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Cordial Labeling of Graphs Using Tribonacci Numbers

Sarbari Mitra and Soumya Bhoumik

(Department of Mathematics, Fort Hays State University, 600 Park Street Hays, USA)

E-mail: s_bhoumik@fhsu.edu

Abstract: We introduce Tribonacci cordial labeling as an extension of Fibonacci cordial labeling, a well-known forms of vertex-labelings. A graph that admits Tribonacci cordial labeling is called Tribonacci cordial graph. In this paper we investigate whether some well-known graphs are Tribonacci cordial.

Key Words: Tribonacci cordial, generalized friendship graph, wheel graph, ring sum, joint sum, Smarandachely cordial k-labeling.

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§1. Introduction

Throughout the paper we assume that G is a simple connected graph of order n.

Definition 1.1 *A function $f : V(G) \to \{0,1\}$ is said to be cordial labeling if the induced function $f^* : E(G) \to \{0,1\}$ defined by*

$$f^*(uv) = |f(u) - f(v)|$$

satisfies the conditions $|v_f(0) - v_f(1)| \leqslant 1$, as well as $|e_f(0) - e_f(1)| \leqslant 1$, where,

$v_f(0) :=$ *number of vertices with label* 0,

$v_f(1) :=$ *number of vertices with label* 1,

$e_f(0) :=$ *number of edges with label* 0,

$e_f(1) :=$ *number of edges with label* 1.

Generally, if there are integers $k \in mathbbZ^+$ such that $|v_f(0) - v_f(1)| \leq k$ or $|e_f(0) - e_f(1)| \leq k$, f is called a Smarandachely vertex cordial k-labeling or Smarandachely edge cordial k-labeling, and G a Smarandache cordial k-labeling graph. Clearly, a Smarandache cordial 1-labeling is nothing else but the cordial labeling of graphs.

The concept of cordial labeling was introduced by Cahit [1] though a variety of vertex labeling. This was further extended to various labeling such as divisor cordial labeling, product cordial labeling, total product cordial labeling, prime cordial labeling etc (See [2] for a dynamic survey). Rokad and Ghodasara introduced Fibonacci cordial labeling [5] and provided a list

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of families of graphs that are Fibonacci cordial. Later this labeling was explored for several other families of graph, (see [3], [4]). Motivated by their work, we investigate *Tribonacci cordial labeling*, which is an extension of the Fibonacci cordial labeling.

Definition 1.2 *The sequence T_n of Tribonacci numbers is defined by the third order linear recurrence relation (for $n \geqslant 0$):*

$$T_{n+3} = T_n + T_{n+1} + T_{n+2}; \quad T_0 = 0, T_1 = T_2 = 1,$$

Definition 1.3 *An injective function $f : V(G) \to \{T_0, T_1, \cdots, T_n\}$ is said to be Tribonacci cordial labeling if the induced function $f^* : E(G) \to \{0, 1\}$ defined by*

$$f^*(uv) = (f(u) + f(v)) \pmod 2$$

satisfies the condition $|e_f(0) - e_f(1)| \leqslant 1$. A graph which admits Tribonacci cordial labeling is called Tribonacci cordial graph.

In this paper we denote the total number of odd edges by $e(1)$ (analogously $e(0)$ for even edges) and $e(1) - e(0)$ will be denoted as $\tilde{e}$.

§2. Main Results

In this section we examine whether some of the trivial graphs like P_n, C_n, K_n are Tribonacci cordial. We can start with a simple observation that for any n, the sequence $\{T_0, T_1, \cdots T_n\}$ has m many evens, where

$$m = \begin{cases} k+1, & \text{if } n = 2k+1; \\ 2k+1, & \text{if } n = 4k \text{ or } 4k+2. \end{cases}$$

Theorem 2.1 *P_n is Tribonacci cordial.*

Proof Let $f : V(P_n) \to \{T_0, T_1, \cdots T_n\}$ be a labeling such that $f(v_i) = T_i$ for all $i = 1, 2, \cdots, n$. Clearly, it implies that the value of $\tilde{e}_{P_n}$ is 0 if n is even and 1 otherwise. □

Theorem 2.2 *For any two assigned Tribonacci labeling (injective) $f : V(C_n) \to \{T_0, T_1, \cdots, T_n\}$ and $g : V(C_n) \to \{T_0, T_1, \cdots, T_n\}$, $\tilde{e}_f - \tilde{e}_g \equiv 0 \pmod 4$.*

Proof Without loss of generality, suppose that f and g are the same Tribonacci labeling except at $v_0 \in V(C_n)$, i.e. $f(v_0) \neq g(v_0)$. Let us consider $e_f(0) = m$ and hence $e_f(1) = n - m$. Also consider v_L and v_R are two adjacent vertices of v_0.

