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Chromatic Polynomial of Smarandache ν_E-Product of Graphs

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Abstract: Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. For a chosen edge set $E \subset E_2$, the *Smarandache ν_E-product* $G_1 \times_{\nu_E} G_2$ of $G_1,\ G_2$ is defined by

$$V(G_1 \times_{\nu_E} G_2) = V_1 \times V_2,$$

$$E(G_1 \times_{\nu_E} G_2) = \{(a,b)(a',b')|a = a', (b,b') \in E_2, \text{or } b = b', (a,a') \in E_1\}$$
$$\cup\{(a,b)(a',b')|(a,a') \in E_1 \text{ and } (b,b') \in E\}.$$

Particularly, if $E = \emptyset$ or E_2, then $G_1 \times_{\nu_E} G_2$ is the *Cartesian product* $G_1 \times G_2$ or *strong product* $G_1 * G_2$ of G_1 and G_2 in graph theory. Finding the chromatic polynomial of Smarandache ν_E-product of two graphs is an unsolved problem in general, even for the Cartesian product and strong product of two graphs. In this paper we determine the chromatic polynomial in the case of the Cartesian and strong product of a tree and a complete graph.

Keywords: Coloring graph, Smarandache ν_E-product graph, strong product graph, Cartesian product graph, chromatic polynomial.

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§1. Introduction

Sabidussi and Vizing defined Graph products first time in [4] [5]. A lot of works has been done on various topics related to graph products, however there are still many open problems [3]. Generally, we can construct Smarandache ν_E-product of graphs G_1 and G_2 for $E \subset E(G_2)$ as follows.

Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. For a chosen edge set $E \subset E_2$, the *Smarandache ν_E-product* $G_1 \times_{\nu_E} G_2$ of $G_1,\ G_2$ is defined by

$$V(G_1 \times_{\nu_E} G_2) = V_1 \times V_2,$$

$$E(G_1 \times_{\nu_E} G_2) = \{(a,b)(a',b')|a = a', (b,b') \in E_2, \text{or } b = b', (a,a') \in E_1\}$$
$$\cup\{(a,b)(a',b')|(a,a') \in E_1 \text{ and } (b,b') \in E\}.$$

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Particularly, if $E = \emptyset$ or E_2, then $G_1 \times_{\nu_E} G_2$ is nothing but the *Cartesian product* $G_1 \times G_2$ or *strong product* $G_1 * G_2$ of G_1 and G_2 in graph theory.

The chromatic polynomial of graph G, $\chi(G,k)$ is the number of different coloring ways, with at most k color. The chromatic number of G is the smallest integer k such that $\chi(G,k)$ be positive [2, 6]. Thus we can determine $\chi(G)$ by calculating $\chi(G,k)$. Let $G-e$ be a graph obtained by deleting an edge e from G. An edge e of G is said to be contracted if it is deleted and its ends are identified. The resulting graph is denoted by $G \cdot e$. The following theorems are well known.

Theorem 1.1([2, 6], Chromatic recurrence) *If G is a simple graph and $e \in E(G)$, then*

$$\chi(G,k) = \chi(G-e,k) - \chi(G \cdot e,k).$$

Theorem 1.2([2]) *If G has n components $G_1, G_2, \cdots, G_n$, then*

$$\chi(G,k) = \chi(G_1,k)\chi(G_2,k)\cdots\chi(G_n,k).$$

Theorem 1.3([2]) *Let G be a graph with subgraphs G_1, G_2 such that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = K_n$. Then*

$$\chi(G,k) = \frac{\chi(G_1,k)\chi(G_2,k)}{\chi(k_n,k)}.$$

Example 1.1 If K_n is a complete graph with n vertices then

$$\chi(K_n,k) = (k)_n = k(k-1)(k-2)\cdots(k-n+1).$$

Example 1.2 If C_n is a cycle graph with n vertices then

$$\chi(C_n,k) = (k-1)^n + (-1)(k-1).$$

§2. Cartesian Product

In this section, we consider the chromatic polynomial of Cartesian product, i.e., Smarandache ν_E-product graph of two graphs G_1, G_2 with $E = \emptyset$.

