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Coupled Fixed Point Results via New Coupled Implicit Contractive Condition in S-Metric Spaces

G. S. Saluja

(H.N. 3/1005, Geeta Nagar, Raipur, Raipur - 492001 (C.G.), India)

E-mail: saluja1963@gmail.com

Abstract: In this paper, we prove a coupled and a common coupled fixed point theorems via newly proposed coupled implicit contractive condition in the framework of S-metric spaces. Also, we give some corollaries of the main result. Furthermore, an illustrative example and an application to the Fredholm integral equation are given. Our results extend, generalize and enrich several results from the existing literature.

Key Words: Coupled fixed point, common coupled fixed point, Smarandachely multifixed point, new coupled implicit contractive condition, S-metric space.

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§1. Introduction

In the history of fixed point theorem and applications, it was Stefan Banach ([3]) who introduced the concept of contraction condition in the year 1922 for obtaining fixed as well as unique fixed point. After Banach's remarkable result a number of researchers started working in the area of developing fixed point theory in the lines of Banach. One can refer B. E. Rhoades [Trans. Am. Math. Soc. 266 (1977)] for various types of contraction as well as non-contraction type conditions which facilitates the contraction map to get unique fixed point.

Later on, authors tried to replace the metric space by generalized metric space. In the literature, there are many generalizations of the metric space exists. One of such generalizations is the generalized metric space or S-metric space. In 2012, Sedghi et al. [33] introduced the notion of a S-metric space which is different from other spaces and proved fixed point theorems in such spaces. They also gave some examples of a S-metric space which shows that the S-metric space is different from other spaces. They built up some topological properties in such spaces and proved some fixed point theorems in the setting of S-metric spaces. In 2014, Sedghi and Dung [34] proved new generalized fixed point theorems such as the *Ćirić's* fixed point result on an S-metric space. In 2017, Özgür and Tas obtained the generalizations of the Banach's contraction principle and the Rhoades's condition on an S-metric spaces (see [20, 21] for more details). In 2017, the same authors proved in [22] some fixed point theorems using new contractive conditions of integral type on a complete S-metric spaces and give some illustrative

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examples to verify the obtained results. They also give an application to Fredholm integral equation. Many results which were proved earlier in metric space are valid in the framework of S-metric spaces.

On the other hand, Bhashkar and Lakshmikantham in [24] introduced the concepts of coupled fixed points and mixed monotone property and illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Later on these results were further extended and generalized by Ciric and Lakshmikantham [7] to coupled coincidence and coupled common fixed point results for nonlinear contractions in partially ordered metric spaces (see, also [5], [6], [16], [18], [19]).

In the year 2011, Aydi [2] proved some coupled fixed point theorems for various contractive type conditions in the setting of partial metric spaces and give some corollaries of the established results. Recently, Saluja [28] proved some common fixed point theorems for contractive type conditions in the setup of complex valued S-metric spaces. Very recently, Saluja [29] proved some coupled fixed point results for contractive type conditions in the framework of complex partial metric spaces (see, also [31]).

In 2018, Popa and Patricu [24] gave the concept of implicit functions in S-metric space which includes most of the existing literature's well-known contractions besides several new ones (see, also [23, 26, 27, 30]).

Recently, Kim [15] studied existence and uniqueness of coupled fixed point for a family of self-mappings satisfying a new coupled implicit relation in the framework of Hilbert space and also proved well-posedness of a coupled fixed point problem.

Motivated by the results and notions mentioned above, the purpose of this paper is to investigate coupled and common coupled fixed point theorems in the setting of S-metric spaces by using newly proposed coupled implicit contractive condition of three variables. Also we give some corollaries of the main results. Furthermore, we give an application to the Fredholm integral equation. Our results extend, generalize and enrich several results from the existing literature.

§2. Preliminaries

Sedghi et al. [33] gave an interesting generalization of D-metric space to S-metric space by formulating its properties as follows:

Definition 2.1([33]) *Let $\mathcal{X}$ be a nonempty set and let $\mathcal{S}\colon \mathcal{X}^3 \to [0,\infty)$ be a function satisfying the following conditions for all x, y, z, $a \in \mathcal{X}$:*

($S1$) $0 < \mathcal{S}(x,y,z)$ for all $x,y,z \in \mathcal{X}$ with $x \neq y \neq z$;

($S2$) $\mathcal{S}(x,y,z) = 0$ if and only if $x = y = z$;

($S3$) $\mathcal{S}(x,y,z) \leq \mathcal{S}(x,x,a) + \mathcal{S}(y,y,a) + \mathcal{S}(z,z,a)$.

Then, the function $\mathcal{S}$ is called an $\mathcal{S}$-metric on $\mathcal{X}$ and the pair $(\mathcal{X},\mathcal{S})$ is called an $\mathcal{S}$-metric space.

Example 2.2 ([33]) Let $\mathcal{X} = \mathbb{R}^n$ and $\|\cdot\|$ a norm on $\mathcal{X}$. Then,

(1) $\mathcal{S}(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an $\mathcal{S}$-metric on $\mathcal{X}$.

(2) $\mathcal{S}(x,y,z) = \|x-z\| + \|y-z\|$ is an $\mathcal{S}$-metric on $\mathcal{X}$.

Example 2.3 ([34]) Let $\mathcal{X} = \mathbb{R}$ be the real line. Then $\mathcal{S}(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an $\mathcal{S}$-metric on $\mathcal{X}$. This $\mathcal{S}$-metric on $\mathcal{X}$ is called the usual $\mathcal{S}$-metric on $\mathcal{X}$.

Example 2.4 ([13]) Let $\mathcal{X}$ be a non-empty set and d be an ordinary metric on $\mathcal{X}$. Then $\mathcal{S}(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in \mathbb{R}$ is an $\mathcal{S}$-metric on $\mathcal{X}$.

Example 2.5 ([35]) Let $\mathcal{X}$ be a non-empty set and d_1, d_2 be two ordinary metrics on $\mathcal{X}$. Then $\mathcal{S}(x,y,z) = d_1(x,z) + d_2(y,z)$ for all $x,y,z \in \mathcal{X}$ is an $\mathcal{S}$-metric on $\mathcal{X}$.

Sedghi et al. [33] proved that the D-metric space is the S-metric space, but in general the converse is not true.

We give some vivid illustrative examples on S-metric spaces as follows:

Example 2.6 ([33]) Let $\mathcal{X} = \mathbb{R}^2$, d is an ordinary metric on $\mathcal{X}$, therefore $\mathcal{S}(x,y,z)$ $= d(x,y) + d(x,z) + d(y,z)$ for all $x,y,z \in \mathbb{R}$ is a $\mathcal{S}$-metric on $\mathcal{X}$. If we connect the points x,y,z by a line, we have a triangle and if we choose a point a mediating this triangle then the inequality $\mathcal{S}(x,y,z) = \mathcal{S}(x,x,a) + \mathcal{S}(y,y,a) + \mathcal{S}(z,z,a)$ holds.

Example 2.7 ([33]) Let $\mathcal{X} = \mathbb{R}$, then $\mathcal{S}(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an $\mathcal{S}$-metric on $\mathcal{X}$. Define a self-map $\mathcal{F}$ on $\mathcal{X}$ by: $\mathcal{F}(x) = \frac{1}{2}\, sinx$, we have

$$\begin{aligned}
\mathcal{S}(\mathcal{F}x, \mathcal{F}x, \mathcal{F}y) &= \left|\frac{1}{2}(sinx - siny)\right| + \left|\frac{1}{2}(sinx - siny)\right| \\
&\leq \left|\frac{1}{2}(x-y)\right| + \left|\frac{1}{2}(x-y)\right| \\
&\leq \frac{1}{2}\big(|(x-y)| + |(x-y)|\big) \\
&= \frac{1}{2}\mathcal{S}(x,x,y)
\end{aligned}$$

for every $x,y \in \mathcal{X}$.

