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A Note on Torian Algebras

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Abstract: In [6], obic algebras were introduced. In this paper, a class of obic algebras is studied. It is shown that with a suitably defined binary relation, this class of obic algebras are partially ordered sets. The partial ordering is used to investigate some of their properties.

Key Words: Torian algebra, obic algebra, harmonic torian algebra, Smarandachely torian algebra.

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§1. Introduction

Algebras of type $(2,0)$ are well known types of algebraic structures. They comprise non-empty sets, some constant element together with a binary operation. In [1], Kim and Kim introduced the notion of BE-algebras. Ahn and So, in [2] and [4] introduced the notions of ideals and upper sets in BE-algebras and investigated related properties.

In [6], obic algebras were introduced. Homomorphisms and krib maps as well as monics of obic algebras were studied. In this paper, a class of obic algebras is studied. It is shown that with a suitably defined binary relation, this class of obic algebras are partially ordered sets. The partial ordering is used to investigate some of their properties.

§2. Preliminaries

Definition 2.1([6]) *A triple $(X;*,0)$; where X is a non-empty set, $*$ a binary operation on X, and 0 a constant element of X is called an obic algebra if the following axioms*

(1) $x*0=x$;
(2) $[x*(y*z)]*x = x*[y*(z*x)]$;
(3) $x*x=0$

hold for all $x,y,z \in X$.

Example 2.1([6]) Consider the multiplicative group $G=\{1,-1,i,-i\}$. Define a binary operation $*$ on G by $a*b=ab^{-1}$. Then $(G;*,1)$ is an obic algebra.

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Lemma 2.1([6]) *Let X be an obic algebra. Then,*

$$x * y = [x * (y * x)] * x$$

hold for all $x, y \in X$.

Definition 2.2([6]) *An obic algebra X is said to have the weak property (WP) if $x * y = 0$ and $y * x = 0$ imply that $x = y$.*

Definition 2.3([6]) *Let $(X; *, 0)$ and $(Y; \circ, 0')$ be obic algebras. A function $f : X \to Y$ is called an obic homomorphism if $f(a * b) = f(a) \circ f(b)$ for all $a, b \in X$.*

Lemma 2.2([6]) *Let $f : X \to Y$ be an obic homomorphism. The equivalence relation $\sim$ defined by $(x \sim y) \Rightarrow f(x) = f(y)$ is a congruence.*

Theorem 2.1([6]) *Let $f : X \to Y$ be an obic homomorphism. Then $(\overline{X}; \diamond, [0])$ is an obic algebra.*

§3. Main Results

Definition 3.1 *An obic algebra X is called torian if $[(x * y) * (x * z)] * (z * y) = 0$ for all $x, y, z \in X$. Otherwise, if there are $x, y, z \in X$ such that $[(x * y) * (x * z)] * (z * y) \neq 0$, such an obic algebra X is called Smarandachely torian.*

Example 3.1 Let $X = \{0, 1\}$. Define a binary operation $*$ on X by the multiplication table below

$*$	0	1
0	0	1
1	1	0

Then, $(X, *, 0)$ is torian.

Lemma 3.1 *Let X be a torian algebra. Then, the following conclusions hold for all $x, y, z \in X$:*

(1) $[0 * (x * y)] * (y * x) = 0$;
(2) $[(x * y) * x] * (0 * y) = 0$;
(3) $[(0 * y) * (0 * z)] * (z * y) = 0$;
(4) $[x * (x * z)] * z = 0$;
(5) $[0 * (x * y)] * (y * x) = [(x * y) * x] * (0 * y)$;
(6) $[0 * (x * y)] * (y * x) = [(0 * y) * (0 * z)] * (z * y)$;
(7) $[0 * (x * y)] * (y * x) = [x * (x * z)] * z$;
(8) $[(x * y) * x] * (0 * y) = [(0 * y) * (0 * z)] * (z * y)$;
(9) $[(x * y) * x] * (0 * y) = [x * (x * z)] * z$;
(10) $[(0 * y) * (0 * z)] * (z * y) = [x * (x * z)] * z$.

Proof The proof follows immediately from definition. □

The following corollary follows from Lemmas 2.1 and 3.1.

