

## Balance Index Set of Caterpillar and Lobster Graphs

Pradeep G.Bhat and Devadas Nayak C

(Department of Mathematics, Manipal Institute of Technology, Manipal University, Manipal-576 104, India)

E-mail: pg.bhat@manipal.edu, devadasnayakc@yahoo.com

**Abstract:** Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . Consider the set  $A = \{0, 1\}$ . A labeling  $f : V(G) \rightarrow A$  induces a partial edge labeling  $f^* : E(G) \rightarrow A$  defined by  $f^*(xy) = f(x)$ , if and only if  $f(x) = f(y)$  for each edge  $xy \in E(G)$ . For  $i \in A$ , let  $v_f(i) = |\{v \in V(G) : f(v) = i\}|$  and  $e_{f^*}(i) = |\{e \in E(G) : f^*(e) = i\}|$ . A labeling  $f$  of a graph  $G$  is said to be friendly if  $|v_f(0) - v_f(1)| \leq 1$ . A friendly labeling is balanced if  $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$ . The balance index set of the graph  $G$ ,  $BI(G)$ , is defined as  $\{|e_{f^*}(0) - e_{f^*}(1)| : \text{the vertex labeling } f \text{ is friendly}\}$ . In this paper, we obtain the balance index set of caterpillar graphs and lobster graphs.

**Key Words:** Friendly labeling, Smarandache friendly labeling, partial edge labeling and balance index set.

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### §1. Introduction

We begin with simple, finite, connected and undirected graph  $G=(V, E)$ . Here the elements of set  $V$  and  $E$  are known as vertices and edges respectively with  $|V| = p$  and  $|E| = q$ . For all other terminologies and notations we follow Harary [1].

**Definition 1.1** A path graph or linear graph is a tree with two or more vertices that contains only vertices of degree 2 and 1.

**Definition 1.2** A caterpillar is a tree in which all the vertices are within distance 1 of a central path.

**Definition 1.3** The graph  $B_{l,m,k}$  is a tree obtained from a path of length  $k$  by attaching the stars  $K_{1,l}$  and  $K_{1,m}$  with its pendent vertices.

**Definition 1.4** A coconut Tree  $CT(m, l)$  is the graph obtained from the path  $P_m$  by appending  $l$  new pendent edges at an end vertex of  $P_m$ .

**Definition 1.5** A lobster graph is a tree in which all the vertices are within distance 2 of a central path.

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**Definition 1.6** A mapping  $f : V(G) \rightarrow \{0, 1\}$  is called friendly labeling of  $G$  if

$$|v_f(0) - v_f(1)| \leq 1,$$

otherwise, a Smarandache friendly labeling of  $G$ , i.e.,  $|v_f(0) - v_f(1)| \geq 2$ .

Lee, Liu and Tan [5] considered a new labeling problem of graph theory. A vertex labeling of  $G$  is a mapping  $f$  from  $V(G)$  into the set  $\{0, 1\}$ . For each vertex labeling  $f$  of  $G$ , a partial edge labeling  $f^*$  of  $G$  is defined in the following way.

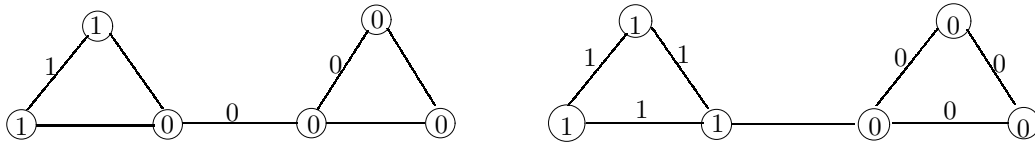
For each edge  $uv$  in  $G$ ,

$$f^*(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 0 \\ 1, & \text{if } f(u) = f(v) = 1 \end{cases}$$

Note that if  $f(u) \neq f(v)$ , then the edge  $uv$  is not labeled by  $f^*$ . Thus  $f^*$  is a partial function from  $E(G)$  into the set  $\{0, 1\}$ . Let  $v_f(0)$  and  $v_f(1)$  denote the number of vertices of  $G$  that are labeled by 0 and 1 under the mapping  $f$  respectively. Likewise, let  $e_{f^*}(0)$  and  $e_{f^*}(1)$  denote the number of edges of  $G$  that are labeled by 0 and 1 under the induced partial function  $f^*$  respectively.

In [3] Kim, Lee, and Ng define the balance index set of a graph  $G$  as  $BI(G) = \{|e_{f^*}(0) - e_{f^*}(1)| : f^* \text{ runs over all friendly labelings } f \text{ of } G\}$ .

