$

Hence the associated Recursive Rough SuperHyperGraph is

$$\big(\underline{R}(T), \overline{R}(T)\big) = \big((W, \underline{R}(X)), (W, \overline{R}(X))\big) = \big((W, \emptyset), (W, \{e_1, e_2\})\big),$$

where

$$T = (W, X) = \big(W, \{e_1\}\big).$$

Since $X = \{e_1\}$ is not a union of R-equivalence classes in $E[W]$, the pair $T = (W, X)$ is an R-rough recursive SuperHyperGraph.

Thus this gives a concrete example of a Recursive Rough SuperHyperGraph.

Theorem 3.6.5 (Well-definedness of Recursive Rough SuperHyperGraphs)**.** *Let*

$$U = \mathrm{RSHG}^{(n,k)} = (V, E)$$

be an (n, k)-recursive SuperHyperGraph on V_0, let $W \subseteq V$, let $X \subseteq E[W]$, and let R be an equivalence relation on $E[W]$. Then both

$$\underline{R}(T) = (W, \underline{R}(X)) \qquad \textit{and} \qquad \overline{R}(T) = (W, \overline{R}(X))$$

are well-defined (n, k)-recursive SuperHyperGraphs on V_0.

Consequently,

$$\big(\underline{R}(T), \overline{R}(T)\big)$$

is a well-defined Recursive Rough SuperHyperGraph.

Proof. Since $U = (V, E)$ is an (n, k)-recursive SuperHyperGraph, by definition one has

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq 2_{V,k} \setminus \{\emptyset\}.$$

Because $W \subseteq V$, it follows immediately that

$$W \subseteq V \subseteq \mathcal{P}^n(V_0),$$

so W is an admissible n-supervertex set over V_0.

Next, by Definition 3.6.1,
$$E[W] = E \cap \left(2_{W,k} \setminus \{\emptyset\}\right),$$
hence every element of $E[W]$ belongs to $2_{W,k} \setminus \{\emptyset\}$. Since R is an equivalence relation on $E[W]$, each equivalence class $[e]_R$ is a subset of $E[W]$.

By definition of lower approximation,
$$\underline{R}(X) = \{\, e \in E[W] : [e]_R \subseteq X \,\},$$
so clearly
$$\underline{R}(X) \subseteq E[W] \subseteq 2_{W,k} \setminus \{\emptyset\}.$$
Similarly, by definition of upper approximation,
$$\overline{R}(X) = \{\, e \in E[W] : [e]_R \cap X \neq \emptyset \,\},$$
and therefore
$$\overline{R}(X) \subseteq E[W] \subseteq 2_{W,k} \setminus \{\emptyset\}.$$

Thus both pairs
$$(W, \underline{R}(X)) \qquad \text{and} \qquad (W, \overline{R}(X))$$
satisfy the defining conditions of an (n, k)-recursive SuperHyperGraph: their vertex set W is contained in $\mathcal{P}^n(V_0)$, and their edge families are contained in $2_{W,k} \setminus \{\emptyset\}$.

Hence
$$\underline{R}(T) = (W, \underline{R}(X)) \quad \text{and} \quad \overline{R}(T) = (W, \overline{R}(X))$$
are both well-defined (n, k)-recursive SuperHyperGraphs. Therefore the pair
$$\left(\underline{R}(T), \overline{R}(T)\right)$$
is a well-defined Recursive Rough SuperHyperGraph. □

Corollary 3.6.6. *Under the assumptions of Theorem 3.6.5, if X is not a union of R-equivalence classes in $E[W]$, then $T = (W, X)$ is an R-rough recursive SuperHyperGraph.*

Proof. This follows directly from Definition 3.6.2, together with the well-definedness established in Theorem 3.6.5. □

Remark 3.6.7. When $k = 0$, one has
$$2_{W,0} = \mathcal{P}(W),$$
so the recursive construction reduces to the ordinary rough hypergraph-type situation on the selected supervertex set W. Thus Recursive Rough SuperHyperGraphs genuinely extend the non-recursive case.

3.7 Recursive Weighted SuperHyperGraph

We next introduce a weighted version of recursive SuperHyperGraphs. The underlying recursive combinatorial structure is retained, while nonnegative weights are assigned to supervertices and recursive superhyperedges.

Definition 3.7.1 (((n,k)-recursive Weighted SuperHyperGraph)**.** Fix a finite nonempty base (ground) set V_0 and integers $n, k \in \mathbb{N} \cup \{0\}$. Let

$$\mathrm{RSHG}^{(n,k)} = (V, E)$$

be an (n,k)-recursive SuperHyperGraph in the sense of Definition 3.1.3.

A (n,k)*-recursive Weighted SuperHyperGraph* is a quadruple

$$\mathrm{RWSHG}^{(n,k)} = (V, E, w_V, w_E),$$

where

$$w_V : V \to \mathbb{R}_{\geq 0}, \qquad w_E : E \to \mathbb{R}_{\geq 0},$$

satisfy the following additional conditions.

(W1) Edge-closure (typed incidence). Every element of every recursive superhyperedge is either a supervertex or a recursive superhyperedge:

$$\forall e \in E,\ \forall x \in e, \quad x \in V \text{ or } x \in E.$$

Equivalently,

$$e \subseteq V \cup E \qquad \text{for all } e \in E.$$

(W2) Well-foundedness and depth bound. Define a binary relation $\prec$ on E by

$$f \prec e \iff f \in e \qquad (f, e \in E).$$

We require that $\prec$ is acyclic and that every chain

$$e_0 \prec e_1 \prec \cdots \prec e_t$$

has length at most k. Equivalently, since E is finite, there exists a rank/height map

$$h : E \to \{0, 1, \ldots, k\}$$

such that

$$f \in e \cap E \implies h(f) < h(e) \qquad \text{for all } e, f \in E.$$

(W3) Vertex-weight map. The map

$$w_V : V \to \mathbb{R}_{\geq 0}$$

assigns a nonnegative weight to each recursive supervertex.

(W4) Edge-weight map. The map

$$w_E : E \to \mathbb{R}_{\geq 0}$$

assigns a nonnegative weight to each recursive superhyperedge.

The quantities

$$w_V(V) := \sum_{v \in V} w_V(v), \qquad w_E(E) := \sum_{e \in E} w_E(e),$$

and

$$w\Big(\mathrm{RWSHG}^{(n,k)}\Big) := w_V(V) + w_E(E)$$

are called, respectively, the *total vertex weight*, the *total edge weight*, and the *total weight* of $\mathrm{RWSHG}^{(n,k)}$.

Remark 3.7.2. The weighted setting does not require an admissibility inequality such as the fuzzy condition $\mu(e) \leq \min_{v \in e} \sigma(v)$. In the weighted case, the recursive structure is carried entirely by the pair (V, E), while the functions w_V and w_E provide quantitative data on vertices and edges.

Example 3.7.3 (A concrete $(1, 1)$-recursive Weighted SuperHyperGraph)**.** Let

$$V_0 = \{a, b, c\}.$$

Define three 1-supervertices by

$$x := \{a\}, \qquad y := \{b\}, \qquad z := \{a, c\},$$

and set

$$V := \{x, y, z\}.$$

Then

$$V \subseteq \mathcal{P}(V_0) = \mathcal{P}^1(V_0).$$

Next, define two recursive superhyperedges by

$$e_1 := \{x, y\}, \qquad e_2 := \{e_1, z\},$$

and let

$$E := \{e_1, e_2\}.$$

For $k = 1$, the depth-1 powerset universe over V is

$$2_{V,1} = \mathcal{P}\big(V \cup \mathcal{P}(V)\big).$$

Since $e_1 = \{x, y\} \subseteq V$, one has

$$e_1 \in \mathcal{P}(V) \subseteq 2_{V,1}.$$

Also, because $e_1 \in \mathcal{P}(V)$ and $z \in V$, it follows that

$$e_2 = \{e_1, z\} \subseteq V \cup \mathcal{P}(V),$$

and hence
$$e_2 \in 2_{V,1}.$$
Therefore,
$$E \subseteq 2_{V,1} \setminus \{\emptyset\},$$
so (V, E) is a $(1, 1)$-recursive SuperHyperGraph.

Now define the vertex-weight map
$$w_V : V \to \mathbb{R}_{\geq 0}$$
by
$$w_V(x) = 1.2, \qquad w_V(y) = 0.8, \qquad w_V(z) = 1.5,$$
and define the edge-weight map
$$w_E : E \to \mathbb{R}_{\geq 0}$$
by
$$w_E(e_1) = 2.0, \qquad w_E(e_2) = 2.7.$$

We verify the required conditions.

(W1) Edge-closure (typed incidence). For the edge $e_1 = \{x, y\}$, both elements x and y belong to V. For the edge $e_2 = \{e_1, z\}$, one element e_1 belongs to E and the other element z belongs to V. Hence every element of every recursive superhyperedge lies in $V \cup E$.

(W2) Well-foundedness and depth bound. The only recursive edge-membership relation among edges is
$$e_1 \prec e_2,$$
because $e_1 \in e_2$, while $e_2 \notin e_1$, $e_1 \notin e_1$, and $e_2 \notin e_2$. Thus $\prec$ is acyclic, and the longest chain has length 1. Equivalently, the rank map
$$h : E \to \{0, 1\}, \qquad h(e_1) = 0, \quad h(e_2) = 1,$$
satisfies
$$f \in e \cap E \Longrightarrow h(f) < h(e).$$

(W3)–(W4) Weight maps. By construction,
$$w_V : V \to \mathbb{R}_{\geq 0} \qquad \text{and} \qquad w_E : E \to \mathbb{R}_{\geq 0}$$
are well-defined maps assigning nonnegative weights to all recursive supervertices and recursive superhyperedges.

Therefore,
$$\mathrm{RWSHG}^{(1,1)} = (V, E, w_V, w_E)$$
is a concrete $(1, 1)$-recursive Weighted SuperHyperGraph.

Its total vertex weight is
$$w_V(V) = w_V(x) + w_V(y) + w_V(z) = 1.2 + 0.8 + 1.5 = 3.5,$$

its total edge weight is

$$w_E(E) = w_E(e_1) + w_E(e_2) = 2.0 + 2.7 = 4.7,$$

and its total weight is

$$w\Big(\mathrm{RWSHG}^{(1,1)}\Big) = w_V(V) + w_E(E) = 3.5 + 4.7 = 8.2.$$

In this example, e_1 is an ordinary recursive superhyperedge on V, whereas e_2 is genuinely recursive because it contains the lower-level edge e_1 as one of its elements.

Theorem 3.7.4 (Well-definedness of Recursive Weighted SuperHyperGraphs)**.** *Let*

$$\mathrm{RWSHG}^{(n,k)} = (V, E, w_V, w_E)$$

be an (n, k)-recursive Weighted SuperHyperGraph in the sense of Definition 3.7.1. Then:

(i) *the pair (V, E) is a well-defined (n, k)-recursive SuperHyperGraph;*

(ii) *the recursive incidence interpretation is well-founded and bounded by depth at most k;*

(iii) *the maps*

$$w_V : V \to \mathbb{R}_{\geq 0} \qquad \textit{and} \qquad w_E : E \to \mathbb{R}_{\geq 0}$$

are well-defined;

(iv) *the totals*

$$w_V(V), \qquad w_E(E), \qquad w\Big(\mathrm{RWSHG}^{(n,k)}\Big)$$

are well-defined nonnegative real numbers.

Proof. We verify the assertions one by one.

(i) By the definition of an (n, k)-recursive SuperHyperGraph,

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq 2_{V,k} \setminus \{\emptyset\}.$$

Hence (V, E) already satisfies the defining structural conditions of a recursive SuperHyperGraph. Therefore the pair (V, E) is well-defined.

(ii) By **(W1)**, every element appearing inside any recursive superhyperedge belongs either to V or to E. Thus the incidence relation has the intended two-sorted interpretation: superedges may contain supervertices and lower-level recursive superhyperedges, but no inadmissible object types.

By **(W2)**, the relation

$$f \prec e \iff f \in e \qquad (f, e \in E)$$

is acyclic, and every $\prec$-chain has length at most k. Equivalently, there exists a rank map $h : E \to \{0, 1, \dots, k\}$ with

$$f \in e \cap E \Longrightarrow h(f) < h(e).$$

Hence recursive containment among edges is well-founded and has bounded depth. Therefore the recursive incidence interpretation is mathematically sound.

(iii) By **(W3)** and **(W4)**, w_V and w_E are functions with domains V and E, respectively, and codomain $\mathbb{R}_{\geq 0}$. Thus they are well-defined maps assigning a unique nonnegative real number to each recursive supervertex and each recursive superhyperedge.

(iv) Since $V \subseteq \mathcal{P}^n(V_0)$ and V_0 is finite, the set V is finite. Because $2_{V,k}$ is constructed by finitely many powerset operations starting from the finite set V, the set $2_{V,k}$ is finite; hence $E \subseteq 2_{V,k} \setminus \{\emptyset\}$ is finite as well.

Therefore

$$\sum_{v \in V} w_V(v) \qquad \text{and} \qquad \sum_{e \in E} w_E(e)$$

are finite sums of nonnegative real numbers, so both belong to $\mathbb{R}_{\geq 0}$ and are well-defined. Their sum

$$w\Big(\mathrm{RWSHG}^{(n,k)}\Big) = w_V(V) + w_E(E)$$

is therefore also a well-defined nonnegative real number.

This proves all four assertions. □

Corollary 3.7.5 (Underlying crisp recursive structure)**.** *Every (n, k)-recursive Weighted SuperHyperGraph*

$$\mathrm{RWSHG}^{(n,k)} = (V, E, w_V, w_E)$$

has a canonical underlying crisp recursive SuperHyperGraph obtained by forgetting the weight maps:

$$\mathsf{Und}\Big(\mathrm{RWSHG}^{(n,k)}\Big) := (V, E).$$

Proof. Immediate from Theorem 3.7.4(i). □

Chapter 4

Hierarchical SuperHyperGraph and Related Concepts

In this chapter, we examine Hierarchical SuperHyperGraphs and related concepts.

4.1 Hierarchical Undirected SuperHyperGraphs

A hierarchical SuperHyperGraph allows vertices drawn from several iterated-powerset levels and permits *mixed-level* edges, while imposing a coherence requirement that guarantees level-to-level consistency.

Definition 4.1.1 (Hierarchical SuperHyperGraph of height r)**.** Let V_0 be a finite nonempty base set, and let $r \in \mathbb{N}_0$. Let $\mathcal{U}_r(V_0)$ be the hierarchical universe from Definition 2.1.5, and let ℓ be the level map from Definition 2.1.6.

A *hierarchical SuperHyperGraph of height* r on V_0 is a pair

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

satisfying the following axioms:

(H1) *Hierarchical supervertex set.* V is a finite nonempty set such that

$$V \subseteq \mathcal{U}_r(V_0).$$

(H2) *Coherence / downward closure.* Whenever $X \in V$ and $\ell(X) \geq 1$, every immediate constituent of X also belongs to V. Equivalently,

$$X \subseteq V.$$

(H3) *Mixed-level superhyperedge family.* E is a finite family of nonempty subsets of V, that is,

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

The elements of E are called *hierarchical superhyperedges.* A hierarchical superhyperedge may contain supervertices from different levels.

Theorem 4.1.2 (Well-definedness of hierarchical SuperHyperGraphs)**.** *Let*

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

satisfy Definition 4.1.1. Then every edge $e \in E$ is a well-defined finite nonempty set of admissible hierarchical supervertices, and the pair (V, E) forms a coherent mixed-level incidence structure over V_0.

Proof. Since $V \subseteq \mathcal{U}_r(V_0)$, every element of V is an admissible hierarchical supervertex of level at most r. Since $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, every edge $e \in E$ is a nonempty subset of V, hence a finite nonempty collection of admissible hierarchical supervertices. The downward-closure axiom guarantees that whenever a selected supervertex has immediate constituents, those constituents are also present in V. Therefore the incidence structure is well-defined and coherent across levels. □

Example 4.1.3 (A concrete hierarchical SuperHyperGraph of height 2)**.** Let

$$V_0 = \{a, b\}, \qquad r = 2.$$

Then

$$\mathcal{P}^{\langle 0\rangle}(V_0) = V_0 = \{a, b\},$$
$$\mathcal{P}^{\langle 1\rangle}(V_0) = \mathcal{P}(V_0) \setminus \{\emptyset\} = \big\{\{a\}, \{b\}, \{a, b\}\big\},$$

and

$$\mathcal{P}^{\langle 2\rangle}(V_0) = \mathcal{P}\big(\mathcal{P}^{\langle 1\rangle}(V_0)\big) \setminus \{\emptyset\}.$$

Define the following hierarchical supervertices:

$$x_0 := a, \qquad x_1 := b,$$
$$x_2 := \{a\}, \qquad x_3 := \{b\}, \qquad x_4 := \{a, b\},$$

and

$$x_5 := \big\{\{a\}, \{a, b\}\big\}.$$

Then

$$x_0, x_1 \in \mathcal{P}^{\langle 0\rangle}(V_0), \qquad x_2, x_3, x_4 \in \mathcal{P}^{\langle 1\rangle}(V_0), \qquad x_5 \in \mathcal{P}^{\langle 2\rangle}(V_0).$$

Now set

$$V := \{x_0, x_1, x_2, x_3, x_4, x_5\}.$$

Clearly,

$$V \subseteq \mathcal{U}_2(V_0) = \mathcal{P}^{\langle 0\rangle}(V_0) \cup \mathcal{P}^{\langle 1\rangle}(V_0) \cup \mathcal{P}^{\langle 2\rangle}(V_0).$$

Next, define the edge family by

$$E := \Big\{\{x_0, x_2\}, \{x_2, x_4, x_5\}, \{x_1, x_3\}, \{x_0, x_4, x_5\}\Big\}.$$

Each element of E is a nonempty subset of V, so

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

We now verify the downward-closure condition. Since

$$x_2 = \{a\} = \{x_0\}, \qquad x_3 = \{b\} = \{x_1\}, \qquad x_4 = \{a, b\} = \{x_0, x_1\},$$

their immediate constituents are all contained in V. Also,

$$x_5 = \big\{\{a\}, \{a, b\}\big\} = \{x_2, x_4\},$$

and both x_2 and x_4 belong to V. Hence, whenever $X \in V$ has level at least 1, all immediate constituents of X are again elements of V.

Therefore,

$$\mathbb{H}^{\langle 2\rangle} = (V, E)$$

is a hierarchical SuperHyperGraph of height 2.

Its layers are

$$V_0 = \{x_0, x_1\} = \{a, b\},$$
$$V_1 = \{x_2, x_3, x_4\} = \big\{\{a\}, \{b\}, \{a, b\}\big\},$$

and

$$V_2 = \{x_5\} = \Big\{\big\{\{a\}, \{a, b\}\big\}\Big\}.$$

This example is genuinely hierarchical, since it contains vertices from three different levels and edges such as

$$\{x_2, x_4, x_5\},$$

which connects level-1 and level-2 vertices in a single hierarchical superhyperedge.

4.2 Fuzzy Hierarchical SuperHyperGraphs

Fuzzy Hierarchical SuperHyperGraphs model multi-level, nested, set-valued vertices and cross-level hyperedges, assigning membership grades to vertices and edges, capturing uncertainty while preserving hierarchical closure.

Definition 4.2.1 (Hierarchical universe and level)**.** Let V_0 be a finite nonempty base set and let $r \in \mathbb{N}$. Define the iterated powersets by

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0),$$

and the hierarchical universe of height r by

$$\mathcal{U}_r(V_0) \;:=\; \bigcup_{k=0}^{r} \Big(\mathcal{P}^k(V_0) \setminus \{\emptyset\}\Big).$$

For $x \in \mathcal{U}_r(V_0)$, define its *level* by

$$\ell(x) \;:=\; \min\{\, k \in \{0, 1, \ldots, r\} :\; x \in \mathcal{P}^k(V_0)\,\}.$$

Definition 4.2.2 (Fuzzy Hierarchical SuperHyperGraph of height r)**.** Let V_0 be a finite nonempty base set and fix $r \in \mathbb{N}$. A *fuzzy hierarchical superhypergraph of height* r on V_0 is a quadruple

$$\widetilde{\mathbb{H}}^{\langle r\rangle} = \big(V,\ E,\ \mu_V,\ \mu_E\big)$$

satisfying the following axioms:

(FH1) **(Hierarchical supervertex set)** V is a finite nonempty set with

$$V \subseteq \mathcal{U}_r(V_0),$$

whose elements are called *hierarchical supervertices*.

(FH2) **(Downward closure / coherence)** If $X \in V$ and $\ell(X) \geq 1$, then every immediate constituent of X is also a supervertex, i.e.,

$$X \subseteq V.$$

(Equivalently, V is downward closed under the membership relation across levels.)

(FH3) **(Superhyperedge family)** E is a finite family of nonempty subsets of V:

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Elements of E are called *hierarchical superhyperedges*; they may join mixed-level supervertices.

(FH4) **(Fuzziness)** $\mu_V : V \to [0,1]$ and $\mu_E : E \to [0,1]$ are membership maps (vertex- and edge-memberships), subject to the standard consistency condition

$$\mu_E(e) \leq \min_{x\in e} \mu_V(x) \qquad (e \in E).$$

We write $V_k := \{x \in V : \ell(x) = k\}$ for the k-th layer $(0 \leq k \leq r)$.

Example 4.2.3 (A concrete fuzzy hierarchical SuperHyperGraph of height 2)**.** Let

$$V_0 = \{a, b\}, \qquad r = 2.$$

Then

$$\mathcal{U}_2(V_0) = \mathcal{P}^{\langle 0\rangle}(V_0) \cup \mathcal{P}^{\langle 1\rangle}(V_0) \cup \mathcal{P}^{\langle 2\rangle}(V_0),$$

where

$$\mathcal{P}^{\langle 0\rangle}(V_0) = \{a, b\},$$
$$\mathcal{P}^{\langle 1\rangle}(V_0) = \mathcal{P}(V_0) \setminus \{\emptyset\} = \big\{\{a\}, \{b\}, \{a, b\}\big\},$$

and

$$\mathcal{P}^{\langle 2\rangle}(V_0) = \mathcal{P}\big(\mathcal{P}^{\langle 1\rangle}(V_0)\big) \setminus \{\emptyset\}.$$

Define the following hierarchical supervertices:

$$x_0 := a, \qquad x_1 := b,$$

$$x_2 := \{a\}, \qquad x_3 := \{b\}, \qquad x_4 := \{a, b\},$$

and

$$x_5 := \big\{\{a\}, \{a, b\}\big\}.$$

Set

$$V := \{x_0, x_1, x_2, x_3, x_4, x_5\}.$$

Then

$$V \subseteq \mathcal{U}_2(V_0).$$

Moreover, the downward-closure condition holds, because

$$x_2 = \{x_0\}, \qquad x_3 = \{x_1\}, \qquad x_4 = \{x_0, x_1\}, \qquad x_5 = \{x_2, x_4\},$$

and all immediate constituents x_0, x_1, x_2, x_4 belong to V.

Now define the hierarchical superhyperedge family by

$$E := \Big\{e_1, e_2, e_3\Big\},$$

where

$$e_1 := \{x_0, x_2\}, \qquad e_2 := \{x_2, x_4, x_5\}, \qquad e_3 := \{x_1, x_3, x_5\}.$$

Clearly, each e_i is a nonempty subset of V, so

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Next, define the fuzzy vertex-membership map

$$\mu_V : V \to [0, 1]$$

by

$$\mu_V(x_0) = 0.9, \qquad \mu_V(x_1) = 0.8,$$
$$\mu_V(x_2) = 0.7, \qquad \mu_V(x_3) = 0.6, \qquad \mu_V(x_4) = 0.75, \qquad \mu_V(x_5) = 0.5.$$

Define the fuzzy edge-membership map

$$\mu_E : E \to [0, 1]$$

by

$$\mu_E(e_1) = 0.65, \qquad \mu_E(e_2) = 0.45, \qquad \mu_E(e_3) = 0.40.$$

We verify the consistency condition.

