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A Study on Hamiltonian Property of Cayley Graphs Over Non-Abelian Groups

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Abstract: The hamiltonian cycles and paths in Cayley graphs naturally arise in computer science in the study of word hyperbolic groups and automatic groups. All Cayley graphs over abelian groups are always hamiltonian. However , for Cayley graphs over non-abelian groups, Chen and Quimpo prove in [1] that Cayley graphs over group of order pq, where p and q primes are Hamiltonian and in [2] that Cayley graphs over hamiltonian groups (i.e., non-abelian groups in which every subgroup is normal) are always hamiltonian. In this paper we investigate the existence of complete hamiltonian cycles and hamiltonian paths in the vertex induced subgraphs of Cayley graphs over non-abelian groups.

Key Words: Cayley graphs, hamiltonian cycles and paths,complete graph,orbit and centralizer of an element in a group,dihedral group.

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§1. Introduction

Let G be a finite non-abelian group and S be a non-empty subset of G. The graph $Cay(G, S)$ is defined as the graph whose vertex set is G and whose edges are the pairs (x, y) such that $sx = y$ for some $s \in S$ and $x \neq y$. Such a graph is called the Cayley graph of G relative to S. The definition of Cayley graphs of groups was introduced by Arthur Cayley in 1978 and the Cayley graphs of groups have received serious attention since then.Since the 1984 survey of results on hamiltonian cycles and paths in Cayley graphs by Witte and Gallian [6], many advances have been made. In this paper, we present a short survey of various results in that direction and make some observations.

§2. Preliminaries

In this section deals with the basic definitions and terminologies of group theory in [4] and [5] and graph theory in [3] which are needed in sequel.

Let G be a group. The orbit of an element x under G is usually denoted by $\bar{x}$ and is defined as $\bar{x} = \{gx/g \in G\}$. Let x be a fixed element of G. The centralizer of an element x in G, $C_G(x)$ is the set of all elements in G that commute with x. In symbols, $C_G(x) = \{g \in G/gx = xg\}$.

A group G act on G by conjugation means $gx = gxg^{-1}$ for all $x \in G$. An element $x \in G$ is called an involution if $x^2 = e$, where e is the identity.Let H be a subgroup of a group G. The subset $aH = \{ah/h \in H\}$ is the left coset of H containing a, while the subset $Ha = \{ha/h \in H\}$ is the right

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coset of H containing a. The notations D_n and Z_n are the dihedral group of order $2n$ and the group of integers modulo n respectively.

A partition of a set S is a collection of non-empty disjoint subset of S whose union is S.

A graph $G = (V, E)$ is said to be connected if there is a path between any two vertices of G. If for each pair of vertices of G there exist a directed path, then it is strongly connected.

Each pair of arbitrary vertices in G can be joined by a directed edge, then it is complete. A subgraph $H = (U, F)$ of a graph $G = (V, E)$ is said to be vertex induced subgraph if F consist of all the edges of G joining pairs of vertices of U.

A hamiltonian path is a path in $G = (V, E)$ which goes through all the vertices in G exactly once. A hamiltonian cycle is a closed hamiltonian path.

§3. Main Results

Theorem 3.1 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ has two disjoint hamiltonian cycles,provided $\bar{x}$ has an element a of order 3 which do not generate $C_G(x)$ but it generates a proper cylic subgroup $\{e, a, b\}$ of $C_G(x)$.*

Proof Since $\bar{x}$ has an element a of order 3 which do not generates $C_G(x)$, we see that $x \neq e$. Let $u \in \{e, a, b\}$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation,we can choose an element $s \in \bar{x}$ such that $s = (ua)a(ua)^{-1}$. Now $su = (ua)a(ua)^{-1}u = (ua)(aa^{-1})(u^{-1}u) = ((ua)e)e = ua$, then there is an edge from u to ua. Again $s(ua) = (ua)a(ua)^{-1}(ua) = ((ua)a)e = ua^2 = ub$, then there is an edge from ua to ub , so there exist a path from u to ub. Again $s(ub) = (ua)a(ua)^{-1}(ub) = (ua)a(a^{-1}u^{-1})(ub) = (ua)(aa^{-1})(u^{-1}u)b = ((ua)e)eb = (ua)eb = uab = ue = u$, then there exist an edge from ub to u.Thus we get a hamiltonian cycle $C_1 : u \to ua \to ub \to u$ in $Cay(C_G(x), \bar{x})$.

