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α-Separation Axioms on Fuzzy Soft T_0 Spaces

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Abstract: The main objective of this article is to introduce four new inferences of fuzzy soft T_0 spaces by using the concept of fuzzy soft topological spaces. We present several new theories and some implications of such spaces. We also show that all these notions preserve some soft invariance properties such as *soft hereditary* and *soft topological* properties.

Key Words: Soft sets, fuzzy soft sets, soft topology, fuzzy soft topology, fuzzy soft open sets, fuzzy soft mapping, image of fuzzy soft mapping.

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§1. Introduction

It is an undeniable fact that the invention of fuzzy sets by Zadeh [1] in 1965 was a groundbreaking event. This type of set is used in control system engineering, image processing, industrial automation, robotics, consumer electronics, and other branches of applied sciences. Besides, it is connected to fuzzy logic giving the opportunity to model under conditions of uncertainty that are vague or not precisely defined, thus succeeding to mathematically solve problems whose statements are expressed in our natural language. Since then, a lot of research has been carried out for generalizing and extending the fuzzy set theory for the purpose of tackling more effectively the existing uncertainty in problems of science, technology, and everyday life.

Consequently, the Russian mathematician Dmtri Molodstov [2] proposed the soft sets in 1999 to overcome the existing difficulty of properly defining the membership function of a fuzzy set. After the introduction of the notion of soft sets, several researchers improved this concept. Maji et al. [3]-[5] presented an application of soft sets in decision-making problems based on the reduction of parameters to keep the optimal choice objects. Pei and Miao [6] showed that soft sets are a class of special information systems.

Topological structures of soft sets were also studied by Sabir and Naz [7]. They defined the

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soft topological spaces over an initial universe with a fixed set of parameters and studied the concepts of soft open sets, soft closed sets, soft closure, soft interior points, soft neighborhood of a point, soft separation axioms as well as their basic properties. Later, Roy and Samanta [8] gave the definition of fuzzy soft topology over the initial universe set. It was further extended by Varol and Aygun [9] and Cetin and Aygun [10].

Sabir Hussain and Bashir Ahmad [11] redefined and explored several properties of soft T_i, $i = 0, 1, 2$, soft regular, soft T_3, soft normal, and soft T_4 axioms using the soft points defined by Zorlutuna [12]. They also discussed some soft invariance properties, namely soft topological property and soft hereditary property. In this work, we newly define in four different ways the notions of Fuzzy Soft T_0 spaces, develop several theories, and discuss various properties, namely hereditary and topological properties.

Throughout this paper, X and Y will be non-empty sets, ϕ will denote the empty set, and E will be the set of all parameters. F_E will denote the soft set, f_A will denote the fuzzy soft set, $\bar{T}$ and τ will represent the soft topology and fuzzy soft topology, respectively.

The rest of this paper is organized as follows: Section 2 presents a brief review of the relevant definitions such as fuzzy sets, soft sets, fuzzy soft sets, soft topology, fuzzy soft topology, fuzzy soft mapping, and the image of fuzzy soft mapping. In Section 3, we develop four ideas of fuzzy soft T_0 spaces, show some implications among them, and introduce several new theories on fuzzy soft T_0 spaces. The concepts of, *good extension*, *hereditary property*, and related theorems are given in Section 4. Finally, Section 5 presents the conclusion and further discussion of this paper.

§2. **Preliminaries**

We recall some basic definitions and known results of soft sets, fuzzy soft sets, operations on fuzzy soft sets, soft topology, fuzzy soft topology, and fuzzy soft mapping.

Definition 2.1([1]) *Let X be a non-empty set and $I = [0, 1]$. A fuzzy set in X is a function $u : X \to I$ which assigns to each element $x \in X$ a degree of membership $u(x) \in I$.*

Definition 2.2([16]) *A pair (F, E) denoted by F_E is called a soft set over X, where F is a mapping given by $F : E \to P(X)$. We denote the family of all soft sets over X by $SS(X, E)$.*

Definition 2.3([16]) *A soft set (F, E) over X is called a null soft set and denoted by $\bar{\phi}$ if $F(e) = \phi$ for every $e \in E$.*

Definition 2.4([16]) *A soft set (F, E) over X is called an absolute soft set and denoted by $\bar{X}$ if $F(e) = X$ for every $e \in E$.*

Definition 2.5([16]) *Let X be an initial universal set, and $A \subseteq E$. Let $\bar{T}$ be a subfamily of the family of all soft sets $S(X)$. We say that the family $\bar{T}$ is a soft topology on X if the following axioms hold:*

(1) $\bar{\phi}_A, \bar{X}_A \in \bar{T}$;

(2) *If* $F_A, G_A \in \bar{T}$, *then* $F_A \cap G_A \in \bar{T}$;

(3) *If* $G_{iA} \in \bar{T}$ *for each* $i \in \Lambda$, *then* $\bigcup_{i\in\Lambda} G_{iA} \in \bar{T}$.