**Example 1.7** Figure 1 shows a graph  $G$  with  $BI(G) = \{0, 1\}$ .



**Figure 1** The friendly labelings of graph  $G$  with  $BI(G) = \{0, 1\}$ .

For a graph with a vertex labeling  $f$ , we denote  $e_{f^*}(X)$  to be the subset of  $E(G)$  containing all the unlabeled edges. In [4] Kwong and Shiu developed an algebraic approach to attack the balance index set problems. It shows that the balance index set depends on the degree sequence of the graph.

**Lemma 1.8**([6]) For any graph  $G$ ,

- (1)  $2e_{f^*}(0) + e_{f^*}(X) = \sum_{v \in v(0)} deg(v)$ ;
- (2)  $2e_{f^*}(1) + e_{f^*}(X) = \sum_{v \in v(1)} deg(v)$ ;
- (3)  $2|E(G)| = \sum_{v \in v(G)} deg(v) = \sum_{v \in v(0)} deg(v) + \sum_{v \in v(1)} deg(v)$ .

**Corollary 1.9**([6]) For any friendly labeling  $f$ , the balance index is

$$e_{f^*}(0) - e_{f^*}(1) = \frac{1}{2} \left( \sum_{v \in v(0)} deg(v) - \sum_{v \in v(1)} deg(v) \right).$$

More details of known results of graph labelings are given in Gallian [2].

In number theory and combinatorics, a partition of a positive integer  $n$ , also called an integer partition, is a way of writing  $n$  as a sum of positive integers. Two sums that differ only in the order of their summands are considered to be the same partition; if order matters then the sum becomes a composition. For example, 4 can be partitioned in five distinct ways:

$$4 + 0, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1.$$

Let  $G$  be any graph with  $p$  vertices. Partition of  $p$  in to  $(p_0, p_1)$ , where  $p_0$  and  $p_1$  are the number of vertices labeled by 0 and 1 respectively.

In [6] Lee, Su and Wang gave the results for balance index set of trees of diameter four. In this paper we obtain balance index set of caterpillar and lobster graphs of diameter  $n$ . To prove our result we are using Lemma 1.8 and Corollary 1.9.

### §2. Balance Index Set of Caterpillar Graphs

Consider the caterpillar graph  $CT(a_1, a_2, a_3, \dots, a_{n-1})$ , where  $a_i, i=1, 2, 3, \dots, n-1$  are the number of vertices adjacent to  $i^{th}$  spine vertices. We name  $n-1$  vertices on the spine as  $u_{a_1}, u_{a_2}, u_{a_3}, \dots, u_{a_{n-1}}$ . Thus for a caterpillar graph there are  $(a_1 + a_2 + a_3 + \dots + a_{n-1})$  number of pendant vertices. The degrees of  $u_{a_1}, u_{a_2}, u_{a_3}, \dots, u_{a_{n-1}}$  are  $a_1 + 1, a_2 + 2, a_3 + 2, \dots, a_{n-2} + 2, a_{n-1} + 1$  respectively. We also name non-spinal vertices adjacent to  $u_{a_1}$  by  $u_{a_1,1}, u_{a_1,2}, u_{a_1,3}, \dots, u_{a_1,a_1}$ . Similarly we name non spinal vertices adjacent to  $u_{a_2}, u_{a_3}, u_{a_4}, \dots, u_{a_{n-1}}$ .

**Theorem 2.1** For  $CT(a_1, a_2, a_3, \dots, a_{n-1})$  of order  $p$  and diameter  $n$ , the balance index is,

$$e_{f^*}(0) - e_{f^*}(1) = \begin{cases} \left\{ \frac{1}{2} \left( l + \sum_{i=1}^{n-1} (-1)^{f(u_{a_i})} a_i \right) \right\}, & \text{if } p \text{ is even} \\ \left\{ \frac{1}{2} \left( l + 1 + \sum_{i=1}^{n-1} (-1)^{f(u_{a_i})} a_i \right) \right\}, & \text{if } p \text{ is odd} \end{cases}$$

where

$$l = \begin{cases} n - 2j - 3, & \text{if } j = i, i - 1, i - 2, \text{ where } i = 2, 3, 4, \dots, \lfloor \frac{n}{2} \rfloor \\ & \text{and } j \text{ number of coefficients of } a_i \text{ are negative} \\ n - 3, & \text{if } f(u_{a_i}) = 0 \text{ for all } i \text{ or } f(u_{a_i}) = \begin{cases} 1, & \text{if } i = 1, n - 1 \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

*Proof* Consider the caterpillar graph  $CT(a_1, a_2, a_3, \dots, a_{n-1})$  of order  $p$  and diameter  $n$ .

**Case 1.**  $n$  is even.

**Subcase 1.1** If  $a_1 + a_2 + a_3 + \dots + a_{n-1}$  is odd, then the number of vertices of  $CT(a_1, a_2, a_3,$

$\dots, a_{n-1}$ ) is  $a_1 + a_2 + a_3 + \dots + a_{n-1} + n - 1$  which is even. Let  $(a_1 + a_2 + a_3 + \dots + a_{n-1}) + n - 1 = 2M$ . For a friendly labeling,  $M$  vertices are labeled 0 and remaining  $M$  vertices are labeled 1.