Since $a \in \bar{x}$ which do not generate $C_G(x)$, then $C_G(x)$ contains at least one element other than $\{e, a, b\}$. Let $u_1 \notin \{e, a, b\}$. Then $su_1 = (ua)a(ua)^{-1}u_1 = (ua)a(a^{-1}u^{-1})u_1 = (ua)(aa^{-1})u^{-1}u_1 = ((ua)e)u^{-1}u_1 = (ua)u^{-1}u_1$.Since u belongs to the subgroup {e,a,b}, we have $ua = au$, therefore $(ua)u^{-1}u_1 = (au)u^{-1}u_1 = a(uu^{-1})u_1 = (ae)u_1 = au_1$. Clearly $au_1 \notin \{e, a, b\}$, for if $au_1 \in \{e, a, b\}$, then $au_1 = u_2 \in \{e, a, b\}$ which implies $u_1 = a^{-1}u_2 \in \{e, a, b\}$ which is a contradiction to our assumption that $u_1 \notin \{e, a, b\}$. So there exist an edge from u_1 to au_1. Again $s(au_1) = (ua)a(ua)^{-1}(au_1) = (ua)a(a^{-1}u^{-1})(au_1) = (ua)(aa^{-1})u^{-1}(au_1) = (ua)eu^{-1}(au_1) = (ua)u^{-1}(au_1) = (au)u^{-1}(au_1) = a(uu^{-1})au_1 = (ae)au_1 = aau_1 = a^2u_1 = bu_1$, as above we can show that $bu_1 \notin \{e, a, b\}$. Thus there exist an edge from au_1 to bu_1 and consequently a path from u_1 to bu_1. Again $s(bu_1) = (ua)a(ua)^{-1}(bu_1) = (ua)a(a^{-1}u^{-1})bu_1 = (ua)eu^{-1}(bu_1) = (au)u^{-1}(bu_1) = (ae)bu_1 = abu_1 = eu_1 = u_1$, then there exist an edge from bu_1 to u_1 . Thus we get another hamiltonian cycle $C_2 : u_1 \to au_1 \to bu_1 \to u_1$ in $Cay(C_G(x)$,$\bar{x})$ which is disjoint from C_1. □

Theorem 3.2 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ has two complete hamiltonian cycles, one with vertex set P_1 and other with vertex set P_2, provided $C_G(x)$ has a partition (P_1, P_2), where $\bar{x}$ has an element a which generates a proper cyclic subgroup $P_1 = \{e, a, b\}$ of $C_G(x)$ and P_2 is the generating set of P_1.*

Proof Since $a \in \bar{x}$ which generates a proper cyclic subgroup $P_1 = \{e, a, b\}$ of $C_G(x)$, by Theorem 3.1, for every $u \in P_1$,we get a complete hamiltonian cycle $C_1 : u \to au \to bu \to u$ in $Cay(P_1, \bar{x})$.Since P_2 is the generating set of P_1, we have $P_2P_2 = P_1, P_2P_1 = P_2, P_1P_2 = P_2$ and $P_1P_1 = P_1$. Let

