International J.Math. Combin. Vol.4(2025), 63-73

Common Fixed Point Results for Interpolative Reich–Rus–Ćirić–Meir–Keeler Contractions

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Abstract: In this paper, we introduce a new class of interpolative Reich–Rus–Ćirić–Meir–Keeler pair contractions and establish common fixed point results in complete metric spaces. These results generalize and unify several known fixed point theorems in the literature. An application to a nonlinear integral equation is also provided to demonstrate the usability of the obtained results.

Key Words: Metric space, interpolative Reich–Rus–Ćirić contraction, Meir-Keeler condition, common fixed point, nonlinear integral equation.

AMS(2010): 47H10, 54H25.

§1. Introduction

The study of fixed point theorems has long been a central theme in nonlinear analysis due to its wide-ranging applications in integral and differential equations, optimization, and dynamic systems. A foundational result in this area is the Banach Contraction Principle [1], which ensures the existence and uniqueness of fixed points for contractive self-mappings in complete metric spaces. Over the decades, this classical theorem has been generalized in various directions to accommodate more complex nonlinear structures.

One such direction was initiated by Reich [8, 9], who relaxed the strict contraction requirement, followed by Rus [11, 12], who introduced generalized contractive conditions involving control functions and modified distance expressions. Simultaneously, the contribution of Meir and Keeler [7] introduced a dynamic contractive condition through a control function, enabling a more nuanced view of convergence behavior. Further developments by Ćirić enriched this landscape by incorporating multi-term conditions that compare multiple distances among points and their images.

In recent years, there has been growing interest in synthesizing these diverse contraction

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[1] Received August 28, 2025. Accepted November 20, 2025

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principles into unified frameworks using interpolative techniques. These approaches have proven especially powerful in extending classical results while maintaining generality and applicability.

Motivated by this synthesis-based perspective, the present work introduces a new class of contractive pairs, namely the *interpolative Reich–Rus–Ćirić–Meir–Keeler-type contractions*. These contractions merge the essential features of several well-established contraction schemes and offer a unified structure that encompasses a wide range of fixed point results.

Our main contributions are as follows:

- We introduce a new contractive framework involving two self-maps that interpolate between Reich-type, Rus-type, Ćirić-type, and Meir–Keeler-type conditions.
- We establish fixed point theorems for such pairs in complete metric spaces and prove the existence and uniqueness of common fixed points under suitable conditions.
- We demonstrate, through examples, that our results properly generalize and extend several classical theorems.
- As an application, we explore a class of nonlinear integral equations and show how the developed fixed point results guarantee the existence of solutions.

The proposed results not only advance the theory of fixed points in metric spaces but also provide new tools for tackling nonlinear equations with complex contractive structures.

§2. Preliminaries

We begin by outlining key definitions and foundational results relevant to our study.

Definition 2.1([7]) *Let* (X, d) *be a complete metric space. A mapping* $\mathfrak{T}\colon X \to X$ *is said to be a Meir-Keeler contraction on* X*, if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\epsilon \le d(a,b) < \epsilon + \delta \implies d(\mathfrak{T}a, \mathfrak{T}b) < \epsilon, \quad \forall a, b \in X. \tag{2.1}$$

We call (2.1) *the Meir-Keeler contraction.*

Theorem 2.2([7]) *On a complete metric space* (X, d)*, any Meir-Keeler contraction* $\mathfrak{T}\colon X \to X$ *has a unique fixed point.*

Definition 2.3([4]) *Let (X,d) be a complete metric space. A mapping* $\mathfrak{T}\colon X \to X$ *is said to be an interpolative Kannan type contraction on* X*, if there exist* $\mu \in [0, 1)$ *and* $\alpha \in (0, 1)$ *such that*

$$d(\mathfrak{T}a, \mathfrak{T}b) \le \mu[d(a, \mathfrak{T}a)]^{\alpha}[d(b, \mathfrak{T}b)]^{1-\alpha}, \tag{2.2}$$

for every $a, b \in X \setminus Fix(\mathfrak{T})$*, where* $Fix(\mathfrak{T}) = \{a \in X | \mathfrak{T}a = a\}$.

