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4-Ordered Hamiltonicity of the Complete Expansion Graphs of Cayley Graphs

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Abstract: In this paper, we prove that the Complete expansion graph $And(k)$ $(k \geq 6)$ is 4-ordered hamiltonian graph by the method of classification discuss.

Key Words: Andr*á*sfai graph, complete expansion graph, k-ordered hamiltonian graph.

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§1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let C be a cycle with given orientation in graph X, $\overrightarrow{C}(\overrightarrow{C} = C)$ with anticlockwise direction and $\overleftarrow{C}$ with clockwise direction. If $x \in V(C)$, then we use x^+ to denote the successor of x on C and x^- to denote its predecessor. Use $C[x, y]$ denote (x, y)-path on C; $C(x, y)$ denote (x, y)-path missing x, y on C. Any undefined notation follows that of [1, 2].

Definition 1.1([1]) *Let G be a group and let C be a subset of G that is closed under taking inverses and does not contain the identity, then the Cayley graph $X(G, C)$ is the graph with vertex set G and edge set $E(X(G, C)) = \{gh : hg^{-1} \in C\}$.*

For a Cayley graph G, it may not be a hamiltonian graph, but a Cayley graph of Abelian group is a hamiltonian graph. $And(k)$ is a family of Cayley graph, which is named by the Hungarian mathematician Andr*á*sfai, it is a k-regular graph with the order $n = 3k - 1$ and it is a hamiltonian graph.

Definition 1.2([1]) *For any integer $k \geq 1$, let $G = Z_{3k-1}$ denote the additive group of integer modulo $3k-1$ and let C be the subset of Z_{3k-1} consisting of the elements congruent to 1 modulo 3. Then we denote the Cayley graph $X(G, C)$ by $And(k)$.*

For convenience, we note $Z_{3k-1} = \{u_0, u_1, \ldots, u_{3k-2}\}$. For $u_i, u_j \in V[And(k)], u_i \sim u_j$ if and only if $j - i \equiv \pm 1 mod 3$. The result are directly by the definition of Andr*á*sfai graph.

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Lemma 1.3 *Let C be any hamiltonian cycle in $And(k)(k \geq 2)$.*

(1) *If $\forall u, x \in V(And(k))$, $u \sim x$ is a chord of C, then $u^- \sim x^-$, $u^+ \sim x^+$.*

(2) *If $\forall u, x, y \in V(And(k))$, $u \sim x$, $u \sim y$ are two chords of C, then $x \sim y^+$.*

The definition of k-ordered hamiltonian graph was given in 1997 by Lenhard as follows.

Definition 1.4([3]) *A hamiltonian graph G of order ν is k-ordered, $2 \leq k \leq \nu$, if for every sequence $(v_1, v_2, \ldots, v_k)$ of k distinct vertices of G, there exists a hamiltonian cycle that encounters $(v_1, v_2, \ldots, v_k)$ in this order.*

Faudree developed above definition into a k-ordered graph.

Definition 1.5([4]) *For a positive integer k, a graph G is k-ordered if for every ordered set of k vertices, there is a cycle that encounters the vertices of the set in the given order.*

It has been shown that $And(k)(k \geq 4)$ is 4-ordered hamiltonian graph by in [5]. The concept of expansion transformation graph of a graph was given in 2009 by A Yongga at first. Then an equivalence definition of complete expansion graph was given by her, that is, the method defined by Cartesian product in [6] as follows.

Definition 1.6([6]) *Let G be any graph and $L(G)$ be the line graph of G. Non-trivial component of $G \square L(G)$ is said complete expansion graph (CEG for short) of G, denoted by $\vartheta(G)$, said the map ϑ be a complete expansion transformation of G.*

The proof of main result in this paper is mainly according to the following conclusions.

Theorem 1.7([1]) *The Cayley graph $X(G, C)$ is vertex transitive.*

Theorem 1.8([5]) *$And(k)(k \geq 4)$ is 4-ordered hamiltonian graph.*

Theorem 1.9([7]) *Every even regular graph has a 2-factor.*

The notations following is useful throughout the paper. For $u \in V(G)$, the clique with the order $d_G(u)$ in $\vartheta(G)$ by u is denoted as $\vartheta(u)$. All cliques are the cliques in $\vartheta(And(k))$ determined by the vertices in $And(k)$, that is maximum Clique. For u, $v \in V(G)$, $\vartheta(u) \sim \vartheta(v)$ means there exist $x \in V(\vartheta(u))$, $y \in V(\vartheta(v))$, `s.t.`$x \sim y$ in $V(\vartheta(G))$, edge (x, y) is said an edge stretching out from $\vartheta(u)$. Use $G_{\vartheta(u)}[x, y; s, t]$ to denote (x, y)-longest path missing s, t in $\vartheta(u)$, where $x, y, s, t \in V(\vartheta(u))$.

