International J.Math. Combin. Vol.1 (2009), 42-45

Cycle-Complete Graph Ramsey Numbers
$r(C_4, K_9), r(C_5, K_8) \leq 33$

M.M.M. Jaradat

(Department of Mathematics, Yarmouk University, Irbid-Jordan)

(Department of Mathematics and Physics, Qatar University, Doha-Qatar)

Email: mmjst4@yu.edu.jo, mmjst4@qu.edu.qa

B.M.N. Alzaleq

(Department of Mathematics, Yarmouk University, Irbid-Jordan)

Email: alzaleq@yahoo.com

Abstract For an integer $k \geq 1$, a cycle-complete graph Smarandache-Ramsey number $r_{s^k}(C_m, K_n)$ is the smallest integer N such that every graph G of order N contains k cycles, C_m, on m vertices or the complement of G contains k complete graph, K_n, on n vertices. If $k = 1$, then the Smarandache-Ramsey number $r_{s^k}(C_m, K_n)$ is nothing but the classical Ramsey number $r(C_m, K_n)$. Radziszowski and Tse proved that $r(C_4, K_9) \geq 30$. Also, By considering the known graph $G = 7K_4$, we have that $r(C_5, K_8) \geq 29$. In this paper we give an upper bound of $r(C_4, K_9)$ and $r(C_5, K_8)$.

Key Words: (Smarandache-)Ramsey number; independent set; cycle; complete graph.

AMS(2000): 05C55, 05C35.

§1. Introduction

Through out this paper we adopt the standard notations, a cycle on m vertices will be denoted by C_m and the complete graph on n vertices by K_n. The minimum degree of a graph G is denoted by $\delta(G)$. An independent set of vertices of a graph G is a subset of $V(G)$ in which no two vertices are adjacent. The independence number of a graph $G, \alpha(G)$, is the size of the largest independent set.

For an integer $k \geq 1$, a Smarandache-Ramsey number $r_{s^k}(H, F)$ is the smallest integer N such that every graph G of order N contains k graph H, or the complement of G contains k graph F. If $k = 1$, then the Smarandache-Ramsey number $r_{s^k}(H, F)$ is nothing but the classical Ramsey number $r(H, F)$. $r(C_m, K_n)$ is called the cycle-complete graph Ramsey number. In one of the earliest contributions to graphical Ramsey theory, Bondy and Erdős [3] proved that for all $m \geq n^2 - 2, r(C_m, K_n) = (m-1)(n-1)+1$. The restriction in the above result was improved by Nikiforov [10] when he proved the equality for $m \geq 4n+2$. Erdős et al. [5] conjectured that $r(C_m, K_n) = (m-1)(n-1)+1$, for all $m \geq n \geq 3$ except $r(C_3, K_3) = 6$. The conjectured were

[1]Received December 12, 2008. Accepted January 16, 2009.

confirmed for some $n = 3, 4, 5$ and 6 (see [2], [6], [12], and [14]). Moreover, in [7] and [8] the conjecture was proved for $m = n = 8$, and $m = 8$ with $n = 7$. Also, the case $n = m = 7$ was proved independently by Baniabedalruhman and Jaradat [1] and Cheng et al. [4].

In a related work, Radziszowski and Tse [11] showed that $r(C_4, K_7) = 22$, $r(C_4, K_8) = 26$ and $r(C_4, K_9) \geq 30$. Also, In [8] Jayawardene and Rousseau proved that $r(C_5, K_6) = 21$. Recently, Schiermeyer [13] and Cheng et al. [4] proved that $r(C_5, K_7) = 25$ and $r(C_6, K_7) = 25$, respectively. In this article we prove the following Theorems:

Theorem A *The complete-cycle Ramsey number $r(C_4, K_9) \leq 33$.*

Theorem B *The complete-cycle Ramsey number $r(C_5, K_8) \leq 33$.*

In the rest of this work, $N(u)$ stands for the neighbor of the vertex u which is the set of all vertices of G that are adjacent to u and $N[u] = N(u) \cup \{u\}$. For a subgraph R of the graph G and $U \subseteq V(G)$, $N_R(U)$ is defined as $(\cup_{u \in U} N(u)) \cap V(R)$. Finally, $\langle V_1 \rangle_G$ stands for the subgraph of G whose vertex set is $V_1 \subseteq V(G)$ and whose edge set is the set of those edges of G that have both ends in V_1 and is called the subgraph of G induced by V_1.

§2. Proof of Theorem A

We prove our result using the contradiction. Suppose that G is a graph of order 33 which contains neither C_4 nor a 9-element independent set. Then we have the following:

1. $\delta(G) \geq 7$. Assume that u is a vertex with $d(u) \leq 6$. Then $|V(G) - N[u]| \geq 33 - 7 = 26$. But $r(C_4, K_8) = 26$. Hence, $G - N[u]$ contains an 8-element independent set. This set with u form a 9-element independent set. That is a contradiction.