Then, the triple $(\bar{X}_A, \bar{T}, A)$ *is called a soft topological space (STS, for short) and the members of* $\bar{T}$ *are called soft open sets (SOS for short). A soft set* F_A *is called soft closed set (SCS, for short) if and only if its complement is a soft open set. That is,* $F_A^c \in \bar{T}$.

Definition 2.6([7]) *A soft topological space* $(F_A, \bar{T}, A)$ *is called soft* T_0 *(*ST_0*) space if for each* $x_1, x_2 \in X$ *with* $x_1 \neq x_2$, *there exists a SOS* $F_A \in \bar{T}$ *such that* $x_1 \in F_A, x_2 \notin F_A$ *or* $x_1 \notin F_A, x_2 \in F_A$.

Definition 2.7([9]) *A fuzzy soft set* f_A *on the universe* X *is a mapping from the parameter set* E *to* I^X, *i.e.,* $f_A : E \to I^X$, *where* $f_A(e) \neq 0_X$ *if* $e \in A \subseteq E$ *and* $f_A(e) = 0_X$ *if* $e \notin A$, *where* 0_X *is the empty fuzzy set on* X.

From now on, we will use $F(X, E)$ instead of the family of all fuzzy soft sets over X. A classical soft set F_A over a universe X can be seen as a fuzzy soft set by using the characteristic function of the set $F_A(e)$:

$$f_A(e)(a) = \chi_{F_A(e)}(a) = \begin{cases} 1, & \text{if } a \in F_A(e), \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.8([13]) *Two fuzzy soft sets* f_A *and* g_B *on* X, *we say that* f_A *is called a fuzzy soft subset of* g_B *and write* $f_A \subseteq g_B$ *if* $f_A(e) \leq g_B(e)$ *for every* $e \in E$.

Definition 2.9([13]) *Two fuzzy soft sets* f_A *and* g_B *on* X *are called equal if* $f_A \subseteq g_B$ *and* $g_B \subseteq f_A$.

Definition 2.10([13]) *Let* $f_A, g_B \in (X, E)$. *Then the union of* f_A *and* g_B *is also a fuzzy soft set* h_C, *defined by* $h_C(e) = f_A(e) \vee g_B(e)$ *for all* $e \in E$, *where* $C = A \cup B$. *Here we write* $h_C = f_A \cup g_B$.

Definition 2.11([13]) *Let* $f_A, g_B \in (X, E)$. *Then the intersection of* f_A *and* g_B *is also a fuzzy soft set* h_C, *defined by* $h_C(e) = f_A(e) \wedge g_B(e)$ *for all* $e \in E$, *where* $C = A \cap B$. *Here we write* $h_C = f_A \cap g_B$.

Definition 2.12([13]) *A fuzzy soft set* f_E *on* X *is called a null fuzzy soft set, denoted by* 0_E, *if* $f_E(E) = 0_X$ *for each* $e \in E$.

Definition 2.13([13]) *A fuzzy soft set* f_E *on* X *is called an absolute fuzzy soft set, denoted by* 1_E, *if* $f_E(E) = 1_X$ *for each* $e \in E$.

Definition 2.14([13]) *Let* $f_A \in (X, E)$. *Then the complement of* f_A *is denoted by* f_A^c *and is defined by* $f_A^c(e) = 1 - f_A(e)$ *for each* $e \in E$.

Definition 2.15([14]) *A fuzzy soft set* g_A *is said to be a fuzzy soft point, denoted by* e_{g_A}, *if for the element* $e \in E$, $g(e) \neq \tilde{\phi}$ *and* $g(e') = \tilde{\phi}$ *for all* $e' \in A - \{e\}$.

Definition 2.16([14]) *A fuzzy soft point e_{g_A} is said to be in a fuzzy soft set h_A, denoted by $e_{g_A} \in h_A$, if for the element $e \in A$, $g(e) \leq h(e)$.*

Definition 2.17([14]) *Let f_A be a fuzzy soft set, $\mathcal{FS}(f_A)$ be the set of all fuzzy soft subsets of f_A, and τ be a subfamily of $\mathcal{FS}(f_A)$. Then τ is called a fuzzy soft topology on f_A if the following conditions are satisfied:*

(1) $\tilde{\phi}_A, f_A \in \tau$;

(2) $f_{1A}, f_{2A} \in \tau \Rightarrow f_{1A} \cap f_{2A} \in \tau$;

(3) *For any index set I, if $f_{iA} \in \tau$ for any $i \in I$, then $\bigcup_{i\in I} f_{iA} \in \tau$.*

Then, the pair (f_A, τ) is called fuzzy soft topological space (FSTS, for short), and the members of τ are called fuzzy soft open sets (FSOS, for short). A fuzzy soft open set g_A is called a fuzzy soft closed set (FSCS, for short) if $g_A^c \in \tau$, where g_A^c is the complement of g_A.

