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Clique Partition of Transformation Graphs

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Abstract: A *clique* in a graph G is a complete subgraph of G. A *clique partition* of G is a collection C of cliques such that each edge of G occurs in exactly one clique in C. The clique partition number $cp(G)$ is the minimum size of a clique partition of G. In this paper upper bounds for the clique partition number of the transformation graphs G^{++-} and G^{+++} for some standard class of graphs is obtained.

Key Words: Transformation graph, clique, clique partition.

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§1. Introduction

All graphs G considered here are finite, undirected and simple. We refer to [1] for unexplained terminology and notations. In 2001 Wu and Meng introduced some new graphical transformations which generalizes the concept of the total graph. As is the case with the total graph, these generalizations referred to as *transformation graphs* G^{xyz} have $V(G)\cup E(G)$ as the vertex set. The adjacency of two of its vertices is determined by adjacency and incidence nature of the corresponding elements in G.

Let α, β be two elements of $V(G)\cup E(G)$. Then associativity of α and β is taken as $+$ if they are adjacent or incident in G, otherwise $-$. Let xyz be a 3-permutation of the set $\{+,-\}$. The pair α and β is said to correspond to x or y or z of xyz if α and β are both in $V(G)$ or both are in $E(G)$, or one is in $V(G)$ and the other is in $E(G)$ respectively. Thus the *transformation graph* G^{xyz} of G is the graph whose vertex set is $V(G)\cup E(G)$ and two of its vertices α and β are adjacent if and only if their associativity in G is consistent with the corresponding element of xyz.

In particular G^{++-} and G^{+++} are defined as:

Definition 1.1 *The transformation graph G^{++-} of G is the graph with vertex set $V(G)\cup E(G)$ in which the vertices u and v are joined by an edge if one of the following holds*

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(1) *both* $u, v \in V(G)$ *and* u *and* v *are adjacent in* G;

(1) *both* $u, v \in E(G)$ *and* u *and* v *are adjacent in* G;

(3) *one is in* $V(G)$ *and the other is in* $E(G)$ *and they are not incident with each other in* G.

Definition 1.2 *The transformation graph* G^{+++} *(total graph) of* G *is the graph with vertex set* $V(G) \cup E(G)$ *in which the vertices* u *and* v *are joined by an edge if one of the following holds*

(1) *both* $u, v \in V(G)$ *and* u *and* v *are adjacent in* G;

(2) *both* $u, v \in E(G)$ *and* u *and* v *are adjacent in* G;

(3) *one is in* $V(G)$ *and the other is in* $E(G)$ *and they are incident with each other in* G.

The transformation graphs are investigated in [2], [3] and [4].

For convenience, the transformation graph G^{xyz} is partitioned into $G^{xyz} = S_x(G) \cup S_y(G) \cup S_z(G)$ where $S_x(G)$, $S_y(G)$ and $S_z(G)$ are the edge-induced subgraphs of G^{xyz}. The edge set of each of which is respectively determined by x, y and z of the permutation xyz. $S_x(G) \cong G$ when x is $+$ and $S_x(G) \cong \overline{G}$ when x is $-$. $S_y(G) \cong L(G)$ when y is $+$ and $S_y(G) \cong \overline{L(G)}$ when y is $-$. When z is $+$, $\alpha, \beta \in V(G^{xyz})$ are adjacent in $S_z(G)$ if they are incident with each other in G. When z is $-$, α, β are adjacent in $S_z(G)$ if they are not incident in G.

A *clique partition* of G is a collection C of cliques such that each edge of G occurs in exactly one clique in C. The clique partition number $cp(G)$ is the minimum size of a clique partition of G.

In this paper the upper bounds for clique partition number of transformation graphs G^{++-} and G^{+++} of some class of graphs such as path, cycle, star, wheel, etc, are obtained.

§2. Clique Partition of P_n^{++-} and C_n^{++-}

We note that the size of P_n^{++-} and C_n^{++-} are $n^2 - n - 1$ and n^2 respectively; the clique numbers of P_n^{++-} and C_n^{++-} is 4. Therefore no clique partition of P_n^{++-} and C_n^{++-} can contain $K_t (t \geqslant 5)$.

