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Complexity of Sequence of Some Families of Graphs and Their Asymptotic Behavior

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Abstract: Calculating and analyzing the number of spanning trees of networks (graphs) is an interesting and important research project in wide variety of fields, such as mathematics, theoretical computer science, chemistry, physics and so on. In this paper, we investigated the number of spanning trees in three sequences of families of graphs of the same average degree $\frac{14}{3}$. We used the electrically equivalent transformations and rules of weighted generating function which avoids the laborious computation of the determinant for counting the number of spanning trees. Finally, we determined the entropy of our studied graphs.

Key Words: Number of spanning trees, electrically equivalent transformations, entropy.

AMS(2010): 05C30,05C50,05C63.

§1. Introduction

Complexity (the number of spanning trees) $\tau(G)$ of a finite connected undirected graph G is the total number of distinct spanning subgraphs of G that are trees. As it's known, the problem of evaluating the number of spanning trees of a finite connected undirected graph has been solved by famous Kirchhoff's matrix-tree theorem [1], the product of all nonzero eigenvalues of the Laplacian matrix of the graph. But, for a large-size graph with thousands of vertices and edges, this problem will become more difficult. How to find out the exact solutions of the number of spanning trees of models has been a demanding and exciting mission, in particular on some real-world networks, and always draws many concerns from various science fields, such as mathematics [2], computer science [3], chemistry [4], physics [5] and so on. Luckily, there has been some useful methods, such as the theory of electrical networks, to find the accurate solution for the number of spanning trees of special graph families, for example lattices, grids, Farey graph and Sierpinski gaskets, see [6], [7], [8], [9], [10].

For a summary of further results for calculating umber of spanning trees of graphs, see [11, 12, 13, 14, 15].

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§2. Electrically Equivalent Transformations

To begin with, we briefly review the electrically equivalent transformation technique introduced in [16,17]. An edge-weighted graph G (with the weight function ω : E(G) $\to [0, \infty)$) can be considered as an electrical network with the weights being the conductances of the corresponding edges. The weighted number of spanning trees in G is defined as: Let G be an edge weighted graph, G' be the corresponding electrically equivalent graph, $\tau(G)$ denotes the weighted number of spanning trees G.

(a) Parallel edges: If two parallel edges with conductances u and v in G are merged into a single edge with conductances $u+v$ in G', then $\tau\left(G'\right) = \tau(G)$;

(b) Serial edges: If two serial edges with conductances u and v in G are merged into a single edge with conductance $\frac{uv}{u+v}$ in G', then $\tau\left(G'\right) = \frac{1}{u+v}\tau(G)$;

(c) $\triangle - Y$ transformation: If a triangle with conductances u, v and w in G is changed into an electrically equivalent star graph with conductances $x = \frac{uv+vw+wu}{u}, y = \frac{uv+vw+wu}{v}$ and $z = \frac{uv+vw+wu}{w}$ in G', then $\tau\left(G'\right) = \frac{(uv+vw+wu)^2}{uvw}\tau(G)$;

(d) $Y - \Delta$ transformation: If a star graph with conductances u, v and w in G is changed into an electrically equivalent triangle with conductances $x = \frac{vw}{u+v+w}, y = \frac{uw}{u+v+w}$ and $z = \frac{uv}{u+v+w}$ in G', then $\tau\left(G'\right) = \frac{1}{u+v+w}\tau(G)$.

In this work, we compute the number of spanning trees of three sequences of graphs of the same average degree $\frac{14}{3}$, we named it Θ_n, Π_n and Σ_n respectively.

§3. Number of Spanning Trees in the Sequences of Θ_n Graph

Consider the sequence of graphs $\Theta_1, \Theta_2, \cdots, \Theta_n$ constructed as shown in Figure 1. According to this construction, the number of total vertices $\left|V\left(\Theta_n\right)\right|$ and edges $\left|E\left(\Theta_n\right)\right|$ are $\left|V\left(\Theta_n\right)\right| = 9n-6$ and $\left|E\left(\Theta_n\right)\right| = 21n-18, n = 1, 2, \cdots$. The average degree of Θ_n is $\dfrac{14}{3}$ in $n \to \infty$.

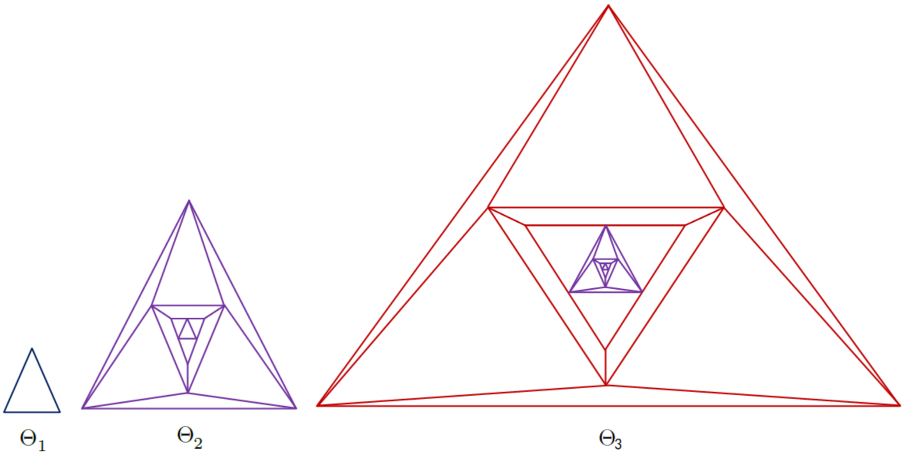


Figure 1. Some sequences of graph Θ_n

Theorem 3.1 *For $n \geq 1$, the number of spanning trees in the sequence of the graph Θ_n is given by* $\dfrac{N_1(\Theta_n)}{M_1(\Theta_n)}$, *where*

$$\begin{aligned} N_1(\Theta_n) &= 4^{n-2}\left((651-142\sqrt{21})(55+12\sqrt{21})^n+(55-12\sqrt{21})^n(651+142\sqrt{21})\right)^2 \\ &\quad \times\left(9-4\sqrt{21}+(33+8\sqrt{21})(6049+1320\sqrt{21})^{n-1}\right)^2, \\ M_1(\Theta_n) &= 147\left(17+(25+4\sqrt{21})(6049+1320\sqrt{21})^{n-1}\right)^2. \end{aligned}$$

Proof We use the electrically equivalent transformation to transform Θ_i to Θ_{i-1}. Figures 2-6 illustrate the graphs $\Theta_1, \Theta_2, G_1 - G_9$ and the transformation process from Θ_2 to Θ_1.

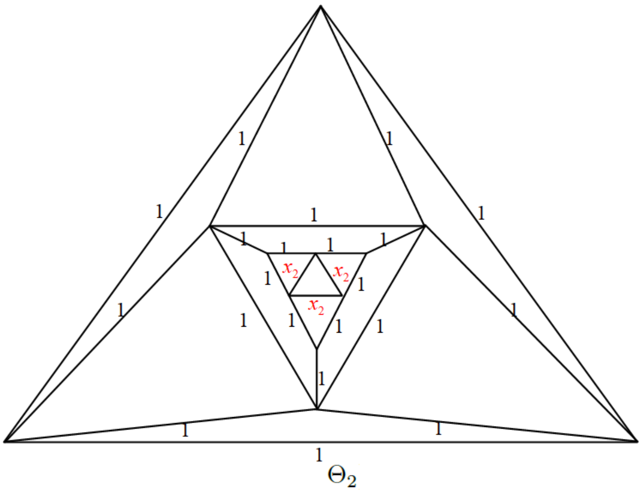


Figure 2.

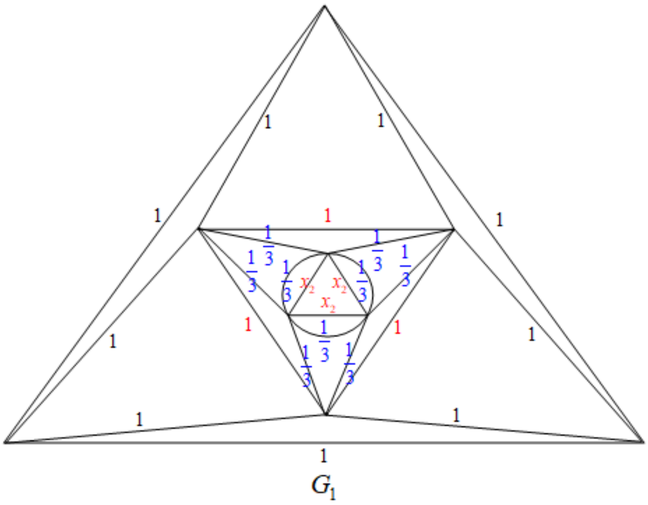
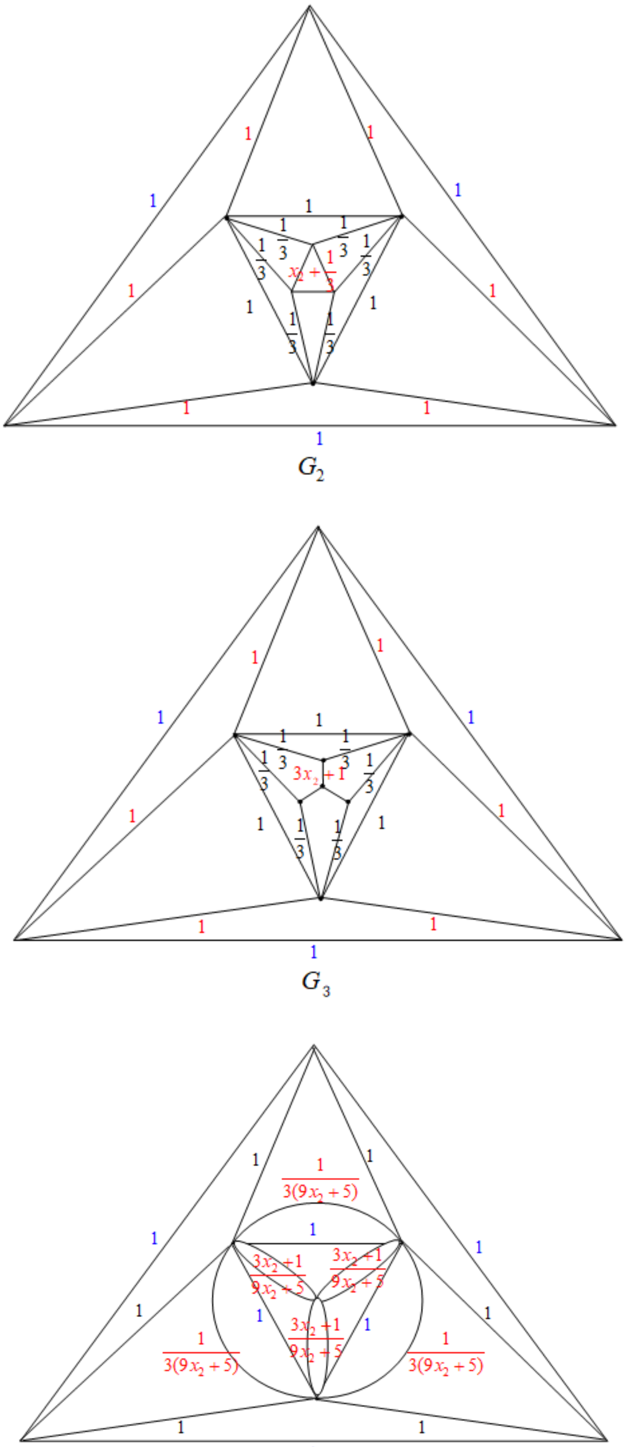


Figure 3.