For

$$e_1 = \{x_0, x_2\},$$

we have

$$\min\{\mu_V(x_0), \mu_V(x_2)\} = \min\{0.9, 0.7\} = 0.7,$$

so

$$\mu_E(e_1) = 0.65 \le 0.7.$$

For

$$e_2 = \{x_2, x_4, x_5\},$$

we have

$$\min\{\mu_V(x_2), \mu_V(x_4), \mu_V(x_5)\} = \min\{0.7, 0.75, 0.5\} = 0.5,$$

so

$$\mu_E(e_2) = 0.45 \leq 0.5.$$

For

$$e_3 = \{x_1, x_3, x_5\},$$

we have

$$\min\{\mu_V(x_1), \mu_V(x_3), \mu_V(x_5)\} = \min\{0.8, 0.6, 0.5\} = 0.5,$$

so

$$\mu_E(e_3) = 0.40 \leq 0.5.$$

Therefore,

$$\widetilde{\mathbb{H}}^{\langle 2\rangle} = \big(V, E, \mu_V, \mu_E\big)$$

is a fuzzy hierarchical SuperHyperGraph of height 2.

Its layers are

$$V_0 = \{x_0, x_1\}, \qquad V_1 = \{x_2, x_3, x_4\}, \qquad V_2 = \{x_5\}.$$

This example is genuinely hierarchical and fuzzy, since it contains vertices from different levels and assigns compatible membership degrees to both vertices and mixed-level superhyperedges.

4.3 Hierarchical Directed SuperHyperGraphs

A hierarchical directed SuperHyperGraph of height r uses supervertices from powerset levels r, enforces downward closure, and allows directed superedges between mixed-level supervertices.

Definition 4.3.1 (Hierarchical directed SuperHyperGraph of height r)**.** Let V_0 be a finite, nonempty base set. For $k \geq 0$ define iterated powersets

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big),$$

and fix an integer $r \geq 0$. Set the hierarchical universe

$$\mathcal{U}_r(V_0) \ := \ \bigcup_{k=0}^{r} \Big(\mathcal{P}^k(V_0) \setminus \{\emptyset\}\Big).$$

For $x \in \mathcal{U}_r(V_0)$ define its level

$$\ell(x) \ := \ \min\{\, k \in \{0, 1, \ldots, r\} : x \in \mathcal{P}^k(V_0)\,\}.$$

A *hierarchical directed SuperHyperGraph of height* r on V_0 is a quadruple

$$\mathbb{H}_{\to}^{\langle r\rangle} = (V, E, \partial^-, \partial^+)$$

such that:

(HD1) (Hierarchical vertex set) V is a finite nonempty set with

$$V \subseteq \mathcal{U}_r(V_0).$$

(HD2) (Directed cross-level superedges) E is a finite nonempty set of directed superedge identifiers and

$$\partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}$$

are tail/head incidence maps. A directed superedge e may join supervertices of *different* levels.

(HD3) (Coherence / downward closure) If $X \in V$ and $\ell(X) \geq 1$, then

$$X \subseteq V.$$

Remark 4.3.2 (Underlying directed hypergraph)**.** Forgetting the level function and viewing vertices simply as elements of a set, any $\mathbb{H}_{\rightarrow}^{\langle r \rangle} = (V, E, \partial^-, \partial^+)$ is a directed hypergraph in incidence form (tails/heads are nonempty subsets of V).

Example 4.3.3 (A concrete hierarchical directed SuperHyperGraph of height 2)**.** Let

$$V_0 = \{a, b\}, \qquad r = 2.$$

Then

$$\mathcal{U}_2(V_0) = \big(\mathcal{P}^0(V_0) \setminus \{\emptyset\}\big) \cup \big(\mathcal{P}^1(V_0) \setminus \{\emptyset\}\big) \cup \big(\mathcal{P}^2(V_0) \setminus \{\emptyset\}\big).$$

Define the following hierarchical supervertices:

$$x_0 := a, \qquad x_1 := b,$$

$$x_2 := \{a\}, \qquad x_3 := \{b\}, \qquad x_4 := \{a, b\},$$

and

$$x_5 := \big\{\{a\}, \{a, b\}\big\}.$$

Set

$$V := \{x_0, x_1, x_2, x_3, x_4, x_5\}.$$

Then

$$x_0, x_1 \in \mathcal{P}^0(V_0), \qquad x_2, x_3, x_4 \in \mathcal{P}^1(V_0) \setminus \{\emptyset\}, \qquad x_5 \in \mathcal{P}^2(V_0) \setminus \{\emptyset\},$$

so

$$V \subseteq \mathcal{U}_2(V_0).$$

Now let

$$E := \{e_1, e_2, e_3\}$$

be a set of directed superedge identifiers. Define the tail and head maps

$$\partial^-, \partial^+ : E \to \mathcal{P}(V) \setminus \{\emptyset\}$$

by

$$\partial^-(e_1) = \{x_0, x_2\}, \qquad \partial^+(e_1) = \{x_4\},$$
$$\partial^-(e_2) = \{x_5\}, \qquad \partial^+(e_2) = \{x_1, x_3\},$$

and

$$\partial^-(e_3) = \{x_2, x_4\}, \qquad \partial^+(e_3) = \{x_5\}.$$

Each tail and head is a nonempty subset of V, so the incidence maps are well-defined.

We next verify the coherence condition. Since

$$x_2 = \{x_0\}, \qquad x_3 = \{x_1\}, \qquad x_4 = \{x_0, x_1\},$$

and

$$x_5 = \{x_2, x_4\},$$

every immediate constituent of every vertex of level at least 1 also belongs to V. Hence the downward-closure condition holds.

Therefore,

$$\mathbb{H}^{\langle 2 \rangle}_{\rightarrow} = (V, E, \partial^-, \partial^+)$$

is a hierarchical directed SuperHyperGraph of height 2.

In this example, the directed superedge e_3 is genuinely hierarchical, since it goes from the level-1 vertices x_2 and x_4 to the level-2 vertex x_5. Thus the structure contains directed cross-level incidence.

4.4 Hierarchical BiDirected SuperHyperGraphs

Hierarchical bidirected superhypergraphs are multi-level bidirected superhypergraphs linked by parent maps, preserving edge projections and balanced incidence signs.

Definition 4.4.1 (Hierarchical bidirected n-SuperHyperGraph)**.** Let S be a nonempty base set and let $n \geq 1$ be an integer. Define the iterated powersets by

$$\mathcal{P}^0(S) := S, \qquad \mathcal{P}^{k+1}(S) := \mathcal{P}\big(\mathcal{P}^k(S)\big) \quad (k \geq 0),$$

and write $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$ for the nonempty powerset.

A *hierarchical bidirected n-SuperHyperGraph of height $L \in \mathbb{N}_0$* is a tuple

$$\mathcal{H}^{(n)}_{\text{hier}} = \Big((V_\ell, E_\ell, \tau_\ell)_{\ell=0}^{L},\ (\pi_\ell)_{\ell=1}^{L}\Big),$$

satisfying the following axioms.

(H1) (Bidirected structure at each level) For every level $\ell \in \{0, 1, \ldots, L\}$,

$$V_\ell \subseteq \mathcal{P}^n(S) \text{ is nonempty}, \qquad E_\ell \subseteq \mathcal{P}^*(V_\ell),$$

and $\tau_\ell : V_\ell \times E_\ell \to \{-1, 0, 1\}$ satisfies

$$\tau_\ell(v, e) = 0 \iff v \notin e, \qquad \sum_{v \in e} \tau_\ell(v, e) = 0 \quad (\forall\, e \in E_\ell).$$

(H2) (Vertex refinement / hierarchy) For every $\ell \in \{1, \ldots, L\}$, $\pi_\ell : V_\ell \twoheadrightarrow V_{\ell-1}$ is a surjection (the *parent map*) such that

$$v \subseteq \pi_\ell(v) \quad (\forall\, v \in V_\ell), \qquad u = \bigcup_{v \in \pi_\ell^{-1}(u)} v \quad (\forall\, u \in V_{\ell-1}).$$

Thus, each coarse n-supervertex is the union of its finer children.

(H3) (Edge compatibility across adjacent levels) For every $\ell \in \{1, \ldots, L\}$ and every $e \in E_\ell$, the projected set

$$\pi_\ell(e) := \{\pi_\ell(v) \mid v \in e\} \subseteq V_{\ell-1}$$

belongs to $E_{\ell-1}$, and the incidence signs are consistent:

$$\tau_\ell(v, e) = \tau_{\ell-1}\big(\pi_\ell(v),\, \pi_\ell(e)\big) \quad (\forall\, v \in V_\ell,\ e \in E_\ell).$$

Example 4.4.2 (A concrete hierarchical bidirected 1-SuperHyperGraph of height 1)**.** Let

$$S = \{a, b, c, d\}, \qquad n = 1, \qquad L = 1.$$

Since $n = 1$, we have

$$\mathcal{P}^1(S) = \mathcal{P}(S).$$

Define the coarse-level vertex set by

$$V_0 := \big\{\{a, b\}, \{c, d\}\big\}.$$

Let

$$u_1 := \{a, b\}, \qquad u_2 := \{c, d\}.$$

Now define the finer-level vertex set by

$$V_1 := \big\{\{a\}, \{b\}, \{c\}, \{d\}\big\}.$$

Write

$$v_1 := \{a\}, \qquad v_2 := \{b\}, \qquad v_3 := \{c\}, \qquad v_4 := \{d\}.$$

Clearly,

$$V_0 \subseteq \mathcal{P}^1(S), \qquad V_1 \subseteq \mathcal{P}^1(S).$$

Next, define the edge families

$$E_0 := \{e_0\}, \qquad E_1 := \{e_1, e_2\},$$

where

$$e_0 := \{u_1, u_2\}, \qquad e_1 := \{v_1, v_3\}, \qquad e_2 := \{v_2, v_4\}.$$

Each edge is a nonempty subset of the corresponding vertex set, so

$$E_0 \subseteq \mathcal{P}^*(V_0), \qquad E_1 \subseteq \mathcal{P}^*(V_1).$$

Define the bidirected incidence maps

$$\tau_0 : V_0 \times E_0 \to \{-1, 0, 1\}, \qquad \tau_1 : V_1 \times E_1 \to \{-1, 0, 1\},$$

by

$$\tau_0(u_1, e_0) = 1, \qquad \tau_0(u_2, e_0) = -1,$$

and $\tau_0(v, e) = 0$ for all other pairs $(v, e) \in V_0 \times E_0$.

At level 1, define

$$\tau_1(v_1, e_1) = 1, \qquad \tau_1(v_3, e_1) = -1,$$
$$\tau_1(v_2, e_2) = 1, \qquad \tau_1(v_4, e_2) = -1,$$

and set $\tau_1(v, e) = 0$ for all remaining pairs $(v, e) \in V_1 \times E_1$.

Then, for each edge, the sign sum is balanced:

$$\sum_{v \in e_0} \tau_0(v, e_0) = 1 + (-1) = 0,$$

$$\sum_{v \in e_1} \tau_1(v, e_1) = 1 + (-1) = 0, \qquad \sum_{v \in e_2} \tau_1(v, e_2) = 1 + (-1) = 0.$$

Now define the parent map

$$\pi_1 : V_1 \twoheadrightarrow V_0$$

by

$$\pi_1(v_1) = u_1, \qquad \pi_1(v_2) = u_1, \qquad \pi_1(v_3) = u_2, \qquad \pi_1(v_4) = u_2.$$

This is surjective. Moreover,

$$v_1 \subseteq u_1, \quad v_2 \subseteq u_1, \quad v_3 \subseteq u_2, \quad v_4 \subseteq u_2,$$

and

$$u_1 = v_1 \cup v_2 = \bigcup_{v \in \pi_1^{-1}(u_1)} v, \qquad u_2 = v_3 \cup v_4 = \bigcup_{v \in \pi_1^{-1}(u_2)} v.$$

Hence the vertex-refinement condition holds.

Finally, check edge compatibility across the two levels. For the edge $e_1 = \{v_1, v_3\}$,

$$\pi_1(e_1) = \{\pi_1(v_1), \pi_1(v_3)\} = \{u_1, u_2\} = e_0 \in E_0.$$

Similarly, for $e_2 = \{v_2, v_4\}$,

$$\pi_1(e_2) = \{\pi_1(v_2), \pi_1(v_4)\} = \{u_1, u_2\} = e_0 \in E_0.$$

Moreover, the incidence signs are preserved:

$$\tau_1(v_1, e_1) = 1 = \tau_0(u_1, e_0), \qquad \tau_1(v_3, e_1) = -1 = \tau_0(u_2, e_0),$$

and likewise

$$\tau_1(v_2, e_2) = 1 = \tau_0(u_1, e_0), \qquad \tau_1(v_4, e_2) = -1 = \tau_0(u_2, e_0).$$

Therefore,

$$\mathcal{H}^{(1)}_{\text{hier}} = \Big((V_\ell, E_\ell, \tau_\ell)_{\ell=0}^{1}, \ (\pi_1) \Big)$$

is a hierarchical bidirected 1-SuperHyperGraph of height 1.

In this example, the coarse edge e_0 is refined into two finer edges, e_1 and e_2, while the bidirected incidence signs remain consistent under the parent projection.

4.5 Hierarchical Uncertain SuperHyperGraph

We next introduce a model-independent uncertainty extension of hierarchical SuperHyperGraphs.

Definition 4.5.1 (Hierarchical Uncertain SuperHyperGraph of height r)**.** Let V_0 be a finite nonempty base set, let $r \in \mathbb{N}_0$, and let M be a fixed uncertain model with degree-domain $\mathrm{Dom}(M)$.

A *Hierarchical Uncertain SuperHyperGraph of height* r on V_0 (of type M) is a quadruple

$$\mathbb{UH}_M^{\langle r \rangle} = \big(V, E, \mu_V, \mu_E\big)$$

satisfying the following axioms:

(UH1) *(Hierarchical supervertex set)* V is a finite nonempty set such that

$$V \subseteq \mathcal{U}_r(V_0),$$

where

$$\mathcal{U}_r(V_0) := \bigcup_{k=0}^{r} \Big(\mathcal{P}^k(V_0) \setminus \{\emptyset\}\Big).$$

The elements of V are called *hierarchical supervertices.*

(UH2) *(Downward closure / coherence)* Whenever $X \in V$ has level at least 1, all of its immediate constituents also belong to V; that is,

$$X \subseteq V.$$

(UH3) *(Hierarchical superhyperedge family)* E is a finite family of nonempty subsets of V:

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

The elements of E are called *hierarchical superhyperedges.* In particular, one edge may join supervertices belonging to different levels.

(UH4) *(Uncertainty-degree assignment on vertices)*

$$\mu_V : V \longrightarrow \mathrm{Dom}(M)$$

is a function assigning to each hierarchical supervertex an admissible uncertainty degree of type M.

(UH5) *(Uncertainty-degree assignment on edges)*

$$\mu_E : E \longrightarrow \mathrm{Dom}(M)$$

is a function assigning to each hierarchical superhyperedge an admissible uncertainty degree of type M.

For each $k \in \{0, \dots, r\}$, the k-th layer is defined by

$$V_k := \{ x \in V : \ell(x) = k \},$$

where

$$\ell(x) := \min\{ k \in \{0, 1, \dots, r\} : x \in \mathcal{P}^k(V_0) \}.$$

We call (V, E) the *underlying crisp hierarchical SuperHyperGraph* of $\mathbb{UH}_M^{\langle r \rangle}$.

Remark 4.5.2. Definition 4.5.1 is the natural hierarchical analogue of the general uncertain-set viewpoint. Unlike the fuzzy case, no universal inequality such as

$$\mu_E(e) \le \min_{x \in e} \mu_V(x)$$

is imposed here, because a general uncertain model M need not carry a canonical order or minimum operation. If the chosen uncertain model has additional algebraic structure, one may impose further model-specific compatibility conditions.

Definition 4.5.3 (Typed uncertainty map)**.** Let

$$\mathbb{UH}_M^{\langle r \rangle} = (V, E, \mu_V, \mu_E)$$

be a Hierarchical Uncertain SuperHyperGraph of height r. Define the typed disjoint union

$$V \sqcup E := (V \times \{0\}) \cup (E \times \{1\}),$$

and define the *typed uncertainty map*

$$\widehat{\mu} : V \sqcup E \longrightarrow \mathrm{Dom}(M)$$

by

$$\widehat{\mu}(x) := \begin{cases} \mu_V(v), & x = (v, 0) \in V \times \{0\}, \\ \mu_E(e), & x = (e, 1) \in E \times \{1\}. \end{cases}$$

Theorem 4.5.4 (Well-definedness of Hierarchical Uncertain SuperHyperGraphs)**.** *Let*

$$\mathbb{UH}_M^{\langle r \rangle} = (V, E, \mu_V, \mu_E)$$

be a Hierarchical Uncertain SuperHyperGraph of height r in the sense of Definition 4.5.1. Then:

(i) *the pair*

$$\mathbb{H}^{\langle r \rangle} := (V, E)$$

is a well-defined hierarchical SuperHyperGraph of height r;

(ii) *the maps*

$$\mu_V : V \to \mathrm{Dom}(M) \qquad \text{and} \qquad \mu_E : E \to \mathrm{Dom}(M)$$

are well-defined uncertainty-degree assignments;

(iii) *the typed disjoint union*

$$V \sqcup E = (V \times \{0\}) \cup (E \times \{1\})$$

is a well-defined finite set;

(iv) *the typed uncertainty map*

$$\widehat{\mu} : V \sqcup E \to \mathrm{Dom}(M)$$

from Definition 4.5.3 is a well-defined function.

Hence $\mathbb{UH}_M^{\langle r\rangle}$ *is a well-defined mathematical structure.*

Proof. We verify each assertion.

(i) By (UH1), V is a finite nonempty subset of $\mathcal{U}_r(V_0)$. By (UH3), E is a finite family of nonempty subsets of V, that is,

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

By (UH2), whenever $X \in V$ has level at least 1, all immediate constituents of X belong to V. Therefore (V, E) satisfies exactly the axioms of a hierarchical SuperHyperGraph of height r. Hence

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

is well-defined.

(ii) By (UH4) and (UH5), μ_V and μ_E are functions with codomain $\mathrm{Dom}(M)$. Thus every hierarchical supervertex and every hierarchical superhyperedge is assigned a unique admissible uncertainty degree of type M. Hence both maps are well-defined.

(iii) Since V and E are finite sets, the Cartesian products $V \times \{0\}$ and $E \times \{1\}$ are finite. Moreover,

$$(V \times \{0\}) \cap (E \times \{1\}) = \emptyset,$$

because their second coordinates differ. Therefore

$$V \sqcup E = (V \times \{0\}) \cup (E \times \{1\})$$

is a well-defined finite set.

(iv) Let $x \in V \sqcup E$. Since $V \sqcup E$ is a disjoint union, either $x = (v, 0)$ for a unique $v \in V$, or $x = (e, 1)$ for a unique $e \in E$, but not both. Hence the case distinction in Definition 4.5.3 assigns exactly one value $\widehat{\mu}(x) \in \mathrm{Dom}(M)$ to each $x \in V \sqcup E$. Therefore

$$\widehat{\mu} : V \sqcup E \to \mathrm{Dom}(M)$$

is a well-defined function.

Combining (i)–(iv), we conclude that

$$\mathbb{UH}_M^{\langle r\rangle} = (V, E, \mu_V, \mu_E)$$

is a well-defined Hierarchical Uncertain SuperHyperGraph of height r. □

Corollary 4.5.5 (Underlying crisp hierarchical structure)**.** *Every Hierarchical Uncertain SuperHyperGraph*

$$\mathbb{UH}_M^{\langle r\rangle} = (V, E, \mu_V, \mu_E)$$

has a canonical underlying crisp hierarchical SuperHyperGraph obtained by forgetting the uncertainty-degree maps:

$$\mathsf{Und}\big(\mathbb{UH}_M^{\langle r\rangle}\big) := (V, E).$$

Proof. Immediate from Theorem 4.5.4(i). □

4.6 Hierarchical Soft SuperHyperGraph

Parameter-dependent hierarchical superhypergraph where each parameter selects a downward-closed multi-level supervertex set together with compatible superhyperedges, yielding a family of structured subconfigurations within one framework.

Definition 4.6.1 (Hierarchical Soft SuperHyperGraph of height r)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r on the base set V_0 in the sense of the Definition, and let A be a nonempty set of parameters.

A *hierarchical soft SuperHyperGraph of height* r over $\mathbb{H}^{\langle r\rangle}$ is a quadruple

$$\mathcal{S}^{\langle r\rangle} = \big(\mathbb{H}^{\langle r\rangle}, F, K, A\big),$$

where

$$F : A \longrightarrow \mathcal{P}(V) \setminus \{\emptyset\}, \qquad K : A \longrightarrow \mathcal{P}(E),$$

such that, for every $a \in A$, the following conditions hold:

(HS1) *(Hierarchical parameter-vertex set)*

$$F(a) \subseteq V.$$

(HS2) *(Downward closure)* whenever $X \in F(a)$ and $\ell(X) \geq 1$, all immediate constituents of X also belong to $F(a)$, i.e.,

$$X \subseteq F(a).$$

(HS3) *(Parameter-edge set)*

$$K(a) \subseteq \{\, e \in E : e \subseteq F(a) \,\}.$$

Equivalently, every edge selected by the parameter a is a hierarchical superhyperedge whose incident vertices all belong to $F(a)$.

For each $a \in A$, the pair

$$\mathbb{H}_a^{\langle r \rangle} := \big(F(a), K(a)\big)$$

is called the *parameter section* of $\mathcal{S}^{\langle r \rangle}$ at a.

Theorem 4.6.2 (Well-definedness of parameter sections)**.** *Let*

$$\mathcal{S}^{\langle r \rangle} = \big(\mathbb{H}^{\langle r \rangle}, F, K, A\big)$$

be a hierarchical soft SuperHyperGraph of height r as in Definition 4.6.1. Then, for every parameter $a \in A$, the parameter section

$$\mathbb{H}_a^{\langle r \rangle} = \big(F(a), K(a)\big)$$

is a well-defined hierarchical SuperHyperGraph of height r on V_0.

Proof. Fix $a \in A$.

First, by (HS1), one has

$$F(a) \subseteq V \subseteq \mathcal{U}_r(V_0),$$

and $F(a) \neq \emptyset$ by the codomain of F. Hence $F(a)$ is a finite, nonempty set of admissible hierarchical supervertices.

Second, by (HS3), every edge $e \in K(a)$ satisfies

$$e \in E \qquad \text{and} \qquad e \subseteq F(a).$$

Since $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, each such e is a nonempty subset of $F(a)$. Therefore

$$K(a) \subseteq \mathcal{P}(F(a)) \setminus \{\emptyset\}.$$

Third, by (HS2), whenever $X \in F(a)$ has level at least 1, all immediate constituents of X also lie in $F(a)$. Thus the required coherence/downward-closure condition is satisfied.

Hence the pair $\big(F(a), K(a)\big)$ satisfies all axioms of a hierarchical SuperHyperGraph of height r on V_0. Therefore $\mathbb{H}_a^{\langle r \rangle}$ is well-defined. □

4.7 Hierarchical Rough SuperHyperGraph

A hierarchical rough superhypergraph approximates multi-level superhyperedge structure through lower and upper hierarchical subconfigurations, capturing granularity, indiscernibility, uncertainty, and approximation within nested relational systems faithfully.

Definition 4.7.1 (Hierarchical partition)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r on V_0. A partition

$$\Pi = \{\, B_\lambda : \lambda \in \Lambda \,\}$$

of V is called a *hierarchical partition* if every block B_λ is nonempty and downward closed, i.e., whenever $X \in B_\lambda$ and $\ell(X) \geq 1$, all immediate constituents of X also belong to B_λ:

$$X \subseteq B_\lambda.$$

Definition 4.7.2 (Lower and upper hierarchical approximations)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r, let Π be a hierarchical partition of V, and let $X \subseteq V$ be a nonempty, downward-closed set.