Definition 2.18([14]) *A fuzzy soft topological space (f_A, τ) is said to be a fuzzy soft T_0 space if for every pair of distinct fuzzy soft points e_{h_A} and e_{g_B}, there exists a fuzzy soft open set containing one but not the other.*

Definition 2.19([9]) *Let (f_A, τ_1) and (g_B, τ_2) be two fuzzy soft topological spaces (FSTS' s), on the two universal sets X and Y, respectively. Then a fuzzy soft mapping $(\varphi, \psi) : (f_A, \tau_1) \to (g_B, \tau_2)$ is called:*

(1) *Fuzzy soft continuous if $(\varphi, \psi)^{-1}(g_B) \in \tau_1$, for all $g_B \in \tau_2$;*

(2) *Fuzzy soft open if $(\varphi, \psi)(f_A) \in \tau_2$, for all $f_A \in \tau_1$;*

(3) *Fuzzy soft closed if $(\varphi, \psi)(f_A)$ is a fuzzy soft closed set of τ_2 for each fuzzy soft closed set f_A of τ_1;*

(4) *Fuzzy soft homeomorphism if (φ, ψ) is bijective, continuous, and open.*

Definition 2.20([13]) *Let $\varphi : X \to Y$ and $\psi : E \to F$ be two mappings, where E and F are parameter sets for the crisp sets X and Y, respectively. Then (φ, ψ) is called a fuzzy soft mapping from (X, E) into (Y, F) and denoted by $(\varphi, \psi) : (X, E) \to (Y, F)$.*

Definition 2.21([13]) *Let f_A and g_B be two fuzzy soft sets over X and Y, respectively, and (φ, ψ) be a fuzzy soft mapping from (X, E) into (Y, F).*

(1) *The image of f_A under the fuzzy soft mapping (φ, ψ), denoted by $(\varphi, \psi)(f_A)$, is defined as*

$$(\varphi, \psi)(f_A)k(y) = \begin{cases} \vee\varphi(x) = y \vee \psi(e) = k f_A(e)(x), & \text{if} \quad \varphi^{-1}(y) \neq \phi, \psi^{-1}(k) \neq \phi; \\ 0, & \text{otherwise,} \end{cases}$$

for all $k \in F$ and $y \in Y$.

(2) *The image of g_B under the fuzzy soft mapping (φ, ψ), denoted by $(\varphi, \psi)^{-1}(g_B)$, is defined as*

$$(\varphi, \psi)^{-1}(g_B)(e)(x) = g_B(\psi(e))(\varphi(x)) \text{ for all } e \in E \text{ and } x \in X.$$

Proposition 2.1([15]) *Let $(\bar{X}_A, \bar{T}, A)$ be a soft topological space over X. Then the collection $T_e = \{F(e) \mid (F, E) \in \bar{T}\}$ for each $e \in E$, defines a topology on X.*

§3. Definitions and Properties of Fuzzy Soft T_0 Spaces

Before we mentioned the definition of fuzzy soft T_0 space, and now in this section we introduce four new ideas of fuzzy soft T_0 spaces, establish some implications among them and develop several new theories on fuzzy soft T_0 spaces. We denote the grade of membership and the grade of non-membership of any point in fuzzy soft set is $\bar{1}$ and $\bar{0}$ respectively. Here $\bar{\alpha}$ means that the grade of membership of any point in fuzzy soft set lies between 0 and 1.

Definition 3.1 *A fuzzy soft topological space (FSTS) (f_A, τ) is called:*

(a) $FST_0(i)$ if for any pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) = \bar{1}$, $f_{1A}(e)(x_2) = \bar{0}$, or $f_{1A}(e)(x_1) = \bar{0}$, $f_{1A}(e)(x_2) = \bar{1}$;

(b) $FST_0(ii)$ if for any pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) = \bar{\alpha}$, $f_{1A}(e)(x_2) = \bar{0}$, or $f_{1A}(e)(x_1) = \bar{0}$, $f_{1A}(e)(x_2) = \bar{\alpha}$ as $0 < \alpha < 1$;

(c) $FST_0(iii)$ if for any pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) > f_{1A}(e)(x_2)$, or $f_{1A}(e)(x_2) > f_{1A}(e)(x_1)$;

(d) $FST_0(iv)$ if for any pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) \neq f_1A(e)(x_2)$ or $f_{1A}(e)(x_2) \neq f_{1A}(e)(x_1)$.