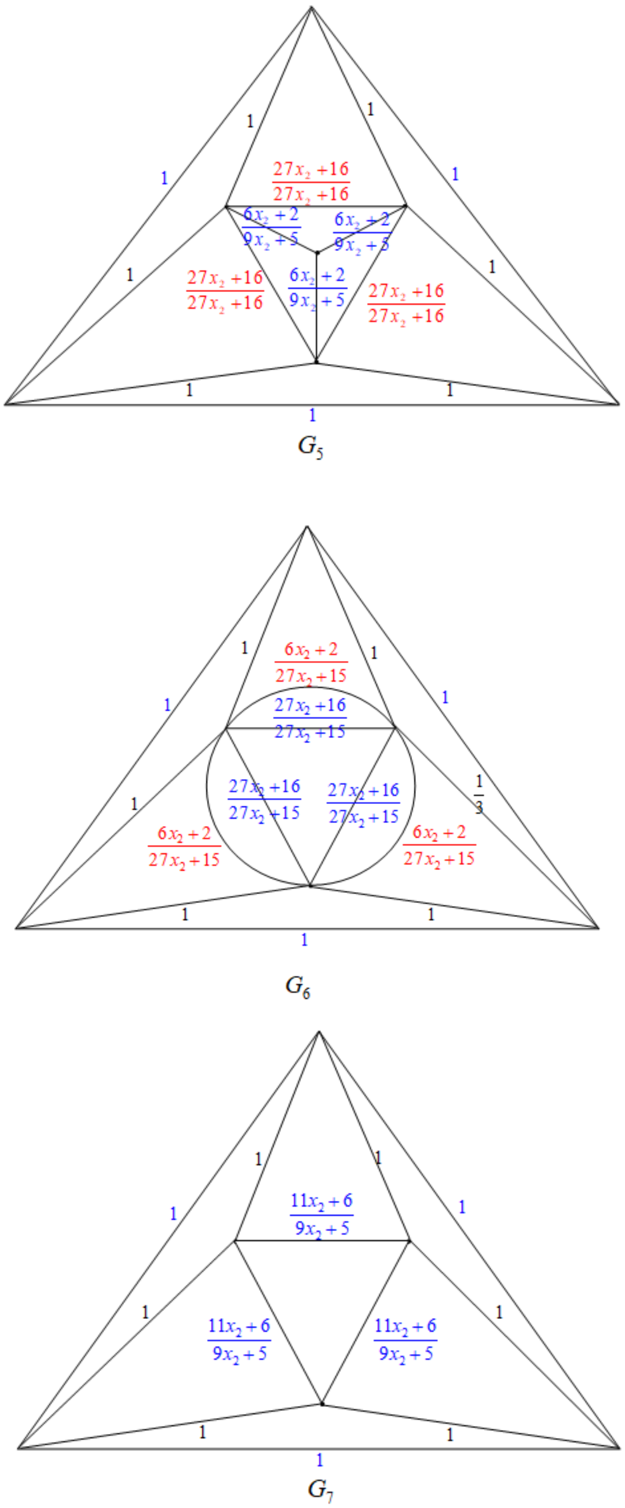


G_2

G_3

G_4

Figure 4.



G_5

G_6

G_7

Figure 5.

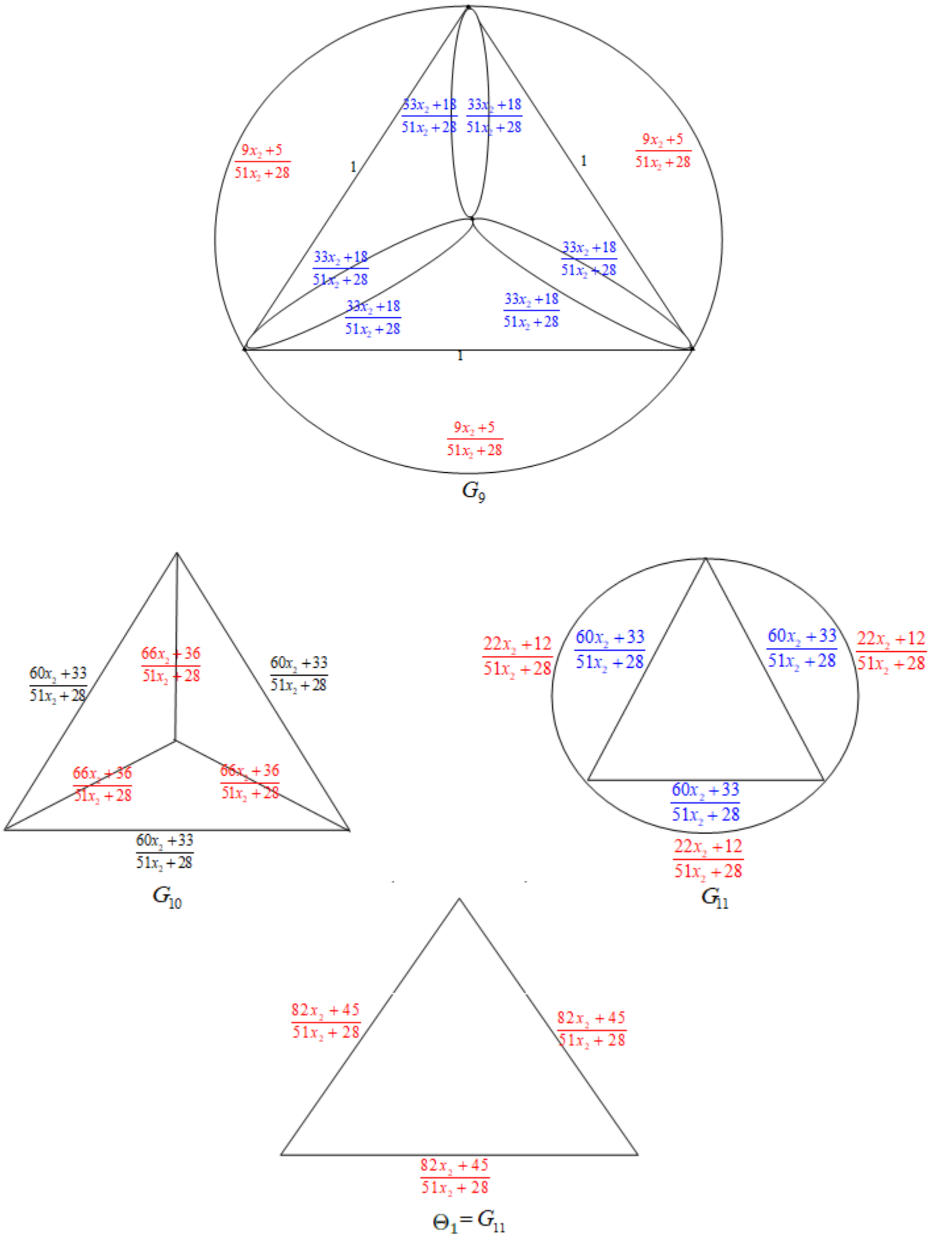


Figure 6. The transformations from θ_2 to θ_1.

Using the properties given in Section 2, we have the following the transformations:

$$\tau\left(G_1\right)=\left[\frac{1}{3}\right]^3 \tau\left(\Theta_2\right), \tau\left(G_2\right)=\tau\left(G_1\right), \tau\left(G_3\right)=\left[9x_2+3\right] \tau\left(G_2\right),$$

$$\tau\left(G_4\right)=\left[\frac{3}{9x_2+5}\right]^3 \tau\left(G_3\right), \tau\left(G_5\right)=\tau\left(G_4\right), \tau\left(G_6\right)=\left[\frac{9x_2+5}{18x_2+6}\right] \tau\left(G_5\right),$$

$$\tau\left(G_7\right)=\tau\left(G_6\right), \tau\left(G_8\right)=9\left[\frac{11 x_2+6}{9 x_2+5}\right] \tau\left(G_7\right), \tau\left(G_9\right)=\left[\frac{9 x_2+5}{51 x_2+28}\right]^3 \tau\left(G_8\right),$$
$$\tau\left(G_{10}\right)=\tau\left(G_9\right), \tau\left(G_{11}\right)=\frac{51 x_2+28}{3\left(66 x_2+36\right)} \tau\left(G_{10}\right) \text{ and } \tau\left(\Theta_1\right)=\tau\left(G_{11}\right).$$

