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Biprimitive Semisymmetric Graphs on $PSL(2,p)$

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Abstract: A simple undirected graph is said to be *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. It is easy to see that every semisymmetric graph is necessarily bipartite, with the two parts having equal size and the automorphism group acting transitively on each of these two parts. A semisymmetric graph is called *biprimitive* if its automorphism group acts primitively on each part. This paper gives a classification of biprimitive semisymmetric graphs arising from the action of the group $PSL(2,p)$ on cosets of A_5, where $p \equiv 1\,(\mathrm{mod}10)$ is a prime. By the way, the structure of the suborbits of $PGL(2,p)$ on the cosets of A_5 is determined.

Keywords: Smarandache multi-group, group, semisymmetric graph, Biprimitive semisymmetric graph, suborbit.

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§1. Introduction

For the group- and graph-theoretic terminology we refer the reader to [1,7]. All graphs considered in this paper are finite, undirected and simple. For a graph X, we use $V(X)$, $E(X)$, $A(X)$ and $Aut(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. If X be a bipartite with bipartition $V(X) = U(X) \cup W(X)$. Set

$$A^+ = \langle g \in A \mid U(X)^g = U(X), W(X)^g = W(X) \rangle .$$

Clearly, if X is connected then either $|A : A^+| = 2$ or $A = A^+$, depending on whether or not there exists an automorphism which interchanges the two parts $U(X)$ and $W(X)$. Suppose G is a subgroup of A^+. Then X is said to be *G-semitransitive* if G acts transitively on both $U(X)$ and $W(X)$, and *semitransitive* if X is A^+-semitransitive. Also X is said to be *biprimitive* if A^+ acts primitively on each part. We call a graph *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. It is easy to see that every semisymmetric graph is a bipartite graph with two parts of equal size and is semitransitive.

The first person who studied semisymmetric graphs was Folkman. In 1967 he constructed

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several infinite families of such graphs and proposed eight open problems (see [6]). Afterwards, Bouwer, Titov, Klin, A.V. Ivanov, A.A. Ivanov and others did much work on semisymmetric graphs (see [2-3,8-10,13]). They gave new constructions of such graphs and nearly solved all of Folkman's open problems. In particular, Iofinova and Ivanov [9] in 1985 classified biprimitive semisymmetric cubic graphs using group-theoretical methods; this was the first classification theorem for such graphs. More recently, following some deep results in group theory which depend on the classification of finite simple groups, some new methods and results in vertex-transitive graphs and semisymmetric graphs have appeared. In [5], for example, the authors give a classification of semisymmetric graphs of order $2pq$ where p and q are distinct primes. It is shown that there are 131 examples of such graphs, which are biprimitive. In [4] a classification is given, of biprimitive semisymmetric graphs arising from the action of the group $PSL(2,p)$, $p \equiv 1\,(\mathrm{mod} 8)$ a prime, on cosets of S_4. In this paper, we will classify all biprimitve graphs arising from the action of the group $PSL(2,p)$, $p \equiv 1\,(\mathrm{mod} 10)$ a prime, on cosets of A_5. To prove the classification theorem, we have to determine the suborbits of $PGL(2,p)$ acts on the cosets of A_5 and such a determination will certainly be useful for other problems.

Throughout the paper, Z_n and D_n denote the cyclic group of order n and the dihedral group of order n, respectively. A semidirect product of the group N by the group H will be denoted by $N:H$. Given a group G and a subgroup H of G, we use $[G:H]$ to denote the set of right cosets of H in G. The action of G on $[G:H]$ is always assumed to be the right multiplication action. More precisely, for $g \in G$, we use $R(g)$ to denote the effect of right multiplication of g on $[G:H]$ and let $R(G) = \{R(g) | g \in G\}$. However, for convenience, in most cases we will identify $R(g)$ with g, except for the special cases to be stated.

A Smarandache multi-group $\mathscr{G}$ is an union of groups $(G_1;\circ_1), (G_2;\circ_2), \cdots, (G_n;\circ_n)$, different two by two for an integer $n \geq 1$. Particularly, if $n = 1$, then $\mathscr{G}$ is just a group. A Smarandache multi-group $\mathscr{G}$ is naturally acting on its underlying graph $G[\mathscr{G}]$. In [5], the authors gave a group-theoretic construction of semitransitive graphs by introducing the definition of so called *bi-coset graph* as following: Let G be a group, let L and R be subgroups of G and let D be a union of double cosets of R and L in G, namely, $D = \cup_i R d_i L$. Define a bipartite graph $X = \mathbf{B}(G,L,R;D)$ with bipartition $V(X) = [G:L] \cup [G:R]$ and edge set $E(X) = \{\{Lg, Rdg\} | g \in G, d \in D\}$. This graph is called the bi-coset graph of G with respect to L, R and D.

Note that in the above construction of semitransitive graphs, if L and R are the same subgroup, then we still use Lg and Rg to denote different vertices in the two parts of $V(X)$. It is proved in [5] that (1) the graph $X = \mathbf{B}(G,L,R;D)$ is a well-defined bipartite graph, and under the right multiplication action on $V(X)$ of G, the graph X is G-semitransitive; (2) every G-semitranstive graph is isomorphic to one of such bi-coset graphs.