Theorem 3.1 *Let (f_A, τ) be a fuzzy soft topological space. Then the above four notions of it form the following implications:*

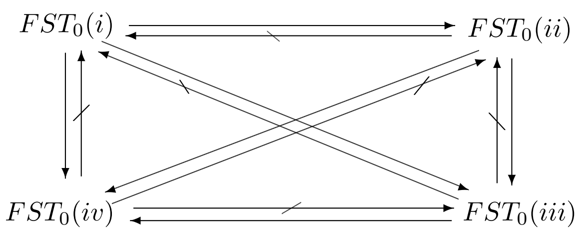

Figure 1 The implications of four notions are shown by a quadrilateral with two diagonals

Proof Let (f_A, τ) be a $FST_0(i)$. Then by definitions, for any pair of $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) = \bar{1}, f_{1A}(e)(x_2) = \bar{0}$, or $f_{1A}(e)(x_1) = \bar{0}, f_{1A}(e)(x_2) = \bar{1}$.

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) = \bar{\alpha},\ f_{1A}(e)(x_2) = \bar{0},\ \text{or} \\ f_{1A}(e)(x_1) = \bar{0},\ f_{1A}(e)(x_2) = \bar{\alpha} \end{cases} \quad \text{as} \quad 0 < \alpha < 1 \tag{1}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) > f_{1A}(e)(x_2), \text{ or} \\ f_{1A}(e)(x_2) > f_{1A}(e)(x_1) \end{cases} \tag{2}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) \neq f_{1A}(e)(x_2), \text{ or} \\ f_{1A}(e)(x_2) \neq f_{1A}(e)(x_1). \end{cases} \tag{3}$$

Hence, from (1), (2), and (3), we see that $FST_0(i) \Rightarrow FST_0(ii) \Rightarrow FST_0(iii) \Rightarrow FST_0(iv)$.

Again, suppose that (f_A, τ) be a $FST_0(i)$. Then by definition, for any pair of $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) = \overline{1}, f_{1A}(e)(x_2) = \overline{0}$, or $f_{1A}(e)(x_1) = \overline{0}, f_{1A}(e)(x_2) = \overline{1}$.

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) > f_{1A}(e)(x_2), \text{ or} \\ f_{1A}(e)(x_2) > f_{1A}(e)(x_1) \end{cases} \tag{4}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) \neq f_{1A}(e)(x_2), \text{ or} \\ f_{1A}(e)(x_2) \neq f_{1A}(e)(x_1). \end{cases} \tag{5}$$

Hence, from (4) and (5), we see that $FST_0(i) \Rightarrow FST_0(iii)$, and $FST_0(i) \Rightarrow FST_0(iv)$.

Finally, let (f_A, τ) be a $FST_0(i)$. Then from (1), for any pair of $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) = \bar{\alpha}, f_{1A}(e)(x_2) = \overline{0}$, or $f_{1A}(e)(x_1) = \overline{0}, f_{1A}(e)(x_2) = \bar{\alpha}$, as $0 < \alpha < 1$.

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) \neq f_{1A}(e)(x_2), \text{ or} \\ f_{1A}(e)(x_2) \neq f_{1A}(e)(x_1). \end{cases} \tag{6}$$

From (6), we see that $FST_0(ii) \Rightarrow FST_0(iv)$. □

None of the reverse implications is true in general, as can be seen in the following counter examples.

Example 3.1 Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ a set of parameters, $A = \{e_1, e_2\} \subset E$, and τ be a fuzzy soft topology on a universal set X generated by $\tau = \{\overline{0}, \overline{1}, f_{1A}\}$ where $f_{1A} = \{e_1 = \{0.6/x_1, 0.7/x_2\}, e_2 = \{0.7/x_1, 0.6/x_2\}\}$. Here $f_{1A}(e_1)(x_1) = 0.6, f_{1A}(e_1)(x_2) = 0.7$ and $f_{1A}(e_2)(x_1) = 0.7, f_{1A}(e_2)(x_2) = 0.6$.

Hence, we observe that (f_A, τ) is $FST_0(iv)$ but not $FST_0(i)$, $FST_0(ii)$, or $FST_0(iii)$. Therefore, $FST_0(iv) \nRightarrow FST_0(i)$, $FST_0(iv) \nRightarrow FST_0(ii)$, and $FST_0(iv) \nRightarrow FST_0(iii)$.

Example 3.2 Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters, $A = \{e_1, e_2\} \subset E$, and τ be a fuzzy soft topology on a universal set X generated by $\tau = \{\overline{0}, \overline{1}, f_{1A}\}$, where $f_{1A} = \{e_1 = \{0.7/x_1, 0.3/x_2\}, e_2 = \{0.3/x_1, 0.7/x_2\}\}$.