We first consider the case that  $u_{a_1}, u_{a_2}, u_{a_3}, \dots, u_{a_{n-1}}$  are all labeled 0, i.e.  $n - 1$  spine vertices are partitioned in to  $(n - 1, 0)$ . Then  $M - (n - 1)$  pendant vertices are labeled 0 and  $M$  pendant vertices are labeled 1. Therefore by Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\ &= \frac{1}{2} [M - (n - 1) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \dots + (a_{n-1} + 1) - M] \\ &= \frac{1}{2} [a_1 + a_2 + a_3 + \dots + a_{n-1} + n - 3]. \end{aligned}$$

If  $n - 1$  spine vertices are partitioned in to  $(n - 2, 1)$ , then  $M - (n - 2)$  pendant vertices are labeled 0 and  $M - 1$  pendant vertices are labeled 1. Two possibilities arise.

(a) If the vertex  $u_{a_1}$  is labeled 1, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\ &= \frac{1}{2} [M - (n - 2) - (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \dots \\ &\quad + (a_{n-1} + 1) - (M - 1)] \\ &= \frac{1}{2} [-a_1 + a_2 + a_3 + \dots + a_{n-1} + n - 3]. \end{aligned}$$

Similarly If  $u_{a_{n-1}}$  is labeled 1, then

$$e_{f^*}(0) - e_{f^*}(1) = \frac{1}{2} [a_1 + a_2 + a_3 + \dots + a_{n-2} - a_{n-1} + n - 3].$$

(b) If one vertex  $u_{a_i}$ ,  $i = 2, 3, 4, \dots, n - 2$  is labeled 1, then  $M - (n - 2)$  pendant vertices are labeled 0 and  $M - 1$  pendant vertices are labeled 1.

Therefore,

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\ &= \frac{1}{2} [M - (n - 2) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) + \dots - (a_i + 2) + \dots \\ &\quad + (a_{n-1} + 1) - (M - 1)] \\ &= \frac{1}{2} [a_1 + a_2 + a_3 + \dots + a_{i-1} - a_i + a_{i+1} + \dots + a_{n-1} + n - 5], \end{aligned}$$

where  $i = 2, 3, 4, \dots, n - 2$ .

If  $n - 1$  spine vertices are partitioned in to  $(n - 1 - i, i)$ , where  $i = 2, 3, 4, \dots, \lfloor \frac{n}{2} \rfloor$ . Then  $M - (n - 1 - i)$  pendant vertices are labeled 0 and  $M - i$  pendant vertices are labeled 1. Three

possibilities arise.

(a) If  $f(u_{a_1}) = f(u_{a_{n-1}}) = 0$ , then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\ &= \frac{1}{2} [M - (n - 1 - i) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) \\ &\quad + \cdots + (a_{n-2} + 2) + (a_{n-1} + 1) - (M - i)] \\ &= \frac{1}{2} [a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2i - 3], \end{aligned}$$

where  $i = 2, 3, 4, \dots, \lfloor \frac{n}{2} \rfloor$  and  $i$  coefficients out of  $a_2, a_3, a_4, \dots, a_{n-2}$  are negative.

(b) If  $f(u_{a_1}) = 0$  and  $f(u_{a_{n-1}}) = 1$ , then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\ &= \frac{1}{2} [M - (n - 1 - i) + (a_1 + 1) + (a_2 + 2) + (a_3 + 2) \\ &\quad + \cdots + (a_{n-2} + 2) - (a_{n-1} + 1) - (M - i)] \\ &= \frac{1}{2} [a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i - 1], \end{aligned}$$

where  $i = 2, 3, 4, \dots, \lfloor \frac{n}{2} \rfloor$  and  $i - 1$  coefficients out of  $a_2, a_3, a_4, \dots, a_{n-2}$  are negative.

(c) If  $f(u_{a_1}) = f(u_{a_{n-1}}) = 1$ , then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\ &= \frac{1}{2} [M - (n - 1 - i) - (a_1 + 1) + (a_2 + 2) + (a_3 + 2) \\ &\quad + \cdots + (a_{n-2} + 2) - (a_{n-1} + 1) - (M - i)] \\ &= \frac{1}{2} [-a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i + 1], \end{aligned}$$

where  $i = 2, 3, 4, \dots, \lfloor \frac{n}{2} \rfloor$  and  $i - 2$  coefficients out of  $a_2, a_3, a_4, \dots, a_{n-2}$  are negative.

**Subcase 1.2** If  $a_1 + a_2 + a_3 + \cdots + a_{n-1}$  is even, then the number of vertices of  $CT(a_1, a_2, a_3, \dots, a_{n-1})$  is  $a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 1$  which is odd. Let  $(a_1 + a_2 + a_3 + \cdots + a_{n-1}) + n - 1 = 2M + 1$ .

