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Enumeration the Number of Spanning Trees of the Sequence of Some Families of Graphs That Have the Same Average Degree

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Abstract: In mathematics one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can create many new graphs from a given set of graphs. In this work, we compute the explicit formulas for the number of spanning trees of sequences of families of graphs of the same average degree four by electrically equivalent transformations and rules of weighted generating function. Finally, we compare the entropy of our graphs with other studied graphs with average degree being four.

Key Words: Number of spanning tree, electrically equivalent transformations, entropy.

AMS(2010): 05C30, 05C50, 05C63.

§1. Introduction

Deriving closed formulae of the number of spanning trees for various graphs has attracted the attention of a lot of researchers. The importance of this research line is in fact due to

(1) Solving some computationally hard problems such as the Steiner tree;
(2) Problem and traveling salesman problem [1];
(3) Counting the number of Eulerian circuits in a graph [2];
(4) Deriving formulas for different type of graphs can be helpful in identifying those graphs that contain the maximum number of spanning trees.

Such an investigation has practical consequences related to network reliability [5,6]. The number of spanning trees $\tau(G)$ of a finite connected undirected graph G is an acyclic $(n-1)$ - edge spanning subgraph. There exist various methods for finding this number. Kirchhoff [7] gave the famous matrix tree theorem: if D is the diagonal matrix of the degrees of G and A denote the adjacency matrix of G, Kirchhoff matrix $L = D - A$ has all of its cofactors equal to $\tau(G)$. Another method to count the complexity of a graph is using Laplacian eigenvalues.

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Let G be a connected graph with k vertices. Kelmans and Chelnoknov [8] derived the following formula

$$\tau(G) \;=\; \frac{1}{k}\prod_{i=1}^{k-1}\mu_i,$$

where $k=\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k = 0$ are the eigenvalues of the Kirchhoff matrix L.

The degeneration of the graph through successive elimination of contraction of its edges represent the core of another way to compute the complexity of a graph [9]. If $G=(V,E)$ is a multigraph with $e \in E$, then $G.e$ is the graph obtained from G by contracting the degree until its endpoints are a single vertex. The formula for computing the number of spanning trees of a multigraph G is given by:

$$\tau(G) \;=\; \tau(G-e)+\tau(G.e)$$

This formula is beautiful but not practically useful (grows exponentially with the size of the graph-may be as many as $2^{|E(G)|}$ terms. For a summary of other results for calculating the umber of the spanning trees of graphs, see [10].

§2. Electrically Equivalent Transformations

Kirchhoff's motivation was study of electrical networks: an edge-weighted graph can be regarded as an electrical network, where weights are the conductance of the respective edges. The effect conductance between two specific vertices x, y can be written as the quotient of (weighted) number of spanning trees and the (weighted) number of so-called thickets, i.e., spanning forests with exactly two components and property that each of the components contains precisely one of the vertices x, y [11-13]. In the following, we list the effect of some simple transformations on the number of spanning trees. Let H be an edge weighted graph, H' be the corresponding electrically equivalent graph, $\tau(H)$ denotes the weighted number of spanning trees H.

(i) *Parallel edges*: If two parallel edges with conductances x and y in H are merged into a single edge with conductances $x+y$ in H', then $\tau(H')=\tau(H)$.

(ii) *Serial edges*: If two serial edges with conductances x and y in H are merged into a single edge with conductance $\frac{xy}{x+y}$ in H', then

$$\tau(H') = \frac{1}{x+y}\tau(H).$$

(iii) $\triangle - Y$ *transformation*: If a triangle with conductances a, b and c in H is changed into an electrically equivalent star graph with conductances

$$x=\frac{ab+bc+ca}{a},\ y=\frac{ab+bc+ca}{b} \text{ and } z=\frac{ab+bc+ca}{c}$$

in H', then

$$\tau(H') = \frac{(ab+bc+ca)^2}{abc}\tau(H).$$

(iv) $Y - \triangle$ transformation: If a star graph with conductances x, y and z in H is changed into an electrically equivalent triangle with conductances

$$a = \frac{yz}{x+y+z}, \ b = \frac{az}{x+y+z} \text{ and } c = \frac{xy}{x+y+z}$$

in H', then

$$\tau\left(H'\right) = \frac{1}{a+b+c}\tau(H).$$

In this work, we compute the number of spanning trees of three sequences of graphs of average degree four based on Tridiminished icosahedron graph we named it $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ and $\mathcal{D}_n$ respectively.

§3. Number of Spanning Trees in the Sequences of $\mathcal{A}_n$ Graph

Consider the sequence of graphs $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ constructed as shown in Figure 1. According to this construction, the number of total vertices $|V(\mathcal{A}_n)|$ and edges $|E(\mathcal{A}_n)|$ are $|V(\mathcal{A}_n)| = 9n-6$ and $|E(\mathcal{A}_n)| = 18n-15$, $n = 1, 2, \cdots$. The average degree of $\mathcal{A}_n$ is in the large n limit which is 4.

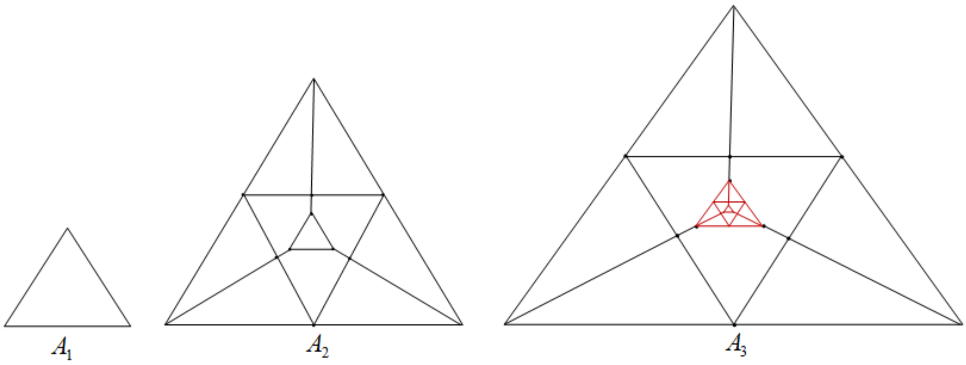


Figure 1. Some sequences of graph $\mathcal{A}_n$

Theorem 3.1 *For any integer $n \geq 1$, the number of spanning trees in the sequence of the graph $\mathcal{A}_n$ is given by*

$$\frac{2^{3n-7}\left(\left(56-17\sqrt{14}\right)\left(23+6\sqrt{14}\right)^n+\left(23-6\sqrt{14}\right)^n\left(56+17\sqrt{14}\right)\right)^2\left(25\left(194+53\sqrt{14}\right)+13\left(\frac{25}{1033+276\sqrt{14}}\right)^n\left(1798+481\sqrt{14}\right)\right)^2}{91875\left(885+220\sqrt{14}-1325^n\left(1033+276\sqrt{14}\right)^{1-n}\right)^2}$$

Proof We use the electrically equivalent transformation to transform $\mathcal{A}_i$ to $\mathcal{A}_{i-1}$. Figures $2-4$ following illustrate the transformation process from $\mathcal{A}_2$ to $\mathcal{A}_1$.

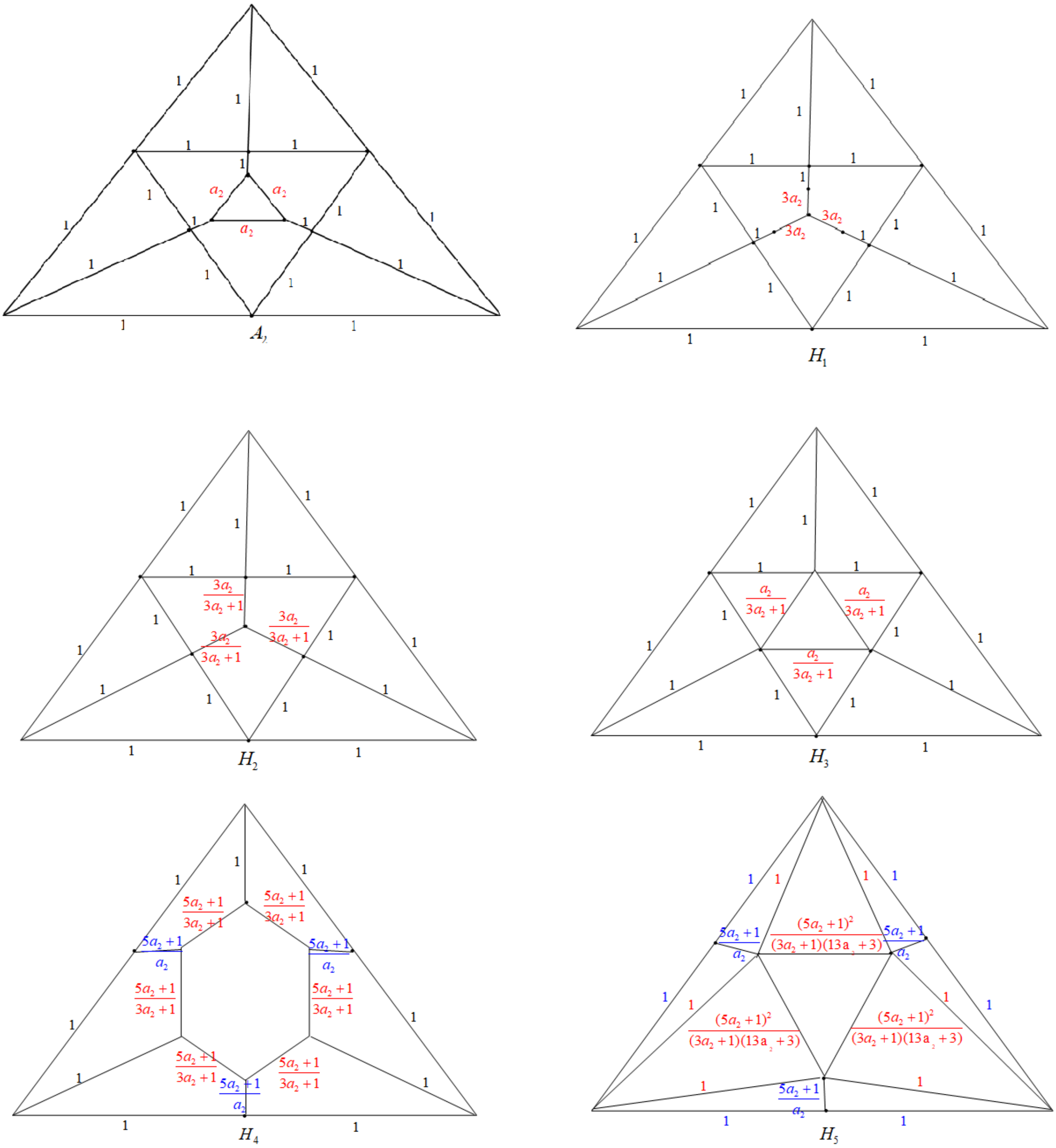

A_2
a_2
a_2
a_2
1
H_1
$3a_2$
$3a_2$
$3a_2$
1
H_2
$\frac{3a_2}{3a_2+1}$
$\frac{3a_2}{3a_2+1}$
$\frac{3a_2}{3a_2+1}$
1
H_3
$\frac{a_2}{3a_2+1}$
$\frac{a_2}{3a_2+1}$
$\frac{a_2}{3a_2+1}$
1
H_4
$\frac{5a_2+1}{3a_2+1}$
$\frac{5a_2+1}{a_2}$
1
H_5
$\frac{(5a_2+1)^2}{(3a_2+1)(13a_2+3)}$
$\frac{5a_2+1}{a_2}$
1

