

3-Product Cordial Labeling of Some Graphs

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Abstract: A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called a 3-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2\}$, where $v_f(i)$ denotes the number of vertices labeled with i , $e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i \pmod{3}$. A graph with a 3-product cordial labeling is called a 3-product cordial graph. In this paper, we establish that the duplicating arbitrary vertex in cycle C_n , duplicating arbitrarily edge in cycle C_n , duplicating arbitrary vertex in wheel W_n , Ladder L_n , Triangular Ladder TL_n and the graph $\langle W_n^{(1)} : W_n^{(2)} : \dots : W_n^{(k)} \rangle$ are 3-product cordial.

Key Words: Cordial labeling, Smarandachely p -product cordial labeling, 3-product cordial labeling, 3-product cordial graph.

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§1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminologies of graph theory as in [2]. The symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G . Let $G(p, q)$ be a graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. A vertex labeling of a graph is a function from the vertex set of the graph to the natural numbers. There are several types of labeling. A detailed survey of graph labeling can be found in [5].

In 1987, Cahit introduced the idea of cordial labelings, a generalization of both graceful and harmonious labelings in [1]. Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. M. Sundaram et al., introduced the concept of product cordial labeling of a graph in [6]. A product cordial labeling of a graph G with the vertex set V is a function f from V to $\{0, 1\}$ such that if each edge uv is

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assigned the label $f(u)f(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.

P.Jeyanthi and A.Maheswari introduced the concept of 3-product cordial labeling [3] and further studied in [4]. They proved that the graphs like path $P_n, K_{1,n}, \langle B_{n,n} : w \rangle$, cycle C_n if $n \equiv 1, 2 \pmod{3}$, $C_n \cup P_n$, $C_m \circ \overline{K_n}$ if $m \geq 3$ and $n \geq 1$, $P_m \circ \overline{K_n}$ if $m, n \geq 1$, W_n if $n \equiv 1 \pmod{3}$ and the middle graph of P_n , the splitting graph of P_n , the total graph of $P_n, P_n[P_2], P_n^2, K_{2,n}$ and the vertex switching of C_n are 3-product cordial graphs. Also they proved that the complete graph K_n is not 3-product cordial if $n \geq 3$. In addition, they proved that if $G(p, q)$ is a 3-product cordial graph, then (i) if $p \equiv 1 \pmod{3}$, then $q \leq \frac{p^2 - 2p + 7}{3}$; (ii) if $p \equiv 2 \pmod{3}$ then $q \leq \frac{p^2 - p + 4}{3}$; (iii) if $p \equiv 0 \pmod{3}$ then $q \leq \frac{p^2 - 3p + 6}{3}$ and if G_1 is a 3-product cordial graph with $3m$ vertices and $3n$ edges and G_2 is any 3-product cordial graph then $G_1 \cup G_2$ is also 3-product cordial graph.

In this paper, we establish that the duplicating arbitrary vertex and duplicating arbitrary edge in cycle C_n , duplicating arbitrary rim vertex in wheel W_n , ladder L_n , triangular ladder TL_n and $\langle W_n^{(1)} : W_n^{(2)} : \dots : W_n^{(k)} \rangle$ are 3-product cordial.

We use the following definitions in the subsequent section.

Definition 1.1 Let G be a graph and let v be a vertex of G . The duplicate graph $D(G, v')$ of G is the graph whose vertex set is $V(G) \cup \{v'\}$ and edge set is $E(G) \cup \{v'x | x \text{ is the vertex adjacent to } v \text{ in } G\}$.

Definition 1.2 Let G be a graph and let $e = uv$ be an edge of G . The duplicate graph $D(G, e' = u'v')$ of G is the graph whose vertex set is $V(G) \cup \{u', v'\}$ and edge set is $E(G) \cup \{u'x, v'y | x \text{ and } y \text{ are the vertices adjacent with } u \text{ and } v \text{ in } G \text{ respectively}\}$.

Definition 1.3 Consider k copies of wheels namely $W_n^{(1)}, W_n^{(2)}, \dots, W_n^{(k)}$. Then, the graph $G = \langle W_n^{(1)} : W_n^{(2)} : \dots : W_n^{(k)} \rangle$ is obtained by joining apex vertex of each $W_n^{(p)}$ and apex of $W_n^{(p-1)}$ to a new vertex x_{p-1} for $2 \leq p \leq k$.