For a friendly labeling, without loss of generality, there are  $M + 1$  vertices labeled 0 and  $M$  vertices labeled 1.

We first consider the case that  $u_{a_1}, u_{a_2}, u_{a_3}, \dots, u_{a_{n-1}}$  are all labeled 0, i.e.  $n - 1$  spine vertices are partitioned in to  $(n - 1, 0)$ . Then  $(M + 1) - (n - 1)$  pendant vertices are labeled 0 and  $M$  pendant vertices are labeled 1.

Therefore,

$$\begin{aligned}
e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\
&= \frac{1}{2} [(M+1) - (n-1) + (a_1+1) + (a_2+2) + (a_3+2) \\
&\quad + \cdots + (a_{n-1}+1) - M] \\
&= \frac{1}{2} [a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2].
\end{aligned}$$

If  $n-1$  spine vertices are partitioned in to  $(n-2, 1)$ , then  $(M+1) - (n-2)$  pendant vertices are labeled 0 and  $M-1$  pendant vertices are labeled 1. Two possibilities arise.

(a) If  $f(u_{a_1}) = 1$ , then

$$\begin{aligned}
e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\
&= \frac{1}{2} [(M+1) - (n-2) - (a_1+1) + (a_2+2) + (a_3+2) \\
&\quad + \cdots + (a_{n-1}+1) - (M-1)] \\
&= \frac{1}{2} [-a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2].
\end{aligned}$$

Similarly, if  $f(u_{a_{n-1}}) = 1$  then

$$e_{f^*}(0) - e_{f^*}(1) = \frac{1}{2} [a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2].$$

(b) If one spine vertex of degree  $a_i, i = 2, 3, 4, \dots, n-2$  is labeled 1, then  $M - (n-2)$  pendant vertices are labeled 0 and  $M-1$  pendant vertices are labeled 1. Therefore,

$$\begin{aligned}
e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\
&= \frac{1}{2} [(M+1) - (n-2) + (a_1+1) + (a_2+2) + (a_3+2) \\
&\quad + \cdots - (a_i+2) + \cdots + (a_{n-1}+1) - (M-1)] \\
&= \frac{1}{2} [a_1 + a_2 + a_3 + \cdots + a_{i-1} - a_i + a_{i+1} + \cdots + a_{n-1} + n - 4],
\end{aligned}$$

where  $i = 2, 3, 4, \dots, n-2$ .

If  $n-1$  spine vertices are partitioned in to  $(n-1-i, i)$ , where  $i = 2, 3, 4, \dots, \lfloor \frac{n}{2} \rfloor$ , then  $(M+1) - (n-1-i)$  pendant vertices are labeled 0 and  $M-i$  pendant vertices are labeled 1. Three possibilities arise.

(a) If  $f(u_{a_1}) = f(u_{a_{n-1}}) = 0$ , then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\ &= \frac{1}{2} [(M+1) - (n-1-i) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) + (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2} [a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2i - 2], \end{aligned}$$

where  $i = 2, 3, 4, \dots, \lfloor \frac{n}{2} \rfloor$  and  $i$  coefficients out of  $a_2, a_3, a_4, \dots, a_{n-2}$  are negative.

(b) If  $f(u_{a_1}) = 0$  and  $f(u_{a_{n-1}}) = 1$ , then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\ &= \frac{1}{2} [(M+1) - (n-1-i) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) - (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2} [a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i], \end{aligned}$$

where  $i = 2, 3, 4, \dots, \lfloor \frac{n}{2} \rfloor$  and  $i-1$  coefficients out of  $a_2, a_3, a_4, \dots, a_{n-2}$  are negative.

(c) If  $f(u_{a_1}) = f(u_{a_{n-1}}) = 1$ , then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left( \sum_{v \in v(0)} \deg(v) - \sum_{v \in v(1)} \deg(v) \right) \\ &= \frac{1}{2} [(M+1) - (n-1-i) - (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) - (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2} [-a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i + 2], \end{aligned}$$

where  $i = 2, 3, 4, \dots, \lfloor \frac{n}{2} \rfloor$  and  $i-2$  coefficients out of  $a_2, a_3, a_4, \dots, a_{n-2}$  are negative.

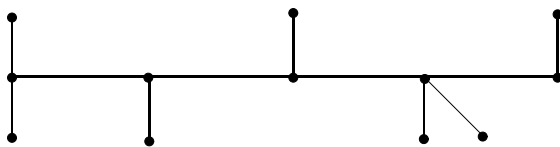
**Case 2.**  $n$  is odd.

**Subcase 2.1** If  $(a_1 + a_2 + a_3 + \cdots + a_{n-1})$  is odd, then the number of vertices of  $CT(a_1, a_2, a_3, \dots, a_{n-1})$  is  $a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 1$  which is odd. Therefore the proof is similar to Subcase 1.2.

