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A Coupled Fixed Point Theorem via Implicit Function in Partially Ordered Partial Metric Spaces and Application

Gurucharan Singh Saluja

(H.N. 3/1005, Geeta Nagar, Raipur, Raipur - 492001 (C.G.), India)

E-mail: saluja1963@gmail.com

Abstract: The aim of this manuscript is to discuss the existence of coupled fixed points in the context of partially ordered partial metric spaces via implicit relations. Moreover, we provide some consequences of the established results. We also state an example to illustrate our work. Our main result extends and generalizes various results in the literature. Especially, our result extends and generalizes the corresponding result of Bhaskar and Lakshmikantham [9] from partially ordered complete metric spaces to partially ordered complete partial metric spaces. Finally, an application to the integral equation is included.

Key Words: Coupled fixed point, implicit function, partial metric space, partially ordered set.

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§1. Introduction

The notion of coupled fixed point for a partially ordered set Ω was introduced by Bhashkar and Lakshmikantham [9]. Several other authors such as Ćirić and Lakshmikantham [10], Sabetghadam et al. [33] and Oleleru et al. [26] have established some coupled fixed point theorems in metric spaces. Afterwards, many researchers have obtained coupled fixed point results for mappings under various contractive conditions in the setting of metric spaces and generalized metric spaces (see [1], [6], [14], [19], [24], [37]).

In 1994, the notion of partial metric space was introduced by Matthews (see, [23]) as part of the study of denotational semantics of dataflow networks. It is well-known that partial metric spaces play an important role in the theory of computation (see, e.g., [15], [21], [32], [36]). The PMS is a generalization of usual metric spaces in which the self-distance need not be zero. Later, Matthews proved the partial metric version of Banach fixed point theorem [8].

Several famous mathematicians have contributed to the development of this research fields. Masiha et al. [22] proved some fixed point results for weakly contractive type mappings in partially ordered partial metric spaces. They applied their results to nonlinear fractional boundary value problem. Altun et al. [5] established some fixed point theorems for generalized contractive type mappings on partial metric spaces. They also proved a homotopy result. Aydi et

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al. [7] introduced the concept of partial Hausdorff metric and they initiated the study of fixed point theory for multi-valued mappings on partial metric spaces using the partial Hausdorff metric and proved an analogous to the well-known Nadler's fixed poin theorem. Heckmann [15] introduced the concept of weak partial metric function and established some fixed point results. Oltra ans Valero [27] generalized the Matthews results in the sense of O'Neil [28] in complete partial metric space (see, also [2], [3], [6], [12], [13], [17], [20], [25]).

The practice of improving contraction conditions in proving fixed point and common fixed point theorems is still in fashion. Recently, with a view to accommodate many contraction conditions, Popa [29] and Popa et al. [31] introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contractive conditions.

In nonlinear analysis, especially in fixed point theory, implicit relations on metric spaces have been investigated highly in many articles (see, e.g., [4], [16], [30], [34] and references therein).

Inspired and motivated by the works of [6, 9, 29] and others, the purpose of this article is to examine the existence of a coupled fixed point theorem for mappings satisfying mixed monotone property in the context of partial metric spaces by using implicit relations. In addition, we provide some consequences of the established result. We also state an example to illustrate our result. Finally, an application to the integral equation is included. Our results extend, generalize and enrich several results in the existing literature.

§2. **Preliminaries**

In this section, we give some definitions and lemmas related to partial metric spaces which will be useful in the proof of our main results.

Definition 2.1([23]) *Let* Ω *be a nonempty set. A partial metric on* Ω *is a function* $p\colon \Omega\times\Omega\to[0,+\infty)$ *such that for all* $v_1, v_2, u\in\Omega$ *the followings are satisfied:*

($p1$) $v_1 = v_2 \Leftrightarrow p(v_1, v_1) = p(v_1, v_2) = p(v_2, v_2)$;
($p2$) $p(v_1, v_1) \leq p(v_1, v_2)$;
($p3$) $p(v_1, v_2) = p(v_2, v_1)$;
($p4$) $p(v_1, v_2) \leq p(v_1, u) + p(u, v_2) - p(u, u)$.

Then, p *is called a partial metric on* Ω *and the pair* (Ω, p) *is called a partial metric space.*

It is clear that if $p(v_1, v_2) = 0$, then from ($p1$), ($p2$), and ($p3$), $v_1 = v_2$. But if $v_1 = v_2$, $p(v_1, v_2)$ may not be 0.

If p is a partial metric on Ω, then the function $d^p\colon \Omega\times\Omega\to[0,+\infty)$ given by

$$d^p(v_1, v_2) = 2p(v_1, v_2) - p(v_1, v_1) - p(v_2, v_2), \tag{2.1}$$

is a usual metric on Ω.

Each partial metric p on Ω generates a T_0 topology τ_p on Ω with the family of open p-balls $\{B_p(y,\varepsilon) : y\in\Omega, \varepsilon>0\}$ where $B_p(y,\varepsilon) = \{z\in\Omega : p(y,z) < p(y,y)+\varepsilon\}$ for all $y\in\Omega$ and

$\varepsilon > 0$. Similarly, closed p-ball is defined as $B_p[y,\varepsilon] = \{z \in \Omega : p(y,z) \leq p(y,y) + \varepsilon\}$ for all $y \in \Omega$ and $\varepsilon > 0$.

Example 2.2([7]) Let $\Omega = [0,+\infty)$ and $p\colon \Omega \times \Omega \to [0,+\infty)$ be given by $p(y,z) = \max\{y,z\}$ for all $y,z \in \Omega$. Then (Ω,p) is a partial metric space.

Example 2.3([7]) Let $\Omega = I$, where I denote the set of all intervals $[y_1,z_1]$ for any real numbers $y_1 \leq z_1$. Let $p\colon \Omega \times \Omega \to [0,\infty)$ be a function such that $p\big([y_1,z_1],[y_2,z_2]\big) = \max\{z_1,z_2\} - \min\{y_1,y_2\}$. Then, (Ω,p) is a partial metric space.

Example 2.4([11]) Let $\Omega = \mathbb{R}$ and $p\colon \Omega \times \Omega \to \mathbb{R}^+$ be given by $p(y,z) = e^{\max\{y,z\}}$ for all $y,z \in \Omega$. Then (Ω,p) is a partial metric space.

Definition 2.5([23]) *Let (Ω,p) be a partial metric space. Then,*

(A) A sequence $\{y_n\}$ converges to a point $y \in \Omega$ if and only if $\lim_{n\to\infty} p(y,y_n) = p(y,y)$;

(B) A sequence $\{y_n\}$ in Ω is called a Cauchy sequence if and only if $\lim_{m,n\to\infty} p(y_m,y_n)$ exists (and finite);

(C) A partial metric space (Ω,p) is said to be complete if every Cauchy sequence $\{y_n\}$ in Ω converges, with respect to τ_p, to a point $y \in \Omega$, such that, $\lim_{m,n\to\infty} p(y_m,y_n) = p(y,y)$;

(D) A mapping $f\colon \Omega \to \Omega$ is said to be continuous at $y_0 \in \Omega$ if for every $\varepsilon > 0$, there exists $\eta > 0$ such that $f\big(B_p(y_0,\eta)\big) \subset B_p\big(f(y_0),\varepsilon\big)$.