Definition 1.4 The ladder graph L_n is defined as the cartesian product of two path graphs.

Definition 1.5 A triangular ladder $TL_n, n \geq 2$ is a graph obtained by completing the ladder TL_n by edges $u_i v_{i+1}$ for $1 \leq i \leq n-1$.

Definition 1.6 Let p be an integer with $p > 1$. A mapping $f : V(G) \rightarrow \{0, 1, 2, \dots, p\}$ is called a Smarandachely p -product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2, \dots, p-1\}$, where $v_f(i)$ denotes the number of vertices labeled with i , $e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i \pmod{p}$. Particularly, if $p = 3$, such a Smarandachely 3-product cordial labeling is called 3-product cordial labeling. A graph with 3-product cordial labeling is called a 3-product cordial graph.

For any real number n , $\lceil n \rceil$ denotes the smallest integer $\geq n$ and $\lfloor n \rfloor$ denotes the greatest integer $\leq n$.

§2. Main Results

Theorem 2.1 *A duplicate graph $D(C_n, e')$ of a cycle C_n is a 3-product cordial graph.*

Proof Let v_1, v_2, \dots, v_n be the vertices of the cycle C_n and let $e = v_1v_2$ and the duplicated edge is $e' = v'_1v'_2$.

Define a vertex labeling $f : V(G) \rightarrow \{0, 1, 2\}$ by considering the following three cases.

Case 1 $n \equiv 0(\text{mod } 3), n > 6$.

$$\text{Define } f(v_i) = \begin{cases} 2 & \text{if } i \equiv 0, 1(\text{mod } 4) \\ 1 & \text{if } i \equiv 2, 3(\text{mod } 4) \end{cases} \text{ for } 1 \leq i \leq n - \left\lceil \frac{n+2}{3} \right\rceil - 1,$$

$$f(v_i) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \text{ for } i = n - \left\lceil \frac{n+2}{3} \right\rceil \text{ and } f(v_i) = 0 \text{ if } n - \left\lceil \frac{n+2}{3} \right\rceil < i \leq n-1;$$

$$f(v_n) = 2, f(v'_1) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}; f(v'_2) = 1.$$

In view of the above labeling pattern we have $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \frac{n+2}{3} \right\rceil, e_f(0) = e_f(1) = e_f(2) = \left\lceil \frac{n+2}{3} \right\rceil$.

Case 2 $n \equiv 1(\text{mod } 3), n > 7$.

$$\text{Define } f(v_i) = \begin{cases} 2 & \text{if } i \equiv 1, 2(\text{mod } 4) \\ 1 & \text{if } i \equiv 0, 3(\text{mod } 4) \end{cases} \text{ for } 1 \leq i \leq n - \left\lceil \frac{n+2}{3} \right\rceil - 2 \text{ when } n \text{ is even}$$

$$\text{and } f(v_i) = \begin{cases} 2 & \text{if } i \equiv 2, 3(\text{mod } 4) \\ 1 & \text{if } i \equiv 0, 1(\text{mod } 4) \end{cases} \text{ when } n \text{ is odd; } f(v_i) = 1 \text{ for } i = n - \left\lceil \frac{n+2}{3} \right\rceil - 1 \text{ and}$$

$$f(v_i) = 0 \text{ if } n - \left\lceil \frac{n+2}{3} \right\rceil \leq i \leq n-1, f(v_n) = 1, f(v'_1) = f(v'_2) = 2.$$

In view of the above labeling pattern we have $v_f(0) = v_f(1) = v_f(2) = \left\lceil \frac{n+2}{3} \right\rceil, e_f(0)-1 = e_f(1) = e_f(2) = \left\lceil \frac{n+2}{3} \right\rceil$.

For $n = 4$, we define the labeling as $f(v'_1) = f(v'_2) = 2, f(v_1) = f(v_2) = 0$ and $f(v_3) = f(v_4) = 1$.