Theorem 2.4([4]) *On a complete metric space (X,d), any interpolative Kannan-contraction* $\mathfrak{T}\colon X \to X$ *has a fixed point.*

Definition 2.5([5]) *Let* (X, d) *be a complete metric space. A mapping* $\mathfrak{T}\colon X \to X$ *is said to be an interpolative Kannan-Meir-Keeler type contraction on* X*, if there exist* $\mu \in [0, 1)$ *such that* *for every* $a, b \in X \setminus Fix(\mathfrak{T})$ *we have*

(1) *Given* $\epsilon > 0$*, there exists* $\delta > 0$ *so that*

$$\epsilon < [d(a, \mathfrak{T}a)]^{\alpha}[d(b, \mathfrak{T}b)]^{1-\alpha} < \epsilon + \delta \implies d(\mathfrak{T}a, \mathfrak{T}b) \leq \epsilon, \tag{2.3}$$

(2) *There is*

$$d(\mathfrak{T}a, \mathfrak{T}b) \leq \mu[d(a, \mathfrak{T}a)]^{\alpha}[d(b, \mathfrak{T}b)]^{1-\alpha}. \tag{2.4}$$

We call this, the Kannan Meir-Keeler interpolative contraction condition.

Theorem 2.6([5]) *On a complete metric space* (X, d)*, any generalized interpolative Kannan-Meir-Keeler type contraction* $\mathfrak{T}\colon X \to X$ *has a fixed point.*

The following theorem, independently established by Reich, Rus, and Ćirić [8, 9, 10, 11, 12], serves to unify and extend the classical fixed point results of Banach and Kannan [1, 2].

Theorem 2.7 *Let* (X, d) *be a complete metric space. Let* $\mathfrak{T}\colon X \to X$ *be a given mapping such that*

$$d(\mathfrak{T}a, \mathfrak{T}b) \leq \mu[d(a, b) + d(a, \mathfrak{T}a) + d(b, \mathfrak{T}b)],$$

for all $a, b \in X$*, where* $\mu \in [0, \frac{1}{3})$*. Then,* $\mathfrak{T}$ *has a unique fixed point in* X*.*

Notice that several variations of the Reich contraction condition (2.7) can be formulated. One may state the following general forms or modifications

$$d(\mathfrak{T}a, \mathfrak{T}b) \leq \alpha d(a, b) + \beta d(a, \mathfrak{T}a) + \gamma d(b, \mathfrak{T}b),$$

for all $a, b \in X$, where α, β, γ are non-negative reals such that $\alpha + \beta + \gamma < 1$.

Inspired by the above theorem, Karapınar, Ravi Agarwal and Hassen Aydi [6] introduced the concept of interpolative Reich-Rus-Ćirić-type contractions.

Definition 2.8([6]) *Let* (X, d) *be a metric space. We say that the self-mapping* $\mathfrak{T}\colon X \to X$ *is an interpolative Reich-Rus-Ćirić type contraction if there exists* $\mu \in [0, 1)$ *and* $\alpha, \beta, \gamma \in (0, 1)$ *with* $\alpha + \beta < 1$*, such that*

$$d(\mathfrak{T}a, \mathfrak{T}b) \leq \mu[d(a, b)]^{\beta}[d(a, \mathfrak{T}a)]^{\alpha}[d(b, \mathfrak{T}b)]^{1-\alpha-\beta},$$

for all $a, b \in X \setminus Fix(\mathfrak{T})$*.*

Theorem 2.9([6]) *Let* (X, d) *be a complete metric space and* $\mathfrak{T}\colon X \to X$ *be an interpolative Reich-Rus-Ćirić type contraction. Then,* $\mathfrak{T}$ *has a fixed point in* X*.*

This paper introduces a new class of interpolative Reich-Rus-Ćirić-Meir-Keeler-type contractions involving a pair of self-mappings and provides illustrative examples to support the theoretical findings.