Define the *lower* and *upper* vertex approximations of X by

$$\underline{\Pi}(X) \;:=\; \bigcup\{\, B \in \Pi : B \subseteq X \,\}, \qquad \overline{\Pi}(X) \;:=\; \bigcup\{\, B \in \Pi : B \cap X \neq \emptyset \,\}.$$

Define the induced lower and upper edge families by

$$\underline{\Pi}_E(X) \;:=\; \{\, e \in E : e \subseteq \underline{\Pi}(X) \,\}, \qquad \overline{\Pi}_E(X) \;:=\; \{\, e \in E : e \subseteq \overline{\Pi}(X) \,\}.$$

Definition 4.7.3 (Hierarchical Rough SuperHyperGraph of height r)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r on V_0, let Π be a hierarchical partition of V, and let $X \subseteq V$ be a nonempty, downward-closed set such that

$$\underline{\Pi}(X) \neq \emptyset.$$

The pair

$$\mathcal{R}^{\langle r\rangle}_{\Pi}(X) \;:=\; \Big(\underline{\mathbb{H}}^{\langle r\rangle}_{\Pi}(X), \overline{\mathbb{H}}^{\langle r\rangle}_{\Pi}(X)\Big)$$

with

$$\underline{\mathbb{H}}^{\langle r\rangle}_{\Pi}(X) := \big(\underline{\Pi}(X), \underline{\Pi}_E(X)\big), \qquad \overline{\mathbb{H}}^{\langle r\rangle}_{\Pi}(X) := \big(\overline{\Pi}(X), \overline{\Pi}_E(X)\big)$$

is called the *hierarchical rough SuperHyperGraph of height* r associated with X (with respect to the hierarchical partition Π).

The structures

$$\underline{\mathbb{H}}^{\langle r\rangle}_{\Pi}(X) \qquad \text{and} \qquad \overline{\mathbb{H}}^{\langle r\rangle}_{\Pi}(X)$$

are called, respectively, the *lower hierarchical approximation SuperHyperGraph* and the *upper hierarchical approximation SuperHyperGraph*.

Theorem 4.7.4 (Well-definedness of lower and upper approximations)**.** *Let*

$$\mathcal{R}_{\Pi}^{\langle r\rangle}(X)=\Big(\underline{\mathbb{H}}_{\Pi}^{\langle r\rangle}(X),\overline{\mathbb{H}}_{\Pi}^{\langle r\rangle}(X)\Big)$$

be a hierarchical rough SuperHyperGraph of height r as in Definition 4.7.3. Then both

$$\underline{\mathbb{H}}_{\Pi}^{\langle r\rangle}(X)=\big(\underline{\Pi}(X),\underline{\Pi}_E(X)\big)$$

and

$$\overline{\mathbb{H}}_{\Pi}^{\langle r\rangle}(X)=\big(\overline{\Pi}(X),\overline{\Pi}_E(X)\big)$$

are well-defined hierarchical SuperHyperGraphs of height r on V_0.

Proof. Since Π is a partition of V, every block $B\in\Pi$ satisfies $B\subseteq V\subseteq\mathcal{U}_r(V_0)$. Hence

$$\underline{\Pi}(X)\subseteq V\subseteq\mathcal{U}_r(V_0),\qquad \overline{\Pi}(X)\subseteq V\subseteq\mathcal{U}_r(V_0).$$

Moreover, $\underline{\Pi}(X)\neq\emptyset$ by assumption, and $\overline{\Pi}(X)\neq\emptyset$ because $X\neq\emptyset$ and each element of X belongs to some block intersecting X.

Next, both $\underline{\Pi}(X)$ and $\overline{\Pi}(X)$ are downward closed. Indeed, they are unions of downward-closed blocks of the hierarchical partition Π. Therefore, whenever Y belongs to one of these vertex sets and $\ell(Y)\geq 1$, all immediate constituents of Y remain in the same block, hence in the same union.

Now consider the lower edge family. By definition,

$$\underline{\Pi}_E(X)=\{\,e\in E: e\subseteq\underline{\Pi}(X)\,\}.$$

Thus every $e\in\underline{\Pi}_E(X)$ is an element of E, hence a nonempty subset of V; moreover $e\subseteq\underline{\Pi}(X)$. Therefore

$$\underline{\Pi}_E(X)\subseteq\mathcal{P}\big(\underline{\Pi}(X)\big)\setminus\{\emptyset\}.$$

Similarly,

$$\overline{\Pi}_E(X)\subseteq\mathcal{P}\big(\overline{\Pi}(X)\big)\setminus\{\emptyset\}.$$

Consequently, each of the pairs

$$\big(\underline{\Pi}(X),\underline{\Pi}_E(X)\big)\qquad\text{and}\qquad\big(\overline{\Pi}(X),\overline{\Pi}_E(X)\big)$$

satisfies the axioms of a hierarchical SuperHyperGraph of height r on V_0: the vertex set is a finite nonempty subset of $\mathcal{U}_r(V_0)$, the edge family consists of nonempty subsets of that vertex set, and the downward-closure condition holds.

Hence both lower and upper approximation structures are well-defined hierarchical SuperHyperGraphs. □

Remark 4.7.5. The partition-based formulation above is equivalent in spirit to a rough-set construction induced by an equivalence relation on the hierarchical vertex set, with the advantage that downward closure is preserved transparently at the level of approximation blocks.

4.8 Hierarchical Weighted SuperHyperGraph

A hierarchical weighted superhypergraph assigns numerical weights to multi-level supervertices and superhyperedges, enabling quantitative analysis of nested, cross-level relational structures within one coherent mathematical framework.

Definition 4.8.1 (Weighted Hierarchical SuperHyperGraph of height r)**.** Let V_0 be a finite nonempty base set, and fix $r \in \mathbb{N}_0$. Let $\mathcal{U}_r(V_0)$ be the hierarchical universe from Definition 2.1.5, and let ℓ be the level map from Definition 2.1.6. A *weighted hierarchical SuperHyperGraph of height* r on V_0 is a quadruple

$$\mathbb{W}^{\langle r \rangle} = \big(V, E, w_V, w_E\big)$$

satisfying the following axioms:

(WH1) *(Hierarchical supervertex set).* V is a finite nonempty set such that

$$V \subseteq \mathcal{U}_r(V_0).$$

The elements of V are called *hierarchical supervertices.*

(WH2) *(Cross-level superhyperedge family).* E is a finite family of nonempty subsets of V, that is,

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

The elements of E are called *hierarchical superhyperedges.* In particular, one edge may contain supervertices from different levels.

(WH3) *(Coherence / downward closure).* Whenever $X \in V$ has level at least 1, all of its immediate constituents are also vertices:

$$X \subseteq V.$$

(WH4) *(Weight maps).* The maps

$$w_V : V \to \mathbb{R}_{\geq 0}, \qquad w_E : E \to \mathbb{R}_{\geq 0}$$

assign, respectively, a nonnegative *vertex weight* to each hierarchical supervertex and a nonnegative *edge weight* to each hierarchical superhyperedge.

For each $k \in \{0, \ldots, r\}$, the k-th layer is defined by

$$V_k \;:=\; \{\, x \in V : \ell(x) = k \,\}.$$

The *total vertex weight*, *total edge weight*, and *total weight* of $\mathbb{W}^{\langle r \rangle}$ are defined by

$$w_V(V) \;:=\; \sum_{x \in V} w_V(x), \qquad w_E(E) \;:=\; \sum_{e \in E} w_E(e), \qquad w(\mathbb{W}^{\langle r \rangle}) \;:=\; w_V(V) + w_E(E).$$

Similarly, the total weight of the k-th layer is

$$w_V(V_k) \;:=\; \sum_{x \in V_k} w_V(x).$$

Remark 4.8.2. The weighted model above is a direct extension of a hierarchical SuperHyperGraph: the hierarchical structure is carried by the pair (V, E), while the maps w_V and w_E provide quantitative information on vertices and edges. Unlike the fuzzy case, no additional admissibility inequality is required in the basic weighted setting.

Theorem 4.8.3 (Well-definedness of Weighted Hierarchical SuperHyperGraphs)**.** *Let*

$$\mathbb{W}^{\langle r\rangle} = \big(V,\ E,\ w_V,\ w_E\big)$$

be a weighted hierarchical SuperHyperGraph of height r on V_0 in the sense of Definition 4.8.1. Then:

(i) *the pair*

$$\mathbb{H}^{\langle r\rangle} := (V, E)$$

is a well-defined hierarchical SuperHyperGraph of height r on V_0;

(ii) *the weight maps*

$$w_V : V \to \mathbb{R}_{\geq 0}, \qquad w_E : E \to \mathbb{R}_{\geq 0}$$

are well-defined functions on the vertex set and edge family, respectively;

(iii) *all finite sums appearing in the definitions of*

$$w_V(V), \quad w_E(E), \quad w(\mathbb{W}^{\langle r\rangle}), \quad w_V(V_k) \ \ (0 \leq k \leq r)$$

are well-defined real numbers.

Proof. We verify each statement in turn.

(i) By (WH1), the set V is a finite nonempty subset of $\mathcal{U}_r(V_0)$. By (WH2), the family E consists of nonempty subsets of V, that is,

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

By (WH3), whenever $X \in V$ and $\ell(X) \geq 1$, all immediate constituents of X also belong to V. Therefore the pair (V, E) satisfies exactly the axioms of a hierarchical SuperHyperGraph of height r. Hence

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

is well-defined.

(ii) By (WH4), w_V assigns to each element $x \in V$ a unique value $w_V(x) \in \mathbb{R}_{\geq 0}$, and w_E assigns to each element $e \in E$ a unique value $w_E(e) \in \mathbb{R}_{\geq 0}$. Hence w_V and w_E are well-defined functions with the indicated domains and codomains.

(iii) Since V is finite by (WH1) and E is finite by (WH2), the sums

$$\sum_{x\in V} w_V(x) \qquad \text{and} \qquad \sum_{e\in E} w_E(e)$$

are finite sums of nonnegative real numbers, and therefore are well-defined elements of $\mathbb{R}_{\geq 0}$.

For each $k \in \{0, \dots, r\}$, the layer

$$V_k = \{ x \in V : \ell(x) = k \}$$

is a subset of the finite set V, hence itself finite. Therefore

$$\sum_{x \in V_k} w_V(x)$$

is also a well-defined nonnegative real number.

Finally, since both $w_V(V)$ and $w_E(E)$ are well-defined real numbers, their sum

$$w(\mathbb{W}^{\langle r \rangle}) = w_V(V) + w_E(E)$$

is again a well-defined real number.

Thus all three assertions hold. □

Corollary 4.8.4. *Every weighted hierarchical SuperHyperGraph of height r has an underlying crisp hierarchical SuperHyperGraph obtained by forgetting the weight maps:*

$$\mathsf{Und}\Big(\mathbb{W}^{\langle r \rangle}\Big) := (V, E).$$

Proof. This follows immediately from Theorem 4.8.3(i). □

4.9 Hierarchical SuperHyperGraph Distance

In view of Table 1.2, two natural distance notions for hierarchical SuperHyperGraphs are the *super-Berge distance* and the *primal distance*. Both are defined on the vertex set of a hierarchical SuperHyperGraph and remain meaningful even when edges join mixed-level supervertices.

Definition 4.9.1 (Incidence graph of a Hierarchical SuperHyperGraph)**.** Let

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Its *incidence graph* is the bipartite graph

$$\mathrm{Inc}\big(\mathbb{H}^{\langle r \rangle}\big) \ := \ \big(V \sqcup E,\ I\big),$$

where

$$I \ := \ \big\{\{x, e\} \subseteq V \sqcup E \mid x \in V,\ e \in E,\ x \in e\big\}.$$

Thus the vertices of $\mathrm{Inc}(\mathbb{H}^{\langle r \rangle})$ are of two types: hierarchical supervertices $x \in V$ and superhyperedges $e \in E$, and x is adjacent to e precisely when $x \in e$.

Definition 4.9.2 (Super-Berge walk, path, and distance)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r, and let $u, v \in V$.

A *super-Berge walk* from u to v is a finite alternating sequence

$$u = x_0,\ e_1,\ x_1,\ e_2,\ \ldots,\ e_m,\ x_m = v$$

such that

$$x_{i-1}, x_i \in e_i \qquad (1 \le i \le m),$$

where $x_0, \ldots, x_m \in V$ and $e_1, \ldots, e_m \in E$.

A *super-Berge path* is a super-Berge walk in which the vertices $x_0, \ldots, x_m$ are pairwise distinct and the edges $e_1, \ldots, e_m$ are pairwise distinct.

The *length* of the above walk/path is m, namely the number of superhyperedges used.

The *super-Berge distance* between u and v is defined by

$$\mathrm{dist}_{SB}(u, v) \ := \ \begin{cases} \min\{\, m \in \mathbb{N}_0 \mid \text{there exists a super-Berge path of length } m \text{ from } u \text{ to } v \,\}, & \text{if such a path ex} \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 4.9.3 (Well-definedness of the super-Berge distance)**.** *Let*

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Then the function

$$\mathrm{dist}_{SB} : V \times V \longrightarrow \mathbb{N}_0 \cup \{\infty\}$$

given in Definition 4.9.2 is well-defined. More precisely, for all $u, v \in V$,

$$\mathrm{dist}_{SB}(u, v) = \frac{1}{2}\, \mathrm{dist}_{\mathrm{Inc}(\mathbb{H}^{\langle r\rangle})}(u, v),$$

where the right-hand side denotes the ordinary graph distance in the incidence graph, with value ∞ when u and v lie in different connected components.

Proof. Fix $u, v \in V$. By the Definition, the set V is finite and

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\},$$

so every edge $e \in E$ is a nonempty subset of V. Therefore the set

$$I = \big\{\{x, e\} \subseteq V \sqcup E : x \in e\big\}$$

is well-defined, and $\mathrm{Inc}(\mathbb{H}^{\langle r\rangle}) = (V \sqcup E, I)$ is a finite bipartite graph.

Now consider an arbitrary super-Berge walk

$$u = x_0,\ e_1,\ x_1,\ e_2,\ \ldots,\ e_m,\ x_m = v.$$

By the defining property $x_{i-1}, x_i \in e_i$, one has

$$\{x_{i-1}, e_i\} \in I \qquad \text{and} \qquad \{e_i, x_i\} \in I \qquad (1 \le i \le m).$$

Hence

$$u = x_0,\ e_1,\ x_1,\ e_2,\ \ldots,\ e_m,\ x_m = v$$

is an ordinary walk in the incidence graph of length $2m$. Conversely, every walk in the incidence graph that starts at $u \in V$ and ends at $v \in V$ must alternate between V-vertices and E-vertices, and therefore has even length $2m$; reading off the alternating sequence yields a super-Berge walk of length m in $\mathbb{H}^{\langle r \rangle}$.

Thus, super-Berge walks between u and v are in bijective correspondence with ordinary walks in $\mathrm{Inc}(\mathbb{H}^{\langle r \rangle})$ between u and v, with the graph-theoretic length exactly twice the super-Berge length. Since the incidence graph is finite, its ordinary graph distance

$$\mathrm{dist}_{\mathrm{Inc}(\mathbb{H}^{\langle r \rangle})}(u, v)$$

is well-defined in $\mathbb{N}_0 \cup \{\infty\}$. Because any u-to-v walk in the incidence graph has even length, the quantity

$$\frac{1}{2}\ \mathrm{dist}_{\mathrm{Inc}(\mathbb{H}^{\langle r \rangle})}(u, v)$$

also belongs to $\mathbb{N}_0 \cup \{\infty\}$.

Therefore $\mathrm{dist}_{SB}(u, v)$ is well-defined, and the stated identity follows. □

Definition 4.9.4 (Primal graph and primal distance)**.** Let

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r.

Its *primal graph* is the simple graph

$$\mathrm{Pr}\big(\mathbb{H}^{\langle r \rangle}\big) = \big(V, E_{\mathrm{Pr}}\big),$$

where

$$E_{\mathrm{Pr}} \ := \ \big\{\{x, y\} \subseteq V \mid x \neq y,\ \exists\, e \in E \text{ such that } \{x, y\} \subseteq e\big\}.$$

Thus two distinct hierarchical supervertices are adjacent in the primal graph whenever they lie together in some common hierarchical superhyperedge.

The *primal distance* between $u, v \in V$ is defined by

$$\mathrm{dist}_{\mathrm{Pr}}(u, v) \ := \ \mathrm{dist}_{\mathrm{Pr}(\mathbb{H}^{\langle r \rangle})}(u, v),$$

that is, the ordinary graph distance between u and v in the primal graph, with value ∞ when u and v lie in different connected components. Equivalently,

$$\mathrm{dist}_{\mathrm{Pr}}(u, v) = \begin{cases} \min\{\, m \in \mathbb{N}_0 \mid \exists\, x_0, x_1, \ldots, x_m \in V,\ x_0 = u,\ x_m = v, \\ \qquad\qquad \forall\, i \in \{1, \ldots, m\}\ \exists\, e_i \in E : \{x_{i-1}, x_i\} \subseteq e_i \,\}, & \text{if such a sequence ex} \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 4.9.5 (Well-definedness of the primal distance)**.** *Let*

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Then the primal graph

$$\mathrm{Pr}\big(\mathbb{H}^{\langle r \rangle}\big) = (V, E_{\mathrm{Pr}})$$

is a well-defined finite simple graph, and consequently the function

$$\mathrm{dist}_{\mathrm{Pr}} : V \times V \longrightarrow \mathbb{N}_0 \cup \{\infty\}$$

from Definition 4.9.4 is well-defined.

Proof. Because V is finite and $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, the set

$$E_{\mathrm{Pr}} = \big\{\{x, y\} \subseteq V \mid x \neq y,\ \exists\, e \in E : \{x, y\} \subseteq e\big\}$$

is a well-defined family of two-element subsets of V. Hence

$$E_{\mathrm{Pr}} \subseteq \binom{V}{2},$$

so $\mathrm{Pr}(\mathbb{H}^{\langle r \rangle}) = (V, E_{\mathrm{Pr}})$ is a simple graph. Since V is finite, the set E_{Pr} is also finite. Therefore the primal graph is a well-defined finite simple graph.

The ordinary graph distance on any finite simple graph is well-defined as an element of $\mathbb{N}_0 \cup \{\infty\}$: it is the minimum length of a graph path between two vertices when such a path exists, and ∞ otherwise. Applying this to the primal graph yields a well-defined function

$$\mathrm{dist}_{\mathrm{Pr}} : V \times V \to \mathbb{N}_0 \cup \{\infty\}.$$

This is exactly the primal distance of Definition 4.9.4. □

Proposition 4.9.6 (Metric property on connected components)**.** *Let*

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Then, on every connected component of the incidence graph, dist_{SB} *is a metric on the set of hierarchical supervertices contained in that component. Likewise, on every connected component of the primal graph,* $\mathrm{dist}_{\mathrm{Pr}}$ *is a metric.*

Proof. By Theorem 4.9.3, dist_{SB} is one-half of the ordinary graph distance in the incidence graph. Restricting to one connected component removes the value ∞, and multiplying a metric by the positive constant $1/2$ preserves the metric axioms. Hence dist_{SB} is a metric on the hierarchical supervertices in that component.

By Theorem 4.9.5, $\mathrm{dist}_{\mathrm{Pr}}$ is the ordinary graph distance on the primal graph; therefore its restriction to any connected component is a metric. □

Remark 4.9.7. The super-Berge distance records movement through alternating vertex–edge incidence, whereas the primal distance records movement only through vertex adjacency induced by common superhyperedges. In general, these two distances need not coincide.

4.10 Hierarchical SuperHyperGraph Labeling

In this section, we introduce three natural labeling notions for hierarchical SuperHyperGraphs: vertex labeling, edge labeling, and total labeling.

Definition 4.10.1 (Canonical finite label set)**.** For each $n \in \mathbb{N}_0$, define

$$[n] \;:=\; \begin{cases} \{1, 2, \ldots, n\}, & n \geq 1, \\ \emptyset, & n = 0. \end{cases}$$

Definition 4.10.2 (Hierarchical SuperHyperGraph vertex labeling)**.** Let

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r on V_0.

A *vertex labeling* of $\mathbb{H}^{\langle r \rangle}$ is a bijection

$$\lambda_V : V \longrightarrow [\,|V|\,].$$

The pair

$$\big(\mathbb{H}^{\langle r \rangle}, \lambda_V\big)$$

is called a *vertex-labeled hierarchical SuperHyperGraph.*

For each $k \in \{0, \ldots, r\}$, the restriction

$$\lambda_V \restriction_{V_k} : V_k \longrightarrow \lambda_V(V_k)$$

is called the *layer-k vertex labeling.*

Theorem 4.10.3 (Well-definedness of vertex labeling)**.** *Let*

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Then a vertex labeling of $\mathbb{H}^{\langle r \rangle}$ is well-defined. More precisely, there exists a bijection

$$\lambda_V : V \to [\,|V|\,].$$

Proof. By the Definition, the set V is finite and nonempty. Hence there exists an enumeration

$$V = \{x_1, x_2, \ldots, x_n\}, \qquad n = |V|.$$

Define

$$\lambda_V(x_i) := i \qquad (1 \leq i \leq n).$$

This assignment is well-defined because each vertex $x \in V$ appears exactly once in the chosen enumeration, so exactly one label is assigned to it.

The map λ_V is injective because distinct vertices receive distinct indices, and it is surjective onto $[n]$ because every integer $i \in \{1, \ldots, n\}$ is attained as $\lambda_V(x_i)$. Therefore λ_V is a bijection

$$\lambda_V : V \to [\,|V|\,].$$

Hence a vertex labeling is well-defined. □

Definition 4.10.4 (Hierarchical SuperHyperGraph edge labeling)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r on V_0.

An *edge labeling* of $\mathbb{H}^{\langle r\rangle}$ is a bijection

$$\lambda_E : E \longrightarrow [\,|E|\,].$$

The pair

$$\big(\mathbb{H}^{\langle r\rangle}, \lambda_E\big)$$

is called an *edge-labeled hierarchical SuperHyperGraph.*

Theorem 4.10.5 (Well-definedness of edge labeling)**.** *Let*

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Then an edge labeling of $\mathbb{H}^{\langle r\rangle}$ is well-defined. More precisely, there exists a bijection

$$\lambda_E : E \to [\,|E|\,].$$

Proof. By the Definition, the family E is finite. If $E = \emptyset$, then $|E| = 0$, so $[\,|E|\,] = [0] = \emptyset$, and the unique map

$$\lambda_E : \emptyset \to \emptyset$$

is a bijection.

Now assume $E \neq \emptyset$. Since E is finite, there exists an enumeration

$$E = \{e_1, e_2, \ldots, e_m\}, \qquad m = |E|.$$

Define

$$\lambda_E(e_j) := j \qquad (1 \leq j \leq m).$$

This map is well-defined because each edge $e \in E$ appears exactly once in the enumeration. It is injective because distinct edges receive distinct labels, and surjective onto $[m]$ because each $j \in \{1, \ldots, m\}$ is attained. Hence λ_E is a bijection

$$\lambda_E : E \to [\,|E|\,].$$

Therefore an edge labeling is well-defined. □

Definition 4.10.6 (Disjoint union of vertices and edges)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph. Define the *typed disjoint union* of vertices and edges by

$$V \sqcup E \ := \ (V \times \{0\}) \cup (E \times \{1\}).$$

Its elements are ordered pairs $(x, 0)$ with $x \in V$ and $(e, 1)$ with $e \in E$. This construction distinguishes vertices from edges even if, as sets, some vertex and some edge happen to have the same underlying set-theoretic form.

Definition 4.10.7 (Hierarchical SuperHyperGraph total labeling)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r on V_0.

A *total labeling* of $\mathbb{H}^{\langle r\rangle}$ is a bijection

$$\lambda_T : V \sqcup E \longrightarrow [\,|V| + |E|\,],$$

where $V \sqcup E$ is the disjoint union from Definition 4.10.6.

The pair

$$\big(\mathbb{H}^{\langle r\rangle}, \lambda_T\big)$$

is called a *totally labeled hierarchical SuperHyperGraph.*

The induced vertex-part and edge-part label maps are

$$\lambda_T^V(x) := \lambda_T(x, 0) \qquad (x \in V),$$

and

$$\lambda_T^E(e) := \lambda_T(e, 1) \qquad (e \in E).$$

Theorem 4.10.8 (Well-definedness of total labeling)**.** *Let*

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Then the total labeling of $\mathbb{H}^{\langle r\rangle}$ is well-defined. More precisely:

(i) *the disjoint union $V \sqcup E$ is a well-defined finite set;*

(ii) *its cardinality is*

$$|V \sqcup E| = |V| + |E|;$$

(iii) *consequently, there exists a bijection*

$$\lambda_T : V \sqcup E \to [\,|V| + |E|\,].$$

Proof. Since V and E are sets, the Cartesian products $V \times \{0\}$ and $E \times \{1\}$ are well-defined sets. Moreover,

$$V \times \{0\} \ \cap\ E \times \{1\} = \emptyset,$$

because the second coordinates are different. Hence

$$V \sqcup E = (V \times \{0\}) \cup (E \times \{1\})$$

is a well-defined disjoint union.