Now we state the main theorem of this paper.

Theorem 1.1 *Let $p \equiv 1(\mathrm{mod}10)$, $G = PSL(2,p)$ and $Q = PGL(2,p)$. Let Y be a biprimitive semisymmetric graph with a subgroup G of $\mathrm{Aut}(Y)$ acting edge-transitively on Y and having A_5 as a vertex stabilizer. Then Y is isomorphic to one of the following graphs:*

(i) *$B(G,L,L;D)$, where $L \cong A_5$ and D is a double coset corresponds to a non-self-paired* *suborbit of G relative to L.*

(ii) $B(G,L,L^\sigma;D)$*, where* σ *is an involution in* $Q\setminus G$ *and* $L\sigma dL$ *corresponds to a non-self-paired suborbit of* Q *relative to* L.

Moreover, each such graph Y *is of order* $\frac{p^3-p}{60}$ *and valency 60, and with the automorphism group* $PSL(2,p)$*. Table 1 lists the total numbers* n_1 *and* n_2 *of nonisomorphic semisymmetric graphs* $B(G,L,L;D)$ *and* $B(G,L,L^\sigma,D)$ *for each of the congruence classes of* p.

TABLE 1.

$p(\text{mod}120)$	n_1	n_2
1	$\frac{p^3-60p^2+1077p-15418}{14400}$	$\frac{p^3-60p^2+1197p-1138}{14400}$
−1	$\frac{p^3-60p^2+1197p-13142}{14400}$	$\frac{p^3-60p^2+1077p+1138}{14400}$
11	$\frac{p^3-60p^2+1197p-7238}{14400}$	$\frac{p^3-60p^2+1077p-5918}{14400}$
−11	$\frac{p^3-60p^2+1077p-8362}{14400}$	$\frac{p^3-60p^2+1197p-7024}{14400}$
31	$\frac{p^3-60p^2+1197p-9238}{14400}$	$\frac{p^3-60p^2+1077p-5518}{14400}$
−31	$\frac{p^3-60p^2+1077p-8762}{14400}$	$\frac{p^3-60p^2+1197p-5042}{14400}$
41	$\frac{p^3-60p^2+1077p-12218}{14400}$	$\frac{p^3-60p^2+1197p-2738}{14400}$
−41	$\frac{p^3-60p^2+1197p-11542}{14400}$	$\frac{p^3-60p^2+1077p-2062}{14400}$
61	$\frac{p^3-60p^2+1077p-11818}{14400}$	$\frac{p^3-60p^2+1197p-4738}{14400}$
−61	$\frac{p^3-60p^2+1197p-9542}{14400}$	$\frac{p^3-60p^2+1077p-2462}{14400}$
71	$\frac{p^3-60p^2+1197p-10838}{14400}$	$\frac{p^3-60p^2+1077p-2318}{14400}$
−71	$\frac{p^3-60p^2+1077p-11962}{14400}$	$\frac{p^3-60p^2+1197p-3442}{14400}$
91	$\frac{p^3-60p^2+1197p-5638}{14400}$	$\frac{p^3-60p^2+1077p-9118}{14400}$
−91	$\frac{p^3-60p^2+1077p-5162}{14400}$	$\frac{p^3-60p^2+1197p-8642}{14400}$
101	$\frac{p^3-60p^2+1077p-8618}{14400}$	$\frac{p^3-60p^2+1197p-6338}{14400}$
−101	$\frac{p^3-60p^2+1197p-7942}{14400}$	$\frac{p^3-60p^2+1077p-5662}{14400}$

§2. Preliminaries

In this section, some preliminary results are given. The first two propositions give some properties of the groups $PSL(2,p)$ and $PGL(2,p)$.

Proposition 2.1 ([11], Lemma 2.1) *Let* p *be an odd prime. Then*

(1) *the maximal subgroups of* $PSL(2,p)$ *are:*

One class of subgroups isomorphic to $Z_p : Z_{\frac{p-1}{2}}$*; one class isomorphic to* D_{p-1}*, when* $p \geqslant 13$*; one class isomorphic to* D_{p+1}*, when* $p \neq 7$*; two classes isomorphic to* A_5*, when* $p \equiv 1(\text{mod}10)$*; two classes isomorphic to* S_4*, when* $p \equiv 1(\text{mod}8)$*; and one class isomorphic to* A_4*, when* $p = 5$ *or* $p \not\equiv 1(\text{mod}8)$.

(2) *The maximal subgroups of* $PGL(2,p)$ *are:*

One class of subgroups isomorphic to $Z_p : Z_{p-1}$*; one class isomorphic to* $D_{2(p-1)}$*, when*

$p \geqslant 7$; *one class isomorphic to* $D_{2(p+1)}$; *one class isomorphic to* S_4, *when* $p = 5$ *or* $p \not\equiv 1(\text{mod}40)$ *and* $p \geqslant 5$; *and one subgroup* $PSL(2,p)$.