Here, $f_{1A}(e_1)(x_1) = 0.7$, $f_{1A}(e_1)(x_2) = 0.3$, and $f_{1A}(e_2)(x_1) = 0.3$, $f_{1A}(e_2)(x_2) = 0.7$.

$$f_{1A}(e_1)(x_1) = 0.7, \quad f_{1A}(e_1)(x_2) = 0.3$$

$$f_{1A}(e_2)(x_1) = 0.3, \quad f_{1A}(e_2)(x_2) = 0.7$$

Hence, we observe that (f_A, τ) is $FST_0(iii)$ but not $FST_0(i)$ and $FST_0(ii)$. Therefore,

$$FST_0(iii) \nRightarrow FST_0(i), \quad \text{and} \quad FST_0(iii) \nRightarrow FST_0(ii).$$

Finally, if we consider $f_{1A} = \{e_1 = \{\bar{\alpha}/x_1, 0/x_2\}, e_2 = \{0/x_1, \bar{\alpha}/x_2\}\}$, where $0 < \alpha < 1$, then we have

$$f_{1A}(e_1)(x_1) = \bar{\alpha}, \quad f_{1A}(e_1)(x_2) = \bar{0}$$

$$f_{1A}(e_2)(x_1) = \bar{0}, \quad f_{1A}(e_2)(x_2) = \bar{\alpha}$$

Thus, (f_A, τ) is $FST_0(ii)$ but not $FST_0(i)$.

Theorem 3.2 *If a fuzzy soft topological space (FSTS) (f_A, τ) is a fuzzy soft T_0 space, then the following statements are equivalent:*

(a) for all $x_1, x_2 \in X$, $x_1 \neq x_2$, and for all $e \in A$, $\overline{f_{1A}}(e)(x_1) \wedge \overline{f_{1A}}(e)(x_2) \leq \bar{1}$;

(b) for all $x_1, x_2 \in X$, $x_1 \neq x_2$, and for all $e \in A$ there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) > \bar{0}, f_{1A}(e)(x_2) = \bar{0}$, or $f_{1A}(e)(x_1) = \bar{0}, f_{1A}(e)(x_2) > \bar{0}$.

Proof $(a) \Rightarrow (b)$: We have from (a) that,

$$\overline{f_{1A}}(e)(x_1) \wedge \overline{f_{1A}}(e)(x_2) \leq \bar{1} \Rightarrow \overline{f_{1A}}(e)(x_1) < \bar{1} \quad \text{or} \quad \overline{f_{1A}}(e)(x_2) < \bar{1}.$$

This implies

$$\bar{1} - \overline{f_{1A}}(e)(x_1) > \bar{0} \quad \text{or} \quad \bar{1} - \overline{f_{1A}}(e)(x_2) > \bar{0}.$$

Let $f_{1A} = \bar{1} - \overline{f_{1A}}$. Then we have

$$f_{1A}(e)(x_1) > \bar{0}, f_{1A}(e)(x_2) = \bar{0} \quad \text{or} \quad f_{1A}(e)(x_1) = \bar{0}, f_{1A}(e)(x_2) > \bar{0},$$

which is (b).

$(b) \Rightarrow (a)$: From (b) we have that, for all $x_1, x_2 \in X$, $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that

$$f_{1A}(e)(x_1) > \bar{0}, f_{1A}(e)(x_2) = \bar{0}, \quad \text{or} \quad f_{1A}(e)(x_1) = \bar{0}, f_{1A}(e)(x_2) > \bar{0}.$$

That implies

$$\bar{1} - f_{1A}(e)(x_1) < \bar{1}, \quad \bar{1} - f_{1A}(e)(x_2) = \bar{0}, \quad \text{or} \quad \bar{1} - f_{1A}(e)(x_2) < \bar{1}, \quad \bar{1} - f_{1A}(e)(x_1) = \bar{0}.$$

Since f_{1A} is a fuzzy soft open set (FSOS), therefore $\bar{1} - f_{1A}$ is a fuzzy soft closed set (FSCS). Hence, we have

$$\overline{f_{1A}}(e)(x_1) < \bar{1} \quad \text{or} \quad \overline{f_{1A}}(e)(x_2) < \bar{1} \ \Rightarrow \ \overline{f_{1A}}(e)(x_1) \wedge \overline{f_{1A}}(e)(x_2) \leq \bar{1},$$

which is (a). □

Theorem 3.3 *Let (f_A, τ, A) be a fuzzy soft topological space (FSTS) over a universal set X, and let $e_{x_1}, e_{x_2} \in f_A$ such that $e_{x_1} \neq e_{x_2}$ as $x_1 \neq x_2$ for every pair $x_1, x_2 \in X$. If there exist FSOS' s (f_{1A}, A) and (f_{2A}, A) such that $e_{x_1} \in (f_{1A}, A)$ and $e_{x_2} \in (f_{1A}, A)^c$ or $e_{x_2} \in (f_{2A}, A)$ and $e_{x_1} \in (f_{2A}, A)^c$, then*