Combining these twelve transformations, we have

$$\tau\left(\Theta_2\right)=4\left(51 x_2+28\right)^2 \tau(\Theta). \tag{3.1}$$

Further

$$\tau\left(\Theta_n\right)=\prod_{i=2}^n 4(51+28)^2 \tau\left(\Theta_1\right)=3 \times(4)^{n-1} x_1^2\left[\prod_{i=2}^n\left(51 x_i+28\right)\right]^2, \tag{3.2}$$

where $x_{i-1}=\frac{82 x_i+45}{51 x_i+28}, i=2,3, \cdots, n$. Its characteristic equation is $51 \mu^2-54 \mu-45=0$, which have two roots roots $\mu_1=\frac{9-4 \sqrt{21}}{17}$ and $\mu_2=\frac{9+4 \sqrt{21}}{17}$. Subtracting these two roots into both sides of $x_{i-1}=\frac{82 x_i+45}{51 x_i+28}$, we get

$$x_{i-1}-\frac{9-4 \sqrt{21}}{17}=\frac{82 x_i+45}{51 x_i+28}-\frac{9-4 \sqrt{21}}{17}=(55+12 \sqrt{21}) \frac{\left(x_i-\frac{9-4 \sqrt{21}}{17}\right)}{\left(51 x_i+28\right)}, \tag{3.3}$$

$$x_{i-1}-\frac{9+4 \sqrt{21}}{17}=\frac{82 x_i+45}{51 x_i+28}-\frac{9+4 \sqrt{21}}{17}=(55-12 \sqrt{21}) \frac{\left(x_i-\frac{9+4 \sqrt{21}}{17}\right)}{\left(51 x_i+28\right)}. \tag{3.4}$$

Let $y_i=\frac{x_i-\frac{9-4 \sqrt{21}}{17}}{x_i-\frac{9+4 \sqrt{21}}{17}}$. Then by Eqs. (3.3) and (3.4), we get $y_{i-1}=(6049+1320 \sqrt{21}) y_i$ and $y_i=(6049+1320 \sqrt{21})^{n-i} y_n$ Therefore $x_i=\frac{(6049+1320 \sqrt{21})^{n-i}\left(\frac{9+4 \sqrt{21}}{17}\right) y_n-\frac{9-4 \sqrt{21}}{17}}{(6049+1320 \sqrt{21})^{n-i} y_n-1}$. Thus

$$x_1=\frac{(6049+1320 \sqrt{21})^{n-1}\left(\frac{9+4 \sqrt{21}}{17}\right) y_n-\frac{9-4 \sqrt{21}}{17}}{(6049+1320 \sqrt{21})^{n-1} y_n-1}. \tag{3.5}$$

Using the expression $x_{n-1}=\frac{82 x_n+45}{51 x_n+28}$ and denoting the coefficients of $82 x_n+45$ and $51 x_n+28$ as σ_n and δ_n we have

$$\begin{aligned}
51 x_n+28 &=\sigma_0\left(82 x_n+45\right)+\delta_0\left(51 x_n+28\right) \\
51 x_{n-1}+28 &=\frac{\sigma_1\left(82 x_n+45\right)+\delta_1\left(51 x_n+28\right)}{\sigma_0\left(82 x_n+45\right)+\delta_0\left(51 x_n+28\right)} \\
51 x_{n-2}+28 &=\frac{\sigma_2\left(82 x_n+45\right)+\delta_2\left(51 x_n+28\right)}{\sigma_1\left(82 x_n+45\right)+\delta_1\left(51 x_n+28\right)}, \\
&\vdots \\
51 x_{n-i}+28 &=\frac{\sigma_i\left(82 x_n+45\right)+\delta_i\left(51 x_n+28\right)}{\sigma_{i-1}\left(82 x_n+45\right)+\delta_{i-1}\left(51 x_n+28\right)},
\end{aligned} \tag{3.6}$$

$$51x_{n-(i+1)}+28=\frac{\sigma_{i+1}\left(82x_n+45\right)+\delta_{i+1}\left(51x_n+28\right)}{\sigma_i\left(82x_n+45\right)+\delta_i\left(51x_n+28\right)}, \tag{3.7}$$

$$\vdots$$

$$51x_2+28=\frac{\sigma_{n-2}\left(82x_n+45\right)+\delta_{n-2}\left(51x_n+28\right)}{\sigma_{n-3}\left(82x_n+45\right)+\delta_{n-3}\left(51x_n+28\right)}$$

Substituting Eq.(4.6) into Eq.(3.2), we obtain

$$\tau\left(\mathrm{E}_n\right)=3\times 4^{n-1}x_1^2\left[\sigma_{n-2}\left(82x_n+45\right)+\sigma_{n-2}\left(51x_n+28\right)\right]^2, \tag{3.8}$$

where $\sigma_0=0,\delta_0=1$ and $\sigma_1=51,\delta_1=28$. By the expression $x_{n-1}=\frac{82x_n+45}{51x_n+28}$ and Eqs. (3.6) and (3.7), we have

$$\sigma_{i+1}=110\sigma_i-\sigma_{i-1};\delta_{i+1}=110\delta_i-\delta_{i-1}. \tag{3.9}$$

The characteristic equation of Eq.(3.9) is $\gamma^2-110\gamma+1=0$ which have two roots $\gamma_1=55+12\sqrt{21}$ and $\gamma_2=55-12\sqrt{21}$. The general solutions of Eq. (3.9) are $\sigma_i=a_1\gamma_1^i+a_2\gamma_2^i;\delta_i=b_1\gamma_1^i+b_2\gamma_2^i$. Using the initial conditions $\sigma_0=0,\delta_0=1$ and $\sigma_1=51,\delta_1=28$, yields

$$\sigma_i=\frac{17\sqrt{21}}{168}(55+12\sqrt{21})^i-\frac{17\sqrt{21}}{168}(55-12\sqrt{21})^i$$

$$\delta_i=\frac{84-9\sqrt{21}}{168}(55+12\sqrt{21})^i+\frac{84+9\sqrt{21}}{168}(55-12\sqrt{21})^i. \tag{3.10}$$

If $x_n=1$, it means that Θ_n without any electrically equivalent transformation. Plugging Eq. (3.10) into Eq.(3.8), we have

$$\begin{aligned}\tau\left(\Theta_n\right) &= 3\times(4)^{n-1}x_1^2\left[\frac{1659+262\sqrt{21}}{42}(55+12\sqrt{15})^{n-2}\right.\\ &\quad\left.+\frac{1659-262\sqrt{21}}{42}(55-12\sqrt{15})^{n-2}\right]^2\end{aligned} \tag{3.11}$$

for integers $n\geq 2$. When $n=1,\tau\left(\Theta_1\right)=3$ which satisfies Eq.(3.11). Therefore, the number of spanning trees in the sequence of the graph Θ_n is given by

$$\begin{aligned}\tau\left(\Theta_n\right) &= 3\times(4)^{n-1}x_1^2\left[\frac{1659+262\sqrt{21}}{42}(55+12\sqrt{15})^{n-2}\right.\\ &\quad\left.+\frac{1659-262\sqrt{21}}{42}(55-12\sqrt{15})^{n-2}\right]^2\end{aligned} \tag{3.12}$$

for integers $n\geq 1$, where

$$x_1=\frac{(6049+1320\sqrt{21})^{n-1}(33+8\sqrt{21})+(9-4\sqrt{21})}{(6049+1320\sqrt{21})^{n-1}(25+4\sqrt{21})+17}, n\geq 1. \tag{3.13}$$

Inserting Eq.(3.13) into Eq.(3.12) we obtain the result. □

§4. Number of Spanning Trees in the Sequences of Π_n Graph

Consider the sequence of graphs $\Pi_1, \Pi_2, \cdots, \Pi_n$ constructed as shown in Figure 7. According to this construction, the number of total vertices $|V(\Pi_n)|$ and edges $|E(\Pi_n)|$ are $|V(\Pi_n)| = 9n-6$ and $|E(\Pi_n)| = 21n-21, n = 1, 2, \ldots$. The average degree of Π_n is in the large n limit which is $\frac{14}{3}$.

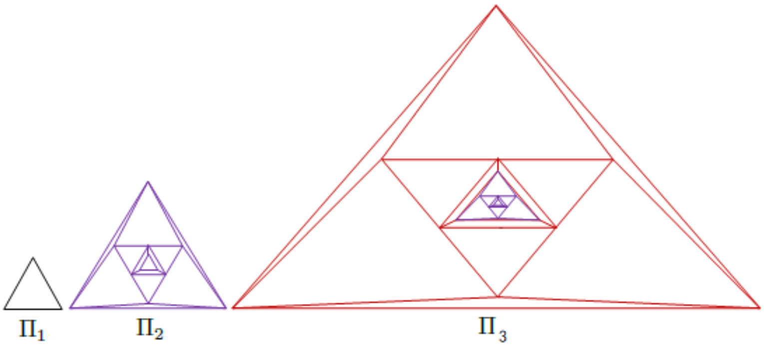


Figure 7. Some sequences of graph Π_n

Theorem 4.1 *For $n \geq 1$, the number of spanning trees in the sequence of Π_n graph is given by $\dfrac{N_2(\Pi_n)}{M_2(\Pi_n)}$, where*

$$\begin{aligned} N_2(\Pi_n) &= 3 \times 4^{n-2}\left(-17(-15+\sqrt{455}) + (655+33\sqrt{455})(8191+384\sqrt{455})^{n-1}\right)^2 \\ &\quad \times \left((18200-853\sqrt{455})(64+3\sqrt{455})^n + (64-3\sqrt{455})^n(18200+853\sqrt{455})\right)^2 \\ M_2(\Pi_n) &= 207025\left(391+(519+16\sqrt{455})(8191+384\sqrt{455})^{n-1}\right)^2. \end{aligned}$$

Proof We use the electrically equivalent transformation to transform Π_i to Π_{i-1}. Figures 8-12 illustrate the graphs $\Pi_1, \Pi_2, G_1 - G_{12}$ and the transformation process from Π_2 to Π_1.