If $f^*(v_L v_0) \equiv 1 \pmod 2$ and $f^*(v_0 v_R) \equiv 0 \pmod 2$ (or vice versa), then it is clear that

$$\tilde{e}_f = \tilde{e}_g.$$

Otherwise without loss of generality, we may assume that

$$f^*(v_L v_0) = f^*(v_0 v_R) \equiv 0 \pmod{2}$$

In this case, clearly $e_g(0)$ will be either m or $m-2$. Respectively, $e_g(1)$ will be either $n-m$ or $n-m+2$. Thus,

$$\tilde{e}_g - \tilde{e}_f = |e_g(0) - e_g(1)| - |e_f(0) - e_f(1)| \equiv 0 \pmod{4}. \qquad \square$$

Corollary 2.3 *For any injective function $f : V(G) \to \{T_0, T_1, \cdots, T_{2m}\}$ on cyclic graph C_{2m}, if $|\tilde{e}| \equiv 2 \pmod 4$, then C_{2m} is not Tribonacci cordial.*

As it is clear from Corollary 2.3, if $n \equiv 2 \pmod 4$, C_n is not Tribonacci cordial under f, then any other function $g : V(G) \to \{T_0, T_1, \cdots, T_{2m}\}$ will not be able to generate $\tilde{e}_g \equiv 0 \pmod 4$. For $n \not\equiv 2 \pmod 4$, we can consider the labeling $f : V(C_n) \to \{T_0, T_1, \cdots T_n\}$ such that $f(v_i) = T_i$ for all $i = 1, 2, \cdots, n$. Clearly it produces odd and even edges alternatively. Thus we have the following theorem.

Theorem 2.4 *C_n is Tribonacci cordial, except $n \equiv 2 \pmod 4$.*

Lemma 2.5 *K_{2m+1} is Tribonacci cordial only for $m \leq 1$.*

Proof First note that the vertex labeling can be chosen from $T_0, T_1, \cdots, T_{2m+1}$, out of which $m+1$ labels are even and thus $m+1$ are odd. Since we only need $2m+1$ many labelings, we drop either an odd or an even Tribonacci number from the list. Without loss of generality, we use all of the Tribonacci numbers except an even one. As there are $m+1$ many odd and m many even vertex labels, $e(1) = m(m+1)$, and $e(0) = \binom{m+1}{2} + \binom{m}{2}$. Hence in order to be Tribonacci cordial we must have

$$|\tilde{e}| = \left| m(m+1) - \binom{m+1}{2} - \binom{m}{2} \right| \leqslant 1.$$

It simplifies to $|m| \leqslant 1$, which is only possible for $m = 0, 1$ and hence, $n = 1, 3$. $\square$

Next we divide the case, n is even, into two categories, namely $n = 4m$, and $4m+2$ to get the following two lemmas.

Lemma 2.6 *K_{4m} is Tribonacci cordial only for $m = 1$.*

Lemma 2.7 *K_{4m+2} is Tribonacci cordial only for $m \leqslant 1$.*

The next theorem follows immediately from the previous lemmas, which provide the complete list of all complete graphs that are Tribonacci cordial.

Theorem 2.8 *A complete list of Tribonacci cordial complete graphs are $K_i, 1 \leq i \leq 4$ and K_6.*

A wheel graph W_n is a graph that contains a cycle of n many vertices such that every vertex of the cycle is connected with another vertex known as the hub, see Figure 1.

Theorem 2.9 *W_n is Tribonacci cordial.*

Proof We start by identifying the vertices of W_n as $V(W_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\}$ where v_i's are the vertices of the cycle in a clockwise manner and v is the hub of the cycle. Such as those shown in Figure 1.

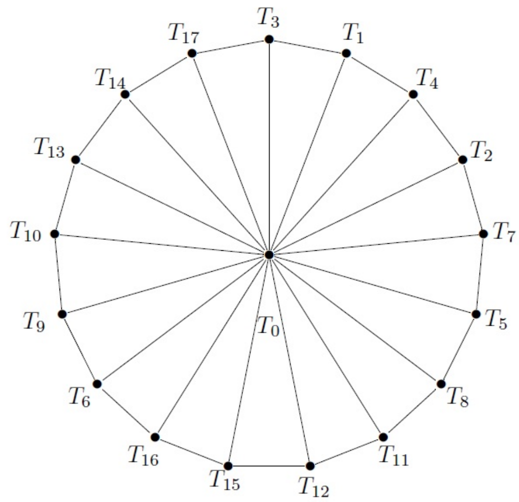

Figure 1. Tribonacci cordial labeling for W_{17} graph

Now, for $n = 4p + q$, where $0 \leqslant q \leqslant 3$, we define

$$p_1 = \begin{cases} p, & \text{if } q = 3; \\ p-1, & \text{otherwise} \end{cases}$$

and $p_2 = p_1 + 2$. Assigns Tribonacci labeling to vertices of the wheel graph W_n as follows:

For $1 \leqslant i \leqslant 2p_1$

$$f(v_i) = \begin{cases} T_{i-2}, & \text{if } i \equiv 0 \pmod 4; \\ T_{i+2}, & \text{if } i \equiv 1 \pmod 4; \\ T_{i-1}, & \text{if } i \equiv 2 \pmod 4; \\ T_{i+1}, & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

For the vertices $2p_1 < i \leqslant n$, define $f(v_i) = T_k$, where,

$$k = \begin{cases} n + (i-3p) - 2\left\lceil \frac{3p-i-2}{2} \right\rceil + 1 - q, & \text{for } 2p_1 + 1 \leqslant i \leqslant 2p_1 + p_2, \text{ and } q = 0,1,2,3; \\ i - 2\left\lfloor \frac{n-i-q}{2} \right\rfloor - 2, & \text{for } 2p_1 + p_2 < i \leqslant n, \text{ and } q = 0,1,2; \\ i - 2\left\lfloor \frac{n-i-2}{2} \right\rfloor - 2, & \text{for } 2p_1 + p_2 < i \leqslant n, \text{ and } q = 3. \end{cases}$$

Then, a simple calculation ensures the validity of the cordiality of the labeling. □

A shell graph is defined as a cycle C_n with $(n-3)$ chords sharing a common vertex, called the apex, see Figure 2. Shell graphs are denoted as $C_{(n,n-3)}$. The vertices of $C_{(n,n-3)}$ are denoted by $\{v_1, v_2, \cdots, v_n\}$, v_1 as the apex.

