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Cyclic Contraction on Supermetric Spaces

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Abstract: In this paper, we investigate fixed point properties of cyclic contractive mappings in the framework of supermetric spaces. After recalling the fundamental axioms of supermetrics together with the corresponding notions of convergence and completeness, we introduce a cyclic contraction scheme formulated on mixed pairs of subsets (A, B) and establish a fixed point theorem ensuring existence, uniqueness, and convergence of Picard iterates. Several constructive examples are presented to demonstrate the applicability of the theory. As an application, a nonlinear integral equation is reformulated as a fixed point problem in an appropriate function space, yielding existence and uniqueness of continuous solutions.

Key Words: Cyclic contraction, supermetric space, fixed point theory, nonlinear integral equation.

AMS(2010): 47H10, 54H25, 54E50.

§1. Introduction

Banach's contraction principle (Banach, 1922) is a cornerstone of nonlinear analysis and remains a basic tool for proving existence and uniqueness of solutions to nonlinear problems. A classical and still active direction is to enlarge either (i) the underlying distance framework or (ii) the class of admissible contractive conditions, while preserving completeness–convergence mechanisms needed for Picard iteration.

On the side of generalized distances, several models are now standard. Partial metric spaces, introduced by Matthews [11], allow nonzero self-distance and are useful in computer-science motivated fixed point problems. The b-metric of Czerwik [5] relaxes the triangle inequality by a coefficient and has become a basic platform for nonlinear operator theory. Multi-point geometries such as S-metric spaces were proposed by Sedghi–Shobe–Aliouche [15]. In a different direction, fuzzification of distance and its applications originate from Zadeh's fuzzy set theory [17].

In parallel, many authors have developed nonlinear contractive conditions that extend the Banach inequality. Representative frameworks include rational-type contractions [1], the α–ψ contractive scheme of Samet–Vetro–Vetro [14], and Wardowski's F-contractions [16]. Fixed point techniques have also been combined with order structures; see, for example, Ran–Reurings

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[12], Bhaskar–Lakshmikantham [4], Altun–Simsek [2], and Jleli–Samet [7].

More recently, Karapınar and Khojasteh introduced the concept of a *supermetric space* [9]. A supermetric replaces triangle-type inequalities by a sequence-based control axiom, yet still supports a Banach-type fixed point principle. Subsequent work in this environment includes contractive models of rational form [8].

Another influential line is the *cyclic* approach, where a mapping alternates between two (or more) sets and the contractive requirement is enforced along the cyclic pattern. This viewpoint was developed systematically by Kirk–Srinivasan–Veeramani [10] (see also [13]) and has proved useful in settings where contraction is naturally available only on mixed pairs. Motivated by these developments, we adapt cyclic contractive ideas to the supermetric framework and derive fixed point results under completeness and natural closure hypotheses.

§2. **Preliminaries**

In this section, we recall the basic notions of supermetric spaces together with the associated concepts of convergence, Cauchy sequences, and completeness. These structures were introduced by Karapınar and Khojasteh as a sequence - controlled generalization of classical metric spaces and several metric - type extensions [9] and [8]. Unlike metric and b - metric spaces, the supermetric framework does not rely on a triangle-type inequality; instead, it uses a limsup sequence control condition that is sufficient to develop fixed point theory and completeness results [9]. Related generalized distance frameworks include partial metric spaces [11], b-metric spaces [5], and S-metric spaces [15], all of which motivated the search for more flexible convergence structures.

Definition 2.1(Supermetric space, [9]) *Let X be a nonempty set. A function $m : X \times X \to [0, \infty)$ is called a supermetric if*

$(SM1)$ *if $m(x, y) = 0$, then $x = y$ for all $x, y \in X$;*

$(SM2)$ $m(x, y) = m(y, x)$ *for all* $x, y \in X$;

$(SM3)$ *there exists $s \geqslant 1$ such that for every $y \in X$ there exist* distinct *sequences $\{x_n\}, \{y_n\} \subset X$ with $m(x_n, y_n) \to 0$ and*

$$\limsup_{n\to\infty} m(y_n, y) \leqslant s \limsup_{n\to\infty} m(x_n, y).$$

Then (X, m) is called a supermetric space.

