

Different Solution Method for Fuzzy Boundary Value Problem with Fuzzy Parameter

Tahir Ceylan

(Department of Mathematic, University of Sinop, Turkey)

Nihat Altinişik

(Department of Mathematics, University of Ondokuz Mays, Turkey)

E-mail: tceylan@sinop.edu.tr, anihat@omu.edu.tr

Abstract: In this paper Titchmark's method is applied for finding the fuzzy eigenvalues and fuzzy eigenfunctions of a fuzzy boundary value problem under generalized Hukuhara differentiability (gH-differentiability). The states of the obtained fuzzy eigenvalues are examined. In addition, this method is illustrated by solving fuzzy problem.

Key Words: gH-derivative, Fuzzy parameter, Fuzzy boundary value problem.

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§1. Introduction

Eigenvalue problems that are formulated by modeling the real case into mathematical models are very important in many fields such as quantum mechanics, control theory, vibration, heat problems. Eigenvalues and eigenfunctions play an important role in the study of ordinary and partial equations. However, in many real world problems, at least some of the parameters are represented by fuzzy rather than crisp numbers. In this case, it is so important to develop classical procedure to fuzzy procedure that would properly treat and find fuzzy eigenvalues and fuzzy differential equations.

The methods used in the solution of fuzzy differential equations have been examined with various studies ([1], [2], [5], [6], [7], [8], [9], [10], [11], [12], [13], [16]). One of the most well-known definitions of difference and derivative for fuzzy set value functions was given by Hukuhara in [10]. By using the H-derivative, Kaleva in [3] started to develop a theory for fuzzy differential equations. Many works have been done by several authers in theoretical and applied fields for fuzzy differential equations with the Hukuhara derivative ([6], [10], [11]). But in some cases this approach suffers certain disadvantages since the diameter of the solutions is unbounded as time t increases ([3], [12]). So here we use gH-difference and gH-derivative to solve FDE under much less restrictive conditions [10]. Solution method of fuzzy boundary value problem with crisp eigenvalue parameter has been studied in ([11], [16]). At first, Buckley found fuzzy

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eigenvalues for matrix containing fuzzy numbers [17]. After, Chiao has studied generalized fuzzy eigenvalues [18].

In this paper we consider the two point fuzzy boundary value problem

$$L = -\frac{d^2}{dx^2}$$

$$L\hat{u} = \hat{\lambda}\hat{u}, \quad x \in [a, b] \quad (1.1)$$

which satisfy the conditions

$$\hat{a}_1\hat{u}(a) = \hat{a}_2\hat{u}'(a) \quad (1.2)$$

$$\hat{b}_1\hat{u}(b) = \hat{b}_2\hat{u}'(b) \quad (1.3)$$

where $\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2$ are positive fuzzy numbers, $\hat{\lambda} > 0$ fuzzy parameter and $\hat{u}(x)$ positive fuzzy function.

§2. Notation and Preliminaries

In this section, we give some concepts and results besides the essential notations which will be used throughout the paper.

Let \hat{u} be a fuzzy subset on \mathbb{R} , i.e. a mapping $\hat{u} : \mathbb{R} \rightarrow [0, 1]$ associating with each real number t its grade of membership $\hat{u}(t)$.

In this paper, the concept of fuzzy real numbers (fuzzy intervals) is considered in the sense of Xiao and Zhu which is defined below:

Definition 2.1([14]) *A fuzzy subset \hat{u} on \mathbb{R} is called a fuzzy real number (fuzzy intervals), whose α -cut set is denoted by $[\hat{u}]_\alpha$, i.e., $[\hat{u}]_\alpha = \{t : \hat{u}(t) \geq \alpha\}$, if it satisfies two axioms:*

(N1) *There exists $r \in \mathbb{R}$ such that $\hat{u}(r) = 1$.*

(N2) *For all $0 < \alpha \leq 1$, there exist real numbers $-\infty < u_\alpha^- \leq u_\alpha^+ < +\infty$ such that $[\hat{u}]_\alpha$ is equal to the closed interval $[u_\alpha^-, u_\alpha^+]$.*

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $F(\mathbb{R})$. $F_K(\mathbb{R})$, the family of fuzzy sets of \mathbb{R} whose α -cuts are nonempty compact subsets of \mathbb{R} . If $\hat{u} \in F(\mathbb{R})$ and $\hat{u}(t) = 0$ whenever $t < 0$, then \hat{u} is called a non-negative fuzzy real number and $F^+(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. For all $\hat{u} \in F^+(\mathbb{R})$ and each $\alpha \in (0, 1]$, real number u_α^- is positive.

The fuzzy real number $\tilde{r} \in F(\mathbb{R})$ defined by

$$\tilde{r} = \begin{cases} 1, & t = r \\ 0, & t \neq r, \end{cases}$$

it follows that \mathbb{R} can be embedded in $F(\mathbb{R})$, that is if $\tilde{r} \in (-\infty, \infty)$, then $\tilde{r} \in F(\mathbb{R})$ satisfies $\tilde{r}(t) = \tilde{0}(t - r)$ and α -cut of \tilde{r} is given by $[\tilde{r}]_\alpha = [r, r], \alpha \in (0, 1]$.

Definition 2.2([2]) *An arbitrary fuzzy number in the parametric form is represented by an ordered pair of functions (u_α^-, u_α^+) , $0 \leq \alpha \leq 1$, which satisfy the following requirements:*

(i) u_α^- is bounded non-decreasing left continuous function on $(0, 1]$ and right-continuous for $\alpha = 0$;

(ii) u_α^+ is bounded non-increasing left continuous function on $(0, 1]$ and right-continuous for $\alpha = 0$;

(iii) $u_\alpha^- \leq u_\alpha^+$, $0 \leq \alpha \leq 1$.

Definition 2.3([2]) *For $\hat{u}, \hat{v} \in F(\mathbb{R})$, and $\lambda \in \mathbb{R}$, the sum $\hat{u} \oplus \hat{v}$ and the product $\lambda \odot \hat{u}$ are defined by*

$$\begin{aligned} [\hat{u} \oplus \hat{v}]^\alpha &= [\hat{u}]^\alpha + [\hat{v}]^\alpha = \{x + y : x \in [\hat{u}]^\alpha, y \in [\hat{v}]^\alpha\}, \quad \forall \alpha \in [0, 1], \\ [\lambda \odot \hat{u}]^\alpha &= \lambda \odot [\hat{u}]^\alpha = \{\lambda x : x \in [\hat{u}]^\alpha\}, \quad \forall \alpha \in [0, 1]. \end{aligned}$$

Define $D : F(\mathbb{R}) \times F(\mathbb{R}) \rightarrow \mathbb{R}^+ \cup \{0\}$ by the equation

$$D(\hat{u}, \hat{v}) = \sup_{0 < \alpha \leq 1} \{ \max[|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|] \}$$

where $[\hat{u}]^\alpha = [u_\alpha^-, u_\alpha^+]$, $[\hat{v}]^\alpha = [v_\alpha^-, v_\alpha^+]$. Then it is easy to show that D is a metric in $F(\mathbb{R})$.

Definition 2.4([9]) *Let $\hat{u}, \hat{v} \in F(\mathbb{R})$. If there exist $\hat{w} \in F(\mathbb{R})$ such that $\hat{u} = \hat{v} \oplus \hat{w}$, then \hat{w} is called the Hukuhara difference of \hat{u} and \hat{v} and it is denoted by $\hat{u} \ominus_h \hat{v}$. If $\hat{u} \ominus_h \hat{v}$ exists, its α -cuts are*

$$[\hat{u} \ominus_h \hat{v}]^\alpha = [u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+]$$

for $\alpha \in [0, 1]$.

Definition 2.5([10]) *The generalized Hukuhara difference of two fuzzy numbers $\hat{u}, \hat{v} \in F(\mathbb{R})$ is defined as follows*

$$[\hat{u} \ominus_{gH} \hat{v}] = \hat{w} \Leftrightarrow \begin{cases} (i) & \hat{u} = \hat{v} \oplus \hat{w} \\ \text{or } (ii) & \hat{v} = \hat{u} \oplus (-1)\hat{w}. \end{cases}$$

In terms of α -cuts we have

$$[\hat{u} \ominus_{gH} \hat{v}]^\alpha = [\min \{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}, \max \{u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+\}]$$

and if the H -difference exists, then $\hat{u} \ominus_h \hat{v} = \hat{u} \ominus_{gH} \hat{v}$; the conditions for the existence of $\hat{w} = \hat{u} \ominus_{gH} \hat{v} \in F(\mathbb{R})$ are

$$\begin{aligned} (i) & \left\{ \begin{array}{l} w_\alpha^- = u_\alpha^- - v_\alpha^- \text{ and } w_\alpha^+ = u_\alpha^+ - v_\alpha^+, \\ \text{with } w_\alpha^- \text{ increasing, } w_\alpha^+ \text{ decreasing, } w_\alpha^- \leq w_\alpha^+ \end{array} \right. \\ (ii) & \left\{ \begin{array}{l} w_\alpha^- = u_\alpha^+ - v_\alpha^+ \text{ and } w_\alpha^+ = u_\alpha^- - v_\alpha^-, \\ \text{with } w_\alpha^- \text{ increasing, } w_\alpha^+ \text{ decreasing, } w_\alpha^- \leq w_\alpha^+ \end{array} \right. \end{aligned}$$

for all $\alpha \in [0, 1]$.

