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Automorphic and Holomorphic Characterization of Crazy Loops

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Abstract: We study the loop variety defined by the identity $(x \cdot (xy)) \cdot z = (y \cdot x) \cdot (x \cdot z)$, which we call *Crazy loops*. We show that Crazy loops are flexible, satisfy left and right inverse properties, and obey the square-symmetric law. For a loop Q with automorphism group $A(Q)$, we characterise when its $A(Q)$-holomorph is itself a Crazy loop: this occurs precisely when the automorphism group satisfies $\alpha^2 \in C_{A(Q)}(\beta)$ for all $\alpha, \beta \in A(Q)$ and when $(x\beta \cdot (x\beta)y)z = yx \cdot xz$ holds in Q for all $x, y, z \in Q$. In particular, it is proved that the $A(Q)$ holomorph $(H, \circ)$ of a loop $(Q, \cdot)$ is a Crazy loop if and only if $(Q, \cdot)$ is a Crazy loop and $(x\beta)^2 \cdot x^\lambda \in N_\rho$. We also describe explicit autotopisms induced by the Crazy identity and show how these restrict to an automorphism of the nucleus. Finally we showed that if $(H, \circ)$ is the $A(Q)$ holomorph of a Crazy loop that is an extra loop and $x\beta \cdot x^\lambda \in N_\lambda$. Then $T(x)^{-1}$ is an automorphism.

Key Words: Loop, Crazy loop, generalized Bol-Moufang type loops, holomorph, automorphism.

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§1. Introduction

An identity $\alpha = \beta$ is said to be of *Bol - Moufang type* (first Bol-Mog type [8]) if it involves three distinct variables in which two variables occur once on each side while the third occurs twice, and the variables appear in the same relative order on both sides [26, 27]. The broader family of *generalized Bol - Moufang* identities arises when the variables do not appear in the same order on both sides of the equal sign. A complete classification of such identities was carried out by Côté, et.al [5], who listed all admissible identities and identified a number of loop varieties defined by them.

Several loop varieties belonging to this scheme have been examined in recent work [6, 18, 20, 21, 23, 25]. Examples include Frute loops, Cheban loops and Mate-1 loops. Among the identities appearing in the classification of Côté et al. is the Crazy identity defined by

$$(x \cdot (xy))z = (yx) \cdot (xz),$$

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The study of holomorphs has provided a powerful framework for analysing loop varieties. Pioneering contributions of Bruck [2] established the role of holomorphs in inverse property loops, while subsequent works by Robinson [28, 29], Huthnance [14], Adeniran [1], Chiboka and Solarin [4], Isere et al.[15] extended holomorphic methods to Bol, extra, weak inverse property loops, generalized Bol loops, conjugacy closed loops and Osborn. More recent developments, particularly those involving WIP and power-associative conjugacy-closed loops [9], have demonstrated that the interaction between loop identities and the automorphism group often encodes the essential algebraic structure of a loop.

Despite this progress, the holomorphic and automorphic structure of Crazy loops has remained completely unexplored. Nothing is known about the behavior of their automorphisms, pseudoautomorphisms, or the internal symmetries that the defining identity imposes. In particular, it is not clear which automorphism-type mappings lift to autotopisms in the holomorph, nor how the Crazy identity shapes the nucleus, inner mappings, or translation structure of the loop.

The aim of this paper is to initiate a systematic study of this variety. We focus on the interaction between the Crazy identity and three central objects: the automorphism group, the pseudoautomorphism group, and the autotopism group of the holomorph. Our goal is to obtain an autotopic characterization of Crazy loops, describe the pseudoautomorphisms they admit, and determine when the holomorph itself inherits the Crazy property.