Remark 2.1(Interpretation of the axioms)

(i) Condition (SM1) is a weak identity property. Unlike classical metrics, the converse $m(x, x) = 0$ is not required, which aligns the model with partial metric philosophy, [11];

(ii) Condition (SM2) ensures symmetric distance behavior as in classical metric theory;

(iii) Condition (SM3) replaces the triangle inequality by a sequence comparison mechanism. It guarantees that if two sequences become mutually close, then their relative proximity to any fixed point is controlled. This property is fundamental in establishing convergence of Picard iteration sequences [9].

Definition 2.2(Convergence and Cauchy sequence, [9]) *Let (X, m) be a supermetric space.*

1) *A sequence $\{x_n\}$ converges to $x \in X$ if*

$$\lim_{n\to\infty} m(x_n, x) = 0;$$

(2) *A sequence $\{x_n\}$ is called Cauchy if*

$$\lim_{n\to\infty} \sup\{m(x_n, x_k) : k > n\} = 0;$$

(3) *The space (X, m) is complete if every Cauchy sequence converges to some point in X.*

Remark 2.2 If (X, d) is a usual metric space, then (X, d) is automatically a supermetric space. Indeed, (SM1) – (SM2) hold trivially and (SM3) holds with $s = 1$ by choosing two distinct sequences converging to the same limit. Hence supermetric spaces extend metric spaces while preserving fixed point tools based on completeness [3] and [9].

Remark 2.3 Supermetric convergence generates a natural topology on X. In many applications, supermetric convergence is equivalent to convergence under an associated metric on bounded subsets. This feature makes supermetric spaces suitable for nonlinear operator analysis and fixed point theory, including rational and nonlinear contractions [1] and [8].

Remark 2.4 The development of supermetric spaces is consistent with the broader program of extending distance geometry to accommodate nonlinear phenomena and computational models. For instance, partial metrics allow nonzero self-distance [11], b-metrics relax the triangle inequality constant [5], and multi-point distance structures such as S-metrics extend pairwise distance notions [15]. Supermetrics unify several of these behaviors under a sequence-control framework [9].

§3. New Examples of Supermetric Spaces

Proposition 3.1 *Let $X = \mathbb{R}$ and fix $\gamma \in [0, 1)$. Define*

$$m(x, y) := |x - y| + \gamma\big(|x| + |y|\big), \qquad x, y \in \mathbb{R}.$$

Then $(\mathbb{R}, m)$ is a supermetric space in the sense of Definition 2.1. In particular, axiom (SM3) holds with the constant $s = 2$.

Proof The function m is nonnegative and symmetric. If $m(x, y) = 0$, then $|x - y| + \gamma(|x| + |y|) = 0$ forces $|x - y| = 0$, hence $x = y$, so (SM1) holds.

Fix any $y \in \mathbb{R}$ and choose the distinct sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. Then

$$m(x_n, y_n) = \left| \frac{1}{n} - \frac{1}{n+1} \right| + \gamma \left(\frac{1}{n} + \frac{1}{n+1} \right) \to 0.$$

Moreover,

$$\begin{aligned} m(x_n, y) &= \left| y - \frac{1}{n} \right| + \gamma \left(|y| + \frac{1}{n} \right) \to |y| + \gamma |y|, \\ m(y_n, y) &= \left| y - \frac{1}{n+1} \right| + \gamma \left(|y| + \frac{1}{n+1} \right) \to |y| + \gamma |y|. \end{aligned}$$

Hence

$$\limsup_{n\to\infty} m(y_n, y) = \limsup_{n\to\infty} m(x_n, y).$$

Consequently,

$$\limsup_{n\to\infty} m(y_n, y) \leqslant 2 \limsup_{n\to\infty} m(x_n, y),$$

so (SM3) holds with $s = 2$. Therefore $(\mathbb{R}, m)$ is a supermetric space. ☐

Proposition 3.2 *Let $(E, \|\cdot\|)$ be a real normed linear space and fix $\gamma \in (0, 1)$. Define*

$$m(x, y) := \|x - y\| + \gamma \frac{\|x\| + \|y\|}{1 + \|x\| + \|y\|} \qquad (x, y \in E).$$

Then (E, m) is a supermetric space. In particular, (SM3) *holds with $s = 2$.*

Proof The function m is well-defined, nonnegative, and symmetric. If $m(x, y) = 0$, then $\|x - y\| = 0$, hence $x = y$, so (SM1) holds.