Remark 2.1 Throughout the rest of this paper, we assume that $\widehat{u} \ominus_{gH} \widehat{v} \in F(\mathbb{R})$ and α -cut representation of fuzzy-valued function $f : (a, b) \rightarrow F(\mathbb{R})$ is expressed by $[f(x)]^\alpha = [(f_\alpha^-)(x), (f_\alpha^+)(x)]$, $x \in (a, b)$ for each $\alpha \in [0, 1]$.

Definition 2.6([10]) Let $x_0 \in (a, b)$ and h be such that $x_0 + h \in (a, b)$, then the gH -derivative of a function $f : (a, b) \rightarrow F(\mathbb{R})$ at x_0 is defined as

$$f'_{gH}(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) \ominus_{gH} f(x_0)}{h}. \quad (2.1)$$

If $f'_{gH}(x_0) \in F(\mathbb{R})$ satisfying (2.1) exists, we say that f is generalized Hukuhara differentiable (gH -differentiable for short) at x_0 .

Definition 2.7([10]) Let $f : [a, b] \rightarrow F(\mathbb{R})$ and $x_0 \in (a, b)$ with $f_\alpha^-(x)$ and $f_\alpha^+(x)$ both differentiable at x_0 . We say that f is $[(i) - gH]$ -differentiable at x_0 if

(i) $[f'_{gH}(x_0)]^\alpha = \left[\left\{ (f_\alpha^-)'(x), (f_\alpha^+)'(x) \right\} \right]$, $\forall \alpha \in [0, 1]$ and f is $[(ii) - gH]$ -differentiable at x_0 if

(ii) $[f'_{gH}(x_0)]^\alpha = \left[\left\{ (f_\alpha^+)'(x), (f_\alpha^-)'(x) \right\} \right]$, $\forall \alpha \in [0, 1]$.

Definition 2.8([4]) The second generalized Hukuhara derivative of a fuzzy function $f : [a, b] \rightarrow F(\mathbb{R})$ at x_0 is defined as

$$f''_{gH}(x_0) = \lim_{h \rightarrow 0} \frac{f'(x_0 + h) \ominus_{gH} f'(x_0)}{h},$$

if $f''_{gH}(x_0) \in F(\mathbb{R})$, we say that $f'_{gH}(x)$ is generalized Hukuhara derivative at x_0 .

Also we say that $f'_{gH}(x)$ is $[(i) - gH]$ -differentiable at x_0 if

$$f'_{i.gH}(x_0, \alpha) = \begin{cases} \left[(f_\alpha^-)''(x_0), (f_\alpha^+)''(x_0) \right], & \text{if } f \text{ is } [(i) - gH] \text{ - differentiable on } (a, b) \\ \left[(f_\alpha^+)''(x_0), (f_\alpha^-)''(x_0) \right], & \text{if } f \text{ is } [(ii) - gH] \text{ - differentiable on } (a, b) \end{cases},$$

for all $\alpha \in [0, 1]$ and that $f'_{gH}(x)$ is $[(ii) - gH]$ -differentiable at x_0 if

$$f'_{ii.gH}(x_0, \alpha) = \begin{cases} \left[(f_\alpha^+)''(x_0), (f_\alpha^-)''(x_0) \right], & \text{if } f \text{ is } [(i) - gH] \text{ - differentiable on } (a, b) \\ \left[(f_\alpha^-)''(x_0), (f_\alpha^+)''(x_0) \right], & \text{if } f \text{ is } [(ii) - gH] \text{ - differentiable on } (a, b) \end{cases},$$

for all $\alpha \in [0, 1]$.

§3. Solution Method of the Two Point Boundary Problem

In this section we concern with fuzzy eigenvalues and fuzzy eigenfunctions of two-point fuzzy boundary value problems. To do this, at first we need to use fuzzy derivatives. So here we use gH -difference and gH -derivative [10]. For the solution of the problem, we apply the solution

method of Titchmark using gH- derivative [15]. Here we accept positive fuzzy functions.

Definition 3.1 Let $\widehat{u} : [a, b] \subset \mathbb{R} \rightarrow F(\mathbb{R})$ be a fuzzy function and

$$\left[\widehat{u}(x, \widehat{\lambda}) \right]^\alpha = [u_\alpha^-(x, \lambda_\alpha^-, \lambda_\alpha^+), u_\alpha^+(x, \lambda_\alpha^-, \lambda_\alpha^+)]$$

be the α -cut representation of the fuzzy function $\widehat{u}(x)$ for all $x \in [a, b]$ and $\alpha \in [0, 1]$. If the fuzzy differential equation (1.1) has the nontrivial solutions such that $u_\alpha^-(x, \lambda_\alpha^-, \lambda_\alpha^+) \neq 0$, $u_\alpha^+(x, \lambda_\alpha^-, \lambda_\alpha^+) \neq 0$ and u_α^- and u_α^+ satisfy Definition 2.1 conditions, then $\widehat{\lambda}$ is fuzzy eigenvalue of (1.1).

Consider the fuzzy eigenvalues of the fuzzy boundary value problem (1.1) – (1.3). Four different solutions are obtained from this problem with generalized Hukuhara derivative as follows.

— If \widehat{u} is (i) – gH differentiable and \widehat{u}' is (ii) – gH differentiable then we say problem (1.1) – (1.3) has (1) – solution.

— If \widehat{u} is (ii) – gH differentiable and \widehat{u}' is (i) – gH differentiable then we say problem (1.1) – (1.3) has (2) – solution.

— If \widehat{u} and \widehat{u}' are (i) – gH differentiable then we say problem (1.1) – (1.3) has (3) – solution.

— If \widehat{u} and \widehat{u}' are (ii) – gH differentiable then we say problem (1.1) – (1.3) has (4) – solution.

For (1)- solution, using the α – cut sets and fuzzy arithmetic we get from (1.1) – (1.3) for $[\widehat{\lambda}]^\alpha = [\lambda_\alpha^-, \lambda_\alpha^+] = [(k_\alpha^-)^2, (k_\alpha^+)^2]$, $k_\alpha^-, k_\alpha^+ > 0$ that

$$\left[- (u_\alpha^-)''(x), - (u_\alpha^+)''(x) \right] = \left[(k_\alpha^-)^2, (k_\alpha^+)^2 \right] [u_\alpha^-(x), u_\alpha^+(x)] \quad (3.1)$$

$$\left[(a_1)_\alpha^-, (a_1)_\alpha^+ \right] [u_\alpha^-(a), u_\alpha^+(a)] = \left[(a_2)_\alpha^-, (a_2)_\alpha^+ \right] \left[(u_\alpha^-)'(a), (u_\alpha^+)'(a) \right] \quad (3.2)$$

$$\left[(b_1)_\alpha^-, (b_1)_\alpha^+ \right] [u_\alpha^-(b), u_\alpha^+(b)] = \left[(b_2)_\alpha^-, (b_2)_\alpha^+ \right] \left[(u_\alpha^-)'(b), (u_\alpha^+)'(b) \right]. \quad (3.3)$$

The general solution of the fuzzy differential equation (3.1) is

$$\left[\widehat{u}(x, \widehat{\lambda}) \right]^\alpha = [u_\alpha^-(x, k_\alpha^-), u_\alpha^+(x, k_\alpha^+)]$$

where

$$u_\alpha^-(x, k_\alpha^-) = C_1(\alpha, k_\alpha^-) u_1(x, k_\alpha^-) + C_2(\alpha, k_\alpha^-) u_2(x, k_\alpha^-)$$

$$u_\alpha^+(x, k_\alpha^+) = C_3(\alpha, k_\alpha^+) u_1(x, k_\alpha^+) + C_4(\alpha, k_\alpha^+) u_2(x, k_\alpha^+).$$