Definition 2.6([23]) *A partial metric space (Ω,p) is said to be complete if every Cauchy sequence $\{y_n\}$ in Ω converges to a point $y \in \Omega$ with respect to τ_p. Furthermore,*

$$\lim_{m,n\to\infty} p(y_m,y_n) = \lim_{n\to\infty} p(y_n,y) = p(y,y).$$

Definition 2.7([9]) *Let $(\Omega,\leq)$ be a partially ordered set. The mapping $H\colon \Omega\times\Omega \to \Omega$ is said to have the mixed monotone property if $H(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x,y \in \Omega$,*

$$x_1,x_2 \in \Omega,\ \ x_1 \leq x_2 \ \Rightarrow\ H(x_1,y) \leq H(x_2,y),$$

and

$$y_1,y_2 \in \Omega,\ \ y_1 \leq y_2 \ \Rightarrow\ H(x,y_1) \geq H(x,y_2).$$

Definition 2.8([9,10]) *An element $(x,y) \in \Omega \times \Omega$ is said to be a coupled fixed point of the mapping $H\colon \Omega \times \Omega \to \Omega$ if $H(x,y) = x$ and $H(y,x) = y$.*

Example 2.9 Let $\Omega = [0,+\infty)$ and $H\colon \Omega \times \Omega \to \Omega$ be defined by $H(x,y) = \frac{x+y}{3}$ for all $x,y \in \Omega$. Then one can easily see that H has a unique coupled fixed point $(0,0)$.

Example 2.10 Let $\Omega = [0,+\infty)$ and $H\colon \Omega \times \Omega \to \Omega$ be defined by $H(x,y) = \frac{x+y}{2}$ for all $x,y \in \Omega$. Then we see that H has two coupled fixed point $(0,0)$ and $(1,1)$, that is, the coupled fixed point is not unique.

Lemma 2.11([6, 23]) (1) *A sequence* $\{y_n\}$ *is Cauchy in a partial metric space* (Ω, p) *if and only if* $\{y_n\}$ *is Cauchy in a metric space* (Ω, d^p) *where*

$$d^p(y,z) = 2p(y,z) - p(y,y) - p(z,z) \ \text{for all} \ y, z \in \Omega.$$

(2) *A partial metric space* (Ω, p) *is complete if a metric space* (Ω, d^p) *is complete, i.e.,*

$$\lim_{n\to\infty} d^p(y_n, y) = 0 \Leftrightarrow p(y,y) = \lim_{n\to\infty} p(y_n, y) = \lim_{n,m\to\infty} p(y_n, y_m).$$

Lemma 2.12([17]) *Let* (Ω, p) *be a partial metric space.*

$(1')$ *If* $y, z \in \Omega$, $p(y,z) = 0$, *then* $y = z$;
$(2')$ *If* $y \neq z$, *then* $p(y,z) > 0$.

One of the characterization of continuity of mappings in partial metric spaces was given by Samet et al. [35] as follows.

Lemma 2.13([35]) *Let* (Ω, p) *be a partial metric space. The function* $F\colon \Omega \to \Omega$ *is continuous if given a sequence* $\{y_n\}_{n\in\mathbb{N}}$ *and* $y \in \Omega$ *such that* $p(y,y) = \lim_{n\to\infty} p(y, y_n)$, *then* $p(Fy, Fy) = \lim_{n\to\infty} p(Fy, Fy_n)$.

Example 2.14([35]) Let $\Omega = [0, +\infty)$ endowed with the partial metric $p\colon \Omega \times \Omega \to [0, +\infty)$ defined $p(y,z) = \max\{y, z\}$ for all $y, z \in \Omega$. Let $F\colon \Omega \to \Omega$ be a non-decreasing function. If F is continuous with respect to the standard metric $d(y,z) = |y - z|$ for all $y, z \in \Omega$, then F is continuous with respect to the partial metric p.

Lemma 2.15([11]) *Let* $y_n \to y$ *as* $n \to \infty$ *in a partial metric space* (Ω, p) *where* $p(y,y) = 0$. *Then* $\lim_{n\to\infty} p(y_n, u) = p(y, u)$ *for all* $u \in \Omega$.

A set of *implicit relations*, denoted by $\mathbb{V}$, is the collection of all continuous functions $V\colon (\mathbb{R}^+)^5 \to \mathbb{R}$ which satisfy:

$(V1)$ $V(t_1, t_2, t_3, t_4, t_5)$ is non-increasing in t_4 and t_5, and
$(V2)$ there exists a function $\psi \in \Psi$ such that

$$V(u, v, w, u+v, u+v) \leq 0 \text{ implies } u \leq v + \psi(w),$$

where Ψ denotes the set of all functions $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with the properties:

(i) ψ is continuous and non-decreasing;
(ii) $\psi(t) < t$ for each $t > 0$ and $\psi(0) = 0$.

Example 2.16 It is easy to check that the following functions are in $\mathbb{V}$.

$(V_{1'})$ $V(t_1, t_2, t_3, t_4, t_5) = t_1 - a\,t_2 - b\,t_3 - c\,t_4 - d\,t_5$, where a, b, c, d are non-negative real numbers such that $a+b+2c+2d<1$;

$(V_{2'})$ $V(t_1,t_2,t_3,t_4,t_5)=t_1-a\max\left\{t_2,t_3,\frac{t_4}{2},\frac{t_5}{2}\right\}$, where $a\in(0,1)$;

$(V_{3'})$ $V(t_1,t_2,t_3,t_4,t_5)=t_1-\psi(\max\{t_2,t_3\})$, where $\psi\in\Psi$.

§3. Main Results

In this section, we shall prove a coupled fixed point theorem via implicit function in the framework of partially ordered partial metric spaces.

Theorem 3.1 *Let $(\Omega,p,\leq)$ be a partially ordered complete partial metric space. Suppose that $H\colon\Omega\times\Omega\to\Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V\in\mathbb{V}$ such that*

$$V\begin{pmatrix} p(H(u,v),H(y,z)),p(u,y),p(v,z), \\ p(H(u,v),u)+p(H(y,z),y),p(H(u,v),y)\end{pmatrix}\leq 0, \tag{3.1}$$

for all $u,v,y,z\in\Omega$ with $u\geq y$ and $v\leq z$. Suppose that either

(a) H is continuous or

(b) Ω has the following property

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u\in\Omega$, then $u_n\leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v\in\Omega$, then $v\leq v_n$ for all n.

If there exist two elements $u_0,v_0\in\Omega$ with $u_0\leq H(u_0,v_0)$ and $v_0\geq H(v_0,u_0)$, then H has a coupled fixed point in Ω.

Proof Let $u_0,v_0\in\Omega$ be such that $u_0\leq H(u_0,v_0)$ and $v_0\geq H(v_0,u_0)$. We construct the iterative sequences $\{u_n\}$ and $\{v_n\}$ in Ω as follows: let $u_1=H(u_0,v_0)$ and $v_1=H(v_0,u_0)$. Then $u_0\leq u_1$ and $v_0\geq v_1$. Again, let $u_2=H(u_1,v_1)$ and $v_2=H(v_1,u_1)$. Since H has the mixed monotone property on Ω, then we have $u_1\leq u_2$ and $v_1\geq v_2$. Continuing the same way as above, we get

$$u_{n+1}=H(u_n,v_n)\ \text{ and }\ v_{n+1}=H(v_n,u_n)\ \text{ for all } n\geq 0, \tag{3.2}$$

and

$$u_0\leq u_1\leq\cdots\leq u_n\leq u_{n+1}\leq\ldots,\quad v_0\geq v_1\geq\cdots\geq v_n\geq v_{n+1}\geq\ldots. \tag{3.3}$$

If there exists $n_0\in\mathbb{N}\cup\{0\}$ such that $u_{n_0}=u_{n_0+1}$ and $v_{n_0}=v_{n_0+1}$, then

$$u_{n_0}=u_{n_0+1}=H(u_{n_0},v_{n_0})\text{ and }v_{n_0}=v_{n_0+1}=H(v_{n_0},u_{n_0}),$$

which concludes that (u_{n_0},v_{n_0}) is a coupled fixed point of H. So, we assume that $u_{n_0}\neq u_{n_0+1}$ or $v_{n_0}\neq v_{n_0+1}$ for all n. By Lemma 2.12 $(2')$, we have $p(u_{n+1},u_n)>0$ and $p(v_{n+1},v_n)>0$ for all n.

