

A New Approach to the Nonhomogeneous Sturm-Liouville Fuzzy Problem with Fuzzy Forcing Function

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Abstract: This paper is on the solution of a nonhomogeneous Sturm-Liouville fuzzy problem with fuzzy forcing function. The problem is studied by a different solution method. Example is solved.

Key Words: Sturm-Liouville theory, fuzzy differential equation, fuzzy boundary value problem, fuzzy eigenvalue.

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§1. Introduction

The Sturm-Liouville theory is used in mathematical physics [1,2]. Sturm-Liouville fuzzy problems with real and fuzzy coefficients were studied by Gültekin Çitil and Altınışık [3,4,5]. Also, fuzzy eigenvalue problems were investigated under the approach of generalized differentiability [6,7] and the fuzzy problem with eigenvalue parameter in the boundary condition was studied [8,9]. On the other hand, Gültekin Çitil investigated the problem with fuzzy eigenvalue parameter and she studied the problem with fuzzy eigenvalue parameter in one of the boundary conditions [10,11].

The aim of this study is to investigate the solution of a nonhomogeneous Sturm-Liouville fuzzy problem with fuzzy forcing function by a different solution method.

§2. Preliminaries

Definition 2.1([12]) *A fuzzy number is a mapping $u: \mathbb{R} \rightarrow [0, 1]$ with the following properties:*

- (1) *u is normal;*
- (2) *u is convex fuzzy set;*
- (3) *u is upper semi-continuous on \mathbb{R} ;*
- (4) *$cl \{x \in \mathbb{R} | u(x) > 0\}$ is compact where cl denotes the closure of a subset.*

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Definition 2.2([13]) Let $U = R$ (where R is the set of real numbers). Let a, b and c real numbers such that $a \leq c \leq b$. A set \tilde{u} with membership function

$$\mu(x) = \begin{cases} \frac{x-a}{c-a}, & a < x < c \\ 1, & x = c \\ \frac{b-x}{b-c}, & c < x < b \\ 0, & \text{otherwise} \end{cases}$$

is called a triangular fuzzy number and is denoted by $\tilde{u} = (a, c, b)$.

Definition 2.3([13]) The classical set $A_\alpha = \{x \in U \mid \mu_{\tilde{A}}(x) \geq \alpha\}$, $0 < \alpha \leq 1$ is called the α -cut of \tilde{A} .

Definition 2.4([13]) For a triangular fuzzy number $\tilde{u} = (a, c, b)$ the α -cuts are intervals $u_\alpha = [\underline{u}_\alpha, \bar{u}_\alpha]$, where

$$\underline{u}_\alpha = a + \alpha(c - a), \bar{u}_\alpha = b + \alpha(c - b).$$

Definition 2.5([13]) Let $F_a(\cdot), F_c(\cdot), F_b(\cdot)$ be continuous functions on an interval I . The fuzzy set \tilde{F} determined by the membership function

$$\mu_{\tilde{F}}(y(\cdot)) = \begin{cases} \alpha, & y = F_a + \alpha(F_c - F_a) \text{ and } 0 \leq \alpha \leq 1 \\ \alpha, & y = F_b + \alpha(F_c - F_b) \text{ and } 0 \leq \alpha \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

is called a triangular fuzzy function and is denoted by $\tilde{F} = (F_a, F_c, F_b)$.

§3. A Nonhomogeneous Sturm-Liouville Fuzzy Problem

Consider the nonhomogeneous Sturm-Liouville fuzzy problem with fuzzy forcing function

$$\tau y := y'' + q(x)y,$$

$$\tau y + \tilde{\lambda}y = \tilde{f}(x), \quad x \in (0, \ell) \quad (3.1)$$

$$Ay(0) + By'(0) = \tilde{\beta} \quad (3.2)$$

$$Cy(\ell) + Dy'(\ell) = \tilde{\delta}, \quad (3.3)$$

where $q(x)$ is positive and continuous function, $A, B, C, D \geq 0$, $A^2 + B^2 \neq 0, C^2 + D^2 \neq 0$, $\tilde{\lambda} = (\underline{\lambda}, \lambda, \bar{\lambda})$ is positive fuzzy eigenvalue, $\tilde{f}(x) = (\underline{f}(x), f(x), \bar{f}(x))$ fuzzy forcing function, $\tilde{\beta} = (\underline{\beta}, \beta, \bar{\beta})$, $\tilde{\delta} = (\underline{\delta}, \delta, \bar{\delta})$ are symmetric triangular fuzzy numbers and $y(x)$ is positive fuzzy function.