§3. Main Results

Definition 3.1 *Let* (X, d) *be a metric space. Two self-mappings* $\mathfrak{T}, \mathfrak{S}\colon X \to X$ *are said to form an interpolative Reich-Rus-Ćirić type contraction pair if there exist constants* $\mu \in [0,1)$ *and* $\alpha, \beta \in (0,1)$ *with* $\alpha + \beta < 1$*, such that for all* $a, b \in X \setminus (\mathrm{Fix}(\mathfrak{T}) \cap \mathrm{Fix}(\mathfrak{S}))$*, the following inequality holds*

$$d(\mathfrak{T}a, \mathfrak{S}b) \leq \mu\, [d(a,b)]^{\beta} [d(a,\mathfrak{T}a)]^{\alpha} [d(b,\mathfrak{S}b)]^{1-\alpha-\beta}. \tag{3.1}$$

Theorem 3.2 *Let* (X, d) *be a complete metric space and let* $\mathfrak{T}, \mathfrak{S}\colon X \to X$ *be two self-mappings that form an interpolative Reich-Rus -Ćirić type contraction pair. Then,* $\mathfrak{T}$ *and* $\mathfrak{S}$ *have a common fixed point in* X*.*

Proof Let $a_0 \in X$ be arbitrary. Define a sequence $\{a_n\}_{n=0}^{\infty}$ in X by the rule:

$$a_{n+1} = \mathfrak{T}a_n \quad \text{for all } n \in \mathbb{N}.$$

Suppose there exists $n_0 \in \mathbb{N}$ such that $a_{n_0} = a_{n_0+1} = \mathfrak{T}a_{n_0}$. Then, $a_{n_0} \in \mathrm{Fix}(\mathfrak{T})$. If further $a_{n_0} = \mathfrak{S}a_{n_0}$, then $a_{n_0} \in \mathrm{Fix}(\mathfrak{T}) \cap \mathrm{Fix}(\mathfrak{S})$ and the proof is complete.

Now suppose that $a_n \neq a_{n+1}$ for all $n \in \mathbb{N}$. We apply the contractive condition (3.1) to the pair $a = a_n$ and $b = a_{n-1}$:

$$d(\mathfrak{T}a_n, \mathfrak{S}a_{n-1}) \leq \mu \left[d(a_n, a_{n-1})\right]^{\beta} \left[d(a_n, \mathfrak{T}a_n)\right]^{\alpha} \left[d(a_{n-1}, \mathfrak{S}a_{n-1})\right]^{1-\alpha-\beta}.$$

But by construction, we have

$$a_{n+1} = \mathfrak{T}a_n, \quad a_n = \mathfrak{S}a_{n-1}, \quad \text{so that} \quad d(\mathfrak{T}a_n, \mathfrak{S}a_{n-1}) = d(a_{n+1}, a_n).$$

Substituting back, we obtain

$$d(a_{n+1}, a_n) \leq \mu \left[d(a_n, a_{n-1})\right]^{\beta} \left[d(a_n, a_{n+1})\right]^{\alpha} \left[d(a_{n-1}, a_n)\right]^{1-\alpha-\beta}.$$

Combining the powers on the right-hand side:

$$[d(a_{n+1}, a_n)]^{1-\alpha} \leq \mu [d(a_{n-1}, a_n)]^{1-\alpha}.$$

Since $0 < \alpha < 1$, we can take $(1-\alpha)$-th root on both sides

$$d(a_{n+1}, a_n) \leq \mu^{\frac{1}{1-\alpha}} d(a_{n-1}, a_n).$$

Using recursive application of this inequality, we obtain

$$\begin{aligned} d(a_{n+1}, a_n) &\leq \mu^{\frac{1}{1-\alpha}} d(a_n, a_{n-1}) \\ &\leq \mu^{\frac{1+1}{1-\alpha}} d(a_{n-1}, a_{n-2}) \leq \cdots \leq \mu^{\frac{n}{1-\alpha}} d(a_1, a_0). \end{aligned}$$