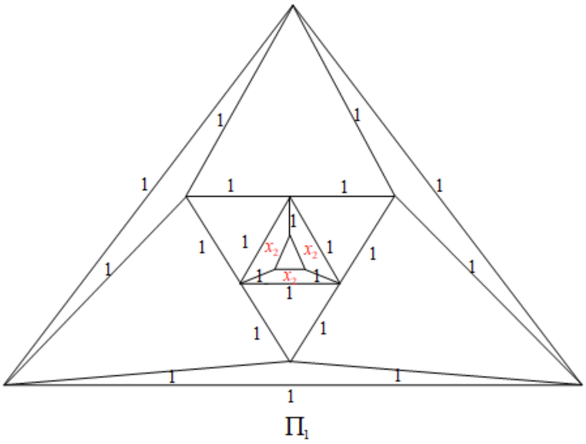
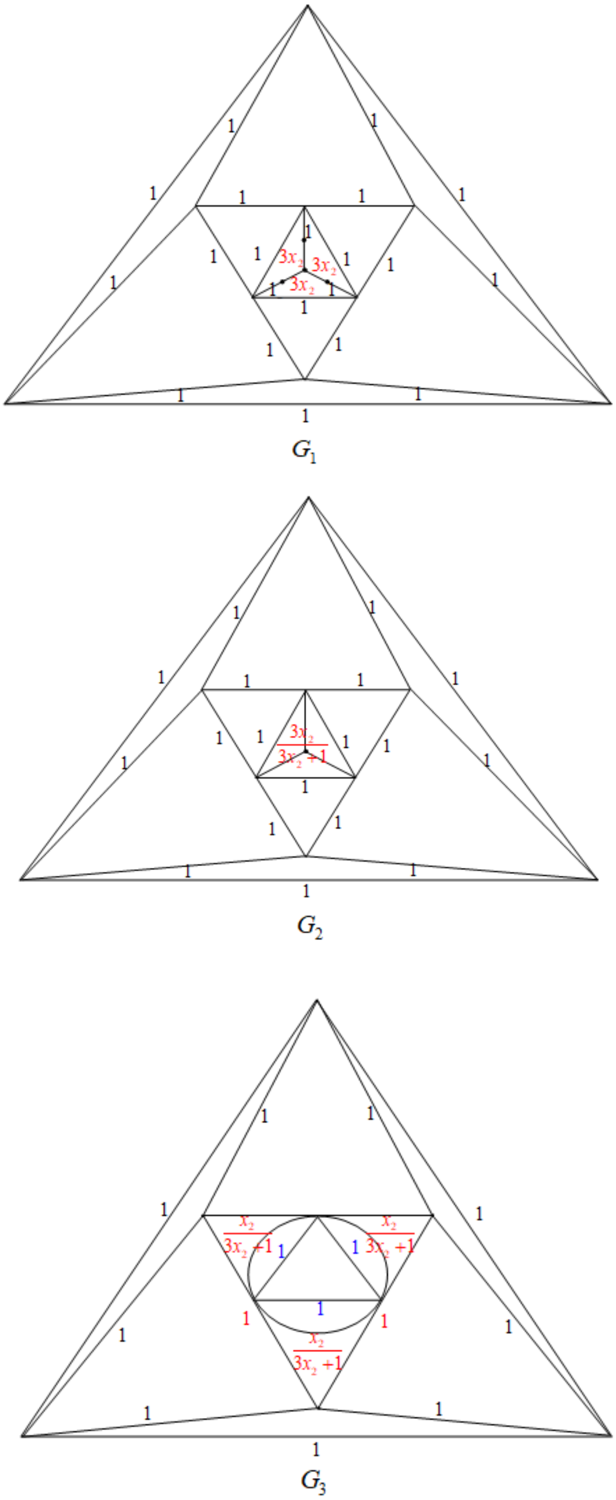


Figure 8.



G_1 G_2 G_3

Figure 9.

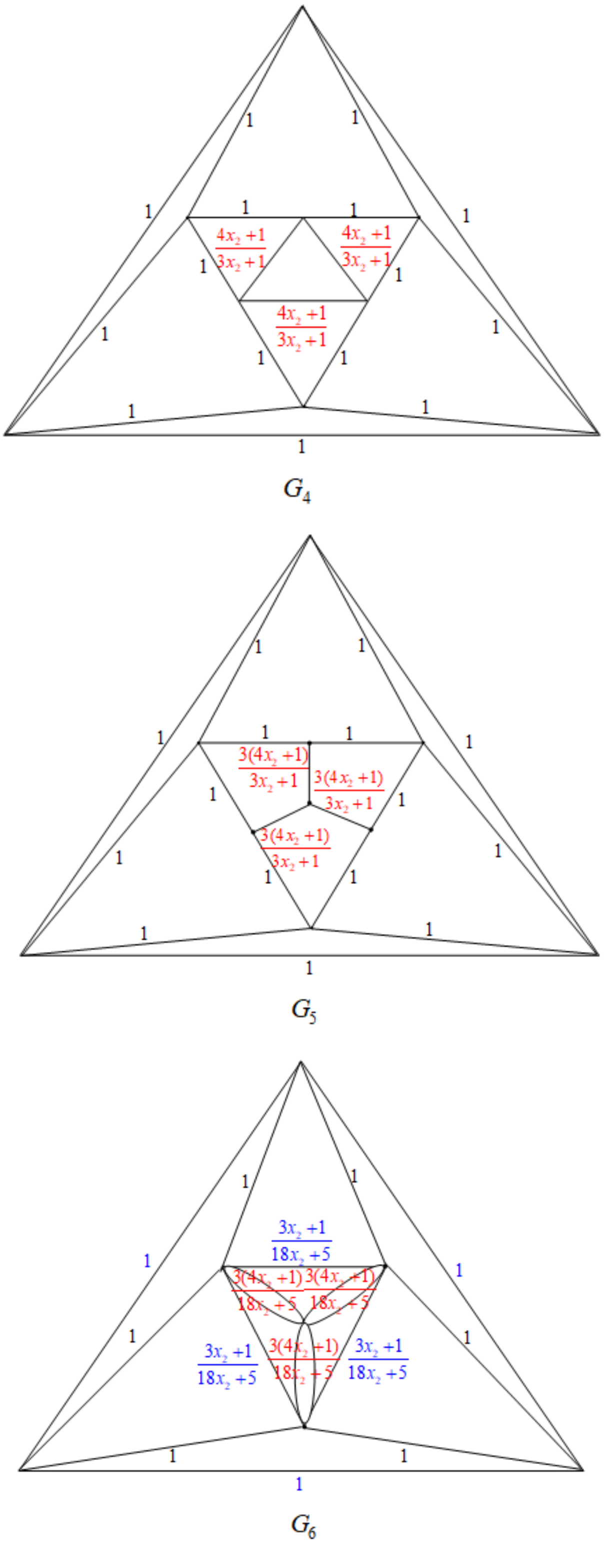
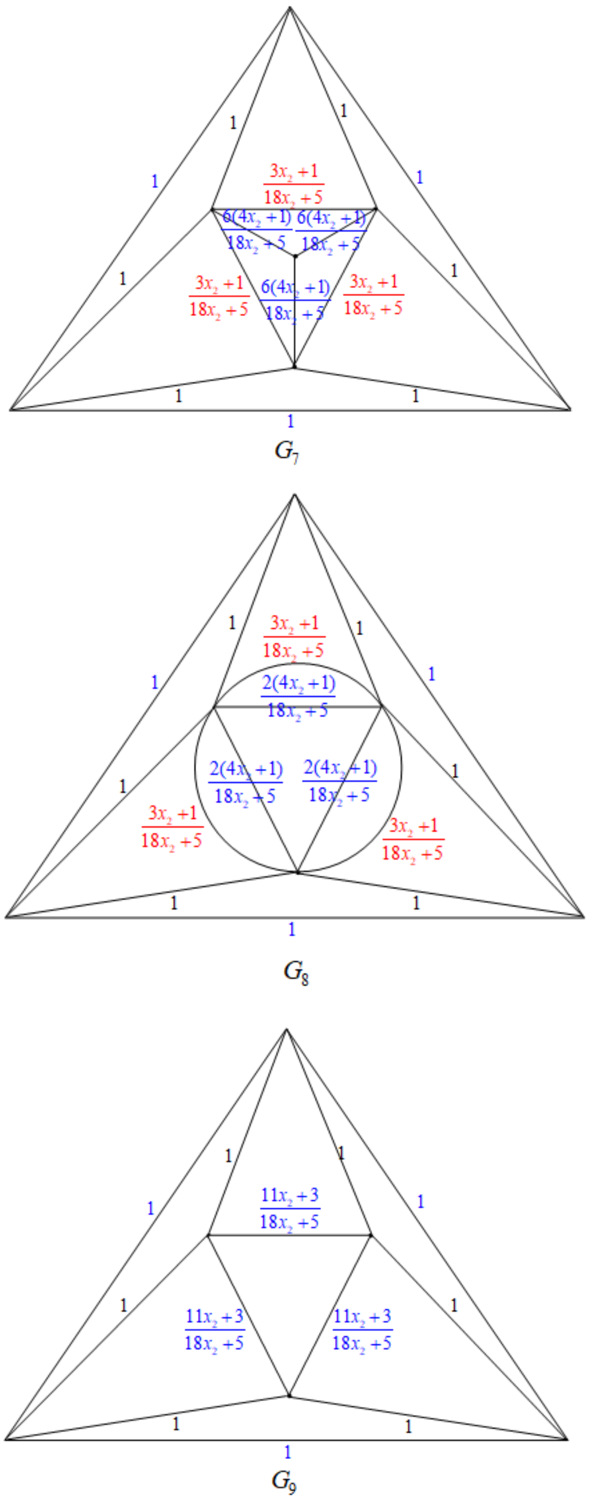

1
$\frac{4x_2+1}{3x_2+1}$
G_4
$\frac{3(4x_2+1)}{3x_2+1}$
G_5
$\frac{3x_2+1}{18x_2+5}$
$\frac{3(4x_2+1)}{18x_2+5}$
G_6

Figure 10.



G_7 G_8 G_9

Figure 11.