Figure 2

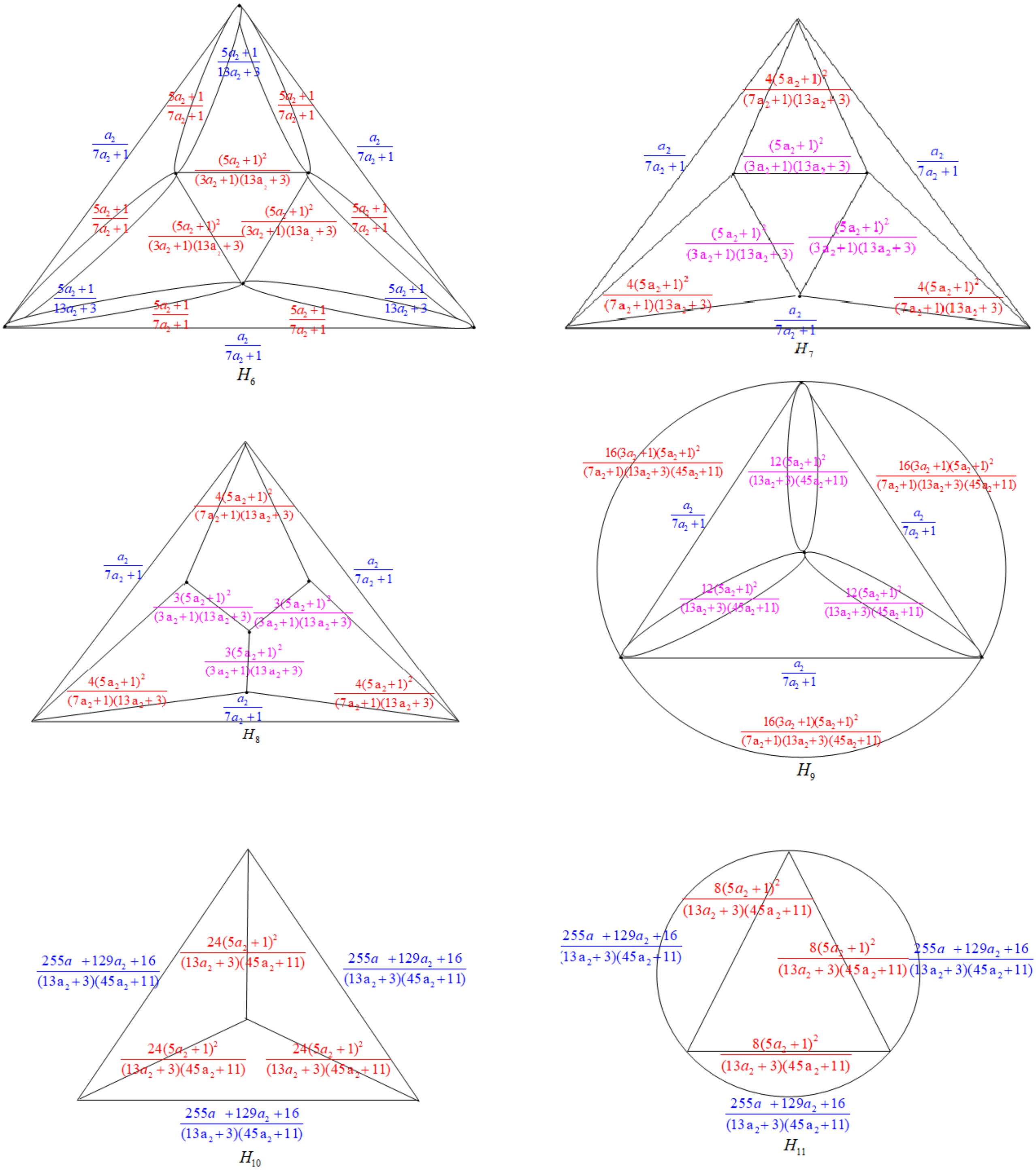

$\frac{5a_2+1}{13a_2+3}$
$\frac{5a_2+1}{7a_2+1}$
$\frac{5a_2+1}{7a_2+1}$
$\frac{a_2}{7a_2+1}$
$\frac{a_2}{7a_2+1}$
$\frac{(5a_2+1)^2}{(3a_2+1)(13a_2+3)}$
$\frac{5a_2+1}{7a_2+1}$
$\frac{5a_2+1}{7a_2+1}$
$\frac{(5a_2+1)^2}{(3a_2+1)(13a_2+3)}$
$\frac{(5a_2+1)^2}{(3a_2+1)(13a_2+3)}$
$\frac{5a_2+1}{13a_2+3}$
$\frac{5a_2+1}{13a_2+3}$
$\frac{5a_2+1}{7a_2+1}$
$\frac{5a_2+1}{7a_2+1}$
$\frac{a_2}{7a_2+1}$
H_6
$\frac{4(5a_2+1)^2}{(7a_2+1)(13a_2+3)}$
$\frac{a_2}{7a_2+1}$
$\frac{(5a_2+1)^2}{(3a_2+1)(13a_2+3)}$
$\frac{a_2}{7a_2+1}$
$\frac{(5a_2+1)^2}{(3a_2+1)(13a_2+3)}$
$\frac{(5a_2+1)^2}{(3a_2+1)(13a_2+3)}$
$\frac{4(5a_2+1)^2}{(7a_2+1)(13a_2+3)}$
$\frac{4(5a_2+1)^2}{(7a_2+1)(13a_2+3)}$
$\frac{a_2}{7a_2+1}$
H_7
$\frac{16(3a_2+1)(5a_2+1)^2}{(7a_2+1)(13a_2+3)(45a_2+11)}$
$\frac{12(5a_2+1)^2}{(13a_2+3)(45a_2+11)}$
$\frac{16(3a_2+1)(5a_2+1)^2}{(7a_2+1)(13a_2+3)(45a_2+11)}$
$\frac{a_2}{7a_2+1}$
$\frac{a_2}{7a_2+1}$
$\frac{12(5a_2+1)^2}{(13a_2+3)(45a_2+11)}$
$\frac{12(5a_2+1)^2}{(13a_2+3)(45a_2+11)}$
$\frac{a_2}{7a_2+1}$
$\frac{16(3a_2+1)(5a_2+1)^2}{(7a_2+1)(13a_2+3)(45a_2+11)}$
H_9
$\frac{4(5a_2+1)^2}{(7a_2+1)(13a_2+3)}$
$\frac{a_2}{7a_2+1}$
$\frac{a_2}{7a_2+1}$
$\frac{3(5a_2+1)^2}{(3a_2+1)(13a_2+3)}$
$\frac{3(5a_2+1)^2}{(3a_2+1)(13a_2+3)}$
$\frac{3(5a_2+1)^2}{(3a_2+1)(13a_2+3)}$
$\frac{4(5a_2+1)^2}{(7a_2+1)(13a_2+3)}$
$\frac{4(5a_2+1)^2}{(7a_2+1)(13a_2+3)}$
$\frac{a_2}{7a_2+1}$
H_8
$\frac{24(5a_2+1)^2}{(13a_2+3)(45a_2+11)}$
$\frac{255a+129a_2+16}{(13a_2+3)(45a_2+11)}$
$\frac{255a+129a_2+16}{(13a_2+3)(45a_2+11)}$
$\frac{24(5a_2+1)^2}{(13a_2+3)(45a_2+11)}$
$\frac{24(5a_2+1)^2}{(13a_2+3)(45a_2+11)}$
$\frac{255a+129a_2+16}{(13a_2+3)(45a_2+11)}$
H_{10}
$\frac{8(5a_2+1)^2}{(13a_2+3)(45a_2+11)}$
$\frac{255a+129a_2+16}{(13a_2+3)(45a_2+11)}$
$\frac{8(5a_2+1)^2}{(13a_2+3)(45a_2+11)}$
$\frac{255a+129a_2+16}{(13a_2+3)(45a_2+11)}$
$\frac{8(5a_2+1)^2}{(13a_2+3)(45a_2+11)}$
$\frac{255a+129a_2+16}{(13a_2+3)(45a_2+11)}$
H_{11}

Figure 3

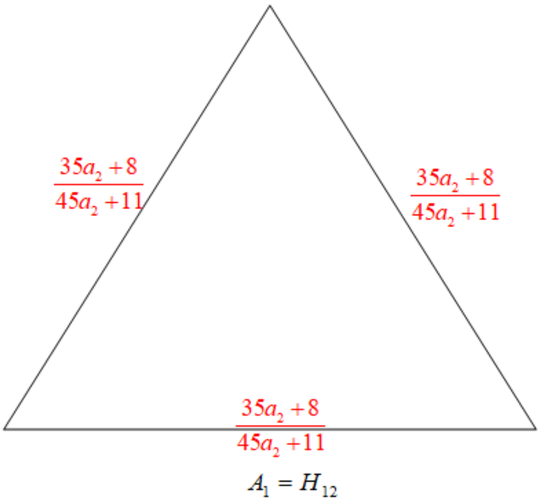


Figure 4

Using the properties given in Section 2, we have the following transformations:

$$\tau(H_1) = 9a_2\tau(\mathcal{A}_2), \qquad \tau(H_2) = \left[\frac{1}{3a_2+1}\right]^3 \tau(H_1),$$

$$\tau(H_3) = \frac{3a_2+1}{9a_2}\tau(H_2), \qquad \tau(H_4) = \left[\frac{(5a_2+1)^2}{a_2(3a_2+1)}\right]^3 \tau(H_3),$$

$$\tau(H_5) = \left[\frac{3a_2+1}{13a_2+3}\right]^3 \tau(H_4), \qquad \tau(H_6) = \left[\frac{a_2}{7a_2+1}\right]^3 \tau(H_5),$$

$$\tau(H_7) = \tau(H_6), \qquad \tau(H_8) = \frac{9(5a_2+1)^2}{(3a_2+1)(13a_2+3)}\tau(H_7),$$

$$\tau(H_9) = \left[\frac{(3a_2+1)(7a_2+1)(13a_2+3)}{(5a_2+1)(45a_2+11)}\right]^3 \tau(H_8), \quad \tau(H_{10}) = \tau(H_9),$$

$$\tau(H_{11}) = \frac{(13a_2+3)(45a_2+11)}{72(5a_2+1)^2}\tau(H_{10}) \qquad \text{and } \tau(\mathcal{A}_1) = \tau(H_{11}).$$

Combining these twelve transformations, we have

$$\tau(\mathcal{A}_2) = 8(45a_2+11)^2\tau(\mathcal{A}_1). \tag{1}$$

Further

$$\tau(\mathcal{A}_n) = \prod_{i=2}^{n} 8(45a_i+11)^2\tau(\mathcal{A}_1) = 3\times 8^{n-1}a_1^2\left[\prod_{i=2}^{n}(45a_i+11)\right]^2, \tag{2}$$

where $a_{i-1} = \frac{35a_i+8}{45a_i+11}, i = 2, 3, \ldots, n$. Its characteristic equation is $45\lambda^2 - 24\lambda - 8 = 0$, which have two roots

$$\lambda_1 = \frac{4-2\sqrt{14}}{15} \quad \text{and} \quad \lambda_2 = \frac{4+2\sqrt{14}}{15}.$$

Subtracting these two roots into both sides of $a_{i-1} = \frac{35a_i+8}{45a_i+11}$, we get

$$a_{i-1} - \frac{4-2\sqrt{14}}{15} = \frac{35a_i+8}{45a_i+11} - \frac{4-2\sqrt{14}}{15} = (23+6\sqrt{14}) \cdot \frac{a_i - \frac{4-2\sqrt{14}}{15}}{45a_i+11}, \tag{3}$$

$$a_{i-1} - \frac{4+2\sqrt{14}}{15} = \frac{35a_i+8}{45a_i+11} - \frac{4+2\sqrt{14}}{15} = (23-6\sqrt{14}) \cdot \frac{a_i - \frac{4+2\sqrt{14}}{15}}{45a_i+11}. \tag{4}$$

Let $b_i = \frac{a_i - \frac{4-2\sqrt{14}}{15}}{a_i - \frac{4+2\sqrt{14}}{15}}$. Then by Eqs. (3) and (4), we get $b_{i-1} = \left(\frac{1033+276\sqrt{14}}{25}\right) b_i$ and $b_i = \left(\frac{1033+276\sqrt{14}}{25}\right)^{n-i} b_n$. Therefore,

$$a_i = \frac{\left(\frac{(033+276\sqrt{14}}{25}\right)^{n-i}\left(\frac{4+2\sqrt{14}}{15}\right) b_n - \frac{4-2\sqrt{14}}{15}}{\left(\frac{1033+276\sqrt{14}}{25}\right)^{n-i} b_n - 1}.$$

