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A Generalization of the Alexander Polynomial

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Abstract: In this paper, we present a generalization of two variables of the Alexander polynomial for a given oriented knot diagram. We define the Alexander polynomial of two variables by an easy method which will be achieved as a result of the interpretation of the crossing point as a particle with input-output spins in the mathematical physics. The classical Alexander polynomial is the case of one of the variables to be equal to 1 in the Alexander polynomial of two variables.

Key Words: Knot polynomials, Alexander polynomial, ambient isotopy.

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§1. Introduction

A knot polynomial is a knot invariant in the form of a polynomial whose coefficients encode some of the properties of a given knot. The Alexander polynomial is the first knot polynomial. It was introduced by J. W. Alexander in 1928 ([1]).

There are several ways to calculate the Alexander polynomial. One of them is the procedure given by Alexander in his paper [1]. This procedure is briefly as follows: Given an oriented diagram of the knot with n crossings. There are $n+2$ regions bounded by the knot diagram. The Alexander polynomial is calculated by using a matrix of size $n \times (n+2)$. The rows of the matrix correspond to crossings, and the columns to the regions. Another one is to calculate from the Seifert matrix ([2]). The Alexander polynomial can also be calculated by using the free derivative defined by Fox [3,4].

Other knot polynomials were not found until almost 60 years later. In the 1960s, J.Conway came up with a skein relation for a version of the Alexander polynomial, usually referred to as the AlexanderConway polynomial [5]. The significance of this skein relation was not realized until the early 1980s, when V. Jones discovered the Jones polynomial [6,7]. This led to the discovery of more knot polynomials, such as the so-called Homfly polynomial [8]. The Homfly polynomial is a generalization of the AlexanderConway polynomial and the Jones polynomial. Soon after Jones' discovery, Louis Kauffman noticed the Jones polynomial could be computed

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by means of a state-sum model, which involved the bracket polynomial, an invariant of framed knots [9-13]. This opened up avenues of research linking knot theory and statistical mechanics.

In recent years, the Alexander polynomial has been shown to be related to Floer homology. The graded Euler characteristic of the knot Floer homology of Ozsváth and Szabó is the Alexander polynomial [14,15].

In this paper, we work on a generalization of two variables of the Alexander polynomial. We define the Alexander polynomial of two variables by an easy method. In the method, the Alexander polynomial of two variables is calculated by using a matrix of size $n \times n$. The rows of the matrix correspond to crossings of the oriented diagram of the knot with n crossings, and the columns to the arcs. The classical Alexander polynomial is the case of one of the variables to be equal to 1 in the Alexander polynomial of two variables.

§2. Alexander Polynomial of Two Variables

A link K of k components is a subset of $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} = S^3$, consisting of k disjoint piecewise simple closed curves; a knot is a link with one component. In fact, two knots (or links) in $\mathbb{R}^3$ can be deformed continuously one into the other if and only if any diagram of one knot can be transformed into a diagram for the knot via a sequence of the Reidmeister moves formed in Figure 1. The equivalence relation on diagrams that is generated by all the Reidmeister moves is called ambient isotopy. In the study, the word knot will be used instead knot and link.

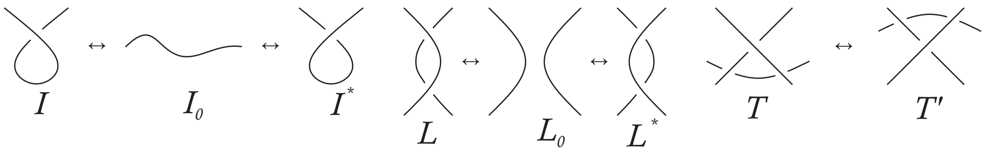

Figure 1

The first Reidemeister move: $I \leftrightarrow I_0$ or $I^* \leftrightarrow I_0$;The second Reidemeister move: $L \leftrightarrow L_0$ or $L^* \leftrightarrow L_0$ and the third Reidemeister move: $T \leftrightarrow T'$.