By Definition, both V and E are finite. Therefore $V \times \{0\}$ and $E \times \{1\}$ are finite, and because they are disjoint,

$$|V \sqcup E| = |V \times \{0\}| + |E \times \{1\}| = |V| + |E|.$$

This proves (i) and (ii).

Since $V \sqcup E$ is finite, it admits an enumeration

$$V \sqcup E = \{\alpha_1, \alpha_2, \ldots, \alpha_{|V|+|E|}\}.$$

Define

$$\lambda_T(\alpha_i) := i \qquad (1 \leq i \leq |V| + |E|).$$

This is well-defined because each element of $V \sqcup E$ occurs exactly once in the enumeration. It is injective by construction, and surjective onto $[\,|V| + |E|\,]$. Hence λ_T is a bijection

$$\lambda_T : V \sqcup E \to [\,|V| + |E|\,].$$

Therefore the total labeling is well-defined. □

Corollary 4.10.9 (Existence of the three basic labelings)**.** *Every hierarchical SuperHyperGraph of height r admits:*

(i) *a vertex labeling;*

(ii) *an edge labeling;*

(iii) *a total labeling.*

Proof. Apply Theorems 4.10.3, 4.10.5, and 4.10.8. □

Remark 4.10.10. The total labeling must be defined on $V \sqcup E$, not on the naive union $V \cup E$. Indeed, a hierarchical supervertex is itself a set-valued object, and a hierarchical superhyperedge is also a set. Thus $V \cap E$ may be nonempty in a purely set-theoretic sense unless one distinguishes the two types by tags.

4.11 Hierarchical SuperHyperGraph Domination

In this section, domination is defined on the hierarchical supervertex set through the natural common-edge adjacency induced by the hierarchical superhyperedge family.

Definition 4.11.1 (Primal adjacency in a Hierarchical SuperHyperGraph)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r on V_0. For two distinct hierarchical supervertices $x, y \in V$, we say that x and y are *adjacent*, and write

$$x \sim_{\mathbb{H}} y,$$

if there exists a hierarchical superhyperedge $e \in E$ such that

$$\{x, y\} \subseteq e.$$

Equivalently, two distinct supervertices are adjacent whenever they belong to a common hierarchical superhyperedge.

Definition 4.11.2 (Neighborhoods)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r, and let $x \in V$.

The *open neighborhood* of x is defined by

$$N_{\mathbb{H}}(x) \;:=\; \{\, y \in V \setminus \{x\} \mid x \sim_{\mathbb{H}} y \,\}.$$

The *closed neighborhood* of x is

$$N_{\mathbb{H}}[x] \;:=\; N_{\mathbb{H}}(x) \cup \{x\}.$$

More generally, for a set $D \subseteq V$, define

$$N_{\mathbb{H}}[D] \;:=\; \bigcup_{x\in D} N_{\mathbb{H}}[x].$$

Definition 4.11.3 (Dominating set)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. A subset $D \subseteq V$ is called a *dominating set* of $\mathbb{H}^{\langle r\rangle}$ if

$$N_{\mathbb{H}}[D] = V.$$

Equivalently, D is dominating if for every $v \in V$, either $v \in D$, or there exists some $x \in D$ and some $e \in E$ such that

$$\{x, v\} \subseteq e.$$

In this case, we say that D *dominates* $\mathbb{H}^{\langle r\rangle}$.

Definition 4.11.4 (Domination number)**.** Let

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Define the family of dominating sets by

$$\mathfrak{D}\big(\mathbb{H}^{\langle r\rangle}\big) \;:=\; \{\, D \subseteq V \mid N_{\mathbb{H}}[D] = V \,\}.$$

The *domination number* of $\mathbb{H}^{\langle r\rangle}$ is

$$\gamma\big(\mathbb{H}^{\langle r\rangle}\big) \;:=\; \min\big\{\, |D| \mid D \in \mathfrak{D}\big(\mathbb{H}^{\langle r\rangle}\big) \,\big\}.$$

Theorem 4.11.5 (Well-definedness of Hierarchical SuperHyperGraph domination)**.** *Let*

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Then the following statements hold:

(i) *for every $x \in V$, the sets $N_{\mathbb{H}}(x)$ and $N_{\mathbb{H}}[x]$ are well-defined finite subsets of V;*

(ii) *for every $D \subseteq V$, the set $N_{\mathbb{H}}[D]$ is a well-defined subset of V;*

(iii) *the family*

$$\mathfrak{D}\big(\mathbb{H}^{\langle r\rangle}\big)$$

of dominating sets is nonempty;

(iv) *the domination number*

$$\gamma\big(\mathbb{H}^{\langle r\rangle}\big)$$

is a well-defined positive integer satisfying

$$1 \le \gamma\big(\mathbb{H}^{\langle r\rangle}\big) \le |V|.$$

Proof. Since

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

is a hierarchical SuperHyperGraph, by definition V is a finite nonempty set and

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

(i) Fix $x \in V$. By Definition 4.11.1,

$$N_{\mathbb{H}}(x) = \{\, y \in V \setminus \{x\} \mid \exists\, e \in E : \{x, y\} \subseteq e \,\}.$$

This is a subset of the finite set $V \setminus \{x\}$, hence it is itself a well-defined finite set. Therefore

$$N_{\mathbb{H}}[x] = N_{\mathbb{H}}(x) \cup \{x\}$$

is also a well-defined finite subset of V.

(ii) Let $D \subseteq V$. Then

$$N_{\mathbb{H}}[D] = \bigcup_{x \in D} N_{\mathbb{H}}[x].$$

Each set $N_{\mathbb{H}}[x]$ is a subset of V by part (i), so their union is again a well-defined subset of V.

(iii) Consider the set $V \subseteq V$. For each $x \in V$, one has

$$x \in N_{\mathbb{H}}[x]$$

by definition of closed neighborhood. Hence

$$V = \bigcup_{x \in V} \{x\} \subseteq \bigcup_{x \in V} N_{\mathbb{H}}[x] = N_{\mathbb{H}}[V].$$

Since every $N_{\mathbb{H}}[x] \subseteq V$, it follows that

$$N_{\mathbb{H}}[V] \subseteq V.$$

Therefore

$$N_{\mathbb{H}}[V] = V,$$

so V is a dominating set. Thus

$$V \in \mathfrak{D}\big(\mathbb{H}^{\langle r \rangle}\big),$$

and the family of dominating sets is nonempty.

(iv) Since $\mathfrak{D}(\mathbb{H}^{\langle r \rangle}) \neq \emptyset$, the set

$$\{\, |D| \mid D \in \mathfrak{D}\big(\mathbb{H}^{\langle r \rangle}\big) \,\}$$

is a nonempty set of positive integers. Moreover, every dominating set D is a subset of the finite set V, so

$$1 \leq |D| \leq |V|.$$

Hence this set of cardinalities is a nonempty finite subset of $\{1, 2, \ldots, |V|\}$, and therefore possesses a minimum element. Consequently,

$$\gamma\big(\mathbb{H}^{\langle r \rangle}\big) = \min\big\{\, |D| \mid D \in \mathfrak{D}\big(\mathbb{H}^{\langle r \rangle}\big) \,\big\}$$

is well-defined and satisfies

$$1 \leq \gamma\big(\mathbb{H}^{\langle r \rangle}\big) \leq |V|.$$

This completes the proof. □

Proposition 4.11.6 (Equivalence with domination in the primal graph)**.** *Let*

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r, and let

$$\Pr\big(\mathbb{H}^{\langle r \rangle}\big) = (V, E_{\Pr})$$

be its primal graph, where

$$E_{\Pr} := \big\{\{x, y\} \subseteq V \mid x \neq y,\ \exists\, e \in E : \{x, y\} \subseteq e\big\}.$$

Then a set $D \subseteq V$ is a dominating set of $\mathbb{H}^{\langle r \rangle}$ if and only if it is a dominating set of $\Pr(\mathbb{H}^{\langle r \rangle})$*. In particular,*

$$\gamma\big(\mathbb{H}^{\langle r \rangle}\big) = \gamma\big(\Pr(\mathbb{H}^{\langle r \rangle})\big).$$

Proof. For distinct $x, y \in V$, the condition

$$x \sim_{\mathbb{H}} y$$

holds if and only if $\{x, y\} \in E_{\mathrm{Pr}}$ by definition of the primal graph. Hence the open and closed neighborhoods of every vertex are the same in $\mathbb{H}^{\langle r \rangle}$ and in $\mathrm{Pr}(\mathbb{H}^{\langle r \rangle})$. Therefore, for every $D \subseteq V$,

$$N_{\mathbb{H}}[D] = V \quad \Longleftrightarrow \quad N_{\mathrm{Pr}(\mathbb{H}^{\langle r \rangle})}[D] = V.$$

Thus the dominating sets coincide in the two structures, and so their minimum cardinalities are equal. □

Remark 4.11.7. This domination notion is vertex-based and uses the natural common-edge adjacency among hierarchical supervertices. Thus, it is the most direct hierarchical analogue of ordinary graph domination.

4.12 Hierarchical SuperHyperTrees

In order to define a tree structure for hierarchical SuperHyperGraphs, we use the natural primal graph associated with common-edge adjacency.

Definition 4.12.1 (Primal graph of a Hierarchical SuperHyperGraph)**.** Let

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r on V_0. Its *primal graph* is the graph

$$\mathrm{Pr}\big(\mathbb{H}^{\langle r \rangle}\big) = \big(V, E_{\mathrm{Pr}}\big),$$

where

$$E_{\mathrm{Pr}} \;:=\; \big\{\{x, y\} \subseteq V \mid x \neq y,\ \exists\, e \in E \text{ such that } \{x, y\} \subseteq e\big\}.$$

Thus two distinct hierarchical supervertices are adjacent in $\mathrm{Pr}(\mathbb{H}^{\langle r \rangle})$ precisely when they belong to a common hierarchical superhyperedge.

Definition 4.12.2 (Hierarchical SuperHyperTree)**.** Let

$$\mathbb{H}^{\langle r \rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. We say that $\mathbb{H}^{\langle r \rangle}$ is a *Hierarchical SuperHyper-Tree* if its primal graph

$$\mathrm{Pr}\big(\mathbb{H}^{\langle r \rangle}\big)$$

is a tree.

Equivalently, $\mathbb{H}^{\langle r \rangle}$ is a Hierarchical SuperHyperTree if the following two conditions hold:

(T1) $\mathrm{Pr}(\mathbb{H}^{\langle r \rangle})$ is connected;

(T2) $\mathrm{Pr}(\mathbb{H}^{\langle r \rangle})$ is acyclic.

Remark 4.12.3. This definition uses the natural primal-type adjacency already induced by the hierarchical superhyperedge family. Hence a Hierarchical SuperHyperTree is a hierarchical SuperHyperGraph whose vertex-adjacency structure is tree-like in the ordinary graph-theoretic sense.

Theorem 4.12.4 (Well-definedness of the primal graph)**.** *Let*

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Then

$$\mathrm{Pr}\big(\mathbb{H}^{\langle r\rangle}\big) = \big(V, E_{\mathrm{Pr}}\big)$$

is a well-defined finite simple graph.

Proof. Since $\mathbb{H}^{\langle r\rangle}$ is a hierarchical SuperHyperGraph, its vertex set V is finite and nonempty, and its edge family satisfies

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Hence every hierarchical superhyperedge is a nonempty subset of V.

Now define

$$E_{\mathrm{Pr}} = \big\{\{x, y\} \subseteq V \mid x \neq y,\ \exists\, e \in E : \{x, y\} \subseteq e\big\}.$$

By construction, each element of E_{Pr} is a two-element subset of V. Therefore

$$E_{\mathrm{Pr}} \subseteq \binom{V}{2}.$$

In particular, no loop $\{x, x\}$ can occur, and no multiple edges arise, because E_{Pr} is a set of unordered pairs. Thus

$$\mathrm{Pr}\big(\mathbb{H}^{\langle r\rangle}\big) = \big(V, E_{\mathrm{Pr}}\big)$$

is a simple graph.

Since V is finite, the set $\binom{V}{2}$ is finite; hence its subset E_{Pr} is also finite. Consequently,

$$\mathrm{Pr}\big(\mathbb{H}^{\langle r\rangle}\big)$$

is a finite simple graph. This proves the claim. □

Theorem 4.12.5 (Well-definedness of Hierarchical SuperHyperTrees)**.** *Let*

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r. Then the statement

"$\mathbb{H}^{\langle r\rangle}$ is a Hierarchical SuperHyperTree"

is well-defined.

More precisely,

$\mathbb{H}^{\langle r\rangle}$ is a Hierarchical SuperHyperTree

if and only if its primal graph

$$\mathrm{Pr}\big(\mathbb{H}^{\langle r\rangle}\big)$$

is a connected acyclic finite simple graph.

Proof. By Theorem 4.12.4,

$$\Pr\bigl(\mathbb{H}^{\langle r\rangle}\bigr)$$

is a well-defined finite simple graph. For finite simple graphs, the notions of connectedness, acyclicity, and tree are standard and well-defined.

Therefore it is meaningful to ask whether

$$\Pr\bigl(\mathbb{H}^{\langle r\rangle}\bigr)$$

is a tree. By Definition 4.12.2, this is exactly the condition for $\mathbb{H}^{\langle r\rangle}$ to be a Hierarchical SuperHyperTree.

Finally, by the standard characterization of trees in finite graph theory, a finite simple graph is a tree if and only if it is connected and acyclic. Applying this characterization to $\Pr(\mathbb{H}^{\langle r\rangle})$, we obtain the stated equivalence. Hence the notion of Hierarchical SuperHyperTree is well-defined. □

Proposition 4.12.6 (Equivalent edge-count characterization)**.** *Let*

$$\mathbb{H}^{\langle r\rangle} = (V, E)$$

be a hierarchical SuperHyperGraph of height r, and let

$$\Pr\bigl(\mathbb{H}^{\langle r\rangle}\bigr) = \bigl(V, E_{\Pr}\bigr)$$

be its primal graph. Then $\mathbb{H}^{\langle r\rangle}$ is a Hierarchical SuperHyperTree if and only if

$$\Pr\bigl(\mathbb{H}^{\langle r\rangle}\bigr) \text{ is connected} \qquad \text{and} \qquad |E_{\Pr}| = |V| - 1.$$

Proof. Because $\Pr(\mathbb{H}^{\langle r\rangle})$ is a finite simple graph by Theorem 4.12.4, the standard characterization of finite trees applies: a finite simple graph is a tree if and only if it is connected and has exactly one fewer edge than vertices. Thus

$$\Pr\bigl(\mathbb{H}^{\langle r\rangle}\bigr) \text{ is a tree}$$

if and only if

$$\Pr\bigl(\mathbb{H}^{\langle r\rangle}\bigr) \text{ is connected} \quad \text{and} \quad |E_{\Pr}| = |V| - 1.$$

Using Definition 4.12.2, this is equivalent to saying that $\mathbb{H}^{\langle r\rangle}$ is a Hierarchical SuperHyperTree. □

Remark 4.12.7. The tree property is imposed on the induced vertex-adjacency structure, not directly on the hierarchical superhyperedge family itself. Thus different hierarchical superhyperedge families may determine the same Hierarchical SuperHyperTree whenever they induce the same primal graph.

Chapter 5

Revisiting Graph Classes Closely Related to SuperHyperGraphs

In addition to Hierarchical SuperHyperGraphs and Recursive SuperHyperGraphs, several other graph classes closely related to SuperHyperGraphs have also been studied. This chapter examines those related concepts and provides comparisons with them.

5.1 Meta-Graph and Iterated Meta-Graph

A meta-graph has graphs as vertices; edges represent labeled relations between those graphs, satisfying relation-defined incidence constraints (cf. [144–148]). A meta-graph is also called a graph of graphs. An iterated meta-graph repeats the construction: vertices are meta-graphs of lower depth, with lifted relations linking them recursively [144]. A similar concept known as the composite graph has also been studied [149–152]. Concepts such as graph clustering can likewise be regarded as operations for constructing a metagraph [153–156].

Definition 5.1.1 (Meta-Graph (Metagraph; graph of graphs))**.** [144] Fix a nonempty universe $\mathcal{G}$ of finite graphs (undirected and loopless by default), and fix a nonempty family $\mathcal{R}$ of binary relations on $\mathcal{G}$, i.e.

$$\mathcal{R} \subseteq \mathcal{P}(\mathcal{G} \times \mathcal{G}).$$

A *Meta-Graph* (or *metagraph*) over $(\mathcal{G}, \mathcal{R})$ is a directed, $\mathcal{R}$-labeled multigraph

$$M = (V, E, s, t, \lambda)$$

such that

$$V \subseteq \mathcal{G}, \qquad s, t : E \to V, \qquad \lambda : E \to \mathcal{R},$$

and the following *incidence constraint* holds:

$$\forall e \in E : \quad \big(s(e), t(e)\big) \in \lambda(e).$$

Elements of V are called *meta-vertices* (each meta-vertex is itself a graph in $\mathcal{G}$), and each $e \in E$ is a *meta-edge* labeled by the relation $\lambda(e)$. If $\mathcal{R} = \{R\}$ is a singleton, the label map may be omitted; if each $R \in \mathcal{R}$ is symmetric, one may view M as an undirected labeled multigraph.

Definition 5.1.2 (Iterated Meta-Graph (depth t)). [144] Fix $(\mathcal{G}, \mathcal{R})$ as in Definition 5.1.1. Define recursively, for $t \in \mathbb{N}_0$, a universe of *level-t objects* $\mathcal{G}^{(t)}$ and a family of *level-t relations* $\mathcal{R}^{(t)}$ as follows:

$$\mathcal{G}^{(0)} := \mathcal{G}, \qquad \mathcal{R}^{(0)} := \mathcal{R}.$$

Assume $\mathcal{G}^{(t)}$ and $\mathcal{R}^{(t)}$ are defined. Let $\mathcal{G}^{(t+1)}$ be the class of all finite metagraphs over $(\mathcal{G}^{(t)}, \mathcal{R}^{(t)})$ (i.e. all tuples $M = (V(M), E(M), s_M, t_M, \lambda_M)$ satisfying Definition 5.1.1 with $\mathcal{G}, \mathcal{R}$ replaced by $\mathcal{G}^{(t)}, \mathcal{R}^{(t)}$).

For each relation $R \in \mathcal{R}^{(t)}$, define its *lift* $R^{\uparrow} \subseteq \mathcal{G}^{(t+1)} \times \mathcal{G}^{(t+1)}$ by

$$(M_1, M_2) \in R^{\uparrow} \quad :\Longleftrightarrow \quad \exists\, x \in V(M_1),\ \exists\, y \in V(M_2) \text{ such that } (x, y) \in R.$$

Set

$$\mathcal{R}^{(t+1)} := \{\, R^{\uparrow} \mid R \in \mathcal{R}^{(t)} \,\}.$$

An *Iterated Meta-Graph of depth* t is then a metagraph

$$M^{(t)} = (V^{(t)}, E^{(t)}, s^{(t)}, t^{(t)}, \lambda^{(t)})$$

over $\big(\mathcal{G}^{(t)}, \mathcal{R}^{(t)}\big)$, i.e.

$$V^{(t)} \subseteq \mathcal{G}^{(t)}, \qquad \lambda^{(t)} : E^{(t)} \to \mathcal{R}^{(t)}, \qquad \forall e \in E^{(t)} :\ \big(s^{(t)}(e), t^{(t)}(e)\big) \in \lambda^{(t)}(e).$$

In particular, depth 0 iterated meta-graphs are ordinary metagraphs, and depth $t \geq 1$ iterated meta-graphs have vertices that are themselves metagraphs, recursively, up to t levels.

A concise comparison between Iterated Meta-Graphs and n-SuperHyperGraphs is presented in Table 5.1.

5.2 n-Filtrated Graph

An n-Filtrated Graph is a graph equipped with n filtration levels, where vertices and edges appear progressively across nested layers.

Definition 5.2.1 (n-Filtrated Graph). Let V_0 be a finite nonempty set and let $n \in \mathbb{N}_0$. An *n-Filtrated Graph over* V_0 is a quadruple

$$\mathrm{FG}^{(n)} = (V, E, \partial, \lambda),$$

where

- $V \subseteq V_0$ is a finite set of vertices;
- E is a finite set of edge identifiers;
- $\partial : E \to \binom{V}{2}$ is an incidence map;

Table 5.1: A concise comparison between Iterated Meta-Graphs and n-SuperHyperGraphs.

Aspect	*Iterated Meta-Graph*	*n-SuperHyperGraph*
Basic form	$M^{(t)} = (V^{(t)}, E^{(t)}, s^{(t)}, t^{(t)}, \lambda^{(t)})$	$\mathrm{SHG}^{(n)} = (V, E)$
Underlying idea	A recursively defined graph-of-graphs structure.	A higher-order graph built by iterating the powerset construction.
Base setting	Starts from a universe of graphs and binary relations.	Starts from a finite nonempty base set V_0.
Vertex type	Vertices are graphs or lower-depth meta-graphs.	Vertices are nested set-valued objects in $\mathcal{P}^n(V_0)$.
Edge type	Directed labeled edges with source and target maps.	Nonempty subsets of the supervertex set.
Incidence	Determined by $$s^{(t)}, t^{(t)} : E^{(t)} \to V^{(t)}$$ together with a relation label $\lambda^{(t)}$.	Determined by set-membership: $$x \in \varepsilon.$$
Directionality	Directed in the basic form.	Usually undirected in the basic form.
Hierarchy mechanism	Obtained recursively from lower-depth meta-graphs.	Obtained by repeated powersets: $$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)).$$
Typical use	Modeling relations among graphs or recursively linked graph systems.	Modeling nested and hierarchical incidence structures.

- $\lambda : V \sqcup E \to \{0, 1, \ldots, n\}$ is a filtration-level map such that

$$\partial(e) = \{u, v\} \implies \lambda(e) \geq \max\{\lambda(u), \lambda(v)\}$$

for every $e \in E$.

For each $k \in \{0, 1, \ldots, n\}$, define

$$V_k := \{v \in V \mid \lambda(v) \leq k\}, \qquad E_k := \{e \in E \mid \lambda(e) \leq k\}.$$

Then

$$G_k := (V_k, E_k, \partial|_{E_k})$$

is called the k-th filtration layer of $\mathrm{FG}^{(n)}$.

For reference, a concise comparison between n-Filtrated Graphs and n-SuperHyperGraphs is presented in Table 5.2.

5.3 MultiGraph and Iterated MultiGraph

A multigraph is a graph allowing parallel edges and loops; formally edges are a multiset with multiplicities between vertices possibly [157–159]. As extensions, concepts such as fuzzy multigraphs [160–162], bipartite multigraphs [163,164], complete multigraphs [165,166], neutrosophic multigraphs [157,167], soft multigraphs [168], and directed multigraphs [169] are known. An iterated multigraph uses iterated multisets as vertex objects, so vertices themselves can be multisets nested to depth n recursively.

Table 5.2: A concise comparison between n-Filtrated Graphs and n-SuperHyperGraphs.

Aspect	*n-Filtrated Graph*	*n-SuperHyperGraph*
Basic form	$\mathcal{F}^{(n)} = (G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n)$	$\mathrm{SHG}^{(n)} = (V, E)$
Underlying idea	A graph with a finite filtration across levels.	A higher-order structure obtained by iterating the powerset construction.
Base setting	Built from ordinary graphs $G_k = (V_k, E_k)$.	Built from a finite nonempty base set V_0.
Vertex type	Vertices remain ordinary graph vertices.	Vertices are set-valued objects in $\mathcal{P}^n(V_0)$.
Edge type	Edges remain ordinary binary edges.	Edges are nonempty subsets of the supervertex set.
Hierarchy mechanism	Hierarchy comes from levelwise filtration.	Hierarchy comes from iterated powersets.
Incidence	Classical graph incidence between two endpoints.	Set-membership incidence between supervertices and superedges.
Typical use	Modeling layered or progressive graph growth.	Modeling nested and hierarchical incidence structures.
Special case	If all levels coincide, it reduces to one ordinary graph.	If $n = 0$, it reduces to a hypergraph-type structure on V_0.

