

## Connected Monophonic Eccentric Domination Number of Corona Product of Some Standard Graphs

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**Abstract:** For any two vertices  $u$  and  $v$  in a connected graph  $G$ , the monophonic distance  $d_m(u, v)$  from  $u$  to  $v$  is defined as the length of a longest  $u - v$  monophonic path in  $G$ . The monophonic eccentricity  $e_m(v)$  of a vertex  $v$  in  $G$  is the maximum monophonic distance from  $v$  to a vertex of  $G$ . A set  $S \subseteq V$  is a *connected monophonic eccentric dominating set* if  $S$  is a monophonic eccentric dominating set and the induced subgraph  $\langle S \rangle$  is connected. The *connected monophonic eccentric domination number*  $\gamma_{cme}(G)$  is the cardinality of a minimum connected monophonic eccentric dominating set of  $G$ . In this paper, we determine the connected monophonic eccentric domination number of corona product of some standard graphs.

**Key Words:** Monophonic eccentric vertex, monophonic eccentric dominating set, Smarandachely dominating on subgraph  $H^+$  or  $H^-$ , monophonic eccentric domination number, connected monophonic eccentric dominating set, connected monophonic eccentric domination number.

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### §1. Introduction

By a graph  $G = (V, E)$  we mean a non-trivial finite undirected graph without loops and multiple edges. The *order* and *size* of  $G$  are denoted by  $p$  and  $q$ , respectively. For basic graph theoretic terminology and results we refer to [1, 4]. For any two vertices  $u$  and  $v$  in a connected graph  $G$ , the *distance*  $d(u, v)$  is the length of a shortest  $u - v$  path in  $G$ . For each vertex  $v$  in  $G$ , define  $d^-(v) = \min \{d(u, v) : u \in V - \{v\}\}$ . A vertex  $u$  ( $\neq v$ ) is called a *neighbor* of  $v$  if  $d(u, v) = d^-(v)$ . A vertex  $v$  is said to *dominate* a vertex  $u$  if  $u = v$  or  $u$  is a neighbor of  $v$ . Since  $d^-(v) = 1$  for all  $v \in V$ , this is equivalent to the standard definition of neighbor. A set  $S$  of vertices of  $G$  is called a *dominating set* if every vertex of  $G$  is dominated by some vertex

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in  $S$ . Equivalently, a set  $S \subseteq V$  is a dominating set of  $G$  if every vertex in  $V - S$  is adjacent to some vertex in  $S$ . A dominating set of  $G$  with minimum cardinality is a *minimum dominating set* and this cardinality is the *domination number*  $\gamma(G)$ . The topic of domination began with Berge in [1] and Ore in [5]. In 1998, a text book devoted to domination written by Teresa et. al. [9].

For any two vertices  $u$  and  $v$  in a connected graph  $G$ , the *detour distance*  $D(u, v)$  is the length of a longest  $u - v$  path in  $G$ . For each vertex  $v$  in  $G$ , define  $D^-(v) = \min \{D(u, v) : u \in V - \{v\}\}$ . A vertex  $u$  ( $\neq v$ ) is called a *detour neighbor* of  $v$  if  $D(u, v) = D^-(v)$ . A vertex  $v$  is said to *detour dominate* a vertex  $u$  if  $u = v$  or  $u$  is a detour neighbor of  $v$ . A set  $S$  of vertices of  $G$  is called a *detour dominating set* if every vertex of  $G$  is detour dominated by some vertex in  $S$ . A detour dominating set of  $G$  with minimum cardinality is a *minimum detour dominating set* and this cardinality is the *detour domination number*  $\gamma_D(G)$ . These concepts were introduced and studied in [3].

A *chord* of a path  $P$  is an edge joining two non-adjacent vertices of  $P$ . A path  $P$  is called a *monophonic path* if it is a chordless path. For any two vertices  $u$  and  $v$  in a connected graph  $G$ , the *monophonic distance*  $d_m(u, v)$  from  $u$  to  $v$  is defined as the length of a longest  $u - v$  monophonic path in  $G$ . The *monophonic eccentricity*  $e_m(v)$  of a vertex  $v$  in  $G$  is  $e_m(v) = \max \{d_m(u, v) : u \in V\}$ . The *monophonic radius*,  $rad_m(G)$  of  $G$  is  $rad_m(G) = \min \{e_m(v) : v \in V\}$  and the *monophonic diameter*,  $diam_m(G)$  of  $G$  is  $diam_m(G) = \max \{e_m(v) : v \in V\}$ . A vertex  $v$  in  $G$  is a *monophonic eccentric vertex* of  $u$  in  $G$  if  $e_m(u) = d_m(u, v)$ . A vertex  $v$  in  $G$  is a *monophonic central vertex* if  $e_m(v) = rad_m(G)$  and the subgraph induced by the monophonic central vertices of  $G$  is the *monophonic center* of  $G$ . The monophonic distance was introduced in [6] and further studied in [7].

Let  $v$  be any vertex of a connected graph  $G$ . The set of all monophonic eccentric vertices of  $v$  is called the *monophonic eccentric neighborhood* of  $v$  and it is denoted by  $N_{e_m}(v)$ . The *monophonic eccentric degree* of a vertex  $v$  is defined as  $deg_{e_m}(v) = |N_{e_m}(v)|$ . The *minimum monophonic eccentric degree*  $\delta_{e_m}(G)$  is defined as  $\delta_{e_m}(G) = \min \{deg_{e_m}(v) : v \in V\}$  and the *maximum monophonic eccentric degree*  $\Delta_{e_m}(G)$  is defined as  $\Delta_{e_m}(G) = \max \{deg_{e_m}(v) : v \in V\}$ . A set  $S \subseteq V$  is a *monophonic eccentric dominating set* if every vertex in  $V - S$  has a monophonic eccentric vertex in  $S$ . The *monophonic eccentric domination number*  $\gamma_{me}(G)$  is the cardinality of a minimum monophonic eccentric dominating set of  $G$ . These concepts were introduced and studied in [10, 11]. A set  $S \subseteq V$  is a *connected monophonic eccentric dominating set* if  $S$  is a monophonic eccentric dominating set and the induced subgraph  $\langle S \rangle$  is connected. The *connected monophonic eccentric domination number*  $\gamma_{cme}(G)$  is the cardinality of a minimum connected monophonic eccentric dominating set of  $G$  [12].