Case 3 $n \equiv 2(\text{mod } 3), n > 5$.

$$\text{Define } f(v_i) = \begin{cases} 2 & \text{if } i \equiv 2, 3(\text{mod } 4) \\ 1 & \text{if } i \equiv 0, 1(\text{mod } 4) \end{cases} \text{ for } 1 \leq i \leq n - \left\lceil \frac{n+2}{3} \right\rceil \text{ when } n \text{ is even and}$$

$$f(v_i) = \begin{cases} 2 & \text{if } i \equiv 0, 1(\text{mod } 4) \\ 1 & \text{if } i \equiv 2, 3(\text{mod } 4) \end{cases} \text{ when } n \text{ is odd; } f(v_i) = 0 \text{ and } f(v_n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

$$\text{for } n - \left\lceil \frac{n+2}{3} \right\rceil < i \leq n-1, f(v'_1) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}, f(v'_2) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

In view of the above labeling pattern we have $v_f(0) = v_f(1) - 1 = v_f(2) = \left\lfloor \frac{n+2}{3} \right\rfloor$ if n is even, $v_f(0) = v_f(1) = v_f(2) - 1 = \left\lfloor \frac{n+2}{3} \right\rfloor$ if n is odd and $e_f(0) = e_f(1) = e_f(2) + 1 = \left\lfloor \frac{n+2}{3} \right\rfloor$.

Thus in each case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. The duplicate $D(C_n, e')$ of a cycle C_n is a 3-product cordial graph. \square

Example for 3-product cordial labeling of the graph G obtained by duplicating the edge v_1v_2 of the cycle C_8 is given in Fig.1.

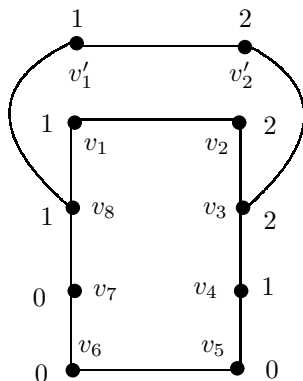


Fig.1

Theorem 2.2 A duplicate graph $D(C_n, v')$ of a cycle C_n is a 3-product cordial graph.

Proof Let v_1, v_2, \dots, v_n be the vertices of cycle C_n and let $v = v_1$ and the duplicated vertex is v'_1 . Define a vertex labeling $f : V(G) \rightarrow \{0, 1, 2, \}$ by $f(v_i) = \begin{cases} 2 & \text{if } i \equiv 2, 3(\text{mod } 4) \\ 1 & \text{if } i \equiv 0, 1(\text{mod } 4) \end{cases}$ for $1 \leq i \leq n - \left\lfloor \frac{n+2}{3} \right\rfloor$, $f(v_i) = 0$ for $n - \left\lfloor \frac{n+2}{3} \right\rfloor < i \leq n - 1$ and $f(v_n) = 1, f(v'_1) = 2$.

In view of the labeling pattern we have $v_f(0) = \frac{n}{3}$,

$$v_f(1) = \begin{cases} \frac{n}{3} + 1 & \text{if } n \text{ is odd} \\ \frac{3}{n} & \text{if } n \text{ is even} \end{cases}, v_f(2) = \begin{cases} \frac{n}{3} + 1 & \text{if } n \text{ is even} \\ \frac{3}{n} & \text{if } n \text{ is odd} \end{cases}$$

Calculation shows that $e_f(0) = e_f(1) = e_f(2) + 1 = \left\lfloor \frac{n+2}{3} \right\rfloor$ if $n \equiv 0(\text{mod } 3)$ and $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lfloor \frac{n+2}{3} \right\rfloor$ and $e_f(0) = e_f(1) = e_f(2) = \left\lfloor \frac{n+2}{3} \right\rfloor$ if $n \equiv 1(\text{mod } 3)$ and $v_f(0) = v_f(1) = v_f(2) = \left\lfloor \frac{n+1}{3} \right\rfloor$ and $e_f(0) - 1 = e_f(1) = e_f(2) = \left\lfloor \frac{n+1}{3} \right\rfloor$ if $n \equiv 2(\text{mod } 3)$. Thus we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence the graph obtained by duplicating an arbitrary vertex of cycle $C_n (n > 4)$ admits 3-product cordial graph. If $n = 3$, we define the vertex labeling as $f(v'_1) = 2, f(v_1) = 0$ and $f(v_2) = f(v_3) = 1$ and if $n = 4$, $f(v'_1) = 2, f(v_1) = 1, f(v_2) = 2, f(v_3) = 0$ and $f(v_4) = 1$. Thus the result follows. \square

Example for the 3-product labeling of the graph G obtained by duplicating the vertex v_1 of the cycle C_6 is given in Fig.2.