Since $u_{n+1}\geq u_n$ and $v_{n+1}\leq v_n$, from equation (3.1) with $u=u_{n+1}$, $v=v_{n+1}$, $y=u_n$

and $z = v_n$, we have

$$V\begin{pmatrix} p(H(u_{n+1}, v_{n+1}), H(u_n, v_n)), p(u_{n+1}, u_n), p(v_{n+1}, v_n), \\ p(H(u_{n+1}, v_{n+1}), u_{n+1}) + p(H(u_n, v_n), u_n), p(H(u_{n+1}, v_{n+1}), u_n) \end{pmatrix} \leq 0,$$

or

$$V\begin{pmatrix} p(u_{n+2}, u_{n+1}), p(u_{n+1}, u_n), p(v_{n+1}, v_n), \\ p(u_{n+2}, u_{n+1}) + p(u_{n+1}, u_n), p(u_{n+2}, u_n) \end{pmatrix} \leq 0. \tag{3.4}$$

By PMS condition ($p4$), we have

$$\begin{aligned} p(u_{n+2}, u_n) &\leq p(u_{n+2}, u_{n+1}) + p(u_{n+1}, u_n) - p(u_{n+1}, u_{n+1}) \\ &\leq p(u_{n+2}, u_{n+1}) + p(u_{n+1}, u_n). \end{aligned} \tag{3.5}$$

By the properties of V and equation (3.5), the inequality (3.4) reduces to

$$V\begin{pmatrix} p(u_{n+2}, u_{n+1}), p(u_{n+1}, u_n), p(v_{n+1}, v_n), \\ p(u_{n+2}, u_{n+1}) + p(u_{n+1}, u_n), p(u_{n+2}, u_{n+1}) + p(u_{n+1}, u_n) \end{pmatrix} \leq 0, \tag{3.6}$$

which yields that

$$p(u_{n+2}, u_{n+1}) \leq p(u_{n+1}, u_n) + \psi(p(v_{n+1}, v_n)). \tag{3.7}$$

Similarly, we can show that

$$p(v_{n+2}, v_{n+1}) \leq p(v_{n+1}, v_n) + \psi(p(u_{n+1}, u_n)). \tag{3.8}$$

By adding equations (3.7)-(3.8) and using the properties of ψ, we have

$$S_n \leq S_{n-1} + \psi(S_{n-1}), \tag{3.9}$$

where $S_n = p(u_{n+2}, u_{n+1}) + p(v_{n+2}, v_{n+1})$.

If there exists $n_1 \in \mathbb{N} \cup \{0\}$ such that $p(u_{n_1+2}, u_{n_1+1}) = 0$, $p(v_{n_1+2}, v_{n_1+1}) = 0$, then $u_{n_1+1} = u_{n_1+2} = H(u_{n_1+1}, v_{n_1+1})$, $v_{n_1+1} = v_{n_1+2} = H(v_{n_1+1}, u_{n_1+1})$ and (u_{n_1+1}, v_{n_1+1}) is a coupled fixed point of H and thus the proof is finished. Suppose, on the contrary, that $p(u_{n_1+2}, u_{n_1+1}) \neq 0$, $p(v_{n_1+2}, v_{n_1+1}) \neq 0$ for all $n \in \mathbb{N}$. Then by the properties of function ψ, we have

$$S_n \leq S_{n-1} + \psi(S_{n-1}) \leq S_{n-1}, \tag{3.10}$$

where S_n is a non-negative sequence and hence convergent to a limit, say S^*. Taking the limit when $n \to \infty$ in equation (3.10), we get

$$S^* \leq S^* + \psi(S^*) \tag{3.11}$$

and consequently, we have $\psi(S^*) = 0$. By the property of function ψ, we obtain $S^* = 0$, that is, $\lim_{n\to\infty} S_n = 0$. Thus

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} p(u_{n+1}, u_n) + p(v_{n+1}, v_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} p(u_{n+1}, u_n) = \lim_{n\to\infty} p(v_{n+1}, v_n) = 0. \tag{3.12}$$

Next, we prove that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{u_n\}$ or $\{v_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ for which we can find subsequences $\{u_{n(k)}\}$, $\{u_{m(k)}\}$ of $\{u_n\}$ and $\{v_{n(k)}\}$, $\{v_{m(k)}\}$ of $\{v_n\}$ with $n(k) > m(k) \geq k$ such that

$$p(u_{n(k)}, u_{m(k)}) \geq \varepsilon, \text{ for all } k = 1, 2, 3, \cdots . \tag{3.13}$$

Furthermore, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies equation (3.13). Then, we have

$$p(u_{n(k)-1}, u_{m(k)}) < \varepsilon. \tag{3.14}$$

Using the triangle inequality, we have

$$\begin{aligned} p(u_{n(k)}, u_{m(k)}) &\leq p(u_{n(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{m(k)}) \\ &\quad - p(u_{n(k)-1}, u_{n(k)-1}) \\ &\leq p(u_{n(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{m(k)}) \\ &< p(u_{n(k)}, u_{n(k)-1}) + \varepsilon. \end{aligned} \tag{3.15}$$

Similarly, we have

$$\begin{aligned} p(v_{n(k)}, v_{m(k)}) &\leq p(v_{n(k)}, v_{n(k)-1}) + p(v_{n(k)-1}, v_{m(k)}) \\ &\quad - p(v_{n(k)-1}, v_{n(k)-1}) \\ &\leq p(v_{n(k)}, v_{n(k)-1}) + p(v_{n(k)-1}, v_{m(k)}) \\ &< p(v_{n(k)}, v_{n(k)-1}) + \varepsilon. \end{aligned} \tag{3.16}$$

From equations (3.13) and (3.15), we have

$$\varepsilon \leq p(u_{n(k)}, u_{m(k)}) \leq p(u_{n(k)}, u_{n(k)-1}) + \varepsilon. \tag{3.17}$$

Letting $k \to \infty$ in equation (3.17) and using equation (3.12), we get

$$\lim_{k\to\infty} p(u_{n(k)}, u_{m(k)}) = \varepsilon. \tag{3.18}$$

Similarly, one can prove that

$$\lim_{k\to\infty} p(v_{n(k)}, v_{m(k)}) = \varepsilon. \tag{3.19}$$

By the triangle inequality, we have

$$\begin{aligned} p(u_{m(k)}, u_{n(k)}) &\leq p(u_{m(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{n(k)}) - p(u_{n(k)-1}, u_{n(k)-1}) \\ &\leq p(u_{m(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{n(k)}), \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} p(u_{m(k)}, u_{n(k)-1}) &\leq p(u_{m(k)}, u_{n(k)}) + p(u_{n(k)}, u_{n(k)-1}) - p(u_{n(k)}, u_{n(k)}) \\ &\leq p(u_{m(k)}, u_{n(k)}) + p(u_{n(k)}, u_{n(k)-1}). \end{aligned} \tag{3.21}$$

Taking the limit as $k \to \infty$ in equations (3.20), (3.21) and using equations (3.12), (3.18), we get

$$\lim_{k \to \infty} p(u_{n(k)-1}, u_{m(k)}) = \varepsilon. \tag{3.22}$$

Again, by triangle inequality, we have

$$\begin{aligned} p(u_{n(k)-1}, u_{m(k)}) &\leq p(u_{n(k)-1}, u_{m(k)-1}) + p(u_{m(k)-1}, u_{m(k)}) \\ &\quad - p(u_{m(k)-1}, u_{m(k)-1}) \\ &\leq p(u_{n(k)-1}, u_{m(k)-1}) + p(u_{m(k)-1}, u_{m(k)}), \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} p(u_{n(k)-1}, u_{m(k)-1}) &\leq p(u_{n(k)-1}, u_{m(k)}) + p(u_{m(k)}, u_{m(k)-1}) \\ &\quad - p(u_{m(k)}, u_{m(k)}) \\ &\leq p(u_{n(k)-1}, u_{m(k)}) + p(u_{m(k)}, u_{m(k)-1}). \end{aligned} \tag{3.24}$$