Theorem 2.10 *A shell graph $C_{(n,n-3)}$ is Tribonacci cordial for an integer $n \geq 4$.*

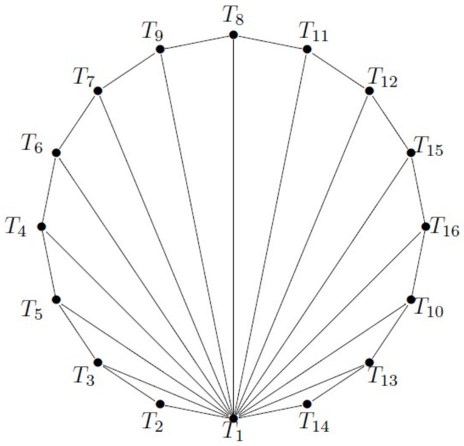

Figure 2. Tribonacci labeling for $C_{16,13}$ graph

Proof Let us rewrite n as $4p + q$, where $0 \leqslant q \leqslant 3$ and $0 \leqslant p \leqslant \lceil n/4 \rceil$. We define p_1 and p_2 as:

$$p_1 = \begin{cases} p, & \text{if } q = 0, 1; \\ p + 1, & \text{if } q = 2, 3 \end{cases}$$

and $p_2 = 2p + 1 - p_1$. The function defined below assigns Tribonacci numbers to the vertices of the wheel graphs. Let $f(v_1) = T_1$, and

$$f(v_i) = \begin{cases} T_{i+1}, & \text{if } i \equiv 0 \pmod 4; \\ T_{i-1}, & \text{if } i \equiv 1 \pmod 4; \\ T_i, & \text{otherwise} \end{cases}$$

for $2 \leqslant i \leqslant 2p_1$. For the vertices $2p_1 < i \leqslant n$, let $f(v_i) = T_k$ where

$$k = \begin{cases} n + (i - 3p) - 2\left\lceil \frac{3p-i-2}{2} \right\rceil - 3 - q, & \text{for } 2p_1 + 1 \leqslant i \leqslant 2p_1 + p_2, \text{ and } q = 0, 1, 2, 3; \\ j - 2\left\lfloor \frac{n-i-q}{2} \right\rfloor - 2, & \text{for } 2p_1 + p_2 < i \leqslant n, \text{ and } q = 0, 1, 2; \\ j - 2\left\lfloor \frac{n-i-2}{2} \right\rfloor - 2, & \text{for } 2p_1 + p_2 < i \leqslant n, \text{ and } q = 3. \end{cases}$$

A simple calculation generates $e_1 = n = e_0$, which ensures that the labeling on W_n is Tribonacci cordial for all integers n. □

The generalized friendship graph $F_{m,n}$ (see Figure 3 for an example) is a collection of n many cycles C_m, meeting at a common vertex. Clearly we can refer friendship graph F_n as $F_{3,n}$. For convenience, let us call the common vertex v the apex, and each cycle a blade of the graph. Consider $V(F_{m,n}) = \{v\} \cup \{v_{i,1}, v_{i,2}, \cdots, v_{i,m}\}_{i=1}^{n}$ where the i^{th} blade is $C_i = \{vv_{i,1}v_{i,2} \cdots v_{i,m}, v\}$. It is evident from the definition that the cardinality of the vertex and edge sets are given by $n(m-1) + 1$ and mn respectively.

Lemma 2.11 *If* $m \equiv 1 \pmod 2$ *and* $n \equiv 2 \pmod 4$*, then* $F_{m,n}$ *is not Tribonacci cordial.*

Proof Note that in any blade, any combination of even and odd Tribonacci labeling on the vertices of the cycle C_m including the apex vertex, generates only even values for e_1. Thus $e_1 \equiv 0 \pmod 2$. Now when $n = 4k + 2$ and $m = 2p + 1$, for some integer $k, p \geqslant 0$, $|E(F_{m,n})| = 8pk + 4(p + k) + 2$. In order to generate Tribonacci cordial labeling for $F_{m,n}$, e_1 must be $4pk + 2(p + k) + 1$, which clearly contradicts that e_1 is even. □

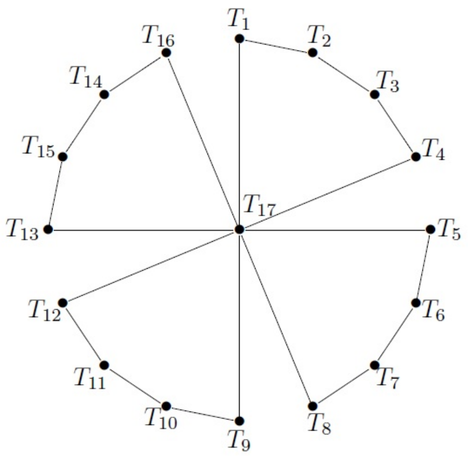

Figure 3. Tribonacci cordial labeling for Friendship graph $F_{5,4}$

Now we investigate whether $F_{m,n}$ is Tribonacci cordial for the remaining values, i.e., for $m \not\equiv 0 \pmod 2$ or $n \not\equiv 2 \pmod 4$. First, we look into the case when $m \equiv 1 \pmod 2$ and $n \not\equiv 2 \pmod 4$ to obtain the following.

Lemma 2.12 $F_{3,n}$ *is Tribonacci cordial if* $n \not\equiv 2 \pmod 4$.