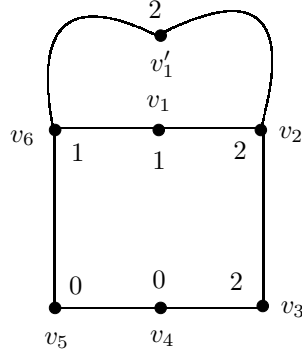


Fig.2

Theorem 2.3 A graph obtained by duplication of arbitrary rim vertex of wheel W_n is 3-product cordial if $n \equiv 0, 2 \pmod{3}$.

Proof Let v_1, v_2, \dots, v_n be the rim vertices of wheel W_n and let c be the apex vertex. Let G be the graph obtained by duplicating arbitrary rim vertex v_1 of wheel. Let v'_1 be the duplicated vertex of v_1 . Define a vertex labeling $f : V(G) \rightarrow \{0, 1, 2\}$ by considering the following two cases.

Case 1 $n \equiv 0 \pmod{3}$.

$$\text{Define } f(v_i) = \begin{cases} 2 & \text{if } i \equiv 2, 3 \pmod{4} \\ 1 & \text{if } i \equiv 0, 1 \pmod{4} \end{cases} \text{ for } 1 \leq i \leq 2 \left\lfloor \frac{n+2}{3} \right\rfloor - 3; f(v_i) = 0 \text{ for } 2 \left\lfloor \frac{n+2}{3} \right\rfloor - 2 \leq i \leq n-1 \text{ and } f(v_n) = 1, f(v'_1) = 2; f(c) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

$$\text{In view of the above labeling pattern we have } v_f(0)+1 = v_f(1) = v_f(2) = \left\lfloor \frac{n+2}{3} \right\rfloor, e_f(0) = e_f(1) = e_f(2) = \left\lfloor \frac{2n+3}{3} \right\rfloor.$$

Case 2 $n \equiv 2 \pmod{3}$.

$$\text{Define } f(v_i) = \begin{cases} 2 & \text{if } i \equiv 2, 3 \pmod{4} \\ 1 & \text{if } i \equiv 0, 1 \pmod{4} \end{cases} \text{ for } 1 \leq i \leq 2 \left\lfloor \frac{n+2}{3} \right\rfloor - 2; f(v_i) = 0 \text{ for } 2 \left\lfloor \frac{n+2}{3} \right\rfloor - 1 \leq i \leq n-1 \text{ and } f(v_n) = 1, f(v'_1) = 2, f(c) = 1.$$

$$\text{In view of the above labeling pattern we have } v_f(0) = v_f(1) - 1 = v_f(2) = \left\lfloor \frac{n+2}{3} \right\rfloor, e_f(0) - 1 = e_f(1) = e_f(2) = \left\lfloor \frac{2n+3}{3} \right\rfloor.$$

Thus in all the case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence the graph obtained by duplicating of arbitrary rim vertex of wheel W_n is 3-product cordial if $n \equiv 0, 2(mod 3)$. \square

Example for the 3-product cordial labeling of the graph G obtained by duplicating the vertex v_1 of the wheel W_9 is given in Fig.3.

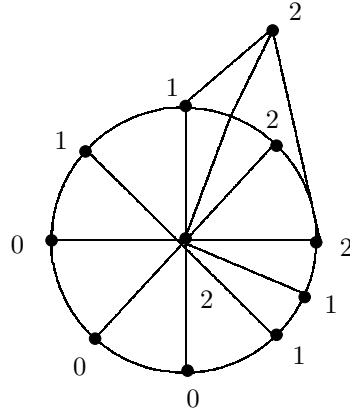


Fig.3

Theorem 2.4 A ladder $L_n = P_n \times P_2$ is 3-product cordial.