Generally, a dominating, detour dominating or monophonic eccentric dominating set  $S$  of graph  $G$  is *Smarandachely dominated on subgraph*  $H \prec G$  if  $G - H$  is not dominating, detour dominating, or monophonic eccentric dominating by set  $S$ , called a Smarandachely dominated on subgraph  $H^-$ , or  $S$  is not a dominating, detour dominating or monophonic eccentric dominating set of graph  $G$  but it is a dominating, detour dominating or monophonic eccentric dominating set  $S$  of graph  $G + H$ , called a Smarandachely dominated on subgraph  $H^-$ . Particularly, if  $H = \emptyset$ , a Smarandachely, detour dominating or monophonic eccentric dominating set

$S$  on subgraph  $H$  is nothing else but dominating, detour dominating or monophonic eccentric dominating by  $S$ . For example, let  $H = P_2$ . If there is an edge  $e \in E(G)$  such that  $G - e$  is not dominating, detour dominating, or monophonic eccentric dominating by set  $S$ , then  $G$  is Smarandachely dominated on a subgraph  $P_2^-$ .

Consider the graph  $G$  given in Figure 1.1. It is easily seen that no 2-element subset of  $G$  is a connected monophonic eccentric dominating set. The sets  $\{v_1, v_2, v_6\}$  and  $\{v_3, v_4, v_5\}$  are the only minimum connected monophonic eccentric dominating set of  $G$  so that  $\gamma_{cme}(G) = 3$ .

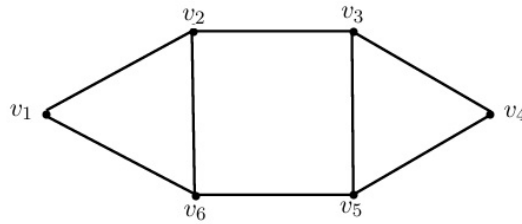


Figure 1.1

The following theorems will be used in the sequel.

**Theorem 1.1**([10]) *If  $G = H + K_p$  or  $K_p + H$ , where  $H$  is any connected graph, then*

$$\gamma_{cme}(G) = \gamma_{cme}(H).$$

**Theorem 1.2**([10]) *Let  $G$  be a cycle of order  $p$  and let  $p \equiv l \pmod{6}$ . Then,*

$$\gamma_{me}(G) = \begin{cases} \left\lceil \frac{p}{3} \right\rceil + 1 & \text{if } l = 2, \\ \left\lceil \frac{p}{3} \right\rceil & \text{otherwise.} \end{cases}$$

**Theorem 1.3**([10]) *Let  $G$  be a wheel of order  $p$  and let  $p \equiv l \pmod{6}$ . Then,*

$$\gamma_{me}(G) = \begin{cases} \frac{p}{3} + 1 & \text{if } l = 3, \\ \left\lceil \frac{p-1}{3} \right\rceil & \text{otherwise.} \end{cases}$$

**Theorem 1.4**([10]) *For the complete graph  $K_p$ ,  $\gamma_{cme}(K_p) = 1$ .*

**Theorem 1.5**([10]) *Let  $G$  be a wheel of order  $p$ .*

(i) *If  $p \leq 9$  and  $p \equiv l \pmod{6}$ , then*

$$\gamma_{cme}(G) = \begin{cases} \frac{p}{3} + 1 & \text{if } l = 3, \\ \left\lceil \frac{p-1}{3} \right\rceil & \text{otherwise.} \end{cases}$$

(ii) If  $p > 9$ , then  $\gamma_{cme}(G) = p - 5$ .

## §2. Connected Monophonic Eccentric Domination Number

The corona of two graphs  $G_1$  and  $G_2$  is the graph  $G = G_1 \circ G_2$  formed from one copy of  $G_1$  and  $|V(G_1)|$  copies of  $G_2$ , where the  $i^{th}$  vertex of  $G_1$  is adjacent to every vertex in the  $i^{th}$  copy of  $G_2$ . The distance related properties of corona was studied in [8] and the domination parameters of corona was studied in [2].

**Theorem 2.1** *Let  $G$  be a connected graph of order  $m$  and let  $H$  be any graph of order  $n$ . Then*

$$\gamma_{cme}(G \circ H) \leq m(1 + \gamma_{me}(H)).$$

*Proof* Let  $H_{i,n}$  be the  $i^{th}$  copy of  $H$  ( $1 \leq i \leq m$ ). Let  $S_i$  be a minimum monophonic eccentric dominating set of  $H_{i,n}$ . In  $G \circ H$ , it is clear that every vertex in  $H_{i,n}$  is monophonic eccentric dominated by a vertex in  $S = \bigcup_{i=1}^m S_i$ . Thus  $S$  is a monophonic eccentric dominating set of  $G \circ H$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = S \cup V(G)$ . Clearly,  $S'$  is a connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G \circ H) \leq m(1 + \gamma_{me}(H))$ .  $\square$

**Remark 2.2** The bounds in Theorem 2.1 is sharp. For the graph  $G = C_r \circ C_s$  where  $s = r + 3$ ,  $\gamma_{cme}(G) = m(1 + \gamma_{me}(C_s))$ .

**Theorem 2.3** *If  $G = P_r \circ C_s$  ( $r \geq 2$ ), then*

$$\gamma_{cme}(G) = \begin{cases} 4 & \text{if } r = 2, 3 \text{ and } s = 3, \\ r + 2 & \text{if } (r = 2, 3 \text{ and } s = 4, 5) \text{ or } (r \geq 4 \text{ and } 3 \leq s \leq \lceil \frac{r+7}{2} \rceil), \\ r + 2 + (2s - r - 8)\gamma_{me}(C_s) & \text{if } r \geq 3 \text{ and } s = \lceil \frac{r+9}{2} \rceil, \lceil \frac{r+11}{2} \rceil, \dots, r+3, \\ r(1 + \gamma_{me}(C_s)) & \text{if } s > r+3. \end{cases}$$

*Proof* Let  $G$  be the corona product of  $P_r$  and  $C_s$ . Let  $u_1, u_2, \dots, u_r$  ( $r \geq 2$ ) be the vertices of  $P_r$  and let  $C_{i,s}: v_{i,1}, v_{i,2}, \dots, v_{i,s}, v_{i,1}$  be the  $i^{th}$  copy of  $C_s$  ( $1 \leq i \leq r$ ). We prove this theorem by considering three cases.

**Case 1.**  $3 \leq s \leq \lceil \frac{r+7}{2} \rceil$ .

**Subcase 1.1**  $r = 2, 3$  and  $s = 3$ .

Let  $S = \{u_1, v_{1,1}, v_{1,2}, v_{1,3}\}$ . It is clear that the vertices  $u_i$  ( $2 \leq i \leq r$ ) and the vertices  $v_{i,j}$  ( $2 \leq i \leq r, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{1,j}$ . Also, the induced subgraph  $\langle S \rangle$  is connected. Hence  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = 4$ .

**Subcase 1.2** ( $r = 2, 3$  and  $s = 4, 5$ ) or ( $r \geq 4$  and  $3 \leq s \leq \left\lceil \frac{r+7}{2} \right\rceil$ ).