§2. Main Results with Proofs

We consider that whether $\vartheta(And(k))$ $(k \geq 4)$ is 4-ordered hamiltonian graph or not in this section.

Theorem 2.1 *$\vartheta(And(k))$ $(k \geq 6)$ is a 4-ordered hamiltonian graph.*

The following lemmas are necessary for the proof of Theorem 2.1.

Lemma 2.2 *For any* $u \in V(And(k))(k \geq 2)$, $\forall x, y \in N(u)$, *there exists a hamiltonian cycle* C *in* $And(k)$, `s.t.` $ux \in E(C)$ *and* $uy \in E(C)$.

Proof Let C_0 is a hamiltonian cycle $u_0 \sim u_1 \sim u_2 \sim \ldots \sim u_{3k-2} \sim u_0$ in $And(k)(k \geq 2)$. For $u \in V(And(k))(k \geq 2)$, $\forall x, y \in N(u)$, then we consider the following cases.

Case 1 $x \sim u \sim y$ on $\overleftarrow{C}_0$. Then $C = C_0$ is that so, since C_0 is a hamiltonian cycle.

Case 2 $x \sim u$ and $u \not\sim y$ on $\overleftarrow{C}_0$ or $x \not\sim u$ and $u \sim y$ on $\overleftarrow{C}_0$. If $x \sim u$ and $u \not\sim y$ on $\overleftarrow{C}_0$, then we can find a hamiltonian cycle C in $And(k)(k \geq 2)$ according to Lemma 1, that is,

$$C = u \sim x \sim \overrightarrow{C}_0(x, y^-) \sim y^- \sim u^- \sim \overleftarrow{C}_0(u^-, y) \sim y \sim u;$$

If $x \not\sim u$ and $u \sim y$ on $\overleftarrow{C}_0$, then we can find a hamiltonian cycle C in $And(k)(k \geq 2)$ according to Lemma 1.3, that is,

$$C = u \sim x \sim \overleftarrow{C}_0(x, u^+) \sim u^+ \sim x^+ \sim \overrightarrow{C}_0(x^+, y) \sim y \sim u.$$

Case 3 $x \not\sim u \not\sim y$ on $\overleftarrow{C}_0$. Then we can find a hamiltonian cycle C in $And(k)(k \geq 2)$ according to Lemma 1.3, that is,

$$C = u \sim x \sim y^+ \sim \overrightarrow{C}_0(y^+, u^-) \sim u^- \sim x^- \sim \overleftarrow{C}_0(x^-, u^+) \sim u^+ \sim \overrightarrow{C}_0[x^+, y] \sim u.$$

For any $u \in V(And(k))(k \geq 2)$, Lemma 2.2 is true since $And(k)$ is vertex transitive. □

Corollary 2.3 *For any two edges which stretch out from any Clique, there exists a hamiltonian cycle in* $\vartheta(And(k))$ *containing them.*

Lemma 2.4 *If* k *is an odd number, then* $And(k)(k \geq 3)$ *can be decomposed into one 1-factor and* $\dfrac{k-1}{2}$ *2-factors.*

Proof $3k - 1$ is an even number, since k is an odd number. There exists one 1-factor M in $And(k)$ by the definition of $And(k)$. According to Theorem 1.9 and the condition of Lemma 2.4 for integers $k \geq 3$, $And(k) - E(M)$ is a $(k-1)$-regular graph with a hamiltonian cycle C_1, $And(k) - E(M) - E(C_1)$ is a $(k-3)$-regular graph with a hamiltonian cycle $C_2, \cdots,$ $And(k) - E(M) - \sum\limits_{i=1}^{\frac{k-1}{2}} E(C_i)$ is an empty graph.

Assume $k = 2r + 1 (r \in Z^+)$, since k is an odd number. First we shall prove the result for $r = 1$, and then by induction on r. If $r = 1$ $(k = 3)$, it is easy to see that $And(k) - E(M)$ is a hamiltonian cycle C_1 by Theorem 1.9 and the analysis form of Lemma 2.4 , so the result is clearly true.

Now, we assume that the result is true if $r = n (r \geq 1, k = 2n + 1)$, that is, $And(2n+1)$ can be decomposed into one 1-factor and n 2-factors. Considering the case of $r = n+1 (k = 2n+3$, we know $And(2n+3)(And[2(n+1)+1])$ can be decomposed into one 1-factor and $n+1$ 2-factors according to the induction.