In view of the [19] the fuzzy problem

$$\left[- (u_\alpha^-)''(x), - (u_\alpha^+)''(x) \right] = \left[(k_\alpha^-)^2, (k_\alpha^+)^2 \right] [u_\alpha^-(x), u_\alpha^+(x)] \quad (3.4)$$

$$\begin{aligned} [u_{\alpha}^{-}(a), u_{\alpha}^{+}(a)] &= [(a_2)_{\alpha}^{-}, (a_2)_{\alpha}^{+}] \\ [(u_{\alpha}^{-})'(a), (u_{\alpha}^{+})'(a)] &= [(a_1)_{\alpha}^{-}, (a_1)_{\alpha}^{+}] \end{aligned} \quad (3.5)$$

has an unique fuzzy solution $[\hat{u}(x, \hat{\lambda})]^{\alpha} = [\hat{\Phi}(x, \hat{\lambda})]^{\alpha}$ for each $x \in [a, b]$ [20]. This solution can be expressed as

$$\begin{aligned} \Phi_{\alpha}^{-}(x, k_{\alpha}^{-}) &= C_{11} \cos((k_{\alpha}^{-})x) + C_{12} \sin((k_{\alpha}^{-})x) \\ \Phi_{\alpha}^{+}(x, k_{\alpha}^{+}) &= C_{13} \cos((k_{\alpha}^{+})x) + C_{14} \sin((k_{\alpha}^{+})x). \end{aligned} \quad (3.6)$$

and we write (3.5) initial conditions in (3.6) such that

$$\begin{aligned} \Phi_{\alpha}^{-}(a, k_{\alpha}^{-}) &= C_{11} \cos((k_{\alpha}^{-})a) + C_{12} \sin((k_{\alpha}^{-})a) = (a_2)_{\alpha}^{-} \\ (\Phi_{\alpha}^{-})'(a, k_{\alpha}^{-}) &= -(k_{\alpha}^{-}) C_{11} \sin((k_{\alpha}^{-})a) + (k_{\alpha}^{-}) C_{12} \cos((k_{\alpha}^{-})a) = (a_1)_{\alpha}^{-}. \end{aligned}$$

From the determinant of the coefficients matrix of the above linear system, we get C_{11} and C_{12} such that

$$\begin{aligned} C_{11} &= \frac{\begin{vmatrix} (a_2)_{\alpha}^{-} & \sin((k_{\alpha}^{-})a) \\ (a_1)_{\alpha}^{-} & (k_{\alpha}^{-}) \cos((k_{\alpha}^{-})a) \end{vmatrix}}{\begin{vmatrix} \cos((k_{\alpha}^{-})a) & \sin((k_{\alpha}^{-})a) \\ -(k_{\alpha}^{-}) \sin((k_{\alpha}^{-})a) & (k_{\alpha}^{-}) \cos((k_{\alpha}^{-})a) \end{vmatrix}} \\ &= (a_2)_{\alpha}^{-} \cos((k_{\alpha}^{-})a) - (a_1)_{\alpha}^{-} \cdot \frac{\sin((k_{\alpha}^{-})a)}{(k_{\alpha}^{-})}, \\ C_{12} &= \frac{\begin{vmatrix} \cos((k_{\alpha}^{-})a) & (a_2)_{\alpha}^{-} \\ -(k_{\alpha}^{-}) \sin((k_{\alpha}^{-})a) & (a_1)_{\alpha}^{-} \end{vmatrix}}{\begin{vmatrix} \cos((k_{\alpha}^{-})a) & \sin((k_{\alpha}^{-})a) \\ -(k_{\alpha}^{-}) \sin((k_{\alpha}^{-})a) & (k_{\alpha}^{-}) \cos((k_{\alpha}^{-})a) \end{vmatrix}} \\ &= (a_2)_{\alpha}^{-} \sin((k_{\alpha}^{-})a) - (a_1)_{\alpha}^{-} \cdot \frac{\cos((k_{\alpha}^{-})a)}{(k_{\alpha}^{-})}. \end{aligned} \quad (3.7)$$

Substituting this (3.7) coefficients the above equations in (3.6), the general solution is obtained as

$$\begin{aligned} \Phi_{\alpha}^{-}(x, k_{\alpha}^{-}) &= \left((a_2)_{\alpha}^{-} \cos((k_{\alpha}^{-})a) - (a_1)_{\alpha}^{-} \cdot \frac{\sin((k_{\alpha}^{-})a)}{(k_{\alpha}^{-})} \right) \cos((k_{\alpha}^{-})x) \\ &+ \left((a_2)_{\alpha}^{-} \sin((k_{\alpha}^{-})a) + (a_1)_{\alpha}^{-} \cdot \frac{\cos((k_{\alpha}^{-})a)}{(k_{\alpha}^{-})} \right) \sin((k_{\alpha}^{-})x). \end{aligned} \quad (3.8)$$

Similarly we find $\Phi_\alpha^+(x, k_\alpha^+)$ as

$$\begin{aligned} \Phi_\alpha^+(x, k_\alpha^+) &= \left((a_2)_\alpha^+ \cos((k_\alpha^+)a) - (a_1)_\alpha^+ \frac{\sin((k_\alpha^+)a)}{(k_\alpha^+)} \right) \cos((k_\alpha^+)x) \\ &\quad + \left((a_2)_\alpha^+ \sin((k_\alpha^+)a) + (a_1)_\alpha^+ \frac{\cos((k_\alpha^+)a)}{(k_\alpha^+)} \right) \sin((k_\alpha^+)x). \end{aligned} \quad (3.9)$$

Again in view of the [19] the fuzzy differential equation (3.4) has an unique fuzzy solution $[\hat{u}(x, \hat{\lambda})]^\alpha = [\hat{\chi}(x, \hat{\lambda})]^\alpha$ satisfying the initial conditions

$$\begin{aligned} [u_\alpha^-(b), u_\alpha^+(b)] &= [(b_2)_\alpha^-, (b_2)_\alpha^+] \\ [(u_\alpha^-)'(b), (u_\alpha^+)'(b)] &= [(b_1)_\alpha^-, (b_1)_\alpha^+] \end{aligned} \quad (3.10)$$

for each $x \in [a, b]$ [20]. This solution can be expressed as

$$\begin{aligned} \chi_\alpha^-(x, k_\alpha^-) &= C_{21} \cos((k_\alpha^-)x) + C_{22} \sin((k_\alpha^-)x) \\ \chi_\alpha^+(x, k_\alpha^+) &= C_{23} \cos((k_\alpha^+)x) + C_{24} \sin((k_\alpha^+)x). \end{aligned} \quad (3.11)$$

and we write (3.10) initial conditions in (3.11) such that

$$\begin{aligned} \chi_\alpha^-(b, k_\alpha^-, k_\alpha^+) &= C_{21} \cos((k_\alpha^-)b) + C_{22} \sin((k_\alpha^-)b) = (b_2)_\alpha^- \\ (\chi_\alpha^-)'(b, k_\alpha^-, k_\alpha^+) &= -(k_\alpha^-) C_{21} \sin((k_\alpha^-)b) + (k_\alpha^-) C_{22} \cos((k_\alpha^-)b) = (b_1)_\alpha^-. \end{aligned}$$

From the determinant of the coefficients matrix of the above linear system, we get C_{21} and C_{22} such that

$$\begin{aligned} C_{21} &= \frac{\begin{vmatrix} (b_2)_\alpha^- & \sin((k_\alpha^-)b) \\ (b_1)_\alpha^- & (k_\alpha^-) \cos((k_\alpha^-)b) \end{vmatrix}}{\begin{vmatrix} \cos((k_\alpha^-)b) & \sin((k_\alpha^-)b) \\ -(k_\alpha^-) \sin((k_\alpha^-)b) & (k_\alpha^-) \cos((k_\alpha^-)b) \end{vmatrix}} \\ &= (b_2)_\alpha^- \cos((k_\alpha^-)b) - (b_1)_\alpha^- \frac{\sin((k_\alpha^-)b)}{(k_\alpha^-)}, \\ C_{22} &= \frac{\begin{vmatrix} \cos((k_\alpha^-)b) & (b_2)_\alpha^- \\ -(k_\alpha^-) \sin((k_\alpha^-)b) & (b_1)_\alpha^- \end{vmatrix}}{\begin{vmatrix} \cos((k_\alpha^-)b) & \sin((k_\alpha^-)b) \\ -(k_\alpha^-) \sin((k_\alpha^-)b) & (k_\alpha^-) \cos((k_\alpha^-)b) \end{vmatrix}} \\ &= (a_2)_\alpha^- \sin((k_\alpha^-)a) - (a_1)_\alpha^- \frac{\cos((k_\alpha^-)a)}{(k_\alpha^-)}. \end{aligned} \quad (3.12)$$

Substituting this (3.12) coefficients the above equations in (3.11), the general solution is obtained as

$$\begin{aligned}\chi_{\alpha}^{-}(x, k_{\alpha}^{-}) &= \left((b_2)_{\alpha}^{-} \cos((k_{\alpha}^{-})b) - (b_1)_{\alpha}^{-} \frac{\sin((k_{\alpha}^{-})b)}{(k_{\alpha}^{-})} \right) \cos((k_{\alpha}^{-})x) \\ &+ \left((b_2)_{\alpha}^{-} \sin((k_{\alpha}^{-})b) + (b_1)_{\alpha}^{-} \frac{\cos((k_{\alpha}^{-})b)}{(k_{\alpha}^{-})} \right) \sin((k_{\alpha}^{-})x).\end{aligned}\quad (3.13)$$