If  $r$  is even, then the vertices  $u_i$  ( $1 \leq i \leq \frac{r}{2}$ ) and  $v_{i,j}$  ( $1 \leq i \leq \frac{r}{2}, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{r,j}$ ; and the vertices  $u_i$  ( $\frac{r+2}{2} \leq i \leq r$ ) and  $v_{i,j}$  ( $\frac{r+2}{2} \leq i \leq r, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{1,j}$ . Then  $S = \{v_{1,j}, v_{r,j}\}$  is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = r + 2.$$

If  $r$  is odd, then the vertices  $u_i$  ( $1 \leq i \leq \frac{r-1}{2}$ ) and  $v_{i,j}$  ( $1 \leq i \leq \frac{r-1}{2}, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{r,j}$ , the vertices  $u_i$  ( $\frac{r+3}{2} \leq i \leq r$ ) and  $v_{i,j}$  ( $\frac{r+3}{2} \leq i \leq r, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{1,j}$ , and the vertices  $u_{\frac{r+1}{2}}$  and  $v_{\frac{r+1}{2},j}$  ( $1 \leq j \leq s$ ) are monophonic eccentric dominated by both the vertices  $v_{1,j}$  and  $v_{r,j}$ . Then  $S = \{v_{1,j}, v_{r,j}\}$  is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = r + 2.$$

**Case 2.**  $r \geq 3$  and  $s = \left\lceil \frac{r+9}{2} \right\rceil, \left\lceil \frac{r+11}{2} \right\rceil, \dots, r+3$ .

**Subcase 2.1**  $r$  is even.

Let  $m = s - \frac{r+8}{2}$ . It can be easily seen that the vertices  $u_i$  ( $1 \leq i \leq \frac{r}{2}$ ) and  $v_{i,j}$  ( $1 \leq i \leq \frac{r}{2} - m, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{r,j}$ . Similarly, the vertices  $u_i$  ( $\frac{r+2}{2} \leq i \leq r$ ) and  $v_{i,j}$  ( $\frac{r+2}{2} + m \leq i \leq r, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{1,j}$ . Let  $S_k$  ( $\frac{r+2}{2} - m \leq k \leq \frac{r}{2} + m$ ) be a minimum monophonic eccentric dominating set of  $C_{k,s}$ . It is clear that, in  $G$ , any vertex in  $C_{k,s}$  ( $\frac{r+2}{2} - m \leq k \leq \frac{r}{2} + m$ ) is monophonic eccentric dominated by a vertex in  $S_k$  and hence

$$S = \left( \bigcup_{k=\frac{r+2}{2}-m}^{\frac{r}{2}+m} S_k \right) \cup \{v_{1,j}, v_{r,j}\}$$

is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$ . Thus

$$\begin{aligned} \gamma_{cme}(G) &= |S'| \\ &= r + 2m \gamma_{me}(C_s) + 2 \\ &= r + 2 + \left[ 2\left(s - \frac{r+8}{2}\right) \right] \gamma_{me}(C_s) \\ &= r + 2 + (2s - r - 8) \gamma_{me}(C_s). \end{aligned}$$

**Subcase 2.2**  $r$  is odd.

Let  $m = s - \frac{r+7}{2}$ . It can be easily verified that the vertices  $u_i$  ( $1 \leq i \leq \frac{r-1}{2}$ ) and  $v_{i,j}$  ( $1 \leq i \leq \frac{r+1}{2} - m, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{r,j}$ . Similarly, the vertices  $u_i$  ( $\frac{r+3}{2} \leq i \leq r$ ) and  $v_{i,j}$  ( $\frac{r+1}{2} + m \leq i \leq r, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{1,j}$ . Also, the vertex  $u_{\frac{r+1}{2}}$  is monophonic eccentric dominated by both the vertices  $v_{1,j}$  and  $v_{r,j}$ . Let  $S_k$  ( $\frac{r+3}{2} - m \leq k \leq \frac{r-1}{2} + m$ ) be a minimum monophonic eccentric dominating set of  $C_{k,s}$ . It is clear that, in  $G$ , any vertex in  $C_{k,s}$  ( $\frac{r+3}{2} - m \leq k \leq \frac{r-1}{2} + m$ ) is monophonic eccentric dominated by a vertex in  $S_k$  and hence

$$S = \left( \bigcup_{k=\frac{r+3}{2}-m}^{\frac{r-1}{2}+m} S_k \right) \cup \{v_{1,j}, v_{r,j}\}$$

is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$ . Thus

$$\begin{aligned} \gamma_{cme}(G) &= |S'| \\ &= r + (2m - 1) \gamma_{me}(C_s) + 2 \\ &= r + 2 + \left[ 2\left(s - \frac{r+7}{2}\right) - 1 \right] \gamma_{me}(C_s) \\ &= r + 2 + (2s - r - 8) \gamma_{me}(C_s). \end{aligned}$$

**Case 3.**  $s > r + 3$ .

If  $r$  is even, then the vertices  $u_i$  ( $1 \leq i \leq \frac{r}{2}$ ) are monophonic eccentric dominated by a vertex  $v_{r,j}$  ( $1 \leq j \leq s$ ) and the vertices  $u_i$  ( $\frac{r+2}{2} \leq i \leq r$ ) are monophonic eccentric dominated by a vertex  $v_{1,j}$  ( $1 \leq j \leq s$ ). If  $r$  is odd, then the vertices  $u_i$  ( $1 \leq i \leq \frac{r-1}{2}$ ) are monophonic eccentric dominated by a vertex  $v_{r,j}$  ( $1 \leq j \leq s$ ), the vertices  $u_i$  ( $\frac{r+3}{2} \leq i \leq r$ ) are monophonic eccentric dominated by a vertex  $v_{1,j}$  ( $1 \leq j \leq s$ ), and the vertex  $u_{\frac{r+1}{2}}$  is monophonic eccentric dominated by both the vertices  $v_{1,j}$  and  $v_{r,j}$  ( $1 \leq j \leq s$ ). Let  $S_k$  ( $1 \leq k \leq r$ ) be a minimum monophonic eccentric dominating set of  $C_{k,s}$ . It is clear that, in  $G$ , any vertex in  $C_{k,s}$  ( $1 \leq k \leq r$ ) is monophonic eccentric dominated by a vertex in  $S_k$  and hence  $S = \bigcup_{k=1}^r S_k$  is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = r + r \gamma_{me}(C_s) = r(1 + \gamma_{me}(C_s)). \quad \square$$

The result of the above theorem contains  $\gamma_{me}(C_s)$  and we can calculate  $\gamma_{me}(C_s)$  using Theorem 1.2.

**Note 2.4** If  $r = 1$ , then  $G = P_1 \circ C_s$  is a wheel. By Theorem 1.5, we have  $\gamma_{cme}(G) =$

$\gamma_{cme}(W_{s+1})$ .