Thus, if k is an odd number, then $And(k)(k \geq 3)$ can be decomposed into one 1-factor and $\dfrac{k-1}{2}$ 2-factors. $\square$

Proof of Theorem 2.1

$\vartheta(And(k))$ is a hamiltonian graph, since $And(k)$ is a hamiltonian graph. So there exists a hamiltonian cycle C_0 in $\vartheta(And(k))$ and a hamiltonian cycle $C_0{}'$ in $And(k)$, such that $C_0 = \vartheta(C_0{}')$, without loss of generality

$$C_0{}' = u_0u_1\dots u_{3k-2}u_0,$$

then

$$C_0 = u_{0,1}u_{0,2}\dots u_{0,k}u_{1,1}u_{1,2}\dots u_{1,k}\dots u_{3k-2,1}u_{3k-2,2}\dots u_{3k-2,k}u_{0,1},$$

where $u_{i,j} \in V(\vartheta(u_i))$, $u_i \in V(And(k))$, $d_{And(k)}(u_i) = k \geq 6$, $i = 0,1,2,\dots,3k-2$, and $u^-_{i,1}$, $u^+_{i,k} \notin V(\vartheta(u_i))$, $u^+_{i,j} = u_{i,j+1}$ $(1 \leq j \leq k-1)$ and $u^-_{i,l} = u_{i,l-1}$ $(2 \leq l \leq k)$. There are three cyclic orders $\forall u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h} \in V[\vartheta(And(k))]$ according to the definition of the ring arrangement of the second kind, as follows: $(u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h})$, $(u_{a,b}, u_{e,f}, u_{c,d}, u_{g,h})$, $(u_{a,b}, u_{c,d}, u_{g,h}, u_{e,f})$(see Fig.1). Let $S = \{(u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h}), (u_{a,b}, u_{e,f}, u_{c,d}, u_{g,h}), (u_{a,b}, u_{c,d}, u_{g,h}, u_{e,f})\}$.

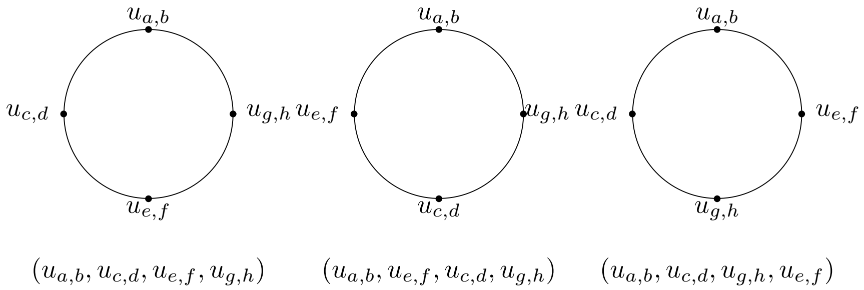

Fig.1 Three cyclic orders

Now, we show that 4-ordered hamiltonicity of $\vartheta(And(k))$ $(k \geq 6)$. In fact, we need to prove that $\alpha \in S$, there exists a hamiltonian cycle containing α. Without loss of generality, hamiltonian cycle C_0 encounters $(u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h})$ in this order. So we just prove: $\forall \beta \in S \setminus (u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h})$, there exists a hamiltonian cycle containing β.

According to the Pigeonhole principle, we consider following cases.

Case 1 If these four vertices $u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h}$ are contained in distinct four Cliques of $\vartheta(And(k))$, respectively. And Theorem 2.1 is true by the result in [5].

Case 2 If these four vertices $u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h}$ are contained in a same Clique of $\vartheta(And(k))$, then $a = c = e = g, b < d < f < h$. Let $S = \{(u_{a,b}, u_{a,d}, u_{a,f}, u_{a,h}), (u_{a,b}, u_{a,f}, u_{a,d}, u_{a,h}), (u_{a,b}, u_{a,d}, u_{a,h}, u_{a,f})\}$.

(1) For $(u_{a,b}, u_{a,d}, u_{a,f}, u_{a,h}) \in S$. C_0 is the hamiltonian cycle that encounters $(u_{a,b}, u_{a,d}, u_{a,f}, u_{a,h})$ in this order, clearly.

(2) For $(u_{a,b}, u_{a,f}, u_{a,d}, u_{a,h}) \in S$. We can find a hamiltonian cycle

$$C = u_{a,b}\overrightarrow{C}_0(u_{a,b}, u^-_{a,d})u^-_{a,d}u_{a,f}\overleftarrow{C}_0(u_{a,f}, u_{a,d})u_{a,d}u^+_{a,f}\overrightarrow{C}_0(u^+_{a,f}, u_{a,h})u_{a,h}\overrightarrow{C}_0(u_{a,h}, u_{a,b})u_{a,b}$$

that encounters $(u_{a,b}, u_{a,f}, u_{a,d}, u_{a,h})$ in this order.