Theorem 2.1 *For a path* P_n $(n \geq 8)$, $cp(P_n^{++-}) \leq n^2 - 6n + 7$.

Proof Consider the path $P_n : v_1 - v_2 - v_3 - \cdots - v_n$. Let $e_i = v_i v_{i+1} (1 \leq i \leq n-1)$ be the edges of P_n. The edge set of P_n^{++-} is partitioned into K_4, K_3 and K_2's. Vertex sets of K_4's and K_3's are listed as elements of the sets B_j.

When $n \equiv 0 (mod\ 4)$,

$B_1 = \{\{v_i,\ v_{i+1},\ e_{i+2},\ e_{i+3}\} : i = 1, 3, 5, \cdots, \frac{n}{2} - 1\}$,
$B_2 = \{\{v_i,\ v_{i+1},\ e_{i-3},\ e_{i-2}\} : i = \frac{n}{2} + 1,\ \frac{n}{2} + 3, \cdots, n-3,\ n-1\}$,
$B_3 = \{\{v_i,\ v_{i+1},\ e_{(\frac{n}{2}+1+i)},\ e_{(\frac{n}{2}+2+i)}\} : i = 2, 4, 6, \cdots, \frac{n}{2} - 4\}$,
$B_4 = \{\{v_i,\ v_{i+1},\ e_{(i-\frac{n}{2}-2)},\ e_{(i-\frac{n}{2}-1)}\} : i = \frac{n}{2} + 4,\ \frac{n}{2} + 6, \cdots, n-2\}$,
$B_5 = \{\{v_{(\frac{n}{2}+2)},\ v_{(\frac{n}{2}+3)},\ e_1,\ e_2\},\ \{v_{(\frac{n}{2}-2)},\ v_{(\frac{n}{2}-1)},\ e_{n-2}, e_{n-1}\},\ \{v_{(\frac{n}{2})},\ v_{(\frac{n}{2}+1)}, e_1\}\}$.

When $n \equiv 2 (mod\ 4)$,

$B_1 = \{\{v_i,\ v_{i+1},\ e_{i+2},\ e_{i+3}\} : i = 1, 3, 5, \cdots, \frac{n}{2} - 2\}$,
$B_2 = \{\{v_i,\ v_{i+1},\ e_{i-3},\ e_{i-2}\} : i = n-1,\ n-3,\ n-5,\ \cdots, \frac{n}{2} - 2\}$,
$B_3 = \{\{v_i,\ v_{i+1},\ e_{(\frac{n}{2}+i)},\ e_{(\frac{n}{2}+i+1)}\} : i = 2, 4, 6, \cdots, \frac{n}{2} - 3\}$,
$B_4 = \{\{v_{(\frac{n}{2}+i)},\ v_{(\frac{n}{2}+i+1)},\ e_{i-1},\ e_i\} : i = 3, 5, 7, \cdots, \frac{n}{2} - 2\}$,
$B_5 = \{\{v_{(\frac{n}{2}-1)},\ v_{(\frac{n}{2})},\ e_1,\ e_2\},\ \{v_{(\frac{n}{2}+1)},\ v_{(\frac{n}{2}+2)},\ e_{n-2},\ e_{n-1}\},\ \{v_{(\frac{n}{2})},\ v_{(\frac{n}{2}+1)},\ e_3\}\}$

When $n \equiv 1, 3 (mod\ 4)$,

$B_1 = \{\{v_i,\ v_{i+1},\ e_{i+2},\ e_{i+3}\} : i = 1, 3, 5 \cdots, n-2\}$,
$B_2 = \{\{v_i,\ v_{i+1},\ e_{i-6},\ e_{i-5}\} : i = n-3,\ n-5,\ n-7, \cdots, 12, 10, 8\}$,
$B_3 = \{\{v_{n-1},\ v_n,\ e_{n-5},\ e_{n-4}\},\ \{v_{n-2},\ v_{n-1},\ e_{n-7},\ e_{n-8}\},\ \{v_6,\ v_7,\ e_1,\ e_2\},$
$\{v_2,\ v_3,\ e_{n-3},\ e_{n-2}\},\ \{v_4,\ v_5,\ e_1\}\}$

In each case there are $n-2$ K_4's and one K_3. These cover all the edges of S_x, S_y and $4n-6$ edges of S_z. The remaining $(n^2 - 7n + 8)$ edges of S_z are covered by K_2's.

