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A Proof of Reciprocity Theorem by Use of Loop Integrals

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Abstract: In this paper, we give a proof of the reciprocity theorem of Ramanujan using loop integrals.

Key Words: Reciprocity theorem, loop integrals, residue calculus.

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§1. Introduction

In his lost notebook [12], Ramanujan recorded the following beautiful reciprocity theorem

$$\rho(a,b)-\rho(b,a)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b,bq/a,q)_\infty}{(-aq,-bq)_\infty}, \tag{1}$$

where

$$\rho(a,b)=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n}$$

and a, b are complex numbers other than 0 and $-q^{-n}$. Throughout this paper, we assume $|q|<1$ and employ the customary notations

$$(a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$
$$(a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty}, \qquad n \text{ is an integer.}$$

The first proof of (1) was given by Andrews [2] by employing his four-variable identity and the well-known Jacobi's triple product identity which is a special case of (1). Somashekara and Fathima [13] used Ramanujan's ${}_1\psi_1$ summation formula and Heine's transformation formula to establish an equivalent version of (1). Bhargava, Somashekara and Fathima [5] provided another proof of (1). Kim, Somashekara and Fathima [10] gave a proof of (1) using only q-binomial theorem. Guruprasad and Pradeep [8] also have devised a proof of (1) using q-binomial theorem.

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Adiga and Anitha [1] established a proof of (1) by the method of analytic continuation. Berndt, Chan, Yeap and Yee [4] found three different proofs of (1). Kang [9] constructed a proof of (1) along the lines of Venkatachaliengar's proof of Ramanujan's ${}_1\psi_1$ summation formula. In [14], Somashekara and Narasimha Murthy gave a proof of (1) using Abel's lemma and Jacobi's triple product identity. Recently, Somashekara, Narasimha Murthy and Shalini [16] have proved an equivalent form of the general identity of Andrews [2] using the parameter augumentation method and employed the same to derive (1). Further, Somashekara and Narasimha Murthy [15] gave a finite form of (1). For more details one may refer the book [3] by Andrews and Berndt .

In 1988, K.Mimachi [11] gave a proof of Ramanujan's ${}_1\psi_1$ summation using loop integrals. Motivated by this, we give a proof of Ramanujan's Reciprocity theorem [12] using loop integrals.

§2. Proof of the Reciprocity Theorem

Making the substitution $a = -cz/q$, $b = c/q$ in (1) we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(cz)_{n+1}} z^n + \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}}{(-c)_n} z^{-n} = \frac{(-qz)_\infty(-1/z)_\infty(q)_\infty}{(-c)_\infty(cz)_\infty}. \tag{2}$$

Using the Heine's transformation [7, eq(III.2), p.359],

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(q)_n(c)_n} z^n = \frac{(c/b)_\infty(bz)_\infty}{(c)_\infty(z)_\infty} \sum_{n=0}^{\infty} \frac{(abz/c)_n(b)_n}{(q)_n(bz)_n} \left(\frac{c}{b}\right)^n \tag{3}$$

in (2) we obtain

$$\sum_{n=-\infty}^{\infty} (-q/c)_n (cz)^n = \frac{(-qz)_\infty(-1/z)_\infty(q)_\infty}{(-c)_\infty(cz)_\infty}, \quad 0 < |z| < \frac{1}{|c|}, \tag{4}$$

where $|q| < 1$, $z \neq 0$, $c \neq -1, -q^{-n}$, $n \in \mathbb{Z}^+$.

Set $C_n := \rho_n e^{i\phi}$ where $\rho_n := \frac{1}{2}|c|\left(|q|^n + |q|^{n+1}\right)$, $0 \leq \phi \leq 2\pi$. Then the series in (4) is defined in $C_n \setminus \{0\}$. Define

$$\begin{aligned} f(t) &:= \frac{(-1/t)_\infty(-tq)_\infty}{(c/t)_\infty(1-tz)}, \\ F(t) &:= \frac{(q)_\infty(-1/t)_\infty(-tq)_\infty}{(-c)_\infty(c/t)_\infty(1-tz)} \\ &= \frac{(q)_\infty}{(-c)_\infty} f(t), \\ I(C) &:= \frac{1}{2\pi i}\int_C f(t)dt \quad \text{and} \\ \underset{t=y}{Res}\ \phi(t) &:= \text{“the residue of } \phi(\text{t}) \text{ at } t = y\text{”}. \end{aligned} \tag{5}$$

The function F(t) has simple poles at $t = cq^j$, $j = 0, 1, 2, \cdots$ and $t = 1/z$. The infinite point ∞ and the origin 0 are essential singularities.

