Cayley Fuzzy Digraph Structure Induced by Groups

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Abstract: In this paper we introduce a class of Cayley fuzzy digraph structure induced by groups. Further many graph properties are expressed in terms of algebraic properties.

Key Words: Fuzzy graph, Cayley digraph structure, vertex transitive graph.

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§1. Introduction

Digraph Structure Let \( V \) be a non-empty set and \( S_1, S_2, \ldots, S_k \) are relations on \( V \) which are mutually disjoint, then \( G' = (V, S_1, S_2, \ldots, S_n) \) is a digraph structure. In addition, if \( S_1, S_2, \ldots, S_k \) are symmetric and irreflexive, then \( G' = (V, S_1, S_2, \ldots, S_k) \) is a graph structure, see [2] for details.

Let \( G \) be a group and \( S_1, S_2, \ldots, S_n \) be mutually disjoint subsets of \( G \). Then the Cayley digraph structure of \( G \) with respect to \( S_1, S_2, \ldots, S_n \) is defined as the digraph structure \( X = (G; E_1, E_2, \ldots, E_n) \), where \( E_i = \{(x, y) : x^{-1}y \in S_i \} \) [1]. In case, a digraph structure with only one connection set is the usual Cayley digraph. So a Cayley digraph structure is a generalization of the Cayley digraph.

Fuzzy Digraph Structure Let \( G' = (V, S_1, S_2, \ldots, S_k) \) be a digraph structure and \( \mu, \rho_1, \rho_2, \ldots, \rho_k \) be fuzzy subsets of \( V, S_1, S_2, \ldots, S_k \) respectively such that \( \rho_i(x, y) \leq \mu(x) \wedge \mu(y) \), for all \( x, y \in V \) and \( i = 1, 2, \ldots, k \). Then \( G = (\mu, \rho_1, \rho_2, \ldots, \rho_k) \) is a fuzzy digraph structure of \( G' \) [8], such that \( \rho_i(x, y) \leq \mu(x) \wedge \mu(y) \), for all \( x, y \in V \) and \( i = 1, 2, \ldots, n \). Then \( G = (\mu, \rho_1, \rho_2, \ldots, \rho_n) \) is a fuzzy digraph structure of \( G' \).

Let \( V \) be a non-empty set, \( \mu \) be fuzzy subset of \( V \) and \( R_1, R_2, \ldots, R_n \) be mutually disjoint fuzzy relations on \( \mu \). Then \( G = (\mu, R_1, R_2, \ldots, R_n) \) is a fuzzy digraph structure on \( V \). In case \( \mu = \chi_V \), where \( \chi_V \) is the characteristic function on \( V \), then the fuzzy digraph structure \((\mu, R_1, R_2, \ldots, R_n)\) is simply denoted by \( G = (V; R_1, R_2, \ldots, R_n) \).

A fuzzy digraph structure \( G = (V; R_1, R_2, \ldots, R_n) \) is called (i) trivial if \( R_i \equiv 0 \) for all \( i \), (ii) reflexive if for all \( x \in V, R_i(x, x) = 1 \) for some \( i \), (iii) symmetric if \( R_i = R_i^{-1} \) for all \( i \), (iv) transitive if for every \( i \) and \( j \), \( R_i \wedge R_j \leq R_k \) for some \( k \), (v) a Hasse diagram if for every positive integer \( m \geq 2 \) and for every \( x_1, x_2, \ldots, x_m \) of \( V \) with \( R_i(x_j, x_{j+1}) > 0 \) for all \( j = 0, 1, 2, \ldots, m-1 \), implies \( R_i(x_0, x_m) = 0 \) for all \( i \), and (vi) complete if for any