Theorem 2.1 *Let K_2 be a complete graph with two vertices and P_n be a path with $n \geq 3$ vertices, then $\chi(P_n \times K_2, k) = (k^2 - 3k + 3)\chi(P_{n-1} \times K_2, k)$.*

Proof If $G_1 = P_{n-1} \times K_2, G_2 = C_4$, we have $G_1 \cup G_2 = P_n \times K_2, G_1 \cap G_2 = K_2$. then by Theorem 1.3, we have

$$\begin{aligned}\chi(P_n \times K_2, k) &= \frac{\chi(P_{n-1} \times K_2, k)\chi(C_4, k)}{\chi(K_2, k)} = \frac{\chi(P_{n-1} \times K_2, k)((k-1)^4 + (-1)(k-1))}{K(k-1)} \\ &= (k^2 - 3k + 3)\chi(P_{n-1} \times K_2, k).\end{aligned}$$

□

By continue above recursive relation, we have following result.

Corollary 2.1 *For $n \geq 3$, $\chi(P_n \times K_2, k) = k(k-1)(k^2 - 3k + 3)^{n-1}$.*

Theorem 2.2 *For each path $P_n,\ n \geq 3$ and complete graph $K_m,\ m \geq 2$,*

$$\chi(P_n \times K_m, k) = (\chi(K_m, k))^n (\sum_{i=0}^{m} \frac{(-1)^i C(m,i)}{\chi(K_i, k)})^{n-1},$$

where $C(m, i)$ is choice of m vertices for i.

Proof If we consider $G_1 = P_{n-1} \times K_m,\ G_2 = P_2 \times K_m$, then $G_1,\ G_2$ are subgraphs of $P_n \times K_m$ such that $G_1 \cup G_2 = P_n \times K_m, G_1 \cap G_2 = K_m$. Therefore, by Theorem 1.3 it follows that,

$$\chi(P_n \times K_m, k) = \frac{\chi(P_{n-1} \times K_m, k)\ \chi(P_2 \times K_m, k)}{\chi(K_m, k)}$$

and with a recursive use of this relation, we have

$$\chi(P_n \times K_m, k) = \frac{\chi(P_2 \times K_m, k)^{n-1}}{\chi(K_m, k)^{n-2}}.$$

Then is sufficient to compute the chromatic polynomial of $P_2 \times K_m$. By using Theorem 1.1 and deleting and contracting the edges of P_2 in this product, at the end, we obtain 2^m graphs. Each of these graphs consist of two copy of K_m which have a $K_i, (0 \leq i \leq m)$ in their intersection, and there is no other edges than these. The chromatic polynomial of these graphs, by using Theorem 1.2 is

$$\phi_i(k) = \frac{(\chi(K_m, k))^2}{\chi(K_i, k)}.$$

Here we define $\chi(K_0, k) = 1$. On the other hand we have a choice of m vertices for i, so the number of these graphs is equal to $C(m, i)$. Since for each i these graphs have $2m - i$ vertices each, thus in summation a coefficient $(-1)^i$ appears,

$$\chi(P_2 \times K_m, k) = \sum_{i=0}^{m} (-1)^i C(m,i)\phi_i(k),$$

then

$$\begin{aligned}\chi(P_n \times K_m, k) &= \frac{\chi(P_2 \times K_m, k)^{n-1}}{\chi(K_m, k)^{n-2}} = \frac{(\sum_{i=0}^{m}(-1)^i C(m,i)\frac{(\chi(K_m,\ k))^2}{\chi(K_i,\ k)})^{n-1}}{\chi(K_m, k)^{n-2}} \\ &= (\chi(K_m, k))^n (\sum_{i=0}^{m} \frac{(-1)^i C(m,i)}{\chi(K_i, k)})^{n-1}.\end{aligned}$$

□

Note that in the steps of this proof we have not used the structure of P_n. But we use only the existence of a vertex of degree one in each step in finding recursive relation. So we can use this argument alternatively for each tree with n vertices. In fact P_n is a special case of tree. Therefore we can obtain a more general result following.