We now show that

$$\sum_{n=-\infty}^{\infty}(-q/c)_n(cz)^n - \frac{(q)_\infty(-qz)_\infty(-1/z)_\infty}{(-c)_\infty(cz)_\infty} = \sum_{j=0}^{\infty}\underset{t=cq^j}{Res}\,F(t) + \underset{t=1/z}{Res}\,F(t). \tag{6}$$

Consider

$$\begin{aligned}\sum_{n=0}^{\infty}(-q/c)_n(cz)^n &= \sum_{n=0}^{\infty}\frac{(-q/c)_\infty}{(-q^{n+1}/c)_\infty}(cz)^n\\ &= (-q/c)_\infty\sum_{n=0}^{\infty}\frac{1}{(-q^{n+1}/c)_\infty}(cz)^n.\end{aligned}$$

On using the special case [7, eq(1.3.15), p.10],

$$\frac{1}{(z)_\infty} = \sum_{n=0}^{\infty}\frac{z^n}{(q)_n}$$

of the well known q-binomial theorem [6], we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(-q/c)_n(cz)^n &= (-q/c)_\infty\sum_{n=0}^{\infty}(cz)^n\sum_{j=0}^{\infty}\frac{(-q^{n+1}/c)^j}{(q)_j}\\ &= (-q/c)_\infty\sum_{j=0}^{\infty}\frac{(-q/c)^j}{(q)_j}\frac{1}{1-czq^j} = \sum_{j=0}^{\infty}\underset{t=cq^j}{Res}\,F(t).\end{aligned} \tag{7}$$

Now, consider

$$\begin{aligned}\underset{t=1/z}{Res}\,F(t) &= \lim_{t\to 1/z}(t-1/z)F(t)\\ &= -\frac{(q)_\infty(-qz)_\infty(-1/z)_\infty}{(-c)_\infty(cz)_\infty}.\end{aligned} \tag{8}$$

Since

$$\sum_{n=-\infty}^{-1}(-q/c)_n(cz)^n = \sum_{n=1}^{\infty}\frac{1}{(-q/cq^n)_n}\frac{1}{(cz)^n}$$

is analytic in $C_n \setminus \{0\}$ as $0 < |z| < 1/|c|$ and from (7) and (8), (6) follows.

Now, it remains to prove that for $|q| < 1$ and $0 < |z| < 1/|c|$,

$$\sum_{j=0}^{\infty}\underset{t=cq^j}{Res}\,F(t) + \underset{t=1/z}{Res}\,F(t) = 0.$$

The simple poles $t = cq^j$, $j = 0, 1, 2, \cdots, n$ and $1/z$ lie in the deleted neighbourhood $C_n \setminus \{0\}$. So, the function is analytic in the region $C_n \setminus \{0\}$ except inside the circles $\gamma_0, \gamma_1, \cdots, \gamma_n, \alpha$ where γ_j is the circle centred at cq^j with very small radius, $0 \le j \le n$ and α is the circle centred at $1/z$ with very small radius.

Thus, by Cauchy's theorem, we have

$$\int_{C_n\setminus\{0\}-\sum_{j=0}^{n}\gamma_j-\alpha} f(t)dt=0.$$

Using the properties of integrals, we have

$$\int_{C_n\setminus\{0\}} f(t)dt=\int_{\sum_{j=0}^{n}\gamma_j} f(t)dt+\int_{\alpha} f(t)dt.$$

Hence

$$\frac{1}{2\pi i}\int_{C_n\setminus\{0\}} f(t)dt=\sum_{j=0}^{n}\underset{t=cq^j}{Res}\, f(t)+\underset{t=1/z}{Res}\, f(t).$$

This yields

$$\sum_{j=0}^{n}\underset{t=cq^j}{Res}\, f(t)+\underset{t=1/z}{Res}\, f(t)=I(C_n\setminus\{0\}),$$

on using (5). Therefore, the problem reduces to proving $I(C_n\setminus\{0\})\longrightarrow 0$ as $n\longrightarrow\infty$. For $n=1,2,3,\cdots$, we have from the definition

$$f(|q|^n t)=\frac{(-1/|q|^n t)_\infty(-|q|^n tq)_\infty}{(c/|q|^n t)_\infty(1-|q|^n tz)}.$$