**Theorem 2.5** *If  $G = P_r \circ W_s$  ( $r \geq 2$ ), then*

$$\gamma_{cme}(G) = \begin{cases} r + 2 & \text{if } 4 \leq s \leq \left\lceil \frac{r+9}{2} \right\rceil, \\ r + 2 + (2s - r - 10)\gamma_{me}(W_s) & \text{if } r \geq 3 \text{ and } s = \left\lceil \frac{r+11}{2} \right\rceil, \left\lceil \frac{r+13}{2} \right\rceil, \dots, r+4, \\ r(1 + \gamma_{me}(W_s)) & \text{if } s > r+4. \end{cases}$$

*Proof* Since  $W_s = C_{s-1} + K_1$ , by Theorem 1.1 we have  $\gamma_{cme}(W_s) = \gamma_{cme}(C_{s-1})$ . Then by Theorem 2.1, the required result can be got.  $\square$

The result of the above theorem contains  $\gamma_{me}(W_s)$  and we can calculate  $\gamma_{me}(W_s)$  using Theorem 1.3.

**Note 2.6** If  $r = 1$ , then  $G = P_1 \circ W_s$ . By Theorem 1.1 we have  $\gamma_{cme}(G) = \gamma_{cme}(W_s)$ .

**Theorem 2.7** *If  $G = P_r \circ K_s$  ( $r \geq 2$ ), then*

$$\gamma_{cme}(G) = \begin{cases} (s+1) \left\lfloor \frac{r}{2} \right\rfloor & \text{if } (2 \leq r \leq 5 \text{ and } 1 \leq s \leq 6 - \left\lfloor \frac{r+4}{2} \right\rfloor) \text{ or } (r \geq 6 \text{ and } s = 1), \\ r + 2 & \text{if } (2 \leq r \leq 5 \text{ and } s > 6 - \left\lfloor \frac{r+4}{2} \right\rfloor) \text{ or } (r \geq 6 \text{ and } s > 1). \end{cases}$$

*Proof* Let  $G$  be the corona product of  $P_r$  and  $K_s$ . Let  $u_1, u_2, \dots, u_r$  ( $r \geq 2$ ) and  $v_{i,1}, v_{i,2}, \dots, v_{i,s}$  be the vertices of  $P_r$  and the vertices of the  $i^{\text{th}}$  copy of  $K_s$  ( $1 \leq i \leq r$ ), respectively. We prove this theorem by considering two cases.

**Case 1.** ( $2 \leq r \leq 5$  and  $1 \leq s \leq 6 - \left\lfloor \frac{r+4}{2} \right\rfloor$ ) or ( $r \geq 6$  and  $s = 1$ ).

Let  $S = \{u_1, v_{1,1}, \dots, v_{1,s}; u_2, v_{2,1}, \dots, v_{2,s}; \dots; u_{\lfloor \frac{r}{2} \rfloor}, v_{\lfloor \frac{r}{2} \rfloor, 1}, \dots, v_{\lfloor \frac{r}{2} \rfloor, s}\}$ . It is clear that the vertices  $u_i$  ( $\lceil \frac{r+1}{2} \rceil \leq i \leq r$ ) and the vertices  $v_{i,1}$  ( $\lceil \frac{r+2}{2} \rceil \leq i \leq r$ ) are monophonic eccentric dominated by the vertex  $v_{1,1}$ . Also, the induced subgraph  $\langle S \rangle$  is connected. Hence  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = 2 \left\lfloor \frac{r}{2} \right\rfloor.$$

**Case 2.** ( $2 \leq r \leq 5$  and  $s > 6 - \left\lfloor \frac{r+4}{2} \right\rfloor$ ) or ( $r \geq 6$  and  $s > 1$ ).

It is clear that the vertices  $u_i$  ( $\lceil \frac{r+1}{2} \rceil \leq i \leq r$ ) and  $v_{i,j}$  ( $\lceil \frac{r+1}{2} \rceil \leq i \leq r, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{1,j}$  ( $1 \leq j \leq s$ ). Also, the vertices  $u_i$  ( $1 \leq i \leq \lfloor \frac{r}{2} \rfloor$ ) and  $v_{i,j}$  ( $1 \leq i \leq \lfloor \frac{r}{2} \rfloor, 1 \leq j \leq s$ ) are monophonic eccentric dominated by a vertex  $v_{r,j}$ . Hence  $S = \{v_{1,j}, v_{r,j}\}$  ( $1 \leq j \leq s$ ) is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = r + 2$ .  $\square$

**Note 2.8** If  $r = 1$ , then  $G = P_1 \circ K_s$  is a complete graph. By Theorem 1.4,  $\gamma_{cme}(G) = 1$ .

**Theorem 2.9** If  $G = P_r \circ K_{1,n}$  ( $2 \leq r \leq 5$ ), then

$$\gamma_{cme}(G) = \begin{cases} (n+2) \left\lfloor \frac{r}{2} \right\rfloor & \text{if } 1 \leq s \leq 5 - \left\lfloor \frac{r+4}{2} \right\rfloor, \\ r+2 & \text{if } s > 5 - \left\lfloor \frac{r+4}{2} \right\rfloor. \end{cases}$$

*Proof* By an argument similar to Theorem 2.7, the required result can be got.  $\square$

**Theorem 2.10** If  $G = P_r \circ K_{1,n}$  ( $r \geq 6$ ), then  $\gamma_{cme}(G) = r+2$ .

*Proof* Let  $G$  be the corona product of  $P_r$  and  $K_{1,n}$ . Let  $u_1, u_2, \dots, u_r$  ( $r \geq 6$ ) and  $v_{i,1}, v_{i,2}, \dots, v_{i,n+1}$  be the vertices of  $P_r$  and the  $i^{th}$  copy of  $K_{1,n}$  ( $1 \leq i \leq r$ ), respectively. It is clear that the vertices  $u_i$  ( $\lceil \frac{r+1}{2} \rceil \leq i \leq r$ ) and  $v_{i,j}$  ( $\lceil \frac{r+1}{2} \rceil \leq i \leq r, 1 \leq j \leq n+1$ ) are monophonic eccentric dominated by a vertex  $v_{1,j}$  ( $1 \leq j \leq n+1$ ). Also, the vertices  $u_i$  ( $1 \leq i \leq \lfloor \frac{r}{2} \rfloor$ ) and  $v_{i,j}$  ( $1 \leq i \leq \lfloor \frac{r}{2} \rfloor, 1 \leq j \leq n+1$ ) are monophonic eccentric dominated by a vertex  $v_{r,j}$ . Hence  $S = \{v_{1,j}, v_{r,j}\}$  ( $1 \leq j \leq n+1$ ) is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = r+2$ .  $\square$