(3) For $(u_{a,b}, u_{a,d}, u_{a,h}, u_{a,f}) \in S$. We can find a hamiltonian cycle

$$C = u_{a,1}\overrightarrow{C}_0(u_{a,1}, u^-_{a,f})u^-_{a,f}u_{a,k}\overleftarrow{C}_0(u_{a,k}, u_{a,f})u_{a,f}\vartheta(C_1)(u_{a,f}, u_{a,1})u_{a,1}$$

that encounters $(u_{a,b}, u_{a,d}, u_{a,h}, u_{a,f})$ in this order by Lemma 2.2 and Corollary 2.3 (see Fig.2).

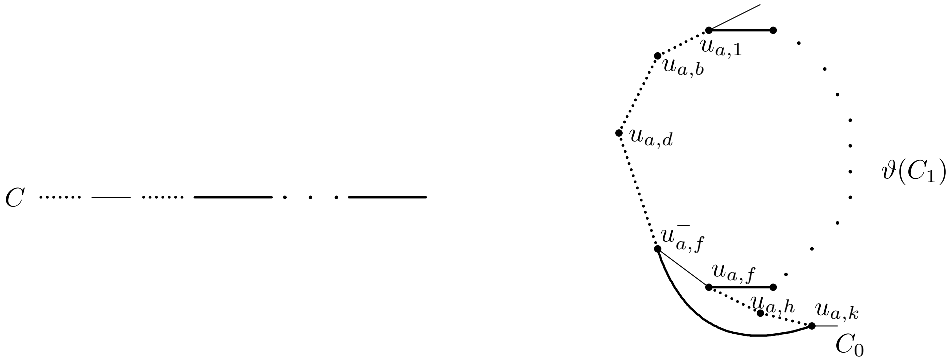

Fig.2 $C = u_{a,1}\overrightarrow{C}_0(u_{a,1}, u^-_{a,f})u^-_{a,f}u_{a,k}\overleftarrow{C}_0(u_{a,k}, u_{a,f})u_{a,f}\vartheta(C_1)(u_{a,f}, u_{a,1})u_{a,1}$

Case 3 If these four vertices $u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h}$ are contained in distinct two Cliques of $\vartheta(And(k))$, without loss of generality, we assume that $u_{a,b}, u_{c,d} \in V(\vartheta(u_a))$ and $u_{e,f}, u_{g,h} \in V(\vartheta(u_e))$ in $\vartheta(And(k))$ or $u_{a,b}, u_{c,d}, u_{e,f} \in V(\vartheta(u_a))$ and $u_{g,h} \in V(\vartheta(u_g))$ in $\vartheta(And(k))$ according to the notations. Let $S = \{(u_{a,b}, u_{a,d}, u_{e,f}, u_{e,h}), (u_{a,b}, u_{e,f}, u_{a,d}, u_{e,h}), (u_{a,b}, u_{a,d}, u_{e,h}, u_{e,f})\}$.

Subcase 3.1 $u_{a,b}, u_{c,d} \in V(\vartheta(u_a))$ and $u_{e,f}, u_{g,h} \in V(\vartheta(u_e))$ in $\vartheta(And(k))$.

(1) For $(u_{a,b}, u_{a,d}, u_{e,f}, u_{e,h}) \in S$. C_0 is the hamiltonian cycle that encounters $(u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h})$ in this order, clearly.

(2) For $(u_{a,b}, u_{e,f}, u_{a,d}, u_{e,h}) \in S$. Let C_1 is a hamiltonian cycle in $And(k)$ or $And(k) - E(M)$, C_2 is a hamiltonian cycle in $And(k) - E(C_1)$ or $And(k) - E(M) - E(C_1)$(see Fig.3). Use $A(C_1)$ to denote a cycle that only through two vertices in $\vartheta(u_i)(i = 1, 2, \ldots, 3k-2)$ and related with $\vartheta(C_1)$, and use $A(C_2)$ to denote the longest cycle missing the vertex on $A(C_1)$ in $\vartheta(And(k))$ or $\vartheta(And(k)) - M$(see Fig.3). We suppose that $P_1 = [x, y]$, $P_2 = [p, u_{a,b}]$ on cycle $A(C_1)$ in $\vartheta(And(k))$ or $\vartheta(And(k)) - M$ and $P_3 = [m, n]$, $P_4 = [s, t]$ on cycle $A(C_2)$ in $\vartheta(And(k)) - A(C_1)$ or $\vartheta(And(k)) - M - A(C_1)$ by Theorem 3[7], the analysis of Lemma 2.4 and the definition of CEG (see appendix). Now, we have a discussion about the position of vertex x, y, p, s and n in $\vartheta(And(k))$.