Similarly we find $\chi_{\alpha}^{+}(x, k_{\alpha}^{-})$

$$\begin{aligned}\chi_{\alpha}^{+}(x, k_{\alpha}^{+}) &= \left((b_2)_{\alpha}^{+} \cos((k_{\alpha}^{+})b) - (b_1)_{\alpha}^{+} \frac{\sin((k_{\alpha}^{+})b)}{(k_{\alpha}^{+})} \right) \cos((k_{\alpha}^{+})x) \\ &+ \left((b_2)_{\alpha}^{+} \sin((k_{\alpha}^{+})b) + (b_1)_{\alpha}^{+} \frac{\cos((k_{\alpha}^{+})b)}{(k_{\alpha}^{+})} \right) \sin((k_{\alpha}^{+})x).\end{aligned}\quad (3.14)$$

Then, from (3.8) – (3.9) and (3.13) – (3.14) we find Wronskian function as

$$\begin{aligned}W(\Phi_{\alpha}^{-}, \chi_{\alpha}^{-})(x, k_{\alpha}^{-}) &= \Phi_{\alpha}^{-}(x, k_{\alpha}^{-})(\chi_{\alpha}^{-})'(x, k_{\alpha}^{-}) - \chi_{\alpha}^{-}(x, k_{\alpha}^{-})(\Phi_{\alpha}^{-})'(x, k_{\alpha}^{-}) \\ &= \left(((a_2)_{\alpha}^{-}(b_1)_{\alpha}^{-} - (a_1)_{\alpha}^{-}(b_2)_{\alpha}^{-}) \cos((k_{\alpha}^{-})(a-b)) \right) \\ &\quad - \left((k_{\alpha}^{-})(a_2)_{\alpha}^{-}(b_2)_{\alpha}^{-} + \frac{(a_1)_{\alpha}^{-}(b_1)_{\alpha}^{-}}{(k_{\alpha}^{-})} \right) \sin((k_{\alpha}^{-})(a-b))\end{aligned}$$

and similarly, we have

$$\begin{aligned}W(\Phi_{\alpha}^{+}, \chi_{\alpha}^{+})(x, k_{\alpha}^{+}) &= \Phi_{\alpha}^{+}(x, k_{\alpha}^{-}, k_{\alpha}^{+})(\chi_{\alpha}^{+})'(x, k_{\alpha}^{-}, k_{\alpha}^{+}) - \chi_{\alpha}^{+}(x, k_{\alpha}^{-}, k_{\alpha}^{+})(\Phi_{\alpha}^{+})'(x, k_{\alpha}^{-}, k_{\alpha}^{+}) \\ &= \left(((a_2)_{\alpha}^{+}(b_1)_{\alpha}^{+} - (a_1)_{\alpha}^{+}(b_2)_{\alpha}^{+}) \cos((k_{\alpha}^{+})(a-b)) \right) \\ &\quad - \left((k_{\alpha}^{+})(a_2)_{\alpha}^{+}(b_2)_{\alpha}^{+} + \frac{(a_1)_{\alpha}^{+}(b_1)_{\alpha}^{+}}{(k_{\alpha}^{+})} \right) \sin((k_{\alpha}^{+})(a-b))\end{aligned}$$

For (2)- solution, we get from (1.1) – (1.3) that

$$\begin{aligned}[-(u_{\alpha}^{-})''(x), -(u_{\alpha}^{+})''(x)] &= [(k_{\alpha}^{-})^2, (k_{\alpha}^{+})^2] [u_{\alpha}^{-}(x), u_{\alpha}^{+}(x)] \\ [(a_1)_{\alpha}^{-}, (a_1)_{\alpha}^{+}] [u_{\alpha}^{-}(a), u_{\alpha}^{+}(a)] &= [(a_2)_{\alpha}^{-}, (a_2)_{\alpha}^{+}] [(u_{\alpha}^{+})'(a), (u_{\alpha}^{-})'(a)] \\ [(b_1)_{\alpha}^{-}, (b_1)_{\alpha}^{+}] [u_{\alpha}^{-}(b), u_{\alpha}^{+}(b)] &= [(b_2)_{\alpha}^{-}, (b_2)_{\alpha}^{+}] [(u_{\alpha}^{+})'(b), (u_{\alpha}^{-})'(b)].\end{aligned}$$

From (1) – solution similarly we find $[\widehat{\Phi}(x, \widehat{\lambda})]_{\alpha}$ and $[\widehat{\chi}(x, \widehat{\lambda})]_{\alpha}$ solution functions such

that

$$\begin{aligned}
 \Phi_{\alpha}^{-}(x, k_{\alpha}^{-}) &= \left((a_2)_{\alpha}^{-} \cos((k_{\alpha}^{-})a) - (a_1)_{\alpha}^{+} \cdot \frac{\sin((k_{\alpha}^{-})a)}{(k_{\alpha}^{-})} \right) \cos((k_{\alpha}^{-})x) \\
 &\quad + \left((a_2)_{\alpha}^{-} \sin((k_{\alpha}^{-})a) + (a_1)_{\alpha}^{+} \frac{\cos((k_{\alpha}^{-})a)}{(k_{\alpha}^{-})} \right) \sin((k_{\alpha}^{-})x) \\
 \Phi_{\alpha}^{+}(x, k_{\alpha}^{+}) &= \left((a_2)_{\alpha}^{+} \cos((k_{\alpha}^{+})a) - (a_1)_{\alpha}^{-} \cdot \frac{\sin((k_{\alpha}^{+})a)}{(k_{\alpha}^{+})} \right) \cos((k_{\alpha}^{+})x) \\
 &\quad + \left((a_2)_{\alpha}^{+} \sin((k_{\alpha}^{+})a) + (a_1)_{\alpha}^{-} \frac{\cos((k_{\alpha}^{+})a)}{(k_{\alpha}^{+})} \right) \sin((k_{\alpha}^{+})x). \tag{3.15}
 \end{aligned}$$

and

$$\begin{aligned}
 \chi_{\alpha}^{-}(x, k_{\alpha}^{-}) &= \left((b_2)_{\alpha}^{-} \cos((k_{\alpha}^{-})b) - (b_1)_{\alpha}^{+} \frac{\sin((k_{\alpha}^{-})b)}{(k_{\alpha}^{-})} \right) \cos((k_{\alpha}^{-})x) \\
 &\quad + \left((b_2)_{\alpha}^{-} \sin((k_{\alpha}^{-})b) + (b_1)_{\alpha}^{+} \frac{\cos((k_{\alpha}^{-})b)}{(k_{\alpha}^{-})} \right) \sin((k_{\alpha}^{-})x) \\
 \chi_{\alpha}^{+}(x, k_{\alpha}^{+}) &= \left((b_2)_{\alpha}^{+} \cos((k_{\alpha}^{+})b) - (b_1)_{\alpha}^{-} \frac{\sin((k_{\alpha}^{+})b)}{(k_{\alpha}^{+})} \right) \cos((k_{\alpha}^{+})x) \\
 &\quad + \left((b_2)_{\alpha}^{+} \sin((k_{\alpha}^{+})b) + (b_1)_{\alpha}^{-} \frac{\cos((k_{\alpha}^{+})b)}{(k_{\alpha}^{+})} \right) \sin((k_{\alpha}^{+})x) \tag{3.16}
 \end{aligned}$$

Then, from (3.15) and (3.16) we find Wronskian function as

$$\begin{aligned}
 W(\Phi_{\alpha}^{-}, \chi_{\alpha}^{-})(x, k_{\alpha}^{-}) &= \Phi_{\alpha}^{-}(x, k_{\alpha}^{-})(\chi_{\alpha}^{-})'(x, k_{\alpha}^{-}) - \chi_{\alpha}^{-}(x, k_{\alpha}^{-})(\Phi_{\alpha}^{-})'(x, k_{\alpha}^{-}) \\
 &= \left((a_2)_{\alpha}^{-} (b_1)_{\alpha}^{+} - (a_1)_{\alpha}^{+} (b_2)_{\alpha}^{-} \right) \cos((k_{\alpha}^{-})(a-b)) \\
 &\quad - \left((k_{\alpha}^{-})(a_2)_{\alpha}^{-} (b_2)_{\alpha}^{-} + \frac{(a_1)_{\alpha}^{+} (b_1)_{\alpha}^{+}}{(k_{\alpha}^{-})} \right) \sin((k_{\alpha}^{-})(a-b))
 \end{aligned}$$

and

$$\begin{aligned}
 W(\Phi_{\alpha}^{+}, \chi_{\alpha}^{+})(x, k_{\alpha}^{+}) &= \Phi_{\alpha}^{+}(x, k_{\alpha}^{+})(\chi_{\alpha}^{+})'(x, k_{\alpha}^{+}) - \chi_{\alpha}^{+}(x, k_{\alpha}^{+})(\Phi_{\alpha}^{+})'(x, k_{\alpha}^{+}) \\
 &= \left(\left((a_2)_{\alpha}^{+} (b_1)_{\alpha}^{-} - (a_1)_{\alpha}^{-} (b_2)_{\alpha}^{+} \right) \cos((k_{\alpha}^{+})(a-b)) \right) \\
 &\quad - \left((k_{\alpha}^{+})(a_2)_{\alpha}^{+} (b_2)_{\alpha}^{+} + \frac{(a_1)_{\alpha}^{-} (b_1)_{\alpha}^{-}}{(k_{\alpha}^{+})} \right) \sin((k_{\alpha}^{+})(a-b))
 \end{aligned}$$