Corollary 3.1 *Let X be a torian algebra. The following conclusions hold for all $x, y, z \in X$:*

(1) $[0 * [x * (y * x)] * x] * [[y * (x * y)] * y] = 0$;
(2) $[[[x * (y * x)] * x] * x] * (0 * y) = 0$;
(3) $[(0 * y) * (0 * z)] * [[z * (y * z)] * z] = 0$;
(4) $[x * [[x * (z * x)] * x]] * z = 0$;
(5) $[0 * [x * (y * x)] * x] * [[y * (x * y)] * y] = [[[x * (y * x)] * x] * x] * (0 * y)$;
(6) $[0 * [x * (y * x)] * x] * [[y * (x * y)] * y] = [(0 * y) * (0 * z)] * [[z * (y * z)] * z]$;
(7) $[0 * [x * (y * x)] * x] * [[y * (x * y)] * y] = [x * [[x * (z * x)] * x]] * z$;
(8) $[[[x * (y * x)] * x] * x] * (0 * y) = [(0 * y) * (0 * z)] * [[z * (y * z)] * z]$;
(9) $[[[x * (y * x)] * x] * x] * (0 * y) = [x * [[x * (z * x)] * x]] * z$;
(10) $[(0 * y) * (0 * z)] * [[z * (y * z)] * z] = [x * [[x * (z * x)] * x]] * z$.

A torian algebra which has the weak property is called a weak property torian algebra (WPTA).

Lemma 3.2 *Let X be a WPTA. Define a relation $\sim$ on X by $x \sim y \Leftrightarrow x * y = 0$ for $x * y \in X$. Then $(X; \sim)$ is a partially ordered set.*

Proof The reflexivity and anti-symmetry follow from definition. Now let $x, y, z \in X$ such that $x \sim y$ and $y \sim z$. Then $x * z = [(x * z) * 0] * 0 = [(x * z) * (x * y)] * (y * z) = 0$. So, the transitivity holds. □

Proposition 3.1 *A torian algebra X is a WPTA if and only if there exists a partial ordering $\sim$ on X such that for all $x, y, z \in X$ hold with:*

(1) $(x * y) * (x * z) \sim (z * y)$;
(2) $[x * (x * y)] \sim y$;
(3) $x * y = 0 \Leftrightarrow x \sim y$.

Proof Suppose X is a WPTA. By Lemma 3.2, X is equipped with a partial ordering. Clearly, $(x * y) * (x * z) \sim (z * y)$ holds. Also, by Lemma 3.1(4), $[x * (x * y)] \sim y$ holds. Clearly, $x * y = 0 \Leftrightarrow x \sim y$ holds.

Conversely, suppose X is a torian algebra with partial ordering $\sim$ satisfying $(x*y)*(x*z) \sim (z * y)$, $[x * (x * y)] \sim y$ and $x * y = 0 \Leftrightarrow x \sim y$ for all $x, y, z \in X$. Let $x, y \in X$ such that $x * y = 0$ and $y * x = 0$. Then $x \sim y$ and $y \sim x$. By anti-symmetry, $x = y$ as required. □

The following corollary follows from Proposition 3.1 and Lemma 2.1.

Corollary 3.2 *A torian algebra X is a WPTA if and only if there exists a partial ordering $\sim$ on X such that for all $x, y, z \in X$ hold with:*

(1) $[[x * (y * x)] * x] * [x * (z * x)] * x \sim [z * (y * z)] * z$;

(2) $[x * [x * (y * x)] * x] \sim y$;
(3) $[x * (y * x)] * x = 0 \Leftrightarrow x \sim y$.

Proposition 3.2 *Let X be a torian algebra with partial ordering $\sim$. Then for all $x, y, z \in X$ hold with:*

(1) $z \sim y \Rightarrow (x * y) \sim (x * z)$;
(2) $x \sim z \Rightarrow (x * y) \sim (z * y)$;
(3) $x \sim y \Rightarrow [0 * (x * z)] \sim (z * y)$.

Proof The proof follows from definition immediately. □

The following corollary follows from Proposition 3.2 and Lemma 2.1.

Corollary 3.3 *Let X be a torian algebra with partial ordering $\sim$. Then, for all $x, y, z \in X$ hold with:*

(1) $z \sim y \Rightarrow [[x * (y * x)] * x] \sim [[x * (z * x)] * x]$;
(2) $x \sim z \Rightarrow [[x * (y * x)] * x] \sim [z * (y * z)] * z$;
(3) $x \sim y \Rightarrow [0 * [x * (z * x)] * x] \sim [[z * (y * z)] * z]$.

Proposition 3.3 *Let X be a torian algebra with partial ordering $\sim$. Then, the following conclusions hold for all $x, y, z \in X$:*

(1) $[0 * (x * y)] \sim (y * x)$;
(2) $[(x * y) * x] \sim (0 * y)$;
(3) $[(0 * y) * (0 * z)] \sim (z * y)$;
(4) $[x * (x * z)] \sim z$.

Proof The proof follows from definition immediately. □

The following corollary follows from Proposition 3.3 and Lemma 2.1.

