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A Note on the Ratio of two Gamma Functions

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Abstract: We consider the quotient of two gamma functions and for it we obtain a simpler expression than the formula deduced by Bagdasaryan [Appl. Maths. and Comput. 256 (2015)].

Key Words: Gamma function, quotient, Pochhammer-Barnes symbol.

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§1. Introduction

Bagdasaryan [1] showed the following result involving the Pochhammer symbol [2-4] and the gamma function [5-8]:

For $k, p, q \in \mathbb{N}, p > q$, we have

$$\frac{\Gamma\left(\frac{1+qk}{p}\right)}{\Gamma\left(\frac{1-(p-q)k}{p}\right)} = \frac{1-(p-q)k}{p} \sum_{k-1=m_2+\cdots+(k-1)m_k} \frac{(-1)^{k-1+m_2+\cdots+m_k}}{p^{k-1+m_2+\cdots+m_k}} \cdot \frac{(k-1+m_2+\cdots+m_k)!}{m_2!\cdots m_k!} \cdot \left[\frac{(q)_2-(q-p)_2}{2!}\right]^{m_2} \cdots \left[\frac{(q)_{k-1}-(q-p)_{k-1}}{(k-1)!}\right]^{m_{k-1}} \left[\frac{(q)_k-(q-p)_k}{k!}\right]^{m_k}, \tag{1}$$

where the sum runs over all partitions of $(k-1)$".

In this note, we obtain a simpler expression for such ratio of gamma functions.

§2. Ratio of Gamma Functions

In fact, we know [7] the property $\Gamma(z+1) = z\Gamma(z)$. Then, it is natural the following sequence

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of products

$$\begin{aligned}\Gamma\left(\frac{1+qk}{p}\right) &= \Gamma\left(\frac{1-p+qk}{p}+1\right)\\ &= \frac{1-p+qk}{p}\Gamma\left(\frac{1-p+qk}{p}\right)\\ &= \frac{1-p+qk}{p}\Gamma\left(\frac{1-2p+qk}{p}+1\right),\\ &= \frac{1-p+qk}{p}\frac{1-2p+qk}{p}\Gamma\left(\frac{1-2p+qk}{p}\right)\\ &= \frac{1-p+qk}{p}\cdots\frac{1-kp+qk}{p}\Gamma\left(\frac{1-kp+qk}{p}\right),\end{aligned} \tag{2}$$

and therefore

$$\begin{aligned}\frac{\Gamma\left(\frac{1+qk}{p}\right)}{\Gamma\left(\frac{1-(p-q)k}{p}\right)} &= \frac{1}{p^k}(1+qk-p)(1+qk-2p)(1+qk-3p)\cdots(1+qk-kp)\\ &= \frac{1}{p^k}\prod_{r=1}^{k}(1+qk-rp)\\ &= \left[\frac{1-p+qk}{p}\right]_k\\ &= \sum_{m=1}^{k} S_k^{(m)}\left(\frac{1-p+qk}{p}\right)^m,\end{aligned} \tag{3}$$

as an alternative to the expression (1), with the participation of the descending factorial function and the Stirling numbers of the first kind [7], [9]. If we observe the sequence (2) it is clear that (3) is valid for p and q arbitrary real numbers with $p \neq 0$, and $k = 1, 2, 3, \cdots$.

Besides, we have the relation [7] following

$$\frac{\Gamma(\beta)}{\Gamma(\beta-k)} = (-1)^k(1-\beta)_k, \tag{4}$$

whose application for $\beta = \frac{1+qk}{p}$ implies the property

$$\frac{\Gamma\left(\frac{1+qk)}{p}\right)}{\Gamma\left(\frac{1-(p-q)k}{p}\right)} = (-1)^k\left(\frac{p-1-qk}{p}\right)_k, \tag{5}$$

which is compatible with (3) because we have the general relation

$$[x]_k = (-1)^k(-x)_k.$$

Consequently, the equalities (3) and (5) are alternatives to (1).

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