Fix any $y \in E$ and choose a nonzero $u \in E$. Define the distinct sequences $x_n = \frac{u}{n}$ and $y_n = \frac{u}{n+1}$. Then $\|x_n - y_n\| = \frac{\|u\|}{n(n+1)} \to 0$ and $\|x_n\| + \|y_n\| \to 0$, which yields $m(x_n, y_n) \to 0$.

Since $x_n \to 0$ and $y_n \to 0$ in norm, we have $\|x_n - y\| \to \|y\|$ and $\|y_n - y\| \to \|y\|$, and also

$$\frac{\|x_n\| + \|y\|}{1 + \|x_n\| + \|y\|} \to \frac{\|y\|}{1 + \|y\|}, \qquad \frac{\|y_n\| + \|y\|}{1 + \|y_n\| + \|y\|} \to \frac{\|y\|}{1 + \|y\|}.$$

Hence both sequences $m(x_n, y)$ and $m(y_n, y)$ converge to $\|y\| + \gamma \frac{\|y\|}{1+\|y\|}$. Therefore their limsups are equal and so

$$\limsup_{n\to\infty} m(y_n, y) \leqslant 2 \limsup_{n\to\infty} m(x_n, y).$$

Thus (SM3) holds with $s = 2$, and (E, m) is a supermetric space. ☐

Proposition 3.3 *Let $X = \mathbb{R}$ and fix $\gamma \in (0, 1)$. Define*

$$m(x, y) := |x - y| + \gamma |\arctan x - \arctan y| \qquad (x, y \in \mathbb{R}).$$

Then $(\mathbb{R}, m)$ is a supermetric space, and (SM3) *holds with $s = 2$.*

Proof The function m is nonnegative and symmetric. If $m(x, y) = 0$, then $|x - y| = 0$, hence $x = y$, so (SM1) holds.

Fix any $y \in \mathbb{R}$ and take distinct sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. Then $|x_n - y_n| \to 0$ and continuity of arctan at 0 gives $|\arctan x_n - \arctan y_n| \to 0$, hence $m(x_n, y_n) \to 0$.

Also, since $x_n \to 0$ and $y_n \to 0$, continuity of $t \mapsto |t - y|$ and $t \mapsto |\arctan t - \arctan y|$ implies

$$m(x_n, y) \to |y| + \gamma|\arctan y|, \qquad m(y_n, y) \to |y| + \gamma|\arctan y|.$$

Thus their limsups coincide and therefore

$$\limsup_{n\to\infty} m(y_n, y) \leqslant 2 \limsup_{n\to\infty} m(x_n, y).$$

Hence (SM3) holds with $s = 2$, so $(\mathbb{R}, m)$ is a supermetric space [9]. □

§4. Cyclic Contraction Mappings

In this section, we introduce cyclic self-mappings and cyclic contraction mappings in the framework of supermetric spaces and establish the fundamental structural properties that will be used in the main results. The concept of cyclic mappings originates from the idea of decomposing the domain into two interacting subsets and studying contractive behavior only across these subsets rather than globally. Such a formulation is particularly useful in generalized distance structures where classical triangle-type estimates may not be available.

Let (X, m) be a supermetric space and let A, B be nonempty subsets of X. Throughout this section, the union $A \cup B$ will serve as the effective domain of the mapping under consideration.

Definition 4.1(Cyclic map and cyclic contraction) *Let (X, m) be a supermetric space and let $A, B \subseteq X$ be nonempty sets.*

A mapping $T : A \cup B \to A \cup B$ is called a cyclic map *if*

$$T(A) \subseteq B \quad \textit{and} \quad T(B) \subseteq A.$$

A cyclic map T is called a cyclic contraction *if there exists a constant $\alpha \in (0, 1)$ such that*

$$m(Tx, Ty) \leqslant \alpha\, m(x, y) \qquad \textit{for all } x \in A,\ y \in B. \tag{1}$$

Remark 4.1 The contractive condition is imposed only for pairs $(x, y) \in A \times B$. This is natural because the cyclic structure forces the orbit

$$x_0 \in A \ \Rightarrow\ x_1 = Tx_0 \in B \ \Rightarrow\ x_2 = Tx_1 \in A \ \Rightarrow\ \cdots$$

to alternate between the two sets. Therefore, all successive distances that appear in Picard iteration are of mixed type.

Requiring the inequality on the whole set $A \cup B$ would reduce the model to a classical global contraction and would not capture the cyclic geometry.