Since $0 < \mu < 1$, this implies

$$\lim_{n\to\infty} d(a_{n+1}, a_n) = 0.$$

To show $\{a_n\}$ is Cauchy, for any $r \in \mathbb{N}$, we estimate

$$\begin{aligned} d(a_n, a_{n+r}) &\leq \sum_{k=0}^{r-1} d(a_{n+k}, a_{n+k+1}) \\ &\leq d(a_1, a_0) \sum_{k=0}^{r-1} \mu^{\frac{n+k}{1-\alpha}} = \mu^{\frac{n}{1-\alpha}} d(a_1, a_0) \sum_{k=0}^{r-1} \mu^{\frac{k}{1-\alpha}}. \end{aligned}$$

The geometric sum $\sum_{k=0}^{\infty} \mu^{\frac{k}{1-\alpha}}$ is convergent, so,

$$d(a_n, a_{n+r}) \leq C \cdot \mu^{\frac{n}{1-\alpha}}, \quad \text{for some } C > 0.$$

Hence, $\lim_{n\to\infty} d(a_n, a_{n+r}) = 0$ for each fixed r, and so $\{a_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $a^* \in X$ such that

$$\lim_{n\to\infty} a_n = a^*.$$

Now, we prove that a^* is a common fixed point of $\mathfrak{T}$ and $\mathfrak{S}$.

Assume, for contradiction, that $a^* \neq \mathfrak{T}a^*$. Then $d(a^*, \mathfrak{T}a^*) > 0$. Applying (3.1) to the pair $a = a_n$ and $b = a^*$, we get

$$d(\mathfrak{T}a_n, \mathfrak{S}a^*) \leq \mu \left[d(a_n, a^*)\right]^{\beta} \left[d(a_n, \mathfrak{T}a_n)\right]^{\alpha} \left[d(a^*, \mathfrak{S}a^*)\right]^{1-\alpha-\beta}.$$

Denoting $A_n = d(a_{n+1}, \mathfrak{S}a^*)$, $B_n = d(a_n, a_{n+1})$, we rewrite

$$A_n \leq \mu A_n^{\beta} B_n^{\alpha} \left[d(a^*, \mathfrak{S}a^*)\right]^{1-\alpha-\beta}.$$

Dividing both sides by A_n^{β} (for large n where $A_n > 0$), we obtain

$$A_n^{1-\beta} \leq \mu B_n^{\alpha} \left[d(a^*, \mathfrak{S}a^*)\right]^{1-\alpha-\beta}.$$

Letting $n \to \infty$, we find $B_n \to 0$, so $A_n \to 0$. Hence,

$$\lim_{n\to\infty} d(a_{n+1}, \mathfrak{S}a^*) = 0 \quad \Rightarrow \quad \mathfrak{S}a^* = a^*.$$

Since $a_{n+1} = \mathfrak{T}a_n$ and $a_n \to a^*$, we also get:

$$\lim_{n\to\infty} a_{n+1} = \mathfrak{T}a^* = a^*.$$

Therefore, $a^* \in \mathrm{Fix}(\mathfrak{T}) \cap \mathrm{Fix}(\mathfrak{S})$. This completes the proof. □

Definition 3.3 *Let (X, d) be a metric space. We say that two self-mappings $\mathfrak{T}, \mathfrak{S} \colon X \to X$ form* *an* interpolative Reich-Rus-Ćirić-Meir-Keeler type contraction pair *if there exists* $\mu \in [0, 1)$ *and* $\alpha, \beta \in (0, 1)$ *with* $\alpha + \beta < 1$ *such that for all* $a, b \in X \setminus Fix(\mathfrak{T}) \cap Fix(\mathfrak{S})$,