Let K be an oriented knot diagram with n crossings. Three arcs of the curve of the oriented diagram K encounters at a crossing. One of these arcs is overpass arc and the other two are underpass arcs that follow one another at the crossing point. Let c_i denote the ith crossing of the oriented diagram K, $i = 1, 2, \cdots, n$. We assume that the arcs s_i, s_j and s_k are encounter at the crossing c_i, see Figure 2.

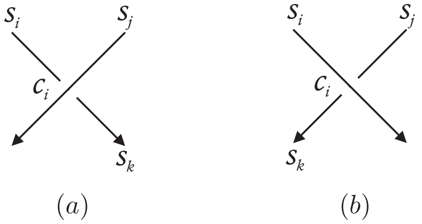

Figure 2 Crossings with positive sign (a) and negative sign (b)

In mathematical physics if we interpret the crossing point as a particle with incoming spins s_i, s_j and outgoing spins s_j, s_k for the crossing in Figure $2a$, then an associated mathematical expression to the crossing point can be regarded as the probability amplitude for this particular combination of spins in and out [13]. We can make a similar comment for the crossing in Figure $2b$, For now it is convenient to consider only the spins. The conservation of spin suggests the rule that

$$s_i + s_j = s_j + s_k.$$

If x and y are algebraic variables, then

$$xs_i + ys_j - xs_j - ys_k = xs_i + (y - x)s_j - ys_k = 0$$

is a assigned equation to the crossing in Figure $2a$. With the same thought, we can assign an equation to the crossing in Figure B. We say the above equation, the crossing equation.

By assigning a crossing equation for each crossing of the oriented diagram K we have a homogeneous system of n equations in n unknowns, and call diagram equation.

Since there are three unknowns (arcs) in a crossing equation, we get zero the coefficient of $(n-3)$ arcs that are not in this crossing equation. Thus, we obtain a coefficients matrix $\mathbf{M}$ of size $n \times n$ of the diagram equation. It is easy to see that the determinant, $|\mathbf{M}|$, of the coefficients matrix $\mathbf{M}$ is zero.

We may then regarded the matrix $\mathbf{M}$ as having entries in the ring $\mathbb{Z}[x, x^{-1}, y, y^{-1}]$ along with its subring $\mathbb{Z}[x, y]$ has the property that any finite set of elements has a greatest common divider. Any integer domain with these properties is called a greatest common divider. Determinants of the minors of size $(n-1) \times (n-1)$ of the matrix $\mathbf{M}$ are equal with multiplying $\mp x^k y^l$, $k, l \in \mathbb{Z}$ that has the greatest common divider.

Definition 2.1 *We will call the Alexander polynomial of two variables that is the greatest common divider of determinants of minors of size* $(n-1) \times (n-1)$ *of the matrix* $\boldsymbol{M}$ *and we'll denote it by* $\nabla(x, y)$.

If $\nabla_{K_1}(x, y)$ and $\nabla_{K_2}(x, y)$ are polynomials that are equal with such a factor, we write $\nabla_{K_1}(x, y) \doteq \nabla_{K_2}(x, y)$. Any one of the minors of size $(n-1) \times (n-1)$ of $\mathbf{M}$ can be taken to be a presentation matrix for $\nabla(x, y)$ and its determinant can be taken to be $\nabla(x, y)$ with multiplying by $\mp x^k y^l$, $k, l \in \mathbb{Z}$, see [4,16].

Example 2.2 We now calculate the Alexander polynomial of two variables of the trefoil knot as an example. Let K be the right-hand diagram trefoil knot drawn in Figure 3. The diagram equation of the knot K is as follows:

$$\begin{aligned}
xs_3 + ys_2 - xs_2 - ys_1 &= -ys_1 + (y - x)s_2 + xs_3 = 0 \\
xs_2 + ys_1 - xs_1 - ys_3 &= (y - x)s_1 + xs_2 - ys_3 = 0 \\
xs_1 + ys_3 - xs_3 - ys_2 &= xs_1 - ys_2 + (y - x)s_3 = 0
\end{aligned}$$

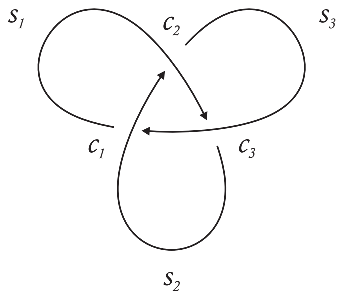

Figure 3 The right-hand trefoil knot.