2. G contains no K_3. Suppose that G contains K_3. Let $\{u_1, u_2, u_3\}$ be the vertex set of K_3. Also, let $R = G - \{u_1, u_2, u_3\}$ and $U_i = N(u_i) \cap V(R)$. Then $U_i \cap U_j = \varnothing$ because otherwise G contains C_4. Also, for each $x \in U_i$ and $y \in U_i$, we have that $xy \notin E(G)$ because otherwise G contains C_4. Now, since $\delta(G) \geq 7$, $|U_i| \geq 5$. Since $r(P_3, K_3) = 5$, as a result either $\langle U_i \rangle_G$ contains P_3 for some $i = 1, 2, 3$ and so G contains C_4 or $\langle U_i \rangle_G$ does not contains P_3 for each $i = 1, 2, 3$ and so each of which contains a 3-element independent set, Thus, three independent set of each consists a 9-element independent set. This is a contradiction.

Now, let u be a vertex of G. Let $N(u) = \{u_1, u_2, \ldots, u_r\}$ where $r \geq 7$. Since G contains no K_3, as a result $\langle N(u) \cup \{u\} \rangle_G$ forms a star. And so,$\{u_1, u_2, \ldots, u_r\}$ is independent. Now, let $N(u_1) = \{v_1, v_2, \ldots, v_k, u\}$ where $k \geq 6$. For the same reasons, $\langle N(u_1) \cup \{u_1\} \rangle_G$ forms a star and so $\{v_1, v_2, \ldots, v_k\}$ is independent. Since G contains no K_3 and no C_4. Then $\{u_2, \ldots, u_r, v_1, v_2, \ldots, v_k\}$ is an independent set. That is a contradiction. The proof is complete. □

§3. Proof of Theorem B

We prove our result by using the contradiction. Assume that G is a graph of order 33 which contains neither C_5 nor an 8-element independent set. By an argument similar to the one in Theorem A and by noting that $r(C_5, K_8) = 25$, we can show that $\delta(G) \geq 8$. Now, we have the following:

1. G contains K_3. Suppose that G does not contain K_3. Let $u \in V(G)$ and $r = |N(u)|$. Then the induced subgraph $< N(u) >_G$ does not contain P_2. Hence $< N(u) >_G$ is a null graph with r vertices. Since $\alpha(G) \leq 7$, as a result $r \leq 7$. Therefore, $8 \leq \delta(G) \leq r \leq 7$. That is a contradiction.

2. G contains $K_4 - e$. Let $U = \{u_1, u_2, u_3\}$ be the vertex set of K_3. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 3$. Since $\delta(G) \geq 8$, $|U_i| \geq 6$ for all $1 \leq i \leq 3$. Now we have the following two cases:

Case 1: $U_i \cap U_j \neq \varnothing$ for some $1 \leq i < j \leq 3$, say $w \in U_i \cap U_j$. Then it is clear that G contains $K_4 - e$. In fact, the induced subgraph $\langle U \cup \{w\}\rangle_G$ contains $K_4 - e$.

Case 2 : $U_i \cap U_j = \varnothing$ for each $1 \leq i < j \leq 3$. Then $\alpha(\langle U_i\rangle_G) \leq 2$, for some $1 \leq i \leq 3$. To see that suppose that $\alpha(\langle U_i\rangle_G) \geq 3$ for each $1 \leq i \leq 3$. Since between any two vertices of U there is a path of order 3, as a result for any $x \in U_i$ and $y \in U_j$, we have $xy \notin E(G), 1 \leq i < j \leq 3$ because otherwise G contains C_5. Therefore, $\alpha(\langle U_1 \cup U_2 \cup U_3\rangle_G) \geq 3 + 3 + 3 = 9$. and so $\alpha(G) \geq 9$, which is a contradiction.
Now, since $|U_i| \geq 6$ and $\alpha(\langle U_i\rangle_G) \leq 2$, for some $1 \leq i \leq 3$ and since $r(K_3, K_3) = 6$ as a result the induced subgraph $\langle U_i\rangle_G$ contains K_3. And so $\langle U_i \cup \{u_i\}\rangle_G$ contains K_4. Hence, G contains $K_4 - e$.

3. G contains K_4. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of $K_4 - e$, where the induced subgraph of $\{u_1, u_2, u_3\}$ is isomorphic to K_3. Without loss of generality we may assume that $u_1u_4, u_2u_4 \in E(G)$. We consider the case where $u_3u_4 \notin E(G)$ because otherwise the result is obtained. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 4$. Then as in **2**, $|U_i| \geq 5$ for $i = 1, 2$ and $|U_i| \geq 6$ for $i = 3, 4$. To this end, we have that $U_i \cap U_j = \varnothing$ for all $1 \leq i < j \leq 4$ except possibly for $i = 1$ and $j = 2$ (To see that suppose that $w \in U_i \cap U_j$ for some $1 \leq i < j \leq 4$ with $i \neq 1$ or $j \neq 2$. Then we consider the following cases:

(1) $i = 3$ and $j = 4$. Then $u_3wu_4u_1u_2u_3$ is a cycle of order 5, a contradiction.

(2) $i = 3$ and $j = 2$. Then $u_3wu_2u_4u_1u_3$ is a cycle of order 5, a contradiction.