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$x, y \in V, R_i(x, y) > 0$, for some $i = 1, 2, \cdots, n$. A walk of length $k$ in a digraph structure is an alternating sequence $W = x_0, e_0, x_1, \cdots, e_k, x_k$, where $e_j = (x_j, x_{j+1})$ and $R_i(e_j) > 0$ for some $i$. A walk $W$ is called a path if all the vertices are distinct. A weak path is a sequence for the walk $W$. A walk is called a circuit if its first and last vertices are the same, but no other vertex is repeated. A weak path is a sequence $x_1, x_2, \cdots, x_m$ of distinct vertices of $V$ such that for $j = 1, 2, \cdots, m - 1$, $R_i \vee R_i^{-1}(x_j, x_{j+1}) > 0$ for some $i = 1, 2, \cdots, n$. Distance between two vertices $x$ and $y$ in $G$ is the length of the shortest path from $x$ to $y$ and is denoted by $d(x, y)$. Diameter of the fuzzy digraph structure $G$, denoted by $d(G)$, is defined by $d(G) = \max_{x,y \in G} d(x, y)$. A fuzzy digraph structure $G = (V; R_1, R_2, \cdots, R_n)$ is called (i) connected (strongly connected) if $y$ is connected to $x$ for all $x, y \in V$, and (ii) weakly connected if any two vertices can be joined by a weak path, that is, the fuzzy digraph structure $G' = (V; R_1 \vee R_1^{-1}, R_2 \vee R_2^{-1}, \cdots, R_n \vee R_n^{-1})$ is connected. A weakly connected fuzzy digraph structure $G = (V; R_1, R_2, \cdots, R_n)$ with out any circuits is called a tree.

The present work is a generalisation of the work in [6] in which Madhavan Namboothiri N.M. et al. introduced a class of Cayley fuzzy graphs induced by groups.

§2. Cayley Fuzzy Digraph Structure

**Definition 2.1** Let $V$ be a group and $\nu_1, \nu_2, \cdots, \nu_n$ be mutually disjoint fuzzy subsets of $V$. Then, Cayley Fuzzy Digraph Structure of $V$ with respect to $\nu_1, \nu_2, \cdots, \nu_n$ is defined as $(V; R_1, R_2, \cdots, R_n)$ where $R_i(x, y) = \nu_i(x^{-1}y)$ and is denoted by CayFD($V; \nu_1, \nu_2, \cdots, \nu_n$). The subsets $\nu_1, \nu_2, \cdots, \nu_n$ are called connection fuzzy subsets of CayFD($V; \nu_1, \nu_2, \cdots, \nu_n$). In case, a Cayley fuzzy digraph structure with only one connection set is usual Cayley fuzzy graph.

**Theorem 2.2** $G = \text{CayFD}(V; \nu_1, \nu_2, \cdots, \nu_n)$ is vertex-transitive.

**Proof** Let $a$ and $b$ be any two arbitrary elements in $G$. Define $\psi : V \to V$ by $\psi(x) = ba^{-1}x$ for all $x \in V$. Clearly, $\psi$ is a bijection onto itself. Furthermore, we have, for each $x, y \in V$,

\[
R_i(\psi(x), \psi(y)) = R_i(ba^{-1}x, ba^{-1}y) \\
= \nu_i((ba^{-1}x)^{-1}(ba^{-1}y)) \\
= \nu_i(x^{-1}y) = R_i(x, y).
\]

Hence, the proof is complete. \(\square\)

**Theorem 2.3** Cayley fuzzy digraph structures are regular.

**Proof** Let $G = \text{CayFD}(V; \nu_1, \nu_2, \cdots, \nu_n)$ be a cayley fuzzy digraph structure. Let $u, v \in V$. Since Cayley fuzzy digraph structures are vertex transitive, there exist an automorphism say, $f$ on $G$ such that, $f(u) = v$ and $R_i(f(x), f(y)) = R_i(x, y)$ for any $x, y \in V$ and $i = 1, 2, \cdots, n$. 
Then the in-degree of $u$,
\[
\text{ind}(u) = \sum_{x \in V} \sum_{i=1}^{n} R_i(x, u) = \sum_{x \in V} \sum_{i=1}^{n} R_i(f(x), f(u)) = \sum_{x \in V} \sum_{i=1}^{n} R_i(f(x), v) = \sum_{y \in V} \sum_{i=1}^{n} R_i(y, v) = \text{ind}(v).
\]
Similarly, we can prove that $\text{outd}(u) = \text{outd}(v)$. Therefore, $G$ is in-regular and out-regular.

