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Conformal Yamabe Soliton and Conformal Gradient Yamabe Soliton on Para-Kenmotsu Manifold

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Abstract: The goal of this article is to study conformal Yamabe soliton and conformal gradient Yamabe soliton on the para-Kenmotsu manifold. Firstly, we have proved some results of para-Kenmotsu manifold when its admit conformal Yamabe soliton. Later, we have worked on conformal gradient Yamabe soliton on the para-Kenmotsu manifold.

Key Words: Yamabe soliton, conformal Yamabe soliton, gradient Yamabe soliton, conformal gradient Yamabe soliton, para-Kenmotsu manifold.

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§1. Introduction

The concept of Yamabe flow was introduced by Hamilton [9] in order to produce Yamabe metrics on compact Riemannian manifolds. The evaluation of the metric g_0 in time t to $g = g(t)$ using the equation is known as Yamabe flow. The equation of this is

$$\frac{\partial}{\partial t}g(t) = -r(t)g(t), \qquad g(0) = 0, t \geq 0,$$

where r is the scalar curvature of the Riemannian metric g. In dimension 2, the Yamabe flow is similar to the Ricci flow. However, the Yamabe flow and the Ricci flow exhibit distinct behaviors at higher dimensions. The Yamabe soliton [1] is a specific solution of the Yamabe flow that moves via a homothetic family of one-parameter difeomorphisms, much like the Ricci soliton [9]. The equation of the Yamabe soliton is

$$\frac{1}{2}\mathcal{L}_X g = (r - \lambda)g,$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X. Many researches have studied on Yamabe soliton such as [5, 6, 8, 11, 17] and many others.

In 2021, Roy, Dey and Bhattacharyya [13] generalized the notation of Yamabe solition and

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they introduced conformal Yamabe soliton which is

$$(\mathcal{L}_X g)(U,V) = \Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big] g(U,V), \tag{0.1}$$

where $\mathcal{L}_X$ denotes the Lie derivative along X, r is the scalar curvature, λ is a constant and p is the time dependent scalar field. $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ confirmed that conformal Yamabe soliton is expanding, steady and shrinking respectively.

When a smooth function f's gradient is represented by X, it can be substituted by Df to create the conformal gradient Yamabe soliton, for which the equation (1.1) takes on the following form

$$\nabla^2 f = \Big\{r - \lambda + (p + \frac{2}{n})\Big\}, \tag{1.1}$$

where, $\nabla^2 f$ is the Hessian of f and this defined as $Hess_f(U,V) = g(\nabla_U Df, V)$, D denotes the gradient [1] operator.

This paper is constructed as follows:

After a brief introduction, we have covered some necessary results of para-Kenmotsu manifold in section two. In section 3, we have worked on conformal Yamabe soliton on para-Kenmotsu manifold. Here we have proved that the scalar r curvature is dependent on p, the soliton vector field X and the Reeb vector field ξ are Killing, X is constant multiple of ξ, the soliton is shrinking, steady and expanding if $p > \frac{34}{3}$, $p = \frac{34}{3}$ and $p < \frac{34}{3}$ respectively and some other results are also proved. In section 4, we have worked on conformal gradient Yamabe soliton.

§2. Preliminaries

An $n-$ dimensional smooth manifold M^n is said to be an almost para-contact manifold ([3], [10], [12]) if it admits an $(1,1)$ tensor field ϕ, a unit vector field ξ, the smooth 1-form η and the pseudo-Riemannian metric g such that

$$\phi^2 U = U - \eta(U)\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \tag{2.1}$$

$$g(U,\xi) = \eta(U), \tag{2.2}$$

$$g(\xi,\xi) = 1, \tag{2.3}$$

$$g(\phi U, \phi V) = -g(U,V) + \eta(U)\eta(V), \tag{2.4}$$

for $\forall U, V \in \chi(M)$, where $\chi(M)$ denotes Lie algebra of smooth vector fields on M.

$$d\eta(U,V) = g(U, \phi V), \tag{2.5}$$

for every $U, V \in \chi(M)$.

An almost para-contact metric manifold is said to be paraKenmotsu manifold if it satisfies

$$(\nabla_U \phi)V = g(\phi U, V) - \eta(V)\phi U, \tag{2.6}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian metric g.

Moreover, in a para-Kenmotsu manifold, we have the following relations [7]

$$\nabla_U \xi = U - \eta(U)\xi, \tag{2.7}$$

$$(\nabla_U \eta)V = g(U, V) - \eta(U)\eta(V), \tag{2.8}$$

$$R(U, V)\xi = \eta(U)V - \eta(V)U, \tag{2.9}$$

$$R(\xi, U)V = \eta(V)U - g(U, V)\xi, \tag{2.10}$$

$$R(\xi, U)\xi = U - \eta(U)\xi,$$

$$S(U, \xi) = -(n-1)\eta(U), \tag{2.11}$$

where Q and R denotes the Ricci operator and the Riemann curvature tensor respectively and $g(QU, V) = S(U, V)$.