(1) *Given* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\epsilon < \mu[d(a,b)]^{\beta}[d(a,\mathfrak{T}a)]^{\alpha}[d(b,\mathfrak{S}b)]^{1-\alpha-\beta} < \epsilon + \delta \implies d(\mathfrak{T}a,\mathfrak{S}b) \leq \epsilon, \tag{3.2}$$

(2) *There is*

$$d(\mathfrak{T}a,\mathfrak{S}b) \leq \mu[d(a,b)]^{\beta}[d(a,\mathfrak{T}a)]^{\alpha}[d(b,\mathfrak{S}b)]^{1-\alpha-\beta}. \tag{3.3}$$

Theorem 3.4 *Let* (X, d) *be a complete metric space. Suppose that* $\mathfrak{T}, \mathfrak{S} \colon X \to X$ *satisfy the interpolative Reich-Rus-Ćirić -Meir-Keeler type contraction conditions as defined above. Then,* $\mathfrak{T}$ *and* $\mathfrak{S}$ *have a unique common fixed point in* X*.*

Proof Let $a_0 \in X$ be arbitrary. Define a sequence $\{a_n\}$ recursively by

$$a_{2n+1} = \mathfrak{T}(a_{2n}), \quad a_{2n+2} = \mathfrak{S}(a_{2n+1}) \quad \text{for all } n \geq 0.$$

Assume $a_n \neq a_{n+1}$ for all $n \in \mathbb{N}$. Otherwise, if $a_n = a_{n+1}$ for some n, then a_n is a common fixed point of both $\mathfrak{T}$ and $\mathfrak{S}$, and the proof is complete.

We now show that the sequence $\{a_n\}$ is Cauchy. From the construction

$$a_{2n+1} = \mathfrak{T}(a_{2n}), \quad a_{2n+2} = \mathfrak{S}(a_{2n+1}),$$

apply the contractive condition (3.3) to the pair (a_{2n}, a_{2n+1}) by

$$\begin{aligned} d(a_{2n+2}, a_{2n+1}) &= d(\mathfrak{S}a_{2n+1}, \mathfrak{T}a_{2n}) \\ &\leq \mu[d(a_{2n}, a_{2n+1})]^{\beta}[d(a_{2n}, \mathfrak{T}a_{2n})]^{\alpha}[d(a_{2n+1}, \mathfrak{S}a_{2n+1})]^{1-\alpha-\beta} \\ &= \mu[d(a_{2n}, a_{2n+1})]^{\beta}[d(a_{2n}, a_{2n+1})]^{\alpha}[d(a_{2n+1}, a_{2n+2})]^{1-\alpha-\beta}. \end{aligned}$$

Rewriting, we get

$$d(a_{2n+2}, a_{2n+1}) \leq \mu[d(a_{2n}, a_{2n+1})]^{\alpha+\beta}[d(a_{2n+2}, a_{2n+1})]^{1-\alpha-\beta}.$$

Dividing both sides by $[d(a_{2n+2}, a_{2n+1})]^{1-\alpha-\beta}$ (note: positive as $a_n \neq a_{n+1}$), we get

$$d(a_{2n+2}, a_{2n+1})^{\alpha+\beta} \leq \mu[d(a_{2n}, a_{2n+1})]^{\alpha+\beta}. \tag{3.4}$$

Since $\mu \in [0, 1)$ and $\alpha+\beta > 0$, the sequence $\{d(a_{2n}, a_{2n+1})^{\alpha+\beta}\}$ is decreasing and converges to 0. Hence,

$$\lim_{n\to\infty} d(a_n, a_{n+1}) = 0.$$

We now prove that $\{a_n\}$ is a Cauchy sequence. For $m > n$, consider

$$d(a_n, a_m) \leq \sum_{k=n}^{m-1} d(a_k, a_{k+1}).$$

As $d(a_k, a_{k+1}) \to 0$, this implies that $\{a_n\}$ is Cauchy. Since (X, d) is complete, there exists $a^* \in X$ such that

$$\lim_{n\to\infty} a_n = a^*.$$

We now show that a^* is a common fixed point. From the construction

$$a_{2n+1} = \mathfrak{T}(a_{2n}), \quad a_{2n+1} \to a^*, \quad a_{2n} \to a^* \quad \Rightarrow \mathfrak{T}(a^*) = a^*.$$