Proposition 2.2 ([5], Lemma 3.9) *Any extension of* $PSL(2,p)$ *by* Z_2 *is isomorphic to either* $PGL(2,p)$ *or* $PSL(2,p) \times Z_2$. *In both cases the extension is split.*

Proposition 2.3 ([5], Lemma 2.3) *The graph* $X = \mathbf{B}(G,L,R;D)$ *is a well-defined bipartite graph. Under the right multiplication action on* $V(X)$ *of* G, *the graph* X *is* G*-semitransitive. The kernel of the action of* G *on* $V(X)$ *is* $\text{Core}_G(L) \cap \text{Core}_G(R)$, *the intersection of the cores of the subgroups* L *and* R *in* G. *Furthermore, we have*

(i) X is G-edge-transitive if and only if $D = RdL$ for some $d \in G$;

(ii) the degree of any vertex in $[G : L]$ (resp. $[G : R]$) is equal to the number of right cosets of R (resp. L) in D (resp. D^{-1}), so X is regular if and only if $|L| = |R|$;

(iii) X is connected if and only if G is generated by elements of $D^{-1}D$;

(iv) $X \cong \mathbf{B}(G, L^a, R^b; D')$ where $D' = \bigcup_i R^b\ (b^{-1}d_i a)L^a$, for any $a, b \in G$.

The next proposition provides one general and three particular conditions, each of which is sufficient for a G-semitransitive graph to be vertex-transitive.

Proposition 2.4 ([5], Lemma 2.6) *Let* $X = \mathbf{B}(G,L,R;D)$. *If there exists an involutory automorphism* σ *of* G *such that* $L^\sigma = R$ *and* $D^\sigma = D^{-1}$, *then* X *is vertex-transitive. In particular,*

(i) *If* G *is abelian and acts regularly on both parts of* X, *then* X *is vertex-transitive. In other words, bi-Cayley graphs of abelian groups are vertex-transitive.*

(ii) *If there exists an involutory automorphism* σ *of* G *such that* $L^\sigma = R$, *and the lengths of the orbits of* L *on* $[G : R]$ *(or the orbits of* R *on* $[G : L]$*) are all distinct, then* X *is vertex-transitive.*

(iii) *If the representations of* G *on the two parts of* X *are equivalent and all suborbits of* G *relative to* L *are self-paired, then* X *is vertex-transitive.*

The link between groups and graphs that we use is the concept of the orbital graph of a permutation group. For the terminology of orbital graph we refer the reader to [12].

The following group theoretical results will be used later.

Proposition 2.5 ([11], Lemma 2.1) *Let* G *be a transitive group on* Ω *and let* $H = G_\alpha$ *for some* $\alpha \in \Omega$. *Suppose that* $K \leqslant G$ *and at least one* G*-conjugate of* K *is contained in* H. *Suppose further that the set of* G*-conjugates of* K *which are contained in* H *form* t *conjugacy classes of* H *with representatives* $K_1, K_2, \cdots, K_t$. *Then* K *fixes* $\sum_{i=1}^{t} |N_G(K_i) : N_H(K_i)|$ *points of* Ω.

Proposition 2.6 ([11], Lemma 2.2) *Let* G *be a primitive permutation group on* Ω, *and let* $H = G_\alpha$ *for some* $\alpha \in \Omega$. *Suppose that* $H = A_5$ *and let* $K_1, \ldots, K_7$ *be seven subgroups of* H *satisfying* $K_1 \cong A_4$, $K_2 \cong D_{10}$, $K_3 \cong D_6$, $K_4 \cong Z_5$, $K_5 \cong Z_3$, $K_6 \cong D_4$ *and* $K_7 \cong Z_2$. *Let* k_i *be the number of points in* Ω *fixed by* K_i, *for* $i = 1, 2, \ldots, 7$. *Then* G *has 1 suborbit of length 1,* $k_1 - 1$ *suborbits of length 5,* $k_2 - 1$ *suborbits of length 6,* $k_3 - 1$ *suborbits of length 10,* $\frac{1}{2}(k_4 - k_2)$ *suborbits of length 12,* $\frac{1}{2}(k_5 - 2k_1 - k_3 + 2)$ *suborbits of length 20,* $\frac{1}{3}(k_6 - k_1)$ *suborbits of length 15,* $\frac{1}{2}(k_7 - 2k_2 - 2k_3 - k_6 + 4)$ *suborbits of length 30, and all the other suborbits have length 60.*

Proposition 2.7 ([11], Lemma 2.3) *Let* $D = D_{2n}$ *be the dihedral group of order* $2n$, *considered* *as a permutation group of degree n generated by $a = (1, 2, \cdots, n)$ and $b = (1)(2, n)(3, n - 1) \cdots (i, n + 2 - i) \cdots$, for any $n \geqslant 2$. Then the nontrival orbitals of D are $\Gamma_i = (1, i)^D = (1, n + 2 - i)^D$, for $2 \leqslant i \leqslant (n + 2)/2$. Each of these orbitals is self-paired. Moreover, for all points i, j, with $i \neq j$, there is an involution in D which interchanges i and j.*

