

Characterization of Locally Dually Flat Special Finsler (α, β) - Metrics

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Abstract: The concept of locally dually flat Finsler metrics originate from information geometry. As we know, (α, β) - metrics defined by a Riemannian metric α and a 1-form β , represent an important class of Finsler metrics .In the year 2014, S. K. Narasimhamurthy ,A.R. Kavyashree and Y. K. Mallikarjun obtained characterization of locally first approximate Matsumoto metric [1].In continuation of the paper we study and characterize locally dually flat for a special Finsler (α, β) metric $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ with isotropic S-curvature, which is not Riemannian.

Key Words: Finsler metric, Riemannian metric;one form metric, S-curvature, locally dually flat metric, locally Minkowskian metric.

AMS(2010): 53C60, 53B40

§1. Introduction

The notion of dually flat metric was first introduced by S. I. Amari and H. Nagaoka, while studying the information geometry on Riemannian spaces [2]. Later, Z. Shen extended the notion of dually flatness to Finsler metrics [7]. Dually flat Finsler metrics form a special important class of Finsler metrics in Finsler information geometry, which play a very important role in studying flat Finsler information structures ([4], [5], [6], [7], and [11]). In 2009, the authors of [4] classified the locally dual flat Randers metrics with almost isotropic flag curvature. Recently, Q. Xia worked on the dual flatness of Finsler metrics of isotropic flag curvature as well as scalar flag curvature ([10], [11]). Also, Q. Xia studied and gave a characterization of locally dually flat (α, β) -metrics on an n-dimensional manifold M ($n \geq 3$) [9]. Further in 2014, the authors of [1] discuss characterization of locally dually flat first approximate Matsumoto metric.

The first example of non-Riemannian dually flat metrics is the Funk metric given by ([4],

¹Received June 13, 2016, Accepted November 8, 2016.

[7]):

$$F = \frac{\sqrt{(1 - |x|^2|y|^2) + \langle x, y \rangle^2}}{1 - |x|^2} \pm \frac{\langle x, y \rangle}{1 - |x|^2}$$

This metric is defined on the unit ball $B^n(\mu) \subseteq R^n$ and is a Randers metric with constant flag curvature $K = \frac{-1}{4}$. This is the only known example of locally dually flat metric with non-zero constant flag curvature up to now.

In this paper, we study and characterize locally dually flat Finsler metric with isotropic S-curvature, which is not Riemannian.

§2. Preliminaries

Let M be an n -dimensional smooth manifold. We denote by TM the tangent bundle of M and by $(x, y) = (x^i, y^j)$ the local coordinates on the tangent bundle TM . A Finsler manifold (M, F) is a smooth manifold equipped with a function $F : TM \rightarrow [0, \infty)$, which has the following properties:

- Regularity: F is smooth in $TM \setminus \{0\}$;
- Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$;
- Strong convexity: the Hessian matrix of $F^2, g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial x^i \partial y^j}$ is positive definite on $TM \setminus \{0\}$.

We call F and the tensor g_{ij} the Finsler metric and the fundamental tensor of M , respectively.

For a Finsler metric $F = F(x, y)$, its geodesic curves are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and given by

$$G^i = \frac{g^{ij}}{4} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \}, \forall y \in T_x M.$$

Definition 2.1 A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a standard coordinate system (x^i, y^i) in TM which satisfies

$$[F^2]_{x^k y^l} y^k = 2[F^2]_{x^l}$$

In this case, the system of coordinates (x^i) is called an adapted local coordinate system. It is easy to see that every locally Minkowskian metric is locally dually flat. But the converse is not generally true [4].

Definition 2.2 A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x, y)$ on an open domain $U \subset R^n$ is locally projectively flat if and only if its

geodesic coefficients G^i are of the form

$$G^i = Py^i$$

where $P : TU = UXR^n \rightarrow R$ is positively homogeneous of degree one, $P(x, \lambda y) = \lambda P(x, y), \forall \lambda > 0$. We call $P(x, y)$ the projective factor of $F(x, y)$.

