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A Novel Result of Coupled Fixed Point in Partially Ordered Partial Metric Spaces with Applications

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Abstract: In this work, we prove, a novel result of coupled fixed point involving rational type contractive condition in the setting of partially ordered complete partial metric spaces. Furthermore, we provide some consequences of the established results. To illustrate the results, an example is provided. Some applications of the main result and its consequences in terms of integral type contractions are also included. Our results extend, generalize and enrich several results from the existing literature (see, e.g., Bhaskar and Lakshmikantham [9] and others).

Key Words: Coupled fixed point, contractive type condition, partial metric space, partially ordered set.

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§1. Introduction

As we known, Matthews [21] in 1994, introduced the notion of partial metric spaces to study the denotational semantics dataflow networks. In this space, the usual metric is replaced by partial metric with an interesting property that the self-distance of any point of space may not be zero. Later, Matthews proved the partial metric version of Banach fixed point theorem [5]. Heckmann [15] introduced the concept of weak partial metric function and established some fixed point results. Oltra and Valero [24] generalized the Matthews results in the sense of O'Neil [25] in complete partial metric space. Abdeljawad et al. [2] considered a general form of the weak ϕ-contraction and established some common fixed point results. Afterwards, many authors focused on partial metric spaces and its topological properties (see, e.g., ([3], [4], [17], [19], [28]).

On the other hand, Bhashkar and Lakshmikantham [9] (2006) introduced the notion of a coupled fixed point and proved some coupled fixed point theorems for mixed monotone mappings in ordered metric spaces and give application in the existence and uniqueness of a solution for periodic boundary value problem (see, also [14]. Later on, Ćirić and Lakshmikantham [11] (2009) investigated some more coupled fixed point theorems in partially ordered sets. Further, many authors have obtained coupled fixed point results for mappings under various contractive

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conditions in the setting of metric spaces and generalized metric spaces (see [4], [13], [16], [22], [23], [26], [27], [29], [32]).

In this paper, we prove a novel result of coupled fixed point involving rational type contraction condition in the setting of partially ordered complete partial metric spaces and provide some consequences of the established result. Moreover, to illustrate the result, an example is provided. Some applications of the main result and its consequences in terms of integral type contractions are also included. Our results extend, generalize and enrich several results from the existing literature (see, e.g., [9] and others).

§2. Preliminaries

In this section, we recall the notion of partial metric space and some of its basic results which will be needed in the sequel.

Definition 2.1([21]) *Let $Y \neq \emptyset$ be a set. A partial metric on Y is a function $p\colon Y\times Y \to [0,+\infty)$ such that for all $x, y, u \in Y$ the followings are satisfied:*

(p1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(p2) $p(x,x) \leq p(x,y)$;
(p3) $p(x,y) = p(y,x)$;
(p4) $p(x,y) \leq p(x,u) + p(u,y) - p(u,u)$.

Then, p is called a partial metric on Y and the pair (Y,p) is called a partial metric space (in short PMS).

It is clear that if $p(x,y) = 0$, then from $(p1)$, $(p2)$, and $(p3)$, $x = y$. But if $x = y$, $p(x,y)$ may not be 0. Furthermore, if p is a partial metric on Y, then the function $d^p\colon Y\times Y \to [0,+\infty)$ given by

$$d^p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{2.1}$$

is a usual metric on Y.

Each partial metric p on Y generates a T_0 topology τ_p on Y with the family of open p-balls $\{B_p(x,\varepsilon) : x \in Y, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in Y : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in Y$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as $B_p[x,\varepsilon] = \{y \in Y : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in Y$ and $\varepsilon > 0$.

Example 2.2 We know some special cases of PMS in the following:

$(*_1)$ Let $Y = [0,+\infty)$ and $p\colon Y \times Y \to [0,+\infty)$ be given by $p(x,y) = \max\{x,y\}$ for all $x, y \in Y$. Then (Y,p) is a partial metric space (see, [1]).

$(*_2)$ Let $I = Y$, where I denote the set of all intervals $[x_1, y_1]$ for any real numbers $x_1 \leq y_1$. Let $p\colon Y\times Y \to [0,\infty)$ be a function such that $p\big([x_1,y_1],[x_2,y_2]\big) = \max\{y_1,y_2\} - \min\{x_1,x_2\}$. Then (Y,p) is a partial metric space (see, [1]).

$(*_3)$ Let $Y = \mathbb{R}$ and $p\colon Y\times Y \to \mathbb{R}^+$ be given by $p(x,y) = e^{\max\{x,y\}}$ for all $x, y \in Y$. Then (Y,p) is a partial metric space (see, [12]).