Taking the limit as $k \to \infty$ in equations (3.23), (3.24) and using equations (3.12), (3.22), we get

$$\lim_{k \to \infty} p(u_{n(k)-1}, u_{m(k)-1}) = \varepsilon. \tag{3.25}$$

Once again using triangle inequality, we have

$$\begin{aligned} p(u_{n(k)-1}, u_{m(k)-1}) &\leq p(u_{n(k)-1}, u_{n(k)}) + p(u_{n(k)}, u_{m(k)-1}) \\ &\quad - p(u_{n(k)}, u_{n(k)}) \\ &\leq p(u_{n(k)-1}, u_{n(k)}) + p(u_{n(k)}, u_{m(k)-1}), \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} p(u_{n(k)}, u_{m(k)-1}) &\leq p(u_{n(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{m(k)-1}) \\ &\quad - p(u_{n(k)-1}, u_{n(k)-1}) \\ &\leq p(u_{n(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{m(k)-1}). \end{aligned} \tag{3.27}$$

Taking the limit as $k \to \infty$ in equations (3.26), (3.27) and using equations (3.12), (3.25), we get

$$\lim_{k \to \infty} p(u_{n(k)}, u_{m(k)-1}) = \varepsilon. \tag{3.28}$$

Since $n(k) > m(k)$, $u_{n(k)-1} \geq u_{m(k)-1}$ and $v_{n(k)-1} \leq v_{m(k)-1}$. From equation (3.1), we have

$$V\begin{pmatrix} p(H(u_{n(k)-1}, v_{n(k)-1}), H(u_{m(k)-1}, v_{m(k)-1})), \\ p(u_{n(k)-1}, u_{m(k)-1}), p(v_{n(k)-1}, v_{m(k)-1}), \\ p(H(u_{n(k)-1}, v_{n(k)-1}), u_{n(k)-1}) \\ +p(H(u_{m(k)-1}, v_{m(k)-1})), u_{m(k)-1}), \\ p(H(u_{n(k)-1}, v_{n(k)-1}), u_{m(k)-1}) \end{pmatrix} \leq 0,$$

or

$$V\begin{pmatrix} p(u_{n(k)}, u_{m(k)}), p(u_{n(k)-1}, u_{m(k)-1}), p(v_{n(k)-1}, v_{m(k)-1}), \\ p(u_{n(k)}, u_{n(k)-1}) + p(u_{m(k)}, u_{m(k)-1}), p(u_{n(k)}, u_{m(k)-1}) \end{pmatrix} \leq 0. \tag{3.29}$$

Letting $k \to \infty$ in equation (3.29), using equations (3.12), (3.18), (3.25) and (3.28), we obtain

$$V(\varepsilon, \varepsilon, \varepsilon, 0, \varepsilon) \leq 0. \tag{3.30}$$

Hence, we find

$$V(\varepsilon, \varepsilon, \varepsilon, 0 + \varepsilon, \varepsilon + 0) \leq V(\varepsilon, \varepsilon, \varepsilon, 0, \varepsilon) \leq 0,$$

which implies $\varepsilon \leq \varepsilon + \psi(\varepsilon)$. Thus, $\psi(\varepsilon) = 0$ and so $\varepsilon = 0$ by the property of ψ. Which is a contradiction. Thus $\{u_n\}$ is a Cauchy sequence. Using the same arguments as above, we can show that $\{v_n\}$ is also a Cauchy sequence. Since Ω is complete, there exist $u, v \in \Omega$ such that

$$\begin{aligned} \lim_{n,m\to\infty} p(u_n, u_m) &= \lim_{n\to\infty} p(u_n, u) = p(u, u), \\ \lim_{n,m\to\infty} p(v_n, v_m) &= \lim_{n\to\infty} p(v_n, v) = p(v, v). \end{aligned} \tag{3.31}$$

Now, we want to show that

$$p(u, u) = 0 = p(v, v).$$

Suppose, on the contrary, that

$$p(u, u) = \mu > 0 \text{ and } p(v, v) = \nu > 0. \tag{3.32}$$

Then, we see that

$$V\begin{pmatrix} p(H(u_{n-1}, v_{n-1}), H(u_{m-1}, v_{m-1})), \\ p(u_{n-1}, u_{m-1}), p(v_{n-1}, v_{m-1}), \\ p(H(u_{n-1}, v_{n-1}), u_{n-1}) \\ +p(H(u_{m-1}, v_{m-1})), u_{m-1}), \\ p(H(u_{n-1}, v_{n-1}), u_{m-1}) \end{pmatrix} \leq 0,$$

or

$$V\begin{pmatrix} p(u_n, u_m), p(u_{n-1}, u_{m-1}), p(v_{n-1}, v_{m-1}), \\ p(u_n, u_{n-1}) + p(u_m, u_{m-1}), p(u_n, u_{m-1}) \end{pmatrix} \leq 0.$$

By using the triangle inequality $(p4)$, we get

$$V\begin{pmatrix} p(u_n,u_m),p(u_{n-1},u_{m-1}),p(v_{n-1},v_{m-1}), \\ p(u_n,u_{n-1})+p(u_m,u_{m-1}), \\ p(u_n,u_{n-1})+p(u_{n-1},u_{m-1}) \end{pmatrix} \leq 0.$$

Letting $n,m\to\infty$ and using equation (3.12), we obtain

$$V(\mu,\mu,0,0,\mu)\leq 0.$$

Hence, we get that

$$V(\mu,\mu,\mu,0+\mu,\mu+0)\leq V(\mu,\mu,\mu,0,\mu)\leq 0,$$

which implies that $\mu\leq\mu+\psi(\mu)$. Thus, $\psi(\mu)=0$ and so $\mu=0$ by the property of ψ. Hence $p(u,u)=0$. By similar fashion, we can show that $p(v,v)=0$.

Now, suppose that the assumption (a) holds. Then, we have

$$\begin{aligned} p(u,H(u_n,v_n)) &\leq p(u,u_{n+1})+p(u_{n+1},H(u_n,v_n)) \\ &\quad -p(u_{n+1},u_{n+1}) \\ &\leq p(u,u_{n+1})+p(u_{n+1},H(u_n,v_n)) \\ &= p(u,H(u_n,v_n))+p(H(u_n,v_n),H(u_n,v_n)). \end{aligned} \tag{3.33}$$

Taking the limit as $n\to\infty$ in equation (3.33), using equation (3.31) and continuity of H, we obtain

$$p(u,H(u,v))=0.$$

Similarly, we can show that

$$p(v,H(v,u))=0.$$

Therefore, $u=H(u,v)$ and $v=H(v,u)$. This shows that (u,v) is a coupled fixed point of H in Ω.

Finally, suppose that assumption (b) holds. Since $\{u_n\}$ is a non-decreasing sequence and $u_n\to u$ as $n\to\infty$ and $\{v_n\}$ is a non-increasing sequence and $v_n\to v$ as $n\to\infty$, by the assumption, we have $u_n\leq u$ and $v_n\geq v$ for all n. From equations (3.1) and (3.31), we have

$$\lim_{n\to\infty}p(u_n,u)=p(u,u)=\lim_{n\to\infty}p(H(u_n,v_n),u), \tag{3.34}$$

and

$$\lim_{n\to\infty}p(v_n,v)=p(v,v)=\lim_{n\to\infty}p(H(v_n,u_n),v). \tag{3.35}$$

We also have

$$V\begin{pmatrix} p(H(u_n,v_n),H(u,v)),p(u_n,u),p(v_n,v), \\ p(H(u_n,v_n),u_n)+p(H(u,v),u),p(H(u_n,v_n),u) \end{pmatrix} \leq 0.$$

Letting $n \to \infty$ and using equations (3.34) and (3.35), we have

$$V(p(u,H(u,v)),0,0,p(u,H(u,v)),p(u,H(u,v))) \leq 0,$$

which implies that $p(u,H(u,v)) \leq 0+\psi(0)=0$. Hence $u=H(u,v)$.