§2. Preliminaries

A loop $(Q, \cdot)$ is a set equipped with a binary operation such that

(1) For every $x \in Q$, the left and right translations

$$L_x(y) = x \cdot y, \qquad R_x(y) = y \cdot x \qquad (y \in Q).$$

are bijections;

(2) There exists an element $e \in Q$ such that

$$ex = xe = x, \;\; \forall x \in Q.$$

Since all translations are bijective, their inverses L_x^{-1} and R_x^{-1} are well defined. We write $x\backslash y = L_x^{-1}(y)$ and $y/x = R_x^{-1}(y)$. For an overview of the theory of loops, readers may check [3, 16, 24, 30]. Generally, a Smarandache loop L is such a loop that a proper subset $A \subset L$ is a subgroup of L respective to the same induced operation. Particularly, if $L = \emptyset$, a Smarandache loop is nothing else but a loop. For the future research, it would be of interest to investigate whether the properties established in this paper can be extended to broader algebraic systems. Specifically, applying these results to Smarandache loops – loops which embed a proper subgroup – could yield interesting algebraic variations.

Definition 2.1 (Nuclei) *The left, middle and right nuclei are*

$$N_\lambda = \{a \in Q : a \cdot (xy) = (ax) \cdot y \ \forall x, y\},$$
$$N_\mu = \{a \in Q : x \cdot (ay) = (xa) \cdot y \ \forall x, y\},$$
$$N_\rho = \{a \in Q : (xy) \cdot a = x \cdot (ya) \ \forall x, y\}$$

and the nucleus $N(Q) = N_\lambda \cap N_\mu \cap N_\rho$.

Remark 2.2 Membership of nuclei admits the usual autotopism description:

$$\begin{aligned} a \in N_\lambda &\iff (L_a, I, L_a) \in \mathrm{Atp}(Q), \quad a \in N_\mu \\ &\iff (R_a^{-1}, L_a, I) \in \mathrm{Atp}(Q), \quad a \in N_\rho \\ &\iff (I, R_a, R_a) \in \mathrm{Atp}(Q). \end{aligned}$$

The commutant of Q is given by

$$C(Q) = \{c : \forall x \in Q, cx = xc\}.$$

A loop Q is said to have the centrum square property if

$$x^2 \in C(Q). \tag{CSP}$$

The center of a loop $Z(Q)$ is defined by $Z(Q) = N(Q) \cap C(Q)$ and a loop has the square symmetric property if it satisfies

$$x \cdot xy = yx \cdot x. \tag{SSP}$$

The left alternative property of a loop is defined as

$$xx \cdot y = x \cdot xy, \tag{LAP}$$

and the right alternative property is given by

$$y \cdot xx = yx \cdot x. \tag{RAP}$$

A loop is an alternative loop if it is left and right alternative. Flexible loops satisfy

$$x \cdot yx = xy \cdot x \tag{FLEX}$$

A loop Q satisfies the left inverse property if

$$x^\lambda \cdot xy = y \tag{LIP}$$

and the right inverse property if

$$xy \cdot y^\rho = x. \tag{RIP}$$

An inverse property loop is a loop that satisfies both the (LIP) and the (RIP) and a loop is a weak inverse property loop if it satisfies any one of the following identities:

$$x(yx)^{\rho} = y^{\rho} \quad or \quad (xy)^{\lambda}x = y^{\lambda}. \tag{WIP}$$

Definition 2.3(Autotopisms, automorphisms, pseudoautomorphisms) *An ordered triple (U, V, W) of bijections of Q is an autotopism if*

$$xU \cdot yV = (x \cdot y)W \qquad (\forall x, y \in Q).$$

The set of all autotopisms forms the group $\mathrm{Atp}(Q)$*.*

A bijection $\alpha : Q \to Q$ is an automorphism when $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$, equivalently $(\alpha, \alpha, \alpha) \in \mathrm{Atp}(Q)$.

A map U is a left pseudoautomorphism with companion $c \in Q$ if

$$(UL_c,\ U,\ UL_c) \in \mathrm{Atp}(Q),$$

where L_c denotes left multiplication by c. Right and middle pseudoautomorphisms are defined analogously.