Thus

$$a_1 = \frac{2\left(\frac{1033+276\sqrt{14}}{25}\right)^{n-1}(194+53\sqrt{14}) - 13(4-2\sqrt{14})}{3\left(\frac{1033+276\sqrt{14}}{25}\right)^{n-1}(177+44\sqrt{14}) - 195}. \tag{5}$$

Using the expression $a_{n-1} = \frac{35a_n+8}{45a_n+11}$ and denoting the coefficients of $35a_n+8$ and $45a_n+11$ as α_n and β_n we have

$$45a_n + 11 = \alpha_0 (35a_n+8) + \beta_0 (45a_n+11),$$

$$45a_{n-1} + 11 = \frac{\alpha_1 (35a_n+8) + \beta_1 (45a_n+11)}{\alpha_0 (35a_n+8) + \beta_0 (45a_n+11)},$$

$$45a_{n-2} + 11 = \frac{\alpha_2 (35a_n+8) + \beta_2 (45a_n+11)}{\alpha_1 (35a_n+8) + \beta_1 (45a_n+11)},$$

$$\vdots$$

$$45a_{n-i} + 11 = \frac{\alpha_i (35a_n+8) + \beta_i (45a_n+11)}{\alpha_{i-1} (35a_n+8) + \beta_{i-1} (45a_n+11)}, \tag{6}$$

$$45a_{n-(i+1)} + 11 = \frac{\alpha_{i+1} (35a_n+8) + \beta_{i+1} (45a_n+11)}{\alpha_i (35a_n+8) + \beta_i (45a_n+11)}, \tag{7}$$

$$\vdots$$

$$45a_2 + 11 = \frac{\alpha_{n-2} (35a_n+8) + \beta_{n-2} (45a_n+11)}{\alpha_{n-3} (35a_n+8) + \beta_{n-3} (45a_n+11)}$$

Substituting Eq.(6) into Eq.(2), we obtain

$$\tau\left(\mathcal{A}_n\right)=3\times 8^{n-1}a_1^2\left[\alpha_{n-2}\left(35a_n+8\right)+\beta_{n-2}\left(45a_n+11\right)\right]^2. \tag{8}$$

where $\alpha_0=0,\beta_0=1$ and $\alpha_1=45,\beta_1=11$. By the expression $a_{n-1}=\frac{35a_n+8}{45a_n+11}$ and Eqs. (6) and (7), we have

$$\alpha_{i+1}=46\alpha_i-25\alpha_{i-1};\beta_{i+1}=46\beta_i-25\beta_{i-1}. \tag{9}$$

The characteristic equation of Eq.(9) is $\mu^2-46\mu+25=0$ which have two roots

$$\mu_1=23+6\sqrt{14}\ \text{ and }\ \mu_2=23-6\sqrt{14}.$$

The general solutions of Eq. (9) are

$$\alpha_i=c_1\mu_1^i+c_2\mu_2^i;\beta_i=d_1\mu_1^i+d_2\mu_2^i.$$

Using the initial conditions $\alpha_0=0,\beta_0=1$ and $\alpha_1=45,\beta_1=11$, yields

$$\begin{aligned}\alpha_i &= \frac{15\sqrt{14}}{56}(23+6\sqrt{14})^i-\frac{15\sqrt{14}}{56}(23-6\sqrt{14})^i;\beta_i\\ &= \left(\frac{7-\sqrt{14}}{14}\right)(23+6\sqrt{14})^i+\left(\frac{7+\sqrt{14}}{14}\right)(23-6\sqrt{14})^i.\end{aligned} \tag{10}$$

If $a_n=1$, it means that $\mathcal{A}_n$ without any electrically equivalent transformation. Plugging Eq. (10) into Eq.(8), we have

$$\begin{aligned}\tau\left(\mathcal{A}_n\right) &= 3\times 8^{n-1}a_1^2\left[\left(\frac{1568+421\sqrt{14}}{56}\right)(23+6\sqrt{14})^{n-2}\right.\\ &\quad\left.+\left(\frac{1568-421\sqrt{14}}{56}\right)(23-6\sqrt{14})^{n-2}\right]^2,\ n\geq 2.\end{aligned} \tag{11}$$

When $n=1,\tau\left(\mathcal{A}_1\right)=3$ which satisfies Eq.(11). Therefore, the number of spanning trees in the sequence of the graph $\mathcal{A}_n$ is given by

$$\begin{aligned}\tau\left(\mathcal{A}_n\right) &= 3\times 8^{n-1}a_1^2\left[\left(\frac{1568+421\sqrt{14}}{56}\right)(23+6\sqrt{14})^{n-2}\right.\\ &\quad\left.+\left(\frac{1568-421\sqrt{14}}{56}\right)(23-6\sqrt{14})^{n-2}\right]^2,\ n\geq 1,\end{aligned} \tag{12}$$

where

$$a_1=\frac{2\left(\frac{1033+276\sqrt{14}}{25}\right)^{n-1}(194+53\sqrt{14})-13(4-2\sqrt{14})}{3\left(\frac{1033+276\sqrt{14}}{25}\right)^{n-1}(177+44\sqrt{14})-195},\ n\geq 1. \tag{13}$$

Inserting Eq. (13) into Eq.(12) we obtain the result. □

§4. Number of Spanning Trees in the Sequences of $\mathcal{B}_n$ Graph

Consider the sequence of graphs $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ constructed shown in Figure 5. According to this construction, the number of total vertices $|V(\mathcal{B}_n)|$ and edges $|E(\mathcal{B}_n)|$ are $|\mathrm{V}(\mathcal{B}_n)| = 9n - 6$ and $|E(\mathcal{B}_n)| = 18n - 15$ for $n = 1, 2, \cdots$ The average degree of $\mathcal{B}_n$ is in the large n limit which is 4.

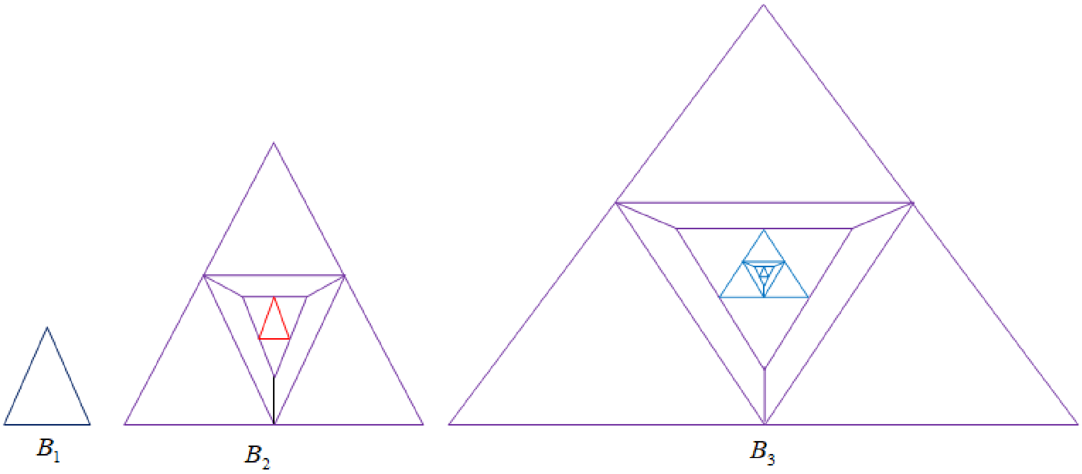


Figure 5. Some sequences of graph $\mathcal{B}_n$

Theorem 4.1 *For $n \geq 1$, the number of spanning trees in the sequence of $\mathcal{B}_n$ graph is given by*

$$\frac{(41+3\sqrt{185})^{2n}\left(72^{3n+4}(259+19\sqrt{185})+(1673+123\sqrt{185})^{n}(-2701+231\sqrt{185})-2(1673-123\sqrt{185})^{n}(1321233+97139\sqrt{185})\right)^{2}}{262848\left(2^{3n+1}(1673+123\sqrt{185})+(21+\sqrt{185})(1673+123\sqrt{185})^{n}\right)^{2}}$$

Proof We use the electrically equivalent transformation to transform $\mathcal{B}_i$ to $\mathcal{B}_{i-1}$. Figures $6-8$ illustrate the transformation process from $\mathcal{B}_2$ to $\mathcal{B}_1$.