Proposition 2.8 ([11], Lemma 2.4) *Let G be a transitive group on Ω and let $H = G_\alpha$ for some $\alpha \in \Omega$. Suppose that G has t conjugacy classes of involutions, say $\mathcal{C}_1, \cdots, \mathcal{C}_t$. Suppose further that a representative u_j in $\mathcal{C}_j$ has N_j cycles of length 2, and that the centralizer of u_j in G has order c_j. Also for a nontrivial self-paired suborbit Δ relative to α and a point $\mathbf{B} \in \Delta$, let $inv(\Delta)$ be the number of involutions in G with a 2-cycle $(\sigma, \mathbf{B})$. Then $\sum_{j=1}^{t} \frac{N_j}{c_j} = \frac{1}{2|H|} \sum_{\Delta=\Delta^*} |\Delta(\alpha)| inv(\Delta)$, where c_j is the order of the centralizer of u_j.*

§3. Proof of Theorem 1.1

Now we begin the proof of Theorem 1.1. From now on we shall assume that $G = PSL(2, p)$ and $Q = PGL(2, p)$, where $p \equiv 1 \,(\mathrm{mod} 10)$. Clearly, $Q = G : \langle\sigma\rangle$ for some involution $\sigma \in Q \setminus G$. Let Y be a semisymmetric biprimitive graph with a subgroup G of $\mathrm{Aut}(Y)$ acting edge-transitively on Y and having A_5 as a vertex stabilizer. Let $U(Y)$ and $W(Y)$ be the bipartition of $V(Y)$. Then $|U(Y)| = |W(Y)| = \frac{p^3 - p}{120}$ and $G_v \cong A_5$ for any $v \in U(Y)$ and $v \in W(Y)$. Now Y is isomorphic to the bi-coset graph $X = B(G, L, R; D)$, where $L \cong R \cong A_5$. With our notation, $V(X) = U(X) \cup W(X) = [G : L] \cup [G : R]$. We will treat the following two cases separately:

(1) Suppose the representations of G on $U(X)$ and $W(X)$ are equivalent. In this case, by Proposition 2.3 (iv), no loss of any generality, we may assume $L = R \cong A_5$. With the completely similar arguments as in [5, Lemma 4.1], we may show that X is semisymmetric if and only if $D^{-1} \neq D$, that is, D corresponds to a non-self-paired suborbit of G relative to L, and two such bi-coset graphs defined (for the same group G) by distinct double cosets D_1 and D_2 are isomorphic if and only if D_1 and D_2 are paired with each other in G, or more precisely, $D_1 = D_2^{-1}$.

(2) Suppose the representations of G on $U(X)$ and $W(X)$ are inequivalent. Let $Q = PGL(2, p) = \langle G, \sigma\rangle$, where $\sigma \in Q \setminus G$ and $\sigma^2 = 1$. By the Proposition 2.1, G has two conjugacy classes of subgroups isomorphic to A_5, which are fused by σ. Therefore, we may let $R = L^\sigma$ so that $X = \mathbf{B}(G, L, L^\sigma; D)$ where $D = L^\sigma dL$ for some $d \in G$. With the similar arguments as in [5, Lemma 4.2], X is semisymmetric if and only if the suborbit $L\sigma dL$ of Q relative to L is not self-paired, and two such graphs $X_1 = \mathbf{B}(G, L, R; D_1)$ and $X_2 = \mathbf{B}(G, L, R; D_2)$ defined by distinct double cosets $D_1 := Rd_1L$ and $D_2 := Rd_2L$ respectively are isomorphic if and only if $D_1' := L\sigma d_1 L$ and $D_2' := L\sigma d_2 L$ are paired with each other in $Q = PGL(2, p)$.

Following the above two cases, we need to determine non-self-paired suborbits of G relative to L and non-self-paired suborbits of Q relative to L which are contained in $[Q : L] \setminus [G : L]$. Noting that the number of non-self-paired suborbits of G relative to L is the same as the number of non-self-paired suborbits of Q relative to L which are contained in $[G : L]$. From now on let $\Omega = [Q : L]$, $\Omega_1 = [G : L]$ and $\Omega_2 = [Q : L] \setminus [G : L]$. We will consider the action of Q on Ω and find all non-self-paired suborbits of Q contained in Ω_1 and in Ω_2 as well. We shall do this only for the case where $G = PSL(2, p)$ and $L = A_5$, $p \equiv 1 (\mathrm{mod} \;\; 120)$, and for the other cases, similar arguments and computations lead to the data listed in Appendix: TABLE $4-1$-TABLE $4-4$.

Let K_i (for $1 \leqslant i \leqslant 7$) be the representatives of the seven conjugacy classes of nontrivial subgroups of L isomorphic to A_4, D_{10}, D_6, Z_5, Z_3, D_4 and Z_2, respectively, and let $K_8 = 1$. Clearly any nontrivial subgroup K of L with a fixed point on Ω must be conjugate to one of these K_i. For each $i \in \{1, \dots, 8\}$, let k_i, k_{i1} and k_{i2} denote the respective numbers of fixed points of K_i in Ω, Ω_1 and Ω_2. Among of all the suborbits with the L-stabilizer K_i, let x_{i1} and x_{i2} denote the respective numbers of the suborbits contained in Ω_1 and Ω_2; let y_i, y_{i1}, $y_{i2} = y_i - y_{i1}$ denote the respective numbers of self-paired suborbits contained in Ω, Ω_1 and Ω_2; and let $h_{i1} = x_{i1} - y_{i1}$ and $h_{i2} = x_{i2} - y_{i2}$ denote the respective numbers of non-self-paired suborbits contained in Ω_1 and Ω_2.