A loop $(Q, \cdot)$ is called an extra loop if Q satisfies one (equivalently all) of the following identities

$$(x \cdot yz)y = xy \cdot zy. \tag{RE}$$

$$yz \cdot yx = y(zy \cdot x). \tag{LE}$$

$$(xy \cdot z)x = x(y \cdot zx) \tag{ME}$$

and extra loops of second Bol -Moufang type have been introduced in [12].

Definition 2.45 *Let* $\mathrm{Aut}(Q)$ *be the automorphism group of Q. The holomorph $H = \mathrm{Aut}(Q) \ltimes Q$ is the semidirect product with multiplication*

$$(\alpha, a) \circ (\beta, b) = (\alpha\beta,\ \alpha(b) \cdot a),$$

where automorphisms act on Q on the left. The neutral element is $(\mathrm{id}, 1)$*.*

§3. **Main Results**

Definition 3.1 *A loop is a Crazy loop if it satisfies*

$$(x(xy))z = (yx)(xz). \tag{1}$$

Proposition 3.2 *In a loop Q, the Crazy identity is equivalent to each of the following:*

$$\begin{aligned} L_{x\cdot xy}L_x^{-1} &= L_{yx}.\\ R_zL_x^2 &= R_{xz}R_x.\\ L_{yx}^{-1}L_{x\cdot xy} &= L_x \end{aligned}$$

Lemma 3.3 *A loop Q is a Crazy loop if and only if $A = (R_xL_x^{-2},\ L_x,\ I)$ is an autotopism of Q.*

Proof Write the crazy identity as $yz = (x\setminus(x\setminus y))x\cdot xz$ and then, the rest is straight forward. □

Lemma 3.4 *In any loop Q, the following are true:*

(1) *Any two of $L_{x^2} = R_x^2$, $L_x^2 = R_x^2$ and $L_{x^2} = L_x^2$ imply the third;*
(2) *Any two of $R_{x^2} = L_x^2$, $L_x^2 = R_x^2$ and $R_{x^2} = R_x^2$ imply the third;*
(3) *Any two of $R_{x^2} = L_x^2$, $L_{x^2} = R_{x^2}$ and $L_{x^2} = L_x^2$ imply the third;*
(4) *Any two of $L_{x^2} = R_{x^2}$, $L_{x^2} = R_x^2$ and $R_{x^2} = R_x^2$ imply the third;*
(5) *$R_{x^2} = R_x^2$, $L_{x^2} = L_x^2$ and $L_x^2 = R_x^2$ imply $R_{x^2} = L_{x^2}$;*
(6) *$R_{x^2} = R_x^2$, $L_{x^2} = L_x^2$ and $R_{x^2} = L_{x^2}$ imply $L_x^2 = R_x^2$;*
(7) *If $x^2 \in C(L)$ and x^2 is in any two of N_λ, N_μ, and N_ρ, then x^2 belongs to all three.*

Proof $(1)-(3)$ and (4) are obvious.

For (5), $L_{x^2} = L_x^2 = R_x^2 = R_{x^2}$, for (6) we have $R_x^2 = R_{x^2} = L_{x^2} = L_x^2$ and for (7), we only consider the case where $x^2 \in C(L)$ and $x^2 \in N_\lambda \bigcap N_\mu$. Then,

$$yz\cdot x^2 = x^2\cdot yz = x^2y\cdot z = yx^2\cdot z = y\cdot x^2z = y\cdot zx^2.$$

Thus, $x^2 \in N_\rho$. □

Lemma 3.5 *Let Q be a Crazy loop. Then,*

$(1) L_x^2 = R_x^2;$	$(2) L_{x^2} = L_x^2;$	$(3) R_xL_x^2 = R_{x^2}R_x;$
$(4) x^\lambda = x^\rho;$	$(5) R_xL_x = L_xR_x;$	$(6) L_x^2 = R_{x^2};$
$(7) R_x^2 = R_{x^2};$	$(8) R_{x^2} = L_{x^2};$	$(9) R_xL_x^2 = L_x^2R_x.$

Proof (1) Setting $z = 1$ in the Crazy identity.
(2) Put $y = 1$ in the Crazy identity.
(3) Replace z with x in the Crazy identity.
(4) Put $z = x^\rho$ in the Crazy identity and use Lemma 3.5(1).
(5) With $z = y$ in the Crazy identity and Lemma 3.5(1).
(6) This is from (3) and (5).
(7) Use Lemma 3.5(1). and Lemma 3.5(6).
(8) Use Lemma 3.5(2). and Lemma 3.5(6).