u_1 be an element in P_2. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose an element $s \in \bar{x}$ such that $s = (ua)a(ua)^{-1}$ for $u \in P_1$. Now $su_1 = (ua)a(ua)^{-1}u_1 = (ua)a(a^{-1}u^{-1})u_1 = (ua)(aa^{-1})(u^{-1}u_1) = (ua)e(u^{-1}u_1) = (ua)(u^{-1}u_1) = (au)(u^{-1}u_1) = a(uu^{-1})u_1 = (ae)u_1 = au_1$. Clearly $au_1 \notin P_1$ since $P_1P_2 = P_2$. So there exist an edge from u_1 to au_1. Again $s(au_1) = (ua)a(ua)^{-1}(au_1) = (ua)a(a^{-1}u^{-1})(au_1) = (ua)(aa^{-1})u^{-1}(au_1) = (ua)eu^{-1}(au_1) = (au)u^{-1}(au_1) = a(uu^{-1})au_1 = (ae)au_1 = (aa)u_1 = a^2u_1 = bu_1$. Here also $bu_1 \notin P_1$,so there exist an edge from au_1 to bu_1 and consequently a path from u_1 to bu_1.Again $s(bu_1) = (ua)a(ua)^{-1}(bu_1) = (ua)a(a^{-1}u^{-1})(bu_1) = (ua)(aa^{-1})u^{-1}(bu_1) = (ua)eu^{-1}(bu_1) = (ua)u^{-1}(bu_1) = (au)u^{-1}(bu_1) = a(uu^{-1})bu_1 = (ae)bu_1 = (ab)u_1 = eu_1 = u_1$, so there exist an edge from bu_1 to u_1. Thus we get another complete hamiltonian cycle $C_2 : u_1 \to au_1 \to bu_1 \to u_1$ in $Cay(P_2, \bar{x})$, which is disjoint from C_1. □

Lemma 3.3 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$,the induced subgraph $Cay(H_1, \bar{x})$of the Cayley graph $Cay(C_G(x), \bar{x})$ is hamiltonian provided $\bar{x}$ has three involutions a, b, c with $ab = ba$, which generates $C_G(x)$ and $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3.*

Proof Since $\bar{x}$ has three involutions a, b, c with $ab = ba$, which generates $C_G(x)$, we see that $C_G(x) = \{e, a, b, ab, cb, a(cb), b(cb), ab(cb), (cb)^2, a(cb)^2, b(cb)^2, ab(cb)^2\}$. Also ,since $H_1 =< b, c >$ is a subgroup of $C_G(x)$,we have $H_1 = \{e, b, cb, b(cb), (cb)^2, b(cb)^2\}$. Let $u \in H_1$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose two involutions s_1 and $s_2 \in \bar{x}$ such that $s_1 = (ub)b(ub)^{-1}$and $s_2 = (uc)c(uc)^{-1}$. Now $s_1u = (ub)b(ub)^{-1}u = (ub)b(b^{-1}u^{-1})u = (ub)(bb^{-1})(u^{-1}u) = ((ub)e)e = ub$, then there is an edge from u to ub. Again $s_2(ub) = (uc)c(uc)^{-1}(ub) = (uc)c(c^{-1}u^{-1})ub = (uc)(cc^{-1})(u^{-1}u)b = (uc)e(eb) = (uc)b = u(cb)$, then there is an edge from ub to $u(cb)$, so there exist a path from u to $u(cb)$. Again $s_1(u(cb)) = (ub)b(ub)^{-1}(u(cb)) = (ub)b(b^{-1}u^{-1})(u(cb)) = (ub)(bb^{-1})(u^{-1}u)(cb) = (ub)e(cb) = ub(cb)$, so there exist an edge from $u(cb)$to $ub(cb)$ and consequently a path from u to $ub(cb)$. Again $s_2(ub(cb)) = (uc)c(uc)^{-1}(ub(cb)) = (uc)(cc^{-1})u^{-1}(ub(cb)) = (uc)c(c^{-1}u^{-1})(ub(cb)) = (uc)e(u^{-1}u)b(cb) = (uc)eb(cb) = u(cb)^2$,then there exist an edge from $ub(cb)$ to $u(cb)^2$ and consequently a path from u to $u(cb)^2$. Again $s_1(u(cb)^2) = (ub)b(ub)^{-1}u(cb)^2 = (ub)b(b^{-1}u^{-1})u(cb)^2 = (ub)(bb^{-1})(u^{-1}u)(cb)^2 = (ub)ee(cb)^2 = (ub)(cb)^2 = ub(cb)^2$, so there exist an edge from $u(cb)^2$ to $ub(cb)^2$. Again $s_2(ub(cb)^2) = (uc)c(uc)^{-1}(ub)(cb)^2 = (uc)c(c^{-1}u^{-1})ub(cb)^2 = (uc)(cc^{-1})(u^{-1}u) \times b(cb)^2 = (uc)e(eb)(cb)^2 = ucb(cb)^2 = u(cb)^3$, so there exist an edge from $ub(cb)^2$ to $u(cb)^3$ and consequently a path from u to $u(cb)^3$. Since $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3, we have $(cb)^3 = e$. Therefore $u(cb)^3 = ue = u$. Thus we get a hamiltonian cycle $C_1 : u \to ub \to u(cb) \to ub(cb) \to u(cb)^2 \to ub(cb)^2 \to u(cb)^3 = ue = u$ in $Cay(H_1, \bar{x})$. In particularly, for $u = e$ we get a hamiltonian cycle $e \to b \to cb \to b(cb) \to (cb)^2 \to b(cb)^2 \to (cb)^3 = e$. □