First we determine the values of x_{i1} and x_{i2}. For $i \in \{1, \dots, 7\}$, these values are given in TABLE 2 and are obtained in the following way. After having determined the respective normalizers of each K_i in L and in G (resp. Q), we apply Proposition 2.5 to calculate k_{i1} (resp. k_i). Then $k_{i2} = k_i - k_{i1}$ can be found also. By Proposition 2.6, we can determine the values of x_{i1} and x_{i2}, $1 \le i \le 7$.

TABLE 2.

i	1	2	3	4	5	6	7
K_i	A_4	D_{10}	D_6	Z_5	Z_3	D_4	Z_2
$N_L(K_i)$	A_4	D_{10}	D_6	D_{10}	D_6	A_4	D_4
$N_G(K_i)$	S_4	D_{20}	D_{12}	D_{p-1}	D_{p-1}	S_4	D_{p-1}
k_{i1}	2	2	2	$\frac{p-1}{10}$	$\frac{p-1}{6}$	2	$\frac{p-1}{4}$
x_{i1}	1	1	1	$\frac{p-1}{20}-1$	$\frac{p-1}{12}-2$	0	$\frac{p-1}{8}-3$
$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p-1)}$	S_4	$D_{2(p-1)}$
k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p-1}{3}$	2	$\frac{p-1}{2}$
k_{i2}	0	0	0	$\frac{p-1}{10}-1$	$\frac{p-1}{6}$	0	$\frac{p-1}{4}$
x_{i2}	0	0	0	$\frac{p-1}{20}$	$\frac{p-1}{12}$	0	$\frac{p-1}{8}$

Finally,

$$\begin{aligned} x_{81} &= \frac{1}{60}\left(\frac{p^3-p}{120} - 1 - \sum_{i=1}^{7} x_{i1}\frac{60}{|K_i|}\right) \\ &= \frac{1}{60}\left(\frac{p^3-p}{120} - 1 - 1\cdot 5 - 1\cdot 6 - 1\cdot 10 - \left(\frac{p-1}{20}-1\right)\cdot 12 \right. \\ &\quad \left. - \left(\frac{p-1}{12}-2\right)\cdot 20 - \left(\frac{p-1}{8}-3\right)\cdot 30\right) \\ &= \frac{p^3 - 723p + 15122}{7200} \end{aligned}$$

and a similar computation gives

$$x_{82} = \frac{1}{60}\left(\frac{p^3-p}{120} - 1 - \sum_{i=2}^{7} x_{i2}\frac{60}{|K_i|}\right)$$

$$= \frac{1}{60}\left(\frac{p^3-p}{120} - 1 - \frac{p-1}{20}\cdot 12 - \frac{p-1}{12}\cdot 20 - \frac{p-1}{8}\cdot 30\right)$$
$$= \frac{p^3-723p+722}{7200}.$$

Next we determine the values of h_{i1} and h_{i2}. We claim all the non-regular suborbits of Q are self-paired, so that $h_{i1} = h_{i2} = 0$ for $1 \leqslant i \leqslant 7$. For example, let $i = 7$ and let Δ be a suborbit with L-stabilizer $K_7 = Z_2$, and take $v \in \Delta$. We consider the action of $N_Q(K_7) \cong D_{2(p-1)}$ on $Fix(K_7)$, the set of fixed points of K_7 on Ω. This action is transitive and the kernel is Z_2. Since $|Fix(K_7)| = \frac{p-1}{2}$, by Proposition 2.7, there exists an element in $N_Q(K_7)$ interchanging $u = L$ and v. So Δ is self-paired, or equivalently, $h_{71} = h_{72} = 0$.

It remains to determine h_{81} and h_{82}, the numbers of non-self-paired suborbits of Q in Ω_1 and in Ω_2 respectively. For these it suffices to calculate y_{81} and y_8, the numbers of self-paired regular suborbits of Q in Ω_1 and in Ω, since $h_{81} = x_{81} - y_{81}$, $h_{82} = x_{82} - y_{82}$ and $y_8 = y_{81} + y_{82}$. By Proposition 2.8, in order to calculate y_{81} (resp. y_8), we need the value of $inv(\Delta)$, which is defined in Proposition 2.8 for all self-paired suborbits Δ of G (resp. Q). Furthermore, to calculate $inv(\Delta)$ we need to know G_{uv} and $G_{\{u,v\}}$ (resp. Q_{uv} and $Q_{\{u,v\}}$), where $u = L$ and $v \in \Delta$.

The lengths l_i ($1 \leqslant i \leqslant 8$) of self-paired suborbits with point stabilizer K_i, the numbers y_{i1} and y_i, the groups G_{uv}, $G_{\{u,v\}}$ and Q_{uv} and $Q_{\{u,v\}}$, and the value of $inv(\Delta)$ for each Δ are listed in the following table.