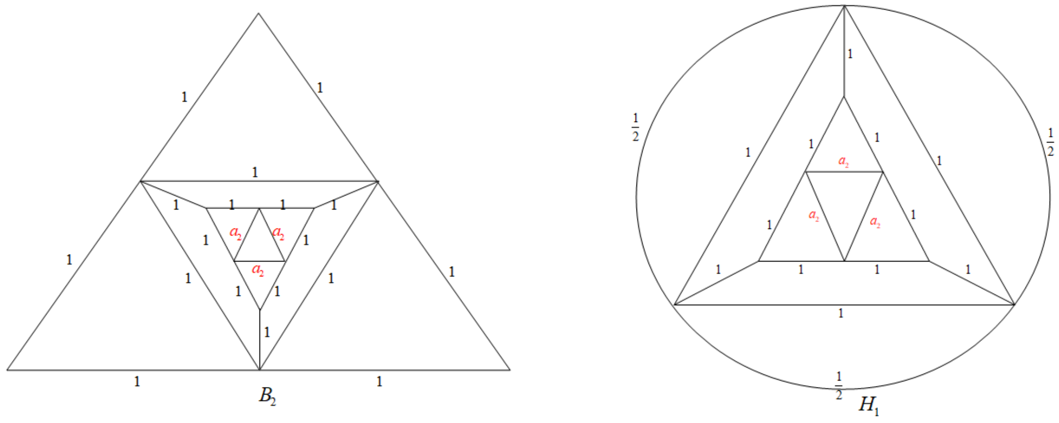


Figure 6

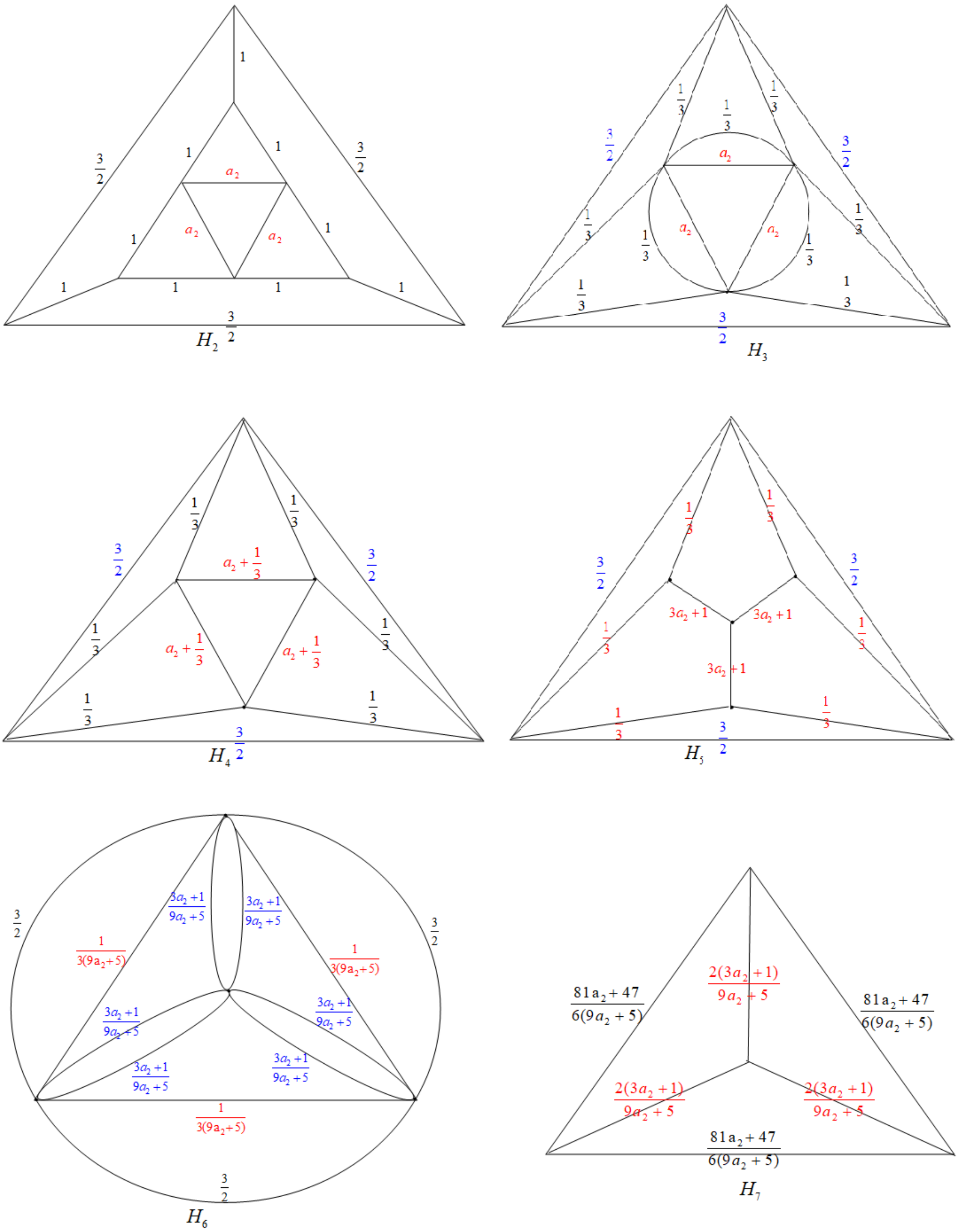
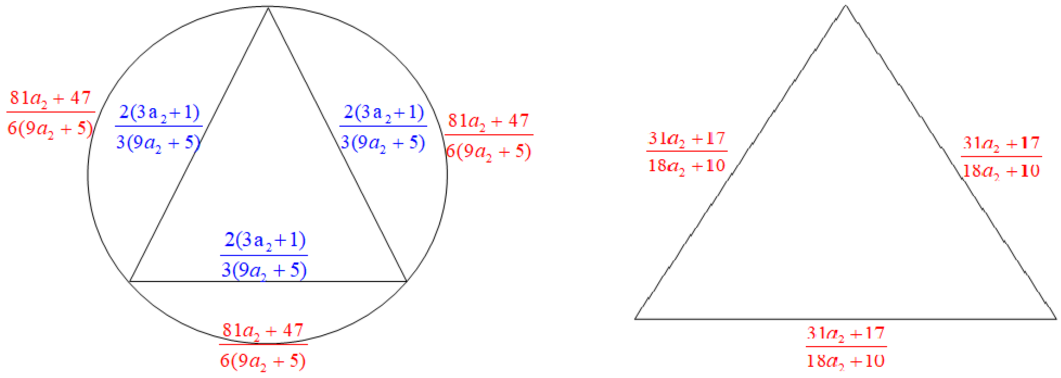

H_2
1
$\frac{3}{2}$
a_2
H_3
$\frac{1}{3}$
a_2
H_4
$a_2+\frac{1}{3}$
H_5
$3a_2+1$
H_6
$\frac{3a_2+1}{9a_2+5}$
$\frac{1}{3(9a_2+5)}$
H_7
$\frac{81a_2+47}{6(9a_2+5)}$
$\frac{2(3a_2+1)}{9a_2+5}$

Figure 7



H_8 B_1

Figure 8

Using the properties given in Section 2, we have the following transformations:

$$\tau(H_1)=\left[\frac{1}{2}\right]^3\tau(\mathcal{B}_2), \qquad \tau(H_2)=\tau(H_1),$$

$$\tau(H_3)=\left[\frac{1}{3}\right]^3\tau(H_2), \qquad \tau(H_4)=\tau(H_3),$$

$$\tau(H_5)=(9a_2+3)\,\tau(H_4),$$

$$\tau(H_6)=\left[\frac{3}{9a_2+5}\right]^3\tau(H_5), \qquad \tau(H_7)=\tau(H_6),$$

$$\tau(H_8)=\frac{9a_2+5}{6(3a_2+1)}\tau(H_7) \text{ and } \quad \tau(\mathcal{B}_1)=\tau(H_8).$$

Combining these nine transformations, we have

$$\tau(\mathcal{B}_2)=4(18a_2+10)^2\,\tau(\mathcal{B}_1). \tag{14}$$

Further

$$\tau(\mathcal{B}_n)=\prod_{i=2}^{n}4(18a_i+10)^2\,\tau(\mathcal{B}_1)=3\times 4^{n-1}a_1^2\left[\prod_{i=2}^{n}(18a_i+10)\right]^2, \tag{15}$$

where $a_{i-1}=\frac{31a_i+17}{18a_i+10}$, $i=2,3,\cdots,n$. Its characteristic equation is $18\lambda^2-21\lambda-17=0$ which have two roots

$$\lambda_1=\frac{7-\sqrt{185}}{12} \quad \text{and} \quad \lambda_2=\frac{7+\sqrt{185}}{12}.$$

Subtracting these two roots into both sides of $a_{i-1}=\frac{31a_i+17}{18a_i+10^2}$, we get

$$a_{i-1}-\frac{7-\sqrt{185}}{12}=\frac{31a_i+17}{18a_i+10}-\frac{7-\sqrt{185}}{12}=(41+3\sqrt{185})\cdot\frac{a_i-\left(\frac{7-\sqrt{185}}{12}\right)}{2(18a_i+10)}, \tag{16}$$

$$a_{i-1} - \frac{7+\sqrt{185}}{12} = \frac{31a_i + 17}{18a_i + 10} - \frac{7+\sqrt{185}}{12} = (41 - 3\sqrt{185}) \cdot \frac{a_i - \left(\frac{7+\sqrt{185}}{12}\right)}{2\,(18a_i + 10)}. \tag{17}$$

Let $b_i = \frac{a_i - \frac{7-\sqrt{185}}{12}}{a_i - \frac{7+\sqrt{185}}{12}}$. Then by Eqs. (16) and (17), we get

$$b_{i-1} = \left(\frac{1673 + 123\sqrt{185}}{8}\right) b_i \quad \text{and} \quad b_i = \left(\frac{1673 + 123\sqrt{185}}{8}\right)^{n-i} b_n.$$

Therefore

$$a_i = \frac{\left(\frac{1673+122\sqrt{185}}{8}\right)^{n-i} \left(\frac{7+\sqrt{185}}{12}\right) b_n - \frac{7-\sqrt{185}}{12}}{\left(\frac{1673+123\sqrt{185}}{8}\right)^{n-i} b_n - 1}.$$

Thus,

$$a_1 = \frac{\left(\frac{1673+123\sqrt{155}}{8}\right)^{n-1} (83 + 7\sqrt{185}) + 4(7 - \sqrt{185})}{3\left(\frac{1673+123\sqrt{185}}{8}\right)^{n-1} (21 + \sqrt{185}) + 48}. \tag{18}$$

Using the expression $a_{n-1} = \frac{31a_n+17}{18a_n+10}$ and denoting the coefficients of $31a_n+17$ and $18a_n+10$ as α_n and β_n, we have

$$18a_n + 10 = \alpha_0\,(31a_n + 17) + \beta_0\,(18a_n + 10)\,,$$

$$18a_{n-1} + 10 = \frac{\alpha_1\,(31a_n + 17) + \beta_1\,(18a_n + 10)}{\alpha_0\,(31a_n + 17) + \beta_0\,(18a_n + 10)},$$

$$18a_{n-2} + 10 = \frac{\alpha_2\,(31a_n + 17) + \beta_2\,(18a_n + 10)}{\alpha_1\,(31a_n + 17) + \beta_1\,(18a_n + 10)},$$

$$\vdots$$

$$18a_{n-i} + 10 = \frac{a_i\,(31a_n + 17) + \beta_i\,(18a_n + 10)}{\alpha_{i-1}\,(31a_n + 17) + \beta_{i-1}\,(18a_n + 10)}, \tag{19}$$

$$18a_{n-(i+1)} + 10 = \frac{\alpha_{i+1}\,(31a_n + 17) + \beta_{i+1}\,(18a_n + 10)}{\alpha_i\,(31a_n + 17) + \beta_i\,(18a_n + 10)}, \tag{20}$$

$$\vdots$$

$$18a_2 + 10 = \frac{\alpha_{n-2}\,(31a_n + 17) + \beta_{n-2}\,(18a_n + 10)}{\alpha_{n-3}\,(31a_n + 17) + \beta_{n-3}\,(18a_n + 10)}.$$

Substituting Eq.(19)into Eq.(15), we obtain

$$\tau\left(\mathcal{B}_n\right) = 3 \times 4^{n-1} a_1^2 \left[\alpha_{n-2}\,(31a_n + 17) + \beta_{n-2}\,(18a_n + 10)\right]^2. \tag{21}$$

where $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 18, \beta_1 = 10$. By the expression $a_{n-1} = \frac{31a_n+7}{18a_n+10}$ and Eqs. (19) and (20), we have

$$\alpha_{i+1} = 41\alpha_i - 4\alpha_{i-1}, \quad \beta_{i+1} = 41\beta_i - 4\beta_{i-1}.$$

Its characteristic equation is $\mu^2 - 41\mu + 4 = 0$ which have two roots

$$\mu_1 = \frac{41 + 3\sqrt{185}}{2} \quad \text{and} \quad \mu_2 = \frac{41 - 3\sqrt{185}}{2}$$

and with the general solution

$$\alpha_i = c_1\mu_1^i + c_2\mu_2^i, \quad \beta_i = d_1\mu_1^i + d_2\mu_2^i.$$

Using the initial conditions $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 18, \beta_1 = 10$, yields

$$\alpha_i = \frac{6\sqrt{185}}{185}\left(\frac{41 + 3\sqrt{185}}{2}\right)^i - \frac{6\sqrt{185}}{185}\left(\frac{41 - 3\sqrt{185}}{2}\right)^i, \tag{22}$$

$$\beta_i = \left(\frac{555 - 21\sqrt{185}}{1110}\right)\left(\frac{41 + 3\sqrt{185}}{2}\right)^i + \left(\frac{555 + 21\sqrt{185}}{1110}\right)\left(\frac{41 - 3\sqrt{185}}{2}\right)^i. \tag{23}$$

If $a_n = 1$, it means that $\mathcal{B}_n$ without any electrically equivalent transformation.

Plugging Eq.(23) into Eq.(21), we have

$$\begin{aligned}\tau(\mathcal{B}_n) \quad &= \quad 3 \times 4^{n-1}a_1^2\left[\left(\frac{518 + 38\sqrt{185}}{37}\right)\left(\frac{41 + 3\sqrt{185}}{2}\right)^{n-2}\right.\\ &\left. + \left(\frac{518 - 38\sqrt{185}}{37}\right)\left(\frac{41 - 3\sqrt{185}}{2}\right)^{n-2}\right]^2, \ n \geq 2.\end{aligned} \tag{24}$$

When $n = 1, \tau(\mathcal{B}_1) = 3$ which satisfies Eq.(24).

Therefore, the number of spanning trees in the sequence of Tridiminished icosahedron graph is given by

$$\begin{aligned}\tau(\mathcal{B}_n) \quad &= \quad 3 \times 4^{n-1}a_1^2\left[\left(\frac{518 + 38\sqrt{185}}{37}\right)\left(\frac{41 + 3\sqrt{185}}{2}\right)^{n-2}\right.\\ &\left. + \left(\frac{518 - 38\sqrt{185}}{37}\right)\left(\frac{41 - 3\sqrt{185}}{2}\right)^{n-2}\right]^2, \ n \geq 1,\end{aligned} \tag{25}$$

where

$$a_1 = \frac{\left(\frac{1673+123\sqrt{185}}{8}\right)^{n-1}(83 + 7\sqrt{185}) + 4(7 - \sqrt{185})}{3\left(\frac{1673+123\sqrt{185}}{8}\right)^{n-1}(21 + \sqrt{185}) + 48}, \ n \geq 1. \tag{26}$$

Inserting Eq.(26) into Eq.(25) we obtain the result. □

§5. Number of Spanning Trees in the Sequences of $\mathcal{C}_n$ Graph

Consider the sequence of graphs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ constructed as shown in Figure 5. According to this construction, the number of total vertices $|V(\mathcal{C}_n)|$ and edges $|E(\mathcal{C}_n)|$ are $|\mathrm{V}(\mathcal{C}_n)| = 9n-6$ and $|E(\mathcal{C}_n)| = 18$ for $n-15, n = 1, 2, \cdots$. The average degree of $\mathcal{C}_n$ is in the large n limit which is 4.