The coefficients matrix $\mathbf{M}_K$ of size 3×3 of this diagram equation is

$$\mathbf{M}_K = \begin{bmatrix} -y & y-x & x \\ y-x & x & -y \\ x & -y & y-x \end{bmatrix}.$$

The determinant, $|\mathbf{M}_k|$ of the coefficients matrix $\mathbf{M}_K$ is zero. Hence, any one of the minors of size 2×2 of the matrix $\mathbf{M}_K$, for instance, ∇_{11} is a presentation matrix and its determinant is $|\nabla_{11}| = -x^2 + xy - y^2$. Thus, the Alexander polynomial of two variables for the knot K; $\nabla_K(x, y) = x^2 - xy + y^2$.

We have $\nabla_K(x, 1) = x^2 - x + 1$ for $y = 1$ (or $\nabla_K(1, y) = 1 - y + y^2$ for $x = 1$). It is the classical Alexander polynomial of the trefoil knot.

The following theorem gives that the Alexander polynomial of two variables is an invariant of the knot.

Theorem 2.3 *If K is an oriented knot diagram, then the Alexander polynomial of two variables, $\nabla_K(x, y)$, of the knot K is an invariant of ambient isotopy.*

Proof In order to prove that the Alexander polynomial is an invariant of ambient isotopy, we must investigate the behavior of $\nabla_K(x, y)$ under the Reidemeister moves given in Figure 1. Here, we shall investigate the behavior of $\nabla_K(x, y)$ under the diagrams given in Figure 4.

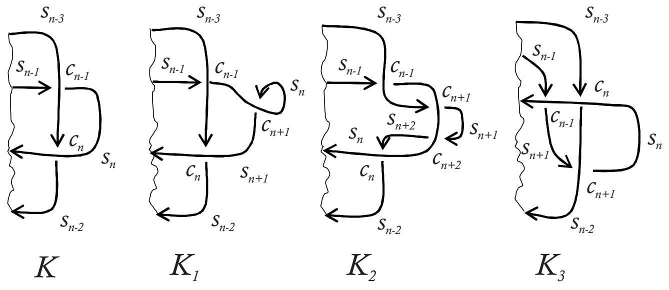

Figure 4

Diagrams for the proof of Theorem 2.3. For the first Reidemeister move: $K \leftrightarrow K_1$; for the second Reidemeister move: $K \leftrightarrow K_2$; for the third Reidemeister move: $K \leftrightarrow K_3$.

Let K be the oriented knot diagram with n crossings given in Figure 4. The diagram equation of the knot K is as follows:

$$\begin{array}{rcrcl}
\cdots & & \cdots & & \cdots \\
xs_{n-1} + ys_{n-3} - xs_{n-3} - ys_n & = & (y-x)s_{n-3} + xs_{n-1} - ys_n & = & 0 \\
xs_{n-3} + ys_n - xs_n - ys_{n-2} & = & xs_{n-3} - ys_{n-2} + (y-x)s_n & = & 0
\end{array}$$

The coefficients matrix $\mathbf{M}_K$ of size $n \times n$ of this diagram equation is

$$\mathbf{M}_K = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & y-x & 0 & x & -y \\ \cdots & x & -y & 0 & y-x \end{bmatrix}.$$

Thus, any one of the minors of size $(n-1) \times (n-1)$ of the matrix $\mathbf{M}_K$, for instance, ∇_{11} is a presentation matrix and its determinant $|\nabla_{11}| = \nabla_K(x, y)$ with multiplying by $\mp x^k y^l$, $k, l \in \mathbb{Z}$.