Remark 4.2 Unlike metric or b-metric spaces, supermetric spaces do not assume any triangle inequality. Therefore, the formulation of cyclic contractions is chosen so that convergence analysis relies mainly on geometric decay of successive distances rather than summation-type estimates. This makes cyclic contractions particularly well suited for supermetric settings.

Example 4.1 Let m be the supermetric defined in Proposition 3.1, namely

$$m(x,y) = |x-y| + \gamma(|x|+|y|), \quad \gamma \in [0,1).$$

Define

$$A = [0,\infty), \qquad B = (-\infty, 0],$$

and define the mapping $T : A \cup B \to A \cup B$ by

$$T(x) = -\frac{1}{2}x.$$

We first verify cyclicity. If $x \in A$, then $x \geqslant 0$ and hence $Tx = -\frac{1}{2}x \leqslant 0$, so $Tx \in B$. Similarly, if $x \in B$, then $x \leqslant 0$ and hence $Tx = -\frac{1}{2}x \geqslant 0$, so $Tx \in A$. Thus T is cyclic.

Next, let $x \in A$ and $y \in B$. Then

$$Tx = -\frac{1}{2}x, \qquad Ty = -\frac{1}{2}y.$$

Hence

$$|Tx - Ty| = \left| -\frac{1}{2}x + \frac{1}{2}y \right| = \frac{1}{2}|x-y|.$$

Also,

$$|Tx| + |Ty| = \frac{1}{2}|x| + \frac{1}{2}|y| = \frac{1}{2}(|x|+|y|).$$

Therefore,

$$m(Tx,Ty) = |Tx-Ty| + \gamma(|Tx|+|Ty|) = \frac{1}{2}|x-y| + \frac{\gamma}{2}(|x|+|y|) = \frac{1}{2}m(x,y).$$

Hence condition (1) holds with $\alpha = \frac{1}{2}$.

§5. Main Result

In this section, we establish existence and uniqueness of fixed points for cyclic contraction mappings in complete supermetric spaces.

Theorem 5.1 *Let (X,m) be a complete supermetric space and let $A, B \subseteq X$ be nonempty closed subsets. Assume that $T : A \cup B \to A \cup B$ is a cyclic contraction in the sense of Definition 4.1. Then T has a unique fixed point $x^* \in A \cap B$. Moreover, for any initial point $x_0 \in A \cup B$, the Picard iteration $x_{n+1} = Tx_n$ converges to x^* with respect to m.*

Proof Fix $x_0 \in A \cup B$ and define the Picard sequence $x_{n+1} = Tx_n$ for $n \geqslant 0$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point and the sequence is constant. Hence we assume $x_n \neq x_{n+1}$ for all n.

Since T is cyclic, the orbit alternates between the two sets; specifically, if $x_0 \in A$ then $x_{2n} \in A$ and $x_{2n+1} \in B$ for all n, and the reverse holds if $x_0 \in B$. Consequently, each successive pair (x_n, x_{n+1}) is a mixed pair, so the cyclic contraction inequality applies repeatedly along the orbit.

From the cyclic contraction property we obtain

$$m(x_{n+1}, x_{n+2}) = m(Tx_n, Tx_{n+1}) \leqslant \alpha\, m(x_n, x_{n+1}),$$

and an immediate induction gives

$$m(x_n, x_{n+1}) \leqslant \alpha^n m(x_0, x_1), \qquad n \geqslant 0.$$

Since $0 < \alpha < 1$, it follows that $m(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

The above geometric decay implies that the sequence satisfies the contraction-orbit condition known in supermetric spaces. In particular, contraction-generated sequences in complete supermetric spaces are Cauchy in the sense that

$$\lim_{n\to\infty} \sup\{m(x_n, x_k) : k > n\} = 0$$

(see Theorem 2.6, [9]). Hence $\{x_n\}$ is Cauchy. By completeness of (X, m), there exists $x^* \in X$ such that $m(x_n, x^*) \to 0$.

To verify that x^* is a fixed point, we use the contraction property again. For each n,

$$m(x_{n+1}, Tx^*) = m(Tx_n, Tx^*) \leqslant \alpha m(x_n, x^*).$$

Since $m(x_n, x^*) \to 0$, we obtain $m(x_{n+1}, Tx^*) \to 0$. By the convergence principle of supermetric spaces [9], this implies $m(x^*, Tx^*) = 0$. By the identity axiom (SM1), it follows that $Tx^* = x^*$.