Definition 5.3.1 (Finite multiset and iterated multiset)**.** Let X be a set. A *finite multiset* on X is a function

$$m : X \to \mathbb{N}_0$$

whose *support*

$$\mathrm{supp}(m) := \{x \in X \mid m(x) > 0\}$$

is finite. We write $\mathsf{M}(X)$ for the set of all finite multisets on X.

For $n \geq 0$, define the *n-fold iterated multiset sets* recursively by

$$\mathsf{M}^0(X) := X, \qquad \mathsf{M}^{n+1}(X) := \mathsf{M}\big(\mathsf{M}^n(X)\big).$$

An element of $\mathsf{M}^n(X)$ is called an *n-fold iterated multiset over X*.

Definition 5.3.2 (MultiGraph (undirected multigraph))**.** Let V be a finite set. Denote by

$$\binom{V}{2}^m := \big\{\{\{u, v\}\} \;\big|\; u, v \in V\big\}$$

the set of *unordered pairs with repetition* (i.e. 2-element multisets), so that $\{\{v, v\}\}$ represents a loop at v.

A *MultiGraph* (undirected multigraph) on V is a pair

$$G = (V, \mu),$$

where $\mu : \binom{V}{2}^m \to \mathbb{N}_0$ is an *edge-multiplicity function*. For $e \in \binom{V}{2}^m$, the value $\mu(e)$ is the number of parallel edges of type e. Equivalently, one may specify a finite multiset $E \in \mathsf{M}\Big(\binom{V}{2}^m\Big)$ and write $G = (V, E)$, where μ is the multiplicity function associated to E.

Remark 5.3.3 (Directed variant)**.** A *directed* multigraph can be defined similarly by a multiplicity map $\mu : V \times V \to \mathbb{N}_0$, where $\mu(u, v)$ counts the number of directed edges from u to v.

Definition 5.3.4 (Iterated MultiGraph of order n)**.** Let X be a nonempty base set and let $n \geq 0$. Set $\mathsf{M}^n(X)$ as in Definition 5.3.1. An *Iterated MultiGraph of order n over X* is an undirected multigraph whose vertex objects are n-fold iterated multisets over X; concretely, it is a pair

$$G^{(n)} = (V^{(n)}, \mu^{(n)})$$

such that:

1. $V^{(n)} \in \mathsf{M}^n(X)$ is an n-fold iterated multiset, interpreted as a *vertex multiset*; let

$$\underline{V}^{(n)} := \mathrm{supp}(V^{(n)}) \subseteq \mathsf{M}^n(X)$$

   be the underlying *set* of distinct vertices.

2. $\mu^{(n)} : \binom{\underline{V}^{(n)}}{2}^m \to \mathbb{N}_0$ is an edge-multiplicity function on unordered pairs (with repetition) of distinct vertices.

Equivalently, one may specify an edge multiset $E^{(n)} \in \mathsf{M}\left(\binom{\underline{V}^{(n)}}{2}^m\right)$ and write $G^{(n)} = (V^{(n)}, E^{(n)})$.

Remark 5.3.5 (Order 0 recovers ordinary multigraphs)**.** When $n = 0$, we have $\mathsf{M}^0(X) = X$, so an Iterated MultiGraph of order 0 is just a multigraph whose vertices lie in the base set X (up to the choice of vertex multiset versus vertex set).

A comparison between Iterated MultiGraphs and n-SuperHyperGraphs is presented in Table 5.3.

Table 5.3: A concise comparison between Iterated MultiGraphs and n-SuperHyperGraphs.

Aspect	*Iterated MultiGraph*	*n-SuperHyperGraph*
Basic form	$G^{(n)} = (V^{(n)}, \mu^{(n)})$	$\mathrm{SHG}^{(n)} = (V, E)$
Underlying idea	A multigraph with recursively defined multiset-based vertices.	A higher-order structure with recursively defined set-based vertices.
Base setting	Built from a base set X using the multiset operator M.	Built from a base set V_0 using the powerset operator $\mathcal{P}$.
Vertex type	Vertices are n-fold iterated multisets.	Vertices are elements of $\mathcal{P}^n(V_0)$.
Edge type	Edges are pairwise and may have multiplicity.	Edges are nonempty subsets of the supervertex set.
Incidence	Essentially pairwise.	Genuinely multiway.
Multiplicity	Fundamental in the basic model.	Not included in the basic model.
Higher-order feature	Higher-order structure appears mainly in the vertex domain.	Higher-order structure appears in both vertices and edges.
Typical use	Modeling repeated pairwise interactions on recursive vertex objects.	Modeling nested, hierarchical, and multiway incidence structures.
Special case	For $n = 0$, it reduces to an ordinary multigraph.	For $n = 0$, it becomes a hypergraph-type structure on V_0.

5.4 Neural Graph

In the literature, graph-based neural models are most commonly studied under the frameworks of graph neural networks and message passing neural networks. Motivated by these perspectives, we introduce the following mathematical notion of a *Neural Graph*.

Definition 5.4.1 (Neural Graph)**.** Let

$$G = (V, E)$$

be a finite simple graph, where $V \neq \emptyset$ is the vertex set and

$$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}.$$

Let $d_0, p, d, q \in \mathbb{N}$. A *Neural Graph* on G is a tuple

$$\mathcal{NG} = \big(G,\, x,\, a,\, h^{(0)},\, (\mathcal{M}_\ell, \mathcal{U}_\ell)_{\ell=0}^{L-1},\, \mathcal{R}\big),$$

where:

(i) $x : V \to \mathbb{R}^{d_0}$ is a vertex-feature map;

(ii) $a : E \to \mathbb{R}^p$ is an edge-feature map;

(iii) $h^{(0)} : V \to \mathbb{R}^d$ is an initial hidden-state map (often obtained from x by an encoder);

(iv) for each layer $\ell \in \{0, 1, \ldots, L-1\}$,

$$\mathcal{M}_\ell : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}^d$$

is a *message function*, and

$$\mathcal{U}_\ell : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$$

is a *state-update function*;

(v) $\mathcal{R}$ is a *readout function* that maps the final collection of vertex states to an output space $\mathbb{R}^q$, and is permutation-invariant with respect to the ordering of vertices.

These data satisfy the layerwise propagation rule: for each $\ell = 0, 1, \ldots, L-1$ and each $v \in V$,

$$m_v^{(\ell)} = \bigoplus_{u \in N_G(v)} \mathcal{M}_\ell\big(h_v^{(\ell)},\, h_u^{(\ell)},\, a(\{u, v\})\big),$$

$$h_v^{(\ell+1)} = \mathcal{U}_\ell\big(h_v^{(\ell)},\, m_v^{(\ell)}\big),$$

where $N_G(v)$ denotes the neighborhood of v in G, and $\bigoplus$ is a permutation-invariant aggregation operator (for example, summation, mean, or maximum).

The final graph-level representation is defined by

$$z_G = \mathcal{R}\big(\{\, h_v^{(L)} : v \in V \,\}\big) \in \mathbb{R}^q.$$

The structure $\mathcal{NG}$ is called a *Neural Graph*.

Remark 5.4.2. A Neural Graph may be interpreted as a graph equipped with neural states and learnable local propagation rules. The message functions encode information flow along edges, the update functions revise vertex states, and the readout function extracts either graph-level or vertex-level outputs.

Remark 5.4.3. If the edge-feature map a is omitted, then one obtains the simplified form

$$m_v^{(\ell)} = \bigoplus_{u \in N_G(v)} \mathcal{M}_\ell\big(h_v^{(\ell)}, h_u^{(\ell)}\big),$$

which is often sufficient in unweighted or unlabeled graph settings.

A concise comparison between Neural Graphs and SuperHyperGraphs is presented in Table 5.4.

Table 5.4: A concise comparison between Neural Graphs and SuperHyperGraphs.

Aspect	*Neural Graph*	*SuperHyperGraph*
Basic form	$\mathcal{NG} = (G, x, a, h^{(0)}, (\mathcal{M}_\ell, \mathcal{U}_\ell), \mathcal{R})$	$\mathrm{SHG}^{(n)} = (V, E)$
Underlying base	A finite simple graph $G = (V, E)$	A finite base set V_0
Vertex type	Ordinary graph vertices	Set-valued supervertices in $\mathcal{P}^n(V_0)$
Edge type	Ordinary pairwise edges	Nonempty subsets of the supervertex set
Main purpose	Neural representation learning on graphs	Higher-order structural modeling
Incidence	Pairwise adjacency	Multiway set-membership incidence
Extra data	Features, hidden states, message/update functions, readout	No neural mechanism in the basic definition
Higher-order aspect	Computation is layered, but the graph structure is still ordinary	Higher-order structure appears in both vertices and edges
Typical output	Learned vertex- or graph-level representations	A mathematical higher-order relational structure

5.5 Hierarchical and Recursive Iterated Meta-Graphs

In this section, we introduce two extensions of Iterated Meta-Graphs: Hierarchical Iterated Meta-Graphs and Recursive Iterated Meta-Graphs. Throughout this section, all object universes are assumed to be set-sized, and all metagraphs considered are finite. This convention avoids set-theoretic ambiguities and ensures that all recursively defined families below are genuine sets.

Definition 5.5.1 (Level-tagged iterated meta-object universe)**.** Let $(\mathcal{G}, \mathcal{R})$ be a base pair as in Definition 5.1.1, and let

$$(\mathcal{G}^{(i)}, \mathcal{R}^{(i)})_{i \geq 0}$$

be the recursively defined iterated meta-object universes and lifted relation families from Definition 5.1.2.

For $r \in \mathbb{N}_0$, define the *level-tagged iterated meta-object universe of height* r by

$$\mathbb{G}_{\leq r} := \bigsqcup_{i=0}^{r} \mathcal{G}^{(i)} = \{\, (i, X) \mid 0 \leq i \leq r,\ X \in \mathcal{G}^{(i)} \,\}.$$

The integer i is called the *level* of (i, X) and is denoted by

$$\ell(i, X) := i.$$

For each $0 \leq i \leq r$ and each relation $R \in \mathcal{R}^{(i)}$, define its level-tagged version

$$\widehat{R}_i \subseteq \mathbb{G}_{\leq r} \times \mathbb{G}_{\leq r}$$

by

$$\big((i, X), (i, Y)\big) \in \widehat{R}_i \iff (X, Y) \in R.$$

Thus $\widehat{R}_i$ only relates objects lying at the same level i.

For $1 \leq i \leq r$, define the *immediate constituent relation*

$$\mathsf{C}_i \subseteq \mathbb{G}_{\leq r} \times \mathbb{G}_{\leq r}$$

by

$$\big((i-1, X), (i, M)\big) \in \mathsf{C}_i \iff M \in \mathcal{G}^{(i)} \text{ and } X \in V(M),$$

where $V(M)$ denotes the vertex set of the metagraph M. Its inverse relation is denoted by

$$\mathsf{C}_i^{-1} := \{\, ((i, M), (i-1, X)) \mid ((i-1, X), (i, M)) \in \mathsf{C}_i \,\}.$$

The *canonical hierarchical relation family of height* r is

$$\mathbb{R}^{\mathrm{H}}_{\leq r} := \{\, \widehat{R}_i \mid 0 \leq i \leq r,\ R \in \mathcal{R}^{(i)} \,\} \cup \{\, \mathsf{C}_i, \mathsf{C}_i^{-1} \mid 1 \leq i \leq r \,\}.$$

Definition 5.5.2 (Hierarchical Iterated Meta-Graph)**.** Let $(\mathcal{G}, \mathcal{R})$ be a base pair as in Definition 5.1.1, and let $r \in \mathbb{N}_0$. A *Hierarchical Iterated Meta-Graph of height* r over $(\mathcal{G}, \mathcal{R})$ is a tuple

$$\mathrm{HIMG}^{\langle r \rangle} = (W, F, s, t, \lambda)$$

satisfying the following conditions.

(i) *Hierarchical meta-vertex set.*

$$W \subseteq \mathbb{G}_{\leq r}$$

is a finite set. Its elements are called *hierarchical iterated meta-vertices.*

(ii) *Hierarchical meta-edge set.* F is a finite set whose elements are called *hierarchical iterated meta-edges.*

(iii) *Source and target maps.* There are maps

$$s, t : F \longrightarrow W.$$

(iv) *Relation-label map.* There is a map

$$\lambda : F \longrightarrow \mathbb{R}^{\mathrm{H}}_{\leq r}.$$

(v) *Typed incidence constraint.* For every $f \in F$,

$$\big(s(f), t(f)\big) \in \lambda(f).$$

(vi) *Downward hierarchical coherence.* If $(i, M) \in W$ and $i \geq 1$, then every immediate constituent of M is also present one level below; that is,

$$X \in V(M) \quad \Longrightarrow \quad (i-1, X) \in W.$$

Theorem 5.5.3 (Well-definedness of Hierarchical Iterated Meta-Graphs)**.** *Let $(\mathcal{G}, \mathcal{R})$ be a set-sized base pair, and let $r \in \mathbb{N}_0$. Then the notion of a Hierarchical Iterated Meta-Graph of height r in Definition 5.5.2 is well-defined.*

More precisely:

(i) *the universe $\mathbb{G}_{\leq r}$ is a well-defined level-tagged set;*

(ii) *the relation family $\mathbb{R}^{\mathrm{H}}_{\leq r}$ is a well-defined family of binary relations on $\mathbb{G}_{\leq r}$;*

(iii) *the incidence condition*

$$\big(s(f), t(f)\big) \in \lambda(f)$$

is meaningful for every $f \in F$;

(iv) *the downward coherence condition is meaningful and finite.*

Proof. Since $(\mathcal{G}, \mathcal{R})$ is set-sized, the recursively constructed families $\mathcal{G}^{(i)}$ and $\mathcal{R}^{(i)}$ are set-sized for every finite i. Hence the finite disjoint union

$$\mathbb{G}_{\leq r} = \bigsqcup_{i=0}^{r} \mathcal{G}^{(i)}$$

is a well-defined set. The use of level tags is essential: even if the same underlying object could be represented at different levels, the tagged objects (i, X) and (j, X) are distinct whenever $i \neq j$.

For each $R \in \mathcal{R}^{(i)}$, the relation $\widehat{R}_i$ is defined as a subset of $\mathbb{G}_{\leq r} \times \mathbb{G}_{\leq r}$. Therefore $\widehat{R}_i$ is a well-defined binary relation on $\mathbb{G}_{\leq r}$. Moreover, for $1 \leq i \leq r$, every $M \in \mathcal{G}^{(i)}$ is, by construction, a finite metagraph over $(\mathcal{G}^{(i-1)}, \mathcal{R}^{(i-1)})$. Hence its vertex set $V(M)$ is well-defined, and so the constituent relation

$$\mathsf{C}_i = \{\, ((i-1, X), (i, M)) \mid X \in V(M) \,\}$$

is also a well-defined binary relation on $\mathbb{G}_{\leq r}$. The same is true for its inverse C_i^{-1}. Thus $\mathbb{R}^{\mathrm{H}}_{\leq r}$ is a well-defined family of binary relations on $\mathbb{G}_{\leq r}$.

Now let

$$\mathrm{HIMG}^{\langle r \rangle} = (W, F, s, t, \lambda)$$

satisfy Definition 5.5.2. Since $s, t : F \to W$ and $W \subseteq \mathbb{G}_{\le r}$, both $s(f)$ and $t(f)$ are elements of $\mathbb{G}_{\le r}$ for every $f \in F$. Since $\lambda(f) \in \mathbb{R}^{\mathrm{H}}_{\le r}$ is a binary relation on $\mathbb{G}_{\le r}$, the statement

$$\big(s(f), t(f)\big) \in \lambda(f)$$

has a definite truth value. Thus the typed incidence constraint is meaningful.

Finally, if $(i, M) \in W$ with $i \ge 1$, then $M \in \mathcal{G}^{(i)}$ is a finite metagraph, and hence $V(M)$ is a finite set. Therefore the condition

$$X \in V(M) \Longrightarrow (i-1, X) \in W$$

is a finite, well-defined condition. Consequently, every clause in Definition 5.5.2 is meaningful and set-theoretically legitimate. Hence the definition is well-defined. □

Corollary 5.5.4 (Level-zero case)**.** *For $r = 0$, a Hierarchical Iterated Meta-Graph of height* 0 *is precisely a Meta-Graph over $(\mathcal{G}, \mathcal{R})$, after identifying each tagged object $(0, X)$ with X.*

Proof. If $r = 0$, then

$$\mathbb{G}_{\le 0} = \{0\} \times \mathcal{G}$$

and

$$\mathbb{R}^{\mathrm{H}}_{\le 0} = \{\, \widehat{R}_0 \mid R \in \mathcal{R} \,\}.$$

There are no constituent relations C_i, because there is no positive level. Hence Definition 5.5.2 reduces to the usual definition of a directed $\mathcal{R}$-labeled metagraph, up to the harmless level tag $(0, \cdot)$. □

Definition 5.5.5 (Recursive Iterated Meta-Graph)**.** Let $(\mathcal{G}, \mathcal{R})$ be a base pair as in Definition 5.1.1. Let $t, k \in \mathbb{N}_0$.

A *Recursive Iterated Meta-Graph of iterated depth t and recursive depth k* over $(\mathcal{G}, \mathcal{R})$ is a tuple

$$\mathrm{RIMG}^{(t,k)} = \Big(W,\ (E_j)_{j=0}^{k},\ (s_j, t_j, \lambda_j)_{j=0}^{k}\Big)$$

constructed as follows.

First, choose a finite set

$$W \subseteq \mathcal{G}^{(t)}.$$

The elements of W are called *top-level iterated meta-vertices*. Put

$$\mathcal{O}_0 := W.$$

Define the initial support map

$$\mathrm{Supp}_0 : \mathcal{O}_0 \longrightarrow \mathcal{P}(W) \setminus \{\emptyset\}$$

by

$$\mathrm{Supp}_0(x) := \{x\} \qquad (x \in W).$$

Assume that, for some $0 \le j \le k$, the finite object set $\mathcal{O}_j$ and the support map

$$\mathrm{Supp}_j : \mathcal{O}_j \longrightarrow \mathcal{P}(W) \setminus \{\emptyset\}$$

have already been defined. For each relation $R \in \mathcal{R}^{(t)}$, define its *j-th recursive support lift*

$$R^{[j]} \subseteq \mathcal{O}_j \times \mathcal{O}_j$$

by

$$(a, b) \in R^{[j]} \quad \Longleftrightarrow \quad \exists x \in \mathrm{Supp}_j(a),\ \exists y \in \mathrm{Supp}_j(b) \text{ such that } (x, y) \in R.$$

Choose a finite set E_j, disjoint from $\mathcal{O}_j$, whose elements are called *recursive meta-edges of rank j*. Choose maps

$$s_j, t_j : E_j \longrightarrow \mathcal{O}_j, \qquad \lambda_j : E_j \longrightarrow \mathcal{R}^{(t)}$$

such that the following incidence condition holds for every $e \in E_j$:

$$\big(s_j(e), t_j(e)\big) \in \lambda_j(e)^{[j]}.$$

If $j < k$, define

$$\mathcal{O}_{j+1} := \mathcal{O}_j \sqcup E_j,$$

and extend the support map to

$$\mathrm{Supp}_{j+1} : \mathcal{O}_{j+1} \longrightarrow \mathcal{P}(W) \setminus \{\emptyset\}$$

by

$$\mathrm{Supp}_{j+1}(a) := \mathrm{Supp}_j(a) \qquad (a \in \mathcal{O}_j),$$

and

$$\mathrm{Supp}_{j+1}(e) := \mathrm{Supp}_j\big(s_j(e)\big) \cup \mathrm{Supp}_j\big(t_j(e)\big) \qquad (e \in E_j).$$

The total recursive meta-edge set is

$$E := \bigsqcup_{j=0}^{k} E_j.$$

An element of E_j is said to have *recursive rank j*.

Theorem 5.5.6 (Well-definedness of Recursive Iterated Meta-Graphs)**.** *Let $(\mathcal{G}, \mathcal{R})$ be a set-sized base pair, and let $t, k \in \mathbb{N}_0$. Then the notion of a Recursive Iterated Meta-Graph* $\mathrm{RIMG}^{(t,k)}$ *in Definition 5.5.5 is well-defined.*

Moreover, the recursive dependency relation among meta-edges is well-founded: if

$$e' \prec e$$

means that e' is used as the source or target of e, then every $\prec$-chain has length at most k.

Proof. Since $(\mathcal{G}, \mathcal{R})$ is set-sized, the iterated universe $\mathcal{G}^{(t)}$ and the relation family $\mathcal{R}^{(t)}$ are well-defined. Thus any finite subset

$$W \subseteq \mathcal{G}^{(t)}$$

is a well-defined set. Hence

$$\mathcal{O}_0 = W$$

is well-defined, and the support map

$$\mathrm{Supp}_0(x) = \{x\}$$

is a well-defined map from $\mathcal{O}_0$ to $\mathcal{P}(W) \setminus \{\emptyset\}$.

Assume inductively that $\mathcal{O}_j$ and

$$\mathrm{Supp}_j : \mathcal{O}_j \to \mathcal{P}(W) \setminus \{\emptyset\}$$

are well-defined. For each $R \in \mathcal{R}^{(t)}$, the lifted relation

$$R^{[j]} = \{\, (a, b) \in \mathcal{O}_j \times \mathcal{O}_j \mid \exists x \in \mathrm{Supp}_j(a),\ \exists y \in \mathrm{Supp}_j(b),\ (x, y) \in R \,\}$$

is a well-defined subset of $\mathcal{O}_j \times \mathcal{O}_j$. Therefore, for each $e \in E_j$, the condition

$$\big(s_j(e), t_j(e)\big) \in \lambda_j(e)^{[j]}$$

has a definite truth value, because $s_j(e), t_j(e) \in \mathcal{O}_j$ and $\lambda_j(e)^{[j]}$ is a binary relation on $\mathcal{O}_j$.

If $j < k$, the next object set

$$\mathcal{O}_{j+1} = \mathcal{O}_j \sqcup E_j$$

is a well-defined finite disjoint union. The extended support map is also well-defined: for old objects $a \in \mathcal{O}_j$, it keeps the already defined support $\mathrm{Supp}_j(a)$; for a new edge $e \in E_j$, it assigns

$$\mathrm{Supp}_j(s_j(e)) \cup \mathrm{Supp}_j(t_j(e)).$$

Both sets in this union are nonempty subsets of W, so their union is again a nonempty subset of W. Hence

$$\mathrm{Supp}_{j+1} : \mathcal{O}_{j+1} \to \mathcal{P}(W) \setminus \{\emptyset\}$$

is well-defined.

By finite induction on $j = 0, 1, \ldots, k$, all object sets, support maps, lifted relations, source maps, target maps, and label maps appearing in Definition 5.5.5 are well-defined.

It remains to verify well-foundedness. A rank-j edge $e \in E_j$ may use only objects of $\mathcal{O}_j$ as its source and target. Since

$$\mathcal{O}_j = W \sqcup E_0 \sqcup \cdots \sqcup E_{j-1}$$

for $j \geq 1$, any recursive edge appearing as a source or target of e must have rank strictly smaller than j. Therefore, along every dependency chain

$$e_0 \prec e_1 \prec \cdots \prec e_m,$$

the ranks strictly increase from left to right. Since the allowed ranks are only

$$0, 1, \ldots, k,$$

one must have $m \leq k$. Thus no directed dependency cycle can occur, and the recursive dependency relation is well-founded with depth bounded by k. □

Corollary 5.5.7 (Rank-zero case)**.** *For $k = 0$, a Recursive Iterated Meta-Graph* $\mathrm{RIMG}^{(t,0)}$ *is an ordinary Iterated Meta-Graph of depth t on the vertex set $W \subseteq \mathcal{G}^{(t)}$.*

Proof. If $k = 0$, then the only recursive meta-edge set is E_0, and

$$\mathcal{O}_0 = W.$$

For every $R \in \mathcal{R}^{(t)}$, the lifted relation $R^{[0]}$ satisfies

$$(a, b) \in R^{[0]} \iff (a, b) \in R$$

for all $a, b \in W$, because $\mathrm{Supp}_0(a) = \{a\}$ and $\mathrm{Supp}_0(b) = \{b\}$. Therefore the incidence condition

$$\big(s_0(e), t_0(e)\big) \in \lambda_0(e)^{[0]}$$

is exactly the ordinary metagraph incidence condition over $(\mathcal{G}^{(t)}, \mathcal{R}^{(t)})$. Hence $\mathrm{RIMG}^{(t,0)}$ is precisely an Iterated Meta-Graph of depth t. □

Remark 5.5.8 (Difference between the two extensions)**.** A Hierarchical Iterated Meta-Graph allows vertices from several iterated levels

$$0, 1, \ldots, r$$

to coexist in one labeled metagraph, together with explicit constituent relations between adjacent levels. By contrast, a Recursive Iterated Meta-Graph fixes one iterated level t for its top-level meta-vertices, but allows higher-rank meta-edges to use lower-rank meta-edges as source or target objects. Thus the former is a mixed-level extension, while the latter is a well-founded recursive-edge extension.