For (3)- solution, we get

$$\begin{aligned}
 \left[-(u_{\alpha}^{+})''(x), -(u_{\alpha}^{-})''(x) \right] &= \left[(k_{\alpha}^{-})^2, (k_{\alpha}^{+})^2 \right] [u_{\alpha}^{-}(x), u_{\alpha}^{+}(x)] \tag{3.17} \\
 \left[(a_1)_{\alpha}^{-}, (a_1)_{\alpha}^{+} \right] [u_{\alpha}^{-}(a), u_{\alpha}^{+}(a)] &= \left[(a_2)_{\alpha}^{-}, (a_2)_{\alpha}^{+} \right] \left[(u_{\alpha}^{-})'(a), (u_{\alpha}^{+})'(a) \right]
 \end{aligned}$$

$$\left[(b_1)_\alpha^-, (b_1)_\alpha^+ \right] \left[u_\alpha^-(b), u_\alpha^+(b) \right] = \left[(b_2)_\alpha^-, (b_2)_\alpha^+ \right] \left[(u_\alpha^-)'(b), (u_\alpha^+)'(b) \right].$$

Using fuzzy arithmetic, fuzzy differential equation (3.17) conversions to the linear system of differential equations such that

$$\begin{aligned} -(u_\alpha^+)''(x) - (k_\alpha^-)^2 u_\alpha^-(x) &= 0 \\ -(u_\alpha^-)''(x) - (k_\alpha^+)^2 u_\alpha^+(x) &= 0. \end{aligned}$$

The general solution of this linear system is

$$\left[\widehat{u}(x, \widehat{\lambda}) \right]^\alpha = \left[u_\alpha^-(x, k_\alpha^-, k_\alpha^+), u_\alpha^+(x, k_\alpha^-, k_\alpha^+) \right]$$

where

$$\begin{aligned} u_\alpha^-(x, k_\alpha^-, k_\alpha^+) &= -C_1 \cosh(k_\alpha^- k_\alpha^+)^{1/2} x - C_2 \sinh(k_\alpha^- k_\alpha^+)^{1/2} x \\ &\quad + C_3 \cos(k_\alpha^- k_\alpha^+)^{1/2} x + C_4 \sin(k_\alpha^- k_\alpha^+)^{1/2} x, \\ u_\alpha^+(x, k_\alpha^-, k_\alpha^+) &= C_1 \cosh(k_\alpha^- k_\alpha^+)^{1/2} x + C_2 \sinh(k_\alpha^- k_\alpha^+)^{1/2} x \\ &\quad + C_3 \cos(k_\alpha^- k_\alpha^+)^{1/2} x + C_4 \sin(k_\alpha^- k_\alpha^+)^{1/2} x. \end{aligned}$$

From (1)– solution similarly we find $\left[\widehat{\Phi}(x, \widehat{\lambda}) \right]^\alpha$ and $\left[\widehat{\chi}(x, \widehat{\lambda}) \right]^\alpha$ solution functions such that

$$\begin{aligned} \Phi_\alpha^-(x, k_\alpha^-, k_\alpha^+) &= - \left(\frac{A_\alpha^+}{2(k_\alpha^- k_\alpha^+)^{1/2}} \right) \cosh(k_\alpha^- k_\alpha^+)^{1/2} x \\ &\quad - \left(\frac{B_\alpha^+}{2(k_\alpha^- k_\alpha^+)^{1/2}} \right) \sinh(k_\alpha^- k_\alpha^+)^{1/2} x \\ &\quad + \left(\frac{A_\alpha^-}{2(k_\alpha^- k_\alpha^+)^{1/2}} \right) \cos(k_\alpha^- k_\alpha^+)^{1/2} x \\ &\quad + \left(\frac{B_\alpha^-}{2(k_\alpha^- k_\alpha^+)^{1/2}} \right) \sin(k_\alpha^- k_\alpha^+)^{1/2} x, \\ \Phi_\alpha^+(x, k_\alpha^-, k_\alpha^+) &= \left(\frac{A_\alpha^+}{2(k_\alpha^- k_\alpha^+)^{1/2}} \right) \cosh(k_\alpha^- k_\alpha^+)^{1/2} x \\ &\quad + \left(\frac{B_\alpha^+}{2(k_\alpha^- k_\alpha^+)^{1/2}} \right) \sinh(k_\alpha^- k_\alpha^+)^{1/2} x \\ &\quad + \left(\frac{A_\alpha^-}{2(k_\alpha^- k_\alpha^+)^{1/2}} \right) \cos(k_\alpha^- k_\alpha^+)^{1/2} x \\ &\quad + \left(\frac{B_\alpha^-}{2(k_\alpha^- k_\alpha^+)^{1/2}} \right) \sin(k_\alpha^- k_\alpha^+)^{1/2} x \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
 \chi_{\alpha}^{-}(x, k_{\alpha}^{-}, k_{\alpha}^{+}) &= -\left(\frac{C_{\alpha}^{+}}{2(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2}}\right) \cosh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} x \\
 &\quad -\left(\frac{D_{\alpha}^{+}}{2(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2}}\right) \sinh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} x \\
 &\quad +\left(\frac{C_{\alpha}^{-}}{2(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2}}\right) \cos(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} x \\
 &\quad +\left(\frac{D_{\alpha}^{-}}{2(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2}}\right) \sin(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} x, \\
 \chi_{\alpha}^{+}(x, k_{\alpha}^{-}, k_{\alpha}^{+}) &= \left(\frac{C_{\alpha}^{+}}{2(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2}}\right) \cosh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} x \\
 &\quad +\left(\frac{D_{\alpha}^{+}}{2(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2}}\right) \sinh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} x \\
 &\quad +\left(\frac{C_{\alpha}^{-}}{2(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2}}\right) \cos(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} x \\
 &\quad +\left(\frac{D_{\alpha}^{-}}{2(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2}}\right) \sin(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} x, \tag{3.19}
 \end{aligned}$$

where,

$$\begin{aligned}
 A_{\alpha}^{+} &= ((a_2)_{\alpha}^{+} - (a_2)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \cosh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} a \\
 &\quad - ((a_1)_{\alpha}^{+} - (a_1)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \sinh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} a, \\
 A_{\alpha}^{-} &= ((a_2)_{\alpha}^{+} + (a_2)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \cos(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} a \\
 &\quad - ((a_1)_{\alpha}^{+} + (a_1)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \sin(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} a, \\
 B_{\alpha}^{+} &= ((a_2)_{\alpha}^{+} - (a_2)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \sinh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} a \\
 &\quad - ((a_1)_{\alpha}^{+} - (a_1)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \cosh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} a, \\
 B_{\alpha}^{-} &= ((a_2)_{\alpha}^{+} + (a_2)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \sin(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} a \\
 &\quad - ((a_1)_{\alpha}^{+} + (a_1)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \cos(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} a, \\
 C_{\alpha}^{+} &= ((b_2)_{\alpha}^{+} - (b_2)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \cosh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} b \\
 &\quad - ((b_1)_{\alpha}^{+} - (b_1)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \sinh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} b, \\
 C_{\alpha}^{-} &= ((b_2)_{\alpha}^{+} + (b_2)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \cos(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} b \\
 &\quad - ((b_1)_{\alpha}^{+} + (b_1)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \sin(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} b, \\
 D_{\alpha}^{+} &= ((b_2)_{\alpha}^{+} - (b_2)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \sinh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} b \\
 &\quad - ((b_1)_{\alpha}^{+} - (b_1)_{\alpha}^{-}) (k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} \cosh(k_{\alpha}^{-}k_{\alpha}^{+})^{1/2} b,
 \end{aligned}$$

$$D_{\alpha}^{-} = ((b_2)_{\alpha}^{+} + (b_2)_{\alpha}^{-}) (k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} \sin (k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} b \\ - ((b_1)_{\alpha}^{+} + (b_1)_{\alpha}^{-}) (k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} \cos (k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} b.$$