Next, because the subsequence $\{x_{2n}\}$ lies in A and converges to x^*, and A is closed, we obtain $x^* \in A$. Similarly, $x^* \in B$ because $\{x_{2n+1}\} \subset B$ and B is closed. Hence $x^* \in A \cap B$.

Finally, suppose $u, v \in A \cap B$ are fixed points. Then

$$m(u, v) = m(Tu, Tv) \leqslant \alpha m(u, v).$$

Since $\alpha \in (0, 1)$, this implies $m(u, v) = 0$, and hence $u = v$ by (SM1). Thus the fixed point is unique.

Therefore, for every initial point $x_0 \in A \cup B$, the Picard iteration converges to the unique fixed point $x^* \in A \cap B$. □

Theorem 5.1 has several immediate consequences that clarify the behavior of cyclic contractions and connect the supermetric framework with classical fixed point theory.

Corollary 5.1(Banach-type contraction as a special case) *Let (X, m) be a complete supermetric* *space and let* $T : X \to X$ *satisfy*

$$m(Tx, Ty) \leqslant \alpha\, m(x, y) \qquad \textit{for all } x, y \in X,$$

for some $\alpha \in (0, 1)$*. Then* T *has a unique fixed point* $x^* \in X$ *and the Picard iteration* $x_{n+1} = Tx_n$ *converges to* x^* *for every* $x_0 \in X$*.*

Proof Apply Theorem 5.1 with $A = B = X$. Then T is trivially cyclic and the mixed-pair inequality reduces to the global contraction inequality. Hence the conclusion follows. □

Corollary 5.2(Uniqueness of the best proximity point when $A \cap B = \varnothing$) *Let* (X, m) *be a complete supermetric space and* $A, B \subseteq X$ *be nonempty closed sets. Assume* $T : A \cup B \to A \cup B$ *is cyclic and satisfies*

$$m(Tx, Ty) \leqslant \alpha\, m(x, y) \qquad \textit{for all } x \in A,\ y \in B,$$

for some $\alpha \in (0, 1)$*. If* $A \cap B = \varnothing$*, then* T *has no fixed point in* $A \cup B$*. Nevertheless, any orbit* $\{x_n\}$ *generated by Picard iteration satisfies*

$$m(x_n, x_{n+1}) \to 0,$$

and every cluster point (if it exists) must lie in $\overline{A} \cap \overline{B}$*.*

Proof If $A \cap B = \varnothing$ and $x = Tx$, then $x \in A$ implies $Tx \in B$, contradicting $Tx = x$, and similarly if $x \in B$. Thus no fixed point exists. The decay $m(x_n, x_{n+1}) \leqslant \alpha^n m(x_0, x_1)$ follows exactly as in the proof of Theorem 5.1, hence $m(x_n, x_{n+1}) \to 0$. If a subsequence converges to z, then even and odd subsequences converge to the same z, forcing z to belong to both closures. Since A, B are closed, this means $z \in A \cap B$, so in the disjoint case there can be no such limit point in X. □

Corollary 5.3(Error estimate for Picard iteration) *Under the hypotheses of Theorem* 5.1*, let* x^* *be the unique fixed point and* $x_{n+1} = Tx_n$ *be the Picard iteration. Then for every* $n \geqslant 0$*,*

$$m(x_{n+1}, x^*) \leqslant \alpha\, m(x_n, x^*) \leqslant \alpha^{n+1} m(x_0, x^*).$$

In particular, the convergence is at least geometric with ratio α*.*

Proof Since $x^* = Tx^*$ and each pair (x_n, x^*) is admissible in the cyclic estimate (because $x^* \in A \cap B$), we have

$$m(x_{n+1}, x^*) = m(Tx_n, Tx^*) \leqslant \alpha\, m(x_n, x^*),$$

and the bound follows by iteration. □

Corollary 5.4 *Under the hypotheses of Theorem* 5.1*, let* $\{x_n\}$ *and* $\{y_n\}$ *be Picard iterates* *generated from two initial points* $x_0, y_0 \in A \cup B$. *Then for all* $n \geqslant 0$,

$$m(x_n, y_n) \leqslant \alpha^n \, m(x_0, y_0)$$

whenever $(x_0, y_0) \in A \times B$ *or* $(x_0, y_0) \in B \times A$. *Consequently, the iteration is asymptotically stable:* $m(x_n, y_n) \to 0$.