Proof For convenience, we redefine vertex v_{ij} as v_k, where $k = 2(i - 1) + j$, and

$$p = \begin{cases} n/2, & \text{if } n \equiv 0 \pmod 4; \\ (n+1)/2, & \text{if } n \equiv 1 \pmod 4; \\ (n-1)/2, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Now we provide the function that assigns Tribonacci numbers to the vertices of the $F_{3,n}$. For $n \equiv 0 \pmod 4$, we label v with T_{2n+1}, and label the rest of the vertices as follow:

$$f(v_k) = \begin{cases} T_{k+1}, & \text{for } 1 \leqslant k \leqslant p \text{ and } \ k \equiv 2 \pmod 4; \\ T_{k-1}, & \text{for } 1 \leqslant k \leqslant p \text{ and } \ k \equiv 3 \pmod 4; \\ T_k, & \text{otherwise .} \end{cases}$$

For $n \equiv 1, 3 \pmod 4$,

$$f(v_k) = \begin{cases} T_{k-1}, & \text{for } 1 \leqslant k \leqslant p; \\ T_k, & \text{for } p \leqslant k \leqslant 2n \end{cases}$$

and finally $f(v) = T_{n-1}$. □

Lemma 2.13 *For $n \not\equiv 2 \pmod 4$, $F_{5,n}$ is Tribonacci cordial.*

Proof Let us define

$$p = \begin{cases} 3n/4, & \text{if } n \equiv 0 \pmod 4; \\ (3n+1)/4, & \text{if } n \equiv 1 \pmod 4; \\ (3n-1)/4, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

The following function assigns Tribonacci labeling to the vertices of the graph $F_{5,n}$. For $1 \leqslant k \leqslant p$, $f(v_k) = T_k$, and for $p+1 \leqslant k \leqslant 4n$ we define

$$f(v_k) = \begin{cases} T_{k+1}, & \text{if } k \equiv 2 \pmod 4; \\ T_{k-1}, & \text{if } k \equiv 3 \pmod 4; \\ T_k, & \text{otherwise} \end{cases}$$

and finally $f(v) = T_{4n+1}$. It can be easily observed that for each blade, excluding the common vertex v, we label two categories of labelings in the following order: viz. p many even-even-odd-odd, and $n-p$ many even-odd-even-odd. Hence the value of $\tilde{e}$ in each blade, of former kind is -1, and 3 on the later (as the common vertex is being labeled by an odd Tribonacci number). Consequently we obtain

$$\tilde{e} = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 4; \\ -1, & \text{if } n \equiv 1 \pmod 4; \\ 1, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$ □

We believe that the following holds.

Conjecture 2.14 *$F_{2k+1,n}$ is Tribonacci cordial for $n \not\equiv 2 \pmod 4$ and any positive integer $k \geqslant 3$.*

Theorem 2.15 *$F_{m,n}$ is Tribonacci cordial if $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 2$ or $m \equiv 0 \pmod 4$ for any n.*

Proof Let $f : V(F_{m,n}) \to \{T_0, T_1, \cdots, T_{n(m-1)+1}\}$ such that $f(v_i) = T_{(m-1)n+2-i}$ and $f(v) = T_1$. It is clear for $m \equiv 0 \pmod 4$, each blade the vertices is getting the labels odd-even-even-odd-$\cdots$-even-odd, which with the odd labeled common vertex result in $\tilde{e} = 0$. Thus we have a Tribonacci cordial graph for any choice of n.

Now for $m \equiv 2 \pmod 4$ where $n \equiv 0 \pmod 2$, we can make the observation that the values of $\tilde{e}$ in each blade are to be $\{-2, 2, 2, -2, -2, 2, 2, -2 \cdots\}$. This clearly implies that that we have a Tribonacci cordial labeling ($\tilde{e} = 0$) when the number of blades are even, i.e., $n \equiv 0 \pmod 2$). This completes the proof. □

Theorem 2.16 *$K_{m,n}$ is Tribonacci cordial for all m, n.*

Proof Let us denote the vertices of $K_{m,n}$ by $\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$. In order to assign Tribonacci labeling to the graph $K_{m,n}$ we consider the following cases.

Case 1. Assume that at least one of m, n is even. Without loss of generality we consider m to be even and use the following labeling: $f(u_i) = T_{i-1}, f(v_i) = T_{m+i-1}$. Note that, $m/2$ even and $m/2$ odd Tribonacci labels are used on one side, which result in $\tilde{e} = 0$, for any assignment of labels on the other side.

Case 2. Consider the case when both m and n are odd. Let $m = 2k_1 + 1$ and $n = 2k_2 + 1$, following the same pattern of labeling as the previous case. It can be easily noted that there are either k_1 and $k_2 + 1$, or $k_1 + 1$ and k_2 even labels used. The former case yields $\tilde{e} = k_1k_2 + (k_1 + 1)(k_2 + 1) - k_1(k_2 + 1) - (k_1 + 1)k_2 = 1$, whereas, later case implies $\tilde{e} = -1$, ensuring the cordiality of the labeling in either one. □

Bistar $B_{m,n}$ is the graph obtained by joining the apex vertices of two copies of star, viz. $K_{1,m}$ and $K_{1,n}$, by an edge. We identify the vertex set as $\{u_i : 1 \leqslant i \leqslant m\} \cup \{v_i : 1 \leqslant i \leqslant n\} \cup \{u, v\}$ where u, v are the apex vertices and u_i, v_i are the pendant vertices connected with u and v respectively. The vertex and edge set cardinalities are given by $m + n + 2$ and $m + n + 1$ respectively.