Therefore $P_n^{++-} = (n-2)K_4 \cup K_3 \cup (n^2 - 7n + 8)K_2$ and $cp(P_n^{++-}) \leq n^2 - 6n + 7$. □

Theorem 2.2 *For a cycle* C_n $(n \geq 8)$, $cp(C_n^{++-}) \leq n^2 - 5n$.

Proof Consider the cycle $C_n : v_1 - v_2 - v_3 - \cdots - v_n - v_1$. Let $e_i = v_i v_{i+1}$ $(1 \leq i \leq n-1)$ and $e_n = v_n v_1$ be the edges of C_n. Edge set of C_n^{++-} is partitioned into K_4's and K_2's. Vertex sets of K_4's are listed as elements of the sets B_j as follows:

When n is even,

$B_1 = \{\{v_i,\ v_{i+1},\ e_j,\ e_k\}:$ for each $i = 1, 3, 5, \cdots, n-3,\ n-1,\ j \equiv i + 2(mod\ n)$ and $k \equiv i + 3(mod\ n)\}$,

$B_2 = \{\{v_i,\ v_j,\ e_k,\ e_l\}\}$ for each $i = 2, 4, 6, \cdots, n-4,\ n-2,\ j \equiv i + 1(mod\ n)\}$ with

$$k \equiv \begin{cases} \frac{n}{2} + 1 + i(mod\ n) & \text{when } \frac{n}{2} \text{ is odd} \\ \frac{n}{2} + i(mod\ n) & \text{when } \frac{n}{2} \text{ is even} \end{cases} \quad \text{and } l \equiv k + 1(mod\ n)$$

When n is odd,

$B_1 = \{\{v_i,\ v_{i+1},\ e_{i+2},\ e_j\}:$ for each $i = 1, 3, 5, \cdots, n-4,\ n-2$ $j \equiv i + 3(mod\ n)\}$,

$B_2 = \{v_i,\ v_{i+1},\ e_j,\ e_k\}:$ for each $i = 2, 4, 6, \cdots, n-7,\ n-5,$ $j \equiv i + 6(mod\ n)$ $k \equiv i + 7(mod\ n)\}$,

$B_3 = \{\{v_n,\ v_1,\ e_6,\ e_7\},\ \{v_{n-3},\ v_{n-2},\ e_2,\ e_3\}, \{v_{n-1},\ v_n,\ e_4,\ e_5\}\}$.

In these sets v_0 and e_0 are taken as v_n and e_n respectively.

In both the cases there are nK_4's. These cover all the edges of S_x, S_y and some edges of S_z. Remaining edges of S_z are listed as K_2's. Therefore $C_n^{++-} = nK_4 \cup (n^2 - 6n)K_2$ and $cp(C_n^{++-}) \leq n^2 - 5n$. □

§3. Clique Partition of G^{++-} with G isomorphic to Comb or Sunlet graphs

The Comb graph $G \cong P_n \odot K_1$ is the graph with path on n vertices and each vertex of path is adjacent to a pendant vertex. The Sunlet graph $S_n \cong C_n \odot K_1$ is a graph with the cycle on n

vertices and each vertex of the cycle is adjacent to a pendant vertex.