Corollary 2.2 *Let T_n be a tree with n vertices and K_m be a complete graph of m vertices, then*

$$\chi(T_n \times K_m, k) = (\chi(K_m, k))^n (\sum_{i=0}^{m} \frac{(-1)^i C(m,i)}{\chi(K_i, k)})^{n-1}.$$

Corollary 2.3 $\chi(C_n \times K_m,\ k) = (k-1)^m (\chi(K_m, k))^n (\sum_{i=0}^{m} \frac{(-1)^i C(m,i)}{\chi(K_i,k)})^{n-1}$.

Proof Let $G_1 = P_n \times K_m,\ G_2 = P_2 \times K_m - E(K_m)$, then $G_1,\ G_2$ are subgraphs of $C_n \times K_m$ such that $G_1 \cup G_2 = C_n \times K_m, G_1 \cap G_2 = K_m$. Therefore, by Theorem 1.3 it follows that

$$\chi(C_n \times K_m, k) = \frac{\chi(P_n \times K_m, k)\ \chi(G_2, k)}{\chi(K_m, k)}$$

But $\chi(G_2, k) = (k-1)^m\ \chi(K_m, k)$ then

$$\begin{aligned}\chi(C_n \times K_m, k) &= \frac{\chi(P_n \times K_m, k)\ (k-1)^m \chi(K_m, k)}{\chi(K_m, k)} \\ &= (k-1)^m (\chi(K_m, k))^n (\sum_{i=0}^{m} \frac{(-1)^i C(m,i)}{\chi(K_i, k)})^{n-1}.\end{aligned}$$

□

Thus for the Cartesian product of a complete graph and a tree and a cycle, the chromatic polynomial is found. However this for two complete graphs is open. If the chromatic polynomial of Cartesian product of two complete graphs of order n is found, we can determine the number of Latin squares of order n. Moreover $\chi(P_n \times P_m)$ is not yet known [1].

§3. **Strong Products**

In this section, we consider the chromatic polynomial of strong product, i.e., Smarandache ν_E-product graph of two graphs $G_1,\ G_2$ with $E = E(G_2)$. We get some theorems for chromatic polynomial of strong products same as Cartesian product.

Theorem 3.1 *Let K_2 be a complete graph with two vertices and P_n be a path with $n \geq 3$ vertices, then*

$$\chi(P_n * k_2, k) = (k^2 - 3k + 3)\chi(P_{n-1} * k_2, k).$$

So we have the following theorem by above recursive relation.

Corollary 3.1 *For $n \geq 3$, $\chi(P_n * K_2, k) = k(k-1)(k^2 - 5k + 6)^{n-1}$.*

Theorem 3.2 *For each path P_n, $n \geq 3$ and complete graph K_m, $m \geq 2$*

$$\chi(P_n * K_m, k) = \prod_{i=0}^{m-1}(k-i)[\prod_{i=m}^{2m-1}(k-i)]^{n-1}.$$

Proof If we consider $G_1 = P_{n-1} * K_m$, $G_2 = P_2 * K_m$, by Theorem 1.3 it follows that

$$\chi(P_n * K_m, k) = \frac{\chi(P_{n-1} * K_m, k)\ \chi(P_2 * K_m, k)}{\chi(K_m, k)}$$

and with a recursive use of this relation we have

$$\chi(P_n * K_m, k) = \frac{(\chi(P_2 * K_m, k)^{n-1}}{\chi(K_m, k)},$$

but $P_2 * K_m = K_{2m}$ and thus

$$\begin{aligned} \chi(P_n * K_m, k) &= \frac{[(k)_{2m}]^{n-1}}{[(k)_m]^{n-2}} = \frac{[k(k-1)\cdots(k-2m+1)]^{n-1}}{[k(k-1)\cdots(k-m+1)]^{n-2}} \\ &= k(k-1)...(k-m+1)[(k-m)(k-m-1)...(k-2m+1)]^{n-1} \\ &= \prod_{i=0}^{m-1}(k-i)[\prod_{i=m}^{2m-1}(k-i)]^{n-1}. \end{aligned}$$

□

Therefore we obtain a more general result as follows.

Corollary 3.2 *Let T_n be a tree with n vertices and K_m be a complete graph with m vertices, then*

$$\chi(T_n * K_m, k) = \prod_{i=0}^{m-1}(k-i)[\prod_{i=m}^{2m-1}(k-i)]^{n-1}.$$

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