(9) Use Lemma 3.5(6). on the right hand side of Lemma 3.5(3).

This completes the proof. □

Remark 3.6 The identity in Lemma 3.5(9). have been used to study power associative conjugacy closed loops with the Weak inverse power associative loops in [7, 10, 11, 13, 17, 19].

Theorem 3.7 *If Q is a Crazy loop, then, (i) $x^{\lambda} \cdot xz = z$; (ii) $yx \cdot x^{\rho} = y$; (iii) $x(yx)^{\rho} = y^{\rho}$; (iv) $N_{\lambda} = N_{\rho} = N_{\mu}$.*

Proof (i) The Crazy identity is equivalent to

$$(x\backslash(x\backslash y)) \cdot xz = yz,$$

put $y = 1$ to get

$$(x\backslash(x^{\rho}))x \ \cdot xz = z. \tag{5}$$

If we set $z = 1$ in (5), we have

$$(x\backslash(x^{\rho}))x = x^{\lambda}. \tag{6}$$

From equations (5) and (6), We have

$$x^{\lambda} \cdot xz = z.$$

(ii) Again if we put $z = x^{\rho}$ in the Crazy identity, we obtain

$$yx = (x \cdot xy)x^{\rho} = (yx \cdot x)x^{\rho} \Leftrightarrow yx \cdot x^{\rho} = y$$

(iii) Any loop satisfying (i) and (ii) will satisfy (iii).

(iv) is an implication of (i) and (ii). □

Theorem 3.8 *Let Q be an extra loop. Then, the following are equivalent:*

(1) *Q is Crazy loop;*

(2) *$T(x) = T^{-1}(x) = T(x^{-1})$ is an automorphism of Q.*

Proof (1) $\Rightarrow$ (2): Let Q be a middle extra loop, then $C = (L_x,\ R_x^{-1},\ R_x^{-1}L_x)$ is an autotopism of Q and suppose Q is also a Crazy loop, we have by Lemma 3.3, $A = (R_xL_x^{-2},\ Lx,\ I)$ is also an autotopism of Q. Therefore, $AC = (R_xL_x^{-1},\ L_xR_x^{-1},\ R_x^{-1}L_x)$. The three components are equal since a Crazy loop is flexible, satisfies the square symmetric property and is an inverse property loop.

(2) $\Rightarrow$ (1): It is similar to the proof of (1) $\Rightarrow$ (2). □

Theorem 3.9([24]) *Let $A = (U,\ V,\ W)$ be the autotopism of a loop Q. Then,*

(1) *If Q is a left inverse property loop, then $A_{\lambda} = (IUI,\ W,\ V)$ is also an autotopism of Q;*

(2) *If Q is a right inverse property loop, then $A_{\lambda} = (W,\ IVI,\ U)$ is also an autotopism of*

Q.

Theorem 3.10 *Let $A(Q)$ be the automorphism group of a loop $(Q, \cdot)$. Then, the $A(Q)$ holomorph $(H, \circ)$ of $(Q, \cdot)$ is a Crazy loop if and only if the following two conditions hold:*