For a comb graph $G \cong P_n \odot K_1$, let $v_i(1 \leq i \leq n)$ denote the vertices of P_n with v_1 and v_n as its end vertices and $e_i = v_i v_{i+1}$ be the edges of P_n and v_i' be the pendant vertices adjacent to each of v_i and $e_i' = v_i v_i'$ be the pendant edges of G. We note that order and size of $V(G^{++-})$ is $4n-1$ and $4n^2-n-3$ respectively and the clique number is 5.

For the sunlet graph $S_n \cong C_n \odot K_1$, let $v_i(1 \leq i \leq n)$ denote the vertices of C_n and v_i' be the pendant vertex adjacent to v_i, $e_i = v_i v_{i+1}$ be the n edges of C_n and $e_i' = v_i v_i'$ be the pendant edges of S_n. We note that order and size of S_n^{++-} is $4n$ and $4n^2+n$ respectively and the clique number is 5.

Theorem 3.1 *Let $G \cong P_n \odot K_1 (n \geq 6)$ be the comb graph. Then $cp(G^{++-}) \leq 4n^2 - 12n + 7$.*

Proof Consider the comb $G \cong P_n \odot K_1$. Edge set of G^{++-} is partitioned into K_5, K_4, K_3 and K_2's. The vertex sets of these cliques are listed as elements of sets B_j are given below:

$B_1 = \{\{v_i,\ v_i',\ e_{i+1},\ e_{i+2}'\} : i = 1, 2, 3, \cdots, n-3\}$,

$B_2 = \{\{v_{n-2},\ v_{n-2}',\ e_{n-1},\ e_n'\},\ \{v_{n-1},\ v_{n-2}',\ e_1,\ e_1'\},\ \{v_n,\ v_n',\ e_1,\ e_2,\ e_2'\}\}$,

$B_3 = \{\{\{v_i,\ v_{i+1},\ e_j\} : i = 1, 2, 3, \cdots, n-5\ ;\ j = i+4\}, \{\{v_i,\ v_{i+1},\ e_j\} : i = n-4, n-3, n-2, n-1;\ j = i-(n-5)\}\}$.

The sets B_1, B_2 and B_3 cover all the edges of S_x, S_y and some edges of S_z while remaining edges of S_z are covered by K_2's.

$B_4 = \{\{\{v_i,\ e_j'\} :$ for each $1 \leq i \leq n; 1 \leq j \leq n$ and $j \neq i+2\}$, $\{\{v_1,\ e_i\} : i = 4\ \ 6 \leq i \leq n-1\}$, $\{\{v_n,\ e_i\} : i = 3\ \ 5 \leq i \leq n-2\}$, $\{v_{n-2},\ e_i\} :\ i = 1\ \ 4 \leq i \leq n-4\}$, $\{\{v_{n-1},\ e_i\} : 4 \leq i \leq n-3\}$, $\{\{v_i,\ e_j\} :$ for each $2 \leq i \leq n-3; 1 \leq j \leq n-1$ *and* $j \neq i-1, i, i+1, i+2, i+3, i+4\}\}$,

$B_5 = \{\{v_i',\ e_j\} :$ for each $1 \leq i \leq n-3; 1 \leq j \leq n-1$ and $j \neq i+1, i+2\}$, $\{\{v_{n-2}',\ e_i\} : i = n$ and $1 \leq i \leq n-2\}, \{\{v_{n-1}',\ e_i\} : 2 \leq i \leq n-1\}$, $\{\{v_n',\ e_i\} : 3 \leq i \leq n-1\}$, $\{\{v_i',\ e_j'\} :$ for each $1 \leq i \leq n-2; 1 \leq j \leq n$ and $j \neq i, i+2\}$, $\{\{v_{n-1}',\ e_j'\} : i = n\ \ 2 \leq i \leq n-2\}$, $\{\{v_n',\ e_i\} : i = 1\ \ 3 \leq i \leq n-1\}\}$.