TABLE 3.

i	l_i	y_{i1}	y_i	$G_{uv} = Q_{uv}$	$G_{\{u,v\}} = Q_{\{u,v\}}$	$inv(\Delta)$
1	5	1	1	A_4	S_4	6
2	6	1	1	D_{10}	D_{20}	6
3	10	1	1	D_6	D_{12}	4
4	12	$\frac{p-1}{20} - 1$	$\frac{p-1}{10} - 1$	Z_5	D_{10}	5
5	20	$\frac{p-1}{12} - 2$	$\frac{p-1}{6} - 2$	Z_3	D_6	3
7	30	$\frac{p-1}{8} - 3$	$\frac{p-1}{4} - 3$	Z_2	D_4	2
8	60	y_{81}	y_8	1	Z_2	1

Next we shall calculate y_{81} and y_8 using Proposition 2.8. We know that Q has two conjugacy classes of involutions. A representative of the first class, say $u_1 \in G$, fixes $\frac{p-1}{2}$ points, and so u_1 contains $N_1 = \frac{\frac{p^3-p}{60} - \frac{p-1}{2}}{2} = \frac{p^3-31p+30}{120}$ cycles of length 2. Further, $C_Q(u_1) \cong D_{2(p-1)}$ has order $c_1 = 2(p-1)$. A representative of the second class, say $u_2 \in Q \setminus G$, has no fixed point and so u_2 contains $N_2 = \frac{p^3-p}{120}$ cycles of length 2. Also $C_Q(u_2) \cong D_{2(p+1)}$ has order $c_2 = 2(p+1)$. By Proposition 2.8 and TABLE 3, we have

$$\frac{p^3-31p+30}{240(p-1)} + \frac{p^3-p}{240(p+1)} = \frac{1}{2\cdot 60}(1\cdot 5\cdot 6 + 1\cdot 6\cdot 6 + 1\cdot 10\cdot 4$$
$$+ \left(\frac{p-1}{10} - 1\right)\cdot 12\cdot 5 + \left(\frac{p-1}{6} - 2\right)\cdot 20\cdot 3 + \left(\frac{p-1}{4} - 3\right)\cdot 30\cdot 2 + 60y_8\right).$$

It follows that $y_8 = \frac{p^2-31p+270}{60}$.

To determine y_{81} and y_{82}, we turn to the group G. Note that G has only one conjugacy class of involutions, and each involution u has precisely $\frac{p-1}{4}$ fixed points in Ω_1 and so has $N = \frac{\frac{p^3-p}{120} - \frac{p-1}{4}}{2} = \frac{p^3-31p+30}{240}$ cycles of length 2. Also $C_G(u) \cong D_{p-1}$ has order $c = p - 1$. By Proposition 2.8 and TABLE 3, we may calculcate $y_{81} = \frac{p^2-30p+509}{120}$. Hence $y_{82} = y_8 - y_{81} = \frac{p^2-32p+31}{120}$ and so $h_{81} = x_{81} - y_{81} = \frac{p^3-60p^2+1077p-15418}{7200}$ and $h_{82} = x_{82} - y_{82} = \frac{p^3-60p^2+1197p-1138}{7200}$.

Hence we find that Q has $\frac{p^3-60p^2+1077p-15418}{7200}$ non-self-paired regular suborbits, which have length 60 and are contained in Ω_1 and Q has $\frac{p^3-60p^2+1197p-1138}{7200}$ non-self-paired regular suborbits, which have length 60 and are contained in Ω_2. So we have $\frac{p^3-60p^2+1077p-15418}{14400}$ semisymmetric graphs X with valency 60 in case (i) and $\frac{p^3-60p^2+1197p-1138}{14400}$ semisymmetric graphs X with valency 60 in case (ii) , as listed in TABLE 1.

Thus we finish the proof of Theorem 1.1. □

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Appendix:

TABLE 4-1

	i	1	2	3	4	5	7	8
	i	1	2	3	4	5	7	8
	K_i	A_4	D_{10}	D_6	Z_5	Z_3	Z_2	1
	$N_L(K_i)$	A_4	D_{10}	D_6	D_{10}	D_6	D_4	
$p \equiv -1(\text{mod}120)$	$N_G(K_i)$	S_4	D_{20}	D_{12}	D_{p+1}	D_{p+1}	D_{p+1}	
	k_{i1}	2	2	2	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	1	1	1	$\frac{p+1}{20}-1$	$\frac{p+1}{12}-2$	$\frac{p+1}{8}-3$	$\frac{p^3-723p+13678}{7200}$
	y_{i1}	1	1	1	$\frac{p+1}{20}-1$	$\frac{p+1}{12}-2$	$\frac{p+1}{8}-3$	$\frac{p^2-32p+447}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-13142}{7200}$
	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p+1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p+1}{3}$	$\frac{p+1}{2}$	
	k_{i1}	0	0	0	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	0	0	0	$\frac{p+1}{20}$	$\frac{p+1}{12}$	$\frac{p+1}{8}$	$\frac{p^3-723p-722}{7200}$
	y_{i2}	0	0	0	$\frac{p+1}{20}$	$\frac{p+1}{12}$	$\frac{p+1}{8}$	$\frac{p^2-30p-31}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p+1138}{7200}$
$p \equiv 11(\text{mod}120)$	$N_G(K_i)$	A_4	D_{10}	D_{12}	D_{p-1}	D_{p+1}	D_{p+1}	
	k_{i1}	1	1	2	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	0	0	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^3-723p+6622}{7200}$
	y_{i1}	0	0	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^2-32p+231}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-7238}{7200}$
	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p+1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p+1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	1	1	0	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	1	1	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^3-723p+6622}{7200}$
	y_{i2}	1	1	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^2-30p+209}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-5918}{7200}$
$p \equiv -11(\text{mod}120)$	$N_G(K_i)$	A_4	D_{10}	D_{12}	D_{p+1}	D_{p-1}	D_{p-1}	
	k_{i1}	1	1	2	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	0	0	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^3-723p+7778}{7200}$
	y_{i1}	0	0	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^2-32p+269}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-8362}{7200}$
	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p-1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p-1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	1	1	0	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	1	1	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^3-723p+7778}{7200}$
	y_{i2}	1	1	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^2-32p+247}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-7042}{7200}$