Corollary 3.4 *Let X be a torian algebra with partial ordering $\sim$. Then the following conclusions hold for all $x, y, z \in X$:*
(1) $[0 * [x * (y * x)] * x] \sim [[y * (x * y)] * y]$;
(2) $[[[x * (y * x)] * x] \sim (0 * y)$;
(3) $[(0 * y) * (0 * z)] \sim [[z * (y * z)] * z]$
(4) $[x * [x * (z * x)] * x] \sim z$.

Proposition 3.4 *Let X be a WPTA. Then, for all $x, y, z \in X$ hold with*

$$x * [x * (x * y)] = x * y.$$

Proof Since X is torian, we have $(x * y) * [x * (x * (x * y))] \sim [x * (x * y)] * y$. Also, $[x * (x * y)] \sim y$ (by Proposition 3.3(4)). So, we now have

$$[x * (x * y)] * y = 0.$$

This gives us $(x * y) * [x * [x * (x * y)]] \sim 0$. Therefore, $(x * y) * [x * [x * (x * y)]] = 0$.

Also, $[x * [x * (x * y)]] * (x * y) = 0$ (by Proposition 3.3(4)). Hence, $x * [x * (x * y)] = x * y$ as required. □

The following corollary follows from Proposition 3.4 and Lemma 2.1.

Corollary 3.5 *Let X be a WPTA. Then, for all $x, y \in X$ hold with:*

$$x * [x * [x * (y * x)] * x] = [[x * (y * x)] * x].$$

Lemma 3.3 *Let X be a WPTA. Then, for all $x, y, z \in X$ hold with*

$$(x * y) * z = (x * z) * y.$$

Proof The proof follows from Proposition 3.3(4) and Proposition 3.1(1). □

Proposition 3.5 *Let X be a WPTA. Then, for all $x, y \in X$ hold with*

$$(0 * x) * (0 * y) = 0 * (x * y).$$

Proof Notice that

$$\begin{aligned}
(0 * x) * (0 * y) &= [[(x * y) * (x * y)] * x] * (0 * y) \\
&= [[(x * y) * x] * (x * y)] * (0 * y) \text{ (by Lemma3.3)} \\
&= [[(x * x) * y] * (x * y)] * (0 * y) \\
&= [(0 * y) * (x * y)] * (0 * y) \\
&= [(0 * y) * (0 * y)] * (x * y) \\
&= 0 * (x * y)
\end{aligned}$$

as required. □

The following corollary follows from Proposition 3.5 and Lemma 2.1.

Corollary 3.6 *Let X be a WPTA. Then, for all $x, y \in X$ hold with*

$$(0 * x) * (0 * y) = 0 * [[x * (y * x)] * x].$$

Proposition 3.6 *Let X be a WPTA. Then, for all $x \in X$ hold with*

$$0 * [x * [0 * (0 * x)]] = 0.$$

Proof Notice that

$$\begin{aligned} 0*[x*[0*(0*x)]] &= (0*x)*[0*[0*(0*x)]] \quad \text{(by Proposition 3.5)} \\ &= (0*x)*(0*x) \quad \text{(by Proposition 3.4)} \\ &= 0 \end{aligned}$$

as required. □

Proposition 3.7 *Let X be a WPTA. Then, for all $x, y, z \in X$ hold with:*

(1) $[0*(y*x)]*(x*y)=0$;
(2) $[(x*y)*(0*y)]*x=0$;
(3) $[(0*y)*(z*y)]*(0*z)=0$;
(4) $[0*(y*x)]*(x*y)=[(x*y)*(0*y)]*x$;
(5) $[0*(y*x)]*(x*y)=[(0*y)*(z*y)]*(0*z)$;
(6) $[(x*y)*(0*y)]*x=[(0*y)*(z*y)]*(0*z)$.

Proof The proof follows from Lemmas 3.1 and 3.3. □

The following corollary follow from Proposition 3.7 and Lemma 2.1.

Corollary 3.7 *Let X be a WPTA. Then, for all $x, y, z \in X$ hold with:*

(1) $[0*[y*(x*y)]*y]*[[x*(y*x)]*x]=0$;
(2) $[[[x*(y*x)]*x]*(0*y)]*x=0$;
(3) $[(0*y)*[z*(y*z)]*z]*(0*z)=0$;
(4) $[0*[y*(x*y)]*y]*[[x*(y*x)]*x]=[[[x*(y*x)]*x]*(0*y)]*x$;
(5) $[0*[y*(x*y)]*y]*[[x*(y*x)]*x]=[(0*y)*[z*(y*x)]*z]*(0*z)$;
(6) $[[[x*(y*x)]*x]*(0*y)]*x=[(0*y)*[z*(y*z)]*z]*(0*z)$.