3.1 Solution Method

Let's divide the problem (3.1)-(3.3) three problems following.

(i) The first problem is

$$\tau y + \lambda y = f(x), \tag{3.4}$$

$$Ay(0) + By'(0) = \beta, \tag{3.5}$$

$$Cy(\ell) + Dy'(\ell) = \delta, \tag{3.6}$$

(ii) The second problem is

$$\tau y = 0, \tag{3.7}$$

$$Ay(0) + By'(0) = (\underline{\beta} - \beta, 0, \bar{\beta} - \beta), \tag{3.8}$$

$$Cy(\ell) + Dy'(\ell) = (\underline{\delta} - \delta, 0, \bar{\delta} - \delta), \tag{3.9}$$

(iii) The third problem is

$$\tau y + (\underline{\lambda} - \lambda, 0, \bar{\lambda} - \lambda) y = (\underline{f}(x) - f(x), 0, \bar{f}(x) - f(x)), \tag{3.10}$$

$$Ay(0) + By'(0) = 0, \tag{3.11}$$

$$Cy(\ell) + Dy'(\ell) = 0. \tag{3.12}$$

We discuss the three problems respectively in the following cases.

Case 1. For the problem (i), consider the boundary value problem

$$\tau y + \lambda y = 0 \tag{3.13}$$

$$Ay(0) + By'(0) = 0, \tag{3.14}$$

$$Cy(\ell) + Dy'(\ell) = 0. \tag{3.15}$$

Let $\varphi_\lambda(x)$ be the solution of differential equation (3.13) satisfying the conditions $y(0) = B$, $y'(0) = -A$ and $\chi_\lambda(x)$ be the solution of differential equation (3.13) satisfying the conditions $y(\ell) = D$, $y'(\ell) = -C$. The eigenvalues of the boundary value problem (3.13) – (3.15) are the zeros of the Wronskian function

$$W(\lambda) = W(\varphi_\lambda, \chi_\lambda)(x) = \varphi_\lambda(x) \chi'_\lambda(x) - \chi_\lambda(x) \varphi'_\lambda(x).$$

Then, if λ is not the eigenvalue of the fuzzy boundary value problem (3.13) – (3.15), since $W(\lambda) \neq 0$, the solutions $\varphi_\lambda(x)$ and $\chi_\lambda(x)$ will be linear independent. According to this, the general solution of the fuzzy differential equation (3.13) is

$$y_\lambda(x) = c_{1\lambda} \varphi_\lambda(x) + c_{2\lambda} \chi_\lambda(x).$$

Using the method variation of parameters, we can search for the general solution of the

differential equation (3.4) as

$$y_\lambda(x) = c_{1\lambda}(x) \varphi_\lambda(x) + c_{2\lambda}(x) \chi_\lambda(x). \quad (3.16)$$

Choosing the functions $\varphi_\lambda(x)$ and $\chi_\lambda(x)$ that

$$c'_{1\lambda}(x) \varphi_\lambda(x) + c'_{2\lambda}(x) \chi_\lambda(x) = 0,$$

$$c'_{1\lambda}(x) \varphi_{\lambda'}(x) + c'_{2\lambda} \chi_{\lambda'}(x) = f(x).$$

From this,

$$c'_{1\lambda}(x) = -\frac{1}{W(\lambda)} \chi_\lambda(x) f(x), \quad c'_{2\lambda}(x) = \frac{1}{W(\lambda)} \varphi_\lambda(x) f(x)$$

are obtained. Thus, we have

$$c_{1\lambda}(x) = \frac{1}{W(\lambda)} \int_x^\ell \chi_\lambda(t) f(t) dt + c_{1\lambda}, \quad c_{2\lambda}(x) = \frac{1}{W(\lambda)} \int_0^x \varphi_\lambda(t) f(t) dt + c_{2\lambda}$$

Substituting these equations in (3.16), the general solution of the differential equation (3.4) is obtained as

$$y_\lambda(x) = \frac{1}{W(\lambda)} \left\{ \varphi_\lambda(x) \int_x^\ell \chi_\lambda(t) f(t) dt + \chi_\lambda(x) \int_0^x \varphi_\lambda(t) f(t) dt \right\} + c_{1\lambda} \varphi_\lambda(x) + c_{2\lambda} \chi_\lambda(x)$$