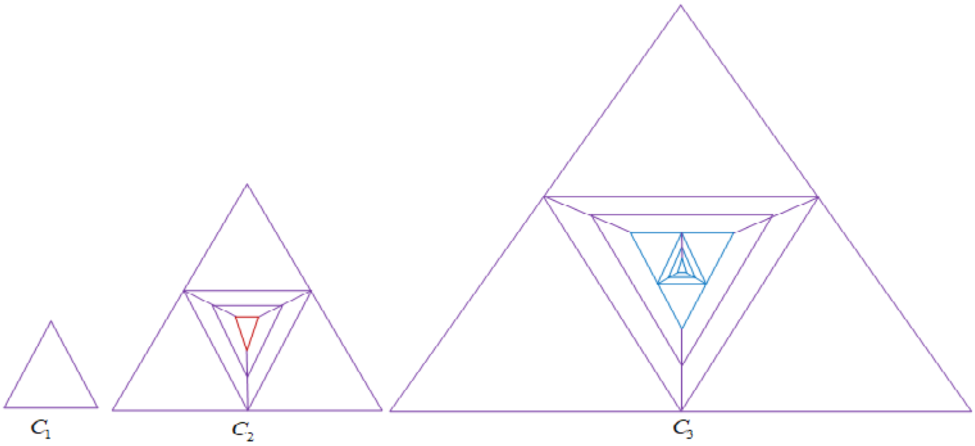


Figure 9. Some sequences of $\mathcal{C}_n$

Theorem 5.1 *For $n \geq 1$, the number of spanning trees in the sequence of $\mathcal{C}_n$ is given by*

$$\frac{32^{-7-n}(61+\sqrt{3705})^{2n}\left(\frac{1}{60}(-45+\sqrt{3705})-\frac{1}{435}\left(\frac{1}{8}(3713-61\sqrt{3705})\right)^{1-n}(600+11\sqrt{3705})\right)^2\left(5681-93\sqrt{3705}+\left(\frac{1}{8}(3713-61\sqrt{3705})\right)^n(5681+93\sqrt{3705})\right)^2}{61009\left(1+\frac{1}{29}2^{1-3n}(131+\sqrt{3705})(3713+61\sqrt{3705})^{n-1}\right)^2}$$

Proof We use the electrically equivalent transformation to transform $\mathcal{C}_i$ to $\mathcal{C}_{i-1}$. Figures $10-12$ illustrate the transformation process from $\mathcal{C}_2$ to $\mathcal{C}_1$.

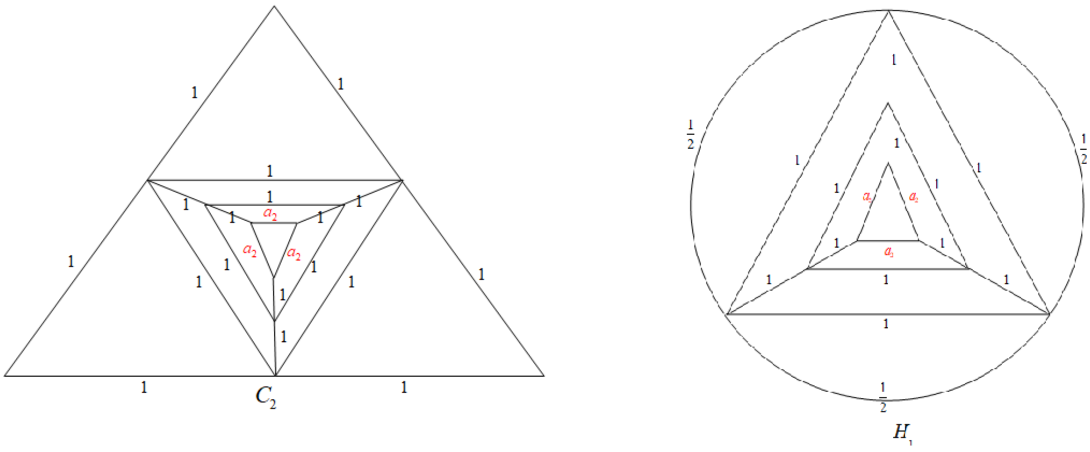
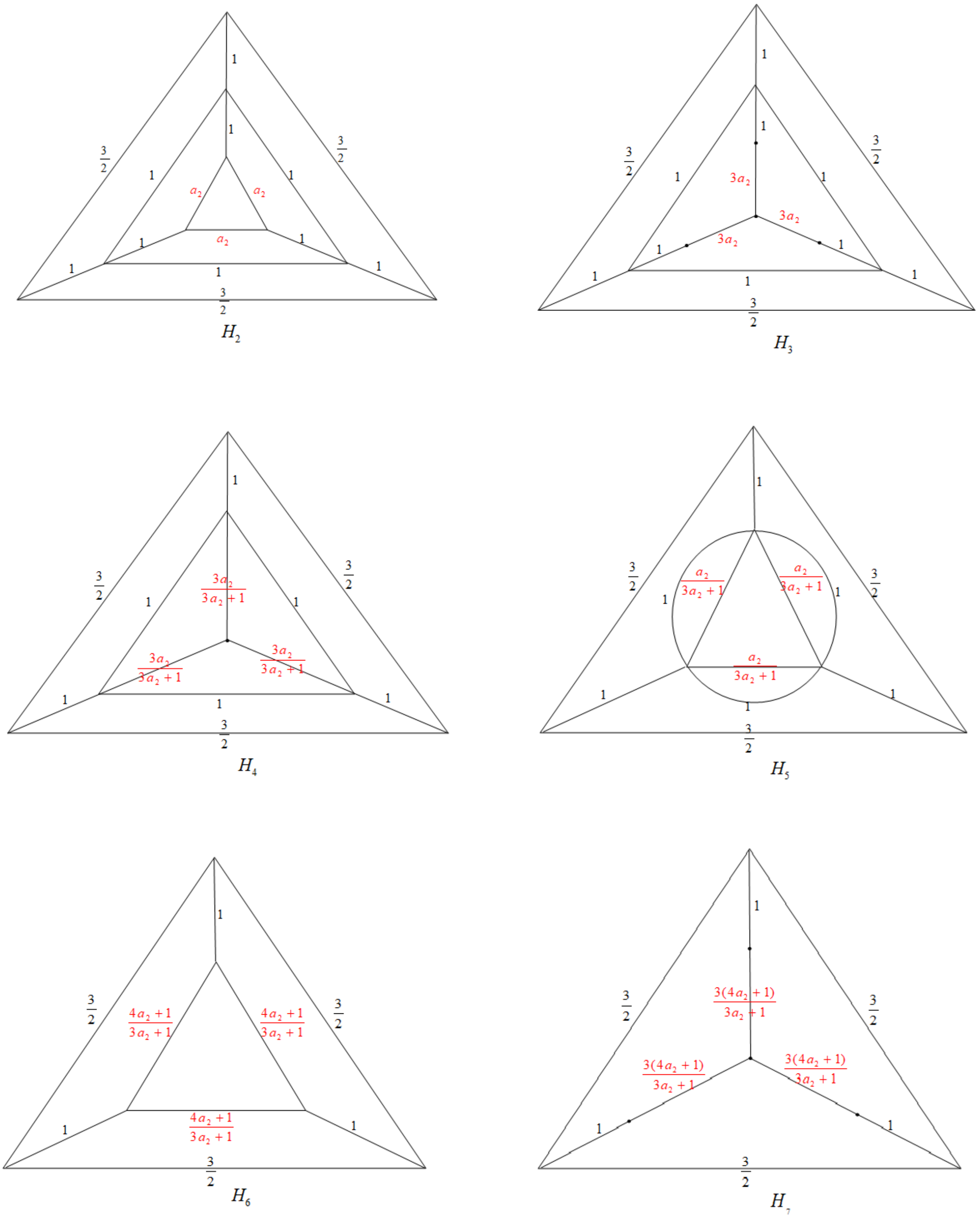


Figure 10



H_2 H_3 H_4 H_5 H_6 H_7

Figure 11

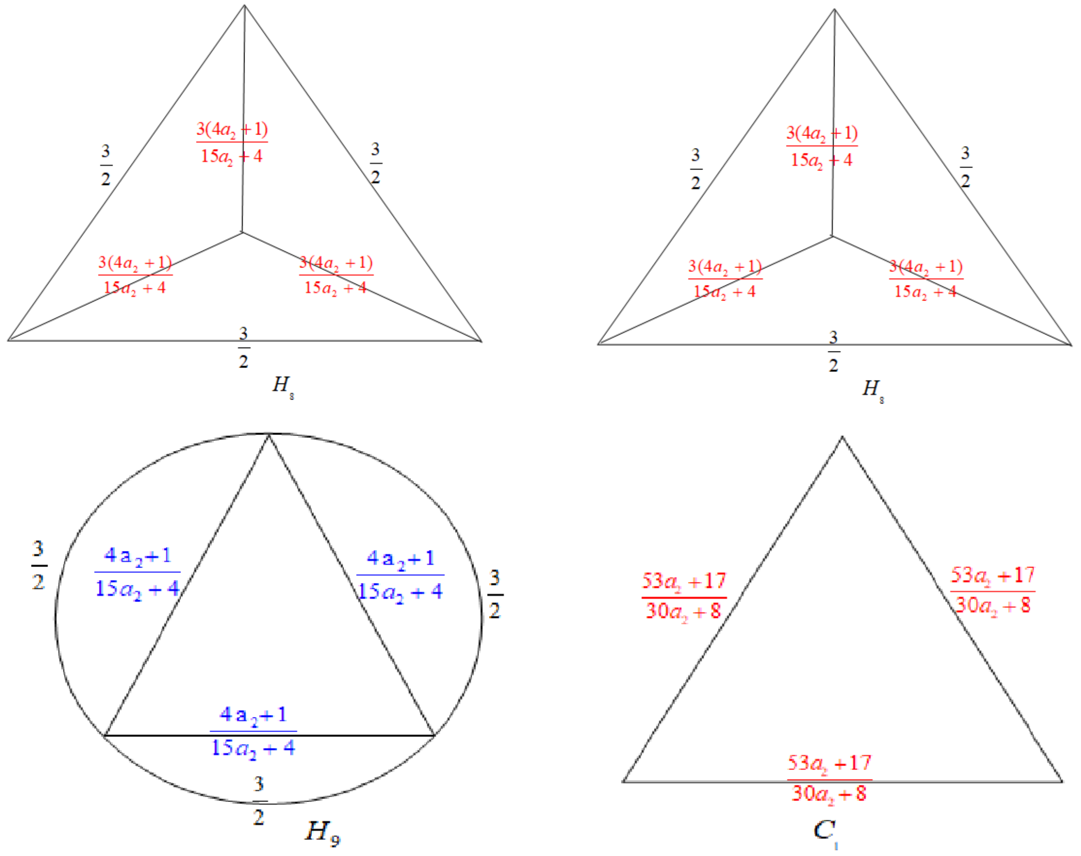


Figure 12

Using the properties given in section 2, we have the following transformations:

$$\tau\left(H_1\right)=\left[\frac{1}{2}\right]^3 \tau\left(\mathcal{C}_2\right), \qquad \tau\left(H_2\right)=\tau\left(H_1\right),$$

$$\tau\left(H_3\right)=9 a_2 \tau\left(H_2\right), \qquad \tau\left(H_4\right)=\left(\frac{1}{3 a_2+1}\right)^2 \tau\left(H_3\right),$$

$$\tau\left(H_5\right)=\frac{3 a_2+1}{9 a_2} \tau\left(H_4\right), \qquad \tau\left(H_6\right)=\tau\left(H_5\right),$$

$$\tau\left(H_7\right)=9\left(\frac{4 a_2+1}{3 a_2+1}\right) \tau\left(H_6\right), \qquad \tau\left(H_8\right)=\left[\frac{\left(3 a_2+1\right)}{15 a_2+4}\right]^3 \tau\left(H_7\right),$$

$$\tau\left(H_9\right)=\left(\frac{15 a_2+4}{9\left(4 a_2+1\right)}\right) \tau\left(H_8\right), \quad \tau\left(\mathcal{C}_1\right)=\tau\left(H_9\right).$$