**Subcase 2.2** If  $a_1 + a_2 + a_3 + \cdots + a_{n-1}$  is even, then the number of vertices of  $CT(a_1, a_2, a_3, \dots, a_{n-1})$  is  $a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 1$  which is even. Therefore the proof is similar to Subcase 1.1.  $\square$

**Example 2.2** Figure 2 shows the caterpillar  $CT(2, 1, 1, 2, 1)$  of diameter 6 and order 12 with

balance index set  $\{0, 1, 2, 3, 4, 5\}$ .



**Figure 2** The caterpillar  $CT(2, 1, 1, 2, 1)$  of diameter 6 and order 12.

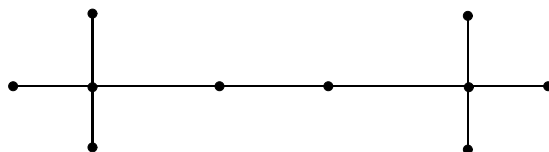
**Corollary 2.3** The balance index set of the graph  $B_{l,m,k}$ ,

$$\begin{aligned}
 BI(B_{l,m,k}) = & \left\{ \left\lfloor \frac{l+m+k}{2} \right\rfloor, \left\lfloor \frac{|l-m+k|}{2} \right\rfloor, \left\lfloor \frac{|-l+m+k|}{2} \right\rfloor, \left\lfloor \frac{l+m+k-2}{2} \right\rfloor \right\} \cup \\
 & \left\{ \left\lfloor \frac{l+m+k-2i}{2} \right\rfloor, \left\lfloor \frac{|l-m+k-2i+2|}{2} \right\rfloor, \left\lfloor \frac{|-l-m+k-2i+4|}{2} \right\rfloor : \right. \\
 & \left. i = 2, 3, 4, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\}.
 \end{aligned}$$

*Proof* The graph  $B_{l,m,k}$  is a caterpillar  $CT(l, 0, 0, \dots, m)$  of diameter  $k + 2$ . Therefore substituting  $n = k + 2$ ,  $a_1 = l$ ,  $a_{n-1} = m$  and  $a_2 = a_3 = a_4 = \dots = a_{n-2} = 0$  in the Theorem 2.1, we get

$$\begin{aligned}
 BI(B_{l,m,k}) = & \left\{ \left\lfloor \frac{l+m+k}{2} \right\rfloor, \left\lfloor \frac{|l-m+k|}{2} \right\rfloor, \left\lfloor \frac{|-l+m+k|}{2} \right\rfloor, \left\lfloor \frac{l+m+k-2}{2} \right\rfloor \right\} \\
 & \cup \left\{ \left\lfloor \frac{l+m+k-2i}{2} \right\rfloor, \left\lfloor \frac{|l-m+k-2i+2|}{2} \right\rfloor, \left\lfloor \frac{|-l-m+k-2i+4|}{2} \right\rfloor : \right. \\
 & \left. i = 2, 3, 4, \dots, \left\lfloor \frac{n}{2} \right\rfloor \right\}. \quad \square
 \end{aligned}$$

**Example 2.4** Figure 3 shows the graph  $B_{3,3,3}$  of diameter 5 and order 10 with balance index set  $\{0, 1, 2, 3, 4\}$ .



**Figure 3** The graph  $B_{3,3,3}$  of diameter 5 and order 10.

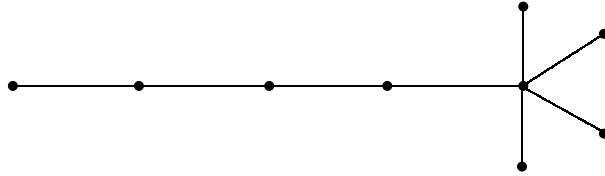
**Corollary 2.5** The balance index set of coconut tree  $CT(m, l)$ ,

$$BI(CT(m, l)) = \left\{ \left\lfloor \frac{l+m-2}{2} \right\rfloor, \left\lfloor \frac{|-l+m-2|}{2} \right\rfloor, \left\lfloor \frac{|l+m-4|}{2} \right\rfloor \right\} \cup \left\{ \left\lfloor \frac{l+m-2i-2}{2} \right\rfloor, \left\lfloor \frac{|-l+m-2i|}{2} \right\rfloor, \left\lfloor \frac{|-l-m-2i+2|}{2} \right\rfloor : i = 2, 3, 4, \dots, \left\lfloor \frac{m}{2} \right\rfloor \right\}.$$