A comparison between Hierarchical Iterated Meta-Graphs and Hierarchical SuperHyperGraphs is presented in Table 5.5.

Table 5.5: Concise comparison between Hierarchical Iterated Meta-Graphs and Hierarchical SuperHyperGraphs.

Aspect	*Hierarchical Iterated Meta-Graph*	*Hierarchical SuperHyperGraph*
Basic objects	Iterated meta-objects $\mathcal{G}^{(i)}$.	Iterated powerset objects $\mathcal{P}^i(V_0)$.
Vertex domain	Vertices may lie in several meta-levels $0, \ldots, r$.	Supervertices may lie in several powerset levels $0, \ldots, r$.
Edge type	Relation-labeled directed meta-edges.	Hyperedges joining nonempty subsets of supervertices.
Hierarchy	Level tags and constituent relations link adjacent meta-levels.	Downward closure links supervertices with lower constituents.
Incidence	Incidence is controlled by admissible meta-relations.	Incidence is set-based and higher-order.
Typical use	Multi-level relations among graph-like objects.	Hierarchical higher-order incidence structures.

A comparison between Recursive Iterated Meta-Graphs and Recursive SuperHyperGraphs is presented in Table 5.6.

Table 5.6: Concise comparison between Recursive Iterated Meta-Graphs and Recursive SuperHyperGraphs.

Aspect	*Recursive Iterated Meta-Graph*	*Recursive SuperHyperGraph*
Basic objects	Meta-vertices are taken from an iterated level $\mathcal{G}^{(t)}$.	Supervertices are taken from $\mathcal{P}^n(V_0)$.
Recursion	Lower-rank meta-edges may become later objects.	Lower recursive edges may occur inside higher edges.
Edge type	Directed, relation-labeled recursive meta-edges.	Recursive superhyperedges with finite higher-order incidence.
Depth parameter	k bounds edge-as-object recursion.	k bounds recursive edge nesting.
Incidence	Source and target are recursive meta-objects.	Edge elements may be vertices or lower edges.
Special case	$k = 0$ gives an Iterated Meta-Graph.	$k = 0$ gives an n-SuperHyperGraph.

5.6 Hierarchical and Recursive Filtrated Graphs

In this section, we introduce two extensions of filtrated graphs: Hierarchical Filtrated Graphs and Recursive Filtrated Graphs. The first one adds an explicit hierarchy among vertices, while the second one allows edges of higher recursive rank to use lower-rank edges as endpoint-like objects. In both cases, the filtration map is required to be compatible with incidence.

Definition 5.6.1 (Hierarchical Filtrated Graph)**.** Let A be a finite nonempty set, whose elements are called *atomic vertices*. Let $n, r \in \mathbb{N}_0$.

A *Hierarchical n-Filtrated Graph of height r* over A is a tuple

$$\mathrm{HFG}^{(n,r)} = (A, V, E, \partial, \lambda, \eta, \mathrm{Supp})$$

satisfying the following conditions.

(i) *Vertex set.* V is a finite set of vertex identifiers, and

$$A \subseteq V.$$

The elements of A are regarded as level-zero, atomic vertices.

(ii) *Edge identifiers.* E is a finite set of edge identifiers, disjoint from V.

(iii) *Graph incidence.* There is an incidence map

$$\partial : E \longrightarrow \binom{V}{2},$$

where

$$\binom{V}{2} = \{\{u, v\} \subseteq V \mid u \neq v\}.$$

Thus every edge has exactly two distinct endpoint vertices.

(iv) *Filtration map.* There is a map

$$\lambda : V \sqcup E \longrightarrow \{0, 1, \ldots, n\}$$

such that, whenever

$$\partial(e) = \{u, v\},$$

one has

$$\lambda(e) \geq \max\{\lambda(u), \lambda(v)\}.$$

(v) *Hierarchy-level map.* There is a map

$$\eta : V \longrightarrow \{0, 1, \ldots, r\}$$

such that

$$\eta(a) = 0 \qquad (a \in A).$$

(vi) *Support map.* There is a map

$$\mathrm{Supp} : V \longrightarrow \mathcal{P}(A) \setminus \{\emptyset\}$$

satisfying

$$\mathrm{Supp}(a) = \{a\} \qquad (a \in A).$$

For $v \in V$, the set $\mathrm{Supp}(v)$ is called the *atomic support* of v.

(vii) *Downward hierarchical coherence.* For every non-atomic vertex $v \in V \setminus A$, there exists a nonempty finite set

$$\mathrm{Ch}(v) \subseteq V$$

of *children* of v such that

$$\eta(w) < \eta(v) \qquad (w \in \mathrm{Ch}(v)),$$

and

$$\mathrm{Supp}(v) = \bigcup_{w \in \mathrm{Ch}(v)} \mathrm{Supp}(w).$$

For each filtration level $k \in \{0, 1, \ldots, n\}$, define

$$V_k := \{v \in V \mid \lambda(v) \leq k\}, \qquad E_k := \{e \in E \mid \lambda(e) \leq k\}.$$

The k-th filtration layer is

$$G_k := (V_k, E_k, \partial|_{E_k}).$$

Theorem 5.6.2 (Well-definedness of Hierarchical Filtrated Graphs)**.** *Let*

$$\mathrm{HFG}^{(n,r)} = (A, V, E, \partial, \lambda, \eta, \mathrm{Supp})$$

be a Hierarchical n-Filtrated Graph of height r in the sense of Definition 5.6.1. Then each filtration layer

$$G_k = (V_k, E_k, \partial|_{E_k}) \qquad (0 \leq k \leq n)$$

is a well-defined finite simple graph. Moreover, the hierarchy induced by η, Supp*, and* Ch *is well-founded and has height at most r.*

Proof. First, V and E are finite sets, and

$$\partial : E \to \binom{V}{2}$$

is a well-defined incidence map. Hence every edge identifier $e \in E$ has a unique unordered pair of distinct endpoints in V.

Fix $k \in \{0, 1, \ldots, n\}$. Since

$$V_k = \{v \in V \mid \lambda(v) \le k\}$$

and

$$E_k = \{e \in E \mid \lambda(e) \le k\},$$

both V_k and E_k are finite subsets of V and E, respectively. It remains to check that

$$\partial|_{E_k} : E_k \to \binom{V_k}{2}$$

is well-defined.

Let $e \in E_k$, and write

$$\partial(e) = \{u, v\}.$$

By the filtration compatibility condition,

$$\lambda(e) \ge \max\{\lambda(u), \lambda(v)\}.$$

Since $e \in E_k$, one has $\lambda(e) \le k$. Therefore

$$\lambda(u) \le k \qquad \text{and} \qquad \lambda(v) \le k.$$

Thus $u, v \in V_k$, and hence

$$\partial(e) = \{u, v\} \in \binom{V_k}{2}.$$

Consequently,

$$\partial|_{E_k} : E_k \to \binom{V_k}{2}$$

is a well-defined map. Therefore

$$G_k = (V_k, E_k, \partial|_{E_k})$$

is a finite simple graph for every k.

Next, consider the hierarchical part. For every vertex $v \in V$,

$$\mathrm{Supp}(v) \in \mathcal{P}(A) \setminus \{\emptyset\},$$

so each vertex has a nonempty atomic support. If $v \in A$, then

$$\mathrm{Supp}(v) = \{v\}$$

by definition. If $v \in V \setminus A$, then there exists a nonempty finite child set $\mathrm{Ch}(v) \subseteq V$ such that

$$\eta(w) < \eta(v) \qquad (w \in \mathrm{Ch}(v)).$$

Thus every step from a non-atomic vertex to one of its children strictly lowers the integer value of η. Since

$$\eta(V) \subseteq \{0, 1, \ldots, r\},$$

there is no infinite descending chain of children. Hence the hierarchy is well-founded, and every child chain has length at most r.

Finally, the identity

$$\mathrm{Supp}(v) = \bigcup_{w \in \mathrm{Ch}(v)} \mathrm{Supp}(w)$$

is meaningful because each $\mathrm{Supp}(w)$ is a nonempty subset of the finite set A. Hence the hierarchical support of every non-atomic vertex is uniquely determined by the supports of lower-level vertices. Therefore the whole structure is well-defined. □

Remark 5.6.3. The map λ controls the filtration level, whereas the map η controls the hierarchy level. These two maps represent different kinds of levels: λ describes when a vertex or edge appears in the filtration, while η describes how high a vertex is in the hierarchical aggregation of atomic vertices.

Definition 5.6.4 (Recursive Filtrated Graph)**.** Let A be a finite nonempty set of atomic vertices, and let $n, k \in \mathbb{N}_0$.

A *Recursive n-Filtrated Graph of recursive depth k* over A is a tuple

$$\mathrm{RFG}^{(n,k)} = \Big(A,\ (\mathcal{O}_i)_{i=0}^{k},\ (E_i)_{i=0}^{k},\ (\partial_i)_{i=0}^{k},\ (\lambda_i)_{i=0}^{k},\ (\mathrm{Supp}_i)_{i=0}^{k}\Big)$$

defined as follows.

(i) *Initial objects.* Set

$$\mathcal{O}_0 := A.$$

The elements of $\mathcal{O}_0$ are called *rank-zero objects.*

(ii) *Initial support.* Define

$$\mathrm{Supp}_0 : \mathcal{O}_0 \longrightarrow \mathcal{P}(A) \setminus \{\emptyset\}$$

by

$$\mathrm{Supp}_0(a) := \{a\} \qquad (a \in A).$$

(iii) *Rank-i recursive edges.* For each $i \in \{0, 1, \ldots, k\}$, E_i is a finite set of edge identifiers, disjoint from $\mathcal{O}_i$. Its elements are called *recursive edges of rank i.*

(iv) *Rank-i incidence.* For each $i \in \{0, 1, \ldots, k\}$, there is an incidence map

$$\partial_i : E_i \longrightarrow \binom{\mathcal{O}_i}{2}.$$

Thus a rank-i edge joins two distinct objects already available at rank i.

(v) *Rank-i filtration map.* For each $i \in \{0, 1, \ldots, k\}$, there is a map

$$\lambda_i : \mathcal{O}_i \sqcup E_i \longrightarrow \{0, 1, \ldots, n\}$$

such that, whenever

$$\partial_i(e) = \{x, y\},$$

one has

$$\lambda_i(e) \geq \max\{\lambda_i(x), \lambda_i(y)\}.$$

(vi) *Recursive enlargement of the object universe.* If $0 \leq i < k$, define

$$\mathcal{O}_{i+1} := \mathcal{O}_i \sqcup E_i.$$

Thus rank-i edges become available as objects at rank $i + 1$.

(vii) *Recursive support extension.* If $0 \le i < k$, define

$$\mathrm{Supp}_{i+1} : \mathcal{O}_{i+1} \longrightarrow \mathcal{P}(A) \setminus \{\emptyset\}$$

by

$$\mathrm{Supp}_{i+1}(x) := \mathrm{Supp}_i(x) \qquad (x \in \mathcal{O}_i),$$

and, for $e \in E_i$ with

$$\partial_i(e) = \{x, y\},$$

by

$$\mathrm{Supp}_{i+1}(e) := \mathrm{Supp}_i(x) \cup \mathrm{Supp}_i(y).$$

(viii) *Compatibility of filtration maps across ranks.* For $0 \le i < k$, the map λ_{i+1} extends λ_i on $\mathcal{O}_i$, and assigns to each old edge $e \in E_i$, now viewed as an object of $\mathcal{O}_{i+1}$, the same filtration level:

$$\lambda_{i+1}(x) = \lambda_i(x) \qquad (x \in \mathcal{O}_i),$$

and

$$\lambda_{i+1}(e) = \lambda_i(e) \qquad (e \in E_i \subseteq \mathcal{O}_{i+1}).$$

The total recursive edge set is

$$E := \bigsqcup_{i=0}^{k} E_i.$$

An element of E_i is said to have *recursive rank* i.

Theorem 5.6.5 (Well-definedness of Recursive Filtrated Graphs)**.** *Let*

$$\mathrm{RFG}^{(n,k)}$$

be a Recursive n-Filtrated Graph of recursive depth k in the sense of Definition 5.6.4. Then the following statements hold.

(i) *For every $i \in \{0, 1, \ldots, k\}$, the rank-$i$ structure*

$$G_i^{\mathrm{rec}} := (\mathcal{O}_i, E_i, \partial_i)$$

is a well-defined finite simple graph on the object set $\mathcal{O}_i$.

(ii) *For every $i \in \{0, 1, \ldots, k\}$ and every $\ell \in \{0, 1, \ldots, n\}$, the filtration layer*

$$G_{i,\ell}^{\mathrm{rec}} := (\mathcal{O}_{i,\ell}, E_{i,\ell}, \partial_i|_{E_{i,\ell}})$$

defined by

$$\mathcal{O}_{i,\ell} := \{x \in \mathcal{O}_i \mid \lambda_i(x) \le \ell\}, \qquad E_{i,\ell} := \{e \in E_i \mid \lambda_i(e) \le \ell\}$$

is a well-defined finite simple graph.

(iii) *The recursive dependency relation is well-founded and has depth at most k.*

Proof. We proceed in three steps.

First, for $i = 0$, one has
$$\mathcal{O}_0 = A,$$
which is finite by assumption. The map
$$\partial_0 : E_0 \to \binom{\mathcal{O}_0}{2}$$
is part of the data, so
$$G_0^{\mathrm{rec}} = (\mathcal{O}_0, E_0, \partial_0)$$
is a finite simple graph on $\mathcal{O}_0$.

Assume now that $\mathcal{O}_i$ is a finite well-defined set. Since E_i is finite and disjoint from $\mathcal{O}_i$, the disjoint union
$$\mathcal{O}_{i+1} = \mathcal{O}_i \sqcup E_i$$
is also a finite well-defined set. Hence, by induction, every $\mathcal{O}_i$ is finite and well-defined. Because each incidence map
$$\partial_i : E_i \to \binom{\mathcal{O}_i}{2}$$
is specified in the definition, every
$$G_i^{\mathrm{rec}} = (\mathcal{O}_i, E_i, \partial_i)$$
is a well-defined finite simple graph. This proves (i).

Second, fix $i \in \{0, 1, \ldots, k\}$ and $\ell \in \{0, 1, \ldots, n\}$. By definition,
$$\mathcal{O}_{i,\ell} = \{x \in \mathcal{O}_i \mid \lambda_i(x) \le \ell\}$$
and
$$E_{i,\ell} = \{e \in E_i \mid \lambda_i(e) \le \ell\}$$
are finite subsets of $\mathcal{O}_i$ and E_i, respectively. It remains to prove that
$$\partial_i|_{E_{i,\ell}} : E_{i,\ell} \to \binom{\mathcal{O}_{i,\ell}}{2}$$
is well-defined.

Let $e \in E_{i,\ell}$, and write
$$\partial_i(e) = \{x, y\}.$$
Since $e \in E_{i,\ell}$, one has
$$\lambda_i(e) \le \ell.$$
By the filtration compatibility condition,
$$\lambda_i(e) \ge \max\{\lambda_i(x), \lambda_i(y)\}.$$
Therefore
$$\lambda_i(x) \le \ell \qquad \text{and} \qquad \lambda_i(y) \le \ell.$$

Thus

$$x, y \in \mathcal{O}_{i,\ell},$$

and consequently

$$\partial_i(e) \in \binom{\mathcal{O}_{i,\ell}}{2}.$$

Hence

$$G_{i,\ell}^{\mathrm{rec}} = (\mathcal{O}_{i,\ell}, E_{i,\ell}, \partial_i|_{E_{i,\ell}})$$

is a well-defined finite simple graph. This proves (ii).

Third, define a dependency relation $\prec$ on the total recursive edge set

$$E = \bigsqcup_{i=0}^{k} E_i$$

as follows. For $e' \in E_j$ and $e \in E_i$, write

$$e' \prec e$$

if e' is used as one of the two endpoint objects of e, that is, if

$$e' \in \partial_i(e).$$

This can happen only when $e' \in \mathcal{O}_i$. Since

$$\mathcal{O}_i = A \sqcup E_0 \sqcup E_1 \sqcup \cdots \sqcup E_{i-1}$$

for $i \geq 1$, any recursive edge appearing as an endpoint of a rank-i edge must have rank strictly smaller than i. Hence, along any chain

$$e_0 \prec e_1 \prec \cdots \prec e_m,$$

the recursive ranks strictly increase from left to right. Since the only available ranks are

$$0, 1, \ldots, k,$$

one must have

$$m \leq k.$$

Therefore no directed dependency cycle is possible, and the recursive dependency relation is well-founded with depth at most k. This proves (iii). □

Corollary 5.6.6 (Ordinary filtrated graph as a special case)**.** *When $k = 0$, a Recursive n-Filtrated Graph of recursive depth* 0 *is exactly an n-Filtrated Graph on the atomic vertex set A.*

Proof. If $k = 0$, then the only object set is

$$\mathcal{O}_0 = A,$$

and the only edge set is E_0. The structure consists of

$$(A, E_0, \partial_0, \lambda_0),$$

where

$$\partial_0 : E_0 \to \binom{A}{2}$$

and

$$\lambda_0 : A \sqcup E_0 \to \{0, 1, \ldots, n\}$$

satisfy

$$\partial_0(e) = \{x, y\} \Longrightarrow \lambda_0(e) \geq \max\{\lambda_0(x), \lambda_0(y)\}.$$

This is precisely the defining data of an n-Filtrated Graph. □

Remark 5.6.7 (Difference between the two extensions)**.** A Hierarchical Filtrated Graph keeps ordinary graph edges, but equips vertices with a hierarchy through the maps

$$\eta : V \to \{0, \ldots, r\} \qquad \text{and} \qquad \mathrm{Supp} : V \to \mathcal{P}(A) \setminus \{\emptyset\}.$$

Thus it models layered aggregation of vertices together with filtration over time or scale.

A Recursive Filtrated Graph, by contrast, allows edges created at lower recursive ranks to become objects that can later serve as endpoints of higher-rank edges. The filtration map still controls when each object or edge appears, while recursive rank controls how deeply edges may be reused as objects.

A comparison between Hierarchical Filtrated Graphs and Hierarchical SuperHyperGraphs is presented in Table 5.7.

Table 5.7: Concise comparison between Hierarchical Filtrated Graphs and Hierarchical SuperHyperGraphs.

Aspect	*Hierarchical Filtrated Graph*	*Hierarchical SuperHyperGraph*
Basic objects	Ordinary vertices equipped with filtration and hierarchy levels.	Supervertices obtained from iterated powerset levels.
Main levels	Filtration level λ and hierarchy level η.	Hierarchical powerset level $0, \ldots, r$.
Vertex structure	Vertices may represent hierarchical aggregations of atomic vertices.	Vertices may be nested set-valued objects.
Edge structure	Ordinary pairwise graph edges.	Hyperedges joining nonempty subsets of supervertices.
Coherence	Child vertices determine the support of higher-level vertices.	Lower-level constituents must appear consistently.
Typical use	Time-, scale-, or layer-dependent hierarchical graphs.	Higher-order hierarchical incidence structures.

A comparison between Recursive Filtrated Graphs and Recursive SuperHyperGraphs is presented in Table 5.8.

5.7 Hierarchical and Recursive Iterated MultiGraphs

In this section, we introduce two extensions of Iterated MultiGraphs: Hierarchical Iterated MultiGraphs and Recursive Iterated MultiGraphs. The first one allows iterated-multiset vertices from several levels to coexist in one multigraph. The second one allows edge-tokens created at lower recursive ranks to become objects that can be used as endpoints at higher recursive ranks.

Table 5.8: Concise comparison between Recursive Filtrated Graphs and Recursive SuperHyperGraphs.

Aspect	*Recursive Filtrated Graph*	*Recursive SuperHyperGraph*
Basic objects	Atomic vertices and recursively promoted edge objects.	Supervertices from $\mathcal{P}^n(V_0)$.
Recursion	Edges may become objects at later ranks.	Lower recursive edges may appear inside higher edges.
Edge form	Pairwise edges between available objects.	Recursive hyperedges with finite higher-order incidence.
Filtration	Each object and edge has a filtration level.	Filtration is not basic unless added separately.
Depth parameter	k bounds edge-as-object promotion.	k bounds recursive edge nesting.
Special case	$k = 0$ gives an ordinary filtrated graph.	$k = 0$ gives an n-SuperHyperGraph.

Throughout this section, X denotes a nonempty base set, and $\mathsf{M}(X)$ denotes the set of all finite multisets on X. For $n \in \mathbb{N}_0$, the n-fold iterated multiset universe is denoted by

$$\mathsf{M}^0(X) := X, \qquad \mathsf{M}^{n+1}(X) := \mathsf{M}(\mathsf{M}^n(X)).$$

For a finite multiset m, its support is denoted by $\mathrm{supp}(m)$. For a set S, we write

$$\binom{S}{2}^m := \{\{\{a, b\}\} \mid a, b \in S\}$$

for the set of unordered pairs with repetition. Thus $\{\{a, a\}\}$ represents a loop at a.

Definition 5.7.1 (Level-tagged iterated multiset universe)**.** Let $r \in \mathbb{N}_0$. The *level-tagged iterated multiset universe of height r over X* is the disjoint union

$$\mathbb{M}_{\le r}(X) := \bigsqcup_{i=0}^{r} \mathsf{M}^i(X) = \{\, (i, a) \mid 0 \le i \le r,\ a \in \mathsf{M}^i(X) \,\}.$$

For $z = (i, a) \in \mathbb{M}_{\le r}(X)$, the integer

$$\ell(z) := i$$

is called the *level* of z.

If $i \ge 1$ and $a \in \mathsf{M}^i(X) = \mathsf{M}(\mathsf{M}^{i-1}(X))$, then a is a finite multiset on $\mathsf{M}^{i-1}(X)$. Its immediate lower-level constituents are the tagged objects

$$\mathrm{Ch}(i, a) := \{\, (i-1, b) \mid b \in \mathrm{supp}(a) \,\} \subseteq \mathbb{M}_{\le r}(X).$$

Definition 5.7.2 (Hierarchical Iterated MultiGraph)**.** Let $r \in \mathbb{N}_0$. A *Hierarchical Iterated MultiGraph of height r over X* is a quadruple

$$\mathrm{HIMuG}^{\langle r\rangle} = (X, r, \nu, \mu)$$

satisfying the following conditions.

(i) *Hierarchical vertex multiset.*

$$\nu \in \mathsf{M}(\mathbb{M}_{\le r}(X)).$$

Its support

$$W := \mathrm{supp}(\nu) \subseteq \mathbb{M}_{\le r}(X)$$

is called the *underlying hierarchical vertex set.*

(ii) *Edge-multiplicity function.* There is a function

$$\mu : \binom{W}{2}^{m} \longrightarrow \mathbb{N}_0.$$

For $p \in \binom{W}{2}^{m}$, the value $\mu(p)$ is the number of parallel edges of endpoint type p.

(iii) *Downward multiset-coherence.* For every vertex $z = (i, a) \in W$ with $i \geq 1$, every lower-level constituent of a appears in W with at least the multiplicity required by a. Equivalently,

$$b \in \operatorname{supp}(a) \quad \Longrightarrow \quad (i-1, b) \in W$$

and

$$\nu(i-1, b) \geq a(b) \qquad \big(b \in \operatorname{supp}(a)\big).$$

Here $a(b)$ denotes the multiplicity of b inside the finite multiset $a \in \mathsf{M}(\mathsf{M}^{i-1}(X))$.