Combining these ten transformations, we have

$$\tau\left(\mathcal{C}_2\right)=2\left(30 a_2+8\right)^2 \tau\left(\mathcal{C}_1\right). \tag{27}$$

Further

$$\tau\left(\mathcal{C}_n\right)=\prod_{i=2}^{n} 2\left(30a_i+8\right)^2 \tau\left(\mathcal{C}_1\right)=3\times 2^{n-1}a_1^2\left[\prod_{i=2}^{n}\left(30a_i+8\right)\right]^2, \tag{28}$$

where $a_{i-1}=\frac{53a_i+14}{30a_i+8}, i=2,3,\cdots,n$. Its characteristic equation is $30\lambda^2-45\lambda-14=0$ which have two roots

$$\lambda_1=\frac{45-\sqrt{3705}}{60} \quad\text{and}\quad \lambda_2=\frac{45+\sqrt{3705}}{60}.$$

Subtracting these two roots into both sides of $a_{i-1}=\frac{53a_i+14}{30a_i+8}$, we get

$$a_{i-1}-\frac{45-\sqrt{3705}}{60}=\frac{53a_i+14}{30a_i+8}-\frac{45-\sqrt{3705}}{60}=(61+\sqrt{3705})\cdot\frac{a_i-\frac{45-\sqrt{3705}}{60}}{2\left(30a_i+8\right)}, \tag{29}$$

$$a_{i-1}-\frac{45+\sqrt{3705}}{60}=\frac{53a_i+14}{30a_i+8}-\frac{45+\sqrt{3705}}{60}=(61-\sqrt{3705})\cdot\frac{a_i-\frac{45+\sqrt{3705}}{60}}{2\left(30a_i+8\right)}. \tag{30}$$

Let $b_i=\frac{a_i-\frac{45-\sqrt{3705}}{60}}{a_i-\frac{45+\sqrt{3705}}{60}}$.Then by Eqs. (29) and (30), we get

$$b_{i-1}=\left(\frac{3713+61\sqrt{3705}}{8}\right)b_i, \quad\text{and}\quad b_i=\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-i}b_n.$$

Therefore,

$$a_i=\frac{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-i}\left(\frac{45+\sqrt{3705}}{60}\right)b_n-\frac{45-\sqrt{3705}}{60}}{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-i}b_n-1}.$$

Thus

$$a_1=\frac{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-1}\left(\frac{600+11\sqrt{3705}}{435}\right)+\left(\frac{45-\sqrt{3705}}{60}\right)}{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-1}\left(\frac{131+\sqrt{3705}}{116}\right)+1}. \tag{31}$$

Using the expression $a_{n-1}=\frac{53a_n+14}{30a_n+8}$ and denoting the coefficients of $53a_n+14$ and $30a_n+8$ as α_n and β_n, we have

$$\begin{aligned}
30a_n+8&=\alpha_0\left(53a_n+14\right)+\beta_0\left(30a_n+8\right)\\
30a_{n-1}+8&=\frac{\alpha_1\left(53a_n+14\right)+\beta_1 30a_n+8)}{\alpha_0\left(53a_n+14\right)+\beta_0\left(30a_n+8\right)}\\
30a_{n-2}+8&=\frac{\alpha_2\left(53a_n+14\right)+\beta_2\left(30a_n+8\right)}{\alpha_1\left(53a_n+14\right)+\beta_1\left(30a_n+8\right)}\\
&\vdots
\end{aligned}$$

$$30a_{n-i}+8=\frac{\alpha_i\left(53a_n+14\right)+\beta_i\left(30a_n+8\right)}{\alpha_{i-1}\left(53a_n+14\right)+\beta_{i-1}\left(30a_n+8\right)}, \tag{32}$$

$$30a_{n-(i+1)}+8=\frac{\alpha_{i+1}\left(53a_n+14\right)+\beta_{i+1}\left(30a_n+8\right)}{\alpha_i\left(53a_n+14\right)+\beta_i\left(30a_n+8\right)}, \tag{33}$$

$$\vdots$$

$$30a_2+8=\frac{\alpha_{n-2}\left(53a_n+14\right)+\beta_{n-2}\left(30a_n+8\right)}{\alpha_{n-3}(53+14)+\beta_{n-3}\left(30a_n+8\right)}$$

Substituting Eq.(31) into Eq.(28), we obtain

$$\tau\left(\mathcal{C}_n\right)=3\times 2^{n-1}a_1^2\left[\alpha_{n-2}\left(53a_n+14\right)+\beta_{n-2}\left(30a_n+8\right)\right]^2, \tag{34}$$

where $\alpha_0=0,\beta_0=1$ and $\alpha_1=30,\beta_1=8$. By the expression $a_{n-1}=\frac{53a_n+14}{30a_n+8}$ and Eqs. (32) and (33), we have

$$\alpha_{i+1}=61\alpha_i-4\alpha_{i-1},\quad \beta_{i+1}=61\beta_i-4\beta_{i-1}. \tag{35}$$

The characteristic equation of Eq. (35) is $\mu^2-61\mu+4=0$ which have two roots

$$\mu_1=\frac{61+\sqrt{3705}}{2}\quad\text{and}\quad\mu_2=\frac{61+\sqrt{3705}}{2}$$

and the general solutions of Eq.(35) are

$$\alpha_i=c_1\mu_1^i+c_2\mu_2^i,\quad \beta_i=d_1\mu_1^i+d_2\mu_2^i.$$

Substituting the initial conditions $\alpha_0=0,\beta_0=1$ and $\alpha_1=30,\beta_1=8$, yields

$$\alpha_i=\frac{2\sqrt{3705}}{247}\left(\frac{61+\sqrt{3705}}{2}\right)^i-\frac{2\sqrt{3705}}{247}\left(\frac{61+\sqrt{3705}}{2}\right)^i;$$

$$\beta_i=\left(\frac{3705-45\sqrt{3705}}{7410}\right)\left(\frac{61+\sqrt{3705}}{2}\right)^i+\left(\frac{3705+45\sqrt{3705}}{7410}\right)\left(\frac{61-\sqrt{3705}}{2}\right)^i. \tag{36}$$

If $a_n=1$, it means that $\mathcal{C}_n$ without any electrically equivalent transformation. Plugging Eq. (36) into Eq.(34), we have

$$\begin{aligned}\tau\left(\mathcal{C}_n\right) &= 3\times 2^{n-1}a_1^2\left[\left(\frac{4693+771\sqrt{3705}}{247}\right)\left(\frac{61+\sqrt{3705}}{2}\right)^{n-2}\right.\\ &\quad\left.+\left(\frac{4693-771\sqrt{3705}}{247}\right)\left(\frac{61-\sqrt{3705}}{2}\right)^{n-2}\right]^2,\ n\geq 2.\end{aligned} \tag{37}$$

When $n=1,\tau\left(\mathcal{C}_1\right)=3$ which satisfies Eq. (37). Therefore, the number of spanning trees in the sequence of $\mathcal{C}_n$ graph is given by

$$\begin{aligned}\tau\left(\mathcal{C}_n\right) &= 3\times 2^{n-1}a_1^2\left[\left(\frac{4693+771\sqrt{3705}}{247}\right)\left(\frac{61+\sqrt{3705}}{2}\right)^{n-2}\right.\\ &\left.+\left(\frac{4693-771\sqrt{3705}}{247}\right)\left(\frac{61-\sqrt{3705}}{2}\right)^{n-2}\right]^2,\ n\geq 1,\end{aligned} \tag{38}$$

where

$$a_1=\frac{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-1}\left(\frac{600+11\sqrt{3705}}{435}\right)+\left(\frac{45-\sqrt{3705}}{60}\right)}{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-1}\left(\frac{131+\sqrt{3705}}{116}\right)+1},n\geq 1. \tag{39}$$

Inserting Eq.(39) into Eq.(38) we obtain the result. □

§6. Number of Spanning Trees in the Sequences of $\mathcal{D}_n$ Graph

Consider the sequence of graphs $\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_n$ constructed as shown in Figure 7. According to this construction, the number of total vertices $|V(\mathcal{D}_n)|$ and edges $|E(\mathcal{D}_n)|$ are $|\mathrm{V}(\mathcal{D}_n)| = 9n-6$ and $|E(\mathcal{D}_n)| = 18n-15,\ n=1,2,\cdots$. The average degree of $\mathcal{D}_n$ is in the large n limit which is 4.

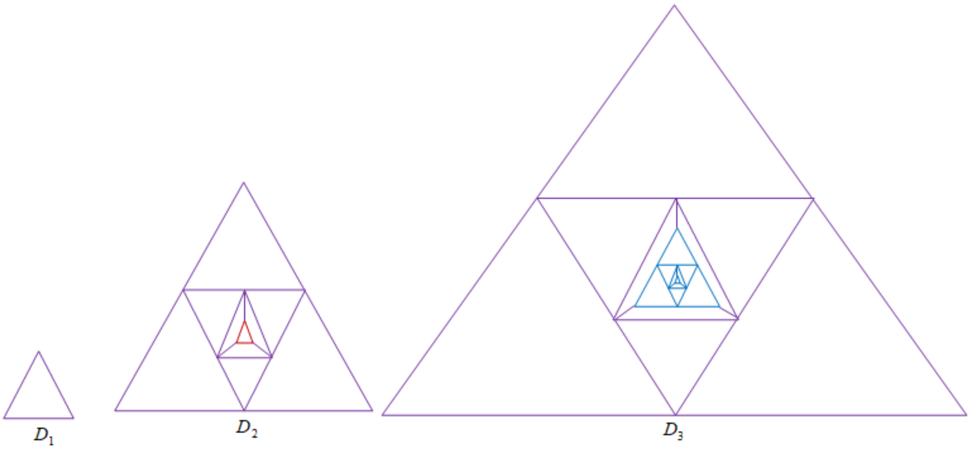
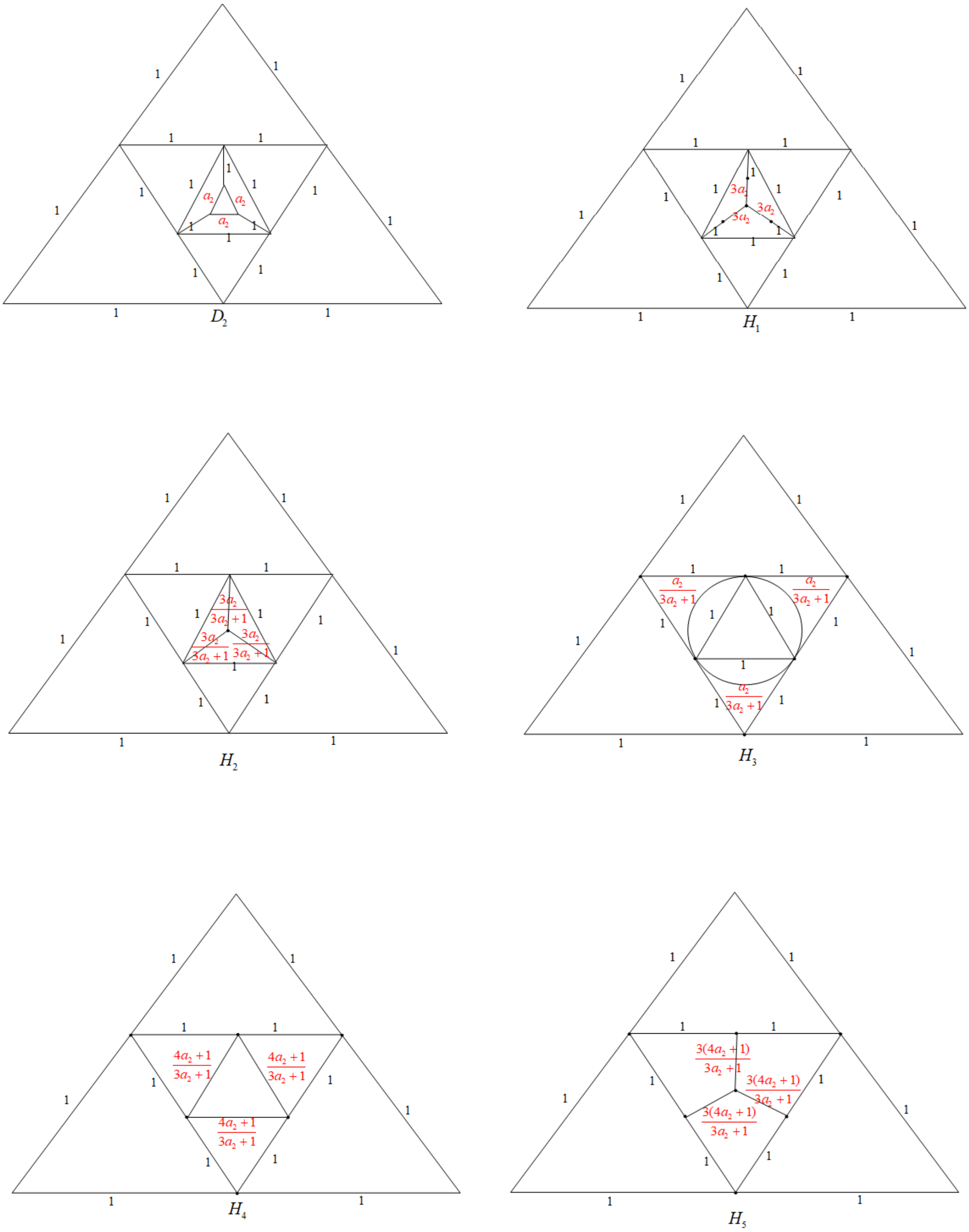