Definition 2.8 *Let $(\mathcal{X},\mathcal{S})$ be an $\mathcal{S}$-metric space. For $r > 0$ and $x \in \mathcal{X}$, we define respectively the open ball $\mathcal{B}_{\mathcal{S}}(x,r)$ and closed ball $\mathcal{B}_{\mathcal{S}}[x,r]$ with center x and radius r as follows:*

$$\mathcal{B}_{\mathcal{S}}(x,r) = \{y \in \mathcal{X} : \mathcal{S}(y,y,x) < r\},$$

$$\mathcal{B}_{\mathcal{S}}[x,r] = \{y \in \mathcal{X} : \mathcal{S}(y,y,x) \leq r\}.$$

Example 2.9 ([34]) Let $\mathcal{X} = \mathbb{R}$. Denote by $\mathcal{S}(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{B}_{\mathcal{S}}(1,2) &= \{y \in \mathbb{R} : \mathcal{S}(y,y,1) < 2\} = \{y \in \mathbb{R} : |y-1| < 1\} \\
&= \{y \in \mathbb{R} : 0 < y < 2\} = (0,2),
\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_{\mathcal{S}}[2,4] &= \{y \in \mathbb{R} : \mathcal{S}(y,y,2) \leq 4\} = \{y \in \mathbb{R} : |y-2| \leq 2\} \\ &= \{y \in \mathbb{R} : 0 \leq y \leq 4\} = [0,4].\end{aligned}$$

Definition 2.10([33], [34]) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space and $A \subset \mathcal{X}$.*

(a_1) *The subset A is said to be an open subset of $\mathcal{X}$, if for every $x \in A$ there exists $r > 0$ such that $\mathcal{B}_{\mathcal{S}}(x,r) \subset A$.*

(a_2) *A sequence $\{y_n\}$ in $\mathcal{X}$ converges to $y \in \mathcal{X}$ if $\mathcal{S}(y_n, y_n, y) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\mathcal{S}(y_n, y_n, y) < \varepsilon$. We denote this by $\lim_{n\to\infty} y_n = y$ or $y_n \to y$ as $n \to \infty$.*

(a_3) *A sequence $\{y_n\}$ in X is called a Cauchy sequence if $\mathcal{S}(y_n, y_n, y_m) \to 0$ as $n, m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $\mathcal{S}(y_n, y_n, y_m) < \varepsilon$.*

(a_4) *The $\mathcal{S}$-metric space $(\mathcal{X}, \mathcal{S})$ is called complete if every Cauchy sequence in $\mathcal{X}$ is convergent in $\mathcal{X}$.*

(a_5) *Let τ be the set of all $A \subset \mathcal{X}$ with the property that for each $x \in A$ and there exists $r > 0$ such that $\mathcal{B}_{\mathcal{S}}(x,r) \subset A$. Then τ is a topology on $\mathcal{X}$ (induced by the $\mathcal{S}$-metric space).*

(a_6) *A nonempty subset A of $\mathcal{X}$ is $\mathcal{S}$-closed if closure of A coincides with A.*

Definition 2.11([33]) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space. A mapping $\mathcal{R}\colon \mathcal{X} \to \mathcal{X}$ is said to be a contraction if there exists a constant $0 \leq b < 1$ such that*

$$\mathcal{S}(\mathcal{R}x, \mathcal{R}y, \mathcal{R}z) \leq b\,\mathcal{S}(x,y,z), \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$.

Remark 2.12 If the $\mathcal{S}$-metric space $(\mathcal{X}, \mathcal{S})$ is complete then the mapping defined as above has a unique fixed point (see [33], Theorem 3.1).

Definition 2.13([33]) *Let $(\mathcal{X}, \mathcal{S})$ and $(\mathcal{Y}, \mathcal{S}')$ be two $\mathcal{S}$-metric spaces. A function $W\colon \mathcal{X} \to \mathcal{Y}$ is said to be continuous at a point $x_0 \in \mathcal{X}$ if for every sequence $\{x_n\}$ in $\mathcal{X}$ with $\mathcal{S}(x_n, x_n, x_0) \to 0$, $\mathcal{S}'(W(x_n), W(x_n), W(x_0)) \to 0$ as $n \to \infty$. We say that W is continuous on $\mathcal{X}$ if W is continuous at every point $x_0 \in \mathcal{X}$.*

Definition 2.14 *Let $\mathcal{X}$ be a non-empty set and let $M, N\colon \mathcal{X} \to \mathcal{X}$ be two self-mappings of $\mathcal{X}$. Then a point $\alpha \in \mathcal{X}$ is called a*

(1) *fixed point of operator M if $M(\alpha) = \alpha$;*
(2) *common fixed point of M and N if $M(\alpha) = N(\alpha) = \alpha$;*
(3) *Smarandachely multifixed point of M and N if $M(\alpha) = \alpha$ or $N(\alpha) = \alpha$.*

Definition 2.15([1]) *Let M and N be single valued self-mappings on a set $\mathcal{X}$. If $z = M(\alpha) = N(\alpha)$ for some $\alpha \in \mathcal{X}$, then α is called a coincidence point of M and N, and z is called a point of coincidence of M and N. We denote the coincidence point of M and N by $C(M,N)$, that is, $C(M,N) = \{\alpha \in \mathcal{X} : M(\alpha) = N(\alpha)\}$.*

Definition 2.16([11]) *Let M and N be single valued self-mappings on a set $\mathcal{X}$. Mappings M and N are said to be commuting if $MNu = NMu$ for all $u \in \mathcal{X}$.*

Example 2.17 Let $\mathcal{X} = [0, \frac{3}{4}]$ and define $M, N\colon \mathcal{X} \to \mathcal{X}$ defined by $M(x) = \frac{x^3}{4}$ and $N(x) = x^4$ for all $x, y \in \mathcal{X}$. Then the mappings M and N have two coincidence points 0 and $\frac{1}{4}$. Clearly, they commute at 0 but not at $\frac{1}{4}$.

Definition 2.18([12]) *Let M and N be single valued self-mappings on a set $\mathcal{X}$. Mappings M and N are said to be weakly compatible if they commute at their coincidence points, i.e., if $M\alpha = N\alpha$ for some $\alpha \in \mathcal{X}$ implies $MN\alpha = NM\alpha$.*

Definition 2.19([11]) *An element $(x, y) \in \mathcal{X} \times \mathcal{X}$ is called*

(b_1) A coupled fixed point ([2]) of the mapping $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ if $F(x, y) = x$ and $F(y, x) = y$;

(b_2) A coupled coincidence point ([7]) of the mappings $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $Q\colon \mathcal{X} \to \mathcal{X}$ if $F(x, y) = Q(x)$ and $F(y, x) = Q(y)$;

(b_3) a common coupled fixed point ([14]) of the mappings $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $Q\colon \mathcal{X} \to \mathcal{X}$ if $x = F(x, y) = Q(x)$ and $y = F(y, x) = Q(y)$.

Example 2.20 Let $\mathcal{X} = [0, +\infty)$ and $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ defined by $F(x, y) = \frac{x+y}{3}$ for all $x, y \in \mathcal{X}$. One can easily see that F has a unique coupled fixed point $(0, 0)$.