The multiset of edges associated with μ is the finite multiset

$$E_\mu \in \mathsf{M}\left(\binom{W}{2}^{m}\right)$$

defined by

$$E_\mu(p) := \mu(p) \qquad (p \in \binom{W}{2}^{m}).$$

Thus one may also write

$$\mathrm{HIMuG}^{\langle r \rangle} = (W, E_\mu)$$

when the base set and the height are clear.

Theorem 5.7.3 (Well-definedness of Hierarchical Iterated MultiGraphs)**.** *Let*

$$\mathrm{HIMuG}^{\langle r \rangle} = (X, r, \nu, \mu)$$

be a Hierarchical Iterated MultiGraph in the sense of Definition 5.7.2. Then the following statements hold.

(i) *The level-tagged vertex universe* $\mathbb{M}_{\leq r}(X)$ *is a well-defined set.*

(ii) *The underlying hierarchical vertex set*

$$W = \operatorname{supp}(\nu)$$

is finite.

(iii) *The edge-multiplicity function*

$$\mu : \binom{W}{2}^{m} \to \mathbb{N}_0$$

defines a well-defined finite undirected multigraph, possibly with loops.

(iv) *The downward multiset-coherence condition is meaningful and finite.*

Proof. For each $i \in \{0, 1, \ldots, r\}$, the iterated multiset universe $\mathsf{M}^i(X)$ is defined recursively from X by repeated application of the finite-multiset operator M. Hence each $\mathsf{M}^i(X)$ is a well-defined set. Since r is finite, the disjoint union

$$\mathbb{M}_{\leq r}(X) = \bigsqcup_{i=0}^{r} \mathsf{M}^i(X)$$

is also a well-defined set. The level tag is important because the same underlying expression may occur at different iterated levels, while (i, a) and (j, a) are distinct whenever $i \neq j$.

Since

$$\nu \in \mathsf{M}(\mathbb{M}_{\leq r}(X)),$$

the support

$$W = \operatorname{supp}(\nu)$$

is finite by the definition of a finite multiset. Therefore

$$\binom{W}{2}^m = \{\{\{u, v\}\} \mid u, v \in W\}$$

is finite and well-defined. A function

$$\mu : \binom{W}{2}^m \to \mathbb{N}_0$$

therefore assigns a definite nonnegative integer multiplicity to each unordered pair with repetition of vertices in W. Consequently, μ determines a finite undirected multigraph on W, with loops allowed by pairs of the form $\{\{u, u\}\}$.

It remains to check the coherence condition. Let $z = (i, a) \in W$ with $i \geq 1$. Since

$$a \in \mathsf{M}^i(X) = \mathsf{M}(\mathsf{M}^{i-1}(X)),$$

the value $a(b)$ is a well-defined nonnegative integer for every $b \in \mathsf{M}^{i-1}(X)$, and the support $\operatorname{supp}(a)$ is finite. Hence the conditions

$$(i-1, b) \in W \qquad \text{and} \qquad \nu(i-1, b) \geq a(b)$$

are meaningful for each $b \in \operatorname{supp}(a)$, and only finitely many such b must be checked. Therefore the downward multiset-coherence condition is well-defined and finite.

Thus every component of $\mathrm{HIMuG}^{\langle r \rangle}$ is mathematically meaningful, and the structure is well-defined. □

Corollary 5.7.4 (Height-zero case)**.** *When $r = 0$, a Hierarchical Iterated MultiGraph of height 0 is an ordinary MultiGraph on a finite multiset of vertices from X, up to the harmless level tag $(0, \cdot)$.*

Proof. If $r = 0$, then
$$\mathbb{M}_{\le 0}(X) = \{0\} \times X.$$
There are no positive-level vertices and hence no downward coherence conditions. Thus Definition 5.7.2 reduces to an undirected multigraph on the finite vertex set
$$W = \operatorname{supp}(\nu) \subseteq \{0\} \times X.$$
Identifying $(0, x)$ with x, this is exactly an ordinary multigraph. □

Definition 5.7.5 (Recursive Iterated MultiGraph)**.** Let $n, k \in \mathbb{N}_0$. A *Recursive Iterated MultiGraph of iterated order* n *and recursive depth* k *over* X is a tuple
$$\mathrm{RIMuG}^{(n,k)} = \Big(X, n, k, \nu_0, (\mathcal{O}_i)_{i=0}^k, (\mu_i)_{i=0}^k, (E_i)_{i=0}^k, (\partial_i)_{i=0}^k, (\mathrm{Supp}_i)_{i=0}^k\Big)$$
defined as follows.

(i) *Initial iterated vertex multiset.*
$$\nu_0 \in \mathsf{M}(\mathsf{M}^n(X)).$$
Set
$$W_0 := \operatorname{supp}(\nu_0) \subseteq \mathsf{M}^n(X).$$

(ii) *Initial object set.*
$$\mathcal{O}_0 := W_0.$$
The elements of $\mathcal{O}_0$ are called *rank-zero objects.*

(iii) *Initial support map.* Define
$$\mathrm{Supp}_0 : \mathcal{O}_0 \longrightarrow \mathsf{M}(W_0)$$
by
$$\mathrm{Supp}_0(x) := \delta_x \qquad (x \in W_0),$$
where δ_x is the singleton multiset on W_0 with multiplicity 1 at x and multiplicity 0 elsewhere.

(iv) *Rank-i edge multiplicities.* For each $i \in \{0, 1, \ldots, k\}$, choose an edge-multiplicity function
$$\mu_i : \binom{\mathcal{O}_i}{2}^m \longrightarrow \mathbb{N}_0.$$

(v) *Rank-i edge tokens.* For each $i \in \{0, 1, \ldots, k\}$, define the finite set of rank-i edge tokens by
$$E_i := \left\{ (i, p, j) \;\middle|\; p \in \binom{\mathcal{O}_i}{2}^m,\ 1 \le j \le \mu_i(p) \right\}.$$
Thus parallel edges are represented by distinct tokens with the same endpoint pair p.

(vi) *Endpoint map.* For $e = (i, p, j) \in E_i$, define

$$\partial_i(e) := p.$$

Hence

$$\partial_i : E_i \longrightarrow \binom{\mathcal{O}_i}{2}^m.$$

(vii) *Recursive enlargement of objects.* If $0 \le i < k$, define

$$\mathcal{O}_{i+1} := \mathcal{O}_i \sqcup E_i.$$

Thus every rank-i edge token becomes an object available at rank $i + 1$.

(viii) *Recursive support extension.* If $0 \le i < k$, extend

$$\mathrm{Supp}_i : \mathcal{O}_i \to \mathsf{M}(W_0)$$

to

$$\mathrm{Supp}_{i+1} : \mathcal{O}_{i+1} \to \mathsf{M}(W_0)$$

as follows. For old objects, set

$$\mathrm{Supp}_{i+1}(x) := \mathrm{Supp}_i(x) \qquad (x \in \mathcal{O}_i).$$

For a new edge token $e \in E_i$, write

$$\partial_i(e) = \{\{a, b\}\}.$$

Then define

$$\mathrm{Supp}_{i+1}(e) := \mathrm{Supp}_i(a) \uplus \mathrm{Supp}_i(b),$$

where $\uplus$ denotes pointwise addition of finite multisets. If $a = b$, then this formula gives

$$\mathrm{Supp}_{i+1}(e) = \mathrm{Supp}_i(a) \uplus \mathrm{Supp}_i(a),$$

so loops are treated with the correct multiplicity.

The total recursive edge-token set is

$$E := \bigsqcup_{i=0}^{k} E_i.$$

An element of E_i is said to have *recursive rank* i.

Theorem 5.7.6 (Well-definedness of Recursive Iterated MultiGraphs)**.** *Let*

$$\mathrm{RIMuG}^{(n,k)}$$

be a Recursive Iterated MultiGraph in the sense of Definition 5.7.5. Then the following statements hold.

(i) *For every $i \in \{0, 1, \ldots, k\}$, the object set $\mathcal{O}_i$ is finite and well-defined.*

(ii) *For every* $i \in \{0, 1, \ldots, k\}$,

$$(\mathcal{O}_i, \mu_i)$$

is a well-defined undirected multigraph, possibly with loops and parallel edges.

(iii) *The edge-token set* E_i *and the endpoint map*

$$\partial_i : E_i \to \binom{\mathcal{O}_i}{2}^m$$

are well-defined for every i.

(iv) *The recursive support map*

$$\mathrm{Supp}_i : \mathcal{O}_i \to \mathsf{M}(W_0)$$

is well-defined for every i.

(v) *The recursive dependency relation among edge tokens is well-founded and has depth at most* k.

Proof. First, since

$$\nu_0 \in \mathsf{M}(\mathsf{M}^n(X)),$$

its support

$$W_0 = \mathrm{supp}(\nu_0)$$

is finite. Hence

$$\mathcal{O}_0 = W_0$$

is a finite well-defined set. The initial support map

$$\mathrm{Supp}_0(x) = \delta_x$$

is well-defined because $\delta_x \in \mathsf{M}(W_0)$ for every $x \in W_0$.

Assume inductively that $\mathcal{O}_i$ is finite and well-defined. Then

$$\binom{\mathcal{O}_i}{2}^m$$

is also finite and well-defined. Therefore a function

$$\mu_i : \binom{\mathcal{O}_i}{2}^m \to \mathbb{N}_0$$

defines an undirected multigraph on the object set $\mathcal{O}_i$, where $\mu_i(p)$ counts the number of parallel edge tokens of endpoint type p.

Since both $\binom{\mathcal{O}_i}{2}^m$ and all integers $\mu_i(p)$ are finite, the set

$$E_i = \left\{ (i, p, j) \;\middle|\; p \in \binom{\mathcal{O}_i}{2}^m,\ 1 \le j \le \mu_i(p) \right\}$$

is finite and well-defined. Moreover, for each $e = (i, p, j) \in E_i$, the assignment

$$\partial_i(e) = p$$

defines a map

$$\partial_i : E_i \to \binom{\mathcal{O}_i}{2}^m.$$

Thus E_i and ∂_i are well-defined.

If $i < k$, then

$$\mathcal{O}_{i+1} = \mathcal{O}_i \sqcup E_i$$

is a finite disjoint union of finite sets. Hence $\mathcal{O}_{i+1}$ is finite and well-defined. This proves by induction that every $\mathcal{O}_i$ is finite and well-defined, and consequently every $(\mathcal{O}_i, \mu_i)$ is a well-defined multigraph.

It remains to check the support maps. Suppose that

$$\mathrm{Supp}_i : \mathcal{O}_i \to \mathsf{M}(W_0)$$

is well-defined. For $x \in \mathcal{O}_i$, the formula

$$\mathrm{Supp}_{i+1}(x) = \mathrm{Supp}_i(x)$$

is meaningful. For $e \in E_i$, write

$$\partial_i(e) = \{\{a, b\}\}.$$

Since $a, b \in \mathcal{O}_i$, the finite multisets

$$\mathrm{Supp}_i(a), \mathrm{Supp}_i(b) \in \mathsf{M}(W_0)$$

are already defined. Their multiset sum

$$\mathrm{Supp}_i(a) \uplus \mathrm{Supp}_i(b)$$

is again an element of $\mathsf{M}(W_0)$. Therefore

$$\mathrm{Supp}_{i+1}(e) = \mathrm{Supp}_i(a) \uplus \mathrm{Supp}_i(b)$$

is well-defined. By induction, all support maps Supp_i are well-defined.

Finally, define a dependency relation $\prec$ on the total edge-token set

$$E = \bigsqcup_{i=0}^{k} E_i$$

as follows. For $e' \in E_j$ and $e \in E_i$, write

$$e' \prec e$$

if e' occurs as one of the two endpoint objects of e. This can happen only when $e' \in \mathcal{O}_i$. But

$$\mathcal{O}_i = W_0 \sqcup E_0 \sqcup E_1 \sqcup \cdots \sqcup E_{i-1} \qquad (i \geq 1).$$

Therefore, if $e' \in E_j$ is an endpoint of $e \in E_i$, then necessarily

$$j < i.$$

Thus recursive rank strictly increases along every dependency chain

$$e_0 \prec e_1 \prec \cdots \prec e_m.$$

Since the only possible ranks are

$$0, 1, \ldots, k,$$

every such chain has length at most k. Hence no directed dependency cycle can occur, and the recursive dependency relation is well-founded.

Therefore all parts of Definition 5.7.5 are well-defined. □

Corollary 5.7.7 (Recursive-depth-zero case)**.** *When $k = 0$, a Recursive Iterated MultiGraph*

$$\mathrm{RIMuG}^{(n,0)}$$

is precisely an Iterated MultiGraph of order n over X, with vertex multiset ν_0 and edge-multiplicity function

$$\mu_0 : \binom{\mathrm{supp}(\nu_0)}{2}^m \to \mathbb{N}_0.$$

Proof. If $k = 0$, then the only object set is

$$\mathcal{O}_0 = \mathrm{supp}(\nu_0) \subseteq \mathsf{M}^n(X),$$

and the only edge-multiplicity function is

$$\mu_0 : \binom{\mathcal{O}_0}{2}^m \to \mathbb{N}_0.$$

No edge-token is reused as an object at a higher recursive rank, because there is no higher rank. Hence the data are exactly those of an Iterated MultiGraph of order n: vertices are n-fold iterated multiset objects over X, and edges are ordinary multigraph edges between such objects, counted by μ_0. □

Remark 5.7.8 (Difference between the two extensions)**.** A Hierarchical Iterated MultiGraph permits vertices from several iterated multiset levels

$$\mathsf{M}^0(X), \mathsf{M}^1(X), \ldots, \mathsf{M}^r(X)$$

to coexist in one multigraph, subject to downward multiset-coherence. Thus its main purpose is mixed-level hierarchical modeling.

A Recursive Iterated MultiGraph fixes the initial vertex type at one iterated order n, but then allows edge tokens to become objects that may serve as endpoints of later edges. Thus its main purpose is recursive reuse of edges as higher-rank objects.

A comparison between Hierarchical Iterated MultiGraphs and Hierarchical SuperHyperGraphs is presented in Table 5.9.

A concise comparison between Recursive Iterated MultiGraphs and Recursive SuperHyperGraphs is presented in Table 5.10.

Table 5.9: Concise comparison between Hierarchical Iterated MultiGraphs and Hierarchical SuperHyperGraphs.

Aspect	*Hierarchical Iterated MultiGraph*	*Hierarchical SuperHyperGraph*
Basic objects	Iterated finite multisets $\mathsf{M}^i(X)$.	Iterated powerset objects $\mathcal{P}^i(V_0)$.
Vertex domain	Vertices may lie in $\bigsqcup_{i=0}^{r} \mathsf{M}^i(X)$.	Vertices may lie in $\bigcup_{i=0}^{r} \mathcal{P}^i(V_0)$.
Vertex multiplicity	Vertex multiplicities are allowed.	Vertices are usually treated as sets.
Edges	Pairwise multiedges with multiplicity.	Hyperedges joining any nonempty vertex subset.
Hierarchy	Lower multiset constituents must appear coherently.	Lower powerset constituents must appear coherently.
Best suited for	Repeated objects, loops, and parallel relations.	Higher-order hierarchical incidence.

Table 5.10: Concise comparison between Recursive Iterated MultiGraphs and Recursive SuperHyperGraphs.

Aspect	*Recursive Iterated MultiGraph*	*Recursive SuperHyperGraph*
Base objects	Initial vertices lie in $\mathsf{M}^n(X)$.	Supervertices lie in $\mathcal{P}^n(V_0)$.
Recursion	Edge tokens become later objects.	Lower edges may become edge elements.
Edge form	Pairwise multiedges, possibly parallel or loops.	Recursive hyperedges with arbitrary finite incidence.
Multiplicity	Explicit via edge multiplicity functions.	Not basic unless separately imposed.
Depth bound	k bounds edge-as-object promotion.	k bounds recursive edge nesting.
Special case	$k = 0$ gives an Iterated MultiGraph.	$k = 0$ gives an n-SuperHyperGraph.

Chapter 6

Conclusions

This book investigated several classes of SuperHyperGraph-based structures, with particular emphasis on Hierarchical SuperHyperGraphs and Recursive SuperHyperGraphs. Recursive SuperHyperGraphs were presented as structures in which n-level supervertices are combined with recursive superhyperedges of bounded depth. The recursive framework allows superhyperedges to contain not only supervertices but also lower-level recursive superhyperedges, while typed incidence closure and well-foundedness conditions prevent ill-defined circular recursion.

Several extensions of recursive SuperHyperGraphs were also examined, including fuzzy, single-valued neutrosophic, uncertain, soft, rough, and weighted versions. These constructions show how recursive incidence structures can be combined with membership degrees, neutrosophic information, general uncertainty models, parameterized families, approximation mechanisms, and numerical weights.

The book also developed the framework of Hierarchical SuperHyperGraphs. In this setting, supervertices from different iterated-powerset levels may coexist in a single structure, and mixed-level superhyperedges may connect objects belonging to different layers. A coherence or downward-closure condition was used to ensure that selected higher-level supervertices remain compatible with their immediate constituents.

Further hierarchical variants were considered, including fuzzy, directed, bidirected, uncertain, hyperfuzzy [170, 171], soft [172], rough [173, 174], and weighted hierarchical SuperHyperGraphs. Related structural notions such as distance, labeling, domination, and tree-like structures were also discussed. These notions provide a starting point for developing a broader theory of multi-level and nested incidence systems.

Finally, related graph-theoretic frameworks, including Meta-Graphs, Iterated Meta-Graphs, Filtrated Graphs, Iterated MultiGraphs, and Neural Graphs, were compared with hierarchical and recursive SuperHyperGraph models. These comparisons help clarify how SuperHyperGraph-based structures relate to other higher-order, multi-level, and representation-oriented graph formalisms.

The present book is mainly theoretical. Future research may develop more detailed structural theorems, algorithms, computational experiments, and applications in areas such as network science, artificial intelligence, knowledge representation, decision modeling, and complex systems.

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Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this book.

Code Availability

No code or software was developed for this study.

Use of Generative AI and AI-Assisted Tools

The authors used generative AI and AI-assisted tools only for limited editorial purposes, such as English grammar checking, stylistic improvement, and LaTeX error checking. These tools were not used to generate mathematical results, fabricate data, manipulate images, or perform any activity that would violate ethical or scholarly standards.

Disclaimer (Others)

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

Appendix (List of Tables)

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Bibliography

[1] Reinhard Diestel. *Graph theory.* Springer (print edition); Reinhard Diestel (eBooks), 2024.

[2] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, pages 3558–3565, 2019.

[3] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra.* Infinite Study, 2020.

[4] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[5] Takaaki Fujita and Florentin Smarandache. Fundamental computational problems and algorithms for superhypergraphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 240, 2024.

[6] N. B. Nalawade, M. S. Bapat, S. G. Jakkewad, G. A. Dhanorkar, and D. J. Bhosale. Structural properties of zero-divisor hypergraph and superhypergraph over $\mathbb{Z}_n$: Girth and helly property. *Panamerican Mathematical Journal*, 35(4S):485, 2025.

[7] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.

[8] Takaaki Fujita. Note for line and total superhypergraphs: Connecting vertices, edges, edges of edges, edges of edges of edges in hierarchical systems. *Journal of Mathematical Analysis and Modeling*, 6(3):11–43, 2025.

[9] Takaaki Fujita and Florentin Smarandache. Topological generalizations of graphs: Integrating hypergraph and superhypergraph perspectives. *Turkish Journal of Mathematics and Computer Science*, 17(2):322–337, 2025.

[10] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[11] Anandhan Prathik, K Uma, and J Anuradha. An overview of application of graph theory. *International Journal of ChemTech Research*, 9(2):242–248, 2016.

[12] Wai-Kai Chen. *Graph theory and its engineering applications*, volume 5. World Scientific Publishing Company, 1997.

[13] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications.* Chapman and Hall/CRC, 2018.

[14] Takaaki Fujita and Florentin Smarandache. Directed n-superhypergraphs incorporating bipolar fuzzy information: A multi-tier framework for modeling bipolar uncertainty in complex networks. *Neutrosophic Sets and Systems*, 88:164–183, 2025.

[15] Salomón Marcos Berrocal Villegas, Willner Montalvo Fritas, Carmen Rosa Berrocal Villegas, María Yissel Flores Fuentes Rivera, Roberto Espejo Rivera, Laura Daysi Bautista Puma, and Dante Manuel Macazana Fernández. Using plithogenic n-superhypergraphs to assess the degree of relationship between information skills and digital competencies. *Neutrosophic Sets and Systems*, 84(1):41, 2025.

[16] Takaaki Fujita. Molecular fuzzy graphs, hypergraphs, and superhypergraphs. *Journal of Intelligent Decision and Computational Modelling*, 1(3):158–171, 2025.

[17] Takaaki Fujita and Florentin Smarandache. Property hypergraphs and property superhypergraphs for data analysis. *European Journal of Pure and Applied Mathematics*, 18(4):6729–6729, 2025.

[18] Takaaki Fujita and Florentin Smarandache. Random n-superhypergraphs: A probabilistic model and generation algorithm for hierarchical networks. *European Journal of Pure and Applied Mathematics*, 18(4):6835–6835, 2025.

[19] Takaaki Fujita and Arif Mehmood. Fuzzy tolerance hypergraphs and fuzzy tolerance superhypergraphs. *Neutrosophic Computing and Machine Learning. ISSN 2574-1101*, 41:46–70, 2025.

[20] Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World.* Infinite Study, 2023.

[21] E. J. Mogro, J. R. Molina, G. J. S. Canas, and P. H. Soria. Tree tobacco extract (*Nicotiana glauca*) as a plithogenic bioinsecticide alternative for controlling fruit fly (*Drosophila immigrans*) using *n*-superhypergraphs. *Neutrosophic Sets and Systems*, 74:57–65, 2024.

[22] Takaaki Fujita. A unified graph-based framework for modeling legal citations: From Graphs to Hyper-Graphs and SuperHyperGraphs. *SciNexuses*, 2:132–143, 2025.

[23] Eduardo Martín Campoverde Valencia, Jessica Paola Chuisaca Vásquez, and Francisco Ángel Becerra Lois. Multineutrosophic analysis of the relationship between survival and business growth in the manufacturing sector of azuay province, 2020–2023, using plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84(1):28, 2025.

[24] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[25] Naganand Yadati, RS Dayanidhi, S Vaishnavi, KM Indira, and G Srinidhi. Knowledge base question answering through recursive hypergraphs. In *Proceedings of the 16th conference of the European chapter of the association for computational linguistics: main volume*, pages 448–454, 2021.

[26] Naganand Yadati. Neural message passing for multi-relational ordered and recursive hypergraphs. *Advances in Neural Information Processing Systems*, 33:3275–3289, 2020.

[27] Takaaki Fujita and Florentin Smarandache. *Representing Higher-Order Networks: A Survey of Graph-Based Frameworks*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[28] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[29] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (V): Revisiting Topological Indices*. Infinite Study, 2026.

[30] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (III): Intersection Graph and Graph Labeling Theory*. Infinite Study, 2026.

[31] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (VII): About Directed SuperHyperGraph*. Infinite Study, 2026.

[32] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (IV): Uncertain Graph Theory*. Infinite Study, 2026.

[33] Yue Gao, Zizhao Zhang, Haojie Lin, Xibin Zhao, Shaoyi Du, and Changqing Zou. Hypergraph learning: Methods and practices. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 44(5):2548–2566, 2020.

[34] Junwu Chen and Philippe Schwaller. Molecular hypergraph neural networks. *The Journal of Chemical Physics*, 160(14), 2024.

[35] Shuyi Ji, Yifan Feng, Donglin Di, Shihui Ying, and Yue Gao. Mode hypergraph neural network. *IEEE Transactions on Neural Networks and Learning Systems*, 2025.

[36] Nicholas Casetti, Pragnay Nevatia, Junwu Chen, Philippe Schwaller, and Connor W Coley. Comment on "molecular hypergraph neural networks"[j. chem. phys. 160, 144307 (2024)]. *The Journal of Chemical Physics*, 161(20), 2024.

[37] Wenjie Du, Shuai Zhang, Zhaohui Cai, Xuqiang Li, Zhiyuan Liu, Junfeng Fang, Jianmin Wang, Xiang Wang, and Yang Wang. Molecular merged hypergraph neural network for explainable solvation gibbs free energy prediction. *Research*, 8:0740, 2025.