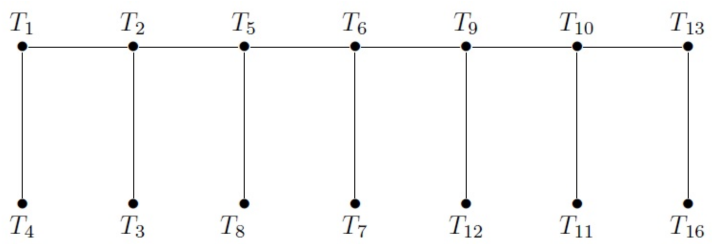

Figure 4. Tribonacci cordial labeling for $P_{\odot}K_1$ graph

Theorem 2.17 *Bistar graph $B_{m,n}$ are Tribonacci cordial.*

Proof Define the function $f : (B_{m,n}) \to \{T_0, T_1, \cdots, T_{m+n+2}\}$ as $f(u_i) = T_{i+1}, f(v_i) = T_{m+i+1}$ which assigns Tribonacci numbers to all pendant vertices. In order to label the apex vertices u, v, consider the following cases.

Case 1. Let at least one of m, n is even. Without loss of generality we can assume that m is even. Label $f(u) = T_0$ and $f(v) = T_1$ if $m+n \equiv 1 \pmod 4$, and vice-versa if $m+n \not\equiv 1 \pmod 4$. It is clear that $m/2$ many vertices are labeled with even (and odd) Tribonacci labels. Now if $m + n \equiv 1 \pmod 4$ then $(n - 1)/2$ many even and $(n + 1)/2$ many odd Tribonacci labels are used on other side. Thus $\tilde{e} = m/2 + 1 + (n - 1)/2 - m/2 - (n + 1)/2 = 0$.

On the other hand if $m+n \not\equiv 1 \pmod 4$, then there are $\lceil n/2 \rceil$ many even and $\lfloor n/2 \rfloor$ many odd Tribonacci labels used on the side of n pendant vertices. Hence $|\tilde{e}| = |m/2 - m/2 + 1 + \lfloor n/2 \rfloor - \lceil n/2 \rceil| \leqslant 1$.

Case 2. Consider both m, n are odd. Once again without loss of generality we will consider three cases: $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$; $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$; $m \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$. For the first case $f(u) = T_1$ and $f(v) = T_0$ and vice-versa on last the two cases. It is very similar to the previous case to verify that this assigns a Tribonacci labeling to the graph $B_{m,n}$. □

Theorem 2.18 *Complete binary trees are Tribonacci cordial.*

Proof Let T be the complete binary tree. We denote the vertices as $\{v_i^j : 1 \leqslant i \leqslant 2^j, 0 \leqslant j \leqslant \ell\}$, where ℓ denoted the levels, and v_i^ℓ is connected to $v_{2i-1}^{\ell+1}$ and $v_{2i}^{\ell+1}$. We define the labeling $f : V(T) \to \{T_0, T_1, \cdots, T_{2n}\}$ as $f(v_i^j) = T_{2^j+i-1}$. If we denote e_k as the edge connecting the vertices the parent $v_{\lceil i/2 \rceil}^{j-1}$ and the child vertex v_i^j, where $k = 2^j + i - 2$, then we observe the following:

$$f^*(e_k) = \begin{cases} 1 \pmod 2, & \text{if } k \equiv 1, 4, 6, 7 \pmod 8; \\ 0 \pmod 2, & \text{if } k \equiv 0, 2, 3, 5 \pmod 8. \end{cases}$$

As the maximum possible value for k is $2(2^\ell - 1)$, we have $e_1 = e_0 = 2^\ell - 1$, which implies $\tilde{e} = 0$. Thus T is Tribonacci cordial. □

Theorem 2.19 P_n^2 *is Tribonacci cordial.*

Proof Let us identify the vertices of the graph as $V(P_n^2) = \{v_1, v_2, \cdots, v_n\}$. We define labeling $f : V(P_n^2) \to \{T_0, T_1, T_2, \cdots, T_n\}$ as follows:

$$f(v_i) = \begin{cases} T_{i-2}, & \text{if } q = 0; \\ T_{i-1}, & \text{if } q = 1, 2; \\ T_i, & \text{if } q = 3, \end{cases}$$

where, $i = 4p + q$. Clearly this labeling generates $e_1 = n$ and $e_0 = n - 1$, hence $\tilde{e} = 1$, which proves the theorem. □

Theorem 2.20 C_n^2 *is Tribonacci cordial only for integers* $n \equiv 0 \pmod 2$.

Proof First we note that, for any two Tribonacci labeling $f : V(C_n^2) \to \{T_0, T_1, \cdots, T_n\}$ and $g : V(C_n^2) \to \{T_0, T_1, \cdots, T_n\}$, $\tilde{e}_f - \tilde{e}_g \equiv 0 \pmod 4$. We omit the proof as it very similar to Theorem 2.2. Now observe that when n is odd, i.e., $n = 2k + 1$, $|E(C_n^2)| = 4k + 2$, thus $\tilde{e}_{C_n^2} \equiv 2 \pmod 4$. For the case n being even, let $v_1, v_2, \cdots, v_n$ be the vertices of the graph C_n^2. We define the following function that assign labelings to the vertices.

$$f(v_i) = \begin{cases} T_{i-2}, & \text{if } q = 0; \\ T_{i-1}, & \text{if } q = 1, 2; \\ T_i, & \text{if } q = 3, \end{cases}$$

where, $i = 4p + q$. Clearly this labeling generates $e_1 = e_0 = n$, hence $\tilde{e} = 0$, which proves the theorem. □