(1) *For every $\alpha, \beta \in A(Q)$,*

$$\alpha^2\beta = \beta\alpha^2;$$

(2) *For all $x, y, z \in Q$ and $\alpha^2 = id,\ \beta \in A(Q)$,*

$$(x\beta \cdot\ (x\beta)y)z = yx \cdot xz. \tag{7}$$

Proof Let $U = (\alpha, x),\ V = (\beta, y),\ W = (\delta, z)$. Then,

$$\begin{aligned}
U \cdot V &= (\alpha\beta,\ x\beta \cdot y), \\
U \cdot (U \cdot V) &= (\alpha^2\beta,\ \ x(\alpha\beta) \cdot\ x\beta \cdot y), \\
(U \cdot (U \cdot V)) \cdot W &= (\alpha^2\beta\delta,\ \ (x\alpha\beta \cdot x\beta \cdot y)\delta \cdot z), \\
V \cdot U &= (\beta\delta,\ y\alpha \cdot x), \\
U \cdot W &= (\alpha\delta,\ x\delta \cdot z), \\
(V \cdot U)\ \cdot (U \cdot W) &= (\beta\alpha^2\delta,\ \ (y\alpha \cdot x)\alpha\delta \cdot\ x\delta \cdot z).
\end{aligned}$$

For $(H, \circ)$ to be a Crazy loop, we must have

$$(U \cdot (U \cdot V)) \cdot W = (V \cdot U)\ \cdot (U \cdot W),$$

i.e.,

$$(\alpha^2\beta\delta,\ \ (x\alpha\beta \cdot x\beta \cdot y)\delta \cdot z) = (\beta\alpha^2\delta,\ \ (y\alpha \cdot x)\alpha\delta \cdot\ x\delta \cdot z).$$

Equating the components gives the first component

$$\alpha^2\beta\delta = \beta\alpha^2\delta.$$

Cancel δ on the right to obtain

$$\alpha^2\beta = \beta\alpha^2$$

and second component

$$(x\alpha\beta \cdot (x\beta \cdot y))\delta \cdot z = (y\alpha \cdot x)\alpha\delta \cdot\ x\delta \cdot z,$$

$$(x\beta\ \cdot (x\alpha^{-1}\beta \cdot y\alpha^{-1}))z\delta^{-1}\alpha^{-1} = (y\alpha \cdot x) \cdot (x\alpha^{-1} \cdot z\delta^{-1}\alpha^{-1}).$$

Thus, the equation (7) holds if $\alpha^2 = id$. □

Corollary 3.11 *Let Q be a Crazy loop and A an automorphism group of Q. Then, the $A(Q)$ holomorph is Crazy if and only if $B = (R_x L^{-2}_{(x\beta)},\ L_x,\ I)$ is an autotopism of Q for all $x \in Q$ and $\beta \in A$.*

Lemma 3.12([14]) *Let Q be a weak inverse property loop and let A be the automorphisms group of Q. Then, the holomorph $A(Q)$ is a weak inverse property loop.*

Lemma 3.13 *Let $(Q, \cdot)$ be a loop and let $A(Q)$ be the automorphism group of Q.*

(*i*) *If $R_x^2 = R_{x^2}$, then $R_{(x\beta)}^2 = R_{(x\beta)^2}$;*

(*ii*) *If $L_x^2 = L_{x^2}$, then $L_{(x\beta)}^2 = L_{(x\beta)^2}$;*

(*iii*) *If $L_x^2 = R_{x^2}$, then $L_{(x\beta)}^2 = R_{(x\beta)^2}$.*

Proof (*i*) By the hypothesis,

$$(y \cdot x) \cdot x = y \cdot x^2.$$

Applying the automorphism β to the RAP identity

$$((y \cdot x) \cdot x)\beta = (y \cdot x^2)\beta.$$

Since β is homomorphism,

$$(y \cdot x)\beta \cdot x\beta = y\beta \cdot (x^2)\beta \Leftrightarrow (y\beta \cdot x\beta) \cdot x\beta = y\beta \cdot ((x\beta) \cdot (x\beta)).$$

$$(y' \cdot x\beta) \cdot x\beta = y' \cdot ((x\beta) \cdot (x\beta)) \Leftrightarrow y' R_{(x\beta)}^2 = y' R_{(x\beta)^2}.$$

(*ii*) and (*iii*) can be prove in a similar way as (*i*). □

Lemma 3.14([22]) *Let Q be a WIPL. If $(U,\ V,\ W) \in Atp(Q)$. Then,*

(1) $(V,\ \lambda W\rho,\ \lambda U\rho) \in Atp(Q)$;

(2) $(\rho W\lambda,\ U,\ \rho V\lambda) \in Atp(Q)$.