Now to prove that $G$ is regular we just need to show that $\text{ind}(1) = \text{outd}(1)$.
\[
\text{ind}(1) = \sum_{x \in V} \sum_{i=1}^{n} R_i(x, 1) = \sum_{x \in V} \nu_i(x^{-1}) = \sum_{x \in V} \nu_i = \sum_{x \in V} \sum_{i=1}^{n} R_i(1, x) = \text{outd}(1).
\]
Therefore, $G$ is regular. $\Box$

**Theorem 2.4** $G = \text{CayF}_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ is a trivial graph if and only if $\nu_i \equiv 0$ for all $i$.

**Proof** By definition, $G$ is trivial if and only if $R_i \equiv 0$ for all $i$. This implies that $\nu_i \equiv 0$ for all $i$. $\Box$

**Theorem 2.5** $G = \text{CayF}_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ is reflexive if and only if $\nu_i(1) = 1$ for some $i$.

**Proof** Assume that $G = \text{CayF}_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ is reflexive. Then for every $x \in V$, $R_i(x, x) = 1$ for some $i$. This implies that $\nu_i(x^{-1}x) = \nu_i(1) = 1$ for some $i$.

Conversely, let $\nu_i(1) = 1$ for some $i$, say $i = k$. This implies that for each $x \in V$, $R_k(x, x) = \nu_k(x^{-1}x) = \nu_k(1) = 1$. That is $G$ is reflexive. $\Box$

**Theorem 2.6** $G = \text{CayF}_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ is symmetric if and only if $\nu_i(x) = \nu_i(x^{-1})$ for all $x \in V$, $i = 1, 2, \cdots, n$.

**Proof** Suppose that $G$ is symmetric. Then for any $x \in V$,
\[
\nu_i(x) = \nu(x^{-1}x^2) = R_i(x, x^2) = R_i^{-1}(x, x^2) = R_i(x^2, x) = \nu_i(x^{-1}x^{-1}x) = \nu_i(x^{-1}).
\]
Therefore, $\nu_i(x) = \nu_i(x^{-1})$.

Conversely, suppose that $\nu_i(x) = \nu_i(x^{-1})$ for all $x \in V$. Then for any $x, y \in V$, $R_i(x, y) = \nu_i(x^{-1}y) = \nu_i((x^{-1}y)^{-1}) = \nu_i(y^{-1}x) = R_i(y, x)$. This implies that, $R$ is symmetric. Hence the proof is complete. $\Box$

**Theorem 2.7** $G = \text{CayF}_D(V; \nu_1, \nu_2, \cdots, \nu_n)$ is transitive if and only if for every $i, j$ and for
any \( x, y \in V, \nu_i(x) \land \nu_j(y) \leq \nu_k(xy) \) for some \( k \).

**Proof** First assume that \( G \) is transitive. That is, for every \( i, j, R_i \circ R_j \leq R_k \) for some \( k \). For \( x, y \in V, \)

\[
\begin{align*}
\nu_i(x) \land \nu_j(y) & \leq \lor \{\nu_i(z) \land \nu_j(z^{-1}(xy)) : z \in V\} \\
& = \lor \{R_i(1, z) \land R_j(z, xy) : z \in V\} \\
& = R_i \circ R_j(1, xy) \\
& \leq R_k(1, xy).
\end{align*}
\]

That is, \( \nu_i(x) \land \nu_j(y) \leq \nu_k(xy) \) for some \( k \).

Now let for any \( x, y \in V \) and \( i, j, \nu_i(x) \land \nu_j(y) \leq \nu_k(xy) \) for some \( k \). Then,

\[
(R_i \circ R_j)(x, y) = \lor \{R_i(x, z) \land R_j(z, y) : z \in V\} \\
= \lor \{\nu_i(x^{-1}z) \land \nu_j(z^{-1}y) : z \in V\} \\
\leq \lor \{\nu_k((x^{-1}z)(z^{-1}y)) : z \in V\} \\
= \nu_k(x^{-1}y) = R_k(x, y).
\]

Thus, \( R_i \circ R_j \leq R_k \) for some \( k \). This completes the proof. \( \square \)

**Theorem 2.8** \( G = CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n) \) is complete if and only if \( \cup \nu_{i0}^+ = V \).