Example 2.21 Let $\mathcal{X} = [0, +\infty)$ and $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be defined by $F(x, y) = \frac{x+y}{2}$ for all $x, y \in \mathcal{X}$. Then we see that F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

Lemma 2.22([33], Lemma 2.5) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space. Then, we have $\mathcal{S}(x, x, y) = \mathcal{S}(y, y, x)$ for all $x, y \in \mathcal{X}$.*

Lemma 2.23([33], Lemma 2.12) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $\mathcal{S}(x_n, x_n, y_n) \to \mathcal{S}(x, x, y)$ as $n \to \infty$.*

Lemma 2.24([9], Lemma 8) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space and A is a nonempty subset of $\mathcal{X}$. Then A is said to be $\mathcal{S}$-closed if and only if for any sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to \infty$, then $x \in A$.*

Lemma 2.25([33]) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space. If $r > 0$ and $x \in \mathcal{X}$, then the ball $\mathcal{B}_{\mathcal{S}}(x, r)$ is an open subset of $\mathcal{X}$.*

Lemma 2.26([34]) *The limit of a sequence $\{x_n\}$ in an $\mathcal{S}$-metric space $(\mathcal{X}, \mathcal{S})$ is unique.*

Lemma 2.27([33]) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space. Then any convergent sequence $\{x_n\}$ in $\mathcal{X}$ is Cauchy.*

In the following lemma we see the relationship between a metric and $\mathcal{S}$-metric.

Lemma 2.28([10]) *Let $(\mathcal{X}, d)$ be a metric space. Then the following properties are satisfied:*

(c_1) *$\mathcal{S}_d(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in \mathcal{X}$ is an $\mathcal{S}$-metric on $\mathcal{X}$.*

(c_2) *$x_n \to x$ in $(\mathcal{X}, d)$ if and only if $x_n \to x$ in $(\mathcal{X}, \mathcal{S}_d)$.*

(c_3) *$\{x_n\}$ is Cauchy in $(\mathcal{X}, d)$ if and only if $\{x_n\}$ is Cauchy in $(\mathcal{X}, \mathcal{S}_d)$.*

(c_4) *$(\mathcal{X}, d)$ is complete if and only if $(\mathcal{X}, \mathcal{S}_d)$ is complete.*

We call the function $\mathcal{S}_d$ defined in Lemma 2.28 (c_1) as the $\mathcal{S}$-metric generated by the metric d. It can be found an example of an $\mathcal{S}$-metric which is not generated by any metric in [10, 21].

Example 2.29 Let $\mathcal{X} = \mathbb{R}$ and the function $\mathcal{S}\colon \mathcal{X}^3 \to [0,\infty)$ be defined as

$$\mathcal{S}(x,y,z) = |x-z| + |x+z-2y|,$$

for all $x,y,z \in \mathbb{R}$. Then the function $\mathcal{S}$ is an $\mathcal{S}$-metric on $\mathcal{X}$ and $(\mathcal{X}, \mathcal{S})$ is an $\mathcal{S}$-metric space. Now, we prove that there does not exists any metric d such that $\mathcal{S} = \mathcal{S}_d$. On the contrary, suppose that there exists a metric d such that

$$\mathcal{S}(x,y,z) = d(x,z) + d(y,z),$$

for all $x,y,z \in \mathbb{R}$. Hence, we obtain

$$\mathcal{S}(x,x,z) = 2d(x,z) = 2|x-z|,$$

and

$$d(x,z) = |x-z|.$$

Similarly, we get

$$\mathcal{S}(y,y,z) = 2d(y,z) = 2|y-z|,$$

and

$$d(y,z) = |y-z|,$$

for all $x,y,z \in \mathbb{R}$. Hence, we have

$$|x-z| + |x+z-2y| = |x-z| + |y-z|,$$

which is a contradiction. Therefore, $\mathcal{S} \neq \mathcal{S}_d$ and $(\mathbb{R}, \mathcal{S})$ is a complete $\mathcal{S}$-metric space.

Now, we introduce a new coupled implicit relation.

Definition 2.30 *Let $\mathbb{R}_+$ (where $\mathbb{R}_+ = [0,\infty)$) be the set of all nonnegative real numbers, Ω be the class of all continuous real valued functions $\omega\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ non-decreasing in the third argument satisfying the following conditions: for $x,y,\alpha,\beta > 0$,*

($CIR1$) $x \leq \omega\Big(\frac{\alpha+\beta}{2}, \frac{x+\alpha}{2}, \frac{y+\beta}{2}\Big)$ *and* $y \leq \omega\Big(\frac{\alpha+\beta}{2}, \frac{y+\beta}{2}, \frac{x+\alpha}{2}\Big)$ *or*

$(CIR2)$ $x \leq \omega\Big(\frac{\alpha+\beta}{2}, 0, 0\Big)$ *and* $y \leq \omega\Big(\frac{\alpha+\beta}{2}, 0, 0\Big)$ *or*

$(CIR3)$ $x \leq \omega\Big(0, \frac{\alpha}{2}, \frac{\beta}{2}\Big)$ *and* $y \leq \omega\Big(0, \frac{\beta}{2}, \frac{\alpha}{2}\Big)$,

there exists a real number $0 < h < 1$ *such that* $x + y \leq h(\alpha + \beta)$.

§3. Main Results

In this section, we prove some unique coupled fixed point and common coupled fixed point theorems for newly proposed coupled implicit contractive condition in the framework of S-metric spaces.

Theorem 3.1 *Let* $(\mathcal{X}, \mathcal{S})$ *be a complete* $\mathcal{S}$*-metric space. Let* $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ *be a mapping satisfying the following contractive condition: for all* $x, y, u, v, z, w \in \mathcal{X}$*:*

$$\begin{aligned}
&\mathcal{S}(F(x,y), F(u,v), F(z,w)) \\
&\leq \omega\Big(\frac{\mathcal{S}(x,u,z) + \mathcal{S}(y,v,w)}{2}, \frac{\mathcal{S}(F(x,y), F(x,y), x) + \mathcal{S}(F(z,w), F(z,w), z)}{2}, \\
&\quad \frac{\mathcal{S}(F(v,u), F(v,u), v) + \mathcal{S}(F(w,z), F(w,z), w)}{2}\Big), \qquad (3.1)
\end{aligned}$$

where $\omega \in \Omega$*. If* F *is continuous, then* F *has a unique coupled point in* $\mathcal{X}$*.*

Proof Choose $x_0, y_0 \in X$. Set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Repeating this process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. Then, from equations (3.1), using $(S2)$ and Lemma 2.22, we have

$$\begin{aligned}
\mathcal{S}(x_n, x_n, x_{n+1}) &= \mathcal{S}(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\leq \omega\Big(\frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(y_{n-1}, y_{n-1}, y_n)}{2}, \\
&\quad \frac{\mathcal{S}(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), x_{n-1}) + \mathcal{S}(F(x_n, y_n), F(x_n, y_n), x_n)}{2}, \\
&\quad \frac{\mathcal{S}(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), y_{n-1}) + \mathcal{S}(F(y_n, x_n), F(y_n, x_n), y_n)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(y_{n-1}, y_{n-1}, y_n)}{2}, \\
&\quad \frac{\mathcal{S}(x_n, x_n, x_{n-1}) + \mathcal{S}(x_{n+1}, x_{n+1}, x_n)}{2}, \\
&\quad \frac{\mathcal{S}(y_n, y_n, y_{n-1}) + \mathcal{S}(y_{n+1}, y_{n+1}, y_n)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(y_{n-1}, y_{n-1}, y_n)}{2}, \\
&\quad \frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})}{2}, \\
&\quad \frac{\mathcal{S}(y_{n-1}, y_{n-1}, y_n) + \mathcal{S}(y_n, y_n, y_{n+1})}{2}\Big). \qquad (3.2)
\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}\mathcal{S}(y_n, y_n, y_{n+1}) &= \mathcal{S}(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq \omega\Big(\frac{\mathcal{S}(y_{n-1}, y_{n-1}, y_n) + \mathcal{S}(x_{n-1}, x_{n-1}, x_n)}{2}, \\ &\qquad \frac{\mathcal{S}(y_{n-1}, y_{n-1}, y_n) + \mathcal{S}(y_n, y_n, y_{n+1})}{2}, \\ &\qquad \frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})}{2}\Big). \qquad (3.3)\end{aligned}$$