**Theorem 2.11** If  $G = P_r \circ K_{m,n}$  ( $r, m, n \geq 2$ ), then  $\gamma_{cme}(G) = r+2$ .

*Proof* By an argument similar to Theorem 2.10, the required result can be got.  $\square$

**Note 2.12** If  $G = P_1 \circ K_{m,n}$ , then

$$\gamma_{cme}(G) = \begin{cases} 1 & \text{if either } m \text{ or } n = 1, \\ 2 & \text{if } m, n \geq 2. \end{cases}$$

**Theorem 2.13** Let  $G = C_r \circ P_1$  ( $r \geq 6$ ) and let  $r \equiv k \pmod{6}$ . Then,

$$\gamma_{cme}(G) = \begin{cases} \left\lfloor \frac{r}{3} \right\rfloor + r - 3 & \text{if } k = 2, 3, \\ \left\lfloor \frac{r}{3} \right\rfloor + r - 4 & \text{if } k = 0, 1, 4 \text{ and } 5. \end{cases}$$

*Proof* Let  $G$  be the corona product of  $C_r$  and  $P_1$ . Let  $u_1, u_2, \dots, u_r$  and  $v_{i,1}$  be the vertices of  $C_r$  and the vertices of the  $i^{th}$  copy of  $P_1$  ( $1 \leq i \leq r$ ), respectively. We prove this theorem by considering six cases.

**Case 1.**  $r \equiv 0 \pmod{6}$ .

Let  $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \dots; v_{r-5,1}, v_{r-4,1}\}$ . It is easily verified that the vertices  $u_{r-1}, v_{r-1,1}, u_3$  and  $v_{3,1}$  are monophonic eccentric dominated by the vertex  $v_{1,1}$ , the vertices  $u_4, v_{4,1}, u_r$  and  $v_{r,1}$  are monophonic eccentric dominated by the vertex  $v_{2,1}, \dots$ , the vertices  $u_{r-1}, v_{r-1,1}, u_{r-3}$  and  $v_{r-3,1}$  are monophonic eccentric dominated by the vertex  $v_{r-5,1}$  and the vertices  $u_r,$

$v_{r,1}$ ,  $u_{r-2}$  and  $v_{r-2,1}$  are monophonic eccentric dominated by the vertex  $v_{r-4,1}$ . It is clear that  $S$  is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_{r-4}\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + r - 4.$$

**Case 2.**  $r \equiv 1(mod 6)$ .

Let  $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \dots; v_{r-6,1}, v_{r-5,1}; v_{r,1}\} \cup \{u_1, u_2, \dots, u_{r-5}; u_r\}$ . By an argument similar to Case 1, it can be easily seen that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + r - 4$ .

**Case 3.**  $r \equiv 2(mod 6)$ .

Let  $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \dots; v_{r-1,1}, v_{r,1}\} \cup \{u_1, u_2, \dots, u_{r-6}; u_{r-1}, u_r\}$ . By an argument similar to Case 1, it can be easily seen that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + 1 + r - 4 = \left\lceil \frac{r}{3} \right\rceil + r - 3.$$

**Case 4.**  $r \equiv 3(mod 6)$ .

Let  $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \dots; v_{r-2,1}, v_{r-1,1}\} \cup \{u_1, u_2, \dots, u_{r-7}; u_{r-2}, u_{r-1}, u_r\}$ . By an argument similar to Case 1, it can be easily seen that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + 1 + r - 4 = \left\lceil \frac{r}{3} \right\rceil + r - 3.$$

**Case 5.**  $r \equiv 4(mod 6)$ .

Let  $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \dots; v_{r-3,1}, v_{r-2,1}\} \cup \{u_1, u_2, \dots, u_{r-8}; u_{r-3}, u_{r-2}, u_{r-1}, u_r\}$ . By an argument similar to Case 1, it can be easily seen that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + r - 4.$$

**Case 6.**  $r \equiv 5(mod 6)$ .

Let  $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \dots; v_{r-4,1}, v_{r-3,1}\} \cup \{u_1, u_2, \dots, u_{r-9}; u_{r-4}, u_{r-3}, \dots, u_r\}$ . By an argument similar to Case 1, it can be easily seen that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + r - 4$ . □

**Theorem 2.14** Let  $G = C_r \circ P_s$  ( $r \geq 8, s \geq 2$ ) and let  $r \equiv k(\text{mod } 8)$ . Then

$$\gamma_{cme}(G) = \begin{cases} \frac{3r-8+k}{2} & \text{if } 2 \leq s \leq r+1 \text{ and } 0 \leq k \leq 3, \\ \frac{3r-k}{2} & \text{if } 2 \leq s \leq r+1 \text{ and } 3 < k < 8, \\ 2r & \text{if } s = r+2, \\ 3r & \text{if } s > r+2. \end{cases}$$

*Proof* Let  $G$  be the corona product of  $C_r$  and  $P_s$ . Let  $u_1, u_2, \dots, u_r$  and  $v_{i,1}, v_{i,2}, \dots, v_{i,s}$  be the vertices of  $C_r$  and the vertices of the  $i^{\text{th}}$  copy of  $P_s$  ( $1 \leq i \leq r$ ), respectively. We prove this theorem by considering three cases.

**Case 1.**  $2 \leq s \leq r+1$ .

**Subcase 1.1**  $r \equiv 0(\text{mod } 8)$ .

Let  $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \dots; v_{r-7,j}, v_{r-6,j}, v_{r-5,j}, v_{r-4,j}\}$  ( $1 \leq j \leq s$ ). It is easily verified that the vertices  $u_{r-1}, v_{r-1,j}, u_3$  and  $v_{3,j}$  are monophonic eccentric dominated by a vertex  $v_{1,j}$ , the vertices  $u_4, v_{4,j}, u_r$  and  $v_{r,j}$  are monophonic eccentric dominated by a vertex  $v_{2,j}$ , the vertices  $u_1, v_{1,j}, u_5$  and  $v_{5,j}$  are monophonic eccentric dominated by a vertex  $v_{3,j}$ , the vertices  $u_2, v_{2,j}, u_6$  and  $v_{6,j}$  are monophonic eccentric dominated by a vertex  $v_{4,j}$ ,  $\dots$ , the vertices  $u_{r-1}, v_{r-1,j}, u_{r-5}$  and  $v_{r-5,j}$  are monophonic eccentric dominated by a vertex  $v_{r-7,j}$ , the vertices  $u_r, v_{r,j}, u_{r-4}$  and  $v_{r-4,j}$  are monophonic eccentric dominated by a vertex  $v_{r-6,j}$ , the vertices  $u_1, v_{1,j}, u_{r-3}$  and  $v_{r-3,j}$  are monophonic eccentric dominated by a vertex  $v_{r-5,j}$  and the vertices  $u_2, v_{2,j}, u_{r-2}$  and  $v_{r-2,j}$  are monophonic eccentric dominated by a vertex  $v_{r-4,j}$ . It is clear that  $S$  is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_{r-4}\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \frac{r}{2} + r - 4 = \frac{3r-8}{2} = \frac{3r-8+k}{2}.$$