TABLE 4-2

	i	1	2	3	4	5	7	8
	K_i	A_4	D_{10}	D_6	Z_5	Z_3	Z_2	1
	$N_L(K_i)$	A_4	D_{10}	D_6	D_{10}	D_6	D_4	
$p \equiv 31(\bmod 120)$	$N_G(K_i)$	S_4	D_{10}	D_6	D_{p-1}	D_{p-1}	D_{p+1}	
	k_{i1}	2	1	1	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	1	0	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-1$	$\frac{p^3-723p+7022}{7200}$
	y_{i1}	1	0	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-1$	$\frac{p^2-32p+271}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-9238}{7200}$
	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p-1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p-1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	0	1	1	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	0	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-2$	$\frac{p^3-723p+7022}{7200}$
	y_{i2}	0	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-2$	$\frac{p^2-30p+209}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-5518}{7200}$
$p \equiv -31(\bmod 120)$	$N_G(K_i)$	S_4	D_{10}	D_6	D_{p+1}	D_{p+1}	D_{p-1}	
	k_{i1}	2	1	1	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	1	0	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-1$	$\frac{p^3-723p+7378}{7200}$
	y_{i1}	1	0	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-1$	$\frac{p^2-30p+269}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-8762}{7200}$
	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p+1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p+1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	0	1	1	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	0	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-2$	$\frac{p^3-723p+7378}{7200}$
	y_{i2}	0	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-2$	$\frac{p^2-32p+207}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-5042}{7200}$
$p \equiv 41(\bmod 120)$	$N_G(K_i)$	S_4	D_{20}	D_6	D_{p-1}	D_{p+1}	D_{p-1}	
	k_{i1}	2	2	1	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	1	1	0	$\frac{p-1}{20}-1$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-2$	$\frac{p^3-723p+11122}{7200}$
	y_{i1}	1	1	0	$\frac{p-1}{20}-1$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-2$	$\frac{p^2-30p+389}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-12218}{7200}$
	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p+1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p+1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	0	0	1	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	0	0	1	$\frac{p-1}{20}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-1$	$\frac{p^3-723p+3922}{7200}$
	y_{i2}	0	0	1	$\frac{p-1}{20}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-1$	$\frac{p^2-32p+111}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-2738}{7200}$
$p \equiv -41(\bmod 120)$	$N_G(K_i)$	S_4	D_{20}	D_6	D_{p+1}	D_{p-1}	D_{p+1}	
	k_{i1}	2	2	1	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	1	1	0	$\frac{p+1}{20}-1$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-2$	$\frac{p^3-723p+10478}{7200}$
	y_{i1}	1	1	0	$\frac{p+1}{20}-1$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-2$	$\frac{p^2-32p+367}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-11542}{7200}$
	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p-1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p-1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	0	0	1	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	0	0	1	$\frac{p+1}{20}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-1$	$\frac{p^3-723p+3278}{7200}$
	y_{i2}	0	0	1	$\frac{p+1}{20}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-1$	$\frac{p^2-30p+89}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-2062}{7200}$