Let

$$\begin{aligned} x &= \mathcal{S}(x_n, x_n, x_{n+1}), \quad y = \mathcal{S}(y_n, y_n, y_{n+1}), \\ \alpha &= \mathcal{S}(x_{n-1}, x_{n-1}, x_n), \quad \beta = \mathcal{S}(y_{n-1}, y_{n-1}, y_n).\end{aligned}$$

Hence from Definition 2.30 ($CIR1$), there exists $0 < h < 1$ such that

$$\begin{aligned}&\mathcal{S}(x_n, x_n, x_{n+1}) + \mathcal{S}(y_n, y_n, y_{n+1}) \\ &\quad \leq h\,[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(y_{n-1}, y_{n-1}, y_n)]. \qquad (3.4)\end{aligned}$$

Set $\mathcal{D}_n = \mathcal{S}(x_n, x_n, x_{n+1}) + \mathcal{S}(y_n, y_n, y_{n+1})$. Then equation (3.4) implies that

$$\mathcal{D}_n \leq h\,\mathcal{D}_{n-1}. \qquad (3.5)$$

Consequently, for each $n \in \mathbb{N}$, we have

$$\mathcal{D}_n \leq h\,\mathcal{D}_{n-1} \leq h^2\,\mathcal{D}_{n-2} \leq \cdots \leq h^n\,\mathcal{D}_0. \qquad (3.6)$$

If $\mathcal{D}_0 = 0$, then $\mathcal{S}(x_0, x_0, x_1) + \mathcal{S}(y_0, y_0, y_1) = 0$. Hence, by condition ($S2$), we get $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$. Thus, (x_0, y_0) is a coupled fixed point of F. Now, we assume that $\mathcal{D}_0 > 0$. For each $m > n$, where $n, m \in \mathbb{N}$, and using ($S3$), we have

$$\begin{aligned}&\mathcal{S}(x_n, x_n, x_m) + \mathcal{S}(y_n, y_n, y_m) \\ &\leq 2\mathcal{S}(x_n, x_n, x_{n+1}) + \mathcal{S}(x_m, x_m, x_{n+1}) \\ &\qquad\qquad + 2\mathcal{S}(y_n, y_n, y_{n+1}) + \mathcal{S}(y_m, y_m, y_{n+1}) \\ &= 2(\mathcal{S}(x_n, x_n, x_{n+1}) + \mathcal{S}(y_n, y_n, y_{n+1})) \\ &\qquad\qquad + \mathcal{S}(x_m, x_m, x_{n+1}) + \mathcal{S}(y_m, y_m, y_{n+1}) \\ &\leq \cdots\cdots\cdots\cdots \\ &\leq 2\Big(\mathcal{D}_n + \mathcal{D}_{n+1} + \cdots + \mathcal{D}_{m-1} + \mathcal{D}_m\Big) \\ &\leq 2(h^n + h^{n+1} + \cdots + h^{m-1} + h^m)\mathcal{D}_0 \\ &\leq 2h^n(1 + h + h^2 + \dots)\mathcal{D}_0 \\ &\leq \Big(\frac{2h^n}{1-h}\Big)\mathcal{D}_0 \to 0 \text{ as } n \to \infty,\end{aligned}$$

since $0 < h < 1$. Thus, $\{x_n\}$ and $\{y_n\}$ are $\mathcal{S}$-Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete, we get $\{x_n\}$ and $\{y_n\}$ are $\mathcal{S}$-convergent to some $c \in \mathcal{X}$ and $d \in \mathcal{X}$ respectively, that is, $\lim_{n\to\infty} x_n = c$ and $\lim_{n\to\infty} y_n = d$. Since F is continuous, then we have

$$\begin{aligned} c &= \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) \\ &= F\Big(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\Big) = F(c,d), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} d &= \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) \\ &= F\Big(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n\Big) = F(d,c). \end{aligned} \tag{3.8}$$

This shows that (c,d) is a coupled fixed point of F.

Now, we show the uniqueness of the coupled fixed point. Assume that (c_1, d_1) is another coupled fixed point of F such that $(c,d) \neq (c_1, d_1)$. Then from equation (3.1), using $(S2)$ and Lemma 2.22, we have

$$\begin{aligned} \mathcal{S}(c,c,c_1) &= \mathcal{S}(F(c,d), F(c,d), F(c_1,d_1)) \\ &\leq \omega\Big(\frac{\mathcal{S}(c,c,c_1)+\mathcal{S}(d,d,d_1)}{2}, \\ &\qquad \frac{\mathcal{S}(F(c,d),F(c,d),c)+\mathcal{S}(F(c_1,d_1),F(c_1,d_1),c_1)}{2}, \\ &\qquad \frac{\mathcal{S}(F(d,c),F(d,c),d)+\mathcal{S}(F(d_1,c_1),F(d_1,c_1),d_1)}{2}\Big) \\ &= \omega\Big(\frac{\mathcal{S}(c,c,c_1)+\mathcal{S}(d,d,d_1)}{2}, \frac{\mathcal{S}(c,c,c)+\mathcal{S}(c_1,c_1,c_1)}{2}, \\ &\qquad \frac{\mathcal{S}(d,d,d)+\mathcal{S}(d_1,d_1,d_1)}{2}\Big) \\ &= \omega\Big(\frac{\mathcal{S}(c,c,c_1)+\mathcal{S}(d,d,d_1)}{2}, 0, 0\Big). \end{aligned} \tag{3.9}$$

Similarly, we have

$$\begin{aligned} \mathcal{S}(d,d,d_1) &= \mathcal{S}(F(d,c), F(d,c), F(d_1,c_1)) \\ &\leq \omega\Big(\frac{\mathcal{S}(d,d,d_1)+\mathcal{S}(c,c,c_1)}{2}, \\ &\qquad \frac{\mathcal{S}(F(d,c),F(d,c),d)+\mathcal{S}(F(d_1,c_1),F(d_1,c_1),d_1)}{2}, \\ &\qquad \frac{\mathcal{S}(F(c,d),F(c,d),c)+\mathcal{S}(F(c_1,d_1),F(c_1,d_1),c_1)}{2}\Big) \\ &= \omega\Big(\frac{\mathcal{S}(d,d,d_1)+\mathcal{S}(c,c,c_1)}{2}, \frac{\mathcal{S}(d,d,d)+\mathcal{S}(d_1,d_1,d_1)}{2}, \\ &\qquad \frac{\mathcal{S}(c,c,c)+\mathcal{S}(c_1,c_1,c_1)}{2}\Big) \\ &= \omega\Big(\frac{\mathcal{S}(d,d,d_1)+\mathcal{S}(c,c,c_1)}{2}, 0, 0\Big) \\ &= \omega\Big(\frac{\mathcal{S}(c,c,c_1)+\mathcal{S}(d,d,d_1)}{2}, 0, 0\Big). \end{aligned} \tag{3.10}$$

Hence from Definition 2.30 ($CIR2$), there exists $0 < h < 1$ such that

$$\mathcal{S}(c,c,c_1) + \mathcal{S}(d,d,d_1) \leq h\,[\mathcal{S}(c,c,c_1) + \mathcal{S}(d,d,d_1)], \tag{3.11}$$

which is a contradiction, since $0 < h < 1$. Hence, we conclude that $\mathcal{S}(c,c,c_1) + \mathcal{S}(d,d,d_1) = 0$, that is, $\mathcal{S}(c,c,c_1) = 0$ and $\mathcal{S}(d,d,d_1) = 0$. Hence by condition ($S2$), we have $c = c_1$ and $d = d_1$. This shows that the coupled fixed point of F is unique. This completes the proof. □

Remark 3.2 If $x = y$ and $\alpha = \beta$ in Definition 2.30, the coupled implicit relation conditions restricted follows implicit relation conditions:

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers, Ω be the class of all continuous real valued functions $\omega\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ non-decreasing in the third argument and satisfying the following conditions: for $x, \alpha > 0$,

($IR1$) $x \leq \omega\Big(\alpha, \frac{x+\alpha}{2}, \frac{x+\alpha}{2}\Big)$, or

($IR2$) $x \leq \omega\big(\alpha, 0, 0\big)$, or

($IR3$) $x \leq \omega\Big(0, \frac{\alpha}{2}, \frac{\alpha}{2}\Big)$,

then, there exists a real number $0 < h < 1$ such that $x \leq h\,\alpha$.