Lemma 2.1([4]) *Let $F = F(x, y)$ be a Finsler metric on an open subset $U \subset R^n$. Then F is locally flat and projectively flat on U if and only if $F_{x^k} = CF_{y^k}$, where C is a constant.*

The S-curvature is a scalar function on TM, which was introduced by Z. Shen to study volume comparison in Riemann-Finsler geometry [3]. The S-curvature measures the average rate of change of $(T_x M, F_x = F|_{T_x M})$ in the direction $y \in T_x M$. It is known that $S = 0$ for Berwald metrics.

Definition 2.3 *A Finsler metric F on an n -dimensional manifold M is said to have isotropic S-curvature if $S = (n + 1)c(x)F$, for some scalar function c on M .*

For a Finsler metric F on an n -dimensional manifold M , the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F = \frac{Vol B^n(1)}{Vol(y^i \in R^n | F(x, y^i \frac{\partial}{\partial x^i})|_x < 1)}$$

Here Vol denotes the Euclidean volumes and $B^n(1)$ denotes the unit ball in R^n . Then the S-curvature is defined by

$$S(y) = \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial (In \sigma_F)}{\partial x^i}$$

where

$$y = y^i \frac{\partial (In \sigma_F)}{(\partial x^i)}|_x \in T_x M [8].$$

For an (α, β) -metric, one can write $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$ and $\phi = \phi(s)$ is a C^∞ function on the interval $(-b_0, b_0)$ with certain regularity properties, $\alpha = \sqrt{(a_{ij}y^i y^j)}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M .

We further denote

$$b_{i|j}\theta^j = db_i - b_j\theta^j_i,$$

where $\theta^i = dx^i$ and $\theta^i_j = \Gamma^j_{ik}dx^k$ denotes the coefficients of the Levi-Civita connection form of α .

Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Clearly, β is closed if and only if $s_{ij} = 0$. An (α, β) -metric is said to be trivial if $r_{ij} = s_{ij} = 0$. We put

$$\begin{aligned} r_{i0} &= r_{ij}y^j, r_{00} = r_{ij}y^i y^j, r_j = r_{ij}b^i, \\ s_{i0} &= s_{ij}y^j, s_j = s_{ij}b^i, s_0 = s_j b^j. \end{aligned}$$

By direct computation, we can obtain a formula for the mean Cartan torsion of an (α, β) -metric as follow:

$$I_i = -\frac{\phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i).$$

Clearly, an (α, β) -metric $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ is Riemannian if and only if $\phi = 0$. Hence, we further we assume that $\phi \neq 0$.

Theorem 2.2([9]) *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an (α, β) -metric on an n -dimensional manifold M^n ($n \geq 3$), where $\alpha = \sqrt{(a_{ij}y^i y^j)}$ is a Riemannian metric and $\beta = b_i(x)y^i \neq 0$. is an 1-form on M . Suppose that F is not Riemannian and $\phi'(s) \neq 0$. Then F is locally dually flat on M if and only if (α, β) and $\phi = \phi(s)$ satisfy*

- (1) $s_{10} = \frac{1}{3}(\beta\theta_l - \theta b_l)$;
- (2) $r_{00} = \frac{2}{3}\theta\beta + \left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\right]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3 b^2)\tau\beta^2$;
- (3) $G'_\alpha = \frac{1}{3}[2\theta + (3k_1 - 2)\tau\beta]y^l + \frac{1}{3}(\theta^l \tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^1$;
- (4) $\tau[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')] = 0$.

where $\tau = \tau(x)$ is a scalar function, $\theta = \theta_i(x)y^i$ is an 1-form on M , $\theta^l = a^{lm}\theta_m$

$$k_1 = \Pi(0), k_2 = \frac{\Pi'(0)}{Q(0)}, k_3 = \frac{1}{6Q^2(0)}[3Q''(0)\Pi'(0) - 6\Pi^2(0) - Q(0)\Pi'''(0)].$$

and

$$Q = \frac{\phi'}{(\phi - s\phi')}, \quad \Pi = \frac{(\phi'^2 + \phi\phi'')}{(\phi(\phi - s\phi'))}.$$

In [4], Cheng-Shen studied the class of (α, β) -metrics of non-Randers type $\phi \neq t_1\sqrt{1 + t_2 s^2} + t_3 s$. with isotropic S-curvature and obtained the following.