Proof Let the vertex set of $L_n = P_n \times P_2$ be $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and the edge set of L_n be $\{u_i u_{i+1} / 1 \leq i \leq n - 1\} \cup \{v_i v_{i+1} / 1 \leq i \leq n - 1\} \cup \{u_i v_i / 1 \leq i \leq n\}$. Then L_n has $2n$ vertices and $3n - 2$ edges.

Define a vertex labeling $f : V(L_n) \rightarrow \{0, 1, 2\}$ by considering the following three cases.

Case 1 $n \equiv 0(mod 3)$, let $n = 3m$.

$$\text{Define } f(u_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq m \\ 2 & \text{for } i = m + j \text{ where } j \equiv 3(mod 4), 1 \leq j \leq 2m \\ 1 & \text{for otherwise} \end{cases}$$

$$f(v_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq m \\ 1 & \text{for } i = m + j \text{ where } j \equiv 0(mod 4), 1 \leq j \leq 2m \\ 2 & \text{for otherwise} \end{cases}$$

In view of the above labeling pattern we have $v_f(0) = v_f(1) = v_f(2) = 2m$, $e_f(0) = 3m, e_f(1) = e_f(2) = 3m - 1$.

Case 2 $n \equiv 1(mod 3)$, let $n = 3m + 1$.

$$\text{Define } f(u_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq m \\ 2 & \text{for } i = m + j \text{ where } j \equiv 3(mod 4), 1 \leq j \leq 2m \text{ and } f(u_n) = 1 \\ 1 & \text{for otherwise} \end{cases}$$

$$f(v_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq m \\ 1 & \text{for } i = m + j \text{ where } j \equiv 0(\text{mod } 4), 1 \leq j \leq 2m + 1 \\ 2 & \text{for otherwise} \end{cases}$$

In view of the above labeling pattern we have $v_f(0) = 2m, v_f(1) = v_f(2) = 2m + 1, e_f(0) = 3m, e_f(1) = \begin{cases} 3m + 1 & \text{if } m \text{ is odd} \\ 3m & \text{if } m \text{ is even} \end{cases}, e_f(2) = \begin{cases} 3m + 1 & \text{if } m \text{ is even} \\ 3m & \text{if } m \text{ is odd.} \end{cases}$

Case 3 $n \equiv 2(\text{mod } 3)$, let $n = 3m + 2$.

$$f(u_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq m + 1 \\ 2 & \text{for } i = m + j \text{ where } j \equiv 3(\text{mod } 4), 2 \leq j \leq 2m + 2 \\ 1 & \text{for otherwise} \end{cases}$$

$$f(v_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq m \\ 1 & \text{for } i = m + j \text{ where } j \equiv 1(\text{mod } 4), 2 \leq j \leq 2m + 2, \\ 2 & \text{for otherwise} \end{cases}$$

$$f(v_{m+1}) = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 2 & \text{if } m \text{ is even} \end{cases}$$

In view of the above labeling pattern we have $v_f(0) = v_f(1) = 2m + 1, v_f(2) = 2m + 2$ and $e_f(0) = 3m + 2, e_f(1) = e_f(2) = 3m + 1$. Thus in each case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence the graph L_n is 3-product cordial. \square

Example for 3-product cordial labeling of the graph G obtained by $L_7 = P_7 \times P_2$ is given in Fig.4.

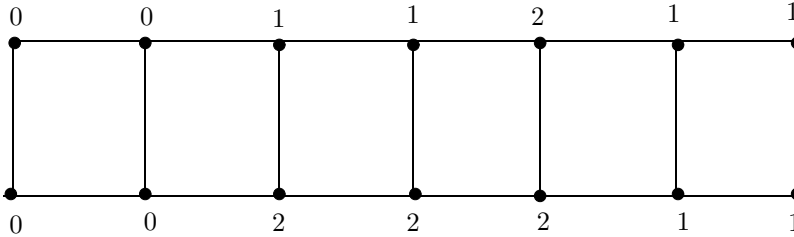


Fig.4

Theorem 2.5 *A triangular Ladder TL_n is 3-product cordial if $n \equiv 1, 2(\text{mod } 3)$.*

Proof Let the vertex set of TL_n be $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and the edge set of TL_n be $\{u_i u_{i+1} / 1 \leq i \leq n - 1\} \cup \{v_i v_{i+1} / 1 \leq i \leq n - 1\} \cup \{u_i v_i / 1 \leq i \leq n\} \cup \{u_i v_{i+1} / 1 \leq i \leq n\}$. Then TL_n has $2n$ vertices and $4n - 3$ edges.