Hence, for $0\le\phi\le 2\pi$,

$$\begin{aligned}|f(\rho_n e^{i\phi})| &= |f(\rho_0|q|^n e^{i\phi})|\\ &= \left|\frac{1}{c}\right|^n\left|\left(\frac{-\rho_0|q|^n e^{i\phi}q}{q^n}\right)_n\left(\frac{-q^n}{|q|^n\rho_0 e^{i\phi}}\right)_\infty(-|q|^n\rho_0 e^{i\phi}q)_\infty\right|\\ &\quad\times\left|\left(\frac{|q|^n\rho_0 e^{i\phi}q}{cq^n}\right)_n\left(\frac{cq^n}{|q|^n\rho_0 e^{i\phi}}\right)_\infty(1-|q|^n tz)\right|^{-1}. \end{aligned}\tag{9}$$

For each factor in the right hand side, we have the following estimates

$$\begin{aligned}\left|\left(\frac{-\rho_0|q|^n e^{i\phi}q}{q^n}\right)_n\right| &= \left|\prod_{j=0}^{n-1}\left(1+\frac{\rho_0|q|^n e^{i\phi}qq^j}{q^n}\right)\right|\\ &\le \prod_{j=0}^{n-1}\left(1+|\rho_0 q^{j+1}|\right)\le\prod_{j=0}^{\infty}\left(1+|\rho_0 q^{j+1}|\right).\end{aligned}\tag{10}$$

$$\left|\left(\frac{-q^n}{|q|^n\rho_0 e^{i\phi}}\right)_\infty\right|=\left|\prod_{j=0}^{\infty}\left(1+\frac{q^n q^j}{|q|^n\rho_0 e^{i\phi}}\right)\right|\le\prod_{j=0}^{\infty}\left(1+\left|\frac{q^j}{\rho_0}\right|\right).\tag{11}$$

$$\begin{aligned}\left|(-|q|^n\rho_0e^{i\phi}q)_\infty\right| &= \left|\textstyle\prod_{j=0}^{\infty}\left(1+|q|^n\rho_0e^{i\phi}qq^j\right)\right| \\ &\leq \textstyle\prod_{j=0}^{\infty}\left(1+|q^{j+1+n}\rho_0|\right) \leq \prod_{j=0}^{\infty}\left(1+|\rho_0q^{j+1}|\right).\end{aligned} \tag{12}$$

$$\begin{aligned}\left|\left(\frac{|q|^n\rho_0e^{i\phi}q}{cq^n}\right)_n\right| &= \left|\prod_{j=0}^{n-1}\left(1-\frac{|q|^n\rho_0e^{i\phi}qq^j}{cq^n}\right)\right| \\ &\geq \prod_{j=0}^{n-1}\left(1-\left|\frac{\rho_0q^{j+1}}{c}\right|\right) \geq \prod_{j=0}^{\infty}\left(1-\left|\frac{\rho_0q^{j+1}}{c}\right|\right) > 0.\end{aligned} \tag{13}$$

$$\left|\left(\frac{cq^n}{|q|^n\rho_0e^{i\phi}}\right)_\infty\right| = \left|\prod_{j=0}^{\infty}\left(1-\frac{cq^nq^j}{|q|^n\rho_0e^{i\phi}}\right)\right| \geq \prod_{j=0}^{\infty}\left(1-\frac{cq^j}{\rho_0}\right) > 0. \tag{14}$$

$$\left|(1-|q|^n\rho_0e^{i\phi}z)\right| \geq 1-|q^n\rho_0z| \geq 1-|\rho_0z| > 0. \tag{15}$$

On using (10)-(15) in (9), it follows that there exists a positive number M such that

$$|f(\rho_ne^{i\phi})| \leq M\left|\frac{1}{c}\right|^n,\ (0\leq\phi\leq2\pi).$$

Hence,

$$\begin{aligned}|I(C_n\setminus\{0\})| &= \left|\frac{1}{2\pi i}\int_{C_n\setminus\{0\}}f(t)dt\right| \leq \frac{\rho_n}{2\pi}\int_0^{2\pi}|f(\rho_ne^{i\phi})||d\phi| \\ &\leq \frac{\rho_n}{2\pi}\underset{0\leq\phi\leq2\pi}{Max}|f(\rho_ne^{i\phi})|2\pi \leq \rho_0|q|^nM\left|\frac{1}{c}\right|^n.\end{aligned}$$

Consequently, for $|c|>1$, $I(C_n\setminus\{0\}) \longrightarrow 0$ as $n\longrightarrow\infty$.

Hence under the condition $|q|<1,\ 0<|z|<1/|c|$ where $c\neq-1,-q^{-n},n\in\mathbb{Z}^+$, we have (4), completing the proof. ☐

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