Theorem 3.2 *Let $(\mathcal{X}, \mathcal{S})$ be a complete $\mathcal{S}$-metric space. Suppose that the mappings $F\colon \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ and $A\colon \mathcal{X} \to \mathcal{X}$ satisfy the the following contractive condition: for all $x, y, u, v, z, w \in \mathcal{X}$:*

$$\begin{aligned}\mathcal{S}(F(x,y), F(u,v), F(z,w)) \leq \omega\Big(&\frac{\mathcal{S}(Ax,Au,Az) + \mathcal{S}(Ay,Av,Aw)}{2},\\ &\frac{\mathcal{S}(F(x,y),F(x,y),Ax) + \mathcal{S}(F(z,w),F(z,w),Az)}{2},\\ &\frac{\mathcal{S}(F(v,u),F(v,u),Av) + \mathcal{S}(F(w,z),F(w,z),Aw)}{2}\Big),\end{aligned} \tag{3.12}$$

where $\omega \in \Omega$. Assume that F and A satisfy the following conditions:

(i) $F(\mathcal{X} \times \mathcal{X}) \subseteq A(\mathcal{X})$;

(ii) $A(\mathcal{X})$ is complete, and

(iii) A is continuous and commutes with F.

Then, F and A have a coupled coincidence point in $\mathcal{X}$. Moreover, if F and A are weakly compatible, then F and A have a unique common coupled fixed point.

Proof Let $x_0, y_0 \in \mathcal{X}$. Since $F(\mathcal{X} \times \mathcal{X}) \subseteq A(\mathcal{X})$, we can choose $x_1, y_1 \in \mathcal{X}$ such that $Ax_1 = F(x_0, y_0)$ and $Ay_1 = F(y_0, x_0)$. Again since $F(\mathcal{X}\times\mathcal{X}) \subseteq A(\mathcal{X})$, we can choose $x_2, y_2 \in \mathcal{X}$ such that $Ax_2 = F(x_1, y_1)$ and $Ay_2 = F(y_1, x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that $Ax_{n+1} = F(x_n, y_n)$ and $Ay_{n+1} = F(y_n, x_n)$. For

$n \in \mathbb{N}$, by equation (3.12), and using Lemma 2.22, we have

$$\begin{aligned}
&\mathcal{S}(Ax_n, Ax_n, Ax_{n+1}) = \mathcal{S}(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\leq \omega\Big(\frac{\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n)}{2}, \\
&\quad \frac{\mathcal{S}(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), Ax_{n-1}) + \mathcal{S}(F(x_n, y_n), F(x_n, y_n), Ax_n)}{2}, \\
&\quad \frac{\mathcal{S}(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), Ay_{n-1}) + \mathcal{S}(F(y_n, x_n), F(y_n, x_n), Ay_n)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n)}{2}, \\
&\quad \frac{\mathcal{S}(Ax_n, Ax_n, Ax_{n-1}) + \mathcal{S}(Ax_{n+1}, Ax_{n+1}, Ax_n)}{2}, \\
&\quad \frac{\mathcal{S}(Ay_n, Ay_n, Ay_{n-1}) + \mathcal{S}(Ay_{n+1}, Ay_{n+1}, Ay_n)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n)}{2}, \\
&\quad \frac{\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ax_n, Ax_n, Ax_{n+1})}{2}, \\
&\quad \frac{\mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n) + \mathcal{S}(Ay_n, Ay_n, Ay_{n+1})}{2}\Big). \qquad (3.13)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\mathcal{S}(Ay_n, Ay_n, Ay_{n+1}) = \mathcal{S}(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\
&\leq \omega\Big(\frac{\mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n) + \mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n)}{2}, \\
&\quad \frac{\mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n) + \mathcal{S}(Ay_n, Ay_n, Ay_{n+1})}{2}, \\
&\quad \frac{\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ax_n, Ax_n, Ax_{n+1})}{2}\Big). \qquad (3.14)
\end{aligned}$$

Let $x = \mathcal{S}(Ax_n, Ax_n, Ax_{n+1})$, $y = \mathcal{S}(Ay_n, Ay_n, Ay_{n+1})$, $\alpha = \mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n)$ and $\beta = \mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n)$. Hence from Definition 2.30 ($CIR1$), there exists $0 < h < 1$ such that

$$\begin{aligned}
&\mathcal{S}(Ax_n, Ax_n, Ax_{n+1}) + \mathcal{S}(Ay_n, Ay_n, Ay_{n+1}) \\
&\leq h\,[\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n)]. \qquad (3.15)
\end{aligned}$$

Set $\mathcal{K}_n = \mathcal{S}(Ax_n, Ax_n, Ax_{n+1}) + \mathcal{S}(Ay_n, Ay_n, Ay_{n+1})$. Then, equation (3.15) implies that

$$\mathcal{K}_n \leq h\,\mathcal{K}_{n-1}. \qquad (3.16)$$

Consequently, for each $n \in \mathbb{N}$, we have

$$\mathcal{K}_n \leq h\,\mathcal{K}_{n-1} \leq h^2\,\mathcal{K}_{n-2} \leq \cdots \leq h^n\,\mathcal{K}_0. \qquad (3.17)$$

If $\mathcal{K}_0 = 0$, then $\mathcal{S}(Ax_0, Ax_0, Ax_1) + \mathcal{S}(Ay_0, Ay_0, Ay_1) = 0$. Hence, by condition ($S2$), we get $Ax_0 = Ax_1 = F(x_0, y_0)$ and $Ay_0 = Ay_1 = F(y_0, x_0)$. Thus, (Ax_0, Ay_0) is a coupled fixed point of F and A. Now, we assume that $\mathcal{K}_0 > 0$. For each $m > n$, where $n, m \in \mathbb{N}$, and using $(S3)$, we have

$$\begin{aligned}
&\mathcal{S}(Ax_n, Ax_n, Ax_m) + \mathcal{S}(Ay_n, Ay_n, Ay_m) \\
&\le 2\mathcal{S}(Ax_n, Ax_n, Ax_{n+1}) + \mathcal{S}(Ax_m, Ax_m, Ax_{n+1}) \\
&\quad +2\mathcal{S}(Ay_n, Ay_n, Ay_{n+1}) + \mathcal{S}(Ay_m, Ay_m, Ay_{n+1}) \\
&= 2(\mathcal{S}(Ax_n, Ax_n, Ax_{n+1}) + \mathcal{S}(Ay_n, Ay_n, Ay_{n+1})) \\
&\quad +\mathcal{S}(Ax_m, Ax_m, Ax_{n+1}) + \mathcal{S}(Ay_m, Ay_m, Ay_{n+1}) \\
&\le \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\le 2\Big(\mathcal{K}_n + \mathcal{K}_{n+1} + \cdots + \mathcal{K}_{m-1} + \mathcal{K}_m\Big) \\
&\le 2(h^n + h^{n+1} + \cdots + h^{m-1} + h^m)\mathcal{K}_0 \\
&\le 2h^n(1 + h + h^2 + \dots)\mathcal{K}_0 \\
&\le \Big(\frac{2h^n}{1-h}\Big)\mathcal{K}_0 \to 0 \text{ as } n \to \infty,
\end{aligned}$$