[38] Julio Cesar Méndez Bravo, Claudia Jeaneth Bolanos Piedrahita, Manuel Alberto Méndez Bravo, and Luis Manuel Pilacuan-Bonete. Integrating smed and industry 4.0 to optimize processes with plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84:328–340, 2025.

[39] Takaaki Fujita. Multi-superhypergraph neural networks: A generalization of multi-hypergraph neural networks. *Neutrosophic Computing and Machine Learning*, 39:328–347, 2025.

[40] Berrocal Villegas Salomón Marcos, Montalvo Fritas Willner, Berrocal Villegas Carmen Rosa, Flores Fuentes Rivera María Yissel, Espejo Rivera Roberto, Laura Daysi Bautista Puma, and Dante Manuel Macazana Fernández. Using plithogenic n-superhypergraphs to assess the degree of relationship between information skills and digital competencies. *Neutrosophic Sets and Systems*, 84:513–524, 2025.

[41] Nelly Hodelín Amable, Elizabeth Esther Vergel De Salazar, Martha Gloria Martínez Isaac, Olivia Catalina Olavarría Sánchez, and Johanna Mariuxi Solís Palma. Representation of motivational dynamics in school environments through plithogenic n-superhypergraphs with family participation. *Neutrosophic Sets and Systems*, 92:570–583, 2025.

[42] Ehab Roshdy, Marwa Khashaba, and Mariam Emad Ahmed Ali. Neutrosophic super-hypergraph fusion for proactive cyberattack countermeasures: A soft computing framework. *Neutrosophic Sets and Systems*, 94:232–252, 2025.

[43] Takaaki Fujita and Arif Mehmood. Superhypergraph attention networks. *Neutrosophic Computing and Machine Learning*, 40(1):10–27, 2025.

[44] Takaaki Fujita and Florentin Smarandache. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 5:577, 2025.

[45] Takaaki Fujita and Florentin Smarandache. Soft directed n-superhypergraphs with some real-world applications. *European Journal of Pure and Applied Mathematics*, 18(4):6643–6643, 2025.

[46] Takaaki Fujita. Directed acyclic superhypergraphs (dash): A general framework for hierarchical dependency modeling. *Neutrosophic Knowledge*, 6:72–86, 2025.

[47] Takaaki Fujita. Metahypergraphs, metasuperhypergraphs, and iterated metagraphs: Modeling graphs of graphs, hypergraphs of hypergraphs, superhypergraphs of superhypergraphs, and beyond. *Current Research in Interdisciplinary Studies*, 4(5):1–23, 2025.

[48] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[49] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[50] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[51] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[52] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

[53] Takaaki Fujita, Ajoy Kanti Das, Sankar Prasad Mondal, and Suman Das. A short note on interval-valued bipolar fuzzy superhypergraphs. *Galoitica: Journal of Mathematical Structures and Applications*, 12(2):51–58, 2025.

[54] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[55] Juanjuan Ding, Wenhui Bai, and Chao Zhang. A new multi-attribute decision making method with single-valued neutrosophic graphs. *International Journal of Neutrosophic Science*, 2021.

[56] M Hamidi and A Borumand Saeid. Accessible single-valued neutrosophic graphs. *Journal of Applied Mathematics and Computing*, 57:121–146, 2018.

[57] Siti Nurul Fitriah Mohamad, Roslan Hasni, Florentin Smarandache, and Binyamin Yusoff. Novel concept of energy in bipolar single-valued neutrosophic graphs with applications. *Axioms*, 10(3):172, 2021.

[58] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, 10:86–101, 2016.

[59] Takaaki Fujita and Florentin Smarandache. A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.

[60] Takaaki Fujita and Florentin Smarandache. *A Dynamic Survey of Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Plithogenic, and Extensional Sets*. Neutrosophic Science International Association (NSIA), 2025.

[61] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[62] P. R. Kavyasree and B. Surender Reddy. N-cubic sets applied to linear spaces. *Kragujevac Journal of Mathematics*, 2022.

[63] Gundeti Soujanya and B. Surender Reddy. N-cubic picture fuzzy linear spaces. *Indian Journal Of Science And Technology*, 2024.

[64] Young Bae Jun, Seok Zun Song, and Seon Jeong Kim. N-hyper sets. *Mathematics*, 2018.

[65] Witold Pedrycz. Shadowed sets: representing and processing fuzzy sets. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 28(1):103–109, 1998.

[66] Mingjie Cai, Qingguo Li, and Guangming Lang. Shadowed sets of dynamic fuzzy sets. *Granular Computing*, 2:85 – 94, 2016.

[67] Gianpiero Cattaneo and Davide Ciucci. Shadowed sets and related algebraic structures. *Fundamenta Informaticae*, 55(3-4):255–284, 2003.

[68] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[69] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[70] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[71] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[72] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pages 305–309, 1994.

[73] Chen Shouyu and Guo Yu. Variable fuzzy sets and its application in comprehensive risk evaluation for flood-control engineering system. *Fuzzy Optimization and Decision Making*, 5(2):153–162, 2006.

[74] Wen-chuan Wang, Dong-mei Xu, Kwok-wing Chau, and Guan-jun Lei. Assessment of river water quality based on theory of variable fuzzy sets and fuzzy binary comparison method. *Water resources management*, 28(12):4183–4200, 2014.

[75] Yunhai Fang, Xilai Zheng, Hui Peng, Huan Wang, and Jia Xin. A new method of the relative membership degree calculation in variable fuzzy sets for water quality assessment. *Ecological indicators*, 98:515–522, 2019.

[76] Vasile Patrascu. A new penta-valued logic based knowledge representation. *arXiv preprint arXiv:1502.05562*, 2015.

[77] Vasile Patrascu. Similarity, cardinality and entropy for bipolar fuzzy set in the framework of penta-valued representation. *arXiv preprint arXiv:1506.02060*, 2015.

[78] Yong-Uk Cho, Young-Bae Jun, and Seok-Zun Song. Bifuzzy ideals of pseudo mv-algebras. *JOURNAL OF APPLIED MATHEMATICS AND COMPUTING*, 22(1/2):475, 2006.

[79] Tadeusz Gerstenkorn and Jacek Mańko. Bifuzzy probabilistic sets. *Fuzzy sets and systems*, 71(2):207–214, 1995.

[80] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[81] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[82] Bui Cong Cuong. Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, 30:409, 2015.

[83] Sankar Das, Ganesh Ghorai, and Madhumangal Pal. Picture fuzzy tolerance graphs with application. *Complex & Intelligent Systems*, 8(1):541–554, 2022.

[84] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[85] Fatma Kutlu Gündoğdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *Journal of intelligent & fuzzy systems*, 36(1):337–352, 2019.

[86] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a γ-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[87] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, 2019.

[88] M Murali Krishna Rao. Tripolar fuzzy interior ideals of γ-semigroup. *Ann. Fuzzy Math. Inform*, 15(2):199–206, 2018.

[89] Shawkat Alkhazaleh. Neutrosophic vague set theory. *Critical Review*, 10:29–39, 2015.

[90] Ashraf Al-Quran, Hazwani Hashim, and Lazim Abdullah. A hybrid approach of interval neutrosophic vague sets and dematel with new linguistic variable. *Symmetry*, 12(2):275, 2020.

[91] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.

[92] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[93] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[94] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[95] Bin Zhu and Zeshui Xu. Some results for dual hesitant fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 26(4):1657–1668, 2014.

[96] José Carlos R Alcantud, Gustavo Santos-García, Xindong Peng, and Jianming Zhan. Dual extended hesitant fuzzy sets. *Symmetry*, 11(5):714, 2019.

[97] Florentin Smarandache. Ambiguous set is a subclass of the double refined indeterminacy neutrosophic set, and of the refined neutrosophic set in general. *Neutrosophic Sets & Systems*, 58, 2023.

[98] Pritpal Singh et al. Ambiguous set theory: A new approach to deal with unconsciousness and ambiguousness of human perception. *Full Length Article*, 5(1):52–2, 2023.

[99] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16, 2024.

[100] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[101] Takaaki Fujita. Review of rough turiyam neutrosophic directed graphs and rough pentapartitioned neutrosophic directed graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:48–79, 2025.

[102] Takaaki Fujita and Florentin Smarandache. Pythagorean, fermatean, and complex turiyam neutrosophic graphs. *SciNexuses*, 2:39–63, 2025.

[103] Prem Kumar Singh, Naveen Surathu, Ghattamaneni Surya Prakash, et al. Turiyam based four way unknown profile characterization on social networks. *Full Length Article*, 10(2):27–7, 2024.

[104] Rama Mallick and Surapati Pramanik. Pentapartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 35:49, 2020.

[105] Radha R. and Stanis Arul Mary. Pentapartitioned neutrosophic pythagorean set. *International Research Journal on Advanced Science Hub*, 2021.

[106] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.

[107] Jiarong Jia. Triple-valued neutrosophic off for public digital cultural service level evaluation in craft and art museums. *Neutrosophic Sets and Systems*, 88(1):44, 2025.

[108] Takaaki Fujita. Triple-valued neutrosophic set, quadruple-valued neutrosophic set, quintuple-valued neutrosophic set, and double-valued indetermsoft set. *Neutrosophic Systems with Applications*, 25(5):3, 2025.

[109] Hongxin Wang. Professional identity formation in traditional chinese medicine students: An educational perspective using triple-valued neutrosophic set. *Neutrosophic Sets and Systems*, 88:83–92, 2025.

[110] Fang Sun, Mifeng Ren, Yujing Shi, and Xuanbai Feng. Quadruple-valued neutrosophic offset for maintenance platforms design quality evaluation of digital monitoring and remote operation in new energy power generation systems. *Neutrosophic Sets and Systems*, 88:940–953, 2025.

[111] Wenwen Meng. Quintuple-valued neutrosophic offset for quality evaluation of cross-border e-commerce talent training based on artificial intelligence. *Neutrosophic Sets and Systems*, 88(1):54, 2025.

[112] Yuan Sun. Assessing the performance of basic education informatization in the digital age using quintuple-valued neutrosophic set. *Neutrosophic Sets and Systems*, 88(1):18, 2025.

[113] Takaaki Fujita, Raed Hatamleh, and Ahmed Heilat. Advanced partitioned neutrosophic offsets, oversets, and undersets: Modeling "neither agree nor disagree" and beyond. *Statistics, Optimization & Information Computing*, 2026.

[114] Takaaki Fujita. Advanced partitioned neutrosophic sets: Formalization of hexa-, hepta-, octa-, nona-, and deca-partitioned structures. *preprint*, 2025.

[115] Giovanni Panti. Multi-valued logics. In *Quantified representation of uncertainty and imprecision*, pages 25–74. Springer, 1998.

[116] Fujita Takaaki and Arif Mehmood. Iterative multifuzzy set, iterative multineutrosophic set, iterative multisoft set, and multiplithogenic sets. *Neutrosophic Computing and Machine Learning*, 41:1–30, 2025.

[117] Muhammad Arshad, Muhammad Saeed, and Atiqe Ur Rahman. Convexity cum concavity on refined fuzzy set with some properties. *Uncertainty Discourse and Applications*, 1(1):140–150, 2024.

[118] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[119] Florentin Smarandache. *Introduction to the multineutrosophic set*. Infinite Study, 2023.

[120] Florentin Smarandache. Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set), pythagorean fuzzy set, spherical fuzzy set, and q-rung orthopair fuzzy set, while neutrosophication is a generalization of regret theory, grey system theory, and three-ways decision (revisited). *Journal of New Theory*, 29:1–31, 2019.

[121] Florentin Smarandache. *Nidus Idearum. Scilogs, XIII: Structure/NeutroStructure/AntiStructure*. Infinite Study, 2024.

[122] Hossein Rashmanlou, Sovan Samanta, Madhumangal Pal, and Rajab Ali Borzooei. Intuitionistic fuzzy graphs with categorical properties. *Fuzzy information and Engineering*, 7(3):317–334, 2015.

[123] Zehui Shao, Saeed Kosari, Muhammad Shoaib, and Hossein Rashmanlou. Certain concepts of vague graphs with applications to medical diagnosis. *Frontiers in physics*, 8:357, 2020.

[124] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[125] Sundas Shahzadi and Muhammad Akram. Coloring of bifuzzy graphs. *Ital. J. Pure Appl. Math*, 36:429–444, 2016.

[126] Muhammad Akram and Karamat H Dar. Interval-valued bifuzzy graphs. In *Proceedings Of The International Conference On Algebra 2010: Advances in Algebraic Structures*, pages 1–10. World Scientific, 2012.

[127] Xinyu Guo, Bingjie Tian, and Xuedong Tian. Hfgnn-proto: Hesitant fuzzy graph neural network-based prototypical network for few-shot text classification. *Electronics*, 11(15):2423, 2022.

[128] Sk Amanathulla, Biswajit Bera, and Madhumangal Pal. Balanced picture fuzzy graph with application. *Artificial Intelligence Review*, 54(7):5255–5281, 2021.

[129] Waqar Arif, Waheed Ahmad Khan, Asghar Khan, and Hossein Rashmanlou. Some indices of picture fuzzy graphs and their applications. *Computational and Applied Mathematics*, 42:1–23, 2023.

[130] M Ramya, S Murali, R Radha, and R Princy. A note on different types of product of fermatean quadripartitioned neutrosophic graphs. *Neutrosophic Sets and Systems*, 97:237–264, 2026.

[131] V Divya and J Jesintha Rosline. Fermatean quadripartitioned neutrosophic fuzzy graph. *Reliability: Theory & Applications*, 20(3 (86)):816–828, 2025.

[132] Sumera Naz, Muhammad Akram, Samirah Alsulami, and Faiza Ziaa. Decision-making analysis under interval-valued q-rung orthopair dual hesitant fuzzy environment. *Int. J. Comput. Intell. Syst.*, 14(1):332–357, 2021.

[133] Takaaki Fujita and Florentin Smarandache. *A compact exploration of turiyam neutrosophic competition graphs*. Infinite Study, 2024.

[134] Said Broumi, D Ajay, P Chellamani, Lathamaheswari Malayalan, Mohamed Talea, Assia Bakali, Philippe Schweizer, and Saeid Jafari. Interval valued pentapartitioned neutrosophic graphs with an application to mcdm. *Operational Research in Engineering Sciences: Theory and Applications*, 5(3):68–91, 2022.

[135] M Myvizhi, Ahmed Abdelhafeez, Mohamed Refaat Abdellah, Mohamed Eassa, Mohamed S Sawah, Hussam Elbehiery, et al. Single-valued neutrosophic graph with heptapartitioend structure. *Neutrosophic Sets and Systems*, 80:728–748, 2025.

[136] R Muthuraj, K Krithika, and S Revathi. Various product on multi fuzzy graphs. *Ratio Mathematica*, 44:231, 2022.

[137] Muhammad Saeed, Atiqe Ur Rahman, and Muhammad Arshad. A study on some operations and products of neutrosophic hypersoft graphs. *Journal of Applied Mathematics and Computing*, 68(4):2187–2214, 2022.

[138] KP Ratheesh. On soft graphs and chained soft graphs. *International Journal of Fuzzy System Applications (IJFSA)*, 7(2):85–102, 2018.

[139] Muhammad Akram, Hafsa M Malik, Sundas Shahzadi, and Florentin Smarandache. Neutrosophic soft rough graphs with application. *Axioms*, 7(1):14, 2018.

[140] Tong He, Yong Chen, and Kaiquan Shi. Weighted rough graph and its application. In *Sixth International Conference on Intelligent Systems Design and Applications*, volume 1, pages 486–491. IEEE, 2006.

[141] Mohammed Alqahtani. Intuitionistic fuzzy quasi-supergraph integration for social network decision making. *International Journal of Analysis and Applications*, 23:137–137, 2025.

[142] Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.

[143] Anam Luqman, Muhammad Akram, and Florentin Smarandache. Complex neutrosophic hypergraphs: new social network models. *Algorithms*, 12(11):234, 2019.

[144] Takaaki Fujita. Molecular metagraph and molecular iterated metagraph in chemistryand biochemistry. *Journal of Applied Mathematics and Symbolic Science*, 1(1):43–57, 2025.

[145] Mathieu Poudret, Agnes Arnould, Jean-Paul Comet, and Pascale Le Gall. Graph transformation for topology modelling. In *International Conference on Graph Transformation*, pages 147–161. Springer, 2008.

[146] Erika Velazquez-Garcia, Ivan Lopez-Arevalo, and Victor Sosa-Sosa. Semantic graph-based approach for document organization. In *Distributed Computing and Artificial Intelligence: 9th International Conference*, pages 469–476. Springer, 2012.

[147] Runzhong Wang, Rui-Xi Wang, Mrunali Manjrekar, and Connor W Coley. Neural graph matching improves retrieval augmented generation in molecular machine learning. *arXiv preprint arXiv:2502.17874*, 2025.

[148] Ben Goertzel. Actpc-chem: Discrete active predictive coding for goal-guided algorithmic chemistry as a potential cognitive kernel for hyperon & primus-based agi. *arXiv preprint arXiv:2412.16547*, 2024.

[149] Chuang Sun, Xuefeng Chen, Ruqiang Yan, and Robert X Gao. Composite-graph-based sparse subspace clustering for machine fault diagnosis. *IEEE Transactions on Instrumentation and Measurement*, 69(5):1850–1859, 2019.

[150] Zhaogeng Liu, Jielong Yang, Xionghu Zhong, Wenwu Wang, Hechang Chen, and Yi Chang. A novel composite graph neural network. *IEEE Transactions on Neural Networks and Learning Systems*, 35(10):13411–13425, 2023.

[151] Pietro Bongini, Niccolo Pancino, Asma Bendjeddou, Franco Scarselli, Marco Maggini, and Monica Bianchini. Composite graph neural networks for molecular property prediction. *International Journal of Molecular Sciences*, 25(12):6583, 2024.

[152] Hao Zhu, Jun Wang, Yin-Ping Zhao, Minhua Lu, and Jun Shi. Contrastive multi-view composite graph convolutional networks based on contribution learning for autism spectrum disorder classification. *IEEE Transactions on Biomedical Engineering*, 70(6):1943–1954, 2022.

[153] Satu Elisa Schaeffer. Graph clustering. *Computer science review*, 1(1):27–64, 2007.

[154] Anton Tsitsulin, John Palowitch, Bryan Perozzi, and Emmanuel Müller. Graph clustering with graph neural networks. *Journal of Machine Learning Research*, 24(127):1–21, 2023.

[155] Stijn Van Dongen. Graph clustering via a discrete uncoupling process. *SIAM Journal on Matrix Analysis and Applications*, 30(1):121–141, 2008.

[156] Yudong Chen, Sujay Sanghavi, and Huan Xu. Improved graph clustering. *IEEE Transactions on Information Theory*, 60(10):6440–6455, 2014.

[157] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Subset Vertex Multigraphs and Neutrosophic Multigraphs for Social Multi Networks*. Infinite Study, 2019.

[158] Brian Alspach and Cun-Quan Zhang. Cycle covers of cubic multigraphs. *Discrete mathematics*, 111(1-3):11–17, 1993.

[159] Landon Butler, Alejandro Parada-Mayorga, and Alejandro Ribeiro. Convolutional learning on multigraphs. *IEEE Transactions on Signal Processing*, 71:933–946, 2023.

[160] Siddhartha Sankar Biswas, Bashir Alam, and MN Doja. Intuitionistic fuzzy shortest path in a multigraph. In *International Conference on Recent Developments in Science, Engineering and Technology*, pages 533–540. Springer, 2017.

[161] Rajab Ali Borzooei and R Almallah. Inverse fuzzy multigraphs and planarity with application in decision-making. *Soft Computing*, 26(4):1531–1539, 2022.

[162] Anthony Shannon and Krassimir Atanassov. On intuitionistic fuzzy multigraphs and their index matrix interpretations. In *2004 2nd International IEEE Conference on'Intelligent Systems'. Proceedings (IEEE Cat. No. 04EX791)*, volume 2, pages 440–443. IEEE, 2004.

[163] Noga Alon. A simple algorithm for edge-coloring bipartite multigraphs. *Information Processing Letters*, 85(6):301–302, 2003.

[164] Fred Galvin. The list chromatic index of a bipartite multigraph. *Journal of Combinatorial Theory, Series B*, 63(1):153–158, 1995.

[165] Michael Tarsi. Decomposition of a complete multigraph into simple paths: nonbalanced handcuffed designs. *Journal of Combinatorial Theory, Series A*, 34(1):60–70, 1983.

[166] Zhihe Liang and Jinping Guo. Decomposition of complete multigraphs into crown graphs. *Journal of Applied Mathematics and Computing*, 32(2):507–517, 2010.

[167] Florentin Smarandache. *Nidus Idearum. Scilogs, III: viva la neutrosophia!*, volume 3. Infinite Study, 2017.

[168] M Baghernejad and Rajab Ali Borzooei. Results on soft graphs and soft multigraphs with application in controlling urban traffic flows. *Soft Computing*, 27(16):11155–11175, 2023.

[169] J Abderramán Marrero, Juan Núñez Valdés, and María Trinidad Villar. Associating hub-directed multigraphs to arrowhead matrices. *Mathematical Methods in the Applied Sciences*, 41(6):2360–2369, 2018.

[170] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[171] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[172] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[173] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[174] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

HyperGraph and SuperHyperGraph Theory with Applications (VIII): Hierarchical SuperHyperGraph, Recursive SuperHyperGraph, and Related Graph Theory

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Hypergraphs generalize ordinary graphs by allowing an edge to connect an arbitrary nonempty subset of the vertex set. SuperHyperGraphs extend this idea further by using iterated powerset constructions, so that vertices themselves may be set-valued, nested, and higher-order objects. This book develops a systematic treatment of two important directions in this framework: Hierarchical SuperHyperGraphs and Recursive SuperHyperGraphs.

First, recursive SuperHyperGraphs are formulated by combining n-level supervertices with recursive superhyperedges of bounded depth. Such recursive superhyperedges may contain both supervertices and lower-level recursive edges, provided that typed incidence and well-foundedness conditions are satisfied. Several related variants are then examined, including fuzzy, single-valued neutrosophic, uncertain, soft, rough, and weighted recursive SuperHyperGraphs.

Second, Hierarchical SuperHyperGraphs are studied as mixed-level structures in which supervertices from different iterated-powerset levels coexist in one incidence system. A coherence condition is imposed so that every selected higher-level supervertex brings its immediate constituents into the same vertex set. Fuzzy, directed, bidirected, uncertain, soft, rough, and weighted hierarchical variants are also introduced.

Finally, the book compares these structures with related graph-theoretic frameworks such as Meta-Graphs, Iterated Meta-Graphs, Filtrated Graphs, Iterated MultiGraphs, Neural Graphs, and several distance, labeling, domination, and tree-like notions. The emphasis is theoretical, and the goal is to provide definitions, examples, comparisons, and foundational observations that may support future applications and computational studies.

Keywords: SuperHyperGraph, HyperGraph, Hierarchical SuperHyperGraph, Recursive SuperHyperGraph, Uncertain Graph, Fuzzy Graph, Neutrosophic Graph

This book develops a comprehensive theoretical framework for the study of **Hierarchical SuperHyperGraphs** and **Recursive SuperHyper-Graphs**, extending classical graph and hypergraph theory toward more expressive higher-order and multi-level relational structures. SuperHyperGraphs generalize hypergraphs through iterated powerset constructions, allowing vertices to be nested, set-valued, and hierarchically organized.

The work introduces recursive SuperHyperGraphs by integrating *n*-level supervertices with recursive superhyperedges of bounded depth, where edges may contain both supervertices and lower-level recursive edges under typed incidence and well-foundedness constraints. Furthermore, the book formulates hierarchical SuperHyperGraphs as mixed-level incidence systems governed by coherence conditions to ensure structural consistency across powerset levels. Several important extensions are explored, including fuzzy, neutrosophic, uncertain, soft, rough, weighted, directed, and bidirected variants.

Theoretical comparisons are also provided with related graph-theoretic models such as Meta-Graphs, Iterated Meta-Graphs, Filtrated Graphs, Iterated MultiGraphs, and Neural Graphs. Emphasizing definitions, formal properties, examples, and structural comparisons, the book establishes a foundational basis for future computational developments and interdisciplinary applications in complex systems, artificial intelligence, network science, and higher-order modeling.

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