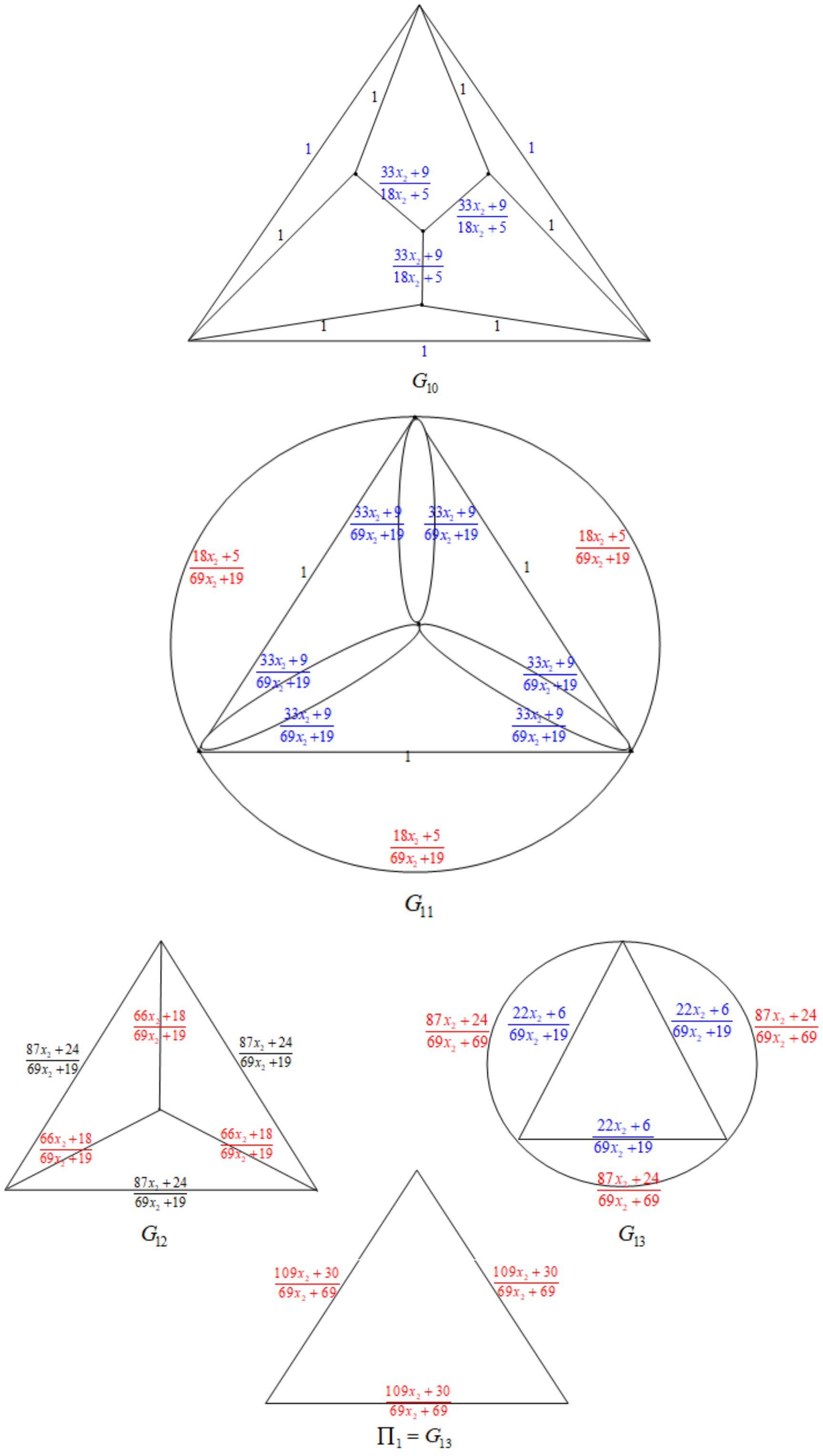


Figure 12. The transformations from Π_2 to Π_1

Using the properties given in Section 2, we have the following the transformations:

$$\begin{aligned}
&\tau\left(G_1\right)=9x_2\tau\left(\Pi_2\right), \tau\left(G_2\right)=\left[\frac{1}{3x_2+1}\right]^3\tau\left(G_1\right), \tau\left(G_3\right)=\frac{3x_2+1}{9x_2}\tau\left(G_2\right),\\
&\tau\left(G_4\right)=\tau\left(G_3\right), \tau\left(G_5\right)=9\left[\frac{4x_2+1}{3x_2+1}\right]\tau\left(G_4\right), \tau\left(G_6\right)=\left[\frac{3x_2+1}{18x_2+5}\right]^3\tau\left(G_5\right),\\
&\tau\left(G_7\right)=\tau\left(G_6\right), \tau\left(G_8\right)=\left[\frac{18x_2+5}{18\left(4x_2+1\right)}\right]\tau\left(G_7\right), \tau\left(G_9\right)=\tau\left(G_8\right),\\
&\tau\left(G_{10}\right)=9\left[\frac{11x_2+3}{18x_2+5}\right]\tau\left(G_9\right), \tau\left(G_{11}\right)=\left[\frac{18x_2+5}{69x_2+19}\right]^3\tau\left(G_{10}\right),\\
&\tau\left(G_{12}\right)=\tau\left(G_{11}\right), \tau\left(G_{13}\right)=\left[\frac{69x_2+19}{18\left(11x_2+3\right)}\right]\tau\left(G_{12}\right) \text{ and } \tau(\Pi)=\tau\left(G_{13}\right).
\end{aligned}$$

Combining these fourteen transformations, we have

$$\tau\left(\Pi_2\right)=4\left(69x_2+19\right)^2\tau\left(\Pi_1\right). \tag{4.1}$$

Further

$$\tau\left(\Pi_n\right)=\prod_{i=2}^{n}4\left(69x_2+19\right)^2\tau\left(\Pi_1\right)=3\times(4)^{n-1}x_1^2\left[\prod_{i=2}^{n}\left(69x_i+19\right)\right]^2, \tag{4.2}$$

where $x_{i-1}=\frac{109x_i+30}{69x_i+19}, i=2,3,\ldots,n$. Its characteristic equation is $23\mu^2-30\mu-10=0$, which have two roots roots $\mu_1=\frac{15-\sqrt{455}}{23}$ and $\mu_2=\frac{15+\sqrt{455}}{23}$. Subtracting these two roots into both sides of $x_{i-1}=\frac{109x_i+30}{69x_i+19}$, we get

$$x_{i-1}-\frac{15-\sqrt{455}}{23}=\frac{109x_i+30}{69x_i+19}-\frac{15-\sqrt{455}}{23}=(64+3\sqrt{455})\frac{x_i-\frac{15-\sqrt{455}}{23}}{69x_i+19}, \tag{4.3}$$

$$x_{i-1}-\frac{15+\sqrt{455}}{23}=\frac{109x_i+30}{69x_i+19}-\frac{15+\sqrt{455}}{23}=(64-3\sqrt{455})\frac{x_i-\frac{15+\sqrt{455}}{23}}{69x_i+19}. \tag{4.4}$$

Let $y_i=\frac{x_i-\frac{15-\sqrt{455}}{23}}{x_i-\frac{15+\sqrt{455}}{23}}$. Then by Eqs. (4.3) and (4.4), we get $y_{i-1}=(8191+384\sqrt{455})y_i$ and $y_i=(8191+ 384\sqrt{455})^{n-i}y_n$. Therefore $x_i=\frac{(8191+384\sqrt{455})^{n-i}\frac{15+\sqrt{455}}{23}y_n-\frac{15-\sqrt{455}}{23}}{(8191+384\sqrt{455})^{n-i}y_n-1}$. Thus

$$x_1=\frac{(8191+384\sqrt{455})^{n-1}\frac{15+\sqrt{455}}{23}y_n-\frac{15-\sqrt{455}}{23}}{(8191+384\sqrt{455})^{n-1}y_n-1}. \tag{4.5}$$

Using the expression $x_{n-1}=\frac{109x_n+30}{69x_n+19}$ and denoting the coefficients of $109x_n+30$ and $69x_n+19$ as σ_n and δ_n we have

$$\begin{aligned}
69x_n+19&=\sigma_0\left(109x_n+30\right)+\delta_0\left(69x_n+19\right)\\
69x_{n-1}+19&=\frac{\sigma_1\left(109x_n+30\right)+\delta_1\left(69x_n+19\right)}{\sigma_0\left(109x_n+30\right)+\delta_0\left(69x_n+19\right)}
\end{aligned}$$

$$69x_{n-2}+19=\frac{\sigma_2\left(109x_n+30\right)+\delta_2\left(69x_n+19\right)}{\sigma_1\left(109x_n+30\right)+\delta_1\left(69x_n+19\right)},$$

$$\vdots$$

$$69x_{n-2}+19=\frac{\sigma_2\left(109x_n+30\right)+\delta_2\left(69x_n+19\right)}{\sigma_1\left(109x_n+30\right)+\delta_1\left(69x_n+19\right)}, \tag{4.6}$$

$$69x_{n-(i+1)}+19=\frac{\sigma_{i+1}\left(109x_n+30\right)+\delta_{i+1}\left(69x_n+19\right)}{\sigma_i\left(109x_n+30\right)+\delta_i\left(69x_n+19\right)}, \tag{4.7}$$

$$\vdots$$

$$69x_2+19=\frac{\sigma_{n-2}\left(109x_n+30\right)+\delta_{n-2}\left(69x_n+19\right)}{\sigma_{n-3}\left(109x_n+30\right)+\delta_{n-3}\left(69x_n+19\right)}$$

Substituting Eq.(4.6) into Eq.(4.2), we obtain

$$\tau\left(\mathrm{E}_n\right)=3\times 4^{n-1}x_1^2\left[\sigma_{n-2}\left(109x_n+30\right)+\sigma_{n-2}\left(69x_n+19\right)\right]^2, \tag{4.8}$$

where $\sigma_0=0,\delta_0=1$ and $\sigma_1=69,\delta_1=19$. By the expression $x_{n-1}=\frac{109x_n+30}{69x_n+19}$ and Eqs. (4.6) and (4.7), we have

$$\sigma_{i+1}=128\sigma_i-\sigma_{i-1};\delta_{i+1}=128\delta_i-\delta_{i-1}. \tag{4.9}$$

The characteristic equation of Eq.(4.9) is $\gamma^2-128\gamma+1=0$ which have two roots $\gamma_1=64+3\sqrt{455}$ and $\gamma_2=64-3\sqrt{455}$. The general solutions of Eq. (4.9) are $\sigma_i=a_1\gamma_1^i+a_2\gamma_2^i;\delta_i=b_1\gamma_1^i+b_2\gamma_2^i$. Using the initial conditions $\sigma_0=0,\delta_0=1$ and $\sigma_1=69,\delta_1=19$, yields

$$\sigma_i=\frac{23\sqrt{455}}{910}(64+3\sqrt{455})^i-\frac{23\sqrt{455}}{910}(64+3\sqrt{455})^i$$

$$\delta_i=\frac{455-15\sqrt{455}}{910}(64+3\sqrt{455})^i+\frac{455+15\sqrt{455}}{910}(64-3\sqrt{455})^i. \tag{4.10}$$

If $x_n=1$, it means that Π_n without any electrically equivalent transformation. Plugging Eq. (4.10) into Eq.(4.8), we haven for all $n\geq 2$

$$\begin{aligned}\tau\left(\Pi_n\right) &= 3\times 4^{n-1}x_1^2\left[\frac{40040+1877\sqrt{455}}{910}(64+3\sqrt{455})^{n-2}\right.\\ &\left.+\frac{40040-1877\sqrt{455}}{910}(64-3\sqrt{455})^{n-2}\right]^2.\end{aligned} \tag{4.11}$$