since $0 < h < 1$. Thus, $\{Ax_n\}$ and $\{Ay_n\}$ are $\mathcal{S}$-Cauchy sequence in $A(\mathcal{X})$. Since $A(\mathcal{X})$ is complete, we get $\{Ax_n\}$ and $\{Ay_n\}$ are $\mathcal{S}$-convergent to some $p_1 \in \mathcal{X}$ and $p_2 \in \mathcal{X}$ respectively. Since A is continuous, we have $\{AAx_n\}$ is $\mathcal{S}$-convergent to Ap_1 and $\{AAy_n\}$ is $\mathcal{S}$-convergent to Ap_2. Also, since A and F are commute, we have

$$AAx_{n+1} = A(F(x_n, y_n)) = F(Ax_n, Ay_n),$$

and

$$AAy_{n+1} = A(F(y_n, x_n)) = F(Ay_n, Ax_n).$$

Therefore,

$$\begin{aligned}
\mathcal{S}(AAx_{n+1}, AAx_{n+1}, F(p_1, p_2)) &= \mathcal{S}(F(Ax_n, Ay_n), F(Ax_n, Ay_n), F(p_1, p_2)) \\
&\le \omega\Big(\frac{\mathcal{S}(AAx_n, AAx_n, Ap_1) + \mathcal{S}(AAy_n, AAy_n, Ap_2)}{2}, \\
&\qquad \frac{\mathcal{S}(F(Ax_n, Ay_n), F(Ax_n, Ay_n), AAx_n) + \mathcal{S}(F(p_1, p_2), F(p_1, p_2), Ap_1)}{2}, \\
&\qquad \frac{\mathcal{S}(F(Ay_n, Ax_n), F(Ay_n, Ax_n), AAy_n) + \mathcal{S}(F(p_2, p_1), F(p_2, p_1), Ap_2)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(AAx_n, AAx_n, Ap_1) + \mathcal{S}(AAy_n, AAy_n, Ap_2)}{2}, \\
&\qquad \frac{\mathcal{S}(AAx_{n+1}, AAx_{n+1}, AAx_n) + \mathcal{S}(F(p_1, p_2), F(p_1, p_2), Ap_1)}{2}, \\
&\qquad \frac{\mathcal{S}(AAy_{n+1}, AAy_{n+1}, AAy_n) + \mathcal{S}(F(p_2, p_1), F(p_2, p_1), Ap_2)}{2}\Big). \qquad (3.18)
\end{aligned}$$

Passing to the limit as $n \to \infty$ in equation (3.18), using Lemmas 2.22, 2.23 and the condition $(S2)$, we obtain

$$\begin{aligned}\mathcal{S}(Ap_1, Ap_1, F(p_1,p_2)) &\leq \omega\Big(\frac{\mathcal{S}(Ap_1, Ap_1, Ap_1) + \mathcal{S}(Ap_2, Ap_2, Ap_2)}{2},\\ &\qquad \frac{\mathcal{S}(Ap_1, Ap_1, Ap_1) + \mathcal{S}(Ap_1, Ap_1, F(p_1,p_2))}{2},\\ &\qquad \frac{\mathcal{S}(Ap_2, Ap_2, Ap_2) + \mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))}{2}\Big)\\ &= \omega\Big(0, \frac{\mathcal{S}(Ap_1, Ap_1, F(p_1,p_2))}{2}, \frac{\mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))}{2}\Big). \end{aligned} \tag{3.19}$$

Similarly, we have

$$\begin{aligned}&\mathcal{S}(AAy_{n+1}, AAy_{n+1}, F(p_2,p_1)) = \mathcal{S}(F(Ay_n, Ax_n), F(Ay_n, Ax_n), F(p_2,p_1))\\ &\leq \omega\Big(\frac{\mathcal{S}(AAy_n, AAy_n, Ap_2) + \mathcal{S}(AAx_n, AAx_n, Ap_1)}{2},\\ &\qquad \frac{\mathcal{S}(F(Ay_n, Ax_n), F(Ay_n, Ax_n), AAy_n) + \mathcal{S}(F(p_2,p_1), F(p_2,p_1), Ap_2)}{2},\\ &\qquad \frac{\mathcal{S}(F(Ax_n, Ay_n), F(Ax_n, Ay_n), AAx_n) + \mathcal{S}(F(p_1,p_2), F(p_1,p_2), Ap_1)}{2}\Big)\\ &= \omega\Big(\frac{\mathcal{S}(AAy_n, AAy_n, Ap_2) + \mathcal{S}(AAx_n, AAx_n, Ap_1)}{2},\\ &\qquad \frac{\mathcal{S}(AAy_{n+1}, AAy_{n+1}, AAy_n) + \mathcal{S}(F(p_2,p_1), F(p_2,p_1), Ap_2)}{2},\\ &\qquad \frac{\mathcal{S}(AAx_{n+1}, AAx_{n+1}, AAx_n) + \mathcal{S}(F(p_1,p_2), F(p_1,p_2), Ap_1)}{2}\Big). \end{aligned} \tag{3.20}$$

Passing to the limit as $n \to \infty$ in equation (3.20), using Lemmas 2.22, 2.23 and the condition $(S2)$, we obtain

$$\begin{aligned}\mathcal{S}(Ap_2, Ap_2, F(p_2,p_1)) &\leq \omega\Big(\frac{\mathcal{S}(Ap_2, Ap_2, Ap_2) + \mathcal{S}(Ap_1, Ap_1, Ap_1)}{2},\\ &\qquad \frac{\mathcal{S}(Ap_2, Ap_2, Ap_2) + \mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))}{2},\\ &\qquad \frac{\mathcal{S}(Ap_1, Ap_1, Ap_1) + \mathcal{S}(Ap_1, Ap_1, F(p_1,p_2))}{2}\Big)\\ &= \omega\Big(0, \frac{\mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))}{2}, \frac{\mathcal{S}(Ap_1, Ap_1, F(p_1,p_2))}{2}\Big). \end{aligned} \tag{3.21}$$

Hence from Definition 2.30 $(CIR3)$, there exists $0 < h < 1$ such that

$$\begin{aligned}&\mathcal{S}(Ap_1, Ap_1, F(p_1,p_2)) + \mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))\\ &\qquad \leq h\,[\mathcal{S}(Ap_1, Ap_1, F(p_1,p_2)) + \mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))],\end{aligned}$$

which is a contradiction, since $0 < h < 1$. Hence, we conclude that

$$\mathcal{S}(Ap_1, Ap_1, F(p_1, p_2)) + \mathcal{S}(Ap_2, Ap_2, F(p_2, p_1)) = 0,$$

that is, $\mathcal{S}(Ap_1, Ap_1, F(p_1, p_2)) = 0$ and $\mathcal{S}(Ap_2, Ap_2, F(p_2, p_1)) = 0$. Hence $Ap_1 = F(p_1, p_2)$ and $Ap_2 = F(p_2, p_1)$. Thus (Ap_1, Ap_2) is a coupled coincidence point of the mappings F and A. Since the pair (F, A) is weakly compatible, so by weak compatibility of the mappings F and A, we have

$$A(F(p_1, p_2)) = F(Ap_1, Ap_2) = Ap_1 \text{ and } A(F(p_2, p_1)) = F(Ap_2, Ap_1) = Ap_2. \tag{3.22}$$

Hence (Ap_1, Ap_2) is a common coupled fixed point of F and A.