Define a vertex labeling $f : V(L_n) \rightarrow \{0, 1, 2\}$ by considering the following two cases.

Case 1 $n \equiv 1(\text{mod } 3)$, let $n = 3k + 1$.

$$f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k + 1 \leq i \leq 3k + 1 \end{cases}, f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 2 & \text{if } k + 1 \leq i \leq 3k + 1 \end{cases}$$

In view of the above labeling pattern we have $v_f(0) + 1 = v_f(1) = v_f(2) = 2k + 1, e_f(0) = e_f(1) = e_f(2) - 1 = 4k$.

Case 2 $n \equiv 2(mod 3)$, let $n = 3k + 2$.

$$f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k + 1 \leq i \leq 3k + 1 \\ 0 & \text{if } i = 3k + 2 \end{cases}, f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 2 & \text{if } k + 1 \leq i \leq 3k + 2 \end{cases}$$

In view of the above labeling pattern we have $v_f(0) = v_f(1) = v_f(2) - 1 = 2k + 1, e_f(0) = e_f(1) + 1 = e_f(2) = 4k + 2$.

Thus in each case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence the graph triangular ladder TL_n is 3-product cordial. \square

Example for the 3-product cordial labeling of the graph G obtained by TL_8 is given in Fig.5.

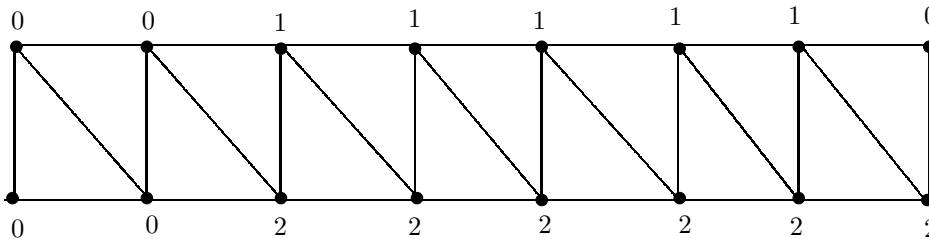


Fig.5

Theorem 2.6 A graph $\langle W_n^{(1)} : W_n^{(2)} : \dots : W_n^{(k)} \rangle$ is 3-product cordial if

- (i) k is multiple of 3 and n is any positive integer;
- (ii) $k \equiv 1(mod 3)$ and $n \equiv 1(mod 3)$.

Proof Let $v_i^{(j)}$ be the rim vertices of $W_n^{(j)}$ and c_j be the apex vertices of $W_n^{(j)}$. Let x_1, x_2, \dots, x_{k-1} be the new vertices. The graph $G = \langle W_n^{(1)} : W_n^{(2)} : \dots : W_n^{(k)} \rangle$ is obtained joining c_j with x_j and $x_{j+1}, 1 \leq j \leq k - 1$. Define a vertex labeling $f : V(G) \rightarrow \{0, 1, 2\}$, we consider the following two cases.

Case 1 $k \equiv 0(mod 3), 1 \leq i \leq n$.

Define

$$f(v_i^{(j)}) = \begin{cases} 0 & \text{for } 1 \leq j \leq \frac{k}{3} \\ 2 & \text{for } \frac{k}{3} < j \leq \frac{2k}{3} \\ 1 & \text{for } \frac{2k}{3} < j \leq k \end{cases}, f(c_j) = \begin{cases} 0 & \text{for } 1 \leq j \leq \frac{k}{3} \\ 1 & \text{for } \frac{k}{3} < j \leq \frac{2k}{3} \\ 2 & \text{for } \frac{2k}{3} < j \leq k \end{cases}$$

and if $k = 3, f(x_j) = \begin{cases} 1 & \text{for } j = \lfloor \frac{k-1}{4} \rfloor + m \text{ where } m \equiv 1(mod 2), 1 \leq m \leq \frac{2k}{3} \\ 2 & \text{for otherwise.} \end{cases}$ and if

$$k > 3, f(x_j) = \begin{cases} 0 & \text{for } 1 \leq j \leq \left\lfloor \frac{k-1}{4} \right\rfloor \\ 1 & \text{for } j = \left\lfloor \frac{k-1}{4} \right\rfloor + m \text{ where } m \equiv 1 \pmod{2}, 1 \leq m \leq \frac{2k}{3} \\ 2 & \text{for otherwise.} \end{cases}$$