(*a*) *(f_A, τ, A) is an FST_0 space;*
(*b*) *(F_A, τ, A) is an ST_0 space;*
(*c*) *(X, τ_e) is a T_0 space.*

Proof Firstly we prove (a). It is clear that $e_{x_2} \in (f_{1A}, A)^c = (f^c_{1A}, A) \implies e_{x_2} \notin (f_{1A}, A)$, which implies that $f_{1A}(e)(x_2) = 0$. Similarly, $e_{x_1} \in (f_{2A}, A)^c = (f^c_{2A}, A) \implies e_{x_1} \notin (f_{2A}, A)$, which implies that $f_{2A}(e)(x_1) = 0$. Thus, we have $e_{x_1} \in (f_{1A}, A), e_{x_2} \notin (f_{1A}, A)$ or $e_{x_2} \in (f_{2A}, A), e_{x_1} \notin (f_{2A}, A)$. This proves that (f_A, τ, A) is an FST_0 space.

Secondly we prove (b), that is (F_A, τ, A) is an ST_0 space. To do this, we define a characteristic function 1_{F_A} such that

$$f_A(e)(x) = 1_{F_A}(e) = \begin{cases} 1 & \text{if } x \in F_A(e) \\ 0 & \text{otherwise.} \end{cases}$$

Let $f_A = (f_{1A}, f_{2A})$ and $F_A = (F_{1A}, F_{2A})$. Now, for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ we have $f_{1A}(e)(x_1) = 1 \implies e_{x_1} \in (F_{1A}, A)$ and $f_{1A}(e)(x_2) = 0 \implies e_{x_2} \notin (F_{1A}, A)$. Therefore, $e_{x_1} \in (F_{1A}, A), e_{x_2} \notin (F_{1A}, A)$ or $e_{x_2} \in (F_{2A}, A), e_{x_1} \notin (F_{2A}, A)$. Thus, (F_A, τ, A) is an ST_0 space.

Finally to prove (c), for any $e \in A$, (X, T_e) is a topological space on X (see Proposition 2.1) and $e_{x_1} \in (F_{1A}, A), e_{x_2} \in (F_{1A}, A)^c$ or $e_{x_2} \in (F_{2A}, A), e_{x_1} \in (F_{2A}, A)^c$. So that $x_1 \in F_{1A}(e), x_2 \notin F_{1A}(e)$ or $x_2 \in F_{2A}(e), x_1 \notin F_{2A}(e)$. Thus, (X, τ_e) is a T_0 space. □

§4. Good Extension, Hereditary and Topological Property

In this section, we discuss some fuzzy soft invariance properties, namely *good extension, hereditary* and *soft topological properties.*

Definition 4.1 *Let $(F_A, \overline{T}, A)$ be a soft topological space and $\tau = \{1_{F_A} : F_A \in \overline{T}\}$, and $1_{F_A} = f_{1A}$. Then (f_A, τ) is the corresponding fuzzy soft topological space of $(F_A, \overline{T}, A)$. Let P be a property of soft topological spaces and FP be its fuzzy soft topological analogue. Then FP is called a 'Good extension' of P if the statement $(F_A, \overline{T}, A)$ has P if and only if (f_A, τ) has FP holds true for every soft topological space $(F_A, \overline{T}, A)$.*

Theorem 4.1 *Let $(F_A, \bar{T}, A)$ be a soft T_0 space and (f_A, τ) be $FST_0(j)$ spaces, where $j = i, ii, iii, iv$. Then $(F_A, \bar{T}, A)$ will be $FST_0(j)$ spaces if and only if $FST_0(j)$ will also be a soft T_0 space.*

Proof Suppose that $(F_A, \bar{T}, A)$ is a soft T_0 (ST_0) space. We prove that $(F_A, \bar{T}, A)$ is $FST_0(j)$ spaces. Since $(F_A, \bar{T}, A)$ is a soft T_0 space, for each $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists a soft open set (SOS) $F_A \in \bar{T}$ such that $x_1 \in F_A$ and $x_2 \notin F_A$, or $x_1 \notin F_A$ and $x_2 \in F_A$. Then, by a characteristic function 1_{F_A}, we have

$$\Rightarrow \begin{cases} 1_{F_A}(e)(x_1) = \bar{1}, \quad 1_{F_A}(e)(x_2) = \bar{0}, \quad \text{or} \\ 1_{F_A}(e)(x_1) = \bar{0}, \quad 1_{F_A}(e)(x_2) = \bar{1}. \end{cases}$$