It's known that the Ricci tensor of a $3-$dimensional para-Kenmotsu manifold is

$$S(U, V) = \frac{1}{2}\Big[(r+2)g(U, V) - (r-6)\eta(U)\eta(V)\Big]. \tag{2.12}$$

Several authors have studied on para-Kenmotsu manifold such as [2, 14, 15, 16] and many others.

§3. Conformal Yamabe Soliton

Theorem 3.1 *If a para-Kenmotsu manifold M^n admits conformal Yamabe soliton (g, ξ, λ, p), then the scalar curvature is dependent on p and the Reeb vector field ξ is Killing.*

Proof If ξ is the Reeb vector field then

$$(\mathcal{L}_\xi g)(U, V) = g(\nabla_U \xi, V) + g(U, \nabla_V \xi).$$

Using (2.7) in the above equation and then applying (2.2), we get

$$(\mathcal{L}_\xi g)(U, V) = 2[g(U, V) - \eta(U)\eta(V)]. \tag{3.1}$$

Again from equation (1.1), we have

$$(\mathcal{L}_\xi g)(U, V) = \Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big]g(U, V). \tag{3.2}$$

Equating (3.1) and (3.2), we get

$$\Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big] g(U,V) = 2[g(U,V) - \eta(U)\eta(V)]. \tag{3.3}$$

Substituting ξ in the place of V in the previous equation, we get

$$\Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big] \eta(U) = 0. \tag{3.4}$$

Since $\eta(U) \neq 0$, it gives

$$r = \lambda - \Big(\frac{p}{2} + \frac{1}{n}\Big), \tag{3.5}$$

where λ is a constant so, the scalar curvature r is dependent on p.

Using (3.5) in (3.2), we get $(\mathcal{L}_\xi g) = 0$. Hence, the Reeb vector field ξ is Killing. □

Theorem 3.2 *Let a* $3-$*dimensional para-Kenmotsu manifold* M^3 *admits conformal Yamabe soliton* (g, ξ, λ, p)*,* ξ *being the Reeb vector field and if the manifold is Ricci symmetric, then*

$$6\lambda - 3p = -34.$$

Proof Using (3.5) in (2.12) for $3-$dimensional, we obtain

$$\begin{aligned} S(U,V) &= \frac{1}{2}\Big[\Big\{\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 2\Big\} g(U,V) \\ &\quad - \Big\{\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 6\Big\}\eta(U)\eta(V)\Big]. \end{aligned} \tag{3.6}$$

Taking covariant derivative of the above equation along Z, we get

$$\begin{aligned} (\nabla_Z S)(U,V) &= -\frac{1}{2}\Big[\Big\{\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 6\Big\} \\ &\quad \times \Big\{\eta(U)(\nabla_Z \eta)V + \eta(V)(\nabla_Z \eta)U\Big]. \end{aligned} \tag{3.7}$$

The manifold is Ricci symmetric i,e, $(\nabla_Z S)(U,V) = 0$, then from (3.7), we get

$$\Big\{\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 6\Big\}\Big\{\eta(U)(\nabla_Z \eta)V + \eta(V)(\nabla_Z \eta)U\Big\} = 0. \tag{3.8}$$

Applying (2.8), in the foregoing equation (3.8), we obtain

$$\Big\{\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 6\Big\}\Big\{g(\phi U, \phi V\Big\} = 0. \tag{3.9}$$

Since $g(\phi U, \phi V) \neq 0$, it yields

$$\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 6 = 0,$$

Hence, from the above

$$6\lambda - 3p = -34. \tag{3.10}$$

This completes the proof. □

Corollary 3.3 *If a* $3-$*dimensional para-Kenmotsu manifold* M^3 *admits conformal Yamabe soliton* (g, ξ, λ, p) *and if the manifold is Ricci symmetric, then the soliton is shrinking if* $p > \frac{34}{3}$*, steady if* $p = \frac{34}{3}$ *and expanding if* $p < \frac{34}{3}$*.*

Proof From equation (3.10), we get

$$6\lambda = 3p - 34.$$

The definition of shrinking, steady and expanding is that $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

So, from the above soliton is shrinking, steady and expanding if $p > \frac{34}{3}$, $p = \frac{34}{3}$ and $p < \frac{34}{3}$ respectively. □

Theorem 3.4 *Let a* $n-$*dimensional para-Kenmotsu manifold admits conformal Yamabe soliton* (g, X, λ, p)*, such that the soliton vector field* X *is pointwise collinear with* ξ*, then* X *is a constant multiple of* ξ *and* X *is a Killing vector field.*