In view of the above labeling pattern we have $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \frac{nk+2k-1}{3} \right\rceil$, $e_f(0) = e_f(1) - 1 = e_f(2) = \left\lfloor \frac{2nk+2k-2}{3} \right\rfloor$.

Case 2 $n \equiv 1 \pmod{3}$, $k \equiv 1 \pmod{3}$ and $1 \leq i \leq n$.

Let $n = 3k_1 + 1$, $k = 3k_2 + 1$ where k_1, k_2 are integer. Define $f(v_i^{(j)}) = 0, 1 \leq j \leq k_2$;

$$f(v_i^{(j)}) = \begin{cases} 0 & \text{for } 1 \leq i \leq k_1 \\ 1 & \text{for } i = k_1 + m \text{ where } m \equiv 2, 3 \pmod{4}, 1 \leq m \leq 2k_1 + 1 \\ 2 & \text{for otherwise} \end{cases}$$

if $j = k_2 + 1$ when n is even and

$$f(v_i^{(j)}) = \begin{cases} 0 & \text{for } 1 \leq i \leq k_1 \\ 2 & \text{for } i = k_1 + m \text{ where } m \equiv 2, 3 \pmod{4}, 1 \leq m \leq 2k_1 + 1 \\ 1 & \text{for otherwise} \end{cases}$$

when n is odd;

$$f(v_i^{(j)}) = \begin{cases} 2 & \text{for } k_2 + 1 < j \leq 2k_2 + 1 \\ 1 & \text{for } 2k_2 + 1 < j \leq 3k_2 + 1 \end{cases}$$

$$f(c_j) = \begin{cases} 0 & \text{for } 0 \leq j \leq k_2 \\ 2 & \text{for } j = k_2 + 1 \\ 1 & \text{for } k_2 + 1 < j \leq 2k_2 + 1 \\ 2 & \text{for otherwise} \end{cases}$$

$$f(x_j) = \begin{cases} 0 & \text{for } 1 \leq j \leq k_2 \\ 1 & \text{for } j = k_2 + m \text{ where } m \equiv 1 \pmod{2}, 1 \leq m \leq 2k_2 \\ 2 & \text{for otherwise} \end{cases}$$

In view of the above labeling pattern we have $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \frac{nk+2k-1}{3} \right\rceil$, $e_f(0) = e_f(1) + 1 = e_f(2) = \left\lfloor \frac{2nk+2k-2}{3} \right\rfloor$. Thus in each case we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence $\langle W_n^{(1)} : W_n^{(2)} : \dots : W_n^{(k)} \rangle$ is 3-product cordial graph if (i) $k \equiv 0 \pmod{3}$, n is any value (ii) $k \equiv 1, n \equiv 1 \pmod{3}$. \square

Example for the 3-product cordial labeling of the graph $\langle W_n^{(1)} : W_n^{(2)} : W_n^{(3)} : \dots : W_n^{(4)} \rangle$ is given in Fig.6.

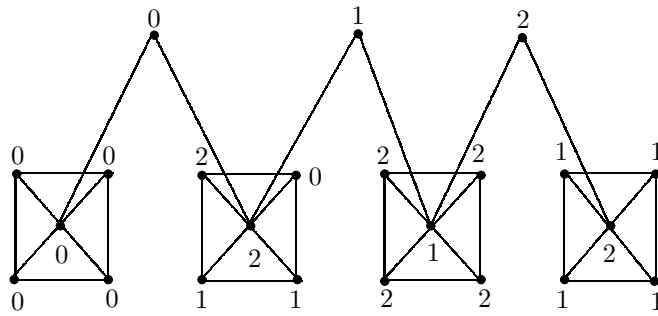


Fig.6

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