**Subcase 1.2**  $r \equiv 1(\text{mod } 8)$ .

Let  $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \dots; v_{r-8,j}, v_{r-7,j}, v_{r-6,j}, v_{r-5,j}\} \cup \{v_{r,j}\}$  ( $1 \leq j \leq s$ ). It is easily verified that the vertices  $u_{r-1}, v_{r-1,j}, u_3$  and  $v_{3,j}$  are monophonic eccentric dominated by a vertex  $v_{1,j}$ , the vertices  $u_4, v_{4,j}, u_r$  and  $v_{r,j}$  are monophonic eccentric dominated by a vertex  $v_{2,j}$ , the vertices  $u_1, v_{1,j}, u_5$  and  $v_{5,j}$  are monophonic eccentric dominated by a vertex  $v_{3,j}$ , the vertices  $u_2, v_{2,j}, u_6$  and  $v_{6,j}$  are monophonic eccentric dominated by a vertex  $v_{4,j}$ ,  $\dots$ , the vertices  $u_{r-1}, v_{r-1,j}, u_{r-6}$  and  $v_{r-6,j}$  are monophonic eccentric dominated by a vertex  $v_{r-8,j}$ , the vertices  $u_r, v_{r,j}, u_{r-5}$  and  $v_{r-5,j}$  are monophonic eccentric dominated by a vertex  $v_{r-7,j}$ , the vertices  $u_{r-8}, v_{r-8,j}, u_{r-4}$  and  $v_{r-4,j}$  are monophonic eccentric dominated by a vertex  $v_{r-6,j}$ , the vertices  $u_{r-7}, v_{r-7,j}, u_{r-3}$  and  $v_{r-3,j}$  are monophonic eccentric dominated by a vertex  $v_{r-5,j}$  and the vertices  $u_2, v_{2,j}, u_{r-2}$  and  $v_{r-2,j}$  are monophonic eccentric dominated by a vertex  $v_{r,j}$ . It is clear that  $S$  is a minimum monophonic eccentric

dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_{r-5}; u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \frac{r+1}{2} + r - 4 = \frac{3r-7}{2} = \frac{3r-8+k}{2}.$$

**Subcase 1.3**  $r \equiv 2 \pmod{8}$ .

Let  $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \dots; v_{r-9,j}, v_{r-8,j}, v_{r-7,j}, v_{r-6,j}\} \cup \{v_{r-1,j}, v_{r,j}\} \cup \{u_1, u_2, \dots, u_{r-6}; u_{r-1}, u_r\}$ . By an argument similar to Subcase 1.2, it is clear that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \frac{r+2}{2} + r - 4 = \frac{3r-6}{2} = \frac{3r-8+k}{2}.$$

**Subcase 1.4**  $r \equiv 3 \pmod{8}$ .

Let  $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \dots; v_{r-10,j}, v_{r-9,j}, v_{r-8,j}, v_{r-7,j}\} \cup \{v_{r-2,j}, v_{r-1,j}, v_{r,j}\} \cup \{u_1, u_2, \dots, u_{r-7}; u_{r-2}, u_{r-1}, u_r\}$ . By an argument similar to Subcase 1.2, it is clear that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \frac{r+3}{2} + r - 4 = \frac{3r-5}{2} = \frac{3r-8+k}{2}.$$

**Subcase 1.5**  $r \equiv 4 \pmod{8}$ .

Let

$$S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \dots; v_{r-11,j}, v_{r-10,j}, v_{r-9,j}, v_{r-8,j}\} \\ \cup \{v_{r-7,j}, v_{r-6,j}, v_{r-5,j}, v_{r-4,j}\} \cup \{u_1, u_2, \dots, u_{r-4}\}.$$

By an argument similar to Subcase 1.2, it can be easily seen that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \frac{r+4}{2} + r - 4 = \frac{3r-4}{2} = \frac{3r-k}{2}.$$

**Subcase 1.6**  $r \equiv 5 \pmod{8}$ .

Let  $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \dots; v_{r-12,j}, v_{r-11,j}, v_{r-10,j}, v_{r-9,j}; v_{r-4,j}, v_{r-3,j}, v_{r-2,j}, v_{r-1,j}\} \cup \{u_1, u_2, \dots, u_{r-9}; u_{r-4}, u_{r-3}, \dots, u_r\}$ . By an argument similar to Subcase 1.2, it can be easily seen that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \frac{r+3}{2} + r - 4 = \frac{3r-5}{2} = \frac{3r-k}{2}.$$

**Subcase 1.7**  $r \equiv 6 \pmod{8}$ .

Let  $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \dots; v_{r-13,j}, v_{r-12,j}, v_{r-11,j}, v_{r-10,j}; v_{r-5,j}, v_{r-4,j}, v_{r-3,j}, v_{r-2,j}\} \cup \{u_1, u_2, \dots, u_{r-10}; u_{r-5}, u_{r-4}, \dots, u_r\}$ . By an argument similar to Sub-

case 1.2, it is clear that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \frac{r+2}{2} + r - 4 = \frac{3r-6}{2} = \frac{3r-k}{2}.$$

**Subcase 1.8**  $r \equiv 7(\text{mod } 8)$ .

Let  $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \dots; v_{r-14,j}, v_{r-13,j}, v_{r-12,j}, v_{r-11,j}; v_{r-6,j}, v_{r-5,j}, v_{r-4,j}, v_{r-3,j}\} \cup \{u_1, u_2, \dots, u_{r-11}; u_{r-6}, u_{r-5}, \dots, u_r\}$ . By an argument similar to Subcase 1.2, it is clear that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \frac{r+1}{2} + r - 4 = \frac{3r-7}{2} = \frac{3r-k}{2}.$$

**Case 2.**  $s = r + 2$ .

Let  $S = \{v_{1,1}, v_{2,1}, \dots, v_{r,1}\}$ . It is easily verified that the vertices  $u_{r-1}, v_{r-1,k}, u_3, v_{3,k}$  ( $2 \leq k \leq s-1$ ) and  $v_{1,s}$  are monophonic eccentric dominated by the vertex  $v_{1,1}$ , the vertices  $u_4, v_{4,k}, u_r, v_{r,k}$  ( $2 \leq k \leq s-1$ ) and  $v_{2,s}$  are monophonic eccentric dominated by the vertex  $v_{2,1}, \dots$ , the vertices  $u_{r-2}, v_{r-2,k}, u_2, v_{2,k}$  ( $2 \leq k \leq s-1$ ) and  $v_{r,s}$  are monophonic eccentric dominated by the vertex  $v_{r,1}$ . It is clear that  $S$  is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = 2r.$$