Theorem 2.3([3]) *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be a non-Riemannian (α, β) -metric on a manifold and $b = \|\beta_x\|_\alpha$. Suppose that $\phi \neq t_1\sqrt{1 + t_2 s^2} + t_3 s$ for any constants $t_1 > 0, t_2$ and t_3 . Then F is of isotropic S-curvature $S = (n + 1)cF$ if and only if one of the following assertions holds*

- (1) β Satisfies

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), s_j = 0, \quad (2.1)$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $c = c(x)$ satisfies

$$\Phi = -2(n + 1)k\frac{\phi\Delta^2}{(b^2 - s^2)}, \quad (2.2)$$

where k is a real constant. In this case, $S = (n + 1)cF$ with $c = k\varepsilon$;

- (2) β satisfies

$$r_{ij} = 0, s_{ij} = 0. \quad (2.3)$$

In this case, $S = 0$, regardless of the choice of a particular ϕ .

§3. Characterization of Locally Dually Flat Finsler (α, β) Metric

Theorem 3.1 *Let $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ be a special Finsler (α, β) metric on a manifold M of dimension $n \geq 3$. Then the necessary and sufficient conditions for F to be locally dually flat on M are the following:*

- (1) $s_{10} = \frac{1}{3}(\beta\theta_l - \theta b_l)$;
- (2) $r_{00} = \frac{2}{3}\theta\beta + \left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l) \right] \alpha^2 + \frac{1}{3}(25 - 30b^2)\tau\beta^2$;
- (3) $G^l_\alpha = \frac{1}{3}[2\theta + 25\tau\beta]y^l + \frac{1}{3}(\theta^l\tau b^l)\alpha^2 + 5\tau\beta^2 b^l$;

where $\tau = \tau(x)$ is a scalar function, $\theta = \theta_k y^k$ is an 1-form on M .

Proof For a Finsler metric $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ we obtain $k_1 = 3, k_2 = 9, k_3 = 10$, and

$$\phi = 1 + s + s^2 + s^3, \phi' = 1 + 2s + 3s^2, \phi'' = 2 + 6s, \phi''' = 6,$$

$$\begin{aligned} \Pi &= \frac{3 + 12s + 18s^2 + 20s^3 + 13s^4}{1 + s - 2s^3 - 3s^4 - 3s^5 - 2s^6}, \\ \Pi(0) &= 3, \quad \Pi'(0) = 9, \quad \Pi''(0) = 18, \quad \Pi'''(0) = 102. \\ Q &= \frac{1 + 2s + 3s^2}{1 - s^2 - 2s^3}, \quad Q' = \frac{2 + 8s + 8s^2 + 8s^3 + 6s^4}{(-1 + s^2 + 2s^3)^2}, \\ Q'' &= \frac{-8(1 + 8s + 9s^2 + 15s^3 + 9s^4 + 6s^5 + 3s^6)}{(-1 + s^2 + 2s^3)^3}, \\ Q(0) &= 1, \quad Q'(0) = 2, \quad Q''(0) = 8, \quad Q'''(0) = 24. \end{aligned}$$

By using the above values in Lemma 2.1, we get

$$[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')] = 0 \quad \text{and} \quad \tau = 0.$$

Then, finally, by substituting k_1, k_2 and k_3 in Lemma 2.1, we infer the claim.

Now, let $\phi = \phi(s)$ be a positive C^∞ function on $(-b_0, b_0)$. For a number $b \in [0, b_0]$, let

$$\Phi = -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'', \quad (3.1)$$

where $\Delta = 1 + sQ + (b^2 - s^2)Q'$. This implies that

$$\Delta = \frac{\phi[1 - 3s^2 - 8s^3 + 2b^2(6 + 2s)]}{(-1 + s^2 + 2s^3)^2}.$$

Then, the equation (3.1) can be written as follows:

$$\Phi = -(Q - sQ')(n + 1)\Delta + (b^2 - s^2)\{(Q - sQ')Q' - (1 + sQ)Q''\}.$$

By using Theorem 2.3, now we will consider a locally dually flat (α, β) -metric with isotropic

S-curvature.