Similarly, one can show that $v=H(v,u)$. Thus in all the above cases, we proved that H has a coupled fixed point in Ω. This completes the proof. □

From Example 2.16 and Theorem 3.1, we obtain the following results.

Corollary 3.2 *Let $(\Omega,p,\leq)$ be a partially ordered complete partial metric space. Suppose that $H\colon \Omega\times\Omega\to\Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V\in\mathbb{V}$ such that*

$$\begin{aligned} p(H(u,v),H(y,z)) \quad \leq \quad & a_1\,p(u,y)+a_2\,p(v,z)+a_3\,[p(H(u,v),u) \\ & +p(H(y,z),y)]+a_4\,p(H(u,v),y) \end{aligned} \tag{3.36}$$

for all $u,v,y,z\in\Omega$ with $u\geq y$ and $v\leq z$, where a_1,a_2,a_3,a_4 are non-negative reals such that $a_1+a_2+2a_3+2a_4<1$. Suppose that either

(a) H is continuous or

(b) Ω has the following property:

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u\in\Omega$, then $u_n\leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v\in\Omega$, then $v\leq v_n$ for all n.

If there exist two elements $u_0,v_0\in\Omega$ with $u_0\leq H(u_0,v_0)$ and $v_0\geq H(v_0,u_0)$, then H has a coupled fixed point in Ω.

Corollary 3.3 *Let $(\Omega,p,\leq)$ be a partially ordered complete partial metric space. Suppose that $H\colon \Omega\times\Omega\to\Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V\in\mathbb{V}$ such that*

$$\begin{aligned} p(H(u,v),H(y,z)) \quad \leq \quad & a\max\Big\{p(u,y),p(v,z),\frac{1}{2}[p(H(u,v),u) \\ & +p(H(y,z),y)],\frac{1}{2}p(H(u,v),y)\Big\} \end{aligned} \tag{3.37}$$

for all $u,v,y,z\in\Omega$ with $u\geq y$ and $v\leq z$, where $a\in(0,1)$ is a constant. Suppose that either

(a) H is continuous or

(b) Ω has the following property:

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u \in \Omega$, then $u_n \leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v \in \Omega$, then $v \leq v_n$ for all n.

If there exist two elements $u_0, v_0 \in \Omega$ with $u_0 \leq H(u_0, v_0)$ and $v_0 \geq H(v_0, u_0)$, then H has a coupled fixed point in Ω.

Corollary 3.4 *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space. Suppose that $H\colon \Omega \times \Omega \to \Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V \in \mathbb{V}$ such that*

$$p(H(u,v), H(y,z)) \leq \psi\big(\max\big\{p(u,y), p(v,z)\big\}\big) \tag{3.38}$$

for all $u, v, y, z \in \Omega$ with $u \geq y$ and $v \leq z$, where $\psi \in \Psi$. Suppose that either

(a) H is continuous or

(b) Ω has the following property:

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u \in \Omega$, then $u_n \leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v \in \Omega$, then $v \leq v_n$ for all n.

If there exist two elements $u_0, v_0 \in \Omega$ with $u_0 \leq H(u_0, v_0)$ and $v_0 \geq H(v_0, u_0)$, then H has a coupled fixed point in Ω.

If we take $a_1 = k$, $a_2 = l$ and $a_3 = a_4 = 0$ where $k, l \in (0,1)$ in Corollary 3.2, then we obtain the following result.

Corollary 3.5 *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space. Suppose that $H\colon \Omega \times \Omega \to \Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V \in \mathbb{V}$ such that*

$$p(H(u,v), H(y,z)) \leq k\, p(u,y) + l\, p(v,z) \tag{3.39}$$

for all $u, v, y, z \in \Omega$ with $u \geq y$ and $v \leq z$, where k, l are non-negative reals such that $k + l < 1$. Suppose that either

(a) H is continuous or

(b) Ω has the following property:

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u \in \Omega$, then $u_n \leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v \in \Omega$, then $v \leq v_n$ for all n.

If there exist two elements $u_0, v_0 \in \Omega$ with $u_0 \leq H(u_0, v_0)$ and $v_0 \geq H(v_0, u_0)$, then H has a coupled fixed point in Ω.

If we take $k = l = m$ where $m \in (0,1)$ in Corollary 3.5, then we obtain the following result.

Corollary 3.6 *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space. Suppose that $H \colon \Omega \times \Omega \to \Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V \in \mathbb{V}$ such that*

$$p(H(u,v), H(y,z)) \leq \frac{m}{2}\left[p(u,y) + p(v,z)\right] \tag{3.40}$$

for all $u, v, y, z \in \Omega$ with $u \geq y$ and $v \leq z$, where $m \in (0,1)$ is a constant. Suppose that either

(a) H is continuous or

(b) Ω has the following property:

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u \in \Omega$, then $u_n \leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v \in \Omega$, then $v \leq v_n$ for all n.

If there exist two elements $u_0, v_0 \in \Omega$ with $u_0 \leq H(u_0, v_0)$ and $v_0 \geq H(v_0, u_0)$, then H has a coupled fixed point in Ω.

Remark 3.7 Corollary 3.6 extends and generalizes Theorems 2.1 and 2.2 of [9] from partially ordered complete metric spaces to partially ordered complete partial metric spaces.

Example 3.8([18]) Let $\Omega = [0, \infty)$ with usual order $\leq$. Then, $(\Omega, p, \leq)$ be a partially ordered partial metric space where $p(u,v) = \max\{u, v\}$. Suppose

$$H(u,v) = \begin{cases} \frac{u-v}{2}, & \text{if } u \geq v, \\ 0, & \text{otherwise,} \end{cases}$$

and $V(t_1, t_2, t_3, t_4, t_5) = t_1 - \frac{1}{2}\max\{t_2, t_3\}$. It is clear that all conditions of Theorem 3.1 are satisfied. Notice that $(0,0)$ is the coupled fixed point of the operator H.

Now, note that if $(\Omega, \leq)$ is a partially ordered set, we endow the product space $\Omega \times \Omega$ with the partial order relation given by

$$(a,b) \leq (f,g) \quad \Leftrightarrow \quad f \geq a \text{ and } g \leq b.$$

We say that two pairs (p,q) and (r,s) are comparable, that is, every pair of elements has either a lower bound or an upper bound.

Theorem 3.9 *In addition to the hypotheses of Theorem 3.1, suppose that, for every $(a,b), (c,d) \in \Omega \times \Omega$, there exists a pair $(n,p) \in \Omega \times \Omega$ such that (n,p) is comparable to (a,b) and (c,d). Then H has a unique coupled fixed point. Moreover $p(t,t) = 0$.*

Proof Suppose that (x,y) and (s,t) are coupled fixed point of H, that is, $x = H(x,y)$, $y = H(y,x)$, $s = H(s,t)$ and $t = H(t,s)$.

Let (α, β) be an element of $\Omega \times \Omega$ comparable to both (x, y) and (s, t). Suppose that $(x, y) \geq (\alpha, \beta)$ (the proof is similar in other cases).