**Case 3.**  $s > r + 2$ .

Let  $S = \{v_{1,1}, v_{2,1}, \dots, v_{r,1}; v_{1,s}, v_{2,s}, \dots, v_{r,s}\}$ . In  $r+2 < s < 2r$ , the vertices  $u_{r-1}, v_{r-1,k}, u_3, v_{3,k}$  ( $s-r+1 \leq k \leq r$ ) and  $v_{1,l}$  ( $r+1 \leq l \leq s$ ) are monophonic eccentric dominated by the vertex  $v_{1,1}$ , the vertices  $u_4, v_{4,k}, u_r, v_{r,k}$  ( $s-r+1 \leq k \leq r$ ) and  $v_{2,l}$  ( $r+1 \leq l \leq s$ ) are monophonic eccentric dominated by the vertex  $v_{2,1}, \dots$ , the vertices  $u_{r-2}, v_{r-2,k}, u_2, v_{2,k}$  ( $s-r+1 \leq k \leq r$ ) and  $v_{r,l}$  ( $r+1 \leq l \leq s$ ) are monophonic eccentric dominated by the vertex  $v_{r,1}$ . Also, the vertices  $v_{1,l}$  ( $1 \leq l \leq s-r$ ) are monophonic eccentric dominated by the vertex  $v_{1,s}$ , the vertices  $v_{2,l}$  ( $1 \leq l \leq s-r$ ) are monophonic eccentric dominated by the vertex  $v_{2,s}, \dots$ , the vertices  $v_{r,l}$  ( $1 \leq l \leq s-r$ ) are monophonic eccentric dominated by the vertex  $v_{r,s}$ .

In  $s \geq 2r$ , the vertices  $u_{r-1}, u_3$  and  $v_{1,l}$  ( $\lfloor \frac{s+3}{2} \rfloor \leq l \leq s$ ) are monophonic eccentric dominated by the vertex  $v_{1,1}$ , the vertices  $u_4, u_r$ , and  $v_{2,l}$  ( $\lfloor \frac{s+3}{2} \rfloor \leq l \leq s$ ) are monophonic eccentric dominated by the vertex  $v_{2,1}, \dots$ , the vertices  $u_{r-2}, u_2$ , and  $v_{r,l}$  ( $\lfloor \frac{s+3}{2} \rfloor \leq l \leq s$ ) are monophonic eccentric dominated by the vertex  $v_{r,1}$ . Also, the vertices  $v_{1,l}$  ( $1 \leq l \leq \lfloor \frac{s+1}{2} \rfloor$ ) are monophonic eccentric dominated by the vertex  $v_{1,s}$ , the vertices  $v_{2,l}$  ( $1 \leq l \leq \lfloor \frac{s+1}{2} \rfloor$ ) are monophonic eccentric dominated by the vertex  $v_{2,s}, \dots$ , the vertices  $v_{r,l}$  ( $1 \leq l \leq \lfloor \frac{s+1}{2} \rfloor$ ) are monophonic eccentric dominated by the vertex  $v_{r,s}$ . Hence, it is clear that  $S$  is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = 3r$ .  $\square$

**Theorem 2.15** *If  $G = C_r \circ P_s$  ( $r \leq 7$ ), then*

$$\gamma_{cme}(G) = \begin{cases} (s+1) \left\lceil \frac{r}{3} \right\rceil & \text{if } 1 \leq s \leq 7 - \left\lceil \frac{r+4}{2} \right\rceil, \\ \frac{3r-1}{2} & \text{if } r = 3, 5 \text{ and } 7 - \left\lceil \frac{r+4}{2} \right\rceil < s < r+2, \\ 8 & \text{if } (r = 4, 6 \text{ and } 7 - \left\lceil \frac{r+4}{2} \right\rceil < s < r+2) \text{ or } (r = 7 \text{ and } 3 \leq s \leq 8), \\ 7 & \text{if } r = 7 \text{ and } s = 2, \\ 2r & \text{if } s = r+2, \\ 3r & \text{if } s > r+2. \end{cases}$$

*Proof* Let  $G$  be the corona product of  $C_r$  and  $P_s$ . Let  $u_1, u_2, \dots, u_r$  and  $v_{i,1}, v_{i,2}, \dots, v_{i,s}$  be the vertices of  $C_r$  and the vertices of the  $i^{\text{th}}$  copy of  $P_s$  ( $1 \leq i \leq r$ ), respectively. We prove this theorem by considering four cases.

**Case 1.**  $1 \leq s \leq 7 - \left\lceil \frac{r+4}{2} \right\rceil$ .

Let  $S = \{u_1, v_{1,1}, \dots, v_{1,s}; u_2, v_{2,1}, \dots, v_{2,s}; \dots; u_{\lceil \frac{r}{3} \rceil}, v_{\lceil \frac{r}{3} \rceil, 1}, \dots, v_{\lceil \frac{r}{3} \rceil, s}\}$ . It is easily verified that every vertex in  $V - S$  has a monophonic eccentric vertex in  $S$  and the induced subgraph  $\langle S \rangle$  is connected. Hence  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = (s+1) \left\lceil \frac{r}{3} \right\rceil$ .

**Case 2.**  $7 - \left\lceil \frac{r+4}{2} \right\rceil < s < r+2$ .

**Subcase 2.1**  $r = 3$  and  $5$ .

If  $r = 3$ , let  $S = \{v_{1,j}, v_{2,j}; u_1, u_2\}$ . If  $r = 5$ , let  $S = \{v_{1,j}, v_{2,j}, v_{4,j}; u_1, u_2, u_3, u_4\}$ . Then by an argument similar to Case 1, it is clear that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = \frac{3r-1}{2}.$$

**Subcase 2.2** ( $r = 4$  and  $6$ ) or ( $r = 7$  and  $3 \leq s \leq 8$ ).

Let  $S = \{u_1, v_{1,j}; u_2, v_{2,j}; u_3, v_{3,j}; u_4, v_{4,j}\}$ . Then by an argument similar to Case 1, it is clear that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = 8$ .

**Subcase 2.3**  $r = 7$  and  $s = 2$ .

Let  $S = \{u_1, v_{1,j}; u_2, v_{2,1}, v_{2,2}; u_3, v_{3,j}\}$ . Then by an argument similar to Case 1, it is clear that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = 7$ .

**Case 3.**  $s = r+2$ .

Let  $S = \{v_{1,1}, v_{2,1}, \dots, v_{r,1}; u_1, u_2, \dots, u_r\}$ . By an argument similar to Case 2 of Theorem 2.14, it can be easily seen that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = 2r$ .