Lemma 3.4 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(aH_1, \bar{x})$ of the Cayley graph $Cay(C_G(x), \bar{x})$ is hamiltonian provided $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$ and $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3.*

Proof Since $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$ and $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3. Then by Lemma 3.3, for every $u \in H_1$, we get a hamiltonian cycle $C_1 : u \to ub \to u(cb) \to ub(cb) \to u(cb)^2 \to ub(cb)^2 \to u(cb)^3 = ue = u$ in $Cay(H_1, \bar{x})$. Since $aH_1 = \{ah/h \in H_1\}$, we see that $aH_1 = \{a, ab, a(cb), ab(cb), a(cb)^2, ab(cb)^2\}$. For if $u = a$ in C_1 , we get another hamiltonian cycle $C_2 : a \to ab \to a(cb) \to ab(cb) \to a(cb)^2 \to ab(cb)^2 \to a(cb)^3 = ae = a$ in $Cay(aH_1, \bar{x})$, which is disjoint from C_1. □

Theorem 3.5 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian provided $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$ and $< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3.*

Proof Since $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$, we see that $C_G(x) = \{e, a, b, ab, cb, a(cb), b(cb), ab(cb), (cb)^2, a(cb)^2, b(cb)^2, ab(cb)^2\}$. Let $H_1 =< b, c >$ be the subgroup of $C_G(x)$ isomorphic to D_3. Then by Lemma 3.3, we get a hamiltonian cycle $C_1 : e \to b \to (cb) \to b(cb) \to (cb)^2 \to b(cb)^2 \to (cb)^3 = e$ in $Cay(H_1, \bar{x})$. Now, consider $aH_1 = \{ah/h \in H_1\}$. Then by Lemma 3.4, we get another hamiltonian cycle $C_2 : a \to ab \to a(cb) \to ab(cb) \to a(cb)^2 \to ab(cb)^2 \to a(cb)^3 = ae = a$ in $Cay(aH_1, \bar{x})$, which is disjoint from C_1, since $aH_1 \cap H_1 = \phi$.

We have $C_G(x) = H_1 \cup aH_1$. By removing the edges $\{e, b\}$ in $Cay(H_1, \bar{x})$ and $\{a, ab\}$ in $Cay(aH_1, \bar{x})$ and adding $\{e, a\}$ and $\{a, ab\}$ we get a hamiltonian cycle $e \to a \to ab(cb)^2 \to a(cb)^2 \to ab(cb) \to a(cb) \to (ab) \to b \to cb \to b(cb) \to (cb)^2 \to b(cb)^2 \to (cb)^3 = e$ in $Cay(C_G(x), \bar{x})$. Thus the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian. □

Theorem 3.6 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian provided $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$ and $C_G(x)$ has a partition (H_1, H_2) where $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3 and H_2 is the generating set of H_1.*