Using the boundary condition (3.5), we have

$$c_{2\lambda} \left(A\chi_\lambda(0) + B\chi'_\lambda(0) \right) = \beta.$$

Also, $A\chi_\lambda(0) + B\chi'_\lambda(0) = W(\lambda)$ and since λ is not eigenvalue, $W(\lambda) \neq 0$. Thus,

$$c_{2\lambda} = \frac{\beta}{A\chi_\lambda(0) + B\chi'_\lambda(0)}.$$

Similarly,

$$c_{1\lambda} = \frac{\delta}{C\varphi_\lambda(\ell) + D\varphi'_\lambda(\ell)}$$

is obtained. Then, the solution of the boundary value problem (3.4) – (3.6) is

$$\begin{aligned} y_\lambda(x) = & \frac{1}{W(\lambda)} \left\{ \varphi_\lambda(x) \int_x^\ell \chi_\lambda(t) f(t) dt + \chi_\lambda(x) \int_0^x \varphi_\lambda(t) f(t) dt \right\} \\ & + \left(\frac{\delta}{C\varphi_\lambda(\ell) + D\varphi'_\lambda(\ell)} \right) \varphi_\lambda(x) + \left(\frac{\beta}{A\chi_\lambda(0) + B\chi'_\lambda(0)} \right) \chi_\lambda(x). \end{aligned} \quad (3.17)$$

Case 2. For the problem (ii), let $y_1(x)$ and $y_2(x)$ be linear independent solutions of the

differential equation $\tau y = 0$. The solution of fuzzy boundary value problem (3.7) – (3.9) is

$$Y_1(x) = (\underline{\beta} - \beta, 0, \bar{\beta} - \beta) w_1(x) + (\underline{\delta} - \delta, 0, \bar{\delta} - \delta) w_2(x), \tag{3.18}$$

where

$$w_1(x) = \frac{y_2(\ell) y_1(x) - y_1(\ell) y_2(x)}{y_1(0) y_2(\ell) - y_1(\ell) y_2(0)}, \quad w_2(x) = \frac{y_1(0) y_2(x) - y_2(0) y_1(x)}{y_1(0) y_2(\ell) - y_1(\ell) y_2(0)}.$$

Case 3. For the problem (iii), let $\tilde{y}(x)$ be the solution of differential equation $\tau y + (\underline{\lambda} - \lambda) y = \underline{f}(x) - f(x)$ and $\tilde{\tilde{y}}(x)$ be the solution of differential equation $\tau y + (\bar{\lambda} - \lambda) y = \bar{f}(x) - f(x)$. Then, the solution of fuzzy boundary value problem (3.10) – (3.12) is

$$Y_2(x) = \left(\min \left\{ \tilde{y}(x), 0, \tilde{\tilde{y}}(x) \right\}, 0, \max \left\{ \tilde{y}(x), 0, \tilde{\tilde{y}}(x) \right\} \right). \tag{3.19}$$

From (3.17), (3.18) and (3.19), fuzzy solution of fuzzy boundary value problem (3.1)-(3.3) is

$$y(x) = y_\lambda(x) + Y_1(x) + Y_2(x).$$

Example 3.1 Consider the fuzzy problem

$$y'' + \tilde{\lambda} y = \tilde{f}(x), \quad x \in (0, 1) \tag{3.20}$$

$$y(0) = \tilde{1}, \tag{3.21}$$

$$y(1) = \tilde{2}, \tag{3.22}$$

where $\tilde{\lambda} = (\lambda + 1, \lambda, \lambda + 3)$ is positive fuzzy eigenvalue, $\tilde{f}(x) = (x - 2, x, x + 2)$ fuzzy forcing function, $\tilde{1} = (0, 1, 2)$, $\tilde{2} = (1, 2, 3)$ are symmetric triangular fuzzy numbers and $y(x)$ is positive fuzzy function.

(i) The first problem is

$$y'' + \lambda y = x, \tag{3.23}$$

$$y(0) = 1, \tag{3.24}$$

$$y(1) = 2. \tag{3.25}$$

Consider the boundary value problem

$$y'' + \lambda y = 0 \tag{3.26}$$

$$y(0) = 0, \tag{3.27}$$

$$y(1) = 0. \tag{3.28}$$

Let

$$\varphi_\lambda(x) = \sin(\sqrt{\lambda}x)$$

be the solution of differential equation (3.26) satisfying the conditions $y(0) = 0$ and

$$\chi_\lambda(x) = \sin(\sqrt{\lambda}) \cos(\sqrt{\lambda}x) - \cos(\sqrt{\lambda}) \sin(\sqrt{\lambda}x)$$

be the solution of differential equation (3.26) satisfying the conditions $y(1) = 0$. From this, making the necessary operations, $W(\varphi_\lambda, \chi_\lambda)(x) = W(\lambda)$ is obtained as