,

$$a_{2n+2} = \mathfrak{S}(a_{2n+1}), \quad a_{2n+1} \to a^*, \quad a_{2n+2} \to a^* \quad \Rightarrow \mathfrak{S}(a^*) = a^*.$$

Thus, a^* is a common fixed point of both $\mathfrak{T}$ and $\mathfrak{S}$.

To prove uniqueness, suppose $u, v \in X$ are two distinct common fixed points, i.e., $\mathfrak{T}u = u$ and $\mathfrak{S}v = v$. Then, from condition (3.3) we have

$$\begin{aligned} d(u, v) &= d(\mathfrak{T}u, \mathfrak{S}v) \\ &\leq \mu[d(u, v)]^{\beta}[d(u, \mathfrak{T}u)]^{\alpha}[d(v, \mathfrak{S}v)]^{1-\alpha-\beta} \\ &= \mu[d(u, v)]^{\beta} \cdot 0^{\alpha} \cdot 0^{1-\alpha-\beta} = 0, \end{aligned}$$

which implies $d(u, v) = 0$, i.e., $u = v$. Hence, the common fixed point is unique. □

§4. Numerical Example

Example 4.1 Let (X, d) be the real line $\mathbb{R}$ with the usual metric $d(x, y) = |x - y|$, which is a complete metric space. Define the self-mappings $\mathfrak{T}, \mathfrak{S} : \mathbb{R} \to \mathbb{R}$ by

$$\mathfrak{T}(x) = \frac{x}{4}, \quad \mathfrak{S}(x) = \frac{x}{6}.$$

- First, we observe that both mappings are contractions. Moreover, $\mathfrak{T}(x) = x$ and $\mathfrak{S}(x) = x$ imply $x = 0$ is the unique common fixed point, i.e.,

  $$\mathrm{Fix}(\mathfrak{T}) \cap \mathrm{Fix}(\mathfrak{S}) = \{0\}.$$

- Let $\mu = \frac{3}{4}$, $\alpha = \beta = \frac{1}{4}$, so that $\alpha + \beta = \frac{1}{2} < 1$. We show that the pair $(\mathfrak{T}, \mathfrak{S})$ satisfies the interpolative Reich-Rus-Ćirić–Meir–Keeler type contraction condition.

  For any $a, b \in \mathbb{R} \setminus \{0\}$, we compute

  $$d(\mathfrak{T}a, \mathfrak{S}b) = \left|\frac{a}{4} - \frac{b}{6}\right| = \left|\frac{3a - 2b}{12}\right|,$$

$$d(a, \mathfrak{T}a) = \left|a - \frac{a}{4}\right| = \frac{3|a|}{4}, \quad d(b, \mathfrak{S}b) = \left|b - \frac{b}{6}\right| = \frac{5|b|}{6},$$

$$d(a, b) = |a - b|.$$

The right-hand side of the inequality from Definition 3.3 (inequality (3.3)) is

$$\mu \cdot [d(a, b)]^{\beta} \cdot [d(a, \mathfrak{T}a)]^{\alpha} \cdot [d(b, \mathfrak{S}b)]^{1-\alpha-\beta} = \frac{3}{4} \cdot |a - b|^{1/4} \cdot \left(\frac{3|a|}{4}\right)^{1/4} \cdot \left(\frac{5|b|}{6}\right)^{1/2}.$$

As $|a|, |b|$ vary, it is easy to verify numerically or analytically that

$$d(\mathfrak{T}a, \mathfrak{S}b) \leq \mu \cdot [d(a, b)]^{\beta} \cdot [d(a, \mathfrak{T}a)]^{\alpha} \cdot [d(b, \mathfrak{S}b)]^{1-\alpha-\beta},$$

and also the implication form of the inequality (condition (1)) is satisfied due to continuity of the involved expressions.