Let $1_{F_A} = f_{1A}$. Therefore,

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) = \bar{1}, \quad f_{1A}(e)(x_2) = \bar{0}, \quad \text{or} \\ f_{1A}(e)(x_1) = \bar{0}, \quad f_{1A}(e)(x_2) = \bar{1}, \end{cases} \tag{7}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) = \bar{\alpha}, \quad f_{1A}(e)(x_2) = \bar{0}, \quad \text{or} \\ f_{1A}(e)(x_1) = \bar{0}, \quad f_{1A}(e)(x_2) = \bar{\alpha}, \quad \text{as } 0 < \alpha < 1, \end{cases} \tag{8}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) > f_{1A}(e)(x_2) \quad \text{or} \\ f_{1A}(e)(x_2) > f_{1A}(e)(x_1), \end{cases} \tag{9}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) \neq f_{1A}(e)(x_2) \quad \text{or} \\ f_{1A}(e)(x_2) \neq f_{1A}(e)(x_1). \end{cases} \tag{10}$$

Hence, from (7), (8), (9), and (10), we see that a soft T_0 space is $FST_0(i)$, $FST_0(ii)$, $FST_0(iii)$, and $FST_0(iv)$ spaces. This implies that a soft T_0 space is $FST_0(j)$ spaces, where $j = i, ii, iii, iv$.

Conversely, assume that (f_A, τ) is $FST_0(j)$ spaces. We will prove that (f_A, τ) is a soft T_0 space. To do this, we will first prove it for $j = i$. Since (f_A, τ) is $FST_0(i)$, by definition, for all $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) = \bar{1}, \quad f_{1A}(e)(x_2) = \bar{0}, \quad \text{or} \\ f_{1A}(e)(x_1) = \bar{0}, \quad f_{1A}(e)(x_2) = \bar{1}. \end{cases}$$

Thus

$$\Rightarrow \begin{cases} f_{1A}^{-1}(e)(\bar{1}) = \{x_1\}, \quad f_{1A}^{-1}(e)(\bar{0}) = \{x_2\}, \quad \text{or} \\ f_{1A}^{-1}(e)(\bar{0}) = \{x_1\}, \quad f_{1A}^{-1}(e)(\bar{1}) = \{x_2\}. \end{cases}$$

Let $f_{1A}^{-1}(\bar{1}) = F_A$. Therefore, $F_A(e) = \{x_1\}$ or $F_A(e) = \{x_2\}$. Hence, for each $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists a soft open set (SOS) $F_A \in \bar{T}$ such that $x_1 \in F_A$ and $x_2 \notin F_A$, or $x_1 \notin F_A$ and $x_2 \in F_A$. Thus, $FST_0(i)$ is ST_0. Similarly, $FST_0(ii)$, $FST_0(iii)$, and $FST_0(iv)$ imply ST_0 space. □

Definition 4.2 *Let (f_A, τ) be a fuzzy soft topological space (FSTS) and $g_A \subset f_A$. Then the fuzzy soft topology $\tau_{g_A} = \{g_A \cap h_A \mid h_A \in \tau\}$ is called the fuzzy soft subspace topology, and (g_A, τ_{g_A}) is called the fuzzy soft subspace of (f_A, τ). A fuzzy soft topological property P is called hereditary if each subspace of a fuzzy soft topological space with property P also has property P.*

Theorem 4.2 *Let (f_A, τ) be a fuzzy soft topological space (FSTS) and $(g_A, \tau_{(g_A)})$ be a subspace of it. Then if (f_A, τ) is $FST_0(j)$, it implies that (g_A, τ_{g_A}) is also $FST_0(j)$, where $j = i, ii, iii, iv$.*

Proof We prove this theorem only for $j = i$. Suppose that (f_A, τ) is $FST_0(i)$. It will be shown that (g_A, τ_{g_A}) is $FST_0(i)$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, and for all $e \in A$ such that $g_A(e)(x_1) = \bar{1}, \quad g_A(e)(x_2) = \bar{0}$. Since (f_A, τ) is $FST_0(i)$, by definition, for all $x_1, x_2 \in X$, $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $h_A \in \tau$ such that either $h_A(e)(x_1) = \bar{1}, \quad h_A(e)(x_2) = \bar{0}$ or $h_A(e)(x_1) = \bar{0}, \quad h_A(e)(x_2) = \bar{1}$. Since $g_A(e)(x_1) = \bar{1}$ and $h_A(e)(x_1) = \bar{1}$, we have $(g_A \cap h_A)(e)(x_1) = 1$ and similarly, $(g_A \cap h_A)(e)(x_2) = 0$ or $(g_A \cap h_A)(e)(x_1) = 0, \quad (g_A \cap h_A)(e)(x_2) = 1$. Hence, (g_A, τ_{g_A}) is $FST_0(i)$. The cases for $j = ii, iii$, and iv can be proved in a similar way. □