Theorem 3.2 *Let $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ be a locally dually flat non-Randers type (α, β) -metric on a manifold M of dimension $n \geq 3$. Suppose that F is of isotropic S curvature $S = (n+1)cF$, where $c = c(x)$ a scalar function is on M . Then F is a locally projectively flat in adapted coordinate system and $G^i = 0$.*

Proof Let $G^i = G^i(x, y)$ and $\overline{G}_\alpha^i = \overline{G}_\alpha^i(x, y)$ denote the coefficients of F and α respectively, in the same coordinate system. By definition, we have

$$G^i = G_\alpha^i + Py^i + Q^i, \quad (3.2)$$

where

$$P = \alpha^{-1}\Theta - 2Q\alpha s_o + r_{00}, \quad (3.3)$$

$$Q^i = \alpha Q s_o^i + \psi - 2Q\alpha s_o + r_{00}b^i, \quad (3.4)$$

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')} = \frac{-1 + s + 12s^2 + 20s^3 + 21s^4 + 3s^5}{2\phi[-1 + 2s^2 + 3s^3 - 2b^2(1 + 2s)]}$$

$$\psi = \frac{1}{2} \frac{\phi''}{((\phi - s\phi') + (b^2 - s^2)\phi'')} = \frac{1 + 3s}{1 - 3s^2 - 8s^3 + b^2(2 + 6s)}$$

First, we suppose that case (i) of Theorem 2.3 holds. It is remarkable that, for a special Finsler (α, β) metric, we have

$$\Delta = \frac{\phi[1 - 3s^2 - 8s^3 + 2b^2(6 + 2s)]}{(-1 + s^2 + 2s^3)^2}$$

It follows that $(-1 + s^2 + 2s^3)^2\Delta$ is a polynomial in s of degree 3. On the other hand we have

$$\phi\Delta^2 = \frac{\phi^2[1 - 3s^2 - 8s^3 + 2b^2(6 + 2s)]^2}{(-1 + s^2 + 2s^3)^4}. \quad (3.5)$$

Hence, if case (2) of Theorem (2.3) holds, then substituting (3.5) we obtain that

$$(b^2 - s^2)(-1 + s^2 + 2s^3)^4\Phi = -2(n+1)k\phi^2[1 - 3s^2 - 8s^3 + 2b^2(6 + 2s)]^2. \quad (3.6)$$

It follows that $(b^2 - s^2)(-1 + s^2 + 2s^3)^4\Phi$ is not a polynomial in s (if $k = 0$, then by considering the Cartan torsion equation, we get a contradiction). Then, we put

$$\phi\Delta^2 = \frac{\overline{\Delta}}{(-1 + s^2 + 2s^3)^4},$$

where

$$\overline{\Delta} = \phi^2[1 - 3s^2 - 8s^3 + 2b^2(6 + 2s)]^2.$$

By assumption, F is a non-Randers type metric. Thus $\overline{\Delta}$ is not a polynomial in s , and then

$(b^2 - s^2)(-1 + s^2 + 2s^3)^4\Phi$ is not a polynomial in s . Now, let us consider another form of Φ

$$\Phi = -(Q - sQ')(n+1)\Delta + (b^2 - s^2)\{(Q - sQ')Q' - (1 + sQ)Q''\},$$

where

$$Q - sQ' = \frac{(1 - 6s^2 - 12s^3 - 15s^4 - 12s^5)}{(-1 + s^2 + 2s^3)^2}.$$

Then,

$$\begin{aligned} \Phi = & \frac{-1}{(-1 + s^2 + 2s^3)^4} \{ \Phi[1 - 15s^2 - 38s^3 - 81s^4 - 108s^5 - 33s^6 - 6s^7 \\ & + n(-1 - 9s^2 - 20s^3 + 3s^4 + 72s^5 + 141s^6 + 156s^7 + 96s^8) \\ & + 2b^2(4(1 + 3s + 9s^2 + 15s^3 + 9s^4 + 6s^5 + 3s^6) \\ & - n(-1 - 3s + 6s^2 + 30s^3 + 51s^4 + 57s^5 + 36s^6))] \}. \end{aligned} \quad (3.7)$$