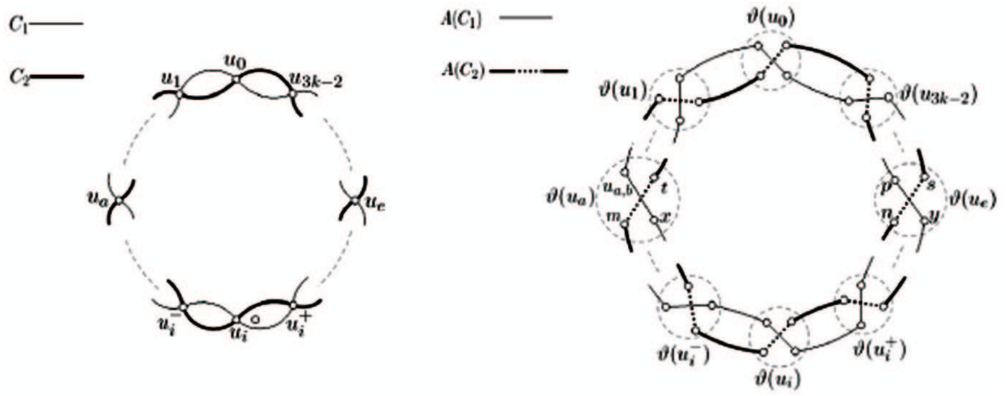

Fig.3

In where, C_1 in $And(k)$ or $And(k)-E(M)$, C_2 in $And(k)-E(C_1)$ or $And(k)-E(M)-E(C_1)$, $A(C_1)$ in $\vartheta(And(k))$ or $\vartheta(And(k))-M$, $A(C_2)$ in $\vartheta(And(k))-A(C_1)$ or $\vartheta(And(k))-A(C_1)-M$.

$$x \neq u_{a,b}, u_{a,d}, \begin{cases} y = u_{e,f}, \begin{cases} p = u_{e,h}, \cdots\cdots\cdots (1) \\ p \neq u_{e,h}, \begin{cases} s, n \neq u_{e,h}, \cdots\cdots\cdots (2) \\ s = u_{e,h}, \cdots\cdots\cdots (3) \\ n = u_{e,h}, \cdots\cdots\cdots (4) \end{cases} \end{cases} \\ y = u_{e,h}, \begin{cases} p = u_{e,f}, \cdots\cdots\cdots (5) \\ p \neq u_{e,f}, \begin{cases} s, n \neq u_{e,f}, \cdots\cdots\cdots (6) \\ s = u_{e,f}, \cdots\cdots\cdots (7) \\ n = u_{e,f}, \cdots\cdots\cdots (8) \end{cases} \end{cases} \\ y \neq u_{e,f}, u_{e,h}, \begin{cases} p = u_{e,f}, \begin{cases} s, n \neq u_{e,h}, \cdots\cdots\cdots (9) \\ s = u_{e,h}, \cdots\cdots\cdots (10) \\ n = u_{e,h}, , \cdots\cdots\cdots (11) \end{cases} \\ p = u_{e,h}, \begin{cases} s, n \neq u_{e,f}, \cdots\cdots\cdots (12) \\ s = u_{e,f}, \cdots\cdots\cdots (13) \\ n = u_{e,f}, \cdots\cdots\cdots (14) \end{cases} \\ p \neq u_{e,f}, u_{e,h}, \begin{cases} s = u_{e,f}, \cdots\cdots\cdots (15) \\ n = u_{e,f}, \cdots\cdots\cdots (16) \\ s = u_{e,h}, \cdots\cdots\cdots (17) \\ n = u_{e,h}, \cdots\cdots\cdots (18) \\ s = u_{e,f}, n = u_{e,h}, \cdots\cdots (19) \\ s = u_{e,h}, n = u_{e,f}, \cdots\cdots (20) \\ s \neq u_{e,f}, n \neq u_{e,h}, \cdots\cdots (21) \end{cases} \end{cases} \end{cases}$$

For cases (1) and (2), we can find a hamiltonian cycle

$$u_{a,b}xP_1(x,y)ysP_4(s,t)tG_{\vartheta(u_a)}[t,m;u_{a,b},x]mP_3(m,n)nG_{\vartheta(u_e)}[n,p;y,s]pP_2(p,u_{a,b})u_{a,b}$$

that encounters $(u_{a,b}, u_{e,f}, u_{a,d}, u_{e,h})$ in this order.