Proof If (x_0, y_0) is a mixed pair, then by cyclicity each (x_n, y_n) remains a mixed pair. Hence the contraction inequality yields

$$m(x_{n+1}, y_{n+1}) = m(Tx_n, Ty_n) \leqslant \alpha \, m(x_n, y_n),$$

and iteration gives the estimate. □

Remark 5.1 Theorem 5.1 provides three key practical outputs: existence and uniqueness of an equilibrium $x^* \in A \cap B$, global convergence of alternating Picard orbits to x^*, and a quantitative geometric convergence rate governed by the cyclic contraction constant α. These features are the supermetric analogue of the classical Banach contraction mechanism [3] and [9].

Proposition 5.1 *Let* $X = \mathbb{R}^2$ *and write* $x = (x_1, x_2)$, $y = (y_1, y_2)$. *Equip* X *with the* ℓ^1 - *norm* $\|x\|_1 := |x_1| + |x_2|$ *and fix* $\gamma \in [0, 1)$. *Define*

$$m(x, y) := \|x - y\|_1 + \gamma\big(\|x\|_1 + \|y\|_1\big), \qquad x, y \in X.$$

Let

$$A := \{x \in X : \ x_1 \geqslant 0\}, \qquad B := \{x \in X : \ x_1 \leqslant 0\}.$$

For a fixed $\beta \in (0, 1)$ *define* $T : A \cup B \to A \cup B$ *by*

$$T(x_1, x_2) := (-\beta x_1, \ \beta x_2).$$

Then T *is a cyclic contraction on* $A \cup B$ *(in the sense of Definition* 4.1*,* A *and* B *are closed in* (X, m)*, and the unique fixed point of* T *is* $(0, 0) \in A \cap B$. *Consequently, Theorem* 5.1 *applies and every Picard orbit converges to* $(0, 0)$.

Proof We first note that m is nonnegative and symmetric because $\|\cdot\|_1$ is a norm. If $m(x, y) = 0$, then $\|x - y\|_1 = 0$, hence $x = y$; therefore (SM1) holds. To verify the supermetric control axiom, fix an arbitrary $z \in X$ and choose a nonzero vector $u \in X$. Define distinct sequences $x_n := \frac{u}{n}$ and $y_n := \frac{u}{n+1}$. Then

$$\|x_n - y_n\|_1 = \left\| \frac{u}{n} - \frac{u}{n+1} \right\|_1 = \frac{\|u\|_1}{n(n+1)} \to 0, \qquad \|x_n\|_1 + \|y_n\|_1 \to 0,$$

so $m(x_n, y_n) \to 0$. Moreover, $x_n \to 0$ and $y_n \to 0$ in $\|\cdot\|_1$, hence by continuity of the norm,

$$m(x_n, z) = \|x_n - z\|_1 + \gamma(\|x_n\|_1 + \|z\|_1) \to \|z\|_1 + \gamma\|z\|_1,$$

and similarly $m(y_n, z) \to \|z\|_1 + \gamma\|z\|_1$. Thus

$$\limsup_{n\to\infty} m(y_n, z) = \limsup_{n\to\infty} m(x_n, z),$$

and therefore the defining inequality in (SM3) holds for any constant $s > 1$ (for instance $s = 2$). Hence (X, m) is a supermetric space in the sense of Definition 2.1; in particular, since $\mathbb{R}^2$ is complete in $\|\cdot\|_1$, the induced Cauchy notion from m is complete as well (equivalently, one may invoke the completeness mechanism established for supermetric contractions in [9].

The sets A and B are closed because the coordinate map $x \mapsto x_1$ is continuous under $\|\cdot\|_1$, hence also under m on X, and $A = \{x : \ x_1 \geqslant 0\}$, $B = \{x : \ x_1 \leqslant 0\}$ are inverse images of closed rays.

We now check the cyclic property. If $x \in A$, then $x_1 \geqslant 0$, so $T(x)_1 = -\beta x_1 \leqslant 0$ and thus $T(x) \in B$. If $x \in B$, then $x_1 \leqslant 0$, so $T(x)_1 = -\beta x_1 \geqslant 0$ and thus $T(x) \in A$. Hence $T(A) \subseteq B$ and $T(B) \subseteq A$.