When $n=1,\tau\left(\Pi_1\right)=3$ which satisfies Eq.(4.11). Therefore, the number of spanning trees in the sequence of the graph Π_n where $n\geq 1$, is given by

$$\begin{aligned}\tau\left(\Pi_n\right) &= 3\times 4^{n-1}x_1^2\left[\frac{40040+1877\sqrt{455}}{910}(64+3\sqrt{455})^{n-2}\right.\\ &\left.+\left(\frac{40040-1877\sqrt{455}}{910}\right)(64-3\sqrt{455})^{n-2}\right]^2,\end{aligned} \tag{4.12}$$

where

$$x_1 = \frac{(8191+384\sqrt{455})^{n-1}(655+33\sqrt{455})+17(15-\sqrt{455})}{(8191+384\sqrt{455})^{n-1}(519+16\sqrt{455})+391}, \ n \geq 1. \tag{4.13}$$

Inserting Eq.(4.13) into Eq.(4.12) we obtain the result. □

§5. Number of Spanning Trees in the Sequences of Σ_n Graph

Consider the sequence of graphs $\Sigma_1, \Sigma_2, \cdots, \Sigma_n$ constructed as shown in Figure 13. According to this construction, the number of total vertices $|V(\Sigma_n)|$ and edges $|E(\Sigma_n)|$ are $|\mathrm{V}(\Sigma_n)| = 9n-6$ and $|E(\Sigma_n)| = 21n-18, n = 1, 2, \cdots$. The average degree of Σ_n is $\dfrac{14}{3}$ in $n \to \infty$.

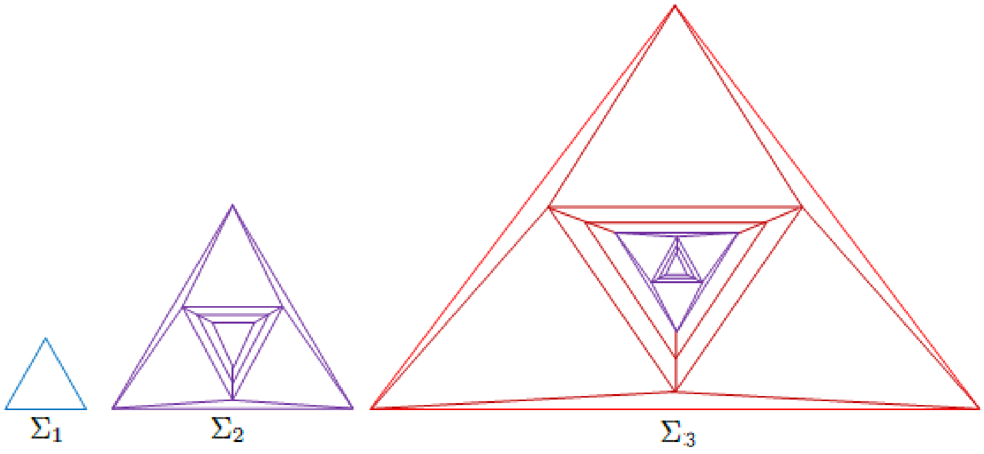


Figure 13. Some sequences of Σ_n

Theorem 5.1 *The number of spanning trees in sequence of* Σ_n *is given by* $\dfrac{N_3(\Sigma_n)}{M_3(\Sigma_n)}$ *for* $n \geq 1$, *where*

$$\begin{aligned} N_3(\Sigma_n) &= 3\times 2^{-3-n}(163+\sqrt{26565})^{2n} \\ &\times \left(-\frac{67}{2}(-117+\sqrt{26565})+\left(\frac{1}{2}(26567-163\sqrt{26565})\right)^{1-n}(9363+62\sqrt{26565})\right)^2 \\ &\times \left(469315-2879\sqrt{26565}+\left(\frac{1}{2}(26567-163\sqrt{26565})\right)^{n}(469315+2879\sqrt{26565})\right)^2, \\ M_3(\Sigma_n) &= 78411025\left(5829+3\times 2^{-n}(4969+19\sqrt{26565})(26567+163\sqrt{26565})^{n-1}\right)^2. \end{aligned}$$

Proof We use the electrically equivalent transformation to transform Σ_i to Σ_{i-1}. Figures 14-17 illustrate the graphs $\Sigma_1, \Sigma_2, G_1 - G_{12}$ and the transformation process from Σ_2 to Σ_1.

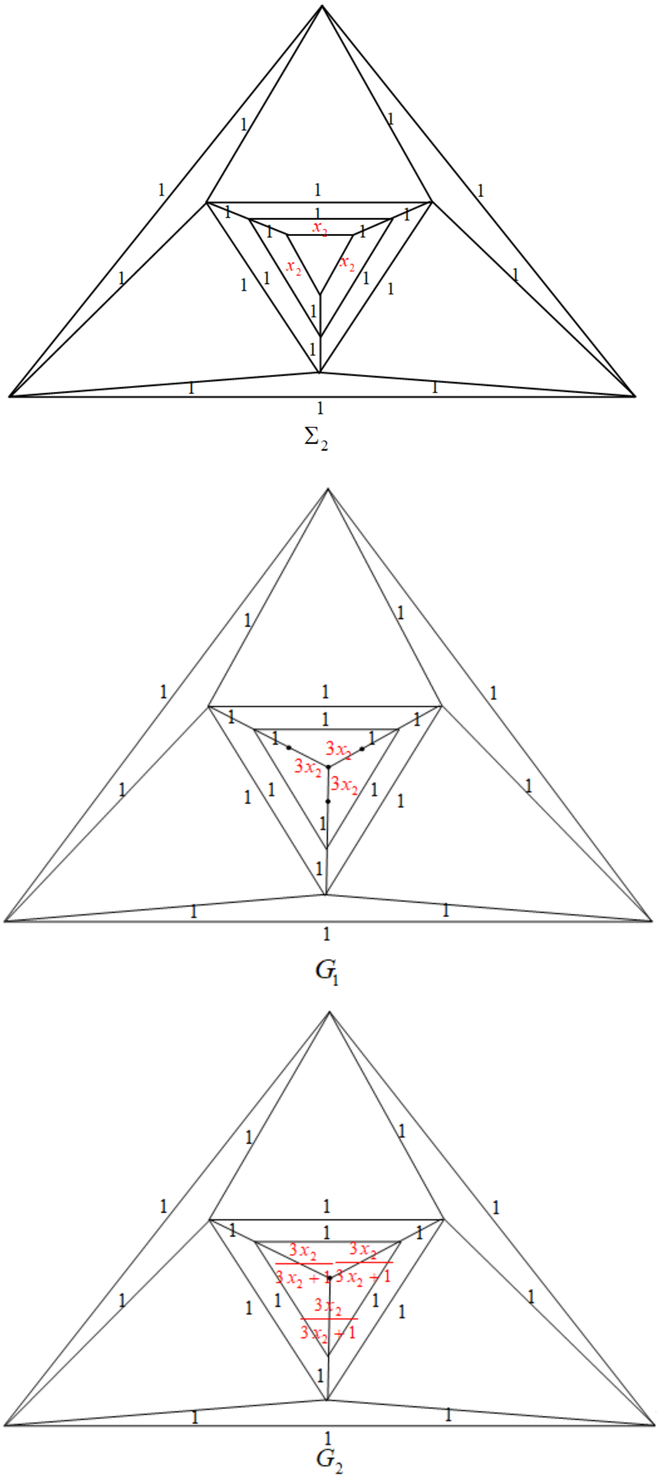

1
x_2
x_2
x_2
Σ_2
1
$3x_2$
$3x_2$
$3x_2$
G_1
1
$\frac{3x_2}{3x_2+1}$
$\frac{3x_2}{3x_2+1}$
$\frac{3x_2}{3x_2+1}$
G_2

Figure 14.

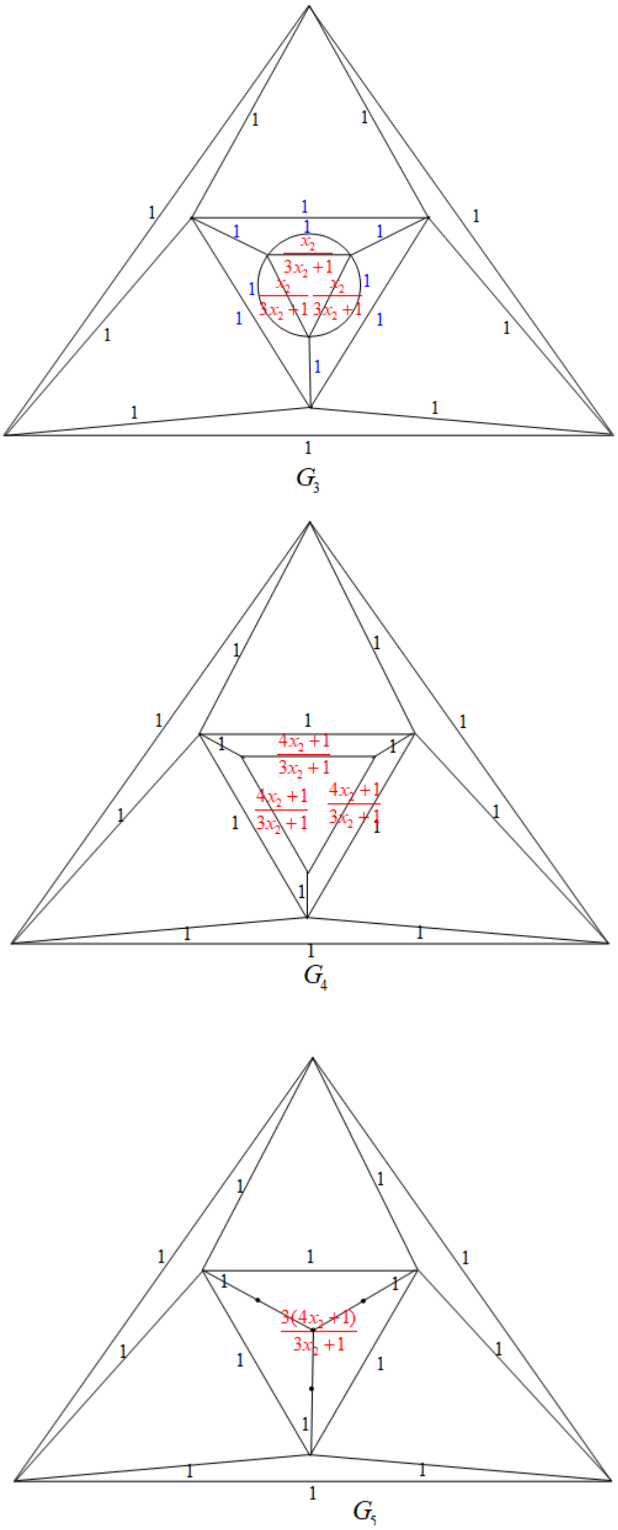

1
1
1
1
1
1
1
1
1
1
1
1
1
1
1
1
1
$\frac{x_2}{3x_2+1}$
$\frac{x_2}{3x_2+1}$
$\frac{x_2}{3x_2+1}$
G_3
$\frac{4x_2+1}{3x_2+1}$
$\frac{4x_2+1}{3x_2+1}$
$\frac{4x_2+1}{3x_2+1}$
G_4
$\frac{3(4x_2+1)}{3x_2+1}$
G_5

Figure 15.