Definition 2.3([21]) *Let (Y,p) be a PMS.*

$(a1)$ *A sequence $\{x_n\}$ converges to a point $x \in Y$ if and only if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$;*

$(a2)$ *A sequence $\{x_n\}$ in Y is called a Cauchy sequence if and only if $\lim_{m,n\to\infty} p(x_m,x_n)$ exists (and finite);*

$(a3)$ *A PMS (Y,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in Y converges, with respect to τ_p, to a point $x \in Y$, such that, $\lim_{m,n\to\infty} p(x_m,x_n) = p(x,x)$;*

$(a4)$ *A mapping $S\colon Y \to Y$ is said to be continuous at $x_0 \in Y$ if for every $\varepsilon > 0$, there exists $c > 0$ such that $S\big(B_p(x_0,c)\big) \subset B_p\big(S(x_0),\varepsilon\big)$.*

Definition 2.4([21]) *A PMS (Y,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in Y converges to a point $x \in Y$ with respect to τ_p. Furthermore,*

$$\lim_{m,n\to\infty} p(x_m,x_n) = \lim_{n\to\infty} p(x_n,x) = p(x,x).$$

Definition 2.5([6, 7, 8]) *Consider a function $\psi\colon [0,+\infty) \to [0,+\infty)$ satisfying*

(i) *ψ is monotone increasing;*

(ii) *$\psi^n(t) \to 0$, as $n \to \infty$;*

(iii) *$\sum_{n=0}^{\infty} \psi^n(t)$ converges for all $t > 0$*

and define

(1) *A function ψ satisfying (i) and (ii) above is called a comparison function;*

(2) *A function ψ satisfying (i) and (iii) above is called a (c)-comparison function.*

Remark 2.6([7, 8]) By definition, we know that

(i) Any (c)-comparison function is a comparison function;

(ii) Every comparison function satisfies $\psi(0) = 0$.

Definition 2.7([9]) *Let $(Y,\leq)$ be a partially ordered set. The mapping $F\colon Y \times Y \to Y$ is said to have the mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x,y \in Y$,*

$$x_1,x_2 \in Y,\ \ x_1 \leq x_2 \ \Rightarrow\ F(x_1,y) \leq F(x_2,y),$$

and

$$y_1,y_2 \in Y,\ \ y_1 \leq y_2 \ \Rightarrow\ F(x,y_1) \geq F(x,y_2).$$

Definition 2.8([9 ,11]) *An element $(x,y) \in Y \times Y$ is said to be a coupled fixed point of the mapping $F\colon Y \times Y \to Y$ if $F(x,y) = x$ and $F(y,x) = y$.*

Example 2.9 Let $Y = [0,+\infty)$ and $F\colon Y \times Y \to Y$ be defined by $F(x,y) = \frac{x+y}{3}$ for all $x,y \in Y$. Then F has a unique coupled fixed point $(0,0)$.

Example 2.10 Let $Y = [0,+\infty)$ and $F\colon Y \times Y \to Y$ be defined by $F(x,y) = \frac{x+y}{2}$ for all

$x, y \in Y$. Then F has two coupled fixed point $(0,0)$ and $(1,1)$, that is, the coupled fixed point is not unique.

Lemma 2.11([4, 20, 21]) *(b1) A sequence $\{x_n\}$ is Cauchy in a PMS (Y,p) if and only if $\{x_n\}$ is Cauchy in a metric space (Y, d^p) where*

$$d^p(x,y) = 2p(x,y) - p(x,x) - p(y,y).$$

(b2) A PMS (Y,p) is complete if a metric space (Y, d^p) is complete, i.e.,

$$\lim_{n\to\infty} d^p(x_n, x) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Lemma 2.12([18]) *Let (Y,p) be a PMS.*

(c1) If $x, y \in Y$, $p(x,y) = 0$, then $x = y$;
(c2) If $x \neq y$, then $p(x,y) > 0$.

One of the characterization of continuity of mappings in partial metric spaces was given by Samet et al. [30] as follows.

Lemma 2.13([30]) *Let (Y,p) be a PMS. The function $R\colon Y \to Y$ is continuous if given a sequence $\{x_n\}_{n\in\mathbb{N}}$ and $x \in Y$ such that $p(x,x) = \lim_{n\to\infty} p(x, x_n)$, then $p(Rx, Rx) = \lim_{n\to\infty} p(Rx, Rx_n)$.*

Example 2.14([30]) Let $Y = [0, +\infty)$ endowed with the partial metric $p\colon Y \times Y \to [0, +\infty)$ defined by $p(x,y) = \max\{x,y\}$ for all $x, y \in Y$. Let $R\colon Y \to Y$ be a non-decreasing function. If R is continuous with respect to the standard metric $d(x,y) = |x - y|$ for all $x, y \in Y$, then R is continuous with respect to the partial metric p.