Proof Since $\bar{x}$ has three involutions a, b, c with $ab = ba$ which generates $C_G(x)$ and $H_1 =< b, c >$ is a subgroup of $C_G(x)$ isomorphic to D_3 , then by Lemma 3.3 we get a hamiltonian cycle $C_1 : e \to b \to cb \to b(cb) \to (cb)^2 \to b(cb)^2 \to (cb)^3 = e$ in $Cay(H_1, \bar{x})$. We have H_2 is the generating set of H_1 and $H_2H_2 = H_1$, $H_2H_1 = H_2$, $H_1H_2 = H_2, H_1H_1 = H_1$. Since $H_1 =< b, c >$ is a subgroup of $C_G(x)$ and the involutions a, b, c in $\bar{x}$ generates $C_G(x)$, we have $a \in H_2$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose two involutions s_1 and s_2 in $\bar{x}$ such that $s_1 = (ab)b(ab)^{-1}$ and $s_2 = (ac)c(ac)^{-1}$. Now $s_1a = (ab)b(ab)^{-1}a = (ab)b(b^{-1}a^{-1})a = ab(bb^{-1})(a^{-1}a) = ((ab)e)e = ab$, so there exist an edge from a to ab in H_2 , since $H_2H_1 = H_2$. Again $s_2(ab) = (ac)c(ac)^{-1}(ab) = (ac)c(c^{-1}a^{-1})ab = (ac)(cc^{-1})(a^{-1}a)b = (ac)eb = (ac)b = a(cb)$. Clearly $a(cb) \notin H_1$, so there exist an edge from ab to $a(cb)$ in H_2. Again $s_1(a(cb)) = (ab)b(ab)^{-1}acb = (ab)(bb^{-1})(a^{-1}a)cb = (ab)ee(cb) = ab(cb)$, so there exist an edge from $a(cb)$ to $ab(cb)$ in H_2. Again $s_2ab(cb) = (ac)c(ac)^{-1}(ab)(cb) = (ac)(cc^{-1})(a^{-1}a)b(cb) = (ac)eeb(cb) = (ac)b(cb) = a(cb)(cb) = a(cb)^2$. Here also $a(cb)^2 \notin H_1$, since $H_2H_1 = H_2$, so there is an edge from $ab(cb)$ to $a(cb)^2$, consequently a path from a to $a(cb)^2$. Again $s_1a(cb)^2 = (ab)b(ab)^{-1}a(cb)^2 = (ab)(bb^{-1})(a^{-1}a)(cb)^2 = (ab)ee(cb)^2 = (ab)(cb)^2$, so there exist an edge from $a(cb)^2$ to $ab(cb)^2$. Again $s_2ab(cb)^2 = (ac)c(ac)^{-1}(ab)(cb)^2 = (ac)(cc^{-1})(a^{-1}a)b(cb)^2 = a(cb)^3 = ae = a$. Thus we get another cycle $C_2 :\ a \to ab \to a(cb) \to ab(cb) \to a(cb)^2 \to ab(cb)^2 \to a(cb)^3 = ae = a$ in $Cay(H_2, \bar{x})$, which is disjoint from C_1. We have $C_G(x) = H_1 \cup H_2$.By removing the edges $\{e, b\}$ in $Cay(H_1, \bar{x})$ and $\{a, ab\}$ in $Cay(H_2, \bar{x})$ and adding the edges $\{e, a\}$ and $\{a, ab\}$ we get a hamiltonian cycle $e \to a \to ab(cb)^2 \to a(cb)^2 \to ab(cb) \to a(cb) \to ab \to b \to cb \to b(cb) \to (cb)^2 \to b(cb)^2 \to (cb)^3 = e$ in $Cay(C_G(x), \bar{x})$. □

Theorem 3.7 *Let G be a finite non-abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph $Cay(C_G(x), \bar{x})$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian provided $C_G(x)$ isomorphic to $Z_{2n+1}, n = 0, 1, 2, \cdots$.*

Proof Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose an element $s \in \bar{x}$ such that $s = (ua)a(ua)^{-1}$. Now $su =$

$(ua)a(ua)^{-1}u = (ua)a(a^{-1}u^{-1})u = (ua)(aa^{-1})(u^{-1}u) = ((ua)e)e = ua$, then there exist an edge from u to ua. Again $s(ua) = (ua)a(ua)^{-1}(ua) = ((ua)a)e = ua^2$, then there is an edge from ua to ua^2, consequently a path from u to ua^2. Continuing in this way, we get a Hamiltonian cycle $u \to ua \to ua^2 \to ua^3 \to \cdots \to ua^{2n+1} = ue = u$ in $Cay(C_G(x), \bar{x})$. $\square$

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