Theorem 4.3 *Let (f_A, τ_1) and (g_B, τ_2) be two fuzzy soft topological spaces (FSTS's) on the two universal sets X and Y, respectively. Let $(\phi, \psi) : (f_A, \tau_1) \to (g_B, \tau_2)$ be a fuzzy soft one-to-one, onto, and continuous map. Then these spaces maintain the following features:*

- *(a)* (f_A, τ_1) *is* $FST_0(i) \iff (g_B, \tau_2)$ *is* $FST_0(i)$;
- *(b)* (f_A, τ_1) *is* $FST_0(ii) \iff (g_B, \tau_2)$ *is* $FST_0(ii)$;
- *(c)* (f_A, τ_1) *is* $FST_0(iii) \iff (g_B, \tau_2)$ *is* $FST_0(iii)$;
- *(d)* (f_A, τ_1) *is* $FST_0(iv) \iff (g_B, \tau_2)$ *is* $FST_0(iv)$.

Proof We prove only (a). Suppose (f_A, τ_1) is $FST_0(i)$. We will prove that (g_B, τ_2) is also $FST_0(i)$. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since (ϕ, ψ) is onto, there exist $x_1, x_2 \in X$ with $x_1 \neq x_2$, such that $\phi(x_1) = y_1$, $\phi(x_2) = y_2$, and $\psi(e) = k$ for all parameters $e \in A$ and for all $k \in B$. Hence, $x_1 \neq x_2$ since ϕ is one-to-one. Since (f_A, τ_1) is $FST_0(i)$, we have that for all $x_1, x_2 \in X$, $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau_1$ such that either $f_{1A}(e)(x_1) = \bar{1}$ and $f_{1A}(e)(x_2) = \bar{0}$, or $f_{1A}(e)(x_1) = \bar{0}$ and $f_{1A}(e)(x_2) = \bar{1}$. Now, there exists an FSOS $(\phi, \psi)(f_{1A}) \in \tau_2$ such that $(\phi, \psi)(f_{1A})_k(y_1) = \bar{1}$ as $f_{1A}(e)(x_1) = \bar{1}$ and $(\phi, \psi)(f_{1A})_k(y_2) = \bar{0}$ as $f_{1A}(e)(x_2) = \bar{0}$. Similarly, $(\phi, \psi)(f_{1A})_k(y_1) = \bar{0}$ and $(\phi, \psi)(f_{1A})_k(y_2) = \bar{1}$. Hence, (g_B, τ_2) is $FST_0(i)$.

Conversely, suppose that (g_B, τ_2) is $FST_0(i)$. We prove that (f_A, τ_1) is $FST_0(i)$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. This implies that $\phi(x_1) \neq \phi(x_2)$ as ϕ is one-to-one. Put $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$. Then $y_1 \neq y_2$. Since (g_B, τ_2) is $FST_0(i)$, there exists an FSOS $g_{1B} \in \tau_2$ such that either $g_{1B}(k)(y_1) = \bar{1}$ and $g_{1B}(k)(y_2) = \bar{0}$, or $g_{1B}(k)(y_1) = \bar{0}$ and $g_{1B}(k)(y_2) = \bar{1}$. Now, there exists an FSOS $(\phi, \psi)^{-1}(g_{1B}) \in \tau_1$ such that $(\phi, \psi)^{-1}(g_{1B})(e)(x_1) = g_{1B}(\psi(e))(\phi(x_1)) = g_{1B}(k)(y_1) = \bar{1}$ and $(\phi, \psi)^{-1}(g_{1B})(e)(x_2) = g_{1B}(\psi(e))(\phi(x_2)) = g_{1B}(k)(y_2) = \bar{0}$. Similarly, $(\phi, \psi)^{-1}(g_{1B})(e)(x_1) = \bar{0}$ and $(\phi, \psi)^{-1}(g_{1B})(e)(x_2) = \bar{1}$. Hence, (f_A, τ_1) is $FST_0(i)$. In the same way, (b), (c) and (d) can be proved. □

§5. Conclusion

This paper develops several new theories based on four new notions of $\alpha - separation$ axioms on fuzzy soft T_0 spaces. The *good extension, soft hereditary* and *soft topological properties* are widely discussed with examples and counter examples, which are extensively applicable in fuzzy logic and fuzzy topology. The similar concepts of such notions and relevant new theories will be investigated for lattice fuzzy soft T_0 spaces in our further study.

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