Assume that (x, y) and (s, t) are comparable, then from inequality (3.1), we have

$$V\begin{pmatrix} p(H(x,y), H(s,t)), p(x,s), p(y,t), \\ p(H(x,y), x) + p(H(s,t), s), p(H(x,y), s) \end{pmatrix} \leq 0,$$

$$V(p(x,s), p(x,s), p(y,t), p(x,x) + p(s,s), p(x,s)) \leq 0,$$

$$V(p(x,s), p(x,s), p(y,t), 0, p(x,s)) \leq 0,$$

or

$$\begin{aligned} &V(p(x,s), p(x,s), p(y,t), 0 + p(x,s), p(x,s) + 0) \\ &\leq V(p(x,s), p(x,s), p(y,t), 0, p(x,s)) \leq 0, \end{aligned}$$

which implies

$$p(x,s) \leq p(x,s) + \psi(p(y,t)). \tag{3.41}$$

By similar fashion, one can show that

$$p(y,t) \leq p(y,t) + \psi(p(x,s)). \tag{3.42}$$

From equations (3.41) and (3.42), we obtain

$$\begin{aligned} p(x,s) + p(y,t) &\leq p(x,s) + p(y,t) + \psi[p(x,s) + p(y,t)] \\ &\Rightarrow p(x,s) + p(y,t) = 0 \\ &\Rightarrow p(x,s) = p(y,t) = 0, \end{aligned}$$

and so, $x = s$ and $y = t$. Thus, $(x, y) = (s, t)$. This shows the uniqueness of coupled fixed point. This completes the proof. □

Theorem 3.10 *In addition to the hypotheses of Theorem* 3.1, *if* u_0, v_0 *are comparable, then the coupled fixed point* $(u, v) \in \Omega \times \Omega$ *satisfies* $u = v$.

Proof Assume that $u_0 \leq v_0$ (a similar argument applies for $v_0 \leq u_0$). Then, by using the mathematical induction

$$u_{n+1} = H(u_n, v_n) \leq H(v_n, u_n) = v_{n+1}.$$

Taking the limit as $n \to \infty$, we have

$$u = \lim_{n \to \infty} u_n \leq \lim_{n \to \infty} v_n = v.$$

From the contractive condition (3.1), we have

$$V\begin{pmatrix} p(H(u,v),H(v,u)),p(u,v),p(v,u), \\ p(H(u,v),u)+p(H(v,u),v),p(H(u,v),v) \end{pmatrix} \leq 0,$$

$$V(p(u,v),p(u,v),p(v,u),p(u,u)+p(v,v),p(u,v)) \leq 0,$$

$$V(p(u,v),p(u,v),p(u,v),p(u,v)+p(u,v),p(u,v)) \leq 0 \quad \big(\text{by (p2),(p3)}\big)$$

or

$$V(p(u,v),p(u,v),p(u,v),p(u,v)+p(u,v),p(u,v)+p(u,v))$$
$$\leq V(p(u,v),p(u,v),p(u,v),p(u,v)+p(u,v),p(u,v)) \leq 0,$$

which implies

$$\begin{aligned} p(u,v) &\leq p(u,v)+\psi(p(u,v)) \\ &\Rightarrow p(u,v)=0 \Rightarrow u=v, \end{aligned}$$

by the property of ψ. This completes the proof. □

Theorem 3.11 *Let $\Omega=[0,1]$. Then $(\Omega,\leq)$ is a partially ordered set with a natural ordering of real numbers. Let $p\colon \Omega\times\Omega\to[0,1]$ be defined by $p(u,v)=|u-v|$ for all $u,v\in\Omega$. Consider the mapping $H\colon \Omega\times\Omega\to[0,1]$ defined by*

$$H(u,v)=\begin{cases} \frac{u^2-v^2+1}{3}, & \text{if } u\leq v, \\ \frac{1}{3}, & \text{if } u>v, \end{cases}$$

for all $u,v\in\Omega$. Then,

(1) *(Ω,p) is a complete partial metric space since (Ω,d^p) is complete;*

(2) *H has the mixed monotone property;*

(3) *H is continuous;*

(4) *$0\leq H(0,1)$ and $1\geq H(1,0)$;*

(5) *there exists a constant $0<m<1$ such that*

$$p\big(H(u,v),H(y,z)\big)\leq \frac{m}{2}\big[p(u,y)+p(v,z)\big]$$

for all $u,v,y,z\in\Omega$ with $u\leq y$ and $v\geq z$. Thus, by Corollary 3.6, H has a coupled fixed point. Moreover, $(\frac{1}{3},\frac{1}{3})$ is the unique coupled fixed point of H.

Proof The proofs of (1) – (4) are obvious.

For any $u\leq y$ and $v\geq z$, we have

$$p(u,y)=y-u, \quad p(v,z)=v-z.$$

The proof of (5) is divided into the following cases.

Case 1. If $y \leq z$. In this case, $u \leq y \leq z \leq v$, and so

$$H(u,v) = \frac{u^2 - v^2 + 1}{3}, \quad H(y,z) = \frac{y^2 - z^2 + 1}{3}.$$

Hence, we get

$$\begin{aligned} p\big(H(u,v), H(y,z)\big) &= p\Big(\frac{u^2 - v^2 + 1}{3}, \frac{y^2 - z^2 + 1}{3}\Big) \\ &= \frac{1}{3}(y^2 - z^2 - u^2 + v^2) = \frac{1}{3}\left[(y^2 - u^2) + (v^2 - z^2)\right] \\ &\leq \frac{1}{3}\left[(y - u) + (v - z)\right] = \frac{1}{3}\left[p(u,y) + p(v,z)\right] \\ &= \frac{m}{2}\left[p(u,y) + p(v,z)\right] \end{aligned}$$

with $m = \frac{2}{3} < 1$.

Case 2. If $y > z$. In this case, $u \leq y \leq v$, and so

$$H(u,v) = \frac{u^2 - v^2 + 1}{3}, \quad H(y,z) = \frac{1}{3}.$$

Hence, we get

$$\begin{aligned} p\big(H(u,v), H(y,z)\big) &= p\Big(\frac{u^2 - v^2 + 1}{3}, \frac{1}{3}\Big) = \frac{1}{3}(v^2 - u^2) \\ &\leq \frac{1}{3}(v^2 - u^2 + y^2 - z^2) = \frac{1}{3}\left[(y^2 - u^2) + (v^2 - z^2)\right] \\ &\leq \frac{1}{3}\left[(y - u) + (v - z)\right] = \frac{1}{3}\left[p(u,y) + p(v,z)\right] \\ &= \frac{m}{2}\left[p(u,y) + p(v,z)\right] \end{aligned}$$

with $m = \frac{2}{3} < 1$.

Case 3. If $u > v$. In this case, $y \leq z \leq v$, and so

$$H(u,v) = \frac{1}{3}, \quad H(y,z) = \frac{y^2 - z^2 + 1}{3}.$$

Hence, we get

$$\begin{aligned} p\big(H(u,v), H(y,z)\big) &= p\Big(\frac{1}{3}, \frac{y^2 - z^2 + 1}{3}\Big) = \frac{1}{3}(y^2 - z^2) \\ &\leq \frac{1}{3}(y^2 - z^2 + v^2 - u^2) = \frac{1}{3}\left[(y^2 - u^2) + (v^2 - z^2)\right] \\ &\leq \frac{1}{3}\left[(y - u) + (v - z)\right] = \frac{1}{3}\left[p(u,y) + p(v,z)\right] \\ &= \frac{m}{2}\left[p(u,y) + p(v,z)\right] \end{aligned}$$

with $m = \frac{2}{3} < 1$.

Thus, in all the above cases, the condition (5) is satisfied. Since $\Omega = [0, 1]$ is a totally ordered set, by Theorem 3.9, $(\frac{1}{3}, \frac{1}{3})$ is the unique coupled fixed point of H. $\square$

§4. An Application to the Integral Equation

In this section, we study the existence of solution of the nonlinear integral equations, as an application of the coupled fixed point theorem proved in the previous section.