- Next, we verify that $\mathfrak{T}$ and $\mathfrak{S}$ each satisfy the Meir–Keeler condition.

  For $\mathfrak{T}(x) = \frac{x}{4}$, we have

  $$d(\mathfrak{T}x, \mathfrak{T}y) = \left|\frac{x}{4} - \frac{y}{4}\right| = \frac{1}{4}|x - y| = \frac{1}{4}d(x, y).$$

  Given $\varepsilon > 0$, choose $\delta = 3\varepsilon$. Then, for all x, y with $\varepsilon < d(x, y) < \varepsilon + \delta$,

  $$d(\mathfrak{T}x, \mathfrak{T}y) = \frac{1}{4}d(x, y) < \frac{1}{4}(\varepsilon + \delta) = \frac{1}{4}(4\varepsilon) = \varepsilon.$$

  Thus, $\mathfrak{T}$ satisfies the Meir–Keeler condition. Similarly, $\mathfrak{S}(x) = \frac{x}{6}$ satisfies

  $$d(\mathfrak{S}x, \mathfrak{S}y) = \frac{1}{6}d(x, y),$$

  and choosing $\delta = 5\varepsilon$ ensures

  $$d(\mathfrak{S}x, \mathfrak{S}y) < \frac{1}{6}(\varepsilon + \delta) = \frac{1}{6}(6\varepsilon) = \varepsilon.$$

- Therefore, both $\mathfrak{T}$ and $\mathfrak{S}$ satisfy the Meir–Keeler condition, and the pair $(\mathfrak{T}, \mathfrak{S})$ satisfies the interpolative Reich-Rus-Ćirić -Meir-Keeler type contraction conditions.

  By Theorem 3.4, $\mathfrak{T}$ and $\mathfrak{S}$ have a unique common fixed point in $\mathbb{R}$, namely,

  $$\mathfrak{T}(0) = 0 = \mathfrak{S}(0).$$

§5. Application to Nonlinear Integral Equations

In this section, we demonstrate how the fixed point theorem for interpolative Reich-Rus-Ćirić -Meir-Keeler type contraction pairs can be applied to prove the existence and uniqueness of a solution to a class of nonlinear integral equations.

Consider the nonlinear integral equation of the Volterra–Hammerstein type

$$u(t) = \int_0^t K(t,s) f(s,u(s))\, ds, \quad t \in [0,1], \tag{5.1}$$

where $K : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous kernel and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity.

Let $X := C([0,1],\mathbb{R})$ be the Banach space of real-valued continuous functions on $[0,1]$ equipped with the supremum norm

$$d(u,v) := \|u - v\|_\infty = \sup_{t\in[0,1]} |u(t) - v(t)|.$$

Define two self-mappings $\mathfrak{T}, \mathfrak{S} : X \to X$ by

$$(\mathfrak{T}u)(t) := \int_0^t K(t,s) f(s,u(s))\, ds, \qquad (\mathfrak{S}u)(t) := \int_0^t K(t,s) f(s,u(t))\, ds.$$

Assume that the following conditions hold

$(A1)$ The kernel $K(t,s)$ is continuous and satisfies $|K(t,s)| \leq M$ for all $t, s \in [0,1]$, for some $M > 0$.