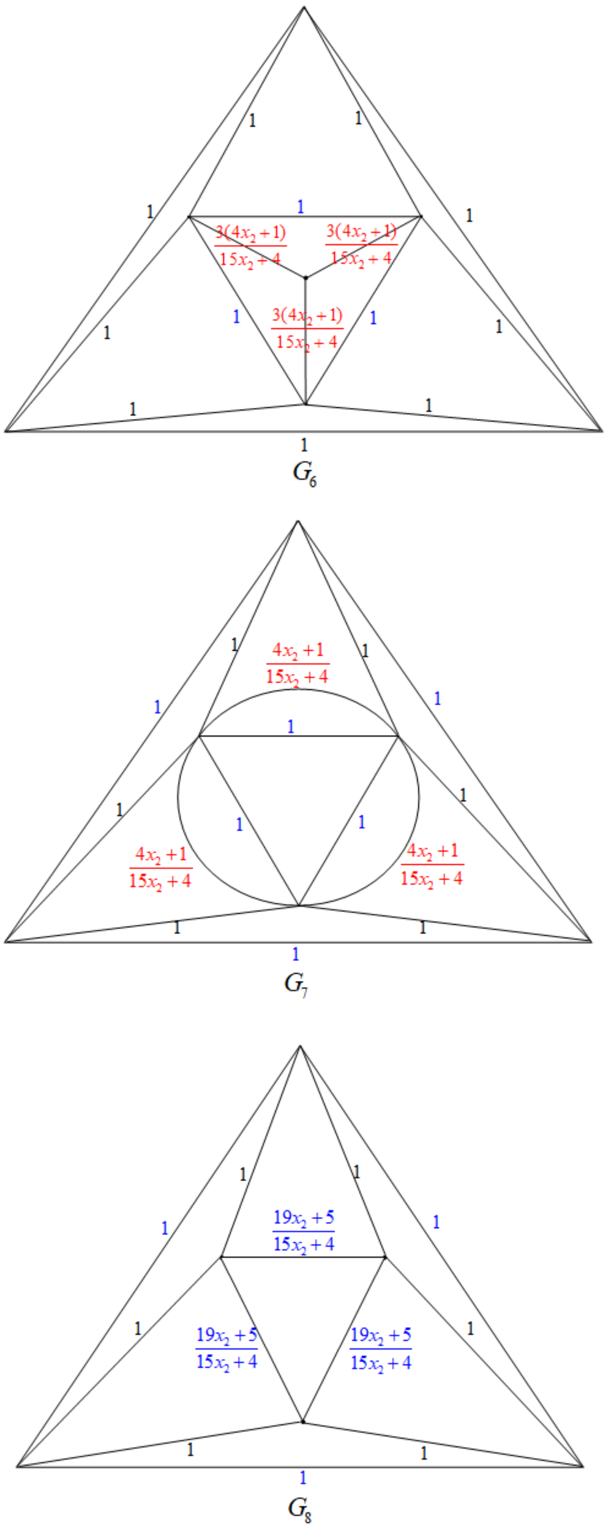

1
1
1
1
$\frac{3(4x_2+1)}{15x_2+4}$
$\frac{3(4x_2+1)}{15x_2+4}$
$\frac{3(4x_2+1)}{15x_2+4}$
1
1
1
1
1
1
1
G_6
1
1
$\frac{4x_2+1}{15x_2+4}$
1
1
1
1
1
1
$\frac{4x_2+1}{15x_2+4}$
$\frac{4x_2+1}{15x_2+4}$
1
1
1
G_7
1
1
$\frac{19x_2+5}{15x_2+4}$
1
1
1
1
$\frac{19x_2+5}{15x_2+4}$
$\frac{19x_2+5}{15x_2+4}$
1
1
1
G_8

Figure 16.

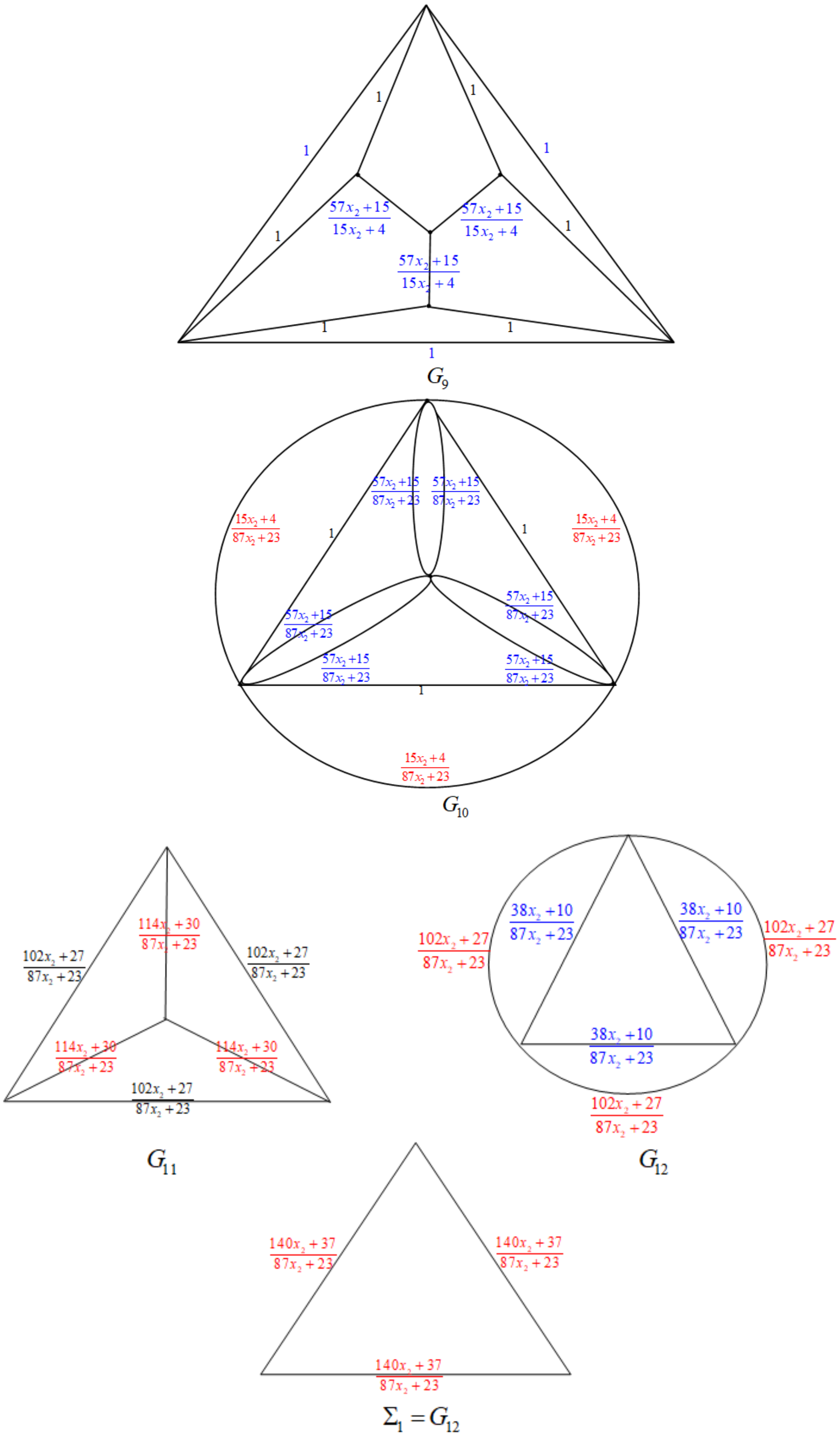


Figure 17. The transformations from Σ_2 to Σ_1

Using the properties given in section 2 , we have the following the transformations:

$$\tau\left(G_1\right)=9x_2\tau\left(\Sigma_2\right),\tau\left(G_2\right)=\left[\frac{1}{3x_2+1}\right]^3\tau\left(G_1\right),\tau\left(G_3\right)=\frac{3x_2+1}{9x_2}\tau\left(G_2\right),$$
$$\tau\left(G_4\right)=\tau\left(G_3\right),\tau\left(G_5\right)=9\left[\frac{4x_2+1}{3x_2+1}\right]\tau\left(G_4\right),\tau\left(G_6\right)=\left[\frac{3x_2+1}{15x_2+4}\right]^3\tau\left(G_5\right),$$
$$\tau\left(G_7\right)=\frac{15x_2+4}{9\left(4x_2+1\right)}\tau\left(G_6\right),\tau\left(G_8\right)=\tau\left(G_7\right),\tau\left(G_9\right)=9\left[\frac{19x_2+5}{15x_2+4}\right]\tau\left(G_8\right),$$
$$\tau\left(G_{10}\right)=\left[\frac{15x_2+4}{87x_2+23}\right]^3\tau\left(G_9\right),\tau\left(G_{11}\right)=\tau\left(G_{10}\right),\tau\left(G_{12}\right)=\frac{87x_2+23}{18\left(19x_2+5\right)}\tau\left(G_{11}\right)\text{ and}$$
$$\tau\left(\Sigma_1\right)=\tau\left(G_{12}\right).$$