Example 2.15([12]) Let $x_n \to x$ as $n \to \infty$ in a PMS (Y,p) where $p(x,x) = 0$. Then $\lim_{n\to\infty} p(x_n, u) = p(x,u)$ for all $u \in Y$.

§3. Main Results

In this section, we shall prove a novel coupled fixed point result involving rational type contraction condition in the setting of partially ordered complete partial metric spaces.

Theorem 3.1 *Let $(Y, p, \leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \geq 0$, for all $a, b, u, v \in Y$, $p(u, F(u,v)) + p(a,u) > 0$ and ψ, a (c)-comparison function, we have*

$$p(F(a,b), F(u,v)) \leq \beta\Big(\frac{p(a, F(a,b))p(a, F(u,v))p(u, F(a,b))}{1 + p(u, F(u,v)) + p(a,u)}\Big) + \psi(p(a,u)). \tag{3.1}$$

If there exist two elements $a_0, b_0 \in Y$ with $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F has a coupled fixed point in Y with $p(z, z) = 0$ for some $z \in Y$.

Proof Choose $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$. We are to prove that a_k is non-decreasing and b_k is non-increasing. That is, for all $k \geq 0$,

$$a_{2k} \leq a_{2k+1} \leq a_{2k+2} \text{ and } b_{2k} \leq b_{2k+1} \leq b_{2k+2}.$$

Firstly, $a_0 \leq F(a_0, b_0) = a_1$ and $b_0 \geq F(b_0, a_0) = b_1$. Then, $a_0 \leq a_1$ and $b_0 \geq b_1$. Again, let $a_2 = F(a_1, b_1)$ and $b_2 = F(b_1, a_1)$. Since F has the mixed monotone property on Y, then we have $a_1 \leq a_2$ and $b_1 \geq b_2$. Repeating the above process, we get two sequences $\{a_k\}$ and $\{b_k\}$ in Y such that $a_{2k+1} = F(a_{2k}, b_{2k})$ and $b_{2k+1} = F(b_{2k}, a_{2k})$ for all $k \geq 0$ and

$$a_0 \leq a_1 \leq \cdots \leq a_{2k} \leq a_{2k+1} \leq \cdots, \quad b_0 \geq b_1 \geq \cdots \geq b_{2k} \geq b_{2k+1} \geq \cdots. \tag{3.2}$$

Now, using equation (3.1) with $a = a_{2k}$, $b = b_{2k}$, $u = a_{2k+1}$ and $v = b_{2k+1}$, we have

$$\begin{aligned} p(a_{2k+1}, a_{2k+2}) &= p\big(F(a_{2k}, b_{2k}), F(a_{2k+1}, b_{2k+1})\big) \\ &\leq \beta\Big(\frac{p(a_{2k}, F(a_{2k}, b_{2k}))p(a_{2k}, F(a_{2k+1}, b_{2k+1}))p(a_{2k+1}, F(a_{2k}, b_{2k}))}{1 + p(a_{2k+1}, F(a_{2k+1}, b_{2k+1})) + p(a_{2k}, a_{2k+1})}\Big) \\ &\quad + \psi(p(a_{2k}, a_{2k+1})) \\ &= \beta\Big(\frac{p(a_{2k}, a_{2k+1})p(a_{2k}, a_{2k+2})p(a_{2k+1}, a_{2k+1})}{1 + p(a_{2k+1}, a_{2k+2}) + p(a_{2k}, a_{2k+1})}\Big) \\ &\quad + \psi(p(a_{2k}, a_{2k+1})) \\ &= \psi(p(a_{2k}, a_{2k+1})). \end{aligned}$$

Therefore,

$$p(a_{2k+1}, a_{2k+2}) \leq \psi(p(a_{2k}, a_{2k+1})) \tag{3.3}$$

and

$$p(b_{2k+1}, b_{2k+2}) \leq \psi(p(b_{2k}, b_{2k+1})). \tag{3.4}$$

Similarly, proceeding as above we obtain

$$p(a_{2k+2}, a_{2k+3}) \leq \psi(p(a_{2k+1}, a_{2k+2})) \tag{3.5}$$

and

$$p(b_{2k+2}, b_{2k+3}) \leq \psi(p(b_{2k+1}, b_{2k+2})). \tag{3.6}$$

Hence, it can be deduced from equations (3.3)-(3.6) that

$$\begin{aligned} p(a_{2k+2}, a_{2k+3}) + p(b_{2k+2}, b_{2k+3}) &\leq \psi(p(a_{2k+1}, a_{2k+2})) + \psi(p(b_{2k+1}, b_{2k+2})) \\ &\leq \psi^2(p(a_{2k}, a_{2k+1})) + \psi^2(p(b_{2k}, b_{2k+1})) \\ &\leq \cdots \leq \psi^n(p(a_0, a_1)) + \psi^n(p(b_0, b_1)). \end{aligned}$$