Now, we show the uniqueness of the common coupled fixed point of F and A. Assume that (Ar_1, Ar_2) is another common coupled fixed point of F and A with $Ap_1 \neq Ar_1$ and $Ap_2 \neq Ar_2$, that is, $(Ap_1, Ap_2) \neq (Ar_1, Ar_2)$. Then by using equation (3.12), using Lemma 2.22 and the condition $(S2)$, we have

$$\begin{aligned}
&S(Ap_1, Ap_1, Ar_1) = S(F(p_1, p_2), F(p_1, p_2), F(r_1, r_2)) \\
&\leq \omega\Big(\frac{\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)}{2}, \\
&\quad \frac{\mathcal{S}(F(p_1, p_2), F(p_1, p_2), Ap_1) + \mathcal{S}(F(r_1, r_2), F(r_1, r_2), Ar_1)}{2}, \\
&\quad \frac{\mathcal{S}(F(p_2, p_1), F(p_2, p_1), Ap_2) + \mathcal{S}(F(r_2, r_1), F(r_2, r_1), Ar_2)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)}{2}, \\
&\quad \frac{\mathcal{S}(Ap_1, Ap_1, Ap_1) + \mathcal{S}(Ar_1, Ar_1, Ar_1)}{2}, \\
&\quad \frac{\mathcal{S}(Ap_2, Ap_2, Ap_2) + \mathcal{S}(Ar_2, Ar_2, Ar_2)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)}{2}, 0, 0\Big).
\end{aligned} \tag{3.23}$$

Similarly, we obtain

$$\begin{aligned}
&S(Ap_2, Ap_2, Ar_2) = S(F(p_2, p_1), F(p_2, p_1), F(r_2, r_1)) \\
&\leq \omega\Big(\frac{\mathcal{S}(Ap_2, Ap_2, Ar_2) + \mathcal{S}(Ap_1, Ap_1, Ar_1)}{2}, \\
&\quad \frac{\mathcal{S}(F(p_2, p_1), F(p_2, p_1), Ap_2) + \mathcal{S}(F(r_2, r_1), F(r_2, r_1), Ar_2)}{2}, \\
&\quad \frac{\mathcal{S}(F(p_1, p_2), F(p_1, p_2), Ap_1) + \mathcal{S}(F(r_1, r_2), F(r_1, r_2), Ar_1)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(Ap_2, Ap_2, Ar_2) + \mathcal{S}(Ap_1, Ap_1, Ar_1)}{2}, \\
&\quad \frac{\mathcal{S}(Ap_2, Ap_2, Ap_2) + \mathcal{S}(Ar_2, Ar_2, Ar_2)}{2}, \\
&\quad \frac{\mathcal{S}(Ap_1, Ap_1, Ap_1) + \mathcal{S}(Ar_1, Ar_1, Ar_1)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)}{2}, 0, 0\Big).
\end{aligned} \tag{3.24}$$

Hence, from Definition 2.30 ($CIR2$), there exists $0 < h < 1$ such that

$$\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)$$
$$\le h\,[\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)],$$

which is a contradiction, since $0 < h < 1$. Hence, we conclude that $\mathcal{S}(Ap_1, Ap_1, Ar_1)$ $+\mathcal{S}(Ap_2, Ap_2, Ar_2)) = 0$, that is, $\mathcal{S}(Ap_1, Ap_1, Ar_1) = 0$ and $\mathcal{S}(Ap_2, Ap_2, Ar_2) = 0$. Hence $Ap_1 = Ar_1$ and $Ap_2 = Ar_2$. This shows that the common coupled fixed point (Ap_1, Ap_2) of the mappings F and A is unique. This completes the proof of Theorem 3.3. □

We define as $\mathcal{T}x = F(x, x)$. Then, we have the following corollary.

Corollary 3.4 *Let $(\mathcal{X}, \mathcal{S})$ be a complete $\mathcal{S}$-metric space. Suppose that the mapping $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition: for all $x, u, z \in \mathcal{X}$:*

$$\mathcal{S}(\mathcal{T}x, \mathcal{T}u, \mathcal{T}z) \le \omega\Big(\mathcal{S}(x, u, z), \frac{\mathcal{S}(\mathcal{T}x, \mathcal{T}x, x) + \mathcal{S}(\mathcal{T}z, \mathcal{T}z, z)}{2}, \frac{\mathcal{S}(\mathcal{T}u, \mathcal{T}u, u) + \mathcal{S}(\mathcal{T}z, \mathcal{T}z, z)}{2}\Big), \tag{3.25}$$

where $\omega \in \Omega$. If $\mathcal{T}$ is continuous, then $\mathcal{T}$ has a unique point in $\mathcal{X}$.

Proof Taking $x = y$, $u = v$ and $z = w$ in Theorem 3.1, then (3.1) coincides with (3.25). Thus, we have the conclusion of the Corollary from Theorem 3.1. □

Next, we give some analogues of coupled fixed point theorems in metric spaces for S-metric spaces by combining Theorem 3.1 with $\omega \in \Omega$ and ω satisfies the conditions ($CIR1$) and ($CIR2$). The following corollary is a generalization and extension of Corollary 2.2 of Aydi [2] from partial metric space to the setting of S-metric space.

Corollary 3.5 *Let $(\mathcal{X}, \mathcal{S})$ be a complete $\mathcal{S}$-metric space. Suppose that the mapping $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition: for all $x, y, u, v, z, w \in \mathcal{X}$:*

$$\mathcal{S}(F(x, y), F(u, v), F(z, w)) \le \frac{k}{2}[\mathcal{S}(x, u, z) + \mathcal{S}(y, v, w)],$$

where $k \in [0, 1)$ is a constant. Then F has a unique coupled fixed point.

Proof The assertion follows using Theorem 3.1 with $\omega(p, q, r) = k\,p$ for some $k \in [0, 1)$ and all $p, q, r \in \mathbb{R}_+$. □

The following corollary is a generalization and extension of Corollary 2.6 of Aydi [2] from partial metric space to the setting of S-metric space.

Corollary 3.6 *Let $(\mathcal{X}, \mathcal{S})$ be a complete $\mathcal{S}$-metric space. Suppose that the mapping $F\colon \mathcal{X} \times \mathcal{X} \to$*

$\mathcal{X}$ satisfying the following contractive condition: for all $x, y, u, v, z, w \in \mathcal{X}$:

$$\mathcal{S}(F(x,y), F(u,v), F(z,w)) \leq \frac{k}{2}[\mathcal{S}(F(x,y), F(x,y), x) + \mathcal{S}(F(z,w), F(z,w), z)],$$

where $k \in [0,1)$ is a constant. Then F has a unique coupled fixed point.

Proof The assertion follows using Theorem 3.1 with $\omega(p,q,r) = k\,q$ for some $k \in [0,1)$ and all $p, q, r \in \mathbb{R}_+$. □

Now, we give an example to validate the result.