*Proof* The coconut tree  $CT(m, l)$  is a caterpillar graph  $CT(0, 0, 0, \dots, m)$  of diameter  $m$ . Therefore substituting  $n = m$ ,  $a_{n-1} = l$  and  $a_1 = a_2 = a_3 = \dots = a_{n-2} = 0$  in the Theorem 2.1, we get

$$BI(CT(m, l)) = \left\{ \left\lfloor \frac{l+m-2}{2} \right\rfloor, \left\lfloor \frac{|-l+m-2|}{2} \right\rfloor, \left\lfloor \frac{|l+m-4|}{2} \right\rfloor \right\} \cup \left\{ \left\lfloor \frac{l+m-2i-2}{2} \right\rfloor, \left\lfloor \frac{|-l+m-2i|}{2} \right\rfloor, \left\lfloor \frac{|-l-m-2i+2|}{2} \right\rfloor : i = 2, 3, 4, \dots, \left\lfloor \frac{m}{2} \right\rfloor \right\}. \quad \square$$

**Example 2.6** Figure 4 shows coconut tree of diameter 5 and order 9 with balance index set  $\{0, 1, 2, 3, 5\}$ .



**Figure 4** The coconut tree of diameter 5 and order 9.

### §3. Balance Index Set of Lobster Graphs

In a caterpillar graph  $CT(a_1, a_2, a_3, \dots, a_{n-1})$ , if  $a_i \neq 0$  for  $i = 2, 3, \dots, n-2$ , then we have  $a_i, i = 2, 3, 4, \dots, n-2$  number of  $P_3$  paths contained the vertex  $u_{a_i}$ . Since  $P_3$  is of length 2, after adding more adjacent edges and vertices to the two end vertices of these paths, the new graph is a lobster graph of diameter  $n$ . We denote the new graph as

$$CT(a_1, a_2, a_3, \dots, a_{n-1})(u_{a_2}(t_{2,1}, t_{2,2}, t_{2,3}, \dots, t_{2,a_2}), (u_{a_3}(t_{3,1}, t_{3,2}, t_{3,3}, \dots, t_{3,a_3})), u_{a_4}(t_{4,1}, t_{4,2}, t_{4,3}, \dots, t_{4,a_4}), \dots, u_{a_{n-2}}(t_{n-2,1}, t_{n-2,2}, t_{n-2,3}, \dots, t_{n-2,a_{n-2}})),$$

where  $t_{i,j}$  is the number of edges and vertices added to the vertex  $u_{a_i,j}$ ,  $i = 2, 3, 4, \dots, n-2$ ,

$j = 1, 2, 3, \dots, a_i$ . Here we have

$$a_1 + a_{n-1} + \sum_{i=2}^{n-2} \sum_{j=1}^{a_i} t_{i,j}$$

pendant vertices.

In order to write the results in an uniform manner we name this lobster graph as

$$\begin{aligned} &CT(d_1, d_2, d_3, \dots, d_{n-2}, d_0)(u_{a_2}(d_{n-1}, d_n, d_{n+1}, \dots, d_{d_2+n-2}), \\ &u_{a_3}(d_{d_2+n-1}, d_{d_2+n}, d_{d_2+n+1}, \dots, d_{d_2+d_3+n-2}), \\ &u_{a_4}(d_{d_2+d_3+n-1}, d_{d_2+d_3+n}, d_{d_2+d_3+n+1}, \dots, d_{d_2+d_3+d_4+n-2}), \dots, \\ &u_{a_{n-2}}(d_{d_2+d_3+\dots+d_{n-3}+n-1}, d_{d_2+d_3+\dots+d_{n-3}+n}, d_{d_2+d_3+\dots+d_{n-3}+n+1}, \dots, d_{d_2+d_3+\dots+d_{n-2}+n-2}). \end{aligned}$$

We also name  $n - 1$  spine vertices by  $v_1, v_2, v_3, \dots, v_{n-2}, v_0$ , the vertices adjacent to  $v_2$  by  $v_{n-1}, v_n, v_{n+1}, \dots, v_{d_2+n-2}$ , adjacent to  $v_3$  by  $v_{d_2+n-1}, v_{d_2+n}, v_{d_2+n+1}, \dots, v_{d_2+d_3+n-2}$ , etc. and adjacent to  $v_{n-2}$  by  $v_{d_2+d_3+d_4+\dots+d_{n-3}+n-1}, v_{d_2+d_3+d_4+\dots+d_{n-3}+n}, v_{d_2+d_3+d_4+\dots+d_{n-3}+n+1}, \dots, v_{d_2+d_3+d_4+\dots+d_{n-2}+n-2}$ .

Thus in this lobster, we have  $d_0 + d_1 + \sum_{i=n-1}^m d_i$  pendant vertices where  $m = \sum_{j=2}^{n-2} d_j + n - 2$ , the degree of  $v_i$  for  $n - 1 \leq i \leq \sum_{j=2}^{n-2} d_j + n - 2$  is  $d_i + 1$  and the degree of  $v_k$  is  $d_i + 2$  for  $k = 2, 3, 4, \dots, n - 2$ .