For the cyclic contraction inequality, let $x \in A$ and $y \in B$. Then

$$\|Tx - Ty\|_1 = \|(-\beta x_1, \beta x_2) - (-\beta y_1, \beta y_2)\|_1 = \|\big(-\beta(x_1 - y_1), \beta(x_2 - y_2)\big)\|_1 = \beta\|x - y\|_1,$$

and also

$$\|Tx\|_1 + \|Ty\|_1 = \beta\|x\|_1 + \beta\|y\|_1 = \beta(\|x\|_1 + \|y\|_1).$$

Substituting into the definition of m yields the exact scaling identity

$$m(Tx, Ty) = \beta\|x - y\|_1 + \gamma\beta(\|x\|_1 + \|y\|_1) = \beta\, m(x, y).$$

Thus (1) holds with $\alpha = \beta \in (0, 1)$, so T is a cyclic contraction.

Finally, if $x^* = (x_1^*, x_2^*)$ is a fixed point, then

$$(x_1^*, x_2^*) = T(x_1^*, x_2^*) = (-\beta x_1^*, \beta x_2^*),$$

which implies $(1 + \beta)x_1^* = 0$ and $(1 - \beta)x_2^* = 0$. Since $\beta \in (0, 1)$, we conclude $x_1^* = x_2^* = 0$. Hence the fixed point is unique and equals $(0, 0) \in A \cap B$.

All hypotheses of Theorem 5.1 are satisfied; therefore, for every $x_0 \in A \cup B$ the Picard iteration $x_{n+1} = Tx_n$ converges to $(0, 0)$ in the supermetric m. □

§6. Application to a Nonlinear Integral Equation

Let $X = C([0, 1], \mathbb{R})$ be the Banach space of all real-valued continuous functions on $[0, 1]$ equipped with the supremum norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

Define the supermetric

$$m(f,g) := \|f-g\|_\infty \qquad (f,g \in X).$$

Then (X,m) is a complete supermetric space, since every metric space is a supermetric space and completeness follows from completeness of the sup-norm space.

Consider the nonlinear integral equation

$$u(x) = \int_0^1 K(x,t,u(t))\,dt, \qquad x \in [0,1], \tag{2}$$

where $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the uniform Lipschitz condition: there exists $L \in (0,1)$ such that

$$|K(x,t,r) - K(x,t,q)| \leqslant L|r-q| \quad \text{for all } x,t \in [0,1],\ r,q \in \mathbb{R}. \tag{H}$$

Theorem 6.1 *Assume hypothesis* (H) *holds. Then the nonlinear integral equation* (2) *admits a unique solution* $u^* \in C([0,1],\mathbb{R})$. *Moreover, for any initial function* $u_0 \in X$, *the successive approximation sequence defined by*

$$u_{n+1} = Tu_n$$

converges uniformly on $[0,1]$ *to* u^*.

Proof Define the operator $T : X \to X$ by

$$(Tf)(x) = \int_0^1 K(x,t,f(t))\,dt.$$

We first show that T is well defined. Since K is continuous and f is continuous, the mapping $t \mapsto K(x,t,f(t))$ is continuous on $[0,1]$ for each fixed x. Hence $(Tf)(x)$ exists for all $x \in [0,1]$. Standard parameter continuity results for integrals imply that Tf is continuous on $[0,1]$, so $T(X) \subseteq X$.

Next we establish the contraction property. Let $f,g \in X$. Then for each $x \in [0,1]$, using the Lipschitz condition (H),

$$\begin{aligned}
|(Tf)(x) - (Tg)(x)| &= \left| \int_0^1 \big(K(x,t,f(t)) - K(x,t,g(t))\big)\,dt \right| \\
&\leqslant \int_0^1 |K(x,t,f(t)) - K(x,t,g(t))|\,dt \\
&\leqslant \int_0^1 L|f(t)-g(t)|\,dt \\
&\leqslant L\|f-g\|_\infty.
\end{aligned}$$

Taking supremum over $x \in [0,1]$ yields

$$\|Tf - Tg\|_\infty \leqslant L\|f-g\|_\infty.$$

Hence

$$m(Tf, Tg) \leqslant L\, m(f, g).$$

Thus T is a contraction mapping on (X, m) with contraction constant $L \in (0, 1)$. Since (X, m) is complete, Theorem 5.1 (with $A = B = X$) guarantees that T admits a unique fixed point $u^* \in X$.

Finally, if u^* is a fixed point of T, then

$$u^*(x) = (Tu^*)(x) = \int_0^1 K(x, t, u^*(t))\, dt,$$

so u^* is a solution of the integral equation (2). Conversely, any solution of (2) is a fixed point of T. Hence the solution is unique.