Theorem 3.15 *Let $(Q, \cdot)$ be a loop and let $A(Q)$ be the group of automorphisms of $(Q, \cdot)$. Then, the $A(Q)$ holomorph $(H, \circ)$ of $(Q, \cdot)$ is a Crazy loop if and only if $(Q, \cdot)$ is a Crazy loop and $(x\beta)^2 \cdot x^\lambda \in N_\rho$.*

Proof Let $(H, \circ)$ be a Crazy loop, then $(Q, \cdot)$ is a Crazy loop since it is isomorphic to subloop of $(H, \circ)$. Now, by Corollary 3.11, $(R_x L_{x\beta}^{-2},\ L_x,\ I)$ is an autotopism of Q, and by Theorem 3.7, Q has WIP, and since the holomorph of a WIP loop is WIP by Lemma 3.12, we see that

$$(I,\ L_{x\beta}^2 R_x^{-1},\ R_x) \tag{8}$$

is an autotopism of Q by Lemma 3.14(2), this is equivalent to

$$y \ \cdot ((x\beta) \cdot (x\beta)(z/x)) = yz \cdot x,$$

with $z = 1$, we have

$$yR((x\beta)^2 \cdot x^\lambda) = yR_x.$$

Therefore, the autotopism in (8) becomes

$$(I,\ R\big((x\beta)^2 \cdot x^\lambda\big),\ R\big((x\beta)^2 \cdot x^\lambda\big)).$$

Thus, $(x\beta)^2 \cdot x^\lambda \in N_\rho$.

Conversely, let Q be a Crazy loop and let each $(x\beta)^2 \cdot x^\lambda \in N_\rho$,. i.e.,

$$(I,\ R\big((x\beta)^2 \cdot x^\lambda\big),\ R\big((x\beta)^2 \cdot x^\lambda\big)). \tag{9}$$

Note that in a Crazy loop,

$$N_\rho = N_\mu = N_\lambda.$$

Let

$$t = (x\beta)^2 \cdot x^\lambda \in N_\mu.$$

Then

$$tx = ((x\beta)^2\ \cdot x^\lambda) \cdot x = (x\beta)^2.$$

Using the middle nuclear property of t,

$$zt \cdot x = z \cdot tx = z \cdot (x\beta)^2.$$

$$zt \cdot x = z \cdot (x\beta)^2. \tag{10}$$

But by Lemma 3.13,

$$\begin{aligned} z \cdot (x\beta)^2 &= (x\beta) \cdot ((x\beta) \cdot z), \\ zt \cdot x &= (x\beta) \cdot ((x\beta) \cdot z), \\ zt &= ((x\beta) \cdot ((x\beta) \cdot z))/x. \end{aligned}$$

Therefore,

$$zt = R_x^{-1} L^2_{(x\beta)}(z).$$

Again, $zt = R_t(z)$, and on the other hand,

$$L^2_{x\beta} R_x^{-1}(z) = (x\beta) \cdot ((x\beta) \cdot (z/x)).$$

Multiply the last expression on the right by x and use the fact that $(x\beta) \in N_\lambda$ if $t \in N_\lambda$,

$$((x\beta) \cdot ((x\beta)(z/x)))x = (x\beta)^2 \cdot (z/x)x = x\beta \cdot (x\beta)z.$$

Therefore,

$$(x\beta) \cdot ((x\beta)(z/x) = ((x\beta) \cdot ((x\beta)z]/x.$$

That is

$$L^2_{x\beta} R_x^{-1} = R_x^{-1} L^2_{(x\beta)}.$$

Thus, $t = L^2_{x\beta} R_x^{-1} = R_x^{-1} L^2_{(x\beta)}$ and therefore the autotopism in (9) becomes

$$(I,\ R(t),\ R(t)) = (I,\ L^2_{x\beta} R_x^{-1},\ L^2_{x\beta} R_x^{-1}). \tag{11}$$

Now equation (10) says $zt \cdot x = z \cdot (x\beta)^2$, if we put $t = x$ and use Lemma 3.13(iii), we obtain

$$zR_x^2 = zL_{(x\beta)}^2 \Leftrightarrow R_x^2 L_{(x\beta)}^{-2} = id \Leftrightarrow (I,\ I,\ R_x^2 L_{(x\beta)}^{-2})$$

is an autotopism.