Thus, it follows that

$$p(a_n, a_{n+1}) + p(b_n, b_{n+1}) \leq \psi^n(p(a_0, a_1)) + \psi^n(p(b_0, b_1)). \tag{3.7}$$

Again, let $n, r \in \mathbb{N}$. Using equation (3.7) inductively and repeated application of triangular inequality, we obtain

$$\begin{aligned}
p(a_n, a_{n+r}) + p(b_n, b_{n+r}) &\leq [p(a_n, a_{n+1}) + p(b_n, b_{n+1})] + [p(a_{n+1}, a_{n+2}) + p(b_{n+1}, b_{n+2})] \\
&\quad + \cdots + [p(a_{n+r-1}, a_{n+r}) + p(b_{n+r-1}, b_{n+r})] \\
&\quad - [p(a_{n+1}, a_{n+1}) + p(b_{n+1}, b_{n+1})] \\
&\quad - [p(a_{n+2}, a_{n+2}) + p(b_{n+2}, b_{n+2})] - \cdots \\
&\quad - [p(a_{n+r-1}, a_{n+r-1}) + p(b_{n+r-1}, b_{n+r} - 1)] \\
&\leq [p(a_n, a_{n+1}) + p(b_n, b_{n+1})] + [p(a_{n+1}, a_{n+2}) + p(b_{n+1}, b_{n+2})] \\
&\quad + \cdots + [p(a_{n+r-1}, a_{n+r}) + p(b_{n+r-1}, b_{n+r})] \\
&\leq \psi^n(p(a_0, a_1)) + \psi^n(p(b_0, b_1)) + \psi^{n+1}(p(a_0, a_1)) + \psi^{n+1}(p(b_0, b_1)) \\
&\quad + \cdots + \psi^{n+r-1}(p(a_0, a_1)) + \psi^{n+r-1}(p(b_0, b_1)) \\
&= \sum_{j=n}^{n+r-1} \psi^j(p(a_0, a_1)) + \sum_{j=n}^{n+r-1} \psi^j(p(b_0, b_1)) \\
&= \sum_{j=0}^{n+r-1} \psi^j(p(a_0, a_1)) - \sum_{j=0}^{n-1} \psi^j(p(a_0, a_1)) \\
&\quad + \sum_{j=0}^{n+r-1} \psi^j(p(b_0, b_1)) - \sum_{j=0}^{n-1} \psi^j(p(b_0, b_1)) \to 0 \ \text{ as } \ j \to \infty
\end{aligned} \tag{3.8}$$

and since ψ is a (c)-comparison function. Hence, $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in a PMS (Y, p) such that $\lim_{m,n\to\infty} p(a_m, a_n) = 0$ and $\lim_{m,n\to\infty} p(b_m, b_n) = 0$ where $m = n + r$ with $m > n \in \mathbb{N}$. Again, since (Y, p) is a complete partial metric space, there exist $a^*, b^* \in Y$ such that

$$p(a^*, a^*) = \lim_{n\to\infty} p(a_n, a^*) = \lim_{n\to\infty} p(a_n, a_n) = 0 \tag{3.9}$$

and

$$p(b^*, b^*) = \lim_{n\to\infty} p(b_n, b^*) = \lim_{n\to\infty} p(b_n, b_n) = 0. \tag{3.10}$$

Now, we have to show that (a^*, b^*) is a coupled fixed point of F, that is, $a^* = F(a^*, b^*)$ and $b^* = F(b^*, a^*)$. Using contractive condition (3.1) again and noting that $\psi(0) = 0$, we have

$$\begin{aligned}
p(F(a^*, b^*), a^*) &\leq p(F(a^*, b^*), a_{2n+1}) + p(a_{2n+1}, a^*) - p(a_{2n+1}, a_{2n+1}) \\
&\leq p(F(a^*, b^*), a_{2n+1}) + p(a_{2n+1}, a^*) \\
&= p(F(a^*, b^*), F(a_{2n}, b_{2n})) + p(a_{2n+1}, a^*) \\
&\leq \beta\Big(\frac{p(a^*, F(a^*, b^*))p(a^*, F(a_{2n}, b_{2n}))p(a_{2n}, F(a^*, b^*))}{1 + p(a_{2n}, F(a_{2n}, b_{2n})) + p(a^*, a_{2n})}\Big) \\
&\quad + \psi(p(a^*, a_{2n})) + p(a_{2n+1}, a^*)
\end{aligned}$$

$$\begin{aligned} &\leq \beta\Big(\frac{p(a^*, F(a^*, b^*))p(a^*, a_{2n+1})p(a_{2n}, F(a^*, b^*))}{1 + p(a_{2n}, a_{2n+1}) + p(a^*, a_{2n})}\Big) \\ &\quad +\psi(p(a^*, a_{2n})) + p(a_{2n+1}, a^*) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $p(F(a^*, b^*), a^*) = 0$ and so $F(a^*, b^*) = a^*$. Similarly, by using inequality (3.1), we can show that $p(F(b^*, a^*), b^*) = 0$ and so $F(b^*, a^*) = b^*$. Hence, (a^*, b^*) is a coupled fixed point of F. This completes the proof. □