Example 3.7 Let $\mathcal{X} = [0,1]$ and the function $\mathcal{S}\colon \mathcal{X}^3 \to [0,\infty)$ be defined as $\mathcal{S}(x,y,z) = |y-z| + |y+z-2x|$ for all $x, y, z \in \mathcal{X}$. Then the function $\mathcal{S}$ is an $\mathcal{S}$-metric on $\mathcal{X}$ and $(\mathcal{X}, \mathcal{S})$ is an $\mathcal{S}$-metric space. Define a map $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ by $F(x,y) = \frac{x}{4} + \frac{y}{8}$ for $x, y \in \mathcal{X}$. Then, we have

$$\begin{aligned}
\mathcal{S}(F(x,y), F(u,v), F(z,w)) &= |F(u,v) + F(z,w) - 2F(x,y)| + |F(u,v) - F(z,w)| \\
&= \Big|\frac{u}{4} + \frac{v}{8} + \frac{z}{4} + \frac{w}{8} - \frac{2x}{4} - \frac{2y}{8}\Big| + \Big|\frac{u}{4} + \frac{v}{8} - \frac{z}{4} - \frac{w}{8}\Big| \\
&= \frac{1}{4}|u+z-2x| + \frac{1}{8}|v+w-2y| + \frac{1}{4}|u-z| + \frac{1}{8}|v-w| \\
&= \frac{1}{4}\big(|u+z-2x| + |u-z|\big) + \frac{1}{8}\big(|v+w-2y| + |v-w|\big) \\
&= \frac{1}{4}\mathcal{S}(x,u,z) + \frac{1}{8}\mathcal{S}(y,v,w) \\
&\leq \frac{1}{4}\big[\mathcal{S}(x,u,z) + \mathcal{S}(y,v,w)\big],
\end{aligned}$$

holds for all $x, y, z, u, v, w \in \mathcal{X}$, where $k = \frac{1}{2} < 1$. It is easy to see that F satisfies all the conditions of Corollary 3.5. Thus F has a unique coupled fixed point, namely $F(0,0) = 0$. Similarly, we can verify the result of Corollary 3.6.

As an application of Corollary 3.5, we find an existence and unique result for a type of the following system of Fredholm integral equations:

$$\begin{aligned}
x(t) &= \int_{\mathcal{E}} \mathcal{H}(t, \alpha, x(\alpha), y(\alpha))d(\alpha) + \beta(t), \quad t, \alpha \in \mathcal{E}, \\
y(t) &= \int_{\mathcal{E}} \mathcal{H}(t, \alpha, y(\alpha), x(\alpha))d(\alpha) + \beta(t), \quad t, \alpha \in \mathcal{E},
\end{aligned} \tag{3.26}$$

where $\mathcal{E}$ is measurable, $\mathcal{H}\colon \mathcal{E} \times \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\beta \in \mathcal{L}^\infty(\mathcal{E})$. Let $\mathcal{X} = \mathcal{L}^\infty(\mathcal{E})$. Now, we define $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ by

$$F(x,y)(t) = \int_{\mathcal{E}} \mathcal{H}(t, \alpha, x(\alpha), y(\alpha))d(\alpha) + \beta(t), \quad t, \alpha \in \mathcal{E}.$$

Obviously, $(x(t), y(t))$ is a solution of the system of Fredholm integral equations (3.26) if and only if $(x(t), y(t))$ is a coupled fixed point of F. Now, we define the function $S\colon \mathcal{X}^3 \to [0, +\infty)$ by

$$\mathcal{S}(x, y, z) = \sup_{\alpha\in\mathcal{E}} |y(\alpha) - z(\alpha)| + \sup_{\alpha\in\mathcal{E}} |y(\alpha) + z(\alpha) - 2x(\alpha)|, \tag{3.27}$$

for all $x, y, z \in \mathcal{X}$. Then the function $\mathcal{S}$ is an $\mathcal{S}$-metric. Now, we show that this $\mathcal{S}$-metric can not be generated by any metric d. We assume that this $\mathcal{S}$-metric is generated by any metric d, that is, there exists a metric d such that

$$\mathcal{S}(x, y, z) = d(x, z) + d(y, z), \tag{3.28}$$

for all $x, y, z \in \mathcal{X}$. Then we get

$$\mathcal{S}(x, x, z) = 2d(x, z) = 2\sup_{\alpha\in\mathcal{E}} |x(\alpha) - z(\alpha)|,$$

and

$$d(x, z) = \sup_{\alpha\in\mathcal{E}} |x(\alpha) - z(\alpha)|. \tag{3.29}$$

Likewise, we obtain

$$\mathcal{S}(y, y, z) = 2d(y, z) = 2\sup_{\alpha\in\mathcal{E}} |y(\alpha) - z(\alpha)|,$$

and

$$d(y, z) = \sup_{\alpha\in\mathcal{E}} |y(\alpha) - z(\alpha)|. \tag{3.30}$$

From equations (3.28), (3.29) and (3.30), we get

$$\sup_{\alpha\in\mathcal{E}} |y(\alpha) - z(\alpha)| + \sup_{\alpha\in\mathcal{E}} |y(\alpha) + z(\alpha) - 2x(\alpha)| = \sup_{\alpha\in\mathcal{E}} |x(\alpha) - z(\alpha)| + \sup_{\alpha\in\mathcal{E}} |y(\alpha) - z(\alpha)|,$$

which is a contradiction. Hence this $\mathcal{S}$-metric is not generated by any metric d. Thus, $(\mathcal{X}, \mathcal{S})$ is a complete $\mathcal{S}$-metric space.

Now, we state and prove our result as follows.

Theorem 3.8 *Suppose the following:*

1. *There exists a continuous function* $\kappa\colon \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *such that*

$$\big|\mathcal{H}(t, \alpha, x(\alpha), y(\alpha)) - \mathcal{H}(t, \alpha, u(\alpha), v(\alpha))\big| \le |\kappa(t, \alpha)|\big[|x(\alpha) - u(\alpha)| + |y(\alpha) - v(\alpha)|\big],$$

for all $x, y, u, v \in \mathcal{X}$ *and* $t, \alpha \in \mathcal{E}$.

2. $$\int_{\mathcal{E}} |\kappa(t, \alpha)| d(\alpha) \le \frac{1}{4}.$$

*Then the integral equation (*3.26*) has a unique solution in* $\mathcal{X}$*.*

Proof Consider

$$
\begin{aligned}
&\mathcal{S}(F(x,y),F(x,y),F(u,v)) = 2\,\big|F(x,y)-F(u,v)\big| \\
&= 2\,\Big|\int_{\mathcal{E}} \mathcal{H}(t,\alpha,x(\alpha),y(\alpha))d(\alpha)+\beta(t) \\
&\quad -\Big(\int_{\mathcal{E}} \mathcal{H}(t,\alpha,u(\alpha),v(\alpha))d(\alpha)+\beta(t)\Big)\Big| \\
&= 2\,\Big|\int_{\mathcal{E}} \big[\mathcal{H}(t,\alpha,x(\alpha),y(\alpha))-\mathcal{H}(t,\alpha,u(\alpha),v(\alpha))\big]d(\alpha)\Big| \\
&\le 2\int_{\mathcal{E}} |\kappa(t,\alpha)|\big[|x(\alpha)-u(\alpha)|+|y(\alpha)-v(\alpha)|\big]d(\alpha) \\
&\le \frac{1}{2}\big[|x(\alpha)-u(\alpha)|+|y(\alpha)-v(\alpha)|\big] \\
&= \frac{1}{4}\big[2(|x(\alpha)-u(\alpha)|+|y(\alpha)-v(\alpha)|)\big] \\
&= \lambda\,\big[\mathcal{S}(x,x,u)+\mathcal{S}(y,y,v)\big]
\end{aligned}
$$

for all $x,y,u,v \in \mathcal{X}$, where $0 \le \lambda = \frac{1}{4} < \frac{1}{2}$. Hence, all the hypothesis of Corollary 3.5 are satisfied and consequently, the integral equation (3.26) has a unique solution. □

§4. Conclusion

In this paper, we prove a unique coupled fixed point and a unique common coupled fixed point theorems under newly proposed coupled implicit relations in the setting of S-metric spaces and give some corollaries of the main results. An illustrative example and an application to the Fredholm integral equation are given. Our results extend and generalize several results from the existing literature.

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