$$W(\lambda) = -\sqrt{\lambda} \sin(\sqrt{\lambda}).$$

From (3.17), the solution of the problem (3.23)-(3.25) is obtained as

$$\begin{aligned} y_\lambda(x) = & \frac{1}{-\sqrt{\lambda} \sin(\sqrt{\lambda})} \left\{ \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \left(1 - x \cos(\sqrt{\lambda}(1-x)) - \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(1-x)) \right) \right. \\ & + \left(\frac{\cos(\sqrt{\lambda}x) \sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \right) \left(x \cos(\sqrt{\lambda}) + \frac{\sin(\sqrt{\lambda})}{\sqrt{\lambda}} \right) \\ & \left. - \frac{1}{\sqrt{\lambda}} \left(x \sin(\sqrt{\lambda}) \cos^2(\sqrt{\lambda}x) + \frac{\cos(\sqrt{\lambda}) \sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \right) \right\} \\ & + \frac{\sin(\sqrt{\lambda}) \cos(\sqrt{\lambda}x) + (2 - \cos(\sqrt{\lambda})) \sin(\sqrt{\lambda}x)}{\sin(\sqrt{\lambda})}. \end{aligned}$$

(ii) The second problem is

$$y'' = 0, \quad (3.29)$$

$$y(0) = (-1, 0, 1), \quad (3.30)$$

$$y(1) = (-1, 0, 1). \quad (3.31)$$

Let $y_1(x) = x$ and $y_2(x) = 1$ be linear independent solutions of the differential equation $y'' = 0$. Then, from (3.18) the solution of fuzzy boundary value problem (3.29) – (3.31) is

$$Y_1(x) = (-1, 0, 1)(1-x) + (-1, 0, 1)x \quad (3.32)$$

(iii) The third problem is

$$y'' + (1, 0, 3)y = (-2, 0, 2), \quad (3.33)$$

$$y(0) = 0, \quad (3.34)$$

$$y(1) = 0. \quad (3.35)$$

The solution of the boundary value problem $y'' + y = -2$, $y(0) = 0$, $y(1) = 0$ is

$$\tilde{y}(x) = 2 \cos(x) + \frac{2(1 - \cos(1))}{\sin(1)} \sin(x) - 2,$$

and the solution of the boundary value problem $y'' + 3y = 2$, $y(0) = 0$, $y(1) = 0$ is

$$\tilde{y}(x) = -2 \cos(\sqrt{3}x) + \frac{2(\cos(\sqrt{3}) - 1)}{\sin(\sqrt{3})} \sin(\sqrt{3}x) + 2.$$

Then, the solution of fuzzy boundary value problem (3.33)-(3.35) is

$$Y_2(x) = \left(\min \left\{ \tilde{y}(x), 0, \tilde{\tilde{y}}(x) \right\}, 0, \max \left\{ \tilde{y}(x), 0, \tilde{\tilde{y}}(x) \right\} \right). \tag{3.36}$$

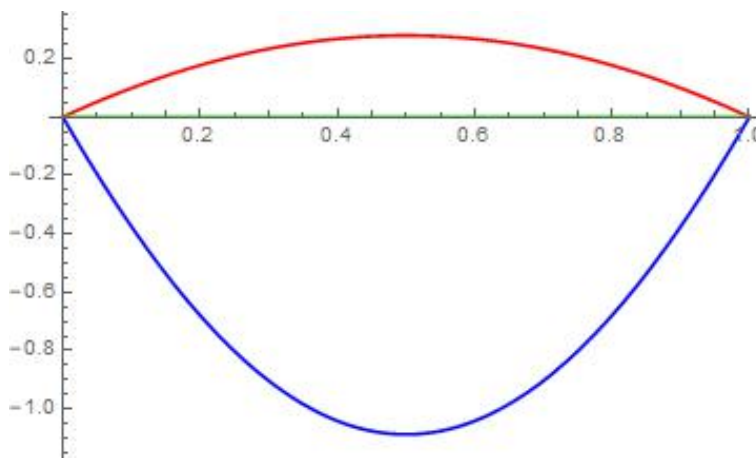


Figure 1 Graphic of (3.36) for $\alpha = 0$ cut (blue and red lines) and $\alpha = 1$ cut (green line)

Consequently, the fuzzy solution of fuzzy boundary value problem (3.20) – (3.22) is

$$y(x) = y_\lambda(x) + Y_1(x) + Y_2(x).$$

§4. Conclusions

In this study, a nonhomogeneous Sturm-Liouville fuzzy problem with fuzzy forcing function is investigated. The problem is solved by a different solution method. Example is solved on studied problem. Examples can be multiplied.

This paper is a new approach to the studied problem.

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