From equations (3.6) and (3.7), the relation $(b^2 - s^2)(-1 + s^2 + 2s^3)^4\Phi$ is a polynomial in s and b of degree 8 and 4 respectively. The coefficient of s^8 is not equal to zero. Hence it is impossible that $\Phi = 0$. Therefore, we can conclude that equation (2.2) does not hold. So, the case (ii) of Theorem 2.3 holds. In this case, we have

$$r_{00} = 0, s_j = 0.$$

In Theorem 3.1(2), by taking $r_{00} = 0$, we obtain (3.8)

$$\left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l) \right] \alpha^2 = \frac{1}{3}\beta[-2\Theta - (25 + 30b^2)\beta\tau]. \quad (3.8)$$

Since α^2 is an irreducible polynomial of y^i , equation (3.8) reduces to the following

$$\tau + \frac{2}{3}(b^2\tau - b_m\theta^m) = 0, \quad (3.9)$$

$$\frac{2}{3}\theta + \frac{1}{3}(25 + 30b^2)\beta\tau = 0, \quad (3.10)$$

where

$$\theta = -\frac{1}{2}(25 + 30b^2)\beta\tau. \quad (3.11)$$

Then, Theorem 3.1(1) yields

$$s_0 = -\frac{1}{3(b^2\tau - \beta b_m\theta^m)}$$

This implies

$$(b^2\tau - \beta b_m\theta^m) = 0$$

From (3.8), (3.9) and (3.11), we obtain

$$\theta = -\frac{1}{2}(25 + 30b^2)\beta\tau. \quad (3.12)$$

From equations (3.9) and (3.12), it follows that $\tau = 0$ and substituting $\tau = 0$ in equation (3.12), we get $\theta = 0$. Thus finally (1), (2) and (3) reduce to the following

$$s_{ij} = 0, \quad G_{\alpha}^l = 0, \quad r_{00} = 0.$$

Since $s_0 = r_{00} = 0$, then equations (3.3) and (3.4) reduce to

$$P = 0 \quad \text{and} \quad Q^i = 0.$$

Then the relation (3.2) becomes $G_{\alpha}^l = 0$, which completes the proof. \square

Theorem 3.3 *Let $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ be a non-Riemannian metric on n -dimensional $n \geq 3$ manifold M . Then F is locally dually flat with isotropic S -curvature. Moreover, $S = (n + 1)CF$ if and only if the structure is locally Minkowskian.*

Proof From Theorem 3.2 we have that $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ is dually flat and projectively flat in any adapted coordinate system. By Lemma 2.1, we infer

$$F_{x^k} = CF_{y^k}.$$

Hence the spray coefficients $G^i = Py^i$ are given by $P = \frac{1}{2}CF$. Since $G^i = 0$, then $P = 0$, and hence $C = 0$. This implies that F_{x^k} and then F is a locally Minkowskian metric in the adapted coordinate system. \square

§4. Conclusions

The authors S. I. Amari and H. Nagaoka ([2]) introduced the notion of dually flat Riemannian metrics, while studying information geometry on Riemannian manifolds. Information geometry emerged from investigating the geometrical structure of a family of probability distributions and was successfully applied to various areas, including statistical inference, control system theorem and multi-terminal information theorem. As known, Finsler geometry is just Riemannian geometry without the quadratic restriction. Therefore, it is natural to extend the construction of locally dually flat metrics to Finsler geometry. In Finsler geometry, Z. Shen [7] extended the notion of locally dually flat metric in Finsler information geometry, which plays a very important role in studying many applications in Finsler information structures. In this article, we study and characterize the locally dually flat a special (α, β) metric $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ with isotropic S -curvature which is not Riemannian.

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