**Theorem 3.1** For a lobster graph of diameter  $n$  and order  $p$ , the balance index is

$$e_{f^*}(0) - e_{f^*}(1) = \begin{cases} \frac{1}{2} \left[ \sum_{i=0}^m (-1)^{f(v_i)} d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} \right], & \text{if } p \text{ is even,} \\ \pm \frac{1}{2} \left[ 1 + \sum_{i=0}^m (-1)^{f(v_i)} d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} \right], & \text{if } p \text{ is odd,} \end{cases}$$

where  $m = \sum_{j=2}^{n-2} d_j + n - 2$ .

*Proof* Let  $G$  be a lobster graph of order  $p$  and diameter  $n$ .

**Case 1.**  $n$  is even.

**Subcase 1.1** If  $\sum_{i=0}^m d_i$  is odd, then the number of vertices equal to  $\sum_{i=0}^m d_i + n - 1$  is even. Let it be  $2M$ . For a friendly labeling, there are  $M$  vertices labeled 0 and remaining  $M$  vertices labeled 1. We first consider the case that  $v_i$  for all  $i$  are labeled 0. Then there are  $M - (n - 1) - \sum_{i=2}^{n-2} d_i$  pendant vertices labeled 0 and remaining  $M$  pendant vertices labeled 1. Then by

Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left[ M - (n-1) + 2(n-3) + d_0 + 1 + d_1 + 1 + \sum_{i=n-1}^m (d_i + 1) - M \right] \\ &= \frac{1}{2} \left[ \sum_{i=n-1}^m d_i + \sum_{j=0}^{n-2} d_j + n - 3 \right] = \frac{1}{2} \left[ \sum_{i=0}^m d_i + n - 3 \right]. \end{aligned}$$

Similarly we assume that there are  $k$  vertices among  $v_i$  for all  $0 \leq i \leq \sum_{j=2}^{n-2} d_j + n - 2$  labeled 0. Then there are  $M - k$  pendant vertices labeled 0 and  $M - \left[ \sum_{j=2}^{n-2} d_j + n - 1 - k \right]$  pendant vertices labeled 1. We define  $P$  to be the set containing all the 0-vertices among  $v_i$  for all  $0 \leq i \leq \sum_{j=2}^{n-2} d_j + n - 2$ . We also name  $N$  to be the set containing all the 1-vertices among  $v_i$  for all  $0 \leq i \leq \sum_{j=2}^{n-2} d_j + n - 2$ . Then by Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left[ M - k + \sum_{v \in P} \deg(v) - \left( M - \left[ \sum_{j=2}^{n-2} d_j + n - 1 - k \right] + \sum_{v \in N} \deg(v) \right) \right] \\ &= \frac{1}{2} \left[ M - k + \left( \sum_{v \in P} (\deg(v) - 1) + 1 \right) - M + \sum_{j=2}^{n-2} d_j \right] \\ &\quad + \left[ n - 1 - k - \left( \sum_{v \in N} (\deg(v) - 1) + 1 \right) \right] \\ &= \frac{1}{2} \left[ M - k + \left( \sum_{v \in P} (\deg(v) - 1) \right) + k - M + \sum_{j=2}^{n-2} d_j \right] \\ &\quad + \left[ n - 1 - k - \left( \sum_{v \in N} (\deg(v) - 1) \right) - \left( \sum_{j=2}^{n-2} d_j + n - 1 - k \right) \right] \\ &= \frac{1}{2} \left[ \sum_{v \in P} (\deg(v) - 1) - \sum_{v \in N} (\deg(v) - 1) \right]. \end{aligned}$$

Also note that

$$\deg(v) - 1 = \begin{cases} d_i, & \text{if } i = 0, 1 \text{ and } n - 3 \leq i \leq \sum_{j=2}^{n-2} d_j + n - 2 \\ d_i + 1, & \text{if } 2 \leq i \leq n - 2 \end{cases}$$

Therefore

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left[ (-1)^{f(v_0)} d_0 + (-1)^{f(v_1)} d_1 + \sum_{i=n-3}^m (-1)^{f(v_i)} d_i + \sum_{j=2}^{n-2} (-1)^{f(v_i)} (d_i + 1) \right] \\ &= \frac{1}{2} \left[ \sum_{i=0}^m (-1)^{f(v_i)} d_i + \sum_{j=2}^{n-2} (-1)^{f(v_i)} \right]. \end{aligned}$$