Multiplying this autotopism with the autotopism in (11), we obtain

$$(I,\ L_{x\beta}^2 R_x^{-1},\ R_x),$$

and since a Crazy loop has WIP, and by Lemma 3.14(1), the last autotopism becomes

$$(R_x L_{(x\beta)}^{-2},\ L_x,\ I).$$

Thus, $(H, \circ)$ is a Crazy loop. □

Corollary 3.16 *Let $(H, \circ)$ be the $A(Q)$ holomorph of a Crazy loop $(Q, \cdot)$. Then, each $\beta \in A(Q)$ is nuclear.*

Theorem 3.17 *Let $(H, \circ)$ be the $A(Q)$ holomorph of a Crazy loop $(Q, \cdot)$. Then,*

(1) $(L_x^2 L_{(x\beta)}^{-2},\ I,\ I)$ *is an autotopism of Q;*

(2) *If $N_\lambda(Q) = \{n \in Q : n \cdot (xy) = (nx) \cdot y,\ \forall x, y \in Q\}$. Then*

$$(L_x^2 L_{(x\beta)}^{-2}) N_\lambda(Q) = N_\lambda(Q);$$

(3) *The restriction $(L_x^2 L_{(x\beta)}^{-2}) \mid N_\lambda : N_\lambda(Q) \mapsto N_\lambda(Q)$ is an automorphism of the subloop $N_\lambda(Q)$.*

Proof (1) Compute the autotopism $A^{-1}B$.

(2) Let $n \in N_\lambda(Q)$ and let $u, v, y \in Q$ such that $y = uv$. From (1), we have

$$L_x^2 L_{(x\beta)}^{-2}(n) \cdot y = ny \Leftrightarrow L_x^2 L_{(x\beta)}^{-2}(n) \cdot uv = n \cdot uv. \tag{12}$$

By the left nucleus property of n, we have

$$n \cdot uv = nu \cdot v. \tag{13}$$

Again by Theorem 3.17(1),

$$L_x^2 L_{(x\beta)}^{-2}(n) \cdot u = nu. \tag{14}$$

We therefore have from the three equations above,

$$(L_x^2 L_{(x\beta)}^{-2}(n) \cdot u) \cdot v = n \cdot uv = L_x^2 L_{(x\beta)}^{-2}(n) \cdot uv. \Leftrightarrow L_x^2 L_{(x\beta)}^{-2}(n) \in N_\lambda.$$

Since $L_x^2 L_{(x\beta)}^{-2}$ is a bijection and its image is equal to $N_\lambda(Q)$. Thus, $L_x^2 L_{(x\beta)}^{-2}) N_\lambda(Q) \subseteq N_\lambda(Q)$.

(3) Let $a, b \in N_\lambda(Q)$, then for arbitrary $y \in Q$, compute

$$(L_x^2 L_{(x\beta)}^{-2}(a) \cdot L_x^2 L_{(x\beta)}^{-2}(b)) \ \cdot y = L_x^2 L_{(x\beta)}^{-2})(a) \ \cdot \ (L_x^2 L_{(x\beta)}^{-2})(b) \cdot y). \tag{15}$$

Applying the autotopism in Theorem 3.17(1) to the product by

$$L_x^2 L_{(x\beta)}^{-2}(b) \cdot y = b \cdot y,$$

equation (15) now becomes

$$(L_x^2 L_{(x\beta)}^{-2})(a) \cdot \ (L_x^2 L_{(x\beta)}^{-2})(b)) \cdot \ y = L_x^2 L_{(x\beta)}^{-2})(a) \ \cdot (b \cdot y). \tag{16}$$