Thus, $G^{++-} = (n-2)K_5 \cup 2K_4 \cup (n-1)K_3 \cup (4n^2 - 14n + 8)K_2$ and hence $cp(G^{++-}) \leq 4n^2 - 12n + 7$. □

Theorem 3.2 *For $S_n \cong C_n \odot K_1 (n \geq 6)$ a sunlet graph, $cp(S_n^{++-}) \leq 4n^2 - 10n$.*

Proof Consider the sunlet graph $S_n \cong C_n \odot K_1$. Edge set of S_n^{++-} is partitioned into K_5, K_3 and K_2's where,

$B_1 = \{\{v_i,\ v_i',\ e_j,\ e_k,\ e_k'\} :$ for each $1 \leq i \leq n,\ j \equiv i+1 (mod\ n),\ k \equiv i+2 (mod\ n)\}$,

$B_2 = \{\{v_i,\ v_j,\ e_k\} :$ for each $1 \leq i \leq n,\ j \equiv i+1 (mod\ n),\ k \equiv i+4 (mod\ n)\}$,

$B_3 = \{\{v_i,\ e_j'\},\ \{v_i',\ e_j'\} :$ for each $1 \leq i \leq n,\ 1 \leq j \leq n$ and $j \neq i, i+2 (mod\ n)\} \cup \{\{v_i',\ e_j\} :$ *for each* $1 \leq i \leq n,\ 1 \leq j \leq n-1$ and $j \neq i+1, i+2 (mod\ n)\} \cup$ $\{\{v_i,\ e_j\} :$ for each $1 \leq i \leq n,\ 1 \leq j \leq n-1$ and $j \neq i-1, i, i+1, i+2, i+3, i+4 (mod\ n)\}$.

Thus, $S_n^{++-} = nK_5 \cup nK_3 \cup (4n^2 - 12n)K_2$ and $cp(S_n^{++-}) \leq 4n^2 - 10n$. □

§4. Clique Partition of Transformation Graphs $K_{1,n}^{++-}$ and W_{n+1}^{++-}

For the star graph $K_{1,n}$, let v_0 be the central vertex, $v_i(1 \leq i \leq n)$ be the pendant vertices and $e_i = v_0v_i$ be the pendant edges. We note that $|V(K_{1,n}^{++-})| = 2n+1$, $|E(K_{1,n}^{++-})| = n(3n-1)/2$ and the clique number is n.

For the wheel graph $W_{n+1} = C_n + K_1$, let v_0 be the central vertex, v_i be the vertices, $e_i = v_0v_i(1 \leq i \leq n)$ be the spokes and $e'_i = v_iv_j(1 \leq i \leq n,\ j = i+1(mod\ n))$ be the hubs of W_{n+1}. Then, $V(W_{n+1}^{++-}) = V(W_{n+1}) \cup E(W_{n+1})$, $|V(W_{n+1}^{++-})| = 3n+1$, $|E(W_{n+1}^{++-})| = 5n(n+1)/2$ and the clique number is n.

Theorem 4.1 *For $n \geq 3$, $cp(K_{1,n}^{++-}) \leq n^2+1$.*

Proof Here $S_y = L(K_{1,n}) \cong K_n$. The clique K_n covers all the edges of S_y; $S_x = K_{1,n}$ and $S_z = nK_{1,n-1}$, which are covered by $n + n(n-1)\ \ K'_2s$.

$\{\{v_0,\ v_i\} : 1 \leq i \leq n\}$ and

$\{\{v_i,\ e_j\} :\ for\ each\ 1 \leq i \leq n,\ 1 \leq j \leq n\ and\ j \neq i\}$

Therefore, $K_{1,n}^{++-} = K_n \cup n^2K_2$ and hence $cp(K_{1,n}^{++-}) \leq n^2+1$. □

Theorem 4.2 *For $n \geq 6$, $cp(W_{n+1}^{++-}) \leq 2n^2 - 6n + 1$.*

Proof The edge set of W_{n+1}^{++-} is partitioned into a K_n, $n\ \ K_4$'s, $2n\ \ K_3$'s and $(2n^2 - 9n)$ K_2's. Here,