**Subcase 1.2.** If  $\sum_{i=0}^m d_i$  is even, then the number of vertices equal to  $\sum_{i=0}^m d_i + n - 1$  is odd. Let it

be  $2M+1$ . For a friendly labeling, there are  $M+1$  vertices labeled 0 and remaining  $M$  vertices labeled 1. We first consider the case that  $v_i$  for all  $i$  are labeled 0. Then there are  $(M+1) - (n-1) - \sum_{i=2}^{n-2} d_i$  pendant vertices labeled 0 and remaining  $M$  pendant vertices labeled 1. Then again by Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left[ (M+1) - (n-1) + 2(n-3) + d_0 + 1 + d_1 + 1 + \sum_{i=n-1}^m (d_i + 1) - M \right] \\ &= \frac{1}{2} \left[ \sum_{i=n-1}^m d_i + \sum_{j=0}^{n-2} d_j + n - 2 \right] = \frac{1}{2} \left[ \sum_{i=0}^m d_i + n - 2 \right]. \end{aligned}$$

Similarly we assume that there are  $k$  vertices among  $v_i$  for all  $0 \leq i \leq \sum_{j=2}^{n-2} d_j + n - 2$  labeled 0.

Then there are  $M+1-k$  pendant vertices labeled 0 and  $M - \left[ \sum_{j=2}^{n-2} d_j + n - 1 - k \right]$  pendant vertices

labeled 1. We define  $P$  to be the set containing all the 0-vertices among  $v_i$  for all  $0 \leq i \leq \sum_{j=2}^{n-2} d_j + n - 2$ .

We also name  $N$  to be the set containing all the 1-vertices among  $v_i$  for all  $0 \leq i \leq \sum_{j=2}^{n-2} d_j + n - 2$ .

Then by Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left[ \left( M+1-k + \sum_{v \in P} \deg(v) \right) - \left( M - \left[ \sum_{j=2}^{n-2} d_j + n - 1 - k \right] + \sum_{v \in N} \deg(v) \right) \right] \\ &= \frac{1}{2} \left[ M+1-k + \left( \sum_{v \in P} (\deg(v) - 1) + 1 \right) - M + \sum_{j=2}^{n-2} d_j \right] \\ &\quad + \left[ n-1-k - \left( \sum_{v \in N} (\deg(v) - 1) + 1 \right) \right] \\ &= \frac{1}{2} \left[ M+1-k + \left( \sum_{v \in P} (\deg(v) - 1) \right) + k - M + \sum_{j=2}^{n-2} d_j \right] \\ &\quad + \left[ n-1-k - \left( \sum_{v \in N} (\deg(v) - 1) \right) - \left( \sum_{j=2}^{n-2} d_j + n - 1 - k \right) \right] \\ &= \frac{1}{2} \left[ 1 + \sum_{v \in P} (\deg(v) - 1) - \sum_{v \in N} (\deg(v) - 1) \right]. \end{aligned}$$

Also note that

$$\deg(v) - 1 = \begin{cases} d_i, & \text{if } i = 0, 1 \text{ and } n-3 \leq i \leq \sum_{j=2}^{n-2} d_j + n - 2 \\ d_i + 1, & \text{if } 2 \leq i \leq n-2 \end{cases}$$

Therefore,

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2} \left[ 1 + (-1)^{f(v_0)} d_0 + (-1)^{f(v_1)} d_1 + \sum_{i=n-3}^m (-1)^{f(v_i)} d_i + \sum_{j=2}^{n-2} (-1)^{f(v_i)} (d_i + 1) \right] \\ &= \frac{1}{2} \left[ 1 + \sum_{i=0}^m (-1)^{f(v_i)} d_i + \sum_{j=2}^{n-2} (-1)^{f(v_i)} \right]. \end{aligned}$$

When a friendly labeling with  $v_f(1) > v_f(0)$ , it produces the negative values of the above balance indexes. Therefore,

$$e_{f^*}(0) - e_{f^*}(1) = \pm \frac{1}{2} \left[ 1 + \sum_{i=0}^m (-1)^{f(v_i)} d_i + \sum_{j=2}^{n-2} (-1)^{f(v_i)} \right].$$

**Case 2.**  $n$  is odd.

**Subcase 2.1** If  $\sum_{i=0}^m d_i$  is odd, then the number of vertices equal to  $\left[ \sum_{i=0}^m d_i \right] + n - 1$  is odd and proof is similar to Subcase 1.2.

**Subcase 2.2** If  $\sum_{i=0}^m d_i$  is even, then the number of vertices equal to  $\left[ \sum_{i=0}^m d_i \right] + n - 1$  is even and proof is similar to Subcase 1.1.

Therefore, for a lobster graph of diameter  $n$  and order  $p$ , the balance index is

$$e_{f^*}(0) - e_{f^*}(1) = \begin{cases} \frac{1}{2} \left[ \sum_{i=0}^m (-1)^{f(v_i)} d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} \right], & \text{if } p \text{ is even} \\ \pm \frac{1}{2} \left[ 1 + \sum_{i=0}^m (-1)^{f(v_i)} d_i + \sum_{i=2}^{n-2} (-1)^{f(v_i)} \right], & \text{if } p \text{ is odd} \end{cases} \quad \square$$

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