**Proof** First assume that \( G \) is complete. That is \( \cup \nu_{i0}^+ = V \times V. \) Clearly, \( \cup \nu_{i0}^+ \subseteq V \). Now let \( x \in V. \) Then \( (1, x) \in \cup \nu_{i0}^+ \) for some \( i. \) That is, \( R_i(1, x) \geq 0, \) which implies, \( \nu_i(x) \geq 0. \) Thus, \( x \in \cup \nu_{i0}^+ \). Therefore, \( V \subseteq \cup \nu_{i0}^+ \). That is, \( \cup \nu_{i0}^+ = V. \)

Conversely, assume \( \cup \nu_{i0}^+ = V. \) Let \( (x, y) \in V \times V. \) Then \( x, y \in V \Rightarrow x^{-1}y \in V \Rightarrow x^{-1}y \in \cup \nu_{i0}^+ \Rightarrow x^{-1}y \in \nu_{i0}^+ \) for some \( i. \) Then, \( \nu_i(x^{-1}y) \geq 0. \) That is, \( R_i(x, y) \geq 0 \) which implies \( (x, y) \in \cup \nu_{i0}^+. \) Hence, \( V \times V \subseteq \cup \nu_{i0}^+. \) Therefore,

\[
\bigcup \nu_{i0}^+ = V \times V.
\]

This completes the proof. \( \square \)

Let \( A_k \) be the set of all elements \( x \in V \) of the form \( x = x_1x_2 \cdots x_k \), where \( x_j \in \nu_{i0}^+ \) for some \( i = 1, 2, \cdots, n. \) Then \( [\vartheta] \) is defined as \([\vartheta] = \bigcup_{k=1}^{n} A_k. \) Let \( B_k \) be the set of all elements \( y \in V \) of the form \( y = y_1y_2 \cdots y_k \), where \( y_j \in (\nu_i \land \nu_i^{-1})_{0}^+ \) for some \( i = 1, 2, \cdots, n. \) Then \( [[\vartheta]] \) is defined as \([[\vartheta]] = \bigcup_{k=1}^{n} B_k. \)

**Theorem 2.9** \( G = CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n) \) is connected if and only if \( V = [\vartheta]. \)

**Proof** First assume that \( G = CayF_D(V; \nu_1, \nu_2, \cdots, \nu_n) \) is connected. Clearly, \( [\vartheta] \subseteq V. \) Now let \( x \in V. \) Then there exists a path from 1 to \( x \) say, \( (1, y_1, y_2, \cdots, y_k = x). \) Then, for
some $i$, $R_i(1,y_l) > 0$, that is, $y_l \in \nu_i^{+}_{i,0}$. Also, $y_{j-1}^{-1}y_j \in \nu_i^{+}_{i,0}$ for $j = 2, 3, \ldots, k$. This implies that $x \in A_k$, since, $x = (1,y_l)(y_1^{-1}y_2)(y_2^{-1}y_3) \cdots (y_k^{-1}y_k)$. Therefore, $x \in \bigcup_{k=1}^{n} A_k = [\nu]$. Hence, $V = [\nu]$.

Conversely, assume that $V = [\nu]$. Let $x, y \in V$. Then $z = x^{-1}y \in V$, implies, $z \in [\nu] = \bigcup_{k=1}^{n} A_k$. Then $z = z_1z_2 \cdots z_k$. Then $1, z_1, z_1z_2, \ldots, z_1z_2 \cdots z_k = z$ is a path from 1 to $z$. Then $x, xz_1, xz_1z_2, \ldots, xz_1z_2 \cdots z_k = xz = y$ is a path from $x$ to $y$, implies $G$ is connected. This completes the proof. □

**Theorem 2.10** $G = \text{CayF}_{D}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is weakly connected if and only if $V = [[\nu]]$.

**Proof** Assume $G$ be weakly connected. Clearly, $[[\nu]] \subseteq V$. Let $x \in V$. Then there exist a weak path say, $1, x_1, x_2, \ldots, x_k = x$ from 1 to $x$. Then, $1x_1 \in (\nu_1 \lor \nu_1^{-1})^+_0$, $x_1^{-1}x_2 \in (\nu_2 \lor \nu_2^{-1})^+_0, \ldots, x_{k-1}^{-1}x_k \in (\nu_k \lor \nu_k^{-1})^+_0$, which clearly implies that $x \in \bigcup_{k} B_k = [[\nu]]$.