Then, from (3.18) and (3.19) we find Wronskian function as

$$W(\Phi_{\alpha}^{-}, \chi_{\alpha}^{-})(x, k_{\alpha}^{-}, k_{\alpha}^{+}) = \Phi_{\alpha}^{-}(x, k_{\alpha}^{-}, k_{\alpha}^{+}) (\chi_{\alpha}^{-})'(x, k_{\alpha}^{-}, k_{\alpha}^{+}) - \chi_{\alpha}^{-}(x, k_{\alpha}^{-}, k_{\alpha}^{+}) (\Phi_{\alpha}^{-})'(x, k_{\alpha}^{-}, k_{\alpha}^{+}), \\ W(\Phi_{\alpha}^{+}, \chi_{\alpha}^{+})(x, k_{\alpha}^{-}, k_{\alpha}^{+}) = \Phi_{\alpha}^{+}(x, k_{\alpha}^{-}, k_{\alpha}^{+}) (\chi_{\alpha}^{+})'(x, k_{\alpha}^{-}, k_{\alpha}^{+}) - \chi_{\alpha}^{+}(x, k_{\alpha}^{-}, k_{\alpha}^{+}) (\Phi_{\alpha}^{+})'(x, k_{\alpha}^{-}, k_{\alpha}^{+}).$$

In these equations k_{α}^{-} and k_{α}^{+} are in the product case. So we can't find k_{α}^{-} and k_{α}^{+} values separately satisfying the equations such that

$$W(\Phi_{\alpha}^{-}, \chi_{\alpha}^{-})(x, k_{\alpha}^{-}, k_{\alpha}^{+}) = 0, \\ W(\Phi_{\alpha}^{+}, \chi_{\alpha}^{+})(x, k_{\alpha}^{-}, k_{\alpha}^{+}) = 0. \quad (3.20)$$

Since we can't find $\widehat{\lambda}$ eigenvalue of (3.17) problem, there is no fuzzy solution for 3-solution. Similarly there is no fuzzy solution for (4)-solution.

In this case for (1.1) – (1.3) fuzzy problem (1) – solution and (2) – solution methods can be applied but (3) –solution and (4) –solution methods cannot be applied.

Theorem 3.1([11]) *The Wronskian functions $W(\Phi_{\alpha}^{-}, \chi_{\alpha}^{-})(x, k_{\alpha}^{-}, k_{\alpha}^{+})$ and $W(\Phi_{\alpha}^{+}, \chi_{\alpha}^{+})(x, k_{\alpha}^{-}, k_{\alpha}^{+})$ are independent of variable x for $x \in (a, b)$, where functions $\Phi_{\alpha}^{-}, \chi_{\alpha}^{-}, \Phi_{\alpha}^{+}, \chi_{\alpha}^{+}$ are the solution of the fuzzy boundary value problem (1.1) – (1.3) and it is show that*

$$[\widehat{W}(\widehat{\lambda})]^{\alpha} = [W_{\alpha}^{-}(k_{\alpha}^{-}, k_{\alpha}^{+}), W_{\alpha}^{+}(k_{\alpha}^{-}, k_{\alpha}^{+})] \quad (3.21)$$

for all $\alpha \in [0, 1]$.

Theorem 3.2([11]) *The fuzzy eigenvalues of the fuzzy boundary value problem (1.1)-(1.3) if and only if consist of the zeros of functions $W_{\alpha}^{-}(k_{\alpha}^{-}, k_{\alpha}^{+})$ and $W_{\alpha}^{+}(k_{\alpha}^{-}, k_{\alpha}^{+})$.*

Example 3.1 Consider the two point fuzzy boundary problem

$$-\widehat{u}'' = \lambda \widehat{u}, \quad x \in [0, 1], \quad (3.22)$$

$$\widehat{1}\widehat{u}(0) = \widehat{2}\widehat{u}'(0), \quad (3.23)$$

$$\widehat{u}(1) = 0, \quad (3.24)$$

where $[1]^{\alpha} = [\alpha, 2 - \alpha]$, $[2]^{\alpha} = [\alpha + 1, 3 - \alpha]$, $[\widehat{\lambda}]^{\alpha} = [\lambda_{\alpha}^{-}, \lambda_{\alpha}^{+}] = [(k_{\alpha}^{-})^2, (k_{\alpha}^{+})^2]$, $k_{\alpha}^{-}, k_{\alpha}^{+} > 0$ and $\widehat{u}(x)$ is positive solution functions. For (1)–solution, using the α –cut sets and fuzzy arithmetic we get from (3.22) – (3.24)

$$[-(u_{\alpha}^{-})''(x), -(u_{\alpha}^{+})''(x)] = [(k_{\alpha}^{-})^2, (k_{\alpha}^{+})^2] [u_{\alpha}^{-}(x), u_{\alpha}^{+}(x)] \quad (3.25)$$

$$[\alpha, 2 - \alpha] [u_{\alpha}^{-}(0), u_{\alpha}^{+}(0)] = [\alpha + 1, 3 - \alpha] [(u_{\alpha}^{-})'(0), (u_{\alpha}^{+})'(0)] \quad (3.26)$$

$$[u_{\alpha}^{-}(1), u_{\alpha}^{+}(1)] = 0. \tag{3.27}$$

Let $[\widehat{\Phi}(x, \widehat{\lambda})]^{\alpha}$ be a solution which is satisfying

$$[u_{\alpha}^{-}(0), u_{\alpha}^{+}(0)] = [\alpha + 1, 3 - \alpha], \quad [(u_{\alpha}^{-})'(0), (u_{\alpha}^{+})'(0)] = [\alpha, 2 - \alpha].$$

initial conditions of fuzzy differential equations (3.25). Then we find $[\widehat{\Phi}(x, \widehat{\lambda})]^{\alpha}$ as

$$\begin{aligned} \Phi_{\alpha}^{-}(x, k_{\alpha}^{-}) &= (\alpha + 1) \cos((k_{\alpha}^{-})x) + \frac{\alpha}{(k_{\alpha}^{-})} \sin((k_{\alpha}^{-})x) \\ \Phi_{\alpha}^{+}(x, k_{\alpha}^{+}) &= (3 - \alpha) \cos((k_{\alpha}^{+})x) + \frac{2 - \alpha}{(k_{\alpha}^{+})} \sin((k_{\alpha}^{+})x) \end{aligned} \tag{3.28}$$

Similarly $[\widehat{\chi}(x, \widehat{\lambda})]^{\alpha}$ be a solution which is satisfying

$$[u_{\alpha}^{-}(1), u_{\alpha}^{+}(1)] = 0, \quad [(u_{\alpha}^{-})'(1), (u_{\alpha}^{+})'(1)] = 1$$

initial conditions of fuzzy differential equations (3.25). Then we find $[\widehat{\chi}(x, \widehat{\lambda})]^{\alpha}$ as

$$\begin{aligned} \chi_{\alpha}^{-}(x, k_{\alpha}^{-}) &= \frac{\cos(k_{\alpha}^{-})}{(k_{\alpha}^{-})} \sin((k_{\alpha}^{-})x) - \frac{\sin(k_{\alpha}^{-})}{(k_{\alpha}^{-})} \cos((k_{\alpha}^{-})x) \\ \chi_{\alpha}^{+}(x, k_{\alpha}^{+}) &= \frac{\cos(k_{\alpha}^{+})}{(k_{\alpha}^{+})} \sin((k_{\alpha}^{+})x) - \frac{\sin(k_{\alpha}^{+})}{(k_{\alpha}^{+})} \cos((k_{\alpha}^{+})x) \end{aligned} \tag{3.29}$$

From Theorem 3.2, fuzzy eigenvalues of the fuzzy problem (3.25) – (3.27) are zeros of the functions $W_{\alpha}^{-}(k_{\alpha}^{-}, k_{\alpha}^{+})$ and $W_{\alpha}^{+}(k_{\alpha}^{-}, k_{\alpha}^{+})$. So we get

$$W_{\alpha}^{-}(k_{\alpha}^{-}) = (\alpha + 1) \cos((k_{\alpha}^{-})) + \frac{\alpha}{(k_{\alpha}^{-})} \sin((k_{\alpha}^{-})) = 0 \tag{3.30}$$

$$W_{\alpha}^{+}(k_{\alpha}^{+}) = (3 - \alpha) \cos((k_{\alpha}^{+})) + \frac{(2 - \alpha)}{(k_{\alpha}^{+})} \sin((k_{\alpha}^{+})) = 0 \tag{3.31}$$

For each $\alpha \in [0, 1]$, if the (k_{α}^{-}) and (k_{α}^{+}) values satisfying (3.30) and (3.31) equations compute with Matlab Program, then eigenvalues of the fuzzy problem (3.25) – (3.27) are obtained. So we show (k_{α}^{-}) values of (3.30) equation with $(k_n)_{\alpha}^{-}, n = 1, 2..$ in Table 1 such that

Table 1. (k_{α}^{-}) eigenvalues corresponding to α

$\alpha \in [0, 1]$	k_0^{-}	k_1^{-}	k_2^{-}	k_3^{-}	k_4^{-}
$\alpha = 0$	1.5708	4.7124	7.8540	10.9956	14.1372
$\alpha = 0.2$	1.6703	4.7475	7.8751	11.0107	14.1489
$\alpha = 0.5$	1.7582	4.7820	7.8962	11.0258	14.1607
$\alpha = 0.8$	1.8114	4.8046	7.9101	11.0358	14.1685
$\alpha = 1$	1.8366	4.8158	7.9171	11.0408	14.1724

and (k_α^+) values of (3.31) with $(k_n)_\alpha^+, n = 1, 2, \dots$ in Table 2 such that