Example 3.2 Let $Y = [0, 4]$ be endowed with the usual partial metric given by $p(x, y) = \max\{x, y\}$ for all $x, y \in Y$. Let $\psi(t) = \frac{1}{2}t$ for all $t \in Y$. Clearly, $\psi(t)$ is a (c)-comparison function. Define $F\colon [0, 4] \times [0, 4] \to [0, 4]$ by $F(a, b) = 3a - 2b$ for all $a, b \in [0, 4]$. Clearly, F has the mixed monotone property.

Let $a_0 = \frac{2}{3}$, $b_0 = \frac{1}{2} \in Y$.

$$F(a_0, b_0) = F\Big(\frac{2}{3}, \frac{1}{2}\Big) = 1 \text{ and } F(b_0, a_0) = F\Big(\frac{1}{2}, \frac{2}{3}\Big) = \frac{1}{6}.$$

Thus,

$$\frac{2}{3} < 1 \text{ and } \frac{1}{2} > \frac{1}{6}.$$

Hence, $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$.

Now, consider the contractive condition (3.1) in Theorem 3.1. Let $a = \frac{4}{3}$, $b = \frac{1}{3}$, $\beta = 2$, $u = \frac{4}{5}$, $v = \frac{1}{5}$ and assume that $a \geq b \geq u \geq v$. Then

$$F(a, b) = F\Big(\frac{4}{3}, \frac{1}{3}\Big) = 3\Big(\frac{4}{3}\Big) - 2\Big(\frac{1}{3}\Big) = 4 - \frac{2}{3} = \frac{10}{3},$$

$$F(u, v) = F\Big(\frac{4}{5}, \frac{1}{5}\Big) = 3\Big(\frac{4}{5}\Big) - 2\Big(\frac{1}{5}\Big) = \frac{12}{5} - \frac{2}{5} = 2,$$

$$p(a, F(a, b)) = p\Big(\frac{4}{3}, \frac{10}{3}\Big) = \max\Big\{\frac{4}{3}, \frac{10}{3}\Big\} = \frac{10}{3},$$

$$p(u, F(u, v)) = p\Big(\frac{4}{5}, 2\Big) = \max\Big\{\frac{4}{5}, 2\Big\} = 2,$$

$$p(u, F(a, b)) = p\Big(\frac{4}{5}, \frac{10}{3}\Big) = \max\Big\{\frac{4}{5}, \frac{10}{3}\Big\} = \frac{10}{3},$$

$$p(a, F(u, v)) = p\Big(\frac{4}{3}, 2\Big) = \max\Big\{\frac{4}{3}, 2\Big\} = 2,$$

$$p(a, u) = p\Big(\frac{4}{3}, \frac{4}{5}\Big) = \max\Big\{\frac{4}{3}, \frac{4}{5}\Big\} = \frac{4}{3}.$$

Now, applying contractive condition (3.1), that is,

$$p(F(a, b), F(u, v)) \leq \beta\Big(\frac{p(a, F(a, b))p(a, F(u, v))p(u, F(a, b))}{1 + p(u, F(u, v)) + p(a, u)}\Big) + \psi(p(a, u)),$$

implies that

$$\begin{aligned} p\Big(\frac{10}{3},2\Big) &= \max\Big\{\frac{10}{3},2\Big\} = \frac{10}{3} = 3.333 \\ &\le 2\Big(\frac{\frac{10}{3}.2.\frac{10}{3}}{1+2+\frac{4}{3}}\Big) + \frac{1}{2}.\frac{4}{3} \\ &= \frac{400}{69} + \frac{2}{3} = \frac{1338}{207} = 6.643, \end{aligned}$$

i.e. $3.333 \le 6.643$.

Since all the assumptions of Theorem 3.1 are satisfied, so F has a coupled fixed point in $Y = [0,4]$.

If we take $\psi(t) = \gamma t$ for all $t > 0$ where $\gamma \in [0,1)$ in Theorem 3.1, then we have the following result.