TABLE 4-3

	i	1	2	3	4	5	7	8
	i	1	2	3	4	5	7	8
	K_i	A_4	D_{10}	D_6	Z_5	Z_3	Z_2	1
	$N_L(K_i)$	A_4	D_{10}	D_6	D_{10}	D_6	D_4	
	$N_G(K_i)$	A_4	D_{20}	D_{12}	D_{p-1}	D_{p-1}	D_{p-1}	
	k_{i1}	1	2	2	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	0	1	1	$\frac{p-1}{20}-1$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{5}{2}$	$\frac{p^3-723p+11522}{7200}$
	y_{i1}	0	1	1	$\frac{p-1}{20}-1$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{5}{2}$	$\frac{p^2-30p+389}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-11818}{7200}$
$p\equiv 61(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p-1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p-1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	1	0	0	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	1	0	0	$\frac{p-1}{20}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{1}{2}$	$\frac{p^3-723p+4322}{7200}$
	y_{i2}	1	0	0	$\frac{p-1}{20}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{1}{2}$	$\frac{p^2-32p+151}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-4738}{7200}$
	$N_G(K_i)$	A_4	D_{20}	D_{12}	D_{p+1}	D_{p+1}	D_{p+1}	
	k_{i1}	1	2	2	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	0	1	1	$\frac{p+1}{20}-1$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{5}{2}$	$\frac{p^3-723p+10078}{7200}$
	y_{i1}	0	1	1	$\frac{p+1}{20}-1$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{5}{2}$	$\frac{p^2-32p+327}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-9542}{7200}$
$p\equiv -61(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p+1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p+1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	1	0	0	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	1	0	0	$\frac{p+1}{20}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{1}{2}$	$\frac{p^3-723p+2878}{7200}$
	y_{i2}	1	0	0	$\frac{p+1}{20}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{1}{2}$	$\frac{p^2-30p+89}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-2462}{7200}$
	$N_G(K_i)$	S_4	D_{10}	D_{12}	D_{p-1}	D_{p+1}	D_{p+1}	
	k_{i1}	2	1	2	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	1	0	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-2$	$\frac{p+1}{8}-2$	$\frac{p^3-723p+10222}{7200}$
	y_{i1}	1	0	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-2$	$\frac{p+1}{8}-2$	$\frac{p^2-32p+351}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-10838}{7200}$
$p\equiv 71(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p+1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p+1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	0	1	0	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	0	1	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}$	$\frac{p+1}{8}-1$	$\frac{p^3-723p+3022}{7200}$
	y_{i2}	0	1	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}$	$\frac{p+1}{8}-1$	$\frac{p^2-30p+89}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-2318}{7200}$
	$N_G(K_i)$	S_4	D_{10}	D_{12}	D_{p+1}	D_{p-1}	D_{p-1}	
	k_{i1}	2	1	2	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	1	0	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-2$	$\frac{p-1}{8}-2$	$\frac{p^3-723p+11378}{7200}$
	y_{i1}	1	0	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-2$	$\frac{p-1}{8}-2$	$\frac{p^2-30p+389}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-11962}{7200}$
$p\equiv -71(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p-1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p-1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	0	1	0	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	0	1	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}$	$\frac{p-1}{8}-1$	$\frac{p^3-723p+4178}{7200}$
	y_{i2}	0	1	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}$	$\frac{p-1}{8}-1$	$\frac{p^2-32p+127}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-3442}{7200}$

TABLE 4-4

	i	1	2	3	4	5	7	8
	K_i	A_4	D_{10}	D_6	Z_5	Z_3	Z_2	1
	$N_L(K_i)$	A_4	D_{10}	D_6	D_{10}	D_6	D_4	
	$N_G(K_i)$	A_4	D_{10}	D_6	D_{p-1}	D_{p-1}	D_{p+1}	
	k_{i1}	1	1	1	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	1	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-\frac{1}{2}$	$\frac{p^3-723p+3422}{7200}$
	y_{i1}	1	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-\frac{1}{2}$	$\frac{p^2-32p+151}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-5638}{7200}$
$p\equiv 91(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p-1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p-1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	1	1	1	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	1	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-\frac{5}{2}$	$\frac{p^3-723p+10622}{7200}$
	y_{i2}	1	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-\frac{5}{2}$	$\frac{p^2-30p+329}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-9118}{7200}$
	$N_G(K_i)$	A_4	D_{10}	D_6	D_{p+1}	D_{p+1}	D_{p-1}	
	k_{i1}	1	1	1	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	1	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-\frac{1}{2}$	$\frac{p^3-723p+3778}{7200}$
	y_{i1}	1	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-\frac{1}{2}$	$\frac{p^2-30p+149}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-5162}{7200}$
$p\equiv -91(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p+1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p+1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	1	1	1	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	1	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-\frac{5}{2}$	$\frac{p^3-723p+10978}{7200}$
	y_{i2}	1	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-\frac{5}{2}$	$\frac{p^2-32p+327}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-8642}{7200}$
	$N_G(K_i)$	A_4	D_{20}	D_6	D_{p-1}	D_{p+1}	D_{p-1}	
	k_{i1}	1	2	1	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	0	1	0	$\frac{p-1}{20}-1$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^3-723p+7522}{7200}$
	y_{i1}	0	1	0	$\frac{p-1}{20}-1$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^2-30p+269}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-8618}{7200}$
$p\equiv 101(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p+1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p-1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	1	0	1	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	1	0	1	$\frac{p-1}{20}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^3-723p+7522}{7200}$
	y_{i2}	1	0	1	$\frac{p-1}{20}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^2-32p+231}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-6338}{7200}$
	$N_G(K_i)$	A_4	D_{20}	D_6	D_{p+1}	D_{p-1}	D_{p+1}	
	k_{i1}	1	2	1	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	0	1	0	$\frac{p+1}{20}-1$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^3-723p+6878}{7200}$
	y_{i1}	0	1	0	$\frac{p+1}{20}-1$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^2-32p+247}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-7942}{7200}$
$p\equiv -101(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p-1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p-1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	1	0	1	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	1	0	1	$\frac{p+1}{20}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^3-723p+6878}{7200}$
	y_{i2}	1	0	1	$\frac{p+1}{20}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^2-30p+209}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-5662}{7200}$