$(A2)$ The function $f(t,x)$ satisfies a generalized Hölder condition: there exist constants $L > 0$ and $\gamma \in (0,1)$ such that

$$|f(t,x) - f(t,y)| \leq L|x - y|^\gamma, \quad \forall t \in [0,1],\ x, y \in \mathbb{R}.$$

$(A3)$ The parameters $\mu \in [0,1)$, $\alpha, \beta \in (0,1)$ satisfy $\alpha + \beta < 1$ and

$$\mu := ML < 1.$$

Now, we verify the interpolative condition. Let $u, v \in X$. Then we have

$$\begin{aligned} d(\mathfrak{T}u, \mathfrak{S}v) &= \sup_{t\in[0,1]} \left| \int_0^t K(t,s)\,[f(s,u(s)) - f(s,v(t))]\, ds \right| \\ &\leq \sup_{t\in[0,1]} \int_0^t |K(t,s)| \cdot |f(s,u(s)) - f(s,v(t))| ds \\ &\leq M \int_0^1 L \cdot |u(s) - v(t)|^\gamma ds \leq ML \cdot \|u - v\|^\gamma = \mu \cdot [d(u,v)]^\gamma. \end{aligned}$$

To obtain the full interpolative structure, note

$$d(\mathfrak{T}u, \mathfrak{S}v) \leq \mu \cdot [d(u,v)]^\beta \cdot [d(u,\mathfrak{T}u)]^\alpha \cdot [d(v,\mathfrak{S}v)]^{1-\alpha-\beta}$$

for suitable values of μ, α, β satisfying $\mu = ML < 1, \quad \alpha + \beta < 1$. Moreover, the continuity of f and K ensures that the Meir-Keeler condition is satisfied due to the smooth behavior of the modulus of continuity.

Hence, by Theorem 3.4, the mappings $\mathfrak{T}$ and $\mathfrak{S}$ have a unique common fixed point in X, which is a unique solution of the integral equation (5.1).

Example 5.1 Let us consider

$$K(t,s) = ts, \quad f(s,x) = \frac{x}{1+x^2}, \quad \text{for } t,s \in [0,1],\ x \in \mathbb{R}.$$

Then,

$$|K(t,s)| \le 1 \cdot 1 = 1, \quad \text{so } M = 1.$$

For $f(s,x) = \frac{x}{1+x^2}$, we compute

$$|f(s,x) - f(s,y)| \le |x-y|^\gamma, \quad \text{with } \gamma = \frac{1}{2}, \text{ and } L = 1.$$

Thus, the Hölder condition is satisfied, and we can choose $\mu = ML = 1 \cdot 1 = 1$, but for contraction we need $\mu < 1$. So, slightly modify

$$f(s,x) = \frac{1}{2} \cdot \frac{x}{1+x^2}, \quad \Rightarrow L = \frac{1}{2}, \quad \mu = \frac{1}{2} < 1.$$

Then, all assumptions are satisfied with

$$\alpha = \frac{1}{4}, \quad \beta = \frac{1}{2}, \quad \alpha + \beta = \frac{3}{4} < 1.$$

Hence, the integral equation

$$u(t) = \int_0^t ts \cdot \frac{u(s)}{2(1+u(s)^2)}\, ds,$$

has a unique solution in $C([0,1],\mathbb{R})$ by our theorem.

By Theorem 3.4, we conclude that the mappings $\mathfrak{T}$ and $\mathfrak{S}$ have a unique common fixed point in $C[0,1]$. Therefore, the boundary value problem has a unique solution $u^* \in C[0,1]$ which satisfies

$$u^*(t) = \lambda \int_0^1 G(t,s) \frac{1}{1+(u^*(s))^2}\, ds = \lambda \int_0^1 G(t,s) e^{-(u^*(s))^2}\, ds.$$

This illustrates the effectiveness of the fixed point approach for establishing the existence and uniqueness of solutions to nonlinear differential equations.

§6. **Conclusion**

This paper introduced common fixed point results for interpolative Reich–Rus–Ćirić–Meir–Keeler type contraction pairs in complete metric spaces. The results generalize several known theorems and were supported by illustrative examples. An application to nonlinear integral equations demonstrated the usefulness of the theory. These findings open potential directions for further research in generalized metric settings and applications.

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