Corollary 3.3 *Let $(Y,p,\le)$ be a partially ordered complete partial metric space. Let $F\colon Y\times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \ge 0$, $\gamma \in [0,1)$ and for all $a,b,u,v \in Y$, where $p(u,F(u,v)) + p(a,u) > 0$, we have*

$$p(F(a,b),F(u,v)) \le \beta\Big(\frac{p(a,F(a,b))p(a,F(u,v))p(u,F(a,b))}{1+p(u,F(u,v))+p(a,u)}\Big) + \gamma\, p(a,u). \tag{3.11}$$

If there exist two elements $a_0, b_0 \in Y$ with $a_0 \le F(a_0,b_0)$ and $b_0 \ge F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z) = 0$ for some $z \in Y$.

Proof It follows from Theorem 3.1 by taking $\psi(t) = \gamma t$ for all $t > 0$ where $\gamma \in [0,1)$. □

If we take $\gamma = 0$ in Corollary 3.3, then we have the following result.

Corollary 3.4 *Let $(Y,p,\le)$ be a partially ordered complete partial metric space. Let $F\colon Y\times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \ge 0$ and for all $a,b,u,v \in Y$, where $p(u,F(u,v)) + p(a,u) > 0$, we have*

$$p(F(a,b),F(u,v)) \le \beta\Big(\frac{p(a,F(a,b))p(a,F(u,v))p(u,F(a,b))}{1+p(u,F(u,v))+p(a,u)}\Big). \tag{3.12}$$

If there exist $a_0, b_0 \in Y$ such that $a_0 \le F(a_0,b_0)$ and $b_0 \ge F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z) = 0$ for some $z \in Y$.

Proof It follows from Corollary 3.3 by taking $\gamma = 0$. □

If we take $\beta = 0$ in Corollary 3.3, then we have the following result.

Corollary 3.5 *Let $(Y,p,\le)$ be a partially ordered complete partial metric space. Let $F\colon Y\times Y \to Y$ be a mapping having the mixed monotone property such that for some $\gamma \in [0,1)$ and for all $a,b,u,v \in Y$, we have*

$$p(F(a,b),F(u,v)) \le \gamma\, p(a,u). \tag{3.13}$$

If there exist $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F has a coupled fixed point in Y with $p(z, z) = 0$ for some $z \in Y$.

If we take $\beta = 0$ in Theorem 3.1, then we have the following result.

Corollary 3.6 *Let $(Y, p, \leq)$ be a partially ordered complete partial metric space. Let $F \colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for all $a, b, u, v \in Y$, $p(u, F(u, v)) + p(a, u) > 0$ and ψ, a (c)-comparison function, we have*

$$p(F(a,b), F(u,v)) \leq \psi(p(a,u)). \tag{3.14}$$

If there exist two elements $a_0, b_0 \in Y$ with $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F has a coupled fixed point in Y.

If we define as $Ta = F(a, a)$ in Corollary 3.5, then we have the following result.

Corollary 3.7([21],Banach's fixed point theorem) *Let (Y, p) be a complete partial metric space. Suppose that the mapping $T \colon Y \to Y$ satisfies the following contractive condition for all $a, u \in Y$:*

$$p(Ta, Tu) \leq \gamma\, p(a, u), \tag{3.15}$$

where $\gamma \in [0, 1)$ is a constant. Then T has a unique fixed point in Y.

Remark 3.8 Theorem 3.1 generalizes and extends the corresponding result of Bhaskar and Lakshmikantham [9] from complete metric space to complete partial metric space.

Remark 3.9 Theorem 3.1 also generalizes and extends the corresponding result of Sabetghadam et al. [31] to the coupled fixed point setting in partially ordered space, the latter consisting of coupled fixed point in partial cone metric space setting.

§4. Applications

In this part, applications of the main result and its consequences in terms of integral type contractions are carried out. Let Ψ denote the set of functions $\varphi \colon [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

(Δ_1) φ is a Lebesgue-integrable function on every compact subset in $[0, +\infty)$ and;

(Δ_2) $\int_0^{\varepsilon} \varphi(\mu) d\mu > 0$, for all $\varepsilon > 0$.

Then, we have the following applications of our results.

Theorem 4.1 *Let $(Y, p, \leq)$ be a partially ordered complete partial metric space. Let $F \colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \geq 0$, $p(u, F(u, v)) +$*

$p(a,u) > 0$ and ψ, a (c)-comparison function, we have

$$\int_0^{p(F(a,b),F(u,v))} \chi(\mu)d\mu \leq \beta \int_0^{\left(\frac{p(a,F(a,b))p(a,F(u,v))p(u,F(a,b))}{1+p(u,F(u,v))+p(a,u)}\right)} \chi(\mu)d\mu + \int_0^{\psi(p(a,u))} \chi(\mu)d\mu \tag{4.1}$$

for all $a,b,u,v \in Y$ such that $a \geq u$ and $b \leq v$, where $\chi \in \Psi$. If there exist $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z) = 0$ for some $z \in Y$.