(3) i, j are not as in the above cases. Then by similar argument as in (2) G contains a C_5. This is a contradiction.

Now, By arguing as in Case 2 of 2, $\alpha(\langle U_2\rangle_G) \leq 1$ or $\alpha(\langle U_i\rangle_G) \leq 2$, for $i = 3$ or 4. And so, the induced subgraph $\langle U_i\rangle_G$ contains K_3 for some $2 \leq i \leq 4$. Thus, G contains K_4.
To this end, let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of K_4. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 4$. Since $\delta(G) \geq 8$, $|U_i| \geq 5$ for all $1 \leq i \leq 4$. Since there is a path of order 4 joining any two vertices of U, as a result $U_i \cap U_j = \varnothing$ for all $1 \leq i < j \leq 4$ (since otherwise, if $w \in U_i \cap U_j$ for some $1 \leq i < j \leq 4$, then the concatenation of the u_i-u_j path of order 4 with u_iwu_j is a cycle of order 5, a contradiction). Similarly, since there is a path of order 3 joining any two vertices of U, as a result for all $1 \leq i < j \leq 4$ and for all $x \in U_i$ and $y \in U_j, xy \notin E(G)$ (otherwise, if there are $1 \leq i < j \leq 4$ such that $x \in U_i$ and $y \in U_j$, and $xy \in E(G)$, then the concatenation of the u_i-u_j path of order 3 with u_ixyu_j is a cycle of

order 5, a contradiction). Also, since there is a path of order 2 joining any two vertices of U, as a result $N_R(U_i) \cap N_R(U_j) = \varnothing, 1 \leq i < j \leq 4$ (otherwise, if there are $1 \leq i < j \leq 4$ such that $w \in N_R(U_i) \cap N_R(U_j)$, then the concatenation of the u_i-u_j path of order 2 with $u_i x w y u_j$ where $x \in U_i$ and $y \in U_j$, and $xw, yw \in E(G)$ is a cycle of order 5, a contradiction). Therefore, $|U_i \cup N_R(U_i) \cup \{u_i\}| \geq \delta(G) + 1$. Thus, $|V(G)| \geq 4(\delta(G) + 1) \geq 4(8 + 1) = 4.9 = 36$. That contradicts the fact that the order of G is 33. □

References

[1] A. Baniabedalruhman and M.M.M. Jaradat, The cycle-complete graph Ramsey number $r(C_7, K_7)$. *Journal of combinatorics, information & system sciences* (Accepted).

[2] B. Bollobás, C. J. Jayawardene, Z. K. Min, C. C. Rousseau, H. Y. Ru, and J. Yang, On a conjecture involving cycle-complete graph Ramsey numbers, *Australas. J. Combin.*, 22 (2000), 63-72.

[3] J.A. Bondy and P. Erdős, Ramsey numbers for cycles in graphs, *Journal of Combinatorial Theory, Series B*, 14 (1973), 46-54.

[4] T.C. E. Cheng, Y. Chen, Y. Zhang and C.T. Ng, The Ramsey numbers for a cycle of length six or seven versus a clique of order seven, *Discrete Mathematics*, 307 (2007), 1047-1053.

[5] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, On cycle-complete graph Ramsey numbers, *J. Graph Theory*, 2 (1978), 53-64.

[6] R.J. Faudree and R. H. Schelp, All Ramsey numbers for cycles in graphs, *Discrete Mathematics*, 8 (1974), 313-329.

[7] M.M.M. Jaradat and B. Alzaliq, The cycle-complete graph Ramsey number $r(C_8, K_8)$. *SUT Journal of Mathematics* 43 (2007), 85-98.

[8] M.M.M. Jaradat and A.M.M. Baniabedalruhman, The cycle-complete graph Ramsey number $r(C_8, K_7)$. *International Journal of pure and applied mathematics*, 41 (2007), 667-677.

[9] C.J. Jayawardene and C. C. Rousseau, The Ramsey number for a cycle of length five versus a complete graph of order six, *J. Graph Theory*, 35 (2000), 99-108.

[10] V. Nikiforov, The cycle-complete graph Ramsey numbers, *Combin. Probab. Comput.* 14 (2005), no. 3, 349–370.

[11] S. P. Radziszowski and K.-K. Tse, A computational approach for the Ramsey numbers $r(C_4, K_n)$, *J. Comb. Math. Comb. Comput.*, 42 (2002), 195-207.

[12] I. Schiermeyer, All cycle-complete graph Ramsey numbers $r(C_n, K_6)$, *J. Graph Theory*, 44 (2003), 251-260.

[13] I. Schiermeyer, The cycle-complete graph Ramsey number $r(C_5, K_7)$, *Discussiones Mathematicae Graph Theory* 25 (2005) 129-139.

[14] Y.J. Sheng, H. Y. Ru and Z. K. Min, The value of the Ramsey number $r(C_n, K_4)$ is $3(n-1)+1$ $(n \geq 4)$, *Ausralas. J. Combin.*, 20 (1999), 205-206.