Theorem 2.21 *For an integer $n \geq 1$, the ladder graph $L_n = P_n \times P_2$ is Tribonacci cordial.*

Proof We start by identifying the vertex set $V(L_n) = \{u_i, v_i : 1 \leqslant i \leqslant n\}$, and the edge set $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{u_iv_i : 1 \leqslant i \leqslant n\}$. The cardinality of the vertex and edges sets are $2n$ and $3n-2$ respectively. As a base case consider $n = 2$. Note that $L_2 = C_4$, which is Tribonacci cordial by Theorem 2.4. Choose the labeling $f(u_1) = T_0, f(u_2) = T_3, f(v_i) = T_i$ for $i = 1, 2$, which gives us $\tilde{e}_{L_2} = 0$. Now generate a Tribonacci cordial labeling for L_3, by assigning $f(u_3) = T_4$ and $f(v_3) = T_5$, which lead us to $\tilde{e}_{L_3} = -1$. Finally, for $n = 4$, label (and relabel) as follows: $f(u_3) = T_6, f(u_4) = T_7$ and $f(u_4) = T_4$, we again get $\tilde{e}_{L_4} = 0$.

Continue labeling the vertices of L_{n+1} from L_n in this fashion, where u_{n+1} and v_{n+1} get the even and odd labeling from $\{T_{2n}, T_{2n+1}\}$ respectively if $n \not\equiv 3 \pmod 4$. When $n \equiv 3 \pmod 4$ in addition to the previous step, we switch the labeling of u_n and u_{n+1}.

Now we show that the above style of labeling always provides Tribonacci cordial labeling for L_n. It can be easily verified that $\tilde{e}_{n+1} = \tilde{e}_n - 1$ for $n \equiv 1, 2 \pmod 4$ and $\tilde{e}_{n+1} = \tilde{e}_n + 1$ for $n \equiv 0, 3 \pmod 4$. Thus L_n is Tribonacci cordial for all n. □

Theorem 2.22 *For any integer $n \geq 1$, the comb graphs $P_n \odot K_1$ are Tribonacci cordial.*

Proof Identify the vertices of a comb graph as $V(G) = \{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ where $u_0, u_1, \cdots, u_n$ are the vertices of the path P_n and $v_1, v_2, \cdots, v_n$ are the attached pendant vertices (see Figure 4). The following function assign Tribonacci cordial labeling to the vertices of the comb graph.

$$f(u_i) = \begin{cases} T_{2i-1}, & \text{if } i \equiv 1 \pmod 2; \\ T_{2i-2}, & \text{if } i \equiv 0 \pmod 2, \end{cases}$$

$$f(v_i) = \begin{cases} T_{2i}, & \text{if } i \equiv 0 \pmod 2 \text{ and } n \equiv 0 \pmod 2; \\ T_{2i+1}, & \text{if } i \equiv 1 \pmod 2 \text{ and } n \equiv 0 \pmod 2; \\ T_{2i-1}, & \text{if } i \equiv 0 \pmod 2 \text{ and } n \equiv 1 \pmod 2; \\ T_{2i+2}, & \text{if } i \equiv 1 \pmod 2 \text{ and } n \equiv 1 \pmod 2 \end{cases}$$

for $i \in \{1, 2, \cdots, n\}$. It can be easily verified that $f^*(u_iu_{i+1}) = 0$ and $f^*(u_iv_i) = 1$ for all i. Thus, for any value of n, $\tilde{e} = e_1 - e_0 = n - (n-1) = 1$. □

Theorem 2.23 *$C_n \odot K_1$ are Tribonacci cordial.*

Proof In the graph $G = C_n \odot K_1$, let $V = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$, where v_i's denote the vertices of C_n and u_i's represent the pendant vertices adjacent to v_i's. The following function assigns Tribonacci cordial labeling to the graph.

$$f(v_i) = \begin{cases} T_{2i-1}, & \text{if } i \equiv 0 \pmod 2; \\ T_{2i-2}, & \text{if } i \equiv 1 \pmod 2, \end{cases}$$

$$f(u_i) = \begin{cases} T_{2i-2}, & \text{if } i \equiv 0 \pmod 2; \\ T_{2i-1}, & \text{if } i \equiv 1 \pmod 2). \end{cases}$$

This completes the proof. □

Theorem 2.24 *Petersen graph is Tribonacci cordial.*

Proof Let v_1, v_2, v_3, v_4, v_5 be the internal vertices and $v_6, v_7, v_8, v_9, v_{10}$ be the external vertices of Petersen graph such that each v_i is adjacent to v_{i+5}, $1 \leqslant i \leqslant 5$. The following function assigns Tribonacci cordial labeling to the vertices of the Petersen graph.

$$f(v_i) = \begin{cases} T_0, & \text{if } i = 10; \\ T_{i+2}, & \text{if } i = 3, 4; \\ T_{i-2}, & \text{if } i = 5, 6; \\ T_i, & \text{otherwise.} \end{cases}$$

This completes the proof. □

Definition 2.25 *the ring sum $G_1 \oplus G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \oplus G_2 = (V_1 \cup V_2, (E1 \cup E_2) - (E_1 \cap E_2))$.*

We now construct a family of graph $G = C_n \oplus K_{1,n}$ where the apex vertex of the star graph $K_{1,n}$ is a member of the graph C_n.

