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Cover Pebbling Number for Square of a Path

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Abstract: Given a graph G and a configuration C of pebbles on the vertices of G, a pebbling step (move) removes two pebbles from one vertex and places one pebble on an adjacent vertex. The cover pebbling number $\gamma(G)$ is the minimum number so that every configuration of $\gamma(G)$ pebbles has the property that after some sequence of pebbling steps(moves), every vertex has a pebble on it. In this paper we determine the cover pebbling number for square of a path.

Key Words: Cover pebbling, square of a path, Smarandachely cover H-pebbling.

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§1. Introduction

The game of pebbling was first suggested by Lagarias and Saks as a tool for solving a number-theoretical conjecture of Erdos. Chung successfully used this tool to prove the conjecture and established other results concerning pebbling numbers. In doing so she introduced pebbling to the literature [1].

Begin with a graph G and a certain number of pebbles placed on its vertices. A pebbling step consists of removing two pebbles from one vertex and placing one on an adjacent vertex. In pebbling, the target is selected, and the goal is to move a pebble to the target vertex. The minimum number of pebbles such that regardless of their initial placement and regardless of the target vertex, we can pebble that target is called the pebbling number of G. In cover pebbling, the goal is to cover all the vertices with pebbles, that is, to move a pebble to every vertex simultaneously. Generally, for a connected subgraph $H < G$, a *Smarandachely cover H-pebbling* is to move a pebble to every vertex in H but not in $G \setminus H$ simultaneously. The minimum number of pebbles required such that, regardless of their initial placement on G, there is a sequence of pebbling steps at the end of which every vertex has at least one pebble on it, is called the cover pebbling number of G. In [2], Crull et al. determine the cover pebbling number of several families of graphs, including trees and complete graphs. Hulbert and Munyan [4] have also announced a proof for the cover pebbling of the n-dimensional cube. In [5], Maggy Tomova and Cindy Wyles determine the cover pebbling number for cycles and certain graph products. In the next section, we determine the cover pebbling number for square of a path.

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§2. The Cover Pebbling Number for square of a Path

Definition([6]) *Let $G = (V(G), E(G))$ be a connected graph. The nth power of G, denoted by G^p, is the graph obtained from G by adding the edge uv to G whenever $2 \leq d(u,v) \leq p$ in G, that is, $G^p = (V(G), E(G) \cup \{uv : 2 \leq d(u,v) \leq p \text{ in } G\})$. If p=1, we define G^1=G. We know that if p is large enough, that is, $p \geq n-1$, then $G^p = K_n$.*

Notation 2.2 The Labeling of P_n^2 is $P_n^2 :\ v_1v_2\cdots v_{n-1}v_n$. Let $p(v_i)$ denote the number of pebbles on the vertex v_i and $p(P_A)$ denote the number of pebbles on the path P_A.

It is easy to see that $\gamma(P_3^2) = 5$ since $P_3^2 \cong K_3$ [2].

Theorem 2.3 *The cover pebbling number of P_4^2 is $\gamma(P_4^2) = 9$.*

Proof Consider the distribution of eight pebbles on v_1. Clearly, we cannot cover at least one of the vertices of P_4^2. Thus,$\gamma(P_4^2) \geq 9$.

Now, consider the distribution of nine pebbles on the vertices of P_4^2. If v_4 has zero pebbles on it, then using at most four pebbles from $P_3^2 : v_1v_2v_3$ we can move a pebble to v_4. After moving a pebble to v_4, P_3^2 contains at least five pebbles and we are done. Next assume that v_4 has at least one pebble. If $p(v_4) \leq$4, then $p(P_3^2) \geq$5 and we are done. If $p(v_4) = 5$ or 6 or 7, clearly we are done. If $p(v_4) \geq$8, then move as many as possible to the vertices of P_3^2 using at most four moves while retaining one or two pebbles on v_4, we cover all the vertices of P_4^2 in these distributions also. Thus, $\gamma(P_4^2) \leq 9$. Therefore, $\gamma(P_4^2) = 9$. □

Theorem 2.4 *The cover pebbling number of P_5^2 is $\gamma(P_5^2) = 13$.*

Proof Consider the distribution of twelve pebbles on v_1. Clearly, we cannot cover at least one of the vertices of P_5^2. Thus,$\gamma(P_5^2) \geq 13$.