Definition 3.2 *A torian algebra X is said to be harmonic if $0*x=x$ for all $x \in X$.*

Corollary 3.8 *Let X be a harmonic WPTA. Then, for all $x, , y, z \in X$ hold with:*

(1) $(y*x)*(x*y)=0$;
(2) $[(x*y)*y]*x=0$;
(3) $[y*(z*y)]*z=0$;
(4) $(y*x)*(x*y)=[(x*y)*y]*x$;
(5) $(y*x)*(x*y)=[y*(z*y)]*z$;
(6) $[(x*y)*y]*x=[y*(z*y)]*z$.

Proof The proof follows from Proposition 3.7. □

Definition 3.3 *Let X be a WPTA. An element $x \in X$ is said to fix 0 if $0*x=0$. If every element in X fixes 0, then X is said to fix 0.*

Proposition 3.8 *Let X be WPTA which fixes 0. Then, for all $x, y \in X$ hold with*

$$(x * y) * x = 0.$$

Proof The proof is straightforward by definition. □

Proposition 3.9 *Let X be a WPTA which fixes 0. Then if $x, y \in X$ such that $x * (x * y) = 0$, then $x = x * y$.*

Proof The proof is straightforward by definition. □

Theorem 3.1 *A WPTA X fixes 0 if and only if $(x * y) * x = 0$ for all $x, y \in X$.*

Proof Notice that $0 = (x*y)*x = (x*x)*y = 0*y$, and the converse follows by Proposition 3.8. □

Proposition 3.10 *Let X be a WPTA. Let $x, y, z \in X$ such that $x * y = x * z$, then,*

$$0 * y = 0 * z.$$

Proof Notice that $(x * y) * x = 0 * y$. Similarly, $(x * z) * x = 0 * z$. Then, the conclusion follows. □

Corollary 3.9 *Let X be a harmonic WPTA. Let $x, y, z \in X$ such that $x * y = x * z$. Then $y = z$.*

Proof The proof follows from Proposition 3.10. □

Proposition 3.11 *A torian algebra X fixes 0 if and only if $x * (0 * y) = x$ for all $x, y \in X$.*

Proof Suppose $x * (0 * y) = x$. Since X is torian, we have

$$\begin{aligned} 0 &= [(0 * x) * (0 * 0)] * (0 * x) \\ &= (0 * x) * (0 * x) = 0 * x \quad \text{(by the hypothesis).} \end{aligned}$$

The converse is obvious. □

Proposition 3.12 *Let X be a WPTA. Then $x * [0 * (0 * x)]$ fixes 0 for any $x \in X$.*

Proof Notice that

$$\begin{aligned} [x * [0 * (0 * x)]] &= (0 * x) * [0 * [0 * (0 * x)]] \quad \text{(by Proposition 3.5)} \\ &= (0 * x) * (0 * x) \quad \text{(by Proposition 3.4)} \\ &= 0 \end{aligned}$$

as required. □

Proposition 3.13 *Let a be a fixed element of a WPTA, X. If $x * a = 0 \Rightarrow x = a$ for any $x \in X$, then $0 * (0 * a) = a$.*

Proof Notice that $[0 * (0 * a)] * a = 0$ by Lemma 3.1(4). So, $0 * (0 * a) = a$ as required. □

Proposition 3.14 *Let a be a fixed element of a WPTA, X. If $0 * (0 * a) = a$, then $0 * x = a$ for some $x \in X$.*

Proof Put $0 * a = x$ in $0 * (0 * a) = a$. Then the conclusion follows. □

Proposition 3.15 *Let a be a fixed element of a WPTA, X. If $0 * x = a$ for some $x \in X$, then $x * a = 0 \Rightarrow x = a$.*

Proof Let $y \in X$ such that $y * a = 0$. Then we have $y * (0 * x) = 0$ and

$$\begin{aligned} a * y &= (0 * x) * y \\ &= [0 * [0 * (0 * x)]] * y \quad \text{(by Proposition 3.4)} \\ &= (0 * y) * [0 * (0 * x)] \quad \text{(by Lemma 3.3)} \\ &= 0 * [y * (0 * x)] \quad \text{(by Proposition 3.5)} \\ &= 0 * (y * a) = 0 * 0 = 0 \end{aligned}$$

Since $a * y = 0$ and $y * a = 0$, then $y = a$ as required. This completes the proof. □

By Propositions 3.13, 3.14 and 3.15, we have the following theorem.

Theorem 3.2 *Let a be a fixed element of a WPTA X. Then the following conclusions are equivalent:*

(1) *$x * a = 0 \Rightarrow x = a$ for any $x \in X$;*
(2) *$0 * (0 * a) = a$;*
(3) *$0 * x = a$ for some $x \in X$.*

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