Theorem 2.26 *The graph $C_n \oplus K_{1,n}$ is Tribonacci cordial, $n \geqslant 3$.*

Proof Start by identifying the vertices of the graph G. Let $V(G) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, \cdots, v_n\}$ is the vertex set of the C_n and $V_2 = \{u = v_1, u_1, u_2, \cdots, u_n\}$ be the vertex set of $K_{1,n}$. Assign Tribonacci numbers to the vertices in the following fashion: $f(v_i) = T_{i-1}$ for $i \in \{1, 2, \cdots, n\}$. However in order to label the pendant vertices, if $n \not\equiv 2 \pmod 4$ then $f(u_i) = T_{n+i-1}$ for $i \in \{1, 2, \cdots, n\}$, whereas

$$f(u_i) = \begin{cases} T_{n+i-1}, & \text{for } 1 \leqslant i \leqslant n-2; \\ T_{n+i}, & \text{for } n-1 \leqslant i \leqslant n \end{cases}$$

if $n \equiv 2 \pmod 4$. It can be easily verified that the above function provides Tribonacci cordial labeling for the graph $C_n \oplus K_{1,n}$. □

The jellyfish graph $J(n, n)$ is obtained from a 4-cycle u_1, u_2, u_3, u_4 by joining u_1 and u_3 with a chord and appending $\{v_i : 1 \leqslant j \leqslant n\}$ and $\{w_i : 1 \leqslant j \leqslant n\}$ pendent vertices from u_4 and u_2 respectively. Consider the following vertex labeling function: $f(u_i) = T_{i-1}$ for$1 \leqslant i \leqslant 4$, $f(w_i) = T_{2i+2}$ for $1 \leqslant i \leqslant n$ and $f(v_i) = T_{2i+n+2}$ for $1 \leqslant i \leqslant n$. It can be easily verified that this function provides a Tribonacci cordial labeling for $J(n, n)$, thus we have the following result.

Theorem 2.27 *$J(n,n)$ is Tribonacci cordial for all n.*

Definition 2.28 *the Joint sum $G_1 \boxplus G_2$ of two graphs G_1 and G_2 is the graph obtained by connecting a vertex of G_1 with a vertex of G_2.*

Theorem 2.29 *The joint sum graph $C_m \boxplus P_n$ is Tribonacci cordial for all values of m, n.*

Proof Define the graph $C_m \boxplus P_n$ as follows:

$V(C_m \boxplus P_n) = \{u_1, u_2, \cdots, u_m\} \cup \{v_1, v_2, \cdots, v_n\}$, where $\{u_1, u_2, \cdots, u_m\}$ are the vertices of the cycle and $\{v_1, v_2, \cdots, v_n\}$ are the vertices of the path. The edges of the cycle are given by $u_i u_{i+1(\text{mod } m)}$ for $1 \leqslant i \leqslant m$ and $v_j v_{j+1}$ for $1 \leqslant i \leqslant n-1$ construct the edges of the path and the edge $u_m v_1$ connects the cycle and the path, see Figure 5 for details.

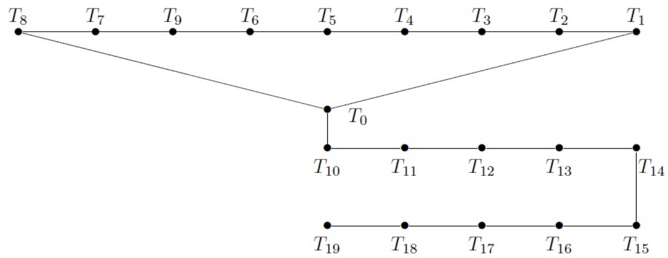

Figure 5. Tribonacci cordial labeling of the joint graph $C_{10} \boxplus P_{10}$

The following vertex labeling generates the Tribonacci Cordial Labeling $f(v_j) = T_{j+m-1}$ for $1 \leqslant j \leqslant n$.

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant m-1; \\ T_0, & \text{for } j = m \end{cases}$$

for all m when $m \not\equiv 2(\text{mod } 4)$. Otherwise $m \equiv 2(\text{mod } 4)$

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant m-4; \\ T_{m-1}, & \text{for } j = m-3; \\ T_{i-1}, & \text{for } m-2 \leqslant i \leqslant m-1; \\ T_0, & \text{for } j = m. \end{cases}$$

This completes the proof. □

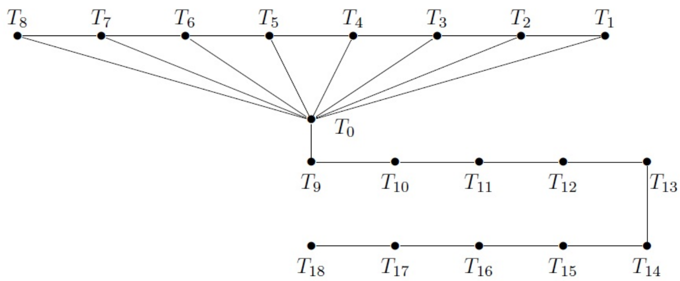

Figure 6. Tribonacci cordial labeling of the joint graph $F_8 \boxplus P_{10}$

Theorem 2.30 *The graph $F_m \boxplus P_n$ is Tribonacci cordial for all m, n.*

Proof We begin identifying the vertices of the graph $F_m \boxplus P_n$. Let $V(F_m \boxplus P_n) = V_1 \cup V_2$, where $V_1 = \{u\} \cup \{u_1, u_2, \cdots, u_m\}$ is the vertex set of F_m (u is hub vertex, which is connected with $u_i, 1 \leqslant i \leqslant m$) and $V_2 = \{v_1, v_2, \cdots, v_n\}$ be the vertex set of P_n, see Figure 6.