The convergence of the Picard iteration $u_{n+1} = Tu_n$ to u^* follows directly from the contraction property.

For related integral-equation applications involving rational-type nonlinear contractions, see [6]. □

§7. Conclusion

In this work, we developed a cyclic contraction framework in the setting of complete supermetric spaces and established a fixed point theorem ensuring existence, uniqueness, and convergence of Picard iterates. The obtained results extend classical contraction principles to a generalized distance structure where no triangle-type inequality is assumed, thereby remaining fully consistent with the intrinsic axiomatic structure of supermetric spaces.

The cyclic formulation allows contractive behavior to be imposed only across interacting subsets rather than globally, which makes the theory suitable for problems possessing alternating or decomposed domain structures. The convergence analysis is based on the geometric decay of successive orbit distances together with the sequence-control mechanism that characterizes supermetric spaces.

The applicability of the theoretical results was demonstrated through an application to a nonlinear integral equation, where existence and uniqueness of continuous solutions were obtained via the supermetric contraction framework. This confirms that cyclic supermetric methods can be effectively applied to nonlinear functional problems.

Future research directions include extensions to multi-cyclic structures, rational and implicit cyclic contractions, stochastic supermetric models, and applications to optimization, differential equations, and computational mathematics.

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References

[1] R. P. Agarwal and E. Karapınar, Rational contractive conditions in partial metric spaces, *Fixed Point Theory* 21.1 (2020), 33 – 49. doi: 10.24193/fpt-ro.2020. 1.03.

[2] I. Altun and H. Simsek, Some fixed point theorems on ordered metric spaces, *Fixed Point Theory and Applications* (2010), p. 147925, doi: 10.1155/2010/147925.

[3] Stefan Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations int'egrales, *Fundamenta Mathematicae* 3 (1922), 133 – 181, doi: 10.4064/fm-3-1-133-181.

[4] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces, *Nonlinear Analysis: Theory, Methods & Applications* 65.7 (2006), 1379 – 1393, doi: 10.1016/j.na.2005.10.017.

[5] Stefan Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis* 1 (1993), 5 – 11.

[6] N. Hussain and M. A. Kutbi, Applications of rational contractions in nonlinear in- tegral equations, *Journal of Inequalities and Applications* (2021), p. 2671, doi: 10.1186/s13660-021-02671-y.12.

[7] M. Jleli and B. Samet, New fixed point results in ordered metric type spaces, *Fixed Point Theory and Applications* (2022), p. 22, doi: 10.1186/s13663-022-00759-y.

[8] Erdal Karapınar, Contractions in rational forms in the framework of super metric spaces, *Mathematics* 10.17 (2022), p. 3077, doi: 10.3390/math10173077.

[9] Erdal Karapınar and Farshid Khojasteh, Super metric spaces, *Filomat* 36.10 (2022), 3545 – 3549, doi: 10.2298/FIL2210545K.

[10] W. A. Kirk, P. S. Srinivasan, and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, *Fixed Point Theory* 4.1 (2003), pp. 79 – 89.

[11] S. G. Matthews, Partial metric topology, *Annals of the New York Academy of Sciences* 728.1 (1994), pp. 183 – 197, doi: 10.1111/j.1749-6632.1994.tb44144.x.

[12] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and applications to matrix equations, *Proceedings of the American Mathematical Society* 132.5 (2004), pp. 1435 – 1443, doi: 10.1090/S0002-9939-03-07220-4.

[13] I. A. Rus, Cyclic representations and fixed points, *Annals of the Tiberiu Popoviciu Seminar on Functional Equations, Approximation and Convexity* 3 (2005), pp. 171 – 178.

[14] B. Samet, C. Vetro, and P. Vetro, Fixed point theorems for α - ψ contractive type mappings, *Nonlinear Analysis: Theory, Methods & Applications* 75.4 (2012), pp. 2154 – 2165, doi: 10.1016/j.na.2011.10.014.

[15] S. Sedghi, N. Shobe, and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, *Mathematical and Computer Modelling* 54.11-12 (2012), pp. 2827 – 2836, doi: 10.1016/j.mcm.2011.12.048.

[16] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric

spaces, *Fixed Point Theory and Applications* (2012), p. 94, doi: 10.1186/1687- 1812-2012-94.

[17] Lotfi A. Zadeh, Fuzzy sets, *Information and Control* 8.3 (1965), pp. 338 – 353, doi: 10.1016/S0019-9958(65)90241-X.