Again apply the autotopism in Theorem 3.17(1) to the product of a and by and use the fact that $a \in N_\lambda$,

$$L_x^2 L_{(x\beta)}^{-2}(a) \cdot (b \cdot y) = a \cdot (b \cdot y) = (a \cdot b) \cdot y. \tag{17}$$

Also, apply the autotopism in Theorem 3.17(1) to the product of ab and y,

$$L_x^2 L_{(x\beta)}^{-2}(ab) \cdot y = (a \cdot b) \cdot y. \tag{18}$$

Combining equations (15), (16), (17) and (18),

$$(L_x^2 L_{(x\beta)}^{-2}(a) \cdot L_x^2 L_{(x\beta)}^{-2}(b)) \ \cdot y = L_x^2 L_{(x\beta)}^{-2}(ab) \cdot y$$

and by right cancellation,

$$(L_x^2 L_{(x\beta)}^{-2}(a) \cdot L_x^2 L_{(x\beta)}^{-2}(b)) = L_x^2 L_{(x\beta)}^{-2}(ab).$$

This completes the proof. □

Corollary 3.18 *Let $(H, \circ)$ be the $A(Q)$ holomorph of a Crazy loop $(Q, \cdot)$. and let $L_x^2 L_{(x\beta)}^{-2}(y) \equiv (y) \ (Mod \ N(Q))$ for all $y \in Q$. Then, $L_x^2 L_{(x\beta)}^{-2}$ is a nuclear automorphism.*

Proof Let $(H, \circ)$ be the $A(Q)$ holomorph of a Crazy loop $(Q, \cdot)$ and let $L_x^2 L_{(x\beta)}^{-2}(y) \equiv (y) \ (Mod \ N(Q))$. Since $N_\lambda = N_\rho = N_\mu$ in a Crazy loop, then $L_x^2 L_{(x\beta)}^{-2}$ acts trivially on the coset of N and induces the identity map on the quotient $Q/N(Q)$. This combined with (3) of Theorem 3.17, shows that $L_x^2 L_{(x\beta)}^{-2}$ restricts to an automorphism of $N(Q)$ and therefore is automorphism on the nucleus. □

Lemma 3.19([29]) *Let $(Q, \cdot)$ be a loop and let $A(Q)$ be a group of automorphism of $(Q, \cdot)$. Then, the $A(Q)$ holomorph $(H, \circ)$ of $(Q, \cdot)$ is an extra loop if and only if $C_\beta = (L_{x\beta}, \ R_x^{-1}, \ R_x^{-1} L_{x\beta})$ is an autotopism of $(Q, \cdot)$.*

Theorem 3.20 *Let $(H, \circ)$ be the $A(Q)$ holomorph of a Crazy loop that is an extra loop. Then, $T(x)^{-1}$ is a left pseudoautomorphism with companion $x\beta \cdot x^\lambda$.*

Proof Let $(H,\circ)$ be the $A(Q)$ holomorph of an extra loop, then

$$C_\beta = (L_{x\beta},\ R_x^{-1},\ R_x^{-1}L_{x\beta})$$

and suppose $(H,\circ)$ is also a Crazy loop, we also have

$$B = (R_xL_{(x\beta)}^{-2},\ L_x,\ I).$$

Then

$$BC_\beta = (R_xL_{(x\beta)}^{-1},\ L_xR_x^{-1},\ R_x^{-1}L_{x\beta}).$$

Since $L_xR_x^{-1}(1) = 1$, we therefore have $T(x)^{-1}$ is left pseudoautomorphism with companion $R_x^{-1}L_{(x\beta)}(1) = L_{(x\beta)}R_x^{-1}(1) = x\beta \cdot x^\lambda$. □

Corollary 3.21 *Let $(H,\circ)$ be the $A(Q)$ holomorph of a Crazy loop. If $(H,\circ)$ is an extra loop and $x\beta \cdot x^\lambda \in N_\lambda$, then $T(x)^{-1}$ is an automorphism.*

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