For cases (3)-(21), we can find a hamiltonian cycle that encounters $(u_{a,b}, u_{e,f}, u_{a,d}, u_{e,h})$ in this order according to the method of (1) and (2).

(3)For cases (2)-(11) and (15)-(21), we can find a hamiltonian cycle that encounters $(u_{a,b}, u_{a,d}, u_{e,h}, u_{e,f})$ in this order according to the method of Case3.1(2).

For case (1), we can find a hamiltonian cycle

$$u_{a,b}G_{\vartheta(u_a)}[u_{a,b},m;t]mP'_3(m,n)nG_{\vartheta(u_e)}[n,p;y,s]pysP'_4(s,t)tu_{a,b}$$

that encounters $(u_{a,b}, u_{a,d}, u_{e,h}, u_{e,f})$ in this order. P'_i is the path which through the all vertices in $\vartheta(u_i)(i = a, \ldots, e)$ and related with $P_i(i = 3, 4)$ in $\vartheta(And(k)) - A(C_1)$ or $\vartheta(And(k)) - M - A(C_1)$(see Fig.4).

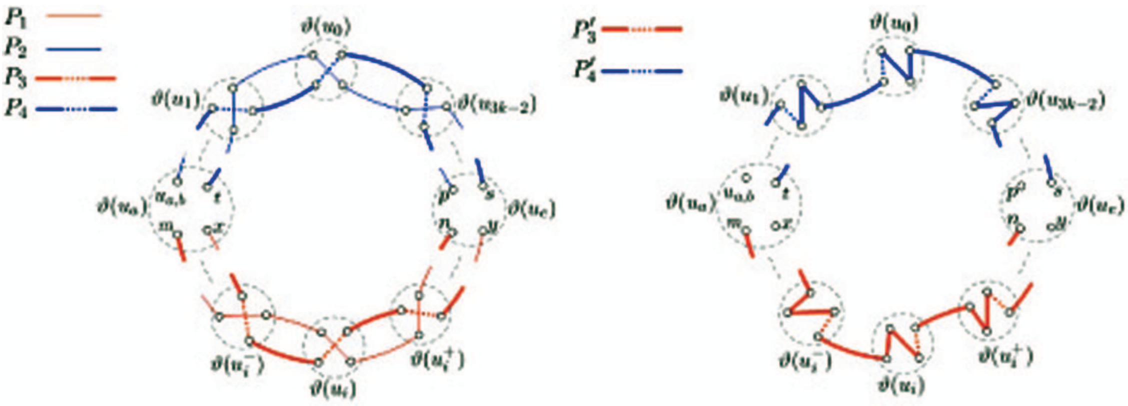

Fig.4

In where, P_1, P_2 in $\vartheta(And(k))$ or $\vartheta(And(k)) - M$, P_3, P_4 in $\vartheta(And(k)) - A(C_1)$ or $\vartheta(And(k)) - M - A(C_1)$, P'_3, P'_4 related with P_3, P_4 in $\vartheta(And(k)) - A(C_1)$ or $\vartheta(And(k)) - M - A(C_1)$.

For 12-14, we can find a hamiltonian cycle that encounters $(u_{a,b}, u_{a,d}, u_{e,h}, u_{e,f})$ in this order according to the method of 1.

Subcase 3.2 $u_{a,b}, u_{c,d}, u_{e,f} \in V(\vartheta(u_a))$ and $u_{g,h} \in V(\vartheta(u_g))$ in $\vartheta(And(k))$. For all condition , we see the result is proved by the method of Subcase 3.1.

Case 4 If these four vertices $u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h}$ are contained in distinct three Cliques of $\vartheta(And(k))$. Without loss of generality, we assume that $u_{a,b}, u_{c,d} \in V(\vartheta(u_a))$, $u_{e,f} \in V(\vartheta(u_e))$ and $u_{g,h} \in V(\vartheta(u_g))$ in $\vartheta(And(k))$.

(1) For $(u_{a,b}, u_{a,d}, u_{e,f}, u_{g,h}) \in S$, C_0 is the hamiltonian cycle that encounters $(u_{a,b}, u_{a,d}, u_{e,f}, u_{g,h})$ in this order, clearly.

