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A Short Note on an Identity of Spivey for Bell Numbers

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Abstract: Spivey obtained an identity for Bell numbers, here we give an elementary proof of it and we show that it gives a recurrence relation for $\sum_{j=0}^{n} j^m S_n^{[j]}$, which shows that these quantities involving the Stirling numbers of the second kind are linear combination of the $B(k)$.

Key Words: Spivey's identity, Bell numbers, Dobinski's formula, Stirling numbers.

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§1. Introduction

Spivey [1]-[4] gave a combinatorial proof of the identity following

$$B(m+n) = \sum_{j=0}^{m} \sum_{k=0}^{n} j^{n-k} \binom{n}{k} S_m^{[j]} B(k), \tag{1}$$

for the Bell numbers [5],[7] and

$$B(n) \equiv \sum_{q=0}^{n} S_n^{[q]}, \tag{2}$$

where $S_n^{[q]}$ is a Stirling number of the second kind [6]-[12].

On the other hand, we know the Dobinski's formula [6], [7], [13]-[15] following

$$\sum_{q=0}^{n} S_n^{[q]} x^q = e^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!} x^k, \quad n \geq 0, \tag{3}$$

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which for $x = 1$ implies the expression

$$B(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}. \tag{4}$$

Spivey [1] comments that the quantities

$$A_r := \sum_{j=0}^{m} j^r S_m^{[j]}, \quad r \geq 0, \tag{5}$$

for a given $m = 0, 1, \ldots$, can be expressed as a linear combination of Bell numbers. In Sec. 2 we use (4) to give an elementary proof of (1), and we deduce a recurrence relation for (5), in harmony with this affirmation of Spivey.

§2. Spivey's Identity

Here we exhibit a simple demonstration of (1). First, we perform the following calculation

$$\sum_{k=j}^{\infty} \frac{k^n}{(k-j)!} = \sum_{q=0}^{\infty} \frac{(q+j)^n}{q!} = \sum_{r=0}^{n} \binom{n}{r} j^{n-r} \sum_{q=0}^{\infty} \frac{q^r}{q!} \overset{(4)}{=} e \sum_{r=0}^{n} \binom{n}{r} j^{n-r} B(r). \tag{6}$$

Besides, let's remember the property ([6], [16])

$$k^m = \sum_{j=0}^{k} \binom{k}{j} j! S_m^{[j]} \tag{7}$$

Then

$$\begin{aligned} B(m+n) &\overset{(4)}{=} \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} k^m \\ &\overset{(7)}{=} \frac{1}{e} \sum_{j=0}^{m} j! S_m^{[j]} \sum_{k=j}^{\infty} \binom{k}{j} \frac{k^n}{k!} \\ &= \frac{1}{e} \sum_{j=0}^{m} S_m^{[j]} \sum_{k=j}^{\infty} \frac{k^n}{(k-j)!} \\ &\overset{(6)}{=} \sum_{j=0}^{m} \sum_{r=0}^{n} j^{n-r} \binom{n}{r} S_m^{[j]} B(r), \end{aligned} \tag{6}$$

in according with (1). □

From (1),

$$B(m+n) = \sum_{r=0}^{n} \binom{n}{r} B(n-r) \sum_{j=0}^{m} j^r S_m^{[j]} \overset{(5)}{=} \sum_{r=0}^{n} \binom{n}{r} B(n-r) A_r, \tag{8}$$

then in this recurrence relation for the quantities (5) we can employ $n = 0, 1, 2, \cdots$ to obtain each A_r as a linear combination of Bell numbers. In fact, for a given integer m,

$$\begin{aligned} A_0 &= B(m), \\ A_1 &= B(m+1) - B(m), \\ A_2 &= B(m+2) - 2B(m+1), \\ A_3 &= B(m+3) - 3B(m+2) + B(m), \text{and so on} \end{aligned} \tag{9}$$

Now, the Euler's operator $\left(x\frac{d}{dx}\right)^m$ ([6],[15],[16],[17]) can be applied to (3) to deduce the following explicit formula for (5),

$$A_r = \sum_{k=0}^{m} \binom{m}{k} \sum_{j=0}^{r} j! S_r^{[j]} S_{m-k}^{[j]} B(k), \tag{10}$$

which is compatible with the values (9); we can consider that (10) is the inversion of (8). The combination of (9) and (10) implies interesting identities. For example,

$$\begin{aligned} \sum_{k=0}^{n-1} 2^{n-k} \binom{n}{k} B(k) &= B(n+2) - B(n+1) - B(n), \\ \sum_{k=0}^{n-1} 3^{n-k} \binom{n}{k} B(k) &= B(n+3) - 3B(n+2) + 2B(n+1) - B(n). \end{aligned} \tag{11}$$

Spivey [18] obtained the following property

$$\sum_{k=0}^{n} (-1)^k k^m S_n^{(k)} = \sum_{j=0}^{m} (-1)^j j! S_m^{[j]} S_{n+1}^{(j+1)}, \tag{12}$$

which can be seen as companion of (10), and for $m = 1$ gives the known relation for the harmonic numbers ([6], [19], [20])

$$H_n = \frac{(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k k S_n^{(k)} = \frac{(-1)^{n+1}}{n!} S_{n+1}^{(2)}, \tag{13}$$

in terms of Stirling numbers of the first kind ([3], [6], [7], [19], [20]).

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