Figure 13. Some sequences of $\mathcal{D}_n$

Theorem 6.1 *For $n\geq 1$, the number of spanning trees in the sequence of $\mathcal{D}_n$ is given by*

$$\frac{\left(59+\sqrt{3477}\right)^{2n}\left(7772^{1+n}\left(50996+865\sqrt{3477}\right)+\left(3479+59\sqrt{3477}\right)^{n}\left(-7924083+129923\sqrt{3477}\right)+851\left(3479-59\sqrt{3477}\right)^{n}\left(96594537+1638137\sqrt{3477}\right)\right)^2}{16119372\left(-8512^{n}\left(3479+59\sqrt{3477}\right)+\left(2861+39\sqrt{3477}\right)\left(3479+59\sqrt{3477}\right)^{n}\right)^2}$$

Proof We use the electrically equivalent transformation to transform $\mathcal{D}_i$ to $\mathcal{D}_{i-1}$. Figure $14-16$ illustrate the transformation process from $\mathcal{D}_2$ to $\mathcal{D}_1$.



D_2 H_1 H_2 H_3 H_4 H_5

Figure 14

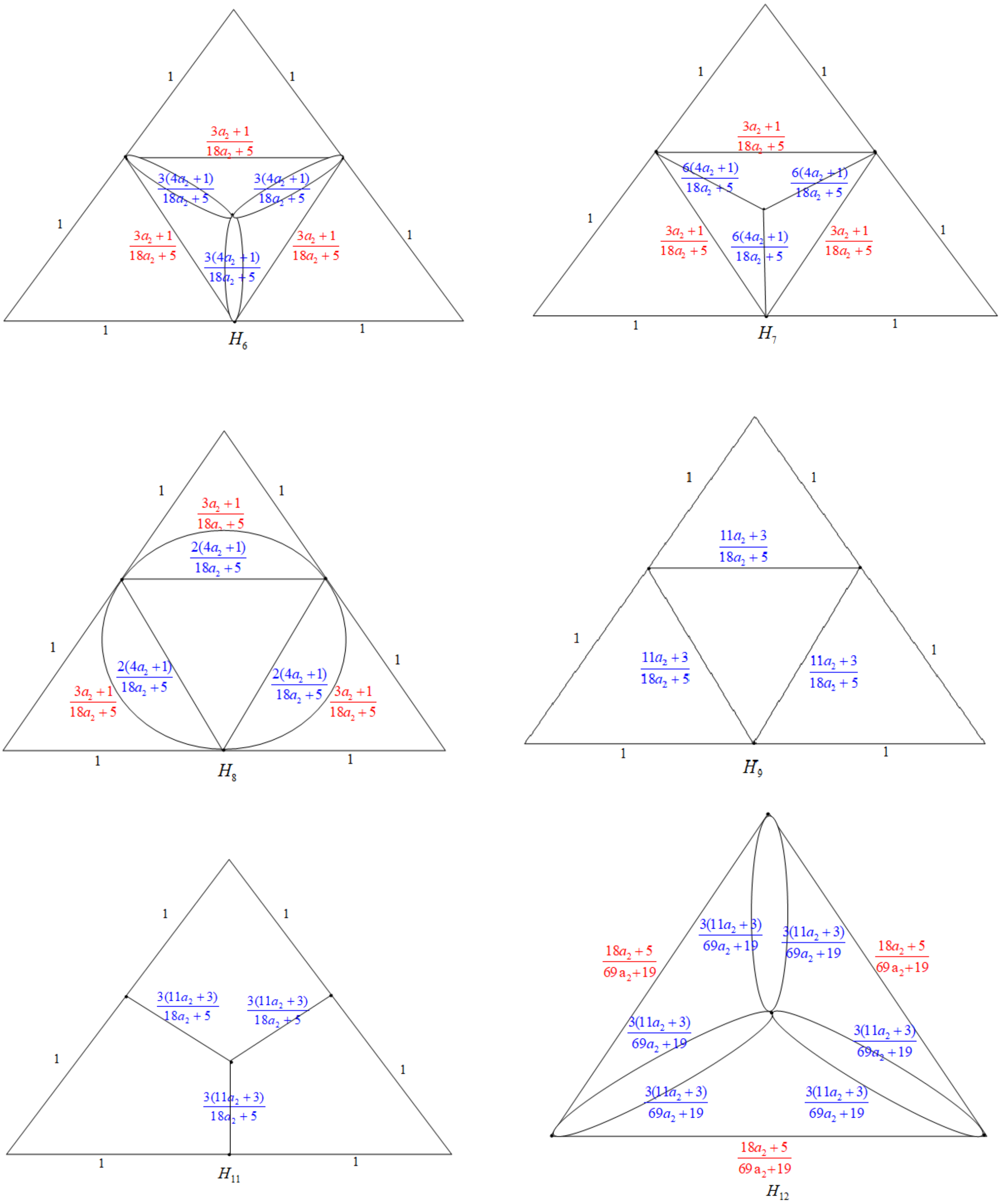

1
1
$\frac{3a_2+1}{18a_2+5}$
$\frac{3(4a_2+1)}{18a_2+5}$
$\frac{3(4a_2+1)}{18a_2+5}$
1
$\frac{3a_2+1}{18a_2+5}$
$\frac{3(4a_2+1)}{18a_2+5}$
$\frac{3a_2+1}{18a_2+5}$
1
1
1
H_6
1
1
$\frac{3a_2+1}{18a_2+5}$
$\frac{6(4a_2+1)}{18a_2+5}$
$\frac{6(4a_2+1)}{18a_2+5}$
1
$\frac{3a_2+1}{18a_2+5}$
$\frac{6(4a_2+1)}{18a_2+5}$
$\frac{3a_2+1}{18a_2+5}$
1
1
1
H_7
1
1
$\frac{3a_2+1}{18a_2+5}$
$\frac{2(4a_2+1)}{18a_2+5}$
1
$\frac{2(4a_2+1)}{18a_2+5}$
$\frac{2(4a_2+1)}{18a_2+5}$
1
$\frac{3a_2+1}{18a_2+5}$
$\frac{3a_2+1}{18a_2+5}$
1
1
H_8
1
1
$\frac{11a_2+3}{18a_2+5}$
1
$\frac{11a_2+3}{18a_2+5}$
$\frac{11a_2+3}{18a_2+5}$
1
1
1
H_9
1
1
$\frac{3(11a_2+3)}{18a_2+5}$
$\frac{3(11a_2+3)}{18a_2+5}$
1
1
$\frac{3(11a_2+3)}{18a_2+5}$
1
1
H_{11}
$\frac{18a_2+5}{69a_2+19}$
$\frac{3(11a_2+3)}{69a_2+19}$
$\frac{3(11a_2+3)}{69a_2+19}$
$\frac{18a_2+5}{69a_2+19}$
$\frac{3(11a_2+3)}{69a_2+19}$
$\frac{3(11a_2+3)}{69a_2+19}$
$\frac{3(11a_2+3)}{69a_2+19}$
$\frac{3(11a_2+3)}{69a_2+19}$
$\frac{18a_2+5}{69a_2+19}$
H_{12}

Figure 15

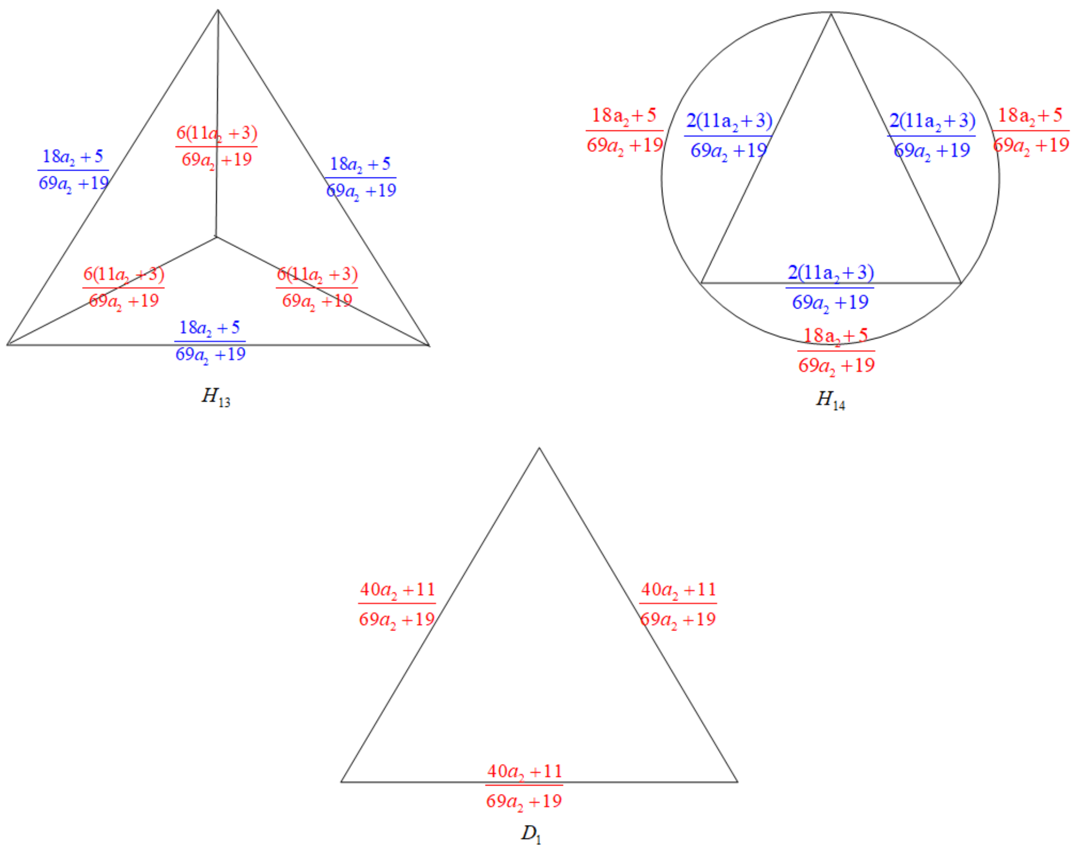


Figure 16

Using the properties given in Section 2, we have the following transformations:

$$\tau(H_1) = 9a_2\tau(\mathcal{D}_2), \qquad \tau(H_2) = \left[\frac{1}{3a_2+1}\right]^3 \tau(H_1),$$

$$\tau(H_3) = \frac{3a_2+1}{9a_2}\tau(H_2), \qquad \tau(H_4) = \tau(H_3),$$

$$\tau(H_5) = 9\left(\frac{4a_2+1}{3a_2}\right)\tau(H_4), \qquad \tau(H_6) = \left[\frac{3a_2+1}{18a_2+5}\right]^3 \tau(H_5),$$

$$\tau(H_7) = \tau(H_6), \qquad \tau(H_8) = \frac{18a_2+5}{18(4a_2+1)}\tau(H_7),$$

$$\tau(H_9) = \tau(H_8), \qquad \tau(H_{10}) = 9\left(\frac{11a_2+3}{18a_2+5)}\right)\tau(H_9),$$

$$\tau(H_{11}) = \left[\frac{18a_2+5}{69a_2+19}\right]^3 \tau(H_{10}), \qquad \tau(H_{12}) = \tau(H_{11}),$$

$$\tau(H_{13}) = \left[\frac{69a_2+19}{18(11a_2+3)}\right]\tau(H_{12}), \quad \tau(\mathcal{D}_1) = \tau(H_{13}).$$

Combining these fourteen transformations, we have

$$\tau\left(\mathcal{D}_2\right)=4\left(69a_2+19\right)^2\tau\left(\mathcal{D}_1\right). \tag{40}$$

Further

$$\tau\left(\mathcal{D}_n\right)=\prod_{i=2}^{n}4\left(69a_i+19\right)^2\tau\left(\mathcal{D}_1\right)=3\times 4^{n-1}a_1^2\left[\prod_{i=2}^{n}\left(69a_i+19\right)\right]^2, \tag{41}$$

where

$$a_{i-1}=\frac{40a_i+11}{69a_i+19}, i=2,3,\ldots,n. \tag{42}$$

Its characteristic equation is $69\lambda^2-21\lambda-11=0$ which have two roots

$$\lambda_1=\frac{21-\sqrt{3477}}{138} \text{ and } \lambda_2=\frac{21+\sqrt{3477}}{138}.$$

Subtracting these two roots into both sides of $a_{i-1}=\frac{40a_i+11}{69a_i+19}$, we get

$$a_{i-1}-\frac{21-\sqrt{3477}}{138}=\frac{40a_i+11}{69a_i+19}-\frac{21-\sqrt{3477}}{138}=(59+\sqrt{3477})\cdot\frac{a_i-\frac{21-\sqrt{3477}}{138}}{2\left(69a_i+19\right)}, \tag{43}$$

$$a_{i-1}-\frac{21+\sqrt{3477}}{138}=\frac{40a_i+11}{69a_i+19}-\frac{21+\sqrt{3477}}{138}=(59-\sqrt{3477})\cdot\frac{a_i-\frac{21+\sqrt{3477}}{138}}{2\left(69a_i+19\right)}. \tag{44}$$

Let $b_i=\frac{a_i-\frac{21-\sqrt{3477}}{138}}{a_i-\frac{21+\sqrt{3477}}{138}}$. Then by Eqs. (42) and (43), we get

$$b_{i-1}=\left(\frac{3479+59\sqrt{3477}}{2}\right)b_i \text{ and } b_i=\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-i}b_n.$$

Therefore,

$$a_i=\frac{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-i}\left(\frac{21+\sqrt{3477}}{138}\right)b_n-\frac{21-\sqrt{3477}}{138}}{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-i}b_n-1}.$$

Thus,

$$a_1=\frac{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-1}\left(\frac{2127+40\sqrt{3477}}{2553}\right)-\left(\frac{21-\sqrt{3477}}{138}\right)}{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-1}\left(\frac{2861+39\sqrt{3477}}{1702}\right)-1}. \tag{45}$$

Using the expression $a_{n-1}=\frac{40a_n+11}{69a_n+19}$ and denoting the coefficients of $40a_n+11$ and $69a_n+19$ as α_n and β_n, we have

$$69a_n+19=\alpha_0\left(40a_n+11\right)+\beta_0\left(69a_n+19\right),$$

$$69a_{n-1}+19=\frac{\alpha_1\left(40a_n+11\right)+\beta_1\left(69a_n+19\right)}{\alpha_0\left(40a_n+11\right)+\beta_0\left(69a_n+19\right)},$$

$$69a_{n-2}+19=\frac{\alpha_2\left(40a_n+11\right)+\beta_2\left(69a_n+19\right)}{\alpha_1\left(40a_n+11\right)+\beta_1\left(69a_n+19\right)},$$

$$\vdots$$

$$69a_{n-i}+19=\frac{\alpha_i\left(40a_n+11\right)+\beta_i\left(69a_n+19\right)}{\alpha_{i-1}\left(40a_n+11\right)+\beta_{i-1}\left(69a_n+19\right)}, \tag{46}$$

$$69a_{n-(i+1)}+19=\frac{\alpha_{i+1}\left(40a_n+11\right)+\beta_{i+1}(69+19)}{\alpha_i\left(40a_n+11\right)+\beta_i\left(69a_n+19\right)}, \tag{47}$$

$$\vdots$$

$$69a_2+19=\frac{\alpha_{n-2}\left(40a_n+11\right)+\beta_{n-2}\left(69a_n+19\right)}{\alpha_{n-3}(40+11)+\beta_{n-3}\left(69a_n+19\right)}.$$

Substituting Eq.(45) into Eq.(41), we obtain

$$\tau\left(\mathcal{D}_n\right)=3\times4^{n-1}a_1^2\left[\alpha_{n-2}\left(40a_n+11\right)+\beta_{n-2}\left(69a_n+19\right)\right]^2, \tag{48}$$

where $\alpha_0=0,\beta_0=1$ and $\alpha_1=69,\beta_1=19$. By the expression $a_{n-1}=\frac{40a_n+11}{69a_n+19}$ and Eqs. (45) and (46), we have

$$\alpha_{i+1}=59\alpha_i-\alpha_{i-1},\quad \beta_{i+1}=59\beta_i-\beta_{i-1}$$

The characteristic equation of Eq.(48) is $\mu^2-59\mu+1=0$ which have two roots

$$\mu_1=\frac{59+\sqrt{3477}}{2}\quad\text{and}\quad\mu_2=\frac{59+\sqrt{3477}}{2}$$

and the general solutions of Eq.(48) are

$$\alpha_i=c_1\mu_1^i+c_2\mu_2^i,\quad \beta_i=d_1\mu_1^i+d_2\mu_2^i.$$

Substituting the initial conditions $\alpha_0=0,\beta_0=1$ and $\alpha_1=69,\beta_1=19$, yields

$$\alpha_i=\frac{23\sqrt{3477}}{1159}\left(\frac{59+\sqrt{3477}}{2}\right)^i-\frac{23\sqrt{3477}}{1159}\left(\frac{59-\sqrt{3477}}{2}\right)^i,$$

$$\beta_i=\left(\frac{1159-7\sqrt{3477}}{2318}\right)\left(\frac{59+\sqrt{3477}}{2}\right)^i+\left(\frac{1159+7\sqrt{3477}}{2318}\right)\left(\frac{59-\sqrt{3477}}{2}\right)^i. \tag{49}$$

If $a_n=1$, it means that $\mathcal{D}_n$ without any electrically equivalent transformation. Plugging Eq. (49) into Eq.(47), we have

$$\begin{aligned}\tau\left(\mathcal{D}_n\right) &= 3\times4^{n-1}a_1^2\left[\left(\frac{50996+865\sqrt{3477}}{1159}\right)\left(\frac{59+\sqrt{3477}}{2}\right)^{n-2}\right.\\ &\left.+\left(\frac{50996-865\sqrt{3477}}{1159}\right)\left(\frac{59-\sqrt{3477}}{2}\right)^{n-2}\right]^2,\ n\geq2.\end{aligned} \tag{50}$$

When $n = 1, \tau(\mathcal{D}_1) = 3$ which satisfies Eq.(50). Therefore the number of spanning trees in the sequence of $\mathcal{D}_n$ graph is given by

$$\begin{aligned}\tau(\mathcal{D}_n) &= 3 \times 4^{n-1} a_1^2 \left[\left(\frac{50996 + 865\sqrt{3477}}{1159} \right) \left(\frac{59 + \sqrt{3477}}{2} \right)^{n-2} \right. \\ &\quad \left. + \left(\frac{50996 - 865\sqrt{3477}}{1159} \right) \left(\frac{59 - \sqrt{3477}}{2} \right)^{n-2} \right]^2, \ n \geq 1, \end{aligned} \tag{51}$$

where

$$a_1 = \frac{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-1} \left(\frac{2127+40\sqrt{3477}}{2553}\right) - \left(\frac{21-\sqrt{3477}}{138}\right)}{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-1} \left(\frac{2861+39\sqrt{3477}}{1702}\right) - 1}, n \geq 1. \tag{52}$$

Inserting Eq.(52) into Eq.(51) we obtain the result. □

§7. Numerical Results

Table 1. illustrates some values of the number of spanning trees in the graphs $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ and $\mathcal{D}_n$.

n	$\tau(\mathcal{A}_n)$	$\tau(\mathcal{B}_n)$	$\tau(\mathcal{C}_n)$	$\tau(\mathcal{D}_n)$
1	3	3	3	3
2	44376	27648	26934	31212
3	732328128	185150208	200050668	434307072
4	12101944579584	1239020203008	1485574848600	6043816558272
5	199991606950244352	8291475833499648	11031866024955312	84105744275374848
6	3304977193903255289856	55486239089142448128	81922542024547792224	1170415440635048951808

§8. Spanning Tree Entropy

After having explicit Formulas for the number of spanning trees of the sequence of the three families of graphs $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ and $\mathcal{D}_n$, we can calculate its spanning tree entropy Z which is a finite number and a very interesting quantity characterizing the network structure, defined as in [14] as

For a graph G,

$$Z(G) = \lim_{n \to \infty} \frac{\ln \tau(G)}{|V(G)|} \tag{53}$$

and particularly,

$$Z(\mathcal{A}_n) = \frac{1}{9}(\ln[8] + 2\ln[23 + 6\sqrt{14}]) = 1.07918497,$$

$$Z(\mathcal{B}_n) = \frac{2}{9}\ln[41 + 3\sqrt{185}] = 0.9787402606,$$

$$Z\left(\mathcal{C}_n\right)=\frac{7\ln[2]}{9}-\frac{2}{9}\ln[61-\sqrt{3705}]=0.9903046082,$$

$$Z\left(\mathcal{D}_n\right)=\frac{2}{9}\ln[59+\sqrt{3477}]=1.060088273$$

Now we compare the value of entropy in our graphs with other graphs. It is clear that the entropy of the $\mathcal{A}_n$ graph is larger than the other three graphs and the entropy of the $\mathcal{B}_n$ graph is smaller than the other three graphs. In addition the entropy of graphs $\mathcal{A}_n$ and $\mathcal{D}_n$ is larger than the fractal scale free lattice [15] which has the entropy 1.040 and the entropy of all four graphs is smaller than two dimensional Sierpinski gasket [16] which has the entropy 1.166 of the same average degree 4.

§9. Conclusions

In this work, we enumerate the number of spanning trees in the sequences of three sequences of graphs of average degree four based on using electrically equivalent transformations. An advantage of this method lies in the avoidance of laborious computation of Laplacian spectra that is needed for a generic method for determining spanning trees.

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