First, label the vertices of the path in following manner:

$f(u) = T_0$ and $f(v_i) = T_{i+m}$ for all $1 \leqslant i \leqslant n$, and for the cycle when $m \equiv 1(\text{mod } 4)$ we have

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant m-3; \\ T_m, & \text{for } j = m-2; \\ T_{i-1}, & \text{for } m-1 \leqslant i \leqslant m. \end{cases}$$

Otherwise, when $m \not\equiv 1(\text{mod } 4)$, $f(u_i) = T_i$ for all $1 \leqslant i \leqslant m$. The vertex labeling defined above generates Tribonacci cordial labeling for the given graph. □

Theorem 2.31 *The graph $C_m \boxplus K_{1,n}$ is Tribonacci cordial for all values of m, n.*

Proof The vertex set of $C_m \boxplus K_{1,n}$ is defined as $V(C_m \boxplus K_{1,n}) = \{u_1, u_2, \cdots, u_m\} \cup \{v, v_1, v_2, \cdots, v_n\}$, where $\{u_1, u_2, \cdots, u_m\}$ are the vertices of the cycle and $\{v_1, v_2, \cdots, v_n\}$ are the pendant vertices of the star graph whereas v is the apex vertex star garph $K_{1,n}$. The edge set of $C_m \boxplus K_{1,n}$ is given by the collection of edges $u_i u_{i+1(\text{mod } m)}$ for $1 \leqslant i \leqslant m$, $u_m v$ and vv_j for $1 \leqslant i \leqslant n$.

The following vertex label assignment ensures that $C_m \boxplus K_{1,n}$ satisfies all the conditions of being a Tribonacci cordial graph.

$f(v_j) = T_{j+m}$ for $1 \leqslant j \leqslant n$, $f(v) = T_m$.

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant m-1; \\ T_0, & \text{for } j = m \end{cases}$$

for all m when $m \not\equiv 2(\text{mod } 4)$. Otherwise for $m \equiv 2(\text{mod } 4)$ $f(v) = T_m$ and

$$f(v_j) = \begin{cases} T_4, & \text{for } j = 1; \\ T_{j+m}, & \text{for } 2 \leqslant j \leqslant n \end{cases}$$

and

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant 3; \\ T_{i+1}, & \text{for } 4 \leqslant i \leqslant m-1; \\ T_0, & \text{for } i = m. \end{cases}$$

This completes the proof. □

Definition 2.32 *The ring sum $G_1 \oplus G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \oplus G_2 = (V_1 \bigcup V_2, (E_1 \bigcup E_2 - E_1 \bigcap E_2))$.*

Theorem 2.33 *The graph $C_m \oplus K_{1,n}$ is Tribonacci cordial for all values of m, n.*

Proof Define the graph $C_m \oplus K_{1,n}$ constituting the vertex set as $V(C_m \oplus K_{1,n}) = \{u_1, u_2, \cdots, u_m\} \cup \{v_1, v_2, \cdots, v_n\}$, where $\{u_1, u_2, \cdots, u_m\}$ and $\{v_1, v_2, \cdots, v_n\}$ are respectively the vertices of the cycle and the star graph. The edges of the cycle are given by $u_i u_{i+1(\text{mod } m)}$ for $1 \leqslant i \leqslant m$ and $u_m v_j$ for $1 \leqslant i \leqslant n$ define the edges of the star graph, see Figure 7.

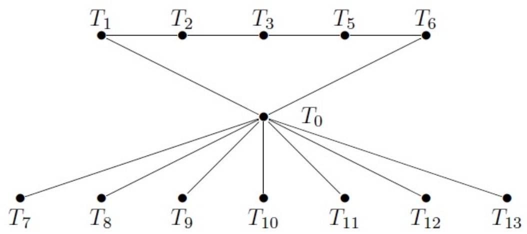

Figure 7. Tribonacci cordial labeling of the ring sum graph $C_6 \oplus K_{1,7}$

The following vertex labeling generates the Tribonacci Cordial Labeling for the given graph. $f(v_j) = T_{j+m-1}$ for $1 \leqslant j \leqslant n$, and

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant m-1; \\ T_0, & \text{for } i = m \end{cases}$$

for all m when $m \not\equiv 2(\text{mod } 4)$. Otherwise (i.e. when $m \equiv 2 \pmod 4$)

$$f(v_j) = \begin{cases} T_4, & \text{for } j = 1; \\ T_{j+m-1}, & \text{for } 2 \leqslant j \leqslant n \end{cases}$$

and

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant 3; \\ T_{i+1}, & \text{for } 4 \leqslant i \leqslant m-1; \\ T_0, & \text{for } j = m. \end{cases}$$

This completes the proof. □

References

[1] I.Cahit, Cordial graphs: A weaker version of graceful and harmonious graphs, *Ars Combinatoria*, 23 (1987), 201–207.

[2] J.A.Gallian, A dynamic survey of graph labelling, *Elect. J. Combinatorics* (2019).

[3] S.Mitra and S.Bhoumik, Fibonacci cordial labeling of some special families of graphs, *Annals of Pure and Applied Mathematics*, 21-2 (2020) 135–140.

[4] A.H.Rokad, Fibonacci cordial labeling of some special graphs, *Oriental Journal of Computer Science and Technology*, 10(4) (2017) 824–828.

[5] A.H.Rokad and G.V.Ghodasara, Fibonacci cordial labeling of some special graphs, *Annals of Pure and Applied Mathematics*, 11(1) (2016) 133–144.