Table 2. (k_α^+) eigenvalues corresponding to α

$\alpha \in [0, 1]$	k_0^+	k_1^+	k_2^+	k_3^+	k_4^+
$\alpha = 0$	1.9071	4.8490	7.9378	11.0558	14.1841
$\alpha = 0.2$	1.8975	4.8443	7.9348	11.0537	14.1825
$\alpha = 0.5$	1.8798	4.8358	7.9295	11.0498	14.1795
$\alpha = 0.8$	1.8566	4.8250	7.9227	11.0449	14.1756
$\alpha = 1$	1.8366	4.8158	7.9171	11.0408	14.1724

When Table 1 and Table 2 are examined, it is seen that the eigenvalues provided the fuzzy conditions from Definition 3.2. In this case $[\hat{\lambda}]^\alpha = [\lambda_\alpha^-, \lambda_\alpha^+] = [(k_\alpha^-)^2, (k_\alpha^+)^2]$ are fuzzy eigenvalues of fuzzy problem (3.25) – (3.27) for (1)–solution. So if we write $(k_n)_\alpha^-$ and $(k_n)_\alpha^+$ eigenvalues in (3.28) and (3.29), then $[\hat{\Phi}_n(x, \hat{\lambda})]^\alpha$ and $[\hat{\chi}_n(x, \hat{\lambda})]^\alpha$ are

$$\begin{aligned} [\hat{\Phi}_n(x, \hat{\lambda})]^\alpha &= [(\Phi_n)_\alpha^-(x, (k_n)_\alpha^-), (\Phi_n)_\alpha^+(x, (k_n)_\alpha^+)] \\ &= \left[(\alpha + 1) \cos((k_n)_\alpha^- x) + \frac{\alpha}{(k_n)_\alpha^-} \sin((k_n)_\alpha^- x), \right. \\ &\quad \left. (3 - \alpha) \cos((k_n)_\alpha^+ x) + \frac{(2 - \alpha)}{(k_n)_\alpha^+} \sin((k_n)_\alpha^+ x) \right] \end{aligned} \quad (3.32)$$

and

$$\begin{aligned} [\hat{\chi}_n(x, \hat{\lambda})]^\alpha &= [(\chi_n)_\alpha^-(x, (k_n)_\alpha^-), (\chi_n)_\alpha^+(x, (k_n)_\alpha^+)] \\ &= \left[\frac{\cos((k_n)_\alpha^-)}{(k_n)_\alpha^-} \sin((k_n)_\alpha^- x) - \frac{\sin((k_n)_\alpha^-)}{(k_n)_\alpha^-} \cos((k_n)_\alpha^- x), \right. \\ &\quad \left. \frac{\cos((k_n)_\alpha^+)}{(k_n)_\alpha^+} \sin((k_n)_\alpha^+ x) - \frac{\sin((k_n)_\alpha^+)}{(k_n)_\alpha^+} \cos((k_n)_\alpha^+ x) \right] \end{aligned} \quad (3.33)$$

To define a valid α –cut set of $[\hat{\Phi}_n(x, \hat{\lambda})]^\alpha$ and $[\hat{\chi}_n(x, \hat{\lambda})]^\alpha$ functions, the conditions

$$\frac{\partial (\Phi_n)_\alpha^-}{\partial \alpha} \geq 0, \frac{\partial (\Phi_n)_\alpha^+}{\partial \alpha} \leq 0 \quad \text{and} \quad (\Phi_n)_\alpha^- \leq (\Phi_n)_\alpha^+ \quad (3.34)$$

$$\frac{\partial (\chi_n)_\alpha^-}{\partial \alpha} \geq 0, \frac{\partial (\chi_n)_\alpha^+}{\partial \alpha} \leq 0 \quad \text{and} \quad (\chi_n)_\alpha^- \leq (\chi_n)_\alpha^+ \quad (3.35)$$

must be satisfied for all $\alpha \in [0, 1]$.

Then for all $\alpha \in [0, 1]$, (3.32) and (3.33) are the eigenfunctions corresponding to $[\hat{\lambda}_n]^\alpha = [\hat{k}_n]^\alpha$ eigenvalues and they must satisfy (3.34) and (3.35) equations. Consider that eigenvalues

of $[\widehat{\Phi}_n(x, \widehat{\lambda})]^\alpha$ and $[\widehat{\chi}_n(x, \widehat{\lambda})]^\alpha$ eigenfunctions depend on α - cut set. So if we change α , then this eigenvalues change and eigenfunctions corresponding to $\widehat{\lambda}$ change.

In particular, we select $(k_2)_{0.2}^- = 7.8751$ in Table 1 and $(k_2)_{0.2}^+ = 7.9348$ in Table 1. for $\alpha = 0.2$. If we substitute this values respectively in (3.32) and (3.33), we have the following figures.

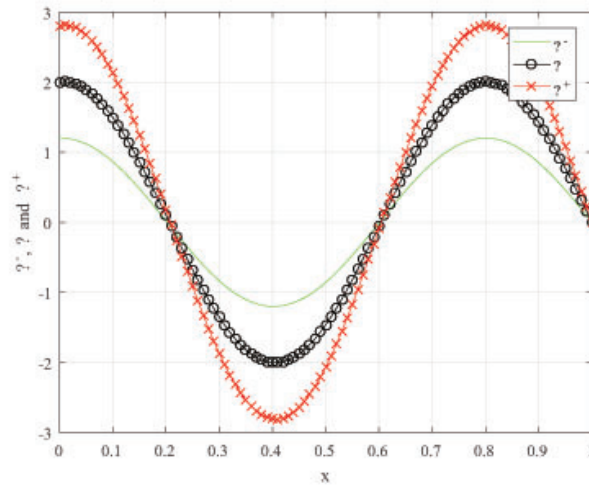


Figure 1 $\widehat{\Phi}(x, k)$ and $\widehat{\Psi}(x, k)$.

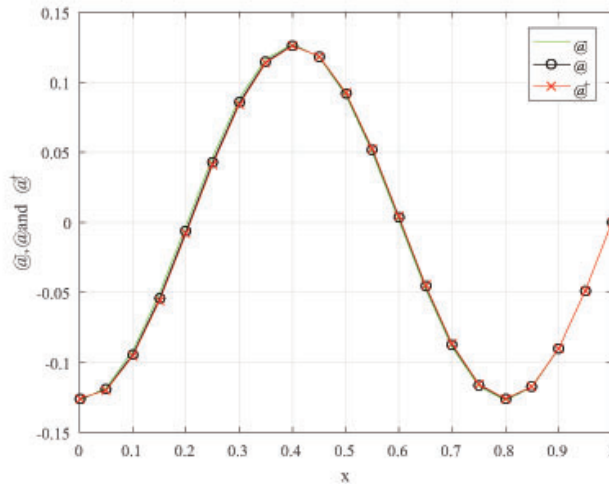


Figure 2 $\widehat{\Phi}(x, k)$ and $\widehat{\Psi}(x, k)$.

From Definition 3.2, we see $[\widehat{\Phi}_n(x, \widehat{\lambda})]^\alpha$ represent a valid fuzzy number for $x \in [0, 0.202]$ and $x \in [0.6063, 1]$ in Figure 1 and we see $[\widehat{\chi}_n(x, \widehat{\lambda})]^\alpha$ represent a valid fuzzy number for $x \in [0.434, 0.601]$ in Figure 2.