If $\psi(t) = \gamma t$ for all $t > 0$, where $\gamma \in [0,1)$ in Theorem 4.1, then we obtain the following result.

Theorem 4.2 *Let $(Y,p,\leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \geq 0$ and $\gamma \in [0,1)$, where $p(u,F(u,v)) + p(a,u) > 0$, we have*

$$\int_0^{p(F(a,b),F(u,v))} \chi(\mu)d\mu \leq \beta \int_0^{\left(\frac{p(a,F(a,b))p(a,F(u,v))p(u,F(a,b))}{1+p(u,F(u,v))+p(a,u)}\right)} \chi(\mu)d\mu + \gamma \int_0^{p(a,u)} \chi(\mu)d\mu \tag{4.2}$$

for all $a,b,u,v \in Y$ such that $a \geq u$ and $b \leq v$, where $\chi \in \Psi$. If there exist $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z) = 0$ for some $z \in Y$.

If $\beta = 0$ in Theorem 4.1, then we obtain the following result.

Theorem 4.3 *Let $(Y,p,\leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that $p(u,F(u,v)) + p(a,u) > 0$ and ψ, a (c)-comparison function, we have*

$$\int_0^{p(F(a,b),F(u,v))} \chi(\mu)d\mu \leq \int_0^{\psi(p(a,u))} \chi(\mu)d\mu, \tag{4.3}$$

for all $a,b,u,v \in Y$ such that $a \geq u$ and $b \leq v$, where $\chi \in \Psi$. If there exist $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z) = 0$ for some $z \in Y$.

If $\gamma = 0$ in Theorem 4.2, then we obtain the following result.

Theorem 4.4 *Let $(Y,p,\leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \geq 0$ and*

$p(u,F(u,v))+p(a,u)>0$, we have

$$\int_0^{p(F(a,b),F(u,v))}\chi(\mu)d\mu\le\beta\int_0^{\left(\frac{p(a,F(a,b))p(a,F(u,v))p(u,F(a,b))}{1+p(u,F(u,v))+p(a,u)}\right)}\chi(\mu)d\mu, \tag{4.4}$$

for all $a,b,u,v\in Y$ such that $a\ge u$ and $b\le v$, where $\chi\in\Psi$. If there exist $a_0,b_0\in Y$ such that $a_0\le F(a_0,b_0)$ and $b_0\ge F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z)=0$ for some $z\in Y$.

If $\beta=0$ in Theorem 4.2, then we obtain the following result.

Theorem 4.5 *Let $(Y,p,\le)$ be a partially ordered complete partial metric space. Let $F\colon Y\times Y\to Y$ be a mapping having the mixed monotone property such that for some $\gamma\in[0,1)$, we have*

$$\int_0^{p(F(a,b),F(u,v))}\chi(\mu)d\mu\le\gamma\int_0^{p(a,u)}\chi(\mu)d\mu, \tag{4.5}$$

for all $a,b,u,v\in Y$ such that $a\ge u$ and $b\le v$, where $\chi\in\Psi$. If there exist $a_0,b_0\in Y$ such that $a_0\le F(a_0,b_0)$ and $b_0\ge F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z)=0$ for some $z\in Y$.

Remark 4.6 Theorem 4.5 extends and generalizes the corresponding result of Branciari [10] from complete metric spaces to partially ordered complete partial metric spaces and coupled fixed point.

§5. Conclusion

In this article, we prove a novel coupled fixed point result for contractive type condition involving rational term in the setting of partially ordered complete partial metric spaces. Moreover, we give some consequences of the established results and provide an illustrative example in support of the established result. Some applications of the main result and its consequences in terms of integral type contractions are also included. The results presented in this paper extend and generalize several results in the existing literature (see, e.g., [9] and others).

References

[1] H. Aydi, M. Abbas and C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, *Topology and Its Appl.*, 159 (2012), No. 14, 3234-3242.

[2] T. Abdeljawad, E, Karap谋nar and K. Tas, Existence and uniqueness of a common fixed point on partial metric spaces, *Appl. Math. Lett.*, 24(11) (2011), 1894-1899.

[3] I. Altun and A. Erduran, Fixed point theorems for monotone mappings on partial metric s-paces, *Fixed Point Theory Appl.*, 2011, Article ID 508730 (2011). doi:10.1155/2011/508730

[4] H. Aydi, Some coupled fixed point results on partial metric spaces, *International J. Math. Math. Sci.*, 2011, Article ID 647091, 11 pages.

[5] S. Banach, Surles operation dans les ensembles abstraits et leur application aux equation integrals, *Fund. Math.*, 3(1922), 133-181.