Combining these thirteen transformations, we get

$$\tau\left(\Sigma_2\right)=2\left(87x_2+23\right)^2\tau\left(\Sigma_1\right). \tag{5.1}$$

Further

$$\tau\left(\Sigma_n\right)=\prod_{i=2}^{n}2\left(87x_2+23\right)^2\tau\left(\Sigma_1\right)=3\times(2)^{n-1}x_1^2\left[\prod_{i=2}^{n}\left(87x_i+23\right)\right]^2, \tag{5.2}$$

where $x_{i-1}=\frac{140x_i+37}{87x_i+23}, i=2,3,\ldots,n$. Its characteristic equation is $87\mu^2-117\mu-37=0$, which have two roots roots $\mu_1=\frac{117-\sqrt{26565}}{174}$ and $\mu_2=\frac{117+\sqrt{26565}}{174}$. Subtracting these two roots into both sides of $x_{i-1}=\frac{140x_i+37}{87x_i+23}$, we get

$$x_{i-1}-\frac{117-\sqrt{26565}}{174}=\frac{140x_i+37}{87x_i+23}-\frac{117-\sqrt{26565}}{174}=(163+\sqrt{26565})\frac{x_i-\frac{117-\sqrt{26565}}{174}}{2\left(87x_i+23\right)}, \tag{5.3}$$

$$x_{i-1}-\frac{117+\sqrt{26565}}{174}=\frac{140x_i+37}{87x_i+23}-\frac{117+\sqrt{26565}}{174}=(163-\sqrt{26565})\frac{x_i-\frac{117+\sqrt{26565}}{174}}{2\left(87x_i+23\right)}. \tag{5.4}$$

Let $y_i=\frac{x_i-\frac{117-\sqrt{26565}}{174}}{x_i-\frac{117+\sqrt{26565}}{174}}$. Then by Eqs. (5.3) and (5.4), we get $y_{i-1}=\frac{26567+163\sqrt{26565}}{2}y_i$ and $y_i=\left(\frac{26567+163\sqrt{26565}}{2}\right)^{n-i}y_n$. Therefore $x_i=\frac{\left(\frac{26567+163\sqrt{26565}}{2}\right)^{n-i}\frac{117+\sqrt{26565}}{174}y_n-\frac{117-\sqrt{26565}}{174}}{\left(\frac{26567+163\sqrt{26565}}{2}\right)^{n-i}y_n-1}$. Thus

$$x_1=\frac{\left(\frac{26567+163\sqrt{26565}}{2}\right)^{n-1}\frac{117+\sqrt{26565}}{174}y_n-\frac{117-\sqrt{26565}}{174}}{\left(\frac{26567+163\sqrt{26565}}{2}\right)^{n-1}y_n-1}. \tag{5.5}$$

Using the expression $x_{n-1}=\frac{140x_n+37}{87x_n+23}$ and denoting the coefficients of $140x_n+37$ and

$87x_n + 23$ as σ_n and δ_n we have

$$\begin{aligned}
87x_n + 23 &= \sigma_0 \left(140x_n + 37\right) + \delta_0 \left(87x_n + 23\right) \\
87x_{n-1} + 23 &= \frac{\sigma_1 \left(140x_n + 37\right) + \delta_1 \left(87x_n + 23\right)}{\sigma_0 \left(140x_n + 37\right) + \delta_0 \left(87x_n + 23\right)} \\
87x_{n-2} + 23 &= \frac{\sigma_2 \left(140x_n + 37\right) + \delta_2 \left(87x_n + 23\right)}{\sigma_1 \left(140x_n + 37\right) + \delta_1 \left(87x_n + 23\right)} \\
&\vdots
\end{aligned}$$

$$87x_{n-i} + 23 = \frac{\sigma_i \left(140x_n + 37\right) + \delta_i \left(87x_n + 23\right)}{\sigma_{i-1} \left(140x_n + 37\right) + \delta_{i-1} \left(87x_n + 23\right)}, \tag{5.6}$$

$$87x_{n-(i+1)} + 23 = \frac{\sigma_{i+1} \left(140x_n + 37\right) + \delta_{i+1} \left(87x_n + 23\right)}{\sigma_i \left(140x_n + 37\right) + \delta_i \left(87x_n + 23\right)}, \tag{5.7}$$

$$\vdots$$

$$87x_2 + 23 = \frac{\sigma_{n-2} \left(140x_n + 37\right) + \delta_{n-2} \left(87x_n + 23\right)}{\sigma_{n-3} \left(140x_n + 37\right) + \delta_{n-3} \left(87x_n + 23\right)}$$

Substituting Eq.(5.6) into Eq.(5.2), we obtain

$$\tau\left(\mathrm{E}_n\right) = 3 \times 2^{n-1} x_1^2 \left[\sigma_{n-2} \left(140x_n + 37\right) + \sigma_{n-2} \left(87x_n + 23\right)\right]^2, \tag{5.8}$$

where $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 87, \delta_1 = 23$. By the expression $x_{n-1} = \frac{140x_n+37}{87x_n+23}$ and Eqs. (5.6) and (5.7), we have

$$\sigma_{i+1} = 163\sigma_i - \sigma_{i-1}; \delta_{i+1} = 163\delta_i - \delta_{i-1}. \tag{5.9}$$

The characteristic equation of Eq.(5.9) is $\gamma^2 - 163\gamma + 1 = 0$ which have two roots $\gamma_1 = \frac{163-\sqrt{26565}}{2}$ and $\gamma_2 = \frac{163+\sqrt{26565}}{2}$. The general solutions of Eq. (5.9) are $\sigma_i = a_1\gamma_1^i + a_2\gamma_2^i; \delta_i = b_1\gamma_1^i + b_2\gamma_2^i$. Using the initial conditions $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 87, \delta_1 = 23$, yields

$$\sigma_i = \frac{29\sqrt{26565}}{8855}\left(\frac{163+\sqrt{26565}}{2}\right)^i - \frac{29\sqrt{26565}}{8855}\left(\frac{163-\sqrt{26565}}{2}\right)^i,$$

$$\delta_i = \frac{26565 - 117\sqrt{26565}}{53130}\left(\frac{163+\sqrt{26565}}{2}\right)^i + \frac{26565 + 117\sqrt{26565}}{53130}\left(\frac{163-\sqrt{26565}}{2}\right)^i. \tag{5.10}$$

If $x_n = 1$, it means that Σ_n without any electrically equivalent transformation. Plugging Eq. (5.10) into Eq.(5.8), we have for all $n \geq 2$

$$\begin{aligned}
\tau\left(\Sigma_n\right) \quad &= \quad 3 \times 2^{n-1} x_1^2 \left[\frac{487025 + 2988\sqrt{26565}}{8855}\left(\frac{163+\sqrt{26565}}{2}\right)^{n-2}\right. \\
&\left. + \frac{487025 - 2988\sqrt{26565}}{8855}\left(\frac{163-\sqrt{26565}}{2}\right)^{n-2}\right]^2.
\end{aligned} \tag{5.11}$$

When $n = 1, \tau(\Sigma_1) = 3$ which satisfies Eq.(5.11). Therefore, the number of spanning trees in the sequence of the graph Σ_n for all $n \geq 1$ is given by

$$\begin{aligned} \tau(\Sigma_n) &= 3 \times 2^{n-1} x_1^2 \left[\frac{487025 + 2988\sqrt{26565}}{8855} \left(\frac{163 + \sqrt{26565}}{2} \right)^{n-2} \right. \\ &\quad \left. + \frac{487025 - 2988\sqrt{26565}}{8855} \left(\frac{163 - \sqrt{26565}}{2} \right)^{n-2} \right]^2, \end{aligned} \tag{5.12}$$

where

$$x_1 = \frac{\left(\frac{26567+16\sqrt{26565}}{2}\right)^{n-1} (9363 + 62\sqrt{26565}) + \frac{67(117-\sqrt{26565})}{2}}{\frac{3}{2}\left(\frac{2x6677+163\sqrt{26555}}{2}\right)^{n-1} (4969 + 19\sqrt{26565}) + 5829}, n \geq 1. \tag{5.13}$$

This completes the proof. □

§6. Numerical Results

Table 1. some values of the number of spanning trees in the graphs Θ_n, Π_n and Σ_n.

n	$\tau(\Theta_n)$	$\tau(\Pi_n)$	$\tau(\Sigma_n)$
1	3	3	3
2	193548	231852	187974
3	9366382128	15192944688	9987870000
4	453257961670848	695563276393152	530695483947096
5	21934059131316880128	65237270132405699328	28197973804093756848
6	1061432982230559089691648	4274867821307268252675072	1498271137983935741049216

§7. Entropy of Spanning Trees

After having explicit Formulas for the number of spanning trees of the sequence of the three families of graphs Θ_n, Π_n and Σ_n, we can calculate its spanning tree entropy Z which is a finite number and a very interesting quantity characterizing the network structure, defined as in [18] as: *For a graph G,*

$$Z(G) = \lim_{n\to\infty} \frac{\ln \tau(G)}{|V(G)|}, \tag{7.1}$$

$$\begin{aligned} Z(\Theta_n) &= \frac{1}{9}(\ln[4] + 2\ln[55 + 12\sqrt{21}]) = 1.198565531, \\ Z(\Pi_n) &= \frac{1}{9}(\ln[4] + 2\ln[64 + 3\sqrt{455}]) = 1.23224809, \\ Z(\Sigma_n) &= \frac{1}{9}(\ln[8] - 2\ln[163 - \sqrt{26565}]) = 1.208952478. \end{aligned}$$

Now we compare the value of entropy in our graphs with other graphs. The entropy of the graph Π_n is larger than the entropy of the graph Θ_n and Σ_n. In addition the entropy of the families Θ_n, Π_n and Σ_n which have the same average degree 14/3 is larger than the entropy of fractal scale free lattice [6] which has the entropy1.040 and 3-prism graph of average degree 4 which has entropy1.0445 [19] and two dimensional Sierpinski gasket [20] which has the entropy 1.166 of the same average degree 4.

§8. Conclusions

In this work, we enumerate the number of spanning trees in the sequences of three sequences of graphs of average degree 14/3 using electrically equivalent transformations. An advantage of this method lies in the avoidance of laborious computation of Laplacian spectra that is needed for a generic method for determining spanning trees.

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