Consider the following nonlinear integral equations

$$\begin{aligned} x(t) &= \mu(t) + \int_0^T R(t,p)g(p, x(p), y(p))dp, \\ y(t) &= \mu(t) + \int_0^T R(t,p)g(p, y(p), x(p))dp, \end{aligned} \tag{4.1}$$

where $t \in I = [0, T]$, with $T > 0$.

We consider the space $\Omega = C(I, \mathbb{R})$ of continuous functions defined in I. Define $p\colon \Omega \times \Omega \to [0, +\infty)$ by

$$p(x, y) = \max_{t \in I} |x(t) - y(t)| \tag{4.2}$$

for all $x, y \in \Omega$. Then (Ω, p) is a complete partial metric space.

Let $\Omega = C(I, \mathbb{R})$ with the natural partial order relation, that is, $x, y \in C(I, \mathbb{R})$,

$$x \leq y \;\Leftrightarrow\; x(t) \leq y(t),\ t \in I.$$

We consider the following assumptions:

(i) the mapping $g\colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\mu\colon I \to \mathbb{R}$ are continuous;

(ii) there exists a continuous $0 \leq m < 1$ such that

$$|g(p, x, y) - g(p, u, v)| \leq \frac{m}{2}\left(|x - u| + |y - v|\right) \tag{4.3}$$

for all $x, y, u, v \in \Omega$ and for all $p \in I$;

(iii) for all $t, p \in I$, there exists a continuous $R\colon I \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{t \in T} \int_0^T R(t,p)dp < 1; \tag{4.4}$$

(iv) there exist $x_0, y_0 \in \Omega$ such that

$$\begin{aligned} x_0(t) &\leq \mu(t) + \int_0^T R(t,p)g(p, x_0(p), y_0(p))dp, \\ y_0(t) &\leq \mu(t) + \int_0^T R(t,p)g(p, y_0(p), x_0(p))dp, \end{aligned} \tag{4.5}$$

where $t \in I$.

Theorem 4.1 *Consider the Corollary* 3.6 *and assume that conditions* (i) - (iv) *are satisfied. Then equation (*4.1*) has a unique solution in* Ω.

Proof Define the mapping $H\colon \Omega^2 \to \Omega$, $(x, y) \to H(x, y)$, where

$$H(x,y)(t) = \mu(t) + \int_0^T R(t,p)g(p,x(p),y(p))dp, \ t \in I \tag{4.6}$$

for all $x, y \in \Omega$ and $t \in I$.

Equation (4.1) can be stated as

$$x = H(x,y) \ \text{ and } \ y = H(y,x). \tag{4.7}$$

For $x, y, u, v \in \Omega$ be such that $x \leq u$ and $y \leq v$ and

$$\begin{aligned} H(x,y)(t) &= \mu(t) + \int_0^T R(t,p)g(p,x(p),y(p))dp \\ &\leq \mu(t) + \int_0^T R(t,p)g(p,u(p),v(p))dp \\ &= H(u,v)(t) \text{ for all } t \in I. \end{aligned} \tag{4.8}$$

From equations (4.2) and (4.3) for all $t \in I$, we have

$$\begin{aligned} p(H(x,y),H(u,v)) &= \max_{t\in I} |H(x,y)(t) - H(u,v)(t)| \\ &\leq \max_{t\in I} \int_0^T R(t,p)\Big|g(p,x(p),y(p)) - g(p,u(p),v(p))\Big|dp \\ &\leq \Big|g(p,x(p),y(p)) - g(p,u(p),v(p))\Big| \\ &\leq \frac{m}{2}(|x(p)-u(p)| + |y(p)-v(p)|) \\ &= \frac{m}{2}[p(x,u) + p(y,v)], \end{aligned}$$

where $0 \leq m < 1$.

So that

$$p(H(x,y),H(u,v)) \leq \frac{m}{2}[p(x,u) + p(y,v)],$$

which is the contractive condition in Corollary 3.6. Thus H has a coupled fixed point in Ω, that is, the system of nonlinear integral equation has a solution. Finally, let (p, q) be a coupled lower and upper solution of the integral equation (4.1), then by assumption (iv) of the Theorem 4.1, we have $p \leq H(p,q) \leq H(q,p) \leq q$. Corollary 3.6 gives us that H has a coupled fixed point, say $(m,n) \in \Omega \times \Omega$. Since $p \leq q$, Theorem 3.10 says us that $m = n$ and this implies $m = H(m,m)$ and m is the unique solution of the integral equation (4.1). □

The aforesaid application is illustrated by the following example.

Example 4.2 Let $\Omega = C([0,1],\mathbb{R})$, $g\colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\mu\colon I \to \mathbb{R}$. Now consider the following functional integral equation

$$\begin{aligned} x(t) &= \frac{t^2}{1+t^4} + \int_0^1 \frac{sin\, p\, 3^{-p} e^{-p}}{9(t+3)} \Big(\frac{|x(p)|}{1+|x(p)|} + \frac{|y(p)|}{1+|y(p)|} \Big) dp \\ y(t) &= \frac{t^2}{1+t^4} + \int_0^1 \frac{sin\, p\, 3^{-p} e^{-p}}{9(t+3)} \Big(\frac{|y(p)|}{1+|y(p)|} + \frac{|x(p)|}{1+|x(p)|} \Big) dp \end{aligned}$$

for all $x, y \in \Omega$ and $t \in I$. Observe that the above equation is a special case of equation (4.1) with

$$\begin{aligned} \mu(t) &= \frac{t^2}{1+t^4}. \\ R(t,p) &= \frac{3^{-p} e^{-p}}{t+3}. \\ g(p,x,y) &= \frac{sin\, p}{9} \Big(\frac{|x(p)|}{1+|x(p)|} + \frac{|y(p)|}{1+|y(p)|} \Big). \\ g(p,y,x) &= \frac{sin\, p}{9} \Big(\frac{|y(p)|}{1+|y(p)|} + \frac{|x(p)|}{1+|x(p)|} \Big). \end{aligned}$$

It is also easily seen that these functions are continuous.

For arbitrary $x, y, u, v \in \Omega$ and for all $p \in I$, we have

$$\begin{aligned} |g(p,x,y) - g(p,u,v)| &= \Big| \frac{sin\, p}{9} \Big(\frac{|x(p)|}{1+|x(p)|} + \frac{|y(p)|}{1+|y(p)|} \Big) \\ &\quad - \frac{sin\, p}{9} \Big(\frac{|u(p)|}{1+|u(p)|} + \frac{|v(p)|}{1+|v(p)|} \Big) \Big| \\ &\leq \frac{1}{9} \big(|x-u| + |y-v| \big) = \frac{m}{2} \big(|x-u| + |y-v| \big). \end{aligned}$$

Therefore, the function g satisfies equation (4.3) with $m = \frac{2}{9} < 1$.

For all $t, p \in I$, there exists $R\colon I \times \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \int_0^1 R(t,p)dp &= \int_0^1 \frac{3^{-p} e^{-p}}{t+3} dp = -\frac{1}{3} \Big(\frac{e^{-1} - 3}{(ln\, 3 + 1)(t+3)} \Big) \\ &= \Big(1 - \frac{1}{3e} \Big) \frac{1}{(ln\, 3 + 1)(t+3)} \leq 1 - \frac{1}{3e} \leq \frac{9}{10} < 1. \end{aligned}$$

We put $x_0(t) = \frac{5t^2}{7(1+t^4)}$ and obtain

$$\begin{aligned} x_0(t) &= \frac{5t^2}{7(1+t^4)} \leq \frac{t^2}{1+t^4} \\ &\leq \frac{t^2}{1+t^4} + \int_0^1 \frac{sin\, p}{9} \Big(\frac{|x(p)|}{1+|x(p)|} + \frac{|y(p)|}{1+|y(p)|} \Big) dp \\ &= \mu(t) + \int_0^T R(t,p) g(p, x_0(p), y_0(p)) dp. \end{aligned}$$

Similarly, we have

$$y_0(t) \leq \mu(t) + \int_0^T R(t,p)g(p, y_0(p), x_0(p))dp.$$

This shows that equation (4.5) holds.