[6] V. Berinde, Iterative approximation of fixed points, Editura Efemeride, 2002.

[7] V. Berinde, Approximation of fixed points, Volume (1912) of *Lecture Notes in Mathematics*, Springer, Berlin, Second edition 2007.

[8] V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, *Maths F. A.*, ar Xiv:1103-5285vi (2012).

[9] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis: TMA*, 65(7) (2006), 1379-1393.

[10] A. Branciari, A fixed point theorem for mappings satisfying a general contractive coof integral type, *Int. J. Math. Math. Sci.*, 29(9) (2002), 531-536.

[11] L. Ciric and V. Lakshmikantham, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis: TMA*, 70(12) (2009), 4341-4349.

[12] S. Chandok, D. Kumar and M. S. Khan, Some results in partial metric space using auxiliary functions, *Applied Math. E-Notes*, 15 (2015), 233-242.

[13] R. O. Gregorio, Coupled fixed point theorems in partially ordered metric spaces, *Adv. Fixed Point Theory*, 9(1) (2019), 17-28.

[14] D. Guo and V. Lakshmikantham, Coupled fixed point of nonlinear operator with application, *Nonlinear Anal. TMA.*, 11 (1987), 623-632.

[15] R. Heckmann, Approximation of metric spaces by partial metric spaces, *Appl. Categ. Structures*, 7, No. 1-2, (1999), 71-83.

[16] Y. Jiang and Feng Gu, Common coupled fixed point results in multivalued metric spaces and applications, *J. Nonlinear Sci. Appl.*, 10 (2017), 1881-1895.

[17] E. Karapınar and I. M. Erhan, Fixed point theorems for operators on partial metric spaces, *Appl. Math. Lett.*, 24(11) (2011), 1900-1904.

[18] E. Karapınar, W. Shatanawi and K. Tas, Fixed point theorems on partial metric spaces involving rational expressions, *Miskolc Math. Notes*, 14 (2013), 135-142.

[19] E. Karapınar and R. P. Agarwal, *Fixed Point Theory in Generalized Metric Spaces*, Springer, 2022.

[20] S. G. Matthews, *Partial Metric Topology*, Research report 2012, Dept. Computer Science, University of Warwick, 1992.

[21] S. G. Matthews, Partial metric topology, *Proceedings of the 8th Summer Conference on Topology and its Applications, Annals of the New York Academy of Sciences*, 728 (1994), 183-197.

[22] H. K. Nashine and W. Shatanawi, Coupled common fixed point theorems for pair of commuting in partially ordered complete metric spaces, *Comput. Math. Appl.*, 62 (2011), 1984-1993.

[23] H. K. Nashine, Coupled common fixed point results in ordered G-metric spaces, *J. Nonlinear Sci. Appl.*, 1 (2012), 1-13.

[24] S. Oltra and O. Valero, Banach's fixed point theorem for partial metric spaces, *Rend. Ist. Mat. Univ. Trieste*, 36(1-2) (2004), 17-26.

[25] S. J. O'Neil, Partial metrics, valuations and domain theory, in: *Proc. 11th summer conference on general topology and application*, in *Annals of the New York Academy of Sciences*, Vol. 806 (1996), 304-315.

[26] M. Paknazar, M. Eshaghi Gordji and S. M. Vaezpour, Coupled fixed point theorems in partially ordered complex valued metric spaces with application, *Int. J. Nonlinear Anal. Appl.*, 13(2) (2022), 13-20.

[27] S. Radenov ́ıć, Coupled fixed point theorems for monotone mappings in partially ordered metric spaces, *Kragujevac J. Maths.*, 38(2) (2014), 249-257.

[28] N. S. Rao, K. Kalyani and T. Gemechu, Fixed point theorems in partially ordered metric spaces with rational expressions, *Inf. Sci. Lett.*, 10(3) (2021), 451-460.

[29] K. Ravibabu, G. N. V. Kishore, Ch. Shrinivasa Rao and Ch. Raghavendra Naidu, Coupled fixed point theorems via mixed monotone property in A_b-metric spaces and applications to integral equations, *J. Siberian Federal Univ. Maths. and Physics*, 15(3) (2022), 343-356.

[30] B. Samet, M. Rajović, R. Lazović and R. Stoiljković, Common fixed point results for nonlinear contractions in ordered partial metric spaces, *Fixed Point Theory Appl.*, 2011, 2011:71.

[31] F. Sabetghadam, H. P. Mashiha and A. H. Sanatpour, Some coupled fixed point theorems in cone metric spaces, *Fixed Point Theory Appl.*, (2009), Article ID 125426, 8 pages.

[32] W. Shatanawi, B. Samet and M. Abbas, Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, *Math. Comput. Modell.* 55 (2012), Nos. 3-4, 680-687.