**Case 4.**  $s > r + 2$ .

Let  $S = \{v_{1,1}, v_{2,1}, \dots, v_{r,1}; v_{1,s}, v_{2,s}, \dots, v_{r,s}; u_1, u_2, \dots, u_r\}$ . By an argument similar to Case 3 of Theorem 2.14, it can be easily seen that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = 3r$ .  $\square$

**Theorem 2.16** *If  $G = C_r \circ C_s$  ( $r \geq 8$ ) and  $r \equiv k \pmod{8}$ , then*

$$\gamma_{cme}(G) = \begin{cases} \frac{3r-8+k}{2} & \text{if } s \leq r+2 \text{ and } 0 \leq k \leq 3, \\ \frac{3r-k}{2} & \text{if } s \leq r+2 \text{ and } 3 < k < 8, \\ r(1 + \gamma_{me}(C_s)) & \text{if } s > r+2. \end{cases}$$

*Proof* Let  $G$  be the corona product of  $C_r$  and  $C_s$ . Let  $u_1, u_2, \dots, u_r$  be the vertices of  $C_r$  and let  $C_{i,s}: v_{i,1}, v_{i,2}, \dots, v_{i,s}, v_{i,1}$  be the  $i^{\text{th}}$  copy of  $C_s$  ( $1 \leq i \leq r$ ), respectively. We prove this theorem by considering two cases.

**Case 1.**  $s \leq r + 2$ .

By an argument similar to Case 1 of Theorem 2.14, the required result can be got.

**Case 2.**  $s > r + 2$ .

Let  $S_k$  ( $1 \leq k \leq r$ ) be a minimum monophonic eccentric dominating set of  $C_{k,s}$ . It is clear that, in  $G$ , any vertex in  $C_{k,s}$  ( $1 \leq k \leq r$ ) is monophonic eccentric dominated by a vertex in  $S_k$ . Also, the vertices  $u_1, u_2, \dots, u_r$  are monophonic eccentric dominated by a vertex in  $S_k$  ( $1 \leq k \leq r$ ) and hence  $S = \bigcup_{k=1}^r S_k$  is a minimum monophonic eccentric dominating set of  $G$ , but the induced subgraph  $\langle S \rangle$  is not connected. Therefore, we consider a set  $S' = \{u_1, u_2, \dots, u_r\} \cup S$ . Clearly,  $S'$  is a minimum connected monophonic eccentric dominating set of  $G$  and so

$$\gamma_{cme}(G) = r(1 + \gamma_{me}(C_s)). \quad \square$$

The result of the above theorem contains  $\gamma_{me}(C_s)$  and we can calculate  $\gamma_{me}(C_s)$  using Theorem 1.2.

**Theorem 2.17** *If  $G = C_r \circ W_s$  ( $r \geq 8$ ) and  $r \equiv k \pmod{8}$ , then*

$$\gamma_{cme}(G) = \begin{cases} \frac{3r-8+k}{2} & \text{if } s \leq r+3 \text{ and } 0 \leq k \leq 3, \\ \frac{3r-k}{2} & \text{if } s \leq r+3 \text{ and } 3 < k < 8, \\ r(1 + \gamma_{me}(W_s)) & \text{if } s > r+3. \end{cases}$$

*Proof* By Theorem 1.1 and by an argument similar to Theorem 2.16, the required result can be got.  $\square$

The result of the above theorem contains  $\gamma_{me}(W_s)$  and we can calculate  $\gamma_{me}(W_s)$  using Theorem 1.3.

**Theorem 2.18** *If  $G = C_r \circ C_s$  ( $r \leq 7$ ), then*

$$\gamma_{cme}(G) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil + r & \text{if } r = 3, 5 \text{ and } 3 \leq s \leq r + 2, \\ 8 & \text{if } r = 4, 6, 7 \text{ and } 3 \leq s \leq r + 2, \\ r(1 + \gamma_{me}(C_s)) & \text{if } s > r + 2. \end{cases}$$

*Proof* Let  $G$  be the corona product of  $C_r$  and  $C_s$ . Let  $u_1, u_2, \dots, u_r$  be the vertices of  $C_r$  and let  $C_{i,s}: v_{i,1}, v_{i,2}, \dots, v_{i,s}, v_{i,1}$  be the  $i^{\text{th}}$  copy of  $C_s$  ( $1 \leq i \leq r$ ), respectively. We prove this theorem by considering two cases.

**Case 1.**  $3 \leq s \leq r + 2$ .

**Subcase 1.1**  $r = 3$  and  $5$ .

If  $r = 3$ , let  $S = \{v_{1,j}, v_{2,j}; u_1, u_2\}$ . If  $r = 5$ , let  $S = \{v_{1,j}, v_{2,j}, v_{4,j}; u_1, u_2, u_3, u_4\}$ . By an argument similar to Case 2 of Theorem 2.15, it can be easily seen that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + r$ .

**Subcase 1.2**  $r = 4, 6$  and  $7$ .

Let  $S = \{u_1, v_{1,j}; u_2, v_{2,j}; u_3, v_{3,j}; u_4, v_{4,j}\}$ . By an argument similar to Case 2 of Theorem 2.15, it clear that  $S$  is a minimum connected monophonic eccentric dominating set of  $G$  and so  $\gamma_{cme}(G) = 8$ .

**Case 2.**  $s > r + 2$ .

By an argument similar to Case 2 of Theorem 2.16, the required result can be got.  $\square$

The result of the above theorem contains  $\gamma_{me}(C_s)$  and we can calculate  $\gamma_{me}(C_s)$  using Theorem 1.2.

**Theorem 2.19** *If  $G = C_r \circ W_s$  ( $r \leq 7$ ), then*

$$\gamma_{cme}(G) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil + r & \text{if } r = 3, 5 \text{ and } 4 \leq s \leq r + 3, \\ 8 & \text{if } r = 4, 6, 7 \text{ and } 4 \leq s \leq r + 3, \\ r(1 + \gamma_{me}(W_s)) & \text{if } s > r + 3. \end{cases}$$

*Proof* By Theorem 1.1 and by an argument similar to Theorem 2.18, the required result can be got.  $\square$

The result of the above theorem contains  $\gamma_{me}(W_s)$  and we can calculate  $\gamma_{me}(W_s)$  using Theorem 1.3.

**Theorem 2.20** If  $G = C_r \circ K_s$  or  $G = C_r \circ K_{m,n}$  and  $r \equiv k \pmod{8}$ , then

$$\gamma_{cme}(G) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil + r & \text{if } r = 3, 5, \\ 8 & \text{if } r = 4, 6, 7, \\ \frac{3r - 8 + k}{2} & \text{if } r \geq 8 \text{ and } 0 \leq k \leq 3, \\ \frac{3r - k}{2} & \text{if } r \geq 8 \text{ and } 3 < k < 8. \end{cases}$$

*Proof* By an argument similar to Case 1 of Theorem 2.18 and Case 1 of Theorem 2.14, the required result can be got.  $\square$

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