Hence, $V \subseteq [[\nu]]$.

Conversely, assume that $V = [[\nu]]$. Let $x, y \in V$, implies $z = x^{-1}y \in V$. Therefore, $z \in [[\nu]]$. Then there exist elements $z_j \in (\nu_j \lor \nu_j^{-1})^+_0$, $j = 1, 2, \ldots, k$, such that $z = z_1z_2 \cdots z_k$, for some $k \in \{1, 2, \ldots, n\}$. Then $1, z_1, z_1z_2, \ldots, z_1z_2 \cdots z_k = z$ is a weak path from 1 to $z$ and hence $x, xz_1, xz_1z_2, \ldots, xz_1z_2 \cdots z_k = xz = y$ is a weak path from $x$ to $y$. Therefore, $G$ is weakly connected. This completes the proof. □

**Theorem 2.11** $G = \text{CayF}_{D}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is partially ordered if and only if

(i) $\nu_i(1) = 1$ for some $i$;

(ii) for every $i, j$ and for any $x, y \in V$, $\nu_i(x) \land \nu_j(y) \leq \nu_k(xy)$ for some $k$;

(iii) $\{x : \nu(x) = \nu(x^{-1})\} = \{1\}$ for all $i = 1, 2, \ldots, n$.

**Theorem 2.12** $G = \text{CayF}_{D}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is quasi-ordered if and only if

(i) $\nu_i(1) = 1$ for some $i$;

(ii) for every $i, j$ and for any $x, y \in V$, $\nu_i(x) \land \nu_j(y) \leq \nu_k(xy)$ for some $k$.

**Theorem 2.13** $G = \text{CayF}_{D}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is a Hasse diagram if and only if $G$ is connected and $\nu_k(x_1x_2 \cdots x_m) = 0$, $k = 1, 2, \ldots, n$, for any collection $x_1, x_2, \ldots, x_m$ of vertices in $V$ with $m \geq 2$ and $\nu_i(x_j) > 0$ for $j = 1, 2, \ldots, m$.

**Proof** Suppose $G$ is a Hasse diagram. Since $\nu_{i,j}(x_j) > 0$ for $j = 1, 2, \ldots, m$, $(1, x_1, x_1x_2, \ldots, x_1x_2 \cdots x_m)$ is a path from 1 to $x_1x_2 \cdots x_m$. Now since $G$ is a Hasse diagram, $R_k(1, x_1x_2 \cdots x_m) = 0$ for all $k$. Therefore $\nu_k(x_1x_2 \cdots x_m) = 0$ for all $k = 1, 2, \ldots, n$.

Conversely suppose, $G$ is connected and $\nu_k(x_1x_2 \cdots x_m) = 0$, $k = 1, 2, \ldots, n$, for any collection $x_1, x_2, \ldots, x_m$ of vertices in $V$ with $m \geq 2$ and $\nu_i(x_j) > 0$ for $j = 1, 2, \ldots, m$. Let
(x_0, x_1, \ldots, x_m) be a path in G from x_1 to x_m, m \geq 2. Then R_i(x_0, x_1) > 0, R_k(x_1, x_2) > 0, \ldots, R_m(x_{m-1}, x_m) > 0 which implies, \nu_1(x_0^{-1}x_1) > 0, \nu_2(x_1^{-1}x_2) > 0, \ldots, \nu_m(x_{m-1}^{-1}x_m) > 0. Thus, by assumption, \nu_k(x_0^{-1}x_1x_1^{-1}x_2x_2^{-1}x_3 \cdots x_{m-1}^{-1}x_m) = \nu_k(x_0^{-1}x_m) = 0. Therefore, R_k(x_0, x_m) = 0 for all k = 1, 2, \ldots, n. Hence, G is a Hasse diagram. This completes the proof.