Hence, the integral equation (4.1) has a unique solution in Ω with $\Omega = C([0,1], \mathbb{R})$.

§5. Conclusion

In this paper, we prove some coupled fixed point theorems via implicit relations in the setting of partially ordered partial metric spaces. Furthermore, we give some consequences of the main result. We provide some illustrative examples to validate the established results. An application to the integral equation is also given. Our results extend and generalize various results in the literature.

References

[1] M. Abbas, M. Ali Khan and S. Radenović, Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, *Appl. Math. Comput.*, 217(1) (2010), 195-202.

[2] M. Abbas, T. Nazir and S. Romaguera, Fixed point results for generalized cyclic contraction mappings in partial metric spaces, *Rev. R. Acad. Cienc. Exactas Fis. Nat.*, Ser. A Mat. RACSAM. 106(1) (2012), 287-297.

[3] T. Abdeljawad, E, Karapınar and K. Tas, Existence and uniqueness of a common fixed point on partial metric spaces, *Appl. Math. Lett.*, 24(11) (2011), 1894-1899.

[4] I. Altun and D. Turkoglu, Some fixed point theorems for weakly compatible multivalued mappings satisfying an implicit relation, *Filomat*, 22 (2008), 13-23.

[5] I. Altun, F. Sola and H. Simsek, Generalized contractions on partial metric spaces, *Topology and its Appl.*, 157 (2010), 2778-2785.

[6] H. Aydi, Some coupled fixed point results on partial metric spaces, *International J. Math. Math. Sci.*, 2011, Article ID 647091, 11 pages.

[7] H. Aydi, M. Abbas and C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, *Topology and Its Appl.*, 159 (2012), No. 14, 3234-3242.

[8] S. Banach, Surles operation dans les ensembles abstraits et leur application aux equation integrals, *Fund. Math.*, 3(1922), 133-181.

[9] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis: TMA*, 65(7) (2006), 1379-1393.

[10] L. Ciric and V. Lakshmikantham, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis: TMA*, 70(12) (2009), 4341-4349.

[11] S. Chandok, D. Kumar and M. S. Khan, Some results in partial metric space using auxiliary functions, *Applied Math. E-Notes*, 15 (2015), 233-242.

[12] Pooja Dhawan and Jatinderdeep Kaur, Some common fixed point theorems in ordered

partial metric spaces via $\mathcal{F}$-generalized contractive type mappings, *Mathematics*, 2019, 7, 193; doi:10.3390/math7020193.

[13] I. Erhan, E. Karapinar and D. Turkoglu, Different types Meir-Keeler contractions on partial metric spaces, *J. Comput. Anal. Appl.*, 14(6) (2012), 1000-1005.

[14] R. O. Gregorio, Coupled fixed point theorems in partially ordered metric spaces, *Adv. Fixed Point Theory*, 9(1) (2019), 17-28.

[15] R. Heckmann, Approximation of metric spaces by partial metric spaces, *Appl. Categ. Structures*, 7, No. 1-2, (1999), 71-83.

[16] N. M. Hung, E. Karapinar and N. V. Luong, Coupled coincidence point theorem in partially ordered metric spaces via implicit relation, *Abstr. Appl. Anal.*, (2012), Art. ID 796964, 2012.

[17] E. Karapinar, W. Shatanawi and K. Tas, Fixed point theorems on partial metric spaces involving rational expressions, *Miskolc Math. Notes*, 14 (2013), 135-142.

[18] N. V. Luong and N. X. Thuan, Coupled fixed points in ordered generalized metric spaces and application to integro-differential equations, (Submitted).

[19] B. Khomdram, Y. Rohen and T. C. Singh, Coupled fixed point theorems in G_b-metric space satisfying some rational contractive conditions, *Springer Plus*, (2016) 5:1261. Doi:10.1186/s40064-016-2925-7.

[20] D. Kumar, S. Sadat, J. R. Lee and C. Park, Some theorems in partial metric spaces using auxiliary functions, *AIMS Maths.*, 6(7) (2021), 6734-6748.

[21] H. P. A. Künzi, H. Pajoohesh and M. P. Schellekens, Partial quasi-metrics, *Theoritical Comput. Sci.*, 365(3) (2006), 237-246.

[22] H. P. Masiha, F. Sabetghadam and N. Shahzad, Fixed point theorems in partial metric spaces with an application, *Filomat*, 27(4) (2013), 617-624.

[23] S. G. Matthews, Partial metric topology, Proceedings of the 8th summer conference on topology and its applications, *Annals of the New York Academy of Sciences*, 728 (1994), 183-197.

[24] H. K. Nashine and W. Shatanawi, Coupled common fixed point theorems for pair of commuting in partially ordered complete metric spaces, *Comput. Math. Appl.*, 62 (2011), 1984-1993.

[25] M. Nazam, Z. Hamid, H. Al Sulami and A. Hussain, Common fixed point theorems in the partial b-metric spaces and an application to the system of boundary value problems, *J. Function Spaces*, Vol. 2021, Art. ID 7777754, https://doi.org/10.1155/2021/7777754.

[26] J. O. Olaleru, G. A. Okeke and H. Akewe, Coupled fixed point theorems for generalized φ-mappings satisfying contractive condition of integral type on cone metric spaces, *Int. J. Math. Model. Comput.*, 2(2) (2012), 87-98.

[27] S. Oltra and O. Valero, Banach's fixed point theorem for partial metric spaces, *Rend. Ist. Mat. Univ. Trieste*, 36(1-2) (2004), 17-26.

[28] S. J. O'Neil, Partial metrics, valuations and domain theory, in: *Proc. 11th Summer Conference on General Topology and Application*, also in *Annals of the New York Academy of Sciences*, Vol.806 (1996), 304-315.

[29] V. Popa, On some fixed point theorems for compatible mappings satisfying an implicit

relation, *Demonstr. Math.*, 32(1) (1999), 157-163.

[30] V. Popa, A general coincidence theorem for compatible multivalued mappings satisfying an implicit relation, *Demonstr. Math.*, 33 (2000), 159-164.

[31] V. Popa, M. Imdad and J. Ali, Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces, *Bull. Malays. Math. Sci. Soc.(2)*, 33(1) (2010), 105-120.

[32] S. Romaguera and M. P. Schellekens, Duality and quasi-normability for complexity spaces, *Appl. General Topology*, 3 (2002), 91-112.

[33] F. Sabetghadamt, H. P. Masiha and A. H. Sanatpour, Some coupled fixed point theorems in cone metric spaces, *Fixed Point Theory Appl.*, Article ID 125426, (2009), 8 pages.

[34] G. S. Saluja, Fixed point theorems on cone S-metric spaces using implicit relation, *Cubo, A Mathematical Journal*, 22(2) (2020), 273-288.

[35] B. Samet, M. Rajović, R. Lazović and R. Stoiljković, Common fixed point results for nonlinear contractions in ordered partial metric spaces, *Fixed Point Theory Appl.*, 2011, 2011:71.

[36] M. P. Schellekens, The correspondence between partial metrics and semivaluations, *Theoritical Comput. Sci.*, 315(1) (2004), 135-149.

[37] W. Shatanawi, B. Samet and M. Abbas, Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, *Math. Comput. Modell.*, 55 (2012), Nos. 3-4, 680-687.