Case 1. The behavior of $\nabla_K(x, y)$ under the first Reidemeister move

The diagram equation of the diagram K_1 with $(n+1)$ crossings given in Figure 4 is as follows:

$$\begin{array}{rcrcl}
\cdots & & \cdots & & \cdots \\
xs_{n-1} + ys_{n-3} - xs_{n-3} - ys_n & = & (y-x)s_{n-3} + xs_{n-1} - ys_n & = & 0 \\
xs_{n-3} + ys_{n+1} - xs_{n+1} - ys_{n-2} & = & xs_{n-3} - ys_{n-2} + (y-x)s_{n+1} & = & 0 \\
xs_n + ys_n - xs_{n+1} - ys_n & = & xs_n - xs_{n+1} & = & 0
\end{array}$$

Hence, we have the following coefficients matrix $\mathbf{M}_{K_1}$ of size $(n-1) \times (n-1)$ of the diagram equation.

$$\begin{aligned}
\mathbf{M}_{K_1} &= \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & y-x & 0 & x & -y & 0 \\ \cdots & x & -y & 0 & 0 & y-x \\ \cdots & 0 & 0 & 0 & x & -x \end{bmatrix} \\
&= \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & y-x & 0 & x & -y & 0 \\ \cdots & x & -y & 0 & y-x & y-x \\ \cdots & 0 & 0 & 0 & 0 & -x \end{bmatrix}.
\end{aligned}$$

Since $|\mathbf{M}_{K_1}| = -x|\mathbf{M}_K| = 0$, the minors of size $(n-1) \times (n-1)$ of $\mathbf{M}_{K_1}$ are equal the corresponding minors size $(n-1) \times (n-1)$ of $\mathbf{M}_K$ and hence, $\nabla_K(x, y) \doteq \nabla_{K_1}(x, y)$. In that case $\nabla_K(x,y)$ is unchanged under the first Reidemeister move.

Case 2. The behavior of $\nabla_K(x,y)$ under the second Reidemeister move

We obtain the following diagram equation from the diagram K_2 with $(n+2)$ crossings given in Figure 4.

$$\begin{array}{rcrcl}
\cdots & & \cdots & & \cdots \\
xs_{n-1}+ys_{n-3}-xs_{n-3}-ys_n & = & (y-x)s_{n-3}+xs_{n-1}-ys_n & = & 0 \\
xs_{n-3}+ys_n-xs_n-ys_{n-2} & = & xs_{n-3}-ys_{n-2}+(y-x)s_n & = & 0 \\
xs_n+ys_{n-2}-xs_{n+1}-ys_n & = & ys_{n-2}+(x-y)s_n-xs_{n+1} & = & 0 \\
xs_{n+1}+ys_n-xs_n-ys_{n+2} & = & (y-x)s_n+xs_{n+1}-ys_{n+2} & = & 0
\end{array}$$

The coefficients matrix $\mathbf{M}_{K_2}$ of size $(n+2)\times(n+2)$ of the diagram equation of K_2 here is as follows.

$$\begin{aligned}
\mathbf{M}_{K_2} &= \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & y-x & 0 & x & -y & 0 & 0 \\
\cdots & x & -y & 0 & y-x & 0 & 0 \\
\cdots & 0 & y & 0 & x-y & -x & 0 \\
\cdots & 0 & 0 & 0 & y-x & x & -y
\end{bmatrix} \\
&= \begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & y-x & 0 & x & -y & 0 & 0 \\
\cdots & x & -y & 0 & y-x & 0 & 0 \\
\cdots & 0 & y & 0 & x-y & -x & 0 \\
\cdots & 0 & y & 0 & 0 & 0 & -y
\end{bmatrix}.
\end{aligned}$$

Since $|\mathbf{M}_{K_2}| = xy|\mathbf{M}_K| = 0$, the minors of size $(n-1)\times(n-1)$ of $\mathbf{M}_{K_2}$ are equal the corresponding minors size $(n-1)\times(n-1)$ of $\mathbf{M}_K$ and $\nabla_K(x,y) \doteq \nabla_{K_2}(x,y)$. Thus $\nabla_K(x,y)$ is unchanged under the second Reidemeister move.