Now, consider the distribution of thirteen pebbles on the vertices of P_5^2. If v_5 has zero pebbles on it, then using at most four pebbles from $P_4^2 : v_1v_2v_3v_4$ we can move a pebble to v_5. After moving a pebble to v_5, P_4^2 contains at least nine pebbles and we are done. Next assume that v_5 has at least one pebble. If $p(v_5) \leq$4, then $p(P_4^2) \geq$9 and we are done. If $p(v_5) = 5$ or 6 or 7, then clearly we are done. If $p(v_5) \geq 8$, then move as many as possible to the vertices of P_4^2 using at most four moves while retaining one or two pebbles on v_5, we cover all the vertices of P_5^2 in these distributions also. Thus, $\gamma(P_5^2) \leq 13$. Therefore, $\gamma(P_5^2) = 13$. □

Theorem 2.5 *The cover pebbling number of P_n^2 is*

$$\gamma(P_n^2) = \begin{cases} 2^{k+2} - 3 & if\ n = 2k+1\ (k \geq 1); \\ 3(2^k - 1) & if\ n = 2k\ (k \geq 2). \end{cases}$$

Proof Consider the following distribution

$$p(v_1) = \begin{cases} 2^{k+2} - 4 & if\ n = 2k+1\ (k \geq 1); \\ 3(2^k) - 4 & if\ n = 2k\ (k \geq 2). \end{cases}$$

and $p(v_i) = 0, i \neq 1$. Notice that we cannot cover at least one of the vertices of P_n^2. Thus,

$$\gamma(P_n^2) \geq \begin{cases} 2^{k+2} - 3 & if\ n = 2k+1\ (k \geq 1); \\ 3(2^k - 1) & if\ n = 2k\ (k \geq 2). \end{cases}$$

. Next, we are going to prove the upper bound by induction on n. Obviously, the result is true for n = 4 and 5, by Theorem 2.3 and Theorem 2.4. So, assume the result is true for m $\leq$ n-1. If v_n has zero pebbles on it, then using at most 2k pebbles from the vertices of $P_{n-1}^2 : v_1 v_2 \cdots v_{n-2} v_{n-1}$ we can cover the vertex v_n. Then P_{n-1}^2contains at least

$$\begin{cases} 3(2^k - 1), & where\ k = \dfrac{n-1}{2}; \\ 2^{k+1} - 3, & where\ k = \dfrac{n}{2} \end{cases}$$

pebbles and we are done by induction. Next, assume that v_n has a pebble on it. If $p(v_n) \leq 2(2^k - 1)$, then

$$p(P_{n-1}^2) \geq \begin{cases} 2^{k+1} - 3 & if\ n\ is\ odd; \\ 2^k - 1 & if\ n\ is\ even. \end{cases}$$

In these both cases, either P_{n-1}^2 has enough pebbles or we can make it by retaining one or two pebbles on v_n and moving as many pebbles as possible from v_n to v_{n-1} or v_{n-2}. So, we are done easily if $p(v_n) \leq 2(2^k - 1)$. Suppose $p(v_n) \geq 2(2^k - 1) + 1$, then by moving as many pebbles as possible to the vertices of P_{n-1}^2, using at most

$$\begin{cases} 2^{k+1} - 2 & if\ n\ is\ odd; \\ 3(2^{k-1}) - 2 & if\ n\ is\ even \end{cases}$$

pebbling steps while retaining one or two pebbles on v_n, and hence we are done. Thus,

$$\gamma(P_n^2) \leq \begin{cases} 2^{k+2} - 3 & if\ n = 2k+1\ (k \geq 1); \\ 3(2^k - 1) & if\ n = 2k\ (k \geq 2). \end{cases}$$

Therefore,

$$\gamma(P_n^2) = \begin{cases} 2^{k+2} - 3 & if\ n = 2k+1\ (k \geq 1); \\ 3(2^k - 1) & if\ n = 2k\ (k \geq 2). \end{cases} \qquad \square$$

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