$B_1 = \{\{e_1,\ e_2,\ e_3,\ \cdots, e_{n-1},\ e_n\}\}$,

$B_2 = \{\{v_i,\ e'_i,\ e'_k,\ e_k\}:$ for each $1 \leq i \leq n,\ j \equiv i+1(mod\ n),\ k \equiv i+2(mod\ n)\}$,

$B_3 = \{\{v_0,\ v_i,\ e'_j\}:$ for each $1 \leq i \leq n,\ j =\equiv i+3(mod\ n)\}$,

$B_4 = \{\{v_i,\ v_j,\ e'_k\}:$ for each $1 \leq i \leq n,\ j \equiv i+1(mod\ n),\ k \equiv i+5(mod\ n)\}$,

$B_5 = \{\{v_i,\ e_j\}:$ for each $1 \leq i \leq n,\ 1 \leq j \leq n$ and $j \neq i, i+2(mod\ n)\}\ \cup\ \{\{v_i,\ e'_j\}:$ for each $1 \leq i \leq n,\ 1 \leq j \leq n$ and $j \neq i-1, i, i+1, i+2, i+3, i+4, i+5(mod\ n)\}$.

(In the above sets v_0, e_0 and e'_0 are taken as v_n, e_n and e'_n.)

Thus $(W_{n+1}^{++-}) = K_n \cup nK_4 \cup 2nK_3 \cup (2n^2 - 9n)K_2$ and hence $cp(W_{n+1}^{++-}) \leq 2n^2 - 6n + 1$.□

§5. Clique Partition of Transformation Graphs P_n^{+++}, C_n^{+++}, $K_{1,n}^{+++}$, W_{n+1}^{+++} and K_n^{+++}

Theorem 5.1 *For $n \geq 3$, $cp(P_n^{+++}) \leq 2n - 3$.*

Proof Consider the path $P_n : v_1 - v_2 - v_3 - \cdots - v_n$. Let $e_i = v_iv_{i+1}$ be the edges of P_n. We note that order, size and clique number of P_n^{+++} are $2n-1$, $4n-5$ and 3 respectively. The edges of subgraphs S_x and S_z are partitioned into K'_3s and that of S_y by K'_2s:

$\{\{v_i,\ v_{i+1},\ e_i\} : 1 \leq i \leq n-1\}$ and $\{\{e_j,\ e_{j+1}\} : 1 \leq j \leq n-2\}$.

Therefore, $P_n^{+++} = (n-1)K_3 \cup (n-2)K_2$ and $cp(P_n^{+++}) \leq 2n - 3$. □

Theorem 5.2 *For* $n \geq 3$, $cp(C_n^{+++}) \leq 2n$.

Theorem 5.3 *For* $n \geq 3$, $cp(K_{1,n}^{+++}) \leq n+1$.

Theorem 5.4 *For* $n \geq 6$, $cp(W_{n+1}^{+++}) \leq 3n+1$.

Proof The order, size and clique number of W_{n+1}^{+++} are $3n+1$, $(n^2+17n)/2$ and $n+1$. The edge set of W_{n+1}^{+++} is partitioned into a K_n, $3nK_3's$. Here,

$B_1 = \{\{e_1,\ e_2,\ e_3,\ \cdots,\ e_{n-1},\ e_n\}\}$,
$B_2 = \{\{e_i,\ e_i',\ e_j'\}:\ \text{for each } 1 \leq i \leq n,\ j \equiv i-1(mod\ n)\}$,
$B_3 = \{\{v_i,\ v_j,\ e_i'\},\ \{v_0,\ v_i,\ e_i\}:\ \text{for each } 1 \leq i \leq n,\ j \equiv i+1(mod\ n)\}$.

Here each edge of subgraphs S_x and S_z are present in exactly one clique of B_3 and each edge of S_y is in exactly one clique of B_1 or B_2. Thus, $W_{n+1}^{+++} = K_n \cup 3nK_3$ and hence $cp(W_{n+1}^{+++}) \leq 3n+1$. □

Theorem 5.5 *For* $n \geqslant 4$, $cp(K_n^{+++}) \leq n+1$.

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