Case 3. The behavior of $\nabla_K(x,y)$ under the third Reidemeister move

We have the following diagram equation from the diagram K_3 with $(n+1)$ crossings given in Figure 4.

$$\begin{array}{rcrcl}
\cdots & & \cdots & & \cdots \\
xs_{n-1}+ys_n-xs_n-ys_{n+1} & = & xs_{n-1}+(y-x)s_n-ys_{n+1} & = & 0 \\
xs_{n-3}+ys_n-xs_n-ys_{n-2} & = & xs_{n-3}-ys_{n-2}+(y-x)s_n & = & 0 \\
xs_{n+1}+ys_{n-2}-xs_{n-2}-ys_n & = & (y-x)s_{n-2}-ys_n+xs_{n+1} & = & 0
\end{array}$$

The coefficients matrix $\mathbf{M}_{K_3}$ of size $(n+1)\times(n+1)$ of the diagram equation of K_3 here is as follows.

$$
\mathbf{M}_{K_3} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 0 & x & y-x & -y \\ \dots & x & -y & 0 & y-x & 0 \\ \dots & 0 & y-x & 0 & -y & x \end{bmatrix}
$$

$$
= \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & y-x & 0 & x & -y & x-y \\ \dots & x & -y & 0 & y-x & 0 \\ \dots & 0 & 0 & 0 & 0 & x \end{bmatrix}.
$$

Since $|\mathbf{M}_{K_3}| = x|\mathbf{M}_K| = 0$, the minors of size $(n-1)\times(n-1)$ of $\mathbf{M}_{K_3}$ are equal the corresponding minors size $(n-1)\times(n-1)$ of $\mathbf{M}_K$ and $\nabla_K(x,y) \doteq \nabla_{K_3}(x,y)$. So $\nabla_K(x,y)$ is unchanged under the third Reidemeister move. Thus proof is completed. □

It is easy to see that, in present of the first and the second Reidemeister moves, the diagram K_3 in Figure 4 is equivalent to the third Reidemeister move, see Figure 5.

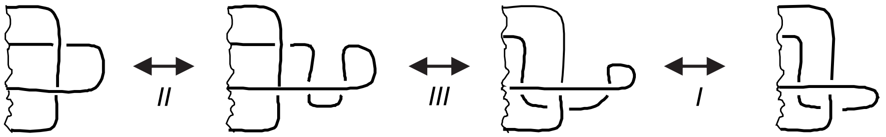

Figure 5 Equivalence of K to K_3 under the third Reidemeister move.

There are different variants, depending on orientation, of the diagrams in Figure 4. Theorem 2.1 can also be proved in the same way for these variants of the diagrams. All possible variants of the diagrams used in the proof of Theorem 2.1 is drawn in Appendix.

Appendix

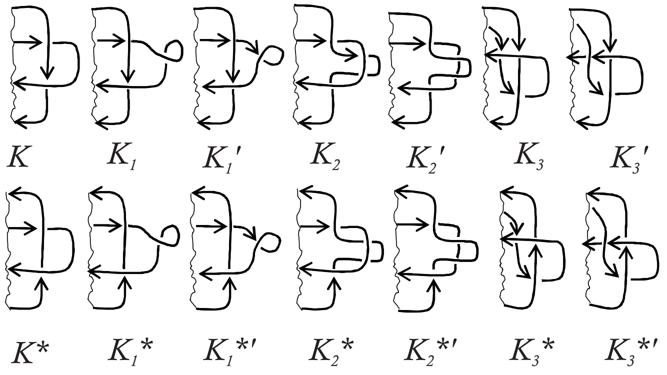

Figure 6

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