**Theorem 2.14** For k = 1, 2, \ldots, n, let A_k be the set of all products of the form \nu_1, \nu_2, \ldots, \nu_k = \{x_1x_2 \cdots x_k : x_j \in \nu_j^+, j = 1, 2, \ldots, k\}. If G = CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n) has finite diameter, then the diameter of G is the least positive integer m such that

\[ G = \bigcup_{A \in A_m} A. \]

**Theorem 2.15** G = CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n) is a tree if and only if V = [[\vartheta]] and 1 \notin [\vartheta].

**Definition 2.16** Let (S, *) be a semigroup. Let A be a fuzzy subset of S. Then A is said to be fuzzy sub-semigroup of S if for all a, b \in S, A(ab) \geq A(a) \wedge A(b).

**Definition 2.17** Let (S, *) be a semigroup and let \nu_1, \nu_2, \ldots, \nu_n be mutually disjoint fuzzy subsets of S. The fuzzy sub-semigroup generated by \nu_1, \nu_2, \ldots, \nu_n is the smallest fuzzy sub-semigroup of S which contains \nu_1, \nu_2, \ldots, \nu_n. Let us denote it by \langle \nu_{(12\cdots n)} \rangle.

**Theorem 2.18** Let (S, *) be a semigroup and let \nu_1, \nu_2, \ldots, \nu_n be mutually disjoint fuzzy subsets of S. Then the fuzzy subset \langle \nu_{(12\cdots n)} \rangle is precisely given by \langle \nu_{(12\cdots n)} \rangle(x) = \bigvee \{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \ldots \wedge \nu_{j_k}(x_k) : x = x_1x_2 \cdots x_k with a finite positive integer k, x_i \in S and \nu_{j_i}(x_i) > 0 for some j_i = 1, 2, \ldots, n\} for any x \in S.

**Proof** Let \nu' be the fuzzy subset of V defined by \nu'(x) = \bigvee \{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \ldots \wedge \nu_{j_m}(x_m) : x = x_1x_2x_3 \cdots x_m, x_j \in \nu_{j_i}^+, m \in \{1, 2, 3, \ldots, n\}\} for any x \in V. If y \in V, by definition of \nu', it is clear that \nu'(y) \geq \nu_{j_k}(y) where j_k \in \{1, 2, \ldots, n\} and \nu_{j_k}(y) \geq 0. Thus, we have \nu_{j_k} \leq \nu' for all j_k. This implies that \nu' contains \nu_1, \nu_2, \ldots, \nu_n. Let x, y \in V. If \nu_{j_k}(x) = 0 or \nu_{j_k}(y) = 0, then \nu_{j_k}(x) \wedge \nu_{j_k}(y) = 0. Then, \nu'(xy) \geq \nu_{j_k}(x) \wedge \nu_{j_k}(y). Again, if \nu_{j_k}(x) \neq 0 and \nu_{j_k}(y) \neq 0, then by definition of \nu', we have \nu'(xy) \geq \nu_{j_k}(x) \wedge \nu_{j_k}(y). Hence \nu' is a fuzzy subsemigroup of V containing \nu_i, i \in \{1, 2, \ldots, n\}. Now let A be any fuzzy subsemigroup of V containing \nu_i, i \in \{1, 2, \ldots, n\}. Then, for any x \in V with x = x_1x_2x_3 \cdots x_m, x_i \in \nu_{j_i}^+, for i = 1, 2, \ldots, n, m \in \{1, 2, 3, \ldots, n\} we have A(x) \geq A(x_1) \wedge A(x_2) \wedge \cdots \wedge A(x_m) \geq \nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \cdots \wedge \nu_{j_m}(x_m), which implies that A(x) \geq \bigvee \{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \ldots \wedge \nu_{j_m}(x_m) : x = x_1x_2x_3 \cdots x_m, x_j \in \nu_{j_i}^+, m \in \{1, 2, 3, \ldots, n\}\} for j_i \in \{1, 2, \ldots, n\} for any x \in V. Therefore, A(x) \geq \nu'(x) for all x \in V. Thus, \nu' = \langle \nu_{(12\cdots n)} \rangle. That is, \langle \nu_{(12\cdots n)} \rangle(x) = \bigvee \{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \ldots \wedge \nu_{j_m}(x_m) : x = x_1x_2x_3 \cdots x_m, x_j \in \nu_{j_i}^+, m \in \{1, 2, 3, \ldots, n\}\} for any x \in V.

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