(2) For $(u_{a,b}, u_{e,f}, u_{a,d}, u_{g,h}) \in S$. Let C_1 is a hamiltonian cycle in $And(k)$ or $And(k) - E(M)$, C_2 is a hamiltonian cycle in $And(k) - E(C_1)$ or $And(k) - E(M) - E(C_1)$, C_3 is a

hamiltonian cycle in $And(k)-E(C_1)-E(C_2)$ or $And(k)-E(M)-E(C_1)-E(C_2)$(see Fig.5). Use $A(C_j)$ to denote a cycle that only through two vertices in $\vartheta(u_i)(i=1,2,\ldots,3k-2)$ and related with $\vartheta(C_j)(j=1,2)$, and use $A(C_3)$ to denote the longest cycle missing the vertex on $A(C_1)$ and $A(C_2)$ in $\vartheta(And(k))$ or $\vartheta(And(k))-M$(see Figure5). We can suppose that $P_1=[u_{c,d},x]$, $P_2=[y,u_{a,b}]$ on cycle $A(C_1)$ in $\vartheta(And(k))$ or $\vartheta(And(k))-M$, $P_3=[m,n]$, $P_4=[p,q]$ on cycle $A(C_2)$ in $\vartheta(And(k))-A(C_1)$ or $\vartheta(And(k))-M-A(C_1)$ and $P_5=[s,t]$, $P_6=[w,z]$ on $A(C_3)$ in $\vartheta(And(k))-\sum\limits_{i=1}^{2}A(C_i)$ or $\vartheta(And(k))-M-\sum\limits_{i=1}^{2}A(C_i)$ by Theorem 3[7], the analysis of Lemma 3 and the definition of CEG (see appendix). Now, we have a discussion about the position of vertex m, q, x, y, p and n in $\vartheta(And(k))$.

$$m,q\neq u_{a,b},u_{a,d},\begin{cases} x=u_{g,h},y\neq u_{g,h},\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)\\ x\neq u_{g,h},\begin{cases} y=u_{g,h},\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)\\ y\neq u_{g,h};\begin{cases} p=u_{g,h},\cdots\cdots\cdots\cdots(3)\\ n=u_{g,h},\cdots\cdots\cdots\cdots(4)\\ p,n\neq u_{g,h}.\ \cdots\cdots\cdots(5)\end{cases}\end{cases}\end{cases}$$

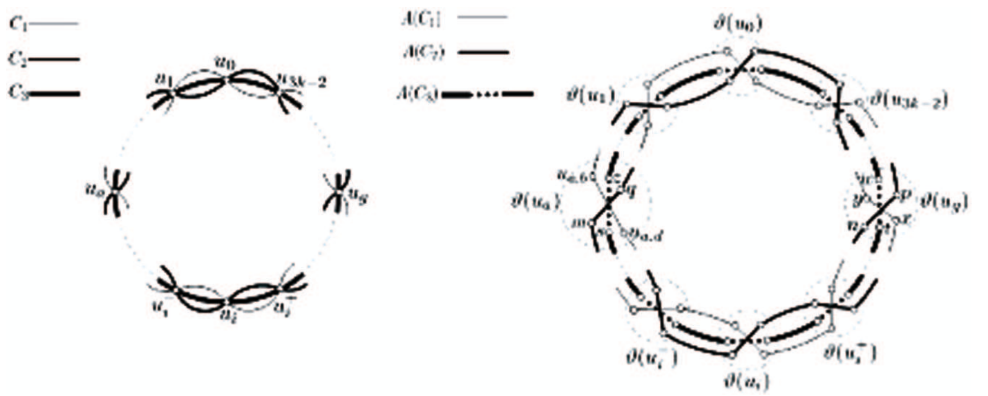

Fig.5

In where, C_1 in $And(k)$ or $And(k)-E(M)$, C_2 in $And(k)-E(C_1)$ or $And(k)-E(M)-E(C_1)$, C_3 in $And(k)-E(C_1)-E(C_2)$ or $And(k)-E(M)-E(C_1)-E(C_2)$, $A(C_1)$ in $\vartheta(And(k))$ or $\vartheta(And(k))-M$, $A(C_2)$ in $\vartheta(And(k))-A(C_1)$ or $\vartheta(And(k))-A(C_1)-M$, $A(C_3)$ in $\vartheta(And(k))-A(C_1)-A(C_2)$ or $\vartheta(And(k))-A(C_1)-A(C_2)-M$.

For case (1), if $u_{e,f}\in V(P_i)$ $(i=2,3,4)$, we can find a hamiltonian cycle that encounters $(u_{a,b},u_{e,f},u_{a,d},u_{g,h})$ in this order according to the method of Subcase 3.1,(2).