Proof Let $X = c\xi$, where c is a function and ξ is the Reeb vector field then

$$(\mathcal{L}_{c\xi} g)(U, V) = g(\nabla_U c\xi, V) + g(U, \nabla_V c\xi).$$

Using (2.7) in the above equation and then applying (2.2), we get

$$(\mathcal{L}_{c\xi} g)(U, V) = (Uc)\eta(V) + (Vc)\eta(U) + 2c\{g(U, V) - \eta(U)\eta(V)\}. \tag{3.11}$$

Again from equation (1.1), we have

$$(\mathcal{L}_{c\xi} g)(U, V) = \Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big] g(U, V). \tag{3.12}$$

Equating (3.11) and (3.12), we get

$$\begin{aligned} \Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big] g(U, V) &= (Uc)\eta(V) + (Vc)\eta(U) \\ &\quad + 2c\{g(U, V) - \eta(U)\eta(V)\}. \end{aligned} \tag{3.13}$$

Putting $V = \xi$, in the previous equation, we obtain

$$(Uc) = \Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big) - \xi c\Big] \eta(U). \tag{3.14}$$

Again, Substituting $U = \xi$ in above equation, we get

$$(\xi c) = \Big[r - \lambda + \Big(\frac{p}{2} + \frac{1}{n}\Big)\Big]. \tag{3.15}$$

Using (3.15) in (3.14) becomes

$$(Uc) = \Big[r - \lambda + \Big(\frac{p}{2} + \frac{1}{n}\Big)\Big]\eta(U). \tag{3.16}$$

Now, taking exterior differentiation of (3.16), we get

$$\Big[r - \lambda + \Big(\frac{p}{2} + \frac{1}{n}\Big)\Big]d\eta = 0. \tag{3.17}$$

Since $d\eta \neq 0$, the above equation becomes

$$\Big[r - \lambda + \Big(\frac{p}{2} + \frac{1}{n}\Big)\Big] = 0. \tag{3.18}$$

Using (3.18) in (3.16) gets

$$Uc = 0,$$

which implies that c is constant.

If we are using (3.18) in (1.1) yields

$$(\mathcal{L}_X g)(U, V) = 0.$$

Hence, X is a Killing vector field. □

§4. **Conformal Gradiant Yamabe Soliton**

Theorem 4.1 *If a $n-$dimensional para-Kenmotsu manifold admits conformal gradient Yamabe soliton with potential function f, then if the scalar curvature is constant then the potential function f is also constant and conversely.*

Proof From equation (1.2), we gets

$$\nabla_U Df = \Big[r - \lambda + \Big(p + \frac{2}{n}\Big)\Big]U. \tag{4.1}$$

Taking covariant differentiation (4.1) along the vector field V, we get

$$\nabla_V \nabla_U Df = (Vr)U + \Big\{r - \lambda + \Big(p + \frac{2}{n}\Big)\Big\}\nabla_V U. \tag{4.2}$$

Interchanging U and V in the above equation, we get

$$\nabla_U \nabla_V Df = (Ur)V + \Big\{r - \lambda + \Big(p + \frac{2}{n}\Big)\Big\}\nabla_U V. \tag{4.3}$$

Again, from (4.1) we have

$$\nabla_{[U,V]}Df = \Big[r - \lambda + \Big(p + \frac{2}{n}\Big)\Big](\nabla_U V - \nabla_V U). \tag{4.4}$$

As is widely known that

$$R(U,V)Df = \nabla_U \nabla_V Df - \nabla_V \nabla_U Df - \nabla_{[U,V]}Df,$$

Using (4.2),(4.3) and (4.4) in the previous equation, we get

$$R(U,V)Df = (Ur)V - (Vr)U. \tag{4.5}$$

Contracting (4.5) over U, we get

$$S(V,Df) = -(n-1)g(V,Dr). \tag{4.6}$$

Substituting, $V = \xi$ and using (2.11) in (4.6), we get $\xi f = \xi r$.
Putting $U = \xi$ in (4.5), we obtain

$$R(\xi,V)Df = (\xi r)V - (Vr)\xi. \tag{4.7}$$

Taking inner product with U, yields

$$g(R(\xi,V)Df,U) = (\xi r)g(U,V) - (Vr)\eta(U). \tag{4.8}$$

From (2.10)

$$g(R(\xi,V)Df,U) = [\eta(U)(Vf) - g(U,V)(\xi f)]. \tag{4.9}$$

As we know,

$$g(R(\xi,V)Df,U) = -g(R(\xi,V)U,Df).$$

So, from equation (4.8) and (4.9) we get,

$$(\xi r)g(V,U) - (Vr)\eta(U) = -[\eta(U)(Vf) - g(U,V)(\xi f)], \tag{4.10}$$

which gives the following after antisymmetrizing

$$(Ur)\eta(V) - (Vr)\eta(U) = (Uf)\eta(V) - (Vf)\eta(U). \tag{4.11}$$

Replacing V by ξ in the previous equation (4.11) and using $\xi f = \xi r$ implies that $Df = Dr$. So, if the scalar curvature is constant, then the potential function is also constant and conversely. This completes the proof. □

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