For (2)–solution, we get from (3.22)-(3.24)

$$\left[- (u_{\alpha}^{-})''(x), - (u_{\alpha}^{+})''(x) \right] = \left[(k_{\alpha}^{-})^2, (k_{\alpha}^{+})^2 \right] [u_{\alpha}^{-}(x), u_{\alpha}^{+}(x)], \quad (3.36)$$

$$[\alpha, 2 - \alpha] [u_{\alpha}^{-}(0), u_{\alpha}^{+}(0)] = [\alpha + 1, 3 - \alpha] \left[(u_{\alpha}^{+})'(0), (u_{\alpha}^{-})'(0) \right], \quad (3.37)$$

$$[u_{\alpha}^{-}(1), u_{\alpha}^{+}(1)] = 0. \quad (3.38)$$

Let $\left[\widehat{\Phi}(x, \widehat{\lambda}) \right]^{\alpha}$ be a solution which is satisfying

$$[u_{\alpha}^{-}(0), u_{\alpha}^{+}(0)] = [\alpha + 1, 3 - \alpha]$$

$$\left[(u_{\alpha}^{+})'(0), (u_{\alpha}^{-})'(0) \right] = [\alpha, 2 - \alpha].$$

initial conditions of fuzzy differential equations (3.36). Then we find $\left[\widehat{\Phi}(x, \widehat{\lambda}) \right]^{\alpha}$ as

$$\begin{aligned} \Phi_{\alpha}^{-}(x, k_{\alpha}^{-}) &= (\alpha + 1) \cos((k_{\alpha}^{-})x) + \frac{2 - \alpha}{(k_{\alpha}^{-})} \sin((k_{\alpha}^{-})x) \\ \Phi_{\alpha}^{+}(x, k_{\alpha}^{+}) &= (3 - \alpha) \cos((k_{\alpha}^{+})x) + \frac{\alpha}{(k_{\alpha}^{+})} \sin((k_{\alpha}^{+})x) \end{aligned} \quad (3.39)$$

Similarly $\left[\widehat{\chi}(x, \widehat{\lambda}) \right]^{\alpha}$ be a solution which is satisfying

$$\begin{aligned} [u_{\alpha}^{-}(1), u_{\alpha}^{+}(1)] &= 0 \\ \left[(u_{\alpha}^{+})'(1), (u_{\alpha}^{-})'(1) \right] &= 1 \end{aligned}$$

initial conditions of fuzzy differential equations (3.36). Then we find $\left[\widehat{\chi}(x, \widehat{\lambda}) \right]^{\alpha}$ as

$$\begin{aligned} \chi_{\alpha}^{-}(x, k_{\alpha}^{-}) &= \frac{\cos(k_{\alpha}^{-})}{(k_{\alpha}^{-})} \sin((k_{\alpha}^{-})x) - \frac{\sin(k_{\alpha}^{-})}{(k_{\alpha}^{-})} \cos((k_{\alpha}^{-})x) \\ \chi_{\alpha}^{+}(x, k_{\alpha}^{+}) &= \frac{\cos(k_{\alpha}^{+})}{(k_{\alpha}^{+})} \sin((k_{\alpha}^{+})x) - \frac{\sin(k_{\alpha}^{+})}{(k_{\alpha}^{+})} \cos((k_{\alpha}^{+})x) \end{aligned} \quad (3.40)$$

From Theorem 3.2, fuzzy eigenvalues of the fuzzy problem (3.36) – (3.38) are zeros of the functions $W_{\alpha}^{-}(k_{\alpha}^{-}, k_{\alpha}^{+})$ and $W_{\alpha}^{+}(k_{\alpha}^{-}, k_{\alpha}^{+})$. So we get

$$W_{\alpha}^{-}(k_{\alpha}^{-}) = (\alpha + 1) \cos((k_{\alpha}^{-})) + \frac{(2 - \alpha)}{(k_{\alpha}^{-})} \sin((k_{\alpha}^{-})) = 0 \quad (3.41)$$

$$W_{\alpha}^{+}(k_{\alpha}^{+}) = (3 - \alpha) \cos((k_{\alpha}^{+})) + \frac{\alpha}{(k_{\alpha}^{+})} \sin((k_{\alpha}^{+})) = 0 \quad (3.42)$$

For each $\alpha \in [0, 1]$, if the (k_{α}^{-}) and (k_{α}^{+}) values satisfying (3.41) and (3.42) equations compute with Matlab Program, then eigenvalues of the fuzzy problem (3.36) – (3.38) are obtained. So we show (k_{α}^{-}) values of (3.41) equation with $(k_n)_{\alpha}^{-}, n = 1, 2, \dots$ in Table 3 such that

Table 3. (k_α^-) eigenvalues corresponding to α

$\alpha \in [0, 1]$	k_0^-	k_1^-	k_2^-	k_3^-	k_4^-
$\alpha = 0$	2.2889	5.0870	8.0962	11.1727	14.2764
$\alpha = 0.2$	2.1746	5.0036	8.0385	11.1295	14.2421
$\alpha = 0.5$	2.0288	4.9131	7.9787	11.0855	14.2074
$\alpha = 0.8$	1.9071	4.8490	7.9378	11.0558	14.1841
$\alpha = 1$	1.8366	4.8158	7.9171	11.0408	14.1724

and (k_α^+) values of (3.31) with $(k_n)_\alpha^+, n = 1, 2 \dots$ in Table 4 such that

Table 4. (k_α^+) eigenvalues corresponding to α

$\alpha \in [0, 1]$	k_0^+	k_1^+	k_2^+	k_3^+	k_4^+
$\alpha = 0$	1.5708	4.7124	7.8540	10.9956	14.1372
$\alpha = 0.2$	1.6150	4.7275	7.8631	11.0021	14.1422
$\alpha = 0.5$	1.6887	4.7544	7.8794	11.0137	14.1513
$\alpha = 0.8$	1.7731	4.7882	7.9000	11.0285	14.1628
$\alpha = 1$	1.8366	4.8158	7.9171	11.0408	14.1724

When Table 3 and Table 4 are examined, it is seen that the eigenvalues didn't provide the fuzzy condition which is $(k_n)_\alpha^- \leq (k_n)_\alpha^+$ from Definition 3.2. In this case $[\hat{\lambda}]^\alpha = [\lambda_\alpha^-, \lambda_\alpha^+] = [(k_\alpha^-)^2, (k_\alpha^+)^2]$ are not fuzzy eigenvalues of fuzzy problem (3.36) – (3.38) for (2)-solution. So we can not write $(k_n)_\alpha^-$ and $(k_n)_\alpha^+$ eigenvalues for $[\hat{\Phi}(x, \hat{\lambda})]^\alpha$ and $[\hat{\chi}(x, \hat{\lambda})]^\alpha$ functions.

For (3)-solution, we get from (3.22)-(3.24)

$$[-(u_\alpha^+)''(x), -(u_\alpha^-)''(x)] = [(k_\alpha^-)^2, (k_\alpha^+)^2] [u_\alpha^-(x), u_\alpha^+(x)] \tag{3.43}$$

$$[\alpha, 2 - \alpha] [u_\alpha^-(0), u_\alpha^+(0)] = [\alpha + 1, 3 - \alpha] [(u_\alpha^+)'(0), (u_\alpha^-)'(0)] \tag{3.44}$$

$$[u_\alpha^-(1), u_\alpha^+(1)] = 0. \tag{3.45}$$

Using fuzzy arithmetic, fuzzy differential equation (3.43) conversions to the linear system of differential equations such that

$$\begin{aligned} -(u_\alpha^+)''(x) - (k_\alpha^-)^2 u_\alpha^-(x) &= 0 \\ -(u_\alpha^-)''(x) - (k_\alpha^+)^2 u_\alpha^+(x) &= 0. \end{aligned}$$

If we solve this linear system we find $[\hat{\Phi}(x, \hat{\lambda})]^\alpha$ and $[\hat{\chi}(x, \hat{\lambda})]^\alpha$ solution functions such

that

$$\begin{aligned}
\Phi_{\alpha}^{-}(x, k_{\alpha}^{-}, k_{\alpha}^{+}) &= -(1 - \alpha) \cosh \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) - \frac{(\alpha - 1)}{(k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}} \sinh \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) \\
&\quad + 2 \cos \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) + \frac{1}{(k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}} \sin \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) \\
\Phi_{\alpha}^{+}(x, k_{\alpha}^{-}, k_{\alpha}^{+}) &= (1 - \alpha) \cosh \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) + \frac{(\alpha - 1)}{(k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}} \sinh \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) \\
&\quad + 2 \cos \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) + \frac{1}{(k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}} \sin \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) \quad (3.46)
\end{aligned}$$

and

$$\begin{aligned}
\chi_{\alpha}^{-}(x, k_{\alpha}^{-}, k_{\alpha}^{+}) &= -\frac{\sin (k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}}{(k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}} \cos \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) \\
&\quad + \frac{\cos (k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}}{(k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}} \sin \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) \\
\chi_{\alpha}^{+}(x, k_{\alpha}^{-}, k_{\alpha}^{+}) &= \frac{\sin (k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}}{(k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}} \cos \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right) \\
&\quad + \frac{\cos (k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}}{(k_{\alpha}^{-} k_{\alpha}^{+})^{1/2}} \sin \left((k_{\alpha}^{-} k_{\alpha}^{+})^{1/2} x \right). \quad (3.47)
\end{aligned}$$

When we write (3.46) and (3.47) functions in the Wronskian functions, we see that k_{α}^{-} and k_{α}^{+} are in the product case. Since we can't find eigenvalue of fuzzy problem(3.43)-(3.45), there is no fuzzy solution for 3-solution.

Similarly there is no fuzzy solution for 4-solution.

§4. Conclusions

The eigenvalue parameter $\hat{\lambda}$ in the fuzzy boundary value problem was considered fuzzy parameter. Solution of the problem was examined by using generalized Hukuhara differentiability concept. The type of [gH]- differentiability of the solution were discussed.

When solutions were analyzed by derivative type, there were (1)-solution and (2)-solution, but (3)-solution and (4)-solution could not be obtained due to lack of fuzzy eigenvalues.

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