If $u_{e,f}\in V(P_1)$, we can find a hamiltonian cycle

$$u_{a,b}qP_4'(q,p)pnP_3'(n,m)mG_{\vartheta(u_a)}[m,s;u_{a,b},q]sP_5'(s,t)tG_{\vartheta(u_g)}[t,y;p,n,t]yP_2'(y,u_{a,b})u_{a,b}\ or$$

$$u_{a,b}mP_3'(m,n)npP_4'(q,p)qG_{\vartheta(u_a)}[q,s;u_{a,b},m]sP_5'(s,t)tG_{\vartheta(u_g)}[t,y;p,n,t]yP_2'(y,u_{a,b})u_{a,b}$$

that encounters $(u_{a,b}, u_{e,f}, u_{a,d}, u_{g,h})$ in this order. There exist some vertices which belong to a same Clique on P_1, P_i and $P_j(i = 3, 4; j = 5, 6)$. And $u_{e,f} \in V(P_i')(i = 3$ or $4)$. P_i' is the path which through the all vertices in $\vartheta(u_i)(i = a, \ldots, g)$ and related with $P_i(i = 5, 6)$ in $\vartheta(And(k)) - \sum_{i=1}^{2} A(C_i)$ or $\vartheta(And(k)) - M - \sum_{i=1}^{2} A(C_i)$, and missing the vertex on P_3', P_4'(refers to Figure4).

For cases (2)-(5), we can find a hamiltonian cycle that encounters $(u_{a,b}, u_{e,f}, u_{a,d}, u_{g,h})$ in this order according to the method of (1).

(3) For cases (1)-(5), we can find a hamiltonian cycle that encounters $(u_{a,b}, u_{a,d}, u_{g,h}, u_{e,f})$ in this order according to the method of Case 4(2). □

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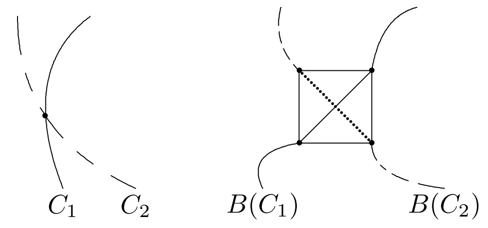
Appendix

By the theorem 1.9, the analysis of Lemma 2.4, the definition of CEG, $And(k)$ and the parity of $k(s \in Z^+)$, we know that

$k = 2s$	$And(k)$	$\vartheta(And(k))$
$s = 1$	C_5	C_{10}
$s = 2$	$And(4) - E(C_1) = C_2$	$\vartheta(And(4)) - B(C_1) = B(C_2)$
$s = 3$	$And(6) - E(C_1) - E(C_2) = C_3$	$\vartheta(And(6)) - B(C_1) - B(C_2) = B(C_3)$
$\vdots$	$\vdots$	$\vdots$
$s = n$	$And(2n) - \sum_{i=1}^{n-1} E(C_i) = C_n$	$\vartheta(And(2n)) - \sum_{i=1}^{n-1} B(C_i) = B(C_n)$



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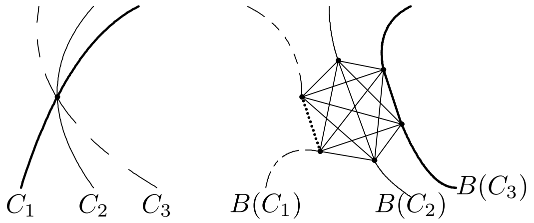
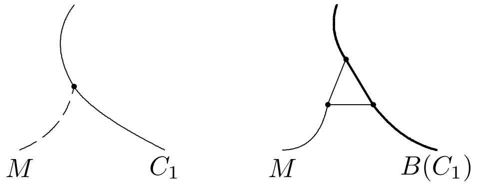


s=3

If k is odd, it should be illustrated that the M's selection method, that is, M satisfy condition $u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h} \notin V(M)$ in $\vartheta(And(k))$. It can be done, because $k \geq 7$.

$k = 2s + 1$	$And(k)$	$\vartheta(And(k))$
$s = 1$	$And(3) - E(M) = C_1$	$\vartheta(And(3)) - M = B(C_21)$
$s = 2$	$And(5) - E(M) - E(C_1) = C_2$	$\vartheta(And(5)) - E(M) - B(C_1) = B(C_2)$
$s = 3$	$And(7) - E(M) - E(C_1) - E(C_2) = C_3$	$\vartheta(And(7)) - M - B(C_1) - B(C_2) = B(C_3)$
$\vdots$	$\vdots$	$\vdots$
$s = n$	$And(2n+1) - E(M) - \sum\limits_{i=1}^{n-1} E(C_i) = C_n$	$\vartheta(And(2n+1)) - M - \sum\limits_{i=1}^{n-1} B(C_i) = B(C_n)$



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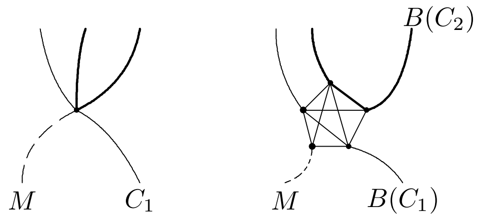

s=2